DE GRUYTER

Yasushi Ishikawa

STOCHASTIC CALCULUS OF VARIATIONS

FOR JUMP PROCESSES

2ND EDITION

STUDIES IN MATHEMATICS 54

Yasushi Ishikawa
Stochastic Calculus of Variations

De Gruyter Studies in Mathematics

Volume 54

Yasushi Ishikawa

Stochastic Calculus of Variations

For Jump Processes

2nd edition

DE GRUYTER

Mathematics Subject Classification 2010
60J25, 60J35, 60G51, 60H07

Author
Prof. Dr. Yasushi Ishikawa
Ehime University
Graduate School of Science and Engineering
Mathematics, Physics, and Earth Sciences
Bunkyo-cho 2-chome
790-8577 Matsuyama
Japan
ishikawa.yasushi.my@math.sci.ehime-u.ac.jp

ISBN 978-3-11-037776-7
e-ISBN (PDF) 978-3-11-037807-8
e-ISBN (EPUB) 978-3-11-039232-6
Set-ISBN 978-3-11-037808-5
ISSN 0179-0986

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Typesetting: le-tex publishing services GmbH, Leipzig
Printing and binding: CPI books GmbH, Leck
♾ Printed on acid-free paper
Printed in Germany

www.degruyter.com

Preface

This book is a concise introduction to the stochastic calculus of variations (also known as Malliavin calculus) for processes with jumps. It is written for researchers and graduate students who are interested in Malliavin calculus for jump processes. In this book, 'processes with jumps' include both pure jump processes and jump-diffusions. The author has tried to provide many results on this topic in a self-contained way; this also applies to stochastic differential equations (SDEs) 'with jumps'. This book also contains some applications of the stochastic calculus for processes with jumps to control theory and mathematical finance.

The field of jump processes is quite wide-ranging nowadays, from the Lévy measure (jump measure) to SDEs with jumps. Recent developments in stochastic analysis, especially Malliavin calculus with jumps in the 1990s and 2000s, have enabled us to express various results in a compact form. Until now, these topics have been rarely discussed in a monograph. Among the few books on this topic, we would like to mention Bichteler–Gravereaux–Jacod (1987) and Bichteler (2002).

One objective of Malliavin calculus (of jump type) is to prove the existence of the density function $p_t(x, y)$ of the transition probability of a jump Markov process X_t probabilistically, especially the very important case where X_t is given by a (Itô, Marcus, Stratonovich ...) SDE, cf. Léandre (1988). Furthermore, granting the existence of the density, one may apply various methods to obtain the asymptotic behaviour of $p_t(x, y)$ as $t \to 0$ where x and y are fixed. The results are known to be different, according to whether $x \neq y$ or $x = y$. We also describe this topic.

The starting point for this book was July 2009, when Prof. R. Schilling invited me to the Technische Universität Dresden, Germany, to teach a short course on Malliavin's calculus for jump processes. He suggested that I expand the manuscript, thus creating a book. Prof. H. Kunita kindly read and commented on earlier drafts of the manuscript. The author is deeply indebted to Professors R. Schilling, M. Kanda, H. Kunita, J. Picard, R. Léandre, C. Geiss, F. Baumgartner, N. Privault and K. Taira.

This book is dedicated to the memory of the late Professor Paul Malliavin.

Matsuyama, December 2012 Yasushi Ishikawa

Preface to the second edition

The present edition is an expanded and revised version of the first edition. Several changes have been added. These changes are based on a set lectures given at Osaka city university and the university of Ryukyus, and on seminars in several universities. These lectures were addressed to graduate and undergraduate students with interests in Probability and Analysis.

In 1970s Professor P. Malliavin has begun analytic studies of the Wiener space; he gave a probabilistic proof of the hypoellipticity result in PDE theory due to L. Hörmander. The method he used on the Wiener space is called stochastic calculus of variations, now called Malliavin calculus. The spirit of Malliavin's work was clarified by J.-M. Bismut, S. Watanabe, I. Shigekawa, D. Nualart and others.

In 1980s and 1990s the movement of stochastic calculus of variations on the Wiener space has been extended to the Poisson space by J.-M. Bismut, J. Jacod, K. Bichteler-J. M. Gravereaux-J. Jacod and others. Due to the development of the theory by them, problems concerning integro-differential operators in the potential theory have come to be resolved. The author has encountered with these techniques and the atmospher in Strassburg, Clermont-Ferrand, La Rochelle and Kyoto.

The main purpose of this monograph is to summarize and explain analytic and probabilistic problems concerning jump processes and jump-diffusion processes in perspective. Our approach to those problems relys largely on the recent developments in the stochastic analysis on the Poisson space and that of SDEs on it. Several perturbation methods on the Poisson space are proposed, each resulting in the integration-by-parts formula of its own types.

The analysis of jump processes has its proper value, since processes with discontinuous trajectories are as natural as processes with continuous trajectories. Professor K. Itô has been interested, and has had a sharp sense, in jump processes (especially in Lévy processes) from the period of his inquiry for the theory of stochastic integration. The theory of stochastic calculus of variations for jump processes is still developing. Its application will cover from economics to mathematical biology, although materials for the latter is not contained in this volume.

It is three years since the first edition has appeared. There have been intense activities focused on stochastic calculus for jump and jump-diffusion processes. The present monograph is an expanded and revised version of the first edition. Changes to the present edition are of two types. One is the necessity to correct typos and small errors, and a necessity for a clearer treatment of many topics to improve the expressions or to give sharp estimates. On the other hand, I have included a new material. In Chapter 3 I have added Section 3.6.5 which treats the analysis of the transition density. In particular it includes a recent development on the Hörmander type hypoellipticity problem for integro-differential operators related to jump-diffusion processes. The notes at the end of the volume are also extended.

During the preparation of the 2nd edition I am indebted to many professors and colleagues in contents and in motivation. Especially, I am indebted to Professors H. Kunita, M. Tsuchiya, E. Hausenblas, A. Kohatsu-Higa, A. Takeuchi, N. Privault and R. Schilling.

A major part of this work was done at Ehime university (1999-) with the aid of Grant-in-Aid for General Scientific Research, Ministry of Education, Culture Sports, Science and Technology, Japan (No. 24540176). I take this opportunity to express my sincere gratitude to all those who related.

The year 2015 is the centennial of the birth of late Professor K. Itô. The development of Itô's theory has been a good example of international cooperation among people in Japan, Europe and U.S. I hope, in this way, we would contribute to peace and development of the world in the future.

Matsuyama, October 2015 — Yasushi Ishikawa

Contents

0 Introduction

A theme of this book is to describe a close interplay between analysis and probability via Malliavin calculus. Compared to other books on this subject, our focus is mainly on jump processes, especially Lévy processes.

The concept of (abstract) Wiener space has been well known since the 1970s. Since then, despite a genuine difficulty with respect to the definition of the (abstract) Wiener space, many textbooks have been published on the stochastic calculus on the Wiener space. It should be noted that it is directly connected to Itô's theory of stochastic differential equations, which Itô invented while inspired by the work of A. N. Kolmogorov. Already at this stage, a close relation between stochastic calculus and PDE theory has been recognised through the transition probability $p_t(x, dy)$, whose density $p_t(x, y)$ is the fundamental solution to Kolmogorov's backward equation.

Malliavin calculus started with the paper [159] by P. Malliavin (cf. [160]). One of the motivations of his paper is the problem of hypoellipticity for operators associated with stochastic differential equations of diffusion type. At the beginning, Malliavin's calculus was not very popular (except for his students, and a few researchers such as Bismut, Jacod, Shigekawa, Watanabe and Stroock) due to its technical difficulties.

Malliavin's paper was presented at the international symposium in Kyoto organised by Prof. K. Itô. At that time, a close relation began between P. Malliavin and the Kyoto school of probability in Japan. The outcome was a series of works by Watanabe [217], Ikeda–Watanabe [82], Shigekawa [199], and others.

The relation between Malliavin calculus for diffusion processes and PDEs has been deeply developed by Kusuoka and Stroock [137–139] and others.

On the other hand, Paul Lévy began his study on additive stochastic processes (cf. [152]). The trajectories of his processes are continuous or discontinuous. The additive processes he studied are now called Lévy processes. The discontinuous Lévy processes have an infinitesimal generator of integro-differential type in the semigroup theory in the sense of Hille–Yosida. Such integro-differential operators have been studied in potential theory, by e.g. Ph. Courrège [45] and Bony–Courrège–Priouret [34]. The latter paper is related to the boundary value problem associated with integro-differential operators.

The theory developed following that of Fourier integral operators and pseudodifferential operators (cf., e.g. [35]).

My first encounter with Malliavin calculus for jump processes was the paper by Léandre [140], where he proves

$$\frac{p(t, x, dy)}{t} = \frac{P(X_t \in dy | X_0 = x)}{t} \sim n(x, dy)$$

as $t \to 0$, if the jump process X_t can reach y by one single jump ($y \neq x$). Here, $n(x, dy)$ denotes the Lévy kernel. This result has been generalised to the case of n jumps, $n = 1, 2, \ldots$ in [85], independently of the work by Picard.

When I started my research in this field, I was inspired by the close relation between the theories of integro-differential operators and jump type Markov processes. Consequently, my own research on the short time asymptotics of the transition density plays an important role in this monograph.

Later, Malliavin calculus has found new applications in the theory of finance.

The presentation of the contents follows the historical development of the theory. The technical requirements of this book are usual undergraduate calculus, probability and abstract measure theory.

Historically, the theory was started by Bismut. The approach by Bismut is based on the Girsanov transform of the underlying probability measure. From an analytic point of view, the main idea is to replace the Radon–Nikodým density function in the Girsanov transform of measures induced by the perturbation of the continuous trajectories by that induced by the perturbation of the discontinuous trajectories. Subsequently, the theory was extended to cover singular Lévy measures using perturbation methods (Picard).

Most textbooks on Malliavin calculus on the Wiener space (e.g. [160, 199]) adopt a functional analytic approach, where the abstract Wiener space and the Malliavin operator appear. I do not use such a setting in this book. This is partly because such a setting is not very intuitive, and partly because the setting cannot directly be transferred to the Poisson space from the Wiener space. This is also discussed in Section 3.4.

In the spirit of potential theory and (nonlocal) integro-differential operators, I have adopted the method of perturbations of trajectories on the Wiener–Poisson space. This perspective fits well to the Markov chain approximation method used in Sections 2.2, 2.3, and to the technique of path-decomposition used in Section 3.6. Hence, it constitutes one of the main themes of this book.

In our approach, both in the Wiener space and in the Poisson space, the main characters are the derivative operator D_t or the finite difference operator $\tilde{D}_u$, and their adjoints δ or $\tilde{\delta}$. The derivative operator is defined to act on the random variable $F(\omega)$ defined on a given probability space $(\Omega, \mathcal{F}, P)$.

In the Wiener space, $\Omega = C_0(\mathbf{T})$ is the space of continuous functions defined on the interval $\mathbf{T}$ equipped with the topology given by the sup-norm. The Malliavin derivative $D_t F(\omega)$ of $F(\omega)$ is then given in two ways, either as a functional derivative or in terms of a chaos expansion, see Section 3.1.1 for details. The definition via chaos expansion is quite appealing since it gives an elementary proof of the Clark–Ocone formula, and since the definition can be carried over to the Poisson space in a natural way; details are stated in Section 3.2.

Here is a short outline of all chapters.

Chapter 1

In Chapter 1, I briefly prepare basic materials which are needed for the theory. Namely, I introduce Lévy processes, Poisson random measures, stochastic integrals, stochastic differential equations (SDE) with jumps driven by Lévy processes, Itô processes, canonical processes, and so on. Some technical issues in the stochastic analysis such as Girsanov transforms of measures, quadratic variation, and the Doléans stochastic exponential are also discussed. The SDEs introduced in Section 1.3 are time independent, i.e. of autonomous (or 'Markovian') type.

In this chapter, technical details on materials concerning SDEs are often referred to citations, as our focus is to expose basic elements for stochastic analysis briefly. Especially, for materials and explanations of diffusion processes, Wiener processes, stochastic integrals with respect to Wiener process, readers can refer to [115].

Chapter 2

The main subject in Chapter 2 is to relate the integration-by-parts procedure in the Poisson space with the (classically) analytic object, that is, the transition density function. I present several methods and perturbations that lead to the integration-by-parts formula on the Poisson space. I am particularly interested in the Poisson space since such techniques on the Wiener space are already introduced in the textbooks on Malliavin calculus. The integration-by-parts formula induces the existence of the smooth density of the probability law or the functional. Analytically, 'existence' and 'smoothness' are two different subjects to attack. However, they are obtained simultaneously in many cases. I present several upper and lower bounds of the transition densities associated with jump processes. Then, I explain the methods to find those bounds. The results stated here are adopted from several papers written by R. Léandre, J. Picard and by myself in the 1990s.

One motivation for Sections 2.2, 2.3 and 2.4 is to provide the short time asymptotic estimates for jump processes from the view point of analysis (PDE theory). Readers will find sharp estimates of the densities which are closely related to the jumping behaviour of the process. Here, the geometric perspectives as polygonal geometry and chain movement approximation will come into play. Compared to the estimation of the heat kernels associated with Dirichlet forms of jump type, such piecewise methods for the solution to SDEs will give more precise upper and lower bounds for transition densities. In Section 2.5, I provide some auxiliary materials.

Chapter 3

In Chapter 3, I study the Wiener, Poisson, and Wiener–Poisson space. Here, I use stochastic analysis on the path space. In Section 3.1, I briefly review stochastic calculus on the Wiener space using the Malliavin–Shigekawa's perturbation. I introduce the derivative operator D and its adjoint δ. In Section 3.2, I discuss stochastic calculus on the Poisson space using Picard's perturbation $\tilde{D}$. In Section 3.3, I introduce perturbations on the Wiener–Poisson space, and define Sobolev spaces on the Wiener–Poisson space based on the norms using these perturbations. A Meyer's type inequality for the adjoint operators on this space is explained in detail. This chapter is the main part in the theoretical aspect of stochastic analysis for processes with jumps.

In Sections 3.5 (General theory) and 3.7 (Itô processes), I define the composition $\Phi \circ F$ of a random variable F on the Wiener–Poisson space with a generalised function Φ in the space $\mathcal{S}'$ of tempered distributions, such as $\Phi(x) = (x-K)^+$ or $\Phi(x) = \delta(x)$. These results are mostly new. In Section 3.6, I investigate the smoothness of the density of the processes defined on the Wiener–Poisson space as functionals of Itô processes, and inquire into the existence of the density.

Chapter 4

Chapter 4 is devoted to applications of the material from the previous chapters to problems in mathematical finance and optimal control. In Section 4.1, I explain applications to asymptotic expansions using the composition of a Wiener–Poisson functional with tempered distributions. In Section 4.1.1, I briefly repeat the material on compositions of the type $\Phi \circ F$ given in Sections 3.5, 3.7. Section 4.1.2 treats the asymptotic expansion of the density, which closely relates to the short time asymptotics stated in Sections 2.2, 2.3. In Section 4.2, I give an application to the optimal consumption problem associated to a jump-diffusion process.

I tried to make the content as self-contained as possible and to provide proofs to all major statements (formulae, lemmas, propositions, …). Nevertheless, in some instances, I decided to refer the reader to papers and textbooks, mostly if the arguments are lengthy or very technical. Sometimes, the proofs are postponed to the end of the current section due to the length. For the readers' convenience, I tried to provide several examples. Also, I have repeated some central definitions and notations.

How to use this book?

Your choices are:

- If you are interested in the relation of Malliavin calculus with analysis, read Chapters 1 and 2
- If you are interested in the basic theory of Malliavin calculus on the Wiener–Poisson space, read Chapters 1 and 3
- If you are interested in the application of Malliavin calculus with analysis, read Chapter 4 in reference with Chapter 1.

I hope this book will be useful as a textbook and as a resource for researchers in probability and analysis.

1 Lévy processes and Itô calculus

> Happy families are all alike; every unhappy family is unhappy in its own way.
>
> Lev Tolstoy, Anna Karenina

In this chapter, we briefly prepare the basic concepts and mathematical tools which are necessary for stochastic calculus with jumps throughout this book. We consider Poisson processes, Lévy processes, and the Itô calculus associated with these processes. Especially, we consider SDEs of Itô and canonical type.

We first introduce Lévy processes in Section 1.1. We provide basic materials to SDEs with jumps in Section 1.2. Then, we introduce SDEs for Itô processes (Section 1.3) in the subsequent section. Since the main objective of this article is to inquire into analytic properties of the functionals on the Wiener–Poisson space, not all of the basic results stated in this chapter are provided with full proofs.

Throughout this book, we shall denote the Itô process on the Poisson space by x_t or $x_t(x)$, and the canonical process by Y_t. The expressions X_t, $X(t)$ are used for both cases of the above, or just in the sense of a general Itô process on the Wiener–Poisson space.

1.1 Poisson random measure and Lévy processes

In this section, we recall the basic materials related to Poisson random measure, Lévy processes and variation norms.

1.1.1 Lévy processes

We denote by $(\Omega, \mathcal{F}, P)$ a probability space where Ω is a set of trajectories defined on $\mathbf{T} = [0, T]$. Here, $T \leq \infty$ and we mean $\mathbf{T} = [0, +\infty)$ in the case $T = +\infty$. In most cases, T is chosen finite. However, the infinite interval $\mathbf{T} = [0, +\infty)$ may appear in some cases. $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is a family of σ-fields on Ω, where $\mathcal{F}_t$ denotes the minimal σ-field, right continuous in t, for which each trajectory $\omega(s)$ is measurable up to time t.

Definition 1.1. A Lévy process $(z(t))_{t\in\mathbf{T}}$ on $\mathbf{T}$ is an m-dimensional stochastic process defined on Ω such that[1]

1. $z(0) = 0$ a.s.
2. $z(t)$ has independent increments (i.e. for $0 \leq t_0 < t_1 < \cdots < t_n$, $t_i \in \mathbf{T}$, the random variables $z_{t_i} - z_{t_{i-1}}$ are independent)

---

1 Here and in what follows, a.s. denotes the abbreviation for 'almost surely'. Similarly, a.e. stands for 'almost every' or 'almost everywhere'.

3. $z(t)$ has stationary increments (i.e. the distribution of $z_{t+h} - z_t$ depends on h, but not on t)
4. $z(t)$ is stochastically continuous (i.e. for all $t \in \mathbf{T} \setminus \{0\}$ and all $\epsilon > 0$ $P(|z(t+h) - z(t)| > \epsilon) \to 0$ as $h \to 0$)
5. $z(t)$ has càdlàg (right continuous on $\mathbf{T}$ with left limits on $\mathbf{T} \setminus \{0\}$) paths.

Here, $m \geq 1$. In case $m = 1$, we also call $z(t)$ a real-valued process.

We denote

$$\Delta z(t) = z(t) - z(t-) .$$

The same notation for Δ will be applied for processes $X_t, M_t, x_t, \ldots$ which will appear later.

We can associate the counting measure N to $z(t)$ in the following way: for $A \in \mathcal{B}(\mathbf{R}^m \setminus \{0\})$, we put

$$N(t, A) = \sum_{0<s\leq t} 1_A(\Delta z(s)), \, t > 0 .$$

Note that this is a counting measure of jumps of z in A up to the time t. As the path is càdlàg, for $A \in \mathcal{B}(\mathbf{R}^m \setminus \{0\})$ such that $\bar{A} \subset \mathbf{R}^m \setminus \{0\}$, we have $N(t, A) < +\infty$ a.s.

A random measure on $\mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$ defined by

$$N((a, b] \times A) = N(b, A) - N(a, A) ,$$

where $a \leq b$ and $\mathbf{T} = [0, T]$, is called a *Poisson random measure* if it follows the Poisson distribution with mean measure $E[N((a, b] \times A)]$, and if for disjoint $(a_1, b_1] \times A_1, \ldots, (a_r, b_r] \times A_r \in \mathcal{B}(\mathbf{T} \times (\mathbf{R}^m \setminus \{0\}))$, $N((a_1, b_1] \times A_1), \ldots, N((a_r, b_r] \times A_r)$ are independent.

Proposition 1.1 (Lévy–Itô decomposition theorem, [192] Theorem I.42). *Let $z(t)$ be a Lévy process. Then, $z(t)$ admits the following representation*

$$z(t) = tc + \sigma W(t) + \int_0^t \int_{|z|<1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|\geq 1} zN(dsdz) ,$$

for a.e. ω for all $t \in \mathbf{T}$. Here, $c \in \mathbf{R}^m$, σ is an $m \times m$-matrix, $(W(t))_{t\in\mathbf{T}}$, $W(0) = 0$ is an m-dimensional standard Wiener process, $N(dtdz)$ is a Poisson random measure with the mean measure $\hat{N}(dtdz) = E[N(dtdz)]$, and $\tilde{N}(dtdz) = N(dtdz) - \hat{N}(dtdz)$. Here, processes $W(t)$ and $t \mapsto (\int_0^t \int_{|z|<1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|\geq 1} zN(dsdz))$ are independent. Furthermore, this representation is unique.

A remarkable point of this result is that the definition of Lévy process contains no assertion for the probability law of the process $z(t)$.

By this proposition, $N(.,.)$ derived from $z(t)$ defines a Poisson random measure on $\mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$. Here, we use the notation of stochastic integrals $\int_0^t \int zN(dsdz)$ and $\int_0^t \int z\tilde{N}(dsdz)$. The precise meaning of these integrals is postponed to Section 1.2.

However, it should be noted here that the Wiener process $W(t)$ and the Poisson random measure $N(dtdz)$ are adapted to the original filtration $(\mathcal{F}_t)$ generated by the Lévy process $z(t)$.

We take the mean measure

$$\mu(A) = E[N(1, A)], A \in \mathcal{B}(\mathbf{R}^m \setminus \{0\}) . \tag{1.1}$$

This (deterministic) measure is called the *Lévy measure* associated to z or to N. Note that μ enjoys

$$\int_{\mathbf{R}^m\setminus\{0\}} (1 \wedge |z|^2)\mu(dz) < +\infty . \tag{1.2}$$

The *compensated Poisson random measure* associated to N is defined by

$$\tilde{N}(dtdz) = N(dtdz) - dt\mu(dz) .$$

In particular, if $\mu(dz)$ satisfies

$$\int_{|z|\geq 1} |z|\mu(dz) < +\infty ,$$

then $z(t)$ can be written in the compact form

$$z(t) = tc' + \sigma W(t) + \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} z\tilde{N}(dsdz) ,$$

where $c' = c + \int_{|z|\geq 1} z\mu(dz)$.

We remark that

$$E[\tilde{N}((s, t] \times A_1)\tilde{N}((s, t] \times A_2)] = \hat{N}((s, t] \times (A_1 \cap A_2)) = (t - s)\mu(A_1 \cap A_2)$$

holds due to the independence property ([127] Proposition 2.1).

A measure μ on $\mathbf{R}^m \setminus \{0\}$ is a Lévy measure associated to some Lévy process if and only if it enjoys the property (1.2). Indeed, we have the following Lévy–Khintchine representation.

Proposition 1.2.

1. *Let z be a Lévy process on $\mathbf{R}^m \setminus \{0\}$. Then,*

$$E[e^{i(\xi, z(t))}] = e^{t\Psi(\xi)} , \quad \xi \in \mathbf{R}^m , \tag{1.3}$$

where

$$\Psi(\xi) = i(c, \xi) - \frac{1}{2}(\xi, \sigma\sigma^T\xi) + \int (e^{i(\xi,z)} - 1 - i(\xi, z)1_{\{|z|<1\}})\mu(dz) . \tag{1.4}$$

Here, $c \in \mathbf{R}^m$, $\sigma\sigma^T$ is a nonnegative definite matrix and μ is a measure which satisfies (1.2).

2. *Given* $c \in \mathbf{R}^m$, *a matrix* $\sigma\sigma^T \geq 0$ *and a* σ*-finite measure* μ *on* $\mathcal{B}(\mathbf{R}^m \setminus \{0\})$ *satisfying* (1.2), *there exists a process* z *for which* (1.3) *and* (1.4) *hold. This process* z *is a Lévy process.*

For the proof, we use formulae

$$E\left[e^{i(\xi,W(t))}\right] = e^{-\frac{1}{2}t(\xi,\sigma\sigma^T\xi)}\,,$$

$$E\left[e^{i(\xi,\int_0^t\int_{|z|<1} z\tilde{N}(dsdz))}\right] = \exp t\left[\int\limits_{|z|<1}(e^{i(\xi,z)} - 1 - i(\xi,z))\mu(dz)\right],$$

and

$$E\left[e^{i(\xi,\int_0^t\int_{|z|\geq 1} zN(dsdz))}\right] = \exp t\left[\int\limits_{|z|\geq 1}(e^{i(\xi,z)} - 1)\mu(dz)\right].$$

Please refer to Theorem 8.1 in [196], and Section 0 in [105]. In the above statement, (a, b) denotes the inner product of a and b.

Let $D_p = \{t \in \mathbf{T}; \Delta z(t) \neq 0\}$. Then, it is a countable subset of $\mathbf{T}$ a.s. Let $A \subset \mathbf{R}^m \setminus \{0\}$. In case $\mu(A) < +\infty$, the process $D_p \ni t \mapsto \sum_{s\leq t,\Delta z(s)\in A}\delta_{(s,\Delta z(s))}$ is called a *Poisson counting measure* associated to the Lévy process $z(t)$ (or, the Lévy measure $\mu(dz)$) taking values in A. The function $D_p \ni t \mapsto p(t) = \Delta z(t)$ is called a *Poisson point process* associated to the Lévy process $z(t)$.

The notion of *Poisson point process* is defined in a general setting using point functions. A point function p is a mapping from D_p to $\mathbf{R}^m \setminus \{0\}$, where D_p is a countable subset of $\mathbf{T}$. The function p defines a counting measure N_p on $\mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$ by

$$N_p((0,t] \times A) = \#\{s \in D_p; s \leq t, p(s) \in A\},\ t > 0,\ A \in \mathcal{B}(\mathbf{R}^m \setminus \{0\})\,.$$

A point process p on $\mathbf{R}^m \setminus \{0\}$ is a random variable p on Ω consisting of point functions. A point process p is called Poisson if $N_p(dtdz)$ is a Poisson random measure on $\mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$.

1.1.2 Examples of Lévy processes

1. Poisson process
   A Poisson process N_t with intensity $\lambda > 0$ is a nonnegative integer-valued process defined on $[0, +\infty)$ which satisfies the following conditions:
   (a) $N_0 = 0$, $\Delta N_t = N_t - N_{t-}$ is 0 or 1
   (b) For $s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$.
   (c) For all t_1, t_2 and all $s > 0$, $N_{t_1+s} - N_{t_1}$ has the same distribution as $N_{t_2+s} - N_{t_2}$.
   (d)
   $$P(N_t = k) = \frac{1}{k!}(\lambda t)^k e^{-\lambda t},\ k = 0, 1, 2, \ldots.$$

We can choose a version which has càdlàg paths. In fact, the property (iv) follows from (i) to (iii), cf. [192] Theorem I.23. We put (iv) for simplicity. The Lévy measure μ of the Poisson process is the point mass $\lambda\delta_{\{1\}}$, and $b = 0$, $\sigma = 0$.
A Poisson process appears quite naturally as a counting process of (discrete) events whose waiting times are independent and identically distributed (i.i.d.) random variables with exponential distribution.

2. Compound Poisson process
Consider a compound Poisson process $Y_t = \sum_{k=1}^{N_t} Y_k$, where (Y_k), $k = 1, 2, \ldots$ are i.i.d. random variables with a common finite distribution μ on $\mathbf{R}^m \setminus \{0\}$ and N_t denotes a Poisson process with the intensity $\lambda > 0$, independent of (Y_i). Then, Y_t has a representation
$$Y_t = \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} zN(dsdz) ,$$
where $N(dsdz)$ denotes a Poisson random measure on $\mathbf{T}\times(\mathbf{R}^m \setminus \{0\})$ with the mean measure $\lambda ds\mu(dz)$.
3. Stable process
A Lévy process such that its Lévy measure μ, given by
$$\mu(dz) = c_\alpha \frac{dz}{|z|^{m+\alpha}} ,$$
is called a *symmetric stable process*, where $\alpha \in (0, 2)$.
If the measure μ is given by
$$\mu(dz) = c'_\alpha a\left(\frac{z}{|z|}\right)\frac{dz}{|z|^{m+\alpha}} ,$$
where $a(\cdot)$ is defined on S^{m-1} and $a(\cdot) \geq 0$, the process is called an *asymmetric stable process*. In case $m = 1$, μ takes the form
$$\mu(dz) = (c_- 1_{\{z<0\}} + c_+ 1_{\{z>0\}})\frac{dz}{|z|^{1+\alpha}} ,$$
where $c_- \geq 0$, $c_+ \geq 0$.
4. Wiener process
A continuous process $W(t)$ is called a Wiener process (or Brownian motion) if
(i) for $0 \leq s < t < +\infty$ $W(t) - W(s)$ is independent of $W(s)$,
(ii) for $0 \leq s < t < +\infty$ $W(t) - W(s)$ has a Gaussian random variable with mean zero and variance $(t - s)M$ for a given nonrandom matrix M.
It is called a standard Wiener process if $M = I$ (identity matrix).
A Wiener process $W(t)$ (on another probability space) such that $W(0) = 0$ satisfies the conditions (1–5) of Definition 1.1. Hence, it is a (continuous) Lévy process.
A Wiener process has a scaling property that if $c > 0$, then $c^{-\frac{1}{2}}W(ct)$ is indistinguishable from $W(t)$ in the sense of distribution.

Now, we proceed by presenting Itô's formula.

Proposition 1.3 (Itô's formula (I), [173] Theorems 9.4, 9.5).

1. *Let $X(t)$ be a real-valued process given by*

$$X(t) = x + tc + \sigma W(t) + \int_0^t \int_{\mathbf{R}\setminus\{0\}} \gamma(z)\tilde{N}(dsdz),\ t \geq 0\ ,$$

where $\gamma(z)$ is such that $\int_{\mathbf{R}\setminus\{0\}} \gamma(z)^2\mu(dz) < \infty$. Let $f : \mathbf{R} \to \mathbf{R}$ be a function in $C^2(\mathbf{R})$, and let

$$Y(t) = f(X(t))\ .$$

Then, the process $Y(t)$, $t \geq 0$ is a real-valued stochastic process which satisfies

$$\begin{aligned} dY(t) = {} & \frac{df}{dx}(X(t))c\,dt + \frac{df}{dx}(X(t))\sigma\,dW(t) + \frac{1}{2}\frac{d^2f}{dx^2}(X(t))\sigma^2\,dt \\ & + \int_{\mathbf{R}\setminus\{0\}} \left[f(X(t)+\gamma(z)) - f(X(t)) - \frac{df}{dx}(X(t))\gamma(z)\right]\mu(dz)\,dt \\ & + \int_{\mathbf{R}\setminus\{0\}} \left[f(X(t-)+\gamma(z)) - f(X(t-))\right]\tilde{N}(dt\,dz)\ . \end{aligned}$$

2. *Let $X(t) = (X^1(t), \ldots, X^d(t))$ be a d-dimensional process given by*

$$X(t) = x + tc + \sigma W(t) + \int_0^t \int \gamma(z)\tilde{N}(dsdz)\ , \quad t \geq 0\ .$$

Here, $c \in \mathbf{R}^d$, σ is a $d \times m$-matrix, $\gamma(z) = [\gamma_{ij}(z)]$ is a $d \times m$-matrix-valued function such that the integral exists, $W(t) = (W^1(t), \ldots, W^d(t))^T$ is an m-dimensional standard Wiener process, and

$$\begin{aligned} \tilde{N}(dtdz) = {} & (N_1(dtdz_1) - 1_{\{|z_1|<1\}}\mu(dz_1)dt, \ldots, N_m(dtdz_m) \\ & - 1_{\{|z_m|<1\}}\mu(dz_m)dt)\ , \end{aligned}$$

where N_j's are independent Poisson random measures with Lévy measures μ_j, $j = 1, \ldots, m$. That is, $X^i(t)$ is given by

$$X^i(t) = x_i + tc_i + \sum_{j=1}^m \sigma_{ij}W_j(t) + \sum_{j=1}^m \int_0^t \int_{\mathbf{R}\setminus\{0\}} \gamma_{ij}(z)\tilde{N}_j(dsdz_j)\ , \quad i = 1, \ldots, d\ .$$

Let $f : \mathbf{R}^d \to \mathbf{R}$ be a function in $C^2(\mathbf{R}^d)$, and let

$$Y(t) = f(X(t))\ .$$

Then, the process $Y(t)$, $t \geq 0$ is a real-valued stochastic process which satisfies

$$\begin{aligned} dY(t) = {} & \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(X(t))c_i\, dt + \sum_{i=1}^{d}\sum_{j=1}^{m} \frac{\partial f}{\partial x_i}(X(t))\sigma_{ij}\, dW_j(t) \\ & + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(X(t))(\sigma\sigma^T)_{ij}\, dt \\ & + \sum_{j=1}^{m} \int_{\mathbf{R}\setminus\{0\}} \Big[f(X(t) + \gamma^j(z_j)) - f(X(t)) \\ & \qquad - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(X(t))\gamma_{ij}(z) \Big] \mu_j(dz_j)\, dt \\ & + \sum_{j=1}^{m} \int_{\mathbf{R}\setminus\{0\}} \Big[f(X(t-) + \gamma^j(z)) - f(X(t-)) \Big] \tilde{N}_j\,(dtdz_j)\,. \end{aligned}$$

Here, γ^j denotes the j-th column of the matrix $\gamma = [\gamma_{ij}]$.

For the precise meaning of the stochastic integrals with respect to $dW(t)$ and $\tilde{N}(dtdz)$, see Section 1.2.

Example 1.1. Let $b = 0$, $\gamma(z) = 0$, $\sigma = 1$ and $f(x) = x^2$. Then, Itô's formula leads to

$$\int_{\mathbf{T}} W(t)\, dW_t = \frac{1}{2}(W(T)^2 - T)\,.$$

1.1.3 Stochastic integral for a finite variation process

A Lévy process $z(t)$ is said to have a finite variation if the total variation

$$|z|_t = \sup_{n\geq 1} \sum_{k=1}^{2^n} \left| z\left(\frac{tk}{2^n}\right) - z\left(\frac{t(k-1)}{2^n}\right) \right| \tag{1.5}$$

is finite a.s. on every compact interval of $[0, +\infty)$. If it is not so, the process is said to have infinite variation.

We introduce the Blumenthal–Getoor index of the Lévy process $z(t)$ by

$$\beta = \inf \left\{ \delta \geq 0;\ \int_{|z|\leq 1} |z|^{\delta} \mu(dz) < +\infty \right\}\,.$$

The index takes values in $[0, 2]$. It is known (cf. [40, 81]) that if $\beta < 1$, then z has a finite variation path a.s., and if $\beta > 1$, then z has an infinite variation path a.s.

We shall define the stochastic integral $\int_s^t f(u, \omega) dz(u, \omega)$ first for the finite variation process and then for the infinite variation process, where f is a bounded, jointly measurable function.

Definition of stochastic integral for finite variation *(FV)* processes

For a *FV* process $z(t)$, we define

$$I(t,\omega) = \int_s^t f(u,\omega)dz(u,\omega)$$

as a Lebesgue–Stieltjes integral (ω-wisely, a.s.), where f is bounded and jointly measurable.

This is a Stieltjes integral of f by $dz(u)$ given ω-wise. In case that $u \mapsto f(u)$ has a continuous path a.s., it is called a Riemann–Stieltjes integral.

If $z(t)$ is a *FV* process and if $f(.,\omega)$ is differentiable a.s., then we have, in fact the usual rule of the change of variables:

$$f(z(t)) - f(z(0)) = \int_0^t f'(z(s))dz(s) \ a.s.$$

For the integral using the infinite variation process, we need the predictable property for $u \mapsto f(u,.)$ and that of semimartingales. For these, see Section 1.2.2.

In the following sections of this chapter, we use the notion of a stochastic differential equation (SDE) with respect to the Lévy process $z(t)$. The precise definition and the properties of the solution are postponed to the next section.

Due to a recent development by T. Lyons [156, 157], there is a possibility to define "stochastic integrals" ω-wisely by using the (iterated) Young integrals of processes of finite or infinite variation not using the integration by semimartingales. The theory is called the rough path theory, and it uses the notion of p-variation norm and spaces. See also [53].

Indeed, similarly to (1.5), we can define the p-variation

$$|z|_t^{(p)} = \sup_{n\geq 1} \sum_{k=1}^{2^n} \left| z\left(\frac{tk}{2^n}\right) - z\left(\frac{t(k-1)}{2^n}\right)\right|^p \tag{1.6}$$

for $p \geq 1$. All components in the series

$$(1, z^1(t), z^2(t), \ldots) \tag{1.7}$$

of the iterated integrals $z^k(t) = \int_{0\leq u_1<\ldots<u_k<t} dz(u_1) \otimes \ldots \otimes dz(u_k)$, $k = 1, 2, \ldots$ are measured by the p-variation norm. The series, viewed as multiplicative functionals of t, are called rough paths. The stochastic integral

$$\int_0^t h(s)dz(s)$$

with respect to the integrator $dz(s)$ of finite p-variation can be embedded into the theory of integration using rough paths.

The space D

Let $\mathbf{T} = [0, T]$, $T < +\infty$. D = $D(\mathbf{T})$ denotes the space of all functions defined on $\mathbf{T}$ with values in $\mathbf{R}^m$ or $\mathbf{R}^d$ that are right continuous on $[0, T)$ and have left limits on $(0, T]$ (càdlàg paths). All discontinuities of an element f in D are of the first kind. Further, for any element f in D, it has at most countably many discontinuities. See [28] Lemma 12.1.

We introduce a topology on $D(\mathbf{T})$ by introducing the Skorohod metric d_T defined by

$$d_{\mathbf{T}}(f, g) = \inf_{\tau} \sup_{t \in \mathbf{T}} \{|f(t) - g(\tau(t))| + |\tau(t) - t|\} ,$$

where τ moves over all strictly increasing, continuous mappings of $\mathbf{T}$ to $\mathbf{T}$ such that $\tau(0) = 0$, $\tau(T) = T$. The topological space $(D(\mathbf{T}), d_{\mathbf{T}})$ is called a *Skorohod space*. The space $(D(\mathbf{T}), d_{\mathbf{T}})$ is separable, and by choosing an equivalent metric $d^{\circ}_{\mathbf{T}}$ it is complete ([28] Section 12).

$D([0, +\infty))$ denotes the space of all càdlàg paths on $[0, +\infty)$. It is a Fréchet space metrisable with the metric

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(1 \wedge d^{\circ}_{[0,n]}(f, g)\right) ,$$

and the topological space $(D([0, +\infty)), d)$ is complete and separable. Here $d^{\circ}_{[0,n]}$ denotes $d^{\circ}_{\mathbf{T}}$ with $\mathbf{T} = [0, n]$.

If we adopt the sup-norm (as in [104])

$$d^{\infty}(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(1 \wedge \sup_{s \in [0,n]} |f(s) - g(s)|\right) ,$$

the space $(D([0, +\infty)), d^{\infty})$ is complete, but it is *not* separable. We will encounter this space again in the next section and in Section 2.5.3.

1.2 Basic materials for SDEs with jumps

In this section, we study stochastic differential equation (SDE) with jumps. We begin with the definitions of the martingale, semimartingale, and the stochastic integral. We treat usual (Itô's) SDEs and Marcus' canonical SDEs. Solutions to these two SDEs are different from each other. The "canonical" integral is introduced by Marcus [164], and has been developed by Kurtz, Pardoux and Protter [134].

1.2.1 Martingales and semimartingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $(\mathcal{F}_t)_{t \in \mathbf{T}}$ of sub σ-fields of $\mathcal{F}$ is called a *filtration* if $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s < t$. A filtration $(\mathcal{F}_t)_{t \in T}$ is said to satisfy the *usual conditions* if $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$ and if it is right continuous. Below, we consider probability spaces equipped with filtrations which satisfy the usual conditions.

A stochastic process $(X_t)_{t\in\mathbf{T}}$ is said to be *adapted* if X_t is $\mathcal{F}_t$-measurable for all t. It is called *progressively measurable* if the function $X_. : [0,t] \times \Omega$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable for all $t \geq 0$. A progressively measurable process is adapted. Conversely, in case $t \mapsto X_t$ is a càdlàg process, if $(X_t)_{t\in\mathbf{T}}$ is adapted then it is progressively measurable. ([14] Exercise 1.3.)

An adapted process M_t having càdlàg (right continuous with left limit) paths is called a *martingale* if it satisfies the following two conditions:

$$(1) \quad M_t \in L^1(P),\, t \in \mathbf{T} \tag{2.1}$$

$$(2) \quad \text{if} \quad s \leq t \quad \text{then} \quad E[M_t|\mathcal{F}_s] = M_s,\ \text{a.s.}, \quad s,t \in \mathbf{T}\,. \tag{2.2}$$

In case that,

$$\text{if} \quad s \leq t \quad \text{then} \quad E[M_t|\mathcal{F}_s] \geq M_s,\ \text{a.s.}, \quad s,t \in \mathbf{T}\,,$$

X_t is called a *submartingale*. In case that

$$\text{if} \quad s \leq t \quad \text{then} \quad E[M_t|\mathcal{F}_s] \leq M_s,\ \text{a.s.}, \quad s,t \in \mathbf{T}\,,$$

X_t is called a *supermartingale*.

A random variable $\mathrm{T} : \Omega \to [0,+\infty]$ is said to be a *stopping time* if the event $\{\mathrm{T} \leq t\} \in \mathcal{F}_t$ for every $t \in \mathbf{T}$. The set of all stopping times is denoted by $\mathcal{T}$. Let $0 = \mathrm{T}_0 \leq \mathrm{T}_1 \leq \ldots \leq \mathrm{T}_n \leq \ldots$ be a sequence of stopping times such that $\mathrm{T}_n \to +\infty$ a.s. An adapted process M_t such that, for some sequence of stopping times as above, $M_{t\wedge \mathrm{T_n}}$ is a martingale for any n is called a *local martingale*. A martingale is a local martingale.

A process X_t is called a *semimartingale* if it can be written as

$$X_t = X_0 + M_t + A_t\ ,$$

where M_t is a local martingale with $M_0 = 0$ and A_t is an adapted càdlàg process of finite variation (FV) with $A_0 = 0$, a.s. In particular, an adapted FV process having a càdlàg path is a semimartingale. A martingale is a semimartingale.

The one-dimensional standard Poisson process N_t is a semimartingale since it can be written as

$$N_t = \tilde{N}_t + t\ ,$$

where $\tilde{N}_t = N_t - t$ is a local martingale, even a martingale.

For a pure jump Lévy process $z(t)$, the compensated Lévy process $\tilde{z}(t) = z(t) - \int_0^t ds \int z\mu(dz)$ is a local martingale if $\int |z|\mu(dz) < +\infty$.

A Lévy process is a semimartingale by the Lévy–Itô decomposition theorem.

If X is a martingale, then there exists a unique modification Y of X which is càdlàg. See the Corollary in [192] Chapter I.2.

1.2.2 Stochastic integral with respect to semimartingales

We would like to define the stochastic (Itô) integral $\int_s^t h(s)dX_u$ by

$$\int_s^t h(u)dX_u = \int_s^t h(s)dM_u + \int_s^t h(s)dA_u \ ,$$

where $h(u)$ is a (locally) bounded predictable process.

A process h is said to be *predictable* if it is measurable with respect to the σ-field $\mathcal{P}$ on $\Omega \times \mathbf{R}_+$. Here, $\mathcal{P}$ denotes the σ-field generated by adapted processes whose trajectories are left continuous with right limits.

As $t \mapsto A_t$ is a finite variation process, the second term on the right-hand side above is given as in the beginning of Chapter 1 (Section 1.1.3). We shall define $\int_s^t h(s)dM_u$ in what follows.

We denote by $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_n$ a sequence of times. An *elementary process* $h(t)$ is a process

$$h(t) = h_0 1_{\{0\}}(t) + \sum_{i=0}^{n-1} h_i 1_{(t_i, t_{i+1}]}(t) \ ,$$

where h_i is $\mathcal{F}_{t_i}$-measurable and $|h_i| < +\infty$ a.s. We denote by $\mathbf{S}$ the set of elementary processes, endowed with the topology given by the uniform convergence in (t, ω). We define the integral $I(h)$ of an elementary process $h \in \mathbf{S}$ with respect to the martingale M having càdlàg path by

$$I(h)(t) = h(0)M_0 + \sum_{i=0}^{n} h_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}) \ .$$

$I(h)$ is called the stochastic integral of h with respect to M.

Then, $I(h)$ has the following properties:

1. If $M_t = W(t)$ (Brownian motion) or $M_t = \tilde{N}_t$ (compensated Poisson process), then $I(h)(t)$ is a martingale. That is,
$$E[I(h)(t)|\mathcal{F}_s] = I(h)(s), s \le t \ .$$
2. $I^2(h)(t) - \int_0^t h^2(s)d[M]_s$ is a martingale.
3. $E[I^2(h)(t)] = E[\int_0^t h^2(s)d[M]_s]$.

Here, $[M]$ denotes the quadratic variation of M (see just below for the definition). For the proof of (1), see [190] Proposition 2.5.7. It follows from (3) that $h \mapsto I(h)$ extends to an isometry from the space of elementary processes equipped with the norm on (progressively measurable) adapted processes in $L^2(\Omega \times [0, +\infty), P \times d[M]_s)$ into $L^2(\Omega, \mathcal{F}, P)$.[2] We state this more precisely in (i)–(iii) below.

2 Two Ps in the two L^2 spaces are distinct. Here, we use the same symbol, supposing that no confusion occurs.

1. We denote by Λ the set

$$\Lambda = \{h \in L^2(\Omega \times [0,+\infty), P \times d[M]_s) ;\ \text{there exists a sequence of elementary functions } h_n \text{ such that } h_n \to h \text{ in } L^2(\Omega \times [0,+\infty), P \times d[M]_s)\} .$$

We can then define for $h \in \Lambda$, that is,

$$I(h) = \lim_{n \to +\infty} I(h_n), I_t(h) = I(h \cdot 1_{[0,t]}) .$$

We denote by **D** the space of adapted processes with càdlàg paths with the Skorohod topology. It can be observed that the process $I(h)$ for $h \in \mathbf{S}$ takes values in **D**.

It is known (cf. [47]) that Λ contains all predictable processes h such that $E[\int_0^\infty h^2(s)d[M]_s] < +\infty$. Hence, for $h \in \Lambda$, the previous properties (1–3) for $I(h)$ hold true.

2. More precisely, we first extend $I(\cdot) : \mathbf{S} \to \mathbf{D}$ to $I(\cdot) : \mathbf{L} \to \mathbf{D}$, where **L** denotes the space of adapted processes with càglàd paths (left continuous paths with right limits) endowed with the topology given by the uniform convergence in probability on compact sets (*ucp*-topology, for short). Here, we say a sequence (h_n) converges to h in the ucp-topology if for each $t > 0$,

$$\sup_{0 \le s \le t} |h_n(s) - h(s)| \to 0$$

in probability.

For the proof of this extension, we use the fact that the elements in **S** are dense in **L** in the ucp-topology, that bounded elements in **L** are dense in **L**, and that the bounded elements in **L** can be approximated by the bounded elements in **S** in the ucp-topology ([192] Theorems II.10, II.11.).

We then extend $I(.) : \mathbf{L} \to \mathbf{D}$ to $I(.) : \Lambda \to \mathbf{D}$. The map I is well-defined for each $h \in \Lambda$.

3. In case $M_t = W(t)$ or $M_t = \tilde{N}_t$, we extend $I(.)$ thus obtained to $I(.) : L^2(\Omega \times [0,+\infty), P \times d[M]_s) \to \mathbf{D}$ by using the L^2-isometry (3) above. Here, we use the fact that Λ is dense in $L^2(\Omega \times [0,+\infty), P \times d[M]_s)$ and the bounded convergence theorem.

To prove this statement, we first approximate an element in $L^2(\Omega \times [0,+\infty), P \times d[M]_s)$ by a sequence of bounded adapted processes in the L^2-norm, and then we approximate the bounded adapted process by a sequence of elementary processes in the L^2 norm. For the precise argument, see [190] Proposition 2.5.3. Indeed, integrals of predictable and adapted versions coincide with each other. The extension $I(h)$ does not depend on the choice of the sequences. To prove this, we need the completeness of space of the square integrable martingales.

The extension $I : L^2(\Omega \times [0,+\infty), P \times d[M]_s) \to \mathbf{D}$ thus obtained is called the *stochastic integral*.

Quadratic variation process

We introduce the quadratic variation process that was previously mentioned. The following result is due to P.-A. Meyer:

Theorem 1.1 ([192] Theorem III.11).
1. *Let X be a càdlàg supermartingale with $X_0 = 0$ such that the class of random variables $\{X_\tau; \tau \in \mathcal{T}\}$ is uniformly integrable. Then, there exists a unique, increasing, predictable process A with $A_0 = 0$ and a uniformly integrable martingale M_t such that $X_t = M_t - A_t$.*
2. *Let X be a càdlàg submartingale with $X_0 = 0$ such that the class of random variables $\{X_\tau; \tau \in \mathcal{T}\}$ is uniformly integrable. Then, there exists a unique, increasing, predictable process A with $A_0 = 0$ and a uniformly integrable martingale M_t such that $X_t = M_t + A_t$.*

Remark 1.1. We have a similar decomposition

$$X_t = M_t - A_t \quad (\text{resp.} \quad X_t = M_t + A_t)$$

without the above mentioned uniform integrability condition, but with replacing M_t to be a *local martingale* ([192] Theorem III.16).

Let M_t be a square integrable martingale, null at 0 and bounded in L^2. Then, by Doob's inequality,

$$E\left[\sup_t M_t^2\right] \leq 4E\left[M_\infty^2\right] < +\infty .$$

Hence, M_t^2 is a submartingale which satisfies the above property. By Theorem 1.1, there exists a unique, increasing, predictable process A with $A_0 = 0$ such that

$$X_t = M_t + A_t .$$

Proposition 1.4 (cf. [194] Theorem IV.26). *Let M be a square integrable martingale such that $M_0 = 0$. Then, there exists a unique increasing process $[M]$, $[M]_0 = 0$, such that*
1. *$M^2 - [M]$ is a uniformly integrable martingale,*
2. *$\Delta[M]_t = (\Delta M_t)^2$ for any t.*

The process $[M]$ is called the *quadratic variation process* of M. An intuitive meaning of $[M]$ is given by

$$[M]_t = \lim_{n\to+\infty} \sum_i \left\{M_{t_i^n} - M_{t_{i-1}^n}\right\}^2$$

where $t_i^n = t \wedge \frac{i}{2^n}$.

By the two decompositions above, we see that A_t in Theorem 1.1(2) coincides with the compensator of $[M]_t$. Namely, the compensator is a predictable FV process, null at 0, such that $[M]_t - A_t$ is a local martingale. We write it by $\langle M\rangle_t$, and call it the conditional quadratic variation, or simply *angle bracket* of M_t.

The processes $[M]_t$ and $\langle M\rangle_t$ coincide if $t \mapsto M_t$ is continuous a.s.

We can decompose

$$M = M^c + M^d ,$$

where M^c is the continuous part and M^d is the purely discontinuous part. The quadratic variation process $[M]_t$ can be decomposed into continuous and discontinuous parts by

$$[M]_t = [M^c]_t + \sum_{0\le s\le t} (\Delta M_s)^2 .$$

Hence, we can decompose

$$[M] = [M]^c + [M]^d .$$

Here, $[M]^c = [M^c]$ and $[M]^d = [M^d]$, where $[M^d] = \sum_{0\le s\le t}(\Delta M_s)^2$.

The property (2) of $I(h)$ above implies that $[I^2(h)]_t = \int_0^t h^2(t)d[M]_t$.

For square integrable martingales M, N such that $M_0 = 0$ and $N_0 = 0$, the quadratic covariance process $[M, N]$ is given by

$$[M, N] = \frac{1}{4}([M + N] - [M - N]) . \tag{2.3}$$

Using this notation, we have

$$[M, N] = [M, N]^c + [M, N]^d .$$

For a continuous semimartingale $X = X_0 + M + A$, we have $[X] = [M]$ since $[A] = 0$. For a general case we put

$$[X]_t = [X^c]_t + \sum_{0\le s\le t} (\Delta X_s)^2 .$$

For two semimartingales X, Y, we put

$$[X, Y] = \frac{1}{4}([X + Y] - [X - Y])$$

as above. An important property for $[X, Y]$ is $d[X, Y] = 0$ if X is a continuous semimartingale and Y is a continuous process of finite variation ([194] p.62).

Let ${}^p\langle M\rangle_t$ be a predictable (previsible) process

$${}^p\langle M\rangle_t = E[[M].|\mathcal{F}_{t-}] .$$

It is called previsible projection of $[M]$. The dual previsible projection $\langle M\rangle_t$ of $[M]$ is similarly defined. See [194] Section VI.21. $\langle M\rangle_t$ is simply called the angle barcket process of M. If M is continuous then $[M]$ and $\langle M\rangle$ coincide with each other.

By the above definition (2.3) and since $I(h)^2(t) - \int_0^t h^2(s)d[M]_s$, $I(g)^2(t) - \int_0^t g^2(s)d[M]_s$ are uniformly integrable martingales, we have

$$[I(h), I(g)]_t = \int_0^t h(s)g(s)d[M]_s$$

(cf. [47] Chapter 3).

In summary, we have the following properties of the stochastic integral $I(h)$, $h \in L^2(\Omega \times [0,+\infty), P \times d[M]_s)$.

Properties of the stochastic integral $I(h)$:

1. For constants α, β and $h, g \in L^2(\Omega \times [0,+\infty), P \times d[M]_s)$,
$$I(\alpha h + \beta g) = \alpha I(h) + \beta I(g), \text{ a.e.}$$
2. For $0 \le t_1 \le t_2 \le t_3$
$$\int_{t_1}^{t_3} h(t)\, dM_t = \int_{t_1}^{t_2} h(t)\, dM_t + \int_{t_2}^{t_3} h(t)\, dM_t \ .$$
3. $t \mapsto I(h)(t)$ is an adapted process.
4. $I(h)(0) = 0$ a.s.
5. If $M_t = W(t)$ or $M_t = \tilde{N}_t$, then $t \mapsto I(h)$ is a martingale, and hence
$$E[I(h)(t)] = 0\ , \quad t > 0\ .$$
6. $$[I(h)]_t = \int_0^t |h(s)|^2 d[M]_s\ , \quad [I(h), I(g)]_t = \int_0^t h(s)g(s)d[M]_s\ .$$

Below, up to the end of this subsection, $(\mathcal{F}_t)$ denotes the filtration generated by the Lévy process $z(t)$ satisfying the usual conditions.

We can introduce the integral with respect to $\tilde{N}$ in terms of the z variable (Poisson random measure) by introducing that by elementary Poisson measures. See [127] Section 2.1. Then, for

$$I(\varphi) = \int \varphi(z)\tilde{N}((s,t] \times dz)\ , \quad I(\psi) = \int \psi(z)\tilde{N}((s,t] \times dz)\ ,$$

we have

$$\langle I(\varphi), I(\psi)\rangle_t = (t-s)\int \varphi(z)\psi(z)\mu(dz)\ .$$

Here, φ and ψ are measurable functions such that

$$\int (|\varphi(z)|^2 + |\psi(z)|^2)\mu(dz) < +\infty\ .$$

We can define an integral

$$\int \int_0^t h(s,z)\tilde{N}(dsdz)$$

by starting from simple predictable processes

$$h(t,z) = \sum_i \psi_i(z) 1_{(t_i,t_{i+1}]}(t) ,$$

and then approximating a measurable $h(t,z)$ such that

$$E\left[\int_0^t \int |h(s,z)|^2 ds\mu(dz)\right] < +\infty . \tag{2.4}$$

Indeed, first we remark

$$\begin{aligned} E\left[\left(\int_s^t \int \psi(z)\tilde{N}(drdz)\right)^2 \middle| \mathcal{F}_s\right] &= \int_s^t \int \psi^2(z)\hat{N}(drdz) \\ &= (t-s)\int \psi^2(z)\mu(dz) , \end{aligned} \tag{2.5}$$

if $\psi(z)$ is $\mathcal{F}_s$-measurable.

For $h(t,z) = \sum_i \psi_i(z)1_{(t_i,t_{i+1}]}(t)$, where ψ_i are $\mathcal{F}_{t_i}$-measurable, we write

$$\int_0^t \int h(s,z)\tilde{N}(dsdz) = \sum_i \psi_i(z)(\tilde{N}((0,t_{i+1}\wedge t]\times dz) - \tilde{N}((0,t_i \wedge t]\times dz)) . \tag{2.6}$$

Then, by (2.5),

$$E\left[\left[\int_0^{\cdot} \int h(s,z)\tilde{N}(ds\,dz)\right]_t\right] = E\left[\int_0^t \int h^2(s,z)\hat{N}(dsdz)\right] .$$

This implies the L^2-isometry

$$E\left[\left|\int_0^t \int h(s,z)\tilde{N}(dsdz)\right|^2\right] = E\left[\int_0^t \int h^2(s,z)\hat{N}(ds\,dz)\right] .$$

We denote by $L^2(\hat{N})$ the set of all predictable functionals $h(s,z)$ satisfying the condition (2.4). The next assertion follows by the standard argument.

Proposition 1.5. *Simple predictable processes h with the property* (2.4) *are dense in* $L^2(\hat{N})$.

By this proposition, any element h in $L^2(\hat{N})$ can be approximated by the stochastic integral of the form (2.6) in $L^2(\hat{N})$.

We have the following martingale representation theorem due to Kunita–Watanabe. This result can be compared in the world of L^2-martingales with the Lévy–Itô decomposition theorem (Proposition 1.1) in the world of Lévy processes.

We assume $m = 1$ for simplicity.

Theorem 1.2 (Kunita–Watanabe representation theorem, cf. [121, 192] Theorem IV.43). *Let M_t be a locally square integrable martingale defined on $(\Omega, \mathcal{F}, P)$. Then, there exist predictable, square integrable processes $\phi(t)$, $\psi(t,z)$ such that*

$$M_t = M_0 + \int_0^t \phi(s)dW(s) + \int_0^t \int \psi(t,z)d\tilde{N}(dsdz) .$$

In the above assertion, we take $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$, where $\mathcal{F}_t$ is the minimal sub σ-field on which $W(s)$ and $\tilde{N}((0,s] \times E)$, $s \le t$ are measurable for each $E \subset \mathbf{R}^m \setminus \{0\}$.

Itô's formula for the Lévy process (Proposition 1.3) can be extended to the following form.

Theorem 1.3 (Itô's formula (II), [192] Theorems II.32, II.33).

1. *Let $X(t)$ be a real-valued semimartingale and let f be a C^2 function on $\mathbf{R}$. Then, $f(X(t))$ is a semimartingale, and it holds that*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s-))dX(s) + \frac{1}{2}\int_0^t f''(X(s-))d[X,X]_s^c + \sum_{0<s\le t} [f(X(s)) - f(X(s-)) - f'(X(s))\Delta X(s)] .$$

2. *Let $X(t) = (X^1(t), \dots, X^d(t))$ be a d-dimensional semimartingale, and let $f : \mathbf{R}^d \to \mathbf{R}$ be a function in $C^2(\mathbf{R}^d)$. Then,*

$$Y(t) = f(X(t))$$

is a semimartingale, and the following formula holds:

$$\begin{aligned} Y(t) - Y(0) = & \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X(s-))dX^i(s) \\ & + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X(s-))d[X^i, X^j]_s^c \\ & + \sum_{0<s\le t}\left[f(X(s)) - f(X(s-)) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X(s-))\Delta X^i(s)\right] . \end{aligned}$$

It gives theoretically a good perspective if we rewrite the formula in the following form, in view of $[X,X] = [X,X]^c + [X,X]^d$. We give it in one dimensional case (1):

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s-))dX(s) + \frac{1}{2}\int_0^t f''(X(s-))d[X,X]_s + \sum_{0<s\le t} [f(X(s)) - f(X(s-)) - f'(X(s))\Delta X(s) - \frac{1}{2}f''(X(s-))(\Delta X(s))^2] .$$

1.2.3 Doléans' exponential and Girsanov transformation

The above formula can be regarded as an equation with respect to the process $X(t)$. Such an equation is called a stochastic differential equation (SDE).

A typical example is Doléans–Dade (local martingale) exponential to the Doléans' equation

$$X_t = 1 + \int_0^t X_{s-} dM_s ,$$

where M is a local martingale, $M_0 = 0$. The solution is

$$\mathcal{E}(M)_t = \exp\left\{M_t - \frac{1}{2}[M^c, M^c]_t\right\} \Pi_{s\leq t}(1 + \Delta M_s) \exp(-\Delta M_s) .$$

If M is a locally square integrable martingale such that $\Delta M_t > -1$ and if it holds that

$$E\left\{\exp\left(\frac{1}{2}\langle M^c\rangle_T + \left[\widetilde{\sum_{t\leq T} f(\Delta M_t)}\right]\right)\right\} < +\infty , \tag{2.7}$$

then $\mathcal{E}(M)_t$ is a martingale for $t \in \mathbf{T}$. Here, $f(x) = (1 + x)\log(1 + x) - x$ and $\tilde{\varphi}$ denotes the compensator of φ. This result is due to Lépingle and Mémin [151] Theorem III.1. In case $(\widetilde{\sum_{t\leq T} f(\Delta M_t)}) = 0$, this condition is called the Novikov condition. We remark the use of the conditional quadratic variation in the condition.

In particular, if $M_t = \lambda W(t)$, $\lambda \in \mathbf{R}$, then

$$\mathcal{E}(\lambda W)_t = \exp\left(\lambda W(t) - \frac{\lambda^2}{2} t\right) .$$

Then, $t \mapsto \mathcal{E}(\lambda W)_t$ is again a local martingale.

Girsanov transformation of measures, martingale exponential (Change of variables)
The following Girsanov transformation of the underlying probability measures plays an essential role in the perturbation of trajectories in Section 2.1.1. We assume the processes are real-valued (cf. [151], see also [173] Theorem 12.21, [66] p. 149).

Theorem 1.4.

1. *Let $\theta(t, z)$, $t \in \mathbf{T}$, $z \in \mathbf{R} \setminus \{0\}$ be a predictable process such that $\theta(t, z) < 1$, and*

$$\int_{\mathbf{T}} \int \{|\log(1 - \theta(t, z))|^2 + \theta(t, z)^2\} \, dt\mu(dz) < +\infty ,$$

and let $u(t)$, $t \in \mathbf{T}$ be a predictable process such that $\int_{\mathbf{T}} u(t)^2 dt < +\infty$.

We put

$$Z_t = \exp\left\{ \int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds + \int_0^t \int \log(1-\theta(s,z))\tilde{N}(dsdz) + \int_0^t \int \{(\log(1-\theta(s,z)) + \theta(s,z))\}ds\mu(dz) \right\} .$$

We further assume that it holds the Lépingle–Mémin condition:

$$E\left[\exp\left(\frac{1}{2}\int_{\mathbf{T}} u^2(t)dt + \int_{\mathbf{T}}\int \left\{(1-\theta(t,z))\log(1-\theta(t,z)) + \theta(t,z)\right\}dt\mu(dz)\right)\right] < +\infty . \quad (2.8)$$

Then, $t \mapsto Z_t$ is a martingale, and

$$E[Z_T] = 1 .$$

Hence $Q(A) = E[1_A . Z_T]$ defines a probability measure on $(\Omega, \mathcal{F})$ such that $Q(A) = E[1_A . Z_t]$ for $A \in \mathcal{F}_t$. That is,

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = Z_t , \quad t > 0 .$$

2. *Let*

$$\tilde{N}_1(dtdz) = \theta(t,z)dt\mu(dz) + \tilde{N}(dtdz)$$

and

$$dW_1(t) = -u(t)dt + dW(t) .$$

Then, $\tilde{N}_1$ is a martingale counting measure with respect to Q, and W_1 is a continuous martingale with respect to Q.

Remark 1.2.

1. To see Z_t is a martingale, we use the Lépingle–Mémin result above. The condition (2.7) is implied by our assumption (2.8). We put

$$U(t) = \int_0^t u(s)dW(s) - \int_0^t \int \theta(s,z)\tilde{N}(dsdz) .$$

Then, $\Delta U(t) = -\theta(t,z) > -1$. We can show by Itô's formula with $f(x) = e^x$ that

$$dZ_t = Z_{t-}dU(t)$$

with $Z_0 = 1$. Hence,

$$Z_t = \mathcal{E}(U)_t$$

by the uniqueness of the Doléans' exponential.

If we assume a uniformity condition in t, that is,

$$|u(t)| \le C \quad \text{and} \quad |\theta(t,z)| \le C|z| \ , \ z \in \text{supp } \mu$$

for some $C > 0$, then Z_t is a *positive martingale*.

2. In Section 2.1.1 below, we introduce a perturbation method using the Girsanov transform of measures. Bismut [30] used the expression for Z_t in terms of SDE above, whereas Bass et al. [15] used the expression $\mathcal{E}(M)_t$.
3. Under the assumption that $(\mathcal{F}_t)$ satisfies the usual conditions, $\tilde{N}_1 = \tilde{N}_t|_{t=1}$ is not necessarily a compensated Poisson random measure with respect to Q, and W_1 is not necessarily a Brownian motion with respect to Q (cf. [26] Warning 3.9.20).

Let $z(t)$ be a one-dimensional Lévy process with the Lévy measure $\mu(dz)$: $z(t) = \int_0^t \int z\tilde{N}(dsdz)$ being a martingale. Here, we assume that supp μ is compact. We put $\theta(t,z) = 1 - e^{\alpha(t)\cdot z}$ where $\alpha(t)$ is fixed below. Then,

$$\begin{aligned}\log(1-\theta(t,z)) + \theta(t,z) &= \alpha(t)\cdot z + 1 - e^{\alpha(t)\cdot z} \\ &= -\left(e^{\alpha(t)\cdot z} - 1 - \alpha(t)\cdot z\right) \ .\end{aligned}$$

Furthermore, we choose $u(t) \equiv 0$. Then,

$$Z_t = \exp\left\{\int_0^t \int \alpha(s)\cdot z\tilde{N}(dsdz) - \int_0^t \int \left(e^{\alpha(s)\cdot z} - 1 - \alpha(s)\cdot z\right) ds\mu(dz)\right\}$$

on $\mathcal{F}_t$. We see $u(t)$ and $\theta(t,z)$ satisfy the above mentioned condition for the uniformity for a bounded $\alpha(t)$. Hence, putting a new measure dQ by $dQ = Z_t dP$ on $\mathcal{F}_t$, the process $z_1(t) = \int_0^t \int z\tilde{N}_1(dsdz)$ is a Lévy process which is a martingale under Q.

Here, we can choose $\alpha(t)$ to be some deterministic function. In particular,

$$\alpha(t) = \frac{d\mathbf{L}}{dq}(\dot{h}(t)) \ ,$$

where $\mathbf{L}(q)$ denotes the Legendre transform of the Hamiltonian associated with the process $z(t)$:

$$H(p) = \log E[e^{p\cdot z(1)}] \ ,$$

and $h(t)$ is an element in the Sobolev space $W^{1,p}(\mathbf{T})$, $p > 2$. This setting is used in the Bismut perturbation in Section 2.1.1.

From now until Section 2.5, we are mainly interested in the processes which are obtained as a solution to the SDE driven by pure jump Lévy processes, and we may not consider those driven by Lévy processes having the diffusion part.

1.3 Itô processes with jumps

In this section, we introduce an Itô SDE driven by a Lévy process, and processes defined by it. A solution to an Itô SDE is called an Itô process.

Let $z(t)$ be a Lévy process, $\mathbf{R}^m$-valued, with Lévy measure $\mu(dz)$ such that the characteristic function ψ_t is given by

$$\psi_t(\xi) = E[e^{i(\xi,z(t))}] = \exp t\left(-\frac{1}{2}(\xi, \sigma\sigma^T\xi) + \int \left(e^{i(\xi,z)} - 1 - i(\xi, z)1_{\{|z|\le 1\}}\right)\mu(dz)\right) .$$

We may write

$$z(t) = (z_1(t), \ldots, z_m(t)) = \sigma W(t) + \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} z(N(dsdz) - 1_{\{|z|\le 1\}}.\mu(dz)ds) ,$$

where $N(dsdz)$ is a Poisson random measure on $\mathbf{T} \times \mathbf{R}^m \setminus \{0\}$ with mean $ds \times \mu(dz)$. We denote the index of the Lévy measure μ by β, that is,

$$\beta = \inf\{\alpha > 0; \int_{|z|\le 1} |z|^\alpha \mu(dz) < +\infty\} .$$

We assume

$$\int_{\mathbf{R}^m\setminus\{0\}} |z|^2\mu(dz) < +\infty \tag{3.1}$$

temporarily for simplicity.

We can write the SDE in a general formulation on Ω. That is, consider an SDE

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t f(X_{s-})dW(s) + \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} g(X_{s-}, z)\tilde{N}(dsdz) . \qquad (*)$$

Here, we assume

$$|b(x)| \le K(1 + |x|) , \quad |f(x)| \le K(1 + |x|) , \quad |g(x, z)| \le K(z)(1 + |x|) ,$$

and

$$|b(x) - b(y)| \le L|x - y| ,$$
$$|f(x) - f(y)| \le L|x - y| ,$$
$$|g(x, z) - g(y, z)| \le L(z)|x - y| .$$

Here, K, L are positive constants, and $K(z)$, $L(z)$ are positive functions, satisfying

$$\int_{\mathbf{R}^m\setminus\{0\}} \{K^p(z) + L^p(z)\}\mu(dz) < +\infty ,$$

where $p \ge 2$. In Section 4.1, we will use the SDE of this form as a financial model.

Theorem 1.5. *Assume x is p-th integrable. Under the assumptions on $b(x)$, $f(x)$, $g(x, z)$ above, the SDE has a unique solution in L^p. The solution is a semimartingale.*

The proof depends on making a successive approximation (X_t^n) of the solution, and on estimating

$$E[\sup_{t \in \mathbf{T}} |X_t^{n+1} - X_t^n|^p] .$$

We omit the detail ([127] Section 3 and [192] Theorem V.6).

To state the notion of the weak solution of $(*)$, we prepare for the enlargement $(\mathcal{F}'_t)$ of $(\mathcal{F}_t)$, where $(\mathcal{F}_t)$ denotes the original filtration generated by $z(t)$. Here, $(\mathcal{F}'_t)$ satisfies the following properties:

1. $\mathcal{F}_t \subset \mathcal{F}'_t$ for each t
2. $(\mathcal{F}'_t)$ satisfies the usual conditions
3. $W(t)$ is a Brownian motion with respect to $\mathcal{F}'_t$
4. $N(dtdz)$ is a Poisson random measure with respect to $\mathcal{F}'_t$
5. x is $\mathcal{F}'_0$-measurable.

If we can find a process $\tilde{X}_t$ which is $\mathcal{F}'_t$-measurable for some $(\mathcal{F}'_t)$ as above, such that $\tilde{X}_0$ and x have the same distribution, and that $\tilde{X}_t$ satisfies $(*)$ for some Lévy process $\tilde{z}(t)$, then it is called a *weak solution*. A solution X_t to $(*)$ is called a *strong solution* if it is an adapted process (to $(\mathcal{F}_t)$) and if it is represented as a functional of the integrator $z(t)$ and x: $X_. = F(x, z(.))$.

A strong solution is a weak solution. Few results are known for the existence of strong solutions in case that $z(t)$ is a general semimartingale. For the uniqueness of the solution (in the strong and weak senses), see [192] Theorems V.6, V.7.

Definition 1.2. We say the pathwise uniqueness of the solution to $(*)$ holds if for any two solutions X^1, X^2 to $(*)$ defined on the same probability space and driven by (the same Brownian motion and) the same Poisson random measure $N(dtdz)$, it holds that

$$P(\sup_{t \in \mathbf{T}} |X_t^1 - X_t^2| = 0) = 1 .$$

There are several conditions known so as that the pathwise uniqueness holds for the solution to the SDE $(*)$. See [13, 80]. See also [210].

Proposition 1.6 (Yamada–Watanabe type theorem).

1. *If the equation $(*)$ has a weak solution and the pathwise uniqueness holds, then it has a strong solution.*
2. *If the equation $(*)$ has two solutions X^1, X^2 on two distinct probability spaces $(\Omega^i, \mathcal{F}^i, P^i)$, $i = 1, 2$, then there exist a probability space $(\Omega, \mathcal{F}, P)$ and two adapted processes $\tilde{X}_t^1, \tilde{X}_t^2$ defined on $(\Omega, \mathcal{F}, P)$ such that the law of $\tilde{X}^i$ coincides with that of X^i, $i = 1, 2$ and $\tilde{X}^i$, $i = 1, 2$ satisfy $(*)$ on $(\Omega, \mathcal{F}, P)$.*

For the proof of this result, see [202] Theorem 137. The proof in [202] is rather complex. In case that the SDE is driven only by the Wiener process, the result is called "Barlow's theorem". See [192] Exercise IV.40 (p. 246 in Second edition).

In what follows, we consider, in particular, the following SDE on the Poisson space with values in $\mathbf{R}^d$ of Itô type

$$x_t(x) = x + \int_0^t b(x_s(x))ds + \sum_{s\le t}^{c} \gamma(x_{s-}(x), \Delta z(s)) , \quad x_0(x) = x . \tag{3.2}$$

Here, $\sum^c$ denotes the compensated sum, that is,

$$\sum_{s\le t}^{c} \gamma(x, \Delta z(s)) = \lim_{\epsilon\to 0} \left\{ \sum_{s\le t, |\Delta z(s)|\ge\epsilon} \gamma(x, \Delta z(s)) - \int_0^t ds \int_{|z|\ge\epsilon} \gamma(x,z)\mu(dz) \right\}$$

(cf. [106] Chapter II). Functions $\gamma(x, z) : \mathbf{R}^d \times \mathbf{R}^m \to \mathbf{R}^d$ and $b(x) : \mathbf{R}^d \to \mathbf{R}^d$ are C^∞-functions whose derivatives of all orders are bounded, satisfying $\gamma(x, 0) = 0$.

Equivalently, we may write $x_t(x)$ as

$$\begin{aligned} x_t(x) = x &+ \int_0^t b'(x_s(x))ds + \int_0^t \int_{|z|\le 1} \gamma(x_{s-}(x), z)\tilde{N}(ds\, dz) \\ &+ \int_0^t \int_{|z|>1} \gamma(x_{s-}(x), z)N(ds\, dz) , \end{aligned} \tag{3.3}$$

where $\tilde{N}$ denotes the compensated Poisson random measure: $\tilde{N}(dsdz) = N(dsdz) - ds\mu(dz)$, $b'(x) = b(x) - \int_{|z|\ge 1} \gamma(x,z)\mu(dz)$, where the integrability of $\gamma(x,z)$ with respect to $1_{\{|z|\ge 1\}} d\mu(z)$ is assumed. We abandon the restriction (3.1) hereafter, and resume condition (1.2).

We assume that the function γ is the following form:

$$\gamma(x, z) = \frac{\partial\gamma}{\partial z}(x, 0)z + \tilde{\gamma}(x, z) \tag{3.4}$$

for some $\tilde{\gamma}(x, z) = o(|z|)$ as $z \to 0$. We assume further
(A.0) that there exists some $0 < \beta < 2$ and positive C_1, C_2 such that as $\rho \to 0$,

$$C_1\rho^{2-\beta} I \le \int_{|z|\le\rho} zz^T \mu(dz) \le C_2 \rho^{2-\beta} I .$$

(A.1)

(a) For any $p \ge 2$ and any $k \in \mathbf{N}^d \setminus \{0\}$,

$$\int |\gamma(x,z)|^p \mu(dz) \le C(1 + |x|)^p , \quad \sup_x \int \left| \frac{\partial^k \gamma}{\partial x^k}(x,z) \right|^p \mu(dz) < +\infty .$$

(b) There exists $\delta > 0$ such that

$$\inf\left\{z^T\left(\frac{\partial\gamma}{\partial z}(x,0)\right)\left(\frac{\partial\gamma}{\partial z}(x,0)\right)^T z; x \in \mathbf{R}^d\right\} \geq \delta|z|^2$$

on $\mathbf{R}^m$.

(A.2) We assume, for some $C > 0$,

$$\inf_{x\in\mathbf{R}^d, z\in\text{supp }\mu}\left|\det\left(I + \frac{\partial\gamma}{\partial x}(x,z)\right)\right| > C\,.$$

Due to the previous result (Theorem 1.5), the SDE (3.3) has a unique solution $x_t(x)$. Furthermore, the condition *(A.2)* guarantees the existence of the flow $\phi_{st}(x)(\omega)\colon \mathbf{R}^d \to \mathbf{R}^d$, $x_s(x) \mapsto x_t(x)$ of diffeomorphisms for all $0 < s \leq t$.

Here, we say $\phi_{st}(x)(\omega) : \mathbf{R}^d \to \mathbf{R}^d$, $x_s(x) \mapsto x_t(x)$ is a flow of diffeomorphisms if it is a bijection a.s. for which $\phi_{st}(x)$ and its inverse are smooth diffeomorphisms a.s. for each $s < t$. We write $\varphi_t(x)$ of $x_t(x)$ if $s = 0$.

We remark that at the jump time $t = \tau$ of $x_t(x)$,

$$x_\tau(x) = x_{\tau-}(x) + \Delta x_\tau(x) = x_{\tau-}(x) + \gamma(x_{\tau-}(x), \Delta z(\tau))\,.$$

Hence,

$$\frac{\partial}{\partial x}\varphi_\tau(x) = \left(I + \frac{\partial}{\partial x}\gamma(x_{\tau-}(x), \Delta z(\tau))\right)\left(\frac{\partial}{\partial x}\varphi_{\tau-}(x)\right)\,,$$

and this implies

$$\left(\frac{\partial}{\partial x}\varphi_\tau(x)\right)^{-1} = \left(\frac{\partial}{\partial x}\varphi_{\tau-}(x)\right)^{-1}\left(I + \frac{\partial}{\partial x}\gamma(x_{\tau-}(x), \Delta z(\tau))\right)^{-1}\,.$$

This describes the movements of the coordinate system induced by $x \mapsto x_\tau(x)$. We observe that in order for the map $x \mapsto \varphi_t(x)$ to be a diffeomorphism a.s., it is necessary that $x \mapsto x+\gamma(x,z)$ is a diffeomorphism for each $z \in$ supp μ. Otherwise, we can not adequately connect the tangent space at $x_{\tau-}(x)$ to that at $x_\tau(x) = x_\tau(x)+\gamma(x_{\tau-}(x), \Delta z(\tau))$. To this end, we assume *(A.2)*.

L^p-estimates

The L^p-estimate for the solution of (3.2) is not easy in general. Here, we provide a simple one. Suppose $x_t(x)$ is given by

$$x_t(x) = x + bt + \int_0^t\int \gamma(z)\tilde{N}(drdz)\,.$$

Under the integrability of $\gamma(z)$ with respect to $1_{\{|z|\geq 1\}}.d\mu(z)$, we have the following L^p estimate for $x_t(x)$.

Proposition 1.7 (cf. [127] Corollary 2.12). *For $p \geq 2$, we have*

$$E\left[\sup_{0\leq s\leq t}|x_s(x)|^p\right] \leq C_p\left\{|x|^p + |b|^p t + t\left(\int |\gamma(z)|^2\mu(dz)\right)^{p/2} + t\int |\gamma(z)|^p\mu(dz)\right\},$$

for some $C_p > 0$.

Jacobian

In order to inquire the flow property of $x_t(x)$, we need the derivative $\nabla x_t(x) = \frac{\partial x_t}{\partial x}(x)$ of $x_t(x)$. It is known that under the conditions *(A.0–A.2)*, the derivative satisfies the following linear equation:

Proposition 1.8. *Under the assumptions (A.0–A.2), the derivative $\nabla x_t(x)$ satisfies the following SDE:*

$$\begin{aligned}\nabla x_t(x) = I &+ \int_0^t \nabla b'(x_{s-}(x))ds\nabla x_{s-}(x)\\ &+ \int_0^t\int_{|z|\leq 1} \nabla\gamma(x_{s-}(x), z)\tilde{N}(dsdz)\nabla x_{s-}(x)\\ &+ \int_0^t\int_{|z|>1} \nabla\gamma(x_{s-}(x), z)N(dsdz)\nabla x_{s-}(x)\,.\end{aligned} \tag{3.5}$$

Proof. We skip the proof for the differentiability since it is considerably long ([127] Theorem 3.3). The SDE for which $\nabla x_t(x)$ should satisfy is given as below.

Let e be a unit vector in $\mathbf{R}^d$, and let

$$X'(t) = \frac{1}{\lambda}(x_t(x+\lambda e) - x_t(x))\,,\quad 0 < |\lambda| < 1\,.$$

Then, $X'(t)$ satisfies the following SDE

$$X'(t) = e + \int_0^t b'_\lambda(s)ds + \int_0^t\int_{|z|\leq 1}\gamma'_\lambda(s,z)\tilde{N}(dsdz) + \int_0^t\int_{|z|>1}\gamma'_\lambda(s,z)N(dsdz)$$

where

$$b'_\lambda(s) = \frac{1}{\lambda}(b'(x_{s-}(x+\lambda e)) - b'(x_{s-}(x)))\,,$$
$$\gamma'_\lambda(s,z) = \frac{1}{\lambda}(\gamma(x_{s-}(x+\lambda e), z) - \gamma(x_{s-}(x), z))\,.$$

We let $\lambda \to 0$. Then, by the chain rule,

$$\lim_{\lambda \to 0} b'_\lambda(s) = \nabla b'(x_{s-}(x)) \cdot \nabla x_{s-}(x) \text{a.s.} ,$$

and

$$\lim_{\lambda \to 0} \gamma'_\lambda(s,z) = \nabla \gamma(x_{s-}(x), z) \cdot \nabla x_{s-}(x) \text{a.s.}$$

for each s. Hence, we have

$$\begin{aligned} \nabla x_t(x) = I &+ \int_0^t \nabla b'(x_{s-}(x)) ds \nabla x_{s-}(x) \\ &+ \int_0^t \int_{|z|\le 1} \nabla \gamma(x_{s-}(x), z) \tilde{N}(dsdz) \nabla x_{s-}(x) \\ &+ \int_0^t \int_{|z|>1} \nabla \gamma(x_{s-}(x), z) N(dsdz) \nabla x_{s-}(x) . \end{aligned}$$

□

The equation (3.5) is fundamental in considering the stochastic quadratic form (an analogue of Malliavin matrix for the jump process). See Chapter 2.

The derivative $\nabla x_t(x)$ depends on the choice of $e \in \mathbf{R}^d$, and is indeed a directional derivative. We denote the Jacobian matrix by the same symbol, namely, $\nabla x_t(x) = \nabla_x x_t(x)$.

Let $m = d$. Here is an expression of $\det \nabla x_t(x)$ which will be useful in the analytic (volume) estimates.

Let

$$\begin{aligned} A_t = &\int_0^t \sum_{i=1}^d \frac{\partial}{\partial x_i} b'^i(x_s(x)) ds \\ &+ \int_0^t \int_{|z|\le 1} \left[\det(I + \nabla \gamma(x_{s-}(x), z)) - 1 - \sum_{i=1}^d \frac{\partial}{\partial x_i} \gamma^i(x_{s-}(x), z) \right] ds \mu(dz) \\ &+ \int_0^t \int_{|z|>1} \left[\det(I + \nabla \gamma(x_{s-}(x), z)) - 1 \right] N(ds, dz) , \\ M_t = &\int_0^t \int_{|z|\le 1} \left[\det(I + \nabla \gamma(x_{s-}(x), z)) - 1 \right] \tilde{N}(dsdz) . \end{aligned}$$

We then have

Lemma 1.1.

$$\det(\nabla x_t(x)) = \exp A_t \cdot \mathcal{E}(M)_t . \tag{3.6}$$

Proof. By Itô's formula applied to $\det \circ \nabla x_t(x)$,

$$
\begin{aligned}
\det(\nabla x_t(x)) = 1 &+ \int_0^t \sum_{i=1}^d \frac{\partial}{\partial x_i} b'^i(x_s(x)) \det(\nabla x_s(x)) ds \\
&+ \int_0^t \int_{|z|\le 1} \left[\det((I + \nabla\gamma(x_{s-}(x), z))\nabla x_{s-}(x)) \right. \\
&\qquad \left. - \det(\nabla x_{s-}(x)) \right] \tilde{N}(dsdz) \\
&+ \int_0^t \int_{|z|>1} \left[\det((I + \nabla\gamma(x_{s-}(x), z))\nabla x_{s-}(x)) \right. \\
&\qquad \left. - \det(\nabla x_{s-}(x)) \right] N(dsdz) \\
&+ \int_0^t \int_{|z|\le 1} \left[\det((I + \nabla\gamma(x_{s-}(x), z))\nabla x_{s-}(x)) - \det(\nabla x_{s-}(x)) \right. \\
&\left. - \sum_{i=1}^d \frac{\partial}{\partial x_i} \gamma^i(x_{s-}(x), z) \det(\nabla x_{s-}(x)) \right] ds\mu(dz) \\
= 1 &+ \int_0^t \det(\nabla x_{s-}(x)) d(A_s + M_s) .
\end{aligned}
$$

Here we used the formula

$$
\frac{d}{dt} \det(A + tBA)|_{t=0} = \operatorname{tr} B \det A
$$

in the right-hand side. Hence, we apply the Doléans' exponential formula and Girsanov's theorem (Theorem 1.4) from Section 1.2.3. In view of $[A, M] = 0$, $[A+M] = [M]$, we have the assertion. □

Using (3.6), it is possible to show the boundedness of $E[\sup_{t\in\mathbf{T}} \det(\nabla x_t(x))^{-p}]$ by a constant which depends on $p > 1$ and the coefficients of the SDE (3.2). (cf. [224] Lemmas 3.1, 3.2.)

Inverse flow

Using the Jacobian $\nabla x_t(x)$, we can show the existence of the inverse flow $x_t^{-1}(x)$ as a representation of the flow property. By the inverse mapping theorem, this is due to the (local) invertibility of the Jacobian $\nabla x_t(x)$.

Associated to the above mentioned process $\nabla x_t(x) = U(t)$:

$$U(t) = \int_0^t \nabla b'(x_{s-}(x))U(s)ds + \int_0^t \int_{|z|\le 1} \nabla\gamma(x_{s-}(x), z)U(s)\tilde{N}(dsdz)$$
$$+ \int_0^t \int_{|z|>1} \nabla\gamma(x_{s-}(x), z)U(s)N(dsdz) \ ,$$

we introduce

$$V(t) = -U(t) + [U, U]_t^c + \sum_{0<s\le t} (I + \Delta U(s))^{-1}(\Delta U(s))^2 \ .$$

Then, $\Psi_t = (\nabla x_t(x))^{-1}$ follows the SDE, satisfying

$$\Psi_t = I + \int_0^t \Psi_{s-} dV(s)$$

([192] Theorem V. 63).

More precisely, this reads for $x_t(x)$, solving (3.3) as

$$(\nabla x_t(x))^{-1} = I - \int_0^t (\nabla x_s(x))^{-1} \nabla b'(x_s(s))ds$$
$$+ \int_0^t \int_{|z|\le 1} (\nabla x_s(x))^{-1}\{\nabla\phi_z(x_s(x))^{-1} - I + \nabla\gamma(x_s(x), z)\}ds\mu(dz)$$
$$+ \int_0^t \int_{|z|\le 1} (\nabla x_{s-}(x))^{-1}\{\nabla\phi_z(x_{s-}(x))^{-1} - I\}\tilde{N}(dsdz)$$
$$+ \int_0^t \int_{|z|>1} (\nabla x_{s-}(x))^{-1}\{\nabla\phi_z(x_{s-}(x))^{-1} - I\}N(dsdz) \ .$$

Here, $\phi_z(x) = x + \gamma(x, z)$.

2 Perturbations and properties of the probability law

> Let me have men about me that are fat; Sleek-headed men and such as sleep o' nights: Yond Cassius has a lean and hungry look; He thinks too much: such men are dangerous.
>
> W. Shakespeare, Julius Caesar, Act 1, Scene 2

In this chapter, we briefly sketch the perturbation methods in the Poisson space and their applications. Namely, we relate Lévy processes (and SDEs) to perturbations for functionals on the Poisson space. The perturbation induces the *integration-by-parts formula* which is an essential tool leading to the existence of the smooth density of the probability law.

In Section 2.1, we reflect several types of perturbations on the Poisson space. In Sections 2.2, 2.3, granting the existence of the density, we present the short time asymptotics from both above and below the density function $p_t(x,y)$ obtained as a result of the integration-by-parts calculation, which uses a perturbation method given in Section 2.1. We can observe distinct differences in the short time bounds of densities from those obtained for diffusion processes with analytic methods.

The existence of the smooth density is closely related to the behaviour of the Fourier transform $\hat{p}(v)$ as $|v| \to \infty$ of the density function $y \mapsto p_t(x,y)$. We will inquire into the existence and behaviour of $p_t(x,y)$ more precisely in Sections 3.3, 3.6, relating to the SDE with jumps.

In Section 2.4, we give a summary on the short time asymptotics of the density functions of various types. Section 2.5 will be devoted to the auxiliary topics concerning the absolute continuity of the infinitely divisible laws and the characterisation of the "support" of the probability laws in the path space.

2.1 Integration-by-parts on Poisson space

We are going to inquire into the details on the absolute continuity and on the smoothness of the transition density for the functionals given as solutions to the SDE of jump-diffusion type, by using Malliavin calculus for processes with jumps. In Malliavin calculus, we make perturbations of the trajectories of processes with respect to the jump times or to the jump size and directions.

Our general aim is to show the integration-by-parts formula for a certain d-dimensional Poisson functional F. The formula is described as follows. For any smooth random variable G, there exists a certain L^p-random variable determined from F and G, denoted by $\mathcal{H}(F,G)$, such that the equality

$$E[\partial\phi(F)G] = E[\phi(F)\mathcal{H}(F,G)] \tag{1.1}$$

holds for any smooth function ϕ. Formula (1.1) is called an *integration-by-parts formula*. We will show in this section that the formula holds for "non-degenerate" Poisson functionals in various cases.

The formula can be applied to proving the existence of the smooth density of the law of F. Indeed, let $p_F(dx)$ denote the law of F. Its Fourier transform $\hat{p}_F(v)$ is written by $E[\phi_v(F)]$, where $\phi_v(x) = \exp(-i(x, v))$. We apply the formula above k times by setting $\phi = \phi_v$ and $G = 1$, and thus

$$E[\partial^k \phi_v(F)] = E[\partial^{k-1}\phi_v(F)\mathcal{H}(F, 1)] \\ = \ldots = E[\phi_v(F)\mathcal{H}(F, \mathcal{H}(F, \cdots, \mathcal{H}(F, 1)))] \;.$$

Since $|\phi_v(F)| \leq 1$ holds for all $v \in \mathbf{R}^d$, the above formula implies that there exists a positive constant C_k such that

$$|E[\partial^k \phi_v(F)]| \leq C_k, \forall v \tag{1.2}$$

for each multi-index k.

Since $\partial^k \phi_v = (-i)^{|k|} v^k \phi_v$, we have

$$|v^k||\hat{p}_F(v)| \leq C_k, \;\forall v$$

for each k, where $\hat{p}_F(v) = E[\phi_v(F)]$. Hence,

$$\left(\frac{1}{2\pi}\right)^d \int e^{i(x,v)} i^{|l|} v^l \hat{p}_F(v)\, dv$$

is well-defined for l such that $|l| < |k| - d$ ([194] Lemma (38.52)).

Then, the theory of the Fourier transform tells us that $p_F(dx)$ has a density $p_F(x)$ and the above integral is equal to $\partial_x^l p_F(x)$. This result is closely related to analysis and the classical potential theory.

The original Malliavin calculus for diffusion processes by Malliavin ([160]) began with this observation, and the aim was to derive the integration-by-parts formula on a probability space. The aim of early works by Bismut ([29, 30]) for jump processes was also the same. In this perspective, one of the most important subjects was to decide which kind of perturbation on the trajectories should be chosen in order to derive the integration-by-parts formula.

Malliavin's approach uses the Malliavin operator L and makes a formalism for the integration-by-parts. It has succeeded in the analysis on the Wiener space, and signified the importance of the integration-by-parts formula. Bismut's approach, on the other hand, uses the perturbation of trajectories by making use of the derivative operator D, and then makes use of the Girsanov transform of the underlying probability measures to show the integration-by-parts formula. We adopt Bismut's approach in this section.

Bismut's approach was originally developed on the Poisson space under the condition that the Lévy measure μ of the base Poisson random measure has a smooth density function. Accordingly, along the idea of the Bismut's approach, the condition has been relaxed by Picard to accept singular Lévy measures where we use the difference operator. See Section 3.3 for Picard's approach.

Malliavin's approach can adopt general functionals on the Wiener (and on the Poisson) space. On the other hand, Bismut's approach, in the formulation treated in this book, can only adopt those Poisson functionals which appear as solutions to SDEs with jumps. For an interpretation of the use of Malliavin's approach to the solutions of SDEs with jumps on the Poisson space, see [25] Section 10.

Remark 2.1. Bismut also considered the approach regarding the Poisson process as a boundary process of a diffusion process in a higher dimensional space, that is, the method of subordination ([32] and [144]). The calculation is a bit messy. We will not adopt this approach.

2.1.1 Bismut's method

In this subsection, we assume the Lévy measure μ has a density with respect to the Lebesgue measure. The Bismut method is the way to make a variation of the intensity, i.e. the density of the Lévy measure. We can use the Girsanov transform of the underlying probability measures of the driving Lévy processes to make perturbations on the jump size and the direction, leading to the integration by parts formula. This method has been initiated by Bismut [30], and has been developed by Bichteler et al. [25, 27] and Bass [15]. Although the idea is simple, the calculation using this method is a bit complicated.

We denote by Ω_1 the set of continuous functions from $\mathbf{T}$ to $\mathbf{R}^m$, by $\mathcal{F}_1$ the smallest σ-field associated to W's in Ω_1, and by P_1 the probability of W for which $W(0) = 0$ and $W(t) := \omega(t)$ behaves as a standard Wiener process (Brownian motion). The triplet $(\Omega_1, \mathcal{F}_1, P_1)$ is called a *Wiener space*.

Let Ω_2 be the set of all nonnegative integer-valued measures on $U = \mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$ such that $\omega(\mathbf{T} \times \{0\}) = 0$, $\omega(\{u\}) \le 1$ for all u, and that $\omega(A) < +\infty$ if $\hat{N}(A) < +\infty$, $A \in \mathcal{U}$. Here, $u = (t, z)$ denotes a generic element of U, $\mathcal{U}$ denotes the Borel σ-field on U, and $\hat{N}(dtdz) = dt\mu(dz)$, where μ is a Lévy measure. Let $\mathcal{F}_2$ be the smallest σ-field of Ω_2 with respect to which $\omega(A) \in \mathcal{F}_2$, $A \in \mathcal{U}$. Let P_2 be a probability measure on $(\Omega_2, \mathcal{F}_2)$ such that $N(dtdz) := \omega(dtdz)$ is a Poisson random measure with intensity measure $\hat{N}(dtdz)$. The space $(\Omega_2, \mathcal{F}_2, P_2)$ is called the *Poisson space*.

Given $\alpha \in (0, 2)$, let $z_j(t)$ be the independent, symmetric jump processes on $\mathbf{R}$ such that $z_j(0) \equiv 0$, $j = 1, \ldots, m$. We assume they have the same law, and that they have the infinitesimal generator

$$L\varphi(x) = \int_{\mathbf{R}\setminus\{0\}} [\varphi(x+z) - \varphi(x)]|z|^{-1-\alpha} dz,\ x \in \mathbf{R},\ \varphi \in C_0^\infty(\mathbf{R})\ .$$

We put $g(z)dz \equiv |z|^{-1-\alpha}dz$, that is, $\mu(dz) = g(z)dz$ is the Lévy measure of $z_j(t)$ having the law of a symmetric α-stable process.

The assumption for the Lévy measure μ to be the α-stable type plays an essential role in this method. See the proof of Lemma 2.2 below.

[A] Integration by parts of order 1

(Step 1)

The symmetric Lévy process $z_j(s)$ has a Poisson random measure representation

$$z_j(s) = \int_0^s \int z\tilde{N}_j(dsdz) ,$$

where $\tilde{N}_j(dsdz)$ is the compensated Poisson random measure on $\mathbf{R}$ associated to $z_j(s)$, with the mean measure

$$ds \times g(z)dz . \tag{1.3}$$

Here,

$$g(z)dz = |z|^{-1-\alpha}dz , \quad \alpha \in (0, 2) .$$

We denote by P the law of $z_j(s)$, $j = 1, \cdots, m$.

Let the process $x_s(x)$ be given by the following linear SDE, that is,

$$\begin{cases} dx_s(x) & = \sum_{j=1}^m X_j(x_{s-}(x))\, dz_j(s) + X_0(x_s(x))ds \\ x_0(x) & \equiv x . \end{cases}$$

Here, the $X_j(x)$s denote d-dimensional smooth vector fields on $\mathbf{R}^d$ whose derivatives of all orders are bounded, $j = 0, 1, \ldots, m$, satisfying

$$\mathrm{Span}(X_1, \ldots, X_m) = T_x(\mathbf{R}^d)$$

at each point $x \in \mathbf{R}^d$.

We assume assumptions *(A.0)* to *(A.2)* in Section 1.3 for $x_t(x)$, with $b(x) = X_0(x)$ and $\gamma(x, z) = \sum_{j=1}^m X_j(x)z_j$. In particular, *(A.2)* takes the following form: for some $C > 0$,

$$\inf_{x\in\mathbf{R}^d, z\in\mathrm{supp}\,\mu} \left|\det\left(I + \frac{\partial X_j(x)}{\partial x} z_j\right)\right| > C , \quad j = 1, \ldots, m .$$

We denote by φ_s the mapping $x \mapsto x_s(x)$.

We use the Girsanov transform of measures, see Section 1.2.3 above (cf. [25] Section 6). Let $v = (v_1, \cdots, v_m)$ be a bounded predictable process on $[0, +\infty)$ to $\mathbf{R}^m$. We consider the perturbation

$$\theta_j^\lambda : z_j \mapsto z_j + \lambda v(z_j)v_j , \quad \lambda \in \mathbf{R} ,\ j = 1, \ldots, m .$$

Here, $\nu(.)$ is a C^2-function such that

$$\nu(z_j) \sim z_j^2$$

for z_j small, $j = 1, \ldots, m$. We consider the above transformation for $|\lambda|$ as sufficiently small.

Let $N_j^\lambda(dsdz)$ be the Poisson random measure defined by

$$\int_0^t \int \phi(z) N_j^\lambda(dsdz) = \int_0^t \int \phi(\theta_j^\lambda(z)) N_j(dsdz)\ , \quad \phi \in C_0^\infty(\mathbf{R})\ ,\ \lambda \in \mathbf{R}\ . \tag{1.4}$$

It is a Poisson random measure with respect to P.

We set $\Lambda_j^\lambda(z) = \{1 + \lambda \nu'(z) v_j\} \frac{g(\theta_j^\lambda(z))}{g(z)}$, and

$$Z_t^\lambda = \exp\left[\sum_{j=1}^m \left\{\int_0^t \int \log \Lambda_j^\lambda(z_j) N_j(dsdz_j) - \int_0^t ds \int (\Lambda_j^\lambda(z_j) - 1) g(z_j) dz_j \right\}\right]\ . \tag{1.5}$$

We will apply m-dimensional version of Theorem 1.4 in Section 1.2. It implies that Z_t^λ is a martingale with mean 1. We define a probability measure P^λ by $P^\lambda(A) = E[Z_t^\lambda 1_A]$, $A \in \mathcal{F}_t$. Then, due to (1.5), the compensator of $N_j(dsdz_j)$ with respect to P^λ is $ds\Lambda_j^\lambda(z_j) g(z_j) dz_j$.

Furthermore, we can compute the compensator of $N_j^\lambda(dsdz_j)$ with respect to P^λ as follows. We have

$$E^{P^\lambda}\left[\int_0^t \int \phi(z_j) N_j^\lambda(dsdz_j)\right] = E^{P^\lambda}\left[\int_0^t \int \phi(\theta_j(z_j)) N_j(dsdz_j)\right]$$
$$= t \int \phi(\theta_j(z_j)) \Lambda_j^\lambda(z_j) g(z_j) dz_j\ . \tag{1.6}$$

Setting $y_j = \theta_j^\lambda(z_j)$, we find by the definition that the above integral is equal to $t \int \phi(y_j) g(y_j) dy_j$. Consequently, the compensator of $N_j^\lambda(dsdz_j)$ with respect to P^λ is equal to $dsg(z_j)dz_j$. Hence, as Poisson random measures, the law of $N_j^\lambda(dsdz_j)$ with respect to P^λ coincides with that of $N_j(dsdz_j)$ with respect to P. Consider the perturbed process $x_s^\lambda(x)$ of $x_s(x)$ defined by

$$\begin{cases} dx_s^\lambda(x) & = \sum_{j=1}^m X_j(x_{s-}^\lambda(x)) dz_j^\lambda(s) + X_0(x_s^\lambda(x)) ds \\ x_0^\lambda(x) & \equiv x\ . \end{cases} \tag{1.7}$$

Here, $z_j^\lambda(t) = \int_0^t \int z N_j^\lambda(dsdz)$. Then, $E^P[f(x_t(x))] = E^{P^\lambda}[f(x_t^\lambda(x))] = E^P[f(x_t^\lambda(x)) Z_t^\lambda]$, and we have $0 = \frac{\partial}{\partial\lambda} E^P[f(x_t^\lambda(x)) Z_t^\lambda]$, $f \in C_0^\infty(\mathbf{R}^d)$. A result of the differentiability of ODE can be applied jump by jump to show that for $|\lambda|$ small, we have

$$\frac{\partial}{\partial\lambda}(f \circ x_t^\lambda(x)) = D_x f(x_t^\lambda(x)) \cdot \frac{\partial x_t^\lambda(x)}{\partial\lambda}, \quad f \in C_0^\infty(\mathbf{R}^d)\ .$$

We would like to differentiate $E[f \circ x_t^\lambda(x)]$ at $\lambda = 0$. To this end, we introduce the following notion.

Definition 2.1. A family $(G^\lambda)_{\lambda \in \Lambda}$ of processes is called F-differentiable at 0 with derivative ∂G if

1. $\sup_{t \in \mathbf{T}} |G^\lambda| \in \cap_{p<\infty} L^p$ for all $\lambda \in \Lambda$ and
2. $\left\| \sup_{t \in \mathbf{T}} \left| G^\lambda - G^0 - \partial G \cdot \lambda \right| \right\|_{L^p} = o(|\lambda|)$ as $\lambda \to 0$ for all $p \geq 1$.

Lemma 2.1.

1. *The family* $(x_t^\lambda(x))$ *is F-differentiable at* $\lambda = 0$.
2. *The family* (Z_t^λ) *is F-differentiable at* $\lambda = 0$.

Proof. (i) By the assumption that $X_0(x), X_j(x), j = 1, \ldots, m$ are smooth vector-valued functions whose derivatives of all orders are bounded, the assumption (A'-2) in [25] Section 6-a is met with $b(x) = X_0(x)$, $c(x, z) = X_j(x) z_j$, $j = 1, \ldots, m$. We use the differentiability result ([25] Theorem 6-24) to assert that $x_t^\lambda(x)$ is F-differentiable at 0.
(ii) As noted in Section 1.2, Z_t^λ is the solution of the SDE

$$dZ_t^\lambda = \sum_{j=1}^m \int Z_{t-}^\lambda \left(1 - \Lambda_j^\lambda\right)(z) \tilde{N}_j(dtdz) .$$

The jump coefficient $\gamma(x, z) = x(1 - \Lambda_j^\lambda)(z)$ and the Lévy measure $\mu(dz) = g(z)dz$ satisfy the conditions *(A.0–A.2)* in Section 1.3. Hence, we have the L^p-estimate (Proposition 1.7) for Z_t^λ for $\lambda \in (-1, 1)$. By applying [25] Theorem 6-24 again, we see that (Z_t^λ) is F-differentiable at $\lambda = 0$. □

As $G^\lambda = x_t^\lambda(x)$ is F-differentiable, we have by (ii) for ∂G

$$\left\| \sup_{t \in \mathbf{T}} \left| \frac{G^\lambda - G^0}{\lambda} - \partial G \right| \right\|_{L^p} = o(1) \quad \text{as } \lambda \to 0.$$

The first term on the left-hand side should be finite for all $\lambda \in \Lambda$ by (i), and ∂G is uniformly integrable with respect to λ in a neighbourhood of $\lambda = 0$. We write

$$H_t^\lambda = \frac{\partial x_t^\lambda(x)}{\partial \lambda}, \quad \lambda \in \Lambda ,$$

and put $H_t = H_t^\lambda|_{\lambda=0}$ hereafter. We give an explicit form of H_t later.

(Step 2)
We can take differentiation $\frac{\partial}{\partial \lambda}$ under the expectation on both sides of the equation

$$E^P[f(x_t(x))] = E^{P^\lambda}[f(x_t^\lambda(x))] = E^P[f(x_t^\lambda(x)) Z_t^\lambda]$$

to have

$$E^P[D_x f(x_t(x)) \cdot H_t] = -E^P[f(x_t(x)) \frac{\partial}{\partial \lambda} Z_t^\lambda|_{\lambda=0}] . \tag{1.8}$$

Here, on the right-hand side of (1.8), from (1.5), we obtain

$$\frac{\partial}{\partial\lambda} Z_t^\lambda|_{\lambda=0} = \sum_{j=1}^m \int_0^t \int \frac{\operatorname{div}[g(\cdot)v_j\nu(\cdot)](z_j)}{g(z_j)} [N_j(dsdz_j) - dsg(z_j)dz_j] \,. \tag{1.9}$$

Here,

$$\int_0^t \int \frac{\operatorname{div}[g(\cdot)v_j\nu(\cdot)](z_j)}{g(z_j)} [N_j(dsdz_j) - dsg(z_j)dz_j]$$
$$= \sum_{s\le t}^{c} \left[v_j \cdot \nu'(\Delta z_j(s)) + v_j \cdot \nu(\Delta z_j(s)) \frac{g'(\Delta z_j(s))}{g(\Delta z_j(s))} \right] .$$

We write $R_t \equiv \frac{\partial}{\partial\lambda} Z_t^\lambda|_{\lambda=0}$ below.

Here, we have

Lemma 2.2. *With a suitable choice of v_j fixed later,*

$$|H_t^{-1}| \in L^p, \quad p \ge 1,\ t > 0 \,. \tag{1.10}$$

This lemma is proved at the end of this subsection.

With the choice of v_j below (given in the proof of Lemma 2.2 below), we can regard H_t as a linear mapping from $\mathbf{R}^d$ to $\mathbf{R}^d$ which is non-degenerate a.s. See (1.24) below. By the inverse mapping theorem, we can guarantee the existence and the differentiability of the inverse of H_t^λ for $|\lambda|$ as small, which we denote by $H_t^{\lambda,-1}$: $H_t^{\lambda,-1} = [H_t^\lambda]^{-1}$.

(Step 3)

We carry out the integration-by-parts procedure for $F_t^\lambda(x) = f(x_t^\lambda(x))H_t^{\lambda,-1}$ as follows. Recall that we have $E[F_t^0(x)] = E[F_t^\lambda(x) \cdot Z_t^\lambda]$. Taking the F-derivation $\frac{\partial}{\partial\lambda}|_{\lambda=0}$ for both sides yields

$$\begin{aligned} 0 = E[D_x f(x_t(x))H_t^{-1}H_t] &+ E\left[f(x_t(x))\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}|_{\lambda=0}\right] \\ &+ E[f(x_t(x))H_t^{-1} \cdot R_t] \,. \end{aligned} \tag{1.11}$$

Here, $\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}$ is defined by $\langle \frac{\partial}{\partial\lambda}H_t^{\lambda,-1}, e\rangle = \operatorname{trace}[e' \mapsto \langle -H_t^{\lambda,-1}(\frac{\partial}{\partial\lambda}H_t^\lambda \cdot e')H_t^{\lambda,-1}, e\rangle]$, $e \in \mathbf{R}^d$, where $\frac{\partial}{\partial\lambda}H_t^\lambda$ is the second F-derivative of $x_t^\lambda(x)$ defined as in [25] Theorem 6-44. As H^λ is F-differentiable at $\lambda = 0$, we put $DH_t = \frac{\partial}{\partial\lambda}H_t^\lambda|_{\lambda=0}$. Then $\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}|_{\lambda=0} = -H_t^{-1}DH_tH_t^{-1}$.

This yields the integration-by-parts formula

$$E[D_x f(x_t(x))] = E[f(x_t(x))\mathcal{H}_t^{(1)}] \tag{1.12}$$

where

$$\mathcal{H}_t^{(1)} = H_t^{-1}DH_tH_t^{-1} - H_t^{-1}R_t \,. \tag{1.13}$$

Lastly, we compute $H_t^{-1}DH_tH_t^{-1}$. By the argument similar to that in [29] p. 477, we have

$$DH_t = \sum_{j=1}^{m} \varphi_t^* \left\{ \sum_{s\le t} (\varphi_s^{*-1}X_j)(x)v'(\Delta z_j(u))v_j(s)v(\Delta z_j(u))v_j(s) \right\}$$
$$= \sum_{j=1}^{m} \varphi_t^* \left\{ \sum_{s\le t} M_j^{(1)}(t,s)v(\Delta z_j(s)) \right\} \quad \text{(say)}. \tag{1.14}$$

Here,

$$\left\| \sum_{s\le t} v(\Delta z_j(s))M_j^{(1)}(t,s) \right\| \in L^p , \quad p \ge 1 , \tag{1.15}$$

$j = 1, \cdots, m$.

Combining (1.10), (1.15) ($t = 1$)

$$E[D_x f(x_1(x))] = E[f(x_1(x))\mathcal{H}_1^{(1)}] . \tag{1.16}$$

This leads our assertion of order 1.

[B] Higher order formulae and smoothness of the density

We can repeat the above argument to obtain the integration-by-parts formulae for higher orders by a direct application of the above method. Instead of describing the detail, we will repeat the argument in **[A]** in a general setting, and paraphrase the argument mentioned in the introduction to Section 2.1.

Let $g(\omega_2)$ be a functional on $(\Omega_2, \mathcal{F}_2, P_2)$, and put

$$G^0 = g(N(dudz)), \; G^\lambda = g(N^\lambda(dudz)) \quad \text{for} \quad \lambda \ne 0 . \tag{1.17}$$

Theorem 2.1. *Suppose that the family $(G^\lambda)_{\lambda\in(-1,1)}$ is F-differentiable at $\lambda = 0$. Then, we have an integration-by-parts formula*

$$E[D_x f(F)G^0] = E[f(F)\mathcal{H}(F, G^0)] \tag{1.18}$$

for any smooth function f. Here,

$$\mathcal{H}(F, G^0) = \left(H_t^{-1}DH_tH_t^{-1} - H_t^{-1}R_t\right)G^0 - H_t^{-1}\partial G , \tag{1.19}$$

where $\partial G = \frac{\partial G^\lambda}{\partial\lambda}|_{\lambda=0}$.

Proof. We put $\Phi^\lambda = f(F^\lambda).H_t^{\lambda,-1}$ with $F^\lambda = x_t^\lambda(x)$. Using Girsanov's theorem (Theorem 1.4), we have

$$E[\Phi^0 G^0] = E[\Phi^\lambda Z_t G^\lambda] , \tag{1.20}$$

where Z_t^λ is given by (1.5).

We take $\lambda = 0$ after derivating the two sides, and obtain

$$0 = E[D_x f(F) H_t^{-1} H_t G^0] + E\left[f(F)\frac{\partial}{\partial\lambda} H_t^{\lambda,-1}|_{\lambda=0} G^0\right]$$
$$+ E[f(F) H_t^{-1} R_t G^0] + E\left[f(F) H_t^{-1}\frac{\partial}{\partial\lambda} G^\lambda|_{\lambda=0}\right]$$

using (1.8), (1.9).

This leads to the integration-by-parts formula (1.18) with

$$\mathcal{H}(F, G^0) = -\left(\frac{\partial}{\partial\lambda} H_t^{\lambda,-1}|_{\lambda=0} + H_t^{-1} R_t\right) G^0 - H_t^{-1}\frac{\partial}{\partial\lambda} G^\lambda|_{\lambda=0}\,. \qquad \square$$

The formula (1.16) is obtained from this theorem by putting $G = 1$.

If we apply the above result with $G^0 = \mathcal{H}(F, 1)$ and $G^\lambda = \mathcal{H}(F^\lambda, 1)$ for $\lambda \neq 0$, then the family $(G^\lambda)_{\lambda\in(-1,1)}$ is F-differentiable at $\lambda = 0$ by (1.10). This leads to the second order formula

$$E[D_x^2 f(F)] = E[D_x f(F)\mathcal{H}(F, 1)] = E[f(F)\mathcal{H}(F, \mathcal{H}(F, 1))]\ .$$

Repeating this argument, we obtain the integration-by-parts formulae of higher orders. We then can prove the smoothness of the density $p_F(x)$ as explained in the introduction to Section 2.1.

Proof of Lemma 2.2

Proof. Part (i) We begin by computing $H_t^\lambda \equiv \frac{\partial x_t^\lambda}{\partial\lambda}$ at $\lambda = 0$.

We recall the SDE (1.7)

$$dG^\lambda = \sum_{j=1}^{m} X_j(G^\lambda(x)) dz_j^\lambda(t) + X_0(G^\lambda) dt, \qquad (1.7)'$$

where $z_j^\lambda(t) = \int_0^t \int z N_j^\lambda(dsdz)$. Recalling the definition of $\theta_j^\lambda(z_j) = z_j + \lambda v(z_j) v_j$, we take differentiation with respect to λ of both sides of (1.7), and put $\lambda = 0$. Then, $H_t = H_t^0 = \partial G$ is obtained as the solution of the following equation, that is,

$$H_t = \sum_{j=1}^{m}\left\{\sum_{s\le t}^{c}\left(\frac{\partial X_j}{\partial x}(x_{s-})\right) H_{s-}\Delta z_j(s) + \sum_{s\le t} X_j(x_{s-}) v_j(s) v(\Delta z_j(s))\right\}$$
$$+ \int_0^t \frac{\partial X_0}{\partial x}(x_{s-}) H_{s-} ds \qquad (1.21)$$

(cf. [25] Theorem 6-24).

Then, H_t is given according to the method of variation of constants, similar to the Duhamel's formula $e^{t(A+B)} = e^{tA}(1 + \int_0^t e^{-sA}Be^{s(A+B)}ds)$ with B corresponding to the second term in the right-hand side of (1.21), by

$$\varphi_t^* \left\{ \sum_{j'=1}^m \sum_{s\le t} \left(\frac{\partial}{\partial x}\varphi_{s-}(x) \right)^{-1} \left\{ \sum_{j=1}^m \left(I + \frac{\partial}{\partial x} X_j(x_{s-}(x))\Delta z_j(s) \right)^{-1} \right\} \right.$$
$$\left. \times\, \nu(\Delta z_{j'}(s)) X_{j'}(x_{s-}(x)) v_{j'}(s) \right\} \tag{1.22}$$

(cf. [25] (6-37), (7-16)).

In the above equation, $\sum_{s\le t}^c$ denotes the compensated sum, and φ_t^* denotes the push-forward $\varphi_t^* Y(x) = (\frac{\partial \varphi_t}{\partial x}(x))Y(x)$ by φ_t, that is, $\varphi_t^* = (\frac{\partial \varphi_t}{\partial x}(x))$. Below we will also use the following notation; φ_t^{*-1} denotes the pull back $\varphi_t^{*-1} Y(x) = (\frac{\partial}{\partial x}\varphi_t(x))^{-1} Y(x_{t-}(x))$ by φ_t, where

$$\left(\frac{\partial}{\partial x}\varphi_t(x) \right)^{-1} = \left(\frac{\partial}{\partial x}\varphi_{t-}(x) \right)^{-1} \left\{ \sum_{j=1}^m \left(I + \frac{\partial}{\partial x} X_j(x_{t-}(x))\Delta z_j(t) \right)^{-1} \right\} .$$

We remark H_t is the F-derivative of $x_t^\lambda(x)$ to the direction of v_j. Notice also that we can regard H_t as a linear functional: $\mathbf{R}^d \to \mathbf{R}$ by (using the above notation)

$$\langle H_t, p \rangle = \left\langle \varphi_t^* \sum_{j=1}^m \sum_{s\le t} \nu(\Delta z_j(s)) (\varphi_s^{*-1} X_j)(x) v_j(s), p \right\rangle , \quad p \in \mathbf{R}^d . \tag{1.23}$$

Furthermore, the process v_j may be replaced by the process $\tilde{v}_j$ with values in $T_x(\mathbf{R}^d)$, which can be identified with the former one by the expression $v_j = \langle \tilde{v}_j, q \rangle$, $q \in \mathbf{R}^d$. That is, $v_j = \tilde{v}_j(q)$.

We put

$$\tilde{v}_j \equiv \left(\frac{\partial}{\partial x}\varphi_s(x) \right)^{-1} X_j(x_{s-}(x))$$

in what follows. Using this expression, H_t defines a linear mapping from $T_x^*(\mathbf{R}^d)$ to $T_x(\mathbf{R}^d)$ defined by

$$q \mapsto H_t(q) = \varphi_t^* \sum_{j=1}^m \sum_{s\le t} \nu(\Delta z_j(s)) (\varphi_s^{*-1} X_j)(x) \tilde{v}_j(q) . \tag{1.24}$$

We shall identify H_t with this linear mapping.

Let $K_t = K_t(x)$ be the stochastic quadratic form on $\mathbf{R}^d \times \mathbf{R}^d$:

$$K_t(p, q) = \sum_{j=1}^m \sum_{s\le t} \nu(\Delta z_j(s)) \langle (\varphi_s^{*-1} X_j)(x), p \rangle \langle q, (\varphi_s^{*-1} X_j)(x) \rangle .$$

Here, $v(z)$ is a function such that $v(z) \sim |z|^2$ for $|z|$ small. For $0 < s < t$, $K_{s,t} = K_{s,t}(\cdot,\cdot)$ is the bilinear form

$$K_{s,t}(p,q) = \sum_{j=1}^{m} \sum_{s<u\le t} v(\Delta z_j(u)) \langle (\varphi_u^{*-1} X_j)(x), p\rangle \langle q, (\varphi_u^{*-1} X_j)(x)\rangle \ . \tag{1.25}$$

That is, $K_t = K_{0,t}$. Let $K_t(p)$ be the quadratic form associated to $K_t(p,q)$:

$$K_t(p) = \sum_{j=1}^{m} \sum_{0<s\le t} v(\Delta z_j(s)) \langle (\varphi_s^{*-1} X_j)(x), p\rangle^2 \ .$$

We associate it with $H_t(q)$ in (1.24) with $\tilde{v}_j(p)$ above. □

Proof. Part (ii) Since $\sup_{s\le t} |\frac{\partial}{\partial x}\varphi_s(x)| \in L^p$ for all $p > 1$ by [87] (1.4), we only have to show

$$E[|K_t^{-1}(v)|^p] \le C(p), \quad v \in S^{d-1} \ .$$

We may put $t = 1$, and we write $K_1(v)$ as K_1.

We shall show

$$P\{K_1^{-1} > \eta^{-1}\} = P\{K_1 < \eta\} = o_{v,k}(\eta^\infty) \tag{1.26}$$

as $\eta \to 0$. Here, and in what follows, for $\eta > 0$ small, we write $f_i(\eta) = o_i(1)$ if $\lim_{\eta\to 0} f_i(\eta) = 0$ uniformly in i, and $f_i(\eta) = o_i(\eta^\infty)$ if $(f_i(\eta)/\eta^p) = o_i(1)$ for all $p > 1$. Furthermore, since $P(\sup_{k\le \frac{1}{\eta}} |(\frac{\partial \varphi_{k\eta}}{\partial x})^{-1}| > \frac{1}{\eta}) = o_k(\eta^\infty)$ (cf. [143] (1.11)), it is sufficient to show, for some $r > 0$,

$$P\{K_{(k+1)\eta}(\bar{v}) - K_{k\eta}(\bar{v}) < \eta^r | \mathcal{F}_{k\eta}\} = o_{v,k}(1) \tag{1.27}$$

for $(k+1)\eta \le 1$. Here $\bar{v} = |(\frac{\partial \varphi_{k\eta}}{\partial x})^{-1}|^{-1} v$.

To this end, it is enough to show

$$P\left\{ \left[\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^{-r_1},\infty)} \left(\int_{|u|>\eta^r} dv_s(k,\eta)(u) \right) \right] \le \eta^{r_2} \middle| \mathcal{F}_{k\eta} \right\} = o_{v,k}(1) \tag{1.28}$$

for some $r_1 > 0$, $r_2 > 0$, $r_1 > r_2$ suitably chosen. Here, $dv_s(u) = dv_s(k,\eta)(u)$ is the Lévy measure of ΔK_s given $\mathcal{F}_{k\eta}$ for $s > k\eta$ (i.e. $ds \times dv_s(u)$ is the compensator of ΔK_s with respect to dP).

Indeed, since

$$Y_t = \exp\left[\sum_{k\eta\le s\le t} 1_{(\eta^r,\infty)}(\Delta K_s) - \int_{k\eta}^{t} ds \int_{|u|>\eta^r} (e^{-1} - 1) dv_s(u) \right]$$

is a martingale (given $\mathcal{F}_{k\eta}$), we have

$$P\left\{\sum_{k\eta\le s\le(k+1)\eta} 1_{(\eta^r,\infty)}(\Delta K_s)=0|\mathcal{F}_{k\eta}\right\}$$

$$\le E\left[\exp\left[\int_{k\eta}^{(k+1)\eta} ds \int_{|u|>\eta^r}(e^{-1}-1)d\nu_s(u)\right]\right.$$

$$: \left\{\omega;\left[\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^{-r_1},\infty)}\left(\int_{|u|>\eta^r} d\nu_s(u)\right)\right]>\eta^{r_2}\right\}\left|\mathcal{F}_{k\eta}\right]$$

$$+E\left[\exp\left[\int_{k\eta}^{(k+1)\eta} ds \int_{|u|>\eta^r}(e^{-1}-1)d\nu_s(u)\right]\right.$$

$$: \left\{\omega;\left[\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^{-r_1},\infty)}\left(\int_{|u|>\eta^r} d\nu_s(u)\right)\right]\le\eta^{r_2}\right\}\left|\mathcal{F}_{k\eta}\right].$$

By (1.28), the right-hand side is inferior to

$$o_k(1)+E\left[\exp\left[\int_{k\eta}^{(k+1)\eta} ds \int_{|u|>\eta^r}(e^{-1}-1)d\nu_s(u)\right]\right.$$

$$: \left\{\omega;\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^{-r_1},\infty)}\left(\int_{|u|>\eta^r} d\nu_s(u)\right)>\eta^{r_2}\right\}\left|\mathcal{F}_{k\eta}\right]$$

$$\le o_{\nu,k}(1)+\exp[(e^{-1}-1)(\eta^{-r_1}\times\eta^{r_2})]=o_{\nu,k}(1)$$

since $\eta^{r_2-r_1}\to\infty$ as $\eta\to 0$. Hence, we have (1.27). □

The proof of (1.28). For each vector field X, we put the criterion processes

$$Cr(s,X,\nu,k,\eta)\equiv\left\langle\left(\frac{\partial\varphi_s}{\partial x}\right)^{-1}X(x_{s-}(x)),\bar{v}\right\rangle,$$

$$Cr(s,\nu,k,\eta)\equiv\sum_{j=1}^{d}|Cr(s,X_j,\nu,k,\eta)|$$

for $s\in[k\eta,(k+1)\eta]$. By $d\nu_s'(X,\nu,k,\eta)$, we denote the Lévy measure of $Cr(s,X,\nu,k,\eta)$.

To get (1.28), it is sufficient to show that for the given $\eta>0$, there exist integers $n=n(\eta)$, $n_1=n_1(\eta)$ such that

$$P\left\{\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^n,\infty)}(Cr(s,\nu,k,\eta))<\eta^{n_1}\middle|\mathcal{F}_{k\eta}\right\}=o_{\nu,k}(1). \tag{1.29}$$

Indeed, consider the event $\{Cr(s, v, k, \eta) \geq c > 0, s \in [k\eta, (k+1)\eta]\}$. Then, we can show on this event

$$\int_{|u|>\eta^r} d\nu_s(k, \eta)(u) \geq C\eta^{-\alpha r/2} \tag{1.30}$$

for $\eta > 0$ small. Here, $\alpha > 0$ is what appeared in the definition of $g(z)$.

This is because $d\nu_t(k, \eta)(u)$ is the sum of transformed measures of $g_j(z)$ by the mapping

$$F_{j,t} : z \mapsto v(z)\langle \varphi_t^{*-1} X_j(x), \bar{v}\rangle^2 .$$

Since $v(z)$ is equal to $|z|^2$ in a neighbourhood of 0,

$$\int_{|z|>\eta^r} \int d\nu_t(k, \eta)(u) \geq C_1 \int_{\eta^{r/2}}^{\infty} g_j(z)dz$$

for each j. By the definition of $g(z)$, the right-hand side is equal to

$$C_1 \int_{\eta^{r/2}}^{\infty} |z|^{-1-\alpha} dz > C\eta^{-\alpha r/2} .$$

Hence, we can choose $r > 0$ such that $C\eta^{-\alpha r/2} \geq \eta^{-r_1}$ for η small, and thus

$$\int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^{-r_1},\infty)} \left(\int_{|u|>\eta^r} d\nu_s(k, \eta)(u) \right) \geq \eta^{r_2} \tag{1.31}$$

for some $r_1 > 0$, $r_2 > r_1$ on this event.

Following (1.29), the probability of the complement of this event is small ($= o_{e,k}(1)$). Hence, (1.28) follows.

To show (1.29), note that it is equivalent to

$$P\left\{ \int_{k\eta}^{(k+1)\eta} ds 1_{(\eta^n,\infty)}(Cr(s, v, k, \eta)) \geq \eta^{n_1} \middle| \mathcal{F}_{k\eta} \right\} = 1 - o_{v,k}(1) . \tag{1.32}$$

This is proved similarly to [87] Section 3, and we omit the detail. □

Remark 2.2. According to [143] Section 3, the result (1.28) follows under a weaker condition *(URH)* (cf. Section 2.5.1). If $Y_t = x_t(x)$ is a canonical process in Section 2.5.1, we do not need the condition *(A.2)*.

2.1.2 Picard's method

Picard's method is also a way to make a variation of the jump intensity, however it is more general. That is, in the previous method for the jump size perturbation, the key assumption is that the Lévy measure μ has a density g which is differentiable. In case that this assumption fails (e.g. the case of the discrete Lévy measure), we have to use a completely different method for the perturbation. Picard [181] has introduced a method of perturbation using the difference operator based on the previous work by Nualart–Vives [171].

We put $\varphi(\rho) = \int_{|z|\le\rho} |z|^2 \mu(dz)$, $\rho > 0$, which is supposed to satisfy the *order condition*

$$\varphi(\rho) \ge c\rho^{\alpha} \quad \text{as} \quad \rho \to 0 \tag{$*$}$$

for some $c > 0$ and $\alpha \in (0, 2)$. Here, we remark, contrary to the Bismut's case, the Lévy measure can be singular with respect to the Lebesgue measure.

Remark 2.3. The order condition is a non-degeneracy condition for the Lévy measure. It is a basic requirement in the stochastic calculus for jump processes by Picard's method. The quantity $2 - \alpha$ is called the characteristic exponent. The power α describes the order of concentration of masses or "weight" of μ around a small neighbourhood of the origin. Indeed, suppose we extend μ to $\bar{\mu} = a\,\delta_{\{0\}} + \mu$ on $\mathbf{R}^m$, $a \ge 0$, and we interpret $a > 0$ as though the continuous (Wiener) component is nonempty. Let $\bar{\varphi}(\rho) = \int_{|z|\le\rho} |z|^2 \bar{\mu}(dz)$. Then, the condition $(*)$ holds for $\bar{\varphi}(\rho)$ with $\alpha = 0$.

In the sequel, we will study a stochastic calculus of variations (Malliavin calculus) on the Poisson space $(\Omega_2, \mathcal{F}_2, P_2)$.

We work on Ω_2 for a moment, and will denote by ω elements of Ω_2 in place of ω_2. In the sequel, we set $u = (t, z)$ and $\hat{N}(du) = \hat{N}(dtdz)$, $\tilde{N}(du) = N(du) - \hat{N}(du)$. Here, μ is a Lévy measure on $\mathbf{R}^m$ such that $\mu(\{0\}) = 0$ and $\int_{\mathbf{R}^m} (|z|^2 \wedge 1)\mu(dz) < +\infty$.

On the measurable space $(U \times \Omega_2, \mathcal{U} \otimes \mathcal{F}_2)$, we put the measures μ^+, μ^- by

$$\mu^+(dud\omega) = N(\omega, du)P(d\omega)\ ,$$

$$\mu^-(dud\omega) = \hat{N}(\omega, du)P(d\omega)\ .$$

Let $|\mu| = \mu^+ + \mu^-$.

Next, we shall introduce the difference operator $\tilde{D}_u$, $u \in U$, acting on the Poisson space as follows.

For each $u = (t, z) = (t, z_1, \ldots, z_m) \in U$, we define a map $\varepsilon_u^- : \Omega_2 \to \Omega_2$ by

$$\varepsilon_u^- \omega(E) = \omega(E \cap \{u\}^c)\ ,$$

and $\varepsilon_u^+ : \Omega_2 \to \Omega_2$ by

$$\varepsilon_u^+ \omega(E) = \omega(E \cap \{u\}^c) + 1_E(u)\ .$$

Here, $\omega(E)$ denotes de value of the random measure on the set E. We write $\varepsilon_u^{\pm}\omega = \omega \circ \varepsilon_u^{\pm}$, respectively. These are extended to Ω by setting $\varepsilon_u^{\pm}(\omega_1, \omega) = (\omega_1, \varepsilon_u^{\pm}\omega)$.

We observe

$$\text{if} \quad u_1 \ne u_2 \quad \text{then} \quad \varepsilon_{u_1}^{\theta_1} \circ \varepsilon_{u_2}^{\theta_2} = \varepsilon_{u_2}^{\theta_2} \circ \varepsilon_{u_1}^{\theta_1},\ \theta_1, \theta_2 \in \{+, -\}$$

and

$$\varepsilon_u^{\theta_1} \circ \varepsilon_u^{\theta_2} = \varepsilon_u^{\theta_1}, \quad \theta_1, \theta_2 \in \{+, -\} .$$

It holds $\varepsilon_u^- \omega = \omega$ a.s. P_2 for $\hat{N}$-almost all u since $\omega(\{u\}) = 0$ holds for almost all ω for $\hat{N}$-almost all u. Also, we have $\varepsilon_u^+ \omega = \omega$ a.s. P_2 for N-almost all u.

In what follows, we denote a random field indexed by $u \in U$ by Z_u. Let $\mathcal{I} \subset \mathcal{U} \times \mathcal{F}_2$ be the sub σ-field generated by the set $A \times E$, $A \in \mathcal{U}$, $E \in \mathcal{F}_{A^c}$. Here $\mathcal{F}_B$ for $B \in U$ denotes the σ-field on Ω_2 generated by $N(C)$, $C \subset B$.

We remark that for Z_u positive, $Z_u \circ \varepsilon_u^+$ and $Z_u \circ \varepsilon_u^-$ are $\mathcal{I}$-measurable by the definition of $\varepsilon_u^{\pm}$.

We cite the following property of the operators ε_u^+, ε_u^-.

Lemma 2.3 ([182] Corollary 1). *Let Z_u be a positive, $\mu^{\pm}$-integrable random field. Then, we have*

$$E\left[\int Z_u N(du)\right] = E\left[\int Z_u \circ \varepsilon_u^+ \hat{N}(du)\right] ,$$
$$E\left[\int Z_u \hat{N}(du)\right] = E\left[\int Z_u \circ \varepsilon_u^- N(du)\right] .$$

The difference operators $\tilde{D}_u$ for a $\mathcal{F}_2$-measurable random variable F is defined by

$$\tilde{D}_u F = F \circ \varepsilon_u^+ - F . \tag{1.33}$$

Since the image of P_2 by ε_u^+ is not absolutely continuous with respect to P_2, $\tilde{D}_u$ is not well-defined for a fixed u. However, it is defined by $\hat{N}(u) \otimes P_2$-almost surely due to Lemma 2.3.

Since $\tilde{D}_u$ is a difference operator, it enjoys the property

$$\tilde{D}_u(FG) = F\tilde{D}_u G + G\tilde{D}_u F + \tilde{D}_u F \tilde{D}_u G ,$$

assuming that the left-hand side is finite a.s. See (2.11).

The next proposition is a key to the integration-by-parts (duality) formula which appears below.

Proposition 2.1 ([182] Theorem 2). *Let Z_u^1, Z_u^2 be processes such that $Z_u^1 \tilde{D}_u Z_u^2$ and $Z_u^2 \tilde{D}_u Z_u^1$ are $|\mu|$-integrable. Then, we have*

$$E\left[\int Z_u^1 \tilde{D}_u Z_u^2 \tilde{N}(du)\right] = E\left[\int Z_u^2 \tilde{D}_u Z_u^1 \tilde{N}(du)\right] = E\left[\int \tilde{D}_u Z_u^1 \tilde{D}_u Z_u^2 \hat{N}(du)\right] .$$

Proof. We see

$$\begin{aligned} E\left[\int Z_u^1 \tilde{D}_u Z_u^2 \tilde{N}(du)\right] &= E\left[\int Z_u^1 \tilde{D}_u Z_u^2 N(du)\right] - E\left[\int Z_u^1 \tilde{D}_u Z_u^2 \hat{N}(du)\right] \\ &= E\left[\int \left(Z_u^1 \circ \varepsilon_u^+\right) \tilde{D}_u Z_u^2 \hat{N}(du)\right] - E\left[\int Z_u^1 \tilde{D}_u Z_u^2 \hat{N}(du)\right] \\ &= E\left[\int \tilde{D}_u Z_u^1 \tilde{D}_u Z_u^2 \hat{N}(du)\right] \end{aligned}$$

due to Lemma 2.3. If we interchange Z_u^1 and Z_u^2, we have another equality. □

This leads to the following proposition.

Proposition 2.2 ([182] Corollary 5). *Let F be a bounded variable, and let Z_u be a random field which is $\mu^\pm$-integrable. Then, we have*

$$(1) \qquad E\left[\int Z_u \tilde{D}_u F N(du)\right] = E\left[F \int (Z_u \circ \varepsilon_u^+)\, \tilde{N}(du)\right]$$

$$(2) \qquad E\left[\int Z_u \tilde{D}_u F \hat{N}(du)\right] = E\left[F \int (Z_u \circ \varepsilon_u^-)\, \tilde{N}(du)\right].$$

In particular, if Z_u is $\mathcal{I}$-measurable,

$$E\left[\int Z_u \tilde{D}_u F \hat{N}(du)\right] = E\left[F \int Z_u \tilde{N}(du)\right].$$

Proof. The assertion (2) is obtained from the previous proposition and Lemma 2.3 by putting $Z_u^1 = F$ and $Z_u^2 = Z_u$. Assertion (1) is obtained similarly by putting $Z_u^1 = Z_u$, $Z_u^2 = F$ and from the assertion (2). □

The adjoint $\tilde{\delta}$ of the operator $\tilde{D} = (\tilde{D}_u)_{u \in U}$ is defined as follows.

We denote by $\mathcal{S}$ the set of the random fields Z_u which are $\mathcal{I}$-measurable, bounded, and of compact support[1]. Let

$$\bar{\mathcal{S}} = \left\{ Z \in L^2(U \times \Omega_2, \mathcal{I}, \mu^-); \quad \text{there exist } Z_n \in \mathcal{S},\, Z_n \to Z \text{ in } \|.\| \right\}.$$

Here, $\|.\|$ is given by

$$\|Z\|^2 = E\left[\int |Z_u|^2 \hat{N}(du)\right] + E\left[\left(\int Z_u \tilde{N}(du)\right)^2\right].$$

For $Z \in \mathcal{S}$, we define $\delta_0(Z)$ by

$$E[F\delta_0(Z)] = E\left[\int Z_u \tilde{D}_u F \hat{N}(du)\right]$$

as an adjoint operator of $\tilde{D}_u$, where F is bounded and of compact support. The last assertion of Proposition 2.1 implies that the mapping $Z \mapsto \int Z_u \tilde{N}(du)$ can be extended to the elements in $\bar{\mathcal{S}}$. Hence, we define $\delta_0(Z)$ as above for $Z \in \bar{\mathcal{S}}$.

Let $Z_u \in L^2(U \times \Omega_2, \mathcal{U} \otimes \mathcal{F}_2, \mu^-)$ be such that $\|Z\| < +\infty$. We denote by $\mathcal{Z}$ the set of all such random fields Z_u. We observe

$$Z_u \circ \varepsilon_u^- \in \bar{\mathcal{S}}.$$

1 It means it is zero outside of a set which is of finite μ^- measure.

Hence, we put

$$\tilde{\delta}(Z) = \delta_0(Z_u \circ \varepsilon_u^-)$$

to extend δ_0 to $\tilde{\delta}$. In view of the last assertion of Proposition 2.2, we are ready to define the adjoint operator.

We set for $Z \in \mathcal{Z}$,

$$\tilde{\delta}(Z) = \int_U Z_u \circ \varepsilon_u^- \tilde{N}(du) \,. \tag{1.34}$$

By Proposition 2.2 and by the definition of $\tilde{\delta}$, the operator satisfies the adjoint property

$$E[F\tilde{\delta}(Z)] = E\left[\int_U \tilde{D}_u F Z_u \hat{N}(du)\right] \tag{1.35}$$

for any bounded $\mathcal{F}_2$-measurable random variable F ([181] (1.12)).

Remark 2.4. The origin of the operators $\varepsilon_u^+, \varepsilon_u^-$ are $\varepsilon_t^+, \varepsilon_t^-$, respectively, used in [172, 182], where $\varepsilon_t^+ = \varepsilon_{(t,1)}^+$ and $\varepsilon_t^- = \varepsilon_{(t,1)}^-$. A main point in those days was to show the duality formula using the operator D_t given by $D_t F = F \circ \varepsilon_t^+ - F \circ \varepsilon_t^-$. Nualart–Vives [172] showed the duality formula (1.35) for D_t by using the chaos decomposition on the Poisson space. We will discuss this topic in Section 3.2.

We have the Meyer's type inequality (Theorem 3.4 in Section 3.3), and the operator $\tilde{\delta}$ is well-defined on the Sobolev space $\mathbf{D}_{k,0,p}$ for $k \geq 0$ and $p \geq 2$. This implies $\tilde{D}_u$ is closable for $p \geq 2$. (For the Sobolev space $\mathbf{D}_{k,l,p}$ over the Wiener–Poisson space, see Section 3.3.)

In the remaining part of this subsection, we shall briefly sketch how to derive the integration-by-parts setting using these operators.

We introduce a linear map $\tilde{Q}_\rho$ by

$$\tilde{Q}_\rho Y = \frac{1}{\varphi(\rho)} \int_{A(\rho)} (\tilde{D}_u F)\tilde{D}_u Y \hat{N}(du) \,. \tag{1.36}$$

Lemma 2.4. *The adjoint of $\tilde{Q}_\rho$ exists and is equal to*

$$\tilde{Q}_\rho^* X = \tilde{\delta}_\rho((\tilde{D}F)^T X) \,, \tag{1.37}$$

where

$$\tilde{\delta}_\rho(Z) = \frac{1}{\varphi(\rho)} \tilde{\delta}(Z 1_{A(\rho)}) = \frac{1}{\varphi(\rho)} \int_{A(\rho)} Z_u \circ \varepsilon_u^- \tilde{N}(du) \,. \tag{1.38}$$

Here $A(\rho) = \{u = (t, z) \in U; |z| \leq \rho\}$.

Let $f(x)$ be a C^2-function with bounded derivatives. We claim a modified formula of integration by parts. Concerning the difference operator $\tilde{D}_u$, we have by the mean value theorem,

$$\tilde{D}_u(f(G)) = (\tilde{D}_u G)^T \int_0^1 \partial f(G + \theta \tilde{D}_u G) d\theta \tag{1.39}$$

for a random variable G on the Poisson space. This implies

$$\tilde{Q}_\rho f(F) = \tilde{R}_\rho \partial f(F) + \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F(\tilde{D}_u F)^T \left(\int_0^1 \{\partial f(F + \theta \tilde{D}_u F) - \partial f(F)\} d\theta \right) \hat{N}(du) . \quad (1.40)$$

Here,

$$\tilde{R}_\rho = \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F(\tilde{D}_u F)^T \hat{N}(du) . \quad (1.41)$$

Use (1.40) and then take the inner product of this with $S_\rho X$. Its expectation yields the following.

Proposition 2.3 ([95] Analogue of the formula of integration by parts). *For any X, we have*

$$E[(X, \partial f(F))] = E[\tilde{Q}_\rho^*(S_\rho X) f(F)] - \frac{1}{\varphi(\rho)} E\left[\left(X, S_\rho \int_{A(\rho)} \tilde{D}_u F(\tilde{D}_u F)^T \left(\int_0^1 \{\partial f(F + \theta \tilde{D}_u F) - \partial f(F)\} d\theta \right) \hat{N}(du) \right)\right] . \quad (1.42)$$

Here, $S_\rho = \tilde{R}_\rho^{-1}$.

Remark 2.5. If F is a Wiener functional, then the formula is written shortly as

$$E[(X, \partial f(F))] = E[Q^*(R^{-1}X) f(F)] = E[\delta((R^{-1}X, DF)) f(F)] . \quad (1.43)$$

See [95] for details.

On the other hand, if $\tilde{R}_\rho$ is not zero or equivalently $\tilde{Q}_\rho$ is not zero, we have a remaining term (the last term of (1.42)). We have this term even if Z_t is a simple Poisson process N_t or its sums. However, if we take $f(x) = e^{i(w,x)}$, $w \in \mathbf{R}^d \setminus \{0\}$, then $\partial f(x) = i e^{i(w,x)} w$ and

$$e^{i(w, F + \theta \tilde{D}_u F)} - e^{i(w,F)} = e^{i(1-\theta)(w,F)} \tilde{D}_u (e^{i(w, \theta F)}) .$$

Hence, we have an expression of the integration-by-parts for the functional

$$E[(X, w) \partial_x (e^{i(w,F)})] = E[(\tilde{Q}_\rho^* + R_{\rho,w}^*) S_\rho X \cdot e^{i(w,F)}], \quad \forall w . \quad (1.44)$$

Here,

$$R_{\rho,w}^* Y = -\frac{i}{\varphi(\rho)} \int_0^1 \left(\tilde{\delta}(e^{i(1-\theta)(w,F)} \chi_\rho \tilde{D} F(\tilde{D} F)^T Y), e^{i(\theta-1)(w,F)} w \right) d\theta . \quad (1.45)$$

We will continue the analysis using this method in Section 3.3.

2.1.3 Some previous methods

Apart from perturbation methods by Bismut and Picard (stated in Sections 2.1.1, 2.1.2) on the Poisson space, there are previously examined perturbation methods which are intuitively more comprehensive. We will briefly recall two of them in this section.

Bismut method (2)

Apart from the Bismut method in Section 2.1.1, a very naive integration by parts formula based on Bismut's idea (perturbation on the jump size and the direction) can be carried out. See P. Graczyk [40, 73] for more details.

In effect, Graczyk has applied this method on a nilpotent group G. Here, we adapt his method to the case $G = \mathbf{R}^d$ for simplicity. A key point of this method is that we can explicitly describe the probability law of the jumps caused by the Lévy process by using the Lévy measure.

Let $m = d$. We consider a jump process given by the stochastic differential equation (SDE)

$$dx_t(x) = \sum_{j=1}^{d} X_j dz_j(t) \,, \quad x_0(x) = x \,.$$

Here, $X_j's$ denote d-dimensional vectors in $\mathbf{R}^d$ which constitute a linear basis, and $z_j(t)$'s are scalar Lévy processes. That is,

$$x_t(x) = Az(t) + x, \quad x_0(x) = x \,.$$

Here, A is a $d \times d$-matrix $A = (X_1, \ldots, X_d)$, and $z(t) = (z_1(t), \ldots, z_d(t))$. By taking a linear transformation, we may assume A is a unitary matrix without loss of generality.

We choose a positive function $\rho \in C^\infty$, $0 \le \rho \le 1$ such that $\rho(z) = 1$ if $|z| \ge 1$, $\rho(z) = 0$ if $|z| \le \frac{1}{2}$. We put $\rho_\eta(z) \equiv \rho(\frac{z}{\eta})$, $\eta > 0$.

We would like to have the integration-by-parts formula

$$E[f^{(\alpha)}(x_T(x))] = E[f(x_T(x)) \cdot M_T^{(\alpha)}], \quad \alpha = 1, 2, \ldots, f \in C^\infty \,.$$

This can be done by elementary calculus as below. We put $T = 1$ for simplicity. In fact, Graczyk [73] has studied a kind of stable process on a homogeneous group. Viewing $\mathbf{R}^d$ as a special case of a nilpotent group, we mimic his calculation to illustrate the idea in the simple case.

(Step 1)
Let $v(z)$ be a C^2-function such that

$$v(z) \sim z^2 \tag{1.46}$$

for $|z|$ small. Choose $\epsilon > 0$ and fix it.

Let $N_j^\epsilon(dsdz)$ be the Poisson counting measure with mean measure

$$ds \times g^\epsilon(z)dz \equiv ds \times \rho_{2\epsilon}(z)g(z)dz \ .$$

We write $z_j^\epsilon(s) = \int_0^s \int zN_j^\epsilon(dsdz)$, $j = 1, \cdots, d$. We denote by $N = N^\epsilon \equiv N_1^\epsilon + \cdots + N_d^\epsilon$ where $N_j^\epsilon = \int_0^1 \int N_j^\epsilon(dsdz)$, and by $S_1, S_2, \cdots$, the jump times of N^ϵ. Let $\langle e_1, \cdots, e_d\rangle$ be a standard basis of $\mathbf{R}^d$, and let $\langle \bar{e}_1, \cdots, \bar{e}_d\rangle = \langle Ae_1, \cdots, Ae_d\rangle$.

Let x_s^ϵ be the solution of the SDE

$$dx_s^\epsilon = \sum_{j=1}^d X_j dz_j^\epsilon(s), \quad x_0^\epsilon \equiv x \ . \tag{1.47}$$

First, we integrate by parts with respect to the process $\{x_s^\epsilon\}$. We shall calculate the expectation of

$$\langle Df(x_1^\epsilon), e\rangle = \sum_{r=1}^d D_r f(x_1^\epsilon)e_r$$

below. Here and in the following calculation, we use Df and $D_r f$ in the sense $(\frac{\partial f}{\partial x_i})_{i=1,\ldots,d}$ and $\frac{\partial f}{\partial x_r}$, respectively.

Since $|\Delta z_j^\epsilon(s)| > \epsilon$ uniformly, we have

$$E\left[\rho_\eta\left(\sum_{r'=1}^d \sum_{s\le 1}\sum_{j=1}^d \nu(\Delta z_j^\epsilon(s))\langle X_j, e_{r'}\rangle^2\right) \times \sum_{r=1}^d \langle Df(x_1^\epsilon), e_r\rangle\right]$$
$$= \sum_{n=1}^\infty P(N=n)E\Big[\rho_\eta\Big(\sum_{r'=1}^d \sum_{k=1}^n \sum_{j=1}^d \nu(\Delta z_j^\epsilon(S_k))$$
$$\times \langle X_j, e_{r'}\rangle^2\Big) \sum_{r=1}^d \langle Df(x_1^\epsilon), e_r\rangle\Big] \ .$$

We introduce the following notations to simplify the argument:

$$(X * \nu)\,(1) = \sum_{r'=1}^d \sum_{s\le 1}\sum_{j=1}^d \nu\left(\Delta z_j^\epsilon(s)\right)\langle X_j, e_{r'}\rangle^2 \ ,$$

$$X * \nu = \sum_{r'=1}^d \sum_{i=1}^n \sum_{j=1}^d \nu\left(\Delta z_j^\epsilon(S_i)\right)\langle X_j, e_{r'}\rangle^2 \ ,$$

$$(X * \nu)_k = \sum_{r'=1}^d \sum_{j=1}^d \nu\left(\Delta z_j^\epsilon(S_k)\right)\langle X_j, e_{r'}\rangle^2 \ .$$

Then, $X * \nu = \sum_{k=1}^n (X * \nu)_k$, and

$$\sum_{r'=1}^d \sum_{i\ne k}^n \sum_{j=1}^d \nu\left(\Delta z_j^\epsilon(S_i)\right)\langle X_j, e_{r'}\rangle^2 = \sum_{i\ne k}(X * \nu)_i \ .$$

In the right-hand side of the above formula, we freeze the variable $\Delta z_j^\epsilon(s_k)$ and view it as $z_\ell \in \operatorname{supp} g^\epsilon$, showing that

$$E[\cdots] = \sum_{k=1}^{n}\sum_{r=1}^{d} E\Big[\Big(\frac{\rho_\eta(X*\nu)}{X*\nu}\Big)(X*\nu)_k\langle Df(x_1^\epsilon), e_r\rangle\Big]$$
$$= \sum_{k=1}^{n}\sum_{r=1}^{d}\sum_{\ell=1}^{d} E\left[\int\Big(\frac{\rho_\eta(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2)}{\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2}\Big)\right.$$
$$\left.\times\left\{\sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2\right\}\langle Df(x_1^\epsilon), e_r\rangle \times \frac{g^\epsilon(z_\ell)}{\int g^\epsilon(z_\ell)dz_\ell}dz_\ell\right].$$

Here, we make an integration by parts on $\mathbf{R}^d$ with respect to z_ℓ, viewing x_1^ϵ as a functional of $z = (z_1, \ldots, z_d)$. To this end, we compare $Df(x_1^\epsilon(z)) = \frac{\partial f}{\partial x}(x_1^\epsilon(z))$ with $\frac{\partial}{\partial z}(f \circ x_1^\epsilon)(z) = Df(x_1^\epsilon(z)) \cdot \frac{\partial x_1^\epsilon}{\partial z}(z)$. Here $\frac{\partial x_1^\epsilon}{\partial z}(z) = A$ if $|z_i| > \epsilon$, $i = 1, \ldots, d$.

Then, we have

$$\int \cdots dz_\ell$$
$$= -\int\left[\frac{\rho'_\eta(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2)}{\left(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2\right)}\right.$$
$$\left.-\frac{\rho_\eta(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2)}{\left(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2\right)^2}\right]$$
$$\times\frac{\partial}{\partial z_\ell}\left\{\sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, \bar{e}_{r'}\rangle^2\right\}$$
$$\times\left\{\sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2\right\} f(x_1^\epsilon)\frac{g^\epsilon(z_\ell)}{\int g^\epsilon(z_\ell)dz_\ell}dz_\ell$$
$$-\int\Big(\frac{\rho_\eta(\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2)}{\sum_{i\neq k}(X*\nu)_i + \sum_{r'=1}^{d}\nu(z_\ell)\langle X_\ell, e_{r'}\rangle^2}\Big)$$
$$\times\sum_{r'=1}^{d}\frac{\nabla_{z_\ell}\{\nu(z_\ell)\langle X_\ell, \bar{e}_{r'}\rangle^2\cdot g^\epsilon(z_\ell)\}}{g^\epsilon(z_\ell)} f(x_1^\epsilon)\frac{g^\epsilon(z_\ell)}{\int g^\epsilon(z_\ell)dz_\ell}dz_\ell\,. \tag{1.48}$$

We suppress below the superscript ϵ for simplicity. Note that we have

$$\nabla_{z_\ell}\{\nu(z_\ell)\langle X_\ell, \bar{e}_{r'}\rangle^2\cdot g(z_\ell)\} = \nabla_{z_\ell}\{\nu(z_\ell)\cdot g(z_\ell)\}\times\langle X_\ell, \bar{e}_{r'}\rangle^2\,. \tag{1.49}$$

We put

$$\Phi_0 = \frac{\rho_\eta((X*\nu)(1))}{(X*\nu)(1)}\,, \tag{1.50}$$

$$\Phi_1 = \left[\frac{\rho'_\eta((X*\nu)(1))}{((X*\nu)(1))} - \frac{\rho_\eta((X*\nu)(1))}{((X*\nu)(1))^2}\right]. \tag{1.51}$$

Then, we have the following expression:

$$E\left[\rho_\eta((X*\nu)(1)) \times \sum_{r=1}^{d} \langle Df(x_1), e_r\rangle\right]$$
$$= -E\left[\Phi_1 \left\{\sum_{k=1}^{N}\sum_{\ell=1}^{d}\sum_{r=1}^{d} \frac{\partial}{\partial z_\ell}\{\nu(z_\ell)\langle X_\ell, \bar{e}_r\rangle^2\}(\Delta z_\ell(S_k))\right.\right.$$
$$\left.\left.\times \left\{\sum_{r'=1}^{d} \nu(\Delta z_\ell(S_k))\langle X_\ell, e_{r'}\rangle^2\right\} f(x_1^\epsilon)\right\}\right]$$
$$- E\left[\Phi_0 \left\{\sum_{k=1}^{N}\sum_{\ell=1}^{d}\sum_{r=1}^{d}\left\{\left(\frac{\nabla_{z_\ell}\{\nu(z_\ell)\cdot g(z_\ell)\}}{g(z_\ell)}\right)(\Delta z_\ell(S_k)) \times \langle X_\ell, \bar{e}_r\rangle^2\right\} f(x_1^\epsilon)\right\}\right] .$$

We put (cf. (1.48), (1.49))

$$\psi_{1,\ell}(z) = \sum_{r=1}^{d} \frac{\partial}{\partial z}\{\nu(z)\langle X_\ell, \bar{e}_r\rangle^2\}(z) ,$$
$$\psi_2(z) = \frac{\nabla_z\{\nu(z)\cdot g(z)\}}{g(z)} . \tag{1.52}$$

Then,

$$\left|E\left[\rho_\eta((X*\nu)(1)) \sum_{r=1}^{d} \langle Df(x_1), e_r\rangle\right]\right|$$
$$\le \|f\|_\infty \left\{E\left[\left|\Phi_1 \sum_{s\le 1}\sum_{\ell=1}^{d} \psi_{1,\ell}(\Delta z_\ell(s))\left\{\sum_{r'=1}^{d} \nu(\Delta z_\ell(s))\langle X_\ell, e_{r'}\rangle^2\right\}\right|\right]\right.$$
$$\left. + E\left[\left|\Phi_0 \sum_{s\le 1}\sum_{\ell=1}^{d}\sum_{r=1}^{d}\left\{\psi_2(\Delta z_\ell(s))\langle X_\ell, \bar{e}_r\rangle^2\right\}\right|\right]\right\} .$$

Hence, we can let $\epsilon \to 0$ to obtain

$$E\left[\rho_\eta((X*\nu)(1)) \sum_{r=1}^{d} \langle Df(x_1), e_r\rangle\right]$$
$$= -E\left[\Phi_1\left(\sum_{s\le 1}\sum_{\ell=1}^{d} \psi_{1,\ell}(\Delta z_\ell(s)) \times \left(\sum_{r'=1}^{d} \nu(\Delta z_\ell(s))\langle X_\ell, e_{r'}\rangle^2\right)\right) f(x_1^\epsilon)\right]$$
$$- E\left[\sum_{r=1}^{d} \Phi_0\left(\sum_{s\le 1}\sum_{\ell=1}^{d} \psi_2(\Delta z_\ell(s)) \times \langle X_\ell, \bar{e}_r\rangle^2\right) f(x_1^\epsilon)\right] . \tag{1.53}$$

In the left-hand side of (1.53), we can show

$$\left[(X*\nu)(1)\right]^{-1} \in L^p, \quad p \ge 1 . \tag{1.54}$$

The proof proceeds similarly to Lemma 2.2, and we omit the detail. Indeed, we consider the form

$$t \mapsto \sum_{r'=1}^{d} \sum_{s \le t} \sum_{j=1}^{d} \nu(\Delta z_j^{\epsilon}(s)) \langle X_j, e_{r'} \rangle^2$$

instead of $K_t(p)$, and the idea of the proof is the same as in Section 2.1.1.

(Step 2, $\eta \to 0$)
By the last expression, we have

$$\begin{aligned} &\left| E\left[\rho_\eta((X * \nu)(1)) \sum_{r=1}^{d} \langle Df(x_1), e_r \rangle \right] \right| \\ &\qquad \le \|f\|_\infty \left\{ E\left[\left| \Phi_1 \sum_{s \le 1} \sum_{\ell=1}^{d} \psi_{1,\ell}(\Delta z_\ell(s)) \left\{ \sum_{r'=1}^{d} \nu(\Delta z_\ell(s)) \langle X_\ell, e_{r'} \rangle^2 \right\} \right| \right] \right. \\ &\qquad\qquad \left. + E\left[\left| \Phi_0 \left\{ \sum_{s \le 1} \sum_{\ell=1}^{d} \sum_{r=1}^{d} \{ \psi_2(\Delta z_\ell(s)) \langle X_\ell, \bar{e}_r \rangle^2 \} \right\} \right| \right] \right\} , \end{aligned}$$

and the right-hand side is independent of $\eta > 0$. By (1.54), $\{\sum_{r'=1}^{d} \sum_{s \le 1} \sum_{j=1}^{d} \nu(\Delta z_j(s)) \langle \cdots \rangle^2\} > 0$ a.e. Hence, by letting $\eta \to 0$, we have

$$\begin{aligned} &E\left[\rho_\eta \left(\sum_{r'=1}^{d} \sum_{s \le 1} \sum_{j=1}^{d} \nu(\Delta z_j(s)) \langle X_j, e_{r'} \rangle^2 \right) \sum_{r=1}^{d} \langle Df(x_1), e_r \rangle \right] \\ &\quad \to E\left[\sum_{r=1}^{d} \langle Df(x_1), e_r \rangle \right] \end{aligned} \tag{1.55}$$

in L^p.

On the other hand, $\sum_{s \le 1} \sum_{\ell=1}^{d} \psi_{1,\ell}(\Delta z_\ell(s)) \langle X_\ell, e_{r'} \rangle^2 \in L^p$, $p > 1$, and $\sum_{s \le 1} \sum_{\ell=1}^{d} \psi_2(\Delta z_\ell(s)) \langle X_\ell, \bar{e}_r \rangle^2 \in L^p$, $p > 1$, since $\nu(\cdot)$ and $\nu'(\cdot)$ are C^∞ and have compact support. Hence, we can let $\eta \to 0$ in the above as before, and we have

$$E\left[\sum_{r=1}^{d} \langle Df(x_1), e_r \rangle \right] = E[f(x_1)\mathcal{H}_1] , \tag{1.56}$$

where

$$\begin{aligned} \mathcal{H}_1 = &\left[(X * \nu)(1) \right]^{-2} \times \sum_{s \le 1} \sum_{\ell=1}^{d} \psi_{1,\ell}(\Delta z_\ell(s)) \sum_{r'=1}^{d} \langle X_\ell, e_{r'} \rangle^2 \\ &- \left[(X * \nu)(1) \right]^{-1} \times \sum_{s \le 1} \sum_{\ell=1}^{d} \sum_{r=1}^{d} \psi_2(\Delta z_\ell(s)) \langle X_\ell, \bar{e}_r \rangle^2 . \end{aligned}$$

We have an integration-by-parts formula of order 1.

Calculation for the integration by parts of order 2 is similar, and we omit the detail. We can proceed to the higher order calculation in a similar way.

Time shifts

Consider a one-dimensional standard Poisson process N_t and denote one of its jump times (jump moment) by T. We consider the shift of the *jump times* of N_t. This works in principle since the inter jump times of the Poisson process has an exponential law which has smooth density. We can compare this method with the previous ones by Bismut which make shifts with respect to the jump size (and direction) of the driving process.

It was first studied by E. Carlen and E. Pardoux [43] in 1990, and subsequently (and presumably, independently) by R. J. Elliot and A. H. Tsoi [56] in 1993. Elliot and Tsoi have obtained the following integration-by-parts formula:

$$E\left[\frac{\partial G}{\partial t}(T)\int_0^T x_s ds\right] = -E\left[\left(\sum_{s\geq 0} x_s \Delta N_s\right) G(T)\right] ,$$

where $G(t)$ denotes some suitable function and T denotes a jump time. Note that the right-hand side $= -E[\delta(u)G(T)]$ in the contemporary notation.

Subsequently, V. Bally et al. [8] has proceeded along this idea. Namely, let $T_1, T_2, \ldots$ be the successive jump times of N_t and let $F = f(T_1, \ldots, T_n)$ be a functional of N. The gradient operator $D = D_t$ is given by

$$D_t F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(T_1, \ldots, T_n) \cdot 1_{\{T_i > t\}} . \tag{1.57}$$

This gradient operator is a differential operator and enjoys the chain rule. This is a main point of this operator, compared to the difference operator $\tilde{D}_u$ which appeared in Section 2.1.2.

However, by operating D again to both sides of (1.57), $D^2 F$ contains the term $D1_{\{T_i>t\}}$, which is singular. This is the price one pays for enjoying the chain rule. As a result, there are some problems with respect to showing the smoothness of the density. Precisely, since we can not expect the iterated use of the operator, we can not apply the higher order derivatives at the same point of time within this theory.

In Bally et al. [8] [9], to avoid the iterated use of the operation by D, a more moderate expression of the gradient operator using the approximation procedure is adopted. We introduce it below.

Let us introduce dumping functions (π_i) which cancel the effect of singularities coming from $D^2F, D^3F, \ldots$ in the manner of (1.57). Namely, let $a_i, b_i, i = 1, 2, \ldots, m$ be random variables on Ω_2 such that $-\infty \leq a_i < b_i \leq +\infty$, and consider weight functions (π_i) such that $0 \leq \pi_i \leq 1$, π_i is smooth on (a_i, b_i), $\pi_i(a_i) = \pi_i(b_i) = 0$, and that $\pi_i > 0$ on (a_i, b_i), $i = 1, 2, \ldots, m$. We remark that the intervals (a_i, b_i) are not necessarily distinct.

For $F = f(T_1, \ldots, T_m)$, we give the dumped derivative by

$$D_i^\pi F = \pi_i \cdot \partial_i f(T_1, \ldots, T_m) . \tag{1.58}$$

Let $D^\pi F = (D_i^\pi F)_{i=1,\ldots,m}$, and call them π derivatives. Then, we can use $D^\pi F$ in place of DF in (1.57) (viewed at $t = T_i$).

One advantage of this notation is that it is a differential operator, and thus it still satisfies the chain rule

$$D_i^\pi \varphi(F) = \varphi'(F) \cdot D_i^\pi F \tag{1.59}$$

for $F = f(T_1, \ldots, T_m)$.

The other advantage is that, due to the dumping effect of π_i at each edge of (a_i, b_i), we can obtain the integration-by-parts formula: for $F_m = f(T_1, \ldots, T_m)$,

$$E[\partial^\beta g(F_m)] = E[g(F_m) H_\beta^m], \quad \beta = 1, 2, \ldots \tag{1.60}$$

where H_β^m is a random variable defined by using D^π.

More precisely, consider a functional $F = f((T_i))$ on Ω_2 of the form

$$f((T_i)) = \sum_{j=1}^{\infty} f^j(T_1, \ldots, T_j) 1_{\{J(\omega)=j\}} .$$

Here, $J(\omega)$ is an integer-valued random variable on Ω_2.

A functional F of such form with some $f \in C_\pi^\infty$ is called a *simple functional*. The set of simple functionals is denoted by S. Here, we put

$$C_\pi^k = \{f \,;\, f \text{ admits partial } \pi \text{ derivatives up to order } k \text{ on } (a_i, b_i) \\ \text{w.r.t. each variable } t_i\} ,$$

and $C_\pi^\infty = \cap_{k=1}^\infty C_\pi^k$. A sequence $U = (U_i)$ of simple functionals $U_i \in S$ is called a *simple process*. The set of simple processes is denoted by $\mathcal{P}$ in this subsection.

The adjoint operator δ of D^π is defined for $U \in \mathcal{P}$ by

$$\delta_i(U) = -\partial_i(\pi_i U_i) + U_i 1_{\{p_J > 0\}} \partial_i^\pi \log p_J$$

and

$$\delta(U) = \sum_{i=1}^{J} \delta_i(U) .$$

Here, p_J denotes the density of the law of $(T_1, \ldots, T_j)$ with respect to the Lebesgue measure on $\{J = j\}$. We assume $\{p_J > 0\}$ is open in $\mathbf{R}^J$ and $p_J \in C_\pi^\infty$ on $\{p_J > 0\}$. We remark that π_i is a localization factor (weight), and thus it is expected that

$$\Pi_{i=1}^{j} \operatorname{supp} \pi_i \subset \operatorname{supp} p_j \quad \text{on } \{J = j\}.$$

Then, we have

Proposition 2.4. *Under the above assumption*

$$E[(D^\pi F, U)] = E[F\delta(U)] \tag{1.61}$$

for $F \in S$, $U \in \mathcal{P}$.

See [9] Proposition 1.

Repeating this calculus and using the chain rule (1.59), we reach the formula (1.60) under the assumption that

$$E[|\gamma(F)|^p] < +\infty \quad \text{for all } p \geq 1 .$$

Here,

$$\gamma(F) = \sigma^{-1}(F)$$

with $\sigma(F) = \sum_{j=1}^{J}(D_j^{\pi}F)^2$. In particular, $H_1^m = \delta(\gamma(F)D^{\pi}F)$ for $\beta = 1$.

The formula (1.60) gives

$$|\hat{p}_{F_m}(v)| \leq |v|^{-\beta} E\left[\left|H_\beta^m\right|\right]$$

by choosing $g(x) = e^{i(x,v)}$. Instead of letting $F_m \to F$ in $L^2(\Omega_2)$ as $m \to +\infty$, we would show

$$\sup_m E\left[\left|H_\beta^m\right|\right] < +\infty$$

with a suitable choice of (π_i). This implies

$$|\hat{p}_F(v)| \leq C(1 + |v|^2)^{-\beta/2}\,, \quad \beta = 1, 2, \ldots$$

as desired.

The time stratification method by A. Kulik [119, 120] can also be regarded as a variation of the time shift method. See also Fournier [64].

In the process of inquiring into the Fock space representation of the Poisson space, Nualart and Vives [171] have introduced the translation operator

$$\Psi_t(\omega) = \omega + \delta_{\{t\}}\,,$$

where ω is a generic element in the Poisson space. This operation can be viewed as adding a new jump time at t to ω associated to $N_{.}$. This operator induces a translation of the functionals by

$$\Psi_t(F)(\omega) = F(\omega + \delta_{\{t\}}) - F(\omega)\,.$$

This operation leads to the difference operator $\tilde{D}_u$ by Picard [181]. Confer Section 2.1.2.

2.2 Methods of finding the asymptotic bounds (I)

In this section and the next, we recall some results stated in Sections 2.1.1, 2.1.2 and, assuming the existence of a transition density $p_t(x, y)$ for the process X_t, we explain various ideas of finding the asymptotic bounds of the density function $p_t(x, y)$ as $t \to 0$ based on methods employed in the proof. Here, X_t denotes a solution of some SDE. The analysis to lead the existence and smoothness of the density function is carried out throughout Chapter 3.

The estimate is closely related, as will be seen in proofs, with fine properties of the trajectories in the short time, and thus with the perturbation methods stated in the previous section. Also, from an analytical point of view, it provides fine properties of the fundamental solution in the short time associated with the integro-differential equation.

In this section, we construct Markov chains which approximate the trajectory of X_t. Throughout this and the next sections, we confine ourselves to the case of the Itô process $X_t = x_t(x)$ given in Section 1.3. In the proof of Theorem 2.3 (b) below, we introduce a supplementary process $x_s(r, S_t, x)$ which is obtained from $x_s(x)$ by directing the times (moments) of big jumps of $x_s(x)$ during $[0, t]$. We will encounter this again in Section 3.6.4.

Our method stated in this and in the next section is not effective to lead the large time ($t > 1$) estimate of the density, and we can not obtain sharp results in such a case. We do not mention this case in this book.

2.2.1 Markov chain approximation

Let $z(s)$ be an $\mathbf{R}^m$-valued Lévy process with Lévy measure $\mu(dz)$: $\int_{\mathbf{R}^m}(|z|^2 \wedge 1)\mu(dz) < +\infty$. That is, the characteristic function ψ_t is given by

$$\psi_t(v) = E[e^{i(v,z(t))}] = \exp(it(v,c) + t\int(e^{i(v,z)} - 1 - i(v,z)1_{\{|z|\leq 1\}})\mu(dz)) \; .$$

We may write $z(s) = cs + \int_0^s \int_{\mathbf{R}^m\setminus\{0\}} z(N(dudz) - du1_{\{|z|\leq 1\}}.\mu(dz))$, where N is a Poisson random measure with mean $du\times\mu(dz)$. Let $\gamma(x,z) : \mathbf{R}^d\times\mathbf{R}^m \to \mathbf{R}^d$ and $b(x) : \mathbf{R}^d \to \mathbf{R}^d$ be C^∞ -functions, whose derivatives of all orders are bounded, satisfying $\gamma(x, 0) = 0$.

We carry out our study in the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, where $\Omega = D([0,+\infty])$ (Skorohod space), $(\mathcal{F}_t)_{t\geq 0}$ = filtration generated by $z(s)$, and P = probability measure on ω of $z(s)$. That is, $\mu(dz) = P(z(s+ds) - z(s) \in dz|z(s))/ds$.

Consider the following SDE studied in Section 1.3, that is,

$$x_t(x) = x + \int_0^t b(x_s(x))ds + \sum_{s\leq t}^{c} \gamma(x_{s-}(x), \Delta z(s)) \; . \tag{2.1}$$

Equivalently, we may write $x_t(x)$ as

$$\begin{aligned} x_t(x) = x &+ \int_0^t b'(x_s(x))ds + \int_0^t \int_{|z|\leq 1} \gamma(x_{s-}(x), z)\tilde{N}(dsdz) \\ &+ \int_0^t \int_{|z|>1} \gamma(x_{s-}(x), z)N(dsdz) \; , \end{aligned} \tag{2.2}$$

where $\tilde{N}$ denotes the compensated Poisson random measure: $\tilde{N}(dsdz) = N(dsdz) - ds\mu(dz)$, $b'(x) = b(x) - \int_{|z|\geq 1} \gamma(x,z)\mu(dz)$, where the integrability of $\gamma(x,z)$ with respect to $1_{\{|z|\geq 1\}}.d\mu(z)$ is assumed. We remark that

$$\gamma(x,z) = \frac{\partial \gamma}{\partial z}(x,0)z + \tilde{\gamma}(x,z) \tag{2.3}$$

for some $\tilde{\gamma}(x,z)$ such that

$$\left| \left(\frac{\partial}{\partial x} \right)^k \tilde{\gamma}(x,z) \right| \leq C_k |z|^\alpha$$

for some $\alpha > \beta \vee 1$ and for $k \in \mathbf{N}^d$ on $\{|z| \leq 1\}$. Here $\beta > 0$ is what appeared in *(A.0)*.

Throughout this section, we assume the following four assumptions; *(A.0~A.2)* in Section 1.3, and

(A.3) (3-a) If $0 < \beta < 1$, we assume that $c = \int_{|z|\leq 1} z\mu(dz)$, $b = 0$ and for all $u \in S^{d-1}$,

$$\int_{\{|z|\leq \rho\}} \langle z,u\rangle^2 1_{\{\langle z,u\rangle > 0\}}(z)\mu(dz) \asymp \rho^{2-\beta} \tag{2.4}$$

as $\rho \to 0$.

(3-b) If $\beta = 1$, then

$$\limsup_{\epsilon\to 0} \left| \int_{\{\epsilon < |z| \leq 1\}} z\mu(dz) \right| < +\infty\,. \tag{2.5}$$

Then, equation (2.1) (resp. (2.2)) has a unique solution. This follows from the fact that (2.1) can be written in the form (in Itô integral)

$$dx_t(x) = d\Theta_t(x_{t-}(x)), \quad x_0(x) = 0\,, \tag{2.6}$$

where

$$\Theta_t(y) = b'(y)t + \int_0^t \left\{ \int_{|z|\leq 1} \gamma(y,z)\tilde{N}(dsdz) + \int_{|z|>1} \gamma(y,z)N(dsdz) \right\}\,,$$

and that equation (2.6) has a unique solution due to [123] Theorem 3.1. Furthermore, due to the inversibility assumption *(A.2)*, there exists a stochastic flow of diffeomorphisms denoted by $\phi_{s,t}(s < t) : \mathbf{R}^d \to \mathbf{R}^d$ such that $x_t(x) = \phi_{s,t}(x_s(x))$, which is inversible ([140] Section 1, [25]; see also [192] Theorem V.65 for the simple case $\gamma(x,z) = X(x)z$).

We cite the following basic result due to Picard [184] using the perturbation stated in Section 2.1.2.

Proposition 2.5 ([184] Theorem 1). *Under the conditions (A.0~A.3), $x_t(x)$ has a C_b^∞-density for each $t > 0$ which we denote by $y \mapsto p_t(x,y)$.*

Methodologies leading to the existence of the smooth density under various conditions have much developed following this result. We shall go a bit into detail in Sections 3.5, 3.6.

The following result is an extension of Theorem 1 in [184] to the above mentioned process.

Theorem 2.2 (general upper bound). *The density $p_t(x, y)$ satisfies the following estimate:*

(a)
$$\sup_{x,y} p_t(x, y) \leq C_0 t^{-\frac{d}{\beta}} \quad \text{as } t \to 0, \text{ for some } C_0 > 0, \tag{2.7}$$
$$p_t(x, x) \asymp t^{-\frac{d}{\beta}} \quad \text{as } t \to 0 \text{ uniformly in } x. \tag{2.8}$$

(b) *For all $k \in \mathbf{N}^d$ there exists $C_k > 0$ such that*
$$\sup_{x,y} \left| p_t^{(k)}(x, y) \right| \leq C_k t^{-\frac{(|k|+d)}{\beta}} \quad \text{as } t \to 0, \tag{2.9}$$

where $p^{(k)}$ denotes the k-th derivative with respect to y.

Below, we give a refinement of this theorem (Theorem 2.3) and provide the proof (Section 2.2.2).

We can give examples that the supremum in (a) is in fact attained on the diagonal $\{x = y\}$ (diagonal estimate):
$$p_t(x, x) \asymp t^{-\frac{d}{\beta}} \quad \text{as } t \to 0$$
uniformly in x. It is well known in the case of a one-dimensional symmetric stable process with index $\beta = \alpha$ that the density function satisfies $p_t(0) \sim Ct^{1/\alpha}$ as $t \to 0$ ([84] Section 2-4).

The estimate above is more exact than the one by Hoh–Jacob [79] using functional analytic methods: there exist $C > 0$, $\nu \in (1, +\infty)$ such that
$$\sup_{x,y} p_t(x, y) \leq Ct^{-\nu} \quad \text{as } t \to 0 .$$

Below, we construct a Markov chain associated to $x_t(x)$. Let $\nu(dz)$ be the probability measure on $\mathbf{R}^d$ given by
$$\nu(dz) = \frac{(|z|^2 \wedge 1)\mu(dz)}{\int (|z|^2 \wedge 1)\mu(dz)} . \tag{2.10}$$

Then, $d\nu \sim d\mu$ (μ and ν are mutually absolutely continuous), and the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ is globally bounded from above.

Consider a series of functions $(A_n)_{n=0}^{\infty}$, $A_n : \mathbf{R}^{d+m\times n} \to \mathbf{R}^d$ defined by $A_0(x) = x$ and $A_{n+1}(x, x_1, \ldots, x_{n+1}) = A_n(x, x_1, \ldots, x_n) + \gamma(A_n(x, x_1, \ldots, x_n), x_{n+1})$. Fix $x \in \mathbf{R}^d$. We put $\mathcal{S}_n$ to be the support of the image measure of $\mu^{\otimes n}$ by the mapping $(x_1, \ldots, x_n) \mapsto A_n(x, x_1, \ldots, x_n)$, and $\mathcal{S} \equiv \bigcup_n \mathcal{S}_n$.

Definition 2.2 (accessible points). Points in $\mathcal{S}$, regarded as points in $\mathbf{R}^d$, are called accessible points. Points in $\bar{\mathcal{S}} \setminus \mathcal{S}$ are called asymptotically accessible points.

Intuitively accessible points are those points which can be reached by $x_t(x)$ by only using a finite number of jumps of $z(s)$. We remark that $\mathcal{S}$ is not necessarily closed, although each $\mathcal{S}_n$ is.

We define for each x the mapping H_x : supp $\mu \to P_x \equiv x + \{\gamma(x,z); z \in \text{supp } \mu\}$ by $z \mapsto x + \gamma(x,z)$. Let $P_x^{(n)} = \{y \in P_{z_{n-1}}; z_1 \in P_x, z_i \in P_{z_{i-1}},\ i = 2, \dots, n-1\}$, $n = 1, 2, \dots (z_0 = x)$. Then, $P_x^{(1)} = P_x$, and $P_x^{(n)}$ can be interpreted as points which can be reached from x by n jumps along the trajectory $x_t(x)$. Given $x, y \in \mathbf{R}^d$ $(y \neq x)$, let $\alpha(x,y)$ be the minimum number l such that $y \in P_x^{(l)}$ if such l exists, and put $\alpha(x,y) = +\infty$ if not. Or equivalently, $\alpha(x,y) = \inf\{n; y \in \cup_{k \le n} \mathcal{S}_k\}$.

To be more precise, we introduce a concrete "singular" Lévy measure of $z(s)$ which has already been described in [209] Example 3.7. Let $\mu(dz) = \sum_{n=0}^{\infty} k_n \delta_{\{a_n\}}(dz)$ be an m-dimensional Lévy measure such that $(a_n; n \in \mathbf{N})$ and $(k_n; n \in \mathbf{N})$ are sequences of points in $\mathbf{R}^d$ and real numbers, respectively, satisfying

(i) $|a_n|$ decreases to 0 as $n \to +\infty$,
(ii) $k_n > 0$,
(iii) $\sum_{n=0}^{\infty} k_n |a_n|^2 < +\infty$.

For this Lévy measure, we can show the unique existence of the solution $x_t(x)$ of (2.2) ([209] Theorems 1.1, 2.1), and the existence of the density under the assumptions *(A.0–A.3)*.

We further assume that

$$N = N(t) \equiv \max\{n; |a_n| > t^{1/\beta}\} \asymp \log\left(\frac{1}{t}\right) . \tag{2.11}$$

Part (b) of the next theorem can be viewed as an extension of Proposition 5.1 in [183].

Theorem 2.3 ([89], Theorem 2). *Assume μ is given as above. Let $y \neq x$.*

(a) *Assume $y \in \mathcal{S}$, that is, $\alpha(x,y) < +\infty$. Then, we have*

$$p_t(x,y) \asymp t^{\alpha(x,y) - d/\beta} \quad \text{as } t \to 0 . \tag{2.12}$$

(b) *Assume $y \in \bar{\mathcal{S}} \setminus \mathcal{S} (\alpha(x,y) = +\infty)$. Suppose $b(x) \equiv 0$ and let $\beta' > \beta$. Then, $\log p_t(x,y)$ is bounded from above by the expression of type $\Gamma = \Gamma(t)$:*

$$\Gamma \equiv -\min \sum_{n=0}^{N} \left(w_n \log\left(\frac{1}{tk_n}\right) + \log(w_n!) \right) + O\left(\log\left(\frac{1}{t}\right) \log\log\left(\frac{1}{t}\right)\right) \tag{2.13}$$

as $t \to 0$. Here, the minimum is taken with respect to all choices of $a_0, \dots, a_N$ by ξ_n for $n = 1, 2, \dots, n_1$ and $n_1 \in \mathbf{N}$ such that

$$|y - A_{n_1}(x, \xi_1, \dots, \xi_{n_1})| \le t^{1/\beta'} , \tag{2.14}$$

where w_n = # of a_n in the choice and $n_1 = \sum_{n=0}^{N} w_n$.

We remark that in finding the above minimum, the conditions (2.11) and (2.14) work complementarily. That is, as $t > 0$ gets smaller, (2.14) becomes apparently more strict, whereas we may use more combinations of a_i's to approximate y due to (2.11). Since $\beta' > \beta$, the condition (2.14) does not prevent $A_{n_1}(x, \xi_1, \dots, \xi_{n_1})$ from attaining y as $t \to 0$, using $a_0, \dots, a_N$ under (2.11). We also remark that if x and a_n are rational points (e.g. $m = 1$, $x = 0$, $a_n = 2^{-n}$), then the result (b) holds for almost all $y \in \bar{\mathbb{S}}(= [0, 1])$ relative to the Lebesgue measure, whereas for $t > 0$, $y \mapsto p_t(x, y)$ is smooth on $\bar{\mathbb{S}}$ due to Proposition 2.5.

2.2.2 Proof of Theorem 2.3

In this subsection, we prove Theorem 2.3. The main idea is to replace the Lévy measure μ by another probability measure ν which is absolutely continuous with respect to μ, and consider another process $\tilde{x}_.$ of pure jump (bounded variation) whose Lévy measure is proportional to ν. The Markov chain which approximates the trajectory of $x_.$ is constructed on the basis of $\tilde{x}_.$.

First, we prove two lemmas which are essential for our estimation. These lemmas are inspired by Picard [184].

Lemma 2.5 (lower bound for accessible points). *Let $(\xi_n)_{n\in\mathbf{N}}$ be the $\mathbf{R}^d$-valued independent and identically distributed random variables obeying the probability law $\nu(dz)$ given by* (2.10)*, independent of $z(s)$. We define a Markov chain $(U_n)_{n\in\mathbf{N}}$ by $U_0 = x$ and $U_{n+1} = U_n + \gamma(U_n, \xi_{n+1})$, $n \in \mathbf{N}$. Assume that for $y \in \mathbf{R}^d$, there exist some $n \geq 1$, $\gamma = \gamma_n \geq 0$ and $c > 0$ such that for all $\epsilon \in (0, 1]$, $P(|U_n - y| \leq \epsilon) \geq c\epsilon^\gamma$. Then, we have*

$$p_t(x, y) \geq C t^{n+(\gamma-d)/\beta} \quad \text{as } t \to 0\,. \tag{2.15}$$

We notice that the lower bound on the right-hand side depends on (n, γ_n). Put $g(x, dz) = d(H_x^* \nu)(z)$, $z \in P_x \setminus \{x\}$, where $H_x^* \nu = \nu \circ H_x^{-1}$. Then, we have an expression of the probability above:

$$P(|U_n - y| \leq \epsilon) = \int_{P_x} \cdots \int_{P_{z_{n-1}}} 1_{\{z_n; |z_n - y| \leq \epsilon\}}(z_n) g(x, dz_1) \dots g(z_{n-1}, dz_n)\,. \tag{2.16}$$

Hence, the condition $P(|U_n - y| \leq \epsilon) \geq c\epsilon^\gamma$ implies:

y can be attained with the singular Lévy measure (dim supp $\nu = 0$) if $\gamma = 0$.

In order to obtain the upper bound, we introduce the perturbation of the chain.

Let $(\varphi_n)_{n\in\mathbf{N}}$ be a series of smooth functions: $\mathbf{R}^d \to \mathbf{R}^d$. We define another Markov chain $(V_n)_{n\in\mathbf{N}}$ by $V_0 = \varphi_0(x)$ and $V_{n+1} = V_n + (\varphi_{n+1} \circ \gamma)(V_n, \xi_{n+1})$. Furthermore, we define the series of real numbers $(\Phi_n)_{n\in\mathbf{N}}$ by

$$\Phi_n \equiv \sup_{k \leq n, y \in \mathbf{R}^d} (|\varphi_k(y) - y| + |\varphi_k'(y) - I|)\,. \tag{2.17}$$

Under these preparations, we have:

Lemma 2.6 (upper bound). *Choose $y \neq x$. Assume there exist a sequence $(\gamma_n)_{n\in\mathbf{N}}$, $\gamma_n \in [0, +\infty]$, and a nondecreasing sequence $(K_n)_{n\in\mathbf{N}}$, $K_n > 0$, such that the following condition holds true for each n and for any $(\varphi_k)_{k=0}^n$ satisfying $\Phi_n \le K_n$: V_n defined as above satisfies with some $C_n > 0$ that*

$$\text{if} \quad \gamma_n < +\infty, \quad \text{then} \quad P(|V_n - y| \le \epsilon) \le C_n \epsilon^{\gamma_n} \quad \text{for all} \quad \epsilon > 0\,, \tag{2.18}$$

and

$$\text{if} \quad \gamma_n = +\infty, \quad \text{then} \quad P(|V_n - y| \le \epsilon) = 0 \quad \text{for} \quad \epsilon > 0 \quad \text{small.} \tag{2.19}$$

Furthermore, we put $\Gamma \equiv \min_n(n + (\gamma_n - d)/\beta)$. Then we have:
1. *If $\Gamma < +\infty$, then $p_t(x, y) = O(t^\Gamma)$ as $t \to 0$.*
2. *If $\Gamma = +\infty$, then for any $n \in \mathbf{N}$ $p_t(x, y) = o(t^n)$ as $t \to 0$.*

Note that Γ depends implicitly on the choice of (K_n). Whereas, for each n, the bigger γ_n gives the better upper bound.

Given the perturbations (φ_j), we define $Q_x^{(0)} = \{\varphi_0(x)\}$ and the sequence $(Q_x^{(n)})$ successively by

$$Q_x^{(n)} \equiv \{z_{n-1} + (\varphi_n \circ \gamma)(z_{n-1}, z);\ z \in \operatorname{supp} \nu,\ z_{n-1} \in Q_x^{(n-1)}\}$$

for $n = 1, 2, \ldots$. Hence, the set $Q_x^{(n)}$ can be interpreted as the points which can be reached from x by V_n, and we have

$$\begin{aligned} &P(|V_n - y| \le \epsilon) \\ &\quad = \int_{P_{\varphi_0(x)}} \cdots \int_{P_{\varphi_{n-1}(z_{n-1})}} 1_{\{z_n;|\varphi_n(z_n)-y|\le\epsilon\}}(z_n) g(\varphi_0(x), dz_1) \ldots g(\varphi_{n-1}(z_{n-1}), dz_n) \end{aligned} \tag{2.20}$$

for $n = 0, 1, 2, \ldots$. We remark that if $y \notin P_x^{(n)}$ $(n \ge 1)$, then by choosing φ_n such that the size Φ_n of perturbations is small enough, we can let $y \notin Q_x^{(n)}$. Therefore, by choosing $K_n > 0$ small and $\epsilon > 0$ small, γ_n may be $+\infty$ in the above.

Proofs of Lemmas 2.5, 2.6, and that of Lemma 2.7 below will be provided in the next subsection.

Proof of statement (a) of Theorem 2.3.
For $n \ge \alpha(x, y)$, we have $P(|U_n - y| \le \epsilon) \ge c$ for $\epsilon \in (0, 1]$ since μ has point masses. Hence, $p_t(x, y) \ge Ct^{\alpha(x,y)-d/\beta}$ by Lemma 2.5.

For the upper bound, if $n < \alpha(x, y)$, then by choosing K_n small, we have $y \notin Q_x^{(n)}$. That is,

$$P(|V_n - y| \le \epsilon) = 0 \quad \text{for } \epsilon > 0 \text{ small,}$$

and we may choose $\gamma_n = +\infty$ in (2.18) and (2.19). On the other hand, if $n \ge \alpha(x, y)$, then we must choose $\gamma_n = 0$ in (2.18) ($\varphi_n = id$ must satisfy it).

Hence, we may choose $\Gamma = \alpha(x,y) - d/\beta$ in the conclusion. These imply

$$p_t(x,y) \asymp t^{\alpha(x,y)-d/\beta} . \tag{2.21}$$

□

Proof of the statement (b) of Theorem 2.3.
We set $\mathcal{S}_{t,k} = ([0,t]^k/\sim)$ and $\mathcal{S}_t = \amalg_{k\geq 0}\mathcal{S}_{t,k}$, where $\amalg_{k\geq 0}$ denotes the disjoint sum and $\sim$ means the identification of the coordinates on the product space $[0,t]^k$ by the permutation.

Let $r = r(t) = t^{1/\beta}$. We denote by $\tilde{z}^r(s)$ the Lévy process having the Lévy measure $\mu.1_{\{|z|>r\}}(z)$. That is, $\tilde{z}^r(s) = \sum_{u\leq s} \Delta z(u) 1_{\{|\Delta z(u)|>r\}}$. The distribution $\tilde{P}_{t,r}$ of the moments (instants) of jumps related to $\tilde{z}^r(s)$ during $[0,t]$ is given by

$$\begin{aligned}\int_{\{\#S_t=k\}} & f(S_t) d\tilde{P}_{t,r}(S_t) \\ &= \left\{ \left(t\int 1_{\{|z|>r\}}(z)\mu(dz) \right)^k \left(\frac{1}{k!}\right) \exp\left(-t\int 1_{\{|z|>r\}}(z)\mu(dz) \right) \right\} \\ &\quad \times \frac{1}{t^k} \int_0^t \cdots \int_0^t f(s_1,\ldots,s_k) ds_1 \cdots ds_k ,\end{aligned} \tag{2.22}$$

where f is a function on $\mathcal{S}_{t,k}$ (a symmetric function on $[0,t]^k$). Given $S_t \in \mathcal{S}_t$, we introduce the process $x_s(r, S_t, x)$ as the solution of the following SDE:

$$\begin{aligned} x_s(r,S_t,x) = x - \int_0^s du \int 1_{\{|z|>r\}}(z)\gamma(x_u(r,S_t,x),z)\mu^r(dz) \\ + \sum_{u\leq s}^{c} \gamma\left(x_{u^-}(r,S_t,x), \Delta z^r(u)\right) + \sum_{s_i\in S_t, s_i\leq s} \gamma(x_{s_i-}(r,S_t,x),\xi_i^r) ,\end{aligned} \tag{2.23}$$

where $(\xi_n^r)_{n\in\mathbf{N}}$ denotes a series of independent and identically distributed random variables obeying the probability law

$$\mu^r(dz) = \frac{1_{\{|z|>r\}}(z).\mu(dz)}{\int 1_{\{|z|>r\}}(z).\mu(dz)} .$$

We remark that $x_s(r,S_t,x)$ is a martingale for each $0 < r < 1$ due to the assumption $b(x) \equiv 0$. We define a new Markov chain $(U_n^r)_{n\in\mathbf{N}}$ by $U_0^r = x$ and $U_{n+1}^r = U_n^r + \gamma(U_n^r, \xi_{n+1}^r)$, $n \in \mathbf{N}$.

We can prove, due to Proposition 2.5, that under *(A.0)* through *(A.3)*, the law of $x_s(r,S_t,x)$ for (ds-a.e.) $s > 0$ has a C_b^∞ -density denoted by $p_s(r,S_t,x,y)$.

Indeed, let $0 \leq s \leq t$. In the case $S_t = \emptyset$, we can choose $1_{\{|z|\leq r\}}.\mu(dz)$ for the measure $\mu(dz)$ in Proposition 2.5. Hence, we have the existence of the density for the law $p_s(r,\emptyset,x,dy)$ of $x_s(r,\emptyset,x)$ $(= x_s(r,S_t,x)|_{S_t=\emptyset})$: $p_s(r,\emptyset,x,dy) = p_s(r,\emptyset,x,y)dy$.

Next, we consider the general case. Since $\tilde{z}^r(s)$ and $z^r(s)$ are independent, we have by the Markov property that the law $p_s(r, S_t, x, dy) \equiv P(x_s(r, S_t, x) \in dy)$ of $x_s(r, S_t, x)$ is represented by

$$\begin{aligned} p_s(r, S_t, x, dy) = \int dz_0' \int_{P_{z_0'}} & p_{s_1}(r, \emptyset, x, z_0') g_r(z_0', dz_1) \\ \times \int dz_1' \int_{P_{z_1'}} & p_{s_2 - s_1}(r, \emptyset, z_1, z_1') g_r(z_1', dz_2) \\ \cdots \int dz_{n_1-1}' \int_{P_{z_{n_1-1}'}} & p_{s_{n_1} - s_{n_1-1}}(r, \emptyset, z_{n_1-1}, z_{n_1-1}') g_r(z_{n_1-1}', dz_{n_1}) \\ \times\, & p_{t-s_{n_1}}(r, \emptyset, z_{n_1}, dy) \end{aligned}$$

if $S_t \in \mathcal{S}_{t,n_1}$. Here, $g_r(x, dz) = P(x + \gamma(x, \xi_i^r) \in dz)$.

On the other hand, once again, we have by the independence of $\tilde{z}^r(s)$ and $z^r(s)$,

$$p_s(x, dy) = \int_{\mathcal{S}_t} p_s(r, S_t, x, dy) d\tilde{P}_{t,r}(S_t) \,,$$

using the factorization of the measure N (cf. [106] p. 71). By Proposition 2.5, the left-hand side has a density $p_s(x, y)$ with respect to dy. Hence, $p_s(r, S_t, x, dy)$ is absolutely continuous with respect to dy ($d\tilde{P}_{t,r}$-a.s.). Hence, we have by the derivation under the integral sign,

$$p_s(x, y) = \int_{\mathcal{S}_t} (p_s(r, S_t, x, dy)/dy)(y) d\tilde{P}_{t,r}(S_t) \,.$$

We denote by $p_s(r, S_t, x, y)$ the derivative $p_s(r, S_t, x, dy)/dy(y)$ which is defined uniquely $d\tilde{P}_{t,r} \otimes dy$-a.e. (Since $d\tilde{P}_{t,r}|_{\mathcal{S}_{t,k}}$ is the uniform distribution on $\mathcal{S}_{t,k}$, $p_s(r, S_t, x, y)$ is defined uniquely as $ds \otimes dy$-a.e.) Since $y \mapsto p_s(x, y)$ is smooth, so is $y \mapsto p_s(r, S_t, x, y)$ ds-a.s., and hence $p_s(r, S_t, x, y)$ is defined as a smooth density ds-a.e.

Thus, by taking $s = t$,

$$p_t(x, y) = \int_{\mathcal{S}_t} p_t(r, S_t, x, y) d\tilde{P}_{t,r}(S_t) = \sum_{k=0}^{\infty} p_t(k, r, x, y) \,, \tag{2.24}$$

where

$$p_t(k, r, x, y) = \int_{\mathcal{S}_{t,k}} p_t(r, S_t, x, y) d\tilde{P}_{t,r}(S_t) \,.$$

Hence,

$$\begin{aligned} p_t(x, y) dy &= E^{\tilde{P}_{t,r}}[P(x_t(r, S_t, x) \in dy)] \\ &= E^{\tilde{P}_{t,r}} E^{(\mu^r)^{\otimes \# S_t}}[P(x_t(r, S_t, x) \in dy | S_t, \xi_1^r, \ldots, \xi_{\# S_t}^r)] \,. \end{aligned} \tag{2.25}$$

For each $S_t \in \mathcal{S}_t$, we put $n_1 = k$ if $S_t \in \mathcal{S}_{t,k}$. We then have

$$\begin{aligned} &P(x_t(r, S_t, x) \in dy | S_t) \\ &= E^{(\mu^r)^{\otimes \# S_t}}[P(x_t(r, S_t, x) \in dy : |y - U^r_{n_1}| \le t^{1/\beta'} | S_t, \xi^r_1, \ldots, \xi^r_{n_1})] \\ &\quad + E^{(\mu^r)^{\otimes \# S_t}}[P(x_t(r, S_t, x) \in dy : |y - U^r_{n_1}| > t^{1/\beta'} | S_t, \xi^r_1, \ldots, \xi^r_{n_1})] \,. \end{aligned} \tag{2.26}$$

First, we have to compute $P(|y - U^r_{n_1}| \le t^{1/\beta'}) \equiv E^{(\mu^r)^{\otimes n_1}}[P(|y - U^r_{n_1}| \le t^{1/\beta'} | \xi^r_1, \ldots, \xi^r_{n_1})]$ for a given $y = x_t(r, S_t, x)$ with $S_t \in \mathcal{S}_{t,n_1}$. We denote, by the random variable W_n, the number of a_n in $(\xi^r_i)_{i=1}^{n_1}$.

Given $n_1 \in \mathbf{N}$, let $(w_n)_{n=0}^N$ and $w_n \in \mathbf{N} \cup \{0\}$ be a series of integers such that $n_1 = \sum_{n=0}^N w_n$. We then have

$$\begin{aligned} &P(\text{for all } n \le N,\, W_n = w_n \text{ and } |y - U^r_{n_1}| \le t^{1/\beta'}) \\ &\qquad \le \prod_{n=0}^{N} \frac{1}{w_n!} (tk_n)^{w_n} e^{-tk_n} \,, \end{aligned}$$

since each W_n is a Poisson random variable with mean tk_n. Hence,

$$\begin{aligned} &\log P\Big(\text{ for all } n \le N,\, W_n = w_n \text{ and } |y - U^r_{n_1}| \le t^{1/\beta'} \Big) \\ &\qquad \le - \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!) + tk_n) \\ &\qquad = - \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!)) + O(1) \,. \end{aligned} \tag{2.27}$$

We introduce the set $\mathcal{W}$ of all $(w_n)_{n=0}^N$ such that for some $n_1 \in \mathbf{N}$, $U^r_{n_1}$ directed by the following condition $(*)$ satisfies $|y - U^r_{n_1}| \le t^{1/\beta'}$, that is,

$(*)$ $\quad w_n = (\#$ of a_n's which appear in $\xi^r_1, \ldots, \xi^r_{n_1})$ and $n_1 = \sum_{n=1}^N w_n$.

Then, from (2.27), it follows that

$$\begin{aligned} &\log P \begin{pmatrix} \text{there exists } (w_n) \in \mathcal{W} \text{ such that for all } n \le N \,, \\ W_n = w_n \text{ and } |y - U^r_{n_1}| \le t^{1/\beta'} \end{pmatrix} \\ &\qquad \le - \min_{\mathcal{W}} \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!)) + O(\log |\mathcal{W}|) \,. \end{aligned} \tag{2.28}$$

On the other hand, we have

$$P(W_n \ge N^3 \text{ and } |y - U^r_{n_1}| \le t^{1/\beta'}) \le Ce^{-N^3} \le Ce^{-c(\log(1/t))^3} \tag{2.29}$$

since W_n is a Poisson variable with mean tk_n. Since this is very small relative to the probability above, we may put the restriction $w_n \le N^3$ and $n_1 \le N^4$. Hence, we may write $|\mathcal{W}| = O((N^3)^N)$, and

$$\log |\mathcal{W}| = O(N \log N) = O(\log(1/t) \log\log(1/t)) \,.$$

Let $x_t(r, \emptyset, x)$ denote the process defined by (2.23) with $S_t = \emptyset$, and $p_t(r, \emptyset, x, y)$ as its density. By Theorem 2.2 (a), $p_t(r, \emptyset, x, y) \le Ct^{-d/\beta}$ as $t \to 0$, and this implies, for a given $S_t \in \mathcal{S}_{t,n_1}, \xi_1^r, \dots, \xi_{n_1}^r$, that $P(x_t(r, S_t, x) \in dy | S_t, \xi_1^r, \dots, \xi_{n_1}^r)/dy = O(t^{-d/\beta})$ as $t \to 0$. Since $z^r(u)$ and (ξ_i^r) are independent, by (2.24) and Fubini's Theorem, we have

$$\log\Bigg(E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} \left[P\left(x_t(r, S_t, x) \in dy : |y - U_{n_1}^r| \le t^{1/\beta'} | S_t, \xi_1^r, \dots, \xi_{n_1}^r \right)\right] \Big/ dy \Bigg)$$
$$\le \log\left(t^{-d/\beta} \exp\left(-\min_{\mathcal{W}} \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!)) + O(\log(1/t)\log\log(1/t)) \right)\right)$$
$$\le -\min_{\mathcal{W}} \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!)) + O(\log(1/t)\log\log(1/t)) . \qquad (2.30)$$

For the second term of (2.26), we have the following lemma, whose proof is given later.

Lemma 2.7. *Given $y = x_t(r, S_t, x)$, $S_t \in \mathcal{S}_{t,n_1}$ and $U_{n_1}^r = A_{n_1}(x, \xi_1^r, \dots, \xi_{n_1}^r)$, there exist $k > 0$ and $C_0 > 0$ such that for every $p > k$,*

$$E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} [P(|y - U_{n_1}^r| > t^{1/\beta'} | S_t, \xi_1^r, \dots, \xi_{n_1}^r)] \le n_1 C_0 \exp[-(p-k)(\log(1/t))^2]$$

as $t \to 0$.

Given $\xi_1^r, \dots, \xi_{n_1}^r$, we have, as above, $E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}} [P(x_t(r, S_t, x) \in dy | S_t, \xi_1^r, \dots, \xi_{n_1}^r)/dy] = O(t^{-d/\beta})$ as $t \to 0$. We integrate this with respect to $(\mu^r)^{\otimes n_1}$ on $\{|y - U_{n_1}^r| > t^{1/\beta'}\}$. Since $z^r(u)$ and (ξ_i^r) are independent, by Lemma 2.7, we then have

$$E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} \left[P\left(x_t(r, S_t, x) \in dy : |y - U_{n_1}^r| > t^{1/\beta'} | S_t, \xi_1^r, \dots, \xi_{n_1}^r \right)\right] \Big/ dy$$
$$\le n_1 C_0' t^{-d/\beta} \exp[-(p-k)(\log(1/t))^2], \text{ as } t \to 0 .$$

We get

$$\log\Bigg(E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} \left[P\left(x_t(r, S_t, x) \in dy : |y - U_{n_1}^r| > t^{1/\beta'} | S_t, \xi_1^r, \dots, \xi_{n_1}^r \right)\right] \Big/ dy \Bigg)$$
$$\le -(p-k)(\log(1/t))^2 + \max_{n_1 \le N^4} \log n_1 + \log C_0' + O(\log(1/t))$$
$$\le -(p-k)(\log(1/t))^2 + O(\log(1/t))$$

since $\max_{n_1 \le N^4} \log n_1 = \log N^4 = O(\log(1/t))$. Since $p > k$ is arbitrary, this can be neglected in view of the right-hand side of (2.30) and (2.26).

After summing up $E^{\tilde{P}_{t,r}|_{\mathbb{S}_{t,n_1}}} E^{(\mu^r)^{\otimes n_1}}[\ldots]$ with respect to $n_1 = 0, \ldots, N^4$, we get

$$\begin{aligned}&\log\left(E^{\tilde{P}_{t,r}|_{\cup_{k\le N^4}\mathbb{S}_{t,k}}} E^{(\mu^r)^{\otimes \# S_t}}\left[P\left(x_t(r, S_t, x) \in dy | S_t, \xi_1^r, \ldots, \xi_{\# S_t}^r\right)\right] \Big/ dy\right)\\ &\le -\min_{\mathcal{W}} \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!))\\ &\quad + O(\log(1/t)\log\log(1/t)) + O(\log(1/t)) . \qquad (2.31)\end{aligned}$$

In view of (2.25), (2.31),

$$\begin{aligned}\log p_t(x, y) \le -\min_{\mathcal{W}} \sum_{n=0}^{N} (w_n \log(1/(tk_n)) + \log(w_n!))\\ + O(\log(1/t)\log\log(1/t)) . \qquad (2.32)\end{aligned}$$

Since there is no difference between the trajectories of the deterministic chain $A_{n_1}(x, \xi_1, \ldots, \xi_{n_1})$ and $U_{n_1}^r$ obtained by using $\{a_n; n = 0, \ldots, N\}$ under (2.11), we have the assertion. □

In Section 2.5.4, we will see the image of (Markov) chain approximations using planar chains.

2.2.3 Proof of lemmas

In this subsection, we give proofs of lemmas 2.5, 2.6, and 2.7 which appeared in the previous subsection.

(A) Proof of Lemma 2.5

To show Lemma 2.5, we prepare several sublemmas.

We recall that μ satisfies *(A.0)*. As stated above, we introduce a probability measure ν by

$$\nu(dz) = \frac{(|z|^2 \wedge 1)\mu(dz)}{\int(|z|^2 \wedge 1)\mu(dz)} .$$

Then, $\mu \sim \nu$. Let $1/c_0$ be the upper bound of $\frac{d\nu}{d\mu}$.

We decompose $z(s)$ into an independent sum

$$z(s) = \tilde{z}(s) + Z(s) ,$$

where $\tilde{z}(s)$ is a Lévy process with the Lévy measure $c_0\nu$ which is a pure jump process, and $Z(s)$ is a Lévy process with the Lévy measure $\mu - c_0\nu$. Let $N = N(t)$ the number of jumps of $\tilde{z}$ before time t, and let $r = r(t) = t^{1/\beta}$.

We further decompose $Z(s)$ into an independent sum $Z(s) = Z^r(s) + \tilde{Z}^r(s)$ as

$$\tilde{Z}^r(s) = \sum_{u\le s} \Delta Z(u) 1_{\{|\Delta Z(u)|>r\}}$$

and

$$Z^r(s) = Z(s) - \tilde{Z}^r(s) .$$

Then, $\tilde{z}, Z^r, \tilde{Z}^r$ are chosen to be independent processes, making $z = \tilde{z} + Z^r + \tilde{Z}^r$.

We introduce three processes $x^r_s(x)$, $x^{(r)}_s(x)$, $\tilde{x}_s(x)$ by using (3.2) in Chapter 1, and replacing $z(.)$ with Z^r, $\tilde{z} + Z^r$, $\tilde{z}$, respectively.

Let

$$\Gamma = \{\omega; \tilde{Z}^r_s \equiv 0,\ s \in [0, t]\} .$$

By *(A.0)*, we have

$$P(\Gamma) > c_t > 0$$

by using the small deviations property ([90], cf. Section 2.5.4). We observe $x_t = x^{(r)}_t$ on Γ. Hence, it is sufficient to estimate the density of $x^{(r)}_t$ at y for the lower bound.

Here, we give three sublemmas.

Sublemma 2.1. *Let $h > 0$ and $k \in \mathbf{N}^d$. Denote by $p_{r,t}(x, y)$ the density of $x^r_t(x)$. Then, $|p^{(k)}_{r,r^\beta h}(x, y)| \le C_{kh} r^{-(|k|+d)}$ for $0 < r \le 1$.*

Proof. Let $h > 0$ and $r > 0$. We put $\bar{x}^r_h(x) \equiv \frac{1}{r} x^r_{r^\beta h}(x)$. $\bar{x}^r_h(x)$ satisfies $\bar{x}^r_h(x) = \int_0^h b_r(\bar{x}^r_{h'}(x))dh' + \sum^c_{h'\le h} \gamma_r(\bar{x}^r_{h'-}(x), \Delta\bar{z}^r(h')) + \frac{1}{r}x$, where $\bar{z}^r(t) \equiv \frac{1}{r} Z^r(r^\beta t)$, $b_r(y) \equiv r^{\beta-1} b(ry)$ and $\gamma_r(x, \zeta) \equiv \frac{1}{r}\gamma(rx, r\zeta)$. Assumption *(A.3)* (3-b) is used here to guarantee that the scaled drift parameter $c_r = r^{\beta-1}(c - \int_{\{r<|\zeta|\le 1\}} \zeta\mu(d\zeta))$ is finite. Then, by the definition of $h \mapsto \bar{x}^r_h(x)$, $P(\bar{x}^r_h(x) \in d(\frac{y}{r}) | \bar{x}^r_0(x) = \frac{x}{r}) = P(x^r_{r^\beta h}(x) \in dy)$.

On the other hand, $y \mapsto b_r(y)$ is C^∞_b (we use here the assumption for the case $\alpha < 1$). $\gamma_r(x, \zeta)$ satisfies the assumptions *(A.0) ~ (A.3)* uniformly in r. Hence, by Proposition 2.5, $(\bar{x}^r_h(x); r > 0)$ have C^∞_b-densities $\bar{p}_h(\frac{x}{r}, y : r)$ which are uniformly bounded with regards to $r > 0$. Hence, by the above relation, $p_{r,r^\beta h}(\frac{x}{r}, y : r) = \frac{1}{r^d}\bar{p}_h(\frac{x}{r}, \frac{y}{r} : r)$. This implies $|p_{r,r^\beta h}(\frac{x}{r}, y : r)| \le C_h r^{-d}$. Estimates for the derivatives with respect to y follow by differentiating the equality. □

The next two sublemmas are cited from [184].

Sublemma 2.2 ([184] Lemma 8). *There exists a matrix-valued function A such that*

$$x^{(r)}_t(x) = \tilde{x}_t(x) + \sum^c_{s\le t} A(s, \tilde{z}(.)) \frac{\partial}{\partial z}\gamma(\tilde{x}_s(x), 0)\Delta Z^r_s + o\left(t^{\frac{1}{\beta}}\right)$$

in case $\beta \ne 1$, and

$$x^{(r)}_t(x) = \tilde{x}_t(x) + \sum^c_{s\le t} A(s, \tilde{z}(.))\left(\frac{\partial}{\partial z}\gamma(\tilde{x}_s(x), 0)\Delta Z^r_s + b(\tilde{x}_s(x))ds\right) + o\left(t^{\frac{1}{\beta}}\right)$$

in case $\beta = 1$. $A(s, \tilde{z})$ and its inverse are bounded by some $C(n)$ on $\{N = n\}$.

Sublemma 2.3 ([184] Lemma 6). *Consider a family of $\mathbf{R}^d$-valued infinitely divisible random variables Θ_t^i indexed by $t > 0$ and $i \in \mathcal{I}$, and another family of $\mathbf{R}^d$-valued random variables Y_t^i. We assume*

(i) Y_t^i *has a C_b^∞ transition density uniformly in (t, i).*
(ii) $\|Y_t^i - \Theta_t^i\|_{L^1} \to 0$ *as $t \to 0$ uniformly in i.*
(iii) *The drift parameter of Θ_t^i is uniformly bounded.*
(iv) *We denote by μ_t^i the Lévy measure of Θ_t^i. There exists a compact $K \subset \mathbf{R}^m$ such that $\operatorname{supp} \mu_t^i \subset K$ uniformly in i, the measure $|x|^2 \mu_t^i(dx)$ is bounded uniformly in (t, i), and μ_t^i satisfies the condition (A.0), or (A.3) (3-a) if $\beta < 1$ uniformly, respectively.*

Then, the density of Y_t^i is bounded away from zero as $t \to 0$ on any compact set uniformly in i.

Proof of Lemma 2.5. (continued)
Put $A_n = \{N = n\} \cap \{T_n \le \frac{t}{2}\}$. Here, $T_1, T_2, \ldots$ are jump times of $\tilde{z}$.

On the set A_n, $\tilde{z}$ has n jumps on $[0, \frac{t}{2}]$ and no jump on $[\frac{t}{2}, t]$. We have

$$P(A_n) \ge ct^n \quad \text{with } c > 0 ,$$

since the left-hand side is greater than $\frac{1}{n!}(\lambda_t)^n e^{-\lambda_t} \ge \frac{1}{n!}(\lambda_t)^n$ with $\lambda_t = \frac{t}{2} \times \nu(\mathbf{R}^d \setminus \{0\})$.

On the other hand, the conditional law of $x_s^{(r)}(x)$ given $(\tilde{z}, x_{t/2}^{(r)}(x))$ coincides with the transition function of $x_s^{(r)}(x)$ on $[\frac{t}{2}, t]$. We apply Sublemma 2.1 to see that the conditional density of $\frac{x_t^{(r)}(x)}{t^{1/\beta}} = \frac{x_t^{(r)}(x)}{r}$ given $\tilde{z}$ is uniformly C_b^1 on A_n. By the definition of $\tilde{x}_t(x)$, the process

$$U_t = \frac{1}{t^{1/\beta}}(x_t^{(r)}(x) - \tilde{x}_t(x))$$

satisfies the same property. By Sublemma 2.2 applied to the process U_t, U_t is equivalent to an infinitely divisible variable

$$\frac{1}{t^{1/\beta}} \left(\int_0^t A(s, \tilde{x}_s(x)) \frac{\partial}{\partial z} \gamma(\tilde{x}_s(x), 0) dZ_s^r \right) + o(1)$$

conditionally on $\tilde{z}(t)$.

We use Sublemma 2.3 (the index i is the path $\tilde{z}$) to obtain that

$$P(U_t \in dz | \tilde{z}) \ge c dz$$

on A_n, z in a compact set. This implies

$$P(x_t^{(r)}(x) \in dy | \tilde{z}) \ge ct^{-d/\beta} dy \quad \text{on} \quad \{|y - \tilde{x}_t(x)| \le t^{1/\beta}\} \cap A_n .$$

This implies

$$P(x_t^{(r)}(x) \in dy) \ge ct^{-d/\beta} P(|\tilde{x}_t(x) - y| \le t^{1/\beta}, A_n) dy . \tag{2.33}$$

On the other hand, each jump $\Delta\tilde{z}(t)$ of $\tilde{z}(t)$ is independent with the common law ν, and the jumps are independent of (T_i). Hence, conditionally on A_n, the variables $\tilde{x}_t(x)$ and U_n have the same law. Thus,

$$\begin{aligned} P(|\tilde{x}_t(x)-y|\le t^{1/\beta}, A_n) &= P(|U_n-y|\le t^{1/\beta}, A_n) \\ &= P(|U_n-y|\le t^{1/\beta})\times P(A_n)\ge ct^{n+\gamma/\beta}\,. \end{aligned} \tag{2.34}$$

From (2.33) and (2.34), we get the conclusion. □

(B) Proof of Lemma 2.6

Note that if y is not an accessible point, then y is not in the support of U_n. By choosing the size of K_n small enough, it is not in the support of V_n. Hence, we can take $\gamma_n=+\infty$ for any n, and hence $\Gamma=+\infty$.

It is sufficient to prove the case $\gamma_n<+\infty$.

To show Lemma 2.6, we again prepare several sublemmas which are cited from [184].

Sublemma 2.4 ([184] Lemma 9). *Consider a family of d-dimensional variables H admitting uniformly C_b^1 densities $p(y)$. Then, for any $q\ge 1$,*

$$p(y)\le C_q\left(\frac{E[|H|^q]+1}{1+|y|^q}\right)^{1/(d+1)}\,.$$

Sublemma 2.5 ([184] Lemma 10). *Consider the pure jump process $\tilde{z}^r(s)$ and $\tilde{x}_s^r(x)$ as defined above for $r=r(t)=t^{1/\beta}$. Let $N=N(t)$ be the number of jumps of $\tilde{z}^r$ before t. Then, for any $\lambda>0$, we have*

$$P(|\tilde{x}_t^r(x)-y|\le\lambda t^{1/\beta})\le C(1+\lambda^L+N^L)t^{\Gamma+\beta/d}$$

for some $L>0$.

Sublemma 2.6 ([184] Lemma 11). *With the same notation as in Sublemma 2.5, for any $k\in\mathbf{N}^d$, the conditional density $\tilde{p}(y)$ of x_t given $\tilde{z}^r$ satisfies*

$$|\tilde{p}^{(k)}(y)|\le C_k t^{-(d+|k|)/\beta}\exp(C_kN)$$

for some $C_k>0$.

Sublemma 2.7 ([184] Lemma 12). *With the same notation as in Sublemma 2.5, for each $q\ge 1$, we have*

$$E[|x_t(x)-\tilde{x}_t^r(x)|^q|\tilde{z}^r]^{1/q}\le c_q t^{1/\beta}\exp(c_qN)\,.$$

Proof of Lemma 2.6. (continued)

Let

$$H=\exp(CN)(x_t-\tilde{x}_t^r)/t^{1/\beta}\,.$$

Here, the constant C is chosen large enough so that the conditional density of H given $\tilde{z}^r$ is uniformly C_b^1, which is possible due to Sublemma 2.6. By Sublemma 2.7, the conditional moments of H given $\tilde{z}^r$ are at most of exponential growth with respect to N. Hence, we apply Sublemma 2.4 to obtain that for each $q \geq 1$,

$$P(H \in dh|\tilde{z}^r)/dh \leq c_q(1+|h|)^{-q}\exp(c_q N)\,.$$

Hence, the conditional density $\tilde{p}(y)$ of $x_t(x)$ satisfies

$$\begin{aligned}\tilde{p}(y) &\leq c_q t^{-d/\beta}\{\exp(c_q N)(1+t^{-1/\beta}|y-\tilde{x}_t^r(x)|\exp(CN))^{-q}\}\\ &\leq c_q t^{-d/\beta}\{\exp(c_q N)(1+t^{-1/\beta}|y-\tilde{x}_t^r(x)|)^{-1}\}^q\end{aligned}$$

due to Sublemma 2.6.

The density $p_t(x,y)$ is the expectation of $\tilde{p}(y)$. To this end, we apply the distribution identity

$$E[Z^q] = q\int_0^\infty u^{q-1}P(Z \geq u)du, \quad Z \geq 0\,.$$

Then, we have

$$\begin{aligned}p_t(x,y) &\leq qc_q t^{-d/\beta}\int_0^\infty u^{q-1}P(1+t^{-1/\beta}|y-\tilde{x}_t^r(x)| \leq \exp(c_q N)/u)du\\ &\leq c_q t^{\Gamma}\int_0^\infty u^{q-1}E\left[(1+N^L+u^{-L}\exp(Lc_q N))\times 1_{\{N\geq(\log u)/c_q\}}\right]du\end{aligned}$$

by Sublemma 2.5.

The variable $N = N(t)$ is a Poisson variable with bounded exponential mean. Hence, the expectation above is uniformly $o(u^{-k})$ for any k as $u \to +\infty$, and uniformly $O(u^{-L})$ as $u \to 0$. Hence, the integral is bounded for $q > L$. □

(C) Proof of Lemma 2.7

We fix $0 < r < 1$, and put $\xi(r,\cdot,x) \equiv x + \gamma(x,\cdot)$ when $\cdot$ is occupied by the random variables ξ_i^r.

Choose $S_t = \emptyset$ and consider the process $s \mapsto x_s(r,\emptyset,x)$ as the solution of (2.23). Given (s_1, y), the solution for $s_2 \geq s_1$ with the initial value z at $s = s_1$ is given by a smooth stochastic semiflow $\phi_{s_1,s_2}(z)$. This is proved in [67]. Assuming $y = x_t(r,S_t,x)$, $S_t \in \mathcal{S}_{t,n_1}$, we shall estimate $|U_{n_1}^r - x_t(r,S_t,x)|$, and show that for $p > k$ and $t \leq 1$,

$$\begin{aligned}&E^{\tilde{P}_{t,r}|\mathcal{S}_{t,n_1}}E^{(\mu^r)^{\otimes n_1}}[P(|U_{n_1}^r - x_t(r,S_t,x)| > t^{1/\beta'}|S_t,\xi_1^r,\ldots,\xi_{n_1}^r)]\\ &\leq n_1 C_0\exp[-(p-k)(\log(1/t))^2]\,.\end{aligned} \tag{2.35}$$

We first recall that $U_{n_1}^r$ and $x_t(r,S_t,x)$ can be represented by

$$U_{n_1}^r = A_{n_1}(x,\xi_1^r,\ldots,\xi_{n_1}^r) = \xi(r,\xi_{n_1}^r,\cdot)\circ\xi\left(r,\xi_{n_1-1}^r,\cdot\right)\circ\cdots\circ\xi(r,\xi_{n_1}^r,x)$$

and

$$x_t(r, S_t, x) = \phi_{t_{n_1},t} \circ \xi\left(r, \xi^r_{n_1}, \cdot\right) \circ \phi_{t_{n_1-1},t_{n_1}} \circ \cdots \circ \xi\left(r, \xi^r_2, \cdot\right) \circ \phi_{t_1,t_2} \circ \xi\left(r, \xi^r_1, \cdot\right) \circ \phi_{0,t_1}(x) .$$

(1) Case $n_1 = 0$. In this case, $S_t = \emptyset$ and $U^r_{n_1} = x$. The process $x_t(r, \emptyset, x)$ is a martingale whose quadratic variation $[x(r, \emptyset, x), x(r, \emptyset, x)]_t$ is bounded by

$$\int_0^t du \int_{-r}^r \{\gamma(x_{u-}(r, \emptyset, x), \zeta)\}^2 \mu(d\zeta) \leq t \cdot \left(2 \sup_{x,\zeta} \left|\frac{\partial \gamma}{\partial \zeta}(x, \zeta)\right|\right)^2 \cdot \int (\zeta^2 \wedge 1)\mu(d\zeta) \leq kt$$

for some $k > 0$, which is uniform in $0 < r < 1$ and x. We have that, for given $p > k$ and $t \leq 1$,

$$P\left(\sup_{0\leq s\leq t,x} |x_s(r, \emptyset, x) - x| > t^{1/\beta'}\right) \leq C_0 \exp[-(p - k)(\log(1/t))^2] \tag{2.36}$$

with C_0 not depending on x, r, and hence the assertion follows for this case.

Indeed, by the upper bound of exponential type ([150] Theorem 13),

$$P\left(\sup_{0\leq s\leq t,x} |x_s(r, \emptyset, x) - x| \geq C\right) \leq 2 \exp\left[-\lambda C + \frac{1}{2}(\lambda S)^2 kt(1 + \exp \lambda S)\right]$$

for $C > 0$, $\lambda > 0$. We choose $C = pS \log(1/t)$ and $\lambda = (\log(1/t))/S$. Then,

$$\begin{aligned} P&\left(\sup_{0\leq s\leq t,x} |x_s(r, \emptyset, x) - x| \geq pS \log(1/t)\right) \\ &\leq 2 \exp[-p(\log(1/t))^2 + (\log(1/t))^2 kt + k(\log(1/t))^2] \\ &\leq C_0 \exp[-(p - k)(\log(1/t))^2] \quad \text{as} \quad t \to 0 . \end{aligned}$$

We choose $S = (1/p)t^{1/\beta}$. Then,

$$\begin{aligned} P&\left(\sup_{0\leq s\leq t,x} |x_s(r, \emptyset, x) - x| \geq t^{1/\beta'}\right) \\ &\leq P\left(\sup_{0\leq s\leq t,x} |x_s(r, \emptyset, x) - x| \geq t^{1/\beta} \log(1/t)\right) \\ &\leq C_0 \exp[-(p - k)(\log(1/t))^2] , \end{aligned}$$

as $t \to 0$. Here, $\beta' > \beta$ and the constant C_0 does not depend on x, r.

(2) Case $n_1 \geq 1$. Assume $S_t = \{t_1, \ldots, t_{n_1}\}$ with $t_1 < \ldots < t_{n_1}$. Given $\xi^r_1, \ldots, \xi^r_{n_1}$, we put

$$I_{n_1} \equiv U^r_{n_1} - x_t(r, S_t, x)$$

and

$$M \equiv \max\left(\sup_{x,\zeta} \left\|\frac{\partial \xi}{\partial x}(r, \zeta, x)\right\|, 1\right) = \max\left(\sup_{x,\zeta} \left\|I + \frac{\partial \gamma}{\partial x}(x, \zeta)\right\|, 1\right) ,$$

which exists by the assumption on γ. Here, $\|\cdot\|$ denotes the norm of the matrix. In rewriting the above, by an elementary calculation, we have

$$\begin{aligned}|I_{n_1}| \le \sup_y |\phi_{t_{n_1},t}(y) - y| + M \sup_y |\phi_{t_{n_1-1},t_{n_1}}(y) - y| \\ + M^2 \sup_y |\phi_{t_{n_1-2},t_{n_1-1}}(y) - y| + \ldots + M^{n_1} \sup_x |\phi_{0,t_1}(x) - x| \,.\end{aligned}$$

We assume $|I_{n_1}| > t^{1/\beta'}$. Then, for some $j \in \{1, \ldots, n_1\}$,

$$\sup_y |\phi_{t_{j-1},t_j}(y) - y| > (1/n_1) M^{-n_1} t^{1/\beta'} \,.$$

Indeed, if for all j, $\sup_y |\phi_{t_{j-1},t_j}(y) - y| \le (1/n_1) M^{-n_1} t^{1/\beta'}$, then

$$\begin{aligned}|I_{n_1}| &\le \sup_y |\phi_{t_{n_1},t}(y) - y| + M \sup_y |\phi_{t_{n_1-1},t_{n_1}}(y) - y| \\ &\quad + M^2 \sup_y |\phi_{t_{n_1-2},t_{n_1-1}}(y) - y| + \ldots + M^{n_1} \sup_x |\phi_{0,t_1}(x) - x| \\ &\le (1/n_1) M^{-n_1} t^{1/\beta'} + (1/n_1) M^{-n_1+1} t^{1/\beta'} + \ldots + (1/n_1) t^{1/\beta'} \le t^{1/\beta'} \,,\end{aligned}$$

which is a contradiction. Hence,

$$\begin{aligned}&E^{\tilde{P}_{t,r}|\mathbb{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} \left[P\left(|I_{n_1}| > t^{1/\beta'} | S_t, \xi_1^r, \ldots, \xi_{n_1}^r | S_t \right) \right] \\ &\quad \le E^{\tilde{P}_{t,r}|\mathbb{S}_{t,n_1}} \left[P(\text{there exists } j \text{ such that} \right. \\ &\qquad\qquad \left. \sup_y |\phi_{t_{j-1},t_j}(y) - y| > (1/n_1) M^{-n_1} t^{1/\beta'}) \right] \\ &\quad \le n_1 P\left(\sup_{0 \le s \le t, x} |x_s(r, \emptyset, x) - x| > (1/n_1) M^{-n_1} t^{1/\beta'} \right) \,.\end{aligned}$$

We choose $\beta'' = \beta''(n_1)$ such that $\beta < \beta'' < \beta'$ and $t^{1/\beta} < t^{1/\beta''} < (1/n_1) M^{-n_1} t^{1/\beta'} < t^{1/\beta'}$ for $t > 0$ small. Then, by the proof of (2.36),

$$\begin{aligned}&E^{\tilde{P}_{t,r}|\mathbb{S}_{t,n_1}} E^{(\mu^r)^{\otimes n_1}} \left[P\left(|I_{n_1}| > t^{1/\beta'} | S_t, \xi_1^r, \ldots, \xi_{n_1}^r \right) \right] \\ &\quad \le n_1 P\left(\sup_{0 \le s \le t, x} |x_s(r, \emptyset, x) - x| > t^{1/\beta''} \right) \\ &\quad \le n_1 C_0 \exp[-(p-k)(\log(1/t))^2]\end{aligned}$$

as $t \to 0$. This proves (2.35). □

2.3 Methods of finding the asymptotic bounds (II)

Assume $x_t(x)$ is an Itô process given by (3.2). In this section, we use an approximation method called the *polygonal geometry* method. Intuitively, we make polygonal lines along with the trajectories of jump paths connecting jump points in between, instead of chains described in the previous section. This intuitive view will help us to understand the support theorems stated in Section 2.5. In contrast to the Markov chain method, this method is applied (at least here) only to the case where the Lévy measure has a density.

By using this method, we can obtain similar results on the short time asymptotic bounds of the density. Statements are simple compared to those obtained by the previous method, although the conditions are more restrictive.

2.3.1 Polygonal geometry

We assume, in particular, the following assumptions on the Lévy measure $\mu(dz)$:

$\mu(dz)$ has a C^∞-density $h(z)$ on $\mathbf{R}^m \setminus \{0\}$ such that

$$\text{supp } h(\cdot) \subset \{z \in \mathbf{R}^m \,;\, |z| \leq c\} \;, \quad \text{where } 0 < c \leq +\infty \;,$$

$$\int_{\mathbf{R}^m \setminus \{0\}} \min\left(|z|^2, 1\right) h(z)dz < +\infty \;,$$

$$\int_{|z|\geq\epsilon} \left(\frac{|\partial h/\partial z|^2}{h(z)}\right) dz < +\infty \quad \text{for } \epsilon > 0 \;, \tag{3.1}$$

and that

$$h(z) = a\left(\frac{z}{|z|}\right) \cdot |z|^{-m-\alpha} \tag{3.2}$$

in a neighbourhood of the origin for some $\alpha \in (0, 2)$, and some strictly positive function $a(\cdot) \in C^\infty(S^{m-1})$.

Here, we are assuming the smoothness of the Lévy measure. This is due to the fact that we employ Bismut's perturbation method. The latter half of the condition in (3.1) is also the requirement from this method. It may seem too restrictive, however, this method will well illustrate the behaviour of the jump trajectories, as will be seen below.

We repeat several notations that appeared in Section 2.2. Set

$$P_x = x + \gamma(x, \text{supp } h) = x + \{\gamma(x, z) \,;\, z \in \text{supp } h\} \;.$$

Then, each P_x is a compact set in $\mathbf{R}^d$ since γ is bounded. For each $y \in \mathbf{R}^d (y \neq x)$, we put

$$\alpha(x, y) \equiv l_0(x, y) + 1 \;.$$

Here, $l_0(x, y)$ denotes the minimum number of distinct points $z_1, \ldots, z_l \in \mathbf{R}^d$ such that

$$z_1 \in P_x, z_i \in P_{z_{i-1}}, \quad i = 2, \ldots, l \text{ and } y \in P_{z_l} \; (z_0 = x) \;. \tag{3.3}$$

We always have $\alpha(x, y) < +\infty$ for each given $x, y \in \mathbf{R}^d (x \neq y)$ by (3.2) and *(A.1)*.

We set

$$g(x,z) = h(H_x^{-1}(z))\ \left|[J\gamma]^{-1}(x, H_x^{-1}(z))\right| \quad \text{for } z \in P_x \setminus \{x\} \tag{3.4}$$

and $g(x,z) = 0$ otherwise. Here, we put $H_x : \text{supp } h \to P_x, z \mapsto x + \gamma(x,z)(= z)$, and $J\gamma = (\partial\gamma/\partial z)(x,z)$ is the Jacobian matrix of γ. The kernel $g(x,z)$ is well-defined and satisfies

$$\int f(z)g(x,z)dz = \int f(x+\gamma(x,z))\, h(z)dz, \quad \text{for} \quad f \in C(P_x)\,. \tag{3.5}$$

That is, $g(x,dz) = g(x,z)dz$ is the Lévy measure of $x_t(x)$; supp $g(x,\cdot) \subset P_x$ by definition.

Remark 2.6. An intuitive view of the polygonal geometry is that one can cover polygonal lines connecting x and $y \in \mathbf{R}^d$ by a sequence of "small balls" P_{z_i}'s with z_i's at the jump points of $x_t(x)$ and connecting those z_i's by piecewise linear lines. Historically, the polygonal geometry method has been employed earlier than the Markov chain method. In the Markov chain approximation setting, assumptions stated in (3.1) are not necessary.

Theorem 2.4 ([85]). *Under the assumptions* (3.1, 3.2) *and (A.1, A.2), we have, for each distinct pair* $(x,y) \in \mathbf{R}^d$ *for which* $\kappa = \alpha(x,y)(< +\infty)$*,*

$$\lim_{t\to 0} \frac{p_t(x,y)}{t^\kappa} = C\,,$$

where

$$\begin{aligned} C &= C(x,y,\kappa) \\ &= \begin{cases} (1/\kappa!)\left\{\int_{P_x} dz_1 \cdots \int_{P_{z_{\kappa-2}}} dz_{\kappa-1} g(x,z_1)\cdots g(z_{\kappa-1},y)\right\}\,, & \kappa \ge 2\,, \\ g(x,y)\,, & \kappa = 1\,. \end{cases} \end{aligned} \tag{3.6}$$

The proof of this theorem will be given below.

2.3.2 Proof of Theorem 2.4

Decomposition of the transition density

In this section we give a decomposition of $p_t(x,y)$, which plays a crucial role. We partly repeat a similar argument to what appeared in Section 2.2.2.

Given $\epsilon > 0$, let $\phi_\epsilon : \mathbf{R}^m \to \mathbf{R}$ be a non-negative C^∞-function such that $\phi_\epsilon(z) = 1$ if $|z| \ge \epsilon$ and $\phi_\epsilon(z) = 0$ if $|z| \le \epsilon/2$. Let $z_t(\epsilon)$ and $z_t'(\epsilon)$ be two independent Lévy processes whose Lévy measures are given by $\phi_\epsilon(z)\, h(z)dz$ and $(1-\phi_\epsilon(z))h(z)dz$, respectively. Then the process $z(t)$ has the same law as that of $z_t(\epsilon) + z_t'(\epsilon)$. Since the process $z_t(\epsilon)$ has finite Lévy measure on $\mathbf{R}^m \setminus \{0\}$, the corresponding Poisson random measure $N_s(\epsilon)$ on $[0,+\infty) \times (\mathbf{R}^m \setminus \{0\})$ counts only finite times in each finite interval $[0,t]$. We set

$\mathcal{S}_{t,k} = ([0,t]^k/\sim)$ and $\mathcal{S}_t = \amalg_{k\geq 0}\mathcal{S}_{t,k}$, where $\amalg_{k\geq 0}$ denotes the disjoint sum and $\sim$ means the identification of the coordinates on the product space $[0,t]^k$ by the permutation. The distribution $\tilde{P}_{t,\epsilon}$ of the moments (instants) of jumps related to $N_s(\epsilon)$ in $[0,t]$ is given by

$$\int_{\{\#S_t=k\}} f(S_t)d\tilde{P}_{t,\epsilon}(S_t) = \left\{\left(t\int \phi_\epsilon(z)h(z)dz\right)^k \left(\frac{1}{k!}\right)\exp\left(-t\int \phi_\epsilon(z)h(z)dz\right)\right\}$$
$$\times \frac{1}{t^k}\int_0^t \cdots \int_0^t f(s_1,\cdots,s_k)ds_1\cdots ds_k\,, \tag{3.7}$$

where f is a function on $\mathcal{S}_{t,k}$ (a symmetric function on $[0,t]^k$).

Let $J(\epsilon)$ be a random variable whose law is $\phi_\epsilon(z)\,h(z)dz/(\int \phi_\epsilon(z)\,h(z)dz)$, and choose a family of independent random variables $J_i(\epsilon)$, $i = 1, 2, \cdots$ having the same law as $J(\epsilon)$. Choose $0 \leq s \leq t < +\infty$. For a fixed $S_t \in \mathcal{S}_t$, we consider the solution of the following SDE with jumps:

$$x_s(\epsilon, S_t, x) = x + \sum_{u\leq s}^{c} \gamma\left(x_{u^-}(\epsilon, S_t, x), \Delta z'_u(\epsilon)\right)$$
$$+ \sum_{s_i\in S_t, s_i\leq s} \gamma(x_{s_i-}(\epsilon, S_t, x), J_i(\epsilon))\,. \tag{3.8}$$

Then, the law of $x_s(\epsilon, S_t, x)$ has a smooth density denoted by $p_s(\epsilon, S_t, x, y)$ (cf. [140] p.87). Then,

$$p_s(x,y) = \int_{\mathcal{S}_t} p_s(\epsilon, S_t, x, y)\, d\tilde{P}_{t,\epsilon}(S_t) \tag{3.9}$$

because $z_t(\epsilon)$ and $z'_t(\epsilon)$ are independent, which is written in a *decomposed form* (by putting $s = t$)

$$p_t(x,y) = \sum_{i=0}^{N-1} p_t(i, \epsilon, x, y) + \bar{p}_t(N, \epsilon, x, y)\,, \tag{3.10}$$

where

$$p_t(i, \epsilon, x, y) = \int_{\mathcal{S}_{t,i}} p_t(\epsilon, S_t, x, y)\, d\tilde{P}_{t,\epsilon}(S_t)$$

and $\bar{p}_t(N, \epsilon, x, y)$ is the remaining term. Then, it follows from the definition and (3.7) that

$$p_t(k, \epsilon, x, y) = (1/k!)\cdot \exp\left(-t\int \phi_\epsilon(z)\,h(z)dz\right)\cdot\left(\int \phi_\epsilon(z)\,h(z)dz\right)^k$$
$$\times \int_0^t \ldots \int_0^t p_t\,(\epsilon, \{s_1, \ldots, s_k\}, x, y)\, ds_k \ldots ds_1\,. \tag{3.11}$$

We denote by $x_s(\epsilon, \emptyset, x)$ the process in (3.8) with $S_t \in \mathcal{S}_{t,0}$, and its density by $p_s(\epsilon, \emptyset, x, y)$.

For the random variable $J(\epsilon)$ introduced above, there exists a density $g_\epsilon(x, z)$ such that $P(x + \gamma(x, J(\epsilon)) \in dz) = g_\epsilon(x, z)dz$. Indeed, $g_\epsilon(x, z)$ is given by

$$g_\epsilon(x, z) = \frac{\phi_\epsilon(H_x^{-1}(z))g(x, z)}{\int \phi_\epsilon(z)\, h(z)dz}\ , \tag{3.12}$$

where $g(x, z)$ is the density of the Lévy measure of $x_t(x)$ (cf. (3.4)). Note that by definition, supp $g_\epsilon(x, \cdot) \subset P_x$, and that $g_\epsilon(x, z)$ is of class C^∞ whose derivatives are uniformly bounded (since $g(x, z)$ only has a singularity at $x = z$). Now, for each $s_1 < \cdots < s_k < t$, we have

$$\begin{aligned} p_t(\epsilon, \{s_1, \ldots, s_k\}, x, y) = \int dz_0' \int_{P_{z_0'}} dz_1 \cdots \int dz_{k-1}' \int_{P_{z_{k-1}'}} dz_k \\ \Big\{ p_{s_1}(\epsilon, \emptyset, x, z_0')g_\epsilon(z_0', z_1)\, p_{s_2-s_1}(\epsilon, \emptyset, z_1, z_1')g_\epsilon(z_1', z_2) \\ \times p_{s_3-s_2}(\epsilon, \emptyset, z_2, z_2') \cdots g_\epsilon(z_{k-1}', z_k)\, p_{t-s_k}(\epsilon, \emptyset, z_k, y) \Big\}\ . \end{aligned} \tag{3.13}$$

Indeed, the increment $x_{s_i+u}(\epsilon, \{s_1, \ldots, s_k\}, x) - x_{s_i}(\epsilon, \{s_1, \ldots, s_k\}, x)$ has the same law as that of $x_u(\epsilon, \emptyset, x) - x$ on $(0, s_{i+1} - s_i)$ for $i = 0, \ldots, k (s_0 = 0, s_{k+1} = t)$, and $x_{(s_i+u)^-}(\epsilon, \{s_1, \ldots, s_k\}, x)$ is going to make a "big jump" (i.e. a jump derived from $J_{i+1}(\epsilon)$) at $u = s_{i+1} - s_i$ according to the law $g_\epsilon\left(x_{(s_{i+1})^-}(\epsilon, \{s_1, \ldots, s_k\}, x), z\right) dz$.

Lower bound

Let $\epsilon_c \equiv \sup\{\epsilon > 0;\ \{|z| < \epsilon\} \subset \text{supp } h\} > 0$, and choose $0 < \epsilon < \epsilon_c$. First, we note that for each $\eta > 0$ and a compact set K, uniformly in $y \in K$, we have

$$\lim_{s \to 0} \int_{\{z; |z-y| \le \eta\}} p_s(\epsilon, \emptyset, z, y)dz = 1\ , \tag{3.14}$$

by Proposition I.2 in Léandre [140]. Then, we have:

Lemma 2.8. *Let $\mathcal{K}$ be a class of nonnegative, equicontinuous, uniformly bounded functions whose supports are contained in a fixed compact set K. Then, given $\delta > 0$, there exists a $t_0 > 0$ such that*

$$\inf_{s \le t} \int f(z) p_{t-s}(\epsilon, \emptyset, z, y)dz \ge f(y) - \delta \tag{3.15}$$

for every $f \in \mathcal{K}$, $y \in K$ and every $t \in (0, t_0)$.

Proof. Given $\delta > 0$, choose $\eta > 0$ so that $|f(z) - f(y)| < \delta/2$ for each $|z - y| < \eta$. Then,

$$\begin{aligned}\int f(z)p_{t-s}(\epsilon, \emptyset, z, y)dz &\geq \int_{\{z;|z-y|\leq\eta\}} f(z)p_{t-s}(\epsilon, \emptyset, z, y)dz \\ &= \int_{\{z;|z-y|\leq\eta\}} (f(z) - f(y))p_{t-s}(\epsilon, \emptyset, z, y)dz + f(y) \\ &\quad \times \int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\epsilon, \emptyset, z, y)dz \\ &\geq (f(y) - \delta/2) \int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\epsilon, \emptyset, z, y)dz\ .\end{aligned}$$

By (3.14), we can choose $t_0 > 0$ so that if $t < t_0$, then

$$\left\{\inf_{y\in K, s\leq t} \int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\epsilon, \emptyset, z, y)dz\right\} (f(y) - \delta/2) \geq f(y) - \delta \quad \text{for all } y \in K\ . \qquad \square$$

Now, we choose an arbitrary compact neighbourhood $\bar{U}(x)$ of x and arbitrary compact sets $K_1, \cdots, K_{k-1}$ of $\mathbf{R}^d$, and set

$$\begin{aligned}\mathcal{K} = \Big\{& g_\epsilon(z'_0, \cdot), g_\epsilon(z'_1, \cdot), \cdots, g_\epsilon(z'_{k-1}, \cdot)\ ; \\ & z'_0 \in \bar{U}(x), z'_1 \in K_1, \cdots, z'_{k-1} \in K_{k-1}\Big\}\ .\end{aligned}$$

Since $\mathcal{K}$ has the property in Lemma 2.8 (cf. (3.15)), it follows from (3.13) that for every $\delta > 0$, there exists $t_0 > 0$ such that for every $0 < t < t_0$,

$$\begin{aligned}& p_t(\epsilon, \{s_1, \ldots, s_k\}, x, y) \\ & \geq \int_{\bar{U}(x)} p_{s_1}(\epsilon, \emptyset, x, z'_0)dz'_0 \int_{K_1} dz'_1 \cdots \int_{K_{k-1}} dz'_{k-1}(g_\epsilon(z'_0, z'_1) - \delta) \\ & \quad \cdots (g_\epsilon(z'_{k-1}, y) - \delta)\ . \end{aligned} \tag{3.16}$$

However, for each fixed $\eta > 0$, we have

$$\lim_{t\to 0} \inf_{s_1\leq t, x\in\mathbf{R}^d} \int_{\{|x-z'_0|\leq\eta\}} p_{s_1}(\epsilon, \emptyset, x, z'_0)dz'_0 = 1\ , \tag{3.17}$$

by (2.13) in [140]. Therefore, for every sufficiently small $t > 0$, it holds that

$$\begin{aligned}p_t(\epsilon, \{s_1, \ldots, s_k\}, x, y) &\geq (1 - \delta) \int_{\bar{U}(x)} \int_{K_1} dz'_1 \int_{K_2} dz'_2 \cdots \\ & \cdots \int_{K_{k-1}} dz'_{k-1}(g_\epsilon(z'_0, z'_1) - \delta)(g_\epsilon(z'_1, z'_2) - \delta) \cdots (g_\epsilon(z'_{k-1}, y) - \delta)\ . \end{aligned} \tag{3.18}$$

Combining (3.11) with (3.18), we have

$$\begin{aligned}&\liminf_{t\to 0}\left(\frac{1}{t^k}\right)p_t(k,\epsilon,x,y)\\&\geq (1/k!)\cdot(1-\delta)\cdot\left(\int \phi_\epsilon(z)\,h(z)dz\right)^k\\&\quad\times\int_{K_1}dz_1'\int_{K_2}dz_2'\cdots\int_{K_{k-1}}dz_{k-1}'\\&\qquad\left\{(g_\epsilon(x,z_1')-\delta)(g_\epsilon(z_1',z_2')-\delta)\cdots(g_\epsilon(z_{k-1}',y)-\delta)\right\}\ .\end{aligned}\tag{3.19}$$

Since $\delta > 0$ and $K_1,\cdots,K_{k-1}$ are arbitrary, and since supp $g_\epsilon(z_{i-1}',\cdot)\subset P_{z_{i-1}'}$, $i = 1,\cdots,k-1$, we have

$$\begin{aligned}&\liminf_{t\to 0}\left(\frac{1}{t^k}\right)p_t(k,\epsilon,x,y)\\&\geq (1/k!)\cdot\left(\int \phi_\epsilon(z)\,h(z)dz\right)^k\\&\quad\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}g_\epsilon(x,z_1)g_\epsilon(z_1,z_2)\cdots g_\epsilon(z_{k-1},y)\ .\end{aligned}\tag{3.20}$$

Hence, it follows from (3.10) that

$$\begin{aligned}&\liminf_{t\to 0}\left(\frac{1}{t^{\alpha(x,y)}}\right)p_t(x,y)\\&\geq (1/\alpha(x,y)!)\cdot\left(\int \phi_\epsilon(z)\,h(z)dz\right)^k\\&\quad\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}g_\epsilon(x,z_1)g_\epsilon(z_1,z_2)\cdots g_\epsilon(z_{k-1},y)\ .\end{aligned}\tag{3.21}$$

Since $\epsilon > 0$ is arbitrary, in view of (3.12), we have

$$\begin{aligned}&\liminf_{t\to 0}\left(\frac{1}{t^{\alpha(x,y)}}\right)p_t(x,y)\\&\geq (1/\alpha(x,y)!)\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}\\&\qquad\{g(x,z_1)g(z_1,z_2)\cdots g(z_{k-1},y)\}\ .\end{aligned}\tag{3.22}$$

The proof for the lower bound is now complete.

Upper bound

The proof of the upper bound $\limsup_{t\to 0}(1/t^{\alpha(x,y)})p_t(x,y)$ is rather delicate and is carried out in the same way as in [140], but it is a little more tedious in our case.

First, we choose and fix $N \geq \alpha(x,y)+1$. Noting that $\sup\{p_t(\epsilon,S_t,x,y);\#S_t\geq 2, x,y\in\mathbf{R}^d\}\leq\tilde{C}(\epsilon)$ (cf. [140] (3.23)), we have:

Lemma 2.9. *For every* $\epsilon > 0$ *and* $t > 0$*, we have*

$$\sup_{y\in\mathbf{R}^d} \bar{p}_t(N,\epsilon,x,y) \le C(\epsilon)\, t^N\ . \tag{3.23}$$

Proof. Recall that

$$\bar{p}_t(N,\epsilon,x,y) = \int_{\mathcal{S}_t\setminus\left(\bigcup_{i=0}^{N-1}\mathcal{S}_{t,i}\right)} p_t(\epsilon,S_t,x,y)\, d\tilde{P}_{t,\epsilon}(S_t)\ .$$

Since

$$\tilde{P}_{t,\epsilon}(\#S_t \ge N) \le C(\epsilon)t^N$$

by the Poisson law, we have

$$\begin{aligned}\bar{p}_t(N,\epsilon,x,y) &\le \sup_{\#S_t\ge N} p_t(\epsilon,S_t,x,y)\tilde{P}_{t,\epsilon}(\#S_t\ge N)\\ &\le \sup\{p_t(\epsilon,S_t,x,y); \#S_t\ge 2, x,y\in\mathbf{R}^d\}C(\epsilon)t^N\\ &\le \tilde{C}(\epsilon)C(\epsilon)t^N\ .\end{aligned}$$

□

Lemma 2.10. *For every* $p > 1$ *and any* $\eta > 0$*, there exists* $\epsilon > 0$ *such that for all* $t \le 1$ *and every* $\epsilon' \in (0,\epsilon)$*,*

$$\sup_{|x-y|\ge\eta} p_t(0,\epsilon',x,y) \le C(\epsilon',\eta)\, t^p\ . \tag{3.24}$$

Combining Lemma 2.9 with Lemma 2.10, we see that it is sufficient to study $p_t(k,\epsilon,x,y)$ for $1 \le k \le \alpha(x,y)$. For a given $\eta > 0$, make a subdivision of the space

$$\mathbf{A} \equiv \left\{(z_0',z_1,\dots,z_{k-1}',z_k)\in\mathbf{R}^d\times\cdots\times\mathbf{R}^d;\ z_1\in P_{z_0'}, z_2\in P_{z_1'},\cdots,z_k\in P_{z_{k-1}'}\right\}$$

as $\mathbf{A} = \bigcup_{i=1}^{2^{k+1}} A_i(\eta)$, where

$$\begin{aligned}A_1(\eta)=\Big\{&(z_0',z_1,\dots,z_{k-1}',z_k)\in\mathbf{R}^d\times\cdots\times\mathbf{R}^d;\ z_1\in P_{z_0'}, z_2\in P_{z_1'},\cdots,z_k\in P_{z_{k-1}'}\\ &\text{and } |x-z_0'|\le\eta, |z_1-z_1'|\le\eta, |z_2-z_2'|\le\eta,\cdots,|z_k-y|\le\eta\Big\}\ ,\\ A_2(\eta)=\Big\{&(z_0',z_1,\dots,z_{k-1}',z_k)\in\mathbf{R}^d\times\cdots\times\mathbf{R}^d;\ z_1\in P_{z_0'}, z_2\in P_{z_1'},\cdots,z_k\in P_{z_{k-1}'}\\ &\text{and } |x-z_0'|\le\eta, |z_1-z_1'|>\eta, |z_2-z_2'|\le\eta,\cdots,|z_k-y|\le\eta\Big\}\ ,\\ &\cdots\cdots\end{aligned}$$

and

$$\begin{aligned}A_{2^{k+1}}(\eta)=\Big\{&(z_0',z_1,\dots,z_{k-1}',z_k)\in\mathbf{R}^d\times\cdots\times\mathbf{R}^d;\ z_1\in P_{z_0'}, z_2\in P_{z_1'},\cdots,\\ &z_k\in P_{z_{k-1}'}\quad\text{and } |x-z_0'|>\eta, |z_1-z_1'|>\eta, |z_2-z_2'|>\eta,\cdots,|z_k-y|>\eta\Big\}\ ,\end{aligned}$$

and we shall classify those divisions into four cases:

$$\begin{aligned}[A] &= \{A_i(\eta); |z_k-y|>\eta\}\ ,\\ [B] &= \Big\{A_i(\eta); |x-z_0'|>\eta \quad\text{and}\quad |z_k-y|\le\eta\Big\}\ ,\\ [C] &= \{A_1(\eta)\} \equiv \Big\{\{|x-z_0'|\le\eta, |z_1-z_1'|\le\eta, |z_2-z_2'|\le\eta,\cdots,\\ &\qquad |z_k-y|\le\eta\}\Big\}\ ,\end{aligned}$$

and

$$[D] = \left\{A_i(\eta); |x - z'_0| \le \eta \text{ and } |z_k - y| \le \eta\right\} \setminus \{A_1(\eta)\} .$$

We put

$$\begin{aligned} &I_{[N],t,\eta}(\epsilon, \{s_1, \ldots, s_k\}, x, y) \\ &= \sum_{A_i(\eta)\in[N]} \int_{A_i(\eta)} \Big\{ p_{s_1}(\epsilon, \emptyset, x, z'_0) g_\epsilon(z'_0, z_1) p_{s_2-s_1}(\epsilon, \emptyset, z_1, z'_1) \\ &g_\epsilon(z'_1, z_2) p_{s_3-s_2}(\epsilon, \emptyset, z_2, z'_2) \times \cdots \times g_\epsilon(z'_{k-1}, z_k) p_{t-s_k}(\epsilon, \emptyset, z_k, y) \Big\} \\ &dz_k dz'_{k-1} \ldots dz_1 dz'_0 , \quad [N] = [A], [B], [C], [D] . \end{aligned} \tag{3.25}$$

Then, in view of (3.13), we have

$$\begin{aligned} &p_t(\epsilon, \{s_1, \ldots, s_k\}, x, y) \\ &\qquad = (I_{[A],t,\eta} + I_{[B],t,\eta} + I_{[C],t,\eta} + I_{[D],t,\eta})(\epsilon, \{s_1, \ldots, s_k\}, x, y) , \end{aligned} \tag{3.26}$$

since $\operatorname{supp}(g_\epsilon(z'_{i-1}, \cdot)) \subset P_{z'_{i-1}}$ for $i = 1, \cdots, k$.

Lemma 2.11. *For any* $(x, y) \in (\mathbf{R}^d \times \mathbf{R}^d) \setminus \triangle$ $(\triangle \equiv \{(x, x); x \in \mathbf{R}^d\})$, *any* $\{s_1, \ldots, s_k\}$, *any* $\eta > 0$ *and any* $p > 1$, *there exists* $\epsilon > 0$ *such that if* $0 < \epsilon' < \epsilon$ *and* $t \le 1$, *then*

$$(I_{[A],t,\eta} + I_{[B],t,\eta} + I_{[D],t,\eta})(\epsilon', \{s_1, \ldots, s_k\}, x, y) \le C(\epsilon', \eta, k, p) t^p .$$

Proof. The proof is essentially the same as that in [140] Proposition III.4, but is a little more complicated.

First, note that there exists $\epsilon > 0$ such that if $0 < \epsilon' < \epsilon$ and $t \le 1$, then

$$P\left\{\sup_{0\le s\le t} |x_s(\epsilon', \emptyset, x) - x| > \eta\right\} \le c(\epsilon', \eta, p) t^p \tag{3.27}$$

([140] Proposition I.4 and Lepeltier–Marchal [150] Lemme 17). Then, we observe

$$I_{[B],t,\eta}(\epsilon', \{s_1, \ldots, s_k\}, x, y) \le C_1(\epsilon', \eta, k, p) t^p .$$

Next, let $G_z : \mathbf{R}^d \to \mathbf{R}^d$, $x \mapsto x + \gamma(x, z)$. If $z \in \operatorname{int}(\operatorname{supp} h)$, then G_z defines a diffeomorphism by *(A.2)*. Let G_z^{-1} denote its inverse mapping, and put $\tilde\gamma(x, z) \equiv G_z^{-1}(x) - x$, $z \in \operatorname{int}(\operatorname{supp} h)$. Since $\gamma(x, z)$ is a bounded, C^∞-function both in x and z, *(A.1)* and *(A.2)* imply that $\tilde\gamma(x, z)$ is also bounded, and C^∞ in $x \in \mathbf{R}^d$ and $z \in \operatorname{int}(\operatorname{supp} h)$. Note that $\tilde\gamma(x, 0) = 0$ since $G_0(x) = x$. For fixed $\epsilon > 0$, put

$$\tilde S_\epsilon = \sup\left\{\left|\tilde\gamma(x, z)\right| ; x \in \mathbf{R}^d, z \in \operatorname{int}(\operatorname{supp}(1 - \phi_\epsilon) \cdot h)\right\} .$$

The following estimate, also obtained by Léandre [140] Proposition I.3, is used in the estimate of $I_{[A],t,\eta}(\epsilon', \{s_1, \ldots, s_k\}, x, y)$:

for every $p > 1$, and every η with $\tilde{S}_\epsilon < \eta$,

$$\limsup_{s\to 0} \sup_{y\in\mathbf{R}^d}(1/s^p) \int_{\{x;|x-y|>\eta\}} p_s(\epsilon, \emptyset, x, y)dx < +\infty . \tag{3.28}$$

Since $\tilde{S}_\epsilon \to 0$ as $\epsilon \to 0$, it follows from (3.25), (3.28) that

$$I_{[A],t,\eta}(\epsilon', \{s_1, \ldots, s_k\}, x, y) \le C_2(\epsilon', \eta, k, p)t^p .$$

Using the inequality (3.27), we can prove a similar estimate for

$$I_{[D],t,\eta}(\epsilon', \{s_1, \ldots, s_k\}, x, y) . \qquad \square$$

Noting Lemma 2.11, we only have to study $I_{[C],t,\eta}(\epsilon', \{s_1, \ldots, s_k\}, x, y)$ for each small $\eta > 0$ and $0 < \epsilon' < \epsilon$. Put $\alpha(x, y) = \kappa(1 \le \kappa < +\infty)$. Then, we have:

Lemma 2.12. *If η is small and $1 \le i < \kappa = \alpha(x, y)$, then*

$$I_{[C],t,\eta}(\epsilon', S_t, x, y) = 0 \textit{ for } S_t \in \mathcal{S}_{t,i} , \tag{3.29}$$

and hence

$$\int_{\mathcal{S}_{t,i}} I_{[C],t,\eta}(\epsilon', S_t, x, y)d\tilde{P}_{t,\epsilon'}(S_t) = 0 . \tag{3.30}$$

Proof. Let $Q_{x,\eta,i}$ and $Q_{x,i}$ be as follows:

$$\begin{aligned} Q_{x,\eta,i} \equiv \Big\{ & z \in \mathbf{R}^d; \exists(z'_0, z_1, z'_1, \cdots, z'_{i-1}, z_i) \in \mathbf{R}^d \times \cdots \times \mathbf{R}^d , \\ & z_1 \in P_{z'_0}, z_2 \in P_{z'_1}, \cdots, z_i \in P_{z'_{i-1}} , \\ & |x - z'_0| \le \eta, |z_1 - z'_1| \le \eta, |z_2 - z'_2| \le \eta, \cdots, |z_i - z| \le \eta \Big\} , \\ Q_{x,i} \equiv \Big\{ & z \in \mathbf{R}^d; \exists(z_1, \cdots, z_{i-1}) \in \mathbf{R}^d \times \cdots \times \mathbf{R}^d , \\ & z_1 \in P_x, z_2 \in P_{z_1}, \cdots, z_{i-1} \in P_{z_{i-2}}, z \in P_{z_{i-1}} \Big\} \\ = & \bigcup \{P_{z_{i-1}}; z_1 \in P_x, \cdots, z_{i-1} \in P_{z_{i-2}}\} . \end{aligned}$$

Here, we put $z_0 = x$. Then, $Q_{x,i}$ is a closed set in $\mathbf{R}^d$ since each P_{z_j} is compact and $z_j \mapsto P_{z_j}$ is continuous. Observe that $Q_{x,\eta,i} \supset Q_{x,i}$ for all $\eta > 0$, and $\bigcap_{\eta>0} Q_{x,\eta,i} = Q_{x,i}$. That is, $y \in Q_{x,i}$ if and only if $y \in Q_{x,\eta,i}$ for all $\eta > 0$. Since supp $(g_{\epsilon'}(z'_j, \cdot)) \subset P_{z'_j}$ for $j = 0, \cdots, i-1$, we observe

$$\begin{aligned} & I_{[C],t,\eta}(\epsilon', S_t, x, y) \\ & \equiv \int_{A_1(\eta)} \Big\{ p_{s_1}(\epsilon', \emptyset, x, z'_0)g_{\epsilon'}(z'_0, z_1)p_{s_2-s_1}(\epsilon', \emptyset, z_1, z'_1) \\ & g_{\epsilon'}(z'_1, z_2)p_{s_3-s_2}(\epsilon', \emptyset, z_2, z'_2) \times \cdots \times g_{\epsilon'}(z'_{i-1}, z_i)p_{t-s_i}(\epsilon', \emptyset, z_i, y) \Big\} \\ & dz_i dz'_{i-1} \ldots dz_1 dz'_0 . \end{aligned} \tag{3.31}$$

In view of the condition in $Q_{x,\eta,i}$, we see that

$$\text{if } y \notin Q_{x,\eta,i} \text{ then } I_{[C],t,\eta}(\epsilon', S_t, x, y) = 0 . \tag{3.32}$$

Recall that $\alpha(x, y) = \kappa$ and $i < \kappa$. By the definition of $\alpha(x, y)$ in Section 2.3.1, we have that $y \notin Q_{x,i}$, which implies $y \notin Q_{x,\eta,i}$ for every sufficiently small $\eta > 0$. Hence, $I_{[C],t,\eta}(\epsilon', S_t, x, y) = 0$ for $S_t \in \mathcal{S}_{t,i}$ if $i < \kappa = \alpha(x, y)$. Thus,

$$\int_{\mathcal{S}_{t,i}} I_{[C],t,\eta}(\epsilon', S_t, x, y) d\tilde{P}_{t,\epsilon'}(S_t) = 0 .$$

□

Lemma 2.13. *Let $\mathcal{K}$ be a class of nonnegative, equicontinuous, uniformly bounded functions whose supports are contained in a fixed compact set K. Let the constants $\eta > 0$ and $\epsilon' > 0$ be those appearing in $I_{[C],t,\eta}(\epsilon', S_t, x, y)$ and (3.32), respectively. Then, for every $\delta > 0$, there exists $t_0 > 0$ such that*

$$\sup_{s \le t} \int_{\{z;|z-y| \le \eta\}} f(z) p_{t-s}(\epsilon', \emptyset, z, y) dz \le f(y) + \delta \tag{3.33}$$

for every $f \in \mathcal{K}$, $y \in K$ and every $t \in (0, t_0)$.

Proof. For a given $\delta > 0$, there exists $\eta_1 = \eta_1(K, \mathcal{K}, \delta) > 0$ such that $|f(z) - f(y)| < \delta/4$ for each $|z - y| < \eta_1$. We may assume $\eta_1 \le \eta$ by choosing small δ. Then, we have

$$\begin{aligned} &\sup_{s \le t} \int_{\{z;|z-y| \le \eta\}} f(z) p_{t-s}(\epsilon', \emptyset, z, y) dz \\ &\le \sup_{s \le t} \int_{\{z;|z-y| \le \eta_1\}} f(z) p_{t-s}(\epsilon', \emptyset, z, y) dz \\ &\quad + \sup_{s \le t} \int_{\{z;\eta_1 < |z-y| \le \eta\}} f(z) p_{t-s}(\epsilon', \emptyset, z, y) dz . \end{aligned} \tag{3.34}$$

The first term can be estimated as in Lemma 2.8 by $f(y) + \delta/2$ for $t \in (0, t_1)$ for some $t_1 > 0$. As for the second term, since $\eta > 0$ is arbitrary in (3.14) and since all f's in $\mathcal{K}$ are uniformly bounded, there exists $t_2 > 0$ such that

$$\sup_{s \le t} \int_{\{z;\eta_1 < |z-y| \le \eta\}} f(z) p_{t-s}(\epsilon', \emptyset, z, y) dz \le \delta/2$$

for every $f \in \mathcal{K}$, $y \in K$ and $t \in (0, t_2)$.

Letting $t_0 \equiv \min\{t_1, t_2\} > 0$, we have the assertion. □

Choose an arbitrary compact neighbourhood $\bar{U}(x)$ of x and arbitrary compact sets $K_1, \cdots, K_{\kappa-1}$ of $\mathbf{R}^d$ such that $\{z; |z-x| \le \eta\} \subset \bar{U}(x)$ and that $Q_{x,\eta,i} \subset K_i$, $i = 1, \cdots, \kappa-1$. Set

$$\begin{aligned} \mathcal{K} = \{&g_{\epsilon'}(z_0', \cdot), g_{\epsilon'}(z_1', \cdot), \cdots, g_{\epsilon'}(z_{\kappa-1}', \cdot) ; \\ &z_0' \in \bar{U}(x), z_1' \in K_1, \cdots, z_{\kappa-1}' \in K_{\kappa-1}\} . \end{aligned}$$

To apply Lemma 2.13, we should be a little more careful since $\epsilon' > 0$ depends on the choice of $\eta > 0$ by Lemma 2.11. Since $\mathcal{K}$ has the property in Lemma 2.13, for a given $\delta > 0$, $\eta > 0$, $\epsilon' > 0$, there exists $t_0 > 0$ such that for every $0 < t < t_0$,

$$\sup_{s \le t} \int_{\{|z_i - z_i'| \le \eta\}} g_{\epsilon'}(z_{i-1}', z_i) p_{t-s}(\epsilon', \emptyset, z_i, z_i') dz_i \le g_{\epsilon'}(z_{i-1}', z_i') + \delta \,, \tag{3.35}$$

for $z_0' \in \bar{U}(x)$, $z_{i-1}' \in K_{i-1}$, $i = 2, \cdots, \kappa (z_\kappa' = y)$.

From (3.35), we have, in view of (3.31), for $0 < t < t_0$,

$$\begin{aligned} I_{[C],t,\eta}&\left(\epsilon', \{s_1, \ldots, s_\kappa\}, x, y\right) \\ &\le \int_{\bar{U}(x)} p_{s_1}(\epsilon', \emptyset, x, z_0') dz_0' \int_{K_1} dz_1' \cdots \int_{K_{\kappa-1}} dz_{\kappa-1}' \\ &\qquad \left\{ (g_{\epsilon'}(z_0', z_1') + \delta) \cdots (g_{\epsilon'}(z_{\kappa-1}', y) + \delta) \right\} . \end{aligned} \tag{3.36}$$

Hence, by (3.17),

$$\begin{aligned} \limsup_{t \to 0} (1/t^\kappa) \int_0^t \cdots \int_0^t &\left(\int \phi_{\epsilon'}(z) \, h(z) dz \right)^\kappa \\ &\times I_{[C],t,\eta}\left(\epsilon', \{s_1, \ldots, s_\kappa\}, x, y\right) ds_\kappa \cdots ds_1 \\ &\le \left(\int \phi_{\epsilon'}(z) \, h(z) dz \right)^\kappa \int_{K_1} dz_1 \int_{K_2} dz_2 \cdots \int_{K_{\kappa-1}} dz_{\kappa-1} \\ &\qquad \{ (g_{\epsilon'}(x, z_1) + \delta)(g_{\epsilon'}(z_1, z_2) + \delta) \cdots (g_{\epsilon'}(z_{\kappa-1}, y) + \delta) \} . \end{aligned} \tag{3.37}$$

Since $\delta > 0$ and $K_1, \cdots, K_{\kappa-1}$ are arbitrary, we have, in view of Lemma 2.9 (with $N = \kappa + 1$), Lemmas 2.10 and 2.11, 2.12 and (3.10), (3.11), (3.26), that is,

$$\begin{aligned} &\limsup_{t \to 0} \left(\frac{1}{t^\kappa} \right) p_t(x, y) \\ &\le (1/\kappa!) \cdot \left(\int \phi_{\epsilon'}(z) \, h(z) dz \right)^\kappa \\ &\quad \times \int_{P_x} dz_1 \int_{P_{z_1}} dz_2 \cdots \int_{P_{z_{\kappa-2}}} dz_{\kappa-1} \{ g_{\epsilon'}(x, z_1) g_{\epsilon'}(z_1, z_2) \cdots g_{\epsilon'}(z_{\kappa-1}, y) \} . \end{aligned} \tag{3.38}$$

Letting $\epsilon' \to 0$, we have, in view of (3.12),

$$\begin{aligned} &\limsup_{t \to 0} \left(\frac{1}{t^\kappa} \right) p_t(x, y) \\ &\le (1/\kappa!) \times \int_{P_x} dz_1 \int_{P_{z_1}} dz_2 \cdots \\ &\quad \times \int_{P_{z_{\kappa-2}}} dz_{\kappa-1} \{ g(x, z_1) g(z_1, z_2) \cdots g(z_{\kappa-1}, y) \} . \end{aligned} \tag{3.39}$$

The proof for the upper bound is now complete.

The statement of Theorem 2.4 is immediate from (3.22) (with $k = \alpha(x, y)$) and (3.39). The condition on the continuity of the Lévy measure has well illustrated the nature of the polygonal method. □

2.3.3 Example of Theorem 2.4 – easy cases

In this subsection, we are concerned with the *constant* in the upper bound of $p_t(x, y)$ as $t \to 0$, mentioned in Theorem 2.4. We shall restrict ourselves to a simple Lévy process to exhibit the virtue of the polygonal method.

Example 2.1. Let $d = m = 2$, and let a smooth radial function η satisfy supp $\eta = \{x; |x| \le 1\}$ and $\eta(x) \equiv 1$ in $\{x; |x| \le 1/2\}$. Put

$$h(z) = \eta(z)|z|^{-2-\alpha},\ \alpha \in (0,1), \quad \text{for } z \in \mathbf{R}^2 \setminus \{0\}\ , \tag{3.40}$$

that is, $h(z)dz$ is the Lévy measure for a *truncated stable process* (cf. [73], Section 3) with index $\alpha \in (0, 1)$. Then, h satisfies (3.1) with $c = 1$ and (3.2). Let $\gamma(x, z) = z$, and let $x_t(x)$ be given by

$$x_t(x) = x + \sum_{s \le t} \Delta z(s)\ . \tag{3.41}$$

Then, $P_x = x + \text{supp}\ h = x + B(1)(B(1) \equiv \{x; |x| \le 1\})$, and $g(x, z)$ is reduced to $g(x, z) = h(z - x)$ (cf. (3.4)).

Let $x_0 = (0, 0)$ and choose $y_0 = (e, 0)$ so that $1 < e < 2$. We then have $\alpha(x_0, y_0) = 2$. The constant $C(x_0, y_0, 2)$ is calculated as follows:

$$C(x_0, y_0, 2) = \int_{P_{x_0}} g(x_0, z) g(z, y_0) dz = \int_{B(1)} h(z - x_0) h(y_0 - z) dz\ . \tag{3.42}$$

The integral (3.42) makes sense. Indeed, if $z = 0$, then $h(y_0 - 0) = h(y_0) = 0$. Since $y_0 \in \{x; |x| > 1\}$, by the continuity of $x \mapsto P_x$, there exists $\delta > 0$ such that if $|z - x_0| < \delta$, and then $y_0 \notin \text{supp}\,(g(z, \cdot))$. That is, $g(z, y_0) = 0$ for $z \in \{z; |z - x_0| < \delta\}$, and this implies that the integral exists.

Example 2.2. Let $m = d$. Let $z_1(t), \cdots, z_d(t)$ be an independent one-dimensional truncated stable processes (stationary martingales with jumps), whose common characteristic function is given by

$$\psi_t(v) = \exp\left[t \int_{\mathbf{R}\setminus\{0\}} (e^{i\langle z, v\rangle} - 1 - i\langle z, v\rangle) h(z) dz \right]\ .$$

That is,

$$z_j(t) = \int_0^{t+} \int_{\mathbf{R}\setminus\{0\}} z \tilde{N}_j(ds dz), \quad j = 1, \cdots, d\ ,$$

where $\tilde{N}_j(dsdz)$'s denote the compensated Poisson random measures on $[0, +\infty) \times (\mathbf{R} \setminus \{0\})$, having the common mean measure $h(z)dz \times ds$: $\tilde{N}_j(dsdz) = N_j(dsdz) - h(z)dz \times ds$. Here, the Lévy measure $h(z)dz$ is given by

$$h(z) = \eta(z)|z|^{-1-\alpha}, \quad \alpha \in (1, 2),\ z \in \mathbf{R} \setminus \{0\}\,, \tag{3.43}$$

where $\eta(z) \in \mathcal{C}_0^\infty(\mathbf{R})$, $0 \le \eta(z) \le 1$, supp $\eta = \{z; |z| \le c\}$, and $\eta(z) \equiv 1$ in $\{z; |z| \le \frac{c}{2}\}$ for some $0 < c < +\infty$.

Let $e_0, e_1, \ldots, e_d$ be constant vectors in $\mathbf{R}^d$ such that $\langle e_1, \cdots, e_d\rangle$ forms a basis of $T_0\mathbf{R}^d$. Given $x \in \mathbf{R}^d$, consider the jump process $x_t(x)$ given by

$$x_t(x) = x + \sum_{j=1}^{d} \sum_{s\le t}^{c} e_j \Delta z_j(s) + e_0 t = x + e_1 z_1(t) + \cdots + e_d z_d(t) + e_0 t\,, \tag{3.44}$$

where $\sum^c$ denotes the compensated sum stated in Section 1.3. As a linear combination of truncated stable processes $z_1(t), \cdots, z_d(t)$ and a uniform motion $e_0 t$, $x_t(x)$ possesses a transition density $p_t(x, y)$ which is C^∞ with respect to $y (y \neq x)$ for $t > 0$ (cf. [143]).

In [84], the lower bound of $p_t(x, y)$ as $t \to 0$ for given distinct x, y is obtained for processes including the above type. However, the analysis to obtain the upper bound for $p_t(x, y)$ is somewhat complicated. This is because in order to get the upper bound, we have to track all the possible trajectories of $x_t(x)$, each having a Lévy measure which is singular with respect to the d-dimensional Lebesgue measure. To do this, we make use of a kind of homogeneity of the process, which is due to the vector fields $e_1, \cdots, e_d$ being constant, and make a combinatorial discussion. For this purpose, we prepare the following notation.

Let

$$\mathcal{I}(d) = \{(i_1, \cdots, i_d); i_1, \cdots, i_d = 0 \text{ or } 1\}\ . \tag{3.45}$$

For $I = (i_1, \cdots, i_d) \in \mathcal{I}(d)$, we put $|I| =$ *the length of* $I = i_1 + \cdots + i_d$. We put $I' = (i_1', \cdots, i_d') \in \mathcal{I}(d)$ for $I = (i_1, \cdots, i_d)$ in the following way: $i_j' = 1$ if $i_j = 0$ and $i_j' = 0$ otherwise. We write $I = (i_1, \cdots, i_d) \le J = (j_1, \cdots, j_d)$ for $I, J \in \mathcal{I}(d)$ if $i_r \le j_r$, $r = 1, \cdots, d$. Consider a sequence $\mathbf{b} = (I_1, \cdots, I_\ell)$ of elements of I's in $\mathcal{I}(d)$. We denote by $\mathcal{S}(k, \ell)$ the following family of sequences:

$$\mathcal{S}(k, \ell) = \left\{\mathbf{b} = (I_1, \cdots, I_\ell); |I_1| + \cdots + |I_\ell| = k, |I_j| \ge 1, j = 1, \cdots, \ell\right\}\ . \tag{3.46}$$

For $\mathbf{b} = (I_1, \cdots, I_\ell) \in \mathcal{S}(k, \ell)$, we put the *weight* $\omega(\mathbf{b})$ and the *α-weight* $\omega_\alpha(\mathbf{b})$ of indices $I_1, \cdots, I_\ell$ as follows: $\omega(\mathbf{b}) = |I_1| + \cdots + |I_\ell|$, $\omega_\alpha(\mathbf{b}) = \sum_{r=1}^{\ell}(|I_r| - |I_r'|/\alpha)$.

Consider the parallelepiped of $\mathbf{R}^d$ built on the basis $\langle e_1, \cdots, e_d\rangle$

$$Q = \left\{t_1 e_1 + \cdots + t_d e_d; t_j \in (\text{supp } h), j = 1, \cdots, d\right\}\ ,$$

and put $P_x = x + Q$. Similarly, define sets

$$Q^I = \{t_{i_1} e_1 + \cdots + t_{i_d} e_d; t_{i_r} \in (\text{supp } h) \text{ if } i_r = 1,\ t_{i_r} = 0 \text{ if } i_r = 0\}$$

and $P_x^I = x + Q^I$ for $I = (i_1, \cdots, i_d) \in \mathcal{I}\,(d)$.

For each $y \in \mathbf{R}^d (y \neq x)$, we put

$$\beta(x, y) \equiv \ell_0(x, y) + 1 . \tag{3.47}$$

Here, $\ell_0(x, y)$ denotes the minimum number of distinct points $z_1, \cdots, z_\ell \in \mathbf{R}^d$ such that $z_1 \in P_x, z_2 \in P_{z_1}, \cdots, z_\ell \in P_{z_{\ell-1}}$ and $y \in P_{z_\ell}$. This means that $y \in \mathbf{R}^d$ can be joined to x by a *polygonal line* with $\ell_0(x, y)$-corners passing along the combination of the parallelepipeds $P_x \cup P_{z_1} \cup \cdots \cup P_{z_{\ell_0(x,y)}}$. Furthermore, we put

$$\begin{aligned}\tilde{\beta}(x, y) = \min \Big\{ \omega(\mathbf{b}); \mathbf{b} = (I_1, \cdots, I_\ell) \in (\mathcal{I}(d))^\ell, \ell = \beta(x, y) ,\\ \text{such that } z_1 \in P_x^{I_1}, z_2 \in P_{z_1}^{I_2}, \cdots, z_{\ell-1} \in P_{z_{\ell-2}}^{I_{\ell-1}} \\ \text{and } y \in P_{z_{\ell-1}}^{I_\ell} \text{ for some } z_1, \cdots, z_{\ell-1} \in \mathbf{R}^d \Big\} ,\end{aligned} \tag{3.48}$$

$$\begin{aligned}\rho_\alpha(x, y) = \min \Big\{ \omega_\alpha(\mathbf{b}); \mathbf{b} = (I_1, \cdots, I_\ell) \in (\mathcal{I}(d))^\ell ,\\ \ell = \beta(x, y), \omega(\mathbf{b}) = \tilde{\beta}(x, y) \Big\} .\end{aligned} \tag{3.49}$$

We remark that $\tilde{\beta}(x, y) \leq d\beta(x, y)$ with $\beta(x, y) < +\infty$ (since $c > 0$), and that $\rho_\alpha(x, y) \leq \tilde{\beta}(x, y)$. In case $c = +\infty$, we always have $\beta(x, y) = 1 (x \neq y)$.

We fix $x \in \mathbf{R}^d$ and $I = (i_1, \cdots, i_d) \in \mathcal{I}(d)$ in what follows. We assume $|I| = b$, and denote by $i_1^*, \cdots, i_b^*$ the nonzero indices i_r arrayed in numerical order. Consider the mapping

$$H_x^I : (\text{supp } h)^{|I|} \longrightarrow P_x^I, \quad (t_{i_1^*}, \cdots, t_{i_b^*}) \longmapsto x + \sum_{r=1}^{b} t_{i_r^*} e_{i_r^*} . \tag{3.50}$$

This introduces the Lévy measure $d\mu_x^I$ on P_x^I by

$$\begin{aligned}\int_{P_x^I} f(z) d\mu_x^I(z) = \int \cdots \int f\left(x + t_{i_1^*} e_{i_1^*} + \cdots + t_{i_b^*} e_{i_b^*}\right) \\ \times h\left(t_{i_1^*}\right) \cdots h\left(t_{i_b^*}\right) dt_{i_1^*} \cdots dt_{i_b^*} ,\end{aligned} \tag{3.51}$$

for $f \in C(P_x^I)$. We abbreviate this by $\mu_x^I(\bar{d}z)$ for simplicity. Here, $\bar{d}z$ denoted the volume element induced on P_x^I by

$$\bar{d}z = |de^{i_1^*} \wedge \cdots \wedge de^{i_b^*}|, \quad z \in P_x^I \tag{3.52}$$

where $(de^{i_1^*}, \cdots, de^{i_b^*})$ denotes the covector field on P_x^I corresponding to $(e_{i_1^*}, \cdots, e_{i_b^*})$. Since $d\mu_x^I(z)$ is the measure transformed from $h(t_1) \otimes \cdots \otimes h(t_b) dt_1 \cdots dt_b$ by the linear mapping

$$H_x^I : (t_1, \cdots, t_b) \longmapsto x + A^{I t}(t_1, \cdots, t_b) ,$$

$(A^I \equiv (e_{i_1^*}, \cdots, e_{i_b^*}))$, it possesses the density with respect to $\bar{d}z$, which is C^∞ with respect to $z \in P_x^I \setminus \{x\}$ and $x \in \mathbf{R}^d$. We shall put

$$g^I(x, z) = \frac{d\mu_x^I}{\bar{d}z}(z) \text{ if } z \in P_x^I \setminus \{x\} , g^I(x, z) = 0 \quad \text{otherwise.} \tag{3.53}$$

Explicitly,

$$g^I(x,z) = (h \otimes \cdots \otimes h)\left(\left(H_x^I\right)^{-1}(z)\right)\left|\left[JH_x^I\right]^{-1}\right|, \quad z \in P_x^I \setminus \{x\}\,. \tag{3.54}$$

Obviously, we have for $I^* = (1, 1, \cdots, 1)$, $g^{I^*}(x,z) = g(x,z)$, the density of the original Lévy measure of $x_t(x)$.

Proposition 2.6. *For each $x, y \in \mathbf{R}^d (y \neq x)$, suppose that the numbers $\lambda = \beta(x,y)$, $\kappa = \tilde{\beta}(x,y)$ and $\rho = \rho_\alpha(x,y)$, defined as above, are finite. If $\kappa \geq d-1$, then we have*

$$\limsup_{t\to 0}\left(\frac{1}{t^\rho}\right) p_t(x,y) \leq C\,.$$

The constant C has an expression

$$C = \sum_{\substack{(I_1,\ldots,I_\ell)\in \mathcal{S}(\kappa,\ell)\,,\\ \lambda\leq\ell\leq\kappa, I_1\leq\cdots\leq I_\ell}} c(I_1,\cdots,I_\ell)\left\{\int_{P_x^{I_1}} \bar{d}z_2 \cdots \int_{P_{z_{\ell-1}}^{I_{\ell-1}}} \bar{d}z_\ell\, g^{I_1}(x,z_2)\cdots g^{I_\ell}(z_\ell,y)\right\}\,.$$

Here, we put $c(I_1,\cdots,I_\ell) = (\kappa!\Pi_{j=1}^d(1/r_j!)).c^*(\kappa,\ell)$, where $r_j \equiv \sharp\{r \in \{1,\cdots,\ell\}; i_{r,j} = 1\}$ for $I_r = (i_{r,1},\cdots,i_{r,d})$ and

$$c^*(\kappa,\ell) = \limsup_{t\to 0}\left(\frac{1}{t^\rho}\right)\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{\ell-1}}^t ds_\ell$$
$$\left\{s_1^{|I_1|-1-|I_1'|/\alpha}(s_2-s_1)^{|I_2|-1-|I_2'|/\alpha}\cdots(s_\ell-s_{\ell-1})^{|I_\ell|-1-|I_\ell'|/\alpha}\right\}\,.$$

The proof of this proposition depends on Léandre's method in [140], and on some discussions in [84, 85]. A crucial idea in our proof is the (intuitive) notion of *polygonal trajectories*. A polygonal trajectory is a trajectory obtained from a piecewise linear line whose segment from a corner point z_i to the next corner point z_{i+1} is parallel to some combination of vectors contained in $\{e_1,\cdots,e_d\}$. It should be noted that in a small time interval, there is a tendency that the process $x_t(x)$ makes use of those trajectories which jump as few times as possible and stay at a point until just before the moment of the next jump.

Intuitively, $\beta(x,y)$ above can be interpreted as the minimum number of jumps by which the trajectory can reach y from x along a polygonal line, and $\tilde{\beta}(x,y)$ corresponds to the weight of such a line. We need another notion of α-weight $\rho_\alpha(x,y)$ of a line for considering the factors which 'stay' in the same corner point at each jump moment, and this gives the real power in t. The decomposition of jumps to those factors which really jump and to those that stay in is carried out in terms of indices I and I'.

Examples of Proposition 2.6

We shall provide some simple examples of Proposition 2.6.

Example 2.3. Let $d = 2$, $c = 1$, $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$, $x_0 = (0, 0)$ and $y_0 = (e, f)$, $0 < e < 1$, $0 < f < 1$. Clearly, $\beta(x_0, y_0) = 1$, $\tilde{\beta}(x_0, y_0) = 2$, and we have $\mathcal{S}(2, 1) = \{((1, 1))\}$, and hence $\rho_\alpha(x_0, y_0) = 2$. We have

$$C = C(x_0, y_0, 2, 1, 2) = c((1, 1))g^{(1,1)}(x_0, y_0) = h(e)h(f) < +\infty .$$

In this case, we know exactly that

$$p_t(x_0, y_0) = \wp_t(e)\wp_t(f) \sim h(e)h(f)t^2 ,$$

as $t \to 0$ (cf. [84]). Hence, the constant is the best possibility.

Example 2.4. Let $d = 2$, $c = 1$, $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$, $x_0 = (0, 0)$ and $y_0 = (e, 0)$, $0 < e < 1$. Clearly, $\beta(x_0, y_0) = \tilde{\beta}(x_0, y_0) = 1$, and we have $\mathcal{S}(1, 1) = \{((1, 0)), ((0, 1))\} \equiv \{(I_1), (I_2)\}$, and hence $\rho_\alpha(x_0, y_0) = 1 - 1/\alpha$. We have

$$\begin{aligned} C = C(x_0, y_0, 1, 1, 1 - 1/\alpha) &= \sum_{(I)\in\mathcal{S}(1,1)} c(I)g^I(x_0, y_0) \\ &= c(I_1)g^{I_1}(x_0, y_0) + c(I_2)g^{I_2}(x_0, y_0) \\ &= \left(\frac{1}{1 - \frac{1}{\alpha}}\right)h(e) < +\infty \end{aligned}$$

since $y_0 \notin P_{x_0}^{I_2} \setminus \{x_0\}$.

In this case, we know exactly

$$p_t(x_0, y_0) = \wp_t(e)\wp_t(0) \sim \frac{1}{\pi}\Gamma\left(1 + \frac{1}{\alpha}\right)h(e)t^{1-\frac{1}{\alpha}} ,$$

as $t \to 0$ (cf. [225], (2.2.11)). Here, $\frac{1}{\pi}\Gamma(1 + \frac{1}{\alpha}) = \frac{1}{\pi}\frac{1}{\alpha}\Gamma(\frac{1}{\alpha}) < \frac{1}{\pi}\Gamma(\frac{1}{2}) < (\frac{1}{1-\frac{1}{\alpha}})$ for all $1 < \alpha < 2$, and hence our constant is overestimated.

Example 2.5. Let $d = 2$, $c = 1$, $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$, $x_0 = (0, 0)$ and $y_0 = (e, 0)$, $1 < e < 2$. Then, $\beta(x_0, y_0) = \tilde{\beta}(x_0, y_0) = 2$. We have

$$\mathcal{S}(2, 2) = \{((1, 0), (1, 0)), ((1, 0), (0, 1)), ((0, 1), (1, 0)), ((0, 1), (0, 1))\} ,$$

and hence $\rho_\alpha(x_0, y_0) = 2 - 2/\alpha$.

We can calculate $C(x_0, y_0, 2, 2, 2 - 2/\alpha)$ as follows:

$$\begin{aligned} C(x_0, y_0, 2, 2, 2 - 2/\alpha) &= \sum_{\substack{(I_1, I_2) \in \mathcal{S}(2,2) \\ I_1 \le I_2}} c(I_1, I_2) \times \left\{ \int_{P_{x_0}^{I_1}} g^{I_1}(x, z_2) g^{I_2}(z_2, y_0) \bar{d}z_2 \right\} \\ &= c\,((1,0),(1,0)) \int_{P_{x_0}^{(1,0)}} g^{(1,0)}(x_0, z_2) g^{(1,0)}(z_2, y_0) \bar{d}z_2 \\ &\quad + c((0,1),(0,1)) \int_{P_{x_0}^{(0,1)}} g^{(0,1)}(x_0, z_2) g^{(0,1)}(z_2, y_0) \bar{d}z_2 \\ &= A + B \quad \text{(say)}\,. \end{aligned}$$

For A, we have, for $z_2 = (u, v)$,

$$\begin{aligned} g^{(1,0)}(x_0, z_2) &= h(u - 0) && \text{if } u \in (-1, 1)\,, \\ &= 0 && \text{otherwise}\,, \\ g^{(1,0)}(z_2, y_0) &= h(e - u) && \text{if } e \in (u - 1, u + 1)\,, \\ &= 0 && \text{otherwise}\,, \end{aligned}$$

and $\bar{d}z_2$ is reduced to du. Hence, $A = c((1,0),(1,0)) \int_{e-1}^{1} h(u)h(e-u)du$.

For B, we have, for $z_2 = (u, v)$,

$$\begin{aligned} g^{(0,1)}(x_0, z_2) &= h(v - 0) && \text{if } v \in (-1, 1)\,, \quad u = 0 \\ &= 0 && \text{otherwise}\,, \\ g^{(0,1)}(z_2, y_0) &= h(-v) && \text{if } v \in (-1, 1)\,, \quad u = e \\ &= 0 && \text{otherwise}\,, \end{aligned}$$

and $\bar{d}z_2$ is reduced to dv. Hence, $B = c((0,1),(0,1)) \int_{-1}^{1} h(v-0) \times 0 dv = 0$ since $y_0 \notin P_{(0,v)}^{(0,1)} \setminus \{(0, v)\}$ for $-1 < v < 1$. Hence,

$$C\,(z_0, y_0, 2, 2, 2 - 1/\alpha) = \left(\frac{1}{1 - \frac{1}{\alpha}} \right) B\left(1 - \frac{1}{\alpha}, 2 - \frac{1}{2}\right) \times \int_{e-1}^{1} h(u)h(e-u)du < +\infty\,.$$

Here, we used

$$\begin{aligned} c((1,0),(1,0)) \equiv \frac{2!}{2!0!} \times \limsup_{t \to 0} \left(\frac{1}{t^{2-2/\alpha}} \right) \times \int_0^t ds_1 \int_{s_1}^t ds_2 \left\{ s_1^{-1/\alpha}(s_2 - s_1)^{-1/\alpha} \right\} \\ = \left(\frac{1}{1 - \frac{1}{\alpha}} \right) B\left(1 - \frac{1}{\alpha}, 2 - \frac{1}{\alpha}\right)\,, \end{aligned}$$

where $B(p, q)$ denotes the Beta-function.

Example 2.6. Let $d = 2$, $c = 1$, $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$, $x_0 = (0,0)$, $y_0 = (\frac{3}{2}, \frac{7}{4})$. Then, $\beta(x_0, y_0) = 2$, $\tilde{\beta}(x_0, y_0) = 4$. We have $\mathcal{S}(4,2) = \{((1,1),(1,1))\}$, and hence $\rho_\alpha(x_0, y_0) = 4$.

We can calculate $C(x_0, y_0, 2, 4, 4)$ as follows:

$$\begin{aligned}
&C(x_0, y_0, 2, 4, 4)\\
&= \sum_{\substack{(I_1,\cdots.I_\ell)\in\mathcal{S}(4,\ell),2\le\ell\le4\\ I_1\le\cdots\le I_\ell}} c(I_1,\cdots,I_\ell)\left\{\int_{P_{x_0}^{I_1}} \bar{d}z_2 \cdots \int_{P_{z_{\ell-1}}^{I_{\ell-1}}} \bar{d}z_\ell g^{I_1}(x,z_2)\cdots g^{I_\ell}(z_\ell, y_0)\right\}\\
&= \sum_{\substack{(I_1,I_2)\in\mathcal{S}(4,2)\\ I_1\le I_2}} c(I_1,I_2)\int_{P_{x_0}^{I_1}} g^{I_1}(x,z_2)g^{I_2}(z_2,y_0)\bar{d}z_2 + \sum_{\substack{(I_1,I_2,I_3)\in\mathcal{S}(4,3)\\ I_1\le I_2\le I_3}} c(I_1,I_2,I_3)\\
&\quad\times\left\{\int_{P_{x_0}^{I_1}} \bar{d}z_2 \int_{P_{z_2}^{I_2}} \bar{d}z_3\, g^{I_1}(x,z_2)\, g^{I_2}(z_2,z_3)\, g^{I_3}(z_3,y_0)\right\}\\
&\qquad+ \sum_{\substack{(I_1,I_2,I_3,I_4)\in\mathcal{S}(4,4)\\ I_1\le I_2\le I_3\le I_4}} c(I_1,I_2,I_3,I_4)\\
&\quad\times\left\{\int_{P_{x_0}^{I_1}} \bar{d}z_2 \int_{P_{z_2}^{I_2}} \bar{d}z_3 \int_{P_{z_3}^{I_3}} \bar{d}z_4 g^{I_1}(x,z_2)g^{I_2}(z_2,z_3)g^{I_3}(z_3,z_4)g^{I_4}(z_4,y_0)\right\}\\
&= A + B + C \quad \text{(say)}.
\end{aligned}$$

We have, for $z_2 = (u, v)$,

$$A = c\left((1,1),(1,1)\right)\int_{\frac{1}{2}}^{1} du \int_{\frac{3}{4}}^{1} dv\, h(u)h(v)h\left(\frac{3}{2}-u\right)h\left(\frac{7}{4}-v\right).$$

We observe

$$\begin{aligned}
&\mathcal{S}(4,3)\cap\{\mathbf{b} = (I_1,I_2,I_3)\in(\mathcal{I}(2))^3\,;I_1\le I_2\le I_3\}\\
&\qquad= \{((1,0),(1,0),(1,1))\,,((0,1),(0,1),(1,1))\}\,,
\end{aligned}$$

and hence for $z_2 = (u,v)$, $z_3 = (w,t)$, we have

$$\begin{aligned}
B &= c((1,0),(1,0),(1,1))\int_{-1}^{1} du \int_{u-1}^{u+1} dw\left\{h(u-0)h(w-u)h\left(\frac{3}{2}-w\right)h\left(\frac{7}{4}\right)\right\}\\
&\quad+ c((0,1),(0,1),(1,1))\int_{-1}^{1} dv \int_{u-1}^{u+1} dt\left\{h(v-0)h(t-v)h\left(\frac{7}{4}-w\right)h\left(\frac{3}{2}\right)\right\}\\
&= 0 + 0
\end{aligned}$$

since $h(\frac{7}{4}) = h(\frac{3}{2}) = 0$. The term C turns out to vanish for a similar reason. Hence,

$$C(x_0, y_0, 2, 4, 4) = \frac{1}{4}\int_{\frac{1}{2}}^{1} du \int_{\frac{3}{4}}^{1} dv h(u)h(v)h\left(\frac{3}{2} - u\right)h\left(\frac{2}{4} - v\right) .$$

Here, we used

$$c((1,1),(1,1)) = \frac{4!}{2!2!} \times \limsup_{t\to 0}\left(\frac{1}{t^4}\right)\int_0^t ds_1 \int_{s_1}^t ds_2 s_1(s_2 - s_1)$$
$$= 6 \times \limsup_{t\to 0}\left(\frac{1}{t^4}\right)\left(\frac{1}{24}t^4\right) = \frac{1}{4} .$$

2.4 Summary of short time asymptotic bounds

By using the integration-by-parts formula and the Fourier formula, we can show the existence of a smooth density for the probability law associated with a process X_t, which derives from the SDE with jumps.

In this section, we provide a summary of various types of the short time bounds of the density function $p_t(x, y)$ with given x, y. Results are basically proved in similar ways to what was stated in the previous sections or in the references therein. Hence, we omit the proof of each case. These bounds are closely related to the short time behaviour of X_t.

2.4.1 Case that $\mu(dz)$ is absolutely continuous with respect to the m-dimensional Lebesgue measure dz

In this subsection, we assume that $x_t(x)$ is given in Section 1.3.

Case that the coefficient $\gamma(x, z)$ is non-degenerate

Proposition 2.7 ([184]). *The density $p_t(x, y)$ satisfies the following estimate:*

$$\text{(a)} \quad \sup_{x,y} p_t(x, y) \le C_0 t^{-\frac{d}{\beta}} \quad \text{as } t \to 0 , \tag{4.1}$$

$$\text{(b)} \quad p_t(x, x) \asymp t^{-\frac{d}{\beta}} \quad \text{as } t \to 0 \text{ uniformly in } x . \tag{4.2}$$

Proposition 2.8 ([85]).

(1) *off-diagonal estimate*

Assume $y \in \mathcal{S}$, that is, $\alpha(x, y) < +\infty$. Then, we have

$$p_t(x, y) \asymp t^{\alpha(x,y)} \quad \text{as } t \to 0 . \tag{4.3}$$

(2) *diagonal estimate*

$$p_t(x,x) \asymp t^{-\frac{d}{\beta}} \quad \text{as } t \to 0 \text{ uniformly in } x \,. \tag{4.2$'$}$$

These results have been mentioned in Theorems 2.2 (a), 2.3, respectively.

Case that the coefficients are degenerate
There is no result at this point.

2.4.2 Case that $\mu(dz)$ is singular with respect to dz

Now, we introduce a concrete "singular" Lévy measure of $z(s)$ which has already been described in [209] (Example 3.7), and in [25] (Section 2). Let $\mu(dz) = \sum_{n=0}^{\infty} k_n \delta_{\{a_n\}}(dz)$ be the d-dimensional Lévy measure such that $(a_n; n \in \mathbf{N})$ and $(k_n; n \in \mathbf{N})$ are sequences of points in $\mathbf{R}^d$ and real numbers, respectively, satisfying
(i) $|a_n|$ decreases to 0 as $n \to +\infty$,
(ii) $k_n > 0$,
(iii) $\sum_{n=0}^{\infty} k_n |a_n|^2 < +\infty$.

For this Lévy measure, we can show the unique existence of the solution $x_t(x)$ or Y_t given in Sections 1.3, 2.5.1 (below), respectively, and the existence of the density functions under some of the assumptions ([126, 181]). We further assume that

$$N = N(t) \equiv \max\{n; |a_n| > t^{1/\beta}\} \asymp \log\left(\frac{1}{t}\right) \,. \tag{4.4}$$

Case that the coefficients are non-degenerate
Proposition 2.9 ([89, 93]). (1) *off-diagonal estimate*
(a) *Assume $y \in \mathcal{S}$, that is, $\alpha(x,y) < +\infty$. Then, we have*

$$p_t(x,y) \asymp t^{\alpha(x,y)-d/\beta} \quad \text{as } t \to 0 \,. \tag{4.5}$$

(b) *Assume $y \in \bar{\mathcal{S}} \setminus \mathcal{S}(\alpha(x,y) = +\infty)$. Suppose $b(x) \equiv 0$ and let $\beta' > \beta$. Then, $\log p_t(x,y)$ is bounded from above by the expression of type $\Gamma = \Gamma(t)$:*

$$\Gamma \equiv -\min \sum_{n=0}^{N} \left(w_n \log\left(\frac{1}{tk_n}\right) + \log(w_n!) \right) + O\left(\log\left(\frac{1}{t}\right) \log\log\left(\frac{1}{t}\right) \right) \,, \tag{4.6}$$

and is bounded from below by the expression of type

$$-(c/\beta)\left(\log\left(\frac{1}{t}\right)\right)^2 + O\left(\log\left(\frac{1}{t}\right)\right)$$

with some $c > 0$ independent of β and y as $t \to 0$.

Here, the minimum in (4.6) is taken with respect to all choices of $a_0, \dots, a_N$ by ξ_n for $n = 1, 2, \dots, n_1$ and $n_1 \in \mathbf{N}$ such that

$$|y - A_{n_1}(x, \xi_1, \dots, \xi_{n_1})| \le t^{1/\beta'} \tag{4.7}$$

where w_n = # of a_n in the choice and $n_1 = \sum_{n=0}^{N} w_n$, where $(A_n)_{n=0}^{\infty}$, $A_n : \mathbf{R}^{d+m\times n} \to \mathbf{R}^d$ are functions defined by

$$\begin{cases} A_0(x) = x\,, \\ A_{n+1}(x, x_1, \dots, x_{n+1}) = A_n(x, x_1, \dots, x_n) + \gamma(A_n(x, x_1, \dots, x_n), x_{n+1})\,. \end{cases}$$

(2) *diagonal estimate*

$$p_t(x, x) \asymp t^{-\frac{d}{\beta}} \quad \textit{as } t \to 0 \textit{ uniformly in } x. \tag{4.2)''$$

These results have been mentioned in Theorems 2.2 (b), 2.3.

Case that the coefficients are degenerate

(The result has only been obtained in the case that X_t is a canonical process as defined in Section 2.5.1. *See* [101] *for a case of Itô process.)*

(1) *off-diagonal estimate*

Consider, in particular, a singular Lévy measure

$$\mu = \sum_{j=1}^{m} \tilde{\mu}_j = \sum_{j=1}^{m} T_j^* \mu_j\,. \tag{4.8}$$

Here, $T_j^* \mu_j = \mu_j \circ T_j^{-1}$, μ_j is a 1-dimensional Lévy measure of form $\sum_{n=1}^{\infty} k_n \delta_{\{\pm\tilde{a}_n\}}(.)$ and $T_j : z_j \mapsto (0, \dots, 0, z_j, 0, \dots, 0)$. We let

$$\mu_j(.) = \sum_{n=0}^{\infty} k_n(\delta_{\{\tilde{a}_n\}}(.) + \delta_{\{-\tilde{a}_n\}}(.)), j = 1, \dots, m \tag{4.9}$$

where

$$k_n = p^{n\beta},\ \beta \in (0, 2), \quad \text{and} \quad \tilde{a}_n = p^{-n}\,.$$

Here, p denotes an arbitrary prime number.

Proposition 2.10 ([91]). *Let μ be given by* (3.8) *above. Let $y \neq x$. Under certain assumptions, we have the following estimates for the density $p_t(x, y)$.*

(a) *Assume $y \in \mathcal{S}$, that is, $\alpha(x, y) < +\infty$. Then, we have*

$$p_t(x, y) \le Ct^{\alpha(x,y)-d/\beta} \quad \textit{as } \ t \to 0\,.$$

(b) *Assume $y \in \bar{\mathcal{S}} \setminus \mathcal{S}(\alpha(x, y) = +\infty)$. Then,* $\log p_t(x, y)$ *is bounded from above by the expression of type*

$$\Gamma(t) + O\left(\log\left(\frac{1}{t}\right)\log\log\left(\frac{1}{t}\right)\right) \tag{4.10}$$

as $t \to 0$.

Here, $\alpha(x, y)$ is defined as above, and $\Gamma(t)$ is given similarly to (4.6). Namely the chain $(A_n)_{n=0}^{\infty}$ above is replaced by $(C_n)_{n=0}^{\infty}$ given below: let $(C_n)_{n\in\mathbf{N}}$, $C_n : \mathbf{R}^{d+m\times n} \to \mathbf{R}^d$ be a deterministic chain given by

$$
\begin{aligned}
C_0(x) &= x\,, \\
C_{n+1}(x; x_1, \ldots, x_{n+1}) &= C_n(x; x_1, \ldots, x_n) \\
&\quad + \left(\sum_{j=1}^{m} x_{n+1}^{(j)} X_j\right)(C_n(x; x_1, \ldots, x_n)) \\
&\quad + \left\{ \mathrm{Exp}\left(\left(\sum_{j=1}^{m} x_{n+1}^{(j)} X_j\right)(C_n(x; x_1, \ldots, x_n))\right) \right. \\
&\qquad \left. - C_n(x; x_1, \ldots, x_n) - \sum_{j=1}^{m} x_{n+1}^{(j)} X_j(C_n(x; x_1, \ldots, x_n)) \right\} \\
&= \mathrm{Exp}\left(\left(\sum_{j=1}^{m} x_{n+1}^{(j)} X_j\right)(C_n(x; x_1, \ldots, x_n))\right). \qquad (4.11)
\end{aligned}
$$

(2) *diagonal estimate*

$$p_t(x, x) \asymp t^{-\frac{d}{\beta}} \quad \text{as } t \to 0 \text{ uniformly in } x\,. \qquad (4.2)'''$$

2.5 Auxiliary topics

In this section, we state some topics which are related, but not directly connected, to Malliavin calculus of jump type. In Section 2.5.1, we state Marcus' canonical processes. In Section 2.5.2, we state the absolute continuity of the infinitely divisible laws in an elementary way. In Section 2.5.3, we state some examples of chain movement approximations of Section 2.2.2. In Section 2.5.4, we state the support theorem for canonical processes. The proofs of the results in Sections 2.5.2, 2.5.4 are carried out by utilising classical analytic methods, and we omit the details.

2.5.1 Marcus' canonical processes

In this section, we introduce Marcus' canonical process ([164]), or geometric process, and the stochastic analysis on it.

Let $X_1, \ldots, X_m$ be C^∞, bounded vector-valued functions (viewed as the vector fields on $\mathbf{R}^d$), whose derivatives of all orders are bounded. We assume that the restricted Hörmander condition for $X_1, \cdots, X_m$ holds, that is,

$$(RH) \qquad \mathrm{Lie}(X_1, \cdots, X_m)(x) = T_x(\mathbf{R}^d) \text{ for all } x \in \mathbf{R}^d\,.$$

We also introduce the space of vectors Σ_k, $k = 0, 1, 2, \ldots$ by

$$\Sigma_0 = \{X_1, \ldots, X_m\}$$

and

$$\Sigma_k = \{[X_i, Y]; i = 1, \ldots, m, Y \in \Sigma_{k-1}\} .$$

Here, $[X, Y]$ denotes the Lie bracket between X and Y.

We say the vectors satisfy the uniformly restricted Hörmander condition if there exist $N_0 \in \mathbf{N}$ and $C > 0$ such that

$$(URH) \qquad \sum_{k=0}^{N_0} \sum_{Y \in \Sigma_k} (v, Y(x))^2 \geq C|v|^2$$

for all $x \in \mathbf{R}^d$ and all $v \in \mathbf{R}^d$.

Let $Y_t(x)$ be an $\mathbf{R}^d$-valued canonical process given by

$$dY_t(x) = \sum_{j=1}^{m} X_j(Y_{t-}(x)) \circ dz_j(t), \quad Y_0(x) = x \tag{5.1}$$

with $z(t) = (z_1(t), \ldots, z_m(t))$.

The above notation can be paraphrased in Itô form as

$$\begin{aligned} dY_t(x) &= \sum_{j=1}^{m} X_j(Y_{t-}(x)) dz_j(t) \\ &+ \left\{ \text{Exp}\left(\sum_{j=1}^{m} \Delta z_j(t) X_j \right)(Y_{t-}(x)) - Y_{t-}(x) - \sum_{j=1}^{m} \Delta z_j(t) X_j(Y_{t-}(x)) \right\} . \end{aligned} \tag{5.2}$$

Here, the second term is a sum of terms of order $O(|\Delta z(s)|^2)$, which in effect takes into account all higher order variations of $z(t)$ at time s, and $\phi(t, x) \equiv \text{Exp}\,(tv)(x)$ is the solution flow of the differential equation

$$\frac{d\phi}{dt}(t, x) = v(\phi(t, x)), \quad \phi(0, x) = x .$$

In (5.1), (5.2), we have omitted the drift (and the diffusion) term(s) for simplicity. In a geometrically intrinsic form on a manifold $M = \mathbf{R}^d$, the process Y_t can be written by

$$dY_t = \text{Exp}\left(\sum_{j=1}^{m} dz_j(t) X_j(Y_{t-}) \right)(x), \quad Y_0(x) = x .$$

According to this model, a particle jumps from x along the integral curve (geometric flow) described by $\phi(s, x)$ instantly, that is, the particle moves along $\phi(s, x)$ during $0 \leq s \leq 1$ *outside of the real time t* and ends up at $\text{Exp}\,(sv)(x)|_{s=1}$. In this sense, a canonical process Y_t is a real jump process with a fictitious auxiliary process, and Y_t

is connected to this continuous process through the fictitious time. The fictitious time can also be viewed as idealized time which man can not observe. In this perspective a canonical process is a continuous process on the idealized time interval which is partially observed.

The equivalence between (5.1) and (5.2) implies that the canonical process is also an Itô process (refer to (5.3) below).

In [134], the process $Y_t(x)$ is called "*Stratonovich*" stochastic integral due to the last correction term on the right-hand side of (5.2). However, it is different from the original Stratonovich integral. In [134], they used this name since two integrals resemble each other in spirit. The difference lies in the fact that $Y_t(x)$ uses a geometric flow as correction term instead of the algebraic average in the original one. We mimic their notation $\circ dz(t)$ in this book. (Some papers use the notation $\diamond dz(t)$ instead of it.) The SDE (5.1) has a unique solution which is a semimartingale ([134] Theorem 3.2). By $p_t(x, dy)$ we denote the transition function of $Y_t(x)$.

The main point of introducing this process is that the solutions are the flows of diffeomorphisms in the sense explained in Section 1.3 (cf. [123] Theorem 3.1, [134] Theorem 3.9). It is shown (cf. [134] Lemma 2.1) that $Y_t(x)$ can be represented as

$$Y_t(x) = x + \sum_{j=1}^{m} \int_0^t X_j(Y_{s-}(x))dz_j(s) + \int_0^t h(s, Y_{s-}(x))d[z]_s^d \tag{5.3}$$

where

$$h(s, x) = \frac{\mathrm{Exp}\,(\sum_{j=1}^m \Delta z_j(s)X_j)(x) - x - \sum_{j=1}^m \Delta z_j(s)X_j(x)}{|\Delta z(s)|^2}$$

is a Lipschitz process in s with a bounded Lipschitz constant, and $[z]_s^d = \sum_{j=1}^m [z_j, z_j]_s^d$ denotes the discontinuous part of the quadratic variation process. An intuitive meaning of this formula is as follows. At each jump time t of z, we open a unit length interval of fictitious time, over which the state $Y_{t-}(x)$ changes continuously along the integral curve $\mathrm{Exp}\,(r\sum_{j=1}^m \Delta z_j(t)X_j)(.)$, $0 \le r \le 1$ to the state

$$Y_t(x) = \mathrm{Exp}\left(\sum_{j=1}^{m} \Delta z_j(t)X_j\right)(Y_{t-}(x)) .$$

Viewing this expression, the canonical process Y_t seems to be near to the Fisk–Stratonovich integral of h (cf. [192] Chapter V.5). However, it differs from the Fisk–Stratonovich integral in that the latter cares only for the continuous part of the quadratic variation process of z, whereas the former also cares for the discontinuous part of it.

Hence, an intuitive view of the canonical process is that as the Itô integral takes care of, up to the second order of discontinuities, $\Delta z(t)$ of the integrator, the canonical process also takes care of the higher orders $(\Delta z(t))^r$, $r = 1, 2, \dots$ of discontinuities in terms of the exponential function in $h(t, x)$. This enables us to naturally treat the jump part geometrically. See the support theorems in Section 2.5.4.

For the Jacobian process $\nabla Y_t(x)$ and its inverse process $(\nabla Y_t(x))^{-1}$, we refer to [95] Lemma 6.2, [128] Lemma 3.1.

For this process, we have the following:

Theorem 2.5 ([128] Theorem 1.1). *Under the uniformly restricted Hörmander condition* (URH) *on* $(X_j,\ j = 1, \ldots, m)$ *and a non-degeneracy condition (A.0) on* μ*, the density of the law of* Y_t *exists for all* $x \in \mathbf{R}^d$ *and all* $t > 0$*:* $p_t(x, dy) = p_t(x, y)dy$.

Equation (5.2) is a coordinate free formulation on SDEs with jumps. See [134] for the precise definition for this type of integrals for semimartingales. We shall call it a *canonical SDE driven by a vector field valued semimartingale* $Y(t) = \sum_{j=1}^m Z^j(t)X_j$, according to Kunita [123, 124, 128].

2.5.2 Absolute continuity of the infinitely divisible laws

As it is called, the infinitely divisible law (ID) on $\mathbf{R}$ is the probability law which has power-roots of any order in the sense of convolutions. That is, π is an ID law if for any n, there exists a probability law π_n such that

$$\pi = \pi_n * \cdots * \pi_n \,. \tag{5.4}$$

We recall the Lévy–Khintchine formula which characterises the Fourier transform of the ID law. The law π is an ID law if and only if there exist $a \in \mathbf{R}$, $b \geq 0$ and a σ-finite measure μ satisfying $\int_{\mathbf{R}\setminus\{0\}}(1 \wedge |x|^2)\mu(dx) < \infty$, for which π can be written as

$$\hat{\pi}(\lambda) = \exp\left[ia\lambda - \frac{b^2\lambda^2}{2} + \int\limits_{\mathbf{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\leq 1\}}\right)\mu(dx)\right] . \tag{5.5}$$

By (5.5), this ID law is the law of X_1. Here, $(X_t)_{t\geq 0}$ is a Lévy process. More precisely, this process is given by the Lévy–Itô decomposition

$$X_t = at + bW_t + \int_0^t \int\limits_{|x|\leq 1} x\tilde{N}(dsdx) + \int_0^t \int\limits_{|x|>1} xN(dsdx) \,. \tag{5.6}$$

Here, W is a real Brownian motion, N is a Poisson measure which has the density $ds\mu(dx)$ on $\mathbf{R}^+ \times \mathbf{R}$.

Here, we see by the infinite divisibility that for a Lévy process X_t such that $X_1 \sim \pi$, every X_t has an infinitely divisible law π_t, and it is given by

$$\hat{\pi}_t(\lambda) = \exp t\left[ia\lambda - \frac{b^2\lambda^2}{2} + \int\limits_{\mathbf{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\leq 1\}}\right)\mu(dx)\right] \tag{5.7}$$

($X_t \sim \pi_t$). As we shall see below, the parameter t is important to inquire into the absolute continuity of the ID law. The next theorem due to W. Döblin characterises the continuity of the ID laws.

Theorem 2.6. *Let π be an ID law whose characteristic function is given by* (5.5). *π is continuous if and only if $b \neq 0$ or μ is an infinite measure.*

Then, we have the following corollary.

Corollary 2.1. *Let π be an ID law whose characteristic function is given by* (5.5). *π is discrete if and only if $b = 0$ and μ is a finite or discrete measure.*

In what follows, we assume π contains no Gaussian part ($b = 0$). For a Lévy measure μ, let $\bar{\mu}(dx) = (|x|^2 \wedge 1)\mu(dx)$.

Theorem 2.7 (Sato [195]). *Let π be an ID law whose characteristic function is given by* (5.5). *Assume $b = 0$ and that μ is an infinite measure. Suppose that there exists some $p \in \mathbf{N}$ such that $\bar{\mu}^{*p}$ is absolutely continuous. Then, π is absolutely continuous.*

This theorem assumes a (very weak) smoothness for the Lévy measure. This assumption is used in stochastic calculus of variations developed by J.-M. Bismut (cf. [160]). A weak point of this theorem is that it excludes the discrete Lévy measure. The following theorem admits the case for a discrete Lévy measure and holds under a weaker assumption on small jumps.

Theorem 2.8 (Kallenberg [111]). *Let π be an ID law whose characteristic function is given by* (5.5), *and let $b = 0$. If*

$$\lim_{\epsilon \to 0} \epsilon^{-2} |\log \epsilon|^{-1} \bar{\mu}(-\epsilon, \epsilon) = +\infty , \tag{5.8}$$

then π has a C^∞ density whose derivatives of all orders decrease at infinity.

We introduce a weaker assumption:

$$\liminf_{\epsilon \to 0} \epsilon^{-2} |\log \epsilon|^{-1} \bar{\mu}(-\epsilon, \epsilon) > 2 . \tag{5.9}$$

Since $\cos x \sim 1 - x^2/2 (x \to 0)$, it follows from the above calculation that

$$|\hat{\pi}(z)| \leq |z|^{-(1+\epsilon)/2} \quad |z| \to \infty ,$$

and hence π is absolutely continuous by a Fourier lemma.

Conversely, let π be the law of X_1. We do not know a priori if $X_{1/2}$ is absolutely continuous under (5.9).

Indeed, we only see by (5.7)

$$|\hat{\pi}_{1/2}(z)| \leq |z|^{-(1+\epsilon)/2} \quad |z| \to \infty ,$$

whereas $\hat{\pi}_{1/2}$ may not be integrable. As we shall see in Theorem 27.23 in [196], it happens that X_1 is absolutely continuous and $X_{1/2}$ is not.

The condition (5.9) is very weak; apart from the condition on the logarithmic rate of convergence of $\bar{\mu}(-\epsilon, \epsilon)$ to 0 as $\epsilon \to 0$, it is quite near to the infiniteness of μ. Indeed, if

$$\liminf_{\epsilon \to 0} \epsilon^{-2} \bar{\mu}(-\epsilon, \epsilon) > 0 \,, \tag{5.10}$$

then μ is an infinite measure. However, the converse of this assertion is false, and there exists an infinite measure which does not satisfy (5.10) nor (5.8).

Proposition 2.11. *For each k, there exists an infinite measure μ such that*

$$\liminf_{\epsilon \to 0} \epsilon^{-(2+k)} \bar{\mu}(-\epsilon, \epsilon) = 0 \,.$$

By considering a sequence which grows sufficiently rapid, we can control the rate of convergence as we wish. The proposition seems to contradict the assertion of Kallenberg (cf. [111] Section 5), however, it is repeatedly stated in the introduction in [169]. This is since the infinite measure satisfies

$$\lim_{\epsilon \to 0} \epsilon^{-2} |\log \epsilon|^r \bar{\mu}(-\epsilon, \epsilon) = +\infty \,, \quad r > 1 \,.$$

Given Theorem 2.6, we can consider the following condition which is more strong with respect to the infiniteness of μ:

$$\int_{|z| \le 1} |z|^\alpha \mu(dz) = +\infty \,, \quad \alpha \in (0, 2) \,.$$

It seems this condition may lead to the absolute continuity of π in the case that (5.10) fails. However, it is not true as we see in the following

Proposition 2.12 (Orey [178]). *For any $\alpha \in (0, 2)$, there exists a Lévy measure μ such that*

$$\int_{|z| \le 1} |z|^\alpha \mu(dz) = +\infty \quad \text{and} \quad \int_{|z| \le 1} |z|^\beta \mu(dz) < \infty \,, \quad \beta > \alpha \tag{5.11}$$

for which the ID law π given by (5.5) *under the condition $b = 0$ is not absolutely continuous.*

It may seem that in Orey's Theorem a special relation between atoms in the Lévy measure and their weights might be preventing π from being absolutely continuous. In reality, there is nothing to do between them. We can show the following counter example.

Proposition 2.13 (Watanabe [218]). *Let c be an integer greater than 2, and let the Lévy measure be*

$$\mu = \sum_{n=0}^{+\infty} a_n \delta_{\{c^{-n}\}} \,.$$

Here, we assume $\{a_n; n \ge 0\}$ to be positive and bounded. Then, the ID law π given by (5.5) *and the condition $b = 0$ is not absolutely continuous.*

In Propositions 2.12, 2.13, the support of μ constitutes a geometric sequence, and the integer character c plays an important role. Proposition 2.13 holds valid even if c is a *Pisot–Vijayaraghavan* number [218]. This is the number $u \in (1, \infty)$ such that for some integer-coefficient polynomial (with the coefficient in the maximal degree 1), it satisfies $F(u) = 0$, where the absolute values of all the roots of $F = 0$ except u are strictly smaller than 1. For example, $u = (\sqrt{5} + 1)/2$ with $F(X) = X^2 - X - 1$. By the theorem of Gauss, Pisot–Vijayaraghavan numbers are necessarily irrational. In the case that the support of the Lévy measure consists of a geometric sequence, several families of algebraic numbers play an important role so that π becomes absolutely continuous. For a recent development of this topic, see [219].

Unfortunately, no necessary-and-sufficient condition is known for μ with respect to this family of numbers so that the corresponding ID law becomes absolutely continuous.

The following criterion is known:

Proposition 2.14 (Hartman–Wintner). *Let μ be a discrete Lévy measure and be infinite, and let π be an ID law given by* (5.5). *Then, π is pure and is either singular continuous or absolutely continuous.*

For the proof of this theorem, see Theorem 27.16 in [196]. (It is a special case focussed on the discrete measure, and the proof is easy.)

Some results in the multidimensional case

To make representations simpler, we study ID laws having no Gaussian part. In $(\mathbf{R}^d, \|.\|)$, let the Fourier transform $\hat{\pi}$ of an ID law π be

$$\hat{\pi}(\lambda) = \exp\left[i\langle a, \lambda\rangle + \int_{\mathbf{R}^d\setminus\{0\}} (e^{i\langle\lambda,x\rangle} - 1 - i\langle\lambda, x\rangle 1_{\{|x|\le 1\}})\mu(dx)\right], \quad \lambda \in \mathbf{R}^d . \tag{5.12}$$

Here, $a \in \mathbf{R}^d$, and π is a positive measure such that $\int(|x|^2 \wedge 1)\mu(dx) < \infty$, and $\langle , \rangle$ is an inner product corresponding to $|.|$. Many results in the previous section also hold true in the multidimensional case since they do not employ the speciality that the dimension is one. For example Theorems 2.6 and 2.7, where $\bar{\mu}(dx) = (|x|^2 \wedge 1)\mu(dx)$, hold true in any dimensions. Conversely, under the assumption that

$$\lim_{\epsilon\to 0} \epsilon^{-2}|\log\epsilon|^{-1}\bar{\mu}(B_\epsilon) = +\infty , \tag{5.13}$$

where B_ϵ denotes a ball with the centre origin and the radius ϵ, Theorem 2.8 is no longer true. This is because

$$|\hat{\pi}(z)| \le \exp - \left[\frac{1}{3} \int_{B_{1/z}} |\langle u, z\rangle|^2 \mu(du)\right] ,$$

and hence we can not appropriately estimate $\langle u, z\rangle$ from below. However, if, instead of (5.13), we have the uniform angle condition

$$\lim_{\epsilon\to 0} \epsilon^{-2} \int_{|\langle v,x\rangle|\le\epsilon} |\langle v, x\rangle|^2 \mu(dx) = +\infty , \quad \text{uniformly in } v \in \mathcal{S}^{d-1} , \tag{5.14}$$

then again by Theorem 2.8, we can prove that π defined by (5.12) has a C^∞-density whose derivatives of all orders vanish at infinity.

A weak point of Theorem 2.8 is that it requires a certain isotrophic property of the measure μ. Indeed, if there exists a hyperplane H such that μ is finite on $\mathbf{R}^d \setminus H$, then π charges a hyperplane with positive probability and hence it is not absolutely continuous.

We say a measure ρ on S^{d-1} is radially absolutely continuous if the support of ρ is not contained in any hyperplane, and if for some positive function g on $S^{d-1} \times \mathbf{R}^+$, it holds that

$$\mu(B) = \int_{S^{d-1}} \rho(d\xi) \int_0^\infty g(\xi, r) 1_B(r\xi) dr \tag{5.15}$$

for all Borel sets $B \in \mathcal{B}(\mathbf{R}^d)$. This decomposition applies for self-decomposable Lévy measures. For details on this important ID law, see [196] Chapter 15. We say μ satisfies the divergence condition if it holds that

$$\int_0^1 g(\xi, r) dr = +\infty \quad \rho - a.e. \tag{5.16}$$

Note that (5.16) is weaker than (5.14) since it just asserts the infiniteness of μ with respect to each direction.

Theorem 2.9 (Sato [195]). *Let π be an ID law for which the characteristic function given by* (5.12) *satisfies* (5.15), (5.16). *Then, π is absolutely continuous.*

See [196] pp. 178–181 for the proof of this theorem. The proof uses induction with respect to the dimension and Theorem 2.7. The condition (5.15) is a very restrictive assumption since it requires the absolute continuity of the radial part of μ through g. We may consider whether this condition can be weakened.

Another weak point of Theorem 2.9 is that it excludes the case that the support consists of curves which are not lines. In order to correspond to this case, we introduce the notion of generalised polar coordinates.

Definition 2.3 (Yamazato [220]). We say μ is absolutely continuous with respect to curves if there exist a finite measure ρ supported on $S \subset \mathcal{S}^{d-1}$ and a positive function g such that for a Borel set $B \in \mathcal{B}(\mathbf{R}^d)$, it holds that

$$\mu(B) = \int_{S^{d-1}} \rho(d\xi) \int_0^\infty g(\xi, r) 1_B(\varphi(\xi, r)) dr .$$

Here, $\varphi : S \times (0, +\infty) \to \mathbf{R}^d \setminus \{0\}$ is a generalised polar coordinate in the sense that φ is a measurable bijection, and $\varphi(\xi, 0+) \equiv 0$, $\varphi(\xi, 1) \equiv \xi$, $\partial_r \varphi(., \xi)$ exists and does not vanish for all ξ. For each vector space H, the set $\{r > 0; \partial_r \varphi \in H\}$ is either an empty set or $(0, \infty)$. For $F_k(\xi_1, \ldots, \xi_k) = \{(r_1, \ldots, r_k); (\varphi(\xi_i, r_i))$ being linearly independent and $(\partial_r \varphi(\xi_i, r_i))$ being linearly dependent $\}$, it holds that $\mathrm{Leb}(F_k(\xi_1, \ldots, \xi_k)) = 0$ for $k \geq 2$ and $\xi_1, \ldots, \xi_k \in S$.

The next result is a partial generalisation of Theorem 2.9.

Theorem 2.10 ([220]). *Let π be an ID law corresponding to a Lévy measure μ which is absolutely continuous with respect to curves. π is absolutely continuous in the following two cases:*
(1) *μ is not supported on any hyperplane, and $\mu(H) = 0$ or $+\infty$ for each vector subspace H.*
(2) *For each hyperplane P, $\mu(\mathbf{R}^d \setminus P)$ is infinite, and $\mu(H) = 0$ for each $(d-2)$-dimensional vector space H.*

Recently, a refinement has been obtained by Yamazato himself. Let $\tilde{\mu}(dx) = \frac{|x|^2}{1+|x|^2} \mu(dx)$. The symbol Λ_H denotes the Lebesgue measure induced on the linear space H by restriction.

Proposition 2.15 ([221] Theorem 1.1). *Assume there exists an integer $r \geq 1$ such that the following two conditions hold:*
(a) *μ is represented as a sum $\mu = \sum_{k=1}^{\infty} \mu_k$, where μ_k is concentrated on a linear subspace H_k and has no mass on any proper linear subspace of H_k and satisfies $(\tilde{\mu}_k)^{*k} << \Lambda_{H_k}$.*
(b) *It holds that*

$$\sum_{k=1}^{\infty} (\mu_k(K^c))^r = +\infty$$

for every $(d-1)$-dimensional linear subspace K of $\mathbf{R}^d$. Then, π is absolutely continuous.

A related work [120] has been done by Kulik.

2.5.3 Chain movement approximation

Below, we provide concrete numerical results concerning Theorem 2.3 (b) for calculating the functional $\Gamma(t)$. In this case, the point y is in $\bar{\mathcal{S}} \setminus \mathcal{S}$ for x in each case. We remark $N(t) \to +\infty$ as $t \to 0$. Hence, we can use, as t goes to 0, a_n's of smaller magnitude in the approximation to y. The corresponding values *Cmin* or *Min* will represent the estimated values $\Gamma(t)$ for each t in the tables. (*Cmin* stands for "compensated minimum", whereas *Min* is used to represent the minimum when the effect of the compensator term $O(\log(\frac{1}{t}) \log\log(\frac{1}{t}))$ is too faint.)

Table 2.1. Finding the minimum in case (a).

β	β'	t	N	*Cmin*	$w_0, \ldots, w_N$
1.5	2.25	0.02	3	2.216128	1 0 3 0
1.5	2.25	0.002	4	5.348213	1 0 3 0 2
1.5	2.25	0.0002	6	17.541525	1 0 3 0 4 0 7
1.0	1.5	0.02	4	5.029539	1 0 3 0 1
1.0	1.5	0.002	6	23.253883	1 0 3 0 5 0 3
1.0	1.5	0.0002	8	31.925063	1 0 4 1 0 0 2 0 5
0.5	0.75	0.02	8	15.405371	1 0 4 1 0 0 1 0 1
0.5	0.75	0.005	10	35.587829	1 0 4 1 0 0 4 0 3 0 1
0.5	0.75	0.003	11	39.123768	1 0 4 1 0 0 5 0 0 0 0 0

Table 2.2. Finding the minimum in case (b).

β	β'	t	N	*Min*	$w_0, \ldots, w_N$
1.5	2.25	0.005	5	23.689470	0 0 2 0 4 0
1.5	2.25	0.002	5	34.708675	0 0 2 0 5 0
1.5	2.25	0.001	6	43.785296	0 1 5 0 0 0 1
1.0	1.5	0.005	7	31.582738	0 1 5 0 1 0 0 0
1.0	1.5	0.002	8	38.913063	0 1 5 0 1 0 0 0 0
1.0	1.5	0.001	9	46.760826	0 1 5 0 1 0 0 0 0 10

This may also be viewed as examples of approximation of the trajectory of $x_t(x)$ as $t \to 0$ by the chain (A_n). We can imagine the asymptotic behaviour of the density function associated with the processes below as $t \to 0$, which is a rapid decrease. In the tables, the last column $w_0, \ldots, w_N$ denotes the number of a_n's for each $n = 0, \ldots, N$ used by the chain to approximate the point y up to the time t.

(a) Let $d = m = 1$, $\gamma(x, \zeta) = \zeta$ (this means $x_t(x)$ is a Lévy process). Let $x = 1$, $y = \sqrt{2}$. We assume $\mu = \sum_{n=0}^{\infty} k_n \delta_{a_n}$ is given by $a_n = (-3)^{-n}$, $k_n = 3^{n\beta}$, $\beta \in (0, 2)$, and assume $\beta' = 1.5\beta$. We calculate Cmin $\equiv \min \sum_{n=0}^{N} (w_n \log(\frac{1}{tk_n}) + \log(w_n!)) - \log(\frac{1}{t}) \log\log(\frac{1}{t})$ and the $(w_n)_{n=0}^{N}$ which attain it (Table 2.1).

(b) Let $d = m = 2$, $\gamma(x, \zeta) = A\zeta$ where $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Let $x = (1, 1)$, $y = (\sqrt{2}, \sqrt{3})$. We assume $\mu = \sum_{n=0}^{\infty} k_n \delta_{a_n}$ is given by $a_n = ((-2)^{-n}, (-3)^{-n})$, $k_n = n^2$, and assume $\beta' = 1.5\beta$. We find Min $\equiv \min \sum_{n=0}^{N} \sum_{i=1}^{d} (w_n \log(\frac{1}{tk_n}) + \log(w_n!))$, and the $(w_n)_{n=0}^{N}$ which attains the minimum (Table 2.2).

(c) Let $d = m = 2$, $\gamma(x, \zeta) = A\zeta$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $x = (1, 1)$, $y = (\sqrt{2}, \sqrt{3})$. We assume $\mu = \sum_{n=0}^{\infty} k_n \delta_{a_n}$ is given by $a_n = ((-2)^{-n}, (-3)^{-n})$, $k_n = n^2$, and assume $\beta' = 1.5\beta$. We find Min $\equiv \min \sum_{n=0}^{N} \sum_{i=1}^{d} (w_n \log(\frac{1}{tk_n}) + \log(w_n!))$, and the $(w_n)_{n=0}^{N}$ which attains the minimum (Table 2.3).

Table 2.3. Finding the minimum in case (c).

β	β'	t	N	*Min*	$w_0, \ldots, w_N$
1.5	2.25	0.005	5	16.916603	1 1 0 0 0 1
1.5	2.25	0.002	5	35.500575	1 1 1 2 0 2
1.5	2.25	0.0001	8	87.584560	1 1 2 4 0 2 0 1 0
1.0	1.5	0.005	7	37.696668	1 1 1 1 0 6 0 0
1.0	1.5	0.002	8	51.635773	1 1 2 4 0 2 0 1 0
1.0	1.5	0.001	9	61.627516	1 1 2 4 0 3 0 0 0 0

2.5.4 Support theorem for canonical processes

In this section, we provide a support theorem for a canonical process Y_t given in Section 2.5.1. We construct “skeletons”indexskeletons φ_t^η for Y_t by using Marcus’ integral. The result is closely related to the trajectories of Markov chains treated in Section 2.2. We assume the condition *(A.0)* for μ and the restricted Hörmander condition *(RH)* for $X_j(x)'s$. In comparison with the proof of short time asymptotic bounds of the transition density stated in the previous section, we can observe the fine behaviours of the jump process.

First, we construct skeletons as follows. Let $u(t) = (u_1(t), \ldots, u_m(t))$, $0 \le t \le T$ be an $\mathbf{R}^m$-valued, piecewise smooth, càdlàg functions having finite jumps. It is decomposed as $u(t) = u^c(t) + u^d(t)$, where $u^c(t)$ is a continuous function and $u^d(t)$ is a purely discontinuous (i.e. piecewise constant except for isolated finite jumps) function.

For $\eta > 0$, we put

$$\mathcal{U}_\eta = \Big\{ u \in \mathrm{D}; u(t) = u^{c,\eta}(t) + u^{d,\eta}(t)\ , \quad u^{d,\eta}(t) = \sum_{s \le t} \Delta u(s).\ 1_{\{\eta<|z|\le 1\}}(\Delta u(s)), u^{c,\eta}(t) = -l_\eta.t \Big\}\ . \tag{5.17}$$

Here, we put $l_\eta = \int_{\{\eta<|z|\le 1\}} z\mu(dz)$, and $\mathrm{D} = D(\mathbf{T}, \mathbf{R}^m)$. Put $\mathcal{U} \equiv \cup_{\eta>0}\mathcal{U}_\eta$. The set $\mathcal{U}$ is called the space of skeletons.

To express $\mathcal{U}$ in the configuration space, we put further

$$\mathcal{V}_\eta = \Big\{ \mathbf{v} = \{v_n,\ n \ge 1\} = \{(t_n, z_n),\ n \ge 1\}\ ; \quad \{t_n\} \text{ is a strictly increasing sequence of } \mathbf{R}^+ \text{ with limit } +\infty, \text{ and } \{z_n\} \text{ a sequence in the support of } \mu \text{ such that } \eta < |z_n| \le 1 \Big\}\ .$$

Note that there is a one-to-one correspondence between $\mathcal{U}_\eta$ and $\mathcal{V}_\eta$ by putting

$$\mathcal{V}_\eta \ni (t_n, z_n) \mapsto u(t) = -l_\eta t + \sum_{t_n \le t} z_n \in \mathcal{U}_\eta\ .$$

Given $u = u^\eta \in \mathcal{U}_\eta$, we put a trajectory $\varphi_t^\eta \in \mathrm{D}' = D(\mathbf{T}, \mathbf{R}^d)$ by

$$d\varphi_t^\eta = \sum_{j=1}^m X_j\left(\varphi_t^\eta\right) \circ du_j(t), \quad \varphi_0^\eta = x\ . \tag{5.18}$$

The solution starting from x at $t = s$ is a piecewise càdlàg smooth function φ_t^η, $t \geq 0$, satisfying for any f which is bounded smooth the equation

$$f\left(\varphi_t^\eta\right) = f(x) + \sum_{j=0}^m \int_s^t X_j f(\varphi_r^\eta) du_j(r) + \sum_{s \leq r \leq t} \left\{ \operatorname{Exp}\left(\sum_{j=1}^m \Delta u_j^d(r) X_j \right) f(\varphi_{r-}^\eta) - f(\varphi_{r-}^\eta) - \sum_{j=1}^m \Delta u_j^d(r) X_j f(\varphi_{r-}^\eta) \right\} . \quad (5.19)$$

holds. The function φ_t^η can be viewed as the image of a skeleton u. We put

$$\mathcal{S}_\eta^x = \{\varphi_t^\eta; \varphi_t^\eta \text{ is as above}\}$$

and $\mathcal{S}^x = \cup_{\eta \in (0,1)} \mathcal{S}_\eta^x$.

For $u, v \in \mathcal{U}$, a Skorohod metric $d_{\mathbf{T}}$ on D is defined by

$$d_{\mathbf{T}}(u, v) = \inf_\lambda \sup_{t \in \mathbf{T}} \{|u(\lambda(t)) - v(t)| + |t - \lambda(t)|\} , \quad (5.20)$$

where the infimum is taken for all homeomorphisms λ of the interval $\mathbf{T}$. We identify $d_{\mathbf{T}}$ with $d_{\mathbf{T}}^\circ$ in Section 1.1.3 so that we regard the space $(\mathrm{D}, d_{\mathbf{T}})$ to be complete. The Skorohod metric on D' is similarly defined and is also denoted by $d_{\mathbf{T}}$. The support of the Lévy process Z is defined by

$$\operatorname{supp} Z = \{u \in \mathrm{D}; \quad \text{for all } \delta > 0 \ [P(\{\omega; d_{\mathbf{T}}(Z, u) < \delta\}) > 0]\} . \quad (5.21)$$

The support of the canonical process Y is similarly defined by

$$\operatorname{supp} Y_\cdot(x) = \{\varphi \in \mathrm{D}'; \quad \text{for all } \epsilon > 0 \ [P(\{\omega; d_{\mathbf{T}}(\Phi(x), \varphi) < \epsilon\}) > 0]\} . \quad (5.22)$$

By $\bar{\mathcal{S}}^x$, we denote the closure of $\mathcal{S}^x$ in $(\mathrm{D}', d_{\mathbf{T}})$.

We define the approximating processes $Z^\eta(t)$, Y_t^η for each $\eta > 0$ as follows. Let

$$Z^\eta(t) = \int_0^t \int_{\eta < |z| \leq 1} z \tilde{N}(dsdz) ,$$

and Y_t^η is given by

$$dY_t^\eta = \sum_{j=1}^m X_j(Y_t^\eta) \circ dZ^{\eta,j}(t), \quad Y_0^\eta = x . \quad (5.23)$$

Furthermore, the complementary process $\tilde{Z}^\eta$

$$\tilde{Z}^\eta(t) = \int_0^t \int_{0 < |z| \leq \eta} z \tilde{N}(dsdz) \quad (5.24)$$

and $\tilde{Y}_t^\eta$ is given by

$$d\tilde{Y}_t^\eta = \sum_{j=1}^m X_j(Y_t) \circ d\tilde{Z}^{\eta,j}(t),\ \tilde{Y}_0^\eta = x\ . \tag{5.25}$$

Note that $dZ(s) = dZ^\eta(s) + d\tilde{Z}^\eta(s)$, and $d[Z]_s = d[Z^\eta]_s + d[\tilde{Z}^\eta]_s$. Hence, we have a decomposition

$$\begin{aligned} Y_t - Y_t^\eta &= \sum_j \int_0^t X_j(Y_{s-})dZ_j(s) - \sum_{j=1}^m \int_0^t X_j(Y_{s-}^\eta)dZ_j^\eta(s) \\ &\quad + \int_0^t h(s, Y_{s-})d[Z]_s - \int_0^t h^\eta(s, Y_{s-}^\eta)d[Z^\eta]_s \\ &= \sum_{j=1}^m \int_0^t \{X_j(Y_{s-}) - X_j(Y_{s-}^\eta)\}dZ_j^\eta(s) \\ &\quad + \int_0^t \{h(s, Y_{s-}) - h^\eta(s, Y_{s-}^\eta)\}d[Z^\eta]_s + \tilde{Y}_t^\eta\ . \end{aligned} \tag{5.26}$$

We now state our basic assumptions concerning "small deviations".

Setting, for every $0 < \rho < \eta$,

$$u_\rho^\eta \ = \int_{\rho\le|z|\le\eta} z\,\mu(dz)\ , \tag{5.27}$$

we say that Z is *quasisymmetric* if for every $\eta > 0$, there exists a sequence $\{\eta_k\}$ decreasing to 0 such that

$$\left|u_{\eta_k}^\eta\right| \longrightarrow 0 \tag{5.28}$$

as $k \to +\infty$. This means that for every η, the compensation involved in the martingale part of $\tilde{Y}^\eta$ is somehow negligible, and of course this is true when Z is really symmetric.

(H.1) For every $\eta > 0$ such that (5.28) does not hold, there exists $\gamma = \gamma^\eta > 1$ and a sequence $\{\eta_k\}$ decreasing to 0 such that

$$\alpha_{\eta_k}^\eta \ = \ o\,(1/|u_{\eta_k}^\eta|)\ ,$$

where α_ρ^η is the angle between the direction u_ρ^η and supp μ on $\{|z| = \gamma\eta\}$, that is, when it can be defined.

Notice first that it always holds in dimension 1 (with $\alpha_{\eta_k}^\eta = 0$). Besides, it is verified in higher dimensions whenever supp μ contains a sequence of spheres whose radius tends to 0 (in particular, a whole neighbourhood of 0), or when the intersection of supp μ with the unit ball coincides with that of a convex cone.

For technical reasons, we also suppose that μ satisfies the following non-degeneracy and scaling condition, which implies the condition *(A.0)* in Section 1.3:

(H.2) There exists $\beta \in [1, 2)$ and positive constants C_1, C_2 such that for any $\rho \le 1$,

$$C_1 \rho^{2-\beta} I \le \int_{|z|\le\rho} zz^* \mu(dz) \le C_2 \rho^{2-\beta} I .$$

Besides, if $\beta = 1$, then

$$\limsup_{\eta\to 0} \left| \int_{\eta\le|z|\le 1} z\,\mu(dz) \right| < +\infty .$$

The inequalities above stand for symmetric positive-definite matrices. This means *(H.2)* demands both the non-degeneracy of the distribution of (point) masses around the origin, and, the way of concentration of masses along the radius. We notice that if $\langle v, .\rangle$ stands for the usual scalar product with v, they are equivalent to

$$\int_{|z|\le\rho} |\langle v, z\rangle|^2 \mu(dz) \asymp \rho^{2-\beta}$$

uniformly for unit vectors $v \in S^{m-1}$. (Here, $\asymp$ means the quotient of the two sides is bounded away from zero and above as $\rho \to 0$.) In particular, $\beta = \inf\{\alpha; \int_{|z|\le 1} |z|^\alpha \mu(dz) < +\infty\}$ (the Blumenthal–Getoor index of Z), and the infimum is not reached. Notice finally that the measure μ may be very singular and have a countable support. The condition *(H.2)* implies the non-degeneracy condition for the matrix B in section 3.5 with $\alpha = 2 - \beta$.

We now state our main theorem:

Theorem 2.11. *Under (H.1), (H.2), we have* supp $Y_.(x) = \bar{\mathbb{S}}^x$. *Here,* $^-$ *means the closure in* $d_{\mathbf{T}}$*-topology.*

The condition *(H.1)* is crucial to the small deviations property: for any $\epsilon > 0$ and any $t > 0$,

$$P\left(\sup_{s\le t} |Y_s| < \epsilon\right) > 0 ,$$

and is related to the distribution of supp μ.

This is an adapted assertion to canonical processes of the support theorem due to Th. Simon [200] for Itô processes. For the proof of this theorem, see [90].

The support theorem helps us a lot in showing the strict positivity of the density function $y \mapsto p_t(x, y)$ ([94] Section 4).

3 Analysis of Wiener–Poisson functionals

> You must take your chance, And either not attempt to choose at all Or swear before you choose, if you choose wrong Never to speak to lady afterward In way of marriage: therefore be advised.
>
> W. Shakespeare, The Merchant of Venice, Act 2, Scene 1

In this chapter, we inquire into the stochastic calculus on the Wiener–Poisson space. The Wiener–Poisson space is the product space of the Wiener and the Poisson spaces. By the Lévy–Itô decomposition theorem (cf. Proposition 1.1), a general Lévy process lives in this space.

In Section 3.1, we study the Wiener space. Stochastic analysis on the Wiener space has been developed by Malliavin, Stroock, Watanabe, Shigekawa, and others, prior to that for the Poisson space. In Section 3.2, we study the Poisson space. Compared to the Wiener space, there exist several tools to analyse the Poisson space, and we shall introduce them accordingly.

Section 3.3 is devoted to the analysis on the Wiener–Poisson space. We construct Sobolev spaces on the Wiener–Poisson space by using the norms induced by several derivatives on it. It turns out in Sections 3.5, 3.7 below that these Sobolev spaces play important roles in showing the asymptotic expansion. In Section 3.4, we consider a comparison with the Malliavin operator (a differential operator) on the Poisson space.

In Sections 3.5, 3.7, we state a theory on the composition of Wiener–Poisson variables with tempered distributions. This composition provides us with a theoretical basis regarding the content of Section 4.1. In Section 3.6, we state results on the density functions associated with the composition theory for Itô processes and on (Hörmander's) hypoellipticity result in the jump-diffusion case.

In place of the integration-by-parts formula in Chapter 2, duality formulas (1.35) in Chapter 2 and (1.11) in Chapter 3 will play crucial roles in this chapter. Throughout this chapter, we denote the Wiener space by $(\Omega_1, \mathcal{F}_1, P_1)$ and the Poisson space by $(\Omega_2, \mathcal{F}_2, P_2)$ in order to distinguish the spaces.

3.1 Calculus of functionals on the Wiener space

In this section, we briefly reflect on the basic facts of the Wiener space. We state the chaos expansion using the Hermite polynomials first. We state this section quite briefly and confine ourselves to those materials which are necessary to state a general theory of the stochastic calculus. Proofs for several propositions are omitted.

Several types of Sobolev norms and Sobolev spaces can be constructed on the Wiener space. However, we will not go into detail on these Sobolev spaces in this section, and postpone it to Section 3.3.

Below, we denote a function of 1-variable by $f(t)$, and a function of n-variables by $f_n(t_1, \ldots, t_n)$. If we write $f_i(t)$, it means it is a function of the variable t with an index i.

We denote by $W(t)$ a 1-dimensional Wiener process (standard Brownian motion) on Ω_1 with filtration $(\mathcal{F}_{1,t})_{t\in\mathbf{T}}$. For $f_n(t_1, \ldots, t_n) \in L^2(\mathbf{T}^n)$, we denote by $I_n(f_n), J_n(f_n)$ the following iterated integrals:

$$I_n(f_n) = \int_{\mathbf{T}^n} f_n(t_1, \ldots, t_n) dW(t_1) \cdots dW(t_n) ,$$

$$J_n(f_n) = \int_{\mathbf{T}} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dW(t_1) \cdots dW(t_n) . \tag{1.1}$$

They are called *Wiener chaos*. The above $J_n(f_n)$ is also written by

$$\int_{0<t_1<\cdots<t_{n-1}<T} f_n(t_1, \ldots, t_n) dW(t_1) \otimes \cdots \otimes dW(t_n) . \tag{1.2}$$

Here, $f_n(t_1, \ldots, t_n)$ can be chosen as a symmetric function on $\mathbf{T}^n$.

We denote by $\tilde{L}^2(\mathbf{T}^n)$ the subspace of $L^2(\mathbf{T}^n)$ consisting of symmetric functions on $\mathbf{T}^n$. If $f_n \in \tilde{L}^2(\mathbf{T}^n)$, then $I_n(f_n) = n! J_n(f_n)$. Furthermore, if $f_n \in L^2(\mathbf{T}^n)$, then $\tilde{f}_n \in \tilde{L}^2(\mathbf{T}^n)$. Here,

$$\tilde{f}_n(t_1, \ldots, t_n) = \frac{1}{n!} \sum_\sigma f_n(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) ,$$

where σ denotes a permutation of $\{1, \ldots, n\}$.

The following chaos expansion is known.

Proposition 3.1. *Any $F \in L^2(\Omega_1) = L^2(\Omega_1, \mathcal{F}_1, P_1)$ can be expressed by*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{1.3}$$

for some (f_n). Here, $f_0 = E[F]$ and $I_0 = id$, and (f_n) are uniquely determined in $L^2(\mathbf{T}^n)$ up to the symmetry among $\{t_1, \ldots, t_n\}$. Especially, (f_n) can be chosen symmetric.

We omit the proof. See [170] Theorems 1.1.1, 1.1.2. The orthogonal projection $\Pi_n(F)$ of F on the n-th Wiener chaos is given by $\Pi_n(F) = I_n(f_n)$. An important property in (1.3) is that $I_n(f_n)$ and $I_m(f_m)$ are orthogonal if $n \neq m$ under the Wiener measure ([170] p. 9). Hence, it is effective in recovering the input from the output developed in the way above. This is called the *Wiener kernel method*.

In order to compute $I_n(f_n)$, we need an orthonormal basis on the Wiener space. The one often used is the Hermite polynomials $(h_n(x))$. The first six Hermite polynomials are

$$h_0(x) = 1 , \qquad h_1(x) = x , \qquad h_2(x) = x^2 - 1 ,$$
$$h_3(x) = x^3 - 3x , \qquad h_4(x) = x^4 - 6x^2 + 3 , \qquad h_5(x) = x^5 - 10x^3 + 15x .$$

The formal definition is due to the generating function $\exp(\theta x - \theta^2/2)$, that is,

$$\sum_{n=0}^{\infty} h_n(x)\frac{\theta^n}{n!} = \exp(\theta x - \theta^2/2)\ .$$

It is interesting to observe that the even order h_n are even functions, and the odd order h_n are odd functions.

By using Hermite polynomials, we can write for $m \in \mathbf{N}$ and for nonnegative integers $\alpha_1, \ldots, \alpha_m$ such that $\alpha_1 + \cdots + \alpha_m = n$,

$$I_n(f_1^{\otimes\alpha_1}\hat{\otimes}\cdots\hat{\otimes}f_m^{\otimes\alpha_m}) = \Pi_{i=1}^m h_{\alpha_i}(I_1(f_i))\ ,$$

where $f_i \in L^2(\mathbf{T})$. Here, $f\hat{\otimes}g$ denotes the symmetrised tensor product of f and g. See [173] Proposition 1.8 for the proof.

In particular, we have

$$\begin{aligned} n! &\int_{0<t_1<\cdots<t_{n-1}<T} f(t_1)\cdots f(t_n)dW(t_1)\ldots dW(t_n) \\ &= \|f\|^n h_n\left(\frac{I_1(f)}{\|f\|}\right), \quad n = 1, 2, \ldots . \end{aligned}$$

Here, $\|f\| = \|f\|_{L^2}$ denotes the L^2-norm on $\mathbf{T}$.

3.1.1 Definition of the Malliavin–Shigekawa derivative D_t

In this subsection, we introduce the Malliavin–Shigekawa derivative D_t on the Wiener space in two ways.

Definition of the derivative D_t (1)

Let $I_1(f) = \int_{\mathbf{T}} f(t)dW(t)$. In case F is given by $F = g(I_1(f_1), \ldots, I_1(f_n))$, where g is a polynomial of n-variables, we define the derivative D_tF by

$$D_tF = \sum_{i=1}^{n} \partial_i g(I_1(f_1), \ldots, I_1(f_n))f_i(t)\ . \tag{1.4}$$

It is also called the gradient operator.

To see the legitimacy of this definition, we observe

$$\begin{aligned} \frac{d}{dt}F(\omega_1 + th)|_{t=0} &= \langle DF, h\rangle \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{g(I_1(f_1) + \epsilon\langle f_1, h\rangle, \ldots, I_1(f_n) + \epsilon\langle f_n, h\rangle) - g(I_1(f_1), \ldots, I_1(f_n))\right\}\ . \end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the paring in $L^2(\mathbf{T})$.

We remark that D_t is a differential operator. That is, it holds

$$D_t(FG) = FD_tG + GD_tF\ .$$

Definition of the derivative D_t (2)

For $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega_1)$, we put

$$D_t F = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad F \in \mathbf{D}_{1,2}^{(1)} . \tag{1.5}$$

Here, we put

$$\mathbf{D}_{1,2}^{(1)} = \left\{ F = \sum_n I_n(f_n); \sum_{n=1}^{\infty} n^2 (n-1)! \|f_n\|^2_{L^2(\mathbf{T}^n)} < +\infty \right\} .$$

That is, $D_t F$ is obtained simply by removing one of the stochastic integrals, inserting the free variable t, and by multiplying n. Here, $D_{\cdot}$ is a mapping from $\mathbf{D}_{1,2}^{(1)}$ to $L^2(\mathbf{T} \times \Omega_1)$, and it is a closable, unbounded operator.

In view of the expression of F in (1.5) as a superposition of higher order movements of $W(t)$ ($n = 1, 2, \ldots$) along the time, it well represents the nature of the "derivative" of F at a fixed time t.

The two definitions (1), (2) of D_t coincide with each other on the Sobolev space $\mathbf{D}_{1,2}^{(1)}$. We sketch the key idea of the proof here.

(i) Case that $f_n(t_1, \ldots, t_n) = f^{\otimes n}(t_1, \ldots, t_n) = f(t_1) \cdots f(t_n)$ for some $f \in L^2(\mathbf{T})$. In this case, we have

$$\begin{aligned} I_n(f^{\otimes n}) &= n! \int_{0<t_1<\cdots<t_{n-1}<T} f(t_1) \cdots f(t_n) dW(t_1) \cdots dW(t_n) \\ &= \|f\|^n h_n \left(\frac{1}{\|f\|} \int_0^T f(t) dW(t) \right) , \quad n = 1, 2, \ldots . \end{aligned} \tag{1.6}$$

Here, $h_n(.)$ is a Hermite polynomial, and hence

$$h_n'(x) = n h_{n-1}(x) . \tag{1.7}$$

Hence,

$$\begin{aligned} D_t I_n(f_n) &= n I_{n-1}(f_n(\cdot, t)) = n I_{n-1}(f^{\otimes n-1}) f(t) \\ &= n \|f\|^{n-1} h_{n-1} \left(\frac{1}{\|f\|} \int_{\mathbf{T}} f(t) dW(t) \right) f(t) \end{aligned} \tag{1.8}$$

for $f^{\otimes n}(t_1, \ldots, t_n) = f(t_1) \cdots f(t_n)$. These imply

$$\begin{aligned} D_t h_n \left(\frac{1}{\|f\|} \int_{\mathbf{T}} f(t) dW(t) \right) &= h_n' \left(\frac{1}{\|f\|} \int_{\mathbf{T}} f(t) dW(t) \right) \cdot \frac{f(t)}{\|f\|} \\ &= n h_{n-1} \left(\frac{1}{\|f\|} \int_{\mathbf{T}} f(t) dW(t) \right) \cdot \frac{f(t)}{\|f\|} . \end{aligned}$$

This leads to

$$D_t\left(\left(\int_{\mathbf{T}} f(t)dW(t)\right)^n\right) = n\left(\int_{\mathbf{T}} f(t)dW(t)\right)^{n-1} \cdot f(t)\,, \quad n = 1, 2, \ldots \tag{1.9}$$

by induction. We observe here the original definition of D_t for polynomials.
(ii) For the general case, we put

$$f_n = \xi_1^{\otimes\alpha_1}\hat{\otimes}\cdots\hat{\otimes}\xi_m^{\otimes\alpha_m}\,,$$

where (ξ_j) are elements from a complete orthonormal system (c.o.n.s.) of $L^2(\mathbf{T})$ and $\alpha_1 + \cdots + \alpha_m = n$. By [102] Theorem 4.1, we have

$$I_n(f_n) = h_{\alpha_1}(I_1(\xi_1))\cdots h_{\alpha_m}(I_1(\xi_m))\,.$$

Since any $f_n \in \tilde{L}^2(\mathbf{T}^n)$ can be approximated by a linear combination of f_n of the above form, we have the assertion.

The statement for the converse direction is obvious in both cases from (1.1) with $I_n(f_n) = n!J_n(f_n)$, and from (1.4) with $D_tI_n(f_n) = nI_{n-1}(f_n(\cdot, t))$ for $f_n = f^{\otimes n}$.

Hence, they coincide on all the polynomials of $I_1(f)$. Since $(h_n(I_1(\xi_j)))$ forms a complete orthonormal system (c.o.n.s.) of $L^2(\Omega_1, P_1)$, this implies that they coincide.

Higher order derivatives $D^k_{t_1,\ldots,t_k}$ are inductively defined for $k = 1, 2, \ldots$

Examples
(1) $g(x) = x,\ f(s) = 1,$

$$W(T) = \int_{\mathbf{T}} 1dW(s) = g\left(\int_{\mathbf{T}} f(s)dW(s)\right)\,.$$

Then,

$$D_tW(T) = (x)' \cdot f(t) = 1\,.$$

(2) $g(x) = x,\ f(s) = s.$

$$D_t\left(\int_{\mathbf{T}} sdW(s)\right) = D_tg\left(\int_{\mathbf{T}} sdW(s)\right) = (x)' \cdot f(t) = t\,.$$

This can be also written as

$$D_t(I_1(f)) = I_0(f(t)) = t\,.$$

In computing the kernels f_n in (1.3), we have

$$f_n = \frac{1}{n!}E[D^nF], \quad n = 0, 1, 2, \ldots.$$

The proof is just as in the case of a Taylor expansion.

For $F = W(t)^3$, we have, for example,

$$f_1(t_1) = E[D_{t_1} W(t)^3] = 3E[W(t)^2]1_{[0,t]}(t_1) = 31_{[0,t]}(t_1)\,,$$
$$f_2(t_1, t_2) = \frac{1}{2!}E[D^2_{t_1,t_2} W(t)^3] = 3E[W(t)]1_{[0,t]}(t_1)1_{[0,t]}(t_2) = 0\,,$$
$$f_3(t_1, t_2, t_3) = \frac{1}{3!}E[D^3_{t_1,t_2,t_3} W(t)^3] = 1_{[0,t]}(t_1)1_{[0,t]}(t_2)1_{[0,t]}(t_2)\,.$$

This implies

$$W_t^3 = \int_0^t 3dW(t_1) + 3! \int_{0<t_1<t_2<t} dW(t_1)dW(t_2)dW(t_3)$$
$$= 3W_t + 6 \int_{0<t_1<t_2<t} dW(t_1)dW(t_2)dW(t_3)\,.$$

For the multivariate case, we have

$$I_n(f_1^{\otimes\alpha_1}\hat{\otimes}\cdots\hat{\otimes}f_m^{\otimes\alpha_m}) = \Pi_{k=1}^m h_{\alpha_k}(I_1(f_k))$$

with $\alpha_1 + \cdots + \alpha_m = n$, $\alpha_j \in \{1, \ldots, n\}$. Here, $f_k \in L^2(\mathbf{T})$.

Clark–Ocone formula

The statement of the Clark–Ocone formula on the Wiener space is as follows. $F \in \mathbf{D}^{(1)}_{1,2}$ can be written by

$$F = E[F] + \int_{\mathbf{T}} \phi(t)dW(t)$$

due to the chaos expansion, where $\phi(t)$ is an adapted process. This $\phi(t)$ can be written as $E[D_tF|\mathcal{F}_{1,t}]$. That is, we have the

Clark–Ocone formula:
Any $F \in L^2(\Omega_1)$ can be represented by

$$F = E[F] + \int_{\mathbf{T}} \phi(t)dW(t)\,. \tag{1.10}$$

For $F \in \mathbf{D}^{(1)}_{1,2}$, it holds that $\phi(t)$ is a.e. equal to the predictable projection of D_tF.

Proof. We write $F = \sum_{n=0}^{\infty} I_n(f_n)$ with $f_n \in \tilde{L}^2(\mathbf{T}^n)$. Then, by the definition (1) of D_t,

$$\begin{aligned}\int_{\mathbf{T}} E[D_t F|\mathcal{F}_{1,t}]dW(t) &= \int_{\mathbf{T}} E\left[\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t))|\mathcal{F}_{1,t}\right] dW(t)\\ &= \int_{\mathbf{T}} \sum_{n=1}^{\infty} nE[I_{n-1}(f_n(\cdot,t))|\mathcal{F}_{1,t}]dW(t)\\ &= \int_{\mathbf{T}} \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,t)\cdot 1_{[0,t]}^{\otimes(n-1)}(\cdot)\right) dW(t)\\ &= \int_{\mathbf{T}} \sum_{n=1}^{\infty} n(n-1)!J_{n-1}\left(f_n(\cdot,t)\cdot 1_{[0,t]}^{\otimes(n-1)}(\cdot)\right) dW(t)\\ &= \sum_{n=1}^{\infty} n!J_n(f_n(.)) = \sum_{n=1}^{\infty} I_n(f_n)\\ &= \sum_{n=0}^{\infty} I_n(f_n) - I_0(f_0) = F - E[F]\ .\end{aligned}$$

Here, $1_{[0,t]}^{\otimes n}(t_1,\ldots,t_n) = 1_{[0,t]}(t_1)\cdots 1_{[0,t]}(t_n)$.

In the above, we used the property

$$E[I_n(f_n)|\mathcal{F}_{1,[0,t]}] = I_n\left(f_n \cdot 1_{[0,t]}^{\otimes n}\right)\ .$$

See [173] Proposition 3.11 for the proof of this property. □

3.1.2 Adjoint operator $\delta = D^*$

We can define the adjoint operator δ of D_t on $L^2(\mathbf{T}\times\Omega_1)$ by

$$E[F\delta(u)] = E\left[\int_{\mathbf{T}} u(t)D_t F dt\right]\ . \tag{1.11}$$

The operator δ is called a *divergence operator*. When $u(t)$ is a process, $\delta(u)$ is also called a *Skorohod integral* of u. In case $u(t)$ is an adapted process, it coincides with the Itô integral.

The operator δ can be characterised as follows using the chaos expansion. Let

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

be a process such that $E[|u(t)|^2] < +\infty$ for $t \in \mathbf{T}$. Here,

$$f_n(\cdot,t) = f_{n+1}(t_1,\ldots,t_n,t_{n+1})|_{t_{n+1}=t}$$

with t fixed.

Then,

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}\left(\tilde{f}_n(\cdot, t)\right) . \tag{1.12}$$

Here, $\tilde{f}_n(\cdot, t)$ is the symmetrisation of f_{n+1} regarding t_{n+1} as a parameter t given by

$$\tilde{f}_{n+1}(t_1, \ldots, t_n, t_{n+1})|_{t_{n+1}=t} .$$

Properties of *δ*

We confine ourselves to the special case where the process $u(t)$ has the form

$$u(t) = Fh(t)$$

where $F \in \mathbf{D}^{(1)}_{1,2}$ and $h \in L^2(\mathbf{T})$. In this case, $\delta(u)$ is calculated by

$$\delta(u) = FI_1(h) - \int_{\mathbf{T}} D_t F h(t) dt .$$

Especially, $\delta(h) = I_1(h)$.

In general it is known that, in case $u(t)$ is a progressively measurable process, $\delta(u)$ coincides with the Itô integral:

$$\delta(u) = \int_{\mathbf{T}} u(t) dW(t) .$$

Furthermore, if $u(t) \in \mathbf{D}^{(1)}_{1,2}$, then $(D_t u(.))_{t\in\mathbf{T}}$ is Skorohod integrable, $I_1(D_t u) \in L^2(\mathbf{T}\times\Omega_1)$, and the following commutativity relation holds:

$$D_t(\delta(u)) = u(t) + \int_{\mathbf{T}} D_t u(s) dW(s) ,$$

or shortly

$$D_t \delta(u) = \delta(D_t u) + u(t) .$$

The proofs of these properties depend on the chaos representation described above. See [170] Section 1.3.

Following Malliavin's approach [160], we can introduce the notion of the Ornstein–Uhlenbeck operator over the Wiener space in the following way. We introduce the Ornstein–Uhlenbeck semigroup $(T_t)_{t\geq 0}$ on $L^2(\Omega_1)$ given by

$$T_t(F) = \sum_{n=0}^{\infty} e^{-nt} \Pi_n(F), \quad F \in L^2(\Omega_1) .$$

Here, $\Pi_n(F)$ denotes the projection of F to the space of n-th Wiener chaos.

Especially, in the case $F = X$, where

$$X = \int_{\mathbf{T}} f(t)dW(t) ,$$

$T_t(F) = p_t$ is given by

$$p_t(u) = \int_{\Omega_1} f\left(e^{-t}u + \sqrt{1 - e^{-2t}}\omega\right) P_1(d\omega) \tag{1.13}$$

where $P_1(du)$ denotes the Wiener measure. To introduce the Sobolev spaces, we use the operator $C = -\sqrt{-L}$ instead of D:

$$CF = \sum_{n=0}^{\infty} -\sqrt{n}\Pi_n(F) .$$

Here, L denotes the infinitesimal generator of the semigroup T_t, called the Ornstein–Uhlenbeck operator, and given by

$$LF = \sum_{n=0}^{\infty} -n\Pi_n(F) .$$

Malliavin [160] and Nualart [170] (Chapter 1) use this notation. However, we prefer to use the derivative operator D_t in the sense of (1) on the Wiener space (Bismut's approach). In fact two approaches give equivalent definitions of the Sobolev space. We return to this topic in Sections 3.3, 3.4.

3.2 Calculus of functionals on the Poisson space

In this section, we state some basic facts on the Poisson space which will be necessary in the later sections. The representation of a (multidimensional) Poisson functional is closely related to the chaos expansion on the Poisson space. Readers can also refer to [190] and [173].

As in the case of the Wiener space, there are two contexts in analysing the Poisson space; either by the gradient operator or by the chaos expansion. We prefer to use the gradient operator $\tilde{D}$ below since it is helpful for constructing the Sobolev spaces in Section 3.3.

3.2.1 One-dimensional case

In this subsection, we denote by $N(dtdz)$ the Poisson random measure with mean measure $dt\delta_{\{1\}}(dz)$, which is associated with the standard Poisson process.

In case of 1-dimensional Poisson space, we can introduce two kinds of "derivative" operators on it.

For $\omega_2 \in \Omega_2$, we denote by $Q(t,\omega_2)$ the random measure on $\mathbf{T}$: $Q(t,\omega_2) = N([0,t]\times\cdot) - t$.

For $f_n(t_1,\dots,t_n)$, we denote by $I_n(f_n)$ the iterated integral

$$n!\int_{t_{n-1}}^{T}\int_{t_{n-2}}^{t_{n-1}}\cdots\int_{0}^{t_2} f_n(t_1,\dots,t_n)dQ(t_1)\cdots dQ(t_n)\,. \tag{2.1}$$

We have the following chaos representation property.

Lemma 3.1 ([1] Theorem 5.3). *Any $F \in L^2(\Omega_2,\mathcal{F}_2,P_2)$ can be expressed by*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.2}$$

for some $f_n \in L^2(\mathbf{T}^n)$.

We introduce two notions of the derivative on the Poisson space.

For $F = \sum_{n=0}^{\infty} I_n(f_n)$, we have

$$D_t^N F = \sum_{n=0}^{\infty} nI_{n-1}(f_n(\cdot,t))\,. \tag{2.3}$$

Although the definition (2.3) is apparently the same as (1.5), the notion is “*far*” in spirit from D_t in the Wiener space. Because, in case

$$\begin{aligned}F &= f(T_1,\dots,T_n) = f(I_1(\delta_{\{T_1\}}),\dots,I_1(\delta_{\{T_n\}}))\\ &= f\left(\int_{\mathbf{T}}\delta_{\{T_1\}}dQ(t_1),\dots,\int_{\mathbf{T}}\delta_{\{T_n\}}dQ(t_n)\right)\end{aligned}$$

is represented by $F = I_n(f_n)$, where f_n is symmetric, we can show

$$D_t^N F = \sum_{k=1}^{n} 1_{(T_{k-1},T_k]}(t)\{f(T_1,\dots,T_{k-1},t,T_k,\dots,T_{n-1}) - f(T_1,\dots,T_n)\}\,. \tag{2.4}$$

Here, T_j denotes the j-th jump moment of N.

Indeed, by putting $g(t_1,\dots,t_n) = \delta_{\{T_1\}}(t_1)\otimes\cdots\otimes\delta_{\{T_n\}}(t_n)$, we extend g to $\tilde{g}(t_1,\dots,t',\dots,t_{n-1}) = \delta_{\{T_1\}}(t_1)\otimes\cdots\otimes\delta_{\{T_{k-1}\}}(t_{k-1})\otimes\delta_{\{t\}}(t')\otimes\delta_{\{T_k\}}(t_k)\otimes\cdots\otimes\delta_{\{T_{n-1}\}}(t_{n-1})$, and to $\breve{g}(t_1,\dots,t_{n-1}) = \delta_{\{T_1\}}(t_1)\otimes\cdots\otimes\delta_{\{T_{k-1}\}}(t_{k-1})\otimes\delta_{\{T_k\}}(t_k)\otimes\cdots\otimes\delta_{\{T_{n-1}\}}(t_{n-1})$. Then,

$$\begin{aligned}D_t^N F &= n(n-1)!\\ &\quad\times\left\{\int_0^T\cdots\int_0^{t_k}\cdots\int_0^{t_2} f_n\circ\tilde{g}(t_1,\dots,t',\dots,t_{n-1})dQ(t_1)\cdots dQ(t')\cdots dQ(t_{n-1})\right.\\ &\qquad\left.-\int_0^T\cdots\int_0^{t_2} f_n\circ\breve{g}(t_1,\dots,t_{n-1})dQ(t_1)\cdots dQ(t_{n-1})\right\}\\ &= f(T_1,\dots,T_{k-1},t,T_k,\dots,T_{n-1}) - f(T_1,\dots,T_n)\end{aligned}$$

in case $T_{k-1} < t \le T_k$. Here, we regard $f_n \circ \check{g}(t_1, \ldots, t_{n-1}) = f_n \circ \check{g}(t_1, \ldots, t_{n-1}, T_n)$.

Note that D_t^N is a *difference operator* instead of a differential operator. According to the notation of [181], D_t^N should be written as $\tilde{D}_{(t,1)}$.

Another operator, $\tilde{D}_t$, is given by

$$\begin{aligned} \tilde{D}_t F &= -\sum_{k=1}^{n} 1_{[0,T_k]}(t)\partial_k f(T_1, \ldots, T_n) \\ &= -\sum_{N_t < k \le n} \partial_k f(T_1, \ldots, T_n) \end{aligned} \tag{2.5}$$

for $F = f(T_1, \ldots, T_n)$ and $f \in C^1(\mathbf{T}^n)$. We remark that a factor "–" is multiplied to the form of the gradient operator D given in Section 2.1.3. This is due to the tradition from [43]. It is consistent with (2.6) below.

Note that this is "*near*" to the original definition of D_t in the Wiener case. Indeed, it is a differential operator, and it satisfies the derivation property $\tilde{D}_t(FG) = G\tilde{D}_t F + F\tilde{D}_t G$, and the chain rule.

If we put $p_n(t) = P(N_t = n) = \frac{1}{n!} t^n e^{-t}$,

$$\begin{aligned} p_n'(t) &= \frac{1}{(n-1)!} t^{n-1} e^{-t} - \frac{1}{n!} t^n e^{-t} \\ &= p_{n-1}(t) - p_n(t) \; . \end{aligned}$$

Using this property, we can directly show, for $F = f(T_1, \ldots, T_n)$,

$$E[D_t^N F|\mathcal{F}_t] = E[\tilde{D}_t F|\mathcal{F}_t] \tag{2.6}$$

(cf. [190] Proposition 7.2.7, [98]). This means they have the same adapted projection.

In view of (2.4) and (2.6), we observe why the time shift method (explained in Section 2.1.3) using the differential operator with respect to the jump times works well on the Poisson space in the one-dimensional case.

Charlier polynomials

In the Poisson space, the role played by Hermite polynomials in the Wiener space will be played by *Charlier polynomials*. Charlier polynomials are a kind of Laguerre polynomials. We introduce them in the following way.

Definition 3.1. Charlier polynomials $C_n(k, t)$ of order n and parameter $t \ge 0$ are defined recursively by

$$\begin{aligned} C_0(k, t) &= 1 \; , \\ C_1(k, t) &= k - 1 \; , \\ C_2(k, t) &= k^2 - (2 + t)k + 1 \; , & k \in \mathbf{R} \; , \quad t \ge 0 \; , \\ C_{n+1}(k, t) &= (k - n - t)C_n(k, t) - ntC_{n-1}(k, t) \; , & n = 1, 2, \ldots \; . \end{aligned}$$

Then, $C_n(k,t)$ satisfies the following properties:

Proposition 3.2.
1. $C_n(k+1,t) - C_n(k,t) = -\frac{\partial C_n}{\partial t}(k,t)$,
2. $C_n(k+1,t) - C_n(k,t) = nC_{n-1}(k,t)$,
3. $C_{n+1}(k,t) = kC_n(k-1,t) - tC_n(k,t)$

for $t > 0$, $k \in \mathbf{N}$.

Proof. See [190] Proposition 6.2.9.

A remarkable property of the Charlier polynomials is the expression

$$C_n(N_t, t) = I_n\left(1_{[0,t]}^{\otimes n}\right) ,$$

where the right-hand side is defined by

$$I_n(f_n) = n! \int_{0<t_1<t_2<\cdots<t_n} f_n(t_1,\ldots,t_n)dQ(t_1)\cdots dQ(t_n)$$

with $f_n = 1_{[0,t]}^{\otimes n}$.

The Charlier polynomials are orthogonal with respect to the Poisson distribution. Indeed,

$$\begin{aligned}\langle C_n(\cdot,t), C_m(\cdot,t)\rangle_{l^2(\mathbf{N})} &= e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}C_n(k,t)C_m(k,t)\\ &= E[C_n(N_t,t)C_m(N_t,t)] = E\left[I_n(1_{[0,t)}^{\otimes n})I_m(1_{[0,t)}^{\otimes m})\right]\\ &= 1_{\{n=m\}}n!t^n .\end{aligned}$$

Using this property, we can make a chaos expansion (unique representation) of some type of Poisson functional by using (multivariate) Charlier polynomials. Refer to [190] Lemma 6.2.10. □

3.2.2 Multidimensional case

We introduce another Poisson random measure for the multidimensional case on Ω_2:

$$N(dtdz) = \sum_{i\in\mathbf{N}} \delta_{(t_i,z_i)}(dtdz) , \quad (t_i,z_i) \in \mathbf{T}\times(\mathbf{R}^m\setminus\{0\}) , \tag{2.7}$$

and

$$\tilde{N}(dtdz) = N(dtdz) - \mu(dz)dt ,$$

where μ is a Lévy measure.

For $h_n = h_n(t_1, z_1, t_2, z_2, \ldots, t_n, z_n) \in L^2(\mathbf{T} \times (\mathbf{R}^m \setminus \{0\}))^{\otimes n}$, let

$$I_0(h) = h \, ,$$

$$I_1(h_1) = \int_{\mathbf{T}} \int_{\mathbf{R}^m \setminus \{0\}} h_1(t,z) \tilde{N}(dtdz) \, ,$$

$$I_n(h_n) = n \int_{\mathbf{T}} \int_{\mathbf{R}^m \setminus \{0\}} I_{n-1}\left(\pi^n_{(t,z)} h_n\right) \tilde{N}(dtdz) \, , \quad n = 2, 3, \ldots \tag{2.8}$$

where

$$\pi^n_{(t,z)} h_n(t_1, z_1, \ldots, t_{n-1}, z_{n-1}) = h_n(t_1, z_1, \ldots, t_{n-1}, z_{n-1}, t, z) 1_{[0,t]}(t_1) \cdots 1_{[0,t]}(t_{n-1}) \, .$$

We recall the difference operator (perturbation due to Carlen–Pardoux, Picard) $\tilde{D}_u$ by

$$\tilde{D}_u F(\omega_2) = F(\omega_2 \circ \varepsilon_u^+) - F(\omega_2) \, ,$$

where $u = (t,z)$. Here, the operator ε_u^+ is given in Section 2.1.2.

Note that if F is represented as

$$F = \sum_{n=0}^{\infty} I_n(h_n) \, ,$$

then $\tilde{D}_u F$ is given by the operation

$$\tilde{D}_{(t,z)} I_n(h_n) = n I_{n-1}(h_n(\cdot, t, z)), \mu(dz)dt - a.e. \, , \tag{2.9}$$

provided that the right-hand side is well-defined.

The idea of the proof is the same as in the case of (2.4). We sketch it below.

We put

$$\Delta_n = \left\{ (t_1, z_1, \ldots, t_n, z_n) \in (\mathbf{T} \times (\mathbf{R}^m \setminus \{0\}))^n \, ; \quad t_i \neq t_j (i \neq j) \right\} .$$

Since $1_{\Delta_n}(t_1, z_1, \ldots, t_n, z_n) \delta_{\{(t,z)\}}(dt_i dz_i) \delta_{\{(t,z)\}}(dt_j dz_j) = 0, \, i, j = 1, \ldots, n,$

$$\begin{aligned} \tilde{D}_{(t,z)} I_n(h_n) &= \tilde{D}_{(t,z)} \int_{\Delta_n} h_n(t_1, z_1, \ldots, t_n, z_n) \tilde{N}(dt_1 dz_1) \cdots \tilde{N}(dt_n dz_n) \\ &= \int_{\Delta_n} h_n(t_1, z_1, \ldots, t_n, z_n) \Pi_{i=1}^n \Big(\omega_2(dt_i dz_i) - dt_i \mu(dz_i) \\ &\qquad + (1 - \omega_2(\{(t,z)\})) \delta_{\{(t,z)\}}(dt_i dz_i) \Big) \\ &\quad - \int_{\Delta_n} h_n(t_1, z_1, \ldots, t_n, z_n) \tilde{N}(dt_1 dz_1) \cdots \tilde{N}(dt_n dz_n) \end{aligned}$$

$$= (1 - \omega_2(\{(t,z)\})) \times \sum_{i=1}^{n} \int_{\Delta_{n-1}} h_n(t_1, z_1, \ldots, t, z, \ldots, t_n, z_n) \Pi_{k \neq i} \tilde{N}(dt_k dz_k)$$
$$= (1 - \omega_2(\{(t,z)\})) \sum_{i=1}^{n} I_{n-1}(h_n(\cdot, t, z, \cdot)) .$$

Since we can take h_n to be symmetric, we have

$$\tilde{D}_{(t,z)} I_n(h_n) = 1_{\{\omega_2(\{(t,z)\})=0\}} n I_{n-1}(h_n(\cdot, t, z, \cdot)) .$$

Since $\hat{N}(\{(t,z)\}) = 0$, we have the assertion. For the precise result, see [190] Proposition 6.4.7.

Furthermore, we have the following "chain rule" in the difference form.

Lemma 3.2. *For F, and φ continuous such that $\varphi(F) \in L^2(\Omega_2)$ and $\varphi(F + \tilde{D}_u F) \in L^2(U \times \Omega_2)$,*

$$\tilde{D}_u \varphi(F) = \varphi(F \circ \varepsilon_u^+) - \varphi(F) = \varphi(F + \tilde{D}_u F) - \varphi(F) . \tag{2.10}$$

The proof uses the Fourier decomposition of $\varphi(F)$ into L^2 components. See [173] Theorem 12.8.

Example 3.1.

(1)

$$\tilde{D}_t N_s = \tilde{D}_t \sum_{k=1}^{\infty} 1_{[T_k, +\infty)}(s) = \sum_{k=1}^{N_s} 1_{(T_k, T_{k+1}]}(t) = 1_{\{t < N_s\}} .$$

(2) In case

$$h(t,z) = z, \quad F = \int_{\mathbf{T}} \int z \tilde{N}(dt dz) = Z(T) ,$$

we have

$$\tilde{D}_{(t,z)} F = I_0(h(\cdot, t, z)) = h(t,z) = z .$$

(3) We have the following property for the operator $\tilde{D}_u$

$$\tilde{D}_u(FG) = F \tilde{D}_u G + G \tilde{D}_u F + \tilde{D}_u F \tilde{D}_u G , \tag{2.11}$$

assuming that the left-hand side is finite a.s. Indeed,

$$\text{L.H.S.} = (FG) \circ \varepsilon_u^+ - FG = (F \circ \varepsilon_u^+) \cdot (G \circ \varepsilon_u^+) - FG$$
$$= (\tilde{D}_u F + F)(\tilde{D}_u G + G) - FG .$$

Choosing $G = F$ and using induction, we have

$$\tilde{D}_u F^n = (F \circ \varepsilon_u^+)^n - F^n = (F + \tilde{D}_u F)^n - F^n . \tag{2.12}$$

(4) For $\frac{G}{F}$, $F \neq 0$, we have

$$\tilde{D}_u \frac{G}{F} = \frac{G \circ \varepsilon_u^+}{F \circ \varepsilon_u^+} - \frac{G}{F} = \frac{FG \circ \varepsilon_u^+ - GF \circ \varepsilon_u^+}{F \cdot F \circ \varepsilon_u^+}$$
$$= \frac{F(\tilde{D}_u G + G) - G(\tilde{D}_u F + F)}{F \cdot F \circ \varepsilon_u^+} = \frac{F\tilde{D}_u G - G\tilde{D}_u F}{F \cdot F \circ \varepsilon_u^+} . \qquad (2.13)$$

For higher order differences for the quotient, we iterate the above formula.

Example 3.2.

(1) Let

$$F = Z(T) = \int_{\mathbf{T}} \int z\tilde{N}(dtdz) .$$

Then,

$$\tilde{D}_{(t,z)} F^2 = (F + \tilde{D}_{(t,z)} F)^2 - F^2$$
$$= (Z(T) + z)^2 - Z(T)^2 = 2Z(T)z + z^2 .$$

(2) Let $G = \exp(F)$, $F = \int_{\mathbf{T}} \int h(t)z\tilde{N}(dtdz)$, $h \in L^2([0, T])$. We then have

$$\tilde{D}_{(t,z)} G = \tilde{D}_{(t,z)}(\exp(F)) = \exp(F + \tilde{D}_{(t,z)} F) - \exp(F)$$
$$= \exp(F + h(t)z) - \exp(F) = \exp(F) \cdot (e^{h(t)z} - 1) = G(e^{h(t)z} - 1) ,$$

that is, the chain rule.

(3) If $m = 1$ and $F = (Z(T) - K)^+$, $K > 0$, then

$$\tilde{D}_{(t,z)} F = ((Z(T) + \tilde{D}_{(t,z)} Z(T)) - K)^+ - (Z(T) - K)^+$$
$$= (Z(T) + z - K)^+ - (Z(T) - K)^+ .$$

3.2.3 Characterisation of the Poisson space

In this subsection, we seek the basis of $L^2(\Omega_2)$.

Let $\mathcal{G}$ be the collection of functions

$$\Bigg\{ g = \sum_{j=1}^{n} h_j(t) 1_{B_j}(z) \in L^2(d\lambda) ;$$
$$B_i \cap B_j = \emptyset (i \neq j) , \quad t \mapsto h_j(t) \text{continuous} , \quad \lambda(B_i) < +\infty \Bigg\} .$$

Here, $\lambda(dtdz) = dt\mu(dz) = \hat{N}(dtdz)$.

Let $\mathcal{M}$ be the subspace of $L^2(\Omega_2)$ spanned by the functions of the form

$$\exp \left\{ \int_{\mathbf{T}} \int g(t,z)\tilde{N}(dtdz) - \int_{\mathbf{T}} \int (e^{g(t,z)} - 1 - g(t,z))\lambda(dtdz) \right\} , \quad g(t,z) \in \mathcal{G} .$$

We remark

$$e^{g(t,z)} - 1 = \sum_{j=1}^{n} (e^{h_j(t)} - 1).\ 1_{B_j}(z)$$

if $g(t,z) = \sum_{j=1}^{n} h_j(t) 1_{B_j}(z) \in \mathcal{G}$, $z \in \cup_{i=1}^{n} B_i$.

Theorem 3.1 (Surgailis [205]). *A function $g \in L^2(d\lambda)$ satisfies*

$$\int_{\{u;g(u)\geq 1\}} e^{2g(s,z)} \lambda(dsdz) < +\infty \tag{2.14}$$

if and only if

$$\exp\left\{ \int_{\mathbf{T}} \int g(s,z) \tilde{N}(dsdz) \right\} \in L^2(\Omega_2)\ .$$

Furthermore, we have

$$\begin{aligned} &\exp\left\{ \int_{\mathbf{T}} \int g(s,z) \tilde{N}(dsdz) - \int_{\mathbf{T}} \int (e^{g(s,z)} - 1 - g(s,z)) \lambda(dsdz) \right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} I_n \left(\bigotimes_{j=1}^{n} (e^{g_j(s,z)} - 1) \right) \end{aligned}$$

for some $g_j \in \mathcal{G}$.

Hence, under (2.14), we can take the “basis” of $L^2(\Omega_2)$ in the form of exponential functionals. For an example of (2.14), assume the case that the tail of the Lévy measure is rapidly decreasing.

Lemma 3.3.

$$\bar{\mathcal{M}}^{dP_2} = L^2(\Omega_2)\ . \tag{2.15}$$

Proof. Let $G \in L^2(\Omega_2)$. Assume

$$E[GH] = 0 \quad \text{for all} \quad H \in \mathcal{M}\ .$$

Then,

$$E\left[G \exp\left\{ \sum_{j=1}^{n} c_j \int_{\mathbf{T}} \int g_j(s,z) \tilde{N}(dsdz) \right\} \right] = 0$$

for all $c_1, \dots, c_n \in \mathbf{R}$, $g_1, \dots, g_n \in \mathcal{G}$, $n \geq 1$. Suppose n and $g_1, \dots, g_n \in \mathcal{G}$ are fixed. Then, the Laplace transform of the signed measure on $\mathbf{R}^n$

$$\nu(B) = E[G 1_B(\tilde{N}(g_1), \dots, \tilde{N}(g_n))]$$

is identically zero. Hence, $\nu \equiv 0$.

Hence, $E[G 1_X] = 0$ for all X which is measurable with respect to the σ-field generated by $\tilde{N}(g)$, $g \in \mathcal{G}$. This implies $G = 0$. □

Characterisation of the Poisson space via chaos decomposition
It is shown that

$$L^2(\Omega_2) = \bigoplus_{n=0}^{\infty} C_n \ ,$$

where

$$C_n = \left\{ I_n(f_n) \ ; \quad f_n \in \otimes^n L^2(d\lambda) \right\} \ .$$

This is because each element g in $\mathcal{M}$ is described as a limit in $L^2(d\lambda)$ of multiple stochastic integrals by Theorem 3.1, and to Lemma 3.3.

Examples in case $m = 1$
(1) $z(t) = \int_0^t \int z\tilde{N}(dsdz)$. Let $y(t) = z^2(t)$. Then, by Itô's formula

$$dy(t) = \int z^2 \mu(dz)dt + \int (2z(t-) + z)z\tilde{N}(dtdz)$$

(cf. $(a+b)^2 - a^2 = 2ab + b^2$). Hence,

$$y^2(t) = z^2(t) = t \int z^2 \mu(dz) + \int_0^t \int (2z(t-) + z)z\tilde{N}(dtdz)$$
$$= t \int z^2 \mu(dz) + \int_0^t \int z^2 \tilde{N}(dtdz) + \int_0^t \int \int_0^{t_2-} \int 2z_1 z_2 \tilde{N}(dt_2 dz_2)\tilde{N}(dt_1 dz_1) \ .$$

(2) Let $z(t) = x_0 + bt + \int_0^t z\tilde{N}(dsdz)$. The local time $L_T(x)$ of $z(t)$ at level x up to the time T is given by

$$L_T(x) = \int_{\mathbf{T}} \delta_x(z(t))dt \ .$$

The local time $L_t(x)$ can be expanded into chaos as ([173] Theorem 14.8)

$$L_t(x) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \frac{1}{n!} \int_0^{\infty} \int \cdots \int_0^{t_2} \int f_n(s_1, z_1, \ldots, s_n, z_n)\tilde{N}(ds_1 dz_1) \cdots \tilde{N}(ds_n dz_n) \ ,$$

where

$$f_n(s_1, z_1, \ldots, s_n, z_n) = \int_{\mathbf{T}} \int (\Pi_{j=1}^n (e^{i\lambda z_j} - 1)h(t, \lambda, x) 1_{\{t > \max(s_j)\}}) d\lambda dt$$

with

$$h(t, \lambda, x) = \exp\left\{ t \int (e^{i\lambda z} - 1 - i\lambda z)\mu(dz) - i\lambda x \right\} \ .$$

Clark–Ocone formula on the Poisson space

Using Theorem 3.1 above, we can derive the *Clark–Ocone formula* on the Poisson space separately, that is,

Theorem 3.2 (cf. [190] Proposition 6.7.1). *For any* $F \in \mathbf{D}_{1,2}^{(2)}$

$$F = E[F] + \int_{\mathbf{T}} \int E[\tilde{D}_{(t,z)} F | \mathcal{F}_t] \tilde{N}(dtdz) \ . \tag{2.16}$$

Here,

$$\mathbf{D}_{1,2}^{(2)} = \left\{ \sum_n I_n(g_n) \ ; \quad \sum_n n^2 (n-1)! \|g_n\|^2_{\otimes L^2(d\lambda)} < +\infty \right\} \ .$$

Proof. Let $F \in \mathbf{D}_{1,2}^{(2)}$. We can choose by Theorem 3.1 F_n such that $F_n \to F$ in $L^2(\Omega_2)$, where

$$F_n = 1 + \sum_{k=1}^{n} \frac{1}{k!} I_k \left(\bigotimes_{j=1}^{k} (e^{g_j(t,z)} - 1) \right)$$

for some $(g_j) = (g_{n,j})$. We put

$$Y_T^{(n)} = \int_{0<t_1<\cdots<t_n<T} \Pi_{j=1}^n (e^{g_j(t_j,z_j)} - 1) \tilde{N}(dt_j dz_j) \ .$$

Then,

$$Y_T^{(n)} = \frac{1}{n!} I_n \left(\bigotimes_{j=1}^{n} (e^{g_j(t,z)} - 1) \right)$$

by the remark at the beginning of Section 3.2.3.

Since the solution to the Doléans–Dade equation

$$Y_t^{(n)} = 1 + \int_0^t Y_{s-}^{(n)} (e^{g_n(s,z)} - 1) \tilde{N}(dsdz) \tag{2.17}$$

is unique, and hence $Y_T^{(n)}$ satisfies

$$\tilde{D}_{t,z} Y_T^{(n)} = (e^{g_n(t,z)} - 1) Y_T^{(n)} \ ,$$

we only have to show the following (2.18) in order to assert

$$E\left[\tilde{D}_{(t,z)} Y_T^{(n)} | \mathcal{F}_t \right] = Y_t^{(n)} (e^{g_n(t,z)} - 1) \ . \tag{2.18}$$

However, since $Y_T^{(n)}$ is a martingale, we have (2.18). □

Remark 3.1. Any $F \in L^2(\Omega_2)$ can be represented by

$$F = E[F] + \int_{\mathbf{T}} \int E[\tilde{D}_{(t,z)} F | \mathcal{F}_t] \tilde{N}(dtdz) \ . \tag{2.19}$$

See Privault [189], Chapter 6, Section 7. See also Løkka [154]; however, this paper is justified only for $d = 1$ and for $F \in \mathbf{D}^{(2)}_{1,2}$. In this paper, he used the functionals

$$\exp\left\{\int_{\mathbf{T}}\int h(t)\gamma(z)\tilde{N}(dtdz) - \int_{\mathbf{T}}\int (e^{h(t)\gamma(z)} - 1 - h(t)\gamma(z))\lambda(dtdz)\right\} \tag{2.20}$$

as the “basis”. Here, $\gamma(z) = e^z - 1\ (z < 0),\ 1 - e^{-z}\ (z > 0)$.

The expression (2.9) implies that $\tilde{D}_u$ is a closable operator on $\mathbf{D}^{(2)}_{1,2}$.

3.3 Sobolev space for functionals over the Wiener–Poisson space

In this section, based on the perturbation operators introduced above, we construct Sobolev spaces over the Wiener, the Poisson, and the Wiener–Poisson spaces.

3.3.1 The Wiener space

The derivative operator

Let $\mathbf{K}_1 = L^2(\mathbf{T}; \mathbf{R}^m)$. For $f = (f^1, \ldots, f^m) \in \mathbf{K}_1$, we set

$$W(f) = \sum_{i=1}^{m} \int_{\mathbf{T}} f^i(s)dW^i(s)\ .$$

By $\mathcal{P}_1$, we denote the collection of random variables X written as

$$X = g(W(f_1), \ldots, W(f_n))\ ,$$

where $g(x_1, \ldots, x_n)$ is bounded $\mathcal{B}(\mathbf{R}^n)$-measurable, smooth in $(x_1, \ldots, x_n)$, $n \in \mathbf{N}$. The Malliavin–Shigekawa derivative of X is an m-dimensional row vector of a stochastic process given as in Section 3.1 by

$$D_t X = \sum_{l=1}^{n} \frac{\partial g}{\partial x_l}(W(f_1), \ldots, W(f_n)) f_l(t)\ . \tag{3.1}$$

The operator $D : L^2(\Omega_1, \mathcal{F}_1, P_1) \to L^2(\Omega_1; \mathbf{K}_1)$ is a closed and unbounded operator. For $(t_1, \ldots, t_j) \in \mathbf{T}^j$, we set $D^j_{t_1,\ldots,t_j} = D_{t_1} \cdots D_{t_j}$. Let l be a non-negative integer and $p \geq 1$. Norms $|\cdot|_{0,l,p}$ for a random variable $X \in \mathcal{P}_1$ are defined by

$$|X|_{0,l,p} := \left(E[|X|^p] + \sum_{j=1}^{l} E\left[\left(\int_{\mathbf{T}^j} \left|D^j_{\mathbf{t}} X\right|^2 d\mathbf{t}\right)^{p/2}\right]\right)^{1/p} \tag{3.2}$$

where $D^j_{\mathbf{t}} = D^j_{t_1,\ldots,t_j}$, and $d\mathbf{t} = dt_1 \cdots dt_j$. Let $\mathbf{D}_{0,l,p}$ be the completion of $\mathcal{P}_1$ with respect to the norm $|\cdot|_{0,l,p}$:

$$\mathbf{D}_{0,l,p} = \bar{\mathcal{P}}_1^{\,|\cdot|_{0,l,p}}\ .$$

Then, $\mathbf{D}_{0,l,p} \subset L^p(\Omega_1, \mathcal{F}_1, P_1)$ and operators D^j, $j = 1, \ldots, l$ are extended to $\mathbf{D}_{0,l,p}$ in an obvious way.

This space is equivalent with those introduced by the norm using the square root of the Ornstein–Uhlenbeck operator L:

$$|F|_{p,2l} = |(I - L)^l F|_p \ ,$$

(cf. [170] Proposition 1.5.3).

Remark 3.2. The two definitions of D_t described in Section 3.1.1 coincide with each other on $\mathbf{D}_{0,1,2}$. See [173] Theorem A.22.

The adjoint operator δ of D has been introduced in Section 3.1.2. The operator δ is a closable operator from $L^2(\Omega_1; \mathbf{K}_1)$ to $L^2(\Omega_1, \mathcal{F}_1, P_1)$. The random variable $\delta(U)$ coincides with the Skorohod integral of $U = \{U_t\}$ with respect to $W(t)$. The operator δ is sometimes called a *divergence operator*. For a process U, we introduce a norm by

$$\|U\|_{0,l,p} := \left(\sum_{j=0}^{l} E\left[\left(\int_{\mathbf{T}^{j+1}} \left| D_{\mathbf{t}}^j U_t \right|^2 d\mathbf{t}dt \right)^{p/2} \right] \right)^{1/p} . \tag{3.3}$$

The following inequality (continuity of the divergence operator), obtained as a corollary to the *Meyer's type inequality*, plays a fundamental role (cf. Nualart [170] Proposition 1.5.4).

Theorem 3.3. *Let $l \geq 0$ and $p > 1$ be fixed. Suppose that $\|U\|_{0,l,p} < +\infty$. Then, there exists a positive constant $c = c(l, p)$ such that*

$$|\delta(U)|_{0,l,p} \leq c\|U\|_{0,l+1,p} \ .$$

See also Lemma 3.4 below.

3.3.2 The Poisson Space

Set $U = \mathbf{T} \times (\mathbf{R}^m \setminus \{0\})$,

$$A(\rho) := \{u \in U \, ; \quad \gamma(u) \leq \rho\} \ ,$$
$$\varphi(\rho) := \int_{A(\rho)} \gamma(u)^2 \, \hat{N}(du) = \int_{\{|z| \leq \rho\}} |z|^2 \, \mu(dz) \ ,$$

where $\gamma(u) = |z|$ for $u = (t, z)$. We repeat that the measure μ satisfies an *order condition* if there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho \to 0} \frac{\varphi(\rho)}{\rho^\alpha} > 0 \ . \tag{3.4}$$

We introduce compound measures $\hat{M}(du)$, $\hat{M}(d\mathbf{u})$ and $\bar{M}(d\mathbf{u}; du)$ in the following way.

Let

$$\hat{M}(du) := \frac{1}{\varphi(1)}\gamma(u)^2 1_{A(1)}(u)\hat{N}(du), \quad \hat{M}(d\mathbf{u}) = \hat{M}(du_1)\cdots\hat{M}(du_k)$$

on the off-diagonal points $\{(\mathbf{u}, u) = (u_1, \ldots, u_k, u); u_i \neq u, i = 1, \ldots, k\}$, and put

$$\bar{M}(d\mathbf{u}; du) = \hat{M}(d\mathbf{u})\hat{M}(du)$$

on $\{(\mathbf{u}, u) = (u_1, \ldots, u_k, u); u_i \neq u, i = 1, \ldots, k\}$.

We put

$$\bar{M}(d\mathbf{u}; du) = \sum_{i=1}^{k} \hat{M}(d\mathbf{u}^{(i)}) \otimes \hat{M}(du).\delta_{\{u_i=u\}}$$

on the k-dimensional diagonal points $\{(\mathbf{u}, u) = (u_1, \ldots, u_k, u);$ for some $i \quad u_i = u\}$, where $\mathbf{u}^{(i)} = \mathbf{u} \setminus \{u_i\}$ viewed as a $(k-1)$-vector.

We have introduced the operator $\tilde{D}_u$ on the random fields defined on $U \times \Omega_2$ in Section 2.1.2. The operator $\tilde{D}$ thus introduced is essentially the same as what has been introduced in Section 3.2.2. The operator $\tilde{D}$ viewed as that $\tilde{D} : L^2(\Omega_2) \to L^2(\Omega_2; \mathbf{K}_2)$ is a closable operator (cf. [181], p. 487 Remark). Here, $\mathbf{K}_2 = L^2(U, \hat{N})$.

Let $\mathbf{u} = (u_1, \ldots, u_k) = ((t_1, z_1), \ldots, (t_k, z_k)) = (\mathbf{t}, \mathbf{z})$. We set $|\mathbf{u}| = |\mathbf{z}| = \max_{1\le i\le k} |z_i|$ and $\gamma(\mathbf{u}) = |z_1|\cdots|z_k|$. We define $\varepsilon^+_{\mathbf{u}} = \varepsilon^+_{u_1} \circ \cdots \circ \varepsilon^+_{u_k}$ and $\tilde{D}_{\mathbf{u}} = \tilde{D}^k_{\mathbf{u}} = \tilde{D}_{u_1}\cdots\tilde{D}_{u_k}$.

Here are some calculation rules concerning $\tilde{D}$. We start from

$$(1) \qquad \tilde{D}(XY) = (\tilde{D}X)Y + X(\tilde{D}Y) + (\tilde{D}X)(\tilde{D}Y)\,,$$

$$(2) \qquad X \circ \varepsilon^+_u = \tilde{D}_u X + X\,.$$

From (1), we deduce

$$\begin{aligned}\tilde{D}^2(XY) = \tilde{D}(\tilde{D}(XY)) &= \tilde{D}\{(\tilde{D}X)Y + X(\tilde{D}Y) + (\tilde{D}X)(\tilde{D}Y)\}\\ &= (\tilde{D}^2X)Y + (\tilde{D}X)(\tilde{D}Y) + (\tilde{D}^2X)\tilde{D}Y\\ &\quad + \tilde{D}X\tilde{D}Y + X\tilde{D}^2Y + \tilde{D}X\tilde{D}^2Y\\ &\quad + (\tilde{D}^2X)\tilde{D}Y + (\tilde{D}X)\tilde{D}^2Y + (\tilde{D}^2X)(\tilde{D}^2Y)\,,\end{aligned}$$

and that

$$\begin{aligned}\tilde{D}(XYZ) = \tilde{D}((XY)Z) &= \tilde{D}(XY)Z + XY\tilde{D}Z + \tilde{D}(XY)\tilde{D}Z\\ &= (\tilde{D}X)YZ + X(\tilde{D}Y)Z + (\tilde{D}X)(\tilde{D}Y)Z + XY\tilde{D}Z\\ &\quad + (\tilde{D}X)Y(\tilde{D}Z) + X(\tilde{D}Y)(\tilde{D}Z) + (\tilde{D}X)(\tilde{D}Y)(\tilde{D}Z)\,.\end{aligned}$$

In general we can write for $\mathbf{u} = (u_1, \ldots, u_k)$

$$\tilde{D}_{\mathbf{u}}\{X_1\cdots X_n\} = \sum_{\mathbf{u}_1,\ldots,\mathbf{u}_n\subset\mathbf{u},\mathbf{u}_1\cup\cdots\cup\mathbf{u}_n=\mathbf{u}} \tilde{D}_{\mathbf{u}_1}X_1\cdots\tilde{D}_{\mathbf{u}_n}X_n\,.$$

Note that the sum can include terms for the empty set. cf. [191] (3.3.4).

Using (1), we can confirm

$$\tilde{D}_u(XY) = (XY) \circ \varepsilon_u^+ - XY = (X \circ \varepsilon_u^+)(Y \circ \varepsilon_u^+) - XY \,.$$

From (2), we deduce

$$\begin{aligned} X \circ \varepsilon_{u_1}^+ \circ \varepsilon_{u_2}^+ &= (X \circ \varepsilon_{u_1}^+) \circ \varepsilon_{u_2}^+ = (\tilde{D}_{u_1} X + X) \circ \varepsilon_{u_2}^+ \\ &= \tilde{D}_{u_2}(\tilde{D}_{u_1} X + X) + \tilde{D}_{u_1} X + X \\ &= \tilde{D}_{u_2}\tilde{D}_{u_1} X + \tilde{D}_{u_2} X + \tilde{D}_{u_1} X + X \,. \end{aligned}$$

Using $\tilde{D}$, let us introduce norms

$$|F|_{k,0,p} := \left(|F|_{0,0,p}^p + \sum_{k'=1}^{k} E\left[\int_{A(1)^{k'}} \left| \frac{\tilde{D}_{\mathbf{u}}^{k'} F}{\gamma(\mathbf{u})} \right|^p \hat{M}(d\mathbf{u}) \right] \right)^{1/p} ,$$

where $k = 1, 2, \ldots$ and $p \geq 1$. For $\varphi \in C_0^\infty(U)$, we set

$$N(\varphi) = \int_U \varphi(t,z) N(dtdz), \quad \tilde{N}(\varphi) = \int_U \varphi(t,z)\tilde{N}(dtdz) \,.$$

Let $\mathcal{P}_2$ be the collection of random variables X written as

$$X = f(\tilde{N}(\varphi_1), \ldots, \tilde{N}(\varphi_n)) \,,$$

where $f(x_1, \ldots, x_n)$ is bounded $\mathcal{B}(\mathbf{R}^n)$-measurable, smooth in $(x_1, \ldots, x_n)$.

Let $\mathbf{D}_{k,0,p}$ be the completion of $\mathcal{P}_2$ with respect to the norm $|\cdot|_{k,0,p}$:

$$\mathbf{D}_{k,0,p} = \bar{\mathcal{P}}_2^{\,|\cdot|_{k,0,p}} \,.$$

Then, $\mathbf{D}_{k,0,p} \subset L^p(\Omega_2, \mathcal{F}_2, P_2)$. Operators $\tilde{D}_{\mathbf{u}}^j$, $\mathbf{u} = (u_1, \ldots, u_j)$, $j = 1, \ldots, l$ are extended to $\mathbf{D}_{k,0,p}$ in an obvious way.

The adjoint operator $\tilde{\delta}$ of $\tilde{D}_u$ in $L^2(\mathbf{T} \times \Omega_2)$ is introduced in Section 2.1.2. It is also called divergence operator.

We introduce another norm for a random field $V = V_u$ such that $V_{(t,0)} = 0$, and V_u is integrable with respect to $\tilde{N} = N - \hat{N}$ as follows:

$$\|V\|_{0,0,p}^{\sim} = E\left[\int_{A(1)} \left| \frac{V_u}{\gamma(u)} \right|^p \hat{M}(du) \right]^{1/p} \tag{3.5}$$

in case $k = 0$, and for $k \geq 1$

$$\|V\|_{k,0,p}^{\sim} = \left\{ \|V\|_{0,0,p}^{\sim p} + \sum_{k'=1}^{k} E\left[\int_{A(1)^{k'} \times A(1)} \left| \frac{\tilde{D}_{\mathbf{u}}^{k'} V_{\mathrm{u}}}{\gamma(\mathbf{u})\gamma(\mathrm{u})} \right|^p \bar{M}(d\mathbf{u}; d\mathrm{u}) \right] \right\}^{1/p} . \tag{3.6}$$

We denote $\chi_\rho = 1_{A(\rho)}$. We have the following inequality:

Theorem 3.4 (Theorem 3.2 in [95]). *Let k be a nonnegative integer and let $p \geq 2$ be an even number. There exists a positive constant $c = c(k,p)$ such that for any $s \geq 2$ and $0 < \rho < 1$, the inequality*

$$|\tilde{\delta}(V\chi_\rho)|_{k,0,p} \leq c\varphi(\rho)^{\frac{1}{2}-\frac{1}{2s}} \|V\chi_\rho\|^{\sim}_{k+p,0,(k+p)s} \tag{3.7}$$

holds for all integrable V.

Proof. Step 1 We first consider the case $k = 0$. Let $\rho > 0$ and $Z_{t,z}$ be the random field mentioned above. Set $\hat{Z}_{\mathbf{u}} = Z_{u_1} \circ \varepsilon^-_{u_1} \cdots Z_{u_p} \circ \varepsilon^-_{u_p}$ and

$$A(\rho) = \{u = (t,z) \in U;\ \ |z| \leq \rho\}\ .$$

Then, we have

$$\tilde{\delta}(Z_u\chi_\rho)^p = \int\limits_{A(\rho)^p} \hat{Z}_{\mathbf{u}}\tilde{N}(du_1)\cdots\tilde{N}(du_p)\ . \tag{3.8}$$

We divide the domain $A(\rho)^p$ of the integral into a disjoint union of subsets as follows. Let $\{I_1,\ldots,I_q\}$ $(q \leq p/2)$ be a family of disjoint subsets of $\{1,2,\ldots,p\}$ such that $|I_h| \geq 2$ for $h = 1,\ldots,q$, where $|I_h|$ denotes the cardinal number of the set I_h. We denote by Δ the set of all such $\{I_1,\ldots,I_q\}$'s and the empty set. We set

$$S_{\{I_1,\ldots,I_q\}} = \Big\{ (u_1,\ldots,u_p) \in A(\rho)^p \Big| \\ u_i = u_j \text{ holds if and only if } i,j \in I_h \text{ for some } I_h \in \{I_1,\ldots,I_q\} \Big\}\ .$$

Then, $S_{\{I_1,\ldots,I_q\}}$ are disjoint with each other and the union of these sets as $\{I_1,\ldots,I_q\}$ runs in Δ is equal to $A(\rho)^p$. Thus, integral (3.7) is written as a sum of the integrals whose domains are the sets $S_{\{I_1,\ldots,I_q\}}$. In the following, we fix a domain $S_{\{I_1,\ldots,I_q\}}$ and consider the integral on this set. Set $J = \{1,\ldots,p\} - \cup_{h=1}^q I_h$. We represent $\mathbf{u} \in S_{\{I_1,\ldots,I_q\}}$ as $(\mathbf{u}_{I_1},\ldots,\mathbf{u}_{I_q},u_J)$, where each $\mathbf{u}_{I_h} = (u_{I_h},\ldots,u_{I_h})$ is an $|I_h|$-dimensional vector with the same component and $u_J \in U^r$ with $r = |J|$. Then, $2q + r \leq p$. Furthermore, $(u_{I_1},\ldots,u_{I_q},u_J) \in S^{q+r}$, where $S^{q+r} = \{(u'_1,\ldots,u'_{q+r}) \in A(\rho)^{q+r} | u'_i \neq u'_j$ for any $i \neq j\}$.

Functions $\hat{Z}_{\mathbf{u}}$, $\mathbf{u} \in A(\rho)^p$, for example, can be regarded as functions of $(u_{I_1},\ldots,u_{I_q},u_J) \in S^{q+r}$. Furthermore, the $|I_h|$ fold product measure $\tilde{N}(du_{I_h})\cdots\tilde{N}(du_{I_h})$ coincides with $N(du_{I_h})$ on the set $S_{\{I_1,\ldots,I_q\}}$. Therefore, the above integral (3.8) is written as

$$\tilde{\delta}(Z_u\chi_\rho)^p = \sum_{\Delta}\int\limits_{S^{q+r}} \hat{Z}_{\mathbf{u}}N(du_{I_1})\cdots N(du_{I_q})\tilde{N}(du_J)\ . \tag{3.9}$$

The expectation of each term on the right-hand side is computed as

$$E\left[\int_{S^{q+r}} \hat{Z}_{\mathbf{u}} N(du_{I_1})\cdots N(du_{I_q})\tilde{N}(du_J)\right]$$
$$= E\left[\int_{S^{q+r}} \tilde{D}_{u_J}\hat{Z}_{\mathbf{u}} \circ \varepsilon^+_{u_{I_1}} \cdots \circ \varepsilon^+_{u_{I_q}} \hat{N}(du_{I_1})\cdots \hat{N}(du_{I_q})\hat{N}(du_J)\right] . \tag{3.10}$$

See Lemma 2.3 in Section 2.1.

Now, set $\hat{W}_u = \frac{Z_u \circ \varepsilon_u^-}{\gamma(u)}$ and $\hat{W}_{\mathbf{u}} = \hat{W}_{u_1} \cdots \hat{W}_{u_p}$. On the domain S^{q+r}, the inequality

$$|\tilde{D}_{\mathbf{u}_J}\hat{Z}_{\mathbf{u}}| = \gamma(u_1)\cdots\gamma(u_p)|\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}}|$$
$$\leq \rho^{p-2q-r}\gamma(u_{I_1})^2\cdots\gamma(u_{I_q})^2\gamma(\mathbf{u}_J)^2\frac{|\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}}|}{\gamma(\mathbf{u}_J)}$$

holds if $|\mathbf{u}| \leq \rho$. Therefore, (3.10) is dominated by

$$c_1\rho^{p-2q-r}E\left[\int_{S^{q+r}} \frac{|\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}} \circ \varepsilon^+_{\mathbf{v}}|}{\gamma(\mathbf{u}_J)}\hat{M}(du_{I_1})\cdots\hat{M}(du_{I_q})\hat{M}(d\mathbf{u}_J)\right] , \tag{3.11}$$

where $c_1 = \varphi(1)^{q+r}$ and $\mathbf{v} = \{u_{I_1}, \ldots, u_{I_q}\}$. We shall estimate $\frac{|\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}}\circ\varepsilon^+_{\mathbf{v}}|}{\gamma(\mathbf{u}_J)}$.

On S^{q+r}, $\hat{W}_{\mathbf{u}}$ is written as $\hat{W}_{\mathbf{u}} = \left(\prod_{i=1}^q \hat{W}_{\mathbf{u}_{I_i}}\right)\left(\prod_{i\in J}\hat{W}_{u_i}\right)$, where $\hat{W}_{\mathbf{u}_{I_i}} = (\hat{W}_{u_{I_i}})^{|I_i|}$. Note the formulas

$$\tilde{D}(XY) = (\tilde{D}X)Y + X(\tilde{D}Y) + (\tilde{D}X)(\tilde{D}Y) ,$$
$$X \circ \varepsilon^+_u = \tilde{D}_u X + X ,$$
$$X \circ \varepsilon^+_{u_1} \circ \varepsilon^+_{u_2} = \tilde{D}_{u_1}\tilde{D}_{u_2}X + \tilde{D}_{u_1}X + \tilde{D}_{u_2}X + X$$

and

$$\begin{aligned}
\tilde{D}_{u_1}(XY)\circ\varepsilon^+_{u_2} &= \tilde{D}_{u_2}\tilde{D}_{u_1}(XY) + \tilde{D}_{u_1}(XY)\\
&= \tilde{D}_{u_2}(\tilde{D}_{u_1}XY + X\tilde{D}_{u_1}Y + \tilde{D}_{u_1}X\tilde{D}_{u_1}Y)\\
&\quad + \tilde{D}_{u_1}XY + X\tilde{D}_{u_1}Y + \tilde{D}_{u_1}X\tilde{D}_{u_1}Y\\
&= (\tilde{D}_{u_2}\tilde{D}_{u_1}X)Y + \tilde{D}_{u_1}X\tilde{D}_{u_2}Y + \tilde{D}_{u_2}\tilde{D}_{u_1}X\tilde{D}_{u_2}Y + (\tilde{D}_{u_2}\tilde{D}_{u_1}Y)X\\
&\quad + \tilde{D}_{u_1}Y\tilde{D}_{u_2}X + \tilde{D}_{u_2}\tilde{D}_{u_1}Y\tilde{D}_{u_2}X\\
&\quad + (\tilde{D}_{u_2}\tilde{D}_{u_1}X)\tilde{D}_{u_1}Y + (\tilde{D}_{u_2}\tilde{D}_{u_1}Y)\tilde{D}_{u_1}X\\
&\quad + (\tilde{D}_{u_2}\tilde{D}_{u_1}X)(\tilde{D}_{u_2}\tilde{D}_{u_1}Y) + (\tilde{D}_{u_1}X)Y + X\tilde{D}_{u_1}Y + \tilde{D}_{u_1}X\tilde{D}_{u_1}Y .
\end{aligned}$$

Then, $|\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}} \circ \varepsilon^+_{\mathbf{v}}|$ is dominated by the sum of terms

$$|\tilde{D}_{\mathbf{v}_l}\tilde{D}_{\mathbf{u}_J}\hat{W}_{\mathbf{u}}|$$

for which $\mathbf{v}_l \subset \mathbf{v}$. Each of these terms is dominated by the sums of $\prod_{i=1}^p |\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} \hat{W}_{u_i}|$, where $\mathbf{u}_{k,i} \subset \mathbf{u}_J$, and $\mathbf{v}_{l,i} \subset \mathbf{v}$, and the summation (denoted by $\sum^*$) is taken for all nonnegative integers k_i, l_i satisfying $k_i \le r$, $l_i \le q$ and $k_1 + \cdots + k_p \ge r$ and $l_1 + \cdots + l_p \ge q$. Furthermore, we have the inequality $\gamma(\mathbf{u}_J) \ge \prod_{i=1}^p \gamma(\mathbf{u}_{k,i})\gamma(\mathbf{v}_{l,i})$ for $|\mathbf{u}_{k,i}| \le 1$, $|\mathbf{v}_{l,i}| \le 1$. Therefore, we have

$$\frac{|\tilde{D}_{\mathbf{u}_J} \hat{W}_{\mathbf{u}} \circ \varepsilon^+_{\mathbf{v}}|}{\gamma(\mathbf{u}_J)} \le \sum^* \prod_{i=1}^p \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} \hat{W}_{u_i}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})} \right| . \tag{3.12}$$

Then, (3.11) is dominated by the sum of the terms

$$\begin{aligned} &c_1 \rho^{p-2q-r} E\left[\int_{S^{q+r}} \left(\prod_{i=1}^p \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} \hat{W}_{u_i}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})} \right| \right) \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q}) \hat{M}(d\mathbf{u}_J) \right] \\ &\le c_2 \rho^{p-2q-r} \varphi(\rho)^{\frac{q+r}{s'}} \\ &\quad \times E\left[\int_{S^{q+r}} \left(\prod_{i=1}^p \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} \hat{W}_{u_i}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})} \right| \right)^s \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q}) \hat{M}(d\mathbf{u}_J) \right]^{1/s} , \end{aligned} \tag{3.13}$$

where $c_2 = c_1 \varphi(1)^{-\frac{q+r}{s}}$. Here, we applied Hölder's inequality with respect to $s, s' > 1$ where $\frac{1}{s} + \frac{1}{s'} = 1$. Since $\varphi(\rho) \ge c\rho^\alpha$ holds for $0 < \rho \le 1$, we have the inequality

$$\begin{aligned} \rho^{p-2q-r} \varphi(\rho)^{\frac{q+r}{s'}} &\le \rho^{\alpha(p/2-q-r/2)} \varphi(\rho)^{(q+r)/s'} \\ &\le c\varphi(\rho)^{p/2-q-r/2+(q+r)/s'} \le c\varphi(\rho)^{\frac{p}{2}(1-1/s)} . \end{aligned}$$

The last inequality follows since $\frac{p}{2} - q - \frac{r}{2} + \frac{q+r}{s'} = \frac{p}{2} - \frac{q}{s} + r(\frac{1}{2} - \frac{1}{s}) \ge \frac{p}{2} - \frac{p}{2s}$ for any $q \le p/2$ and $s \ge 2$. Therefore, (3.13) is dominated by the sum of

$$c_3 \varphi(\rho)^{\frac{p}{2}(1-1/s)} \prod_{i=1}^p E\left[\int_{A(\rho)^{l_i+k_i+1}} \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} \hat{W}_{u_i}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})} \right|^{ps} \bar{M}(d\mathbf{v}_{l,i}, d\mathbf{u}_{k,i}; du_i) \right]^{1/ps} .$$

Since $l_i + k_i \le p$ and since $\frac{\tilde{D}_{\mathbf{u}} Z_u}{\gamma(\mathbf{u})\gamma(u)} = \frac{\tilde{D}_{\mathbf{u}} \hat{W}_u}{\gamma(\mathbf{u})}$ a.s., the above is dominate by

$$c_4 \varphi(\rho)^{\frac{p}{2}(1-1/s)} (\|Z\chi_\rho\|^{\sim}_{p,0,ps})^p .$$

We have thus shown the inequality (3.7) in the case $k = 0$.

Step 2 Next, consider the case $k = 1$. Note the commutation relation of $\tilde{\delta}$ and $\tilde{D}$:

$$\tilde{D}\tilde{\delta}(Z\chi_\rho) = \tilde{\delta}(\tilde{D}Z\chi_\rho) + Z\chi_\rho .$$

(The commutation relation is explained simply by using the chaos decomposition. See [190] Proposition 4.1.4.) Then, we get

$$E\left[\int_U \left|\frac{\tilde{D}_u\tilde{\delta}(Z\chi_\rho)}{\gamma(u)}\right|^p \hat{M}(du)\right]$$
$$\leq 2^p\left\{\int_U E\left[\left|\tilde{\delta}\left(\frac{\tilde{D}_u Z\chi_\rho}{\gamma(u)}\right)\right|^p\right]\hat{M}(du) + E\left[\int_U \left|\frac{Z_u\chi_\rho}{\gamma(u)}\right|^p \hat{M}(du)\right]\right\} .$$

We can compute the integrand $E[|\tilde{\delta}(\frac{\tilde{D}_u Z\chi_\rho}{\gamma(u)})|^p]$ similarly as before. Then, the first term on the right-hand side is dominated by the sums of the terms

$$c\varphi(\rho)^{\frac{p+1}{2}(1-1/s)}\prod_{i=1}^p E\left[\int_{S^{q+r}\times A(\rho)}\left|\frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}}\tilde{D}^{k_i}_{\mathbf{u}_{k,i}}\left(\frac{\tilde{D}_u Z_{u_i}}{\gamma(u_i)\gamma(u)}\right)}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})}\right|^{(p+1)s}\right.$$
$$\left.\times\bar{M}(du_{I_1},\cdots,du_{I_q},d\mathbf{u}_J;du)\right]^{1/(p+1)s}$$
$$\leq c'\varphi(\rho)^{\frac{p+1}{2}(1-1/s)}(\|Z\chi_\rho\|^{\sim}_{p+1,0,(p+1)s})^p .$$

The second term is dominated by $\varphi(\rho)^{1/s'}(\|Z\chi_\rho\|^{\sim}_{0,0,ps})^p \leq (\varphi(\rho)^{1/2-1/2s}\|Z\chi_\rho\|^{\sim}_{0,0,ps})^p$. Therefore, the inequality (3.7) holds in the case $k = 1$.

Step 3 Next, consider the case $k = 2$. Again, by the commutation relation, we have

$$\tilde{D}_{u_2}\tilde{D}_{u_1}\tilde{\delta}(Z\chi_\rho) = \tilde{D}_{u_2}(\tilde{D}_{u_1}(\tilde{\delta}Z\chi_\rho)) = \tilde{D}_{u_2}(\tilde{\delta}(\tilde{D}_{u_1}Z\chi_\rho) + \tilde{D}_{u_1}Z_{u_1}\chi_\rho)$$
$$= \tilde{\delta}(\tilde{D}_{u_2}\tilde{D}_{u_1}Z\chi_\rho) + \tilde{D}_{u_1}Z_{u_2}\chi_\rho + \tilde{D}_{u_2}\tilde{D}_{u_1}Z_{u_1}\chi_\rho .$$

Therefore, the calculation is obtained as a combination of the above cases.

Repeating this argument, we get (3.7) for any k. The proof is complete. □

Remark 3.3. The norms $|F|_{k,0,p}$, $\|V\|^{\sim}_{k,0,p}$ above on the Poisson space are not unique candidates. We can also consider a BMO-type norm

$$|||F|||_{k,0,p} = \left(|F|^p_{0,0,p} + \frac{1}{k}\sum_{k'=1}^k E\left[\int_{A(1)^{k'}}\left|\tilde{D}^{k'}_{\mathbf{u}}F\right|^p\hat{M}(d\mathbf{u})\right]\right)^{1/p} ,$$

which is more weak than $|F|_{k,0,p}$. For a random field $V = V_u$, we can take

$$|||V|||^{\sim}_{k,0,p} = \left(\int_{A(1)}\frac{|||V_u|||^p_{k,0,p}}{\gamma(u)^p}\hat{M}(du)\right)^{1/p} .$$

Under these settings, we have another expression for Theorem 3.4.

3.3.3 The Wiener–Poisson space

We set $(\Omega, \mathcal{F})$ to be the product measure space (i.e. $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$), and denote $\omega = (\omega_1, \omega_2) \in \Omega$. We consider the product probability measure $P = P_1 \otimes P_2$ on $(\Omega, \mathcal{F})$. The completion of $\mathcal{F}$ with respect to P is denoted by $\bar{\mathcal{F}}$. Sub σ-fields $\mathcal{F}_1 \otimes \{\emptyset, \Omega_2\}$ and $\{\emptyset, \Omega_1\} \otimes \mathcal{F}_2$ are identified with $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The space $(\Omega, \mathcal{F}, P)$ is called the *Wiener–Poisson space*. We write $W(t)(\omega) = W(t)(\omega_1) = \omega_1(t)$, $N(dtdz)(\omega) = N(dtdz)(\omega_2) = \omega_2(dtdz)$.

We define $\mathbf{K} := \mathbf{K}_1 \oplus \mathbf{K}_2$. Then, $\mathbf{K}$ is a Hilbert space with the inner product

$$(h_1, h_2) = (f_1, f_2)_{\mathbf{K}_1} + (g_1, g_2)_{\mathbf{K}_2}$$

where $h_i = f_i \oplus g_i \in \mathbf{K}$. We regard the operators D_t and $\tilde{D}_u$ as $D_t \oplus id$ and $id \oplus \tilde{D}_u$, respectively. Notice that the operators $\varepsilon_u^{\pm}$ are also extended to Ω by setting $\varepsilon_u^{\pm}(\omega_1, \omega_2) = (\omega_1, \varepsilon_u^{\pm}\omega_2)$. Let us introduce the operator $\mathcal{D}_{(t,u)}$ as

$$\mathcal{D}_{(t,u)} := D_t \oplus \tilde{D}_u \ : \ L^2(\Omega) \to L^2(\Omega\, ;\, \mathbf{K})\, ;$$

for $X = \sum_{i=1}^{k} X_1^{(i)} X_2^{(i)} \in \mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$ where $X_1^{(i)} \in L^2(\Omega_1)$ and $X_2^{(i)} \in L^2(\Omega_2)$, we have

$$\mathcal{D}_{(t,u)} X = \sum_{i=k} \left(D_t X_1^{(i)} X_2^{(i)} \oplus X_1^{(i)} \tilde{D}_u X_2^{(i)} \right)$$

if $X_1^{(i)}$ and $X_2^{(i)}$ are in the domain of D and of $\tilde{D}$, respectively. The operator $\mathcal{D}_{(t,u)}$ is a closed and unbounded operator.

For $\mathbf{t} = (t_1, \ldots, t_k)$, $\mathbf{u} = (u_1, \ldots, u_k)$, let

$$\mathcal{D}_{(\mathbf{t},\mathbf{u})} = \mathcal{D}_{(t_k,u_k)} \cdots \mathcal{D}_{(t_2,u_2)} \mathcal{D}_{(t_1,u_1)}\ ,$$

where each component on the right-hand side is given by the above.

Let $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$. The spaces $\mathcal{P}_1, \mathcal{P}_2$ identify with $\mathcal{P}_1 \otimes 1$, $1 \otimes \mathcal{P}_2$, respectively. For $p \geq 2$,

$$\mathbf{D}_{k,l,p} = \bar{\mathcal{P}}^{|\cdot|_{k,l,p}}\ ,$$

where

$$|F|_{k,l,p} := \left(|F|_{0,l,p}^p + \sum_{k'=1}^{k} \sum_{l'=0}^{l} E\left[\int_{A(1)^{k'}} \left(\int_{\mathbf{T}^{l'}} \left| \frac{D_{\mathbf{t}}^{l'} \tilde{D}_{\mathbf{u}}^{k'} F}{\gamma(\mathbf{u})} \right|^2 d\mathbf{t} \right)^{p/2} \hat{M}(d\mathbf{u}) \right] \right)^{1/p}.$$

The operator $\mathcal{D}_{(t,u)}$ is continuously extended to $\mathbf{D}_{k,l,p}$. Let

$$\mathbf{D}_\infty = \cap_{k,l=0}^{\infty} \cap_{p \geq 2} \mathbf{D}_{k,l,p}\ .$$

We regard the operators δ and $\tilde{\delta}$ as $\delta \otimes id$ and $id \otimes \tilde{\delta}$. We regard the spaces $\mathrm{Dom}\,(\delta)$ and $\mathrm{Dom}\,(\tilde{\delta})$ as $\mathrm{Dom}\,(\delta) \otimes L^2(\Omega_2)$ and $L^2(\Omega_1) \otimes \mathrm{Dom}\,(\tilde{\delta})$, respectively. We define the operator $\bar{\delta}$ by

$$\bar{\delta} = \delta \oplus \tilde{\delta}\ .$$

Then, duality equalities (1.35) and (1.11) yield:

Proposition 3.3. *For $Z = U_t \oplus V_u \in L^2(\Omega\,;\mathbf{K})$ such that $V_{(t,0)} = 0$, suppose that $U_t \in$ Dom (δ) and $V_u \in$ Dom $(\tilde{\delta})$. Then, for any $H \in \mathbf{D}_{1,1,2}$, we have*

$$E[H\bar{\delta}(Z)] = E\left[\int D_t H\, U_t dt + \int \tilde{D}_u H\, V_u\, \hat{N}(du)\right] . \tag{3.14}$$

We introduce norms $\|\ \|_{k,l,p}$, $\|\ \|^{\sim}_{k,l,p}$. For $U_t \in L^2(\Omega;\mathbf{K}_1)$, we define $\|U\|_{k,l,p}$ to be

$$\|U\|_{k,l,p} := \left(\|U\|^p_{0,l,p} + \sum_{k'=1}^{k}\sum_{l'=0}^{l} E\left[\int_{A(1)^{k'}} \left(\int_{\mathbf{T}^{l'+1}} \left|\frac{D^{l'}_{\mathbf{t}}\tilde{D}^{k'}_{\mathbf{u}}U_t}{\gamma(\mathbf{u})}\right|^2 d\mathbf{t}\right)^{p/2} \hat{M}(d\mathbf{u})\right]\right)^{1/p} . \tag{3.15}$$

For $V_u \in L^2(\Omega;\mathbf{K}_2)$ such that $V_{(t,0)} = 0$, we define $\|V\|^{\sim}_{k,l,p}$ to be

$$\|V\|^{\sim}_{0,l,p} = \sum_{l'=0}^{l} E\left[\int_{A(1)}\left[\int\left|\frac{D^{l'}_{\mathbf{t}}V_u}{\gamma(u)}\right|^2 d\mathbf{t}\right]^{\frac{p}{2}}\hat{M}(du)\right]^{1/p} \tag{3.16}$$

in case $k = 0$, and for $k \geq 1$,

$$\|V\|^{\sim}_{k,l,p} = \left\{\|V\|^{\sim p}_{0,0,p} + \sum_{k'=1}^{k}\sum_{l'=0}^{l} E\left[\int_{A(1)^{k'}\times A(1)}\left[\int\left|\frac{D^{l'}_{\mathbf{t}}\tilde{D}^{k'}_{\mathbf{u}}V_{\mathrm{u}}}{\gamma(\mathbf{u})\gamma(\mathrm{u})}\right|^2 d\mathbf{t}\right]^{\frac{p}{2}}\bar{M}(d\mathbf{u};d\mathrm{u})\right]\right\}^{1/p} .$$

For $Z = U_t \oplus V_u$, we define the norm $\|Z\|_{k,l,p}$ to be

$$\|Z\|_{k,l,p} := \left(\|U\|^2_{k,l,p} + \|V\|^{\sim}_{k,l,p}{}^2\right)^{\frac{1}{2}} .$$

For $n \geq 0$, $p \geq 2$,

$$\mathbf{D}^{\sim(1)}_{k,l,p} = \left\{U_t \in L^2(\Omega;\mathbf{K}_1)\,;\ \|U\|_{k,l,p} < +\infty\right\} ,$$
$$\mathbf{D}^{\sim(2)}_{k,l,p} = \left\{V_u \in L^2(\Omega;\mathbf{K}_2)\,;\ V_{(t,0)} = 0,\ \|V\|^{\sim}_{k,l,p} < +\infty\right\} ,$$
$$\mathbf{D}^{\sim}_{k,l,p} = \mathbf{D}^{\sim(1)}_{k,l,p} \oplus \mathbf{D}^{\sim(2)}_{k,l,p} ,$$

and

$$\mathbf{D}^{\sim}_{\infty} = \cap_{k,l=0}^{\infty}\cap_{p\geq 2}\mathbf{D}^{\sim}_{k,l,p} .$$

The next lemma verifies the continuity property of adjoint operators.

Lemma 3.4. *Let k, l be nonnegative integers and let $p \geq 2$ be an even number.*

(i) *Suppose that $U = \{U_t\} \in \mathbf{D}^{\sim(1)}_{k.l,p}$. Then, there exists a positive constant $c = c(k,l,p)$ such that*

$$|\delta(U)|_{k,l,p} \leq c\|U\|_{k,l+1,p} . \tag{3.17}$$

(ii) *Suppose that* $V = \{V_u\} \in \mathbf{D}^{\sim (2)}_{k.l,p}$. *Then, there exists a positive constant* $c = c(k, l, p)$ *such that for any* $s \geq 2$ *and* $0 < \rho < 1$, *the inequality*

$$|\tilde{\delta}(V\chi_\rho)|_{k,l,p} \leq c\varphi(\rho)^{\frac{1}{2}-\frac{1}{2s}} \|V\chi_\rho\|^{\sim}_{k+p,l,(k+p)s} \,. \tag{3.18}$$

Proof for (i). If $Z_t = G\tilde{Z}_t$, where G is $\mathcal{F}_2$-measurable and $\tilde{Z}_t$ is $\mathcal{F}_1$-measurable. Then Meyer's type inequality implies

$$|\delta(G\tilde{Z})|^p_{0,l-1,p} \leq c^p \|G\tilde{Z}\|^p_{0,l,p}, \quad a.s.\ \omega_2\ \ (P_2)\,. \tag{3.19}$$

Taking the expectation with respect to P_2, we obtain the same inequality for $Z = G\tilde{Z}$. Observe that any $\{Z_t\}$ can be approximated by a sequence $\{Z^n_t\}$ which are written as linear sums of $G\hat{Z}_t$ mentioned above. Then the inequality holds for any $\mathcal{F}$-measurable process Z_t. Note that $\tilde{D}$ and δ are commutative. Repeating this argument to $\{\tilde{D}^k Z\}$, we can show that there exists a positive constant c such that

$$|\delta(Z)|_{k,l,p} \leq c\|Z\|_{k,l+1,p}\,, \tag{3.20}$$

holds. □

Proof for (ii). In view of Theorem 3.4, the assertion for the case $l \geq 1$ for each k remains to be shown.

Step 1 First, suppose $k = 0$ and $l = 1$. Since D and $\tilde{\delta}$ are commutative, we have $D_t\tilde{\delta}(Z) = \tilde{\delta}(D_t Z)$. Therefore, we have

$$E\left[\left(\int_{\mathbf{T}} |D_t\tilde{\delta}(Z)|^2 dt\right)^{p/2}\right] = \int \cdots \int_{\mathbf{T}^{p/2}} E[\tilde{\delta}(D_{t_1}Z)^2 \cdots \tilde{\delta}(D_{t_{p/2}}Z)^2] dt_1 \cdots dt_{p/2}\,. \tag{3.21}$$

The integrand $\tilde{\delta}(D_{t_1}Z)^2 \cdots \tilde{\delta}(D_{t_{p/2}}Z)^2$ can be written similarly as (3.9). Set $D_t\hat{W}_u = \frac{D_t Z_u}{\gamma(u)} \circ \varepsilon^-_u$. Then, $E[\tilde{\delta}(D_{t_1}Z)^2 \cdots \tilde{\delta}(D_{t_{p/2}}Z)^2]$ is dominated by sums of the following terms:

$$c_1\rho^{p-2q-r} \int_{S^{q+r}} E\left[\prod_{i=1}^{p/2} \left|\frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} D_{t_i}\hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})}\right| \right.$$
$$\left. \times \left|\frac{\tilde{D}^{l'_i}_{\mathbf{v}'_{l,i}} \tilde{D}^{k'_i}_{\mathbf{u}'_{k,i}} D_{t_i}\hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})}\right|\right] \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q})\hat{M}(d\mathbf{u}_J)\,, \tag{3.22}$$

where $\mathbf{v}_{l,i}, \mathbf{v}'_{l,i} \subset \mathbf{v}$ and $\mathbf{u}_{k,i}, \mathbf{u}'_{k,i} \subset \mathbf{u}_J$. Integrate the above by $dt_1 \cdots dt_{p/2}$ over $\mathbf{T}^{p/2}$, and use Fubini's theorem and then the Schwarz inequality. Then, it is dominated by

$$c'_1\rho^{p-2q-r} \int_{S^{q+r}} E\left[\prod_{i=1}^{p/2} \left(\int_{\mathbf{T}} \left|\frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} D_{t_i}\hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})}\right|^2 dt_i\right)^{1/2}\right.$$
$$\left. \times \left(\int_{\mathbf{T}} \left|\frac{\tilde{D}^{l'_i}_{\mathbf{v}'_{l,i}} \tilde{D}^{k'_i}_{\mathbf{u}'_{k,i}} D_{t_i}\hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})}\right|^2 dt_i\right)^{1/2}\right] \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q})\hat{M}(d\mathbf{u}_J)$$

$$\leq c_2' \varphi(\rho)^{p/2(1-1/s)} \prod_{i=1}^{p/2} \left(\int_{S^{q+r}} E\left[\left(\int_{\mathbf{T}} \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} D_{t_i} \hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})} \right|^2 dt_i \right)^{\frac{ps}{2}} \right] \right.$$

$$\left. \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q}) \hat{M}(d\mathbf{u}_J) \right)^{\frac{1}{ps}}$$

$$\times \left(\int_{S^{q+r}} E\left[\left(\int_{\mathbf{T}} \left| \frac{\tilde{D}^{l_i'}_{\mathbf{v}'_{l,i}} \tilde{D}^{k_i'}_{\mathbf{u}'_{k,i}} D_{t_i} \hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})} \right|^2 dt_i \right)^{ps/2} \right] \hat{M}(du_{I_1}) \cdots \hat{M}(du_{I_q}) \hat{M}(d\mathbf{u}_J) \right)^{1/ps}$$

$$\leq c_3' (\varphi(\rho)^{1/2-1/2s} \| Z\chi_\rho \|^{\sim}_{p,1,ps})^p \,. \tag{3.23}$$

Therefore, the inequality (3.18) holds for $k = 0$ and $l = 1$.

Step 2 Next, for the case $k = 1$, $l = 1$. Using $D_t\tilde{\delta}(Z) = \tilde{\delta}(D_t Z)$, $\tilde{D}\tilde{\delta}(Z\chi_\rho) = \tilde{\delta}(\tilde{D}Z\chi_\rho) + Z\chi_\rho$, we can write

$$D_t \tilde{D}\tilde{\delta}(Z\chi_\rho) = D_t(\tilde{\delta}(\tilde{D}Z\chi_\rho) + Z\chi_\rho) = \tilde{\delta}(D_t\tilde{D}Z\chi_\rho) + D_t Z\chi_\rho \,. \tag{3.24}$$

Hence,

$$\int_{\mathbf{T}} \left| \frac{D_t \tilde{D}_u \tilde{\delta}(Z\chi_\rho)}{\gamma(u)} \right|^2 dt \leq 4 \left(\int_{\mathbf{T}} \left| \tilde{\delta}\left(\frac{D_t\tilde{D}_u Z_u \chi_\rho}{\gamma(u)} \right) \right|^2 dt + \int_{\mathbf{T}} \left| \frac{D_t Z_u \chi_\rho}{\gamma(u)} \right|^2 \right) .$$

In order to estimate the term

$$E\left[\int_{A(\rho)} \left(\int_{\mathbf{T}} \left| \tilde{\delta}\left(\frac{D_t\tilde{D}_u Z_u \chi_\rho}{\gamma(u)} \right) \right|^2 dt \right)^{p/2} \hat{M}(du) \right] ,$$

we prepare the following.

We choose a permutation σ of $\{1, \ldots, p\}$ and fix it, and write $(u_{\sigma(1)}, \ldots, u_{\sigma(p)})$ by $(u_1, \ldots, u_p)$, where $u_i \in A(\rho)$. We divide $(u_1, \ldots, u_p)$ into 2 cosets, and denote it by $(u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}, \ldots, u_{p/2,1}, u_{p/2,2})$. We put $\Delta_2^{(i)} = (u_{i,1}, u_{i,2})$, $i = 1, \ldots, p/2$ and

$$\Delta_2 = \{\Delta_2^{(1)}, \ldots, \Delta_2^{(p/2)}\} .$$

Let q_2 denotes the number k such that $u_{k,1} = u_{k,2}$ in Δ_2. Furthermore, let $J_2 = \{u_1, \ldots, u_p\} \setminus \cup_i \{\Delta_2^{(i)}; u_{i,1} = u_{i,2}\}$, and $r_2 = p - 2q_2$, coinciding with the cardinal of the set J_2.

Let

$$S_2^{q_2+r_2} = \{(u_1, \ldots, u_{q_2+r_2}) ;$$
$$\text{if } u_l = u_{k_l, i_l} \text{ then } u_{k,1} \neq u_{k,2} \text{ for all } k = k_l, l = 1, \ldots, q_2 + r_2\} .$$

For the first term on the right-hand side of (3.24), we begin by estimating

$$\int_{A(\rho)} \left(\int_{\mathbf{T}} \left| \frac{\tilde{\delta}(D_t\tilde{D}_{u'} Z\chi_\rho)}{\gamma(u')} \right|^2 dt \right)^{p/2} \hat{M}(du') , \tag{3.25}$$

where

$$|\tilde{\delta}(D_t\tilde{D}_{u'}Z\chi_\rho)|^2 = \tilde{\delta}^2(D_t\tilde{D}_{u'}Z\chi_\rho) = \int_{\Delta_2^{(i)}} (D_t\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1,u_2)} M(du_1; du_2)\,.$$

Here,

$$(D_t\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1,u_2)} = (D_t\tilde{D}_{u'}Z_{u_1}\chi_\rho \circ \varepsilon^-_{u_1})(D_t\tilde{D}_{u'}Z_{u_2}\chi_\rho \circ \varepsilon^-_{u_2})\,,$$

and $M(du_1; du_2) = \tilde{N}(du_1)\tilde{N}(du_2)$ on $\{(u_1, u_2); u_1 \neq u_2\}$ and $M(du_1; du_2) = N(du_1)$ on $\{(u_1, u_2); u_1 = u_2\}$. Hence, (3.25) is equal to the sum of

$$\int_{A(\rho)} \Pi_{i=1}^{p/2} \left(\int_{\mathbf{T}} \int_{\Delta_2^{(i)}} \left| \frac{(D_t\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1,u_2)}}{\gamma(u')^2} \right| M(du_1; du_2)dt \right) \hat{M}(du') \tag{3.26}$$

with respect to the choice of Δ_2. We shall calculate

$$E\left[\int_{A(\rho)} \left(\int_{\mathbf{T}} \int_{\Delta_2^{(i)}} \left| \frac{(D_t\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1,u_2)}}{\gamma(u')^2} \right| M(du_1; du_2)dt \right)^{p/2} \hat{M}(du') \right]. \tag{3.27}$$

We write $\hat{W}_{u_i} = \frac{Z_{u_i}\circ\varepsilon^-_{u_i}}{\gamma(u_i)}$, $i = 1, 2$. Note that (3.27) can be written as

$$\int_{\Delta_2^{(1)}} E\left[\left(\int_{\mathbf{T}} \int_{A(\rho)^2} \left| \frac{(D_{t_1}\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1^{(1)},u_2^{(1)})}}{\gamma(u')^2} \right| M(du_1^{(1)}; du_2^{(1)})dt_1 \right) \cdots \right.$$
$$\left. \cdots \left(\int_{\mathbf{T}} \int_{\Delta_2^{(p/2)}} \left| \frac{(D_{t_{p/2}}\tilde{D}_{u'}Z\chi_\rho\widehat{)}_{(u_1^{(p/2)},u_2^{(p/2)})}}{\gamma(u')^2} \right| M(du_1^{(p/2)}; du_2^{(p/2)})dt_{p/2} \right) \right] \hat{M}(du')\,.$$

Note that (3.10) holds. In view of (3.22), the above is bounded by

$$c_1 \int_{\mathbf{T}^{p/2}} \int_{S_2^{q_2+r_2}\times A(\rho)} E\left[\prod_{i=1}^{p/2} \left| \frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}} \tilde{D}^{k_i}_{\mathbf{u}_{k,i}} D_{t_i}\tilde{D}_{u'}\hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})\gamma(u')} \right| \left| \frac{\tilde{D}^{l'_i}_{\mathbf{v}'_{l,i}} \tilde{D}^{k'_i}_{\mathbf{u}'_{k,i}} D_{t_i}\tilde{D}_{u'}\hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})\gamma(u')} \right| \right]$$
$$\hat{M}(du_1)\cdots\hat{M}(du_{q_2+r_2})\hat{M}(du')dt_1\cdots dt_{p/2}\,, \tag{3.28}$$

where $\mathbf{v}_{l,i}, \mathbf{v}'_{l,i} \subset S_2^{q_2+r_2} \setminus J_2$ and $\mathbf{u}_{k,i}, \mathbf{u}'_{k,i} \subset J_2$.

Calculating as above, (3.28) is dominated by

$$c_1'\rho^{p_2-2q_2-r}\int_{S^{q_2+r_2}\times A(\rho)} E[\prod_{i=1}^{p/2}\left(\int_{\mathbf{T}}\left|\frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}}\tilde{D}^{k_i}_{\mathbf{u}_{k,i}}D_{t_i}\tilde{D}_{u'}\hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})\gamma(u')}\right|^2 dt_i\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbf{T}}\left|\frac{\tilde{D}^{l_i'}_{\mathbf{v}'_{l,i}}\tilde{D}^{k_i'}_{\mathbf{u}'_{k,i}}D_{t_i}\tilde{D}_{u'}\hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})\gamma(u')}\right|^2 dt_i\right)^{\frac{1}{2}}]\bar{M}(du_1\cdots du_{q_2+r_2};du')\,. \quad (3.29)$$

We use Hölder's inequality, and (3.29) is bounded by

$$c_1''\varphi(\rho)^{\frac{p}{2}(1-\frac{1}{s})}\prod_{i=1}^{p/2}(\|Z\chi_\rho\|^{\sim 2}_{1+p,1,ps}) = c_1''\left(\varphi(\rho)^{\frac{1}{2}-\frac{1}{2s}}\|Z\chi_\rho\|^{\sim 2}_{1+p,1,ps}\right)^p\,.$$

Summing up with respect to the permutations σ and divided by ${}_pC_2$, we have the upper bound with respect to the first term.

For the second term, we see

$$E\left[\int_{A(\rho)}\left(\int_{\mathbf{T}}\left|\frac{D_tZ_u\chi_\rho}{\gamma(u)}\right|^2 dt\right)^{p/2}\hat{M}(du)\right]$$
$$\le \varphi(\rho)^{1/s'}(\|Z\chi_\rho\|^{\sim}_{0,1,ps})^p \le \left(\varphi(\rho)^{\frac{1}{2}(1-\frac{1}{s})}\|Z\chi_\rho\|^{\sim}_{1,1,ps}\right)^p$$

as above.

Step 3 Next, consider the case $k=0$, $l=2$.

Since $D_{t_1}D_{t_2}\tilde{\delta}(Z)=\tilde{\delta}(D_{t_1}D_{t_2}Z)$,

$$E\left[\left(\int_{\mathbf{T}^2}\left|D_{t_1}D_{t_2}\tilde{\delta}(Z\chi_\rho)\right|^2 dt_1dt_2\right)^{p/2}\right]$$
$$=\int_{\mathbf{T}^2}\cdots\int_{\mathbf{T}^2}E\left[\tilde{\delta}^2(D_{t_1}D_{t_2}Z\chi_\rho)\cdots\tilde{\delta}^2(D_{t_{p/2-1}}D_{t_{p/2}}Z\chi_\rho)\right]dt_1\cdots dt_p\,. \quad (3.30)$$

The integrand $\tilde{\delta}^2(D_{t_1}D_{t_2}Z\chi_\rho)\cdots\tilde{\delta}^2(D_{t_{p/2-1}}D_{t_{p/2}}Z\chi_\rho)$ can be written similarly as (3.9). Then, similarly to the case $k=0$, $l=1$, (3.30) can be bounded from above by

$$c_3\rho^{p-2q-r}\int_{S^{q+r}}E\left[\prod_{i=1}^{p/2}\left(\int_{\mathbf{T}^2}\left|\frac{\tilde{D}^{l_i}_{\mathbf{v}_{l,i}}\tilde{D}^{k_i}_{\mathbf{u}_{k,i}}D_{t_{i,1}}D_{t_{i,2}}\hat{W}_{u_{2i-1}}}{\gamma(\mathbf{v}_{l,i})\gamma(\mathbf{u}_{k,i})}\right|^2 dt_{i,1}dt_{i,2}\right)^{1/2}\right.$$
$$\left.\left(\int_{\mathbf{T}^2}\left|\frac{\tilde{D}^{l_i'}_{\mathbf{v}'_{l,i}}\tilde{D}^{k_i'}_{\mathbf{u}'_{k,i}}D_{t'_{i,1}}D_{t'_{i,2}}\hat{W}_{u_{2i}}}{\gamma(\mathbf{v}'_{l,i})\gamma(\mathbf{u}'_{k,i})}\right|^2 dt'_{i,1}dt'_{i,2}\right)^{1/2}\right]\hat{M}(du_{I_1})\cdots\hat{M}(du_{I_q})\hat{M}(d\mathbf{u}_J)$$
$$\le c_3'(\varphi(\rho)^{1/2-1/2s}\|Z\chi_\rho\|^{\sim}_{p,2,ps})^p\,.$$

Therefore, we have the assertion for $k = 0$, $l = 2$.

Step 4 Next, consider the case $k = 1$, $l = 2$. By the commutation relation, we have

$$D_{t_2}D_{t_1}\tilde{D}\tilde{\delta}(Z\chi_\rho) = D_{t_2}D_{t_1}(\tilde{\delta}(\tilde{D}Z\chi_\rho) + Z\chi_\rho) = \tilde{\delta}(D_{t_2}D_{t_1}\tilde{D}Z\chi_\rho) + D_{t_2}D_{t_1}Z\chi_\rho \ .$$

Hence, the calculation is similar to the above case.

Step 5 In general, $D_{\mathbf{t}}^l\tilde{D}_{\mathbf{u}}^k Z\chi_\rho$ can be written as a sum of terms of the form

1. $\tilde{\delta}(D_{\mathbf{t}}^l\tilde{D}_{\mathbf{u}}^{k'} Z\chi_\rho)$, $k' \le k$, and
2. $D_{\mathbf{t}}^l\tilde{D}_{\mathbf{u}}^{k'} Z\chi_\rho$, $k' \le k$

by using the commutation relation. Terms of the form (2) can be estimated directly by the norm $\|Z\|_{k+p,l,(k+p)s}$. For the term of the form (1),

$$\int_{\mathbf{T}^l} \left|\tilde{\delta}(D_{\mathbf{t}}^l\tilde{D}_{\mathbf{u}}^{k'} Z\chi_\rho)\right|^2 dt_1 \cdots dt_l$$

can be calculated as in Step 2, and taking the expectation in terms of $\tilde{\delta}$, we have the desired estimate. The proof is complete.

End of proof of Lemma 3.4. □

From Lemma 3.4, we immediately have

Theorem 3.5. *Let* $0 < \rho < 1$. *For* $U = \{U_t\} \in \mathbf{D}_{k.l,p}^{\sim(1)}$, *and* $V = \{V_u\} \in \mathbf{D}_{k.l,p}^{\sim(2)}$, *we set* $Z_\rho = U \oplus V\chi_\rho$. *For any* $k, l \in \mathbf{N}$, *and any even number* $p \ge 2$, *there exists a positive constant* $C = C(k, l, p)$ *such that for any* $s \ge 2$,

$$|\bar{\delta}(Z_\rho)|_{k,l,p} \le C\left(\|U\|_{k,l+1,p} + \varphi(\rho)^{\frac{1}{2}-\frac{1}{2s}}\|V\chi_\rho\|^{\sim}_{k+p,l,(k+p)s}\right) \tag{3.31}$$

holds.

We denote by $\mathbf{D}'_{k,l,p}$ the analytic adjoint space of $\mathbf{D}_{k,l,p}$. That is, the normed space of continuous linear forms on $\mathbf{D}_{k,l,p}$ with the adjoint norm $|\cdot|'_{k,l,p}$ given by

$$|\Phi|'_{k,l,p} = \sup_{|G|_{k,l,p}=1} |\langle\Phi, G\rangle| \ .$$

Here, $\langle\cdot,\cdot\rangle$ denotes the paring representing the bounded liner functional $\langle\Phi,\cdot\rangle$. We put

$$\mathbf{D}'_{\infty,l,p} = \cup_{k=0}^{\infty}\mathbf{D}'_{k,l,p},\ \mathbf{D}'_{k,\infty,p} = \cup_{l=0}^{\infty}\mathbf{D}'_{k,l,p},\ \mathbf{D}'_{\infty} = \cup_{k,l=0}^{\infty}\cup_{p\ge 2}\mathbf{D}'_{k,l,p} \ .$$

Spaces

$$\mathbf{D}^{\sim}_{k,l,p}{}',\ \mathbf{D}^{\sim}_{\infty,l,p}{}' = \cup_{k=0}^{\infty}\mathbf{D}^{\sim}_{k,l,p}{}',\ \mathbf{D}^{\sim}_{k,\infty,p}{}' = \cup_{l=0}^{\infty}\mathbf{D}^{\sim}_{k,l,p}{}',\ \mathbf{D}^{\sim}_{\infty}{}' = \cup_{k,l=0}^{\infty}\cup_{p\ge 2}\mathbf{D}^{\sim}_{k,l,p}{}'$$

are defined similarly.

We can regard $F \in \mathbf{D}_{k,l,p}$ as a linear continuous functional:

$$G \mapsto E[FG]$$

whenever the right-hand side is finite. The norm of this linear continuous functional induced by F is

$$|F|'_{k,l,p} = \sup_{|G|_{k,l,p}=1} |E[GF]| \,.$$

Here, E denotes the expectation with respect to P.

The following condition plays a crucial role in verifying that F has a smooth density.

Definition 3.2 (The condition *(ND)*). We say that a random variable F satisfies the condition *(ND)* if for all $p \geq 1$, $k \geq 0$, there exists $\beta \in (\frac{\alpha}{2}, 1]$ such that

$$\sup_{\rho\in(0,1)} \sup_{\substack{v\in\mathbf{R}^d,\\ |v|=1}} \sup_{\tau\in A^k(\rho)} E\left[\left|\left((v, \Sigma v) + \varphi(\rho)^{-1} \int_{A(\rho)} |(v, \tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F|\leq\rho^\beta\}} \hat{N}(du)\right)^{-1} \circ \varepsilon_\tau^+\right|^p\right] < +\infty \,, \quad (3.32)$$

where Σ is the Malliavin's covariance matrix $\Sigma = (\Sigma_{i,j})$, where $\Sigma_{i,j} = \int_{\mathbf{T}} (D_t F_i, D_t F_j) dt$.

The part $\left((v, \Sigma v) + \varphi(\rho)^{-1} \int_{A(\rho)} \cdots \hat{N}(du)\right)$ is regarded as a Malliavin matrix on the Wiener–Poisson space. We remark that we require the stability of the Malliavin matrix under the secondary perturbation by ε_τ^+ of the Malliavin matrix.

Intuitively, this condition implies that in each direction, at least one of the two derivatives is non-degenerate in probability. The part $(\cdots)^{-1}$ is what we can regard as the inverse of the Malliavin matrix composed with diffusion and small jumps in the Wiener–Poisson space. In the condition *(ND)* above, we require the finiteness of it in the L^p-sense under additional outer perturbations by $\tau \in A(\rho)^k$ of any order $k = 1, 2, \ldots$. The reader may remember the outer perturbation of the Markov chain introduced in Section 2.2.2 by using the functions (φ_n).

The condition above will be used in the composition of F with distributions in Section 3.5.

We finally remark that under the condition *(ND)*, we can take an even number k_0 such that $\frac{1}{k_0} < \frac{2\beta}{\alpha} - 1$. We put

$$q_0 = 1 - \frac{\alpha}{2\beta}\left(1 + \frac{1}{k_0}\right) > 0 \,. \quad (3.33)$$

This quantity will play a crucial role in Sections 3.5, 3.6.

3.4 Relation with the Malliavin operator

On the Wiener space, there are two approaches which lead to the integration-by-parts formula: one is using the (Malliavin–Shigekawa) derivative operator D_t and the other is using Malliavin operators due to Stroock and Malliavin. One can proceed in an analogous way on the Poisson space by using the Malliavin operator. We remark that we then require that the Lévy measure μ has a smooth density function: $\mu(dz) = g(z)dz$.

Let $\mathcal{R}$ denote the set of all functions of form $\Phi = f(N(\varphi_1), \dots, N(\varphi_n))$, where $f \in C^2(\mathbf{R}^n)$ is having derivatives of polynomial growth, and φ_i are test functions on U.

A linear operator L on $\mathcal{R} \subset \cap_{p<+\infty} L^p$ is called a *Malliavin operator* if it satisfies the following:

(1)

$$E[\phi L\psi] = E[\psi L\phi], \quad \phi, \psi \in \mathcal{R} .$$

(2) the bilinear operator defined by

$$\Gamma(\phi, \psi) = L(\phi\psi) - \phi L\psi - \psi L\phi$$

is nonnegative. (This operator is called a *carré-du-champ* operator.)

(3) For $\Phi = (\phi^1, \dots, \phi^n) \in \mathcal{R}^n$ and $F \in C^2(\mathbf{R}^n)$,

$$L(F \circ \Phi) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F \circ \Phi \cdot L(\phi_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} F(\Phi)\Gamma(\phi^i, \phi^j) .$$

More precisely, L is given on the Poisson space for $\Phi = f(N(\varphi_1), \dots, N(\varphi_n))$ by

$$L\Phi = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(N(\varphi_1), \dots, N(\varphi_n)) N(\rho \Delta_z \varphi_i + D_z \rho \cdot (D_z \varphi_i)^T) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(N(\varphi_1), \dots, N(\varphi_n)) N(\rho D_z \varphi_i \cdot (D_z \varphi_j)^T) \quad (4.1)$$

(cf. [25] (9-2)). Here, $D_z = \frac{\partial}{\partial z}$, Δ_z denotes the Laplacian with respect to the z-variable, $\rho(z)$ is some function such that $\rho(z) \in \cap_{p\geq 1} L^p$ and $D_z \rho(z) \in \cap_{p\geq 1} L^p$, and $N(\psi) = \int_{\mathbf{T}} \int \psi(t, z) N(dtdz)$.

In this way, on the Poisson space, we can introduce the derivative operator $\bar{\mathcal{D}}_{(t,z)}$ for $\Phi = f(N(\varphi_1), \dots, N(\varphi_n))$ by

$$\bar{\mathcal{D}}_{(t,z)} \Phi = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (N(\varphi_1), \dots, N(\varphi_n)). \frac{\partial}{\partial z} \varphi_i(t, z) . \quad (4.2)$$

One can then show that the operator $\Gamma(\phi, \psi)$, given by

$$\Gamma(\phi, \psi) = \int_{\mathbf{T}} \int \bar{\mathcal{D}}_{(t,z)} \phi (\bar{\mathcal{D}}_{(t,z)} \psi)^T \rho(z) N(dtdz)$$

is a carré-du-champ operator (cf. [25] Section 12-3)). This setting will result in the derivation of the integration-by-parts formula.

On the other hand, we also have the *difference operator* $\tilde{D}_u$ on the Poisson space as introduced in Section 2.1.2 by

$$\tilde{D}_u \Phi = \Phi \circ \varepsilon_u^+ - \Phi . \quad (4.3)$$

By the mean value theorem,

$$\tilde{D}_u\Phi = \sum_{i=1}^{n}\int_0^1 \frac{\partial f}{\partial x_i}(N(\varphi_1+\theta\tilde{D}_u\varphi_1),\dots,N(\varphi_n+\theta\tilde{D}_u\varphi_n))d\theta\cdot\tilde{D}_u N(\varphi_i), \quad u=(t,z). \quad (4.4)$$

This operator will also result in the derivation of an analogue of the integration-by-parts formula (cf. [95] (3.4.16)).

We shall compare (4.2) and (4.4). The operators $\bar{\mathcal{D}}$ and $\tilde{D}$ are apparently similar. However, $\tilde{D}_u f$ is estimated by the mean value of $f(.)$, and the norm

$$\int \left|\frac{\tilde{D}_u\Phi}{\gamma(u)}\right|^p \hat{M}(du) = \int \left|\frac{\Phi\circ\varepsilon_u^+ - \Phi}{\gamma(u)}\right|^p \hat{M}(du) \quad (4.5)$$

is taken with respect to the mean value of $\Phi\circ\varepsilon_u^+$ and Φ, instead of the differential value which is associated with the derivative operator $\bar{\mathcal{D}}_{(t,z)}$. Hence, consequences in the analysis based on two approaches on the Poisson space (i.e. one is based on $\bar{\mathcal{D}}_{(t,z)}$ and Γ above, the other is based on $\tilde{D}_u$) may not strictly coincide with each other in general. Especially regarding the case that the Lévy measure μ being singular (e.g. $\mu = \sum_{k=1}^{\infty} c_k\delta_{\{a_k\}}$), the operator $\bar{\mathcal{D}}_{(t,z)}$ may not apply since we can not take the derivative of $\mu(dz)$.

On the other hand, using the framework of (1)–(3) above, one can develop the analysis of the transition semigroup in terms of the carré-du-champ operator. Namely, by utilising the theory of nonlocal Dirichlet forms. One can show the existence of the density for the transition semigroup. We will not treat this topic in this book. See, for example, [37].

3.5 Composition on the Wiener–Poisson space (I) – general theory

In this section, we make a composition $T\circ F$ of a Wiener–Poisson functional F with a tempered distribution T. One motivation for doing it is to make asymptotic expansions for functionals $F = F(\epsilon)$ defined on the Wiener–Poisson space, where $F(\epsilon)$ depends on a small parameter ϵ.

The density function $p_F(x)$ of F can formally be defined by $E[\delta_x(F)] = \langle\delta_x(F), 1\rangle$, where Dirac's delta function δ_x is an element of the tempered distributions $\mathcal{S}'$.

To this end, we construct, in Section 3.5.1, a series of spaces $\mathcal{S}_{2m}$ in $\mathcal{S}'$ which exhaust $\mathcal{S}'$, that is, $\cup_{m\geq 1}\mathcal{S}_{-2m} = \mathcal{S}'$. We remark that $\mathcal{S}_{2m}$ is a weighted Sobolev space, for which the degree of the smoothness and that of decrease at infinity are symmetric with respect to the weight. In Section 3.5.2, we provide a sufficient condition so that the non-degeneracy condition *(ND)* holds in terms of the kernels.

3.5.1 Composition with an element in $\mathcal{S}'$

We denote by $\mathcal{S}$ the space of rapidly decreasing C^∞-functions defined on $\mathbf{R}^d$ after L. Schwartz. For $\varphi \in \mathcal{S}$, we introduce a new norm, that is,

$$\|\varphi\|_{2m} = \left(\int \sum_{i+j\leq m} \left\{ \left| (1-\Delta)^j (1+|y|^2)^i \varphi \right|^2 \right\} dy \right)^{\frac{1}{2}} \tag{5.1}$$

for $m = 1, 2, \ldots$ We let $\mathcal{S}_{2m}$ be the completion of $\mathcal{S}$ with respect to this norm. We remark $\mathcal{S} \subset \mathcal{S}_{2m}$, $m = 1, 2, \ldots$

We introduce the dual norm $\|\psi\|_{-2m}$ of $\|\varphi\|_{2m}$ by

$$\|\psi\|_{-2m} = \sup_{\varphi\in\mathcal{S}_{2m}, \|\varphi\|_{2m}=1} |(\varphi, \psi)| \;, \tag{5.2}$$

where $(\varphi, \psi) = \int \varphi(x)\bar{\psi}(x)dx$. We denote by $\mathcal{S}_{-2m}$ the completion of $\mathcal{S}$ with respect to the norm $\|\psi\|_{-2m}$. Furthermore,

$$\mathcal{S}_\infty = \cap_{m\geq 1}\mathcal{S}_{2m}, \quad \mathcal{S}_{-\infty} = \cup_{m\geq 1}\mathcal{S}_{-2m} \;.$$

We denote by $\mathcal{S}'$ the space of tempered distributions (the dual space of $\mathcal{S}$). We cite the following representation theorem.

Proposition 3.4 ([163] Theorem 2.14). *For each $\Phi \in \mathcal{S}'$, there exist $k, m \in \mathbf{N}$ and $(f_\alpha), f_\alpha \in L^2(\mathbf{R}^d)$ such that*

$$\Phi = (1+|x|^2)^k \sum_{|\alpha|\leq m} D^\alpha f_\alpha(x) \;.$$

Using this result, we can prove the following:

Proposition 3.5. *We have*

$$\mathcal{S} = \mathcal{S}_\infty, \quad \mathcal{S}' = \mathcal{S}_{-\infty} \;.$$

Proof. For the second statement, we prove $\mathcal{S}' \subset \mathcal{S}_{-\infty}$ since $\mathcal{S}_{-2m} \subset \mathcal{S}'$ for each $m = 1, 2, \ldots$

Let $|\varphi|_{2m}$ be the norm defined on $\mathcal{S}$ by

$$|\varphi|_{2m} = \left(\sum_{i\leq m} \int \left| (1-\Delta)^i \varphi(y) \right|^2 dy \right)^{1/2} \;. \tag{5.3}$$

This norm is equivalent, by the Plancharel equality, to the norm $|\varphi|_{2m}^{\sim}$ given by

$$|\varphi|_{2m}^{\sim} = \left(\sum_{i\leq m} \int \left(1+|v|^2\right)^i |\hat{\varphi}(y)|^2 dy \right)^{1/2} \;. \tag{5.4}$$

We denote by H_{2m} the completion of $\mathcal{S}$ by this norm. Let H_{-2m} be the dual space of H_{2m} with the dual norm $|\varphi|_{-2m}^{\sim}$. Since $|\varphi|_{2m}^{\sim} \leq \|\varphi\|_{2m}$, we have $\mathcal{S}_{2m} \subset H_{2m}$ for $m = 1, 2, \ldots$ This implies $H_{-2m} \subset \mathcal{S}_{-2m}$.

Let Φ be any element of $\mathcal{S}'$. By the proposition above (Proposition 3.4), Φ can be decomposed as

$$\Phi = (1 + |x|^2)^k \Phi' \tag{5.5}$$

for some $k, m \in \mathbf{N}$ and $\Phi' \in H_{-2m}$. We may assume $k < m$ by choosing a large m. Take any $\varphi \in \mathcal{S}_{2m}$, then $(1 + |x|^2)^k \varphi \in H_{2(m-k)}$.

Since $\langle (1 + |x|^2)^k \Phi', \varphi \rangle = \langle \Phi', (1 + |x|^2)^k \varphi \rangle$, we have

$$\begin{aligned} \|(1 + |x|^2)^k \Phi'\|_{-2m} &\leq |\Phi'|_{-2(m-k)} \sup_{\|\varphi\|_{2m}=1} |(1 + |x|^2)^k \varphi|_{2(m-k)} \\ &\leq C|\Phi'|_{-2(m-k)} < +\infty \,. \end{aligned}$$

Hence, $\Phi = (1 + |x|^2)^k \Phi' \in \mathcal{S}_{-2(m-k)}$. This proves the assertion.

The first statement follows directly from the definition of $\mathcal{S}$. □

Remark 3.4. Watanabe [216] used the norm

$$|||\varphi|||_{2m} = |(1 + |x|^2 - \Delta)^m \varphi|_\infty \tag{5.6}$$

instead of $\|.\|_{2m}$ in the analysis on the Wiener space. We can show further that $\mathcal{S}'$ can be decomposed into the sum of subspaces characterised by this norm as

$$\mathcal{S}' = \cup_{m \geq 1} \tilde{C}^{-2m} \,.$$

Here,

$$\tilde{C}^{-2m} = \Big\{ g \in \mathcal{S}'; A^{-m} g \in \hat{C}\,,\ \text{there exists } g_n \in \mathcal{S} \\ \text{such that } |A^{-m} g_n - A^{-m} g|_\infty \to 0 (n \to +\infty) \Big\}\,,$$

where $\hat{C}$ is a space of continuous functions decreasing to 0 at infinity, and $A = (1 + |x|^2 - \Delta)$. This gives one way to define the composition of a Wiener–Poisson functional with an element in $\mathcal{S}'$ (cf. [223]).

Here, on the other hand, we use the Fourier transform on the Wiener–Poisson space based on the L^2 theory to define the composition. Thus, we use the (weighted) Sobolev norm as above.

We define the Fourier transform of $\varphi \in \mathcal{S}$ by

$$\hat{\varphi}(v) = \mathcal{F}\varphi(x) = \left(\frac{1}{2\pi}\right)^d \int e^{-i(v,y)} \varphi(y) dy \,.$$

The Fourier transform for $\psi \in \mathcal{S}'$ is properly defined. It is well known that

$$\mathcal{F}\mathcal{S} = \mathcal{S}, \quad \mathcal{F}\mathcal{S}' = \mathcal{S}' \,.$$

If a characteristic function $\phi(v) = E[e^{i(v,F)}]$ of a random variable F is smooth and of polynomial decay (rapidly decreasing) at infinity, then it belongs to $\mathcal{S}$, and hence the density function $p_F(x) = \mathcal{F}\phi(x)$ of F exists and belongs to $\mathcal{S}$.

In what follows, we put

$$\psi_G(v) = E[e^{i(v,F)}G] \tag{5.7}$$

for $G \in \mathbf{D}_\infty$. We also write $e_v(x) = e^{i(v,x)}$ in what follows.

We recall the condition *(ND)* for a Wiener–Poisson functional F given in Section 3.3.3 (Definition 3.2).

Proposition 3.6. *Let $F \in \mathbf{D}_\infty$ and assume the condition (ND). Then, for any $n \in \mathbf{N}$, there exist $k, l \in \mathbf{N}$, $p \geq 2$ and $C = C_{k,l,p} > 0$ such that*

$$|E[Ge_v(F)]| \leq C(1 + |v|^2)^{-\frac{1}{2}nq_0}|G|_{k,l,p}|F|^n_{k,l,p} \sup_{|v|>1} |Q^F(v)^{-1}|^n_{k,l,p} \tag{5.8}$$

for $|v| \geq 1$. Here, we put $B_v = A(|v|^{-\frac{1}{\beta}})$,

$$Q^F(v) = \left(v', \sum v'\right) + \frac{1}{|v|^2\varphi(|v|^{-\frac{1}{\beta}})} \int_{B_v} \left|e^{i(v,\tilde{D}_u F)} - 1\right|^2 \hat{N}(du)\ ,$$

and $\sum = (\int_{\mathbf{T}}(D_tF)^i(D_tF)^j dt)$, where $q_0 > 0$ is what appeared in (3.33).

Remark 3.5. Under the condition *(ND)*, $\sup_{|v|>1} \sup_{\mathbf{u}\in B_v^k} E[|Q^F(v)^{-1} \circ \varepsilon^+_{\mathbf{u}}|^p]$ is finite for all $p \geq 1$, $k = 0, 1, 2, \ldots$ since there exists $0 < c < 1$ such that

$$\int_{B_v} |e^{i(v,\tilde{D}_u F)} - 1|^2 \hat{N}(du) \geq c \int_{B_v} |(v, \tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F| \leq \frac{1}{|v|}\}} \hat{N}(du)$$

for $|v| > 1$.

Proof. Let

$$Z^F(v) = Z^F_t(v) = -i\frac{(v, D_tF)}{|v|^2 Q^F(v)}\ ,$$

$$\tilde{Z}^F(v) = \tilde{Z}^F_u(v) = \frac{(e^{-i(v,\tilde{D}_u F)} - 1).\ 1_{B_v}(u)}{|v|^2\varphi(|v|^{-\frac{1}{\beta}})Q^F(v)}$$

where $B_v = A(|v|^{-\frac{1}{\beta}})$.

We set $\mathbf{Z}^F(v) = Z^F(v) \oplus \tilde{Z}^F(v)$. Since

$$e_v(F) = ((Z^F(v)D_t \oplus \tilde{Z}^F(v)\tilde{D}_u), e_v(F))_{L^2(\mathbf{T})\oplus L^2(\hat{N})}\ ,$$

we have

$$E[Ge_v(F)] = E[e_v(F)\{\delta(Z^F(v)G) + \tilde{\delta}(\tilde{Z}^F(v)G)\}]\ .$$

Hence,

$$E[e_v(F)G] = E[e_v(F)\bar{\delta}(\mathbf{Z}^F\bar{\delta}(\mathbf{Z}^F \cdots \bar{\delta}(\mathbf{Z}^F\bar{\delta}(G\mathbf{Z}^F))\cdots))]\ .$$

Here,

$$\bar{\delta}(\mathbf{Z}^F(v)) = \delta(Z^F(v)) + \tilde{\delta}(\tilde{Z}^F(v))\ .$$

Here, we use the following:

Lemma 3.5. *Under the condition (ND), for any* $k, l \in \mathbf{N}$, $p \geq 2$, *there exists* $C > 0$ *such that*

$$|\delta(Z^F(v)G)|_{k,l,p} \leq \frac{C}{|v|}|F|_{k,l+2,3p} \sup_{|v|>1} |Q^F(v)^{-1}|_{k,l+1,3p}|G|_{k,l+1,3p} \,,$$
$$|\tilde{\delta}(\tilde{Z}^F(v)G)|_{k,l,p} \leq \frac{C}{|v|}\varphi\left(|v|^{-\frac{1}{\beta}}\right)^{-\frac{1}{2}(1+\frac{1}{k_0})} |F|_{k',l,p'} \sup_{|v|>1} |Q^F(v)^{-1}|_{k',l,p'}|G|_{k',l,p'} \,. \tag{5.9}$$

Here $k' = k + p + 1$, $p' = 3(k + p)k_0$.

Proof of Lemma 3.5
The first inequality follows from the extended Meyer's type inequality (Lemma 3.4), and from Hölder's inequality:

$$|XYZ|_{k,l+1,p} \leq c|X|_{k,l+1,3p}|Y|_{k,l+1,3p}|Z|_{k,l+1,3p} \,.$$

For the second inequality, we use Lemma 3.4 (2) in Section 3.3 as $V\chi_\rho = \tilde{Z}^F(v)G = \tilde{Z}^F(v)G1_{\{|u|\leq|v|^{-1/\beta}\}}$. Then, we have

$$|\tilde{\delta}(\tilde{Z}^F(v)G)|_{k,l,p} \leq c\varphi\left(|v|^{-\frac{1}{\beta}}\right)^{\frac{1}{2}-\frac{1}{2s}} \|\tilde{Z}^F(v)G\|^{\sim}_{k+p,l,(k+p)k_0} \,.$$

Here, we use a Hölder's inequality

$$\|\tilde{Z}^F(v)G\|^{\sim}_{k+p,l,(k+p)k_0} \leq \frac{C}{|v|^2\varphi(|v|^{-\frac{1}{\beta}})}\|\tilde{F}_u(v)\|^{\sim}_{k+p,l,p'}|Q^F(v)^{-1}|_{k',l,p'}|G|_{k',l,p'} \,,$$

where $\tilde{F}_u(v) = (e^{i(v,\tilde{D}_uF)} - 1)1_{B_v(u)}$.

We use the mean value theorem

$$\tilde{D}_{u'}\psi(G) = (\tilde{D}_{u'}G)\int_0^1 \partial\psi(G + \theta\tilde{D}_{u'}G)d\theta$$

for $G = (v, \tilde{D}_uF)$ and $\psi(x) = e^{ix} - 1$. Furthermore, we parametrise $u = |v|^{-1/\beta}\tilde{u}$ for sufficiently large $|v|$. As $|u| \leq |v|^{-1/\beta}$ with $\frac{\alpha}{2} < \beta \leq 1$, we have $|v|^{-1/\beta} \leq |v|^{-1}$, and

$$\int_{B_v(u)} |\tilde{D}_{u'}(e^{i(v,\tilde{D}_uF)} - 1)|^2\hat{N}(du)$$
$$= \int_{B_v(u)} \left| |v|\left(\frac{v}{|v|}, \tilde{D}_{u'}\tilde{D}_uF\right)\int_0^1 \partial\psi(\tilde{D}_uF + \theta\tilde{D}_{u'}(v, \tilde{D}_uF))d\theta\right|^2 \hat{N}(du)$$
$$\leq |v|^2 \cdot |v|^{-1} \int_{|\tilde{u}|\leq 1} \left|\left(\frac{v}{|v|}, \tilde{D}_{u'}\tilde{D}_uF\right)\right|^2 \hat{N}(d\tilde{u}) \,.$$

Hence, $\|\tilde{F}_u(v)\|^{\sim}_{k+p,l,p'} \leq c|v||F|_{k+p+1,l,p'}$. Combining this with the above, we have the assertion for the second inequality. □

The above inequalities can be regarded as Poincaré inequalities in the Wiener and Poisson spaces.

Due to this lemma, we can show

$$|\bar{\delta}(G\mathbf{Z}^F)|_{k,l,p}$$
$$\leq C\frac{1}{|v|}\left(1+\varphi\left(|v|^{-\frac{1}{\beta}}\right)^{-\frac{1}{2}(1+\frac{1}{k_0})}\right)|F|_{k',l',p'} \times |Q^F(v)^{-1}|_{k',l',p'}|G|_{k',l'p'} .$$

Observe

$$\frac{1}{|v|}+\frac{1}{|v|}\varphi\left(|v|^{-\frac{1}{\beta}}\right)^{-\frac{1}{2}(1+\frac{1}{k_0})} \leq C|v|^{-q_0} .$$

Combining these inequalities, and using this estimate repeatedly, we have the conclusion of Proposition 3.6. □

Remark 3.6. If we restrict the range of $\mathbf{u} \in A(1)^k$ to $\mathbf{u} \in A(\rho(v))^k$, where $\rho(v)$ is a function of v in the form of a negative power of $|v|$, in the norms $|.|_{k,l,p}$ and $\|.\|^{\sim}_{k,l,p}$, we can relate $\rho(v)$ to the decay order of $|v|$ in the right hand side of (5.8) in place of $|v|^{-q_0}$. Confer with [77] Lemma 4.4.

We have the following proposition (cf. [130] Lemma 2.14]):

Proposition 3.7. *Let $F \in \mathbf{D}_\infty$ satisfy the (ND) condition. For any m, there exist $k, l, p > 2$ and $C_m > 0$ such that*

$$|\varphi \circ F|'_{k,l,p} \leq C_m\left(\sum_{\beta\leq m}|(1+|F|^2)^\beta|_{k,l,p}\right)\|\varphi\|_{-2m} \tag{5.10}$$

for $\varphi \in \mathcal{S}$.

Proof. We have

$$\begin{aligned} E\left[\varphi(F)G\right] &= E\left[\int e^{i(v,F)}\hat{\varphi}(v)dv \cdot G\right] \\ &= \int \hat{\varphi}(v)E\left[e^{i(v,F)} \cdot G\right]dv = \int \hat{\varphi}(v)\psi_G(v)dv . \end{aligned}$$

Step 1 We assert $\psi_G \in \mathcal{S}_\infty$. Indeed, we observe

$$(1-\Delta)^\beta\psi_G(v) = E\left[G(1+\sum_{j=1}^d(F_j)^2)^\beta e^{i(v,F)}\right] .$$

Here, by Proposition 3.6, for each n, there exist k, l, p so that the right-hand side is dominated by

$$C\left|\left(1+\sum_{j=1}^{d}(F_j)^2\right)^{\beta}\right|_{k,l,p/2}|G|_{k,l,p/2}\left(1+|v|^2\right)^{-nq_0/2}$$
$$\leq C\left|\left(1+|F|^2\right)^{\beta}\right|_{k,l,p}|G|_{k,l,p}\left(1+|v|^2\right)^{-nq_0/2}\ .$$

Here, $q_0 \in (0,1)$ is chosen independent of n in (3.33).

Hence,

$$|(1+|v|^2)^{nq_0/2}(1-\Delta)^{\beta}\psi_G(v)| \leq C|(1+|F|^2)^{\beta}|_{k,l,p}|G|_{k,l,p}\ .$$

We move $n \leq 2m/q_0$, then by (5.1), we observe

$$\|\psi_G\|_{2m} \leq C_m(\sum_{\beta\leq m}|(1+|F|^2)^{\beta}|_{k,l,p})|G|_{k,l,p}, \quad m = 1, 2, \ldots$$

with some $k = k_m$, $l = l_m$, $p = p_m$. This implies $\psi_G \in \mathcal{S}_\infty$.

Step 2 The assertion (5.10) holds.

We observe

$$\begin{aligned}|\varphi\circ F|'_{k,l,p} &= \sup_{|G|_{k,l,p}=1}|E[\varphi\circ FG]|\\ &\leq \sup_{|G|_{k,l,p}=1}\left|\int\psi_G(v)\hat{\varphi}(v)dv\right|\\ &\leq \sup_{|G|_{k,l,p}=1}\|\psi_G\|_{2m}\|\hat{\varphi}\|_{-2m}\\ &\leq C_m(\sum_{\beta\leq m}|(1+|F|^2)^{\beta}|_{k,l,p})\|\varphi\|_{-2m}\ .\end{aligned}$$

Hence, we have (5.10). □

The above formula implies that for $\Phi \in \mathcal{S}_{-2m}$ and $F \in \mathbf{D}_\infty$, we can define the composition $\Phi\circ F$ as an element of $\mathbf{D}'_\infty$. Since $\cup_{m\geq1}\mathcal{S}_{-2m} = \mathcal{S}'$, we can define the composition $\Phi\circ F$ for $\Phi \in \mathcal{S}'$ and $F \in \mathbf{D}_\infty$.

Definition 3.3. Suppose that F satisfies (*ND*). For a distribution $T \in \mathcal{S}'$, the composite $T\circ F$ is the linear functional on $\mathbf{D}_\infty$ which is defined by

$$\langle T\circ F\ , G\rangle = {}_{\mathcal{S}'}\langle\mathcal{F}T\ , E[Ge^{i\cdot F}]\rangle_{\mathcal{S}}, \quad G \in \mathbf{D}_\infty\ .$$

Due to the proof in Proposition 3.6, we have the following result for $\psi_G(v)$:

Proposition 3.8. *Let $F \in \mathbf{D}_\infty$ satisfy the condition (ND), and let $G \in \mathbf{D}_\infty$. For any $n \in \mathbf{N}$, there exist $k, l \in \mathbf{N}$, $p > 2$ and $C > 0$ such that*

$$|\psi_G(v)| \leq C(1+|v|^2)^{-\frac{q_0}{2}n}|F|^n_{k,l,p}|G|_{k,l,p}, \quad |v| \geq 1\ . \tag{5.11}$$

Choose $G = 1$. As stated in the introduction to Section 2.1, the above proposition implies that F has a smooth density $p_F(y)$.

3.5.2 Sufficient condition for the composition

Let F be a Wiener–Poisson variable.

Condition (R) We say F satisfies the condition *(R)* if for any positive integer $k = 0, 1, 2, \ldots$ and $p > 1$, the derivatives satisfy

$$\sup_{t\in\mathbf{T},\mathbf{u}\in A(1)^k} E\left[\sum_{i=1}^m \sup_{|z|\le 1} |\partial_{z_i}\tilde{D}_{t,z}F \circ \varepsilon_{\mathbf{u}}^+|^p + \sum_{i,j=1}^m \sup_{|z|\le 1} |\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}F \circ \varepsilon_{\mathbf{u}}^+|^p\right] < +\infty .$$

We recall $\varphi(\rho) = \int_{|z|<\rho} |z|^2\mu(dz)$, and assume that the Lévy measure satisfies the order condition in (3.4). Let B be the infinitesimal covariance, that is, a non-negative symmetric matrix which satisfies

$$(v, Bv) \le (v, B_\rho v), \quad v \in \mathbf{R}^m \setminus \{0\} \text{ for } 0 < \rho < \rho_0 ,$$

for some $\rho_0 > 0$. Here B_ρ is a matrix

$$B_\rho = \frac{1}{\varphi(\rho)} \int_{|z|\le\rho} zz^T\mu(dz) .$$

We remark that such a matrix B exists (eventually one may take $B \equiv 0$). However we do not know we can choose $B > 0$ (positive definite) even if $B_\rho > 0$ for each $0 < \rho < \rho_0$. One sufficient condition for this is due to Picard–Savona [185]: there exist $0 < \alpha < 2$ and $c, C > 0$ such that

$$c\rho^\alpha|u|^2 \le \int_{|z|\le\rho} (u, z)^2\mu(dz) \le C\rho^\alpha|u|^2$$

holds for any $u \in \mathbf{R}^m$ and $0 < \rho < 1$. Indeed, if we choose $u = e_j$, $j = 1, \ldots, m$ (unit vectors), then

$$mc\rho^\alpha \le \int_{|z|\le\rho} |z|^2\mu(dz) \le mC\rho^\alpha .$$

Hence

$$(u, B_\rho u) = \frac{1}{\varphi(\rho)} \int_{|z|\le\rho} (u, z)^2\mu(dz) \ge \frac{c\rho^\alpha|u|^2}{mC\rho^\alpha} \ge c'|u|^2$$

for some $c' > 0$.

Even if it exists positive, the choice of B may not be unique. In this case we choose one and fix it.

We now require sufficient conditions for the condition *(ND)*. To this end, we introduce

$$R = \int_{\mathbf{T}} D_t F (D_t F)^T dt = \Sigma^F .$$

$$\tilde{K} = \int_{\mathbf{T}} (\partial \tilde{D}_{t,0} F) B (\partial \tilde{D}_{t,0} F)^T dt .$$

$$\tilde{K}_\rho = \int_{\mathbf{T}} \partial \tilde{D}_{t,0} F B_\rho (\partial \tilde{D}_{t,0} F)^T dt, \quad \rho > 0 .$$

Here, $\partial \tilde{D}_{t,0} F = \partial_z \tilde{D}_{t,z} F|_{z=0}$.

Furthermore,

$$Q(v) = (v', (R + \tilde{K}) v') ,$$

$$Q_\rho(v) = (v', (R + \tilde{K}_\rho) v') ,$$

where $v' = \frac{v}{|v|}$, $v \in \mathbf{R}^d$, $\rho > 0$.

Definition 3.4.

(1) We say F satisfies the *(ND2)* condition if $R + \tilde{K}$ is invertible and if for any integer k and $p > 1$ such that

$$\sup_{v \in \mathbf{R}^d, |v|=1, \mathbf{u} \in A(1)^k} E[(v, (R + \tilde{K}) \circ \varepsilon_{\mathbf{u}}^+ v)^{-p}] < +\infty .$$

(2) We say F satisfies the *(ND3)* condition if $R + \tilde{K}$ is invertible and if for any integer k and any $p > 1$, there exists $\rho_0 > 0$ such that

$$\sup_{0<\rho<\rho_0} \sup_{v \in \mathbf{R}^d, |v|=1, \mathbf{u} \in A(1)^k} E[|(v, (R + \tilde{K}_\rho) \circ \varepsilon_{\mathbf{u}}^+ v)^{-p}|] < +\infty .$$

Lemma 3.6. *Assume F satisfies the condition (R). Then, we have the following.*

(i) *The condition (ND2) implies the condition (ND3).*

(ii) *The condition (ND3) implies the condition (ND).*

Proof. (i) Indeed,

$$\begin{aligned} &(v, (R + \tilde{K}_\rho) v) \\ &= \left(v, \left(\int_{\mathbf{T}} \left\{ D_t F D_t F^T + (\partial \tilde{D}_{t,0} F) \frac{1}{\varphi(\rho)} \int_{|z|\le\rho} z z^T \mu(dz) (\partial \tilde{D}_{t,0} F)^T \right\} dt \right) v \right) \\ &\ge c \left(v, \left(\int_{\mathbf{T}} \left\{ D_t F D_t F^T + (\partial \tilde{D}_{t,0} F) B (\partial \tilde{D}_{t,0} F)^T \right\} dt \right) v \right) \\ &\ge C' (v, (R + \tilde{K}) v) \end{aligned}$$

for $0 < \rho < \rho_0$, for some $C' > 0$.

(ii) Let

$$T(v,\rho) = T_\rho(v) = (v', Rv') + \frac{1}{\varphi(\rho)} \int_{A(\rho)} (v', \tilde{D}_u F)^2 1_{\{|\tilde{D}_u F| \le \rho^\beta\}} \hat{N}(du)$$

where $v' = v/|v|$.

We shall prove that there exists $C > 0$ such that for some $\delta_0' > 0$,

$$\sup_{0<\rho<\delta_0'} E\left[\left|T_\rho(v)^{-1} \circ \varepsilon_{\mathbf{u}}^+\right|^p\right] \le C \sup_{0<\rho<\delta_0'} E\left[\left|Q_\rho(v)^{-1} \circ \varepsilon_{\mathbf{u}}^+\right|^p\right] \tag{5.12}$$

holds for all $\mathbf{u}$.

If (5.12) is verified, it implies the condition *(ND)*. Indeed, if the condition *(ND3)* holds, then there exists $\rho_0 = \delta_0' > 0$ such that the right-hand side of (5.12) is finite. Therefore, (left-hand side of (5.12)) $< +\infty$. Hence, we have the condition *(ND)*.

We fix v and write T_ρ, Q_ρ for $T_\rho(v)$, $Q_\rho(v)$, respectively. Take $\delta_0 > 0$. We introduce a random variable $\tau = \tau(v, \mathbf{u})$ for $v \in S^{d-1}$ and $|\mathbf{u}| \le 1$ by

$$\tau = \inf\left\{\rho \in (0,\delta_0)\,; |T_\rho \circ \varepsilon_{\mathbf{u}}^+ - Q_\rho \circ \varepsilon_{\mathbf{u}}^+| \ge \frac{1}{2} Q_\rho \circ \varepsilon_{\mathbf{u}}^+\right\}\ .$$

We write $\tau = \delta_0$ if $\{\ldots\} = \emptyset$.

We have

$$T_{\tau\wedge\rho} \circ \varepsilon_{\mathbf{u}}^+ \ge \frac{1}{2} Q_\rho \circ \varepsilon_{\mathbf{u}}^+\ .$$

Since $\rho \mapsto \varphi(\rho) T_\rho$ is continuous and nondecreasing, we have

$$T_\rho \circ \varepsilon_{\mathbf{u}}^+ \ge \frac{\varphi(\tau\wedge\rho)}{\varphi(\rho)} T_{\tau\wedge\rho} \circ \varepsilon_{\mathbf{u}}^+\ .$$

Hence,

$$T_\rho \circ \varepsilon_{\mathbf{u}}^+ \ge \frac{1}{2}\frac{\varphi(\tau\wedge\rho)}{\varphi(\rho)} Q_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^+ a.s\ .$$

By the Schwarz inequality, we have

$$\left(E\left[T_\rho^{-p} \circ \varepsilon_{\mathbf{u}}^+\right]\right)^{\frac{1}{p}} \le 2\left(E\left[Q_{\rho\wedge\tau}^{-2p} \circ \varepsilon_{\mathbf{u}}^+\right]\right)^{\frac{1}{2p}} \cdot \left(E\left[\left(\frac{\varphi(\rho)}{\varphi(\tau\wedge\rho)}\right)^{2p}\right]\right)^{\frac{1}{2p}}\ .$$

We claim

$$\sup_{|v|=1}\ \sup_{|\mathbf{u}|\in A(\rho)^k} \left(E\left[\left(\frac{\varphi(\rho)}{\varphi(\tau\wedge\rho)}\right)^{2p}\right]\right)^{\frac{1}{2p}} < +\infty\,, \quad k = 1, 2, \ldots\ . \tag{5.13}$$

If this is verified, then we have the assertion (5.12).

To show (5.13), it is sufficient to show that for any $p > 1$, there exists a positive constant $c_p > 1$ such that for all v and $\mathbf{u}$,

$$P(\tau \le \rho) \le c_p \rho^{4p}\,, \quad 0 < \rho < \delta_0\ .$$

On the set

$$E(\rho) = \left\{ \sup_{|(t,z)|<\rho} |\tilde{D}_{(t,z)}F| < \rho^{\beta} \right\} ,$$

we have

$$|T_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+} - Q_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+}| \\ \leq \frac{1}{\varphi(\rho \wedge \tau)} \int_{A(\rho\wedge\tau)} \left|(v, \tilde{D}_{t,z}F)^2 \circ \varepsilon_{\mathbf{u}}^{+} - (v, \partial\tilde{D}_{t,0}F.z)^2 \circ \varepsilon_{\mathbf{u}}^{+}\right| \hat{N}(du), \quad u = (t, z) .$$

Due to the condition *(R)*, the integrand is dominated by the term

$$|z|^3 \Phi(t, z, \mathbf{u}) ,$$

where $\Phi(t, z, \mathbf{u})$ is a nonnegative random variable satisfying

$$\sup_{t\in\mathbf{T}} \sup_{|\mathbf{u}|\leq 1} E\left[\sup_{|z|\leq\delta_0} \Phi(t, z, \mathbf{u})^{4p} \right] < +\infty$$

for any $p > 1$.

We put

$$\Psi = |T_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+} - Q_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+}| .$$

Then, the above implies that for any $p > 1$, there exists $c_1 > 0$

$$E\left[\Psi^{4p}.\, 1_{E(\rho)}\right] \leq c_1 \rho^{4p} T^{4p} \tag{5.14}$$

if $\rho < \delta_0$ for any v and $\mathbf{u}$.

Then, combining the condition *(ND3)* and (5.14), we have

$$E\left[\left(\frac{\Psi}{Q_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+}} \right)^{4p} \right] \leq c_2 \rho^{4p}$$

for $\rho < \delta_0$ for all u and v. Hence, by Chebyschev's inequality,

$$P\left(E(\rho) \cap \left(\Psi \geq \frac{1}{2} Q_{\rho\wedge\tau} \circ \varepsilon_{\mathbf{u}}^{+} \right) \right) \leq c_3 \rho^{4p}, \quad \rho < \delta_0 .$$

On the other hand,

$$E(\rho)^c = \left\{ \sup_{|(t,z)|<\rho} \left|\tilde{D}_{(t,z)}F\right| \geq \rho^{\beta} \right\} .$$

Then, again by Chebyschev's inequality,

$$P(E(\rho)^c) \leq E\left[\left(\frac{\sup_{|(t,z)|<\rho} |\tilde{D}_u F|}{\rho^{\beta}} \right)^{2p'} \right] \leq c_4 \rho^{2(1-\beta)p'} \leq c_4 \rho^{4p}, \quad \rho < \delta_1$$

for some $\delta_1 > 0$ which are sufficiently small, where $p' = \frac{2p}{1-\beta}$. Here, we again used the condition *(R)*. These imply $P(\tau \le \rho) \le (c_3 + c_4)\rho^{4p}$ for $\rho < \delta_0' = \delta_0 \wedge \delta_1$.

We put $m(x) = P(\tau \le x)$. Then $m(x) \le Cx^{4p}$. By the order condition we have $\varphi(\rho) \ge c\rho^{\alpha}$. Hence

$$E\left[\left(\frac{1}{\varphi(\rho \wedge \tau)}\right)^{2p} 1_{\tau \le \rho}\right] = \int_0^{\rho} (\frac{1}{\varphi(x)})^{2p} m(dx)$$
$$\le \int_0^{\rho} C_1 \left(\frac{1}{x}\right)^{2\alpha p} m(dx) \le C_2(\rho) .$$

This implies

$$E\left[\left(\frac{\varphi(\rho)}{\varphi(\rho \wedge \tau)}\right)^{2p}\right]^{1/2}$$
$$\le P(\tau \ge \rho) + \varphi(\rho)^p E\left[\left(\frac{1}{\varphi(\rho \wedge \tau)}\right)^{2p} 1_{\tau \le \rho}\right]^{1/2} < +\infty, \quad 0 < \rho < 1 .$$

Hence, we have the assertion (5.13). □

Proposition 3.9. *We assume the conditions (R) and (ND2).*

(1) *For each n, there exist $C > 0$ and k, l, p such that*

$$|E[Ge_v(F)]| \le C(1 + |v|^2)^{-\frac{1}{2}nq_0} |G|_{k,l,p} |F|_{k,l,p}^n \sup_{|v|>1} |\{Q^F(v)\}^{-1}|_{k,l,p}, \quad |v| \ge 1 .$$

Here,

$$Q^F(v) = (v', \Sigma^F v') + \frac{1}{|v|^2 \varphi(|v|^{-\frac{1}{\beta}})} \int_{B_v} |e^{i(v,\tilde{D}_u F)} - 1|^2 \hat{N}(du) .$$

(2)

$$\sup_{|v|>1} |\{Q^F(v)\}^{-1}|_{k,l,p} < +\infty$$

for each k, l, p.

Proof. By Lemma 3.6, the condition *(ND)* holds. Then, the remaining part of the proof for (1) is the same as that of Proposition 3.6.

Next, we show the finiteness of $|Q^F(v)^{-1}|_{k,l,p}$. Indeed, under the condition *(ND)*, $\sup_{|v|>1} \sup_{\mathbf{u} \in B_v^k} E[|Q^F(v)^{-1} \circ \varepsilon_{\mathbf{u}}^+|^p]$ is finite for all $p \ge 1$, $k, l = 0, 1, 2, \ldots$ since there exists $0 < c < 1$ such that

$$\int_{B_v} \left|e^{i(v,\tilde{D}_u F)} - 1\right|^2 \hat{N}(du) \ge c \int_{B_v} |(v, \tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F| \le \frac{1}{|v|}\}} \hat{N}(du)$$

for $|v| > 1$. This proves the assertion of Proposition 3.9. □

3.6 Smoothness of the density for Itô processes

In this section, we choose F to be X_t, where $t \mapsto X_t$ denotes an Itô process from Section 1.3. We seek conditions that induce the condition *(ND)* for F. Normally, the existence of big jumps of the driving process for X_t will disturb the condition *(ND)* to hold. To this end, we decompose the trajectory of X_t into parts with small jumps and parts having big jumps. The reader will find a technique based on the same idea which is used in Section 2.2. The case for canonical SDEs will be treated in Section 4.1.1 below.

In this and the next sections, we assume that supp μ is compact in $\mathbf{R}^m$ and that $T < +\infty$.

3.6.1 Preliminaries

To this end, we divide the jumps of the driving Lévy process into "big" and "small" parts. We consider the process X'_t which is given as a solution to an SDE driven only by small jumps. We put $F' = X'_t$, and show that the condition *(NDB)* (see below) implies the condition *(ND)* for F'.

We also show the existence of smooth density for $F = X_t$ under such conditions. For the proof, we use an analogue of the integration by parts formula.

We recall an Itô process X_t given in Section 1.3 by the SDE

$$X_t = x + \int_0^t b(X_r)dr + \int_0^t \sigma(X_r)dW(r) + \int_0^t \int g(X_{r-}, z)\tilde{N}(drdz) .$$

Here, the coefficients b, σ and g are infinitely times continuously differentiable with respect to x and z, and are assumed to satisfy the conditions in Section 1.3. Furthermore, in this section, all derivatives of b, σ, g of all orders (≥ 1) are assumed to be bounded. Especially, the function g is assumed to satisfy

$$|\nabla_x^\alpha g(x, z)| \leq L^\alpha(z)$$

for some positive functions $L^\alpha(z)$ such that $\lim_{z\to 0} L^\alpha(z) = 0$, and that $\int L^\alpha(z)^p \mu(dz) < +\infty$ for all $p \geq 2$ and all multi-indices α.

Under these conditions, the SDE above has a unique solution under any initial condition x.

The flow condition means

$$\inf_{x,z\in \text{supp } \mu} |\det(I + \nabla g(x, z))| > C > 0 .$$

A sufficient condition for the flow condition is
Condition (D):

$$\sup_{x,z\in \text{supp } \mu} |\nabla g(x, z)| \leq c \tag{6.1}$$

for some $0 < c < 1$.

This condition corresponds to the condition *(A.2)* in Section 1.3 with g replacing γ. The condition *(D)* implies that the map $x \mapsto x + g(x,z)$ is a diffeomorphism for all z, and this, in turn, results in that $t \mapsto (x \mapsto X_t(x))$ is a flow of diffeomorphisms a.s. (cf. [192] Theorem V.10.65). Here, $X_t(x)$ signifies that the process X_t starts from x at $t = 0$.

The Lévy measure μ is said to have a weak drift if the limit $\lim_{\epsilon\to 0}\int_{\epsilon<|z|<1} z_i \mu(dz)$ exists, $i = 1, \dots, m$. We assume μ has a weak drift.

We first show that F', composed only of small jumps, satisfies the condition *(ND)* under a positivity condition *(NDB)* (Theorem 3.6 below). We have shown in Proposition 3.8 above that, under the condition *(ND)*, for any $G \in \mathbf{D}_\infty$ and any $n \in \mathbf{N}$, there exists $C > 0$ such that

$$|E[Ge_v(F')]| \le C(1 + |v|^2)^{-\frac{1}{2}nq_0} \tag{6.2}$$

for all $|v| \ge 1$, where $q_0 > 0$ is what appeared in Proposition 3.8.

For a general F, we recover it from F' by adding big jumps as finite perturbations. We state that the estimate (6.2) is valid for general F in Proposition 3.12 below.

To show these assertions, we choose and fix a small $\delta_0 > 0$ so that (6.1) holds for $|z| < \delta_0$. That is,

$$\sup_{x,|z|\le\delta_0} |\nabla g(x,z)| \le c\,.$$

Constants involved in the calculation below will not be uniform in δ_0, however this does not cause difficulty since we do not tend to $\delta_0 \to 0$. Quite often, it will be convenient to take $\delta_0 = 1$. We decompose the Lévy measure μ into the sum $\mu' + \mu''$, where

$$\mu'(dz) = 1_{(0,\delta_0]}(|z|)\mu(dz)\,,$$
$$\mu''(dz) = 1_{(\delta_0,+\infty)}(|z|)\mu(dz)\,.$$

Accordingly, we decompose $N(dtdz)$ as $N'(dtdz) + N''(dtdz)$.

We decompose the probability space $(\Omega, \mathcal{F}, P)$ as follows: $\Omega = \Omega' \times \Omega''$, $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$, $P = P' \times P''$. Here, $\Omega = \Omega_2$, $\mathcal{F} = \mathcal{F}_2$ and $P = P_2$.

Let z''_t be a Lévy process given by $N''(dtdz)$ defined on $(\Omega'', \mathcal{F}'', P'')$.

We denote by X'_t the solution to the SDE

$$X'_t = x + \int_0^t \left(b(X'_r) - \int_{1<|z|} g(X'_r, z)\mu(dz) \right) dr$$
$$+ \int_0^t \sigma(X'_r)dW(r) + \int_0^t \int_{|z|\le 1} g(X'_r, z)\tilde{N}(drdz).1_{\{|z|\le 1\}}\,.$$

Here, the existence of $\int_{1<|z|} g(x,z)\mu(dz)$ is assumed for each x. We write $X'_{s,t} = X'_t \circ (X'_s)^{-1}$ for $s < t$. Then, X_t is a solution to the SDE driven by

$$X'_t \oplus \int_0^t \int_{|z|>1} g(x,z)N(drdz)\,. \tag{6.3}$$

That is,

$$X_t = X'_t \circ \varepsilon^+_{\mathbf{u}} = X'_{t_n,t} \circ \phi_{z_n} \circ X'_{t_{n-1},t_n} \circ \cdots \circ \phi_{z_1} \circ X'_{t_1} ,$$

where $\mathbf{u} = ((t_1, z_1), \ldots, (t_n, z_n))$, $t_1 < t_2 < \cdots < t_n$, denoting the moments of big jumps such that $|z_i| > 1$ and $\phi_z(x) = x + g(x, z)$. Thus, the function ϕ plays the role of a connector.

Since $\sigma(x), g(x, z)$ satisfy the linear growth condition with respect to x, we can prove the Condition **(R)** for $F' = X'_T$.

Lemma 3.7. *For any positive integer k and $p > 1$, the derivatives satisfy*

$$\sup_{t\in\mathbf{T},\mathbf{u}\in A(1)^k} E\left[\sum_{i=1}^m \sup_{|z|\le 1} |\partial_{z_i}\tilde{D}_{t,z}X' \circ \varepsilon^+_{\mathbf{u}}|^p + \sum_{i,j=1}^m \sup_{|z|\le 1} |\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}X' \circ \varepsilon^+_{\mathbf{u}}|^p\right] < +\infty , \quad (6.4)$$

where $X' = X'_T$.

Proof. We see

$$\tilde{D}_u X' = X'_{t,T} \circ \phi_z(X'_{t-}(x)) - X'_{t,T}(X'_{t-}(x)) .$$

Hence, $\partial_{z_i}\tilde{D}_{t,z}X'$, $\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}X'$ are written by using the flow property as

$$\partial_{z_i}\tilde{D}_{t,z}X' = \nabla X'_{t,T}(X'_{t-}(x))\partial_{z_i}g(X'_{t-}(x), z) ,$$
$$\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}X' = \nabla X'_{t,T}(X'_{t-}(x))\partial_{z_i}\partial_{z_j}g(X'_{t-}(x), z) ,$$

respectively. Hence, we see that

$$E\left[\left(\sum_{i=1}^m \sup_{|z|\le 1}\left|\partial_{z_i}\tilde{D}_{t,z}X' \circ \varepsilon^+_{\mathbf{u}}\right| + \sum_{i,j=1}^m \sup_{|z|\le 1}\left|\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}X' \circ \varepsilon^+_{\mathbf{u}}\right|\right)^p\right]$$

is dominated by

$$CE\left[\left|\nabla X'_{t,T}(X'_t \circ \varepsilon^+_{\mathbf{u}})\right|^{2p}\right]^{\frac{1}{2}} \times E\left[\left(1 + |X'_t \circ \varepsilon^+_{\mathbf{u}}|\right)^{2p}\right]^{\frac{1}{2}} ,$$

where $C > 0$ is independent of t, $\mathbf{u}$. Here we use Sobolev inequality to show the boundedness of $\sup_{|z|\le 1} |\partial_{z_i}\tilde{D}_{t,z}X' \circ \epsilon^+_{\mathbf{u}}|, \ldots$

Next we show for each k, p,

$$\sup_{\mathbf{u}\in A(1)^k} E\left[|\nabla X'_{t,T}(x) \circ \varepsilon^+_{\mathbf{u}}|^{2p}\right] \le C'$$

and

$$\sup_{\mathbf{u}\in A(1)^k} E\left[|X'_t \circ \varepsilon^+_{\mathbf{u}}|^{2p}\right] \le C'(1 + |x|)^{2p}$$

for some $C' > 0$.

We consider the case $t = 0$, $X'_{t,T}(x) = X'_T(x)$ only, since the discussion is similar in other cases. We put $\xi_T = X'_T(x) \circ \varepsilon^+_{\mathbf{u}}$. Notations ξ_t, $\xi_{t,T}$ are defined similarly.

We recall the original SDE for X'_t, and denote it simply by

$$X'_t = x + \int_0^t \mathcal{X}(X'_{r-}, dr) .$$

First we assume $\sigma(x)\sigma(x)^T > 0$. The Malliavin–Shigekawa derivative of ξ_T satisfies the following linear SDE

$$D_t \xi_T = \sigma(\xi_t) + \int_t^T \nabla \mathcal{Y}(\xi_{r-}, dr).D_t \xi_{r-} ,$$

where $\mathcal{Y}(x) = \mathcal{X} \circ \varepsilon^+_{\mathbf{u}}$. See [25] (6-25).

On the other hand,

$$\nabla \xi_{t,T}(\xi_t) = I + \int_t^T \nabla \mathcal{Y}(\xi_{r-}, dr) \nabla \xi_{t,r-}(\xi_t) .$$

Comparing these two, due to the uniqueness of the solution to the linear SDE, the solution to the above SDE is given by

$$D_t \xi_T(x) = \nabla \xi_{t,T}(\xi_t(x)) \sigma(\xi_t(x)) .$$

Since $x \mapsto \sigma(x)$ is of linear growth, for any $k \in \mathbf{N}$ and $p > 2$ there exists a positive constant C such that the solution $D_t \xi_T(x)$ satisfies

$$\sup_{\mathbf{u} \in A(1)^k} E[|D_t \xi_T(x)|^p] \le C(1 + |x|)^p .$$

This proves the second assertion.

As $\nabla \xi_t(x)$ satisfies the linear SDE above, we apply the argument for $\nabla^2 \xi_t(x) = \nabla(\nabla \xi_t(x))$. Then we obtain the first assertion.

Next we consider the case $\sigma(x)\sigma(x)^T = 0$. Calculation using the operator $\tilde{D}_u$ is similar. See [101] Lemma 4.5. Hence, we have (6.4). □

3.6.2 Big perturbations

Let $u_1 = (t_1, z_1) \in U''_1 = \mathbf{T} \times \{z; |z| > 1\}$. Then,

$$X'_t \circ \varepsilon^+_{(t_1,z_1)} = X'_{t_1,t} \circ \phi_{t_1,z_1} \circ X'_{t_1}(x) ,$$

where

$$\phi_{t_1,z_1}(x) = x + g(x, z_1) .[1]$$

1 The operation by ϕ does not depend on t. We write $\phi_{t,z}$ to indicate the moment of the operation.

For $\mathbf{u} = (u_1, \ldots, u_n) = ((t_1, z_1), \ldots, (t_n, z_n)) \in U_1'' \times \cdots \times U_1''$, we put

$$X_t' \circ \varepsilon_{\mathbf{u}}^+ = X_{t_n,t}' \circ \phi_{t_n,z_n} \circ X_{t_{n-1},t_n}' \circ \cdots \circ \phi_{t_1,z_1} \circ X_{t_1}' .$$

Based on this decomposition, we introduce the conditional randomization as follows. We write

$$X_t = X_t' \circ \varepsilon_q^+ = X_t' \circ \varepsilon_{(q(u_1),\ldots,q(u_n))}^+ , \quad t_n \le t < t_{n+1} , \tag{6.5}$$

where $q(u) = (q(u_1), \ldots, q(u_n))$ is a Poisson point process with the counting measure $N''(dtdz)$ and $t_n = \inf\{t; N''((0,t] \times \mathbf{R}^m) \ge n\}$. The reader can recall a similar decomposition of the density in Section 2.3.2 ((3.16)).

As noted above, we first analyse the small jump part X' of X, and then perturb it by big jumps due to $q(u)$.

Let $F' = X_T'(x)$. Then,

$$F' \circ \varepsilon_{t,z}^+ = X_{t,T}' \circ \phi_{t,z} \circ X_{t-}'(x) .$$

Hence,

$$\begin{aligned} \tilde{D}_{t,z} F' &= X_{t,T}' \circ \phi_{t,z}(X_{t-}') - X_{t,T}'(X_{t-}') \\ &= \nabla X_{t,T}'(\phi_{t,\theta z}(X_{t-}')) \nabla_z \phi_{t,\theta z}(X_{t-}') \end{aligned}$$

for some $\theta \in (0,1)$.

This implies $F' \in \mathbf{D}_{1,0,p}$ by Lemma 3.7, and finally we observe $F' \in \mathbf{D}_\infty$. Furthermore, we have

$$\partial_z \tilde{D}_{t,z} F'|_{z=0} = \nabla X_{t,T}'(X_{t-}') G(X_{t-}') .$$

Here, $G(x) = \nabla_z g(x,z)|_{z=0}$.

In view of this, $\tilde{K}_\rho, R$ in Section 3.5.2 turn into

$$\tilde{K}_\rho' = \int_{\mathbf{T}} \nabla X_{t,T}'(X_{t-}') G(X_{t-}') B_\rho G(X_{t-}')^T \nabla X_{t,T}'(X_{t-}')^T dt ,$$

and

$$R' = \int_{\mathbf{T}} \nabla X_{t,T}'(X_{t-}') \sigma(X_{t-}') \sigma(X_{t-}')^T \nabla X_{t,T}'(X_{t-}')^T dt ,$$

respectively. Then,

$$R' + \tilde{K}_\rho' = \int_{\mathbf{T}} \nabla X_{t,T}'(X_{t-}') C_\rho(X_{t-}') \nabla X_{t,T}'(X_{t-}')^T dt ,$$

where

$$C_\rho(x) = \sigma(x)\sigma(x)^T + G(x) B_\rho G(x)^T .$$

Furthermore, $\tilde{K}$ turns into

$$\tilde{K}' = \int_{\mathbf{T}} \nabla X'_{t,T}(X'_{t-})G(X'_{t-})BG(X'_{t-})^T \nabla X'_{t,T}(X'_{t-})^T dt \, .$$

Then,

$$R' + \tilde{K}' = \int_{\mathbf{T}} \nabla X'_{t,T}(X'_{t-})C(X'_{t-})\nabla X'_{t,T}(X'_{t-})^T dt \, .$$

Here,

$$C(x) = \sigma(x)\sigma(x)^T + G(x)BG(x)^T \, .$$

We remark $C_\rho(x) \geq G(x)B_\rho G(x)^T$, and that

$$R' + \tilde{K}' \leq R' + \tilde{K}'_\rho, \quad 0 < \rho < \rho_0 \, . \tag{6.6}$$

Let

$$\tilde{Q}'(v) = (v', (R' + \tilde{K}')v') \, ,$$
$$\tilde{Q}'_\rho(v) = (v', (R' + \tilde{K}'_\rho)v') \, ,$$

corresponding to $\tilde{Q}(v)$, $\tilde{Q}_\rho(v)$ in Section 3.5 where $v' = \frac{v}{|v|}$.

Definition 3.5. We say F satisfies the condition *(NDB)* if there exists $C > 0$ such that

$$(v, C(x)v) \geq C|v|^2 \tag{6.7}$$

holds for all x.

Theorem 3.6. *Assume the condition (NDB) holds for F'. Then, the condition (ND) in Section* 3.3.3 *holds for F'.*

Due to Proposition 3.7, the composition $\Phi \circ F'$ of $\Phi \in \mathcal{S}'$ and F' is justified. The proof of this theorem consists of the following two propositions (Propositions 3.10, 3.11).

Proposition 3.10. *Assume the conditions (NDB) and (D). Then, for each $N \in \mathbf{N}$, the family $\{X'_t(x)\}_{x \in \{|x| \leq N\}}$ satisfies the condition (ND2) for $p > 2$.*

Proof. The matrix $C(x)$ satisfies (6.7). By the expression of $R' + \tilde{K}'_\rho$, by Jensen's inequality $((\frac{1}{t}\int_0^t f(s)ds)^{-1} \leq \frac{1}{t}\int_0^t f(s)^{-1}ds$ for $f(.) > 0)$ and (6.6), we have

$$\begin{aligned}\tilde{Q}'_\rho(v)^{-1} \circ \varepsilon^+_{\mathbf{u}} &\leq \tilde{Q}'(v)^{-1} \circ \varepsilon^+_{\mathbf{u}} \\ &\leq \frac{1}{CT^2} \int_{\mathbf{T}} \left|(\nabla_x X'_{t,T}(X'_t(x)) \circ \varepsilon^+_{\mathbf{u}})^{-1}\right|^2 dt, \quad 0 < \rho < \rho_0 \, .\end{aligned}$$

Hence,

$$E\left[\tilde{Q}'_\rho(v)^{-p} \circ \varepsilon^+_{\mathbf{u}}\right] \leq \frac{1}{CT^2} \int_{\mathbf{T}} E\left[\left|(\nabla_x X'_{t,T}(X'_t(x)) \circ \varepsilon^+_{\mathbf{u}})^{-1}\right|^{2p}\right]^{\frac{1}{2}} dt \, .$$

Here, we have

$$\sup_{\mathbf{u}\in A(1)^k} E\left[\left|\left(\nabla_x X'_{t,T}(X'_t(x))\circ\varepsilon^+_{\mathbf{u}}\right)^{-1}\right|^{2p}\right]^{\frac{1}{2}} < +\infty \tag{6.8}$$

for $k = 1, 2, \dots$. Hence, we have the assertion.

We prove (6.8) below. We introduce a matrix-valued process $U^{x,t}(u)$, $t \le u \le T$ by

$$U^{x,t}(u) = \int_t^u \nabla b(X'_{t,r-}(x))dr + \int_t^u \nabla\sigma(X'_{t,r-}(x))dW(r)$$
$$+ \int_u^t \int \nabla g(X'_{t,r-}(x), z)\tilde{N}(drdz) .$$

Then, the matrix-valued process

$$\Phi^{x,t}(u) \equiv \nabla X'_{t,u}(x)$$

satisfies the linear SDE

$$\Phi^{x,t}(u) = I + \int_t^u dU^{x,t}\Phi^{x,t}(r-) .$$

That is, Φ is the Doléans' exponential (stochastic exponential) $\mathcal{E}(U)^{t,x}$.

Since b, σ and g are assumed to be C_b^∞ functions, the coefficients $\nabla b(x)$, $\nabla\sigma(x)$ of $U^{x,t}(u)$ are bounded, and $|\nabla g(x,z)| \le K^1(z)$, where $K^1(z)$ satisfies $\int K^1(z)^p\mu(dz) < +\infty$ for any $p \ge 2$. Then, by the uniqueness of the Doléans' exponential and by the direct expression of it ([192] Section V.3),

$$\sup_{x\in\mathbf{R}^d, t\in[0,T], t\le u\le T} E[|\Phi^{x,t}(u)|^p] < +\infty, \quad p > 2 .$$

We observe

$$I + \Delta U^{x,t}(r) = I + \nabla g(X'_{t,r}(x), z), \quad z = q(r) .$$

Therefore, the matrix $I + \Delta U^{x,t}(r)$ is invertible by the condition *(D)*.

Let $V^{x,t}(u)$ be the process given by

$$V^{x,t}(u) = -U^{x,t}(u) + [U^{x,t}, U^{x,t}]^c_u + \sum_{t\le r\le u}(I + \Delta U^{x,t}(r))^{-1}(\Delta U^{x,t}(r))^2 . \tag{6.9}$$

Here, $[U^{x,t}, U^{x,t}]^c_u$ denotes the continuous part of the matrix-valued process $(\sum_{k=1}^d [(U^{x,t})^{ik}, (U^{x,t})^{kj}]_u)$.

We introduce another matrix-valued process $\Psi^{x,t}(u)$, $t \le u \le T$ by the SDE

$$\Psi^{x,t}(u) = I + \int_t^u \Psi^{x,t}(r-)dV^{x,t}(r) .$$

That is, $\Psi^{x,t}(u) = \mathcal{E}(V^{x,t})_u$. The process $\Psi^{x,t}(u)$ is the inverse of $\Phi^{x,t}(u)$, that is, $\Psi^{x,t}(u) = (\nabla X'_{t,u}(x))^{-1}$ ([192] Corollary to Theorem II 38).

We can write $\Psi^{x,t}(u)$ in the form

$$\Psi^{x,t}(u) = \int_t^u \Psi^{x,t}(r-)\nabla b'(X'_{t,r-}(x))dr + \int_t^u \Psi^{x,t}(r-)\nabla\sigma(X'_{t,r-}(x))dW(r) + \int_t^u \int \Psi^{x,t}(r-)\nabla h(X'_{t,r-}(x), z)\tilde{N}(drdz) ,$$

where $\nabla b'$ is the drift part determined by (6.9), and h is given by the relation $I + h = (I + g)^{-1}$. As the coefficients $\nabla b'(x)$, $\nabla\sigma(x)$ and $\nabla h(x, z)$ are bounded functions (due to the condition *(D)*), we have

$$\sup_{x\in\mathbf{R}^d, t\in[0,T], t<u<T} E[|\Psi^{x,t}(u)|^p] < +\infty, \quad p > 2 .$$

By using the argument in [95] (Lemma 6.3 and (6.17)), we have the assertion (6.8). □

Proposition 3.11. *The condition (ND2) for F' implies the condition (ND) for F'.*

Proof. This proposition follows directly from Lemma 3.6 and (6.4). □

In view of Propositions 3.10, 3.11, we see that the condition *(ND)* holds for $F' = X'_t$ under *(NDB)* and the assertion in Theorem 3.6 holds.

3.6.3 Concatenation (I)

In this and the next subsection, we show the assertion (6.2) with F' replaced by $F = X_t$.

Proposition 3.12. *Under the condition (NDB), for any $G \in \mathbf{D}_\infty$ and any $n \in \mathbf{N}$, there exists $C > 0$ such that*

$$|E[Ge_v(F)]| \le C(1 + |v|^2)^{-\frac{1}{2}nq_0} \tag{6.10}$$

for all $|v| \ge 1$, where $q_0 > 0$ is what appeared in Proposition 3.8.

Choosing $G \equiv 1$, this assertion implies that F has a smooth density function p_F under the condition *(NDB)*. See the introduction of Section 2.1

To show Proposition 3.12, we first claim the following proposition, where F' satisfies the condition *(D)*. We put

$$E_v = \{\sup_{u\in A(|v|^{-1/\beta})} |\tilde{D}_u F'| \le |v|^{-1}\} = E(|v|^{-\frac{1}{\beta}}) .$$

Proposition 3.13. *Assume the condition (NDB). Then, for any $n \in \mathbf{N}$, there exist $k, l \in \mathbf{N}$, $p \ge 2$ and $C = C_{k,l,p} > 0$ such that*

$$|E[Ge_v(F').1_{E_v}]| \le C(1 + |v|^2)^{-\frac{1}{2}nq_0}|G|_{k,l,p}\{|F'|^n_{k,l,p} \times \sup_{v,0<\rho<\rho_0} |\tilde{Q}'^{-1}_\rho(v)|^n_{k,l,p}\}, \quad |v| \ge \rho^{-\beta} .$$

We remark that there exists $0 < c < 1$ such that $Q^{F'}(v) \geq c\tilde{Q}'_{|v|^{-\frac{1}{\beta}}}(v)$ if $|\tilde{D}_u F'| \leq |v|^{-1}$, where $Q^{F'}(v)$ is given in Proposition 3.6 (Section 3.5.1) for $F = F'$. Then, this proposition is proved as Proposition 3.6.

According to (6.5),

$$(R' + \tilde{K}'_\rho) \circ \varepsilon_q^+ = R + \tilde{K}_\rho \quad \text{a.s. } dP\,. \tag{6.11}$$

Since F' and N'' are independent, by the above proposition, we have

$$\begin{aligned} E[Ge_v(F).1_{E_v}] &= E''\left[E'[Ge_v(F' \circ \varepsilon_q^+).1_{E_v}]\right] \\ &\leq C(1+|v|^2)^{-\frac{1}{2}nq_0}|G|_{k,l,p} \\ &\quad \times E''\left[\left\{|F' \circ \varepsilon_q^+|_{k,l,p}^{2n} + \sup_{v,0<\rho<\rho_0}\left|\tilde{Q}'^{-1}_\rho(v) \circ \varepsilon_q^+\right|_{k,l,p}^{2n}\right\}\right]\,. \end{aligned} \tag{6.12}$$

Our objective below is to show the finiteness of the right-hand side of (6.12) under the conditions *(NDB)*. If (6.12) holds, choosing $G \equiv 1$, we have the exponential decay of the characteristic function.

In (6.12) we see

$$\int |F' \circ \varepsilon_q^+|_{k,l,p}^{2n} dP''(q) \leq \left(\int |F' \circ \varepsilon_q^+|_{k,l,p}^{p} dP''(q)\right)^{\frac{2n}{p}} = |F|_{k,l,p}^{2n}\,.$$

This norm is finite for all k, l, p. We can check it by a direct calculation as in Proposition 3.16 below.

It remains to show

$$\sup_{v,0<\rho<\rho_0} \int \left|\tilde{Q}'^{-1}_\rho(v) \circ \varepsilon_q^+\right|_{k,l,p}^{2n} dP''(q) < +\infty \tag{6.13}$$

under *(NDB)*.

To prove this assertion, we begin with the case $q(u) = \emptyset$.

Proposition 3.14.

$$\begin{aligned} \left|\tilde{Q}'_\rho(v)^{-1}\right|_{k,l,p} &\leq C_{k,l,p}|\tilde{Q}'_\rho(v)|_{k,l,2(k+1)(l+1)p} \\ &\quad \times \left\{\sum_{(\mathbf{u}_j)} \Pi_{j=1}^s \left(1 + E\left[\int \left|\tilde{Q}'_\rho(v)^{-1} \circ \varepsilon_{\mathbf{u}_j}^+\right|^{2(k+1)(l+1)p} \hat{M}(d\mathbf{u}_j)\right]^{\frac{1}{2(k+1)p}}\right)\right\}\,, \end{aligned}$$

where $\mathbf{u}_j$ *runs over all subsets of* $\mathbf{u} = \{u_1, \ldots, u_k\}$, $u_j \in A(1)$ *such that* $\mathbf{u}_1 \cup \cdots \cup \mathbf{u}_s = \mathbf{u}$.

Proof. Case $(k, l) = (0, 1)$. Since $D_t\tilde{Q}'(v)^{-1} = -\frac{D_t\tilde{Q}'(v)}{\tilde{Q}'^2(v)}$, we have

$$\left|\tilde{Q}'_\rho(v)^{-1}\right|_{0,1,p} \leq C_{0,1,p}|\tilde{Q}'_\rho(v)|_{0,1,2p}|\tilde{Q}'(v)^{-1}|_{0,0,4p}^2\,.$$

Case $(k,l)=(1,0)$. Since $\tilde{D}_u\tilde{Q}'(v)^{-1}=-\frac{\tilde{D}_u\tilde{Q}'(v)}{\tilde{Q}'(v)\tilde{Q}'(v)\circ\varepsilon_u^+}$, we have

$$\begin{aligned}\left|\tilde{Q}'_\rho(v)^{-1}\right|_{1,0,p} &\le C_{1,0,p}|\tilde{Q}'_\rho(v)|_{1,0,2p}\left(E\left[\left|\tilde{Q}'_\rho(v)^{-1}\right|^{4p}\right]\right)^{\frac{1}{4p}}\\ &\quad\times\left(E\left[\left(\int\left|\tilde{Q}'_\rho(v)^{-1}\circ\varepsilon_u^+\right|^{2p}\hat{M}(du)\right)^2\right]\right)^{\frac{1}{4p}}\\ &\le C'_{1,0,p}|\tilde{Q}'_\rho(v)|_{1,0,2p}\left(E\left[\left|\tilde{Q}'_\rho(v)^{-1}\right|^{4p}\right]\right)^{\frac{1}{4p}}\\ &\quad\times\left(1+E\left[\int\left|\tilde{Q}'_\rho(v)^{-1}\circ\varepsilon_u^+\right|^{4p}\hat{M}(du)\right]\right)^{\frac{1}{4p}}.\end{aligned}$$

In what follows, we write Q for $\tilde{Q}'_\rho$ for simplicity. In the general case,

$$D_{\mathbf{t}}^lQ^{-1}=\sum_r(-1)^r\frac{D_{\mathbf{t}_1}^{l_1}Q\cdots D_{\mathbf{t}_r}^{l_r}Q}{Q^{r+1}},$$

where the sum is taken with respect to all the choices of $(l_1,\ldots,l_r)$, $l_j\ge 1$ such that $l_1+\cdots+l_r=l$. Here, $|\mathbf{t}|=l$ and $|\mathbf{t}_j|=l_j$, $j=1,\ldots,r$.

Hence, by the formula in Section 3.3.2 with $n=r+1$,

$$\begin{aligned}\tilde{D}_{\mathbf{u}}^kD_{\mathbf{t}}^lQ^{-1}&=\sum_{\mathbf{u}_0\cup\mathbf{u}_1\cup\cdots\cup\mathbf{u}_r=\mathbf{u}}\tilde{D}_{\mathbf{u}_1}D_{\mathbf{t}_1}^{l_1}Q\cdots\tilde{D}_{\mathbf{u}_r}D_{\mathbf{t}_r}^{l_r}Q\tilde{D}_{\mathbf{u}_0}Q^{-(r+1)}\\ &=\sum_r\sum_s\operatorname{sgn}(r,s)Y_{k_0,\ldots,k_s}^{l'_1,\ldots,l'_r}Q^{-(r+1)}\circ\varepsilon_{\mathbf{u}_0}^+.\end{aligned}$$

Here, $Y=Y_{k_0,\ldots,k_s}^{l'_1,\ldots,l'_r}$ is written by

$$Y_{k_0,\ldots,k_s}^{l'_1,\ldots,l'_r}=\left(D_{\mathbf{t}'_1}^{l'_1}Q\cdots D_{\mathbf{t}'_r}^{l'_r}Q\right)\cdot D_{\mathbf{t}'_1}^{l'_1}Q\circ\varepsilon_{\mathbf{u}_1}^+\cdots D_{\mathbf{t}'_s}^{l'_s}Q\circ\varepsilon_{\mathbf{u}_s}^+,$$

and $\sum_s$ denotes the sum with respect to all the choices of $\mathbf{u}_0,\ldots,\mathbf{u}_s\subset\{u_1,\ldots,u_k\}=\mathbf{u}$ for which $|\mathbf{u}_j|=k_j$, $j=0,\ldots,s$. We remark $r\le l$ and $s\le k$.

To see this, we can calculate, for $k=2$, $\mathbf{u}=(u_1,u_2)$, $\tilde{D}_u\tilde{Q}'(v)^{-1}$ as

$$\begin{aligned}\tilde{D}_{u_2}\tilde{D}_{u_1}Q^{-1}(v)&=\tilde{D}_{u_2}\left(-\frac{\tilde{D}_{u_1}Q}{Q\cdot Q\circ\varepsilon_{u_1}^+}\right)\\ &=-\tilde{D}_{u_2}\tilde{D}_{u_1}Q\cdot(Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1}-\tilde{D}_{u_1}Q\tilde{D}_{u_2}(Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1}\\ &\quad-\tilde{D}_{u_2}\tilde{D}_{u_1}Q\cdot\tilde{D}_{u_2}(Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1}\\ &=-\mathrm{I}-\mathrm{II}-\mathrm{III}\quad\text{(say)}.\end{aligned}$$

Here,

$$\begin{aligned}\mathrm{II}&=\tilde{D}_{u_1}Q\cdot\left((Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1}\circ\varepsilon_{u_2}^+-(Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1}\right)\\ &=\tilde{D}_{u_1}Q\cdot(Q\circ\varepsilon_{u_2}^+\cdot Q\circ\varepsilon_{u_1}^+\circ\varepsilon_{u_2}^+)^{-1}-\tilde{D}_{u_1}Q\cdot(Q\cdot Q\circ\varepsilon_{u_1}^+)^{-1},\end{aligned}$$

and terms I, III are calculated similarly. Hence, terms $\tilde{D}_{u_1}Q$, $\tilde{D}_{u_2}\tilde{D}_{u_1}Q$ arise from the term $\tilde{D}_{\mathbf{u}}Q^{-1}$. Further, let $A = Q$, $B = Q^{-(r+1)}$. Then, we have

$$\begin{aligned}\tilde{D}_{u_2}\tilde{D}_{u_1}(D_tA \cdot B) = &\tilde{D}_{u_2}\tilde{D}_{u_1}D_tA \cdot B + \tilde{D}_{u_1}D_tA \cdot \tilde{D}_{u_2}B + \tilde{D}_{u_2}\tilde{D}_{u_1}D_tA \cdot \tilde{D}_{u_2}B\\ &+ \tilde{D}_{u_2}D_tA \cdot \tilde{D}_{u_1}B + D_tA \cdot \tilde{D}_{u_2}\tilde{D}_{u_1}B + \tilde{D}_{u_2}D_tA \cdot \tilde{D}_{u_2}\tilde{D}_{u_1}B\\ &+ \tilde{D}_{u_2}\tilde{D}_{u_1}D_tA \cdot \tilde{D}_{u_1}B + \tilde{D}_{u_1}D_tA \cdot \tilde{D}_{u_2}\tilde{D}_{u_1}B\\ &+ \tilde{D}_{u_2}\tilde{D}_{u_1}D_tA \cdot \tilde{D}_{u_2}\tilde{D}_{u_1}B\ .\end{aligned}$$

Noting that $\tilde{D}_{u_1}B = B \circ \varepsilon^+_{u_1} - B$, we have the above expression for Y.

We have

$$E\left[\int\left(\int_{\mathbf{T}^l}\left|\frac{Y^{l'_1,\dots,l'_r}_{k_0,\dots,k_s}}{\gamma(\mathbf{u})}\right|^2 d\mathbf{t}\right)^p \hat{M}(d\mathbf{u})\right]^{1/2p} \le |Q|_{k,l,(s+1)(r+1)p} \le C|Q|_{k,l,(k+1)(l+1)p}\ .$$

Here, $C > 0$ does not depend on v.

On the other hand,

$$\begin{aligned}&E\left[\int\left|Q^{-(r+1)} \circ \varepsilon^+_{\mathbf{u}_0}\right|^{2p} \hat{M}(d\mathbf{u}_0)\right]^{1/2p}\\ &\le \sum_{(u'_j)} \Pi^s_{j=1} E\left[\int\left|Q^{-1} \circ \varepsilon^+_{\mathbf{u}'_j}\right|^{2(s+1)(r+1)p} \hat{M}(d\mathbf{u}'_j)\right]^{1/2(s+1)p}\ .\end{aligned}$$

These prove the assertion. □

Due to the assumption **(NDB)**, we have desired finiteness in the case $q(u) = \emptyset$ by Proposition 3.10.

Using Proposition 3.14, we have the following result for $F' = X'_t(x)$ which is more precise than Proposition 3.6 in Section 3.5.1.

Proposition 3.15. *Let F' and G satisfy the same assumption as Proposition* 3.13. *We have then for each n there exist $k, l \in \mathbf{N}$, $p, p' > 2$ and $C > 0$ such that for $|v| > 1$*

$$\begin{aligned}|E[Ge_v(F')]| \le\ & C(1 + |v|^2)^{-\frac{1}{2}nq_0}\Big(|F'|^{3n}_{k+1,l+1,2(l+1)(k+1)p}\\ &\times\left\{\sum_{(\mathbf{u}_j)} \Pi^s_{j=1}(1 + E[\int \sup_{0<\rho<\rho_0}(\tilde{Q}'_\rho(v))^{-2(l+1)(k+1)p} \circ \varepsilon^+_{\mathbf{u}_j}\hat{M}(d\mathbf{u}_j)]^{\frac{1}{2(k+1)p}})\right\}^n\\ &+ \tilde{\Phi}_{p'}(F')\Big)|G|_{k,l,p}, \quad |v| > \rho^{-\beta}\ .\end{aligned} \tag{6.14}$$

Here $\tilde{\Phi}_p(F') = \sup_{t\in\mathbf{T}} \Phi_{p,t}(F')$ where

$$\Phi_{p,t}(F') = \sup_{0<\rho<\rho_0} \frac{1}{\rho^p} E[\sup_{0<r<t,|z|<\rho} |\tilde{D}_u F'|^p]\ ,$$

and $\mathbf{u}_j$ runs over all subsets of $\{u_1, \dots, u_k\}$, $\mathbf{u}_j \in A(1)$, $j = 1, \dots, s$ such that $|\mathbf{u}_1| + \dots + |\mathbf{u}_s| = k$, $|\mathbf{u}_j| \ge 1$ for $s \le k$.

Proof. We recall

$$Q^{F'}(v) = (v', Rv') + \frac{1}{|v|^2 \varphi(|v|^{-\frac{1}{\beta}})} \int_{B_v} |e^{i(v,\tilde{D}_u F')} - 1|^2 \hat{N}(du) ,$$

and

$$\tilde{Q}'_\rho(v) = (v', (R' + \tilde{K}'_\rho)v')$$

where $v' = \frac{v}{|v|}$, $v \in \mathbf{R}^d$, $\rho > 0$.

We start from Proposition 3.6

$$|E[Ge_v(F')]| \le C(1 + |v|^2)^{-\frac{1}{2}nq_0} |G|_{k,l,p} |F'|^n_{k,l,p} \sup_{|v|>1} |Q^{F'}(v)^{-1}|^n_{k,l,p} , \tag{6.15}$$

for $|v| \ge 1$.

We need a bound for $E[Q^{F'}(v)^{-p}]$. We have, since $Q^{F'}(v) \ge Q^{F'}(v).1_{E_v} \ge c\,\tilde{Q}'_{|v|^{-\frac{1}{\beta}}}(v)$ for some $0 < c < 1$,

$$E[Q^{F'}(v)^{-p}.1_{E_v}] \le c^{-p} E[\tilde{Q}'_{\tilde{\rho}}(v)^{-p}] , \tag{6.16}$$

where $\tilde{\rho} = |v|^{-\frac{1}{\beta}}$.

Hence by Proposition 3.14,

$$|Q^{F'}(v)^{-1}.1_{E_v}|_{k,l,p} \le C_{k,l,p} |Q^{F'}(v)|_{k,l,2(k+1)(l+1)p} \times \left\{ \sum_{(\mathbf{u}_j)} \Pi_{j=1}^s \left(1 + E\left[\int |Q^{F'}(v)^{-1} \circ \varepsilon^+_{\mathbf{u}_j}|^{2(k+1)(l+1)p} \hat{M}(d\mathbf{u}_j) \right]^{\frac{1}{2(k+1)p}} \right) \right\} . \tag{6.17}$$

In the expression (6.17), first we have

$$\sup_{|v|\ge 1} |Q^{F'}(v)|_{k,l,2(k+1)(l+1)p} \le C|F'|^2_{k+1,l+1,2(k+1)(l+1)p} . \tag{6.18}$$

See [77] (6.9).

Second, we treat the the expression $E[\ldots]$ in (6.17).

$$\begin{aligned} &E[\int Q^{F'}(v)^{-\tilde{p}} \circ \varepsilon^+_{\mathbf{u}_j} \hat{M}(d\mathbf{u}_j)]^{\frac{1}{2(k+1)p}} \\ &\le E[\int \sup_{0<\rho<\rho_0} \tilde{Q}'_\rho(v)^{-\tilde{p}} \circ \varepsilon^+_{\mathbf{u}_j} \hat{M}(d\mathbf{u}_j)]^{\frac{1}{2(k+1)p}} , \end{aligned} \tag{6.19}$$

where we choose $\tilde{p} = 2(k+1)(l+1)p$. This implies

$$|Q^{F'}(v)^{-1}.1_{E_v}|^n_{k,l,p} \le C|F'|^{2n}_{k+1,l+1,2(k+1)(l+1)p} \times \left\{ \sum_{(\mathbf{u}_j)} \Pi_{j=1}^s \left(1 + E\left[\int \sup_{0<\rho<\rho_0} \tilde{Q}'_\rho(v)^{-\tilde{p}} \circ \varepsilon^+_{\mathbf{u}_j} \hat{M}(d\mathbf{u}_j) \right]^{\frac{1}{2(k+1)p}} \right) \right\}^n . \tag{6.20}$$

We have also

$$E[e^{(v,F')}G(1-1_{E_v})]$$
$$\le P(\sup_{u\in[0,t]\times A(|v|^{-1/\beta})}|\tilde{D}_uF'|\ge |v|^{-1})|G|_{0,0,2}$$
$$\le \frac{|v|^{-p'/\beta}}{|v|^{-p'}}\Phi_{p',t}(F')|G|_{0,0,2}$$
$$\le |v|^{-2(1-\frac{\alpha}{2\beta})n}\Phi_{p',t}(F')|G|_{0,0,2}\le |v|^{-2q_0n}\Phi_{p',t}(F')|G|_{0,0,2}$$

holds if $p'=p'_n$ is chosen so that $p'\ge\frac{(2-\frac{\alpha}{\beta}n)}{(\frac{1}{\beta}-1)}$. Here we used Chebyshev's inequality.

Combining these we have the assertion. □

Remark 3.7. We can show $\tilde{\Phi}_p(F')<+\infty$ for each $p>2$.

Indeed, we recall $F'=X'_t$ and we write $X'_{s,t}=X'_t\circ(X'_s)^{-1}$. We prove

$$\sup_{x,t\in\mathbf{T}}\Phi_{p,t}(F')<+\infty\,.$$

We write $\tilde{D}_uF'$ by $\tilde{D}_{r,z}X'_t$.

Using the process $Y^z_s(x)$, where

$$Y^z_{t-r}=X'_{0,t-r}\circ\phi_{r,z}(X'_{-r,0}(x))-X'_{-r,t-r}(x)\,,$$

the law of $\{Y^z_{t-r};0<r<t,|z|<1\}$ coincides with the law of $\{\tilde{D}_{r,z}X'_{0,t},0<r<t,|z|<1\}$.

The process Y^z_{t-r} satisfies the integral equation

$$Y^z_{t-r}(x)=Y^z_s+\int_0^{t-r}b(X'_{-r,0})ds+\int_0^{t-r}\sigma(X'_{-r,s})dW(s)$$
$$+\int_0^{t-r}\int G(s,z')\tilde{N}(dsdz')+\int_0^{t-r}\int G(s,z')ds\mu(dz')\,,$$

where

$$Y^z_0=\phi_{r,z}(X'_{-r,0})-X'_{-r,0}=g(X'_{-r,0},z)$$

and

$$G(s,z')=g(X'_{0,s}\circ\phi_{r,z}\circ X'_{-r,0},z)-g(X'_{-r,s},z')\,.$$

The fourth term of the right hand side is a martingale. Therefore, by Doob's inequality, we have for some $c>0$

$$E[\sup_{0<r<t} |Y^z_{t-r}|^p] \le c \left\{ E[|Y^z_0|^p] + E\left[\left(\int_0^t |b(X'_{-r,s})|ds\right)^p\right] \right.$$
$$+ E\left[\left(\int_0^t |\sigma(X'_{-r,s})|^2 dW(s)\right)^{p/2}\right] + E\left[\left|\int_0^t \int G(s,z')\tilde{N}(dsdz')\right|^p\right]$$
$$\left. +E\left[\left(\int_0^t \int |G(s,z')|^2 ds\mu(dz')\right)^{p/2}\right]\right\} .$$

Each term of the right hand side can be estimated directly, and we can show that

$$\sup_{x,t\in\mathbf{T}} E[\sup_{0<r<t} |Y^z_{t-r}(x)|^p] < +\infty .$$

Since $z \mapsto Y^z_{t-r}$ is differentiable, and since any moments of the derivatives are uniformly bounded with respect to x and t, we can show that there exists a positive constant c' such that

$$\sup_{x,t\in\mathbf{T}} E[\sup_{0<r<t} |Y^z_{t-r}(x) - Y^{z'}_{t-r}(x)|^p] \le c'|z-z'|^p$$

for any $|z| \le 1$, $|z'| \le 1$, by a similar argument. Similarly, we have

$$\sup_{x,t\in\mathbf{T}} \sup_{|z|\le 1} E[\sup_{0<r<t} |\partial^{\mathbf{i}}_z Y^z_{t-r}(x)|^p] < +\infty ,$$

and

$$\sup_{x,t\in\mathbf{T}} E\left[\int_{|z|\le 1} \sup_{0<r<t} |\partial^{\mathbf{i}}_z Y^z_{t-r}(x)|^p dz\right] < +\infty$$

for $|\mathbf{i}| = 1, 2$, where $\mathbf{i}$ denotes a multi-index. By Sobolev's theorem there exists a positive constant C such that

$$\sup_{x,t\in\mathbf{T}} E[\sup_{0<r<t} \sup_{|z|\le 1} |\partial^{\mathbf{i}}_z Y^z_{t-r}(x)|^p] \le C ,$$

for $|\mathbf{i}| = 1$. Since $Y^0_{t-r}(x) = 0$, this implies

$$\sup_{x,t\in\mathbf{T}} E\left[\sup_{0<r<t} \sup_{|z|\le 1} \left|\frac{Y^z_{t-r}(x)}{z}\right|^p\right] \le C .$$

This implies

$$\sup_{x,t\in\mathbf{T}} \sup_{0<\rho<\rho_0} \frac{1}{\rho^p} E[\sup_{0<r<t} \sup_{|z|\le 1} |\tilde{D}_{r,z} X'_{t-r}(x)|^p] < +\infty .$$

This proves the assertion.

This completes the proof of Proposition 3.12 in case $q(u) = \emptyset$.

3.6.4 Concatenation (II) – the case that *(D)* may fail

Finally, we consider the case that F may not satisfy the condition *(D)*. That is, the case $q(u) \neq \emptyset$. We still assume that supp μ *is compact in* $\mathbf{R}^m$ *and that* $T < +\infty$.

To verify (6.13) in this general case, we construct a process $\tilde{X}'(x, S_t, t)$ as follows. Here again, we encounter the methodology in Section 2.3.2. By $S_t = \{s_1, \dots, s_{k'}\}$, we denote the jump times of z''_s in $[0, t]$.

Let $\mathcal{S}_{t,k'} = [0, t]^{k'} / \sim$, where $\sim$ means the identification of $(s_1, \dots, s_{k'})$ by permutation. Let $\mathcal{S}_t = \amalg_{k' \geq 0} \mathcal{S}_{t,k'}$.

The law $\tilde{P}_t$ of the moments of the big jumps related to z''_s is given by

$$\int f(S_t) d\tilde{P}_t(S_t) \tag{6.21}$$

$$= \left(t \int \mu''(dz)\right)^{k'} \frac{1}{k'!} \exp\left(-t \int \mu''(dz)\right) \times \frac{1}{t^{k'}} \int \cdots \int f(s_1, \dots, s_{k'}) ds_1 \cdots ds_{k'}$$

$$= \left(\int \mu''(dz)\right)^{k'} \frac{1}{k'!} \exp\left(-t \int \mu''(dz)\right) \times \int \cdots \int f(s_1, \dots, s_{k'}) ds_1 \cdots ds_{k'}$$

if $\# S_t = k'$, where f is a symmetric function on $[0, t]^{k'}$.

Let $S_t \in \mathcal{S}_t$. We introduce a new process $\tilde{X}'(x, S_t, t)$ by the SDE:

$$\begin{aligned} \tilde{X}'(x, S_t, t) = x &+ \int_0^t \left\{ b(\tilde{X}'(x, S_t, r)) - \int g(\tilde{X}'(x, S_t, r), z)\mu_1(dz) \right\} dr \\ &+ \int_0^t \sigma(\tilde{X}'(x, S_t, r)) dW(r) + \int_0^t \int g(\tilde{X}'(x, S_t, r-), z) \tilde{N}'(dr dz) \\ &+ \sum_{s_i \in S_t, s_i \leq t} g(\tilde{X}'(x, S_t, s_i-), \xi_i) . \end{aligned}$$

Here, (ξ_i) are independent and identically distributed, obeying the probability law

$$\mu''_1(dz) = \frac{\mu''(dz)}{\int \mu''(dz)} .$$

We put

$$\tilde{F}' = \tilde{X}'(x, S_t, t) ,$$

identifying $q(u) = ((t_1, z_1), \dots, (t_{k'}, z_{k'}))$ with $(S_t, (\xi_i))$. Then, $\tilde{F}'$ can be regarded as $F' \circ \varepsilon^+_{q(u)}$.

By using this notation, we can identify $\tilde{Q}'_\rho(v) \circ \varepsilon^+_q$ with $\tilde{Q}'_\rho(v)(S_t)$. Hence, our objective is to show:

Proposition 3.16. *Assume the condition (NDB). Then,*

$$\sup_{|v| \geq 1} \left(\int E' \left[\int \left| \tilde{Q}'_\rho(v)^{-1}(S_t) \circ \varepsilon^+_{\mathbf{u}} \right|^{2(k+1)p'} \hat{M}(d\mathbf{u}) \right] dP''(q) \right) < +\infty ,$$

$$\mathbf{u} \in A(1)^k , \quad k = 1, 2, \dots , \tag{6.22}$$

for each even number $p' \geq 2$. *Here, the expectation with respect to the measure* P' *is denoted by* E'.

Proof of Proposition 3.16. *Step 1.* In case $k = 1$, with $u_1 = (t_1, z_1)$,

$$\tilde{F}' \circ \varepsilon^+_{(t_1,z_1)} = \tilde{X}'_{t_1,t} \circ \phi_{t_1,z_1} \circ \tilde{X}'_{t_1}(x) ,$$

where

$$\phi_{t,z}(x) = x + g(x, z) .$$

We remark

$$\phi_{t,0}(x) = x, \quad \nabla_z \phi_{t,z}(x) = \nabla_z g(x, z) .$$

Hence,

$$\begin{aligned}\tilde{D}_{t_1,z_1}\tilde{X}' &= \tilde{X}'_{t_1,t} \circ \phi_{t_1,z_1} \circ \tilde{X}'_{t_1-}(x) - \tilde{X}'_{t_1,t} \circ \tilde{X}'_{t_1-}(x) \\ &= \nabla\tilde{X}'_{t_1,t}(\phi_{t_1,\theta z_1}(\tilde{X}'_{t_1-}))\nabla_z \phi_{t_1,\theta z_1}(\tilde{X}'_{t_1})z_1\end{aligned}$$

for $\theta \in (0, 1)$, and

$$\partial\tilde{D}_{t_1,0}\tilde{X}' = \nabla\tilde{X}'_{t_1,t_1}(\tilde{X}'_{t_1-})\nabla_z g(\tilde{X}'_{t_1}, 0) = \nabla\tilde{X}'_{t_1,t}(\tilde{X}'_{t_1-})G(\tilde{X}'_{t_1}) .$$

We see $\tilde{X}' \in \mathbf{D}_{1,0,p}$, and finally $\tilde{X}' \in \mathbf{D}_\infty$.

Hence,

$$\begin{aligned}\tilde{Q}'_\rho(v)(S_t) \circ \varepsilon^+_{u_1} - \tilde{Q}'_\rho(v)(S_t) \geq \frac{1}{2}\Bigg(v', \int_{\mathbf{T}} \Bigg\{\nabla\tilde{X}'_{t_1,t}G(\tilde{X}'_{t_1})z_1\Bigg(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \\ +G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Bigg)z_1^T G(\tilde{X}'_{t_1})^T(\nabla\tilde{X}'_{t_1,t})^T\Bigg\}dtv'\Bigg) , \quad |z_1| < 1 .\end{aligned}$$

Then, since $\tilde{Q}'_\rho(v)(S_t) \geq 0$,

$$\begin{aligned}\tilde{Q}'_\rho(v)(S_t)^{-1} \circ \varepsilon^+_{u_1} \leq c\Bigg(v', \Bigg(\int_{\mathbf{T}} \nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1})z_1\Bigg(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \\ +G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Bigg)z_1^T G(\tilde{X}'_{t_1})^T(\nabla\tilde{X}'_{t_1,t}(x))^T dt\Bigg)v'\Bigg)^{-1} .\end{aligned}$$

Here, we have

$$E\left[\left|\left(\nabla\tilde{X}'_{t_1,t}(x)\right)^{-1}\right|^p\right] \leq C\exp\{c_p(t - t_1)\} . \tag{6.23}$$

Indeed, let $\phi_{s,t}$ denote the stochastic flow generated by $\tilde{X}'(x, S_t, t)$. Consider the reversed process

$$Z_s = \phi^{-1}_{(t-s)-,t} .$$

Then, $Z_s(S_t)$ satisfies the (backward) SDE between big jumps

$$Z_s(S_t) = y - \int_s^{s_{i+1}} \left\{ b(Z_u(S_t)) + \int (-g_0(Z_u(S_t), z) + \sum_{j=1}^m z_j V_j(Z_u(S_t)))\mu'(dz) \right\} du$$
$$- \int_s^{s_{i+1}} \sigma(Z_u(S_t)) dW_u + \int_s^{s_{i+1}} \int g_0(Z_{u-}(S_t), z)\tilde{N}'(dudz) \,,$$
$$Z_{s_{i+1}-} = y, \; s_i < s < s_{i+1} \,.$$

Here, g_0 is such that $x \mapsto x + g_0(x, z)$ is the inverse map of $x \mapsto x + g(x, z)$ for μ' a.e. z, and $V_j(x) = \frac{\partial}{\partial z_j} g(x, z)|_{z=0}$. Then, [67] Theorem 1.3 implies the assertion (6.23).

The finiteness of the term

$$E'\left[\int \left| (v', \left(\int_{\mathbf{T}} \nabla \tilde{X}'_{t_1,t}(x) G(\tilde{X}'_{t_1}) z_1 \Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \right.\right.\right.$$
$$\left.\left.\left. + G(\tilde{X}'_t) B_\rho G(\tilde{X}'_t)^T \Big) z_1^T G(\tilde{X}'_{t_1})^T (\nabla \tilde{X}'_{t_1,t}(x))^T dt \right) v')^{-1} \right|^{p'} \hat{M}(du_1) \right]$$

for all $|v'| = 1$ holds as follows.

It is sufficient to show the finiteness of

$$E'\left[\int \left| \left(\int_{\mathbf{T}} \nabla \tilde{X}'_{t_1,t}(x) G(\tilde{X}'_{t_1}) z_1 \Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \right.\right.\right.$$
$$\left.\left.\left. + G(\tilde{X}'_t) B_\rho G(\tilde{X}'_t)^T \Big) z_1^T G(\tilde{X}'_{t_1})^T (\nabla \tilde{X}'_{t_1,t}(x))^T dt \right)^{-1} \right|^{p'} \hat{M}(du_1) \right] \,.$$

We have

$$\left(\int_{\mathbf{T}} \nabla \tilde{X}'_{t_1,t}(x) G(\tilde{X}'_{t_1}) z_1 \left(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \right.\right.$$
$$\left.\left. + G(\tilde{X}'_t) B_\rho G(\tilde{X}'_t)^T \right) z_1^T G(\tilde{X}'_{t_1})^T \left(\nabla \tilde{X}'_{t_1,t}(x) \right)^T dt \right)^{-1}$$
$$\le \frac{1}{T^2} \int_{\mathbf{T}} \left(\nabla \tilde{X}'_{t_1,t}(x) G(\tilde{X}'_{t_1}) z_1 \Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \right.$$
$$\left. + G(\tilde{X}'_t) B_\rho G(\tilde{X}'_t)^T \Big) z_1^T G(\tilde{X}'_{t_1})^T (\nabla \tilde{X}'_{t_1,t}(x))^T \right)^{-1} dt \,.$$

Here, we used the Jensen's inequality $(\frac{1}{T}\int_{\mathbf{T}} f(s)ds)^{-1} \le \frac{1}{T}\int_{\mathbf{T}} f(s)^{-1}ds$ for a positive function f. We integrate the right-hand side with respect to $\hat{M}(du_1)$. Then, we have

$$\int \left| \left(\int_{\mathbf{T}} \nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1})z_1\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \right.\right.$$
$$\left.\left. +G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Big)z_1^T G(\tilde{X}'_{t_1})^T(\nabla\tilde{X}'_{t_1,t}(x))^T dt\right)^{-1}\right| \hat{M}(du_1)$$
$$\le \frac{1}{T^2}\int\int_{\mathbf{T}}\Big(\nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1})z_1\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T$$
$$+ G(\tilde{X}'_t)BG(\tilde{X}'_t)^T\Big)z_1^T(\nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1}))^T\Big)^{-1} dt\hat{M}(du_1)\,.$$

Let $\lambda_1 > 0$ denote the minimum eigen value of $C(x)$. Due to *(NDB)*, we may assume it does not depend on x. Then,

$$E'\left[\frac{1}{T^2}\int\int_{\mathbf{T}}\Big(\nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1})z_1\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T\right.$$
$$\left.+G(\tilde{X}'_t)BG(\tilde{X}'_t)^T\Big)z_1^T(\nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1}))^T\Big)^{-1} dt\hat{M}(du_1)\right]$$
$$\le E'\left[\int\frac{1}{CT^2}\int_{\mathbf{T}}\lambda_1^{-1}|z_1|^2 dt\hat{M}(du_1)\right]$$
$$\le \frac{1}{T^2}\lambda_1^{-1}C'T^2E'[1] = C'\lambda_1^{-1} < +\infty\,, \quad \text{a.e. } P''$$

uniformly in v by *(NDB)*. Here, $C' > 0$ does not depend on ρ. From this, it follows that

$$E'\left[\int\left|\left(\int_{\mathbf{T}}\nabla\tilde{X}'_{t_1,t}(x)G(\tilde{X}'_{t_1})z_1\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T\right.\right.\right.$$
$$\left.\left.\left.+G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Big)z_1^T G(\tilde{X}'_{t_1})^T(\nabla\tilde{X}'_{t_1,t}(x))^T dt\right)^{-1}\right|^{p'}\hat{M}(du_1)\right] < +\infty\,, \quad \text{a.e. } P''$$

uniformly in v for $p' \ge 2$.

These imply

$$E'[\sup_v \int |\tilde{Q}'_\rho(v)(S_t)^{-1}\circ\varepsilon^+_{u_1}|^{p'}\hat{M}(du_1)]$$
$$\le C(q(u))\hat{M}(A(1)) < +\infty\,, \quad \text{a.e. } P''$$

by Proposition 3.9, where $p' \ge 2$, $0 < t \le T$.

We shall integrate the right-hand side with respect to the measure $d\tilde{P}_t(S_t)\otimes\mu_1''$. We put $C_1(t) = E^{\mu_1''}[C(q(u))]$. Then, $C_1(t) \le C_1(T) < +\infty$ since μ_1'' is a bounded measure and since supp μ is compact.

For the general case $\#S_t = k'$, we proceed as above by replacing $\mu_1''(dz_1)$ with $\mu_1''(dz_1)\otimes\cdots\otimes\mu_1''(dz_{k'})$. Since (ξ_i) are independent and identically distributed, having a bounded law, we have

$$E^{(\mu_1'')^{\otimes k'}}[C(q(u))] \le C_1(t)^{k'}, \quad k' = 1, 2, \ldots$$

where $q(u) = ((t_1, z_1), \ldots, (t_{k'}, z_{k'}))$, $0 < t_1 < \cdots < t_{k'} < t$. Summing up with respect to the Poisson law, we have, in view of (6.21),

$$\begin{aligned}\sum_{k'=0}^{\infty} \frac{1}{k'!} c^{k'} e^{-tc} C_1(t)^{k'} t^{k'} &= \sum_{k'=0}^{\infty} \frac{1}{k'!} (tcC_1(t))^{k'} e^{-tc} \\ &\le \sum_{k'=0}^{\infty} \frac{1}{k'!} (tcC_1(t))^{k'} e^{-tcC_1(t)} e^{tcC_1(t)} \le e^{tcC_1(t)} \\ &\le e^{TcC_1(T)},\end{aligned}$$

where we put $c = \mu_1''(\{|z| \ge 1\})$. Hence, we have the assertion (6.22) for $k = 1$.
Step 2. Similarly, in the general case, for $\mathbf{u} = (u_1, \ldots, u_k) = ((t_1, z_1), \ldots, (t_k, z_k))$,

$$\tilde{F}' \circ \varepsilon_{\mathbf{u}}^{+} = \tilde{X}'_{t_k,t} \circ \phi_{t_k,z_k} \circ \tilde{X}'_{t_{k-1},t_k} \circ \cdots \cdots \phi_{t_1,z_1} \circ \tilde{X}'_{0,t_1}.$$

Hence,

$$\begin{aligned}\tilde{D}_{\mathbf{u}}\tilde{F}' = \tilde{F}' \circ \varepsilon_{\mathbf{u}}^{+} - \tilde{F}' &= \nabla\tilde{X}'_{t_k,t}(\phi_{t_k,\theta_k z_k}(I_{k-1}))\nabla_z\phi_{t_k,\theta_k z_k}(I_{k-1})\cdots \\ &\cdots\nabla\tilde{X}'_{t_2,t_3}(\phi_{t_2,\theta_2 z_2}(I_1))\nabla_z\phi_{t_2,\theta_2 z_2}(I_1) \\ &\cdot\nabla\tilde{X}'_{t_1,t_2}(\phi_{t_1,\theta_1 z_1}(\tilde{X}'_{t_1-}))\nabla_z\phi_{t_1,\theta_1 z_1}(\tilde{X}'_{t_1-})z_1 \otimes\cdots\otimes z_k,\end{aligned}$$

$\theta_i \in (0, 1)$, $i = 1, \ldots, k$. Here,

$$I_1 = \nabla\tilde{X}'_{t_1,t_2}(\phi_{t_1,\theta_1 z_1}(\tilde{X}'_{t_1-}))\nabla_z\phi_{t_1,\theta_1 z_1}(\tilde{X}'_{t_1-})$$

and I_i are defined inductively by

$$I_i = \nabla\tilde{X}'_{t_i,t_{i+1}}(\phi_{t_i,\theta_i z_i}(I_{i-1}))\nabla_z\phi_{t_i,\theta_i z_i}(I_{i-1}), \quad i = 2, \ldots, k.$$

Hence,

$$\begin{aligned}\partial\tilde{D}_{(\mathbf{t},0)}\tilde{F}' &= \partial_{\mathbf{z}}\tilde{D}_{\mathbf{u}}\tilde{F}'|_{\mathbf{z}=0} \\ &= \nabla\tilde{X}'_{t_k,t}(J_{k-1})G(J_{k-1})\cdots\nabla\tilde{X}'_{t_2,t_3}(J_1)G(J_1)\nabla\tilde{X}'_{t_1,t_2}(\tilde{X}'_{t_1-})G(\tilde{X}'_{t_1-}).\end{aligned}$$

Here,

$$J_1 = \nabla\tilde{X}'_{t_1,t_2}(\tilde{X}'_{t_1-})G(\tilde{X}'_{t_1-}),$$

and J_i is defined inductively as

$$J_i = \nabla\tilde{X}'_{t_i,t_{i+1}}(J_{i-1})G(J_{i-1}) \,, \quad i = 2, \dots, k \,.$$

We have

$$\begin{aligned}
&\tilde{Q}'_\rho(v)(S_t) \circ \varepsilon^+_{\mathbf{u}} - \tilde{Q}'_\rho(v)(S_t) \\
&\geq \frac{1}{2}\Bigg(v', \int_{\mathbf{T}} \Bigg\{\partial_{\mathbf{z}}\tilde{D}_{\mathbf{u}}\tilde{F}'\Bigg|_{\mathbf{z}=0} z_1 \otimes \cdots \otimes z_k\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T \\
&\quad + G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Big)\left(\partial_{\mathbf{z}}\tilde{D}_{\mathbf{u}}\tilde{F}'\big|_{\mathbf{z}=0} z_1 \otimes \cdots \otimes z_k\right)^T\Bigg\} dtv'\Bigg) \,, \\
&\quad |z_1| < 1, \dots, |z_k| < 1 \,.
\end{aligned}$$

Then, since $\tilde{Q}'_\rho(v)(S_t) \geq 0$,

$$\begin{aligned}
\tilde{Q}'_\rho(v)(S_t)^{-1} \circ \varepsilon^+_{\mathbf{u}} \leq c^k \Bigg(v', \Bigg(\int_{0<t_1<\ldots<t_k<t<T} &\nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots \\
\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1})z_1 \otimes \cdots \otimes z_k&\Big(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T + G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\Big)(z_1 \otimes \cdots \otimes z_k)^T \\
\Big(\nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots &\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1})\Big)^T dt_1\ldots dt_k\Bigg)v'\Bigg)^{-1} \,.
\end{aligned}$$

We have, as in the case $k = 1$,

$$\begin{aligned}
&\Bigg(\int_{0<t_1<\ldots<t_k<t<T} \nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1}) \\
&\qquad z_1 \otimes \cdots \otimes z_k\left(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T + G(\tilde{X}'_t)B_\rho G(\tilde{X}'_t)^T\right)(z_1 \otimes \cdots \otimes z_k)^T \\
&\qquad \left(\nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1})\right)^T dt_1\ldots dt_k\Bigg)^{-1} \\
&\leq \frac{1}{T^{2k}}\int_{0<t_1<\ldots<t_k<t<T}\Bigg(\nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1}) \\
&\qquad z_1 \otimes \cdots \otimes z_k\left(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T + G(\tilde{X}'_t)BG(\tilde{X}'_t)^T\right)(z_1 \otimes \cdots \otimes z_k)^T \\
&\qquad \left(\nabla\tilde{X}'_{t_k,t}G(\tilde{X}'_{t_k})\nabla\tilde{X}'_{t_{k-1},t_k}G(\tilde{X}'_{t_{k-1}})\cdots\nabla\tilde{X}'_{t_1,t_2}(x)G(\tilde{X}'_{t_1})\right)^T\Bigg)^{-1} dt_1\ldots dt_k \,.
\end{aligned}$$

Hence,

$$E'\left[\int \left|(\cdots)^{-1}\right| \hat{M}(d\mathbf{u})\right]$$
$$\leq \mathcal{C}_{k,\rho} E'\left[\frac{1}{T^{2k}} \int_{0<t_1<\ldots<t_k<t<T} \lambda_1^{-k}|z_1|^2 \cdots |z_k|^2 \right.$$
$$\nabla\tilde{X}'_{t_k,t} G(\tilde{X}'_{t_k}) \nabla\tilde{X}'_{t_{k-1},t_k} G(\tilde{X}'_{t_{k-1}}) \cdots \nabla\tilde{X}'_{t_1,t_2}(x) G(\tilde{X}'_{t_1})$$
$$\left(\sigma(\tilde{X}'_t)\sigma(\tilde{X}'_t)^T + G(\tilde{X}'_t) B G(\tilde{X}'_t)^T\right)$$
$$\left.\left(\nabla\tilde{X}'_{t_k,t} G(\tilde{X}'_{t_k}) \nabla\tilde{X}'_{t_{k-1},t_k} G(\tilde{X}'_{t_{k-1}}) \cdots \nabla\tilde{X}'_{t_1,t_2}(x) G(\tilde{X}'_{t_1})\right)^T dt\right]$$
$$\leq C_{k,p}\lambda_1^{-k} < +\infty\,, \quad \text{a.e. } P''$$

uniformly in v.

Calculating as in the case that $k = 1$, these amount to imply

$$\sup_{|v|\geq 1} E'\left[\int \left|\tilde{Q}'_\rho(v)(S_t)^{-1} \circ \varepsilon^+_{\mathbf{u}}\right|^{2(k+1)p'} \hat{M}(d\mathbf{u})\right] \leq C_k(q(u))\hat{M}(A(1)^k) < +\infty\,, \quad \text{a.e. } P''$$

for $p' \geq 2$. Integrating the right-hand side with respect to $d\tilde{P}_t(S_t) \otimes (\mu''_1)^{\otimes \# S_t}$, we have the assertion (6.22). This concludes that the assertion (6.13) makes sense, and the assertion (6.2) holds for general F. □

End of proof of Proposition 3.12.

3.6.5 More on the density

In this subsection we treat two special topics concerning the density function associated to the SDE of jump type. First one is Hörmander type hypoellipticity theorem associated to jump-diffusion processes. The second one is the assertion that the density function is in $\mathcal{S}$.

Existence of density under (UH) condition

We treat linear SDE and prove the existence of the smooth density under a condition on degenerate vector fields which appear as 'coefficients' of driving processes in the SDE. The topic is special since the degeneracy is focused on the coefficients rather than the driving processes themselves. To this end we need some knowledge from the geometry in $\mathbf{R}^d$. We denote the vector fields on $\mathbf{R}^d$ by V_j in place of X_j, according to the custom in the geometry.

For a technical reason we assume the Lévy measure has a special form:

$$\mu(dz) = |z|^{-m-\alpha} dz\,, \tag{6.24}$$

in some neighbourhood of $\mathbf{0}$, where $0 < \alpha < 2$. Here we are continuing the assumption that supp μ is compact.

We consider the SDE of Itô type

$$dX_t = b(X_t, t)dt + \sum_{j=1}^{m} V_j(X_t, t)dW(t) + \int g(X_{t-}, t, z)\tilde{N}(dtdz) . \tag{6.25}$$

Here the time-dependent coefficients $b(x, t)$, $V_j(x, t)$ and $g(x, t, z)$ satisfy the conditions for $b(x)$, $f(x)$ and $g(x, z)$ in Section 1.3 globally in t, respectively. The dependence of these terms on t plays an important role below.

Assume also the condition **(D)**. The order condition for μ is satisfied, and the existence of

$$b_0(x, t) = \lim_{\delta\to 0} \int_{|z|>\delta} g(x, t, z)\mu(dz) < +\infty . \tag{6.26}$$

follows, both from (6.24).

We put

$$V_0(x, t) = b(x, t) - \frac{1}{2}\sum_{i,j} \frac{\partial \sigma^{\cdot j}}{\partial x_i}(x, t)\sigma^{ij}(x, t) - b_0(x, t) .$$

Then the solution X_t of (6.25) can also be written as one given by

$$dX_t = V_0(X_t, t)dt + \sum_{j=1}^{m} V_j(X_t, t) \circ dW(t) + \lim_{\delta\to 0} \int_{|z|>\delta} g(X_{t-}, t, z)N(dtdz) . \tag{6.27}$$

Here $V_j(t) = V_j(x, t) = \sigma^{\cdot j}$ is a vector field which has been introduced in Section 3.6.1, $j = 1, \ldots, m$. Further we introduce vector fields $\tilde{V}_j(x, t)$ by

$$\tilde{V}_j(x, t) = \sum_{k=1}^{m} \tilde{\sigma}^{\cdot k}(x, t)\tau_{kj} ,$$

where

$$\tilde{\sigma}^{ij}(x, t) = G^{ij}(x, t) = \partial_{z_j} g^i(x, t, z)|_{z=0} ,$$

and τ_{ij} are given by $(\tau_{ij}) = B^{1/2}$ (square root of the non-negative matrix B), where B appeared in Section 3.6.2.

We introduce spaces of vector fields as follows.

$$\tilde{\Sigma}_0 = \{V_j(t), \tilde{V}_j(t), j = 1, \ldots, m\} ,$$
$$\tilde{\Sigma}_k = \left\{ \frac{\partial}{\partial t} V(t) + [V_0(t), V(t)], [V_j(t).V(t)], [\tilde{V}_j(t).V(t)], j = 1, \ldots, m, V(t) \in \tilde{\Sigma}_{k-1} \right\} ,$$
$$k = 1, 2, \ldots$$

Here $[\cdot, \cdot]$ denotes the Lie bracket. Namely, $[X_j, X_k] = X_j^{\nabla} X_k - X_k^{\nabla} X_j$, where $X_j^{\nabla} X_k$ denotes the covariant derivative of X_k in the direction of X_j.

Remark 3.8. If $V_j(x,t) = V_j(x)$, $j = 1, \ldots, m$ then $\frac{\partial}{\partial t}V(t) = 0$.

We introduce two non-degeneracy conditions, which are generarizations of the conditions (RH) and (URH), respectively. The difference is whether the drift vector and its brackets are used or not.

We say the vectors satisfy the condition (SH) (*strong Hörmander condition*) if there exists $N_0 \in \mathbf{N}$ such that

$$(SH) \quad \mathrm{span}(\cup_{k=0}^{N_0} \tilde{\Sigma}_k) = T_x\mathbf{R}^d$$

for all $x \in \mathbf{R}^d$ and $t \in \mathbf{T}$.

We say the vectors satisfy the condition (UH) (*uniform Hörmander condition*) if there exist $N_0 \in \mathbf{N}$ and $C > 0$ such that

$$(UH) \quad \sum_{k=0}^{N_0} \sum_{V \in \tilde{\Sigma}_k} (v, V(x,t))^2 \geq C|v|^2$$

for all $x \in \mathbf{R}^d$, $t \in \mathbf{T}$ and $v \in \mathbf{R}^d$.

Remark 3.9. The condition (URH), introduced in Section 2.5.1, implies (UH) and the condition (UH) implies (SH). If $N_0 = +\infty$ and $V_0 = 0$, then the condition (SH) is reduced to (RH) with $V_j(t) = V_j$, $j = 1, \ldots, m$.

Theorem 3.7. *Assume the condition* (6.24) *and* ***(D)****. If the condition (UH) holds, then the solution of the SDE* (6.25) *satisfies* ***(ND)****; hence there exists a smooth density.*

We fix $\delta > 0$. We put

$$V_0'(x,t) = b(x,t) - \frac{1}{2}\sum_{i,j} \frac{\partial \sigma^{\cdot j}}{\partial x_i}(x,t)\sigma^{ij}(x,t) - \int_{|z|>\delta} g(x,t,z)\mu(dz) .$$

The solution X_t of (6.25) can also be written as the one given by

$$dX_t = V_0'(X_t,t)dt + \sum_{j=1}^m V_j(X_t,t) \circ dW(t) + \int_{|z|\leq\delta} g(X_{t-},t,z)\tilde{N}(dtdz) + \int_{|z|>\delta} g(X_{t-},t,z)N(dtdz) . \tag{6.28}$$

Let X_t' be the truncated process given as a solution of the SDE driven by the Wiener process and by small jumps, given by

$$dX_t' = V_0'(X_t',t)dt + \sum_{j=1}^m V_j(X_t',t) \circ dW(t) + \int_{|z|\leq\delta} g(X_{t-}',t,z)\tilde{N}(dtdz) . \tag{6.29}$$

By the condition **(D)**, $x \mapsto X_t'(x)$ satisfies the flow property.

Let $q^\circ(t)$ be the Poisson point process associated to $N(dtdz)$, and let $q(t)$ be the restriction of $q^\circ(t)$ to the subdomain $D_q = \{t \in \mathbf{T}; |q^\circ(t)| > \delta\}$. Then it holds $X_t(x) = X'_t(x) \circ \varepsilon_q^+$ a.s. Hence if $D_q = \{0 < \tau_1 < \tau_2 < \dots\}$ then

$$X_t(x) = X'_{\tau_{n-1},t} \circ \phi_{\tau_{n-1},q(\tau_{n-1})} \circ \cdots \circ \phi_{\tau_1,q(\tau_1)} \circ X'_{\tau_1}(x)$$

for $\tau_{n-1} < t < \tau_n$. Here $\phi_{t,z}(x) = x + g(x,t,z)$.

A modified family of vector fields Υ'_k, $k = 0, 1, 2, \dots$ is given as follows.

$$\Upsilon'_0 = \tilde{\Sigma}_0 = \{V_j(t), \tilde{V}_j(t), j = 1, \dots, m\},$$

$$\Upsilon'_k = \{\mathcal{L}' V(t), [V_j(t), V(t)], [\tilde{V}_j(t), V(t)],\ j = 1, \dots, m,\ V \in \Upsilon'_{k-1}\},$$

$$k = 1, 2, \dots$$

Here

$$\mathcal{L}' V(t) = \frac{\partial}{\partial t} V(t) + [V'_0(t), V(t)] + \frac{1}{2} \sum_{j=1}^{m} [V_j(t), [V_j(t), V(t)]]$$

$$+ \int_{0<|z|\le\delta} \{\phi_{t,z}^{*-1} V(t) - V(t) - [V^g(t), V(t)]\} \mu(dz),$$

where $\phi_{t,z}^{*-1} V(t)$ is the pull-back of $V(x,t)$ by the mapping $\phi_{t,z}$: $\phi_{t,z}^{*-1} V(x,t) = (\frac{\partial}{\partial x} \phi_{t,z}(x))^{-1} V(\phi_{t,z}(x), t)$, and $V^g(t)$ is the vector field such that the coefficients coincide with $g(x,t,z)$. The operator $\mathcal{L}'$ creates an extended drift vector associated to $V(t)$ in the Lie algebra generated by $V_j(t)$, $\tilde{V}_j(t)$.

The following condition $(\mathrm{MUH})_\delta$ (*modified uniform Hörmander condition*) is a generalization of the condition **(NDB)**.

Definition 3.6. We say the condition $(\mathrm{MUH})_\delta$ holds if there exist $N_1 \in \mathbf{N}$ and $C_1 > 0$ such that

$$\sum_{k=0}^{N_1} \sum_{V \in \Upsilon'_k} (v, V(x,t))^2 \ge C_1 |v|^2$$

for all $x \in \mathbf{R}^d$, $t \in \mathbf{T}$ and $v \in \mathbf{R}^d$.

Here $0 < \delta < \delta_0$, and $\delta_0 > 0$ is fixed temporarily.

Remark 3.10. (1) The choice of families of vector fields $(\Upsilon'_k)_k$ may not be unique. If $V_0(x,t)$ is not void, we can choose families of vector fields $(\mathcal{L})_k$ as in Kulik [118], making use mainly of the parts $[V_0(t), \tilde{V}_j(t)]$ and $\mathcal{L}' V(t)$.

(2) In case there is no jump part, Bismut [29] proved the existence of a smooth density under a general Hörmander condition. In case there is no diffusion part, Léandre [143] proved the existence of a smooth density under the condition (URH).

Proof of Theorem 3.7. This theorem is a corollary of the following two assertions.
(A) (UH) implies $(\mathrm{MUH})_\delta$ for $0 < \delta < \delta_0$.
(B) $(\mathrm{MUH})_\delta$ implies (ND).
Then the statement of Theorem is due to Proposition 3.8.

Proof of assertion (A). We construct a new families of vector fields $(\Upsilon_k^0)_k$. The family $(\Upsilon_k^0)_k$ is defined as Υ_k', replacing $\mathcal{L}'$ by $\mathcal{L}^0$ given as follows:

$$\mathcal{L}^0 V(t) = \frac{\partial}{\partial t} V(t) + [V_0(t), V(t)] + \frac{1}{2}\sum_{j=1}^{m} [V_j(t), [V_j(t), V(t)]] \ .$$

Then it holds that for any $n \in \mathbf{N}$

$$\text{linear span of } \cup_{k=0}^{n} \tilde{\Sigma}_k \subset \text{linear span of } \cup_{k=0}^{2n+1} \Upsilon_k^0 \ . \tag{6.30}$$

Indeed, if $V \in \tilde{\Sigma}_0 = \Upsilon_0$ then $\frac{\partial}{\partial t} V(t) + [V_0, V] = \mathcal{L}^0 V - \frac{1}{2}\sum_{j=1}^{m}[V_j, [V_j, V]]$ is in the linear span of $\cup_{k=0}^{3} \Upsilon_k^0$. This implies that the linear span of $\tilde{\Sigma}_0 \cup \tilde{\Sigma}_1 \subset$ linear span of $\cup_{k=0}^{3} \Upsilon_k^0$. Repeating this argument we have (6.30).

Due to the assumption (6.24) on the existence of the weak drift the linear span of $\cup_{k=0}^{2n+1} \Upsilon_k^0$ is contained in the linear span of $\cup_{k=0}^{2n+1} \Upsilon_k'$ assuming the former is full rank.

Hence if the bilinear form associated to V's in $\tilde{\Sigma}_k$, $k = 0, \ldots, n_0$ is uniformly positive definite, then the bilinear form associated to V's in Υ_k', $k = 0, \ldots, n_1$ is also uniformly positive definite for $n_1 = 2n_0 + 1$. This proves the assertion. □

Proof of assertion (B). We define $\Theta_V(x, t)$ for $V \in \cup_{k=0}^{N_1} \Upsilon_k'$ and $\Theta(x, t)$ by

$$\Theta_V(x, t) = X_t^{*-1} V(x, t)\{X_t^{*-1} V(x, t)\}^T, \ \Theta(x, t) = \sum_{k=0}^{N_1} \sum_{V \in \Upsilon_k'} \Theta_V(x, t) \ .$$

Here where $X_t^{*-1} V(x, t)$ is the pull-back of $V(x, t)$ by the mapping $x \mapsto X_t(x)$: $X_t^{*-1} V(x, t) = (\frac{\partial}{\partial x} X_t(x))^{-1} V(X_t(x), t)$. Then the assertion (B) for (6.25) is due to the following proposition. The proof of the assertion (B) is complete granting the following proposition. □

Proposition 3.17. *(i) If the condition $(\mathrm{MUH})_\delta$ holds, then*

$$\Pi(x) = \nabla X_t(x) \Theta(x, t) (\nabla X_t(x))^T$$

is invertible a.s., and it satisfies

$$\sup_{|x|<N} \sup_{v \in S^{d-1}, \mathbf{u} \in A(1)^k} E[(v, \Pi(x) v)^{-p} \circ \varepsilon_{\mathbf{u}}^+] < +\infty \tag{6.31}$$

for some $p > 1$.

(ii) Granting that (6.31) *holds, the **(ND)** condition holds.*

Proof of Proposition 3.17 *(i).*
[I] We first prove the assertion for $X_t = X'_t$. Functionals $\Theta_V(x,t)$, $\Theta(x,t)$, $\Pi(x)$ for $X_t = X'_t$ are denoted by $\Theta'_V(x,t)$, $\Theta'(x,t)$, $\Pi'(x)$ respectively. Choose $\beta > 0$ so that $\beta > \max(\frac{4}{2-\alpha}, 8)$, where α is the degree in the order condition. Choose arbitrary $h \in \mathbf{N}$ and fix it.

(Step 1) Case $k = 0$. First we show

$$\sup_{|x|<N} \sup_{v\in S^{d-1}, \mathbf{u}\in A(1)^k} E[(v, \Theta'(x)v)^{-p} \circ \varepsilon^+_{\mathbf{u}}] < +\infty \tag{6.32}$$

for $k = 0$.

A basic idea is to use dilation, that is, re-scaling the space parameter according to the rank of the Lie algebra. See [167]. To realize this, the step of dilation is chosen to be $\frac{1}{\beta}$, where β is given as above. We introduce events E_i, $i = 0, \ldots, n_1$ and F by

$$E_i = \left\{ \sum_{V\in \Upsilon'_i} \int_{\mathbf{T}} |Y_V(t)|^2 dt < \epsilon^{\beta^{-2i}} \right\}, \quad i = 0, 1, \ldots, n_1 ,$$

and

$$F = E_0 \cap E_1 \cap \cdots \cap E_{n_1} .$$

Here $Y_V(t) = Y_V(v,t) = v^T {X'}^{*-1}_t V(x,t)$ and $0 < \epsilon < \epsilon_1$, where $\epsilon_1 = \min_{0\le i\le n_1} (\frac{1}{k_i})^{\beta^{2i+1}}$. Here k_i is the number of elements of Υ'_i.

We have a decomposition

$$E_0 = (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{n_1-1} \cap E_{n_1}^c) \cup F ,$$

where $E_0 = \{(v, \Theta'(x)v) < \epsilon\}$. Hence

$$P(E_0) \le \sum_{i=0}^{n_1-1} P(E_i \cap E_{i+1}^c) + P(F) . \tag{6.33}$$

We have for some $C > 0$

$$\sup_{|x|\le N} \sup_{v\in S^{d-1}} P(E_i \cap (E_{i+1})^c) \le C\epsilon^h, \quad 0 < \epsilon < \epsilon_1 \; i = 1, \ldots, n_1 - 1 . \tag{6.34}$$

We have also for some $C_1 > 0$

$$\sup_{|x|\le N} \sup_{v\in S^{d-1}} P(F) \le C\epsilon^{\beta^{-2n_1}h}, \quad 0 < \epsilon < \epsilon_1 . \tag{6.35}$$

These assertions are due to the finitude of n_1 in $(MUH)_\delta$. We prove them below.

By (6.34), (6.35),

$$\sup_{|x|\le N} \sup_{v\in S^{d-1}} P(E_0) < C'\epsilon^{\beta^{-2n_1}h}, \quad 0 < \epsilon < \epsilon_1 . \tag{6.36}$$

Since $E_0 = \{(v, \Theta'(x)v) < \epsilon\}$, we have

$$\sup_{|x|\le N} \sup_{v\in S^{d-1}} P((v, \Pi'(x)v)^{-p} \circ \varepsilon^+_q) < C'\epsilon^{\beta^{-2n_1}h}, \quad 0 < \epsilon < \epsilon_1 ,$$

for $p < h\beta^{-2n_1}/2$. Indeed, (6.36) implies

$$\sup_{|x|\leq N} E[|\det \Theta'(x)|^{-2p/d}] < +\infty .$$

As $|\det \Pi'(x)| = |\det \nabla X'(x)|^2 |\det \Theta'(x)|$, and since

$$\sup_{|x|\leq N} E[|\det \nabla X'(x)|^{-4p/d}] < +\infty$$

due to the flow property, we have the assertion above.

Hence we have the assertion (6.32) in case $k = 0$.

Proof of assertions (6.34), (6.35). For the proof of (6.34), we use inequalities of Norris type. We fix i. We observe that $Y_V(t)$ satisfies

$$dY_V(t) = a_V(t)dt + f_V(t)dW(t) + g_V(t,z)\tilde{N}(dtdz) ,$$

where

$$a_V(t) = v^T X_t^{*-1} \mathcal{L}' V(x,t) ,$$

$$f_V(t) = (v^T X_t^{*-1}[V_1(t), V(t)](x), \cdots, v^T X_t^{*-1}[V_m(t), V(t)](x)) ,$$

and

$$g_V(t,z) = v^T X_t^{*-1}\{\phi_{t,z}^{*-1}\} .$$

Given $V \in \Upsilon_i'$ and $0 < \epsilon < \epsilon_1$, put

$$A_V(\epsilon) = \left\{ \int_{\mathbf{T}} |f_V(t)|^2 dt < \epsilon^{\beta^{-2i}} \right\} ,$$

$$B_V(\epsilon) = \left\{ \int_{\mathbf{T}} |a_V(t)|^2 dt + \int_{\mathbf{T}} |f_V(t)|^2 dt + \int_{\mathbf{T}} |\tilde{g}_V(t)|^2 dt < \epsilon^{\beta^{-2(i+1)}} \cdot \frac{1}{k_i} \right\} ,$$

and $$C_V(\epsilon) = \left\{ \int_{\mathbf{T}} |a_V(t)|^2 dt + \int_{\mathbf{T}} |f_V(t)|^2 dt + \int_{\mathbf{T}} |\tilde{g}_V(t)|^2 dt < \epsilon^{\beta^{-(2i+1)}} \right\} ,$$

where

$$|\tilde{g}_V(t)|^2 = \sum_{j=1}^m |v^T X_t^{*-1}[\tilde{V}_j(t), V(t)](x)|^2 .$$

Then

$$E_i \cap E_{i+1}^c \subset \cup_{V\in\Upsilon_i'}\{A_V(\epsilon) \cap B_V(\epsilon)^c\}$$
$$\subset \cup_{V\in\Upsilon_i'}\{A_V(\epsilon) \cap C_V(\epsilon)^c\} .$$

We put $\tilde{\epsilon} = \epsilon^{\beta^{-(2i+1)}}$ so as that $\tilde{\epsilon}^{\beta} = \epsilon^{\beta^{-2i}}$, and apply Lemma 3.8 below (with $\gamma = (x,v)$) below for $E^\gamma(\tilde{\epsilon}) = A_V(\epsilon)$ and $F^\gamma(\tilde{\epsilon}) = C_V(\epsilon)$ with $\tilde{p} = \beta^{2i+1}h$.

Then we have

$$\sup_{|x|\le N, v\in S^{d-1}} P(A_V(\epsilon)\cap C_V(\epsilon)^c) < C\tilde{\epsilon}^{\tilde{p}} \le C\epsilon^h, \quad 0<\epsilon<\epsilon_1 .$$

Therefore

$$\sup_{|x|\le N, v\in S^{d-1}} P(E_i\cap (E_{i+1})^c) \le C\epsilon^h, \quad 0<\epsilon<\epsilon_1 .$$

Here $C = C_i$. Since $h \in \mathbf{N}$ is arbitrary, we choose $C = \max_{i=0,\dots,n_1} C_i$ and the assertion (6.34) follows.

For the proof of (6.35) we proceed as follows.

Let $K(x,v) = \sum_{k=0}^{n_1}\int_{\mathbf{T}}\sum_{V\in\Upsilon'_k}|Y_V(t)|^2dt$. If $\omega\in F=F^{x,v}$ then $K(x,v)(\omega) < (n_1+1)e^{\beta^{-2n_1}}$ by definition. Therefore $F^{x,v}\subset\{K(x,v)<(n_1+1)e^{\beta^{-2n_1}}\}$.

On the other hand, we have by (MUH)$_\delta$ there exists $c_1>0$ such that

$$\begin{aligned} K(x,v) &\ge \int_0^{T_h(x)} v^T X_t'^{*-1} H(x,t) X_t'^{*-1} v\,dt \\ &\ge c_1 \int_0^{T_h(x)} |v^T\nabla X_t(x)^{-1}|^2 dt \\ &\ge c_1(M_h+1)^{-2}T_h(x) . \end{aligned}$$

Here $H(x,t) = \sum_{k=0}^{n_1}\sum_{V\in\Upsilon'_k} V(x,t)V(x,t)^T$, and $T_h(x)$ is the stopping time of order h with respect to N and $k=0$. That is,

$$T_h(x) = \inf\{t\in\mathbf{T}; |X_t'(x)-x|\ge M_h \text{ or } |\nabla X_t'(x)|\ge M_h\}\wedge T ,$$

where $M_h = (N+1)(K+1)^h$ and $K>0$ is such that

$$\sup_{x,t}\frac{|g(x,t,z)|+|\nabla_x g(x,t,z)|}{1+|x|}\le K .$$

Therefore if we choose γ so that $(n_1+1)e^{\beta^{-2n_1}} = c_1(M_h+1)^{-2}\gamma$, we have

$$\{K(x,v)<(n_1+1)e^{\beta^{-2n_1}}\}\subset\{T_h(x)<\gamma\} .$$

Consequently,

$$P(F^{x,v})\le P(T_h(x)<\gamma)\le c_h\gamma^h = C_1 e^{\beta^{-2n_1}h}$$

for all $|x|\le N$, $v\in S^{d-1}$. Here we used the lemma 3.9 below. This proves the assertion (6.35). □

Lemma 3.8. *Let $Y^\gamma(t)$, $t\in\mathbf{T}$ be a semimartingale given by the following SDE:*

$$Y^\gamma(t) = y^\gamma + \int_0^t a^\gamma(s)ds + \sum_{j=1}^m\int_0^t f_j^\gamma(s)dW^j(s)$$

$$+\int_0^t \int_{|z|\le\delta} g^\gamma(s,z)\tilde{N}(dsdz) + \int_0^t \int_{|\delta|>\delta} g^\gamma(s,z)N(dsdz)\ .$$

Let $\epsilon > 0$. We introduce events $E^\gamma(\epsilon), F^\gamma(\epsilon)$ by

$$E^\gamma(\epsilon) = \left\{ \int_{\mathbf{T}} |Y^\gamma_{t-}|^2 dt < \epsilon^\beta \right\}\ ,$$

$$F^\gamma(\epsilon) = \left\{ \int_{\mathbf{T}} \{|a^\gamma(t)|^2 + |f^\gamma(t)|^2 + |\tilde{g}^\gamma(t)|^2\} dt < \epsilon \right\}\ .$$

Here $\beta > 0$ such that $\beta > \max(\frac{4}{2-\alpha}, 8)$ and $\tilde{g}^\gamma(t) = \partial g^\gamma(t) B^{1/2}$, where $\partial g^\gamma(t) = \frac{\partial}{\partial z} g^\gamma(t,z)|_{z=0}$.

We further assume $a^\gamma(t)$ above satisfies the SDE

$$a^\gamma(t) = y^\gamma + \int_0^t b^\gamma(s)ds + \int_0^t \sigma^\gamma(s)dW(s)$$

$$+\int_0^t \int_{|z|\le\delta} h^\gamma(s,z)\tilde{N}(dsdz) + \int_0^t \int_{|\delta|>\delta} h^\gamma(s,z)N(dsdz)\ .$$

We put

$$\theta^\gamma = \|a^\gamma\|^2 + \|b^\gamma\|^2 + \|f^\gamma\|^2 + \|\sigma^\gamma\|^2 + \|g^\gamma\|^2 + \|h^\gamma\|^2 + \|\partial g^\gamma\|^2 + \|\partial h^\gamma\|^2 + \|\frac{r^\gamma}{|z|^2}\|^4 + \|\frac{s^\gamma}{|z|^2}\|^4\ .$$

Here $r^\gamma = g^\gamma - \sum_{j=1}^m z_j \partial g^\gamma$, $s^\gamma = h^\gamma - \sum_{j=1}^m z_j \partial h^\gamma$, and $\|\cdot\|$ denotes the sup-norm such as

$$\|f^\gamma\| = \sup_{t\in\mathbf{T}} |f^\gamma(t)|,\ \|r^\gamma\| = \sup_{t\in\mathbf{T}, |z|\le\delta} |r^\gamma(t,z)|, \ldots .$$

Assume $E[(\theta^\gamma)^p] < +\infty$ for any $p > 1$. Then for each $p > 1$ there exists a $C_p > 0$ such that

$$\sup_\gamma P(E^\gamma(\epsilon) \cap F^\gamma(\epsilon)^c) < C_p \epsilon^p \tag{6.37}$$

for any $\epsilon > 0$.

The proof of this lemma is due to Komatsu–Takeuchi's estimate [116] and a subtle calculation. We give the proof at the end of this subsubsection.

The following lemma is used to prove assertion (6.35) above.

Lemma 3.9 ([131]). *Let $N > 1$, $k \in \mathbf{N}$. Then for any $h \in \mathbf{N}$ there exists a $C_h > 0$ such that*

$$\sup_{|x|\le N, u\in A(1)^k} P(T_h(x) \circ \varepsilon_u^+ < \epsilon) \le C_h \epsilon^h$$

for $0 < \epsilon < 1$.

The proof of this lemma depends on the decomposition of the event according to the measurament of displacement $X_t'(x) - x$ or $\nabla X_t'(x) - I$. See [131] Lemma 3.1 for the proof.

We continue the proof of Proposition 3.17 (i).

(Step 2) Case $k \geq 1$.

We consider the case $k = 1$. Let $u = (s_1, z_1)$ and put $\bar{Y}_V(t) = v^T X_t^{*-1} V(x,t) \circ \varepsilon_u^+$, $\bar{Y}(t) = \sum_{k=0}^{n_1} \sum_{V \in \Upsilon_k'} Y_V(t)$. We introduce events

$$\bar{E}_i = \left\{ \sum_{V \in \Upsilon_i'} \int_{\mathbf{T}} |\bar{Y}_V(t)|^2 dt < \epsilon^{\beta^{-2i}} \right\} \;,$$

$i = 0, 1, \ldots, n_1$. We observe

$$\bar{Y}_V(t) = v^T X_t^{*-1} V(x,t), \quad (s_1 > t)$$
$$\bar{Y}_V(t) = v^T X_t^{*-1} \phi_u^{*-1} V(x,t), \quad (s_1 < t)\;.$$

Here if $s_1 < t$ then $X_t^{*-1} \phi_u^{*-1} V(x,t)$ satisfies

$$\begin{aligned} X_t^{*-1}\phi_u^{*-1} V(x,t) &= X_s^{*-1} V(x,s_1) + \int_{s_1}^t X_r^{*-1} \mathcal{L}'\{\phi_u^{*-1} V(x,r)\} dr \\ &+ \sum_{j=1}^m \int_{s_1}^t X_r^{*-1} \phi_u^{*-1} [V_j(r), V(r)](x) dW^j(r) \\ &+ \int_{s_1}^t \int_{|z|\le\delta} X_r^{*-1}\{\phi_{r,z}^{*-1}\phi_u^{*-1} V(x,r) - \phi_u^{*-1} V(x,r)\}\tilde{N}(drdz) \\ &+ \int_{s_1}^t \int_{|z|>\delta} X_r^{*-1}\{\phi_{r,z}^{*-1}\phi_u^{*-1} V(x,r) - \phi_u^{*-1} V(x,r)\} N(drdz)\;. \end{aligned}$$

Then as in Step 1 we can show

$$\sup_{|x|\le N} \sup_{v \in S^{d-1}} P(\bar{E}_i \cap (\bar{E}_{i+1})^c) \le C\epsilon^h, \quad i = 1, \ldots, n_1 - 1 \tag{6.38}$$

for $0 < \epsilon < \epsilon_1$.

We put $\bar{F} = \bar{E}_1 \cap \cdots \cap \bar{E}_{n_1}$. We shall prove

$$\sup_{|x|\le N} \sup_{v \in S^{d-1}} P(\bar{F}) \le C\epsilon^{\beta^{-2n_1} h}\;. \tag{6.39}$$

To this end, set $\bar{K}(x,v) = \int_{\mathbf{T}} |\bar{Y}(t)|^2 dt$. Then it is written

$$\begin{aligned} \bar{K}(x,v) &= \int_0^{s_1} v^T X_t'^{*-1} H(x,t) (X_t'^{*-1})^T v dt \\ &+ \int_{s_1}^T v^T X_t'^{*-1} \phi_u^{*-1} H(x,t) (X_t'^{*-1} \phi_u^{*-1})^T v dt\;. \end{aligned}$$

By the assumption $(\text{MUH})_\delta$, there exists $c_1 > 0$ such that

$$v^T H(x,t)v \geq c_1|v|^2, \quad v^T \phi_u^{*-1} H(x,t) {\phi_u^{*-1}}^T v \geq c_1|v|^2 ,$$

for $t \in \mathbf{T}$, $|x| \leq M_h$, $u \in A(1)$. Here $H(x,t)$ and M_h are as above. Then we have (6.39) as in Step 1.

Using (6.38), (6.39) we can show

$$\sup_{|x|\leq N} \sup_{v\in S^{d-1}} P(\bar{E}_0) < C'' \epsilon^{\beta^{-2n_1} h}$$

for $0 < \epsilon < \epsilon_1$. We have the assertion (6.32) for $k = 1$ as in Step 1.

We can verify the case $k \geq 2$ similarly.

[II] Next we consider the case $X_t(x) = X_t'(x) \circ \varepsilon_q^+$. We write $\Pi(x) = \Pi'(x) \circ \varepsilon_q^+$. We show the case $k = 0$ only, since the cases for $k \geq 1$ are similar to the above.

We have

$$\sup_{|x|\leq N} \sup_{v\in S^{d-1}} P(E_i \circ \varepsilon_q^+ \cap (E_{i+1} \circ \varepsilon_q^+)^c) \leq C\epsilon^h, \quad i = 1, \ldots, n_1 - 1 \tag{6.34'}$$

as in (6.34) (first part).

We have also

$$\sup_{|x|\leq N} \sup_{v\in S^{d-1}} P(F \circ \varepsilon_q^+) \leq C\epsilon^{\beta^{-2n_1} h}, \quad 0 < \epsilon < \epsilon_1. \tag{6.35'}$$

Indeed, put $K'(x,v) = \sum_{i=0}^{n_1} \int_{\mathbf{T}} \sum_{V\in\Upsilon_i'} |Y_V(t)|^2 dt$. Then by $(\text{MUH})_\delta$

$$K'(x,v) \geq C_1 \int_{\mathbf{T}} |v^T \nabla X_t'|^2 dt .$$

Hence

$$K'(x,v)^{-1} \leq \frac{1}{C_1 T^2 |v|^2} \int_{\mathbf{T}} |\nabla X_t'|^2 dt$$

by Jensen's inequality. By Chebyschev's inequality

$$E[K'(x,v)^{-1} \circ \varepsilon_q^+] \leq \frac{1}{(C_1 T^2 |v|^2)^h} E\left[\left(\int_{\mathbf{T}} |\nabla X_t' \circ \varepsilon_q^+|^2 dt\right)^h\right] \epsilon^h$$

for $h > 1$.

Since $F \subset \{K'(x,v) < (n_1+1)\epsilon^{\beta^{-2n_1}}\}$,

$$P(F \circ \varepsilon_q^+) \leq P(K'(x,v) \circ \varepsilon_q^+ < (n_1+1)\epsilon^{\beta^{-2n_1}}) \leq C' \epsilon^{\beta^{-2n_1} h}$$

for $|x| < N$, $v \in S^{d-1}$. Hence we have (6.35)$'$.

By (6.34)$'$, (6.35)$'$, we have in view of (6.33)

$$\sup_{|x|\leq N} \sup_{v\in S^{d-1}} E[(v, \Theta'(x)v)^{-p} \circ \varepsilon_q^+] < C' \epsilon^{\beta^{-2n_1} h}, \quad 0 < \epsilon < \epsilon_1 , \tag{6.40}$$

for $p < h\beta^{-2n_1}/2$.

Hence

$$\sup_{|x|\leq N}\sup_{v\in S^{d-1}} E[(v,\Pi'(x)v)^{-p}\circ\varepsilon_q^+] < C'\epsilon^{\beta^{-2n_1}h}, \quad 0<\epsilon<\epsilon_1\,, \tag{6.41}$$

for $p < h\beta^{-2n_1}/2$. Indeed, (6.40) implies

$$\sup_{|x|\leq N, u\in A(1)} E[|\det\Theta'(x)|^{-2p/d}\circ\varepsilon_q^+] < +\infty\,.$$

As $|\det\Pi'(x)| = |\det\nabla X'(x)|^2|\det\Theta'(x)|$, and since

$$\sup_{|x|\leq N, u\in A(1)} E[|\det\nabla X'(x)|^{-4p/d}\circ\varepsilon_q^+] < +\infty$$

due to the flow property, we have the assertion (6.41).

As $\Pi(x) = \Pi'(x)\circ\varepsilon_q^+$ a.s., we have the assertion (6.32) for $p < h\beta^{-2n_1}/2$.

Since $h\in\mathbf{N}$ is arbitrary, we have the assertion for all $p>1$. □

Proof of Proposition 3.17 *(ii)*. The assertion follows as in Theorem 3.6 with $\Theta(x,t)$ replacing $C(x)$. □

End of proof of Theorem 3.7 □

Proof of Lemma 3.8. In order to prove this lemma we need the following estimate due to Komatsu–Takeuchi.

To apply it we introduce below a parameter v, and assume it satisfies $\frac{1}{16} < v < \frac{1}{8}$ temporary, due to a technical reason. After the sublemma below is confirmed with this restriction, we then choose instead $\frac{1}{24} < v < \frac{1}{12}$, $\frac{1}{36} < v < \frac{1}{18}$ and so on, and we apply the (modified) sublemma consecutively. Finally we see the assertion is valid for all $0 < v < \frac{1}{8}$.

Sublemma 3.1 ([116] Theorem 3). *Let v be an arbitrary number such that $0 < v < \frac{1}{8}$. There exist a positive random variable $\mathcal{E}(\lambda,\gamma)$ satisfying $E[\mathcal{E}(\lambda,\gamma)]\leq 1$, and a positive constants C, C_0, C_1, C_2 independent of λ,γ such that the following inequality holds on the set $\{\theta^\gamma\leq\lambda^{2v}\}$ for all $\lambda>1$ and γ:*

$$\begin{aligned}
&\lambda^4\int_{\mathbf{T}}\left\{|Y^\gamma(t)|^2\wedge\frac{1}{\lambda^2}\right\}dt + \frac{1}{\lambda^v}\log\mathcal{E}(\lambda,\gamma) + C \\
&\geq C_0\lambda^{1-4v}\int_{\mathbf{T}}|a^\gamma(t)|^2dt + C_1\lambda^{2-2v}\int_{\mathbf{T}}|f^\gamma(t)|^2dt \\
&\quad + C_2\lambda^{2-2v}\int_{\mathbf{T}}\int\left\{|g^\gamma(t,z)|^2\wedge\frac{1}{\lambda^2}\right\}dt\,\mu(dz)\,.
\end{aligned} \tag{6.42}$$

To prove Lemma 3.8, we choose $0<\eta<1$ so small that

$$2-2v-\alpha(1+\eta) > \frac{4}{\beta}$$

holds, where β is what appeared in the statement of Lemma 3.8. Then we can choose an $r > 1$ such that

$$\frac{\beta}{4} > r > \frac{1}{2 - 2\nu - \alpha(1+\eta)} \vee \frac{1}{1-4\nu} .$$

The restriction above of the value of ν reflects here to the restriction on the value of α, so that $0 < \alpha < 2(1-\nu) < \frac{15}{8} < 2$. However, as we noted above, ν can be chosen smaller consecutively, and the argument is valid for all given α such that $0 < \alpha < 2$.

First we consider the last term of RHS of (6.42). As $g^\gamma(t,0) = 0$, the function $z \mapsto g^\gamma(t,z)$ can be written as $g^\gamma(t,z) = \partial g^\gamma(t)z + r^\gamma(t,z)$ for some $r^\gamma(t,z)$. Then

$$|g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2} \geq \frac{1}{2}|g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2} - (r^\gamma(t,z))^2 ,$$

and hence for $0 < \kappa < \lambda$

$$\int \left(|g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2}\right)\mu(dz) \geq \int_{|z|\leq\frac{\kappa}{\lambda}} \left(|g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2}\right)\mu(dz)$$

$$\geq \frac{1}{2}\int_{|z|\leq\frac{\kappa}{\lambda}} \left(|\partial g^\gamma(t)\frac{z}{|z|}|^2 \wedge \frac{1}{|z|^2\lambda^2}\right)|z|^2\mu(dz) - \left(\frac{\kappa}{\lambda}\right)^2 \int_{|z|\leq\frac{\kappa}{\lambda}} \left(\frac{|r^\gamma(t,z)|}{|z|^2}\right)^2 |z|^2\mu(dz)$$

$$\geq \frac{1}{2}\varphi\left(\frac{\kappa}{\lambda}\right)\int_{|z|\leq\frac{\kappa}{\lambda}} \left(|\partial g^\gamma(t)\frac{z}{|z|}|^2 \wedge \frac{1}{\kappa^2}\right)\bar{\mu}_{\kappa/\lambda}(dz) - \varphi\left(\frac{\kappa}{\lambda}\right)\left(\frac{\kappa}{\lambda}\right)^2 \left\|\frac{r^\gamma}{|z|^2}\right\|^2 ,$$

where $\varphi(\rho) = \int_{|z|\leq\rho}|z|^2\mu(dz)$ and $\bar{\mu}_\rho(dz)$ is a probability measure given by $\bar{\mu}_\rho(dz) = \frac{|z|^2}{\varphi(\rho)}1_{\{|z|\leq\rho\}}\mu(dz)$.

We put $\lambda = \epsilon^{-r}$ and $\kappa = \epsilon^{\eta r}$. Then

$$\frac{\kappa}{\lambda} = \epsilon^{(1+\eta)r} \tag{6.43}$$

and $\varphi(\frac{\kappa}{\lambda}) \geq C_4\epsilon^{\alpha(1+\eta)r}$ by the order condition. Since $\|\frac{r^\gamma}{|z|^2}\|^4 \leq \lambda^{2\nu}$ on $\{\theta^\gamma \leq \lambda^{2\nu}\}$, the last term of (6.42) dominates

$$C_4\epsilon^{\alpha(1+\eta)r}|\partial g^\gamma(t)B_{\epsilon^{(1+\eta)r}}\partial g^\gamma(t)^T| - C_5\epsilon^{(2+\alpha)(1+r)r} ,$$

on $\{\theta^\gamma \leq \lambda^{2\nu}\}$ due to (6.43). Here B_ρ is the covariance matrix of the Lévy measre μ on $\{|z| \leq \rho\}$.

Hence (6.42) implies

$$\begin{aligned}
&\epsilon^{-4r}\int_{\mathbf{T}}\{|Y^\gamma(t)|^2 \wedge \epsilon^{2r}\}dt + \epsilon^{r\nu}\log\mathcal{E}(\epsilon^{-r},\gamma) + C \\
&\geq C_0\epsilon^{-r(1-4\nu)}\int_{\mathbf{T}}|a^\gamma(t)|^2dt + C_1\epsilon^{-r(2-2\nu)}\int_{\mathbf{T}}|f^\gamma(t)|^2dt \\
&\quad + C_2C_4\epsilon^{-r(2-2\nu)}\epsilon^{\alpha(1+\eta)r}\int_{\mathbf{T}}|\partial g^\gamma(t)B\partial g^\gamma(t)^T|dt - C_2C_5\epsilon^{-(2-2\nu)}\epsilon^{(2+\alpha)(1+r)r} .
\end{aligned}$$

Here B denotes the infinitesimal covariance.

We set

$$\rho = r \min\{1 - 4v, 2 - 2v - \alpha(1 + \eta)\} - 1$$

according to the choice of r above. Then $\rho > 0$. The above inequality yields

$$\epsilon^{-4r} \int_{\mathbf{T}} \{|Y^\gamma(t)|^2 \wedge \epsilon^{2r}\} dt + \epsilon^{rv} \log \mathcal{E}(\epsilon^{-r}, \gamma) + C$$

$$\geq C_6 \epsilon^{-(1+\rho)} \int_{\mathbf{T}} \{|a^\gamma(t)|^2 + |f^\gamma(t)|^2 + |\tilde{g}^\gamma(t)|^2\} dt - C_7 \epsilon^{-(1+\rho)} \epsilon^{r'} .$$

on $\{\theta^\gamma \leq \epsilon^{-rv}\}$, where $r' = 2(1 + \eta)r > 1$.

We put

$$G_1^\gamma = \{\theta^\gamma > \epsilon^{-rv}\} = \{\theta^{\gamma/rv} > \epsilon^{-1}\} ,$$

$$G_2^\gamma = \{\theta^\gamma \leq \epsilon^{-rv}\} \cap \left\{ \int_{\mathbf{T}} \{|Y^\gamma(t-)|^2 \wedge \epsilon^{2r}\} dt < \epsilon^\beta \right\} \cap \left\{ \int_{\mathbf{T}} \{|a^\gamma(t)|^2 + |f^\gamma(t)|^2 + |\tilde{g}^\gamma(t)|^2\} dt > \epsilon \right\} .$$

Then it holds

$$E^\gamma(\epsilon) \cap F^\gamma(\epsilon)^c \subset G_1^\gamma \cup G_2^\gamma .$$

Therefore the left hand side of (6.37) is dominated by $\sup_\gamma \{P(G_1^\gamma) + P(G_2^\gamma)\}$.

We will calculate them seperately. We have by Chebyschev's inequality

$$\sup_\gamma P(G_1^\gamma) \leq \epsilon^p E[(\sup_\gamma \theta^\gamma)^{p/rv}] \leq c_p \epsilon^p .$$

On the other hand, due to (6.42)

$$G_2^\gamma \subset \{\mathcal{E}(\epsilon^{-r}, \gamma)^{\epsilon^{rv}} \geq \exp(-\epsilon^{\beta-4r} + C_6 \epsilon^{-\rho} - C_7 \epsilon^{-\rho} \epsilon^{r'-1} - C)\} .$$

By Chebyschev's inequality again

$$\sup_\gamma P(G_2^\gamma) \leq e^C \exp(\epsilon^{\beta-4r} - C_6 \epsilon^{-\rho} + C_7 \epsilon^{-\rho} \epsilon^{r'-1}) \times E[\mathcal{E}(\epsilon^{-r}, \gamma)^{\epsilon^{rv}}] .$$

We have

$$\epsilon^{\beta-4r} + C_7 \epsilon^{-\rho} \epsilon^{r'-1} < \frac{C_6}{2} \epsilon^{-\rho}$$

holds for $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$. Since $E[\mathcal{E}(\epsilon^{-r}, \gamma)^{\epsilon^{rv}}]$ with $0 < \epsilon^{rv} < 1$, we have for any $p > 1$

$$\sup_\gamma P(G_2^\gamma) \leq e^C \exp\left(-\frac{C_6}{2} \epsilon^{-\rho}\right) \leq c'_p \epsilon^p$$

for any $0 < \epsilon < \epsilon_0$. These prove the assertion (6.37). □

Remark 3.11. In the above expression, the symbol $\mathcal{E}(\lambda, \gamma)$ is a phantom. In reality it is written as an exponential of Wiener and Poisson integrals. See [116] Lemma 5.1.

Density in $\mathcal{S}$

By Proposition 3.12, F has a smooth density $p_F(y)$ under the condition *(NDB)*. In case that $F \in \mathbf{D}_\infty$ and it satisfies the condition *(ND)*, we can show it is also rapidly decreasing.

Indeed, let $\varphi(v) = E[e_v(F)]$ be the characteristic function of F. Then, $\varphi(v)$ is represented by

$$\varphi(v) = \int e^{i(v,y)} p_F(y)dy .$$

Hence,

$$p_F(y) = \left(\frac{1}{2\pi}\right)^d \int e^{-i(v,y)} \varphi(v)dv ,$$

and

$$(1 + |y|^2)^j \nabla^\alpha p_F(y) = \left(\frac{1}{2\pi}\right)^d (-i)^{|\alpha|} \int e^{-i(v,y)} (1 - \Delta)^j v^\alpha \varphi(v)dv , \tag{6.44}$$

if the right-hand side is finite. Here, α is a multi-index, $v^\alpha = v_1^{\alpha_1} \cdots v_d^{\alpha_d}$ and $j = 0, 1, 2, \ldots$

We show it is legitimate below. Since $\partial_v^\alpha e_v(F) = i^{|\alpha|} F^\alpha e_v(F)$, and since

$$(1 - \Delta)^j v^\alpha \varphi(v) = E[(1 - \Delta)^j v^\alpha e_v(F)] = \sum_{i=0}^{j} {}_jC_i E[(-\Delta)^i \{v^\alpha e_v(F)\}] ,$$

we have

$$|(1 - \Delta)^j v^\alpha \varphi(v)| \le C(\alpha, j)(1 + |v|^2)^{-nq_0/2+|\alpha|}$$

by Proposition 3.9 ($G \equiv 1$) and due to that $F \in L^p$ for all $p > 1$. Choosing a sufficiently large n (depending on α and j), we see that the right-hand side of (6.17) is finite.

This implies that $p_F(y)$ is rapidly decreasing as $|y| \to +\infty$.

Showing this property of the density function is characteristic to our method using Fourier analysis. We continue the analysis on the density in Section 4.1.2.

3.7 Composition on the Wiener–Poisson space (II) – Itô processes

Let $F = X_t$, where X_t is a general Itô process given in Section 3.6.1. We again assume the order condition for the Lévy measure μ. Formulae (6.10) and (6.12) suggest that for any $n \in \mathbf{N}$, there exist some $k = k_n$, $l = l_n$, $p = p_n$ and $C > 0$ such that

$$\sup_{|G|_{k,l,p}=1} |E[Ge_v(F)]| \le C(1 + |v|^2)^{-\frac{q_0}{2} n} . \tag{7.1}$$

Indeed, calculating as in Theorem 3.2 in [77] for $F = F' \circ \varepsilon_q^+$, and putting $G \equiv 1$, we can show:

Proposition 3.18. *For any $n \in \mathbf{N}$ there exist some $k, l \in \mathbf{N}$, $p > 2$ such that it holds*

$$|e_v(F)|'_{k,l,p} \le C'(1 + |v|^2)^{-\frac{q_0}{2} n} \tag{7.2}$$

for some $C = C_{k,l,p,n}$ under the condition (NDB).

We give a sketch of the proof.

We put

$$U'_t(v) = \frac{-(i\frac{v}{|v|^2}, D_t F')}{Q'(v)} ,$$
$$V'_u(v) = \frac{|v|^{-2}\varphi(|v|^{-\frac{1}{\beta}})^{-1}(e_v(-\tilde{D}_u F') - 1)\chi_{B(v)}}{Q'(v)} ,$$

where

$$Q'(v) := (v', \Sigma v') + \frac{1}{|v|^2\varphi(|v|^{-\frac{1}{\beta}})} \int_{A(\rho(v))} |e_v(\tilde{D}_u F) - 1|^2 \hat{N}(du) ,$$

where $v' = \frac{v}{|v|}$.

As $\gamma(\tau) \le 1$ and $\gamma(q(u)) > 1$,

$$Q(v)(S_t) \circ \varepsilon^+_\tau = Q'(v) \circ \varepsilon^+_q \circ \varepsilon^+_\tau .$$

Hence, to show (7.2), we first write $F = F' \circ \varepsilon^+_{q(u)}$ and show the estimate for F' viewing $q(u)$ as a parameter, and then integrate it with respect to the big jumps $q(u) = (S_t, (\xi_i))$. Here, we have:

Lemma 3.10. *Suppose that F' satisfies (ND). Then, we have, for any $k, l \in \mathbf{N}$ and $p \ge 2$,*

$$\sup_{|v|\ge 1}(|v| \, \|U'(v)\|_{k,l,p}) < +\infty ,$$
$$\sup_{|v|\ge 1}(|v| \, \varphi(|v|^{-\frac{1}{\beta}})\|V'(v)\|^{\sim}_{k,l,p}) < +\infty . \tag{7.3}$$

This lemma follows as in Lemma 4.4 in [77]. The use of this lemma is justified by Theorem 3.6.

Let

$$Z' = U' \oplus V' .$$

Then, we have

$$D_{(t,u)} e_v(F') = ((iv, DF') \oplus (e_v(\tilde{D}F') - 1)) e_v(F') .$$

Hence, one can check the identity

$$e_v(F') = \left\langle U' \oplus V' \, , D_{(t,u)} e_v(F') \right\rangle_{\mathbf{K}} ,$$

where $\mathbf{K} = L^2(\mathbf{T}, \mathbf{R}^m) \oplus L^2(U, \hat{N})$.

This leads to

$$E^{P'}[G e_v(F')] = E^{P'}[\langle GZ' \, , D_{(t,u)} e_v(F') \rangle_{\mathbf{K}}] = E^{P'}[\bar{\delta}(GZ') e_v(F')] .$$

Iterating this argument, we have

$$E^{P'}[G e_v(F')] = E^{P'}[\bar{\delta}(GZ') e_v(F')] = E^{P'}[\bar{\delta}(Z'\bar{\delta}(Z' \cdots \bar{\delta}(Z'\bar{\delta}(GZ')))) e_v(F')] .$$

Hence, in order to show (7.2), it is sufficient to show that for each n, there exist $k, l \in \mathbf{N}$, an even number $p \geq 2$, $k', l' \in \mathbf{N}$, an even number $p' \geq 2$, and $C > 0$ such that

$$\sup_{|G|_{k',l',p'}=1} \Big| \overbrace{\bar{\delta}\left(Z' \circ \varepsilon_q^+ \bar{\delta}\left(Z' \circ \varepsilon_q^+ \cdots \bar{\delta}\left(Z' \circ \varepsilon_q^+ \bar{\delta}\left(GZ' \circ \varepsilon_q^+\right)\right)\right)\right)}^{n} \Big|_{k,l,p} \leq C(1+|v|^2)^{-\frac{q_0}{2}n} . \tag{7.4}$$

Indeed, for $n = 1$, we have by the inequality (Theorem 3.5) that

$$\begin{aligned}
&|\bar{\delta}(GZ'(v))|_{k,l,p} \\
&\leq C\left(\|GU'(v)\|_{k,l+1,p} + \varphi\left(|v|^{-\frac{1}{\beta}}\right)^{\frac{1}{2}(1-\frac{1}{k_0})} \|GV'(v)\chi_{A(\rho(v))}\|^{\sim}_{k+p,l,(k+p)k_0} \right) \\
&\leq C|G|_{k+p,l+1,2p,\rho(v)} \\
&\quad \cdot \left(\|U'(v)\|_{k+p,l+1,2p} + \varphi(|v|^{-\frac{1}{\beta}})^{\frac{1}{2}(1-\frac{1}{k_0})} \|V'(v)\chi_{A(\rho(v))}\|^{\sim}_{k+p,l+1,2(k+p)k_0} \right) \\
&\leq C'|G|_{k+p,l+1,2p}|v|^{-q_0} ,
\end{aligned}$$

where in the second inequality, we have used the Schwarz inequality, and in the last one, we have used the previous lemma (Lemma 3.10) and the order condition. Hence, we have the assertion for $k' = k + p$, $l' = l + 1$ and $p' = 2p$.

Suppose that the assertion holds for $n - 1$. Then, we have

$$\begin{aligned}
&\Big| \overbrace{\bar{\delta}(Z'\bar{\delta}(Z' \cdots \bar{\delta}(Z'\bar{\delta}(GZ'))))}^{n} \Big|_{k,l,p} \\
&\leq C\left(\|\delta(U')\|_{k,l+1,p} + \varphi\left(|v|^{-\frac{1}{\beta}}\right)^{\frac{1}{2}(1-\frac{1}{k_0})} \|\tilde{\delta}(V')\|^{\sim}_{k+p,l,(k+p)k_0} \right) \\
&\quad \cdot | \overbrace{\bar{\delta}(Z' \cdots \bar{\delta}(Z'\bar{\delta}(GZ')))}^{n-1} |_{k,l+1,2p} \\
&\leq C|G|_{k',l',p'}|v|^{-q_0}|v|^{-q_0(n-1)} \leq C|v|^{-q_0 n} ,
\end{aligned}$$

where the second inequality follows from the assumption of the induction. Hence, we have the assertion for F' by induction.

Since $F = F' \circ \varepsilon^+_{q(u)}$ and since $\tilde{P}_{t,1}(S_t) \otimes (\mu'')^{\otimes \# S_t}$ is a bounded measure, we integrate the above inequality with respect to this measure and obtain (7.4). This proves the assertion (7.2) for F. □

Hence, although F (instead of F') may not satisfy the condition *(ND)*, we can define the composition $\Phi \circ F$ of F with $\Phi \in \mathcal{S}'$ under *(NDB)* due to (7.2). This leads to the asymptotic expansion of the density (Section 4.1.2) under a condition equivalent to *(NDB)*.

4 Applications

> Incessant is the change of water where the stream glides on calmly: the spray appears over a cataract, yet vanishes without a moment's delay. But none of them has resisted the destructive work of time.
>
> Kamo no Chomei (1155–1216), Hojoki/The Ten Foot Square Hut (Japanese classic)

In this chapter, we provide some applications of jump processes and their stochastic analysis given in the previous sections.

In Section 4.1, we give an asymptotic expansion of functionals which can be viewed as models of financial assets. Namely, we apply composition techniques developed in Section 3.5 to a functional $F(\epsilon)$ of the solution $X_t = S_t(\epsilon)$ to the canonical SDE with parameter $\epsilon > 0$. A notion of the *asymptotic expansion* in the space $\mathbf{D}'_\infty$ is introduced, and is applied to $F(\epsilon)$ as $\epsilon \to 0$. We take Φ in the space $\mathcal{S}'$ of tempered distributions. We shall justify the asymptotic expansion

$$\Phi \circ F(\epsilon) \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \quad \text{in} \quad \mathbf{D}'_\infty .$$

The expansions are based on formal Taylor expansions with respect to the power of ϵ. However, they are justified quantitatively using the three parameter norms introduced in Section 3.3.

Taking the expectation, the above gives the asymptotic expansion of the value

$$E[\Phi \circ F(\epsilon)] \sim E[f_0] + \epsilon E[f_1] + \epsilon^2 E[f_2] + \cdots .$$

Especially in case $\Phi(x) = e_v(x) = e^{i(v,x)}$, this result gives the asymptotic expansion of the density function

$$p(\epsilon, y) \sim p_0(y) + \epsilon p_1(y) + \epsilon^2 p_2(y) + \cdots ,$$

which is closely related to the short time asymptotic estimates stated in Theorems 2.2, 2.4 and Proposition 2.6 in Sections 2.2, 2.3. We treat this topic in Section 4.1.2. In Section 4.1.3, we provide some examples of the asymptotic expansion which often appear in applications in the theory of finance. In Section 4.1, we denote the solution of the SDE by S_t or $S_t(\epsilon)$ instead of X_t, according to the custom in the financial theory.

Section 4.2 is devoted to the application of a finance theory for jump processes to the optimal consumption problem. Although the problem is one-dimensional, it provides us with a good example of the interplay between the probability theory (stochastic analysis) and the PDE theory (boundary value problem).

4.1 Asymptotic expansion of the SDE

In this subsection, we apply composition techniques developed in Sections 3.5, 3.7 to a functional $F(\epsilon)$ of the solution $S_t(\epsilon)$ to the canonical SDE with parameter $\epsilon > 0$.

In economic models, the process which represents the asset price is often complex. We would like to analyse the behaviour of it by inferring from simple ones. In our model below, $S_t(1) = S_t(\epsilon)|_{\epsilon=1}$ is the target process, and $\epsilon > 0$ is an approximation parameter (coefficient to random factors).

We expand in Section 4.1.1 the composition $\Phi \circ S_t(\epsilon)$ with $\Phi \in \mathcal{S}'$ by

$$\Phi \circ S_t(\epsilon) \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots .$$

Each term f_j in the above has a proper financial meaning according to the order j. Based on this expansion, the expansion of the transition density function (Section 4.1.2) will be helpful in practical applications.

We consider an m-dimensional Lévy process Z_t given by

$$Z_t = bt + \sigma W(t) + \int_0^t \int_{|z|\le 1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|>1} zN(dsdz) ,$$

where $b \in \mathbf{R}^m$ and σ is an $m \times m$ matrix such that $\sigma\sigma^T = A$.

Let $\epsilon \in (0, 1]$ be a small parameter. Let $S(\epsilon)$ be a d-dimensional jump process depending on ϵ, given as the solution to the canonical SDE

$$dS_t(\epsilon) = a_0(S_t(\epsilon)) + \epsilon \sum_{i=1}^m a_i(S_t(\epsilon)) \circ dZ_t^i, \quad S_0(\epsilon) = x .$$

That is, $S_t(\epsilon)$ is given by

$$\begin{aligned} S_t(\epsilon) &= S_0(\epsilon) + \int_0^t a_0(S_{s-}(\epsilon))ds + \epsilon \sum_{i,j=1}^m \int_0^t a_i(S_{s-}(\epsilon))\sigma_{ij} \circ dW^j(s) \\ &+ \epsilon \Bigg\{ \int_0^t \int_{|z|\le 1} (\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))\tilde{N}(dsdz) + \int_0^t \int_{|z|>1} (\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))N(dsdz) \\ &\qquad + \int_0^t \int_{|z|\le 1} (\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon) - \sum_{i=1}^m z_i a_i(S_{s-}(\epsilon)))\hat{N}(dsdz) \Bigg\} . \end{aligned} \tag{1.1}$$

Here, $a_i(x)$, $i = 0, 1, \ldots, m$ are bounded C^∞ functions from $\mathbf{R}^d$ to $\mathbf{R}^d$ with bounded derivatives of all orders, and $\circ dW^j(s)$ denotes the Stratonovich integral with respect to the Wiener process $W^j(t)$. Here, ϕ_s^z denotes a one parameter group of diffeomorphisms (integral curve) generated by $\sum_{i=1}^m z_i a_i(.)$, and $\phi_1^z = \phi_s^z|_{s=1}$. We may also write $\mathrm{Exp}\,(t \sum_{i=1}^m z_i a_i)$ for ϕ_t^z. Obviously, the coefficient function $\phi_1^z(x)$ depends on $a_i(x)'s$. See Section 2.5.1 for details on canonical SDEs.

We remark that our asset model above adapts well to the *geometric Lévy* model (cf. [69] Section 2, [175] Section 2.2). The jump term represents the discontinuous fluctuation of the basic asset with parameter ϵ. In our model, each jump size of the driving

Poisson random measure N is not necessarily small, but the total fluctuation given as the stochastic integral of N is controlled by the small parameter $\epsilon > 0$.

Remark 4.1. By transforming the Stratonovich integral into the Itô integral, (1.1) can be written simply as

$$S_t(\epsilon) = S_0(\epsilon) + \int_0^t \tilde{a}_0(S_{s-}(\epsilon))ds + \epsilon \sum_{i,j=1}^m \int_0^t a_i(S_{s-}(\epsilon))\sigma_{ij} dW^j(s) + \epsilon \int_0^t \int (\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))\tilde{N}(dsdz) , \quad (1.2)$$

where

$$\tilde{a}_0(x) = a_0(x) + \frac{1}{2}\epsilon \sum_{i,j=1}^m \nabla a_i(x) a_i(x) \sigma_{ij}^2 + \epsilon \left\{ \int_{|z|>1} (\phi_1^z(x) - x)\mu(dz) + \int_{|z|\le 1} (\phi_1^z(x) - x - \sum_{i=1}^m z_i a_i(x))\mu(dz) \right\} .$$

We also remark that our model is slightly different from those studied in [114, 132] and [222], where the drift coefficient $\tilde{a}_0$ also on depends ϵ when transformed into Itô form.

It is known that (1.1) has a unique global solution for each ϵ. We sometimes omit ϵ and simply write S_t for $S_t(\epsilon)$.

One of our motivations in mathematical finance is to make a composition $\Phi(S_T(\epsilon))$ with some functional Φ appearing in the European call or put options, and lead an asymptotic expansion

$$\Phi(S_T(\epsilon)) \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

in some function space. Note that the above is not a formal expansion, but it is an expansion in terms of the Sobolev norms introduced in Section 3.3.

As expanded above, errors from the deterministic factor are arranged in terms of the order of the power of ϵ. As $\epsilon \to 0$, each coefficient of ϵ^j will provide a dominant factor at its power j in the model.

We recall $m \times m$ matrices B_ρ and B by

$$B_\rho = \frac{1}{\varphi(\rho)} \int_{|z|\le\rho} zz^T \mu(dz) ,$$

$$B = \liminf_{\rho\to 0} B_\rho ,$$

provided that the Lévy measure satisfies the order condition. We set $B = 0$ if μ does not satisfy the order condition. The Lévy process is called *non-degenerate* if the matrix

$A+B$ is invertible. We define a $d\times m$ matrix $C(x)$ by $(a_1(x),\ldots,a_m(x))$. We choose the same quantity $q_0 = 1-\frac{\alpha}{2\beta}(1+\frac{1}{k_0}) > 0$ as in (3.33) (Section 3.3) and fix it.

4.1.1 Analysis on the stochastic model

We introduce a basic assumption on $A+B$, where $A=\sigma\sigma^T$:
Assumption (AS) There exists $c>0$ such that for all $x\in\mathbf{R}^d$, it holds that

$$\left(v, C(x)(A+B)C(x)^T v\right) \ge c|v|^2 \ .$$

Remark 4.2. (1) This assumption corresponds to *(NDB)* in Definition 3.5 (Section 3.6).

(2) Assume $d=1$. If $S_0(\epsilon)=x>0$ and $A>0$, then the (Merton) model stated in (1.1) can be interpreted to satisfy the assumption *(AS)* with $C(x)=x$ and $g(x,z)=\phi_1^z(x)-x$. Indeed, in this case, the process $S_t(\epsilon)$ hits $\{0\}$ only by drift and the diffusion factors, and once it hits $\{0\}$, it stays there afterwards.

We denote by $S_t(\epsilon)$ the solution to (1.1) starting from $S_0(\epsilon)=x$ at $t=0$. It is known that it has a modification such that $S_t(\epsilon):\mathbf{R}^d\to\mathbf{R}^d$ is a C^∞ diffeomorphism a.s. for all $t>0$ (cf. Fujiwara–Kunita [67]).

In order to simplify the notation, we introduce the following assumption:
Assumption (AS2)

$$\int\limits_{|z|>1} |\phi_1^z(x)-x|\mu(dz) < +\infty \quad \text{for all } x\in\mathbf{R}^d \ .$$

Then, we have $|\int_{\mathbf{R}^m\setminus\{0\}}\{\phi_1^z(x)-x-\sum_{i=1}^m z_i a_i(x)1_{\{|z|\le 1\}}\}\mu(dz)|<+\infty$ for all $x\in\mathbf{R}^d$.

We note that $S_t(0)$ is given by the ODE $S_t(0)=S_0(0)+\int_0^t a_0(S_s(0))ds$, $S_0(0)=x$. We put

$$F_t(\epsilon)=\frac{1}{\epsilon}(S_t(\epsilon)-S_t(0)) \ ,$$

and let

$$F(\epsilon)=F_T(\epsilon) \ ,$$

where $T<+\infty$. In the sequel, we sometimes suppress ϵ in $S_t(\epsilon)$ for notational simplicity.

We can calculate the stochastic derivatives easily. Indeed,

$$D_tF=\left(\nabla S_{t,T}(S_{t-})\left(\sum_{i=1}^m a_i(S_{t-})\sigma_{ij}\right),\ j=1,\ldots,m\right), \quad a.e.\ \ dtdP \ ,$$

where $\nabla S_{t,T}(x)$ is the Jacobian matrix of the map $S_{t,T}(x):\mathbf{R}^d\to\mathbf{R}^d$, and $S_{s,t}=S_t\circ S_s^{-1}$.

Further, since

$$\begin{aligned}\frac{d}{ds}S_{t,T}(x+\epsilon(\phi_s^z(x)-x)) &= \nabla S_{t,T}(x+\epsilon(\phi_s^z(x)-x))\frac{d}{ds}(x+\epsilon(\phi_s^z(x)-x))\\ &= \nabla S_{t,T}(x+\epsilon(\phi_s^z(x)-x))\epsilon\sum_{i=1}^m z_i a_i(\phi_s^z(x)) \ ,\end{aligned}$$

we have

$$\tilde{D}_{t,z}F = \frac{1}{\epsilon}(S_{t,T}(S_{t-} + \epsilon(\phi_1^z(S_{t-}) - S_{t-})) - S_{t,T}(S_{t-}))$$
$$= \nabla S_{t,T}(S_{t-} + \epsilon(\phi_\theta^z(S_{t-}) - S_{t-}))\sum_{i=1}^m z_i a_i(\phi_\theta^z(S_{t-})) \,,$$

by the mean value theorem, where $\theta \in (0, 1)$. Due to the above, we have

$$\partial\tilde{D}_{t,0}F(\epsilon) = \partial_z\tilde{D}_{t,z}F(\epsilon)\Big|_{z=0} = \nabla S_{t,T}(S_{t-})(a_1(S_{t-}), \ldots, a_m(S_{t-})), \quad a.e.\ \ dtdP \,.$$

Repeating this argument, we find $F \in \mathbf{D}_\infty$. (See Section 3.3 for the definition of the Sobolev space $\mathbf{D}_\infty$.)

Due to [192] Theorem V.60 we have:

Lemma 4.1 (cf. [95] Lemma 6.2). *Assume (AS) and (AS2). The Jacobian matrix $\nabla S_{s,t}$ is invertible a.s. Both $|\nabla S_{s,t}|$ and $|(\nabla S_{s,t})^{-1}|$ (norms of these matrices) belong to L^p for any $p > 1$. Further, there exist positive constants c_p and C_p such that*

$$E[|\nabla S_{s,t}|^p] + E[|(\nabla S_{s,t})^{-1}|^p] \le C_p \exp c_p(t-s) \tag{1.3}$$

holds for any x and $s < t$, and $\epsilon > 0$ sufficiently small.

Furthermore, we have:

Lemma 4.2. *For any $k \in \mathbf{N}$, there exist positive constants c_p' and C_p' such that*

$$\sup_{\mathbf{u}\in A(1)^k} E[|(\nabla S_{s,t} \circ \varepsilon_{\mathbf{u}}^+)^{-1}|^p] \le C_p' \exp c_p'(t-s) \,, \tag{1.4}$$

with any $p \ge 2$, holds for any x and $s < t$, and $\epsilon > 0$ sufficiently small.

Remark 4.3. In [95], we have assumed that the support of μ is compact. However, the assertion can be extended to $\mathbf{R}^m$ under the assumption *(AS2)*.

Proof of Lemma 4.2. We will show $\sup_{\mathbf{u}\in A(1)^k} |(\nabla S_{s,t} \circ \varepsilon_{\mathbf{u}}^+)^{-1}| \in L^p$, $p \ge 2$. For $k = 1$, we write

$$(\nabla S_{s,t} \circ \varepsilon_{\mathbf{u}}^+)^{-1} = (\nabla S_{s,t})^{-1} \circ \varepsilon_{\mathbf{u}}^+ = \tilde{D}_{\mathbf{u}}(\nabla S_{s,t})^{-1} + (\nabla S_{s,t})^{-1} \,.$$

For general $\mathbf{u} = (u_1, \ldots, u_k)$, $(\nabla S_{s,t})^{-1} \circ \varepsilon_{\mathbf{u}}^+$ is a sum of terms $\tilde{D}_{u_{i_1},\ldots,u_{i_l}}(\nabla S_{s,t})^{-1}$, where $i_1, \ldots, i_l \in \{1, \ldots, k\}$, $l \le k$.

As $|(\nabla S_{s,t})^{-1}| \in L^p$ by Lemma 4.1, we will show $\sup_{\mathbf{u}\in A(1)^k} |\tilde{D}_{\mathbf{u}}(\nabla S_{s,t})^{-1}| \in L^p$, $\mathbf{u} = (u_1, \ldots, u_k)$ for $k = 1, 2, \ldots$.

We recall that the inverse matrix $(\nabla S_{s,t})^{-1}$ satisfies a.s. the SDE

$$
\begin{aligned}
(\nabla S_{s,t})^{-1} = I - &\int_s^t (\nabla S_{s,r-})^{-1}\nabla\tilde{a}_0(S_{r-})dr \\
&- \epsilon \sum_{i,j=1}^m \int_s^t (\nabla S_{s,r-})^{-1}\nabla a_i(S_{r-})\sigma_{ij}dW^j(r) \\
&+ \epsilon \int_s^t \int (\nabla S_{s,r-})^{-1}\left\{(\nabla\phi_1^z(S_{r-}))^{-1} - I\right\}\tilde{N}(drdz)\ . \qquad (1.5)
\end{aligned}
$$

Let $k = 1$ and let $u = (s_1, z_1)$. Since $\tilde{D}_u(\nabla S_{t-})^{-1} = 0$ if $t < s_1$, we assume $t \geq s_1$. Then, $\tilde{D}_u(\nabla S_{t-})^{-1}$ should satisfy

$$
\begin{aligned}
\tilde{D}_u(\nabla S_t)^{-1} = (\nabla\phi_1^z(S_{s_1}))^{-1} - I - &\int_{s_1}^t \tilde{D}_u((\nabla S_{r-})^{-1}\cdot\nabla\tilde{a}_0(S_{r-}))dr \\
&- \epsilon \sum_{i,j=1}^m \int_{s_1}^t \tilde{D}_u((\nabla S_{r-})^{-1}\cdot\nabla a_i(S_{r-}))\sigma_{ij}dW^j(r) \\
&+ \epsilon \int_{s_1}^t \int \tilde{D}_u\{(\nabla S_{r-})^{-1}(\nabla\phi_1^z(S_{r-}))^{-1} - I)\}\tilde{N}(drdz)\ .
\end{aligned}
$$

We will show $|\tilde{D}_u(\nabla S_t)^{-1}| \in L^p$. Let $\tau_n, n = 1, 2, \ldots$ be a sequence of stopping times such that $\tau_n = \inf\{t > s; |\tilde{D}_u(\nabla S_{s,t})^{-1}| > n\}$ ($= T$ if the set $\{\cdots\}$ is empty). In the following discussion, we denote the stopped process $\tilde{D}_u(\nabla S_{s,t\wedge\tau_n})^{-1}$ as $\tilde{D}_u(\nabla S_t)^{-1}$. However, constants appearing in the inequalities do not depend on n, nor on u.

Drift term We have

$$
\begin{aligned}
&\tilde{D}_u((\nabla S_{r-})^{-1}\cdot\nabla\tilde{a}_0(S_{r-})) \\
&\quad = \tilde{D}_u(\nabla S_{r-})^{-1}\cdot\nabla\tilde{a}_0(S_{r-}) + (\nabla S_{r-})^{-1}\cdot\tilde{D}_u\nabla\tilde{a}_0(S_{r-}) + \tilde{D}_u(\nabla S_{r-})^{-1}\cdot\tilde{D}_u\nabla\tilde{a}_0(S_{r-})\ ,
\end{aligned}
$$

and

$$
\tilde{D}_u(\nabla\tilde{a}_0(S_{r-})) = (\tilde{D}_u S_{r-})^T\nabla^2\tilde{a}_0(S_{r-} + \theta\tilde{D}_u S_{r-})
$$

for some $\theta \in (0, 1)$ by the mean value theorem. Since $\nabla\tilde{a}_0$ is a bounded function, we have the inequality

$$
\begin{aligned}
&E\left[\left|\int_s^t \tilde{D}_u(\nabla S_{r-})^{-1}\nabla\tilde{a}_0(S_{r-})dr\right|^p\right] \\
&\qquad\qquad \leq cE\left[\left|\int_s^t |\tilde{D}_u(\nabla S_{r-})^{-1}|dr\right|^p\right] \leq c'\int_s^t E[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p]dr\ .
\end{aligned}
$$

Similarly, since $\nabla^2 \tilde{a}_0$ is a bounded function,

$$E\left[\left|\int_s^t (\nabla S_{r-})^{-1} \tilde{D}_u \nabla \tilde{a}_0(S_{r-}) dr\right|^p\right] \le cE\left[\left|\int_s^t |(\nabla S_{r-})^{-1}||\tilde{D}_u S_{r-}| dr\right|^p\right]$$
$$\le c' \int_s^t E[|(\nabla S_{r-})^{-1}|^p |\tilde{D}_u S_{r-}|^p] dr$$
$$\le c'' \int_s^t (E[|(\nabla S_{r-})^{-1}|^{p_1}])^{p/p_1} (E[|\tilde{D}_u S_{r-}|^{p_2}])^{p/p_2} dr$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $p_1 > 1$, $p_2 > 1$. Since $E[|\tilde{D}_u S_{r-}|^{p_2}] < +\infty$ for all $p_2 \ge 1$, we have

$$E\left[\left|\int_s^t (\nabla S_{r-})^{-1} \tilde{D}_u \nabla \tilde{a}_0(S_{r-}) dr\right|^p\right] \le c''' \int_s^t E[|(\nabla S_{r-})^{-1}|^p] dr\,.$$

As for the third term,

$$E\left[\left|\int_s^t \tilde{D}_u (\nabla S_{r-})^{-1} \tilde{D}_u \nabla \tilde{a}_0(S_{r-}) dr\right|^p\right] \le cE\left[\left|\int_s^t |\tilde{D}_u (\nabla S_{r-})^{-1}||\tilde{D}_u S_{r-}| dr\right|^p\right]$$
$$\le c' \int_s^t E[|\tilde{D}_u (\nabla S_{r-})^{-1}|^p |\tilde{D}_u S_{r-}|^p] dr\,.$$

Hence, we have

$$E\left[\left|\int_s^t \tilde{D}_u (\nabla S_{r-})^{-1} \tilde{D}_u \nabla \tilde{a}_0(S_{r-}) dr\right|^p\right] \le c'' \int_s^t E[|\tilde{D}_u (\nabla S_{r-})^{-1}|^p] dr$$

as above.

Diffusion term We have

$$\tilde{D}_u((\nabla S_{r-})^{-1} \cdot \nabla a_i(S_{r-})) =$$
$$\tilde{D}_u (\nabla S_{r-})^{-1} \cdot \nabla a_i(S_{r-}) + (\nabla S_{r-})^{-1} \cdot \tilde{D}_u \nabla a_i(S_{r-}) + \tilde{D}_u (\nabla S_{r-})^{-1} \cdot \tilde{D}_u \nabla a_i(S_{r-})\,.$$

Then,

$$E\left[\left|\int_s^t \tilde{D}_u (\nabla S_{r-})^{-1} \cdot \nabla a_i(S_{r-}) \sigma_{ij} dW^j(r)\right|^p\right] \le c \int_s^t E[|\tilde{D}_u (\nabla S_{r-})^{-1}|^p] dr\,.$$

We have

$$\tilde{D}_u(\nabla a_i(S_{r-})) = (\tilde{D}_u S_{r-})^T \nabla^2 a_i(S_{r-} + \theta' \tilde{D}_u S_{r-})$$

for some $\theta' \in (0,1)$ as above. Hence,

$$E\left[\left|\int_s^t (\nabla S_{r-})^{-1}\cdot \tilde{D}_u \nabla a_i(S_{r-})\sigma_{ij}dW^j(r)\right|^p\right] \le c' \int_s^t E[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p]dr$$

as above. The third term is bounded similarly.

Jump term We have

$$\begin{aligned}\tilde{D}_u\{(\nabla S_{r-})^{-1}(\nabla\phi_1^z(S_{r-}))^{-1} - I)\} &= \tilde{D}_u(\nabla S_{r-})^{-1}.(\nabla\phi_1^z(S_{r-}))^{-1} - I)\\ &+ (\nabla S_{r-})^{-1}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1} - I) + \tilde{D}_u(\nabla S_{r-})^{-1}.\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1} - I)\ .\end{aligned}$$

Then,

$$\begin{aligned}&E\left[\left|\int_{s_1}^t\int \tilde{D}_u(\nabla S_{r-})^{-1}(\nabla\phi_1^z(S_{r-}))^{-1} - I)\tilde{N}(drdz)\right|^p\right]\\ &\le cE\left[\left|\int_s^t\int |\tilde{D}_u(\nabla S_{r-})^{-1}(\nabla\phi_1^z(S_{r-}))^{-1} - I)|^2\hat{N}(drdz)\right|^{p/2}\right]\\ &\quad + c'E\left[\int_s^t\int |\tilde{D}_u(\nabla S_{r-})^{-1}(\nabla\phi_1^z(S_{r-}))^{-1} - I)|^p\hat{N}(drdz)\right]\ . \qquad (1.6)\end{aligned}$$

We remark

$$\sup_x \int |(\nabla\phi_1^z(x))^{-1} - I|^2\mu(dz) < +\infty, \quad \sup_x \int |(\nabla\phi_1^z(x))^{-1} - I|^2\mu(dz) < +\infty\ .$$

Hence, the right-hand side of (1.6) is dominated by $c''\int_s^t E[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p]dr$.

We remark $(\nabla\phi_1^z)^{-1} = \nabla((\phi_1^z)^{-1}) = \nabla(\phi_1^{-z})$ by the inverse mapping theorem. Hence,

$$\begin{aligned}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1} - I) &= \tilde{D}_u(\nabla\phi_1^{-z}(S_r)) = \nabla\phi_1^{-z}(S_{s_1,r}\circ\phi_1^{-z}(S_r)) - \nabla\phi_1^{-z}(S_{s_1,r})\\ &= \nabla^2\phi_1^{-z}(S_{s_1,r}\circ\phi_{\theta''}^{-z}(S_r))\cdot\nabla S_{s_1,r}(\phi_{\theta''}^{-z}(S_r))\otimes\sum_{i=1}^m(-z_i)a_i(\phi_{\theta''}^{-z}(S_r))\\ &= -(\nabla S_{s_1,r}(\phi_{\theta''}^{-z}(S_r)))^T\nabla^2\phi_1^{-z}(S_{s_1,r}\circ\phi_{\theta''}^{-z}(S_r))\cdot\sum_{i=1}^m z_i a_i(\phi_{\theta''}^{-z}(S_r))\end{aligned}$$

for some $\theta'' \in (0,1)$. By this expression,

$$E\left[\int_s^t\int |(\nabla S_{r-})^{-1}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1} - I)|^p\hat{N}(drdz)\right]$$

$$\leq cE\left[\int_s^t\int |(\nabla S_{r-})^{-1}|^p\cdot|(\nabla S_{s_1,r}(\phi_{\theta''}^{-z}(S_r)))|^p\right.$$
$$\left.\times|\nabla^2\phi_1^{-z}(S_{s_1,r}\circ\phi_{\theta''}^{-z}(S_r))|^p\cdot|\sum_{i=1}^m z_i a_i(\phi_{\theta''}^{-z}(S_r))|^p\hat{N}(drdz)\right].$$

We have $E[\sup_y|\nabla S_{s_1,r}(y)|^p]<+\infty$, $\sup_x\int|\nabla^2\phi_1^{-z}(x)|^p\mu(dz)<+\infty$, and

$$\sup_{x,\theta}\int|\sum_{i=1}^m z_i a_i(\phi_\theta^{-z}(x))|^p\mu(dz)<+\infty,\quad p\geq 2\,.$$

Hence, the right-hand side of the above formula is dominated by $c'\int_s^t E[|(\nabla S_{r-})^{-1}|^p]dr$.

Finally,

$$E\left[\left|\int_s^t\int\tilde{D}_u(\nabla S_{r-})^{-1}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1}-I)\hat{N}(drdz)\right|^p\right]$$
$$\leq cE\left[\left|\int_s^t\int|\tilde{D}_u(\nabla S_{r-})^{-1}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1}-I)|^2\hat{N}(drdz)\right|^{p/2}\right]$$
$$+cE\left[\int_s^t\int|\tilde{D}_u(\nabla S_{r-})^{-1}\tilde{D}_u(\nabla\phi_1^z(S_{r-}))^{-1}-I)|^p\hat{N}(drdz)\right].$$

Due to similar arguments as above, both terms are dominated by

$$c\int_s^t E[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p]dr\,.$$

Since we know $E[|(\nabla S_{r-})^{-1}|^p]<+\infty$ for all $p>1$, the whole terms are dominated by

$$c\int_s^t E[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p]dr\,.$$

Hence, by Lemma 4.1,

$$E\left[|\tilde{D}_u(\nabla S_{t-})^{-1}|^p\right]\leq c+\epsilon c\int_s^t E\left[|\tilde{D}_u(\nabla S_{r-})^{-1}|^p\right]dr\,.$$

Hence,

$$E[|\tilde{D}_u(\nabla S_{t-})^{-1}|^p]\leq c\exp\epsilon c(t-s)$$

by the Gronwall's lemma. As the constants can be chosen, not depending on n nor on $\mathbf{u}$, we let $n\to+\infty$ to obtain the assertion for $k=1$.

Repeating this argument, we have the assertion of the lemma. □

By the differentiability of $z \mapsto \phi_s^z$, we can see that $\tilde{D}_{t,z}F$ is twice continuously differentiable with respect to $z = (z_1, \ldots, z_m)$ a.s.

Writing $S_t(\epsilon) = X_t$ in the form in Section 3.6, $\sigma(x)$, $b(x)$ and $g(x,z)$ are given by $\sigma(x) = \epsilon(a_1(x), \ldots, a_m(x))\sigma$, $\tilde{a}_0(x)$ and $g(x,z) = \epsilon(\phi_1^z(x) - x) = \epsilon(\mathrm{Exp}\,(\sum_{i=1}^m z_i a_i)(x) - x)$, respectively. Since all derivatives of $a_i(x)$s are bounded and satisfy the linear growth condition, we are in the same situation as in Section 3.6.1.

The condition *(NDB)* is satisfied by *(AS)*, and $S_t(\epsilon)$ enjoys the flow property since it is a canonical process. Regarding $F(\epsilon) = \frac{1}{\epsilon}(S_T(\epsilon) - S_T(0))$ where $S_T(0) = \mathrm{Exp}\,(Ta_0)(x)$ is deterministic, we see, by Lemma 3.7 from Chapter 3, the condition *(R)* holds for $F(\epsilon)$ for each $\epsilon > 0$. That is, for any positive integer k and $p > 1$, derivatives satisfy $0 < \epsilon < 1$

$$\sup_{t\in\mathbf{T},\mathbf{u}\in A(1)^k} E\left[\sum_{i=1}^m \sup_{|z_i|\le 1} |\partial_{z_i}\tilde{D}_{t,z}F(\epsilon)\circ\varepsilon_{\mathbf{u}}^+|^p + \sum_{i,j=1}^m \sup_{|z|\le 1} |\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}F(\epsilon)\circ\varepsilon_{\mathbf{u}}^+|^p\right] < +\infty\,. \quad (1.7)$$

We put the Malliavin covariance $R(\epsilon)$ concerning the Wiener space depending on $\epsilon > 0$ by

$$R(\epsilon) = \int_{\mathbf{T}} D_t F(\epsilon)(D_t F(\epsilon))^T dt\,. \quad (1.8)$$

The Malliavin covariance $R(\epsilon)$ is then represented by

$$R(\epsilon) = \int_{\mathbf{T}} \nabla S_{t,T}(S_{t-})C(S_{t-})AC(S_{t-})^T \nabla S_{t,T}(S_{t-})^T dt, \quad a.s.dP\,. \quad (1.9)$$

We put

$$\tilde{K}(\epsilon) = \int_{\mathbf{T}} (\partial\tilde{D}_{t,0}F(\epsilon))B(\partial\tilde{D}_{t,0}F(\epsilon))^T dt\,. \quad (1.10)$$

Then, the covariance $\tilde{K}(\epsilon)$ is written as

$$\tilde{K}(\epsilon) = \int_{\mathbf{T}} \nabla S_{t,T}(S_{t-})C(S_{t-})B(C(S_{t-}))^T \nabla S_{t,T}(S_{t-})^T dt, \quad a.s.dP\,. \quad (1.11)$$

Furthermore, we put

$$\tilde{K}_\rho(\epsilon) = \int_{\mathbf{T}} \partial\tilde{D}_{t,0}F(\epsilon)B_\rho(\partial\tilde{D}_{t,0}F(\epsilon))^T dt, \quad a.s.dP \quad (1.12)$$

for $\rho > 0$. Note that $\tilde{K}_\rho(\epsilon)$ can be written as

$$\tilde{K}_\rho(\epsilon) = \int_{A(\rho)} \frac{1}{\varphi(\rho)} \nabla S_{t,T}(S_{t-})C(S_{t-})zz^T(C(S_{t-}))^T \nabla S_{t,T}(S_{t-})^T \hat{N}(du)\,. \quad (1.13)$$

We introduce the following condition to guarantee the existence of the composition $\Phi \circ F(\epsilon)$ with $\Phi \in \mathcal{S}'$ uniformly in ϵ.

Definition 4.1. We say $F(\epsilon) = (F^1, \ldots, F^d)$ satisfies the condition *(UND)* (*uniformly non-degenerate*) if for all $p \geq 1$ and any integer k, there exists $\beta \in (\frac{\alpha}{2}, 1]$ such that it holds that

$$\limsup_{\epsilon \to 0} \sup_{\rho \in (0,1)} \sup_{\substack{v \in \mathbf{R}^d \\ |v|=1}} \operatorname{ess\,sup}_{\mathbf{u} \in A^k(\rho)} E[|((v, \Sigma(\epsilon)v) + \varphi(\rho)^{-1} \int_{A(\rho)} |(v, \tilde{D}_u F(\epsilon))|^2 1_{\{|\tilde{D}_u F(\epsilon)| \leq \rho^\beta\}} \hat{N}(du))^{-1} \circ \varepsilon_{\mathbf{u}}^+|^p] < +\infty\,,$$

where $\Sigma(\epsilon) = (\Sigma_{i,j}(\epsilon))$, $\Sigma_{i,j}(\epsilon) = \int_{\mathbb{T}} D_t F^i(\epsilon) D_t F^j(\epsilon) dt$.

In [77], we put $1_{\{|(v, \tilde{D}_u F(\epsilon))| \leq \rho^\beta\}}$ in place of $1_{\{|\tilde{D}_u F(\epsilon)| \leq \rho^\beta\}}$. Here, we have strengthened the condition.

We are treating a series of random variables with parameter ϵ defined on the Wiener–Poisson space. We set the topology among them by using the order of ϵ as it tends to 0 in terms of the $|\cdot|_{k,l,p}$ norms. We define the asymptotic expansion in $\mathbf{D}_\infty$ and in $\mathbf{D}'_\infty$ in terms of the order of ϵ.

Definition 4.2. Let $G(\epsilon)$ be a Wiener–Poisson variable. We write

$$G(\epsilon) = O(\epsilon^m) \quad \text{in} \quad \mathbf{D}_\infty$$

if $G(\epsilon) \in \mathbf{D}_\infty$ for $\epsilon \in (0, 1]$, and if for all $k, l \geq 0$ and for any $p \geq 2$, it holds

$$\limsup_{\epsilon \to 0} \frac{|G(\epsilon)|_{k,l,p}}{\epsilon^m} < \infty\,.$$

Definition 4.3.

(1) We say that $F(\epsilon)$ has the asymptotic expansion

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j \quad \text{in} \quad \mathbf{D}_\infty$$

if it holds that

(i) $F(\epsilon), f_0, f_1, \cdots \in \mathbf{D}_\infty$ for any ϵ.

(ii) For any nonnegative integer m,

$$F(\epsilon) - \sum_{\nu=0}^{m} \epsilon^\nu f_\nu = O(\epsilon^{m+1}) \quad \text{in} \quad \mathbf{D}_\infty\,.$$

(2) We say that $\Phi(\epsilon) \in \mathbf{D}'_\infty$ has an asymptotic expansion

$$\Phi(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \Phi_\nu \quad \text{in} \quad \mathbf{D}'_\infty$$

if for any $m \geq 0$, there exist $k = k(m) \geq 0$, $l = l(m) \geq 0$ and $p \geq 2$ such that for some $\Phi(\epsilon), \Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_m \in \mathbf{D}'_{k,l,p}$, it holds that

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon^{m+1}} |\Phi(\epsilon) - \sum_{\nu=0}^{m} \epsilon^j \Phi_\nu|_{k,l,p}{}' < +\infty\,.$$

Remark 4.4. In (1), the topology for $(F(\epsilon))$ is a weak topology given as an inductive limit in m of those given by the strong convergence with respect to the $|\cdot|_{k,l,p}$-norm up to the order $m = 1, 2, 3, \ldots$, where the index (k, l, p) is arbitrary. On the other hand, in (2), the topology for $(\Phi(\epsilon))$ is a weak topology given as an inductive limit in m of those given by the $*$-weak topology induced on $\mathbf{D}'_{k,l,p}$ (the topology on $\mathbf{D}'_{k,l,p}$ of pointwise convergence with respect to the $|\cdot|'_{k,l,p}$-norm) up to the order $m = 1, 2, 3, \ldots$, where the index (k, l, p) is as above.

Lemma 4.3. (1) *If $F(\epsilon) \in \mathbf{D}_\infty$ and $G(\epsilon) \in \mathbf{D}_\infty$, $\epsilon \in (0, 1]$, are such that*

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j \quad \text{in} \quad \mathbf{D}_\infty$$

and

$$G(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j g_j \quad \text{in} \quad \mathbf{D}_\infty ,$$

then $H(\epsilon) = F(\epsilon)G(\epsilon)$ satisfies

$$H(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j h_j \quad \text{in} \quad \mathbf{D}_\infty$$

and h_j's are obtained by the formal multiplication:

$$h_0 = g_0 f_0,\ h_1 = g_0 f_1 + g_1 f_0,\ h_2 = g_0 f_2 + g_1 f_1 + g_2 f_0,\ \ldots$$

(2) *If $F(\epsilon) \in \mathbf{D}_\infty$ and $\Phi(\epsilon) \in \mathbf{D}'_\infty$, $\epsilon \in (0, 1]$, such that*

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j \quad \text{in} \quad \mathbf{D}_\infty$$

and

$$\Phi(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \Phi_j \quad \text{in} \quad \mathbf{D}'_\infty ,$$

then $\Psi(\epsilon) = F(\epsilon)\Phi(\epsilon)$ satisfies

$$\Psi(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \Psi_j \quad \text{in} \quad \mathbf{D}'_\infty$$

and Ψ_j's are obtained by the formal multiplication:

$$\Psi_0 = f_0 \Phi_0,\ \Psi_1 = \Phi_0 f_1 + \Phi_1 f_0,\ \Psi_2 = \Phi_0 f_2 + \Phi_1 f_1 + \Phi_2 f_0,\ \ldots$$

The proof of this lemma is similar to that of [82] Proposition 9.3(i), (iii).

The following is the main assertion.

Theorem 4.1 ([77] Theorem 5.5). *Suppose $F(\epsilon)$ satisfies the condition (UND), and that $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in $\mathbf{D}_\infty$. Then, for all $\Phi \in \mathcal{S}'$, we have $\Phi \circ F(\epsilon) \in \mathbf{D}'_\infty$ has an asymptotic expansion in $\mathbf{D}'_\infty$:*

$$\Phi \circ F(\epsilon) \sim \sum_{m=0}^{\infty} \sum_{|\mathbf{n}|=m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} \Phi) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}}$$
$$\sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \cdots \quad in \quad \mathbf{D}'_\infty .$$

Here, Φ_0, Φ_1, Φ_2 are given by the formal expansion

$$\Phi_0 = \Phi \circ f_0, \qquad \Phi_1 = \sum_{i=1}^{d} f_1^i (\partial_{x_i} \Phi) \circ f_0 ,$$

$$\Phi_2 = \sum_{i=1}^{d} f_2^i (\partial_{x_i} \Phi) \circ f_0 + \frac{1}{2} \sum_{i,j=1}^{d} f_1^i f_1^j (\partial^2_{x_i x_j} \Phi) \circ f_0 ,$$

$$\Phi_3 = \sum_{i=1}^{d} f_3^i (\partial_{x_i} \Phi) \circ f_0 + \frac{2}{2!} \sum_{i,j=1}^{d} f_1^i f_2^j (\partial^2_{x_i x_j} \Phi) \circ f_0 + \frac{1}{3!} \sum_{i,j,k=1}^{d} f_1^i f_1^j f_1^k (\partial^3_{x_i x_j x_k} \Phi) \circ f_0$$

$$\cdots$$

In case $m = d = 1$, the condition *(UND)* holds under the assumption that $\inf_{x \in \mathbf{R}} a(x) > 0$. See [77] Theorem 7.4. This theorem is proved at the end of this subsection.

Before applying this theorem directly, we introduce another non-degeneracy condition:

Definition 4.4. We say $F(\epsilon)$ satisfies the condition *(UND2)* if $R(\epsilon) + \tilde{K}(\epsilon)$ is invertible and if for any integer k and $p > 1$ such that

$$\limsup_{\epsilon \to 0} \sup_{|v|=1, \mathbf{u} \in A(1)^k} E[(v, (R(\epsilon) + \tilde{K}(\epsilon)) \circ \varepsilon_{\mathbf{u}}^+ v)^{-p}] < +\infty . \tag{1.14}$$

We claim:

Proposition 4.1. *The assumptions (AS), (AS2) imply the condition (UND2).*

Proof. By (1.9), (1.11) and by *(AS)*, we have

$$(v, (R(\epsilon) + \tilde{K}(\epsilon)) \circ \varepsilon_{\mathbf{u}}^+ v) \geq C \int_{\mathbf{T}} |v^T (\nabla S_{t,T} \circ \varepsilon_{\mathbf{u}}^+)|^2 dt .$$

Since we have, by the Jensen's inequality,

$$\left(\frac{1}{t} \int_0^t f(s) ds \right)^{-1} \leq \frac{1}{t} \int_0^t f(s)^{-1} ds$$

for a positive function $f(.)$,

$$(v,(R(\epsilon)+\tilde{K}(\epsilon))\circ\varepsilon_{\mathbf{u}}^{+}v)^{-1}\le\frac{1}{CT^2}\int_{\mathbf{T}}|(v,(\nabla S_{t,T}\circ\varepsilon_{\mathbf{u}}^{+}))|^{-2}dt$$
$$\le\frac{1}{CT^2|v|^2}\int_{\mathbf{T}}|(\nabla S_{t,T}\circ\varepsilon_{\mathbf{u}}^{+})^{-1}|^2dt\,.$$

In the last inequality, we used the inequality $|v^T\nabla S_{t,T}\circ\varepsilon_{\mathbf{u}}^{+}|^{-1}\le|(\nabla S_{t,T}\circ\varepsilon_{\mathbf{u}}^{+})^{-1}|/|v|$.

Hence, by taking v for $|v|=1$,

$$\sup_{|v|=1,\mathbf{u}\in A(1)^k}E[(v,(R(\epsilon)+\tilde{K}(\epsilon))\circ\varepsilon_{\mathbf{u}}^{+}v)^{-p}]^{1/p}$$
$$\le\frac{1}{CT^2}\int_{\mathbf{T}}\sup_{\mathbf{u}\in A(1)^k}E\left[|(\nabla S_{t,T}\circ\varepsilon_{\mathbf{u}}^{+})^{-1}|^{2p}\right]^{1/p}dt\,.$$

The last member is dominated by a positive constant $C_{p,k}>0$ by Lemma 4.2. □

Furthermore, we have:

Proposition 4.2. *The condition (UND2) for $F(\epsilon)$ implies the condition (UND) for $F(\epsilon)$.*

Combining Propositions 4.1 and 4.2, we can use the theory of composition given in Section 3.5, and lead to the asymptotic expansion using Theorem 4.1 under the non-degeneracy of the coefficients.

To prove the above proposition, we prepare the following notion.

Definition 4.5. We say $F(\epsilon)$ satisfies the condition *(UND3)* if $R(\epsilon)+\tilde{K}(\epsilon)$ is invertible and if for any integer k and any $p>1$, there exists $\rho_0>0$ such that

$$\limsup_{\epsilon\to0}\sup_{0<\rho<\rho_0}\sup_{|v|=1,\mathbf{u}\in A(1)^k}E[|(v,(R(\epsilon)+\tilde{K}_\rho(\epsilon))\circ\varepsilon_{\mathbf{u}}^{+}v)^{-p}|]<+\infty\,. \tag{1.15}$$

The proof of Proposition 4.2 consists of the following two lemmas.

Lemma 4.4. *For $F(\epsilon)$, the condition (UND2) implies the condition (UND3).*

Lemma 4.5. *The condition (UND3) for $F(\epsilon)$ implies the condition (UND) for $F(\epsilon)$.*

Proof of Lemma 4.4. The proof proceeds in a similar way to Lemma 3.6 (i) (Section 3.5.2), replacing $\tilde{K}'$, $\tilde{K}'_\rho$ with $\tilde{K}(\epsilon)$, $\tilde{K}_\rho(\epsilon)$, respectively. Indeed,

$$(v,(R(\epsilon)+\tilde{K}_\rho(\epsilon))v)$$
$$=\left(v,\left(\int_{\mathbf{T}}\left\{\nabla S_{t,T}(S_{t-})C(S_{t-})\left(A+\frac{1}{\varphi(\rho)}\int_{|z|\le\rho}zz^T\mu(dz)\right)(C(S_{t-}))^T\nabla S_{t,T}(S_{t-})^T\right\}\right)v\right)$$
$$\ge C'(v,(R(\epsilon)+\tilde{K}(\epsilon))v)$$

for $0<\rho<\rho_0$, for some $C'>0$. Then, we use the uniformity with respect to $\epsilon\in(0,\epsilon_0)$ in (1.15). □

Proof of Lemma 4.5. The proof also proceeds as in Lemma 3.6 (ii). Here, we introduce a stopping time $\tau = \tau(v, \mathbf{u}, \epsilon)$ by

$$\tau = \inf \left\{ \rho \in (0, \delta_0); |R_\rho(\epsilon) \circ \varepsilon^+_{\mathbf{u}} - T_\rho(\epsilon) \circ \varepsilon^+_{\mathbf{u}}| \geq \frac{1}{2} T_\rho(\epsilon) \circ \varepsilon^+_{\mathbf{u}} \right\} ,$$

instead of the τ in the proof of Lemma 3.10. Then, we use the uniformity with respect to ϵ in *(UND3)*. □

We claim first that $S_t(\epsilon)$ can be expanded as

$$S_t(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n s_n(t) \quad \text{in} \quad \mathbf{D}_\infty \tag{1.16}$$

and then show the explicit formula for $s_n(t)$. This leads to the nontrivial asymptotic expansion

$$F_t(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n f_n(t) \quad \text{in} \quad \mathbf{D}_\infty ,$$

where $F_t(\epsilon)$ has appeared above.

To this end, we would put

$$S_t^{(n)}(\epsilon) = \frac{\partial^n S_t(\epsilon)}{\partial \epsilon^n} , \quad n = 1, 2, 3, \ldots$$

and denote

$$A_n(t) = S_t^{(n)}(\epsilon)|_{\epsilon=0} , \quad n = 1, 2, 3, \ldots$$

If these terms are calculated properly in $\mathbf{D}_\infty$, we will have

$$S_t(\epsilon) \sim S_t(0) + \epsilon A_1(t) + \epsilon^2 \frac{1}{2!} A_2(t) + \cdots \tag{1.17}$$

and hence

$$F_t(\epsilon) \sim A_1(t) + \epsilon \frac{1}{2!} A_2(t) + \epsilon^2 \frac{1}{3!} A_3(t) + \cdots \tag{1.18}$$

To verify the claim (1.16), we prepare the following lemmas.

Lemma 4.6. *Let* $0 \leq s_0 \leq T$. *Let* $Y_{s_0,t}(\epsilon)$, $0 < s_0 < t \leq T$ *be a semimartingale of the form*

$$Y_{s_0,t}(\epsilon) = Y_{s_0,s_0}(\epsilon) + \int_{s_0}^{t} g_r(\epsilon) dr + \int_{s_0}^{t} f_r(\epsilon) dW(s) + \int_{s_0}^{t} \int h_r(\epsilon, z) \tilde{N}(dsdz) ,$$

where $Y_{s_0,s_0}(\epsilon)$ *is* $\mathcal{F}_{s_0}$*-measurable for each* $0 < \epsilon < 1$, *and* $g_r(\epsilon), f_r(\epsilon), h_r(\epsilon, z)$ *are predictable processes for each* $(\epsilon, z) \in (0, 1) \times \mathbf{R}^m$. *Suppose that for each* $p \geq 2$,

$$\sup_{0<\epsilon<1} E[|Y_{s_0,s_0}(\epsilon)|^p] < +\infty$$

and that

$$\sup_{0<\epsilon<1} E[\sup_{s_0<r\le T} |Y_{s_0,r}(\epsilon)|^2] < +\infty\,.$$

Further, suppose that there exists a nonnegative predictable process $\eta_t(\epsilon)$, $s_0 < t \le T$, such that for each $t \in (s_0, T]$,

$$|g_t(\epsilon)| \le C\left(\sup_{s_0<r\le t} |Y_{r-}(\epsilon)| + \eta_t(\epsilon)\right), \tag{1.19}$$

$$|f_t(\epsilon)| \le C\left(\sup_{s_0<r\le t} |Y_{r-}(\epsilon)| + \eta_t(\epsilon)\right), \tag{1.20}$$

and

$$\int |h_t(\epsilon, z)|^p \mu(dz) \le C_p\left(\sup_{s_0<r\le t} |Y_{r-}(\epsilon)| + \eta_t(\epsilon)\right)^p \tag{1.21}$$

for any $p \ge 2$. Then, we have

$$\left\|\sup_{s_0<r\le T} |Y_r(\epsilon)|\right\|_p \le C_p\left(\|Y_{s_0,s_0}\|_p + E\left[\int_{s_0}^{T} |\eta_r(\epsilon)|^p dr\right]^{1/p}\right).$$

Proof of this lemma follows from the similar argument in Lemma 7.1 of [76] and we omit the details.

Lemma 4.7. *For each $p \ge 2$, there exists $C_p > 0$ such that*

$$E\left[\sup_{0<s\le T} |S_s(\epsilon) - S_s(\epsilon_1)|^p\right] \le C_p|\epsilon - \epsilon_1|^p \tag{1.22}$$

and

$$E\left[\sup_{0<s\le T}\left|\frac{S_s(\epsilon) - S_s(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{S_s(\epsilon) - S_s(\epsilon_2)}{\epsilon - \epsilon_2}\right|^p\right] \le C_p|\epsilon_1 - \epsilon_2|^p\,. \tag{1.23}$$

Proof. One can check that the process $S(\epsilon) - S(\epsilon_1)$ satisfies the condition in Lemma 4.6 with $Y_t = S_t(\epsilon) - S_t(\epsilon_1)$, $s_0 = 0$, $Y_{s_0,s_0} = 0$ and $\eta_t(\epsilon) = |\epsilon - \epsilon_1|(1 + S_{t-}(\epsilon)|)$. Hence, Lemma 4.6 shows (1.22).

We shall prove (1.23);

$$\begin{aligned} Y_t := {} & \frac{S_t(\epsilon) - S_t(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{S_t(\epsilon) - S_t(\epsilon_2)}{\epsilon - \epsilon_2} \\ = {} & \int_0^t g_r dr + \int_0^t f_r dW(r) + \int_0^t \int h_r(z)\tilde{N}(drdz)\,, \end{aligned}$$

where

$$g_r = \frac{\tilde{a}_0(S_r(\epsilon)) - \tilde{a}_0(S_r(\epsilon_1))}{\epsilon - \epsilon_1} - \frac{\tilde{a}_0(S_r(\epsilon)) - \tilde{a}_0(S_r(\epsilon_2))}{\epsilon - \epsilon_2},$$

$$f_r = \sum_{i,j=1}^{m} \frac{\epsilon a_i(S_r(\epsilon)) - \epsilon_1 a_i(S_r(\epsilon_1))}{\epsilon - \epsilon_1}\sigma_{ij} - \sum_{i,j=1}^{m} \frac{\epsilon a_i(S_r(\epsilon)) - \epsilon_2 a_i(S_r(\epsilon_2))}{\epsilon - \epsilon_2}\sigma_{ij},$$

$$h_r(z) = \sum_{i=1}^{m} \frac{\epsilon a_i(S_{r-}(\epsilon))z_i - \epsilon_1 a_i(S_{r-}(\epsilon_1))z_i}{\epsilon - \epsilon_1} - \sum_{i=1}^{m} \frac{\epsilon a_i(S_{r-}(\epsilon))z_i - \epsilon_2 a_i(S_{r-}(\epsilon_2))z_i}{\epsilon - \epsilon_2}.$$

Put

$$\eta_t = \left(1 + \left|\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right|\right)|S_{t-}(\epsilon_1) - S_{t-}(\epsilon_2)|$$

from (1.22), and thus we have

$$E\left[\sup_{0<s\le T} |\eta_s|^p\right] \le C|\epsilon_1 - \epsilon_2|^p.$$

Hence, all we have to do is to show that g_t, f_t, and $h_t(z)$ satisfy the conditions in Lemma 4.6. We shall only prove (1.20) since we can similarly show (1.19) and (1.21), that is,

$$\begin{aligned} |f_r| \le{}& \sum_{i,j=1}^{m} |\sigma_{ij}| \left|\epsilon_1 \frac{a_i(S_{t-}(\epsilon)) - a_i(S_{t-}(\epsilon_1))}{\epsilon - \epsilon_1} - \epsilon_2 \frac{a_i(S_{t-}(\epsilon)) - a_i(S_{t-}(\epsilon_2))}{\epsilon - \epsilon_2}\right| \\ \le{}& \sum_{i,j=1}^{m} |\sigma_{ij}| \Bigg|\Bigg\{\left(\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right)^T \\ & \times \int_0^1 [\nabla a_i(S_{t-}(\epsilon) + \theta(S_{t-}(\epsilon) - S_{t-}(\epsilon_1)))]\, d\theta \\ & + \left(\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right)^T \int_0^1 [-\nabla a_i(S_{t-}(\epsilon_2) + \theta(S_{t-}(\epsilon) - S_{t-}(\epsilon_2))) \\ & + \nabla a_i((S_{t-}(\epsilon_1) + \theta(S_{t-}(\epsilon) - S_{t-}(\epsilon_1)))] d\theta\Bigg\}\Bigg| \\ & + \sum_{i,j=1}^{m} |\sigma_{ij}||\epsilon_1 - \epsilon_2| \left|\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right| \|\nabla a_i\|_\infty. \end{aligned}$$

The mean value theorem shows that the first term is dominated by

$$\begin{aligned} & \|\nabla a_i\|_\infty \left|\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right| \\ & \quad + \|\nabla^2 a_i\|_\infty \left|\frac{S_{t-}(\epsilon) - S_{t-}(\epsilon_2)}{\epsilon - \epsilon_2}\right| |S_{t-}(\epsilon_1) - S_{t-}(\epsilon_2)| \\ & \le C(|Y_{t-}| + \eta_t). \end{aligned}$$

The second term on the right-hand side in the above is also dominated by η_t. Hence, f_t satisfies the condition in Lemma 4.6. □

By this lemma, we can use Lemma 1.1 in [67] to see $\epsilon \mapsto S_t(\epsilon)$ is continuously differentiable on $(0, 1]$. Inductively, we can see that $\epsilon \mapsto S_t(\epsilon)$ is infinitely times differentiable. Hence, $S_t^{(n)}(\epsilon) = \frac{\partial^n S_t(\epsilon)}{\partial \epsilon^n}$ is well-defined for $n = 1, 2, \dots$.

By using these preliminary results, we can show the following proposition.

Proposition 4.3. *For all $n \geq 1$ all $k, l \geq 0$ and $p \geq 2$, we have*

$$\sup_{0<\epsilon<1} |S_t^{(n)}(\epsilon)|_{k,l,p} < +\infty\,. \tag{1.24}$$

In particular, A_n in (1.18) *are in L^p, $n = 1, 2, 3, \dots$.*

Proof. We use induction.

Case $n = 0$.

The assertion holds, that is, first, since it holds that for each $k \geq 0$, $l \geq 0$ and $p \geq 1$,

$$E\left[\int_{\mathbf{T}} |D_{t'}^l S_t(\epsilon)|^p \, dt'\right] < +\infty\,,$$

$$E\left[\int_{A(1)^k} |\frac{\tilde{D}_{\mathbf{u}}^k S_t(\epsilon)}{\gamma(\mathbf{u})}|^p \hat{M}(d\mathbf{u})\right] < +\infty$$

for each $\epsilon > 0$, by Theorem 2.2.1 in [170] and by the proof of [181] Lemma 3.3, respectively. Secondly, it holds since the continuity property holds with respect to $0 < \epsilon < 1$ as given in Lemma 4.8.

Case $n \geq 1$.

Step 1 Assume the assertion (1.24) holds for $n \geq 0$ with $k, l = 0, 1, 2, \dots$ and with $p \geq 2$. We shall show

$$\sup_{0<\epsilon<1} |S_t^{(n+1)}(\epsilon)|_{k,l,p} < +\infty, \quad k, l = 0, 1, 2, \dots .$$

To this end, we put

$$\begin{aligned}
F_s^1(\epsilon) &= \frac{d^{n+1}}{d\epsilon^{n+1}} \tilde{a}_0(S_s(\epsilon)) - \nabla \tilde{a}_0(S_s(\epsilon)) S_s^{(n+1)}(\epsilon)\,, \\
F_s^2(\epsilon, z) &= \frac{d^{n+1}}{d\epsilon^{n+1}} \left(\epsilon(\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))\right) \\
&\quad - \epsilon\left(\sum_{i=1}^m z_i \nabla a_i(S_{s-}(\epsilon)) S_{s-}^{(n+1)}(\epsilon) - S_{s-}^{(n+1)}(\epsilon)\right)\,,
\end{aligned}$$

$$F_s^3(\epsilon) = \frac{d^{n+1}}{d\epsilon^{n+1}}\left(\epsilon \sum_{i,j=1}^{m} a_i(S_s(\epsilon))\right)\sigma_{ij} - \epsilon \sum_{i,j=1}^{m} \nabla a_i(S_s(\epsilon)).S_{s-}^{(n+1)}(\epsilon)\sigma_{ij}\ .$$

By decomposing the first term, $F_{s-}^2(\epsilon, z)$ can be written as a linear sum of terms

$$\sum_{i=1}^{m} z_i \nabla^l a_i(S_{s-}(\epsilon))(S_s^{(1)})^{\otimes l_1} \otimes \cdots \otimes (S_s^{(n)})^{\otimes l_n}, \quad l = 0, \ldots, n$$

and

$$\epsilon \sum_{i=1}^{m} z_i \nabla^{l'}(S_{s-}(\epsilon))(S_s^{(1)})^{\otimes l_1} \otimes \cdots \otimes (S_s^{(n)})^{\otimes l_n}, \quad l' = 1, \ldots, n+1 \tag{1.25}$$

for $l_1, \ldots, l_n \in \{0, \ldots, n\}$. Similarly, $F_s^3(\epsilon)$ can be written as a linear sum of terms

$$\sum_{i,j=1}^{m} \nabla^l a_i(S_s(\epsilon))(S_s^{(1)})^{\otimes l_1} \otimes \cdots \otimes (S_s^{(n)})^{\otimes l_n}\sigma_{ij}, \quad l = 0, \ldots, n$$

and

$$\epsilon \sum_{i,j=1}^{m} \nabla^{l'} a_i(S_s(\epsilon))(S_s^{(1)})^{\otimes l_1} \otimes \cdots \otimes (S_s^{(n)})^{\otimes l_n}\sigma_{ij}, \quad l' = 1, \ldots, n+1 \tag{1.26}$$

for $l_1, \ldots, l_n \in \{0, \ldots, n\}$.

Hence, by the assumption of the induction

$$\sup_{0<\epsilon<1}\sup_{0<s<T}\int_{A(1)^k} E\left[\left|\int_{\mathbf{T}^l} \frac{D_t^l \tilde{D}_u^k F_s^2(\epsilon, z)}{\gamma(u)} dt\right|^p\right]\hat{M}(du) < +\infty, \quad k, l = 0, 1, 2, \ldots, \tag{1.27}$$

$$\sup_{0<\epsilon<1}\sup_{0<s<T}\int_{A(1)^k} E\left[\int_{\mathbf{T}^l}\left|\frac{D_t^l \tilde{D}_u^k F_s^3(\epsilon)}{\gamma(u)}\right|^p dt\right]\hat{M}(du) < +\infty, \quad k, l = 0, 1, 2, \ldots.$$

Step 2 We write $u = (s_1, z_1) \in A(1)$. Since $\tilde{D}_u S_t^{(n+1)}(\epsilon) = 0$ for $s_1 > t$, it remains to show

$$\sup_{0<\epsilon<1}\sup_{0<t<T}\int_{A(1)\cap((0,t]\times\mathbf{R}^m\setminus\{0\})}\left\|\frac{\tilde{D}_{(s_1,z_1)}S_t^{(n+1)}(\epsilon)}{|z_1|}\right\|_p \hat{M}(du) < +\infty$$

in order to show the assertion for $k = 1, l = 0$.

To this end, let $\tilde{S}_{s_1,t}(\epsilon, z_1) = \tilde{D}_u S_t^{(n+1)}(\epsilon)$, $s_1 \le t$. By the change of variables formula, we have $\tilde{D}_u f(F) = (\tilde{D}_u F)^T(\int_0^1 f'(F + \theta\tilde{D}_u F)d\theta)$. Hence, we observe that $\tilde{S}_{s_1,t}(\epsilon)$

satisfies

$$\begin{aligned}\tilde{S}_{s_1,t}(\epsilon, z_1) &= \tilde{D}_u F^2_{s_1}(\epsilon, z_1) + \epsilon \sum_{i=1}^m z_i \nabla a_i(S_{s_1}(\epsilon)) \cdot S^{(n+1)}_{s_1}(\epsilon) + \tilde{Y}_{s_1,t} \\ &\quad + \int_{s_1}^t \int_0^1 (\tilde{D}_u S_s(\epsilon))^T \nabla^2 \tilde{a}_0 (S_{s-}(\epsilon) + \theta \tilde{D}_u S_s(\epsilon)) \tilde{S}_{s_1,s-}(\epsilon, z_1)) d\theta ds \\ &\quad + \int_{s_1}^t \int \int_0^1 \epsilon \sum_{i=1}^m (\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i (S_{s-}(\epsilon) + \theta \tilde{D}_u S_{s-}(\epsilon)) \tilde{S}_{s_1,s-}(\epsilon, z_1) z_i d\theta \tilde{N}(dsdz) \\ &\quad + \int_{s_1}^t \int_0^1 \epsilon \sum_{i,j=1}^m (\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i (S_{s-}(\epsilon) + \theta \tilde{D}_u S_{s-}(\epsilon)) \tilde{S}_{s_1,s-}(\epsilon, z_1) \sigma_{ij} d\theta dW^j(s) ,\end{aligned} \tag{1.28}$$

where

$$\begin{aligned}\tilde{Y}_{s_1,t} = \int_{s_1}^t &\Bigg\{ \tilde{D}_u (F^1_{s-}(\epsilon)) + \nabla \tilde{a}_0 (S_{s-}(\epsilon)) \tilde{S}_{s_1,s-}(\epsilon) \\ &\quad + \int_0^1 (\tilde{D}_u S_s(\epsilon))^T \nabla^2 \tilde{a}_0 (S_{s-}(\epsilon) + \theta \tilde{D}_u S_{s-}(\epsilon)) S^{(n+1)}_{s-}(\epsilon) d\theta \Bigg\} ds \\ &+ \int_{s_1}^t \int \Bigg\{ \tilde{D}_u (F^2_{s-}(\epsilon)) + \epsilon \sum_{i=1}^m z_i (\nabla a_i (S_{s-}(\epsilon) \tilde{S}_{s_1,s-}(\epsilon) \\ &\quad + \int_0^1 (\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i (S_{s-}(\epsilon) + \theta \tilde{D}_u S_{s-}(\epsilon)) S^{(n+1)}_{s-}(\epsilon) d\theta) \Bigg\} \tilde{N}(dsdz) \\ &+ \int_{s_1}^t \Bigg\{ \tilde{D}_u (F^3_{s-}(\epsilon)) + \epsilon \sum_{i,j=1}^m \sigma_{ij} (\nabla a_i (S_s(\epsilon)) \tilde{S}_{s_1,s-}(\epsilon) \\ &\quad + \int_0^1 (\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i (S_{s-}(\epsilon) + \theta \tilde{D}_u S_{s-}(\epsilon)) S^{(n+1)}_{s-}(\epsilon)) \Bigg\} dW(s) .\end{aligned}$$

We recall that $F^2_{s-}(\epsilon, z)$, $F^3_s(\epsilon)$ can be written as a sum of the terms of the form (1.25), (1.26), respectively. We write it below as $\tilde{D}^0_u G = G$ and $S^{(0)}_t(\epsilon) = S_t(\epsilon)$ for abbreviation. We apply Lemma 4.6 for $Y = \tilde{Y}$ with $s_0 = s$ and

$$\eta_t = \left(\left| \tilde{D}_u S^{(0)}_t(\epsilon) \right| + \sum_{k_j \in \{0,1\}, 1 \le k_0 + \ldots + k_n \le n} \left| \tilde{D}^{k_0}_u S^{(0)}_t(\epsilon) \cdots \tilde{D}^{k_n}_u S^{(n)}_t(\epsilon) \right| \right) \times |S^{(n+1)}_t(\epsilon)| . \tag{1.29}$$

Then, we have

$$\sup_{0 < s_1 < t \le T} \|\tilde{Y}_{s_1,t}\|_p \le CE \left[\int_{s_0}^T |\eta_r|^p dr \right]^{1/p} < +\infty . \tag{1.30}$$

Here, we have used the assumption of the induction at the last inequality.

Step 3 By the smoothness of $\tilde{a}_0, a_1, \ldots, a_m$ and by the assumption of the induction, we have

$$|(\tilde{D}_u S_s(\epsilon))^T \nabla^2 \tilde{a}_0(S_{s-}(\epsilon) + \theta \tilde{D}_u S_s(\epsilon))\tilde{D}_u S_{s-}^{(n+1)}(\epsilon)| \le C|\tilde{D}_u S_s(\epsilon)||\tilde{S}_{s_1,s-}(\epsilon, z_1)| ,$$

$$\int\int_0^1 |(\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i(S_{s-}(\epsilon) + \theta \tilde{D}_u S_s(\epsilon)) z_i \tilde{D}_u S_{s-}^{(n+1)}(\epsilon)|^p d\theta \mu(dz)$$
$$\le C|\tilde{D}_u S_s(\epsilon)|^p |\tilde{S}_{s_1,s-}(\epsilon, z_1)|^p, \quad p \ge 2 ,$$

and

$$|\sigma_{ij}(\tilde{D}_u S_s(\epsilon))^T \nabla^2 a_i(S_{s-}(\epsilon) + \theta \tilde{D}_u S_s(\epsilon))\tilde{D}_u S_{s-}^{(n+1)}(\epsilon)|$$
$$\le C|\sigma_{ij}||\tilde{D}_u S_s(\epsilon)||\tilde{S}_{s_1,s-}(\epsilon, z_1)| .$$

It follows from (1.27) that

$$\sup_{s_1 \le t} \int_{A(1)} \left\| \frac{F_{s_1-}^2(\epsilon, z_1)}{\gamma(u)} + \epsilon \sum_{i=1}^m \frac{z_i}{\gamma(u)} \nabla a_i(S_{s_1}(\epsilon)) S_{s_1}^{(n+1)}(\epsilon) \right\|_p \hat{M}(du) < +\infty . \tag{1.31}$$

Hence, we apply a similar argument to Lemma 4.6 with $s_0 = s_1$, $\eta_t = \tilde{Y}_{s_1,t} + |\tilde{D}_u S_t(\epsilon)|.|\tilde{S}_{s_1,t}(\epsilon, z_1)|$, and

$$Y_{s_0,s_0} = F_{s_1}^2(\epsilon, z_1) + \epsilon \sum_{i=1}^m z_i \nabla a_i(S_{s_1}(\epsilon)) S_{s_1}^{(n+1)}(\epsilon), \quad Y_{s_1,t} = \tilde{S}_{s_1,t}(\epsilon, z_1) ,$$

and we have

$$\sup_{0<\epsilon<1} E\left[\int_{A(1)} \sup_{s_1<t\le T} |\frac{\tilde{S}_{s_1,t}(z_1,\epsilon)}{\gamma(u)}|^p \hat{M}(du)\right]$$
$$\le C_p \left(\left\| \int_{A(1)} \left\{ \frac{F_{s_1}^2(\epsilon, z_1)}{\gamma(u)} + \epsilon \sum_{i=1}^m \frac{z_i}{\gamma(u)} \nabla a_i(S_{s_1}(\epsilon)) S_{s_1}^{(n+1)}(\epsilon) \right\} \hat{M}(du) \right\|_p^p \right.$$
$$\left. + \int_{s_1}^t E\left[\sup_{r<s} \int_{A(1)} |\frac{\eta_r}{\gamma(u)}|^p \hat{M}(du)\right] ds \right) .$$

This implies by (1.30), (1.31) that

$$\left|\sup_{s<t} \left|S_{s-}^{(n+1)}(\epsilon)\right|\right|_{1,0,p} \le C + K \int_{s_1}^t \left|\sup_{r<s} |S_{r-}(\epsilon)|\right|_{1,0,p_1} \left|\sup_{r<s} \left|S_{r-}^{(n+1)}(\epsilon)\right|\right|_{1,0,p_2} ds ,$$

where $p_1 > 1$, $p_2 > 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. By the assumption of the induction, $|\sup_{r<s} |S_{r-}(\epsilon)||_{1,0,p_1}$ is uniformly finite, where $p_2 > 1$ can be chosen arbitrarily close to p. Hence,

$$\left|\sup_{s<t}\left|S_{s-}^{(n+1)}(\epsilon)\right|\right|_{1,0,p} \leq C + K' \int_{s_1}^{t} \left|\sup_{r<s}\left|S_{r-}^{(n+1)}(\epsilon)\right|\right|_{1,0,p} ds\ .$$

This implies

$$\sup_{0<\epsilon<1}\sup_{0<t<T}|S_t^{(n+1)}(\epsilon)|_{1,0,p} < +\infty\ .$$

Step 4 Repeating this argument, we have

$$\sup_{0<\epsilon<1}\sup_{0<t<T}|S_t^{(n+1)}(\epsilon)|_{k,0,p} < +\infty\ , \quad k = 0, 1, 2, \ldots\ .$$

We also have the assertion

$$\sup_{0<\epsilon<1}\sup_{0<t<T}|S_t^{(n+1)}(\epsilon)|_{0,l,p} < +\infty\ , \quad l = 0, 1, 2, \ldots$$

due to Theorem 7.1 in [161].

Repeating this argument, we obtain the assertion. □

From this proposition, we see that $S_t^{(n)}(\epsilon)\ , \quad n = 1, 2, \ldots$ can be calculated as a solution to the SDE which derives from (1.2):

$$S_t^{(n)}(\epsilon) = \int_0^t \frac{d^n}{d\epsilon^n}\tilde{a}_0(S_{s-}(\epsilon))ds + \sum_{i,j=1}^{m}\int_0^t \frac{d^n}{d\epsilon^n}(\epsilon a_i(S_{s-}(\epsilon)))\sigma_{ij}dW^j(s)$$
$$+ \int_0^t\int \frac{d^n}{d\epsilon^n}\{\epsilon(\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))\}\tilde{N}(dsdz),\ S_0^{(n)}(\epsilon) = 0\ , \quad n = 1, 2, \ldots$$

(cf. [25] (5-25)).

In particular, $S_t^{(1)}(\epsilon)$ satisfies the following SDE:

$$S_t^{(1)}(\epsilon) = \int_0^t (\nabla\tilde{a}_0(S_{s-}(\epsilon))S_{s-}^{(1)}(\epsilon) + h_1(S_{s-}(\epsilon)))ds + \sum_{i,j=1}^{m}\int_0^t a_i(S_{s-}(\epsilon))\sigma_{ij}dW^j(s)$$
$$+ \epsilon\sum_{i,j=1}^{m}\int_0^t \nabla a_i(S_{s-}(\epsilon))S_{s-}^{(1)}(\epsilon)\sigma_{ij}dW^j(s)$$
$$+ \int_0^t\int(\phi_1^z(S_{s-}(\epsilon)) - S_{s-}(\epsilon))\tilde{N}(dsdz)$$
$$+ \epsilon\int_0^t\int\{\nabla\phi_1^z(S_{s-}(\epsilon) - I)S_{s-}^{(1)}(\epsilon)\}\tilde{N}(dsdz),\ S_0^{(1)}(\epsilon) = 0\ .$$

Here,

$$h_1(x) = (\tilde{a}_0 - a_0)|_{\epsilon=1}(x)\ .$$

Putting $\epsilon = 0$, $f_0 = S_T^{(1)}(0) = S_T^{(1)}(\epsilon)|_{\epsilon=0}$ is given by the following SDE:

$$S_T^{(1)}(0) = \int_{\mathbf{T}} (\nabla a_0(S_{s-}(0))S_{s-}^{(1)}(0) + h_1(S_{s-}(0)))ds + \sum_{i,j=1}^{m} \int_{\mathbf{T}} a_i(S_{s-}(0))\sigma_{ij}dW^j(s)$$
$$+ \int_{\mathbf{T}} \int (\phi_1^z(S_{s-}(0)) - S_{s-}(0))\tilde{N}(dsdz), \ S_0^{(1)}(0) = 0 \,. \quad (1.32)$$

We recall that $t \mapsto S_t(0)$ is deterministic, and compare the SDE above with (1.2) with $\epsilon = 1$. As we assume *(AS)* and *(AS2)*, we see that f_0 satisfies the condition *(ND)* by Propositions 3.10, 3.11.

Proposition 4.4 (cf. Hayashi [76]). *We have*

$$S_t(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \frac{1}{n!} S_t^{(n)}(0) \quad (1.33)$$

in $\mathbf{D}_\infty$.

Proof. Since

$$S_t(\epsilon) - \sum_{n=0}^{m} \epsilon^n \frac{1}{n!} S_t^{(n)}(0) = \frac{\epsilon^{m+1}}{m!} \int_0^1 (1-\theta)^m S_t^{(m+1)}(\epsilon\theta)d\theta \,,$$

we have by Proposition 4.2

$$\left| S_t(\epsilon) - \sum_{n=0}^{m} \epsilon^n \frac{1}{n!} S_t^{(n)}(0) \right|_{k,l,p} \leq C_m \sup_{0<\epsilon<1} \left| S_t^{(m+1)}(\epsilon) \right|_{k,l,p} \epsilon^{m+1} \,,$$

and hence the assertion. □

The flow property of $S_t(\epsilon)$ follows in Proposition 4.2 with $g(x,z) = \phi_1^z(x) - x$, and assumption *(AS)* corresponds to the condition *(NDB)*. Hence, condition *(UND)* holds for $S_t(\epsilon)$ (and for $F_t(\epsilon)$) for $\epsilon \in (0,1]$, due to Propositions 3.10, 3.11.

Hence, we can make the composition $\Phi \circ F_t(\epsilon)$ in $\mathbf{D}'_\infty$ for $\Phi \in \mathcal{S}'$, and by Theorem 4.1, we have its asymptotic expansion for $\Phi \in \mathcal{S}'$. For example, taking Φ to be $e_v(.)$, we will have an asymptotic expansion of the characteristic function, leading to the asymptotic expansion for the density function for $F_t(\epsilon)$.

We give some examples of the composition in Section 4.1.3.

Proof of Theorem 4.1.
(Step 1)
We first give a proof for the case $\Phi \in H_{-\infty}$. Let $\Phi \in H_{-2s_0}$. We prepare the following lemma.

Lemma 4.8 ([77] Lemma 5.6). *Assume $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in $\mathbf{D}_\infty$, and that $F(\epsilon)$ satisfies the uniform condition (ND). Put*

$$R_m(\xi, \epsilon) = e_\xi(F(\epsilon)) - \sum_{|\mathbf{n}| \le m} \frac{e_\xi(f_0)}{\mathbf{n}!}(i\xi, F(\epsilon) - f_0)^{\mathbf{n}} .$$

Then, for any $m \ge 1$ and $s > 0$, there exist natural numbers $k \ge 1$, $l \ge 0$ and a real number $r > 2$ such that

$$\limsup_{\epsilon \to 0} \int_{\{|\xi| \ge 1\}} \sup_{|G|_{k,l,r}=1} \frac{|E[GR_m(\cdot, \epsilon)]|}{\epsilon^{m+1}}(1 + |\xi|)^s d\xi < \infty .$$

Using this lemma, we shall prove Theorem 4.1. For the proof of this lemma, see [77] Section 6.

Fix $m \in \mathbf{N}$. By the definition of the composition (Def. 3.5.6), we have

$$\begin{aligned}
&|\Phi \circ F(\epsilon) - \sum_{|\mathbf{n}| \le m} \left(\frac{1}{\mathbf{n}!}\partial^{\mathbf{n}}\Phi\right) \circ f_0 (F(\epsilon) - f_0)^{\mathbf{n}}|'_{k,l,p} \\
&= \sup_{|G|_{k,l,p}=1} \left| \left\langle \sum_{|\mathbf{n}| \le m} (\frac{1}{\mathbf{n}!}\partial^{\mathbf{n}}\Phi) \circ f_0 (F(\epsilon) - f_0)^{\mathbf{n}}, G \right\rangle \right| \\
&= \sup_{|G|_{k,l,p}=1} \left| \sum_{|\mathbf{n}| \le m} \frac{1}{\mathbf{n}!} \left\langle \mathcal{F}\partial^{\mathbf{n}}\Phi, E[G(F(\epsilon) - f_0)^{\mathbf{n}} \cdot e_\xi(f_0)] \right\rangle \right| \\
&= \sup_{|G|_{k,l,p}=1} \left| \left\langle \mathcal{F}\Phi, E[GR_m(\xi, \epsilon)] \right\rangle \right| .
\end{aligned}$$

The right-hand side of the last equality is dominated by

$$\le |\Phi|_{H_{-2s_0}} \sup_{|G|_{k,l,p}=1} \left[\int (1 + |\xi|^2)^{s_0} |E[GR_m(\cdot, \epsilon)]|^2 d\xi \right]^{\frac{1}{2}} .$$

Thus, the lemma above proves the assertion of Theorem 4.1.

(Step 2)
Next, we prove the theorem in the general case $\Phi \in \mathcal{S}'$. As stated in Section 3.5.1, Φ can be decomposed as

$$\Phi = (1 + |x|^2)^k \Phi'$$

for some $k, s_0 \in \mathbf{N}$ and $\Phi' \in H_{-2s_0}$. We assume $k < s_0$ by choosing large s_0. Choose $s_0' \ge s_0 + k$, and regard $\Phi' \in H_{-2s_0'}$.

We repeat the calculation in Step 1 with Φ replaced by $(1 + |x|^2)^k \Phi'$. As

$$\|(1 + |x|^2)^k \Phi'\|_{-2s_0'} \le C|\Phi'|_{H_{-2(s_0'-k)}}$$

(see the proof of Proposition 3.5), the last term in Step 1 is replaced by

$$C|\Phi'|_{H_{-2s_0}} \cdot | \sup_{|G|_{k,l,p}=1} |E[GR_m(\cdot, \epsilon)]||_{H_{2s_0}} .$$

Hence, the assertion is proved.

Computing the coefficients of each power of ϵ, we obtain the explicit form of $\Phi_0, \Phi_1, \ldots$. □

4.1.2 Asymptotic expansion of the density

As an application, we state the asymptotic expansion of the density $p(\epsilon, y)$ of $F(\epsilon) = F_T(\epsilon)$.

Since $S_T(0)$ is deterministic, $F(\epsilon) = \frac{1}{\epsilon}(S_T(\epsilon) - S_T(0))$ has an asymptotic expansion

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$$

in $\mathbf{D}_\infty$ by Proposition 4.4.

We put $F(0) = S_T^{(1)}(\epsilon)|_{\epsilon=0} = f_0$. By Proposition 4.3, $F(0) \in \mathbf{D}_\infty$, and by Proposition 4.4,

$$|F(\epsilon) - F(0)|_{k,l,p} \to 0 \quad (\epsilon \to 0)$$

for all k, l, p. Furthermore, by the expression (1.32) in Section 4.1.1, $F(0)$ also satisfies conditions *(AS)*, *(AS2)*, and hence the condition *(ND)* by Propositions 3.10, 3.11. By Proposition 3.12, $F(0)$ has a rapidly decreasing smooth density $p(y)$.

Since $S_T(\epsilon)$ satisfies the conditions *(AS)*, *(AS2)*, $S_T(\epsilon)$ given above satisfies the condition *(UND2)* by Proposition 4.1. Combining this with the fact that $F(0)$ above satisfies the condition *(ND)*, we see $F(\epsilon) = \frac{1}{\epsilon}(S_T(\epsilon) - S_T(0))$ satisfies the condition *(UND2)*. By Proposition 3.12, $F(\epsilon)$ has a rapidly decreasing smooth density $p(\epsilon, y)$ for each ϵ sufficiently small.

We have the following proposition.

Proposition 4.5. *We assume assumption (R) and condition (UND2).*

(1) *For each ϵ and each n, there exist $C > 0$ and k, l, p such that*

$$|E[Ge_v(F(\epsilon))]| \leq C(1 + |v|^2)^{-\frac{1}{2}nq_0}|G|_{k,l,p}|F(\epsilon)|^n_{k,l,p} \sup_{|v|>1} |\{Q^{F(\epsilon)}(v)\}^{-1}|_{k,l,p} .$$

Here,

$$Q^{F(\epsilon)}(v) = (v', \Sigma^{F(\epsilon)} v') + \frac{1}{|v|^2 \varphi(|v|^{-\frac{1}{\beta}})} \int_{B_v} |e^{i(v, \tilde{D}_u F(\epsilon))} - 1|^2 \hat{N}(du) .$$

(2)

$$\limsup_{\epsilon \to 0} \sup_{|v|>1} |\{Q^{F(\epsilon)}(v)\}^{-1}|_{k,l,p} < +\infty$$

for each k, l, p.

The proof proceeds similarly to Proposition 3.9. Here, the second assertion follows from condition *(UND2)*. We omit the details.

We denote by φ^ϵ, φ the characteristic functions of $F(\epsilon)$, $F(0)$ respectively:

$$\varphi^\epsilon(v) = E[e_v(F(\epsilon))], \quad \varphi(v) = E[e_v(F(0))] .$$

As stated at the end of Section 3.6,

$$p(\epsilon, y) = \left(\frac{1}{2\pi}\right)^d \int e^{-i(v,y)} \varphi^\epsilon(v) dv$$

and

$$(1 + |y|^2)^j \nabla^\alpha p(\epsilon, y) = \left(\frac{1}{2\pi}\right)^d (-i)^{|\alpha|} \int e^{-i(v,y)} (1 - \Delta)^j v^\alpha \varphi^\epsilon(v) dv \ .$$

Here, α is a multi-index, $v^\alpha = v_1^{\alpha_1} \cdots v_d^{\alpha_d}$, and $j = 0, 1, 2, \ldots$. The same calculation proceeds to $\varphi(v)$, $p(y)$, that is,

$$(1 + |y|^2)^j \nabla^\alpha p(y) = (\frac{1}{2\pi})^d (-i)^{|\alpha|} \int e^{-i(v,y)} (1 - \Delta)^j v^\alpha \varphi(v) dv \ .$$

Proposition 4.6. *For each $j \in \mathbf{N}$, $N \in \mathbf{N}$ and each multi-index β, we have*

$$\lim_{\epsilon \to 0} \sup_{|y| \le N} (1 + |y|^2)^j |\nabla_y^\alpha p(\epsilon, y) - \nabla_y^\alpha p(y)| = 0 \ .$$

Proof. By Proposition 4.5, we have for n sufficiently large

$$|(1 - \Delta)^j v^\alpha \varphi^\epsilon(v)| \le C(\epsilon)(1 + |v|^2)^{-nq_0/2 + |\alpha|} \ ,$$

where

$$\sup_\epsilon C(\epsilon) < +\infty \ .$$

Due to the above expressions on φ^ϵ and φ,

$$\sup_y |(1 + |y|^2)^j \nabla^\alpha (p(\epsilon, y) - p(y))| \le \left(\frac{1}{2\pi}\right)^d \int |(1 - \Delta)^j v^\alpha (\varphi^\epsilon(v) - \varphi(v))| dv \ .$$

The left-hand side goes to 0 as $\epsilon \to 0$ due to the Lebesgue's convergence theorem by choosing n sufficiently large so that $(nq_0)/2 > d + 1 + |\alpha|$ in Proposition 3.8. □

Remark 4.5. More naively, the density $p(\epsilon, y)$ has an expression

$$p(\epsilon, y) = E[\delta_y(F(\epsilon))] \ ,$$

using $\delta_y(.) \in \mathcal{S}'$. See [82] Section V.10.

We can show further that $p(\epsilon, y)$ has an asymptotic expansion

$$p(\epsilon, y) \sim \sum_{j=0}^{\infty} \epsilon^j p_j(y)$$

as $\epsilon \to 0$.

Proposition 4.7. *For each $m = 1, 2, \dots$ and $N \in \mathbf{N}$, we have*

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon^{m+1}} \sup_{|y|\le N} \left| p(\epsilon, y) - \sum_{j=0}^{m} \epsilon^j p_j(y) \right| < +\infty \tag{1.34}$$

for some rapidly decreasing smooth $p_j(y)$, $j = 0, 1, 2, \dots$.

Proof. First, we show that the $p_j(y)$'s above are smooth. Indeed, take $\Phi(.) = e_\xi(.)$ and apply Theorem 4.1. Then, we have

$$e_\xi(F(\epsilon)) \sim \Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots \quad \text{in} \quad \mathbf{D}'_\infty ,$$

where

$$\Phi_0 = e_\xi(f_0) , \quad \Phi_1 = \sum_{i=1}^{d} f_1^i (\partial_{x_i} e_\xi) \circ f_0$$

$$\Phi_2 = \sum_{i=1}^{d} f_2^i (\partial_{x_i} e_\xi) \circ f_0 + \frac{1}{2} \sum_{i,j=1}^{d} f_1^i f_1^j (\partial^2_{x_i x_j} e_\xi) \circ f_0 ,$$

$$\Phi_3 = \sum_{i=1}^{d} f_3^i (\partial_{x_i} e_\xi) \circ f_0 + \frac{2}{2!} \sum_{i,j=1}^{d} f_1^i f_2^j (\partial^2_{x_i x_j} e_\xi) \circ f_0 + \frac{1}{3!} \sum_{i,j,k=1}^{d} f_1^i f_1^j f_1^k (\partial^3_{x_i x_j x_k} e_\xi) \circ f_0 , \tag{1.35}$$

$$\cdots$$

and $f_j \in \mathbf{D}_\infty$. Here,

$$\partial_x^s e_\xi(\cdot) = i^{|s|} \xi^s e_\xi(\cdot) \in \mathcal{S}' , \tag{1.36}$$

and f_0 satisfies *(ND)*, hence the composition $\partial_x^s e_\xi \circ f_0$ is justified as an element in $\mathbf{D}'_\infty$. Here, we used Lemma 4.3 for the well-definedness of each Φ_j.

By Proposition 3.18 (Section 3.7),

$$|E[e_\xi(f_0)]| \le C_n (1 + |\xi|^2)^{-\frac{q_0}{2} n}$$

for any $n \in \mathbf{N}$. More precisely, let m be any nonnegative integer. Due to (1.36), Lemma 4.7, and by Proposition 3.8 (Section 3.5), for each n, there exist $k = k(n)$, $l = l(n)$, $p = p(n)$ and $C = C_n > 0$ such that

$$\begin{aligned} &\left| E\left[f_1^{\otimes s_1} \otimes \cdots \otimes f_m^{\otimes s_m} \partial_x^s e_\xi(f_0) \right] \right| \\ &\quad \le C(1 + |\xi|^2)^{-\frac{q_0}{2} n} |f_0|_{k,l,p} |f_1^{\otimes s_1} \otimes \cdots \otimes f_m^{\otimes s_m}|^n_{k,l,p} \\ &\quad \le C'(1 + |\xi|^2)^{-\frac{q_0}{2} n} |f_1|^n_{k,l,sp} \cdots |f_m|^n_{k,l,sp} |f_0|_{k,l,p} , \end{aligned} \tag{1.37}$$

where $s_i \ge 0$, $s = s_1 + \dots + s_m \le m$ and we used Hölder's inequality. We apply this estimate to (1.35). Then, we observe that for each n, there exists $C_j = C_j(n) > 0$ such that

$$|E[\Phi_j]| \le C_j (1 + |\xi|^2)^{-\frac{q_0}{2} n} , \quad j = 0, 1, \dots, m . \tag{1.38}$$

This proves that $p_j(y)$ are smooth.

Next, we show the assertion (1.34). Since

$$p(\epsilon, y) = \left(\frac{1}{2\pi}\right)^d \int e^{-i(y,\xi)} E[e_\xi(F(\epsilon))] d\xi \ ,$$

$$p(\epsilon, y) - \sum_{j=0}^{m} \epsilon^j p_j(y) = \left(\frac{1}{2\pi}\right)^d \int e^{-i(y,\xi)} E\left[e_\xi(F(\epsilon)) - \sum_{j=0}^{m} \epsilon^j \Phi_j \right] d\xi \ , \quad (1.39)$$

we put

$$g_m(\xi) = e_\xi(F(\epsilon)) - \sum_{j=0}^{m} \epsilon^j \Phi_j \ .$$

We divide the region of integration in (1.39) into two parts.

(i) $\{|\xi| < 1\}$

By the expression in Theorem 4.1,

$$\begin{aligned} g_m(\xi) &= e_\xi(F(\epsilon)) - \sum_{|\mathbf{n}| \le m}^{m} \frac{1}{\mathbf{n}!} e_\xi(f_0)(i\xi, F(\epsilon) - f_0)^{\mathbf{n}} \\ &= \sum_{|\mathbf{n}| \ge m+1}^{\infty} \frac{1}{\mathbf{n}!} e_\xi(f_0)(i\xi, F(\epsilon) - f_0)^{\mathbf{n}} \ . \end{aligned}$$

Hence,

$$\begin{aligned} E[|g_m(\xi)|] &\le \epsilon^{m+1} \sum_{j=m+1}^{\infty} \frac{1}{j!} E\left[\left| (i\xi, \frac{1}{\epsilon}(F(\epsilon) - f_0)) \right|^j \right] \\ &\le \epsilon^{m+1} \sum_{j=m+1}^{\infty} \frac{1}{j!} |\xi|^j E\left[\left| \frac{1}{\epsilon}(F(\epsilon) - f_0) \right|^j \right] \ . \end{aligned}$$

Since $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in $\mathbf{D}_\infty$,

$$\limsup_{\epsilon \to 0} E\left[\left| \frac{1}{\epsilon}(F(\epsilon) - f_0) \right| \right] < K_1$$

for some $K_1 > 0$. Similarly, since $F(\epsilon)^j \sim \sum_{l=0}^{\infty} \epsilon^l f_l^{(j)}$ in $\mathbf{D}_\infty$, $j = 1, 2, \ldots$,

$$\limsup_{\epsilon \to 0} E\left[\left| \frac{1}{\epsilon}(F(\epsilon) - f_0) \right|^j \right] < K_2^j, \quad j = m+1, m+2, \ldots$$

for some $K_2 > 0$ for ϵ small. Here, we used Hölder's inequality.

Hence, the series is absolutely convergent and is bounded by $\epsilon^{m+1} h_m(\xi)$ for $0 < \epsilon < \epsilon_0$, where $h_m(\xi)$ is continuous and $\epsilon_0 > 0$ (depending on m).

Hence,

$$E[|g_m(\xi)|] \le \epsilon^{m+1} h_m(\xi), \quad 0 < \epsilon < \epsilon_0, \ |\xi| < 1 \ . \quad (1.40)$$

(ii) $\{|\xi| \geq 1\}$

We use Lemma 4.8 in Section 4.1.1. By Lemma 4.8, for any $s > 0$,

$$|E[g_m(\xi)]| \leq C_s \epsilon^{m+1}(1+|\xi|^2)^{-\frac{s}{2}}, \quad \epsilon \in (0, \epsilon_1), |\xi| \geq 1 \tag{1.41}$$

for some $\epsilon_1 > 0$ (depending on m).

Hence, by (1.40) and (1.41),

$$\left| p(\epsilon, y) - \sum_{j=0}^{m} \epsilon^j p_j(y) \right|$$
$$\leq \left(\frac{1}{2\pi}\right)^d \left\{ \int_{\{|\xi|<1\}} \epsilon^{m+1} h_m(\xi) d\xi + \int_{\{|\xi|\geq 1\}} \epsilon^{m+1} C'(1+|\xi|^2)^{-\frac{s}{2}} d\xi \right\} \tag{1.42}$$

for $0 < \epsilon < \epsilon_0 \wedge \epsilon_1$, for some $C' = C'(\epsilon_1, s, m)$ uniformly in y. Since s is arbitrary, taking $\sup_{|y|\leq N}$ on both sides, we have the assertion.

End of proof of Proposition 4.7. □

4.1.3 Examples of asymptotic expansions

In this section, we give some concrete examples of the asymptotic expansions $\Phi \circ F(\epsilon)$ for several $\Phi \in \mathcal{S}'$ verified in Section 4.1.2.

Assume that a functional $F(\epsilon)$ of $S_T(\epsilon)$ has an asymptotic expansion

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j g_j \quad \text{in} \quad \mathbf{D}_\infty \ .$$

Then, by Theorem 4.1, we have the asymptotic expansion

$$\Phi(F(\epsilon)) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n}{dx^n} \Phi \right) \circ g_0 \cdot (F(\epsilon) - g_0)^n \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \ .$$

Here, the f_j's are calculated as in Theorem 4.1.

(1) Let $d = 1$. $\Phi = \delta_{\{0\}} \in \mathcal{S}'$.

We recall

$$\langle \Phi \circ g, G \rangle = \langle \hat{\Phi}, GE[e^{i(v,g)}] \rangle_v$$

by the definition of the composition. In particular,

$$E[\Phi(g)] = \langle \Phi(g), 1 \rangle = \langle \hat{\Phi}, GE[e^{i(v,g)}] \rangle_v \ .$$

Here, $\hat{\Phi}$ denotes the Fourier transform of Φ:

$$\hat{\Phi}(v) = \mathcal{F}\Phi(v) = \frac{1}{2\pi} \int e^{-i(v,x)} \Phi(x) dx \ .$$

We have

$$\mathcal{F}\delta_{\{0\}} = \frac{1}{2\pi} .$$

Hence,

$$\langle \delta_{\{0\}} \circ g, G \rangle = \left\langle \mathcal{F}\delta_{\{0\}}, E[Ge^{i(v,g)}] \right\rangle = \left\langle \frac{1}{2\pi}, E[Ge^{i(v,g)}] \right\rangle ,$$

and

$$\begin{aligned} E[\delta_{\{0\}} \circ g] &= \langle \delta_{\{0\}} \circ g, 1 \rangle \\ &= \frac{1}{2\pi} \langle 1, E[e^{i(v,g)}] \rangle = \frac{1}{2\pi} \int E[e^{i(v,g)}] dv . \end{aligned}$$

Similarly, denoting $\delta = \delta_{\{0\}}$, we have

$$\mathcal{F}\delta' = iv\mathcal{F}\delta = iv\frac{1}{2\pi} , \quad \mathcal{F}\delta'' = -\frac{1}{2\pi}v^2, \ \ldots .$$

Then,

$$\begin{aligned} f_0 &= \Phi \circ g_0 = \delta \circ g_0 , \\ f_1 &= g_1 \Phi' \circ g_0 = g_1(\delta' \circ g_0) , \\ f_2 &= g_2 \Phi' \circ g_0 + \frac{1}{2!} g_1^2 \Phi'' \circ g_0 = g_2 \delta' \circ g_0 + \frac{1}{2!} g_1^2 \delta'' \circ g_0 , \\ & \ldots , \end{aligned}$$

and hence

$$\begin{aligned} E[\delta \circ F(\epsilon)] &= \langle \delta \circ F(\epsilon), 1 \rangle \qquad (1.43) \\ &\sim \langle \delta \circ g_0, 1 \rangle + \epsilon \langle g_1 \delta' \circ f_0, 1 \rangle + \epsilon^2 \{ \langle \tfrac{1}{2} g_1^2 \delta'' \circ g_0, 1 \rangle + \langle g_2 \delta' \circ g_0, 1 \rangle \} + \cdots . \end{aligned}$$

Here,

$$\begin{aligned} \langle \delta \circ g_0, 1 \rangle &= \langle \hat{\delta}, E[e^{i(v,g_0)}] \rangle = \frac{1}{2\pi} \int E[e^{i(v,g_0)}] dv , \\ \langle g_1 \delta' \circ g_0, 1 \rangle &= \langle \widehat{\delta'}, E[g_1 e^{i(v,g_0)}] \rangle_v = \left\langle \frac{i}{2\pi} v, E[g_1 e^{i(v,g_0)}] \right\rangle_v , \\ \left\langle \frac{1}{2} g_1^2 \delta'' \circ g_0, 1 \right\rangle &= \langle \widehat{\delta''}, E[g_1^2 e^{i(v,g_0)}] \rangle_v = -\left\langle \frac{1}{2\pi} v^2, E\left[\frac{1}{2} g_1^2 e^{i(v,g_0)} \right] \right\rangle_v , \end{aligned}$$

and

$$\langle g_2 \delta' \circ g_0, 1 \rangle = \langle \widehat{\delta'}, E[g_2 e^{i(v,g_0)}] \rangle_v = \left\langle \frac{i}{2\pi} v, E[g_2 e^{i(v,g_0)}] \right\rangle_v ,$$

$$\ldots .$$

In summary, we have the asymptotic expansion for $\delta(F(\epsilon))$ as above.

If we take $F(\epsilon) - x$ instead of $F(\epsilon)$, $E[\delta_{\{0\}} \circ (F(\epsilon) - x)] = E[\delta_{\{x\}} \circ F(\epsilon)]$ denotes the density $p(x, \epsilon)$ of $F(\epsilon)$. The above formula gives an asymptotic expansion of $p(x, \epsilon)$ with respect to $\epsilon > 0$.

(2) Let $d = 1$, $\Phi(x) = Y(x) = 1_{\{x>0\}}$.
We have

$$\mathcal{F}Y(v) = -\frac{i}{2\pi}\frac{1}{v - i0} = -\frac{i}{2\pi}\left(\text{p.v.}\frac{1}{v} + i\pi\delta\right) .$$

Here, p.v. denotes the *Cauchy's principal value* defined by

$$\left\langle \text{p.v.}\frac{1}{v}, \varphi \right\rangle = \int_{|v|\le 1} \frac{\varphi(v) - \varphi(0)}{v} dv + \int_{|v|\ge 1} \frac{\varphi(v)}{v} dv .$$

Similarly to (1), we have

$$\begin{aligned} \mathcal{F}Y' &= iv\mathcal{F}Y = v\frac{1}{2\pi}\left(\text{p.v.}\frac{1}{v} + i\pi\delta\right) = \frac{1}{2\pi} , \\ \mathcal{F}Y'' &= -v^2\mathcal{F}Y = v^2\frac{i}{2\pi}\left(\text{p.v.}\frac{1}{v} + i\pi\delta\right) = v\frac{i}{2\pi} , \\ &\cdots . \end{aligned}$$

Hence, we can calculate, as in (1),

$$\begin{aligned} \langle Y \circ g_0, 1\rangle &= \langle \hat{Y}, E[e^{i(v,g_0)}]\rangle = -\frac{i}{2\pi}\left\langle \left(\text{p.v.}\frac{1}{v} + i\pi\delta\right), E[e^{i(v,g_0)}] \right\rangle_v , \\ \langle g_1 Y' \circ g_0, 1\rangle &= \langle \widehat{Y'}, E[g_1 e^{i(v,g_0)}]\rangle_v = \left\langle \frac{1}{2\pi}v, E[g_1 e^{i(v,g_0)}] \right\rangle_v , \\ \left\langle \frac{1}{2}g_1^2 Y'' \circ g_0, 1\right\rangle &= \left\langle \widehat{Y''}, E[g_1^2 e^{i(v,g_0)}] \right\rangle_v = \left\langle \frac{i}{2\pi}v, E[\frac{1}{2}g_1^2 e^{i(v,g_0)}] \right\rangle_v , \end{aligned}$$

and

$$\langle g_2 Y' \circ g_0, 1\rangle = \langle \widehat{Y'}, E[g_2 e^{i(v,g_0)}]\rangle_v = \left\langle \frac{1}{2\pi}, E[g_2 e^{i(v,g_0)}] \right\rangle_v .$$

These make an asymptotic expansion

$$\begin{aligned} &E[Y \circ F(\epsilon)] = \langle Y \circ F(\epsilon), 1\rangle \\ &\quad \sim \langle Y \circ g_0, 1\rangle + \epsilon\langle g_1 Y' \circ f_0, 1\rangle + \epsilon^2\left\{\left\langle \frac{1}{2}g_2 Y'' \circ g_0, 1\right\rangle + \langle g_1^2 Y' \circ g_0, 1\rangle\right\} + \cdots . \end{aligned} \tag{1.44}$$

(3) Let $d = 1$ and $F_t(\epsilon) = \frac{S_t(\epsilon) - S_t(0)}{\epsilon}$.
We consider the local time of $F_t(\epsilon)$ at x via Donsker's delta function:

$$L_T(x) = \int_{\mathbf{T}} \delta_{\{0\}}(F_t(\epsilon) - x)dt .$$

We remark that

$$\delta_{\{x\}} \circ F_t(\epsilon) = \delta_{\{0\}}(\cdot - x) \circ F_t(\epsilon) = \delta_{\{0\}}(F_t(\epsilon) - x) .$$

Then, by (1),

$$\begin{aligned}\delta_{\{x\}}(F_t(\epsilon)) \sim \delta \circ (g_0(t) - x) + \epsilon\delta' \circ (g_0(t) - x)g_1(t) \\ + \epsilon^2 \left\{\frac{1}{2}g_1^2(t)\delta'' \circ (g_0(t) - x) + g_2(t)\delta' \circ (g_0(t) - x)\right\} + \cdots \\ = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots .\end{aligned} \tag{1.45}$$

Hence, we have an asymptotic expansion

$$\begin{aligned}L_T(x, \epsilon) &= \int_{\mathbf{T}} \delta_{\{0\}}(F_t(\epsilon) - x)dt \\ &\sim \int_{\mathbf{T}} dt\{f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots\} \\ &= \int_{\mathbf{T}} f_0(t)dt + \epsilon \int_{\mathbf{T}} f_1(t)dt + \epsilon^2 \int_{\mathbf{T}} f_2(t)dt + \cdots\end{aligned}$$

using the expressions in (1).

(4) Let $d = 1$. Let $\Phi_1(x) = (x - K)_+ = (x - K)Y(x - K) \in \mathcal{S}_{-1}$. Here, $K > 0$.

We write $\Phi_1(x) = \Psi_1(x - K)$, where $\Psi_1(y) = y \cdot Y(y)$. Here, we have $\mathcal{F}\Phi_1(v) = \mathcal{F}[\Psi_1(\cdot - K)](v) = e^{-iKv}\mathcal{F}\Psi_1(v)$, where $\mathcal{F}g = \hat{g}$.

Hence,

$$\begin{aligned}\mathcal{F}\Psi_1(v) &= \mathcal{F}[xY(x)](v) \\ &= i\mathcal{F}[(-ix)Y(x)] = i\frac{d}{dv}\mathcal{F}[Y(x)] \\ &= i\frac{d}{dv}\frac{-i}{2\pi}\left(\text{p.v.}\frac{1}{v} + i\pi\delta\right) = \frac{1}{2\pi}\left(\frac{d}{dv}\left(\text{p.v.}\frac{1}{v}\right) + i\pi\delta'\right) .\end{aligned}$$

This makes

$$\mathcal{F}[\Phi_1](v) = e^{-iKv}\frac{1}{2\pi}(\frac{d}{dv}(\text{p.v.}\frac{1}{v}) + i\pi\delta') .$$

Similarly, in the case that $\Phi_2(x) = (x - K)_+^2 = \Psi_2(x - K) \in \mathcal{S}_{-2}$, where $\Psi_2(y) = y^2 \cdot Y(y)$, $K > 0$, we have the following calculation:

$$\begin{aligned}\mathcal{F}[\Psi_2](v) &= \mathcal{F}[x^2Y(x)](v) = -\mathcal{F}[(-i)^2Y(x)](v) \\ &= -\left(\frac{d^2}{dv^2}\right)\mathcal{F}[Y](v) = \frac{i}{2\pi}\left(\frac{d^2}{dv^2}\left(\text{p.v.}\frac{1}{v}\right) + i\pi\delta''\right) .\end{aligned}$$

Here, we have the derivatives of $\text{p.v.}\frac{1}{v} = (v_+^{-1} - v_-^{-1})$. Here, x_+^λ, x_-^λ are defined by

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda\varphi(x)dx ,$$

$$(x_-^\lambda, \varphi) = (x_+^\lambda, \varphi(-\cdot)) ,$$

respectively. These are defined for $\lambda \in \mathbf{C} \setminus \{-1, -2, \ldots\}$, however, $x_+^\lambda - x_-^\lambda$, $x_+^\lambda + x_-^\lambda$ are well-defined at $\lambda = -1, -2$, respectively. Furthermore, we have

$$\frac{d}{dx} x_-^{-n} = n x_-^{-n-1} - \frac{\delta^{(n)}}{n!}(x) ,$$
$$\frac{d}{dx} x_+^{-n} = -n x_+^{-n-1} + \frac{(-1)^n}{n!}\delta^{(n)}(x) .$$

Hence,

$$\begin{aligned}\frac{d}{dv}\text{p.v.}\frac{1}{v} &= \frac{d}{dv} v_+^{-1} - \frac{d}{dv} v_-^{-1} \\ &= -v_+^{-2} - \delta' - (v_-^{-2} - \delta') = -(v_+^{-2} + v_-^{-2}) .\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d^2}{dv^2}\text{p.v.}\frac{1}{v} &= -\frac{d}{dv}(v_+^{-2} + v_-^{-2}) \\ &= -\left\{-2v_+^{-3} + \frac{1}{2}\delta'' + 2v_+^{-3} - \frac{1}{2}\delta''\right\} = 2(v_+^{-3} - v_-^{-3}) .\end{aligned}$$

In summary,

$$\mathcal{F}\Phi_1 = -e^{-iKv}\frac{1}{2\pi}(v_+^{-2} + v_-^{-2} - i\pi\delta')$$
$$\mathcal{F}\Phi_2 = e^{-iKv}\frac{i}{2\pi}(2(v_+^{-3} - v_-^{-3}) + i\pi\delta'') .$$

For the derivatives, we have

$$\mathcal{F}[\Phi_i'] = iv\mathcal{F}[\Phi_i], \quad \mathcal{F}[\Phi_i''] = (iv)^2\mathcal{F}[\Phi_i] = -v^2\mathcal{F}[\Phi_i], \ \ldots .$$

We remark also

$$v \cdot v_+^{-1} = Y(v), \qquad v \cdot v_-^{-1} = Y(-v) ,$$
$$v^2 \cdot v_+^{-2} = Y(v), \qquad v^2 \cdot v_-^{-2} = Y(-v) ,$$

and

$$v \cdot v_+^{-2} = v_+^{-1}, \quad v \cdot v_-^{-2} = v_-^{-1}, \ \ldots .$$

Calculating as above, for $i = 1, 2$, we have

$$f_0 = \Phi_i \circ g_0 ,$$
$$f_1 = g_1\Phi_i' \circ g_0 ,$$
$$f_2 = g_2\Phi_i' \circ g_0 + \frac{1}{2!}g_1^2\Phi_i'' \circ g_0 ,$$
$$\ldots ,$$

and hence

$$E[\Phi_i \circ F(\epsilon)] = \langle \Phi_i \circ F(\epsilon), 1\rangle \tag{1.46}$$

$$\sim \langle \Phi_i \circ g_0, 1\rangle + \epsilon\langle g_1\Phi_i' \circ f_0, 1\rangle + \epsilon^2\{\langle \tfrac{1}{2}g_1^2\Phi_i'' \circ g_0, 1\rangle + \langle g_2\Phi_i' \circ g_0, 1\rangle\} + \cdots\,, \quad i = 1, 2\,.$$

Here,

$$\begin{aligned}
&\langle \Phi_1 \circ g_0, 1\rangle \\
&\quad = \left\langle \hat{\Phi}_1, E[e^{i(v,g_0)}]\right\rangle_v = -\left\langle e^{-iKv}\frac{1}{2\pi}(v_+^{-2} + v_-^{-2} - i\pi\delta'), E[e^{i(v,g_0)}]\right\rangle_v\,, \\
&\left\langle g_1\Phi_1' \circ g_0, 1\right\rangle \\
&\quad = \left\langle \hat{\Phi_1'}, E[g_1 e^{i(v,g_0)}]\right\rangle_v = -i\left\langle e^{-iKv}\frac{1}{2\pi}(v_+^{-1} + v_-^{-1} - i\pi v\delta'), E[g_1 e^{i(v,g_0)}]\right\rangle_v\,, \\
&\left\langle \frac{1}{2}g_1^2\Phi_1'' \circ g_0, 1\right\rangle \\
&\quad = \frac{1}{2}\left\langle e^{-iKv}\frac{1}{2\pi}(Y(v) + Y(-v) - i\pi v^2\delta'), E[g_1^2 e^{i(v,g_0)}]\right\rangle_v\,,
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle g_2\Phi_1' \circ g_0, 1\right\rangle \\
&\quad = \left\langle \hat{\Phi_1'}, E[g_2 e^{i(v,g_0)}]\right\rangle_v = -i\left\langle e^{-iKv}\frac{1}{2\pi}(v_+^{-1} + v_-^{-1} - i\pi v\delta'), E[g_2 e^{i(v,g_0)}]\right\rangle_v\,, \\
&\ldots\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\langle \Phi_2 \circ g_0, 1\rangle \\
&\quad = \left\langle \hat{\Phi}_2, E[e^{i(v,g_0)}]\right\rangle_v = \left\langle e^{-iKv}\frac{i}{2\pi}(2(v_+^{-3} - v_-^{-3}) + i\pi\delta''), E[e^{i(v,g_0)}]\right\rangle_v\,, \\
&\left\langle g_1\Phi_2' \circ g_0, 1\right\rangle \\
&\quad = \left\langle \widehat{\Phi_2'}, E[g_1 e^{i(v,g_0)}]\right\rangle_v \\
&\quad = -\left\langle e^{-iKv}\frac{1}{2\pi}(2(v_+^{-2} - v_-^{-2}) + i\pi v\delta''), E[g_1 e^{i(v,g_0)}]\right\rangle_v\,, \\
&\left\langle \frac{1}{2}g_1^2\Phi_2'' \circ g_0, 1\right\rangle \\
&\quad = -\frac{i}{4\pi}\left\langle e^{-iKv}(2(v_+^{-1} - v_-^{-1}) + i\pi v^2\delta''), E[g_1^2 e^{i(v,g_0)}]\right\rangle_v\,,
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle g_2 \Phi_1' \circ g_0, 1 \right\rangle \\
&\quad = \left\langle \widehat{\Phi_2'}, E[g_2 e^{i(v,g_0)}] \right\rangle_v \\
&\quad = -\left\langle e^{-iKv} \frac{1}{2\pi}(2(v_+^{-2} - v_-^{-2}) + i\pi v \delta''), E[g_2 e^{i(v,g_0)}] \right\rangle_v , \\
&\ldots .
\end{aligned}$$

4.2 Optimal consumption problem

In this section, we will consider an optimal consumption problem associated to the jump-diffusion process as an application of the analysis of jump-diffusion processes.

The process $z(t)$ below models some kind of asset price (e.g. stock price, price of precious metals, price of immobile property,...) whose trajectory may not be continuous. The owner of this asset would like to consume it based on the dividends optimally before it ruins.

The objectives here are to find the expected future value of this asset at $t = 0$: $z(0) = z$, and to seek the optimal consumption policy. In this section, we use classical stochastic analysis as a tool instead of the stochastic analysis of variations.

We construct a Hamilton–Jacobi–Bellman (HJB) equation with a one-sided boundary condition and solve it by the penalty method.

4.2.1 Setting of the optimal consumption

We assume an asset process $z(t)$ evolves according to the one-dimensional SDE of jump-diffusion type

$$\begin{aligned}
dz(t) &= \{f(z(t)) - \tilde{\mu} z(t) - c(t)\}dt - \sigma z(t) dW(t) \\
&\quad + z(t-) \int_{|\zeta|<1} (e^{\zeta} - 1)\tilde{N}(dt d\zeta) + z(t-) \int_{|\zeta|\ge 1} (e^{\zeta} - 1) N(dt d\zeta), \quad z(0) = z \ge 0 \qquad (2.1)
\end{aligned}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$. Here, $\{W(t)\}$ denotes the standard Brownian motion (Wiener process), and $N(dt d\zeta)$ denotes a Poisson random measure on $[0, +\infty) \times \mathbf{R}$ with the mean measure $dt\mu(d\zeta)$, and $\tilde{N}(dt d\zeta) = N(dt d\zeta) - dt\mu(d\zeta)$ denotes the compensated random measure. That is, $\mu(d\zeta)$ is a measure satisfying $\int_{\mathbf{R}\setminus\{0\}} (1 \wedge |z|^2)$ $\mu(d\zeta) < +\infty$. Especially, it can be a singular measure such as a sum of point masses.

The trajectory may hit $(-\infty, 0]$ due to the drift part $\{f(z(t)) - \tilde{\mu} z(t) - c(t)\}$. Thus, we set the stopping time

$$\tau_z = \inf\{t \ge 0; z(t) \le 0\} .$$

The boundary condition states that the process $z(t)$ must stay at $\{0\}$ after τ_z.

We assume $\tilde{\mu} \in \mathbf{R}$, $\sigma \geq 0$, and that the growth function $f(z)$ satisfies

$$f(z) : \textit{Lipschitz continuous}, \ \textit{increasing and concave}, \ f(0) = 0 \, . \tag{2.2}$$

Our motivation is to maximise the expected utility

$$J(\mathbf{c}) = E\left[\int_0^{\tau_z} e^{-\beta t} U(c(t))dt\right] \tag{2.3}$$

over the class $\mathcal{C}$, with the above-mentioned condition that (2.1) has a nonnegative solution $z(t)$ a.s. for $z(0) = z \geq 0$. Here, the set $\mathcal{C}$ denotes the set of nonnegative consumption policies $\mathbf{c} = \{c(t)\}$ such that it is a nonnegative *adapted* càdlàg process satisfying

$$\int_0^t c(s)ds < \infty, \quad \forall t \geq 0, \ \text{a.s.} \tag{2.4}$$

The (potential) function $U(.)$ is regarded as a utility function following the so-called Gossen's (Inada's) law *(B.3)* below which relates to the consumption rate $c(t)$, so that the hasty investor would like to maximise his or her utility, and β denotes the dumping rate of the utility as time goes by. The optimal value of $J(\mathbf{c})$ as a function of $z = z(0)$ is called the *value function* and is denoted by $v(z)$:

$$v(z) = \sup_{\mathbf{c} \in \mathcal{C}} J(\mathbf{c}) \, .$$

Intuitively, we would like to maximise the expected utility (2.3) as long as the asset process (2.1) remains positive a.s.

We state the setting more precisely. We assume

$$\text{supp } \mu \subset [0, +\infty) \tag{$B.1$}$$

$$\int_{|\zeta| \geq 1} (e^{\zeta} - 1)\mu(d\zeta) < +\infty \, , \tag{$B.2$}$$

and put

$$\tilde{r} = \int_{\mathbf{R} \setminus \{0\}} (e^{\zeta} - 1 - \zeta \cdot 1_{|\zeta|<1})\mu(d\zeta) \, ,$$

which plays the role of intrinsic drift.

The assumption *(B.1)* means that, starting from $z > 0$, the process $z(t)$ diffuses around $\{z\}$ (when $\sigma > 0$), increases (goes rightward) by jumps, or moves by drift. It dies when it reaches $\{0\}$ at $t = \tau_z$ due to the drift. Otherwise, it might remain in $(0, \infty)$. Technically, we impose *(B.1)* since we admit the degeneracy of the second order term ($\sigma = 0$). Otherwise, this condition will be avoidable. See [71].

An intuitive interpretation for *(B.1)* is that the asset is based on a saving account which bears dividends (sporadic incomes) whose return per unit is described by $\frac{\Delta z(t)}{z(t)}$. An example in economics associated with this problem is described in the Appendix (ii).

Under *(B.2)*, $z(t)$ can be written by

$$dz(t) = \{f(z(t)) - \mu z(t) - c(t)\}dt - \sigma z(t)dW(t) + z(t-)\int (e^{\zeta} - 1)\tilde{N}(dtd\zeta) \,,$$
$$z(0) = z \geq 0 \,, \quad (2.5)$$

where $\mu = \tilde{\mu} + \int_{|\zeta|<1}(e^{\zeta} - 1 - \zeta)\mu(d\zeta) - \tilde{r}$. Note that $e^{\zeta} - 1 \geq 0$ and that by writing $e^{\zeta}z = z + (e^{\zeta} - 1)z$, the third term on the right-hand side of (2.5) is of the Lévy–Khintchine form associated with the jump $(e^{\zeta} - 1)z$.

To find the value function $v(z)$, we consider the one-dimensional Hamilton–Jacobi–Bellman (HJB for short) equation of integro-differential type on $[0, +\infty)$:

$$Lv(z) + \tilde{U}(v'(z)) = 0, \quad z > 0,\ v(0) = 0 \quad (2.6)$$

(cf. (2.8) below).

Here, L is an integro-differential operator given by

$$Lv(z) = -\beta v(z) + \frac{1}{2}\sigma^2 z^2 v'' + (f(z) - \mu z)v'$$
$$+ \int \{v(z + \gamma(z,\zeta)) - v(z) - v'(z)\cdot\gamma(z,\zeta)\}\mu(d\zeta) \,, \quad (2.7)$$

where $\beta > 0$, $\sigma \geq 0$, $\gamma(z,\zeta) = z(e^{\zeta} - 1)$, $\mu \in \mathbf{R}$, and $f(z)$ is a function on $[0, +\infty)$ satisfying (2.2). We write

$$Lv = -\beta v + L_0 v$$

in the sequel.

The symbol $\tilde{U}(x)$ is the Legendre transform of (the negative potential) $-U(-x)$, i.e.

$$\tilde{U}(x) = \max_{c>0}\{U(c) - cx\} \,.$$

Further, $U(c)$ is assumed to have the following properties:

$$U \in C([0,\infty)) \cap C^2((0,\infty)), \quad U(c) : \text{strictly concave and increasing on} [0,\infty) \,,$$
$$U'(c) : \text{strictly decreasing}, \quad U'(\infty) = U(0+) = 0, \quad U'(0+) = U(\infty) = \infty \,. \quad (B.3)$$

However, a difficulty arises in the case (2.7) where there exists a degeneracy in the HJB equation. Namely, the second order term will degenerate at $z = 0$, or even the coefficient σ may be identically zero. To avoid this difficulty and obtain the value function, we use an analytic method. Namely, we first construct $v(z)$ as a weak solution, and then show the uniqueness and the existence of it as the solution.

Since L satisfies the positive maximum principle, L can be viewed as a pseudodifferential operator with the symbol $a(x, \xi)$ given by

$$a(x,\xi) = a_1(x,\xi) + a_2(x,\xi) ,$$

where

$$a_1(x,\xi) = -\beta - \frac{1}{2}\sigma^2 x^2 \xi^2 + i(f(x) - \mu x)\xi$$
$$a_2(x,\xi) = \int \{e^{i\xi\gamma(x,\zeta)} - 1 - i\xi \cdot \gamma(x,\zeta)\}\mu(d\zeta) .$$

The symbol of L_0 is given by $(a_1(x,\xi) + \beta) + a_2(x,\xi)$.

By *(B.1)*, the process $z(t)$ has no negative jumps. In this setting, it is known that the "trace" $v(0+) = \lim_{z\to 0+} v(z)$ exists finite for the original nonlocal boundary value problem on $\mathbf{R}$

$$Lv(z) + \tilde{U}(v'(z)) = 0,\ z > 0, \quad v(z) = 0,\ z \le 0 . \tag{2.8}$$

By this reason, it suffices to consider the equation (2.8) replacing v with $v.1_{[0,+\infty)}$, interpreting $v(z) = 0$, $z \le 0$ with $v(0) = v(0+) = 0$, which is (2.6). The above property for L of being able to take the trace safely at the boundary is called the *transmission property*. See [83, 206].

Finally, we remark that we can rewrite (2.6) as

$$\left(\beta + \frac{1}{\epsilon}\right) v(z) = L_0 v(z) + \tilde{U}(v'(z)) + \frac{1}{\epsilon} v(z),\ z > 0 \tag{2.9}$$
$$v(0) = 0 ,$$

for $\epsilon > 0$ which is chosen later (see the proof of Lemma 4.10). Whereas, we remark that when comparing (2.6) and (2.9), $\epsilon > 0$ is merely an apparent parameter. We shall show later that the solution $v = v_\epsilon$ is approximated by the solution $u = u_{M,\epsilon}$ of

$$\left(\beta + \frac{1}{\epsilon}\right) u(z) = L_0 u + \tilde{U}_M(u'(z)) + \frac{1}{\epsilon} u(z),\ z > 0 \tag{2.10}$$
$$u(0) = 0 ,$$

where $\tilde{U}_M(x) \equiv \max\{U(c) - cx; 0 < c \le M\}$ and $M > 0$.

4.2.2 Viscosity solutions

We solve the problem in terms of viscosity solutions. In this subsection, we shall prepare notions which are necessary below, and show (existence and) uniqueness results for the viscosity solution v of (2.6).

First, we introduce the notion of superjet $J^{2,+}v(z)$ and subjet $J^{2,-}v(z)$ of v at z as follows:

$$J^{2,+}v(z) = \left\{(p,q) \in \mathbf{R}^2; \limsup_{y\to z} \frac{v(y)-v(z)-p(y-z)-\frac{1}{2}q|y-z|^2}{|y-z|^2} \le 0\right\} ,$$

$$J^{2,-}v(z) = \left\{(p,q) \in \mathbf{R}^2; \liminf_{y\to z} \frac{v(y)-v(z)-p(y-z)-\frac{1}{2}q|y-z|^2}{|y-z|^2} \ge 0\right\} .$$

Let

$$F(z,u,p,q,B^1(z,u,p),B_1(z,u,p)) = -\beta u + \frac{1}{2}\sigma^2 z^2 q + (f(z)-\mu z)p + B^1(z,u,p) + B_1(z,u,p) + \tilde{U}(p) ,$$

and

$$\tilde{F}(z,u,p,q,B^1(z,u,p),B_1(z,u,p)) = \max\{F(z,u,p,q,B^1(z,u,p),B_1(z,u,p)), -p\} ,$$

where

$$B^1(z,u,p) = \int_{|\zeta|>1} \{u(z+\gamma(z,\zeta)) - u(z) - p\cdot\gamma(z,\zeta)\}\mu(d\zeta) ,$$

and

$$B_1(z,u,p) = \int_{|\zeta|\le 1} \{u(z+\gamma(z,\zeta)) - u(z) - p\cdot\gamma(z,\zeta)\}\mu(d\zeta) .$$

The following definition is due to [11] and [175].

Definition 4.6. Let a function $v \in C([0,\infty))$ satisfy $v(0) = 0$.

(1) The function v is called a viscosity subsolution of (2.6) if for all $z \in (0,\infty)$ and all $\phi \in C^2((0,\infty))$ such that if $v(z) = \phi(z)$ and if z is a local maximum point of $v - \phi$, the following relation holds:

$$\tilde{F}(z,v,\phi'(z),\phi''(z),B^1(z,v,\phi'(z)),B_1(z,\phi,\phi'(z))) \ge 0, \quad z > 0 .$$

(2) The function v is called a viscosity supersolution of (2.6) if for all $z \in (0,\infty)$ and all $\phi \in C^2((0,\infty))$ if $v(z) = \phi(z)$ and if z is a local maximum point of $v - \phi$, the following relation holds:

$$\tilde{F}(z,v,\phi'(z),\phi''(z),B^1(z,v,\phi'(z)),B_1(z,\phi,\phi'(z))) \le 0, \quad z > 0 .$$

(3) If v is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Remark 4.6. The direction of the inequality in (1) and in (2) above depends on the definition of the form of second order operator F cf. [11]. For alternative definitions of viscosity solutions, see [20] and [5].

To show the statements below, we need a comparison theorem for jump-diffusions. We cite it as a lemma which follows as a corollary to [179] Theorem 3.1.

Let $z^1(t), z^2(t)$ be processes in $\mathbf{R}$ given by (2.1) with the initial conditions $z^1(0) = z_1$, $z^2(0) = z_2$, and the drifts $b_1(t,z) = f_1(z) - \mu_1 z - c(t)$, $b_2(t,z) = f_2(z) - \mu_2 z - c(t)$, respectively. We put no boundary condition for $z^i(t)$'s. We assume (2.2), (2.4), *(B.1)* and *(B.2)*.

Lemma 4.9 ([179] Theorem 3.1). *Under the conditions above on $\sigma, c(t)$ and on $f = f_1, f_2$, we have the comparison result*

$$(z_2 \geq z_1 \quad \textit{and} \quad b_2(t,z) \geq b_1(t,z)) \Rightarrow z^2(s) \geq z^1(s) \quad \textit{for} \quad s \in [t,T], \quad a.s.$$

for any $t \leq T$.

Indeed, conditions (H3.1), (H3.2), (2.15) in [179] are met with $\sigma(t,z) = \sigma$, $b(t,z) = f(z) - \mu z - c(t)$, and $\gamma(t,z,\zeta) = z(e^{\zeta} - 1)$. However, we have to use this result adequately for our setting since the above result is stated for the processes on $[0,T]$ where $T > 0$ is fixed, whereas we put the boundary condition $z^i(t) = 0$ for $t \geq \tau_{z^i}$, $i = 1,2$, which corresponds to the Dirichlet boundary condition $v(z) = 0$, $z \leq 0$ in (2.8). To this end, we proceed in the following way.

Since $T > 0$ is arbitrary in the above, we let $T \to +\infty$ and we use it only for $0 \leq s < \tau_{z^1} \wedge \tau_{z^2}$ as long as $\tau_{z^1} \wedge \tau_{z^2}$ is finite. Then, we have our assertion of the comparison for our processes for $0 < s < \tau_{z^1} \wedge \tau_{z^2}$ directly from the lemma since (due to *(B.1)*) the boundary condition has no effect with respect to the jump term up to $\tau_{z^1} \wedge \tau_{z^2}$. For the case $\tau_{z^1} \leq s \leq \tau_{z^2}$, on the other hand, we observe $z^2(s) > 0$ whereas $z^1(s) = 0$, and hence the assertion.[1]

This lemma can also be shown by a calculation similar to [82] Section VI.1.

HJB equation

We note that (2.10) is the HJB equation associated with the optimisation problem

$$u_M(z) = \sup_{\mathbf{c} \in \mathcal{C}_M} E\left[\int_0^{\tau_z} e^{-(\beta + \frac{1}{\epsilon})t} \{U(c(t)) + \frac{1}{\epsilon} u_M(z(t))\} dt\right], \quad M > 0\,, \tag{2.11}$$

where $\mathcal{C}_M$ denotes the class of all nonnegative, integrable, $\mathcal{F}_t$-adapted processes $\mathbf{c} \in \mathcal{C}$ such that $0 \leq c(t) \leq M$ for all $t \geq 0$. By (2.8), we have $z(t) = z(t \wedge \tau_z) \geq 0$ for each

1 In [179], it is assumed $\int_Z \mu(d\zeta) < +\infty$ for $Z \subset \mathbf{R} \setminus \{0\}$. However, this condition can be replaced by $\int_{Z_n} \mu(d\zeta) < +\infty$ for each compact $Z_n \subset \mathbf{R} \setminus \{0\}$ such that $Z_n \to \mathbf{R} \setminus \{0\}$. See the argument on p. 378 of [179].

$\mathbf{c} \in \mathcal{C}_M$ because $c(t)$ is identified with $c(t)1_{\{t\le\tau_z\}}$ in (2.6). Taking κ so that

$$0 < \kappa \le \mu \,, \tag{2.12}$$

we can choose a constant $A > 0$ by concavity such that

$$f(z) - \kappa z < A \,. \tag{2.13}$$

We assume

$$\kappa + \mu < \beta. \tag{B.4}$$

Furthermore, we observe by *(B.3)* and (2.13) that the linear function

$$\varphi(z) \equiv z + B \tag{2.14}$$

satisfies

$$\begin{aligned} &-\beta\varphi(z) + L_0\varphi(z) + \tilde{U}(\varphi'(z)) \\ &\le -\beta B + A + \tilde{U}(1) < 0, \quad z \ge 0 \end{aligned} \tag{2.15}$$

for some constant $B > 0$. Let $\mathcal{B}$ denote the space

$$\begin{aligned} \mathcal{B} = \{h; h \quad &\text{is measurable on} \quad [0, +\infty) \quad \text{and satisfies that there exists} \quad C_\rho > 0 \\ &\text{for any} \quad \rho > 0 \quad \text{such that } |h(z) - h(\tilde{z})| \le C_\rho|z - \tilde{z}| + \rho(\varphi(z) + \varphi(\tilde{z}))\,, \\ &z, \tilde{z} \in [0, \infty)\}\,. \end{aligned} \tag{2.16}$$

We put the norm $\|h\| = \sup_{z\ge 0} |h(z)|/\varphi(z) < \infty$ on $\mathcal{B}$. The space $\mathcal{B}$ is a Banach space.

Remark 4.7. Functions h in $\mathcal{B}$ can also be regarded as those defined on $\mathbf{R}$ by extending to be zero outside of $[0, +\infty)$. This extension is irrelevant to the boundary value at 0 due to the transmission property from the right. See the argument around (2.8) above.

Lemma 4.10. *We assume that there exists a concave function* $\psi \in \mathcal{B} \cap C^2((0, +\infty))$ *such that*

$$\begin{aligned} &-\beta\psi(z) + L_0\psi(z) + \tilde{U}(\psi'(z)) \le 0\,, \quad z > 0\,, \\ &\qquad \psi'(z) > 0, z > 0 \quad \textit{and} \quad \psi(0) = 0\,. \end{aligned} \tag{2.17}$$

Under (B.3), (2.2)*, and (B.4), for each* $M > 0$*, there exists a unique solution* $u = u_M \in \mathcal{B}$ *of* (2.11) *for any* $\epsilon > 0$.

Remark 4.8. The condition (2.17) refers to the existence of a C^2 function related to the viscosity supersolution of (2.11). The existence of such a function, given L_0, depends on the form of $\tilde{U}$. Analytically, we assume the existence of a Lyapunov function associated to the equation. Instead of providing a sufficient condition for the existence of $\psi \in \mathcal{B} \cap C^2$ to (2.17), we give an example for it after the proof.

Proof. We divide the proof into three steps.

Step 1 We first show that

$$\sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}\psi(z(t))\}dt\right] \le \psi(z)\,, \tag{2.18}$$

$$\sup_{\mathbf{c}\in\mathcal{C}_M} E\left[e^{-(\beta+\frac{1}{\epsilon})\tau}|z(\tau) - \tilde{z}(\tau)| \cdot 1_{\{\tau_{\tilde{z}}\le\tau_z\}}\right] \le 2|z - \tilde{z}|\,, \tag{2.19}$$

for any stopping time τ, where $\{\tilde{z}(t)\}$ is the solution of (2.1) subject to $\mathbf{c} \in \mathcal{C}_M$ with $\tilde{z}(0) = \tilde{z}$. It is easily seen that

$$E\left[\int_0^t (z(s)\psi'(z(s)))^2\,ds\right] \le CE\left[\int_0^t z(s)^2\,ds\right] < \infty\,.$$

This yields that

$$M_\epsilon(t) := \int_0^t e^{-(\beta+\frac{1}{\epsilon})s}\psi'(z(s))\sigma z(s)dW(s) + \int_0^t\int e^{-(\beta+\frac{1}{\epsilon})s}$$
$$\{\psi(z(s-) + \gamma(z(s-),\zeta)) - \psi(z(s-)) - \psi'(z(s-))\gamma(z(s-),\zeta)\}\tilde{N}(dsd\zeta)$$

is a local martingale on $[0, \tau_z]$ conditioned that $\tau_z < +\infty$. Hence, by (2.17) and (2.5), Itô's formula gives

$$\begin{aligned}
0 &\le E[e^{-(\beta+\frac{1}{\epsilon})(t\wedge\tau_z)}\psi(z(t\wedge\tau_z))]\\
&= \psi(z) + E\Bigg[\int_0^{t\wedge\tau} e^{-(\beta+\frac{1}{\epsilon})s}\Big\{-\Big(\beta + \frac{1}{\epsilon}\Big)\psi(z(s))\\
&\quad + (f(z(s)) - \mu z(s) - c(s))\psi'(z(s)) + \frac{1}{2}\sigma^2 z(s)^2\psi''(z(s))\\
&\quad + \int\{\psi(z(s-) + \gamma(z(s-),\zeta)) - \psi(z(s-)) - \psi'(z(s-))\cdot\gamma(z(s-),\zeta)\}\mu(d\zeta)\Big\}\,ds\\
&\quad + M_\epsilon(t\wedge\tau_z)\Bigg]\\
&\le \psi(z) - E\left[\int_0^{t\wedge\tau_z} e^{-(\beta+\frac{1}{\epsilon})s}\left\{U(c(s)) + \frac{1}{\epsilon}\psi(z(s))\right\}ds\right]\,.
\end{aligned} \tag{2.20}$$

Thus, we deduce (2.18).

Next, we show (2.19). We set $Z(t) = z(t) - \tilde{z}(t)$. (The notation $Z(t)$ is temporarily used in this subsection, as long as it makes no confusion with the original Lévy process.) Let $K(t)$ be the process given by

$$dK(t) = (\kappa + \mu)K(t)dt - \sigma K(t)dW(t) + K(t-)\int(e^\zeta - 1)\tilde{N}(dtd\zeta), \quad K(0) = |z - \tilde{z}|\,,$$

which is nonnegative a.s. Then, by Lemma 4.9 $|Z(t)| \le K(t) a.s.$, $0 < t \le \tau_z \wedge \tau_{\tilde{z}}$.

We choose $\epsilon > 0$ and fix it. Then, we have

$$\begin{aligned}
E\left[e^{-(\beta+\frac{1}{\epsilon})\tau}|z(\tau\wedge\tau_z)-\tilde{z}(\tau\wedge\tau_{\tilde{z}})|.\,1_{\{\tau\le\tau_{\tilde{z}}\le\tau_z\}}\right]
&= E\left[e^{-(\beta+\frac{1}{\epsilon})\tau}|Z(\tau)|.\,1_{\{\tau\le\tau_{\tilde{z}}\le\tau_z\}}\right]\\
&\le E\left[e^{-(\beta+\frac{1}{\epsilon})\tau}K(\tau)\right]\\
&= |z-\tilde{z}|E\Bigg[\exp\Bigg\{-\left(\beta+\frac{1}{\epsilon}\right)\tau+(\kappa+\mu)\tau-\sigma W(\tau)-\frac{1}{2}\sigma^2\tau\\
&\quad+\int_0^\tau\int\zeta\tilde{N}(dt\,d\zeta)-\tau\int(e^\zeta-1-\zeta)\mu(d\zeta)\Bigg\}\Bigg]\\
&\le |z-\tilde{z}|E[\exp\{-\sigma W(\tau)-\frac{1}{2}\sigma^2\tau+\int_0^\tau\int\zeta\tilde{N}(dt\,d\zeta)-\tau\int(e^\zeta-1-\zeta)\mu(d\zeta)\}]\\
&= |z-\tilde{z}|\,.
\end{aligned}$$

Next, we set $\hat{z}(t)=z(t\vee\tau_{\tilde{z}})$. Then, we have on $\{\tau_{\tilde{z}}<+\infty\}$,

$$\begin{aligned}
\hat{z}(t) &= z(\tau_{\tilde{z}})+\int_{\tau_{\tilde{z}}}^{t\vee\tau_{\tilde{z}}}[f(z(r))-\mu z(r)]dr-\int_{\tau_{\tilde{z}}}^{t\vee\tau_{\tilde{z}}}c(r)dr-\int_{\tau_{\tilde{z}}}^{t\vee\tau_{\tilde{z}}}\sigma z(r)dW(r)\\
&\quad+\int_{\tau_{\tilde{z}}}^{t\vee\tau_{\tilde{z}}}\int z(r-)(e^\zeta-1)\tilde{N}(drd\zeta)\\
&\le z(\tau_{\tilde{z}})+\int_{\tau_{\tilde{z}}}^{t}[f(\hat{z}(r))-\mu\hat{z}(r)]dr-\int_{\tau_{\tilde{z}}}^{t}\sigma\hat{z}(r)dW(r)\\
&\quad+\int_{\tau_{\tilde{z}}}^{t}\int\hat{z}(r-)(e^\zeta-1)\tilde{N}(drd\zeta)\quad a.s.
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{z}(t\wedge\tau_{\tilde{z}}) &\le z(\tau_{\tilde{z}})+\int_{\tau_{\tilde{z}}}^{t\wedge\tau_z}(\kappa+\mu)\hat{z}(r)dr-\int_{\tau_{\tilde{z}}}^{t\wedge\tau_z}\sigma\hat{z}(r)dW(r)\\
&\quad+\int_{\tau_{\tilde{z}}}^{t\wedge\tau_z}\int\hat{z}(r-)(e^\zeta-1)\tilde{N}(drd\zeta)
\end{aligned}$$

$$= z(\tau_{\tilde z}) + \int_{\tau_{\tilde z}}^{t} (\kappa+\mu)\hat z(r)1_{\{r\le\tau_z\}}dr - \int_{\tau_{\tilde z}}^{t} \sigma\hat z(r)1_{\{r\le\tau_z\}}dW(r)$$
$$+ \int_{\tau_{\tilde z}}^{t}\int \hat z(r-)(e^{\zeta}-1)1_{\{r\le\tau_z\}}\tilde N(drd\zeta)$$
$$\le K(\tau_{\tilde z}) + \int_{\tau_{\tilde z}}^{t} (\kappa+\mu)\hat z(r\wedge\tau_z)dr - \int_{\tau_{\tilde z}}^{t} \sigma\hat z(r\wedge\tau_z)dW(r)$$
$$+ \int_{\tau_{\tilde z}}^{t}\int \hat z((r\wedge\tau_z)-)(e^{\zeta}-1)\tilde N(drd\zeta)$$

on $\{\tau_{\tilde z} < \infty, \tau_{\tilde z} \le \tau_z\}$, a.s.

Since

$$K(t\vee\tau_{\tilde z}) = K(\tau_{\tilde z}) + \int_{\tau_{\tilde z}}^{t} (\kappa+\mu)K(r\vee\tau_{\tilde z})dr - \int_{\tau_{\tilde z}}^{t} \sigma K(t\vee\tau_{\tilde z})dW(r)$$
$$+ \int_{\tau_{\tilde z}}^{t}\int K((r\vee\tau_{\tilde z})-)(e^{\zeta}-1)\tilde N(drd\zeta)$$

on $\{\tau_{\tilde z} < \infty, \tau_{\tilde z} \le \tau_z\}$, by the argument just after Lemma 4.9, we have

$$\hat z(t\wedge\tau_z) \le K(t\vee\tau_{\tilde z})$$

on $\{t < \tau_z, \tau_{\tilde z} \le \tau_z, \tau_{\tilde z} < +\infty\}$. Then,

$$0 \le \hat z(t) \le K(t) \quad \text{on} \quad \{\tau_{\tilde z} \le t \le \tau_z\} \;\; a.s.$$

Therefore, we have

$$\begin{aligned} &E\left[e^{-(\beta+\frac{1}{\epsilon})\tau}|z(\tau\wedge\tau_z) - \tilde z(\tau\vee\tau_{\tilde z})|1_{\{\tau_{\tilde z}\le\tau\le\tau_z\}}\right] \\ &\quad = E[e^{-(\beta+\frac{1}{\epsilon})\tau}|z(\tau)-\tilde z(\tau)|1_{\{\tau_{\tilde z}\le\tau\le\tau_z\}}] \\ &\quad = E[e^{-(\beta+\frac{1}{\epsilon})\tau}\hat z(\tau)1_{\{\tau_{\tilde z}\le\tau\le\tau_z\}}] \\ &\quad \le E[e^{-(\beta+\frac{1}{\epsilon})\tau}K(\tau)] \le |z-\tilde z| \,. \end{aligned}$$

These imply (2.19).

Step 2 Next, we define for each $\epsilon > 0$, $M > 0$,

$$T_M h(z) = \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}\left\{U(c(t)) + \frac{1}{\epsilon}h(z(t))\right\}dt\right] \quad \text{for} \quad h\in\mathcal{B}\,, \tag{2.21}$$

and show that T_M is a contraction

$$T_M : \mathcal{B}_\varphi \to \mathcal{B}_\varphi \tag{2.22}$$

where

$$\mathcal{B}_\varphi = \{h \in \mathcal{B} : 0 \le h \le \varphi, h(0) = 0\} .$$

By (2.18), it is easy to see that $T_M h(0) = 0 \le T_M h \le \varphi$ and that $\|T_M h\| < \infty$ for $h \in \mathcal{B}_\varphi$. Since $z(0) \ge 0$, we note by (2.5) that $z(t) = 0$ if $t > \tau_z$. Hence, by (2.16), we have

$$\begin{aligned}
|T_M h(z) - T_M h(\tilde{z})| &\le \left| \sup_{\mathbf{c}} E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}h(z(t))\}dt \right.\right. \\
&\quad \left.\left. - \int_0^{\tau_{\tilde{z}}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}h(\tilde{z}(t))\}dt \right]\right| \\
&\le \sup_{\mathbf{c}} E\left[\int_{\tau_z\wedge\tau_{\tilde{z}}}^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}U(c(t))dt \right. \\
&\quad \left. + \int_0^{\infty} e^{-(\beta+\frac{1}{\epsilon})t}\frac{1}{\epsilon}|h(z(t)) - h(\tilde{z}(t))|dt \right] \\
&\le \sup_{\mathbf{c}} E\left[\int_{\tau_z\wedge\tau_{\tilde{z}}}^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}U(c(t))dt \right] \\
&\quad + \frac{C_\rho}{\epsilon}\sup_{\mathbf{c}} E\left[\int_0^{\infty} e^{-(\beta+\frac{1}{\epsilon})t}|z(t) - \tilde{z}(t)|dt \right] \\
&\quad + \frac{\rho}{\epsilon}\sup_{\mathbf{c}} E\left[\int_0^{\infty} e^{-(\beta+\frac{1}{\epsilon})t}\{\varphi(z(t)) + \varphi(\tilde{z}(t))\}dt \right] \\
&\equiv J_1 + J_2 + J_3, \text{ say} .
\end{aligned}$$

By (2.16), we can take sufficiently small $\epsilon > 0$ such that

$$\begin{aligned}
E[\sup_t |M_\epsilon(t)|] \le C\Bigg(&\sigma E\left[\int_0^{\infty} e^{-2(\beta+\frac{1}{\epsilon})s} z(s)^2 ds \right]^{1/2} \\
&+E[\int_0^{\infty}\int e^{-2(\beta+\frac{1}{\epsilon})s}|\{\psi(z(s-) + \gamma(z(s-), \zeta)) \\
&-\psi(z(s-)) - \psi'(z(s-))\gamma(z(s-), \zeta)\}|^2 \mu(d\zeta)ds]^{1/2} \Bigg) < +\infty .
\end{aligned}$$

Then, by the optional stopping theorem,

$$E[M_\epsilon(\tau_z \wedge \tau_{\tilde{z}})] = 0 .$$

By using (2.17), according to the same line as (2.20), we have in view of (2.6),

$$\begin{aligned}J_1 &\le E[e^{-(\beta+\frac{1}{\epsilon})(\tau_z\wedge\tau_{\tilde z})}\psi(z(\tau_z\wedge\tau_{\tilde z})) - e^{-(\beta+\frac{1}{\epsilon})\tau_z}\psi(z(\tau_z))] \\ &= E[\{e^{-(\beta+\frac{1}{\epsilon})(\tau_z\wedge\tau_{\tilde z})}\psi(z(\tau_z\wedge\tau_{\tilde z})) - e^{-(\beta+\frac{1}{\epsilon})\tau_z}\psi(z(\tau_z))\}1_{\{\tau_{\tilde z}<\tau_z\}}] \\ &\le E[\{e^{-(\beta+\frac{1}{\epsilon})\tau_{\tilde z}}\psi(z(\tau_{\tilde z})) - e^{-(\beta+\frac{1}{\epsilon})\tau_{\tilde z}}\psi(\tilde z(\tau_{\tilde z}))\}1_{\{\tau_{\tilde z}<\tau_z\}}] \,.\end{aligned}$$

Indeed, in case $z < \tilde z$, we have $z(t) \le \tilde z(t)$ a.s. by Lemma 4.9. Hence $1_{\{\tau_{\tilde z}<\tau_z\}} = 0$, and $J_1 \le 0$. In case $z > \tilde z$, we have $z(t) \ge \tilde z(t)$ a.s. Then, $\tau_{\tilde z} \le \tau_z$, and $e^{-(\beta+\frac{1}{\epsilon})\tau_z} \le e^{-(\beta+\frac{1}{\epsilon})\tau_{\tilde z}}$. As $z(\tau_z) = 0, \tilde z(\tau_{\tilde z}) = 0$, we have $\psi(z(\tau_z)) = \psi(\tilde z(\tau_{\tilde z})) = \psi(0) = 0$. Hence, the right-hand side is dominated by

$$\begin{aligned}\sup_{\mathbf{c}} E[e^{-(\beta+\frac{1}{\epsilon})\tau_{\tilde z}}\{C_\rho|z(\tau_{\tilde z}) - \tilde z(\tau_{\tilde z})| + \rho(\varphi(z(\tau_{\tilde z})) + \varphi(\tilde z(\tau_{\tilde z})))\}1_{\{\tau_{\tilde z}<\tau_z\}}] \\ \le 2C_\rho|z - \tilde z| + \rho(\varphi(z) + \varphi(\tilde z))\end{aligned}$$

by using that $\psi \in \mathcal{B}_\varphi$ and by (2.18), (2.19).

Moreover, by (2.19),

$$J_2 \le 2C_\rho|z - \tilde z|/\epsilon \,.$$

Also, we recall that $\tilde z(t) = 0$ if $t > \tau_{\tilde z}$. Hence, by (2.19),

$$J_3 \le \rho(\varphi(z) + \varphi(0) + \varphi(\tilde z) + \varphi(0)) \le 2\rho(\varphi(z) + \varphi(\tilde z)) \,.$$

Therefore, we get $T_M h \in \mathcal{B}_\varphi$, which implies (2.22).

Step 3 Now, we have

$$E[\varphi(z(t\wedge\tau_z))] \le \varphi(z) \,.$$

Indeed, by Itô's formula, we have

$$E[z(t)] \le z + \int_0^t E[f(z(s)) - \mu z(s)]ds + E[\text{martingale}] \,.$$

Since f is strictly concave, $E[f(z(s))] \le f(E[z(s)])$. Hence, by the assumption (2.12), (2.13), we can conclude.

Hence,

$$\begin{aligned}|T_M h_1(z) - T_M h_2(z)| &= \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}\frac{1}{\epsilon}|h_1(z(t)) - h_2(z(t))|dt\right] \\ &\le \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t}\frac{1}{\epsilon}\|h_1 - h_2\|\varphi(z(t))dt\right] \\ &\le \frac{1}{\beta\epsilon + 1}\|h_1 - h_2\|\varphi(z) < \|h_1 - h_2\|\varphi(z) \,.\end{aligned}$$

Therefore, by the contraction mapping theorem, T_M has a fixed point $u_M \in \mathcal{B}_\varphi$. This completes the proof. □

Example of (2.17). We assume

$$U(c) = \frac{1}{r}c^r (0 < r < 1) .$$

Then, $\tilde{U}(x) = \frac{1-r}{r} x^{\frac{r}{r-1}}$, $x > 0$. Furthermore, we see that

$$\frac{1}{2}\sigma^2 r(r-1) + (\kappa - \mu)r \le 0 < \beta .$$

Then, we can choose $R > 0$ for which

$$\beta - \{\tfrac{1}{2}\sigma^2 r(r-1) + (\kappa - \mu)r + (1-r)R^{\frac{1}{r-1}}\} \ge 0$$

holds.

We choose $\psi(z) = \frac{R}{r}z^r$. Then, ψ satisfies $\psi'(z) = Rz^{r-1} > 0$, $z > 0$ and $\psi(0) = 0$. Since $\psi''(z) = R(r-1)z^{r-2}$ is continuous on $(0, +\infty)$, ψ is in $\mathcal{B} \cap C^2$.

Furthermore, we have

$$\begin{aligned}
&\psi(z + \gamma(z,\zeta)) - \psi(z) - \psi'(z)\cdot\gamma(z,\zeta) \\
&\quad = \frac{R}{r}\{(z + \gamma(z,\zeta))^r - z^r - rz^{r-1}\cdot\gamma(z,\zeta)\} \\
&\quad = R\{(z + \theta\gamma(z,\zeta))^{r-1} - z^{r-1}\}\gamma(z,\zeta) \le 0
\end{aligned}$$

since $r < 1$ where $\theta \in (0,1)$ is given by the mean value theorem. Hence, the integral

$$\begin{aligned}
&\int_0^\infty \{\psi(z + \gamma(z,\zeta)) - \psi(z) - \psi'(z)\cdot\gamma(z,\zeta)\}\mu(d\zeta) \\
&\qquad = R\int_0^\infty \{(z + \theta\gamma(z,\zeta))^{r-1} - z^{r-1}\}\gamma(z,\zeta)\mu(d\zeta) \le 0, \quad z > 0 .
\end{aligned}$$

This implies

$$\begin{aligned}
&-\beta\psi(z) + L_0\psi(z) + \tilde{U}(\psi'(z)) \\
&\qquad \le -\beta\frac{R}{r}z^r + \frac{1}{2}\sigma^2 R(r-1)z^r + R(f(z) - \mu z)z^{r-1} + \frac{1-r}{r}R^{\frac{r}{r-1}}z^r .
\end{aligned}$$

The right-hand side of the above is dominated by

$$\frac{R}{r}\{-\beta + \tfrac{1}{2}\sigma^2 r(r-1) + (\kappa - \mu)r + (1-r)R^{\frac{1}{r-1}}\}z^r \le 0$$

on $(0, +\infty)$. Hence, the condition (2.17) is met.

Theorem 4.2. *We assume (B.3),* (2.2)*, (B.4) and* (2.17)*. Then,* $u = u_M \in \mathcal{B}$ *of* (2.11) *is a viscosity solution of* (2.10) *and is concave.*

Proof. In this proof, we assume the *dynamic programming principle* (Bellman principle) that $u(z)$ is continuous on $[0,\infty)$ and satisfies

$$u(z) = \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\int_0^{\tau_z\wedge\tau} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}u(z(t))\}dt + e^{-(\beta+\frac{1}{\epsilon})\tau_z\wedge\tau}u(z(\tau_z\wedge\tau))\right] \quad (*)$$

for any bounded stopping time τ.

By the above, we assume u is in $C([0,\infty))$. Then, we prove that u is a viscosity solution of (2.10) in the following way.

(a) *u is a viscosity supersolution.*

Let $z \in O \subset (0,\infty)$, where O is an open interval. Choose any $\phi \in C^2(O)$ such that $\phi(z) = u(z) = c_0$ and $u \geq \phi$ in O. We can assume $c_0 = 0$ without loss of generality.

Let $m \in [0,\infty)$. Choose $\epsilon > 0$ so that $z - \epsilon m \in O$. Since the jumps of $z(t)$ are only rightward, $z \mapsto u(z)$ is nondecreasing. Hence,

$$\phi(z) = u(z) \geq u(z-\epsilon m) \geq \phi(z-\epsilon m)\,.$$

This implies $-m\phi'(z) \leq 0$. Hence,

$$F(z,u,p,q,B^1(z,v,p),B_1(z,\phi,p)) \leq 0$$

for $p = \phi'(z)$, $q = \phi''(z)$.

Take $c(t) \equiv c \in \mathcal{C}_M$. Let τ_r be the exit time of $z(t)$ from $\bar{O}_r(z) = \{z'; |z-z'| \leq r\} \subset (0,\infty)$. By the dynamic programming principle $(*)$,

$$u(z) \geq E\left[\int_0^{t\wedge\tau_r} e^{-(\beta+\frac{1}{\epsilon})s}\{U(c(s)) + \frac{1}{\epsilon}u(z(s))\}ds + e^{-(\beta+\frac{1}{\epsilon})t\wedge\tau_r}u(z(t\wedge\tau_r))\right].$$

We replace u by ϕ by $u(z) = \phi(z)$ and by using the ordering $u \geq \phi$ in O. Then, we have, by the Itô's formula and (2.10),

$$\begin{aligned}
0 &\geq E\left[\int_0^{t\wedge\tau_r} e^{-(\beta+\frac{1}{\epsilon})s}\{U(c(s)) + \frac{1}{\epsilon}u(z(s))\}ds + e^{-(\beta+\frac{1}{\epsilon})t\wedge\tau_r}u(z(t\wedge\tau_r))\right] - \phi(z)\\
&\geq E[\int_0^{t\wedge\tau_r} e^{-(\beta+\frac{1}{\epsilon})s}\{U(c) + L\phi(z(s)) + \frac{1}{\epsilon}\phi(z(s)) - c\phi'(z(s))\}ds\\
&\geq \min_{y\in\bar{O}_r(z)}[L\phi(y) + \frac{1}{\epsilon}\phi(y) - c\phi'(y) + U(c)] \times E\left[\frac{1}{\beta+\frac{1}{\epsilon}}(1-e^{-(\beta+\frac{1}{\epsilon})(t\wedge\tau_r)})\right].
\end{aligned}$$

Dividing the last inequality by t and taking successively the limits as $t \to 0$ and $r \to 0$, we have $L\phi(z) - c\phi'(z) + U(c) \leq 0$. Maximising the right-hand side over $c \in \mathcal{C}_M$, we have

$$F(z,u,p,q,B^1(z,v,p),B_1(z,\phi,p)) \leq 0, \quad z > 0$$

and

$$\tilde{F}(z, u, p, q, B^1(z, v, p), B_1(z, \phi, p)) \leq 0, \quad z > 0$$

with $p = \phi'(z)$, $q = \phi''(z)$. Hence, u is a subersolution of (2.10).

(b) *u is a viscosity subsolution.*

Let $z \in O \subset (0, \infty)$. Let $\phi \in C^2(O)$ be any function such that $\phi(z) = u(z)$ and $u \leq \phi$ on O.

Assume that the subsolution inequality for ϕ fails at z. Then, there exists an $\epsilon > 0$ such that $F(\phi) = F(z, u, p, q, B^1, B_1) \leq -\epsilon$ and $-\phi(y) \leq -\epsilon$ on some ball $\bar{O}_r(z) \subset O$ with $p = \phi'(z)$, $q = \phi''(z)$.

We apply the inequality that there exists a constant $\eta > 0$ and $t_0 > 0$ such that

$$E\left[e^{-(\beta+\frac{1}{\epsilon})t}\phi(z(t \wedge \tau^{r,\mathbf{c}})) + \int_0^{t\wedge\tau^{r,\mathbf{c}}} e^{-(\beta+\frac{1}{\epsilon})s}\{U(c(s)) + \frac{1}{\epsilon}u(z(s))\}ds\right] \leq \phi(z) - \eta t$$

holds for all $t \in (0, t_0]$ and any $\mathbf{c} \in \mathcal{C}$ ([110] Lemma 4.4.3, [99] Proposition 2.5). Here, $\tau^{r,\mathbf{c}}$ denotes the exit time of $z(t)$ from $\bar{O}_r(z)$ by $\mathbf{c}$.

Fix $t \in (0, t_0)$. By the dynamic programming principle $(*)$, we have that there exists $\mathbf{c} \in \mathcal{C}_M$ such that

$$u(z) \leq E\left[e^{-(\beta+\frac{1}{\epsilon})t}\phi(z(t \wedge \tau^{r,\mathbf{c}})) + \int_0^{t\wedge\tau^{r,\mathbf{c}}} e^{-(\beta+\frac{1}{\epsilon})s}\{U(c(s)) + \frac{1}{\epsilon}u(z(s))\}ds\right] + \frac{1}{2}\eta t$$

for each $\tau = \tau^{r,\mathbf{c}}$.

As $u \leq \phi$ in O, we have

$$u(z) \leq \phi(z) - \eta t + \frac{1}{2}\eta t = \phi(z) - \frac{1}{2}\eta t\,.$$

This is a contradiction.

To see the concavity of u, we also recall $T_M h(z)$ of (2.21). We prove below $T_M h(z)$ is concave if h is concave. Moreover, by induction, $T_M^n h$ is concave for any $n \geq 1$. By Lemma 4.10, we have

$$T_M^n h \to u \quad \text{as} \quad n \to \infty\,.$$

We can choose $h(z) = 0(z) \in \mathcal{B}_\varphi$, where $0(z)$ denotes the function identically equal to zero. Then, we can conclude the assertion.

To this end, we fix $h \in \mathcal{B}_\varphi$ and assume it is concave.

Choose distinct points z_1, z_2 in $(0, \infty)$, and choose $z_1(t), z_2(t)$ starting from these points, respectively. For each $\delta > 0$, we choose $c_1(t), c_2(t)$ in $\mathcal{C}_M$, corresponding to

$z_1(t), z_2(t)$, respectively, satisfying

$$T_M h(z_i) - \delta < E\left[\int_0^{\tau_i} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_i(t)) + \frac{1}{\epsilon}h(z_i(t))\}dt\right], \quad i = 1, 2 .$$

Let $0 < \lambda < 1$, and we put $c_\lambda(t) = \lambda c_1(t) + (1-\lambda)c_2(t)$. We shall show below that $c_\lambda(t) \in \mathcal{C}_M$, and that

$$\begin{aligned} &E\left[\int_0^{\tau_{z^\circ}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_\lambda(t)) + \frac{1}{\epsilon}h(z^\circ(t))\}dt\right] \\ &\qquad \geq \lambda E\left[\int_0^{\tau_{z_1}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_1(t)) + \frac{1}{\epsilon}h(z_1(t))\}dt\right] \\ &\qquad\qquad + (1-\lambda)E\left[\int_0^{\tau_{z_2}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_2(t)) + \frac{1}{\epsilon}h(z_2(t))\}dt\right] . \end{aligned}$$

Here, $z^\circ(t)$ is the solution to the following SDE:

$$\begin{aligned} &dz^\circ(t) = \\ &\quad (f(z^\circ(t)) - \mu z^\circ(t) - c_\lambda(t))dt - \sigma z^\circ(t)dW(t) + z^\circ(t)\int(e^\zeta - 1)\tilde{N}(dt\,d\zeta) , \\ &\qquad z^\circ(0) = \lambda z_1 + (1-\lambda)z_2 . \end{aligned}$$

We put $z_\lambda(t) = \lambda z_1(t) + (1-\lambda)z_2(t)$, and $\tau_{z_\lambda} = \tau_{z_1} \vee \tau_{z_2}$. We then have

$$0 \leq z_\lambda(t) \leq z^\circ(t), \quad a.s.$$

Indeed, using the concavity and Lipschitz continuity of $f(.)$,

$$\begin{aligned} E[(z_\lambda(t) - z^\circ(t))^+] &\leq E\left[\int_0^t (f(z_\lambda(s)) - f(z^\circ(s))).\ 1_{\{z_\lambda(s)\geq z^\circ(s)\}}ds\right] \\ &\leq C_f \int_0^t E[(z_\lambda(s) - z^\circ(s))^+]ds . \end{aligned}$$

Here, C_f denotes the Lipschitz constant. By Gronwall's inequality, we have the assertion. This implies $c_\lambda(t) \in \mathcal{C}_M$.

By this comparison result, using the monotonicity of and the concavity of $U(.)$, $h(.)$,

$$\begin{aligned} T_M h(\lambda z_1 + (1-\lambda) z_2) &\geq E\left[\int_0^{\tau_{z^\circ}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_\lambda(t)) + \frac{1}{\epsilon}h(z^\circ(t))\}dt\right] \\ &\geq E\left[\int_0^{\tau_{z_\lambda}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_\lambda(t)) + \frac{1}{\epsilon}h(z_\lambda(t))\}dt\right] \\ &\geq \lambda E\left[\int_0^{\tau_{z_\lambda}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_1(t)) + \frac{1}{\epsilon}h(z_1(t))\}dt\right] \\ &\quad + (1-\lambda)E\left[\int_0^{\tau_{z_\lambda}} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c_2(t)) + \frac{1}{\epsilon}h(z_2(t))\}dt\right]. \end{aligned}$$

These imply that $T_M h(z)$ is concave. Therefore, this yields that u is concave. □

Remark 4.9. The proof of the Bellman principle $(*)$ in the general setting (i.e. without continuity or measurability assumptions) is not easy. Occasionally, one is advised to decompose the equality into two parts:
(i)

$$u(z) \leq \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\limsup_{z'\to z} \int_0^{\tau_{z'}\wedge\tau} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}u(z(t))\}dt \right. \\ \left. + e^{-(\beta+\frac{1}{\epsilon})\tau_{z'}\wedge\tau} u(z(\tau_{z'}\wedge\tau))\right],$$

and the converse
(ii)

$$u(z) \geq \sup_{\mathbf{c}\in\mathcal{C}_M} E\left[\liminf_{z'\to z} \int_0^{\tau_{z'}\wedge\tau} e^{-(\beta+\frac{1}{\epsilon})t}\{U(c(t)) + \frac{1}{\epsilon}u(z(t))\}dt \right. \\ \left. + e^{-(\beta+\frac{1}{\epsilon})\tau_{z'}\wedge\tau} u(z(\tau_{z'}\wedge\tau))\right].$$

The proof for (i) is relatively easy. The proof for (ii) is difficult if it is not assumed to be continuous.

This principle is proved in the diffusion case in [110] Theorem 4.5.1, and in a two-dimensional case for some jump-diffusion process in [96] Lemma 1.5.

The next result provides a verification theorem for finding the value function in terms of the viscosity solution for (2.6).

Theorem 4.3. *We assume (B.3),* (2.2)*, (B.4) and* (2.17)*. Then, there exists a concave viscosity solution v of* (2.6) *such that* $0 \le v \le \varphi$*.*

Proof. Let $0 < M < M'$. By (2.21), we have $T_M h \le T_{M'} h$ for $h \in \mathcal{B}$. Hence, u_M must have a weak limit, which is denoted by v:

$$u_M \uparrow v \quad \text{as } M \to \infty .$$

It is clear that v is concave and then continuous on $(0, \infty)$. We shall show $v(0+) = 0$, where the existence of the left-hand side is guaranteed by the transmission property.

It follows from (2.21) and (2.18) that

$$v(z) \le \sup_{\mathbf{c} \in \mathcal{C}} E\left[\int_0^{\tau_z} e^{-(\beta + \frac{1}{\epsilon})t} \{U(c(t)) + \frac{1}{\epsilon} v(z(t))\} dt \right] \le \varphi(z), \quad z \ge 0 .$$

Hence, for any $\rho > 0$, there exists $\mathbf{c} = (c(t)) \in \mathcal{C}$ which depends on z such that

$$v(z) - \rho < E\left[\int_0^{\tau_z} e^{-(\beta + \frac{1}{\epsilon})t} U(c(t)) dt \right] + E\left[\int_0^{\tau_z} e^{-(\beta + \frac{1}{\epsilon})t} \frac{1}{\epsilon} v(z(t)) dt \right] .$$

First, let

$$N(t) \equiv z \exp\left((\kappa + \mu)t - \sigma W(t) - \frac{1}{2}\sigma^2 t + \int_0^t \int \zeta \tilde{N}(ds d\zeta) - \int_0^t \int (e^{\zeta} - 1 - \zeta)\mu(d\zeta) ds \right) .$$

Since $z(t) \le N(t)$ for each $t > 0$ by the Gronwall's inequality, and since

$$E\left[\sup_{0 \le s \le t} N^2(s) \right] < +\infty$$

for each $t > 0$, we see by (2.9) that

$$E[e^{-\{(\beta + \frac{1}{\epsilon})(\tau_z \wedge s)\}} z(\tau_z \wedge s)] = z + E\left[\int_0^{\tau_z \wedge s} e^{-(\tau_z \wedge s)t} \{-(\beta + \frac{1}{\epsilon}) z(t) + f(z(t)) - \mu z(t) - c(t)\} dt \right] .$$

By *(B.4)*, we then choose $\epsilon > 0$ so that $f'(0+) + \mu < \beta + 1/\epsilon$. Then, by letting $s \to +\infty$ and $z \to 0$, we get

$$E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t} c(t)dt\right] \le z + E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t} f(z(t))dt\right] \le z + E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t} \kappa N(t)dt\right] \to 0\,.$$

On the other hand, by the concavity *(B.3)* of $U(.)$, for each $\rho > 0$, there exists $C_\rho > 0$ such that

$$U(x) \le C_\rho x + \rho \tag{2.23}$$

for $x \ge 0$. Then, letting $z \to 0$ and then $\rho \to 0$, we obtain

$$E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t} U(c(t))dt\right] \le C_\rho E\left[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\epsilon})t} c(t)dt\right] + \frac{\rho}{\beta+\frac{1}{\epsilon}} \to 0\,.$$

Thus, by the dominated convergence theorem,

$$v(0+) \le E\left[\int_0^{\infty} e^{-(\beta+\frac{1}{\epsilon})t} \frac{1}{\epsilon} v(0+)dt\right] \le \frac{1}{\beta\epsilon + 1} v(0+)\,,$$

which implies $v(0+) = 0$. Thus, $v \in C([0,\infty))$.

By Dini's theorem, u_M converges to v locally and uniformly on $[0,\infty)$. Therefore, by the standard stability results (cf. [10] Proposition II.2.2), we deduce that v is a viscosity solution of (2.9) and then of (2.6). □

Comparison

We give a comparison theorem for the viscosity solution v of (2.6).

Theorem 4.4. *Let f_i, $i = 1, 2$ satisfy* (2.2) *and let $v_i \in C([0,\infty))$ be the concave viscosity solution of* (2.6) *for f_i in place of f such that $0 \le v_i \le \varphi$. Suppose*

$$f_1 \le f_2\,. \tag{2.24}$$

Then, under (B.3), (2.2)*, (B.4) and* (2.17)*, we have*

$$v_1 \le v_2\,.$$

Proof. Let

$$\varphi_\nu(z) \equiv z^\nu + B .$$

By (2.12), *(B.4)*, we first note that there exists $1 < \nu < 2$ such that

$$-\beta + \frac{1}{2}\sigma^2\nu(\nu-1) + \kappa\nu - \mu\nu < 0 . \tag{2.25}$$

Hence, by (2.12), (2.13), $\varphi_\nu(z)$ satisfies

$$\begin{aligned} -\beta\varphi_\nu(z) + \frac{1}{2}\sigma^2 z^2\varphi_\nu''(z) + (f_2(z) - \mu z)\varphi_\nu'(z) - \int\{\varphi_\nu'(z)\cdot\gamma(z,\zeta)\}\mu(d\zeta) \\ \le (-\beta + \frac{1}{2}\sigma^2\nu(\nu-1) + \kappa\nu - \mu\nu)z^\nu + A\nu z^{\nu-1} - \beta B < 0 , \quad z \ge 0 \end{aligned} \tag{2.26}$$

for a suitable choice of $B > 0$.

Suppose that $v_1(z_0) - v_2(z_0) > 0$ for some $z_0 \in (0,\infty)$. Then, there exists $\eta > 0$ such that

$$\sup_{z\ge 0}[v_1(z) - v_2(z) - 2\eta\varphi_\nu(z)] > 0 .$$

Since, by the concavity of v, it holds that

$$v_1(z) - v_2(z) - 2\eta\varphi_\nu(z) \le \varphi(z) - 2\eta\varphi_\nu(z) \to -\infty \quad \text{as} \quad z \to +\infty .$$

It implies that we can find some $\bar{z} \in (0,\infty)$ such that

$$\sup_{z\ge 0}[v_1(z) - v_2(z) - 2\eta\varphi_\nu(z)] = v_1(\bar{z}) - v_2(\bar{z}) - 2\eta\varphi_\nu(\bar{z}) > 0 .$$

Define

$$\Psi_n(z,y) = v_1(z) - v_2(y) - \frac{n}{2}|z-y|^2 - \eta(\varphi_\nu(z) + \varphi_\nu(y))$$

for each $n > 0$. It is clear that

$$\Psi_n(z,y) \le \varphi(z) + \varphi(y) - \eta(\varphi_\nu(z) + \varphi_\nu(y)) \to -\infty \quad \text{as} \quad z + y \to +\infty .$$

Hence, we can find $(z_n, y_n) \in [0,\infty) \times [0,\infty)$ such that

$$\Psi_n(z_n, y_n) = \sup\{\Psi_n(z,y) : (z,y) \in [0,\infty)\times[0,\infty)\} .$$

Since

$$\begin{aligned} \Psi_n(z_n,y_n) &= v_1(z_n) - v_2(y_n) - \frac{n}{2}|z_n - y_n|^2 - \eta(\varphi_\nu(z_n) + \varphi_\nu(y_n)) \\ &\ge v_1(\bar{z}) - v_2(\bar{z}) - 2\eta\varphi_\nu(\bar{z}) > 0 , \end{aligned} \tag{2.27}$$

we can deduce

$$\begin{aligned} \frac{n}{2}|z_n - y_n|^2 &\le v_1(z_n) - v_2(y_n) - \eta(\varphi_\nu(z_n) + \varphi_\nu(y_n)) \\ &\le \varphi(z_n) + \varphi(y_n) - \eta(\varphi_\nu(z_n) + \varphi_\nu(y_n)) < M \end{aligned}$$

for some $M > 0$. Thus, we deduce that the sequences $\{z_n + y_n\}$ and $\{n|z_n - y_n|^2\}$ are bounded by some constant, and

$$|z_n - y_n| \to 0 \; as \; n \to \infty \, .$$

Moreover, we can find

$$z_n \to \hat{z} \in [0, \infty) \quad \text{and} \quad y_n \to \hat{z} \in [0, \infty) \quad \text{as} \;\; n \to \infty$$

for some $\hat{z} \in [0, +\infty)$, taking a subsequence if necessary. By the definition of (z_n, y_n), we recall that

$$\begin{aligned}\Psi_n(z_n, y_n) &= v_1(z_n) - v_2(y_n) - \frac{n}{2}|z_n - y_n|^2 - \eta(\varphi_v(z_n) + \varphi_v(y_n)) \\ &\geq v_1(z_n) - v_2(z_n) - 2\eta\varphi_v(z_n) \, ,\end{aligned}$$

and thus

$$\frac{n}{2}|z_n - y_n|^2 \leq v_2(z_n) - v_2(y_n) + \eta(\varphi_v(z_n) - \varphi_v(y_n)) \to 0 \quad \text{as} \;\; n \to \infty \, . \tag{2.28}$$

Hence, $n|z_n - y_n|^2 \to 0$ as $n \to \infty$.

Passing to the limit in (2.27), we get

$$v_1(\hat{z}) - v_2(\hat{z}) - 2\eta\varphi_v(\hat{z}) > 0 \quad \text{and} \quad \hat{z} > 0 \, . \tag{2.29}$$

Next, let $V_1(z) = v_1(z) - \eta\varphi_v(z)$ and $V_2(y) = v_2(y) + \eta\varphi_v(y)$. Applying Ishii's lemma (cf. [46], [61] Section V.6) to

$$\Psi_n(z, y) = V_1(z) - V_2(y) - \frac{n}{2}|z - y|^2 \, ,$$

we obtain $q_1, q_2 \in \mathbf{R}$ such that

$$\begin{aligned}&(n(z_n - y_n), q_1) \in \bar{J}^{2,+}V_1(z_n) \, , \\ &(n(z_n - y_n), q_2) \in \bar{J}^{2,-}V_2(y_n) \, , \\ &-3n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} q_1 & 0 \\ 0 & -q_2 \end{pmatrix} \leq 3n\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\end{aligned} \tag{2.30}$$

where

$$\bar{J}^{2,\pm}V_i(z) = \left\{ (p, q) : \begin{array}{c} \exists z_r \to z \, , \\ \exists (p_r, q_r) \in J^{2,\pm}V_i(z_r) \, , \\ (V_i(z_r), p_r, q_r) \to (V_i(z), p, q) \end{array} \right\} , \quad i = 1, 2 \, .$$

Recall that

$$\begin{aligned}J^{2,+}v_1(z) &= \{(p + \eta\varphi_v'(z), q + \eta\varphi_v''(z)) : (p, q) \in J^{2,+}V_1(z)\} \, , \\ J^{2,-}v_2(y) &= \{(p - \eta\varphi_v'(y), q - \eta\varphi_v''(y)) : (p, q) \in J^{2,-}V_2(y)\} \, .\end{aligned}$$

Hence, by (2.30),

$$(p_1, \bar{q}_1) := (n(z_n - y_n) + \eta\varphi'_v(z_n), q_1 + \eta\varphi''_v(z_n)) \in \bar{J}^{2,+}v_1(z_n)\,,$$
$$(p_2, \bar{q}_2) := (n(z_n - y_n) - \eta\varphi'_v(y_n), q_2 - \eta\varphi''_v(y_n)) \in \bar{J}^{2,-}v_2(y_n)\,.$$

By the definition of viscosity solutions, we have

$$\begin{aligned}
&-\beta v_1(z_n) + \frac{1}{2}\sigma^2 z_n^2 \bar{q}_1 + (f_1(z_n) - \mu z_n)p_1 \\
&\qquad + B^1(z_n, v_1, p_1) + B_1(z_n, \phi_1, \phi'_1) + \tilde{U}(p_1) \geq 0\,, \\
&-\beta v_2(y_n) + \frac{1}{2}\sigma^2 y_n^2 \bar{q}_2 + (f_2(y_n) - \mu y_n)p_2 \\
&\qquad + B^1(y_n, v_2, p_2) + B_1(y_n, \phi_2, \phi'_2) + \tilde{U}(p_2) \leq 0\,.
\end{aligned}$$

Here, the C^2 functions ϕ_1, ϕ_2 satisfy that $\phi'_1(z_n) = p_1 = n(z_n - y_n) + \eta\varphi'_v(z_n)$, $v_1 - \phi_1$ has a global maximum at z_n, and $\phi'_2(y_n) = p_2 = n(z_n - y_n) - \eta\varphi'_v(y_n)$, $v_2 - \phi_2$ has a global minimum at y_n, respectively. Furthermore, due to [5] Lemma 2.1, there exists a monotone sequence of functions $(\phi_{1,k}(z))$, $\phi_{1,k}(z) \in C^2$ such that $v_1(z_n) = \phi_{1,k}(z_n)$, and $v_1(z) \leq \phi_{1,k}(z) \leq \phi_1(z)$, $\phi_{1,k} \downarrow v_1(z)(k \to \infty)$ for $z = z_n + \gamma(z_n, \zeta)$, $\zeta \geq 0$. We can also take such a sequence $(\phi_{2,k})(z)$ of C^2 functions such that $v_2(z) \geq \phi_{2,k}(z) \geq \phi_2(z)$, $\phi_{2,k} \uparrow v_2(z)(k \to \infty)$ for $z = y_n + \gamma(y_n, \zeta)$, $\zeta \geq 0$.

Putting these inequalities together, we get

$$\begin{aligned}
&\beta[v_1(z_n) - v_2(y_n)] \\
&\leq \frac{1}{2}\sigma^2(z_n^2\bar{q}_1 - y_n^2\bar{q}_2) \\
&\quad + \{(f_1(z_n) - \mu z_n)p_1 - (f_2(y_n) - \mu y_n)p_2\} + \{\tilde{U}(p_1) - \tilde{U}(p_2)\} \\
&\quad + \int_{|\zeta|>1} \{(v_1(z_n + \gamma(z_n, \zeta)) - v_2(y_n + \gamma(y_n, \zeta))) - (v_1(z_n) - v_2(y_n)) \\
&\quad - (p_1 \cdot \gamma(z_n, \zeta) - p_2 \cdot \gamma(y_n, \zeta))\}\mu(d\zeta) \\
&\quad + \int_{|\zeta|\leq 1} \{(\phi_1(z_n + \gamma(z_n, \zeta)) - \phi_2(y_n + \gamma(y_n, \zeta))) \\
&\quad - (\phi_1(z_n) - \phi_2(y_n)) - (\phi'_1(z_n) \cdot \gamma(z_n, \zeta) - \phi'_2(y_n) \cdot \gamma(y_n, \zeta))\}\mu(d\zeta) \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5\,, \quad \text{say.}
\end{aligned}$$

We consider the case where there are infinitely many $z_n \geq y_n \geq 0$. By (2.28) and (2.30), it is easy to see that

$$\begin{aligned}
I_1 &\leq \frac{1}{2}\sigma^2\{3n|z_n - y_n|^2 + \eta(z_n^2\varphi''_v(z_n) + y_n^2\varphi''_v(y_n))\} \\
&\to \sigma^2\eta\hat{z}^2\varphi''_v(\hat{z}) \quad \text{as } n \to \infty\,.
\end{aligned}$$

By monotonicity and $p_1 \geq p_2$,

$$I_3 \leq 0\,.$$

By (2.24), we have $f_1(z_n)(z_n - y_n) \le f_2(z_n)(z_n - y_n)$. Hence,

$$\begin{aligned} I_2 \le (f_2(z_n) - f_2(y_n))n(z_n - y_n) - \mu n(z_n - y_n)^2 \\ + \eta\{(f_1(z_n) - \mu z_n)\varphi_v'(z_n) + (f_2(y_n) - \mu y_n)\varphi_v'(y_n)\} \\ \to \eta(f_1(\hat z) + f_2(\hat z) - 2\mu\hat z)\varphi_v'(\hat z) \quad \text{as } n \to \infty\,. \end{aligned}$$

As for the term I_4, we have

$$\begin{aligned} I_4 \le \int_{|\zeta|>1} \{(\phi_1(z_n + \gamma(z_n,\zeta)) - \phi_1(z_n)) - (\phi_2(y_n + \gamma(y_n,\zeta)) - \phi_2(y_n)) \\ - (p_1 \cdot \gamma(z_n,\zeta) - p_2 \cdot \gamma(y_n,\zeta))\}\mu(d\zeta) \\ \to -2\eta\varphi_v'(\hat z) \int_{|\zeta|>1} \gamma(\hat z,\zeta)\mu(d\zeta)\,. \end{aligned}$$

Similarly,

$$\begin{aligned} I_5 \le \int_{|\zeta|\le 1} \{(\phi_1(z_n + \gamma(z_n,\zeta)) - \phi_1(z_n)) - (\phi_2(y_n + \gamma(y_n,\zeta)) - \phi_2(y_n)) \\ - (\gamma(z_n,\zeta)\phi_1'(z_n) - \gamma(y_n,\zeta)\phi_2'(y_n))\}\mu(d\zeta) \\ \to -2\eta \int_{|\zeta|\le 1} \varphi_v'(\hat z)\gamma(\hat z,\zeta)\mu(d\zeta)\,. \end{aligned}$$

Thus, by (2.26),

$$\begin{aligned} &\beta[v_1(\hat z) - v_2(\hat z)] \\ &\quad \le 2\eta\left\{\frac{1}{2}\sigma^2\hat z^2\varphi_v''(\hat z) + (f_2(\hat z) - \mu\hat z)\varphi_v'(\hat z) - \int \varphi_v'(\hat z)\cdot\gamma(\hat z,\zeta)\mu(dz)\right\} \\ &\quad \le 2\eta\beta\varphi_v(\hat z) \end{aligned}$$

due to (2.25). This is contrary to (2.29). Suppose that there are infinitely many $y_n \ge z_n \ge 0$. By concavity, we have $v_i(y_n) \ge v_i(z_n)$, $i = 1, 2$, and thus

$$v_1(y_n) - v_2(z_n) \ge v_1(z_n) - v_2(y_n)\,.$$

Hence, the maximum of $\Psi_n(z, y)$ is attained at (y_n, z_n). Interchanging y_n and z_n in the above argument, we again get a contradiction as above. Thus, the proof is complete. □

Remark 4.10. In view of (2.5), we observe that the value of $v(z)$ will increase in the presence of the jump term (i.e. $\tilde r > 0$).

4.2.3 Regularity of solutions

In this subsection, we shall show the smoothness of the viscosity solution v of (2.6) in case $\sigma > 0$.

Theorem 4.5. *Under (B.3),* (2.2), *(B.4),* (2.17) *and that* $\sigma > 0$, *we have* $v \in C^2((0,\infty))$ *and* $v'(0+) = \infty$.

Proof. We divide the proof into six steps.
Step 1 By the concavity of v, we recall a classical result of Alexandrov (cf. [2] and [36, 61]) to see the Lebesgue measure of $(0,\infty) \setminus \mathcal{D} = 0$, where

$$\mathcal{D} = \{z \in (0,\infty) : v \text{ is twice differentiable at } z\} .$$

By the definition of twice differentiability, we have

$$(v'(z), v''(z)) \in J^{2,+}v(z) \cap J^{2,-}v(z), \quad \forall z \in \mathcal{D} ,$$

and hence

$$\beta v(z) = L_0 v + \tilde{U}(v'(z)) .$$

Let $d^{\pm}v(z)$ denote the right-hand and left-hand derivatives respectively:

$$d^+v(z) = \lim_{y \downarrow z} \frac{v(y) - v(z)}{y - z}; \quad d^-v(z) = \lim_{y \uparrow z} \frac{v(y) - v(z)}{y - z} .$$

Define $r^{\pm}(z)$ by

$$\begin{aligned}\beta v(z) = \frac{1}{2}\sigma^2 z^2 r^{\pm}(z) + (f(z) - \mu z)d^{\pm}v(z) \\ + \int \{v(z + \gamma(z,\zeta)) - v(z) - d^{\pm}v(z) \cdot \gamma(z,\zeta)\}\mu(d\zeta) + \tilde{U}(d^{\pm}v(z)) \\ \text{for all } z \in (0,\infty) . \quad (**)\end{aligned}$$

Since $d^+v = d^-v = v'$ on $\mathcal{D}$, we have $r^+ = r^- = v''\, a.e.$ by the formula above. Furthermore, $d^+v(z)$ is right continuous by definition, and so is $r^+(z)$. Hence, it is easy to see by the mean value theorem that

$$v(y) - v(z) = \int_z^y d^+v(s)ds ,$$

$$d^+v(s) - d^+v(z) = \int_z^s r^+(t)dt, \qquad s > z .$$

Thus, we get

$$\begin{aligned}R(v;y) &:= \{v(y) - v(z) - d^+v(z)(y - z) - \frac{1}{2}r^+(z)|y - z|^2\}/|y - z|^2 \\ &= \int_z^y (d^+v(s) - d^+v(z) - r^+(z)(s - z))ds/|y - z|^2 \\ &= \int_z^y \left\{ \int_z^s (r^+(t) - r^+(z))dt \right\} ds/|y - z|^2 \to 0 \quad \text{as } y \downarrow z .\end{aligned}$$

Step 2 We show that $\tilde{U}(z)$ is strictly convex. Suppose it were not true. Then,

$$\tilde{U}(z_0) = \xi \tilde{U}(z_1) + (1-\xi)\tilde{U}(z_2) \tag{2.31}$$

for some $z_1 \neq z_2$, $0 < \xi < 1$ and $z_0 = \xi z_1 + (1-\xi)z_2$. Let m_i, $i = 0, 1, 2$ be the maximiser of $\tilde{U}(z_i)$. By a simple manipulation, $m_i = (U')^{-1}(z_i)$. It is clear (since $m_0 \neq m_i$) that

$$\tilde{U}(z_i) = U(m_i) - m_i z_i \geq U(m_0) - m_0 z_i, \quad i = 1, 2\ . \tag{2.32}$$

By (2.31), we observe that the equality in (2.32) must hold. Hence, we have $m_0 = m_1 = m_2$ and then $z_1 = z_2$. This is a contradiction.

Step 3 We claim that $v(z)$ is differentiable at $z \in (0, \infty) \setminus \mathcal{D}$. It is a well-known fact in convex analysis (cf. [44] Example 2.2.5) that

$$\partial v(z) = [d^+ v(z), d^- v(z)], \qquad \forall z \in (0, \infty)\ ,$$

where $\partial v(z)$ denotes the generalised gradient of v at z and $d^+ v(x)$, $d^- v(x)$ are given above. Suppose $d^+ v(z) < d^- v(z)$, and set

$$\hat{p} = \xi d^+ v(z) + (1-\xi) d^- v(z)\ ,$$
$$\hat{r} = \xi r^+(z) + (1-\xi) r^-(z)\ , \quad 0 < \xi < 1\ .$$

If it were true that

$$\limsup_{y \to z} R(v; y) > 0\ ,$$

then we can find a sequence $y_n \to z$ such that $\lim_{n\to\infty} R(v; y_n) > 0$. By virtue of the result in Step 1, we may consider that $y_n \leq y_{n+1} < z$ for every n, taking a subsequence if necessary. Hence, by the definition of $R(v; y)$,

$$\lim_{n\to\infty} \frac{v(y_n) - v(z) - d^+ v(z)(y_n - z)}{|y_n - z|} \geq 0\ .$$

This leads to the contradiction $d^+ v(z) \geq d^- v(z)$. Thus, we have $\limsup_{y\to z} R(v; y) \leq 0$, and $(d^+ v(z), r^+(z)) \in J^{2,+} v(z)$. Similarly, we observe that $(d^- v(z), r^-(z)) \in J^{2,+} v(z)$. By the convexity of $J^{2,+} v(z)$, we get

$$(\hat{p}, \hat{r}) \in J^{2,+} v(z)\ .$$

Now, by Step 2, we see that

$$\tilde{U}(\hat{p}) < \xi \tilde{U}(d^+ v(z)) + (1-\xi)\tilde{U}(d^- v(z))\ ,$$

and hence, by $(**)$,

$$\beta v(z) > \frac{1}{2}\sigma^2 z^2 \hat{r} + (f(z) - \mu z)\hat{p}$$
$$+ \int \{v(z + \gamma(z,\zeta)) - v(z) - \hat{p} \cdot \gamma(z,\zeta)\}\mu(d\zeta) + \tilde{U}(\hat{p})\ .$$

On the other hand, by the definition of the viscosity solution,

$$\beta v(z) \leq \frac{1}{2}\sigma^2 z^2 q + (f(z) - \mu z)p + \int \{v(z + \gamma(z,\zeta) - v(z) - p \cdot \gamma(z,\zeta)\}\mu(d\zeta) + \tilde{U}(p)\,, \quad \forall (p,q) \in J^{2,+}v(z)\,.$$

This is a contradiction. Therefore, we deduce that $\partial v(z)$ is a singleton, and this implies v is differentiable at z.

Step 4 We claim that $v'(z)$ is continuous on $(0,\infty)$. Let $z_n \to z$ and $p_n = v'(z_n) \to p$. Then, by the concavity

$$v(y) \leq v(z) + p(y - z), \quad \forall y\,.$$

Hence, we see that p belongs to $D^+v(z)$, where

$$D^+v(z) = \left\{p \in \mathbf{R} : \limsup_{y\to z} \frac{v(y) - v(z) - p(y - z)}{|y - z|} \leq 0\right\}\,.$$

Since $\partial v(z) = D^+v(z)$ and $\partial v(z)$ is a singleton by Step 3, we deduce $p = v'(z)$. This implies the assertion.

Step 5 We set $u = v'$. Since it holds that

$$\beta v(z_n) = \frac{1}{2}\sigma^2 z_n^2 u'(z_n) + (f(z_n) - \mu z_n)u(z_n) + \int \{v(z_n + \gamma(z_n,\zeta) - v(z_n) - u(z_n) \cdot \gamma(z_n,\zeta)\}\mu(d\zeta) + \tilde{U}(u(z_n))\,, \quad z_n \in \mathcal{D}\,,$$

the sequence $\{u'(z_n)\}$ converges uniquely as $z_n \to z \in (0,\infty) \setminus \mathcal{D}$, and u is Lipschitz near z (cf. [44]). We recall that $\partial u(z)$ coincides with the convex hull of the set, that is,

$$\partial u(z) = \left\{q \in \mathbf{R} : q = \lim_{n\to\infty} u'(z_n), z_n \in \mathcal{D} \to z\right\}\,.$$

Then,

$$\beta v(z) = \frac{1}{2}\sigma^2 z^2 q + (f(z) - \mu z)u(z) + \int \{v(z + \gamma(z,\zeta)) - v(z) - u(z) \cdot \gamma(z,\zeta)\}\mu(d\zeta) + \tilde{U}(u(z))\,, \quad \forall q \in \partial u(z)\,.$$

Hence, we observe that $\partial u(z)$ is a singleton, and then $u(z)$ is differentiable at z. The continuity of $u'(z)$ follows immediately as above. Thus, we conclude that $u \in$

$C^1((0,\infty))$, and hence $(0,\infty) \setminus \mathcal{D}$ is empty.

Step 6 Suppose $v'(z) \le 0$. Then, by *(B.3)*, we have

$$\tilde{U}(v'(z)) = \infty .$$

This is contrary to the equality $(**)$ in view of the conclusion in Step 3. Thus, we deduce $v'(0+) > 0$, and by *(B.3)* again, $v'(0+) = \infty$. □

4.2.4 Optimal consumption

In this subsection, we combine the optimal consumption policy $\mathbf{c}^*$ with the optimisation problem (2.3). We assume $\sigma > 0$ throughout this subsection in order to guarantee the existence of a C^2-solution.

Lemma 4.11. *Under (B.3),* (2.2)*, (B.4) and* (2.17)*, we have*

$$\liminf_{t\to\infty} E[e^{-\beta t} v(z(t))] = 0 \tag{2.33}$$

for every $\mathbf{c} \in \mathcal{C}$.

Proof. By (2.15), we can choose $0 < \beta' < \beta$ such that

$$-\beta'\varphi(z) + \frac{1}{2}\sigma^2 z^2 \varphi''(z) + (f(z) - \mu z)\varphi'(z) \tag{2.34}$$

$$+ \int \{\varphi(z + \gamma(z,\zeta)) - \varphi(z) - \varphi'(z)\cdot\gamma(z,\zeta)\}\mu(d\zeta) + \tilde{U}(\varphi'(z)) \le 0. \qquad z \ge 0.$$

Here, $\varphi(z) = z + B$. Hence, Itô's formula gives

$$E[e^{-\beta t}\varphi(z(t))] \le \varphi(z) + E\left[\int_0^t e^{-\beta s}(-\beta + \beta')\varphi(z(s))ds\right] ,$$

from which

$$(\beta - \beta')E\left[\int_0^\infty e^{-\beta s}\varphi(z(s))ds\right] < \infty .$$

Thus,

$$\liminf_{t\to\infty} E[e^{-\beta t}\varphi(z(t))] = 0 .$$

By Theorem 4.3,

$$v(z) \le \varphi(z) ,$$

which implies (2.33). □

Now, we consider the equation:

$$dz^*(t) = (f(z^*(t)) - \mu z^*(t) - c^*(t))dt + \sigma z^*(t)dW(t) + z^*(t-)\int(e^\zeta - 1)\tilde{N}(dtd\zeta)\,, \quad z^*(0) = z > 0\,, \qquad (2.35)$$

where

$$c^*(t) = (U')^{-1}(v'(z^*(t-)))1_{\{t\le\tau_{z^*}\}}\,. \qquad (2.36)$$

Here, $c^*(t)$ is the consumption rate which maximises the Hamilton function (associated with L) operated to v. For a typical case $U(c) = \frac{1}{r}c^r$ ($r \in (0, 1)$),

$$c^*(t) = (v'(z^*(t-)))^{\frac{1}{r-1}}.\ 1_{\{t\le\tau_{z^*}\}}\,.$$

Lemma 4.12. *Under (B.3),* (2.2)*, (B.4) and* (2.17)*, there exists a unique solution* $z^*(t) \ge 0$ *of* (2.35).

Proof. Let $G(z) = f(z^+) - \mu z - (U')^{-1}(v'(z^+))$. Here, $z^+ = \max(z, 0)$. Since $G(z)$ is Lipschitz continuous and $G(0) = 0$, there exists a solution $\chi(t)$ of

$$d\chi(t) = G(\chi(t))dt + \sigma\chi(t)dW(t) + \chi(t-)\int(e^\zeta - 1)\tilde{N}(dtd\zeta), \quad \chi(0) = z > 0\,. \qquad (2.37)$$

Define

$$z^*(t) \equiv \chi(t \wedge \tau_\chi) \ge 0\,.$$

We note by the concavity of f that $G(z) \le C_1z^+ + C_2$ for some $C_1, C_2 > 0$. Then, we apply the comparison theorem (Lemma 4.9) to (2.37) and to $\hat{z}(t)$ given by

$$d\hat{z}(t) = (C_1\hat{z}(t)^+ + C_2)dt + \sigma\hat{z}(t)dW(t) + \hat{z}(t-)\int(e^\zeta - 1)\tilde{N}(dtd\zeta)\,,$$
$$\hat{z}(0) = z > 0 \qquad (2.38)$$

to obtain $0 \le z^*(t) \le \hat{z}(t)$ for all $0 < t < \tau_{z^*} \wedge \tau_{\hat{z}}$. Furthermore, it follows from the definition that $\tau_{z^*} = \tau_\chi$, and hence

$$dz^*(t) = 1_{\{t\le\tau_{z^*}\}}\Big[f(z^*(t)) - \mu z^*(t) - c^*(t)dt + \sigma z^*(t)dW(t) + z^*(t-)\int(e^\zeta - 1)\tilde{N}(dtd\zeta)\Big]\,.$$

Therefore, $z^*(t)$ solves (2.35).

To prove the uniqueness, let $\tilde{z}(t)$ be another solution of (2.35). We notice that the function $z \mapsto G(z)$ is locally Lipschitz continuous on $(0, \infty)$. By using Theorem 1.5 in Section 1.3, we can get

$$z^*(t \wedge \tau_{z^*} \wedge \tau_{\tilde{z}}) = \tilde{z}(t \wedge \tau_{z^*} \wedge \tau_{\tilde{z}})\,.$$

Hence,

$$\tau_{z^*} = \tau_{\tilde{z}}, \qquad z^*(t) = \tilde{z}(t) \quad \text{for} \quad t \le \tau_{z^*}\,.$$

This implies $z^*(t) = \tilde{z}(t)$ for all $0 \le t < \tau_{z^*}$. ☐

We assume the SDE (2.35) has a strong solution. As it will be expected from (2.6), the optimal consumption policy is given as follows.

Theorem 4.6. *Let the assumptions of Theorem* 4.5 *hold. Then, an optimal consumption* $\mathbf{c}^* = \{c^*(t)\}$ *is given by* (2.36).

Proof. We set $\zeta = s \wedge \tau_{z^*}$ for any $s > 0$, and let $\zeta_n = \zeta \wedge \inf\{t > 0; |z^*(t)| < \frac{1}{n} \text{ or } |z^*(t)| > n\}$. By (2.6), (2.35) and Itô's formula, we have

$$
\begin{aligned}
&E[e^{-\beta\zeta_n} v(z^*(\zeta_n-))] \\
&= v(z) + E\Big[\int_0^{\zeta_n} e^{-\beta t}\Big\{-\beta v(z) + v'(z) 1_{\{t<\tau_{z^*}\}}[f(z) - \mu z - c^*(t)] + \frac{1}{2}\sigma^2 z^2 v''(z) \\
&\quad + \int \{v(z + \gamma(z,\zeta)) - v(z) - v'(z)\cdot\gamma(z,\zeta)\}\mu(d\zeta)\Big|_{z=z^*(t)}\Big\} dt + M(\zeta_n)\Big] \\
&= v(z) - E\left[\int_0^{\zeta_n} e^{-\beta t} U(c^*(t)) dt\right],
\end{aligned}
$$

where $\zeta_n = \zeta \wedge \tau_n$ for some localising sequence of stopping times $\tau_n \uparrow \infty$ of the local martingale $M(t)$ with $M(0) = 0$. From (2.34), (2.38) and Doob's inequalities for martingales, it follows that

$$
\begin{aligned}
&E\left[\sup_n e^{-\beta\zeta_n} v(z^*(\zeta_n-))\right] \\
&\quad \le E\left[\sup_{0\le r\le s} e^{-\beta r}\varphi(z^*(r-))\right] \\
&\quad \le \varphi(z) + 2C\Bigg(\sigma E\left[\int_0^s (e^{-\beta t} z^*(t))^2 dt\right]^{1/2} \\
&\qquad + E\Bigg[\int_0^s \int e^{-2\beta t}|\varphi(z^*(t-) + \gamma(z^*(t-),\zeta)) \\
&\qquad - \varphi(z^*(t-)) - \varphi'(z^*(t-))\cdot\gamma(z^*(t-),\zeta)|^2 \mu(d\zeta) dt\Bigg]^{1/2}\Bigg) \\
&\quad \le \varphi(z) + 2C\Bigg(\sigma E\left[\int_0^s \hat{z}(t)^2 dt\right]^{1/2} + E\Bigg[\int_0^s \int e^{-2\beta t}|\varphi(\hat{z}(t-) + \gamma(\hat{z}(t-),\zeta)) \\
&\qquad - \varphi(\hat{z}(t-)) - \varphi'(\hat{z}(t-))\cdot\gamma(\hat{z}(t-),\zeta)|^2 \mu(d\zeta) dt\Bigg]^{1/2}\Bigg) < +\infty .
\end{aligned}
$$

Therefore, letting $n \to \infty$, by the dominated convergence theorem, we get

$$E[e^{-\beta\zeta}v(z^*(\zeta-))] = v(z) - E\left[\int_0^{\zeta} e^{-\beta t}U(c^*(t))dt\right] .$$

By (2.33), we have

$$\begin{aligned}&\liminf_{s\to\infty} E[e^{-\beta(s\wedge\tau_{z^*})}v(z^*(s\wedge\tau_{z^*}-))]\\&= \liminf_{s\to\infty} E[e^{-\beta s}v(z^*(s-)) : \tau_{z^*} \geq s] = 0 .\end{aligned}$$

Passing to the limit, we deduce

$$v(z) = E\left[\int_0^{\tau_{z^*}} e^{-\beta t}U(c^*(t))dt\right] = J(\mathbf{c}^*) .$$

By the same calculation as above, we can obtain

$$J(\mathbf{c}) = E\left[\int_0^{\tau_z} e^{-\beta t}U(c(t))dt\right] \leq v(z), \quad \mathbf{c} \in \mathcal{C} .$$

Due to the comparison theorem (Theorem 4.3), we remark that the solution $v(z)$ of (2.6) must be unique. Hence, we have the assertion. The proof is complete. □

4.2.5 Historical sketch

The origin of the topic goes back to Merton [162]; See Appendix (ii). As for the jump-diffusion type control problem, the paper [212] has studied a model (X_t) given by

$$dX_t = \mu dt + \sigma dW(t) - dZ_t - dK_t, \quad X_{0-} = x .$$

Here, Z_t denotes a càdlàg (jump) process corresponding to the company's spending (paying *dividends* to the stock holders), $\sigma > 0$ and K_t corresponds to the activity to invest. The expected utility is measured up to time T in terms of Z_t by

$$E\left[\int_{\mathbf{T}} e^{-\beta t}dZ_t\right]$$

instead of the consumption rate $c(t)$ composed in the utility function. The paper [49] studies, in a similar framework, the control in switching between paying dividends and investments. The paper [74] studies the same model X_t, but the expected utility is measured by

$$E\left[U(\int_{\mathbf{T}} e^{-\beta t}dZ_t)\right] ,$$

where $U(.)$ is a utility function. Here, they measure the gross amount of dividends up to time T by $U(.)$ instead of the consumption rate $c(t)$, which makes some difference in the interpretation. In these papers, the main perturbation term is the diffusion (Brownian motion). On the other hand, Framstad [65] has studied a model (X_t) given by

$$dX_t = X_{t-}(\mu(X_t)dt + \sigma(X_t)dW(t) + \int \eta(X_{t-}, \zeta)\tilde{N}(dtd\zeta)) - dH_t$$

in the Wiener–Poisson framework. Here, H_t denotes a càdlàg process corresponding to the total amount of harvest up to time t, and $\sigma(.) \geq 0$. Under the setting that the expected utility is measured by

$$E\left[\int_{t_0}^{\infty} e^{-\beta t} dH_t\right],$$

he describes that the optimal harvesting policy is given by a single value x^*, which plays a role of a barrier at which one reflects the process downward. The paper [74] also leads to a similar conclusion. A related recent paper is [3].

For recent development concerning the Maximum principle in the jump-diffusion setting, see [176].

Appendix

> A friend of Diagoras tried to convince him of the existence of the gods, by pointing out how many votive pictures tell about people being saved from storms at sea by dint of vows to the gods, to which Diagoras replied that there are nowhere any pictures of those who have been shipwrecked and drowned at sea.
>
> Marcus Tullius Cicero (106–43, BC), De natura decorum III 89

(i) Notes

For basic materials which are supposed to be preliminary in this book a reader can see [39] and [194].

Chapter 1

Composition of Chapter 1 is compact. For details and proofs on basic materials concerning stochastic processes, stochastic integrals, and SDE of diffusion type, see the classical [82]. The materials in Section 1.1 are well known. See [196]. The materials in Section 1.2 are mainly due to [127, 192] and [47]. The materials in Section 1.3 are also due to [127]. For the precise on the stochastic flow appeared in Section 1.3, see [4] Section 6.4. Additionally, the equivalence between (3.2) and (3.3) is justified by Theorem 6.5.2 in [4]. For SDEs on manifolds, see [4, 58].

Chapter 2

There are two approaches to stochastic processes. One is based on macroscopic view using functional analytic approach, by e.g., Kolmogorov, Malliavin–Stroock; the other is based on microscopic view tracking each trajectory, by e.g., Itô, Bismut and Picard. We adopted the latter one.

Section 2.1.1 is due to [87]. Section 2.1.2 is due to [181, 182] and [95]. A part of Section 2.1.3 is due to [56] and to [8, 9]. Historically, the Bismut's method (2) stated in Section 2.1.3 was developed before the one stated in Section 2.1.1. It has evolved to cover the variable coefficient case in a fairly general setting. See [116]. There are some attempts to analyse jump processes as boundary processes (e.g. [32]), however these results have not been much fruitful.

Sections 2.2, 2.3 are based on [85, 89, 143, 183] and [184]. Historically, the polygonal method stated in Section 2.3 was developed before the Markov chain method in Section 2.2. In those days the author did not know the idea to seperate the big jump part of the driving Lévy process by the magnitude $O(t^{1/\beta})$, and used simply the constant threshold ϵ. The proof of Sublemma 2.3 in Section 2.2 is closely related to the strict positivity of the densities for infinitely divisible random variables, mentioned at the end of Section 2.5.4. Upper bound of the density for not necessary short, but still bounded, time interval is given in [185].

Section 2.5.2 is adapted from [97]. As for numerical results stated in section 2.5.3, calculating the constant $C = C(x, y, \kappa)$ in Theorem 2.4 numerically is not an easy task in general. Especially if the function $g(x, y)$ has singular points, numerical calculation of iterated integrals in (3.6) is known to be heavy. Section 2.5.4 is due to [90, 91].

Chapter 3

For details on the iterated integral expressions of Wiener (-Poisson) variables treated in Section 3.1, see [190]. For the Charlier polynomials stated in Section 3.2, also see [190]. As for the well-definedness of multiple integrals with respect to Lévy processes, see [112].

The contents in Section 3.3 are mainly based on [95]. The definition of the norms $|F|_{k,l,p}$, $\|U\|_{k,l,p}$ in Section 3.3.3 is based on the idea that we take the projection of the Wiener factor of F first, and then measure the remaining Poisson factor after, both in the sense of L^p norm. We can consider other norms as well on the Wiener–Poisson space. Indeed, by taking another Sobolev norm for the Wiener factor, it is possible that the framework of showing the estimate for $\tilde{\delta}(V)$ can be carried over to show the estimate of $\delta(U)$, where it has been carried out using Meyer's type inequality (Theorem 3.3) in the framework of Ornstein–Uhlenbeck semi-group. Up to present, there seems to exist no decisive version of the definition for the norms on it. Section 3.4 is due to [25].

Our theory on *stochastic* analysis stated in Sections 3.5–3.7, and basic properties of Lévy processes stated in Chapter 1, are deeply related with the theory of Fourier analysis. The contents of Sections 3.5, 3.7 are due to [77, 95, 130]. In Section 3.5.1 the iteration procedure using the functional $e_v(F) = e^{i(v,F)}$ works effectively. It will be another virtue of Malliavin calculus, apart from the integration-by-parts formula in the Wiener space. Indeed, as stated in the latter half of Section 3.6.5, employing the Fourier method it has turned out that the integration-by-parts formula is not a main tool in showing the smoothness of the transition density.

The origin of the theory stated in Sections 3.5–3.7 is [181]. Between [95] and [77] there has been a quest for finding a proper object in the Poisson space, which may correspond to Malliavin matrix in the Wiener space. The theory is also closely related to [170] Section 2.1.5. A related topic is treated in [173] Chapter 14.

The contents of Section 3.6 are inspired by [130]. The result can be extended to the non-Markovian process:

$$X_t = \int_0^t b(X_r, r)dr + \int_0^t \sigma(X_r, r)dW_r + \int_0^t \int g(X_r, r, z)\tilde{N}(drdz) .$$

The paper [130] tries to analyse SDEs of this type. The method used in Section 3.6.4 is new.

The former part of Section 3.6.5 is mainly due to [131]. This part is added in the second edition, so the SDE here is based on the setting that the vector fields depend on t. There are cases where the density function exists despite that the stochastic coefficients are fairly degenerate and the (ND) condition fails for the solution of a SDE. We are often required delicate estimates in such cases, see e.g., [50]. The extension of the theory stated in the former half of this subsection to the case for other types of Lévy measures is not well known. The inclusion of this subsection is also due to author's recent interest in Itô process on manifolds, as Itô process with big jumps does not satisfy the condition **(D)** in general.

Chapter 4

Section 4.1 is based on [77]. A related work is done by Yoshida [223] and by Kunitomo and Takahashi [132, 133]. The result in Section 4.1.2 is new. Section 4.1.3 is also new.

Section 4.2 is based on [96, 99]. The existence of 'trace' at the boundary for the value function can be justified analytically using the theory of pseudo-differential operators [206]. The control theory with respect to jump processes is developing. See, e.g. [110] and Kabanov's recent papers.

During 2013–2015 the author and Dr. Yamanobe considered an application of Malliavin calculus to stochastic action potential model in axons [100]. But the content is not included in this volume.

(ii) Ramsey theory

We are concerned with a one-sector model of optimal economic growth under uncertainty in the neoclassical context. It was originally presented by Ramsey [193] and has been refined by Cass [41]. See also [63].

Define the following quantities:

$$\begin{aligned} y(t) &= \text{ labour supply at time } t \geq 0\ , \\ x(t) &= \text{capital stock at time } t \geq 0\ , \\ \lambda &= \text{ the constant rate of depreciation (written as } \beta \text{ in the text)},\ \lambda \geq 0\ , \\ F(x, y) &= \text{production function producing} \\ &\qquad \text{the commodity for the capital stock } x \geq 0 \text{ and the labour force } y \geq 0\ . \end{aligned}$$

We now state the setting in the model. Suppose that the labour supply $y(t)$ at time t, and the capital stock $x(t)$ at t are governed by the following stochastic differential equations

$$dy(t) = ry(t)dt + \sigma y(t)dW(t)\,, \qquad y(0) = y > 0, \quad r \neq 0,\ \sigma \geq 0\,, \tag{A.1}$$

$$\begin{aligned} dx(t) &= (F(x(t), y(t)) - \lambda x(t) - c(t)y(t))dt + x(t-) \int_{|\zeta|<1} (e^{\zeta} - 1)\tilde{N}(dtd\zeta) \\ &\quad + x(t-) \int_{|\zeta|\geq 1} (e^{\zeta} - 1)N(dtd\zeta)\,, \quad x(0) = x > 0\,. \end{aligned} \tag{A.2}$$

An intuitive explanation is as follows. Suppose there exists a farm (or a company, a small country, ...) that makes economical activities based on labour and capital. At time t, the farm initiates production which is expressed by $F(x(t), y(t))$. At the same time, he or she has to consume the capital at the rate $c(t)$, and the capital may depreciate as time goes by. The second and third terms on the right-hand side of (A.2) correspond to a random fluctuation in capital.

Under this interpretation, we may even assume that the labour supply $y(t)$ is constant, and consider the optimal consumption under the randomly fluctuated capital. This assumption has projected economical meaning since the labour supply is not easy to control (i.e. make decrease) in short time.

This type of problem is called the stochastic Ramsey problem. For reference, see [70]. We denote by $V(x, y)$ the value function associated with $x(t)$ and $y(t)$. The HJB equation associated with this problem reads as follows:

$$\begin{aligned} \beta V(x, y) &= \frac{1}{2}\sigma^2 y^2 V_{yy}(x, y) + ryV_y(x, y) + (F(x, y) - \lambda x)V_x(x, y) \\ &\quad + \int \{V(e^{\zeta}x, y) - V(x, y) - V_x(x, y) \cdot x(e^{\zeta} - 1)\}\mu(d\zeta) \\ &\quad + \tilde{U}(V_x(x, y)y)\,, \qquad V(0, y) = 0\,, \quad x > 0\,, \quad y > 0\,. \end{aligned} \tag{A.3}$$

We seek the solution $V(x, y)$ in a special form, namely,

$$V(x, y) = v(z), \quad z = \frac{x}{y}\,. \tag{A.4}$$

Then, (A.3) turns into the form (2.6)–(2.7) in Section 4.2 with $f(z) = F(z^+, 1)$, $\mu = r - \tilde{r} + \lambda - \sigma^2$. Here, we assume the homogeneity of F with respect to y: $F(x, y) = F(\frac{x}{y}, 1)y$. We can show that $V(x, y)$ defined as above is a viscosity solution of (A.3), and that if $\sigma > 0$, then it is in $C^2((0, \infty) \times (0, \infty))$. This derives the optimal consumption function in a way of Section 4.2.4. The condition $z(t) \geq 0$, $t \geq 0$ a.s. implies that the economy can sustain itself.

Although it is not implied in general that we can find the solution of (A.3) as in (A.4) in the setting (A.1)–(A.2), under the same token as above (i.e. the labour supply is constant), the problem will correspond to what we have studied in the main text. That is, we can identify the value function in the form $v(z) = V(z, 1)$.

Bibliography

[1] Aase K, Øksendal B, Privault N, Uboe J. White noise generalizations of the Clark–Haussmann–Ocone theorem with application to mathematical finance. Finance Stoch 2001, 4, 465–496.
[2] Alexandroff A. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it (Russian), Leningrad State Univ Annals [Uchenye Zapiski] Math 1939, 6, 3–35.
[3] Albrecher H, Lautscham V. From ruin to bankruptcy for compound Poisson surplus processes. ASTIN Bulletin 2013, 43, 213–243.
[4] Applebaum D. Lévy processes and stochastic calculus. 2nd edn. Cambridge, UK, Cambridge University Press, 2009.
[5] Arisawa M. A remark on the definitions of viscosity solutions for the integro-differential equations with Lévy operators, J Math Pures Appl 2008, 89, 567–574.
[6] Albeverio S, Kondratiev Yu, Röckner M. Analysis and geometry on configuration space. J Funct Anal 1998, 157, 242–291.
[7] Azencott R. Grand deviations et applications. In: Henneqin PL, ed. Ecole d'Eté de Probabilités de Saint-Flour 8, Lecture Notes in Math 774. Berlin, Germany, Springer-Verlag, 1980, 1–176.
[8] Bally V, Bavouzet M-P, Messaoud M. Integration by parts formula for locally smooth laws and applications to sensitivity computations. Ann App Probab 2007, 17, 33–66.
[9] Bally V, Clément E. Integration by parts formula and application to equations with jumps. Probab Theory Relat Fields 2011, 151, 613–657.
[10] Bardi M, Capuzzo-Dolcetta I. Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations. Boston, USA, Birkhauser, 1997.
[11] Barles G, Imbert C. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. Ann Inst H Poincaré Anal Non Linéaire 2008, 25, 567–585.
[12] Barlow MT, Nualart D. Lectures on probability theory and statistics. In: Bernard P, ed. Lectures from the 25th Saint-Flour Summer School, Lecture Notes in Mathematics 1690. Berlin, Germany, Springer-Verlag, 1998, 123–227.
[13] Bass R. Stochastic differential equations driven by symmetric stable processes. In: Amema J, Émery M, Ledoux M, Yor M, eds. Séminaire de Probabilites 36, Lecture Notes in Mathematics 1801. Berlin, Germany, Springer-Verlag, 2003, 302–313.
[14] Bass R. Stochastic processes. Cambridge, UK, Cambridge University Press, 2011.
[15] Bass R, Cranston M. The Malliavin calculus for pure jump processes and applications to local time. Ann Probab 1986, 14, 490–532.
[16] Ben Arous G. Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale. Ann Inst Fourier Grenoble 1989, 39, 73–99.
[17] Ben Arous G. Flots et series de Taylor stochastiques. Probab Th Rel Fields 1989, 81, 29–77.
[18] Ben Arous G, Léandre R. Décroissance exponentielle du noyau de la chaleur sur la diagonale (I). Probab Th Rel Fields 1991, 90, 175–202.
[19] Ben Arous G, Léandre R. Décroissance exponentielle du noyau de la chaleur sur la diagonale (II). Probab Th Rel Fields 1991, 90, 377–402.
[20] Benth F, Karlsen K, Reikvam K. Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach. Finance Stoch 2001, 5, 275–303.
[21] Benth F, Karlsen K, Reikvam K. Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution. Finance Stoch 2001, 5, 447–467.
[22] Berg C, Frost G. Potential theory on locally compact Abelian groups. Berlin, Germany, Springer-Verlag, 1975.

[23] Bertoin J. Lévy processes. Cambridge, UK, Cambridge University Press, 1996.

[24] Bertoin J, Lindner A, Maller RA. On continuity properties of the law of integrals of Lévy process. In: Donati-Martin C, Émery M, Rouault A, Stricker C, eds. Séminaire de Probabilites 41, Lecture Notes in Mathematics 1934. Berlin, Germany, Springer-Verlag, 2008, 137–159.

[25] Bichteler K, Gravereaux JM, Jacod J. Malliavin Calculus for Processes with Jumps. New York, USA, Gordon and Breach Science Publishers, 1987.

[26] Bichteler K. Stochastic integration with jumps. Encyclopedia of Mathematics and its Applications 89. Cambridge, UK, Cambridge University Press, 2002.

[27] Bichteler K, Jacod J. Calcul de Malliavin pour les diffusions avec sauts: existence d'une densite dans le case unidimensionel. In: Azéma J, Yor M, eds. Séminaire de Probabilités 17, Lecture Notes in Math 986. Berlin, Germany, Springer-Verlag, 1983, 132–157.

[28] Billingsley P. Convergence of Probability Measures. New York, USA, Wiley-Interscience, 1999.

[29] Bismut JM. Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Z Wahrsch Verw Gebiete 1981, 56, 469–505.

[30] Bismut JM. Calcul des variations stochastiques et processus de sauts. Z Wahrsch Verw Gebiete 1983, 63, 147–235.

[31] Bismut JM. Large deviation and the Malliavin calculus. In: Coates J, Helgason S, eds. Progress in Mathematics 45. Birkhäuser, Boston, USA, 1984.

[32] Bismut JM. Jump processes and boundary processes. In: Itô K, ed. Proceeding Taniguchi Symp 1982. Tokyo, Japan, North-Holland/Kinokuniya, 1984, 53–104.

[33] Blumenthal RM, Getoor RK. Sample functions of stochastic processes with stationary independent increments. J Math Mech 1961, 10, 493–516.

[34] Bony JM, Courrège P, Priouret P. Semi-group de Feller sur une variété à bord compacte et problème aux limites intégro-différentiels du second ordre donant lieu au principle du maximum. Ann Inst Fourier (Grenoble) 1968, 18, 369–521.

[35] Bony JM. Fourier integral operators and Weyl–Hörmander calculus. In: Bony JM, Morimoto M, eds. New trends in microlocal analysis. Tokyo, Japan, Springer, 1997, 3–21.

[36] Borwein JM, and Noll D. Second order differentiability of convex functions in Banach spaces. Trans Amer Math Soc 1994, 342, 43–81.

[37] Bouleau N, Denis L. Application of the lent particle method to Poisson-driven SDEs. Probab Theory Related Fields 2011, 151, 403–433.

[38] Bouleau N, Hirsch F. Dirichlet forms and analysis on Wiener space. Berlin, Germany, Walter de Gruyter, 1991.

[39] Breiman L. Probability. Classics in applied mathematics. Philadelphia, USA, SIAM, 1992.

[40] Byczkowski T, Graczyk P. Malliavin calculus for stable processes on Heisenberg group. Probab Math Statist 1992, 13, 277–292.

[41] Cass D. Optimal growth in an aggregative model of capital accumulation. Review of economic studies 1965, 32, 233–240.

[42] Castell F. Asymptotic expansion of stochastic flows. Probab Th Rel Fields 1993, 96, 225–239.

[43] Carlen E, Pardoux E. Differential calculus and integration by parts on Poisson space. Math Appl 1990, 59, 63–73.

[44] Clarke FH. Optimization and Nonsmooth Analysis. New York, USA, John Wiley, 1983.

[45] Courrège Ph. Noyaux de convolution singuliers opérant sur les fonctions höldériennes et noyaux de convolution régularisants. Séminaire de Probabilités 1. Strassburg, France, Univ. Strassburg, 1967, 34–51.

[46] Crandall MG, Ishii H, Lions PL. User's guide to viscosity solutions of second order partial differential equations. Bull Amer Math Soc 1992, 27, 1–67.

[47] Dana RA, Jeanblanc M. Financial markets in continuous time. Translated from the 1998 French original by Anna Kennedy, Springer Finance. Berlin, Germany, Springer-Verlag, 2003.

[48] Davydov YA, Lifshits MA, Smorodina NV. Local properties of distributions of stochastic functionals. Translations Math Monographs 173. Providence, USA, AMS, 1998.
[49] Décamps JP, Villeneuve S. Optimal dividend policy and growth option. Finance Stoch 2007, 11, 3–27.
[50] Delarue F, Menozzi S. Density estimates for a random noise propagating through a chain of differential equations. J Funct Anal 2010, 259, 1577–1630.
[51] Dellacherie C, Maisonneuve B, Meyer PA. Probabilités et Potentiel. Paris, France, Hermann, 1992.
[52] Doss H, Priouret P. Petites perturbations de systèmes dynamiques avec réflexion. In: Azéma J, Yor M, eds. Séminaire de Proba 17, Lecture Notes in Math 986. Berlin, Germany, Springer-Verlag, 1983, 353–370.
[53] Dudley R, Norvaisa R. Differentiability of six operators on nonsmooth functions and *p*" variation. Lecture Notes in Mathematics 1703. Berlin, Germany, Springer-Verlag, 1999.
[54] Dupis P, Ellis RS, Weiss A. Large deviations for Markov processes with discontinuous statistics, I: General upper bounds. Ann Probab 1991, 19, 1280–1297.
[55] Dziubański J. Asymptotic behaviour of densities of stable semigroups of measures. Probab Th Rel Fields 1991, 87, 459–467.
[56] Elliot RJ, Tsoi AH. Integration by parts for Poisson processes. J Multivariate Anal 1993, 44, 179–190.
[57] Erickson KB, Maller RA. Finiteness of integrals of functionals of Lévy processes. Proc London Math Soc 2007, 94, 386–420.
[58] Émery M. Stochastic calculus in manifolds. Universitext. Berlin, Germany, Springer-Verlag, 1989.
[59] Ethier SN, Kurtz TG. Markov Processes. New York, USA, John Wiley and Sons, 1986.
[60] Fefferman C, Phong DH. Subelliptic eigenvalue problems. In: Beckner W, ed. Proceeding of the Conference on harmonic analysis in honor of Antonio Zygmund. Belmont, USA, Wadsworth, 1983, 590–606.
[61] Fleming WH, Soner HM. Controlled Markov Processes and Viscosity Solutions. New York, USA, Springer-Verlag, 1993.
[62] Fleming WH, Soner HM. Asymptotic expansion for Markov processes with Lévy generators. Applied Math Opt 1989, 19, 203–223.
[63] Foldes L. The optimal consumption function in a Brownian model of accumulation Part A: The consumption function as solution of a boundary value problem. J Econ Dynamics and Control 2001, 25, 1951–1971.
[64] Fournier N. Smoothness of the law of some one-dimensional jumping S.D.E.s with non-constant rate of jump. Electron J Probab 2008, 13, 135–156.
[65] Framstad NC. Optimal harvesting of a jump diffusion population and the effect of jump uncertainty. SIAM J Control Optim 2003, 42, 1451–1465.
[66] Freidlin MI, Ventcel AD. Random perturbation of dynamical system. Grundlehren der math. Wissenschaften 260. Berlin, Germany, Springer-Verlag, 1984.
[67] Fujiwara T, Kunita H, Stochastic differential equations for jump type and Lévy processes in endomorphisms group, J Math Kyoto Univ 1985, 25, 71–106.
[68] Fujiwara T, Kunita H. Canonical SDE's based on semimartingales with spatial parameters; Part I Stochastic flows of diffeomorphisms, Part II Inverse flows and backwards SDE's. Kyushu J Math 1999, 53, 265–300; 301–331.
[69] Fujiwara T, Miyahara Y. The minimal entropy martingale measures for geometric Lévy processes. Finance Stoch 2003, 7, 509–531.
[70] Gapen T, Cosimano F. Solving Ramsey problems with nonlinear projection models. Stud Nonlinear Dyn Econom 2005, 9, Article 3.

[71] Garroni NG, Menaldi JL. Green functions for second order parabolic integro-differential problems. Essex, UK, Longman Scientific & Technical, 1992.
[72] Geman D, Horowitz J. Occupation times for smooth stationary processes. Ann Probab 1973, 1, 131–137.
[73] Graczyk P. Malliavin calculus for stable processes on homogeneous groups. Studia Math 1991, 100, 183–205.
[74] Grandits P, Hubalek F, Schachmayer W, Zigo M. Optimal expected exponential utility of dividend payments in a Brownian risk model. Scand Actuar J 2007, 2007, 73–107.
[75] Guo X, Pham H. Optimal partially reversible investment with entry decision and general production function. Stochastic Process Appl 2005, 115, 705–736.
[76] Hayashi M. Asymptotic expansion for functionals of Poisson random measure. PhD Thesis. Kobe, Japan, Kobe University, 2008.
[77] Hayashi M, Ishikawa Y. Composition with distributions of Wiener–Poisson variables and its asymptotic expansion. Math Nach 2012, 285, 619–658.
[78] Hewitt E, Stromberg K. Real and abstract analysis. New York, USA, Springer-Verlag, 1969.
[79] Hoh W, Jacob N. Some Dirichlet forms generated by pseudo differential operators. Bull Sci Math 1992, 116, 383–398.
[80] Hopfner R. An extension of the Yamada–Watanabe condition for pathwise uniqueness to stochastic differential equations with jumps. Elect Comm in Probability 2009, 14, 447–456.
[81] Hudson WN, Mason JD. Variational sums for additive processes. Proc Amer Math Soc 1976, 55, 395–399.
[82] Ikeda N, Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd edn. Amsterdam, Netherlands, North-Holland, 1989.
[83] Ishikawa Y. A small example of non-local operators having no transmission property. Tsukuba J Math 2001, 25, 399–411.
[84] Ishikawa Y. On the lower bound of the density for jump processes in small time. Bull. Sc. math 1993, 117, 463–483.
[85] Ishikawa Y. Asymptotic behavior of the transition density for jump-type processes in small time. Tohoku Math J 1994, 46, 443–456.
[86] Ishikawa Y. On the upper bound of the density for truncated stable processes in small time. Potential Analysis 1997, 6, 11–37.
[87] Ishikawa Y. Large deviation estimate of transition densities for jump processes. Ann IHP Probabilités 1997, 33, 179–222.
[88] Ishikawa Y, Léandre R. Diagonal estimates of transition densities for jump processes in small time. In: Ustunel A, ed. Stochastic analysis and related topics 6, Progr Probab 42. Basel, Swiss, Birkhäuser, 1998, 251–273.
[89] Ishikawa Y. Density estimate in small time for jump processes with singular Lévy measures. Tohoku Math J 2001, 53, 183–202.
[90] Ishikawa Y. Support theorem for jump processes of canonical type. Proc Japan Acad 2001, 77, 79–83.
[91] Ishikawa Y. Existence of the density for a singular jump process and its short time properties. Kyushu J Math 2001, 55, 267–299.
[92] Ishikawa Y. Density estimate in small time for jump processes with singular Lévy measures. Funct Differ Equ 2001, 8, 273–285.
[93] Ishikawa Y. Exponential type decrease of the density for jump processes with singular Lévy measures in small time. In: Capar U, Ustunel A, eds. Stochastic analysis and related topics 8, Progr Probab 53. Basel, Swiss, Birkhäuser, 2003, 129–149.

[94] Ishikawa Y. Analysis of jump processes and its application to optimal control. In: Akahori J, Ogawa S, Watanabe S, eds. Stochastic processes and applications to mathematical finance. River Edge, NJ, USA, World Scientific, 2004, 97–118.
[95] Ishikawa Y, Kunita H. Malliavin calculus on the Wiener–Poisson space and its application to canonical SDE with jumps. Stochastic Process. Appl 2008, 116, 1743–1769.
[96] Ishikawa Y. Optimal control problem associated with jump processes. Appl Math Optim 2004, 50, 21–65.
[97] Ishikawa Y, Simon Th. Absolute continuity and continuity of random variables concerning Lévy processes (in Japanese). Cooperative research report 213. Tokyo, Japan, Institute of Statistical Mathematics, 2008, 104–132.
[98] Ishikawa Y. Malliavin calculus applied to mathematical finance and a new formulation of the integration-by-parts. RIMS Kokyuroku 1620, Kyoto, Japan, Kyoto University, 2009, 67–80.
[99] Ishikawa Y. Optimal consumption control problem associated with jump-diffusion processes. Preprint, 2012.
[100] Ishikawa Y, Yamanobe T. Asymptotic expansion of a nonlinear oscillator with a jump-diffusion process. Preprint, 2015.
[101] Ishikawa Y, Kunita H, Tsuchiya M. Smooth density and its short time estimate for jump process determined by SDE. Preprint, 2015.
[102] Itô K. Multiple Wiener integral. J. Math. Soc. Japan 1951, 3, 157–169.
[103] Itô K. Spectral type of the shift transformation of differential processes with stationary increments. Trans Amer Math Soc 1956, 81, 253–263.
[104] Itô K. Kakuritsu-ron, I–III. (in Japanese) [Probability theory, I–III.] 2nd edn. Tokyo, Japan, Iwanami Shoten, 1983.
[105] Itô K. Stochastic Processes: Lectures given at Aarhus University, Berlin, Germany, Springer, 2004.
[106] Jacod J. Calcul stochastique et problèm des martingales. Lect Notes in Math 714. Berlin, Germany, Springer-Verlag, 1975.
[107] Jacod J. Equations diffrentielles stochastiques linaires: la methode de variation des constantes. In: Azéma J, Yor M, eds. Seminar on Probability 16, Lecture Notes in Math 920. Berlin, Germany, Springer, 1982, 442–446.
[108] Jacob N, Schilling R. Lévy-type processes and pseudodifferential operators. In: Barndorff-Nielsen O, ed. Lévy processes, Boston, USA, Birkhäuser, 2001, 139–168.
[109] Jacod J, Shiryaev AN. Limit theorems for stochastic processes. Berlin, Germany, Springer-Verlag, 1987.
[110] Kabanov Y, Safarian M. Markets with Transaction Costs. Berlin, Springer, 2009.
[111] Kallenberg O. Splitting at backward times in regenerative sets. Ann Probab 1981, 9, 781–799.
[112] Kallenberg O, Szulga J. Multiple integration with respect to Poisson and Lévy processes. Probab Th Rel Fields 1989, 83, 101–134.
[113] Kamien MI, Schwartz NL. Dynamic Optimization. 2nd edn. Amsterdam, Netherlands, North-Holland, 1991.
[114] Kim YJ, Kunitomo N. Pricing Options under Stochastic Interest Rates: A New Approach. Asia pacific financial markets 1999, 6, 49–70.
[115] Karatzas I, Shreve SE. Brownian motion and stochastic calculus. 2nd edn. New York, USA, Springer, 1991.
[116] Komatsu T, Takeuchi T. On the smoothness of pdf of solutions to SDE of jump type, Int. J. Diff. Eq. Appl. 2 (2001), 141–197.
[117] Komatsu T, Takeuchi T. Simplified probabilistic approach to the Hörmander theorem, Osaka J. Math. 38 (2001), 681–691.

[118] Kulik A. On a regularity of distributions for solution of SDE of a jump type with arbitrary Levy measures of the noise, Ukrain. Mat. Zh. 57 (2005), 1261–1283.

[119] Kulik A. Stochastic calculus of variations for general Lévy processes and its applications to jump-type SDEs with non-degenerate drift. arXiv:math (accessed October 1 2015, at http://arxiv.org/abs/math.PR/0606427), 2007.

[120] Kulik A. Absolute continuity and convergence in variation for distributions of functionals of Poisson point measure. J Theoret Probab 2011, 24, 1–38.

[121] Kunita H, Watanabe S. On square integrable martingales. Nagoya Math J 1967, 30, 209–245.

[122] Kunita H. Stochastic flows acting on Schwartz distributions. J Theor Prabab 1994, 7, 247–278.

[123] Kunita H. Stochastic differential equations with jumps and stochastic flows of diffeomorphisms. In: Ikeda N, ed. Itô's stochastic calculus and probability theory. Tokyo, Japan, Springer-Verlag, 1996, 197–211.

[124] Kunita H. Canonical SDEs based on Lévy processes and their supports. In: Cranel H, Gundlach M, eds. Stochastic Dynamics. Berlin, Germany, Springer-Verlag, 1999, 283–304.

[125] Kunita H. Stochastic flows with self-similar properties. In: Davies IM, Truman A, Elworthy KD, eds. Stochastic Analysis and Applications. Singapore, World Scientific, 1996, 286–300.

[126] Kunita H. Asymptotic self-similarity and short time asymptotics of stochastic flows. J Math Sci Univ Tokyo 1997, 4, 595–619.

[127] Kunita H. Stochastic differential equation based on Lévy processes and stochastic flows of diffeomorphisms. In: Rao, MM, ed. Real and stochastic analysis. Boston, USA, Birkhauser, 2004, 305–373.

[128] Kunita H. Smooth density of canonical stochastic differential equation with jumps, Astérisque 2009, 327, 69–91.

[129] Kunita H. Itô's stochastic calculus: its surprising power for applications. Stochastic Process. Appl. 2010, 120, 622–652.

[130] Kunita H. Analysis of nondegenerate Wiener–Poisson functionals and its application to Itô's type SDE with jumps. Sankhya 2011, 73-A, 1–45.

[131] Kunita H, Nondegenerate SDE's with jumps and ypoelliptic properties, J. Math. Soc. Japan 2013, 65, 993–1035.

[132] Kunitomo N, Takahashi A. The asymptotic expansion approach to the valuation of interest rate contingent claims. Math. Finance 2011, 11, 117–151.

[133] Kunitomo N, Takahashi A. Applications of the asymptotic expansion approach based on Malliavin–Watanabe calculus in financial problems. In: Akahori J, Ogawa S, Watanabe S, eds. Stochastic processes and applications to mathematical finance, River Edge, USA, World Sci Publ, 2004, 195–232.

[134] Kurtz TG, Pardoux E, Protter P. Stratonovich stochastic differential equations driven by general semimartingales. Ann Inst Henri Poincaré 1995, 31, 351–377.

[135] Kusuoka S. Theory and application of Malliavin calculus (in Japanese). Sugaku (Math Soc Japan) 1989, 41, 154–165.

[136] Kusuoka S. Stochastic analysis as infinite dimensional analysis (in Japanese). Sugaku (Math Soc Japan) 1994, 45, 289–298.

[137] Kusuoka S, Stroock D. Applications of the Malliavin calculus. I. In: Itô K, ed. Stochastic analysis. Amsterdam, Netherlands, North-Holland, 1984, 271–306.

[138] Kusuoka S, Stroock D. Applications of the Malliavin calculus. II. J Fac Sci Univ Tokyo Sect IA Math 1985, 32, 1–76.

[139] Kusuoka S, Stroock D. Applications of the Malliavin calculus. III. J Fac Sci Univ Tokyo Sect IA Math 1987, 34, 391–442.

[140] Léandre R. Densité en temps petit d'un processus de sauts. In: Azéma J, Meyer PA, Yor M, eds. Séminaire de Proba 21, Lecture Notes in Math 1247. Berlin, Germany, Springer-Verlag, 1987, 81–99.
[141] Léandre R. Minoration en temps petit de la densité d'une diffusion dégénérée, J Funct Anal 1987, 74, 399–414.
[142] Léandre R. Integration dans la fibre associée a une diffusion dégénérée, Probab Th Rel Fields 1987, 76, 341–358.
[143] Léandre R. Régularité de processus de sauts dégénérés (II). Ann Inst H Poincaré Probabilités 1988, 24, 209–236.
[144] Léandre R. Calcul des variations sur un brownien subordonné. In: Azema J, Meyer PA, Yor M, eds. Séminaire de Probabilites 22, Lect Notes in Math 1321, Berlin, Germany, Springer, 1988, 414–434.
[145] Léandre R. Dévelopment asymptotic de la densité d'une diffusion dégénérée. Forum Math 1992, 4, 45–75.
[146] Léandre R. Volume de sous-riemanniennes et explosion du noyau de la chaler au sens de Stein. In: Azema J, Meyer PA, Yor M, eds. Séminaire de Proba 23, Lecture Notes in Math 1372. Berlin, Germany, Springer-Verlag, 1989, 428–447.
[147] Léandre R. Applications quantitatives et geometriques du calcul de Malliavin. In: Métivier M, Watanabe S, eds. Stochastic Analysis, Lecture Notes in Math 1322. Berlin, Germany, Springer-Verlag, 1988, 109–133.
[148] Léandre R, Russo F. Estimation de Varadhan pour des diffusions à deux paramètres. Probab Th Rel Fields 1990, 84, 429–451.
[149] Léandre R. A simple proof for a large deviation theorem. In: Nualart D, ed. Proceeding of Barcelona Seminar on Stochastic Analysis, Progr Probab 32. Basel, Swiss, Birkhäuser, 1993, 72–76.
[150] Lepeltier JP, Marchal B. Probl'ešme des martingales et équations différentielles associées à un opérateur intégro-différentiel. Ann Inst H Poincaré Probabilités 1976, 12, 43–103.
[151] Lépingle D, Mémin J. Sur l'integrabilité des martingales exponentielles. Z Wahrsch Verw Gebiete 1978, 42, 175–203.
[152] Lévy P. Processus stochastiques et mouvement Brownien, Suivi d'une note de M. Loève. 2nd edn. Paris, France, Gauthier-Villars & Cie, 1965.
[153] Lifschits MA. An application of the stratification method to the study of functionals of processes with independent increments. Theory Probab Appl 1984, 29, 753–764.
[154] Løkka A. Martingale representation of functionals of Lévy processes. Stochastic Anal Appl 2004, 22, 867–892.
[155] Lungu EM, Øksendal B. Optimal harvesting from a population in a stochastic crowded environment. Math Biosci 1997, 145, 47–75.
[156] Lyons T, Qian Z. System Control and Rough Paths. Oxford, UK, Oxford Univ Press, 2003.
[157] Lyons T, Caruana M, Lévy T. Differential equations driven by rough paths. Lecture Notes in Math 1908. Berlin, Germany, Springer, 2007.
[158] Ma ZM, Röckner M, Zhang TS. Approximation of arbitrary Dirichlet processes by Markov chains. Ann Inst H Poincaré Probab Statist 1998, 34, 1–22.
[159] Malliavin P. Stochastic calculus of variation and hypoelliptic operators. In: Itô K, ed. Proceedings of the International Symposium on Stochastic Differential Equations. New York, USA, Wiley, 1978, 195–263.
[160] Malliavin P. Stochastic analysis. Berlin, Germany, Springer-Verlag, 1997.
[161] Malliavin P, Thalmaier A. Stochastic calculus of variations in mathematical finance. Berlin, Germany, Springer-Verlag, 2006.

[162] Merton RC. An asymptotic theory of growth under uncertainty. Rev Econ Studies 1975, 42, 375–393.
[163] Mizohata S. The theory of partial differential equations. Cambridge, UK, Cambridge Univ Press, 1973.
[164] Marcus SI. Modeling and approximation of stochastic differential equations driven by semimartingales. Stochastics 1980/81, 4, 223–245.
[165] Morimoto H. Stochastic Control and Mathematical Modeling. Cambridge, UK, Cambridge Univ. Press, 2010.
[166] Nagai H. Stochastic differential equation (in Japanese). Tokyo, Japan, Kyoritsu Shuppan, 1999.
[167] Nagel A, Stein EM, Wainger S. Balls and metrics defined by vector fields I: Basic properties. Acta Math 1985, 155, 103–147.
[168] Norris JR. Integration by parts for jump processes. In: Azéma J, Meyer PA, Yor M, eds. Séminaire de Proba. 22, Lecture Notes in Math 1321. Berlin, Germany, Springer-Verlag, 1988, 271–315.
[169] Nourdin I, Simon Th. On the absolute continuity of Lévy processes with drift. Ann Probab 2006, 34, 1035–1051.
[170] Nualart D. The Malliavin calculus and related topics. 2nd edn. Berlin, Germany, Springer-Verlag, 2006.
[171] Nualart D, Vives J. Anticipative calculus for the Poisson process based on the Fock space. In: Azéma J, Meyer PA, Yor M, eds. Séminaire de Probabilités 24, Lecture Notes in Math 1426. Berlin, Germany, Springer 1990, 154–165.
[172] Nualart D, Vives J. A duality formula on the Poisson space and some applications. In: Bolthausen E, Dozzi M, Russo R, eds. Seminar on Stochastic Analysis, Random Fields and Applications. Basel, Swiss, Birkhauser, 1995.
[173] Di Nunno G, Øksendal B, Proske F. Malliavin calculus for Lévy processes with applications to finance (Corr. 2nd printing). Berlin, Germany, Springer, 2009.
[174] Øksendal B. Stochastic differential equations. 6th edn. Berlin, Germany, Springer, 2007.
[175] Øksendal B, Sulem A. Applied Stochastic Control of Jump Processes. 2nd edn. Berlin, Germany, Springer-Verlag, 2007.
[176] Øksendal B, Sulem A. Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. SIAM J. Control Optim 2009, 48, 2945–2976.
[177] Ocone D. Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. Stochastics 1984, 12, 161–185.
[178] Orey S. On continuity properties of infinitely divisible functions. Ann Math Statist 1968, 39, 936–937.
[179] Peng S, Zhu X. Necessary and sufficient condition for comparison theorem of 1" dimensional stochastic differential equations. Stochastic Process Appl 2006, 116, 370–380.
[180] Pham H. Optimisation et Contrôle Stochastic Appliqués à la Finance. Berlin, Germany, Springer, 2007.
[181] Picard J. On the existence of smooth densities for jump processes. Probab Th Relat Fields 1996, 105, 481–511. Erratum: Probab Th Relat Fields 2010, 147, 711–713.
[182] Picard J. Formules de dualité sur l'espace de Poisson. Ann Inst Henri Poincaré 1996, 32, 509–548.
[183] Picard J. Density in small time for Lévy processes. ESAIM Probab Statist 1997, 1, 358–389 (electronic).
[184] Picard J. Density in small time at accessible points for jump processes. Stochastic Process Appl 1997, 67, 251–279.

[185] Picard J, Savona C. Smoothness of the law of manifold-valued Markov processes with jumps. Bernoulli 2013, 19(5A), 1880–1919.
[186] Prat JJ. Résolution de l'équation de Schrödinger par intégration stochastique du champ magnétique. Probab Th Rel Fields 1993, 95, 217–235.
[187] Privault N. Chaotic and variational calculus in discrete and continuous time for the Poisson process. Stochastics Stochastic Rep 1994, 51, 83–109.
[188] Privault N. A pointwise equivalence of gradients on configuration spaces. C R Acad sci Paris 1998, 327, 677–682.
[189] Privault N. An extension of stochastic calculus to certain non-Markovian processes. unpublished note, Evry, France, Univ d'Evry 1997.
[190] Privault N. Stochastic analysis in discrete and continuous settings with normal martingales. Lecture Notes in Math 1982. Berlin, Germany, Springer-Verlag, 2009.
[191] Privault N. Stochastic analysis of Poisson random measures. Lecture note at Satellite summer school to the 7th international conference on Lévy processes 2013. Bedlewo, Poland.
[192] Protter Ph. Stochastic Integration and Differential Equations. 2nd edn. Berlin, Germany, Springer-Verlag, 2005.
[193] Ramsey FP. A mathematical theory of saving. Economic Journal 1928, 38, 543–559.
[194] Rogers LCG, Williams D. Diffusions, Markov Processes, and Martingales: Itô Calculus. Cambridge, UK, Cambridge Univ Press, 2000.
[195] Sato K. Absolute continuity of multivariate distributions of class L. J Multivariate Anal 1982, 12, 89–94.
[196] Sato K. Additive Processes (in Japanese). Tokyo, Japan, Kinikuniya, 1990; English version: Lévy processes and infinitely divisible distributions. Cambridge, UK Cambridge University Press, 1999.
[197] Sato K, Yamazato M. Operator-selfdecomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. Stoch Proc Appl 1964, 17, 73–100.
[198] Sharpe MJ. Support of convolution semigroups and densities. In: Heyer H, ed. Probability measures on groups and related structures. Singapore, World Scientific Publishing, 1995, 363–369.
[199] Shigekawa I. Stochastic analysis. (Translated from the 1998 Japanese original by the author). Providence, USA, American Mathematical Society, 2004.
[200] Simon Th. Théorème de support pour processus à sauts. C R Acad sci Paris 1999, 328, 1075–1080.
[201] Simon Th. Support of a Marcus equation in dimension 1. Electron Comm Probab 2000, 5, 149–157.
[202] Situ R. Theory of stochastic differential equations with jumps and applications. New York, USA, Springer, 2005.
[203] Solé J, Utzet F, Vives J. Chaos expansions and Malliavin calculus for Lévy processes. In: Benth F, Di Nunno G, Lindstrom T, Øksendal B, eds. Stochastic analysis and applications. Berlin, Germany, Springer, 2007, 595–612.
[204] Stoyanov JM. Counterexamples in Probability. New York, USA, John Wiley ans Sons, 1987.
[205] Surgailis D. On multiple Poisson stochastic integrals and associated Markov semigroups. Probab Math Statist 1984, 3, 217–239.
[206] Taira K. Semigroups, boundary value problems and Markov processes. 2nd edn. Heidelberg, Germany, Springer, 2014.
[207] Takanobu S. Multiple stochastic integrals appearing in the stochastic Taylor expansions. J Math Soc Japan 1995, 47, 67–92.
[208] Takanobu S. Diagonal estimate of transition probability densities of certain degenerate diffusion processes. J Funct Anal 1990, 91, 221–236.

[209] Tsuchiya M. Lévy measure with generalized polar decomposition and the associated SDE with jumps. Stochastics and Stochastic Rep 1992, 38, 95–117.
[210] Tsuchiya T. On the pathwise uniqueness of solutions of stochastic differential equations driven by multi-dimensional symmetric α stable class. J Math Kyoto Univ 2006, 46, 107–121.
[211] De Valliere D, Kabanov Y. Consumption-investment problem with transaction costs for Lévy driven price processes. Preprint, 2010.
[212] La Vath V, Pham H, Villeneuve S. A mixed singular/switching control problem for a dividend policy with reversible technology investment. Ann Appl Probab 2008, 18, 1164–1200.
[213] Varopoulos N, Saloff-Coste L, Coulhon T. Analysis and Geometry on Groups. Cambridge, UK, Cambridge University Press, 1992.
[214] Ventsel AD. Rough limit theorems on large deviations for Markov stochastic processes I, II. Theory Probab Appl 1976, 21, 227–242, 499–512.
[215] Watanabe S. Stochastic analysis and its application (in Japanese). Sugaku (Math. Soc. Japan) 1990, 42, 97–110.
[216] Watanabe S. Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels. Ann Probab 1987, 15, 1–39.
[217] Watanabe S. Lectures on Stochastic Differential Equations and Malliavin Calculus. Tata institute of fundamental researc. Berlin, Germany, Springer-Verlag, 1984.
[218] Watanabe T. Absolute continuity of some semi-selfdecomposable distributions and self-similar measures. Probab Theory Related Fields 2000, 117, 387–405.
[219] Watanabe T. Temporal change in distribution properties of Lévy processes. In: Barndorff-Nielsen O, ed. Lévy Processes: Theory and Applications. Boston, USA, Birkhauser, 2001, 89–107.
[220] Yamazato M. Absolute continuity of transition probabilities of multidimensional processes with stationary independent increments. Theory Probab Appl 1995, 39, 347–354.
[221] Yamazato M. Absolute continuity conditions for multidimensional infinitely divisible distributions and their applications. Theory Probab Appl 2012, 56, 299–312.
[222] Yoshida N. Asymptotic expansion for statistics related to small diffusions. J Japan Statist Soc 1992, 22, 139–159.
[223] Yoshida N. Conditional expansions and their applications. Stochastic Process Appl 2003, 107, 53–81.
[224] Zhang X. Degenerate irregular SDEs with jumps and application to integro-differential equations of Fokker–Planck type. Electron J Probab 2013, 55, 1–25 (electronic).
[225] Zolotarev VM. One-dimensional stable distributions. Translations of A.M.S. 65. Providence, USA, A.M.S., 1986.

List of symbols

(symbol, first appeared section)

Norms

$|F|_{k,l,p}$, 3.3.3
$|F|'_{k,l,p}$, 3.3.3
$\gamma(u)$, 3.3.2
$|\varphi|_{2m}$, 3.5.1
$|\varphi|^{\sim}_{2m}$, 3.5.1
$\|\varphi\|_{2m}$, 3.5.1
$\|\psi\|_{-2m}$, 3.5.1
$\|U\|_{k,l,p}$, 3.3.3
$\|V\|^{\sim}_{k,l,p}$, 3.3.3

Spaces

$\mathbf{T}$, 1.1.1
$A(\rho)$, 2.1.2
B_v, 3.5.1
$\mathcal{B}$, $\mathcal{B}_\varphi$, 4.2.2
$\tilde{C}^{-2m}$, 3.5.1
$\mathcal{C}$, 4.2.1
$\mathcal{C}_M$, 4.2.2
$\mathbb{D}$, 1.1.3
$\mathbf{D}$, 1.2.2
$\mathbf{D}^{(1)}_{1,2}$, 3.1.1
$\mathbf{D}^{(2)}_{1,2}$, 3.2.3
$\mathbf{D}_{k,l,p}$, 3.3.3
$\mathbf{D}^{\sim\,(1)}_{k,l,p}$, 3.3.3
$\mathbf{D}^{\sim\,(2)}_{k,l,p}$, 3.3.3
$\mathbf{D}^{\sim}_{k,l,p}$, 3.3.3
$\mathbf{D}'_{k,l,p}$, 3.3.3
$\mathbf{D}_\infty$, 3.3.3
$\mathbf{D}'_\infty$, 3.3.3
H_{2m}, 3.5.1
H_{-2m}, 3.5.1
$\mathbf{K}_1$, 3.3.1
$\mathbf{K}_2$, 3.3.2
$\mathbf{K}$, 3.3.3
Ω_1, 2.1.1
Ω_2, 2.1.1
$\mathcal{P}_1$, 3.3.1
$\mathcal{P}_2$, 3.3.2
$\mathcal{S}$, 3.5.1
$\mathcal{S}_{2m}$, 3.5.1
$\mathcal{S}'$, 3.5.1
U, 2.1.1

Functions

$e_v(x) = e^{i(v,x)}$, 3.5.1
$\varphi(\rho) = \int_{|z|\leq\rho} |z|^2\mu(\mathrm{d}z)$, 3.5.2
$\psi_G(v)$, 3.5.1
$U^F(v)$, $V^F(v)$, 3.5.1
$Z^F(v)$, 3.5.1
$\tilde{Z}^F(v)$, 3.5.1

Operators

$C(x)$, $C_\rho(x)$, 3.6.2
D_t, 3.3.3
$\tilde{D}_u$, 2.1.2
$\mathcal{D}_{(t,u)}$, 3.3.3
$\tilde{\mathcal{D}}_{(t,z)}$, 3.4
D^N_t, 3.2.1
$\tilde{D}_t$, 3.2.1
δ, 3.1.2
$\tilde{\delta}$, 2.1.2
$\bar{\delta}$, 3.3.3
ε^+_u, ε^-_u, 2.1.2
$\tilde{K}$, 3.5.2
$\tilde{K}_\rho$, 3.5.2
$\tilde{K}'$, 3.5.2
$\tilde{K}'_\rho$, 3.5.2
$\tilde{K}(\epsilon)$, 4.1.1
$\tilde{K}_\rho(\epsilon)$, 4.1.1
$[M]$, 1.2.2
$Q^F(v)$, 3.5.1
$Q(v)$, 3.5.2
$Q_\rho(v)$, 3.5.2
R, 3.5.2
$R(\epsilon)$, 4.1.1
$T_\rho(v)$, 3.5.2
$[X, Y]$, 1.2.2

Processes

$\mathcal{E}(M)_t$, 1.2.3
$S_t(\epsilon)$, 4.1
$x_t(x)$, 1.3
$x_s(r, S_t, x)$, 2.2.2
X_t, 1.1.1
X'_t, 3.6.1
$\tilde{X}'(x, S_t, t)$, 3.6.4
$z(t)$, 1.1.1
$Z_s(S_t)$, 3.6.4

Measures

$\mu(\mathrm{d}z)$, 1.1.1
$\bar{M}(\mathrm{d}u)$, 3.3.2
$\hat{M}(\mathrm{d}u)$, 3.3.2
$N(\mathrm{d}s\mathrm{d}z)$, 1.1.1
$\hat{N}(\mathrm{d}s\mathrm{d}z)$, 1.1.1
$\tilde{N}(\mathrm{d}s\mathrm{d}z)$, 1.1.1
π, 2.5.1

Index

De Gruyter Studies in Mathematics

Volume 62
Ulrich Krause
Positive Dynamical Systems in Discrete Time, 2015
ISBN 978-3-11-036975-5, e-ISBN 978-3-11-036569-6, Set-ISBN 978-3-11-036571-9

Volume 61
Francesco Altomare, Mirella Cappelletti Montano, Vita Leonessa, Ioan Rasa
Markov Operators, Positive Semigroups and Approximation Processes, 2015
ISBN 978-3-11-037274-8, e-ISBN 978-3-11-036697-6, Set-ISBN 978-3-11-036698-3

Volume 60
Vladimir A. Mikhailets, Alexandr A. Murach, Peter V. Malyshev
Hörmander Spaces, Interpolation, and Elliptic Problems, 2014
ISBN 978-3-11-029685-3, e-ISBN 978-3-11-029689-1, Set-ISBN 978-3-11-029690-7

Volume 59
Jan de Vries
Topological Dynamical Systems, 2014
ISBN 978-3-11-034073-0, e-ISBN 978-3-11-034240-6, Set-ISBN 978-3-11-034241-3

Volume 58
Lubomir Banas, Zdzislaw Brzezniak, Mikhail Neklyudov, Andreas Prohl
Stochastic Ferromagnetism: Analysis and Numerics, 2014
ISBN 978-3-11-030699-6, e-ISBN 978-3-11-030710-8, Set-ISBN 978-3-11-030711-5

Volume 57
Dmitrii S. Silvestrov
American-Type Options: Stochastic Approximation Methods, Volume 2, 2015
ISBN 978-3-11-032968-1, e-ISBN 978-3-11-032984-1, Set-ISBN 978-3-11-032985-8

www.degruyter.com