

CONTROLLABILITY OF DYNAMIC SYSTEMS

The Green's Function Approach

**Ara S. Avetisyan
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Controllability of Dynamic Systems

Controllability of Dynamic Systems:

*The Green's Function
Approach*

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A. A.: To Academician of NAS of Armenia
Sergei A. Ambartsumyan, on the occasion of
his 95th birthday

As. Kh.: To my dearest mom

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Preface

Controllability, i.e., the ability of a system to be transmitted from a given initial state to a required terminal state by an admissible control within a finite time, is one of the most crucial characteristics of control systems. Controllability is of two main types: exact and approximate. The choice depends on how precisely the required terminal state is implemented. The existence of powerful computers and efficient numerical tools for linear and nonlinear equations allows the exact or approximate controllability analysis to be carried out numerically even for extremely complicated systems. Such analysis can bring with it a burdensome computational cost. However, in some special cases a simplifying step can considerably speed up the analysis. Such a step, for example, could be the determination of explicit dependence between state and control functions, i.e., a solution of the state constraints. While this is to some extent possible for linear systems, for nonlinear systems it is much more complicated.

The main motivation behind presenting our work to a wider scientific community is to illustrate how efficiently the Green's function method can be applied in controllability analysis of both linear and nonlinear dynamic systems and possibly initiate new studies in this di-

PREFACE

rection. We illustrate this idea by a basic analysis of a few typical examples. Even though the examples were picked intuitively rather than systematically, they include some of the common issues which usually make the controllability analysis complicated: coordinate dependent material characteristics, unbounded domains, uncertainty in internal or external parameters, higher dimensions, specific non-linearities, etc. We also address the problem of determination of resolving control functions in an explicit form, which speeds up the controllability analysis further.

Generally speaking, due to, for example, modeling inaccuracies, random issues, uncertainties, etc., even by means of highly precise production technologies it is practically impossible to implement the desired state exactly. The terminal state implemented by the “best” choice of admissible control is more often “sufficiently” close to the desired state, rather than coinciding with it exactly. That is the motivation behind paying the most attention specifically to the *approximate* controllability. Nevertheless, possibilities of exact implementation of required states are shown as well.

In order to make the reading more engaging, the book is completely free of rigorous mathematical statements such as lemmas, propositions, theorems, as well as redundant long proofs, which can be found in the cited references. The focus is concentrated on the *ways* the developed approach can be applied in dealing with particular control problems. Thus, the book is mostly intended for researchers who are focused on the applications of control, as well as for engineers attempting to apply control techniques in their practice. It can also be useful for postgraduate students in mechanics, physics, engineering, applied mathematics, etc.

Even though we study the controllability of linear and nonlinear differential equations arising in many applied areas of science, including physics, production, mechanical, aerospace, civil and chemical engineering, hydrodynamics, information processing and transfer, communications, etc., we refrain from giving specific recommendations regarding particular real-life objects, processes or phenomena. The reason is very simple: we stand on mathematical ground and look at all those systems from a mathematical viewpoint. Nevertheless, we hope that the book will capture the interest of applied scientists towards the approach which will lead to its real-life implementations.

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Asatur Zh. Khurshudyan
Yerevan, Shanghai, 2018

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It would be hardly possible to write this book without the continual support and care of my wife, unceasingly helping me throughout the whole process of writing. I always admire her unwavering patience and bright mind.

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List of Symbols

All notations and symbols are consistent over the book. Therefore, in order to assist readers who do not engage the book linearly, the main notions, functions, and symbols occurring in the book are defined separately here.

| Symbol | Name & Definition |
|---------------------------|---|
| \mathbb{R} | Real numbers |
| \mathbb{R}^+ | Positive real numbers |
| \mathbb{N} | Natural numbers |
| $m, n, k \dots$ | $m, n, k \in \mathbb{N}$ |
| Ω | An open subset of \mathbb{R}^n |
| $\overset{\circ}{\Omega}$ | Interior of Ω |
| $\partial\Omega$ | Boundary of Ω $\partial\Omega = \Omega \setminus \overset{\circ}{\Omega}$ |
| $\overline{\Omega}$ | Closure of Ω $\overline{\Omega} = \Omega \cup \partial\Omega$ |
| $\text{supp}(\cdot)$ | Support of a function $\text{supp}(f) = \overline{\{\mathbf{x}, f(\mathbf{x}) \neq 0\}}$ |

LIST OF SYMBOLS

| Symbol | Name & Definition |
|--------------|--|
| δ | Dirac delta function $\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1,$ $\int_{-\infty}^{\infty} f(x) \delta(x - \xi) dx = f(\xi) \quad \forall \xi \in \mathbb{R}$ |
| δ | Multidimensional Dirac function $\delta(\mathbf{x}) = \prod_{k=1}^n \delta(x_k), \quad \mathbf{x} = (x_1, \dots, x_n)$ |
| δ_m^n | Kronecker symbol $\delta_m^n = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$ |
| θ | Heaviside function $\theta(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0, \end{cases}$ $\theta(x) = \int_{-\infty}^x \delta(\xi) d\xi$ |
| sign | Sign function $\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$ $\text{sign}(x) = 2\theta(x) - 1$ |

| Symbol | Name & Definition |
|-----------------|--|
| χ_{Ω} | Indicator function $\chi_{\Omega}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega, \\ 0, & \text{else,} \end{cases}$ $\chi_{[a,b]}(x) = \theta(x - a) - \theta(x - b)$ |
| rect | Rectangular function $\text{rect}(x) = \begin{cases} 0, & x > \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 1, & x < \frac{1}{2}, \end{cases}$ $\text{rect}(x) = \theta\left(\frac{1}{4} - x^2\right) = \theta\left(x + \frac{1}{2}\right) - \theta\left(x - \frac{1}{2}\right)$ |
| erf | Gauss error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp[-\xi^2] d\xi$ |
| \mathbf{x} | State Variable $\mathbf{x} \in \Omega$ |
| t | Time $t \in \mathbb{R}^+$ |
| \mathbf{w} | State function $\mathbf{w} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ |
| \mathbf{W} | State space Appropriate Hilbert space |
| \mathbf{w}_0 | Initial state $\mathbf{w}_0 : \Omega \rightarrow \mathbb{R}^m$ |

LIST OF SYMBOLS

| Symbol | Name & Definition |
|-------------------------------------|--|
| \mathbf{w}_T | Required terminal state $\mathbf{w}_T : \Omega \rightarrow \mathbb{R}^m$ |
| $\overline{\mathbf{W}}$ | Restriction of \mathbf{W} for fixed t Corresponding Hilbert space: $\mathbf{w}_0, \mathbf{w}_T \in \overline{\mathbf{W}}$ |
| $\ \cdot\ _{\overline{\mathbf{W}}}$ | Norm in $\overline{\mathbf{W}}$ |
| \mathbf{u} | Distributed control $\mathbf{u} : \mathbb{R}^+ \rightarrow \mathbb{R}^k$ |
| \mathbf{U} | Distributed control space Appropriate Hilbert space |
| \mathbf{u}_b | Boundary control $\mathbf{u}_b : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ |
| \mathbf{U}_b | Boundary control space Appropriate Hilbert space |
| \mathcal{U} | Set of admissible controls $\mathcal{U} = \{\mathbf{u} \in \mathbf{U}, \mathbf{u} \leq \epsilon, \text{ o.p.c.}\}$ |
| \mathcal{U}_{res} | Set of resolving controls |
| \mathbf{f} | External influence $\mathbf{f} : \mathcal{U} \times \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ |
| \mathbf{F} | Influence space Appropriate Hilbert space |
| L^1 | L^1 space $L^1(\Omega) = \left\{ \mathbf{f} : \Omega \rightarrow \mathbb{R}^m, \int_{\Omega} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} < \infty \right\}$ |
| $\ \cdot\ _{L^1}$ | L^1 -norm $\ \mathbf{f}\ _{L^1(\Omega)} = \int_{\Omega} \mathbf{f}(\mathbf{x}) \, d\mathbf{x}$ |

| Symbol | Name & Definition |
|--------------------------|---|
| L^1_{loc} | L^1_{loc} space $L^1_{loc}(\Omega) = \left\{ \mathbf{f} : \Omega \rightarrow \mathbb{R}^m, \int_{\Omega} \mathbf{f}(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x} < \infty \right\}$ for each infinitely differentiable function $\boldsymbol{\varphi} : \Omega \rightarrow \mathbb{R}^m$ with $\text{supp}(\boldsymbol{\varphi}) \subseteq \Omega$ |
| L^2 | L^2 space $L^2(\Omega) = \left\{ \mathbf{f} : \Omega \rightarrow \mathbb{R}^m, \left[\int_{\Omega} \mathbf{f}(\mathbf{x}) ^2 \, d\mathbf{x} \right]^{\frac{1}{2}} < \infty \right\}$ |
| $\ \cdot\ _{L^2}$ | L^2 -norm $\ \mathbf{f}\ _{L^2(\Omega)} = \left[\int_{\Omega} \mathbf{f}(\mathbf{x}) ^2 \, d\mathbf{x} \right]^{\frac{1}{2}}$ |
| L^2_{ν} | Weighted L^2 space $L^2_{\nu}(\Omega) = \left\{ \mathbf{f} : \Omega \rightarrow \mathbb{R}^m, \left[\int_{\Omega} \mathbf{f}(\mathbf{x}) ^2 \nu(\mathbf{x}) \, d\mathbf{x} \right]^{\frac{1}{2}} < \infty \right\}$ |
| $\ \cdot\ _{L^2_{\nu}}$ | Weighted L^2 -norm $\ \mathbf{f}\ _{L^2_{\nu}(\Omega)} = \left[\int_{\Omega} \mathbf{f}(\mathbf{x}) ^2 \nu(\mathbf{x}) \, d\mathbf{x} \right]^{\frac{1}{2}}$ |
| L^{∞} | L^{∞} space $L^{\infty}(\Omega) = \left\{ \mathbf{f} : \Omega \rightarrow \mathbb{R}^m, \sup_{\mathbf{x} \in \Omega} \mathbf{f}(\mathbf{x}) < \infty \right\}$ |
| $\ \cdot\ _{L^{\infty}}$ | L^{∞} or sup-norm $\ \mathbf{f}\ _{L^{\infty}(\Omega)} = \sup_{\mathbf{x} \in \Omega} \mathbf{f}(\mathbf{x}) $ |
| \mathcal{D} | State operator $\mathcal{D} : \mathbf{W} \rightarrow \mathbf{F}$ |

LIST OF SYMBOLS

| Symbol | Name & Definition |
|-----------------|---|
| \mathcal{B} | Boundary operator $\mathcal{B} : \mathbf{W} \rightarrow \mathbf{U}_b$ |
| \mathcal{I} | Operator of initial conditions $\mathcal{I} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ |
| \mathcal{T} | Operator of terminal conditions $\mathcal{T} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ |
| \mathcal{R}_T | Residue $\mathcal{R}_T(\mathbf{u}) = \ \mathcal{T}[\mathbf{w}] - \mathbf{w}_T\ _{\overline{\mathbf{W}}}$ |
| Δ | Laplace operator $\Delta \mathbf{w} = \sum_{k=1}^n \frac{\partial^2 \mathbf{w}(\mathbf{x})}{\partial x_k^2}$ |
| $ \cdot $ | Euclidean norm $ \mathbf{x} = \sqrt{\sum_{k=1}^n x_k^2} \text{ for } \mathbf{x} \in \mathbb{R}^n$ |

1

Introduction

“Everything must be made as simple as possible. But not simpler.”

– Albert Einstein

This introductory chapter begins by laying out the main concepts and problems the book is dealing with (Section 1.1), and then proceeds to the mathematical foundations of the solution technique including the description of the Green’s function method for both linear and nonlinear systems (Sections 1.2 and 1.3). The technique itself is then described in detail in Section 1.4. The goal for the chapter is to set out the issues in an understandable manner for researchers without special mathematical training (especially for engineers and applied scientists) so it is bereft of mathematical formalism.

All the notations that are not common or well-known are explained within the text directly after being used.

1.1 Controllability

The majority of dynamic systems, e.g., vehicles, aircrafts, robots, production equipment, financial and biological processes, etc., are somehow controlled by a predetermined program or influence often referred to as *control programs* or simply *controls*. Controls are designed and implemented by special *controllers* attached to the system. Controllers have limited capabilities, so that restrained types of controls can be elaborated. Controls that a controller is able to use in design are often referred to as *admissible controls*.

Dynamic systems may need to be controlled for a variety of different purposes. Put simply, the aim of a control is to provide a stable transition of a dynamic system from a given state to a required terminal one within a fixed amount of time. Additional requirements, such as constraints on the state at intermediate time instants, constraints on control, mixed constraints, etc., may be considered as well. For a given system, situations may occur such that, even for a fixed range of system parameters, prescribed initial and terminal states, and a fixed control time, it is impossible to develop an admissible control providing the desired state transition. That is why before exploiting control systems, an overall examination of the ability to accommodate the desired state in the required time by means of attached controllers is carried out. This property is called *controllability*.

There are two main types of controllability—exact and approximate. A system is called *exactly* controllable if by a specific choice of admissible controls it can be transitioned from a given state to a required state exactly in a finite amount of time. There is a huge body of references devoted to the development of methods of exact control-

lability analysis and to the establishment of exact controllability for particular types of dynamic systems. For general introduction into the subject, the reader is referred to [1–28] and references therein.

1.1.1 Exact Controllability

Consider the mathematical interpretation of controllability. Let the state of a dynamic system be characterized by a vector-function $\mathbf{w} : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$, $k, n, m \in \mathbb{N}$, satisfying the state equations of the system and possibly some other constraints such as boundary and initial conditions, intermediate state constraints, etc. In what follows, the system of such constraints on \mathbf{w} is referred to as state constraints.

The simplest problem of control now can be formulated as follows. By appropriate choice of the control vector $\mathbf{u} : \mathbb{R}^+ \rightarrow \mathbb{R}^k$, being chosen from a given set of admissible controls, denoted here by \mathcal{U} , the system is required to achieve the given terminal state $\mathbf{w}_T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ within a required finite time $T > 0$. The controllability of the system is verified by evaluating the mismatch of the desired state and the state implemented by a specific choice of $\mathbf{u} \in \mathcal{U}$, i.e., the residue

$$\mathcal{R}_T(\mathbf{u}) = \|\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})\|_{\overline{\mathbf{W}}}. \quad (1.1)$$

Here the subscript T at the residue is a short form of denoting the obvious dependence $\mathcal{R} = \mathcal{R}(\cdot, T)$.

The choice of the norm in (1.1) usually depends on the regularity of external influences, the consistency of boundary and initial data, etc. Very often L^1 , L^2 and L^∞ norms are considered, given by

$$\mathcal{R}_T(\mathbf{u}) = \int_{\Omega} |\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})| d\mathbf{x},$$

$$\mathcal{R}_T(\mathbf{u}) = \left[\int_{\Omega} |\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})|^2 d\mathbf{x} \right]^{\frac{1}{2}},$$

$$\mathcal{R}_T(\mathbf{u}) = \sup_{\mathbf{x} \in \Omega} |\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})|,$$

respectively. Here $|\cdot|$ denotes the Euclidean norm.

Then, the system is called *exactly controllable* if there exists $\mathbf{u} \in \mathcal{U}$ such that at a prescribed instant T ,

$$\mathcal{R}_T(\mathbf{u}) = 0. \tag{1.2}$$

Note also that (1.1) is the simplest expression for the residue. More complicated forms of (1.1) are usually considered in applied problems. For example, if the aim of a control is the implementation of the terminal condition

$$\mathcal{T}[\mathbf{w}] = \mathbf{w}_T \text{ in } \Omega$$

at the instant $t = T$, then the residue (1.1) takes the form

$$\mathcal{R}_T(\mathbf{u}) = \|\mathcal{T}[\mathbf{w}](\mathbf{u}, \mathbf{x}) - \mathbf{w}_T(\mathbf{x})\|_{\overline{\mathbf{W}}}.$$

In many dynamic control problems, especially in those arising in engineering (e.g. motion ceasing, vibration suspension, material heating, etc.) the case of $\mathbf{w}_T \equiv 0$, i.e., transmitting the system to the equilibrium state, is the primary consideration. If a corresponding admissible control exists, then the system is usually referred to as *null-controllable* (sometimes *null-exact-controllable*).

Remark 1.1. *It is noteworthy that for linear systems the null-controllability is equivalent to the ordinary exact controllability in the sense*

that the function

$$\tilde{\mathbf{w}}(\mathbf{u}, \mathbf{x}, t) = \mathbf{w}(\mathbf{u}, \mathbf{x}, t) - \mathbf{w}_T(\mathbf{x})$$

satisfies the same state equation as \mathbf{w} but with a different right-hand side and a homogeneous terminal condition. Moreover, in the case of consistent boundary, initial, and terminal data, the form of boundary and initial conditions are preserved.

1.1.2 Approximate Controllability

Nevertheless, depending on controller capabilities, it may turn out that for any given initial state an admissible control providing the desired state in a given time does not exist, i.e., the control system is not exactly controllable. The lack of exact controllability may occur because of many reasons [29–42], such as topology [30], unreachable terminal states, error in the approximation of the solution, unboundedness of the domain [33], uncertainty in internal or external parameters [35], type of boundary condition [42], etc. In other words, lack of controllability means that, for prescribed initial and terminal states and a fixed control time T , the equality (1.2) does not hold exactly for any $\mathbf{u} \in \mathcal{U}$, i.e., the residue

$$\mathcal{R}_T(\mathbf{u}) > 0 \text{ on } \mathcal{U}.$$

Usually, by extending the set of admissible controls, e.g., complementing it by sliding modes [43], a non-controllable system may become controllable (see, for instance, [44–46] and references therein). On the other hand, this extension may result in, for instance, instability of the system. More specifically, it might happen that in order to ensure exact controllability, it is necessary to complement the set of admissi-

1. INTRODUCTION

ble controls by impulsive impacts expressed by the Dirac function and its derivatives. Being efficient in general, such controls can lead to a discontinuity of the state function and may cause infinite growth in the total energy of the system. That is why control systems are elaborated only after a comprehensive theoretical analysis on simultaneous controllability, stability, reliability, flexibility, etc.¹.

It might also be the case that, even though (1.2) does not hold exactly for prescribed initial and terminal states, and, given control time T by any choice of the admissible control, at least for one admissible control the implemented state is sufficiently close in the sense of the distance in $\overline{\mathbf{W}}$ to the given state, i.e., it is in a small ε -neighborhood of the desired state (see Fig. 1.1). In other words,

$$\mathcal{R}_T(\mathbf{u}) \leq \varepsilon \tag{1.3}$$

with *required* precision $\varepsilon > 0$. If for given inputs there exists an appropriate admissible control then the system is called *approximately* controllable.

Exact controllability implies approximate controllability since exactly controllable systems are approximately controllable with arbitrarily small precision ε . However, the reverse is not correct. Thus, the class of approximately controllable systems is wider than that of exactly controllable systems, in the sense that on \mathcal{U} inequality (1.3) is satisfied more likely than equality (1.2). Moreover, the set of terminal states that are reachable approximately is apparently much wider than the set that are reachable exactly.

Evaluation and verification of (1.2) or (1.3) on \mathcal{U} may be sophisticated especially when \mathcal{U} has a complex structure or when the state

¹The full list of features depends on a particular system or its purpose of use.

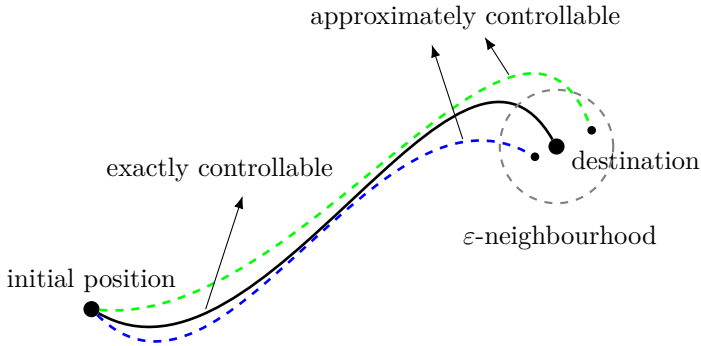


Figure 1.1: Schematic presentation of exact and approximate controllabilities

constraints have a complicated form and, in particular, are nonlinear. Despite the amount of existing references, the derivation of easy-to-verify criteria for exact and/or approximate controllability of a particular control system is still a very challenging and important problem in modern control theory.

For further introductions to the concept, as well as existing mathematical approaches for providing (1.3), the reader is referred to [20, 47–55] and references therein. See also [56–62] for particular systems.

1.1.3 Choice of the Set of Admissible Controls

In practice, the set of admissible controls for a particular system consists of programs that the attached controller is able to design and

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implement. However, in dealing with theoretical models, this set is usually allowed to include a wide class of functions². On one hand, this allows exploration of new opportunities leading to an improvement of the control system under study and accordingly an enrichment of the controller capability; on the other hand, it makes the theoretical study much more sophisticated.

Because of this, the set of admissible controls is chosen in such a way that the solution of the state constraints exists in \mathbf{W} and is unique, so that \mathcal{R}_T is well defined. Depending on specific requirements on \mathbf{w} , the admissible controls may be restricted by some additional constraints, such as sign-constantness, boundedness, differentiability up to a required order, specific decay rate, etc.

In many applications, largely arising from engineering, the aim of the control is not just to implement a required terminal state at a given instant T , but rather to ensure that in all consequent instances $t > T$ the state of the system will continue to be stably close to the achieved one. Such requirements are appropriate in motion control, structural heating, material forming, vibration suspension, damage healing, electromagnetic radiation, etc. In such cases, when the desired terminal state is achieved, the controller stops elaborating controls. In other words, admissible controls operate within the interval $[0, T]$ and

$$\mathbf{u} \equiv 0 \quad \text{as } t > T.$$

Mathematically this fact is expressed as

$$\text{supp}(\mathbf{u}) \subseteq [0, T], \tag{1.4}$$

²In particular, \mathcal{U} can contain all bounded, continuous or even measurable functions. Moreover, in some exceptional cases, \mathcal{U} can be complemented by switching regimes and sliding modes.

where $\text{supp}(\mathbf{u})$ denotes the support of \mathbf{u} . Functions with (1.4) are called *compactly supported*. The idea of compactly supported control was introduced in the pioneering work [63] and was developed further in [66–71]. In what follows it is always assumed that admissible controls are compactly supported in $[0, T]$.

Then, in general, the set of admissible controls can be defined as

$$\mathcal{U} = \{\mathbf{u} \in \mathbf{U}, |\mathbf{u}| \leq \epsilon, \text{supp}(\mathbf{u}) \subseteq [0, T], \text{ o.p.c.}\}$$

with prescribed ϵ . Here o.p.c. means *other possible constraints*.

Other possible constraints that are imposed on admissible controls may be derived in the case of boundary control. Such a constraint is derived when a consistency between boundary and initial and terminal data is required. The resulting constraints are discrete equalities given at the instants $t = 0$ and $t = T$ of the form

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}(T) = \mathbf{u}_T. \quad (1.5)$$

They are usually considered as boundary conditions for admissible controls. Then, \mathcal{U} is complemented with (1.5).

Despite all possible constraints, depending on the inner structure of \mathbf{U} or \mathbf{U}_b , the set \mathcal{U} may still be too large, causing the verification of (1.2) or (1.3) on it to be very computationally costly. However, if admissible controls satisfying all possible constraints are found explicitly, computational costs can be reduced.

1.1.4 Resolving Controls

Obviously, not all admissible controls ensure controllability. Moreover, as mentioned above, it might happen that there does not exist

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an admissible control ensuring exact or even approximate controllability of a particular system at all. The admissible controls that provide controllability in a given T are called *resolving*. The set of all resolving controls is denoted by \mathcal{U}_{res} . In general, $\mathcal{U}_{res} \subsetneq \mathcal{U}$. In this terminology, the lack of controllability means that $\mathcal{U}_{res} = \emptyset$.

In many problems, especially about approximate controllability, an inequality type constraint of the form $|\mathbf{u}| \leq \tilde{\epsilon}$ can be derived with an estimated $\tilde{\epsilon}$, which, in the case when $\tilde{\epsilon} \geq 0$, is sufficient for approximate controllability. In such cases, all admissible controls

$$\mathbf{u} \in \tilde{\mathcal{U}}_{res} = \{\mathbf{u} \in \mathcal{U}, |\mathbf{u}| \leq \tilde{\epsilon}\}$$

ensure the approximate controllability of the system.

Nevertheless, since the constraint is only a sufficient condition, then, in general, $\tilde{\mathcal{U}}_{res} \subsetneq \mathcal{U}_{res}$. In some exceptional cases it might happen that $\tilde{\epsilon} \geq \epsilon$, so that $\mathcal{U} \subseteq \tilde{\mathcal{U}}_{res}$, i.e., all admissible controls provide approximate controllability. On the other hand, when $\tilde{\mathcal{U}}_{res} = \emptyset$, for proper controllability analysis the residue must be evaluated on the whole \mathcal{U} and (1.2) or (1.3) must be checked directly.

1.1.5 Methods of Finding Resolving Controls

The verification of (1.2) or (1.3) is enough only for establishing whether a particular system is exactly or approximately controllable or not. When dealing with real-life systems it is equally important to find admissible controls providing either type of controllability explicitly, which would significantly reduce computational costs.

There exist at least two ways for explicit representation of resolving controls. The first way is to use some norm minimizing algorithm

for the minimization of (1.1). As a general statement, the problem is formulated as a constrained minimization problem

$$\|\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})\|_{\overline{\mathbf{W}}} \xrightarrow{\mathbf{u} \in \mathcal{U}} \min,$$

where \mathbf{w} is subjected to state constraints. Additional constraints on the control, such as cost functional, equality type constraints like (1.5), etc., can be considered as well. If the minimum is attained and is zero, then

$$\mathbf{u} = \operatorname{argmin} \|\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x})\|_{\overline{\mathbf{W}}} \in \mathcal{U}$$

provides exact controllability of the system at T . If the minimum is attained and is not equal to zero but remains smaller than a required precision ε then the system is approximately controllable at T with precision ε .

Several efficient numerical tools exist for norm minimization including fast gradient methods, the momentum method, nonlinear programming, etc., and their modifications (see, for instance, [72–74]). The norm minimization approach has an advantage that besides explicit representation for minimizers, in cases when (1.2) does not hold, it is possible to identify how precisely (1.3) does hold. Nevertheless, the approach requires derivatives of \mathcal{R}_T and may require high computational costs.

The second approach is based on the evaluation of the constraint

$$\mathbf{w}(\mathbf{u}, \mathbf{x}, T) - \mathbf{w}_T(\mathbf{x}) = \mathbf{0} \quad \text{a.e. in } \Omega, \quad (1.6)$$

which is equivalent to (1.2). If \mathbf{w} is determined from the state constraints explicitly then the set of the resolving controls is defined as

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follows:

$$\mathcal{U}_{res} = \{\mathbf{u} \in \mathcal{U}, (1.6)\}.$$

If (1.6) is luckily solved for \mathbf{u} exactly then the resolving controls ensure exact controllability of the system. However, this only happens in a very few special cases. In general, (1.6) is solved for \mathbf{u} approximately, providing controls that ensure approximate controllability of the system.

Moreover, as a matter of fact, the resolving controls are not unique, so that this approach allows one to consider problems of optimal choice, i.e., problems of finding $\mathbf{u} \in \mathcal{U}_{res}$ extremizing some cost functional $\kappa : \mathbf{U} \rightarrow \mathbb{R}$. Compared with the previous approach, in this case it is not so costly once the set \mathcal{U}_{res} is constructed. On the other hand, an explicit solution of (1.6) can be difficult, especially for nonlinear state constraints, large n and m , etc.

The main disadvantage of the approach is in the difficulty of deriving the explicit form of the state function \mathbf{w} . Nevertheless, since there exist handbooks like [75, 76] containing exact solutions to many nonlinear differential equations, this approach can still be applied for the analysis of various systems arising in mechanics and physics.

Apparently, the explicit representation of resolving controls strongly depends on the solvability of the constraints eventually obtained for controls. Sometimes, it is easier and more efficient to heuristically choose admissible controls which contain a set of free parameters that satisfy some of the constraints. The remaining part of the constraints can be satisfied by appropriate choice of the free parameters (see Section 2.3 below for details). If such a representation is derived, then the controllability analysis will be greatly simplified.

1.2 The Green's Function Method for Linear Systems

Enormous technological progress over the last century has led to the design of new machines and equipment with impressive features. This has also provided opportunities for a significant improvement of existing equipment. A very important step in the pathway from design to production of such machines is theoretical modeling. The development of theoretical modeling tools have had to increase in coverage in order to match the complementary increase in sophistication of production requirements.

Theoretical models use some idealized hypotheses and specific constructions which allow one to reproduce the main functions of the modeled system. Depending on how precise those structures and hypotheses express the real system, the model is considered to be acceptable for specific purposes (or not). After deriving the theoretical model, some constitutive relations between the characteristics of the system being studied are used to derive its mathematical model.

Mathematical models usually provide algebraic, functional, differential, difference, integral, or mixed types of constraints, consisting of relations between the system's internal and external characteristics. Those constraints contain equations, inequalities, inclusions, and their coupled systems. The model is usually complemented by some additional constraints needed to detach a specific solution and make the complete system of constraints consistent³. Then, the mathematical model provides a necessary base for the quantitative and qualitative

³For example, if the mathematical model consists of differential equations, then those constraints are Cauchy and/or boundary conditions.

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analysis of the theoretical model, allowing the ability to explore the specific issues for which the system is modeled.

Two main kind of methods exist for analyzing mathematical models: exact and approximate, depending on what type of calculations they involve. Commonly speaking, exact methods are more preferable, since they provide complete description of the dependence between unknown and known quantities. Nevertheless, such methods are only applicable to a very restricted classes of models. Therefore, approximate methods are used more often.

The choice of a specific method of analysis depends first of all on the type of functional description provided by the mathematical model. Specifically, it depends on the aims and required accuracy of the analysis. In particular, if the required accuracy is not very high, a rough approximation may be used to reduce computational costs. Conversely, higher accuracy can often require enormous computational costs, and the model analysis is likewise time consuming.

The complexity of the mathematical model plays a crucial role in deciding how one analyzes a particular model. The simplest mathematical models are those providing linear constraints, also referred to as linear models. The corresponding analysis is called *linear analysis*. Even though they only allow the ability to capture the basic character of the system, linear models are easier to analyze. This analysis is facilitated by the presence of various well-developed mathematical methods for both rigorous and approximate analysis of linear models⁴.

The mathematical model of most linear dynamic systems as a rule

⁴There exists a huge body of references devoted to linear analysis. In what follows only models described by differential equations are considered. Some very well known and commonly used methods for differential equations are listed in [77, 78]. See also the references therein.

consists of initial-boundary value problems for differential equations. Among well-known methods for *linear* non-homogeneous differential equations is the method of using the Green's function. There is a huge body of references on the theory and applications of the method. The reader is referred to [77–92] and references therein, which is not even close to being a full list of references on the subject. Nevertheless, they contain all the necessary details needed for applying the Green's function technique to the analysis of linear systems.

The concept of the Green's function, which since its invention has led to an abundance of very important results in physics and engineering, dates back to George Green [93], who used it as an auxiliary tool to determine electric fields against an electrostatic background. The main advantage of this approach is that it allows one to establish a closed form dependence of unknown quantities against all internal and external parameters of the system, thus making the linear analysis much easier.

In order to illustrate the use of the Green's function in the analysis of linear models, let the mathematical model of a dynamic system be described by the following abstract differential equation:

$$\mathcal{D}[\mathbf{w}] = \mathbf{f}(\mathbf{x}, t) \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.7)$$

where $\Omega \subseteq \mathbb{R}^n$ is the domain of the system⁵, $\mathcal{D} : \mathbf{W} \rightarrow \mathbf{F}$ is a *linear* differential operator, and the vector $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ represents the resultant of all external signals or influences interacting with the system.

⁵ Ω can be bounded or unbounded.

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Boundary and initial conditions are also assumed to be prescribed⁶:

$$\mathcal{B}[\mathbf{w}] = \mathbf{w}_b(t) \quad \text{in } \partial\Omega \times \mathbb{R}^+, \quad (1.8)$$

$$\mathcal{I}[\mathbf{w}] = \mathbf{w}_0(\mathbf{x}) \quad \text{in } \Omega \times \{0\}. \quad (1.9)$$

Here $\mathcal{B} : \mathbf{W} \rightarrow \mathbf{U}_b$ and $\mathcal{I} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ are linear operators, and the vector-functions $\mathbf{w}_b : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ and $\mathbf{w}_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are given. It is assumed that \mathbf{f} , \mathbf{w}_b , and \mathbf{w}_0 satisfy the existence and uniqueness conditions of a solution $\mathbf{w} \in \mathbf{W}$.

In particular problems, it is often easier to operate with homogeneous initial and boundary conditions. It turns out that by introducing a new state function, it is always possible to include the boundary and initial data into the right-hand side of (1.7). To this end distributions are usually involved as well as the filtering property of the Dirac delta function, provided that

$$\int_{\Omega} \mathcal{B}^e[\mathbf{w}](\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{x}_b) d\mathbf{x} = \mathbf{w}_b(t) \quad \text{in } \mathbb{R}^+ \quad \text{for all } \mathbf{x}_b \in \partial\Omega$$

and

$$\int_0^t \mathcal{I}^e[\mathbf{w}](\mathbf{x}, t) \delta(t) dt = \mathbf{w}_0(\mathbf{x}) \quad \text{in } \Omega$$

are taken into account. Here $\mathcal{B}^e[\cdot]$ and $\mathcal{I}^e[\cdot]$ are the extensions of the operators $\mathcal{B}[\cdot]$ and $\mathcal{I}[\cdot]$ to $\overline{\Omega} \times \mathbb{R}^+$, respectively.

Then, as a result of algebraic transformations, (1.7)-(1.9) are reduced to

$$\mathcal{D}[\tilde{\mathbf{w}}] = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}_b(\mathbf{w}_b) + \mathbf{g}_i(\mathbf{w}_0) \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.10)$$

⁶If Ω is unbounded, instead of boundary conditions, growth conditions on \mathbf{w} at infinity are usually prescribed.

$$\mathcal{B}[\tilde{\mathbf{w}}] = \mathbf{0} \quad \text{in } \partial\Omega \times \mathbb{R}^+, \quad (1.11)$$

$$\mathcal{I}[\tilde{\mathbf{w}}] = \mathbf{0} \quad \text{in } \Omega \times \{0\}, \quad (1.12)$$

where $\tilde{\mathbf{w}}$ is the new state function.

The explicit form of \mathbf{g}_b and \mathbf{g}_i depends on the form of the operators \mathcal{D} , \mathcal{B} , and \mathcal{I} , and contains the Dirac function and its derivatives⁷. Since these operators are linear, then \mathbf{g}_b is linear in \mathbf{w}_b and \mathbf{g}_i is linear in \mathbf{w}_0 , i.e.,

$$\mathbf{g}_b(\mathbf{w}_b) = \mathbf{g}_{b0}(\mathbf{x}) \mathbf{w}_b(t), \quad \mathbf{g}_i(\mathbf{w}_0) = \mathbf{g}_{i0}(t) \mathbf{w}_0(\mathbf{x}). \quad (1.13)$$

Precisely, they are the functions \mathbf{g}_{b0} and \mathbf{g}_{i0} that contain the Dirac function and its derivatives.

Then, the solution to the initial-boundary value problem

$$\mathcal{D}[\mathbf{G}] = \delta(\mathbf{x} - \boldsymbol{\xi}) \delta(t - \tau) \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.14)$$

$$\mathcal{B}[\mathbf{G}] = \mathbf{0} \quad \text{in } \partial\Omega \times \mathbb{R}^+, \quad (1.15)$$

$$\mathcal{I}[\mathbf{G}] = \mathbf{0} \quad \text{in } \Omega \times \{0\}, \quad (1.16)$$

$\mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau)$, is called the *Green's function* of the system (1.10)-(1.12).

There are different approaches for the derivation of the general solution of (1.14)-(1.16). In general, any \mathbf{G} satisfying (1.14)-(1.16) is a distribution rather than a proper function. That is the main reason that \mathbf{G} is usually sought by methods suitable for distributions. Among such methods are the method of integral transforms, the Fourier method of variable separation, the method of eigenvalue expansion, etc. The

⁷For particular forms and applications of such transformations see [64,65], and for their applications in control and optimization see [46,66–71,94]. See also Subsection 1.2.1 below.

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choice of a particular method depends on, first of all, the domain Ω . Particularly, when Ω is semi-infinite, then the Laplace integral transform is usually used, while if it is infinite, the Fourier integral transform is more suitable. The separation of variables and the eigenvalue expansion are often applied when Ω is finite. In some exceptional cases a special combination of those methods needs to be applied. For instance, in dynamic problems the Laplace integral transform is applied with respect to the time variable t . Further, if Ω is bounded, then the method of variables separation can be involved. Otherwise, Fourier transform can be applied. For a more thorough introduction into existing methods for the derivation of the Green's function, see [85, 88].

Assuming that the generalized solution of (1.14)-(1.16) is somehow found, then the general solution of (1.10)-(1.12) is given by the Green's representation formula

$$\begin{aligned}\tilde{w}(\mathbf{x}, t) &= \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{f}(\boldsymbol{\xi}, \tau) d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_b(\mathbf{w}_b(\tau)) d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_i(\mathbf{w}_0(\boldsymbol{\xi})) d\boldsymbol{\xi} d\tau \quad \text{in } \bar{\Omega} \times \mathbb{R}^+, \end{aligned}$$

or, taking (1.13) into account,

$$\begin{aligned}\tilde{w}(\mathbf{x}, t) &= \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{f}(\boldsymbol{\xi}, \tau) d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_{b0}(\boldsymbol{\xi}) \mathbf{w}_b(\tau) d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_{i0}(\tau) \mathbf{w}_0(\boldsymbol{\xi}) d\boldsymbol{\xi} d\tau \quad \text{in } \bar{\Omega} \times \mathbb{R}^+. \end{aligned}$$

Thus, the solution to (1.10)-(1.12) is expressed in terms of the right-hand side, boundary and initial data, and other system characteristics included in \mathbf{G} . It is now evident that, by means of the Green's representation formula, linear analysis becomes much simpler for different expressions of external signals and influences, as well as boundary and initial data. Moreover, this formula allows the exploration of new features that may occur in the case of specific combinations of system data.

Since the functions \mathbf{g}_{b0} and \mathbf{g}_{i0} are expressed in terms of distributions, then the integrals

$$\begin{aligned} \mathbf{G}_b(\mathbf{x}, t, \tau) &= \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_{b0}(\boldsymbol{\xi}) \, d\boldsymbol{\xi}, \\ \mathbf{G}_i(\mathbf{x}, \boldsymbol{\xi}, t) &= \int_0^t \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{g}_{i0}(\tau) \, d\tau \end{aligned} \tag{1.17}$$

are eventually expressed in terms of the Green's function and its derivatives⁸, and the form of the Green's formula

$$\begin{aligned} \tilde{\mathbf{w}}(\mathbf{x}, t) &= \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{f}(\boldsymbol{\xi}, \tau) \, d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \mathbf{G}_b(\mathbf{x}, t, \tau) \mathbf{w}_b(\tau) \, d\tau + \\ &+ \int_{\Omega} \mathbf{G}_i(\mathbf{x}, \boldsymbol{\xi}, t) \mathbf{w}_0(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \quad \text{in } \bar{\Omega} \times \mathbb{R}^+ \end{aligned} \tag{1.18}$$

becomes more convenient.

Note that the vector-functions \mathbf{f} , \mathbf{w}_b , and \mathbf{w}_0 must be chosen in such a way that the integrals in (1.18) are well defined, so that the solution $\tilde{\mathbf{w}} \in \mathbf{W}$ exists and is unique.

⁸See Subsection 1.2.1 below.

1.2.1 Particular Cases

In order to be more illustrative, consider some particular cases⁹. First, consider the one-dimensional ordinary differential equation

$$\frac{d^2x}{dt^2} + k^2 \cdot x = 0 \quad \text{in } \mathbb{R}^+$$

describing the free vibrations of a spring with stiffness $c = \frac{k^2}{m}$ and mass m in linear approximation.

Assume that at the initial instant $t = 0$, the following Cauchy conditions are given:

$$x(0) = x_0, \quad \dot{x}(0) = \left. \frac{dx}{dt} \right|_{t=0} = x_0^1,$$

where x_0 and x_0^1 are given constants. Then, it is shown that the new state function

$$\tilde{x}(t) = \theta(t - 0)x(t)$$

satisfies the non-homogeneous equation

$$\frac{d^2\tilde{x}}{dt^2} + k^2 \cdot \tilde{x} = x_0\dot{\delta}(t) + x_0^1\delta(t),$$

with homogeneous Cauchy conditions

$$\tilde{x}(0) = 0, \quad \dot{\tilde{x}}(0) = 0.$$

⁹The aim of this section is to show how the boundary and initial data are included into the right-hand side of the governing equation and how then the Green's formula (1.18) works for particular linear state constraints. Thus, the full derivation procedure of the Green's function, which can be found elsewhere, is skipped.

Here the dot above the Dirac function means the first derivative in terms of distributions, and $t - 0$ in the argument of the Heaviside function denotes that the point $t = 0$ is approached from the right. It is evident that in the sense of distributions

$$\dot{\theta}(t - 0) = \delta(t).$$

After applying the Laplace transform, the Green's function, i.e., the solution of

$$\begin{aligned} \frac{d^2 G}{dt^2} + k^2 \cdot G &= \delta(t - \tau) \quad \text{in } \mathbb{R}^+, \\ G(0, \tau) &= 0, \quad \left. \frac{dG}{dt} \right|_{t=0} = 0 \quad \text{for all } \tau \in \mathbb{R}^+ \end{aligned}$$

is derived as [78]

$$G(t, \tau) = \frac{1}{k} \theta(t - \tau) \sin[k(t - \tau)].$$

Note that this formula makes sense even in the limiting case $k \rightarrow 0$. Therefore, the transformation (1.13) becomes

$$\mathbf{g}_i(x_0, x_0^1) = \mathbf{g}_{i0}(t) \begin{pmatrix} x_0 \\ x_0^1 \end{pmatrix} \quad \text{with } \mathbf{g}_{i0}(t) = \begin{pmatrix} \dot{\delta}(t) & 0 \\ 0 & \delta(t) \end{pmatrix},$$

and (1.17) results in

$$\begin{aligned} G_{i11}(t) &= \int_0^t G(t, \tau) \dot{\delta}(\tau) d\tau = - \left. \frac{\partial G(t, \tau)}{\partial \tau} \right|_{\tau=0} = \\ &= \theta(t - 0) \cos(kt), \end{aligned}$$

$$G_{i22}(t) = \int_0^t G(t, \tau) \delta(\tau) d\tau = G(t, 0) = \frac{1}{k} \theta(t - 0) \sin(kt).$$

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Here it is taken into account that for any differentiable function f

$$\int_0^t f(t-\tau) \delta(\tau) d\tau = -\left. \frac{\partial f(t-\tau)}{\partial \tau} \right|_{\tau=0},$$

and that in the sense of distributions

$$\begin{aligned} \frac{\partial G(t, \tau)}{\partial \tau} &= -\frac{1}{k} \delta(t-\tau) \sin[k(t-\tau)] - \theta(t-\tau) \cos[k(t-\tau)] = \\ &= -\theta(t-\tau) \cos[k(t-\tau)]. \end{aligned}$$

Then,

$$\tilde{x}(t) = x_0 G_{i11}(t) + x_0^1 G_{i22}(t) = \theta(t-0) \left[x_0 \cos(kt) + \frac{x_0^1}{k} \sin(kt) \right],$$

so that

$$x(t) = x_0 \cos(kt) + \frac{x_0^1}{k} \sin(kt) \quad \text{in } \mathbb{R}^+.$$

Consider the one-dimensional homogeneous heat equation in an infinite domain

$$\frac{\partial \Theta}{\partial t} = d \frac{\partial^2 \Theta}{\partial x^2} \quad \text{in } \mathbb{R} \times \mathbb{R}^+$$

with the initial condition

$$\Theta(x, 0) = \Theta_0(x) \quad \text{in } \mathbb{R}.$$

Here Θ describes the temperature distribution in an infinite, sufficiently thin rod, the material of which has the thermal diffusivity of d , and Θ_0 is a prescribed function.

It is obvious that in this case the new state function

$$\tilde{\Theta}(x, t) = \theta(t-0) \Theta(x, t)$$

reduces the above Cauchy problem to the following one:

$$\frac{\partial \tilde{\Theta}}{\partial t} = d \frac{\partial^2 \tilde{\Theta}}{\partial x^2} + \Theta_0(x) \delta(t),$$

$$\tilde{\Theta}(x, 0) = 0.$$

The corresponding Green's function is the solution of the Cauchy problem

$$\frac{\partial G}{\partial t} = d \frac{\partial^2 G}{\partial x^2} + \delta(x - \xi) \delta(t - \tau) \quad \text{in } \mathbb{R} \times \mathbb{R}^+,$$

$$G = 0 \quad \text{in } \mathbb{R}.$$

The problem domain consists of both an infinite piece and a semi-infinite piece. Therefore, the Fourier transform must be applied with respect to $x \in \mathbb{R}$ coordinate, while for the time-variable $t \in \mathbb{R}^+$, the Laplace transform is more suitable. Eventually, the Green's function is found to be

$$G(x, \xi, t, \tau) = \frac{1}{\sqrt{4\pi d(t - \tau)}} \theta(t - \tau) \exp \left[-\frac{(x - \xi)^2}{4d(t - \tau)} \right].$$

At this, the transformation (1.13) results in

$$g_b(\Theta_0) = g_{b0}(t) \Theta_0(x) \quad \text{with } g_{b0}(t) = \delta(t),$$

and (1.17) provides

$$G_i(x, \xi, t) = G(x, \xi, t, 0) = \frac{1}{\sqrt{4\pi dt}} \theta(t - 0) \exp \left[-\frac{(x - \xi)^2}{4dt} \right].$$

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Eventually, the general solution reads as

$$\begin{aligned}\tilde{\Theta}(x, t) &= \int_0^t \int_{-\infty}^{\infty} G(x, \xi, t, \tau) \delta(\tau) \Theta_0(\xi) d\xi d\tau = \\ &= \int_{-\infty}^{\infty} G_i(x, \xi, t) \Theta_0(\xi) d\xi = \\ &= \frac{1}{\sqrt{4\pi dt}} \theta(t-0) \int_{-\infty}^{\infty} \Theta_0(\xi) \exp\left[-\frac{(x-\xi)^2}{4dt}\right] d\xi,\end{aligned}$$

so that

$$\Theta(x, t) = \frac{1}{\sqrt{4\pi dt}} \int_{-\infty}^{\infty} \Theta_0(\xi) \exp\left[-\frac{(x-\xi)^2}{4dt}\right] d\xi \quad \text{in } \mathbb{R} \times \mathbb{R}^+.$$

In the case of a one-dimensional homogeneous wave equation on the semi-axis

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+,$$

where c is the velocity of wave propagation, the boundary and initial conditions

$$\begin{aligned}w(0, t) &= w_b(t) \quad \text{in } \mathbb{R}^+, \\ w(x, 0) &= w_0(x), \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = w_0^1(x) \quad \text{in } \mathbb{R}^+\end{aligned}$$

are included into its right-hand side by the transformation

$$\tilde{w}(x, t) = \theta(t-0) \theta(x-0) w(x, t).$$

As a result, the wave equation transforms into

$$\frac{\partial^2 \tilde{w}}{\partial t^2} = c^2 \frac{\partial^2 \tilde{w}}{\partial x^2} - c^2 w_b(t) \delta'(x) + [w_0(x) \dot{\delta}(t) + w_0^1(x) \delta(t)].$$

The Green's function is derived by applying the Laplace transform twice:

$$G(x, \xi, t, \tau) = \frac{1}{2c} \theta(x - \xi) \theta(t - \tau) \theta(c(t - \tau) - |x - \xi|).$$

In this case the transformation (1.13) results in

$$g_b(w_b) = g_{b0}(x) w_b(t) \quad \text{with} \quad g_{b0}(x) = \delta'(x),$$

$$\mathbf{g}_i(w_0, w_0^1) = \mathbf{g}_{i0}(t) \begin{pmatrix} w_0 \\ w_0^1 \end{pmatrix} \quad \text{with} \quad \mathbf{g}_{i0}(t) = \begin{pmatrix} \dot{\delta}(t) & 0 \\ 0 & \delta(t) \end{pmatrix},$$

and (1.17) provides

$$G_b(x, \xi, t) = -c^2 \int_0^\infty G(x, \xi, t, \tau) \delta'(\xi) d\xi = c^2 \frac{\partial G(x, \xi, t, \tau)}{\partial \xi} \Big|_{\xi=0},$$

$$G_{i11}(x, \xi, t) = \int_0^t G(x, \xi, t, \tau) \delta(\tau) d\tau = G(x, \xi, t, 0),$$

$$G_{i22}(x, \xi, t) = \int_0^t G(x, \xi, t, \tau) \dot{\delta}(\tau) d\tau = - \frac{\partial G(x, \xi, t, \tau)}{\partial \tau} \Big|_{\tau=0}.$$

Eventually, the solution of the wave equations reads as

$$\begin{aligned} \tilde{w}(x, t) &= \int_0^t G_b(x, \xi, t) w_b(\tau) d\tau + \\ &+ \int_0^\infty [G_{i11}(x, \xi, t) w_0^1(\xi) - G_{i22}(x, \xi, t) w_0(\xi)] d\xi. \end{aligned}$$

Finally, consider the homogeneous two-dimensional plate equation

$$D\Delta\Delta w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{in} \quad [0, l_1] \times [0, l_2] \times \mathbb{R}^+$$

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describing free transverse (bending) vibrations of sufficiently thin rectangular plates. Here Δ is the two-dimensional Laplace operator, D is the bending stiffness, ρ is the material density, and h is the small constant thickness of the plate.

Let the plate be simply supported through all its edges so that the boundary conditions are given as follows:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, l_1 \quad \text{in } [0, l_2] \times \mathbb{R}^+,$$
$$w = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at } y = 0, l_2 \quad \text{in } [0, l_1] \times \mathbb{R}^+.$$

The following initial conditions are given:

$$w(x, y, 0) = w_0(x, y), \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = w_0^1(x, y) \quad \text{in } [0, l_1] \times [0, l_2],$$

which are included into the right-hand side of the plate equation by the transformation

$$\tilde{w}(x, y, t) = \theta(t - 0) w(x, y, t).$$

If the boundary conditions were not homogeneous, then a similar transform must be done with respect to x and y variables. Then, the plate equation transforms into

$$D\Delta\Delta\tilde{w} + \rho h \frac{\partial^2 \tilde{w}}{\partial t^2} = \rho h [w_0(x, y) \dot{\delta}(t) + w_0^1(x, y) \delta(t)].$$

The boundary conditions preserve their form, and the initial conditions are transformed to

$$\tilde{w}(x, y, 0) = 0, \quad \left. \frac{\partial \tilde{w}}{\partial t} \right|_{t=0} = 0.$$

In this case, since the plate is finite, the separation of variables is more appropriate, providing

$$G(x, \xi, y, \eta, t, \tau) = \frac{8}{\alpha l_1 l_2} \theta(t - \tau) \times \sum_{n,m=1}^{\infty} \varphi_n(x) \varphi_n(\xi) \phi_m(y) \phi_m(\eta) \psi_{nm}(t),$$

where

$$\begin{aligned} \varphi_n(x) &= \sin\left(\frac{\pi n}{l_1} x\right), \quad \phi_m(y) = \sin\left(\frac{\pi m}{l_2} y\right), \\ \psi_{nm}(t) &= \exp[-\lambda_{mn}(t - \tau)], \\ \lambda_{mn} &= \alpha \left[\left(\frac{\pi n}{l_1}\right)^2 + \left(\frac{\pi m}{l_2}\right)^2 \right], \quad \alpha = \sqrt{\frac{D}{\rho h}}. \end{aligned}$$

The transformation (1.13) then results in

$$\mathbf{g}_i(w_0, w_0^1) = \mathbf{g}_{i0}(t) \begin{pmatrix} w_0 \\ w_0^1 \end{pmatrix} \quad \text{with} \quad \mathbf{g}_{i0}(t) = \begin{pmatrix} \dot{\delta}(t) & 0 \\ 0 & \delta(t) \end{pmatrix},$$

and (1.17) becomes

$$\begin{aligned} G_{i11}(x, \xi, y, \eta, t) &= \rho h \int_0^t G(x, \xi, y, \eta, t, \tau) \dot{\delta}(\tau) d\tau = \\ &= -\rho h \left. \frac{\partial G(x, \xi, y, \eta, t, \tau)}{\partial \tau} \right|_{\tau=0}, \\ G_{i22}(x, \xi, y, \eta, t) &= \rho h \int_0^t G(x, \xi, y, \eta, t, \tau) \delta(\tau) d\tau = \\ &= \rho h G(x, \xi, y, \eta, t, 0). \end{aligned}$$

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Thus, the general solution is expressed as follows:

$$\begin{aligned}\tilde{w}(x, y, t) = & \int_0^{l_1} \int_0^{l_2} G_{i11}(x, \xi, y, \eta, t) w_0(\xi, \eta) d\xi d\eta + \\ & + \int_0^{l_1} \int_0^{l_2} G_{i22}(x, \xi, y, \eta, t) w_0^1(\xi, \eta) d\xi d\eta.\end{aligned}$$

1.3 The Green's Function Method for Non-linear Systems

As mentioned in the beginning of Section 1.2, linear models provide the simplest constraints, which are appropriate only for basic theoretical analysis. In some cases, even with the inaccuracy of linear models they are still sufficiently useful. However, when a more precise analysis is required, linear models are insufficient and nonlinear models must be used. For example, the Euler-Bernoulli model provides a linear differential equation for analyzing transverse deflections of thin beams, which is valid until the beam is subject to loads causing transverse deflections of infinitesimally small magnitude compared with the thickness of the beam. But, if the acting loads cause deflections comparable with the thickness of the beam, then, for proper analysis, geometrically nonlinear beam theories yielding nonlinear differential equations must be considered.

On the other hand, the higher accuracy provided by using nonlinear models is counterbalanced by a higher computational cost required for their analysis. The continuous development and improvement of efficient numerical schemes, as well as the enormous capability of today's computers, allow the approximation of regular nonlinear differential equations within several minutes. Singularities occurring in the equation or initial and/or boundary conditions, the complexity of the domain, the dimension of the system, etc., of course, can considerably increase the computational cost. However, appropriate modifications in the chosen numerical scheme sometimes allow the problem to be handled within a reasonable amount of time. Nevertheless, if the analysis must be carried out for several model parameters varying in different

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ranges, computations can take much longer.

Similar to the case of linear models, the analysis of nonlinear models can be simplified when the explicit solution of the state constraints are somehow found or the approximation scheme contains semi-analytical steps. However, there is not any systematic approach for exact solutions of nonlinear equations in general. On the other hand, there exist various semi-analytical methods for analyzing specific types of nonlinear models. Because the choice of the method strictly depends on the peculiarities of the problem under consideration, such methods can be efficiently involved for analyzing various specific applied nonlinear models.

Among such methods are the Adomian decomposition method [95, 96], the cell-to-cell mapping method [97], the homotopy analysis method [98, 99], the group-theoretical approach [100, 101], the inverse scattering transform [102], generalized separation of variables [76, 103, 104], the simplest equation method [105], singularity analysis [106], etc. See also [107] for an inclusive introduction into nonlinear models and [108, 109] for alternative methods for the analysis of nonlinear systems, as well as [110] for a wider review. See also the interesting article [111].

Several previous attempts have been made to extend the concept of the Green's function from linear systems, for which it is initially developed, to nonlinear ones. Some of them are described further in this section. In the majority of case studies of nonlinear models by means of the Green's function method, the solution of the nonlinear state constraints is approximated by the solution of their linearized version (see, for instance, [112–118] and references therein). The disadvantage of this approach is that the linear approximation may not include some essential features of the nonlinear model. Therefore, in general, this

approach is valid only for a very basic analysis. On the other hand, the Adomian decomposition and homotopy analysis methods, for example, involve a special expansion of the nonlinear term (Adomian polynomials), so that the nonlinear problem is split into a set of linear problems. Then, using the Green's formula at each iteration step, eventually a similar representation for the approximate solution of the nonlinear problem can be derived.

Other approaches use the Green's function solution of an auxiliary problem to invert the nonlinear state constraints and derive their integral formulation [119–127]. Extensive numerical analysis in [126] shows the efficiency of the integral formulation approach over the usual discretizing schemes like finite differences. This approach is successfully applied in the study of nonlinear reaction-diffusion processes [128–131]. In some exceptional cases, the Green's function for nonlinear state constraints is also obtained from fully nonlinear considerations [87, 132–146].

Without going deep into all the existing approaches, only two of them are described here in detail with specific particular cases.

1.3.1 Cacuci's Approach

The first approach, developed by D. G. Cacuci et al. in [147–151], introduces the notion of the so-called dual operator, which is shown to be the generalization of the adjoint operator, defined for linear systems, to the nonlinear case. Then, the dual operator is used to define the so-called advanced and retarded propagators, which are shown to be the generalizations of the Green's function to the nonlinear systems. Finally, those propagators are used to derive a closed-form integral representation for the solution of a nonlinear problem, analogous to

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the Green's representation formula for linear systems.

For a more detailed description of Cacuci's approach, consider the abstract nonlinear differential equation

$$\mathcal{D}[\mathbf{w}] = \mathbf{f}(\mathbf{x}, t) \quad \text{in } \Omega \times \mathbb{R}^+ \quad (1.19)$$

subject to the nonlinear boundary conditions

$$\mathcal{B}[\mathbf{w}] = \mathbf{w}_b(t) \quad \text{in } \partial\Omega \times \mathbb{R}^+ \quad (1.20)$$

and the nonlinear initial conditions

$$\mathcal{I}[\mathbf{w}] = \mathbf{w}_i(\mathbf{x}) \quad \text{in } \Omega \times \{0\}. \quad (1.21)$$

Here $\mathcal{D} : \mathbf{W} \rightarrow \mathbf{F}$, $\mathcal{B} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$, and $\mathcal{I} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ are nonlinear operators¹⁰ acting on the state vector function¹¹ $\mathbf{w} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$, $\mathbf{f} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a given vector-function, and $\mathbf{w}_b : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ and $\mathbf{w}_i : \Omega \rightarrow \mathbb{R}^m$ are the given boundary and initial data.

In the nonlinear case it is also possible to include the boundary and initial data (1.20) and (1.21) into the state equation (1.19) (see Section 1.2 for the linear case). The corresponding transformations are also expressed in terms of the Dirac function and its derivatives, but have a more complicated form compared with the linear case. Following Cacuci, formally denote that inclusion by

$$\mathcal{D}[\tilde{\mathbf{w}}] + \delta\mathcal{B}[\tilde{\mathbf{w}}] + \delta\mathcal{I}[\tilde{\mathbf{w}}] = \mathbf{f}(\mathbf{x}, t) + \delta\mathbf{w}_b(t) + \delta\mathbf{w}_i(\mathbf{x}) \quad \text{in } \overline{\Omega} \times \mathbb{R}^+. \quad (1.22)$$

¹⁰In Cacuci's approach \mathcal{D} may be a differential, integral, or even multiplicative operator, or a combination of them. Nevertheless, in order to remain in the scope of this book, assume that \mathcal{D} is a nonlinear differential operator.

¹¹In [149] the state space is $\mathbf{W} = L^2$.

Here $\tilde{\mathbf{w}}$ is the new state vector satisfying (1.22) and the homogeneous boundary and initial conditions

$$\mathcal{B}[\tilde{\mathbf{w}}] = \mathbf{0} \quad \text{in } \partial\Omega \times \mathbb{R}^+, \tag{1.23}$$

$$\mathcal{I}[\tilde{\mathbf{w}}] = \mathbf{0} \quad \text{in } \Omega \times \{0\}. \tag{1.24}$$

Without stopping on the details of the method, let us mention only the final representation formula for the general solution of (1.22)-(1.24), which reads as

$$\begin{aligned} \tilde{\mathbf{w}}(\mathbf{x}, t) = & \int_0^t \int_{\bar{\Omega}} \mathbf{G}(\tilde{\mathbf{w}}(\mathbf{x}, t), \mathbf{x}, \boldsymbol{\xi}, t, \tau) [\mathbf{f}(\boldsymbol{\xi}, \tau) + \\ & + \delta \mathbf{w}_b(\tau) + \delta \mathbf{w}_i(\boldsymbol{\xi})] d\boldsymbol{\xi} d\tau \quad \text{in } \bar{\Omega} \times \mathbb{R}^+. \end{aligned} \tag{1.25}$$

Here \mathbf{G} is the so-called backward propagator defined as the unique solution (in the sense of distributions) of

$$\{\mathcal{D}_0[\tilde{\mathbf{w}}] + \delta \mathcal{B}_0[\tilde{\mathbf{w}}] + \delta \mathcal{I}_0[\tilde{\mathbf{w}}]\} \mathbf{G} = \delta(\mathbf{x} - \boldsymbol{\xi}) \delta(t - \tau) \quad \text{in } \bar{\Omega} \times \mathbb{R}^+.$$

Above,

$$\begin{aligned} \mathcal{D}_0[\tilde{\mathbf{w}}] \mathbf{v} &= \int_0^1 \mathcal{D}'[\epsilon \tilde{\mathbf{w}}] \mathbf{v} d\epsilon, \\ \delta \mathcal{B}_0[\tilde{\mathbf{w}}] \mathbf{v} &= \int_0^1 \delta \mathcal{B}'[\epsilon \tilde{\mathbf{w}}] \mathbf{v} d\epsilon, \quad \delta \mathcal{I}_0[\tilde{\mathbf{w}}] \mathbf{v} = \int_0^1 \delta \mathcal{I}'[\epsilon \tilde{\mathbf{w}}] \mathbf{v} d\epsilon, \end{aligned}$$

where the prime denotes the Gâteaux derivative defined by

$$\mathcal{D}'[\tilde{\mathbf{w}}] \mathbf{v} = \left. \frac{d\mathcal{D}[\tilde{\mathbf{w}} + \epsilon \mathbf{v}]}{d\epsilon} \right|_{\epsilon=0}.$$

It is proved that the representation formula (1.25) is unique. An

analogous formula holds for the forward propagator denoted by G^* .

1.3.1.1 Particular Case: The Generalized Burgers-Korteweg-de Vries Equation

Consider the one-dimensional generalized Burgers-Korteweg-de Vries equation

$$\frac{\partial w}{\partial t} + \eta \frac{\partial w}{\partial x} w^p \frac{\partial w}{\partial x} - \nu \frac{\partial^2 w}{\partial x^2} + \alpha \frac{\partial^3 w}{\partial x^3} = 0 \text{ in } (0, 1) \times \mathbb{R}^+,$$

where η , ν , and α are positive constants, and $p \in \mathbb{N}$. In particular, when $\eta = \nu = 0$ and $p = \alpha = 1$, the equation is reduced to the Burgers' equation, while when $\eta = \alpha = 0$ and $p = \nu = 1$, it is reduced to the Korteweg-de Vries equation.

It is assumed that the periodic boundary conditions

$$w(0, t) = w(1, t) = 0, \quad \left. \frac{\partial w}{\partial x} \right|_{x=0,1} = 0, \quad \left. \frac{\partial^2 w}{\partial x^2} \right|_{x=0,1} = 0 \text{ in } \mathbb{R}^+$$

are given. The initial state

$$w(x, 0) = w_0(x) \text{ in } [0, 1]$$

is also assumed to be given, where w_0 is a prescribed function.

Then, the Cacuci's approach provides [151]

$$w(x, t) = \int_0^1 w_0(\xi) G^*(\xi, 0, x, t) d\xi - \frac{1}{1+p} \int_0^t \int_0^1 G_b^*(\xi, \tau, x, t) \left[(w^0)^p w(\xi, \tau) - w^{p+1}(\xi, \tau) \right] d\xi d\tau,$$

where $w^0 \in L^2[0, 1]$ is an arbitrary function, and G^* is the forward propagator given by

$$G^*(\xi, \tau, x, t) = \theta(t - \tau) \left[1 + 2 \sum_{n=1}^{\infty} \varphi_n(x - \xi, t - \tau) \right],$$

$$G_b^*(\xi, \tau, x, t) = \frac{\partial G^*(\xi, \tau, x, t)}{\partial \xi},$$

$$\varphi_n(x, t) = \cos \left[2\pi n \left(x + \left(4\pi^2 n^2 \alpha - \eta - \frac{(w^0)^2}{1+p} \right) t \right) \right].$$

It is shown in [150] that this solution is consistent with the exact one, so that the approach is advantageous.

1.3.2 Frasca's Approach

The second approach is developed by M. Frasca in [152, 153] for specific applications in quantum field theory. This approach formally uses the Green's representation formula for linear equations, i.e., the convolution of the Green's function and the right-hand side, to approximate the solution of a nonlinear problem. At this, the nonlinear Green's function is derived from the corresponding *nonlinear* equation. To incorporate the approximation error, a scaling factor is introduced into the right-hand side of the equation for the Green's function. Even though the Frasca's approach is developed for a nonlinear differential equation of a particular form, it would be beneficial to explore possibilities of extending this approach for nonlinear equations of other or maybe general forms.

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Consider the second order nonlinear ordinary differential equation

$$\frac{d^2\mathbf{w}}{dt^2} + \mathbf{N}(\mathbf{w}, t) = \mathbf{f}(t) \quad \text{in } \mathbb{R}^+. \quad (1.26)$$

Here $\mathbf{w} : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is the state vector-function of a dynamic system, $\mathbf{N} : \mathbb{R}^m \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a generic non-linearity, and $\mathbf{f} : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a given vector-function.

For simplicity assume that the homogeneous¹² Cauchy conditions

$$\mathbf{w}(0) = \mathbf{0}, \quad \left. \frac{d\mathbf{w}}{dt} \right|_{t=0} = \mathbf{0} \quad (1.27)$$

are given. Then, the general solution of (1.26), (1.27) is approximately given by the Green's representation formula

$$\mathbf{w}(t) \approx \int_0^t \mathbf{G}(t - \tau) \mathbf{f}(\tau) d\tau \quad \text{in } \mathbb{R}^+. \quad (1.28)$$

Here \mathbf{G} is the formal Green's function of (1.26), (1.27), defined as the fundamental solution of

$$\frac{d^2\mathbf{G}}{dt^2} + \mathbf{N}(\mathbf{G}, t) = \mathbf{s}\delta(t) \quad \text{in } \mathbb{R}^+, \quad (1.29)$$

$$\mathbf{G}|_{t=0} = 0, \quad \left. \frac{d\mathbf{G}}{dt} \right|_{t=0} = 0. \quad (1.30)$$

Above $\mathbf{s} \in \mathbb{R}^m$ is a scaling factor and is chosen to minimize the error of approximation. Corresponding error estimates in the case of particular non-linearities are derived in [153]. See also Paragraph 1.3.2.2 below.

Thus, if \mathbf{N} has such a form for which the Cauchy problem (1.29),

¹²The approach in principle can be extended to the case of non-homogeneous Cauchy conditions.

(1.30) admits an exact solution, then the solution of (1.26), (1.27) can be approximated by (1.28). Luckily, equations of the form (1.29) are exactly solvable for quite general forms of \mathbf{N} . For corresponding cases see, for instance, [82, 155–157].

Furthermore, Frasca has shown that (1.28) is the first term in the so-called short time expansion (see [153] for details) of the exact solution of (1.26), (1.27). More specifically,

$$\mathbf{w}(t) = \int_0^t \mathbf{G}(t - \tau) \mathbf{f}(\tau) \, d\tau + \sum_{k=1}^{\infty} a_k \int_0^t (t - \tau)^k \mathbf{G}(t - \tau) \mathbf{f}(\tau) \, d\tau,$$

where a_k are defined by calculating the derivatives

$$\left. \frac{d^k \mathbf{w}}{dt^k} \right|_{t=0}.$$

The series is truncated once the required accuracy of approximation is achieved.

1.3.2.1 Particular Cases

Several particular forms of \mathbf{N} admitting explicit an solution of (1.29), (1.30) are considered in this paragraph. Consider for simplicity the one-dimensional case. The exact solution of (1.29) can be obtained, for instance, in the case of a quadratic non-linearity

$$N(w, t) = aw^2 \tag{1.31}$$

with constant $0 \neq a \in \mathbb{R}$. Then (1.29) takes the form

$$\frac{d^2 G}{dt^2} + aG^2 = s\delta(t) \quad \text{in } \mathbb{R}^+,$$

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the exact solution of which is

$$G(t) = -\frac{1}{c}\theta(t)\wp(ct + c_1; 0, c_2). \quad (1.32)$$

Here,

$$c = \left(-\frac{a}{6}\right)^{\frac{1}{3}},$$

c_1 and c_2 are integration constants determined from the Cauchy conditions (1.30), and \wp is the Weierstrass elliptic function (p -function) defined as

$$\wp(t; \omega_1, \omega_2) = \frac{1}{t^2} + \sum_{n^2+m^2 \neq 0} \left[\frac{1}{(t + \omega_1 m + \omega_2 n)^2} - \frac{1}{(\omega_1 m + \omega_2 n)^2} \right].$$

Note that the Green's function strongly depends on sign a . Indeed, the conditions sign $a < 0$, $c > 0$ make G to be a real-valued function. Otherwise, when sign $a > 0$, it is apparent that G becomes a complex-valued function.

It is also possible to derive an exact solution in the case where

$$N(w, t) = \frac{a}{w}$$

with constant $0 \neq a \in \mathbb{R}$. Then, G is determined from

$$\frac{d^2 G}{dt^2} + \frac{a}{G} = s\delta(t) \quad \text{in } \mathbb{R}^+,$$

the exact solution of which is

$$G(t) = c_1\theta(t)\exp[-\varphi^2(t; c_1, c_2)],$$

$$\varphi(t; c_1, c_2) = \operatorname{erf}^{-1} \left[-\sqrt{\frac{2a}{\pi} c_1^2 (t + c_2)^2} \right],$$

where erf^{-1} is the inverse function to the Gauss error function, and c_1 and c_2 are integration constants determined from the Cauchy conditions (1.30). The Green's function in this case also depends on sign a .

Another exactly integrable case for (1.26) is provided in the case of cubic non-linearity

$$N(w, t) = w^3.$$

This case corresponds to the homogeneous Duffing equation without terms characterizing damping and linear stiffness, which describes forced oscillations in physically and geometrically nonlinear media. Then (1.29), (1.30) is reduced to

$$\frac{d^2 G}{dt^2} + G^3 = s\delta(t) \quad \text{in } \mathbb{R}^+,$$

$$G|_{t=0} = 0, \quad \left. \frac{dG}{dt} \right|_{t=0} = 0,$$

the exact solution of which reads as [152]

$$G(t) = 2^{\frac{1}{4}} \theta(t) \cdot \operatorname{sn} \left[\frac{t}{2^{\frac{1}{4}}}, i \right].$$

Here sn is the Jacobi snoidal function, and i is the imaginary unit. In numerical computations it is usually convenient to use its Fourier series given by

$$\operatorname{sn}(t, i) \approx \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{(-1)^n \exp \left[-\pi n - \frac{\pi}{2} \right]}{1 + \exp[-2\pi n - \pi]} \sin \left[\frac{2n+1}{2K} \pi t \right],$$

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where

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 + \sin^2 \tau}} d\tau \approx 1.3103.$$

In the case of exponential non-linearity

$$N(w, t) = \exp w,$$

Eq. (1.29) is reduced to

$$\frac{d^2 G}{dt^2} + \exp G = s\delta(t) \quad \text{in } \mathbb{R}^+.$$

Its exact solution is

$$G(t) = \theta(t) \cdot \ln \left[\frac{1}{2} c_1 \left(1 - \tanh^2 \left(\frac{1}{2} \sqrt{c_1 (t + c_2)^2} \right) \right) \right].$$

Here as well the integration constants c_1 and c_2 must be determined from the Cauchy conditions (1.30).

Another exactly solvable case is provided when

$$N(w, t) = \sin w.$$

This case corresponds to the reduced sine-Gordon equation, having many applications in various fields of science. Then, (1.29) and (1.30) are reduced to

$$\frac{d^2 G}{dt^2} + \sin w = s\delta(t) \quad \text{in } \mathbb{R}^+,$$

$$G|_{t=0} = 0, \quad \left. \frac{dG}{dt} \right|_{t=0} = 0,$$

the exact solution of which reads as [152]

$$G(t) = 2\theta(t) \cdot \operatorname{am} \left[\frac{t}{\sqrt{2}}, \sqrt{2} \right].$$

Here am is the Jacobi amplitude function

$$\operatorname{am} (t, \sqrt{2}) = \arcsin \left[\operatorname{sn} (t, \sqrt{2}) \right].$$

To simplify the computations, its Fourier series given by

$$\begin{aligned} \operatorname{am} (t, \sqrt{2}) &\approx \frac{(1+i)\pi}{4K} t + \\ &+ 2 \sum_{n=0}^{\infty} \frac{1}{n} \frac{\exp \left[-\frac{(1+i)\pi n}{2} \right]}{1 + \exp [-(1+i)\pi n]} \sin \left[\frac{(1+i)\pi n}{2K} t \right] \end{aligned}$$

is used.

It is noteworthy that in the case of non-linearity

$$N(w, t) = f(w) \left(\frac{dw}{dt} \right)^3,$$

where f is any strictly positive (strictly negative), twice integrable function, the Green's function is found to be [158]

$$G(t) = \theta(t) \left[\int_0^w \left(c_2 + \int_0^z f(w) dw \right) dz \right]^{-1} (t + c_1).$$

Here the superscript -1 means the inverse function, and c_1 and c_2 are integration constants. In particular, when

$$f(w) = w^3,$$

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then

$$G(t) = \theta(t) \cdot \frac{t + c_2 - c_1 W(\varphi(t; c_1, c_2))}{c_1},$$

where W is the Lambert function and

$$\varphi(t; c_1, c_2) = \frac{1}{c_1} \exp\left[\frac{t + c_2}{c_1}\right].$$

Note that the equation (1.26) is a coordinate reduced version of various applied partial differential equations, describing, as a rule, nonlinear vibrations in solids or fluids [154]¹³. Indeed, consider, for instance, the one-dimensional nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = \alpha \frac{\partial}{\partial x} \left[\exp[\lambda x] \frac{\partial w}{\partial x} \right] + N(w)$$

with generic nonlinear function N . Using the method of generalized variable separation and denoting by

$$\chi^2 = a_1 \left[\frac{\exp[-\lambda x]}{\alpha \lambda^2} - \frac{(t + a_2)^2}{4} \right],$$

where a_1 and a_2 are arbitrary constants, the wave equation is reduced to the nonlinear ordinary differential equation [103]

$$\frac{d^2 w}{d\chi^2} + \frac{4}{c_1} N(w) = 0.$$

Thus, using the explicit solutions given in the handbooks [75, 76], solutions of the corresponding nonlinear *partial* differential equations

¹³Specifically, it describes nonlinear vibrations of a pendulum, longitudinal wave propagation in physically or geometrically nonlinear elastic rods, nonlinear electromagnetic oscillations, etc., and also has applications in quantum field theory.

can also be approximated by (1.28).

1.3.2.2 Numerical Error of Frasca's Approximation

In order to trust the validity of the approximation (1.28), comparisons between approximate and exact solutions are carried out for different right-hand sides f . For simplicity, consider the one-dimensional case. Let, for instance, the non-linearity be given by (1.31) with $a = 1$. Then, the Green's function is given by (1.32).

A slight modification in the approximation scheme is done by introducing two scaling parameters s_1 and s_2 , where s_1 enters into the Green's function equation:

$$\frac{d^2 G}{dt^2} + G^2 = s_1 \delta(t),$$

while s_2 regulates the approximation

$$w(t) \approx s_2 \int_0^t G(t - \tau) f(\tau) d\tau. \quad (1.33)$$

To measure the error of approximation, the logarithmic error

$$\text{Er}(t) = \log_{10} |w_{\text{app}}(t) - w_{\text{exact}}(t)|$$

is computed, evaluating the difference between the numerical solution w_{app} and the exact solution w_{exact} in degrees of 10.

Fig. 1.2 expresses the behaviour of Er against $t \in [0, 1]$ when $f(t) = \delta(t)$. As expected, the approximation error is quite small. More specifically, $\text{Er} \leq -6.55$. Less accurate but still efficient approximation is provided in the case when $f(t) = \theta(t)$. Fig. 1.3 shows that $\text{Er} \leq -3.5$. For harmonic perturbations the error is still small: $\text{Er} \leq -4.25$ (see

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Fig. 1.4). Further, for exponential and polynomial sources, the estimate $\text{Er} \leq -2.5$ is observed (see Fig. 1.5). For a logarithmic source, $\text{Er} \leq -4$ (see Fig. 1.6). The efficiency of (1.33) is also observed for other types of non-linearities. The error of approximation by (1.28) in the case of (1.31) is shown in Tab. 1.1 for different right-hand sides f .

| $f(t)$ | min Er | max Er | s_1 | s_2 |
|---------------------|--------|--------|---------|---------|
| $\tilde{\delta}(t)$ | -6.63 | -6.55 | 2 | 1 |
| $\theta(t)$ | -8 | -3.5 | 2.1475 | 0.93107 |
| $\sin(t)$ | -9.5 | -4.25 | 2.7807 | 0.72126 |
| $\exp(t)$ | -6 | -2.5 | 197.1 | 0.01 |
| $1 + t + t^2 + t^3$ | -6.1 | -2.5 | 27.5783 | 0.07149 |
| $\ln(1 + t)$ | -9.1 | -4 | 2.5925 | 0.7743 |

Table 1.1: Minimal and maximal logarithmic errors of approximation for various source functions: quadratic potential

The approximation error can be reduced further by involving proceeding terms of the short time expansion above.

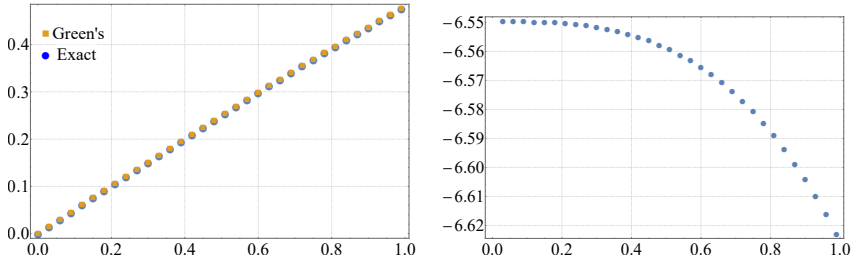


Figure 1.2: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = \delta(t)$

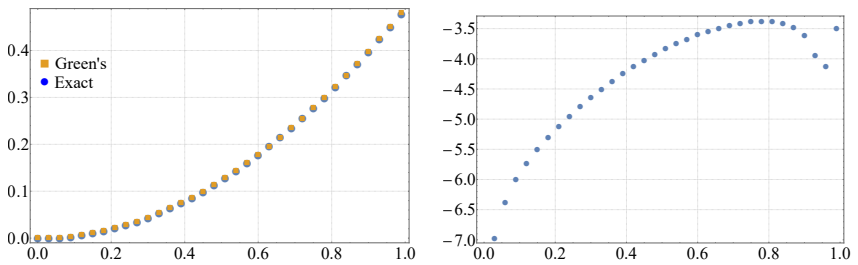


Figure 1.3: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = \theta(t)$

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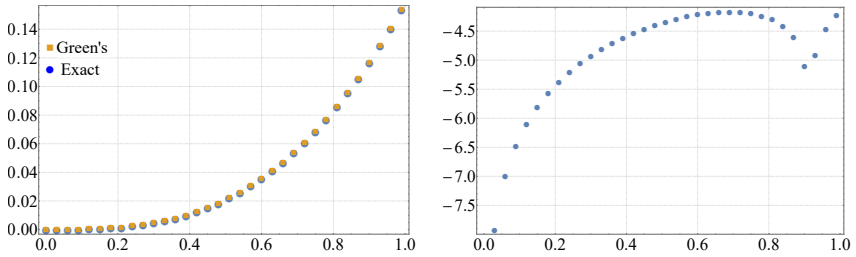


Figure 1.4: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = \sin(t)$

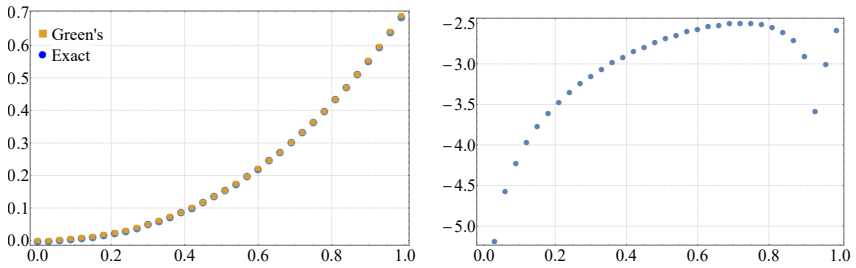


Figure 1.5: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = \exp(t)$

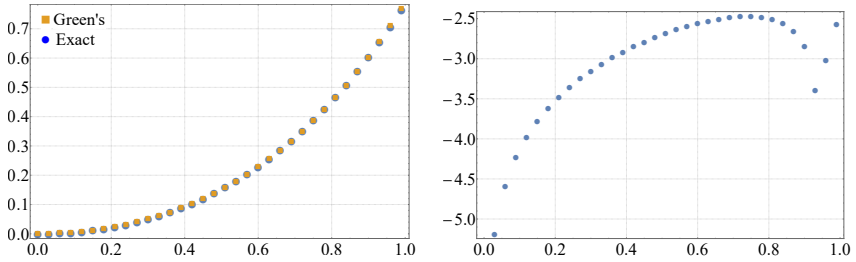


Figure 1.6: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = 1 + t + t^2 + t^3$

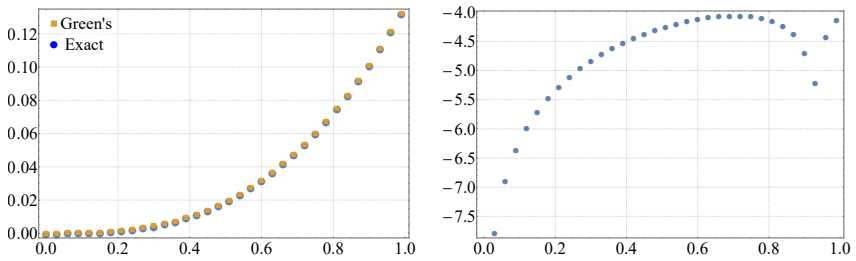


Figure 1.7: Discrete plot of exact and approximate solutions (left) and Er (right) for $f(t) = \ln(1 + t)$

1.4 The Green's Function Method in Controllability Analysis

Several attempts to invoke the Green's function method exist in the controllability analysis of *linear* dynamic systems (see, for instance, [159–175] and references therein). Recently, a new attempt which is also valid in the case of *nonlinear* systems has been reported in [94, 176] by the authors of the present book. The main idea of the approach is to use the Green's function to establish a link between the residue and the control function.

The algorithm can be split into several simple steps.

- i*) First, the solution of the state constraints is represented by the Green's formula (see (1.18) and (1.25)) and is evaluated at the terminal instant $t = T$, at which the controllability of the system is required.
- ii*) Second, the evaluated expression is substituted into the residue (1.1), making the dependence $\mathcal{R}_T = \mathcal{R}_T(\mathbf{u})$ explicit¹⁴.
- iii*) Eventually, one of the methods described in Subsection 1.1.5 can be utilized to satisfy (1.2) or (1.3) and to represent the resolving controls explicitly.

It is noteworthy that in handbooks [75, 76, 82] and in other references containing exact solutions of nonlinear differential equations, a lot of nonlinear equations are solved either in parametric form or implicitly. It is strongly believed that a similar technique can be developed in

¹⁴Be it either a boundary, a distributed, or a mixed type control problem.

that case when the Green's formula in the first step above is replaced by another type of explicit solution.

Among the advantages of the method is that it addresses the two main challenges mentioned in Section 1.1 and therefore it can be efficiently applied for either exact or approximate controllability analyses. Thanks to the existing extensions of the Green's function for nonlinear equations, dynamic systems with both linear and nonlinear state equations can be studied. On the other hand, because of some algebraic and geometric complexities, the determination of the Green's function even for some linear systems can be a difficult task. It is also an advantage of the approach that it can still be utilized in approximate controllability analysis in cases when only the upper bounds of the Green's function are able to be obtained. For relevant works refer to [177–188] and references therein.

The case studies carried out in Chapters 2 and 3 indicate that the approach is very efficient for systems with variable characteristics, localized controls, unbounded domains, uncertainties, higher dimensions, and generic non-linearities. Moreover, the approach is still applicable in cases when the state nonlinear equation is exactly linearizable. In such cases the linearizing transformation provides a nonlinear relation between the solution of the nonlinear equation and the Green's function of the linearized equation. At this, the controllability analysis is quite straightforward.

The approach also has several drawbacks. In the case when the exact controllability is studied and the residue has the simplest form (1.1), the derivation of restrictions on the control is quite straightforward. However, when the residue has a complicated form, the derivation of restrictions themselves can be complicated. When approximate

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controllability is studied, the approach provides conditions which are merely sufficient. This means that if those conditions are not satisfied on the set of admissible controls, the residue must be implemented and (1.3) must be checked directly. For residues with complicated forms that procedure may exhibit a high computational cost.

Despite the existence of earlier attempts, this approach is original and distinctive because of its usefulness for the establishment of *approximate* controllability, as well as for the ability to analyze the controllability of *nonlinear* systems.

2

Linear Systems

This chapter is devoted to specific applications of the approach proposed in Section 1.4 for *linear* systems. Even though linear models generally describe real-life objects poorly, their mathematical models consist of simpler (linear) relations. Mathematical techniques of linear analysis are well-developed and are easy to apply even in the case of complicated coupled systems, which makes the study of control problems for linear systems easier.

This chapter is organized as follows. The general problem for a system of abstract differential equations is described in Section 2.1 and the resolving system in the general case is obtained in Section 2.2. Some particular cases are considered in Subsections 2.2.1 and 2.2.2, and resolving equations for the exact and approximate controllability are respectively derived. Section 2.3 discusses several special types of resolving control regimes for exact and approximate controllability analysis. Some particular cases are considered in Section 2.4.

2.1 Problem Statement

In what follows, the simplest problem of control theory is considered. The system under consideration is a system with the controlled state $\mathbf{w} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ governed by *linear* state constraints¹, an initial state $\mathbf{w}_0 : \Omega \rightarrow \mathbb{R}^m$, and a desired terminal state $\mathbf{w}_T : \Omega \rightarrow \mathbb{R}^m$. The control time $T > 0$ and the set of admissible controls \mathcal{U} are prescribed. The problem is to figure out whether the system is exactly or approximately controllable by means of admissible controls.

Let the state of the system be described by the linear differential equation

$$\mathcal{D}[\mathbf{w}] = \mathbf{f}(\mathbf{u}, \mathbf{x}, t) \quad \text{in } \Omega \times \mathbb{R}^+. \quad (2.1)$$

Here $\mathcal{D} : \mathbf{W} \rightarrow \mathbf{F}$ is the state operator, and $\mathbf{f} : \mathcal{U} \times \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a given vector-function. It is assumed that $\mathbf{f} \in \mathbf{F}$ satisfies the existence and uniqueness conditions of the solution of (2.1) in \mathbf{W} under the boundary and initial conditions

$$\mathcal{B}[\mathbf{w}] = \mathbf{u}_b(t) \quad \text{in } \partial\Omega \times \mathbb{R}^+, \quad (2.2)$$

and

$$\mathcal{I}[\mathbf{w}] = \mathbf{w}_0(\mathbf{x}) \quad \text{in } \Omega \times \{0\}, \quad (2.3)$$

respectively. Above, $\mathcal{B} : \mathbf{W} \rightarrow \mathbf{U}_b$ and $\mathcal{I} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ are *linear* operators.

Either \mathbf{u} or \mathbf{u}_b can be considered as a control function². Then the

¹Unlike Section 1.1, here the dependence of \mathbf{w} on \mathbf{u} is made explicit through the state constraints, so that \mathbf{u} does not occur in the argument of \mathbf{w} .

²In principle, the case of combined control can also be considered. Then, the resolving systems derived in Section 2.2, as well as the particular solutions obtained in Section 2.3 below can be adapted correspondingly.

problem is mathematically formulated as follows: find such an admissible control \mathbf{u} or \mathbf{u}_b that, at the given instant $T > 0$, the following terminal condition is provided:

$$\mathcal{T}[\mathbf{w}] = \mathbf{w}_T \text{ in } \Omega \times \{T\}. \quad (2.4)$$

Here $\mathcal{T} : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ is a *linear* operator as well. The choice of $\mathcal{T}[\cdot]$ strongly depends on the specific requirements imposed by the study. More often the case when

$$\mathcal{T}[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}, T) \text{ in } \Omega$$

is considered.

Remark 2.1. *As it is mentioned in Remark 1.1, since the system (2.1)-(2.3) is linear, without a loss of generality the equivalent problem of providing the equilibrium terminal state, i.e., $\mathbf{w}_T \equiv 0$, can be considered.*

Then, the residue between the required state and the state implemented at $t = T$ by a choice of an admissible control reads as

$$\mathcal{R}_T(\mathbf{u}, \mathbf{u}_b) = \|\mathcal{T}[\mathbf{w}] - \mathbf{w}_T\|_{\overline{\mathbf{W}}}. \quad (2.5)$$

The problem can now be reformulated as follows: figure out whether there exists a $\mathbf{u} \in \mathcal{U}$ (or $\mathbf{u}_b \in \mathcal{U}$) such that the residue (2.5) satisfies

$$\mathcal{R}_T(\mathbf{u}, \mathbf{u}_b) = 0, \quad (2.6)$$

or, if it is not the case,

$$\mathcal{R}_T(\mathbf{u}, \mathbf{u}_b) \leq \varepsilon, \quad (2.7)$$

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with a *given* precision $\varepsilon > 0$.

For a time-continuous transition of the state function, the consistency between the boundary, initial, and terminal data is required³. In terms of (2.2)-(2.4), the consistency implies the following restrictions on the boundary control:

$$\begin{aligned}\mathcal{B}[\mathcal{I}[\mathbf{w}]] &= \mathcal{B}[\mathbf{w}_0] = \mathbf{u}_b(0), \\ \mathcal{I}[\mathcal{B}[\mathbf{w}]] &= \mathcal{I}[\mathbf{u}_b] = \mathbf{w}_0(\mathbf{x}) \Big|_{\mathbf{x} \in \partial\Omega}, \\ \mathcal{B}[\mathcal{T}[\mathbf{w}]] &= \mathcal{B}[\mathbf{w}_T] = \mathbf{u}_b(T), \\ \mathcal{T}[\mathcal{B}[\mathbf{w}]] &= \mathcal{T}[\mathbf{u}_b] = \mathbf{w}_T(\mathbf{x}) \Big|_{\mathbf{x} \in \partial\Omega}.\end{aligned}\tag{2.8}$$

Let the set of admissible controls be of the form

$$\mathcal{U} = \{\mathbf{u} \in \mathbf{U}, |\mathbf{u}| \leq \varepsilon, \text{supp}(\mathbf{u}) \subseteq [0, T]\}$$

with prescribed $\varepsilon > 0$. In the case of boundary control, the space \mathbf{U} must be substituted by \mathbf{U}_b , and \mathcal{U} must be complemented by (2.8).

³It is noteworthy that in the case of approximate controllability, the boundary and terminal data can be inconsistent.

2.2 Resolving System

The second approach described in Subsection 1.1.5 is applied in this section for the derivation of constraints on admissible controls ensuring exact or approximate controllability.

Assume for simplicity that

$$\begin{aligned} \mathcal{B}[\mathbf{w}](t) &= \mathbf{w}(\mathbf{x}, t) \Big|_{\mathbf{x} \in \partial\Omega}, \quad \mathcal{I}[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}, 0), \\ \mathcal{T}[\mathbf{w}](\mathbf{x}) &= \mathbf{w}(\mathbf{x}, T). \end{aligned}$$

Then, the solution of (2.1)-(2.3) can be represented as

$$\begin{aligned} \mathbf{w}(\mathbf{x}, t) &= \int_0^t \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t, \tau) \mathbf{f}(\mathbf{u}, \boldsymbol{\xi}, \tau) \, d\boldsymbol{\xi} d\tau + \\ &+ \int_0^t \mathbf{G}_b(\mathbf{x}, t, \tau) \mathbf{u}_b(\tau) \, d\tau + \\ &+ \int_{\Omega} \mathbf{G}_i(\mathbf{x}, \boldsymbol{\xi}, t) \mathbf{w}_0(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \quad \text{in } \bar{\Omega} \times \mathbb{R}^+, \end{aligned} \tag{2.9}$$

where \mathbf{G} is the Green's function of (2.1)-(2.3), and \mathbf{G}_b and \mathbf{G}_i are defined by (1.17). Evaluating the state function (2.9) at $t = T$ and substituting into the residue (2.5), the following is obtained

$$\begin{aligned} \mathcal{R}_T(\mathbf{u}, \mathbf{u}_b) &= \left\| \int_0^T \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, T, \tau) \mathbf{f}(\mathbf{u}, \boldsymbol{\xi}, \tau) \, d\boldsymbol{\xi} d\tau + \right. \\ &+ \int_0^T \mathbf{G}_b(\mathbf{x}, T, \tau) \mathbf{u}_b(\tau) \, d\tau + \\ &+ \left. \int_{\Omega} \mathbf{G}_i(\mathbf{x}, \boldsymbol{\xi}, T) \mathbf{w}_0(\boldsymbol{\xi}) \, d\boldsymbol{\xi} - \mathbf{w}_T(\mathbf{x}) \right\|_{\bar{\mathbf{W}}}. \end{aligned} \tag{2.10}$$

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Then, the exact controllability of (2.1)-(2.3), i.e., the equality (2.5), is equivalent to

$$\int_0^T \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, T, \tau) \mathbf{f}(\mathbf{u}, \boldsymbol{\xi}, \tau) \, d\boldsymbol{\xi} \, d\tau + \int_0^T \mathbf{G}_b(\mathbf{x}, T, \tau) \mathbf{u}_b(\tau) \, d\tau = \mathbf{M}_T(\mathbf{x}) \quad \text{a.e. in } \overline{\Omega}, \quad (2.11)$$

where⁴

$$\mathbf{M}_T(\mathbf{x}) = \mathbf{w}_T(\mathbf{x}) - \int_{\Omega} \mathbf{G}_i(\mathbf{x}, \boldsymbol{\xi}, T) \mathbf{w}_0(\boldsymbol{\xi}) \, d\boldsymbol{\xi},$$

which provides a system of constraints on admissible controls. Therefore, the full set of resolving controls is defined by

$$\mathcal{U}_{res} = \{\mathbf{u} \in \mathcal{U}, (2.11)\}.$$

Consider now the approximate controllability of (2.1)-(2.3). By virtue of the Minkowski inequality, the estimate for the residue (2.10)

$$\mathcal{R}_T(\mathbf{u}, \mathbf{u}_b) \leq \mathcal{R}_{T0} + \mathcal{R}_{T1}(\mathbf{u}_b) + \mathcal{R}_{T2}(\mathbf{u})$$

holds uniformly with

$$\begin{aligned} \mathcal{R}_{T0} &= \|\mathbf{M}_T\|_{\overline{\mathbf{W}}}, \\ \mathcal{R}_{T1}(\mathbf{u}_b) &= \left\| \int_0^T \mathbf{G}_b(\mathbf{x}, T, \tau) \mathbf{u}_b(\tau) \, d\tau \right\|_{\overline{\mathbf{W}}}, \\ \mathcal{R}_{T2}(\mathbf{u}) &= \left\| \int_0^T \int_{\Omega} \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, T, \tau) \mathbf{f}(\mathbf{u}, \boldsymbol{\xi}, \tau) \, d\boldsymbol{\xi} \, d\tau \right\|_{\overline{\mathbf{W}}}. \end{aligned}$$

⁴Here the subscript T indicates the dependence of \mathbf{M} on T .

Thus, the inequality

$$\mathcal{R}_{T_0} + \mathcal{R}_{T_1}(\mathbf{u}_b) + \mathcal{R}_{T_2}(\mathbf{u}) \leq \varepsilon \text{ on } \mathcal{U} \quad (2.12)$$

will be sufficient for the approximate controllability. Being merely a sufficient condition, the inequality (2.12) is easier to verify compared with the direct evaluation and verification of the more general criterion (2.7).

A weaker, but more easily verifiable criterion can be obtained, taking into account that for the given initial and terminal data, \mathcal{R}_{T_0} is fixed. Further, if the control is carried out only by the boundary function \mathbf{u}_b , then \mathcal{R}_{T_2} is also fixed. Furthermore, by virtue of the obvious estimate

$$\mathcal{R}_{T_1}(\mathbf{u}_b) \leq \int_0^T \|\mathbf{G}_b(\mathbf{x}, T, \tau)\|_{\overline{\mathbf{W}}} |\mathbf{u}_b(\tau)| d\tau \leq \varepsilon \gamma_T,$$

where

$$\gamma_T = \int_0^T \|\mathbf{G}_b(\mathbf{x}, T, \tau)\|_{\overline{\mathbf{W}}} d\tau,$$

the inequality

$$\mathcal{R}_{T_0} + \mathcal{R}_{T_2}(\mathbf{u}) + \varepsilon \gamma_T \leq \varepsilon \quad (2.13)$$

will be sufficient for approximate controllability of the system as well. Therefore, admissible controls

$$\mathbf{u}_b \in \tilde{\mathcal{U}}_{res} = \{\mathbf{u} \in \mathcal{U}, \quad |\mathbf{u}| \leq \tilde{c}_T\}$$

with

$$\tilde{c}_T \leq \frac{\varepsilon - \mathcal{R}_{T_0} - \mathcal{R}_{T_2}(\mathbf{u})}{\gamma_T}, \quad (2.14)$$

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ensure the approximate controllability of (2.1)-(2.3) as soon as

$$\varepsilon - \mathcal{R}_{T_0} - \mathcal{R}_{T_2}(\mathbf{u}) \geq 0. \quad (2.15)$$

A similar estimate can be obtained in the case when a distributed control with \mathbf{u} is considered.

However, it might be the case that within a reasonable range of characteristics of a particular system, the inequality (2.15) does not hold, leading to

$$\tilde{\mathcal{U}}_{res} = \emptyset.$$

In such cases, the inequality (2.12) must be evaluated and verified. If for given data (2.12) does not hold as well, then, for a proper approximate controllability analysis, (2.7) with (2.10) must be verified directly.

As with (2.12), the condition (2.13) is also merely sufficient, providing restrictions on the initial and terminal states, right-hand side, control time, and other system characteristics included in the expression of \mathbf{G} . Therefore, before involving any procedure for providing (2.7), it would be reasonable to verify (2.13) first. Note that since \mathcal{R}_{T_0} is determined by means of the initial and terminal data, while \mathcal{R}_{T_2} is evaluated by means of the right-hand side, (2.15) can be in principle provided by a specific choice of the given data. Therefore, the value of $\tilde{\varepsilon}$ can be regulated in a desired range by varying the value of γ_T .

The consistency of the boundary and initial and terminal data (cf. (2.8)) provides the following discrete constraints on the boundary control:

$$\mathbf{u}_b(0) = \mathbf{w}_0(\mathbf{x})|_{\partial\Omega}, \quad \mathbf{u}_b(T) = \mathbf{w}_T(\mathbf{x})|_{\partial\Omega}, \quad (2.16)$$

by which \mathcal{U} must be complemented.

2.2.1 Particular Cases: Exact Controllability

In the general case the resolving system (2.11) is a fully coupled system of equations⁵ with respect to the control function. Therefore, the explicit determination of the resolving controls can be complicated. However, in some particular cases it is possible to reduce the complexity of the system.

In the one-dimensional case let $\Omega = (a, b)$, $a < b \in \mathbb{R}$. Then, (2.11) takes the form

$$\int_0^T \int_a^b G(x, \xi, T, \tau) f(u, \xi, \tau) \, d\xi \, d\tau + \int_0^T G_b(x, T, \tau) u_b(\tau) \, d\tau = M_T(x) \tag{2.17}$$

$$\text{a.e. in } [a, b],$$

where

$$M_T(x) = w_T(x) - \int_a^b G_i(x, \xi, T) w_0(\xi) \, d\xi,$$

and G_i and G_b are the one-dimensional analogues of (1.17).

When the control is carried out merely via the boundary function u_b acting at $x = a$, i.e., when

$$w(a, t) = u_b(t), \quad w(b, t) \equiv 0 \quad \text{in } \mathbb{R}^+,$$

the resolving equation (2.17) is reduced to the linear constraint

$$\int_0^T G_b(x, T, \tau) u_b(\tau) \, d\tau = M_{Tb}(x) \quad \text{a.e. in } [a, b], \tag{2.18}$$

⁵Note that, in general, the constraint is *linear* in boundary control and is non-linear in distributed control.

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where

$$M_{Tb}(x) = M_T(x) - \int_0^T \int_a^b G(x, \xi, T, \tau) f(u, \xi, \tau) d\xi d\tau.$$

The compatibility conditions between the boundary, initial, and terminal data are supposed to be satisfied. Then, the set of admissible controls must be complemented by the constraints

$$u_b(0) = w_0(a), \quad u_b(T) = w_T(a), \quad (2.19)$$

reduced from (2.16). Thus,

$$\mathcal{U} = \{u \in \mathbf{U}_b, |u| \leq \epsilon, \text{supp}(u) \subseteq [0, T], (2.19)\}.$$

In this case, the set of resolving controls is

$$\mathcal{U}_{res} = \{u \in \mathcal{U}, (2.18)\}.$$

On the other hand, in the case of distributed control, the resolving equation (2.17) is reduced to the following nonlinear constraint:

$$\int_0^T \int_a^b G(x, \xi, T, \tau) f(u, \xi, \tau) d\xi d\tau = M_{Td}(x) \quad (2.20)$$

a.e. in $[a, b]$,

where

$$M_{Td}(x) = M_T(x) - \int_0^T G_b(x, T, \tau) u_b(\tau) d\tau.$$

The exact solvability of (2.20) strongly depends on the dependence of

f on u . Particularly, when f is linear in u , i.e., if, for given functions f_0 and f_1 ,

$$f(u, x, t) = f_0(x, t)u(t) + f_1(x, t), \quad (2.21)$$

then (2.20) results in the linear constraint

$$\int_0^T G_0(x, T, \tau)u(\tau) d\tau = M_{Td1}(x) \quad \text{a.e. in } [a, b], \quad (2.22)$$

where

$$G_0(x, T, \tau) = \int_a^b G(x, \xi, T, \tau) f_0(\xi, \tau) d\xi$$

and

$$M_{Td1}(x) = M_{Td}(x) - \int_0^T \int_a^b G(x, \xi, T, \tau) f_1(\xi, \tau) d\xi d\tau.$$

The functions f_0 and f_1 must be chosen in such a way that the integrals in G_0 and M_{Td1} make sense, and therefore (2.22) is well defined.

In cases when the dependence of f on u is nonlinear, it is hardly possible to solve (2.20) exactly. Therefore, numerical methods are usually involved.

2.2.2 Particular Cases: Approximate Controllability

Restrict the consideration to the one-dimensional case. Since the system is linear, without any loss of generality, let the aim of the control be the transmitting of the system to rest in a given T , i.e., providing $w_T \equiv 0$ by a specific choice of admissible boundary controls. Assume,

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in addition, that

$$|w_0| \leq \alpha \quad \text{in } [a, b]$$

and

$$|f| \leq \beta \quad \text{in } \mathcal{U} \times [a, b] \times \mathbb{R}^+$$

hold uniformly. Then, introducing the notations

$$g_i(\xi, T) = \|G_i(x, \xi, T)\|_{\overline{\mathbf{W}}}, \quad g(\xi, T, \tau) = \|G(x, \xi, T, \tau)\|_{\overline{\mathbf{W}}},$$

it is then easy to obtain the following estimates according to the definition of the norm:

$$\begin{aligned} \mathcal{R}_{T_0} &= \left\| \int_a^b G_i(x, \xi, T) w_0(\xi) \, d\xi \right\|_{\overline{\mathbf{W}}} \leq \\ &\leq \int_a^b \|G_i(x, \xi, T)\|_{\overline{\mathbf{W}}} |w_0(\xi)| \, d\xi = \\ &= \int_a^b g_i(\xi, T) |w_0(\xi)| \, d\xi \leq \alpha \int_a^b g_i(\xi, T) \, d\xi, \\ \mathcal{R}_{T_2}(u) &= \left\| \int_0^T \int_a^b G(x, \xi, T, \tau) f(u, \xi, \tau) \, d\xi d\tau \right\|_{\overline{\mathbf{W}}} \leq \\ &\leq \int_0^T \int_a^b \|G(x, \xi, T, \tau)\|_{\overline{\mathbf{W}}} |f(u, \xi, \tau)| \, d\xi d\tau \leq \\ &\leq \beta \int_0^T \int_a^b g(\xi, T, \tau) \, d\xi d\tau. \end{aligned}$$

It is proved that, in general, g_i and g are at least integrable [83], i.e.,

$$\alpha_{T_0} := \int_a^b g_i(\xi, T) \, d\xi < \infty,$$

and

$$\beta_{T0} := \int_0^T \int_a^b g(\xi, T, \tau) d\xi d\tau < \infty,$$

providing

$$\mathcal{R}_{T0} \leq \alpha \cdot \alpha_{T0}, \quad \mathcal{R}_{T2}(u) \leq \beta \cdot \beta_{T0}.$$

On the other hand, as it has already been established in the beginning of Section 2.2,

$$\mathcal{R}_{T1}(u_b) \leq \epsilon \cdot \gamma_T \quad \text{on } \mathcal{U},$$

therefore for the approximate controllability of the system it is sufficient that

$$\alpha \cdot \alpha_{T0} + \epsilon \cdot \gamma_T + \beta \cdot \beta_{T0} \leq \epsilon. \tag{2.23}$$

Here α , β , ϵ , and ϵ are given constants, while α_{T0} , γ_T , and β_{T0} depend on the system characteristics (these may be, for instance, material parameters, system geometry, boundedness, etc.). It follows from (2.23) that as long as

$$\epsilon - \alpha \cdot \alpha_{T0} - \beta \cdot \beta_{T0} \geq 0,$$

any admissible boundary control

$$u_b \in \tilde{\mathcal{U}}_{res} = \{u \in \mathcal{U}, |u| \leq \tilde{\epsilon}_T\}$$

with

$$\tilde{\epsilon}_T = \frac{\epsilon - \alpha \cdot \alpha_{T0} - \beta \cdot \beta_{T0}}{\gamma_T}$$

ensures the approximate controllability of the system at T .

Further, under the assumptions made above, consider the case of distributed control. Let the aim continue to be providing null-controllability

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of the system. Assume that the right-hand side f is linear in u and (2.21) holds with

$$|f_0| \leq \beta_1, \quad |f_1| \leq \beta_2 \quad \text{in } [a, b] \times \mathbb{R}^+.$$

Then, by virtue of the Minkowski inequality, the following estimate holds on \mathcal{U} :

$$\begin{aligned} \mathcal{R}_{T_2}(u) &= \left\| \int_0^T \int_a^b G(x, \xi, T, \tau) (f_0(\xi, \tau) u(\tau) + \right. \\ &\quad \left. + f_1(\xi, \tau)) d\xi d\tau \right\|_{\overline{\mathbf{W}}} \leq \\ &\leq \int_0^T \int_a^b \|G(x, \xi, T, \tau)\|_{\overline{\mathbf{W}}} |f_0(\xi, \tau) u(\tau) + f_1(\xi, \tau)| d\xi d\tau \leq \\ &\leq (\beta_1 \cdot \epsilon + \beta_2) \int_0^T \int_a^b g(\xi, T, \tau) d\xi d\tau = \\ &= (\beta_1 \cdot \epsilon + \beta_2) \beta_{T_0}. \end{aligned}$$

For simplicity assume that

$$u_b \equiv 0 \quad \text{in } \mathbb{R}^+,$$

providing

$$\mathcal{R}_{T_1}(u_b) = 0.$$

Therefore, for approximate controllability of the system it is sufficient that

$$\alpha \cdot \alpha_{T_0} + (\beta_1 \cdot \epsilon + \beta_2) \beta_{T_0} \leq \epsilon. \tag{2.24}$$

Here the fixed constants are α , ϵ , β_1 , and β_2 . Then, the inequality (2.24)

provides a restriction on system parameters included in the quantities α_{T_0} and β_{T_0} , for which the system is approximately controllable. On the other hand, from (2.24) it follows that any admissible control

$$u \in \tilde{\mathcal{U}}_{res} = \{u \in \mathcal{U}, \quad |u| \leq \tilde{\epsilon}_T\}$$

with

$$\tilde{\epsilon}_T = \frac{\varepsilon - \alpha \cdot \alpha_{T_0} - \beta_2 \cdot \beta_{T_0}}{\beta_1 \cdot \beta_{T_0}},$$

ensures approximate controllability of the system at T as long as

$$\varepsilon - \alpha \cdot \alpha_{T_0} - \beta_2 \cdot \beta_{T_0} \geq 0.$$

2.3 Particular Solutions of the Resolving System

Resolving systems define the set of resolving controls implicitly. The evaluation of those systems on the whole \mathcal{U} can still require high computational cost. The determination of the general solution of resolving systems will significantly reduce that cost. Nevertheless, the explicit determination of resolving controls can be quite complicated. On the other hand, as it is shown below, it is possible to derive various hierarchies of admissible controls that satisfy the resolving system.

2.3.1 Exact Controllability

Generally, the exact controllability of a particular system in one space dimension⁶ is equivalent to the equality type constraint of the form

$$\int_0^T K_T(u, x, \tau) d\tau = f_T(x) \quad \text{a.e. in } [a, b], \quad (2.25)$$

where K_T and f_T are given functions.

Remark 2.2. *In practice, the explicit determination of resolving controls from (2.25) can be quite complicated. Particularly, both sides of the resolving equation depend on x , while the control, be it distributed or boundary, should merely depend on t . Therefore, (2.25) cannot be viewed, e.g., as a Fredholm integral equation of the first kind. Nevertheless, in cases when*

$$K_T(u, x, \tau) = f_T(x) g_T(u, \tau) \quad \text{in } [a, b],$$

⁶The technique can be straightforwardly adapted to the case of higher dimensions.

the equation (2.25) becomes a Fredholm integral equation of the first kind

$$\int_0^T g_T(u, \tau) d\tau = 1,$$

which can be solved efficiently for generic kernels g (refer to [189] for details). For particular solutions see also Subsection 2.3.2 below.

An efficient way of solving (2.25) can be developed based on the expansion of K_T and f_T into a series of orthogonal functions. Let $\{\varphi_n\}_{n=1}^\infty$ be a family of orthogonal over the interval $[a, b]$ functions ⁷, i.e.,

$$\int_a^b \varphi_m(x) \varphi_n(x) dx = \delta_m^n,$$

where δ_m^n is the Kronecker delta. Denote by f_{Tn} and K_{Tn} the expansion coefficients of the functions f_T and K_T , respectively, so that

$$f_T(x) = \sum_{n=1}^\infty f_{Tn} \varphi_n(x)$$

with

$$f_{Tn} = \int_a^b f_T(x) \varphi_n(x) dx,$$

and

$$K_T(u, x, \tau) = \sum_{n=1}^\infty K_{Tn}(u, \tau) \varphi_n(x)$$

with

$$K_{Tn}(u, \tau) = \int_a^b K_T(u, x, \tau) \varphi_n(x) dx.$$

Then, since $\{\varphi_n\}_{n=1}^\infty$ are orthogonal, the equation (2.25) is equivalent

⁷In general, depending on the system domain, regularity of K_T and f_T , etc., families of functions orthogonal with weight may need to be considered.

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to the infinite system

$$\int_0^T K_{Tn}(u, \tau) d\tau = f_{Tn}, \quad n \in \mathbb{N}. \quad (2.26)$$

In specific problems, depending on the required accuracy of computations, the decay rate of K_{Tn} , and f_{Tn} , the infinite system (2.26) is truncated and studied for some finite N .

Remark 2.3. *Note that (2.26) or its truncated version can be treated in different ways. On one hand it is an infinite system of Fredholm integral equations of the first kind, which can be efficiently solved numerically [189–191]. On the other hand, it can be treated as an infinite-dimensional problem of moments. The treating of (2.26) as a problem of moments has several advantages, among which are availability of explicit L^p -optimal solutions⁸ for $1 \leq p \leq \infty$, necessary and sufficient conditions for solution existence, etc. (see [192] for the nonlinear case and [67] for the linear case). Other treatments are also possible.*

Another way to resolve (2.26) is the heuristic determination of controls. Namely, based on some considerations, say, the physical treatment of the problem, controller capabilities, etc., the control function is chosen to have a specific form containing a set of free parameters. This function is then substituted into the system (2.26) or its truncated version. Then, a discrete system of, in general, nonlinear algebraic equations is derived with respect to the set of free parameters. Eventually, different order numerical methods can be involved to approximate the solution of the derived system. See, for instance, [193, 194] for details.

⁸A solution of (2.26) in $L^p[0, T]$ with

$$\|u\|_{L^p[0, T]} \rightarrow \min.$$

Consider some particular cases. Let the infinite system (2.26) be truncated for some finite $N \in \mathbb{N}$. Then, the control function can be sought, for instance, in the form of the trigonometric sum

$$u(t) = \sum_{m=1}^M u_m \sin(\omega_m t + \gamma_m), \quad (2.27)$$

where $M \in \mathbb{N}$ and u_m, ω_m , and γ_m are free parameters chosen to satisfy (2.26) exactly. Substituting (2.27) into the truncated part of the system (2.26), the system of nonlinear equations

$$\mathbf{K}_T(u_m, \omega_m, \gamma_m) = \mathbf{f}_T \quad (2.28)$$

is obtained. Here,⁹

$$\mathbf{K}_T = (\tilde{K}_{T1} \dots \tilde{K}_{TN})^T, \quad \mathbf{f}_T = (f_{T1} \dots f_{TN})^T,$$

$$\tilde{K}_{Tn}(u_m, \omega_m, \gamma_m) = \int_0^T K_{Tn} \left(\sum_{m=1}^M u_m \sin(\omega_m \tau + \gamma_m), \tau \right) d\tau.$$

Consider also the piecewise-constant regime

$$u(t) = \sum_{m=1}^M u_m \theta(t - t_m), \quad (2.29)$$

corresponding to a jump from the constant regime $u(t_m)$ to $u(t_{m+1})$ when t switches from t_m to t_{m+1} . Here u_m and t_m are free parameters chosen to satisfy (2.26) exactly. At this, the instants t_m satisfy the

⁹Note that in proceeding formulas the notation $F(x_m)$ means that the function F depends on all $x_m, m = 1, \dots, M$.

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inequality constraints

$$0 \leq t_1 < t_2 < \dots < t_M \leq T,$$

foreclosing the overlap of different regimes.

In this case, (2.26) is reduced to

$$\mathbf{K}_T(u_m, t_m) = \mathbf{f}_T, \quad (2.30)$$

with

$$\tilde{K}_{Tn}(u_m, t_m) = \int_0^T K_{Tn} \left(\sum_{m=1}^M u_m \theta(\tau - t_m), \tau \right) d\tau.$$

Furthermore, consider the impulsive regime

$$u(t) = \sum_{m=1}^M u_m \delta(t - t_m), \quad 0 \leq t_1 < t_2 < \dots < t_M \leq T \quad (2.31)$$

formally describing instantaneous impacts of magnitude u_m at the instants $t = t_m$.

In this case also the free parameters are u_m and t_m . Substituting (2.31) into (2.26), the following nonlinear system is derived:

$$\mathbf{K}_T(u_m, t_m) = \mathbf{f}_T, \quad (2.32)$$

with

$$\tilde{K}_{Tn}(u_m, t_m) = \int_0^T K_{Tn} \left(\sum_{m=1}^M u_m \delta(\tau - t_m), \tau \right) d\tau.$$

In many applications the piecewise-continuous control

$$u(t) = \sum_{m=1}^M u_m(t) \chi_{[t_m, t_{m+1}]}(t), \quad 0 \leq t_1 < \dots < t_M < t_{M+1} = T$$

is considered, corresponding to switches between the time-dependent regimes u_m . At this, any of the regimes (2.27), (2.29), and (2.31) can be used as u_m . In this case the resolving equation is reduced to

$$\mathbf{K}_T(u_m, t_m) = \mathbf{f}_T,$$

with

$$\tilde{K}_{Tn}(u_m, t_m) = \int_0^T K_{Tn} \left(\sum_{m=1}^M u_m(\tau) \chi_{[t_m, t_{m+1}]}(\tau), \tau \right) d\tau.$$

In the case of boundary control, K_T is linear in u , that is

$$K_T(u, x, \tau) = K_{0T}(x, \tau) u(\tau) + K_{1T}(x, \tau)$$

for given functions K_{0T} and K_{1T} .

In addition, u satisfies the boundary conditions

$$u(0) = u_0, \quad u(T) = u_T. \tag{2.33}$$

Then, (2.28), (2.30), and (2.32) are reduced to linear systems for u_m . Indeed, in the case of, e.g., (2.31), the resulting system for the free parameters will be

$$\mathbf{K}_{0T}(u_m, t_m) = \mathbf{f}_T + \mathbf{K}_{1T},$$

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where

$$\mathbf{K}_{0T} = (\tilde{K}_{0T1} \dots \tilde{K}_{0TN})^T, \quad \mathbf{K}_{1T} = (\tilde{K}_{1T1} \dots \tilde{K}_{1TN})^T,$$

and

$$\begin{aligned} & \int_0^T \left[K_{0Tn}(\tau) \sum_{m=1}^M u_m \delta(\tau - t_m) + K_{1Tn}(\tau) \right] d\tau = \\ &= \sum_{m=1}^M u_m \int_0^T K_{0Tn}(\tau) \delta(\tau - t_m) d\tau + \int_0^T K_{1Tn}(\tau) d\tau = \\ &= \sum_{m=1}^M u_m K_{0Tn}(t_m) + \int_0^T K_{1Tn}(\tau) d\tau := \\ &:= \tilde{K}_{0Tn}(u_m, t_m) + \tilde{K}_{1Tn}. \end{aligned}$$

Here K_{0Tn} and K_{1Tn} are the expansion coefficients of K_{0T} and K_{1T} into a series of $\{\varphi_n\}_{n=1}^\infty$, respectively. At this, (2.33) implies additional restrictions on u_m : when $t_1 = 0$ and $t_M = T$,

$$u_1 = u_0 \quad \text{and} \quad u_M = u_T.$$

Otherwise, (2.31) can be applied in the case when $u_0 = u_T = 0$.

Remark 2.4. *In general, the N -dimensional system (2.28) contains $3M$ unknowns, therefore it might be irresolvable. Nevertheless, if some of the free parameters, say ω_m and/or γ_m , $m = 1, \dots, M$, are prescribed, then (2.28) can become solvable. Moreover, in the case of boundary control, (2.28) is reduced to a linear system for u_m . Therefore, if ω_m and γ_m are prescribed and $M = N$, u_m are found straightforwardly. Otherwise, i.e., when $M \neq N$, for finding specific solutions, the techniques of nonlinear programming [195] must be utilized.*

The same reasoning applies also to the systems (2.30) and (2.32) containing, in general, $2M$ unknowns.

Remark 2.5. Any solution derived from the truncated version of the infinite-dimensional system (2.26) is approximate, which means that (2.25), in general, is satisfied approximately. In that case, the residue (2.10) is not exactly zero, i.e., (2.6) is not satisfied exactly. Thus, truncation of (2.26), which simplifies the explicit representation of resolving controls, eventually leads to the fulfillment of the approximate controllability condition (2.7).

2.3.2 Approximate Controllability

Assume that the system under study is linear in control. Then its approximate controllability is reduced to the evaluation of an integral equation of the form

$$\int_0^T K_T(\tau) |u(\tau)| d\tau = f_T \quad (2.34)$$

for a positive constant f_T , a positive and bounded kernel K_T , and $u \in \mathcal{U}$. In the case of boundary control, the control function is constrained by the discrete constraints

$$u(0) = u_0, \quad u(T) = u_T. \quad (2.35)$$

It is easy to see that one of the obvious solutions is the constant function

$$u = u^o = \text{const}, \quad (2.36)$$

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leading in (2.34) to

$$|u^o| = \frac{f_T}{k_T} \quad \text{with} \quad k_T = \int_0^T K_T(\tau) \, d\tau.$$

When the control is carried out by the boundary data, this regime is applicable only in the case when $u_0 = u_T = u^o \neq 0$.

In vibration control problems, time-harmonic controls of the form

$$u(t) = u^o \sin(\omega t + \gamma) \tag{2.37}$$

are usually considered, reducing (2.34) to the nonlinear equation

$$|u^o| \int_0^T K_T(\tau) |\sin(\omega\tau + \gamma)| \, d\tau = f_T \tag{2.38}$$

with respect to the free parameters u^o , ω , and γ . In the case of boundary control, (2.35) provides the additional constraint

$$u_0 \sin(\omega T + \gamma) = u_T \sin(\gamma).$$

A particular solution of (2.34) can be constructed using the quasi-polynomial control

$$u(t) = \sum_m \sum_n u_{mn} t^m (T-t)^n, \tag{2.39}$$

where u_{mn} and $m, n \in \mathbb{R}^+$ are free parameters satisfying the system (2.34). Assume, for simplicity, that $\text{sign } u \geq 0$. Then, (2.34) provides

$$\sum_m \sum_n u_{mn} P_T^{m,n} = f_T, \tag{2.40}$$

where

$$P_T^{m,n} = \int_0^T K_T(\tau) \tau^m (T - \tau)^n d\tau.$$

Evidently, when m and n are prescribed, (2.40) becomes a linear equation for u_{mn} .

In the case of boundary control, in order to satisfy (2.35) by the quasi-polynomial regime, the term

$$u^0(t) = u_0 + [u_T - u_0] \frac{t}{T}$$

must be added to (2.39). Then, (2.40) is reduced to

$$\sum_m \sum_n u_{mn} P_T^{m,n} = f_T - u_0 P_T^{0,0} - \frac{u_T - u_0}{T} P_T^{1,0}.$$

A fast verification of approximate controllability is provided by the trigonometric regime

$$u(t) = \sum_{m=1}^M \sum_n u_{mn} \left[1 - \cos\left(\frac{2\pi mt}{T}\right) \right]^n, \quad (2.41)$$

where u_{mn} , m , $M \in \mathbb{N}$, and $n \in \mathbb{R}^+$ are the free parameters to satisfy (2.34). Assuming that $\text{sign } u \geq 0$, for the free parameters the nonlinear equation

$$\sum_{m=1}^M \sum_n u_{mn} C_T^{m,n} = f_T \quad (2.42)$$

is derived, where

$$C_T^{m,n} = \int_0^T K_T(\tau) \left[1 - \cos\left(\frac{2\pi m\tau}{T}\right) \right]^n d\tau.$$

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In the case of boundary control, in order to satisfy (2.35), the term u^0 above should be added to the control. The resulting system reads as follows:

$$\sum_{m=1}^M \sum_n u_{mn} C_T^{m,n} = f_T - u_0 C_T^{0,0} - \frac{u_T - u_0}{T} C_T^{1,0}.$$

Other particular solutions appropriate for the physical treatment of the problem are also possible. In many applied problems it becomes necessary to involve sliding modes [43]. As an example consider the switching or piecewise constant control regime

$$u(t) = \sum_{m=1}^M u_m \theta(t - t_m) \quad (2.43)$$

subject to the inequality type constraints $0 \leq t_1 < \dots < t_M \leq T$.

Here u_m , t_m , and M are free parameters. Note that piecewise continuous regimes with $u_m = u_m(t)$ are also often considered. In such cases, any of the continuous regimes above can be considered as u_m .

Assuming that sign $u \geq 0$, (2.34) can be reduced to

$$\sum_{m=1}^M \Theta_T^m(u_m, t_m) = f_T, \quad (2.44)$$

where

$$\begin{aligned} \Theta_T^m(u_m, t_m) &= \int_0^T K_T(\tau) u_m(\tau) \theta(\tau - t_m) d\tau = \\ &= \int_{t_m}^T K_T(\tau) u_m(\tau) d\tau. \end{aligned}$$

Note that in the case of boundary control, the additional constraints

$$u_1 = u_0, \quad \sum_{m=1}^M u_m(T) = u_T$$

are derived when $t_1 = 0$. Otherwise, the term $u_0\theta(t - 0)$ should be added to (2.43).

Another application of the sliding mode control is the so-called optimal stopping regime [196] usually given as

$$u(t) = u^o(t)\theta(t^o - t). \tag{2.45}$$

In this case, the equality (2.34) must be satisfied by an appropriate choice of u^o and t^o . In addition, t^o satisfies the inequality type constraint

$$0 < t^o \leq T.$$

Obviously, in this case,

$$\text{supp}(u) \subseteq [0, t^o].$$

Furthermore, (2.34) is reduced to

$$\Theta_T(u^o, t^o) = f_T, \tag{2.46}$$

where

$$\begin{aligned} \Theta_T(u^o, t^o) &= \int_0^T K_T(\tau) |u^o(\tau)| \theta(t^o - \tau) d\tau = \\ &= \int_0^{t^o} K_T(\tau) |u^o(\tau)| d\tau. \end{aligned}$$

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Therefore, if t^o is prescribed and $u^o = \text{const}$, then

$$|u^o| = \frac{f_T}{\Theta_T^0} \quad \text{with} \quad \Theta_T^0 = \int_0^{t^o} K_T(\tau) \, d\tau.$$

In the case of boundary control, if $t^o < T$, only the first condition in (2.35) can be satisfied by u^o , and u_T necessarily should be zero. Otherwise, if $t^o = T$, both conditions must be satisfied by u^o . When $u^o = \text{const}$, the latter makes sense if and only if $u_0 = u_T = u^o$.

The set of reachable terminal states can be significantly extended, by complementing \mathcal{U} with impulsive actions of the form

$$u(t) = \sum_{m=1}^M u_m \delta(t - t_m), \quad (2.47)$$

subject to the inequality type constraints

$$0 \leq t_1 < \dots < t_M \leq T.$$

The free parameters are u_m , t_m , and M . Additionally, if all $u_m \geq 0$, then (2.34) is reduced to the nonlinear constraint

$$\sum_{m=1}^M u_m D_T(t_m) = f_T, \quad (2.48)$$

where

$$D_T(t_m) = \int_0^T K_T(\tau) \delta(\tau - t_m) \, d\tau = K_T(t_m).$$

In the case of boundary control, if $t_1 = 0$ and $t_M = T$, (2.35) is satisfied only when

$$u_1 = u_0, \quad u_N = u_T.$$

Otherwise, (2.47) is applicable if and only if

$$u_0 = 0, \quad u_N = u_T,$$

or

$$u_1 = u_0, \quad u_N = 0,$$

or

$$u_0 = 0, \quad u_T = 0,$$

respectively.

Remark 2.6. *Concerning the determination of the free parameters from (2.38), (2.40), (2.42), (2.44), (2.46), and (2.48) recall Remark 2.4. Apparently, those control regimes can also be involved in the case when the system is nonlinear in control.*

Remark 2.7. *In all the cases above, together with the constraints derived above, the free parameters must also satisfy the inequality type constraint*

$$|u| \leq \epsilon \quad \text{in} \quad [0, T].$$

This means that the techniques of nonlinear programming likely must be applied.

Remark 2.8. *In order to preserve the restriction $\text{supp}(u) \subseteq [0, T]$, the control regimes above must be multiplied by $\chi_{[0, T]}(t)$.*

Remark 2.9. *In practice, it might be the case that the substitution of any of the aforementioned particular solutions does not provide (2.34) exactly. Basically, for the approximate controllability it is sufficient*

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that the inequality

$$\int_0^T K_T(\tau) |u(\tau)| d\tau \leq f_T$$

holds by a choice of the free parameters, which in some sense is an easier task. Particularly, for instance, in the case of (2.37), from (2.38) it follows that for the approximate controllability of the system it suffices to take

$$|u^0| \leq \frac{f_T}{k_T}.$$

2.4 Examples

Some particular problems are considered in this section to test the approach described in Section 1.4 and reveal its advantages and drawbacks. Primary attention is paid to approximate controllability, even though the ways of analyzing exact controllability are discussed as well. The examples contain some specific issues that usually complicate the controllability analysis, such as variable system characteristics, an unbounded system domain, uncertainty in external parameters, point sources, etc.

2.4.1 Finite Thin Inhomogeneous Rod Heated from Both Ends

Let a sufficiently thin rod of length l be heated from both ends $x = 0$ and $x = l$. Assume that the rod is thermo-isolated from the external medium, which means that the heat transferring from the rod edges is exchanged strictly within the rod. Let the thermal conductivity and volumetric heat capacity of the rod, denoted here by κ and ν , respectively, be functions of the x -coordinate given by

$$\kappa(x) = \kappa_0 \chi(x), \quad \nu(x) = \frac{\nu_0}{\chi(x)}.$$

Here κ_0 and ν_0 are positive constants characterizing the thermal conductivity and volumetric heat capacity of a homogeneous material, and χ is a strictly positive and piecewise continuous in $[0, l]$ function describing the material distribution law within the rod. Evidently,

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$\text{supp}(\chi) = [0, l]$. The above assumption provides

$$\kappa(x) \cdot \nu(x) = \kappa_0 \nu_0 = \text{const},$$

which considerably simplifies the study.

The aim of the control is to choose boundary heating regimes, such that in a required time the temperature distribution within the rod becomes uniform and admits a given value.

The temperature distribution in the rod, denoted here by Θ , obeys the one-dimensional heat equation with coordinate dependent coefficients

$$\nu(x) \frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x} \left[\kappa(x) \frac{\partial \Theta}{\partial x} \right] \quad \text{in } (0, l) \times \mathbb{R}^+. \quad (2.49)$$

The boundary and initial conditions are as follows:

$$\Theta(0, t) = \Theta(l, t) = u_b(t) \quad \text{in } \mathbb{R}^+, \quad (2.50)$$

$$\Theta(x, 0) = \Theta_0(x) \quad \text{in } [0, l]. \quad (2.51)$$

Mathematically, the problem is to find such an admissible control, $u_b \in \mathcal{U}$, that in the required T provides the terminal condition

$$\Theta(x, T) \equiv \Theta_T = \text{const} \quad \text{in } [0, l]$$

exactly or approximately. In other words, it is required to find such a $u_b \in \mathcal{U}$ that the residue

$$\mathcal{R}_T(u_b) = \|\Theta(x, T) - \Theta_T\|_{L^2_{\nu}[0, l]}$$

satisfies

$$\mathcal{R}_T(u_b) = 0, \quad (2.52)$$

or, if that does not hold,

$$\mathcal{R}_T(u_b) \leq \varepsilon, \tag{2.53}$$

with a required precision $\varepsilon > 0$.

In the above definition of the residue, a weighted L^2 space is considered, where the weight is the coefficient ν of the heat equation¹⁰.

In addition, it is assumed that the boundary, initial, and terminal data are consistent, so that the boundary control satisfies

$$u_b(0) = \Theta_0(0), \quad u_b(T) = \Theta_T. \tag{2.54}$$

The set of admissible controls in this case is considered to be

$$\mathcal{U} = \{u_b \in L^2[0, T], \quad \text{supp}(u_b) \subseteq [0, T], \quad (2.54)\}.$$

According to the Green's representation formula, the general solution of (2.49)-(2.51), evaluated at the instant $t = T$ reads as

$$\begin{aligned} \Theta(x, T) &= \int_0^l \Theta_0(\xi) G(x, \xi, T) \nu(\xi) d\xi + \\ &+ \int_0^T G_b(x, T - \tau) u_b(\tau) d\tau \quad \text{in } [0, l], \end{aligned} \tag{2.55}$$

in which [77]

$$G(x, \xi, t) = \sum_{n=1}^{\infty} \frac{1}{\|\varphi_n\|_{L^2_\nu[0, l]}^2} \varphi_n(x) \varphi_n(\xi) \exp[-\lambda_n^2 t],$$

¹⁰The reason will become clear from the proceeding calculations.

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$$\begin{aligned}
 G_b(x, t) &= - \int_0^l G(x, \xi, t) [\kappa(0) \delta'(\xi) + \kappa(l) \delta'(\xi - l)] d\xi = \\
 &= \left[\kappa(\xi) \frac{\partial G(x, \xi, t)}{\partial \xi} \right] \Big|_{\xi=0} + \left[\kappa(\xi) \frac{\partial G(x, \xi, t)}{\partial \xi} \right] \Big|_{\xi=l} = \\
 &= \sum_{n=1}^{\infty} \frac{\kappa(0) \varphi'_n(0) + \kappa(l) \varphi'_n(l)}{\|\varphi_n\|_{L^2_\nu[0,l]}^2} \varphi_n(x) \exp[-\lambda_n^2 t].
 \end{aligned}$$

Above λ_n^2 and φ_n are the eigenvalues and corresponding eigenfunctions of the following Sturm-Liouville problem:

$$\frac{d}{dx} \left[\kappa(x) \frac{d\varphi}{dx} \right] + \lambda^2 \nu(x) \varphi = 0 \quad \text{in } (0, l), \quad (2.56)$$

$$\varphi(0) = \varphi(l) = 0. \quad (2.57)$$

It is noteworthy that (2.56), (2.57) admits an exact solution for a sufficiently large class of distribution functions χ [82].

Consider, in particular, the case of null-controllability, i.e., $\Theta_T = 0$. Then, (2.52) is equivalent to

$$\int_0^T G_b(x, T - \tau) u_b(\tau) d\tau + \int_0^l \Theta_0(\xi) G(x, \xi, T) \nu(\xi) d\xi = 0$$

or

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\exp[-\lambda_n^2 T]}{\|\varphi_n\|_{L^2_\nu[0,l]}^2} \varphi_n(x) \left[\alpha_n \int_0^T \exp[\lambda_n^2 \tau] u_b(\tau) d\tau + \right. \\
 \left. + \int_0^l \Theta_0(\xi) \varphi_n(\xi) \nu(\xi) d\xi \right] = 0,
 \end{aligned}$$

where $\alpha_n = \kappa(0) \varphi'_n(0) + \kappa(l) \varphi'_n(l)$.

Further, since by the definition, $\{\varphi_n\}_{n=1}^{\infty}$ are orthogonal over the

interval $[0, l]$ with the weight ν , then the last equality is equivalent to the infinite-dimensional system

$$\alpha_n \int_0^T \exp [\lambda_n^2 \tau] u_b (\tau) d\tau = -\psi_n, \quad n \in \mathbb{N}, \quad (2.58)$$

where

$$\psi_n = \int_0^l \Theta_0 (\xi) \varphi_n (\xi) \nu (\xi) d\xi.$$

Then, the set of resolving controls will be

$$\mathcal{U}_{res} = \{u \in \mathcal{U}, (2.58)\}.$$

Depending on the asymptotic behavior of λ_n and ψ_n for large n , (2.58) can be truncated by some finite N (refer to Section 2.3).

Consider the approximate null-controllability of the rod. Then, by virtue of the Minkowski and Cauchy-Schwartz inequalities, the residue can be estimated as follows:

$$\begin{aligned} \mathcal{R}_T (u_b) &\leq \left\| \int_0^T G_b (x, T - \tau) u_b (\tau) d\tau \right\|_{L^2_\nu [0, l]} + \|M_T\|_{L^2_\nu [0, l]} \leq \\ &\leq \int_0^T g (T - \tau) |u_b (\tau)| d\tau + \|M_T\|_{L^2_\nu [0, l]}, \end{aligned}$$

where

$$\begin{aligned} g (t) &= \|G_b (x, t)\|_{L^2_\nu [0, l]} = \sqrt{\int_0^l G_b^2 (x, t) \nu (x) dx} = \\ &= \sqrt{\sum_{n=1}^\infty \frac{\alpha_n^2}{\|\varphi_n\|_{L^2_\nu [0, l]}^2} \exp [-2\lambda_n^2 t]}, \end{aligned}$$

$$\begin{aligned} M_T(x) &= \int_0^l \Theta_0(\xi) G(x, \xi, T) \nu(\xi) d\xi = \\ &= \sum_{n=1}^{\infty} \frac{\exp[-\lambda_n^2 T]}{\|\varphi_n\|_{L^2_\nu[0,l]}^2} \varphi_n(x) \psi_n. \end{aligned}$$

Here it is taken into account that

$$\left[\sum_n a_n \right]^2 = \sum_n \left[a_n^2 + a_n \cdot \sum_{k \neq n} a_k \right], \tag{2.59}$$

$$\int_0^l \varphi_n^2(x) \nu(x) dx = \|\varphi_n\|_{L^2_\nu[0,l]}^2,$$

as well as the fact that the functions $\{\varphi_n\}_{n=1}^\infty$ are orthogonal over the interval $[0, l]$ with the weight ν . Furthermore, straightforward calculations lead to

$$\|M_T\|_{L^2_\nu[0,l]}^2 = \sum_{n=1}^{\infty} \frac{\exp[-2\lambda_n^2 T]}{\|\varphi_n\|_{L^2_\nu[0,l]}^2} \psi_n^2.$$

Eventually, for the approximate null-controllability of the rod it is sufficient to find such a control $u_b \in \mathcal{U}$ that

$$\int_0^T g(T - \tau) |u_b(\tau)| d\tau \leq \varepsilon - \|M_T\|_{L^2_\nu[0,l]}, \tag{2.60}$$

as far as

$$\tilde{\varepsilon}_T = \varepsilon - \|M_T\|_{L^2_\nu[0,l]} \geq 0. \tag{2.61}$$

Apparently, since M_T has an exponential decay in T , then there exists such a $T > 0$ that (2.61) holds. On the other hand, as it was mentioned in Subsection 2.3.2, for fixed T , (2.61) can be provided by a particular choice of system characteristics. To make this obvious, consider particular inhomogeneities of the rod. More specifically, consider

the cases when

$$\chi(x) = \exp(\alpha x), \quad \alpha \in \mathbb{R},$$

and

$$\chi(x) = \left(\frac{x}{l} + \beta\right)^{-\alpha}, \quad \beta > 0, \quad \alpha \in \mathbb{R}.$$

Setting $\alpha = 0$ at any step of the preceding calculations, the case of a homogeneous rod will be obtained. It is obvious that the hardening or the softening of the rod material in both cases depend on the sign of α .

In both cases, the general solution of (2.56) reads as

$$\varphi(x) = c_1 \sin \left[\frac{\lambda}{\sqrt{d}} \phi(x) \right] + c_2 \cos \left[\frac{\lambda}{\sqrt{d}} \phi(x) \right] \quad \text{in } [0, l], \quad (2.62)$$

where in the first case

$$\phi(x) = \frac{1}{\alpha} \exp(-\alpha x),$$

while in the second case

$$\phi(x) = \frac{l}{\alpha + 1} \left(\frac{x}{l} + \beta\right)^{\alpha+1}.$$

Here,

$$d = \frac{\kappa_0}{\nu_0}$$

is the constant thermal diffusivity of a material with the thermal conductivity κ_0 and the volumetric heat capacity ν_0 . It measures the rate of the heat transfer from areas with higher temperature to those with lower temperature. Therefore, the larger the d is, the faster the heat transfers.

Involving the boundary conditions (2.57), in both cases of inhom-

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generality it is derived that c_1 remains arbitrary, while

$$c_2 = -c_1 \tan \left[\frac{\lambda}{\sqrt{d}} \phi(0) \right],$$

and λ_n are the roots of the equation

$$\sin \left[\frac{\lambda}{\sqrt{d}} \phi(l) \right] - \cos \left[\frac{\lambda}{\sqrt{d}} \phi(l) \right] \tan \left[\frac{\lambda}{\sqrt{d}} \phi(0) \right] = 0.$$

After simple transformations, this equation becomes

$$\sin \left[\frac{\lambda}{\sqrt{d}} (\phi(l) - \phi(0)) \right] = 0,$$

admitting the solution

$$\lambda_n = \sqrt{d} \frac{\pi n}{\phi(l) - \phi(0)}, \quad n \in \mathbb{N}, \quad (2.63)$$

as long as $\phi(l) - \phi(0) \neq 0$. Note that this holds in both cases of inhomogeneity considered above.

Since c_1 is arbitrary, let $c_1 = 1$. Then, (2.62) is reduced to

$$\varphi_n(x) = \sin \left[\frac{\phi(x) - \phi(0)}{\phi(l) - \phi(0)} \cdot \pi n \right].$$

Moreover, if in particular $\Theta_0 = \varphi_N$ for a fixed $N \in \mathbb{N}$, then using the orthogonality of the functions $\{\varphi_n\}_{n=1}^{\infty}$ over the interval $[0, l]$ with the weight ν once again, by virtue of (2.63), $\|M_T\|_{L^2_{\nu}[0, l]}^2$ is straightforwardly reduced to

$$\|M_T\|_{L^2_{\nu}[0, l]}^2 = \exp[-2\lambda_N^2 T] = \exp \left[-\frac{2\pi^2 N^2 d T}{(\phi(l) - \phi(0))^2} \right].$$

Therefore, indeed, by a specific choice of material characteristics d , α , and β , it is in principle possible to provide the inequality

$$\tilde{\varepsilon}_T \geq 0$$

for a given ε .

Then, any admissible control

$$u_b \in \tilde{\mathcal{U}}_{res} = \left\{ u_b \in \mathcal{U}, \quad |u_b| \leq \frac{\tilde{\varepsilon}_T}{\tilde{\varepsilon}_T} \right\}$$

with

$$\tilde{\varepsilon}_T = \int_0^T g(T - \tau) d\tau,$$

provides approximate null-controllability of the rod at T .

For numerical analysis consider the case when

$$\chi(x) = \exp(\alpha x) \quad \text{with} \quad \alpha = \frac{1}{l}.$$

Assume that the bulk material of the rod (i.e., the material with κ_0 and ν_0) is made from copper¹¹. Moreover, let

$$l = 0.1, \quad \Theta_0 = \varphi_3, \quad \varepsilon = 10^{-3}.$$

Then,

$$\tilde{\varepsilon}_T \geq 0 \quad \text{for} \quad T \geq 42.$$

First consider the case of the constant control regime (2.36). Then, $u^o = 10^{-4}$ ensures the approximate null-controllability of the rod for

¹¹Computations show that different materials do not significantly affect the solution, but only cause slight changes in numerical values of control parameters. That is why the consideration is restricted only by one material.

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$T \geq 90$ (see Fig. 2.3). Moreover, it turns out that regimes with $u^\circ < 10^{-4}$ may provide (2.60) for $42 \leq T < 90$.

Now consider the quasi-polynomial control regime (2.39). Then, the approximate null-controllability of the rod is provided for $T = 110$ when $u_{11} = 5 \cdot 10^{-7}$, $m = n = 1$ (see Fig. 2.4).

On the other hand, involving the trigonometric control regime (2.41), it is possible to reduce the control time to $T = 70$ (see Fig. 2.5).

It turns out that regimes like switching control (2.43) or optimal stopping control (2.45) or their mixed variants can be quite efficient in the sense that the approximate null-controllability of the rod can be achieved with the same precision, but with less intensity of control and for smaller T . For example, in the case when

$$u_b(t) = 2 \cdot 10^{-5} [\theta(t - 25) - \theta(t - 30) + \\ + \theta(t - 45) - \theta(t - 50) + \theta(t - 65) - \theta(t - 70)],$$

the rod is approximately null-controllable in $T = 70$ (see Fig. 2.6).

The residue (2.53) is evaluated for the control regimes considered above (see Tab. 2.1). The smallest value of $\mathcal{R}_T = 2.5 \cdot 10^{-4}$ for the residue guarantees the quasi-polynomial control regime in $T = 110$. On the other hand, the trigonometric and switching regimes may provide $\mathcal{R}_T \leq 5 \cdot 10^{-4}$ already for $T = 70$.

| Control regime | Residue | Control time |
|------------------|---------------------|--------------|
| Constant | $3 \cdot 10^{-4}$ | 90 |
| Quasi-polynomial | $2.5 \cdot 10^{-4}$ | 110 |
| Trigonometric | $5 \cdot 10^{-4}$ | 70 |
| Switching | $5 \cdot 10^{-4}$ | 70 |

Table 2.1: Residue (2.53) for different control regimes: $\chi(x) = \exp(\alpha x)$

2.4.2 Semi-Infinite Rod Heated by a Point Source

Let a sufficiently thin semi-infinite rod be thermo-isolated from the external medium. Since it is proved that the rod is not exactly null-controllable by means of L^2 boundary controls [33], let the rod be heated by a point source with a controllable intensity $u = u(t)$ placed at an inner point $x = x_0$, $0 < x_0 < \infty$, of the rod. The aim of the control is to determine admissible intensities for the source, which heats the rod to a required temperature in a given finite time T .

The temperature distribution in the rod, denoted here by Θ , obeys the one-dimensional heat equation

$$\frac{\partial \Theta}{\partial t} = d \frac{\partial^2 \Theta}{\partial x^2} + u(t)v(x) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+, \quad (2.64)$$

with the conditions of thermo-isolation

$$\Theta(0, t) = \lim_{x \rightarrow \infty} \Theta(x, t) \equiv 0 \quad \text{in } \mathbb{R}^+. \quad (2.65)$$

Here d is the thermal diffusivity of the rod.

Remark 2.10. *In this and related controllability problems, the function v in the right-hand side of (2.64), describing the shape of the heaters, plays a very crucial role. It turns out that the set of the reachable terminal states strongly depends on v . Moreover, as it is shown below, the null-controllability of the rod can be achieved only for very specific forms of v .*

Very often the case of heating with point sources is considered (see, e.g., [28, 165]). In such cases, v is expressed in terms of the Dirac function and its derivatives, so that the right-hand side of (2.64) becomes a distribution. The practical implementation of point sources is

usually carried out by means of L^1_{loc} sequences converging in the sense of distributions to the Dirac function. Examples of such sequences include [64, 65]

$$\delta_\zeta(x) = \frac{1}{\pi|\zeta|} \exp\left[-\left(\frac{x}{\zeta}\right)^2\right], \quad \delta_\zeta(x) = \frac{1}{\pi} \frac{\zeta}{\zeta^2 + x^2} \quad \text{as } \zeta \rightarrow 0.$$

For small ζ , $v_\zeta = \delta_\zeta$ in some approximation describes a point heater (see Fig. 2.1).

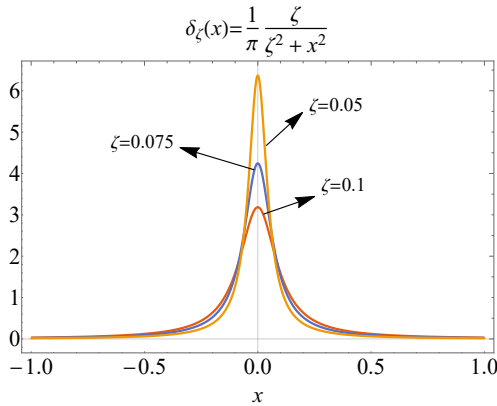


Figure 2.1: δ_ζ for decreasing values of ζ

The temperature distribution at the initial instant $t = 0$ is given by

$$\Theta(x, 0) = \Theta_0(x) \quad \text{in } \mathbb{R}^+. \tag{2.66}$$

Mathematically, the problem is to choose such a distributed heating regime $u \in \mathcal{U}$, that in a given T provides a desired terminal distribution

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of the temperature in the rod given by

$$\Theta(x, T) = \Theta_T(x) \quad \text{in } \mathbb{R}^+, \quad (2.67)$$

for a prescribed function $\Theta_T \in L^\infty(\mathbb{R}^+)$.

The set of admissible controls is considered to be

$$\mathcal{U} = \{u \in L^\infty[0, T], |u| \leq \epsilon, \text{ supp}(u) \subseteq [0, T]\}.$$

Since the rod is semi-infinite and the heat transfers with a finite speed, then, generally speaking, for exact satisfaction of the terminal condition (2.67) it is required to consider infinitely large T (refer to the reasoning made in [46]). That is why in this case the consideration of the approximate controllability is more appropriate.

In other words, it is required to find a $u \in \mathcal{U}$, such that

$$\mathcal{R}_T(u) = \|\Theta(x, T) - \Theta_T(x)\|_{L^\infty(\mathbb{R}^+)} \leq \epsilon, \quad (2.68)$$

with a given precision $\epsilon > 0$.

Assume that the boundary and initial data are consistent, so that

$$\Theta_0(0) = \lim_{x \rightarrow \infty} \Theta_0(x) = 0.$$

For the general solution of (2.64)-(2.66) evaluated at $t = T$, the Green's formula provides

$$\begin{aligned} \Theta(x, T) = & \int_0^\infty G(x, \xi, T) \Theta_0(\xi) d\xi + \\ & + \int_0^T G_v(x, T - \tau) u(\tau) d\tau \quad \text{in } \mathbb{R}^+, \end{aligned} \quad (2.69)$$

where [77]

$$G(x, \xi, t) = \frac{1}{\sqrt{4\pi dt}} \left[\exp \left[-\frac{(x - \xi)^2}{4dt} \right] - \exp \left[-\frac{(x + \xi)^2}{4dt} \right] \right],$$

and

$$G_v(x, t) = \int_0^\infty G(x, \xi, t) v(\xi) d\xi. \tag{2.70}$$

Note that the initial temperature Θ_0 and the heater shape v must be chosen in such a way that the improper integrals in (2.69) and (2.70) are well defined.

Further, since the Minkowski inequality provides

$$\mathcal{R}_T(u) \leq \int_0^T g(T - \tau) |u(\tau)| d\tau + \|M_T\|_{L^\infty(\mathbb{R}^+)},$$

where

$$g(t) = \|G_v(x, t)\|_{L^\infty(\mathbb{R}^+)},$$

and

$$M_T(x) = \int_0^\infty G(x, \xi, T) \Theta_0(\xi) d\xi - \Theta_T(x),$$

then, for the fulfillment of (2.68), i.e., for the approximate controllability of the rod, it suffices that

$$\int_0^T g(T - \tau) |u(\tau)| d\tau + \|M_T\|_{L^\infty(\mathbb{R}^+)} \leq \varepsilon. \tag{2.71}$$

Evidently, as far as

$$\tilde{\varepsilon}_T = \varepsilon - \|M_T\|_{L^\infty(\mathbb{R}^+)} \geq 0 \tag{2.72}$$

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holds, then any admissible control satisfying

$$u \in \tilde{\mathcal{U}}_{res} = \left\{ u \in \mathcal{U}, |u| \leq \frac{\tilde{\varepsilon}_T}{\tilde{\varepsilon}_T} \right\}$$

with

$$\tilde{\varepsilon}_T = \int_0^T g(T - \tau) d\tau,$$

ensures the approximate controllability of the rod at T . The particular solutions from Section 2.3 can be involved to satisfy (2.71).

Let in particular

$$v(x) = \delta(x - 0.5), \quad \Theta_0(x) = \theta(x) - \theta(x - 1),$$

$$\Theta_T \equiv 0, \quad \varepsilon = 10^{-3}.$$

In that case,

$$M_T(x) = \frac{\sqrt{\pi}}{2} \left[2 \operatorname{erf} \left[\frac{x}{\sqrt{4\pi dT}} \right] - \operatorname{erf} \left[\frac{x-1}{\sqrt{4\pi dT}} \right] - \operatorname{erf} \left[\frac{x+1}{\sqrt{4\pi dT}} \right] \right],$$

which (for fixed x) decreases as T increases. Then, the inequality (2.72) holds for very large values of T , specifically, for $T \geq 5.16 \cdot 10^5$. The same behaviour is observed for a number of other initial conditions. Therefore, in such cases (2.68) must be verified directly.

On the other hand, for the initial temperature of the form

$$\Theta_0(x) = \exp \left[-\frac{(x-0.1)^2}{\zeta^2} \right] \quad \text{with } \zeta = 10^{-3},$$

the inequality (2.72) is satisfied already for $T \geq 213.18$. Let $T = 240$.

Then, the control

$$\begin{aligned}
 u(t) = u^o & \left(1 - \cos \left[\frac{2\pi m}{T} t \right] \right)^n \left[- [\theta(t) - \theta(t - 60)] + \right. \\
 & + [\theta(t - 90) - \theta(t - 120) + \theta(t - 150) - \theta(t - 180)] + \\
 & \left. + [\theta(t - 200) - \theta(t - 240)] \right], \tag{2.73}
 \end{aligned}$$

with $m = 5$, $n = 0.3$ and $u^o = 10^{-4}$ ensures the approximate null-controllability of the rod with $\mathcal{R}_T \leq 8 \cdot 10^{-4}$ (see Fig. 2.7).

Implementing (2.68) directly, it is possible to achieve the approximate null-controllability of the rod for smaller values of T . Indeed, let $T = 150$. Then, the switching control

$$\begin{aligned}
 u(t) = u^o & [\theta(15 - t) - \theta(t - 30) + \theta(t - 45)] - \\
 & - u^o [\theta(t - 60) - \theta(t - 90) - \theta(t - 100) + \theta(t - 120)], \tag{2.74}
 \end{aligned}$$

with $u^o = 10^{-4}$ ensures the approximate null-controllability of the rod with $\mathcal{R}_T \leq 8 \cdot 10^{-4}$ (see Fig. 2.8).

2.4.3 Elastic Beam Subjected to a Dynamic Load with Uncertainty

Consider an elastic finite beam of length l subjected to a concentrated dynamical load of intensity P . The exact location of the load application, denoted by x_0 , is not known, however it is given that $x_0 \in (l_0, l_1) \subset (0, l)$ for prescribed l_0 and l_1 . Assume that $x = l$ end of the beam is simply supported, while at $x = 0$ only the bending moment is fixed, and the transverse deflection is controlled. The aim is to find boundary controls providing controllability of the beam in a given time T . It is supposed that the load vanishes at some $0 < T_0 < T$.

Since there exists an uncertainty in the system, it is hardly possible to provide exact controllability in a finite time [35, 197]. That is why the approximate controllability of the beam is studied.

Assume also that the beam is sufficiently thin and the load intensity varies in such a range that the beam undergoes merely infinitesimal strains. In that case, the Euler-Bernoulli assumptions can be involved (see [198] for details). Then, limiting the consideration by the linear elasticity, the transverse deflection of the beam, denoted here by w , satisfies the fourth order differential equation¹²

$$EJ \frac{\partial^4 w}{\partial x^4} + \rho S \frac{\partial^2 w}{\partial t^2} = P(t) \chi_{[0, T_0]}(t) \delta(x - x_0) \quad \text{in } (0, l) \times \mathbb{R}^+, \quad (2.75)$$

subject to the boundary conditions

$$w(0, t) = u_b(t), \quad w(l, t) = \frac{\partial^2 w}{\partial x^2} \Big|_{x=0, l} = 0 \quad \text{in } \mathbb{R}^+. \quad (2.76)$$

Here E is the Young's modulus, ρ is the density, J is the cross-sectional

¹²Regarding the right-hand side, recall Remark 2.10.

moment of inertia, and S is the cross-sectional area of the beam. The quantity EJ measures the resistance of the beam against a bending load and is called bending stiffness.

At the initial time instant $t = 0$ the state of the beam is given by

$$w(x, 0) = w_0(x), \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = w_{01}(x) \quad \text{in } [0, l]. \quad (2.77)$$

Mathematically, the problem is to find such a boundary control function u_b , that provides the inequalities

$$\mathcal{R}_T(u_b) = \|w(x, T) - w_T(x)\|_{L^2[0, l]} \leq \varepsilon, \quad (2.78)$$

$$\mathcal{R}_{T1}(u_b) = \left\| \left. \frac{\partial w}{\partial t} \right|_{t=T} - w_{T1}(x) \right\|_{L^2[0, l]} \leq \varepsilon_1 \quad (2.79)$$

simultaneously in a given finite T with required precisions $\varepsilon, \varepsilon_1 > 0$ and given functions $w_T, w_{T1} \in L^2[0, l]$. As in many applications, here it is assumed that $w_T = w_{T1} \equiv 0$, i.e., it is required to suspend the vibrations of the beam.

The initial and boundary data are supposed to be consistent:

$$\begin{aligned} u_b(0) &= w_0(0), \quad \dot{u}_b(0) = w_{01}(0), \\ u_b(T) &= w_T(0), \quad \dot{u}_b(T) = w_{T1}(0). \end{aligned} \quad (2.80)$$

Then, the set of admissible controls is considered to be

$$\mathcal{U} = \{u \in L^2[0, T], |u| \leq \varepsilon, \text{supp}(u) \subseteq [0, T], (2.80)\}.$$

Unlike the previous two cases, where only one inequality (c.f., for example, (2.53) or (2.68)) must have been provided, in this case the

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two inequalities (2.78) and (2.79) must be provided simultaneously.

For the further analysis it is convenient to use the dimensionless quantities

$$\frac{x}{l}, \quad \frac{ct}{l}, \quad \frac{w}{l}, \quad \frac{Pl^2}{EJ},$$

for the coordinate, time, deflection, and load intensity, respectively. Here c is the speed of the elastic wave propagation in the beam:

$$c = \sqrt{\frac{E}{\rho}}.$$

New symbols for those quantities are not introduced, in order to make the reading convenient.

Then, the general solution of (2.75)-(2.77) according to the Green's representation formula is given by

$$\begin{aligned} w(x, t) = & \int_0^1 \left[w_0(\xi) \frac{\partial G(x, \xi, t)}{\partial t} + w_{01}(\xi) G(x, \xi, t) \right] d\xi + \\ & + \int_0^t \int_0^1 P(\tau) \chi_{[0, T_0]}(\tau) \delta(\xi - x_0) G(x, \xi, t - \tau) d\xi d\tau + \\ & + \int_0^t u_b(\tau) G_b(x, t - \tau) d\tau \quad \text{in } [0, 1] \times \mathbb{R}^+. \end{aligned} \tag{2.81}$$

Here [77]

$$\begin{aligned} G(x, \xi, t) = & \frac{2\alpha}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \varphi_n(x) \varphi_n(\xi) \psi_n(t), \\ \varphi_n(x) = & \sin(\pi n x), \quad \psi_n(t) = \sin\left(\frac{\pi^2 n^2}{\alpha} t\right), \quad \alpha^2 = \frac{Sl^2}{J}, \end{aligned}$$

$$G_b(x, t) = \int_0^1 \delta'''(\xi) G(x, \xi, t) d\xi = - \left. \frac{\partial^3 G(x, \xi, t)}{\partial \xi^3} \right|_{\xi=0} = 2\pi\alpha \sum_{n=1}^{\infty} n\varphi_n(x) \psi_n(t).$$

Differentiation of (2.81) with respect to t once provides a similar representation formula for the beam particle velocity:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= \int_0^1 \left[w_0(\xi) \frac{\partial^2 G(x, \xi, t)}{\partial t^2} + w_{01}(\xi) \frac{\partial G(x, \xi, t)}{\partial t} \right] d\xi + \\ &+ \int_0^t \int_0^1 P(\tau) \chi_{[0, T_0]}(\tau) \delta(\xi - x_0) \frac{\partial G(x, \xi, t - \tau)}{\partial t} d\xi d\tau \\ &+ \int_0^t u_b(\tau) \frac{\partial G_b(x, t - \tau)}{\partial t} d\tau \quad \text{in } [0, 1] \times \mathbb{R}^+. \end{aligned} \tag{2.82}$$

For the derivation of (2.82) the Leibniz rule

$$\frac{\partial}{\partial t} \int_0^t f(t, \tau) d\tau = f(t, t) + \int_0^t \frac{\partial f(t, \tau)}{\partial t} d\tau$$

is taken into account together with the obvious relation

$$G(\cdot, \cdot, 0) \equiv 0.$$

Evaluating the expressions (2.81) and (2.82) at $t = T$ and substituting into the inequalities (2.78) and (2.79), respectively, by virtue of the Minkowski and Cauchy-Schwartz inequalities, the following estimates are derived

$$\mathcal{R}_T(u_b) \leq \int_0^T g(T - \tau) |u_b(\tau)| d\tau + \|M_T\|_{L^2[0,1]}, \tag{2.83}$$

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$$\mathcal{R}_{T1}(u_b) \leq \int_0^T g_1(T - \tau) |u_b(\tau)| d\tau + \|M_{1T}\|_{L^2[0,1]}, \quad (2.84)$$

where

$$g(t) = \|G_b(x, t)\|_{L^2[0,1]}, \quad g_1(t) = \left\| \frac{\partial G_b(x, t)}{\partial t} \right\|_{L^2[0,1]},$$

$$\begin{aligned} M_T(x, x_0) &= \int_0^1 \left[w_0(\xi) \frac{\partial G(x, \xi, t)}{\partial t} + w_{01}(\xi) G(x, \xi, t) \right] \Big|_{t=T} d\xi + \\ &+ \int_0^{T_0} \int_0^1 P(\tau) \delta(\xi - x_0) G(x, \xi, T - \tau) d\xi d\tau = \\ &= \int_0^1 \left[w_0(\xi) \frac{\partial G(x, \xi, t)}{\partial t} + w_{01}(\xi) G(x, \xi, t) \right] \Big|_{t=T} d\xi + \\ &+ \int_0^{T_0} P(\tau) G(x, x_0, T - \tau) d\tau, \end{aligned}$$

$$\begin{aligned} M_{1T}(x, x_0) &= \int_0^1 \left[w_0(\xi) \frac{\partial^2 G(x, \xi, t)}{\partial t^2} + w_{01}(\xi) \frac{\partial G(x, \xi, t)}{\partial t} \right] \Big|_{t=T} d\xi + \\ &+ \int_0^{T_0} \int_0^1 P(\tau) \delta(\xi - x_0) \frac{\partial G(x, \xi, T - \tau)}{\partial t} \Big|_{t=T} d\tau = \\ &= \int_0^1 \left[w_0(\xi) \frac{\partial^2 G(x, \xi, t)}{\partial t^2} + w_{01}(\xi) \frac{\partial G(x, \xi, t)}{\partial t} \right] \Big|_{t=T} d\xi + \\ &+ \int_0^{T_0} P(\tau) \frac{\partial G(x, x_0, T - \tau)}{\partial t} \Big|_{t=T} d\tau. \end{aligned}$$

Thus, the right-hand sides of the inequalities (2.83) and (2.84) depend on the uncertain value of $x_0 \in (l_0, l_1)$ through the norms of the functions M_T and M_{1T} . Therefore, the boundary control needs to be determined to compensate for the “worst” influence of x_0 on $\|M_T\|_{L^2[0,1]}$ and $\|M_{1T}\|_{L^2[0,1]}$.

Computing the corresponding time derivatives of the Green’s func-

tion above,

$$\frac{\partial G(x, \xi, t)}{\partial t} = 2 \sum_{n=1}^{\infty} \varphi_n(x) \varphi_n(\xi) \psi_{1n}(t),$$

$$\frac{\partial^2 G(x, \xi, t)}{\partial t^2} = -\frac{2\pi^2}{\alpha} \sum_{n=1}^{\infty} n^2 \varphi_n(x) \varphi_n(\xi) \psi_n(t),$$

$$\frac{\partial G_b(x, t)}{\partial t} = 2\pi^3 \sum_{n=1}^{\infty} n^3 \varphi_n(x) \psi_{1n}(t),$$

where

$$\psi_{1n}(t) = \cos\left(\frac{\pi^2 n^2}{\alpha} t\right),$$

the functions g , g_1 and M_T , M_{1T} can be evaluated as follows:

$$g(t) = \sqrt{\int_0^1 G_b^2(x, t) dx} = \sqrt{2\pi\alpha} \sqrt{\sum_{n=1}^{\infty} n^2 \psi_n^2(t)},$$

$$g_1(t) = \sqrt{\int_0^1 \left(\frac{\partial G_b(x, t)}{\partial t}\right)^2 dx} = \sqrt{2\pi^3} \sqrt{\sum_{n=1}^{\infty} n^6 \psi_{1n}^2(t)},$$

and

$$M_T(x, x_0) = \sum_{n=1}^{\infty} A_n(x_0, T) \varphi_n(x),$$

$$M_{1T}(x, x_0) = \sum_{n=1}^{\infty} A_{1n}(x_0, T) \varphi_n(x),$$

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Here

$$\begin{aligned}
 A_n(x_0, T) &= 2w_{0n}\psi_{1n}(T) + \frac{2\alpha}{\pi^2 n^2} w_{01n}\psi_n(T) + \\
 &\quad + \frac{2\alpha}{\pi^2 n^2} P_n(T) \varphi_n(x_0), \\
 A_{1n}(x_0, T) &= -\frac{2\pi^2 n^2}{\alpha} w_{0n}\psi_n(T) + 2w_{01n}\psi_{1n}(T) + \\
 &\quad + 2P_{1n}(T) \varphi_n(x_0),
 \end{aligned}$$

and

$$\begin{aligned}
 w_{0n} &= \int_0^1 w_0(\xi) \varphi_n(\xi) d\xi, \\
 w_{01n} &= \int_0^1 w_{01}(\xi) \varphi_n(\xi) d\xi, \\
 P_n(T_0, T) &= \int_0^{T_0} P(\tau) \psi_n(T - \tau) d\tau, \\
 P_{1n}(T_0, T) &= \int_0^{T_0} P(\tau) \psi_{1n}(T - \tau) d\tau.
 \end{aligned}$$

Moreover, taking into account the summation formula (2.59) and that in this case

$$\|\varphi_n\|_{L^2[0,1]}^2 = \frac{1}{2},$$

the direct integration leads to

$$\begin{aligned}
 \|M_T\|_{L^2[0,1]}^2 &= \frac{1}{2} \sum_{n=1}^{\infty} A_n^2(x_0, T), \\
 \|M_{1T}\|_{L^2[0,1]}^2 &= \frac{1}{2} \sum_{n=1}^{\infty} A_{1n}^2(x_0, T).
 \end{aligned} \tag{2.85}$$

Remark 2.11. *It is evident from the expressions for A_n and A_{1n} that when $x_0 \searrow 0$ or $x_0 \nearrow 1$, then M_T and M_{1T} become independent on x_0 . It is also obvious that the “worst” influence x_0 has on $\|M_T\|_{L^2[0,1]}$ and $\|M_{1T}\|_{L^2[0,1]}$ when $x_0 \rightarrow \frac{1}{2}$, i.e., when the load is acting at the mid-point of the beam. This is a consequence which should be expected.*

On the other hand, assuming that for a given constant P_0

$$|P(t)| \leq P_0 \quad \text{in } [0, T_0],$$

the lower and upper bounds of the integrals in P_n and P_{1n} are estimated as follows:

$$|P_n(T_0, T)| \leq \frac{2P_0\alpha}{\pi^2 n^2}, \quad |P_{1n}(T_0, T)| \leq \frac{4P_0\alpha}{\pi^2 n^2}.$$

It is taken into account that

$$\int_0^{T_0} |\psi_n(T - \tau)| d\tau = \frac{\alpha}{\pi^2 n^2} [\psi_{1n}(T - T_0) \text{sign}[\psi_n(T - T_0)] - \psi_{1n}(T) \text{sign}[\psi_n(T)]],$$

$$\int_0^T |\psi_{1n}(T - \tau)| d\tau = \frac{\alpha}{\pi^2 n^2} [2 - \psi_n(T - T_0) \text{sign}[\psi_{1n}(T - T_0)] + \psi_n(T) \text{sign}[\psi_{1n}(T)]].$$

Therefore the Minkowski inequality provides

$$A_n^2(x_0, T) \leq A_{0n}^2(T) + \left(\frac{4P_0\alpha^2}{\pi^4 n^4}\right)^2,$$

$$A_{1n}^2(x_0, T) \leq A_{01n}^2(T) + \left(\frac{4P_0\alpha}{\pi^2 n^2}\right)^2,$$

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where

$$A_{0n}(T) = 2w_{0n}\psi_{1n}(T) + \frac{2\alpha}{\pi^2 n^2} w_{01n}\psi_n(T),$$

$$A_{01n}(T) = -\frac{2\pi^2 n^2}{\alpha} w_{0n}\psi_n(T) + 2w_{01n}\psi_{1n}(T).$$

Substituting those estimates into (2.85) and taking into account the values of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450},$$

the norms in (2.85) are estimated as follows:

$$\|M_T\|_{L^2[0,1]}^2 \leq \frac{1}{2} \sum_{n=1}^{\infty} A_{0n}^2(T) + \frac{8}{4725} P_0^2 \alpha^4,$$

$$\|M_{1T}\|_{L^2[0,1]}^2 \leq \frac{1}{2} \sum_{n=1}^{\infty} A_{01n}^2(T) + \frac{16}{90} P_0^2 \alpha^2.$$

Thus, as far as

$$\tilde{\varepsilon}_T = \varepsilon - \sqrt{\frac{1}{2} \sum_{n=1}^{\infty} A_{0n}^2(T) + \frac{8}{4725} P_0^2 \alpha^4} \geq 0, \quad (2.86)$$

$$\tilde{\varepsilon}_{1T} = \varepsilon_1 - \sqrt{\frac{1}{2} \sum_{n=1}^{\infty} A_{01n}^2(T) + \frac{16}{90} P_0^2 \alpha^2} \geq 0, \quad (2.87)$$

any admissible control

$$u_b \in \tilde{\mathcal{U}}_{res} = \left\{ u \in \mathcal{U}, \quad |u| \leq \min \left\{ \frac{\tilde{\varepsilon}_T}{\tilde{\varepsilon}_T}, \frac{\tilde{\varepsilon}_{1T}}{\tilde{\varepsilon}_{1T}} \right\} \right\}$$

with

$$\tilde{\epsilon}_T = \int_0^T g(T - \tau) d\tau, \quad \tilde{\epsilon}_{1T} = \int_0^T g_1(T - \tau) d\tau,$$

provides the approximate null-controllability of the beam at the required T for any $x_0 \in (l_0, l_1)$.

For numerical analysis let the cross section of the beam be rectangular, providing

$$S = bh, \quad J = \frac{bh^3}{12}, \quad \alpha^2 = 12 \frac{l^2}{h^2}, \quad \frac{Pl^2}{EJ} = \frac{P}{Eh^2} \alpha^2,$$

where b is the width and h is the thickness of the beam. Moreover, let

$$w_0(x) = w_{01}(x) = \varphi_1(x), \quad P = P_0 = \text{const}, \quad T_0 = 1,$$

$$\frac{l}{h} = 10, \quad \frac{P_0}{Eh^2} \alpha^2 = 1.26 \cdot 10^{-4}, \quad \varepsilon = 10^{-2}.$$

Then, the inequalities (2.86) and (2.87) are simultaneously satisfied for $6.41 \leq T \leq 6.54$ (see Fig. 2.2). Fix $T = 6.5$. Then,

$$\tilde{\epsilon}_T \approx 720, \quad \tilde{\epsilon}_{1T} \approx 165.14,$$

so that admissible controls with

$$|u_b| \leq \frac{\tilde{\epsilon}_{1T}}{\tilde{\epsilon}_T} \approx 5.4 \cdot 10^{-5}$$

ensure approximate null-controllability of the beam in $T = 6.5$.

It is observed that when $\frac{l}{h} > 10$, then for simultaneous satisfaction of the inequalities (2.86) and (2.87) for at least one $T > 0$ it is necessary that $\frac{P_0}{Eh^2} \alpha^2 < 2.98 \cdot 10^{-4}$ corresponding to infinitesimal strains. For smaller values of the last fraction, there exist several values of T for

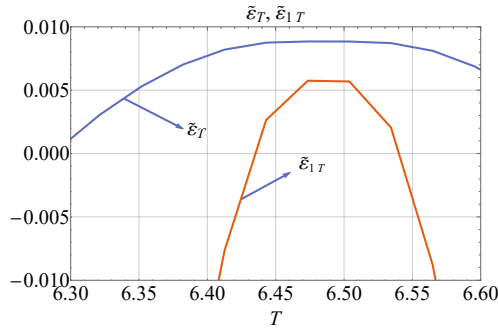


Figure 2.2: Plots of $\tilde{\varepsilon}_T$ and $\tilde{\varepsilon}_{1T}$ against T

which (2.86) and (2.87) are satisfied simultaneously. On the other hand, considering smaller values for $\frac{l}{h}$ may lead to a distortion in the Euler-Bernoulli assumptions and make the beam model above invalid.

A more comprehensive analysis requires direct verification of (2.86) and (2.87). However, computations show that when $P_0 \sim 10^5$, then $w \sim 10^{-5}$ when the load is active and $w \sim 10^{-9}$ when it vanishes (see Fig. 2.9). Therefore the required precision must be at least $\varepsilon \sim 10^{-10}$. On the other hand, the residual axial stress arising in the beam after the load vanishes accepts larger values and can serve as a controllability criterion. Thus, as a residue consider the quantity

$$\bar{\mathcal{R}}_T(u) = \left\| \left\| E \frac{\partial^2 w}{\partial x^2} \Big|_{t=T} - \sigma_0 \right\|_{L^2[0,l]} \right\|, \quad (2.88)$$

where

$$\sigma_x(x, T) = E \frac{\partial^2 w}{\partial x^2} \Big|_{t=T}, \quad x \in [0, l],$$

is the axial stress of the beam evaluated at $t = T$ and σ_0 is a prescribed

threshold value. Thus, $\overline{\mathcal{R}}_T$ evaluates how close the axial stress is to a given threshold at the instant T .

For a more specific analysis, consider a beam of length $l = 1 \text{ m}$ ¹³ and of a square cross-section with dimensions $0.1 \text{ m} \times 0.1 \text{ m}$. Therefore

$$J = \frac{b^4}{12} \approx 8.3 \cdot 10^{-6} \text{m}^4.$$

Let the material of the beam be made from copper, i.e., $E = 1.1 \cdot 10^{11} \text{ N/m}^2$, $\rho = 8960 \text{ kg/m}^3$. Limit the computations to the case when $x_0 = \frac{l}{2}$ (recall Remark 2.11). Assume that

$$P(t) = P_0 \sin(\pi t) [\theta(t - 0) - \theta(t - T_0)],$$

with $T_0 = 1 \text{ s}$. Set $T = 1.2 \text{ s}$.

Note that the maximal value of $\sigma_x(x, T) = 300 \text{ N/m}^2$ (see Fig. 2.10 (right)). In order to reduce this value, the threshold value is set to $\sigma_0 = 200 \text{ N/m}^2$ and the optimal stopping boundary control regime

$$u(t) = u^o [\theta(t - 0) - \theta(t - t^o)] \sin(\omega_1 t + \omega_2)$$

is chosen to ensure the inequality

$$\overline{\mathcal{R}}_T(u) \leq \varepsilon \tag{2.89}$$

for (2.88) with the precision $\varepsilon = 10^{-5}$. Above, u^o , t^o , ω_1 , and ω_2 are free parameters. At this, t^o is constrained by $0 < t^o \leq T$. Then, the

¹³Here the dimensions are introduced to make the evaluation more realistic.

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inequality (2.89) holds with $\varepsilon = 10^{-5}$ when

$$u^o = 10^{-9}, \quad \omega_1 = 1, \quad \omega_2 = 0, \quad t^o = 0.85.$$

Moreover, Fig. 2.11 shows that besides ensuring (2.89), the considered control regime reduces the total displacement of the beam almost twice.

Consider now the switching regime

$$u(t) = u_1 [\theta(t - 0) - \theta(t - t_1)] + u_2 [\theta(t - t_2) - \theta(t - t_3)],$$

where u_1 , u_2 , t_1 , t_2 , and t_3 are free parameters. Let the threshold value in this case be $\sigma_0 = 150 \text{ N/m}^2$. Then,

$$u_1 = u_2 = 10^{-9}, \quad t_1 = 0.25, \quad t_2 = 0.4, \quad t_3 = 0.65,$$

provide (2.89) with $\varepsilon = 10^{-5}$. Note that the total displacement in this case is also reduced almost twice (see Fig. 2.12).

2.4.4 Square Heated from the Boundary

Let a sufficiently thin square sheet which occupies in the Cartesian coordinate system Oxy the flat domain $\Omega = [0, a] \times [0, a]$ be heated from its boundary. Assume that the upper and lower surfaces of the sheet are thermo-isolated, so that the heat is exchanged exceptionally within the plate. The aim of the control is to determine a heating regime at the boundary such that the temperature distribution in the sheet in a required finite time T becomes uniform and admits a given constant value.

The heat distribution in the sheet in time, denoted here by Θ , is described by the two-dimensional heat equation

$$\frac{\partial \Theta}{\partial t} = d\Delta \Theta \quad \text{in } \Omega \times \mathbb{R}^+,$$

where d is the thermal diffusivity of the sheet material. The boundary conditions are

$$\begin{aligned} \Theta(x, 0, t) = \Theta(x, a, t) &= u_b(t)v(x) \quad \text{in } [0, a] \times \mathbb{R}^+, \\ \Theta(0, y, t) = \Theta(a, y, t) &= u_b(t)v(y) \quad \text{in } [0, a] \times \mathbb{R}^+. \end{aligned}$$

Here v is the distribution of the heaters along the sheet edges, satisfying $v > 0$ and $\text{supp}(v) \subseteq [0, a]$. The case $v \equiv 1$ corresponds to the uniform distribution of the heaters along the boundary¹⁴.

The initial temperature distribution in the square sheet is given by

$$\Theta(x, y, 0) = \Theta_0(x, y) \quad \text{in } \Omega.$$

¹⁴A more general problem with different heating regimes on different edges of the sheet, as well as with different distribution functions can be considered for achieving a better result. However, with the symmetry assumption, the study is considerably simplified.

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Mathematically, the problem is to find such a boundary control $u_b \in \mathcal{U}$, that in the required time T provides a uniform terminal temperature distribution

$$\Theta(x, y, T) \equiv \Theta_T = \text{const} \quad \text{in } \Omega$$

exactly or approximately for a given Θ_T . In other words, it is required to find such an admissible control $u_b \in \mathcal{U}$, that the residue

$$\mathcal{R}_T(u_b) = \|\Theta(x, y, T) - \Theta_T\|_{L^2(\Omega)}$$

satisfies

$$\mathcal{R}_T(u_b) = 0 \tag{2.90}$$

or

$$\mathcal{R}_T(u_b) \leq \varepsilon \tag{2.91}$$

with a required precision $\varepsilon > 0$.

The control function is constrained by the compatibility of the initial, terminal, and boundary data, implying

$$\begin{aligned} u_b(0)v(x) &= \Theta_0(x, 0) = \Theta_0(x, a) \quad \text{in } [0, a], \\ u_b(0)v(y) &= \Theta_0(0, y) = \Theta_0(a, y) \quad \text{in } [0, a], \\ u_b(T) &= \Theta_T. \end{aligned} \tag{2.92}$$

In general, (2.92) makes sense only in the case when

$$\frac{\Theta_0(x, 0)}{v(x)} = \frac{\Theta_0(x, a)}{v(x)} = \frac{\Theta_0(0, y)}{v(y)} = \frac{\Theta_0(a, y)}{v(y)} = \text{const} := \Theta_{00} \quad \text{in } \partial\Omega.$$

Then, (2.92) provides

$$u_b(0) = \Theta_{00}, \quad u_b(T) = \Theta_T. \tag{2.93}$$

As a set of admissible controls consider

$$\mathcal{U} = \{u \in L^2 [0, T], |u| \leq \epsilon, \text{supp}(u) \subseteq [0, T], (2.93)\}.$$

In this case, the Green's formula evaluated at $t = T$ results in

$$\begin{aligned} \Theta(x, y, T) &= \int_{\Omega} \Theta_0(\xi, \eta) G(x, \xi, y, \eta, T) \, d\xi d\eta + \\ &+ \int_0^T G_b(x, y, T - \tau) u_b(\tau) \, d\tau \quad \text{in } \bar{\Omega}, \end{aligned}$$

where [77]

$$G(x, \xi, y, \eta, t) = \frac{4}{a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_m(x) \varphi_m(\xi) \varphi_n(y) \varphi_n(\eta) \psi_{mn}(t),$$

$$\begin{aligned} G_b(x, y, t) &= d \int_{\Omega} G(x, y, \xi, \eta, t) [v(\eta) \delta'(\xi - a) - v(\eta) \delta'(\xi) + \\ &+ v(\xi) \delta'(\eta - a) - v(\xi) \delta'(\eta)] \, d\eta d\xi = \\ &= d \int_0^a v(\eta) \frac{\partial G(x, y, \xi, \eta, t)}{\partial \xi} \Big|_{\xi=a}^{\xi=0} \, d\eta + \\ &+ d \int_0^a v(\xi) \frac{\partial G(x, y, \xi, \eta, t)}{\partial \eta} \Big|_{\eta=a}^{\eta=0} \, d\xi = \\ &= \frac{4\pi d}{a^3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{mn} \varphi_m(x) \varphi_n(y) \psi_{mn}(t), \end{aligned}$$

$$\alpha_{mn} = m(1 - (-1)^m) \gamma_n + n(1 - (-1)^n) \gamma_m, \quad \gamma_n = \int_0^a v(\eta) \varphi_n(\eta) \, d\eta,$$

$$\varphi_m(x) = \sin\left(\frac{\pi m}{a} x\right), \quad \psi_{mn}(t) = \exp\left[-d \frac{\pi^2}{a^2} (m^2 + n^2) t\right],$$

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$$f(x) \Big|_{x=a}^{x=0} = f(0) - f(a).$$

Since $\{\varphi_m\}_{m=1}^{\infty}$ are orthogonal over the interval $[0, a]$, then from the final expression for G_b it follows that (2.90) is equivalent to the infinite system

$$\frac{4\pi d}{a^3} \alpha_{mn} \int_0^T \psi_{mn}(T - \tau) u_b(\tau) d\tau = M_{Tmn}, \quad m, n \in \mathbb{N}, \quad (2.94)$$

with

$$\begin{aligned} M_{Tmn} &= \int_{\Omega} M_T(x, y) \varphi_m(x) \varphi_n(y) dx dy, \\ M_T(x, y) &= \int_{\Omega} \Theta_0(\xi, \eta) G(x, \xi, y, \eta, T) d\xi d\eta - \Theta_T = \\ &= \frac{4}{a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu_{Tmn} \varphi_m(x) \varphi_n(y) - \Theta_T, \end{aligned}$$

where

$$\mu_{Tmn} = \psi_{mn}(T) \int_{\Omega} \Theta_0(\xi, \eta) \varphi_m(\xi) \varphi_n(\eta) d\xi d\eta.$$

Thus,

$$M_{Tmn} = \frac{4}{a^2} \mu_{Tmn} - \frac{a^2 \Theta_T}{\pi^2 mn} (1 - (-1)^m) (1 - (-1)^n).$$

It is evident that M_{Tmn} decreases very fast with individual or simultaneous increase of m and n , which means that (2.94) can be truncated and considered for some finite M and N . Eventually, the determination of u_b is reduced to a finite-dimensional system of algebraic equations. For further details and particular solutions of (2.94), refer to Subsection 2.3.1.

It is noteworthy that there might exist solutions of (2.94), which do

not satisfy (2.93). Indeed, assume that $\Theta_T = 0$, and

$$\Theta_0(x, y) = \varphi_M(x) \varphi_M(y), \quad v = \varphi_M,$$

for a fixed odd M . Then

$$\alpha_{mn} = 4M \delta_m^M \delta_n^M, \quad M_{Tmn} = \frac{4}{a^2} \mu_{Tmn} = \frac{4}{a^2} \psi_{MM}(T) \delta_m^M \delta_n^M,$$

and therefore (2.94) consists of only one equation

$$\int_0^T \psi_{MM}(-\tau) u_b(\tau) d\tau = \frac{a}{\pi M d}.$$

Here it is taken into account that

$$\psi_{MM}(T - \tau) = \psi_{MM}(T) \cdot \psi_{MM}(-\tau).$$

Evidently, the function

$$u_b(t) = u^o \cdot \psi_{MM}(t) \quad \text{with} \quad u^o = \frac{a}{\pi M T d},$$

belongs to $L^2[0, T]$ and satisfies the equation above, but it does not ensure the exact controllability of the plate, since the second condition in (2.93) is satisfied exactly only when $T \rightarrow \infty$.

On the other hand, for the approximate null-controllability of the square, it is necessary and sufficient that the inequality

$$\left\| \int_0^T G_b(x, y, T - \tau) u_b(\tau) d\tau + M_T(x, y) \right\|_{L^2(\Omega)} \leq \varepsilon \quad (2.95)$$

holds on \mathcal{U} . Then, by virtue of the Minkowski and Cauchy-Schwartz

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inequalities, the fulfillment of the inequality

$$\int_0^T g(T - \tau) |u_b(\tau)| d\tau \leq \varepsilon - \|M_T\|_{L^2(\Omega)} \quad (2.96)$$

for at least one $u_b \in \mathcal{U}$ is sufficient for the fulfillment of (2.95) as far as

$$\tilde{\varepsilon}_T = \varepsilon - \|M_T\|_{L^2(\Omega)} \geq 0.$$

Here

$$g(t) = \|G_b(x, y, t)\|_{L^2(\Omega)} = \frac{2\pi d}{a^2} \sqrt{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{mn}^2 \psi_{mn}^2(t)},$$

and

$$\begin{aligned} \|M_T\|_{L^2(\Omega)} &= \sqrt{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} M_{Tmn}^2 \|\varphi_m\|_{L^2[0,a]}^2 \|\varphi_n\|_{L^2[0,a]}^2} = \\ &= \frac{a}{2} \sqrt{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} M_{Tmn}^2}. \end{aligned}$$

In the derivation above, the summation formula (2.59) and

$$\|\varphi_n\|_{L^2[0,a]}^2 = \frac{a}{2}.$$

are taken into account.

In that case, any admissible control satisfying

$$u_b \in \tilde{\mathcal{U}}_{res} = \left\{ u \in \mathcal{U}, \quad |u| \leq \frac{\tilde{\varepsilon}_T}{\tilde{\varepsilon}_T} \right\}$$

with

$$\tilde{\epsilon}_T = \int_0^T g(T - \tau) d\tau,$$

is a resolving control ensuring the approximate null-controllability of the square.

Particular solutions presented in Subsection 2.3.2 can be involved to satisfy (2.96).

For numerical analysis let

$$a = 0.1, \quad v \equiv 1, \quad \Theta_0(x, y) = 10\varphi_1(x)\varphi_1(y), \quad \Theta_T = 0, \quad \varepsilon = 10^{-3}.$$

In addition, assume that the material of the square is made from copper. Then, the inequality

$$\tilde{\epsilon}_T = \varepsilon - \frac{20}{a} \exp\left[-2d\frac{\pi^2}{a^2}T\right] \geq 0$$

holds for $T \geq 57$.

It turns out that involving the constant control regime with $u^o = -10^{-4}$, it is possible to provide the approximate null-controllability of the square in $T = 150$ (see Fig. 2.13).

For achieving a higher precision for less T , control regimes other than constant can be involved. Indeed, involving the quasi-polynomial regime (2.39), the approximate null-controllability of the square can be ensured for $T = 120$ (see Fig. 2.14).

On the other hand, the trigonometric regime (2.41) can provide a smaller value for the residue in (2.91), but for larger T . Specifically, (2.41) with $u^o = 10^{-4}$, $m = 7$ and $n = 1.5$ provides $\mathcal{R}_T \leq 3 \cdot 10^{-4}$ in $T = 180$ (see Fig. 2.15).

The residue (2.53) is evaluated for the control regimes considered

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above (see Tab. 2.2). The smallest value for the residue guarantees the trigonometric control regime, but in $T = 180$. On the other hand, the quasi-polynomial control regime provides $\mathcal{R}_T \leq 3.5 \cdot 10^{-4}$ for $T = 120$.

| Control regime | Residue | Control time |
|------------------|---------------------|--------------|
| Constant | $9 \cdot 10^{-4}$ | 150 |
| Quasi-polynomial | $3.5 \cdot 10^{-4}$ | 120 |
| Trigonometric | $3 \cdot 10^{-4}$ | 180 |

Table 2.2: Residue (2.91) for different control regimes

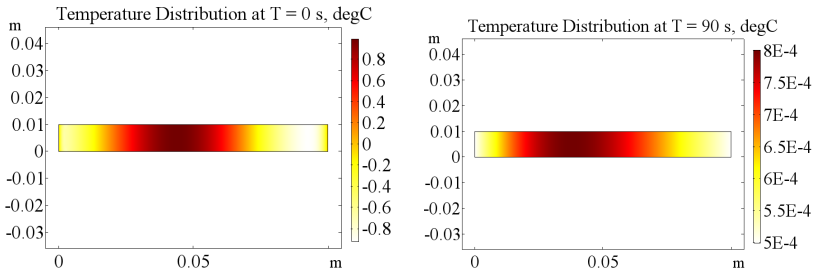


Figure 2.3: Initial (left) and implemented terminal (right) distributions of temperature in the rod with $\chi(x) = \exp(\alpha x)$ when $u_b \equiv 10^{-4}$

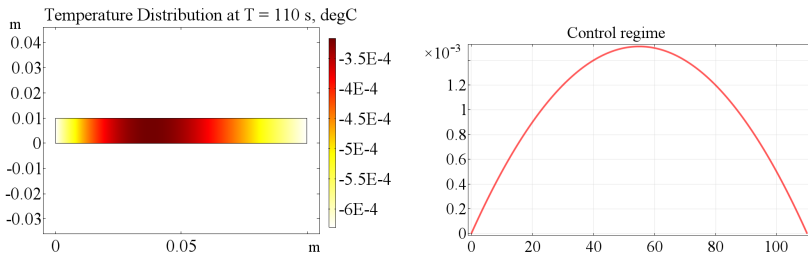


Figure 2.4: Implemented terminal state (left) for quasi-polynomial control regime (right) with $u^o = 5 \cdot 10^{-7}$, $m = n = 1$

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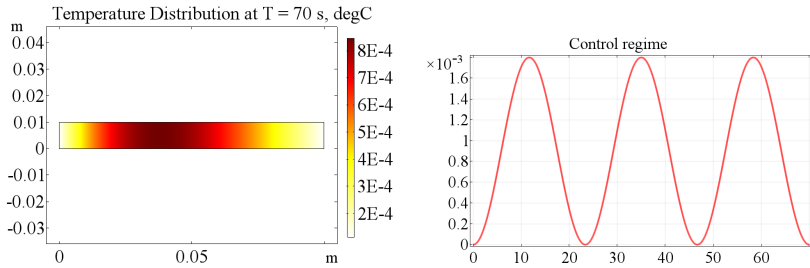


Figure 2.5: Implemented terminal state (left) for trigonometric control regime (right) with $u^o = 9 \cdot 10^{-4}$, $m = 3$, $n = 1$

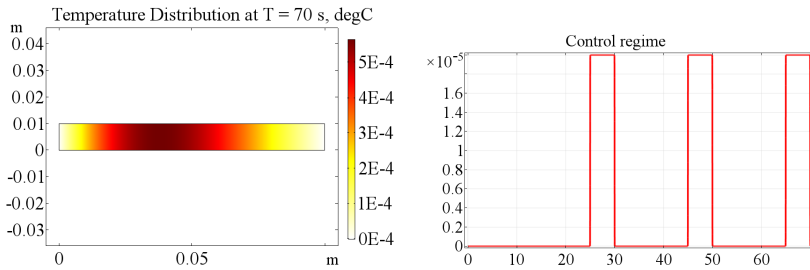


Figure 2.6: Implemented terminal state (left) for switching control regime (right) with $u^o = 2 \cdot 10^{-5}$, $t_1 = 25$, $t_2 = 30$, $t_3 = 45$, $t_4 = 50$, $t_5 = 65$, $t_6 = 70$

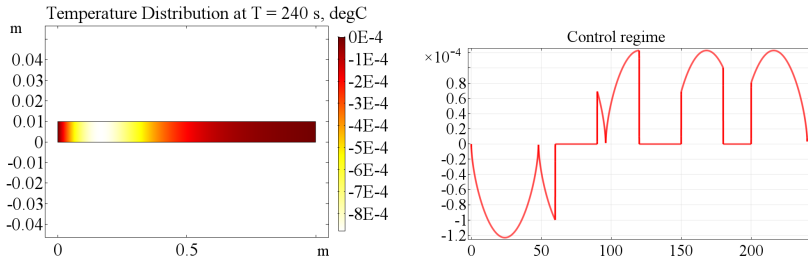


Figure 2.7: Implemented terminal state (left) for the switching control regime (2.73) (right)

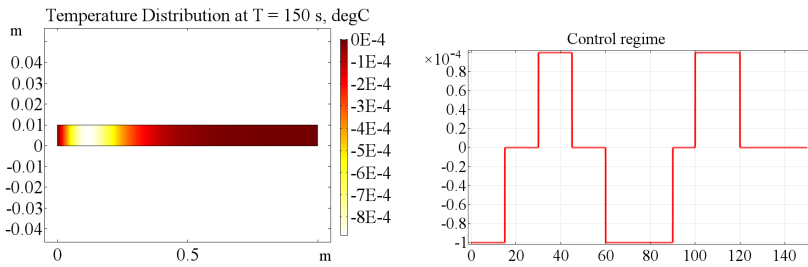


Figure 2.8: Implemented terminal state (left) for the switching control regime (2.74) (right)

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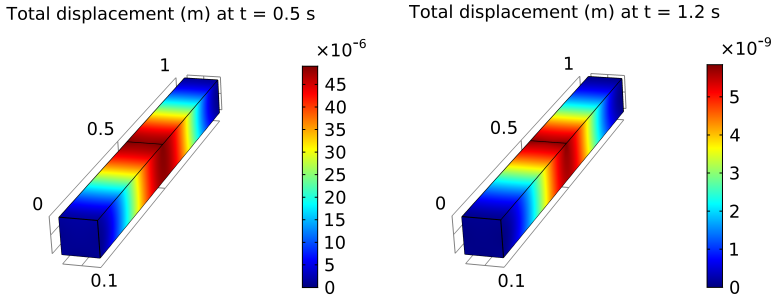


Figure 2.9: Maximal uncontrolled displacement at $t = 0.5$ s (left) and at $t = T$ (right)

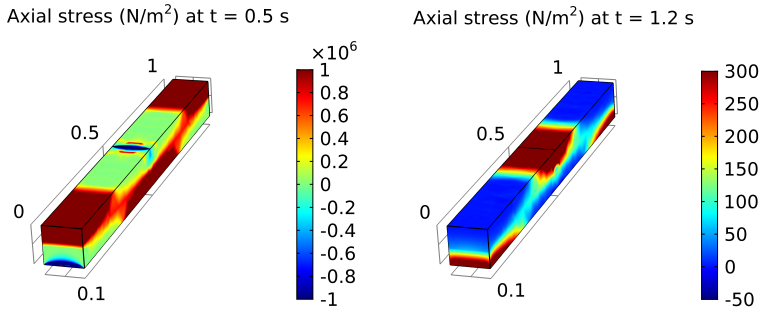


Figure 2.10: Maximal uncontrolled axial stress at $t = 0.5$ s (left) and at $t = T$ (right)

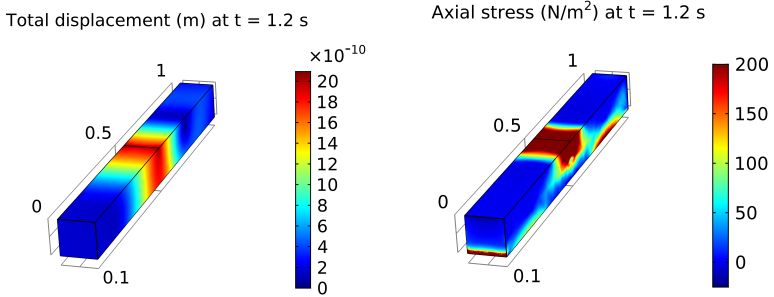


Figure 2.11: Controlled total displacement (left) and axial stress of the beam at $t = T$: optimal stopping control

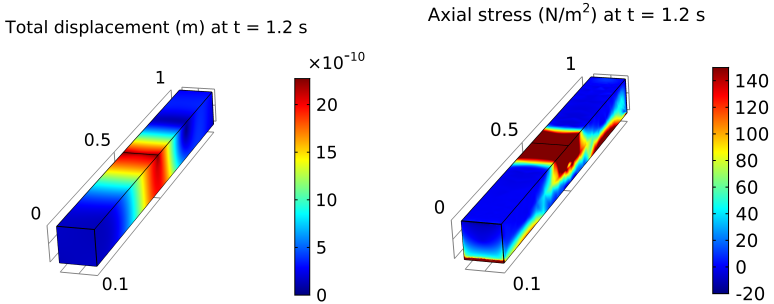


Figure 2.12: Controlled total displacement (left) and axial stress of the beam at $t = T$: switching control

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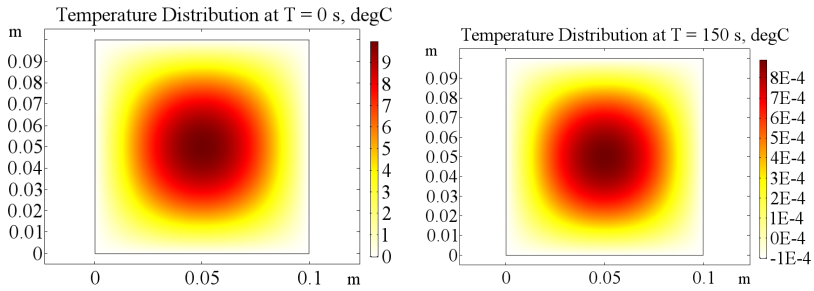


Figure 2.13: Initial (left) and implemented terminal (right) distributions of temperature in the square when $u_b \equiv -10^{-4}$

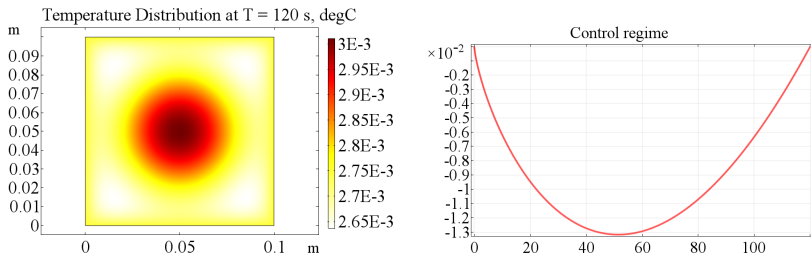


Figure 2.14: Implemented terminal state (left) for quasi-polynomial control regime (right) with $u^o = -10^{-5}$, $m = 0.75$ and $n = 1$

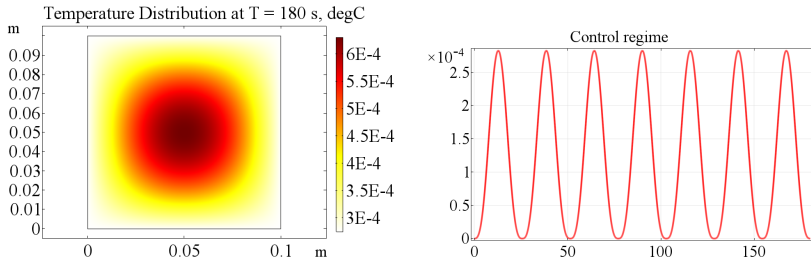


Figure 2.15: Implemented terminal state (left) for trigonometric control regime (right) with $u^o = 10^{-4}$, $m = 7$ and $n = 1.5$

3

Nonlinear Systems

Nonlinear models generally describe real-life phenomena more precisely than linear ones. Moreover, in some cases linear models can become totally useless. That is why in modern control theory primary attention is paid to the control of *nonlinear* models. Numerical methods for nonlinear models are more costly, therefore making explicit solutions especially worthwhile. As it was shown in Section 1.3, a kind of Green's representation formula is valid for an exact (Cacuci's approach) and approximate (Frasca's approach) solution of nonlinear models. The latter is used below for the derivation of approximate controllability conditions for nonlinear systems with specific dynamics.

This chapter is organized as follows. The problem is described in Section 3.1 and the resolving constraints are obtained in Section 3.2. Particular nonlinear systems are studied on approximate controllability in Section 3.3. The exact and approximate controllability of Burgers' equation is studied in Section 3.4 using exact linearization.

3.1 Problem Statement

Consider the simplest problem about providing a given terminal state in a given finite time for a system with nonlinear dynamics of a specific form. Let the system be governed by the second order ordinary nonlinear differential equation (cf. (1.26))

$$\frac{d^2\mathbf{w}}{dt^2} + \mathbf{N}(\mathbf{w}, t) = \mathbf{f}(\mathbf{u}, t) \quad \text{in } \mathbb{R}^+, \quad (3.1)$$

subjected to the Cauchy condition

$$\mathcal{I}[\mathbf{w}] = \mathbf{w}_b \quad \text{at } t = 0. \quad (3.2)$$

Above, $\mathbf{N} : \mathbf{W} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a generic nonlinear vector-function, $\mathbf{f} : \mathbf{U} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is a given vector-function representing the controller influence on the system, and $\mathbf{w}_b \in \mathbb{R}^m$ is a given constant vector.

The system (3.1) is a reduced version of nonlinear partial differential equations that are linear in $\frac{\partial^2 \tilde{\mathbf{w}}}{\partial t^2}$. Indeed, using the generalized separation of variables, the equation

$$\frac{\partial^2 \tilde{\mathbf{w}}}{\partial t^2} + \tilde{\mathbf{N}}\left(\frac{\partial^k \tilde{\mathbf{w}}}{\partial \mathbf{x}^k}, \mathbf{x}, t\right) = \tilde{\mathbf{f}}(\mathbf{u}, \mathbf{x}, t), \quad k = 0, 1, \dots, \quad (3.3)$$

in principle can be reduced to (3.1). See [103, 104] for details.

In the general treatment, if the variable t represents the time, systems like (3.3) describe wave phenomena in different media. That is why they are often referred to as oscillating systems.

The set of admissible controls is considered to be

$$\mathcal{U} = \{\mathbf{u} \in \mathbf{U}, |\mathbf{u}| \leq \epsilon, \text{supp}(\mathbf{u}) \subseteq [0, T]\}.$$

Mathematically, the problem is to find such admissible controls $\mathbf{u} \in \mathcal{U}$ that provide the terminal condition

$$\mathcal{T}[\mathbf{w}] = \mathbf{w}_T \quad \text{at } t = T, \quad (3.4)$$

in a given finite time T for a prescribed constant vector $\mathbf{w}_T \in \mathbb{R}^m$.

The exact or approximate controllability in this case is verified in terms of the residue

$$\mathcal{R}_T(\mathbf{u}) = |\mathcal{T}[\mathbf{w}] - \mathbf{w}_T|.$$

The nature of exact and approximate controllability described in Chapter 2 for linear systems applies here as well. Here,

$$\mathcal{R}_T(\mathbf{u}) = 0 \quad \text{on } \mathcal{U} \quad (3.5)$$

implies exact controllability, and

$$\mathcal{R}_T(\mathbf{u}) \leq \varepsilon \quad \text{on } \mathcal{U} \quad (3.6)$$

with a given precision $\varepsilon > 0$ implies approximate controllability.

In this chapter, only the case when

$$\mathcal{T}[\mathbf{w}] = \mathbf{w}(T)$$

is considered. The solution procedure can be adapted correspondingly.

3.2 Resolving System

To simplify the derivations, assume that the Cauchy condition (3.2) is of the form

$$\mathbf{w}(0) = 0, \quad \left. \frac{d\mathbf{w}}{dt} \right|_{t=0} = 0.$$

Then, the solution of (3.1) can be approximated by the formal Green's representation formula

$$\mathbf{w}(t) \approx \int_0^t \mathbf{G}(t, \tau) \mathbf{f}(\mathbf{u}, \tau) d\tau \text{ in } \mathbb{R}^+. \quad (3.7)$$

Following the algorithm described in Section 1.4, evaluate the solution at $t = T$ and substitute into the residue to derive

$$\mathcal{R}_T(\mathbf{u}) \approx \left| \int_0^T \mathbf{G}(T, \tau) \mathbf{f}(\mathbf{u}, \tau) d\tau - \mathbf{w}_T \right|.$$

Then, the exact satisfaction of (3.5) is equivalent to the system of restrictions

$$\int_0^T \mathbf{G}(T, \tau) \mathbf{f}(\mathbf{u}, \tau) d\tau = \mathbf{w}_T. \quad (3.8)$$

In this case, the set of resolving controls will be

$$\mathcal{U}_{res} = \{\mathbf{u} \in \mathcal{U}, (3.8)\}.$$

The particular solutions obtained in Subsection 2.3.1 can be involved for explicit representation of the resolving controls.

However, note that since (3.7) is only an approximation to the solution of the nonlinear system (3.1), (3.2), then, in general, for every admissible control $u \in \mathcal{U}$ and any terminal state (3.4) the residue sat-

isfies

$$\mathcal{R}_T(\mathbf{u}) > 0.$$

In other words, even the exact solution of (3.8) ensures the approximate controllability of the system.

On the other hand, since the triangle inequality provides the estimate

$$\left| \int_0^T \mathbf{G}(T, \tau) \mathbf{f}(\mathbf{u}, \tau) \, d\tau - \mathbf{w}_T \right| \leq \left| \int_0^T \mathbf{G}(T, \tau) \mathbf{f}(\mathbf{u}, \tau) \, d\tau \right| + |\mathbf{w}_T|,$$

then, obviously, the inequality

$$\left| \int_0^T \mathbf{G}(T, \tau) \mathbf{f}(\mathbf{u}, \tau) \, d\tau \right| \leq \varepsilon - |\mathbf{w}_T| \tag{3.9}$$

will be sufficient for approximate controllability of the system, as soon as

$$\tilde{\varepsilon}_T = \varepsilon - |\mathbf{w}_T| \geq 0.$$

Note that, for instance, in the case of approximate null-controllability, the last inequality always holds true.

Furthermore, assuming that for a given constant $\epsilon_0 > 0$

$$|\mathbf{f}(\mathbf{u}, t)| \leq \epsilon_0 \quad \text{in } \mathcal{U} \times \mathbb{R}^+,$$

then from the left-hand side of (3.9) it can be derived that

$$\tilde{\epsilon}_0 \leq \frac{\tilde{\varepsilon}}{g_T},$$

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where

$$g_T = \int_0^T |\mathbf{G}(T, \tau)| d\tau,$$

is sufficient for the approximate controllability of the system. In other words, any admissible control satisfying

$$\mathbf{u} \in \tilde{\mathcal{U}}_{res} = \{\mathbf{u} \in \mathcal{U}, |\mathbf{f}(\mathbf{u}, t)| \leq \tilde{\epsilon}_0\},$$

ensures the approximate controllability of the system. For the explicit determination of the resolving controls in this case, the particular solutions obtained in Subsection 2.3.2 can be involved.

Remark 3.1. *In general, the solution of the system (3.8) is not unique providing opportunities for an optimal choice among the resolving controls. Usually, a cost functional of the form*

$$\mathcal{C}[u] = \int_0^T W(\mathbf{u}, t) dt,$$

is considered with a given density function W having a specific interpretation in the particular problem under study.

3.3 Examples

Several particular examples are considered in this section in order to demonstrate how the proposed algorithm works in the case of nonlinear systems. For simplicity, the consideration is limited to the one-dimensional case. In addition, it is supposed that the systems are linear in controls. All the studied systems and their Green's functions are borrowed from existing references (see Section 1.3).

3.3.1 Pendulum with Cubic Non-Linearity

Consider in (3.1) the cubic non-linearity

$$N(w, t) = w^3.$$

In this case, the approximation (3.7) evaluated at $t = T$ implies

$$w(T) \approx \int_0^T G(T - \tau) u(\tau) d\tau, \quad (3.10)$$

where (see Subsection 1.3.2)

$$G(t) = 2^{\frac{1}{4}} \theta(t) \cdot \operatorname{sn} \left[2^{-\frac{1}{4}} t, i \right].$$

The aim of the control is to determine admissible controls providing (3.5) or (3.6) for the given T and w_T^1 . First, consider the constant regime (2.36) with

$$u^o = w_T.$$

¹Without losing the generality one can assume that $w_T = 1$, because otherwise dividing both sides of the equality (3.10) by w_T , its form is preserved.

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In this case, the quantity

$$\Delta(T) = 1 - \int_0^T G(T - \tau) d\tau$$

measures the error between the desired and implemented terminal states. Fig. 3.1 (left) shows that Δ is a periodic function of T . Moreover, in the case of constant control $u^o = w_T$, there exist four different values of T in the range $T \in (1, 10)$, for which $\Delta \equiv 0$. Namely, $T = 1.37$, $T = 3.88$, $T = 6.61$ and $T = 9.12$. Note again that even though the equality holds exactly, it still implies the approximate controllability of the system.

Consider now the constant control regime with $u^o = \alpha w_T$, $\alpha \in \mathbb{R}$. Evidently, the case

$$\alpha = \frac{1}{I(T)},$$

trivially defines resolving controls ensuring approximate controllability of the system. Moreover, computations show that the integral

$$I(T) = \int_0^T G(T - \tau) d\tau$$

is non-negative for all $T \in [1, 10]$ (see Fig. 3.1 (right)), which means that when $\alpha \leq 0$, it follows that $\Delta(T) > 1$. Therefore, in that case the system is not approximately controllable. Thus, the system is approximately controllable if and only if

$$\alpha \geq \frac{1}{I(T)}.$$

When the case of the harmonic control (2.37) with $u^o = \alpha w_T$, $\alpha \in \mathbb{R}$, is considered, the error between the desired and implemented

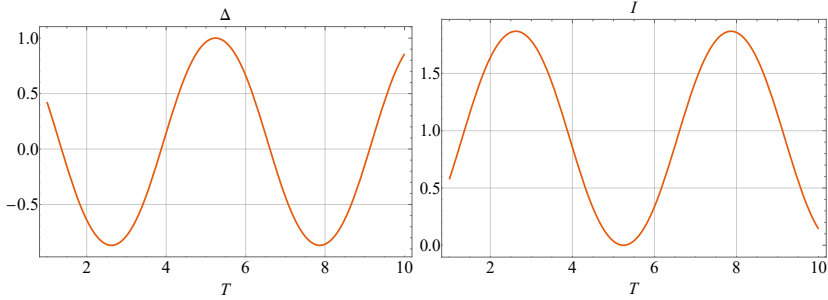


Figure 3.1: Dependence Δ (left) and I (right) against the control time T for a constant control $u^o = w_T$

terminal states becomes

$$\Delta(T) = 1 - \alpha \int_0^T G(T - \tau) \sin(\omega\tau + \gamma) d\tau.$$

The aim is to find such real triples (α, ω, γ) for which

$$\mathcal{R}_T(u) \approx |\Delta(T)| \leq \varepsilon$$

with a given precision ε .

Computations show that the integral

$$I(T) = \int_0^T G(T - \tau) \sin(\omega\tau + \gamma) d\tau$$

satisfies

$$|I(T)| \leq 3$$

in the range $T \in [1, 10]$ for all couples (ω, γ) (see Fig. 3.2 (right)).

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Therefore, the fulfillment of the equality

$$\Delta(T) = 0$$

is possible, i.e., the system is approximately controllable if and only if

$$|\alpha| \geq \frac{1}{3}.$$

Without loss of generality assume that $\alpha = 1$. Then, for instance, in the case when $\omega = \frac{\pi}{2}$, $\gamma = 0$, the equation $\Delta(T) = 0$ has four roots (see Fig. 3.2 (left)): $T = 1.77$, $T = 2.72$, $T = 5.89$ and $T = 7.69$.

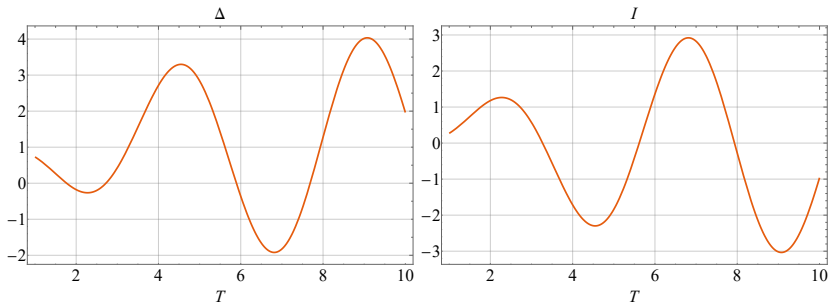


Figure 3.2: Dependence of Δ (left) and I (right) against the control time T for the harmonic control with $u^o = w_T$, $\omega = \frac{\pi}{2}$, $\gamma = 0$

In some physical problems, the usage of the switching or piecewise constant control regimes (2.43) may be required. Let the control consist of 3 modes, i.e.,

$$u(t) = u_1\theta(t - t_1) + u_2\theta(t - t_2) + u_3\theta(t - t_3). \quad (3.11)$$

The aim is to find such constants $\{u_n, t_n\}_{n=1}^3$, partially restricted by the constraint

$$0 \leq t_1 < t_2 < t_3 \leq T,$$

that the error between the desired and implemented terminal states, which is evaluated by

$$\begin{aligned} \Delta(T) = 1 - \int_0^T G(T - \tau) [u_1\theta(\tau - t_1) + \\ + u_2\theta(\tau - t_2) + u_3\theta(\tau - t_3)] d\tau, \end{aligned}$$

satisfies

$$\Delta(T) = 0 \quad \text{or} \quad |\Delta(T)| \leq \varepsilon$$

for a given ε .

By the definition of the Heaviside function

$$\begin{aligned} \Delta(T) = 1 - u_1 \int_{t_1}^T G(T - \tau) d\tau - u_2 \int_{t_2}^T G(T - \tau) d\tau - \\ - u_3 \int_{t_3}^T G(T - \tau) d\tau. \end{aligned}$$

Note that since the integrals

$$I_n(T) = \int_{t_n}^T G(T - \tau) d\tau, \quad n = 1, 2, 3,$$

are non-negative (see Fig. 3.3), one possible solution of $\Delta(T) = 0$ is

$$u_1 = \frac{1}{I_1(T)}, \quad u_2 = \frac{1}{I_2(T)}, \quad u_3 = -\frac{1}{I_3(T)}, \quad (3.12)$$

for arbitrary T . At this, t_1 , t_2 , and t_3 may be fixed.

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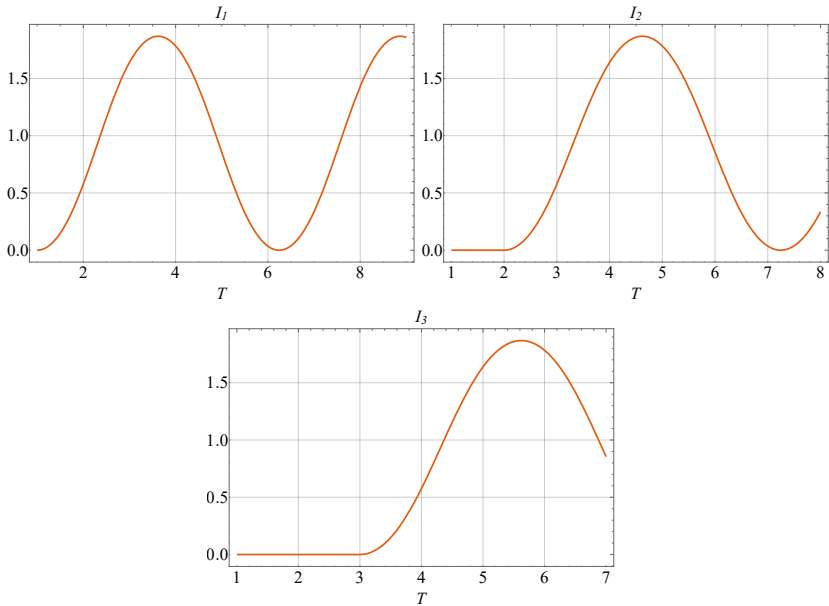


Figure 3.3: Dependence of I_n against the control time T for the switching control with $t_1 = 1$, $t_2 = 2$, $t_3 = 3$

Excluding from consideration the cases when $T = 6.24$ for which $u_1 \rightarrow \infty$, and $T = 7.24$ for which $u_2 \rightarrow \infty$, for any $T \geq t_3$ the system is approximate controllable for (3.11) with (3.12). Let $T = 4$, then (3.12) provides $u_1 = 0.56$, $u_2 = 0.61$ and $u_3 = -1.74$. Moreover, it is evident from Fig. 3.4 that in that case the equation $\Delta(T) = 0$ has four roots when $T \in [1, 10]$. Other solutions are also possible.

Wide physical applications also have the discrete control regime (2.47). The filtering property of the Dirac function considerably sim-

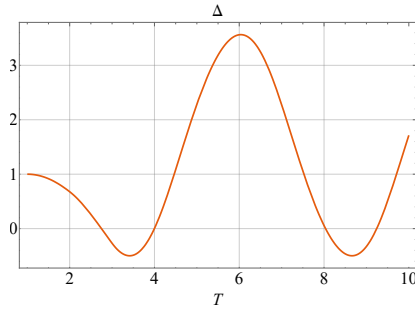


Figure 3.4: Dependence of Δ against the control time T for the switching control

plifies the analysis. Indeed, let

$$u(t) = u_1\delta(t - t_1) + u_2\delta(t - t_2) + u_3\delta(t - t_3), \quad (3.13)$$

where

$$0 \leq t_1 < t_2 < t_3 \leq T.$$

Then, by virtue of the filtering property of the Dirac function the error takes the simple form

$$\Delta(T) = 1 - u_1G(T - t_1) - u_2G(T - t_2) - u_3G(T - t_3).$$

Obviously, the regime (3.13) with

$$u_1 = \frac{1}{G(T - t_1)}, \quad u_2 = \frac{1}{G(T - t_2)}, \quad u_3 = -\frac{1}{G(T - t_3)}, \quad (3.14)$$

satisfies $\Delta(T) = 0$ for all t_1, t_2, t_3 , and $T \geq t_3$, i.e., any discrete control (3.13) with (3.14) ensures the approximate controllability of the system

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for arbitrary $T > t_3$, excluded perhaps the finitely many isolated points where

$$I_n(T) = G(T - t_n) = 0, \quad n = 1, 2, 3$$

(see Fig. 3.5 (left)). In particular, fixing $t_1 = 1$, $t_2 = 2$, $t_3 = 3$ and $T = 4$, it is obtained that

$$u_1 = -2.23, \quad u_2 = 1.37, \quad u_3 = 0.93.$$

Moreover, Fig. 3.5 (right) shows that the equation $\Delta(T) = 0$ has three roots when $T \in [1, 10]$.

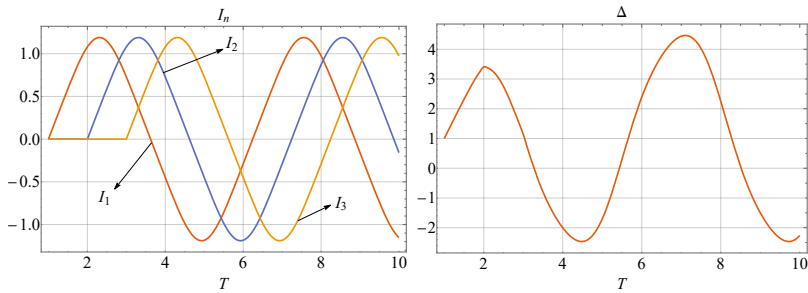


Figure 3.5: Dependence of I_n (left) and Δ (right) against the control time T for the discrete control

3.3.2 Sine-Gordon Equation

Now consider the case when

$$N(w, t) = \sin w,$$

which corresponds to the space-reduced sine-Gordon equation. In this case (see Subsection 1.3.2)

$$G(t) = 2\theta(t) \cdot \operatorname{am} \left[\frac{t}{\sqrt{2}}, \sqrt{2} \right].$$

Considering the constant control regime

$$u^o = \alpha w_T$$

for $\alpha \in \mathbb{R}$, the error takes the form

$$\Delta(T) = 1 - \alpha \int_0^T G(T - \tau) d\tau.$$

Since the integral

$$I(T) = \int_0^T G(T - \tau) d\tau$$

is non-negative (see Fig. 3.6 (left)), the equation $\Delta(T) = 0$ has roots when $T \in [1, 10]$ if and only if

$$\alpha \geq \frac{1}{I(T)}.$$

Moreover,

$$\alpha = \frac{1}{I(T)}$$

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ensures the approximate controllability of the system for arbitrary $T \in [1, 10]$. Furthermore, in some cases, the equation $\Delta(T) = 0$ may have more than one root. Specifically, when $\alpha = 1$, the function $\Delta(T)$ turns to zero in $[1, 10]$ twice: $T = 1.24$ and $T = 8.68$ (see Fig. 3.6 (right)).

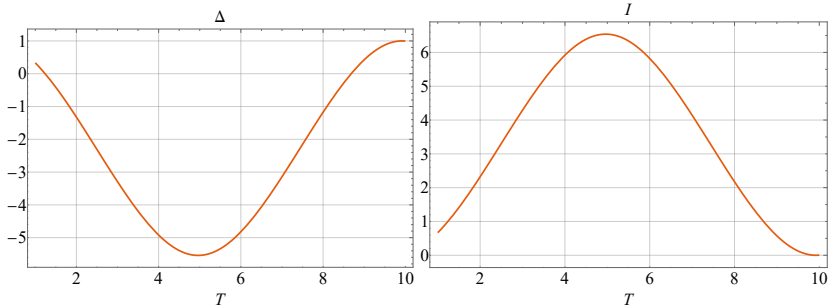


Figure 3.6: Dependence Δ (left) and I (right) against control time T for constant control with $\alpha = 1$

The analysis for other control regimes is similar to that performed in Subsection 3.3.1.

3.3.3 Approximation Error Analysis

In order to estimate how efficient the approximation (3.7) is in controllability analysis, a comparison between the derived solution and numerical solution of the state equation with predetermined right-hand side is carried out. To this end the Green's formula (3.7) for particular forms of N considered in Subsections 3.3.1 and 3.3.2 is evaluated at $t = T$. Here T resolves the corresponding equation $\Delta(T) = 0$. In the case when it has several roots, only the smallest root is considered.

For the analysis, the scaling parameter introduced in (1.29) turns to be very efficient. The relative errors are given in Tab. 3.1 and 3.2.

| Control regime | Relative error | Scaling factor |
|----------------|---------------------|----------------|
| Constant | $1.9 \cdot 10^{-5}$ | -2.38 |
| Harmonic | $2.7 \cdot 10^{-5}$ | -4.12 |
| Switching | $7 \cdot 10^{-6}$ | 0.85 |
| Discrete | $7.1 \cdot 10^{-6}$ | -0.89 |

Table 3.1: Relative error measures of approximation (3.7) for $N(w, t) = w^3$

| Control regime | Relative error | Scaling factor |
|----------------|---------------------|---------------------|
| Constant | $7.9 \cdot 10^{-7}$ | -1.34 |
| Harmonic | $6.5 \cdot 10^{-9}$ | -6.305 |
| Switching | $2.4 \cdot 10^{-6}$ | -1.58 |
| Discrete | $4.3 \cdot 10^{-5}$ | $1.5 \cdot 10^{-3}$ |

Table 3.2: Relative error measures of approximation (3.7) for $N(w, t) = \sin w$

3.4 Burgers' Equation

There is a large class of nonlinear ordinary and partial differential equations that can be exactly linearized using special algebraic transformations (some of them can be found, for example, in [199–203] and references therein). If the state equation of a system under study is nonlinear, but can be exactly linearized by such a transformation, then the general solution of the linearized equation can be represented via the Green's representation formula and therefore the controllability analysis of the system can be carried out as it was described in Section 1.4. Such transformations exist for various equations in physics and mechanics [75]. The procedure is demonstrated on the well known Burgers' equation below.

The so-called Hopf-Cole transformation

$$w = -2d \frac{1}{\Theta} \frac{\partial \Theta}{\partial x}, \quad (3.15)$$

proposed in [204] and [205] independently, allows the nonlinear Burgers' equation

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} = d \frac{\partial^2 w}{\partial x^2}, \quad (3.16)$$

to be reduced to the linear heat equation

$$\frac{\partial \Theta}{\partial t} = d \frac{\partial^2 \Theta}{\partial x^2}.$$

The Burgers' equation arises in various branches of applied mathematics and mechanics, such as fluid mechanics and turbulence, nonlinear acoustics, gas dynamics, traffic flow: the coefficient d is treated accordingly.

Consider the Burgers' equation (3.16) in \mathbb{R} . Let at the initial instant $t = 0$ it be given that

$$w(x, 0) = w_0(x) \quad \text{in } \mathbb{R},$$

where $w_0 \in L^\infty(\mathbb{R})$ is a given function. Then, the evaluation of (3.15) at $t = 0$ provides the following initial condition² for Θ :

$$\Theta(x, 0) = \exp \left[-\frac{1}{2d} \int_{-\infty}^x w_0(\xi) d\xi \right] := \Theta_0(x) \quad \text{in } \mathbb{R}.$$

Here it is taken into account that

$$\left. \frac{\partial \Theta(x, t)}{\partial x} \right|_{t=0} = \frac{\partial \Theta(x, 0)}{\partial x}.$$

Consider now the non-homogeneous heat equation

$$\frac{\partial \Theta}{\partial t} = d \frac{\partial^2 \Theta}{\partial x^2} + f(x, t) \tag{3.17}$$

in the particular case when the source function is given by

$$f(x, t) = u(t)v(x).$$

Here, as usual, u denotes the control function, and v denotes its distribution on x -axis.

Let the problem be in explicit representation of admissible controls $u \in \mathcal{U}$ with

$$\mathcal{U} = \{u \in L^\infty[0, T], |u| \leq \epsilon, \text{ supp}(u) \subseteq [0, T]\},$$

²In the case of finite domains, boundary conditions are transformed in a similar way.

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for which within a required time T the terminal state

$$w(x, T) = w_T(x) \quad \text{in } \mathbb{R} \quad (3.18)$$

is provided for (3.15) exactly or approximately for a desired $w_T \in L^\infty(\mathbb{R})$. In other words, the problem is to find a distributed control $u \in \mathcal{U}$, such that for the given T , the residue

$$\mathcal{R}_T(u) = \|w(x, T) - w_T(x)\|_{L^\infty(\mathbb{R})}$$

defined for (3.15) evaluated on the solution of (3.17) satisfies

$$\mathcal{R}_T(u) = 0, \quad (3.19)$$

or if that is not the case

$$\mathcal{R}_T(u) \leq \varepsilon \quad (3.20)$$

with a given precision $\varepsilon > 0$.

Since (3.15) provides an explicit dependence of w on u , the approach described in Section 1.4 can be applied. The general solution of (3.17) is given by the Green's formula (see Section 1.2) as follows:

$$\begin{aligned} \Theta(x, t) &= \int_{-\infty}^{\infty} \Theta_0(\xi) G(x - \xi, t) d\xi + \\ &+ \int_0^t \int_{-\infty}^{\infty} f(\xi, \tau) G(x - \xi, t - \tau) d\xi d\tau \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \end{aligned} \quad (3.21)$$

Here G is the Green's function of (3.17), defined as [77]

$$G(x, t) = \frac{1}{\sqrt{4\pi dt}} \exp\left[-\frac{x^2}{4dt}\right].$$

Then, the controlled flow will be given in terms of (3.21) as follows:

$$w(x, t) = -\frac{2d}{\Theta(x, t)} \frac{\partial \Theta(x, t)}{\partial x} \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \quad (3.22)$$

Straightforward calculations allow \mathcal{R}_T to be expressed in terms of Θ . Indeed, by denoting

$$\Theta_T(x) = \exp \left[-\frac{1}{2d} \int_{-\infty}^x w_T(\xi) d\xi \right], \quad (3.23)$$

where w_T is given by (3.18), and evaluating the solution (3.22) at the terminal instant $t = T$, \mathcal{R}_T can be transformed to

$$\begin{aligned} \mathcal{R}_T(u) &= \|w(x, T) - w_T(x)\|_{L^\infty(\mathbb{R})} = \\ &= 2d \left\| \frac{1}{\Theta(x, T)} \frac{\partial \Theta(x, T)}{\partial x} - \frac{1}{\Theta_T(x)} \frac{d\Theta_T(x)}{dx} \right\|_{L^\infty(\mathbb{R})} = \\ &= 2d \left\| \frac{\partial}{\partial x} \ln \left| \frac{\Theta(x, T)}{\Theta_T(x)} \right| \right\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Therefore, according to the definition of norm, (3.19) is equivalent to

$$\frac{\partial}{\partial x} \ln \left| \frac{\Theta(x, T)}{\Theta_T(x)} \right| = 0 \quad \text{a.e. in } \mathbb{R}.$$

The integration with respect to x once leads to

$$\frac{\Theta(x, T)}{\Theta_T(x)} = c,$$

where c is an arbitrary constant. Therefore, considering the residue

$$\tilde{\mathcal{R}}_T(u) = \|\Theta(x, T) - \Theta_T(x)\|_{L^\infty(\mathbb{R})} \quad (3.24)$$

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with Θ_T defined as in (3.23), it becomes obvious that the equality

$$\tilde{\mathcal{R}}_T(u) = 0$$

is equivalent to (3.19). Thus, the exact controllability of the Burgers' equation is equivalent to the exact controllability of the linear heat equation.

For computations assume that

$$w_0 \equiv 0, \quad w_T(x) = \text{rect}\left(\frac{x}{2}\right), \quad v(x) = \chi_{[-1,1]}(x),$$

where $\chi_{[-1,1]}$ is the indicator function of $[-1, 1]$, and rect is the rectangular function. Then,

$$\Theta_0 \equiv 1, \quad \Theta_T(x) = \exp\left[-\frac{1}{2d}[(x+1)\theta(x+1) - (x-1)\theta(x-1)]\right].$$

Following the algorithm of Section 1.4, (3.21) is evaluated at $t = T$,

$$\begin{aligned} \Theta(x, T) &= \int_{-\infty}^{\infty} G(x - \xi, T) d\xi + \int_0^T G_d(x, T - \tau) u(\tau) d\tau = \\ &= \sqrt{4dT} + \int_0^T G_d(x, T - \tau) u(\tau) d\tau, \end{aligned}$$

and substituted into the expression of $\tilde{\mathcal{R}}_T$ (3.24). Here

$$\begin{aligned} G_d(x, t) &= \int_{-\infty}^{\infty} G(x - \xi, t) v(\xi) d\xi = \int_{-1}^1 G(x - \xi, t) d\xi = \\ &= \frac{1}{2} \left[\text{erf}\left[\frac{x+1}{\sqrt{4dt}}\right] - \text{erf}\left[\frac{x-1}{\sqrt{4dt}}\right] \right]. \end{aligned}$$

First, check the possibility of the exact controllability. The resolving

equation, equivalent to (3.19) is

$$\int_0^T G_d(x, T - \tau) u(\tau) \, d\tau = \Theta_T(x) - \sqrt{4dT} \quad \text{a.e. in } \mathbb{R}. \quad (3.25)$$

In this case $w_T \geq 0$ in \mathbb{R} , meaning that Θ_T has an exponential decay in x . Since G_d also has an exponential decay in x , it allows to consider (3.25) in $[-l, l]$ for some constant $l > 1$. Expanding G_d and Θ_T into Fourier series and taking into account that both are even functions, it is derived

$$\Theta_T(x) = \frac{1}{2}\Theta_{T0} + \sum_{n=1}^{\infty} \Theta_{Tn} \cos\left(\frac{\pi n}{l}x\right),$$

with

$$\Theta_{Tn} = \frac{2}{l} \int_0^l \Theta_T(x) \cos\left(\frac{\pi n}{l}x\right) \, dx,$$

and

$$G_d(x, t) = \frac{1}{2}G_{d0}(t) + \sum_{n=1}^{\infty} G_{dn}(t) \cos\left(\frac{\pi n}{l}x\right),$$

with

$$G_{dn}(t) = \frac{2}{l} \int_0^l G_d(x, t) \cos\left(\frac{\pi n}{l}x\right) \, dx.$$

The resolving system is formed by equating the coefficients of cosines' for corresponding n in both sides of the equation above, resulting in

$$\begin{aligned} \int_0^T G_{d0}(T - \tau) u(\tau) \, d\tau &= \Theta_{T0} - 4\sqrt{dT}, \\ \int_0^T G_{dn}(T - \tau) u(\tau) \, d\tau &= \Theta_{Tn}, \quad n \in \mathbb{N}. \end{aligned} \quad (3.26)$$

For particular solutions of (3.26), see Subsection 2.3.1 and Remark 2.3.

For simplicity, restrict the consideration of the Fourier expansions

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above by $N = 50$. Fig. 3.7 expresses the logarithmic error function

$$\text{Er}(x) = \log_{10} \left| \Theta_T(x) - \frac{1}{2}\Theta_{T0} - \sum_{n=1}^{50} \Theta_{Tn} \cos\left(\frac{\pi n}{l}x\right) \right| \quad \text{in } [-l, l],$$

for $l = 2$, $N = 50$. It is evident that $N = 50$ provides a good approximation for Θ_T . The error decreases very fast for larger l .

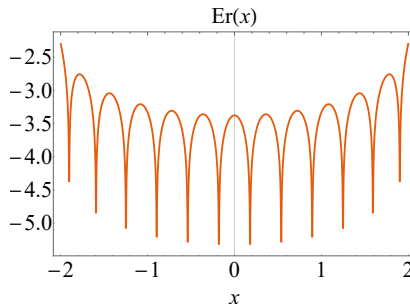


Figure 3.7: Logarithmic error of approximation of Θ_T by its Fourier series expansion for $N = 50$

Further, in order to derive a consistent system of algebraic equations, let the control be given in the form

$$u(t) = \frac{1}{2}u_0 + \sum_{n=1}^{50} u_n \cos\left(\frac{\pi n}{T}t\right), \quad (3.27)$$

where u_n , $n = 0, 1, \dots, 50$, are unknown constants that need to be determined. Substituting u into the resolving system (3.25) truncated with $N = 50$, the following system of linear algebraic equations is derived:

$$\mathbf{A}\mathbf{u} = \mathbf{\Theta}_T, \quad (3.28)$$

where

$$\mathbf{A} = \{A_{mn}\}_{m,n=0}^N, \quad A_{mn} = \int_0^T G_{dm}(T - \tau) \cos\left(\frac{\pi n}{T}\tau\right) d\tau,$$

$$\mathbf{u} = \{u_0 \dots u_N\}^T, \quad \Theta_T = \{\Theta_{T0} \dots \Theta_{TN}\}^T.$$

Solving (3.28) with respect to the unknowns u_n and substituting them into (3.27), the absolute error between $w(x, T)$ and w_T is plotted in Fig. 3.8 (left). The corresponding control regime is plotted as well (see Fig. 3.8 (right)). It turns out that at $T = 0.1$ the maximal absolute error is on the order of $7 \cdot 10^{-2}$. The maximal mismatch (known also as Gibb’s phenomenon) occurs at the vertexes $x = \pm 1$, which can be reduced by considering larger values for N .

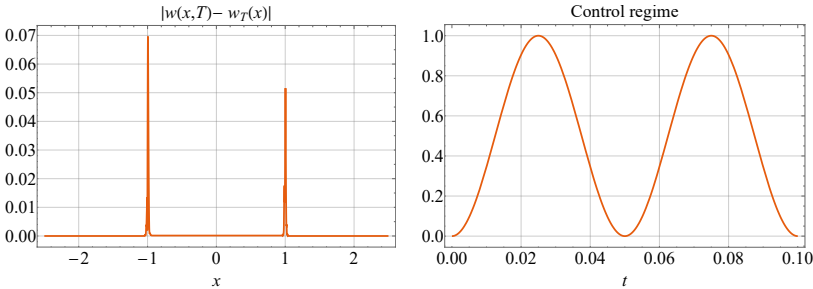


Figure 3.8: Absolute difference between the implemented and required terminal states (left) and the corresponding control regime (3.27)

Apparently, the equivalence between the exact controllabilities established above does not occur in the case of approximate controllability. Nevertheless, the Minkowski inequality provides for the residue

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with the estimate

$$\mathcal{R}_T(u) \leq 2d \left\| \frac{1}{\Theta(x, T)} \frac{\partial \Theta(x, T)}{\partial x} \right\|_{L^\infty(\mathbb{R})} + \|w_T\|_{L^\infty(\mathbb{R})}.$$

Further, since both Θ and $\frac{\partial \Theta}{\partial x}$ are uniformly bounded in $\mathbb{R} \times [0, T]$, then using the obvious inequality³

$$\|g_1 \cdot g_2\|_{L^\infty(\mathbb{R})} \leq \|g_1\|_{L^\infty(\mathbb{R})} \cdot \|g_2\|_{L^\infty(\mathbb{R})},$$

as well as the solution (3.21) evaluated at $t = T$, the following estimate for the residue is obtained:

$$\mathcal{R}_T(u) \leq 2d\mathcal{R}_{T0}(u) \cdot \mathcal{R}_{T1}(u) + \|w_T\|_{L^\infty(\mathbb{R})}.$$

Here

$$\begin{aligned} \mathcal{R}_{T0}(u) &= \left\| G_i(x, T) + \int_0^T G_d(x, T - \tau) u(\tau) d\tau \right\|_{L^\infty(\mathbb{R})}^{-1}, \\ G_i(x, T) &= \int_{-\infty}^{\infty} \Theta_0(\xi) G(x - \xi, T) d\xi, \\ \mathcal{R}_{T1}(u) &= \left\| \frac{\partial G_i(x, T)}{\partial x} + \int_0^T \frac{\partial G_d(x, T - \tau)}{\partial x} u(\tau) d\tau \right\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Thus, as soon as

$$2d\mathcal{R}_{T0}(u) \cdot \mathcal{R}_{T1}(u) + \|w_T\|_{L^\infty(\mathbb{R})} \leq \varepsilon,$$

then (3.20) holds, i.e., the Burgers' equation is approximately control-

³This directly follows from the generalized Hölder inequality as p goes to infinity.

lable. Therefore, when the inequality

$$\tilde{\varepsilon}_T := \varepsilon - \|w_T\|_{L^\infty(\mathbb{R})} \geq 0, \tag{3.29}$$

holds⁴, any admissible control

$$u \in \tilde{\mathcal{U}}_{res} = \left\{ u \in \mathcal{U}, \mathcal{R}_{T_0}(u) \cdot \mathcal{R}_{T_1}(u) \leq \frac{\tilde{\varepsilon}_T}{2d} \right\}$$

is a resolving control, i.e., it ensures the approximate controllability of the Burgers' equation at the given T with the required accuracy ε .

Furthermore, using the Minkowski inequality, it is obtained

$$\mathcal{R}_{T_0}(u) \leq c_0 + \int_0^T g_0(T - \tau) |u(\tau)| \, d\tau,$$

and

$$\mathcal{R}_{T_1}(u) \leq c_1 + \int_0^T g_1(T - \tau) |u(\tau)| \, d\tau,$$

where

$$c_0 = \|G_i(x, T)\|_{L^\infty(\mathbb{R})}, \quad c_1 = \left\| \frac{\partial G_i(x, T)}{\partial x} \right\|_{L^\infty(\mathbb{R})},$$

$$g_0(t) = \|G_d(x, t)\|_{L^\infty(\mathbb{R})}, \quad g_1(t) = \left\| \frac{\partial G_d(x, t)}{\partial x} \right\|_{L^\infty(\mathbb{R})}.$$

Therefore, as soon as (3.29) holds, the inequality

$$\int_0^T [g_1(T - \tau) - g_0(T - \tau)] |u(\tau)| \, d\tau \leq \frac{\tilde{\varepsilon}_T}{2d} c_0 - c_1$$

⁴Note that it always holds in the case of approximate null-controllability.

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is also sufficient for the approximate controllability of the Burgers' equation.

For a more specific analysis, consider a flow governed by (3.16) in a thin infinite layer. Assume that the flow source is located at $x = -1$ section of the layer and generates a harmonic flux

$$\varphi(t) = [\theta(t - 0) - \theta(t - 1)] \exp(i\pi t),$$

which vanishes when $t > 1$. Therefore, the Burgers' equation must be complemented by the initial condition

$$w(x, 0) \equiv 0 \quad \text{in} \quad [-1, 1],$$

and boundary condition

$$w(-1, t) = \begin{cases} \varphi(t), & t \in [0, 1], \\ 0, & \text{else.} \end{cases}$$

Let in particular

$$v(x) = \text{rect}(2x).$$

The problem is to find such an admissible control $u \in \mathcal{U}$, that ensures the approximate null-controllability of the flow, i.e.,

$$\mathcal{R}_T(u) = \|w(x, T)\|_{L^\infty[-1,1]} \leq \varepsilon$$

with $\varepsilon = 10^{-4}$. Obviously, the desired T must satisfy $T < T_0$, where T_0 satisfies

$$\|w(x, T_0)\|_{L^\infty[-1,1]} \leq \varepsilon,$$

when $u \equiv 0$. Restricting the consideration by $d = 1$, it is obtained that

$T_0 = 10.35$ (see Fig. 3.9).

Involving the distributed control regime (2.37), it is possible to achieve the approximate null-controllability of the flow for $T = 9.2$ when $u^o = -1$, $\omega = 2\pi$ and $\gamma = 0$ (see Fig. 3.10). Furthermore, it is evident from Fig. 3.11 that when $u^o = 1$, $\omega = \pi$ and $\gamma = -\frac{\pi}{2}$, the flow is approximately null-controllable at $T = 7.15$.

On the other hand, the same accuracy can be achieved by using a single impulsive action

$$u(t) = u^o \delta(t - t^o),$$

where $0 < t^o < T_0$. Indeed, Fig. 3.12 shows that when $u^o = 1$ and $t^o = 1$, the flow is approximately null-controllable at $T = 8.5$.

Furthermore, involving the switching regime (2.43), it is possible to achieve the approximate null-controllability of the flow for smaller values of T . Choosing $u_1 = -u_2 = 0.1$, $t_1 = 0$ and $t_2 = 1.2$, the approximate null-controllability is achieved for $T = 6$ (see Fig. 3.13).

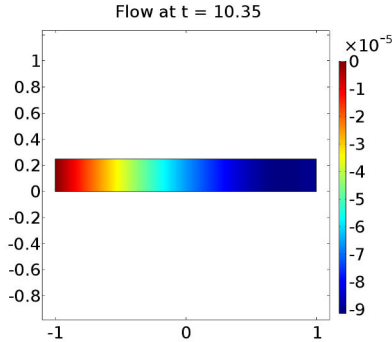


Figure 3.9: Uncontrolled flow in the layer at $t = 10.35$

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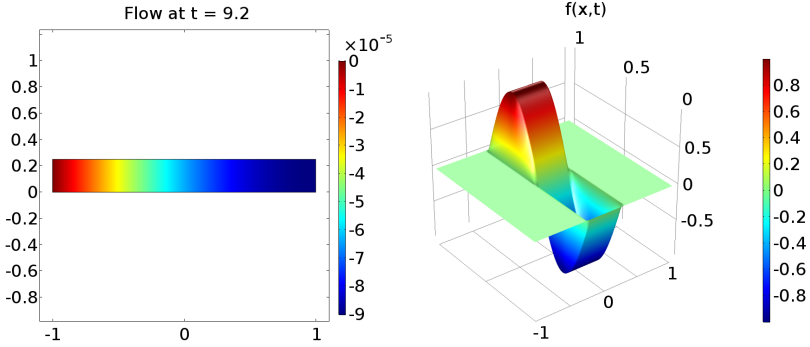


Figure 3.10: Flow in the layer at $t = 9.2$ (left) and corresponding time-harmonic control regime (right)

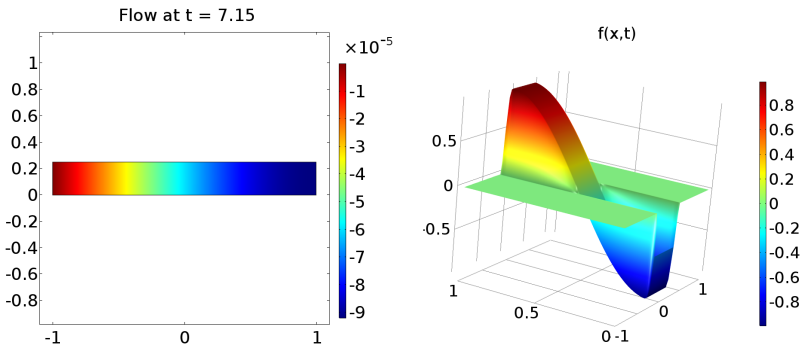


Figure 3.11: Flow in the layer at $t = 7.15$ (left) and corresponding time-harmonic control regime (right)

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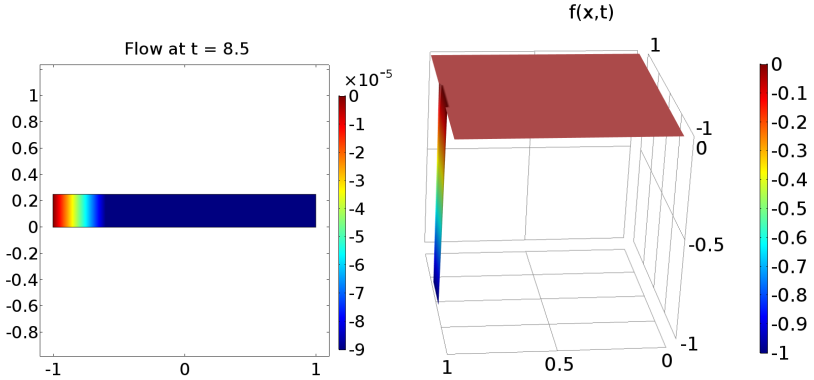


Figure 3.12: Flow in the layer at $t = 8.5$ (left) and corresponding impulsive control regime (right)

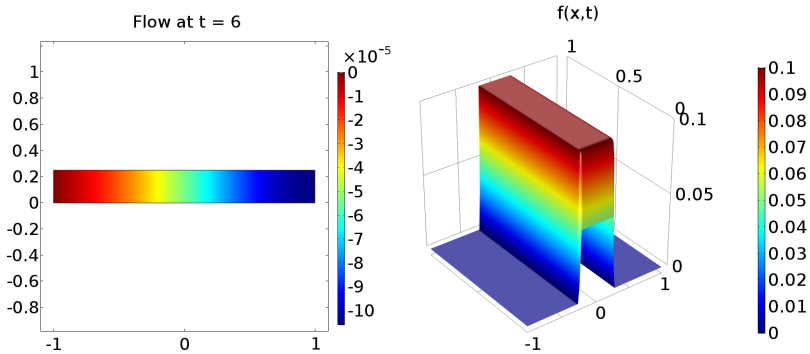


Figure 3.13: Flow in the layer at $t = 8.5$ (left) and corresponding impulsive control regime (right)

Summary

Along with stability and reliability, controllability, i.e., the ability to produce controls providing a desired terminal state within a specified time, is one of the most crucial features of control systems. In order to ensure proper functionality, control systems are verified to possess the mentioned properties (and perhaps some additional ones as well). The usual analysis of controllability consists of evaluation of the residue between the desired state and the state implemented by a specific choice of an admissible control at the required instant. Depending on the value of the residue obtained on the set of admissible controls, the controllability is classified as exact or approximate. Thus, if the residue is exactly zero, the system is referred to as *exactly controllable*. If this does not hold, but the residue remains smaller than a required precision, then the system is called *approximately controllable*.

There exist several approaches for verifying particular systems on exact or approximate controllability. One of the approaches is based on the minimization of the residue on the set of admissible controls. By solving the minimization problem, it is possible not only to figure out whether the system is exactly controllable or not, but to estimate to what extent the system is approximately controllable. Moreover, the

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approach allows the determination of the set of resolving controls explicitly, which is also a valuable advantage. However, the minimization procedure can be of a high computational cost, e.g., in higher dimensions, for systems with complicated state constraints, for a large set of admissible controls, in the case of discontinuous state transitions, etc. Another approach consists in a heuristic representation of resolving controls which contain a set of free parameters. Those parameters are eventually determined by substituting the controls into the residue and solving the resulting nonlinear algebraic equations. Compared with the previous approach, this one operates with explicitly defined controls, instead of searching through the set of admissible controls. Therefore, it incurs a smaller computational cost.

The controllability analysis can be simplified if the explicit solution of the state constraints is somehow found, since in that case the dependence of the residue on the control function is explicit. Taking into account that the notion of the Green's function is extended from linear to nonlinear systems, the Green's representation formula can serve that purpose. Indeed, evaluating the state function represented by means of the Green's formula at the required instant and substituting into the residue, either of the aforementioned approaches can be applied to study the controllability and to define the set of resolving controls explicitly. The advantage of the Green's function method in exact controllability analysis is conditioned by the availability of extensive handbooks containing explicit Green's functions for numerous known equations and their coupled systems. In approximate controllability analysis the Green's function method is more advantageous due to the existence of the upper and lower bounds for the Green's function of some "irregular" systems, for which the explicit derivation of

the Green's function is complicated.

Case studies show that the developed approach works well for linear systems with complicated state constraints, such as partial differential equations with variable coefficients, equations defined in unbounded domains, equations containing uncertainties, higher dimensional systems. At this, the approach is efficient both in exact and approximate controllability analysis. By means of expansion into series of orthogonal functions, the exact controllability analysis is reduced to an infinite-dimensional system of nonlinear equality type constraints on the control function. Verifying that the expansion coefficients decrease sufficiently fast, the infinite-dimensional system is usually truncated and studied as a finite-dimensional problem of moments. The algorithm for finding the L^p optimal solution of the finite-dimensional nonlinear problem of moments is quite straightforward. Therefore, optimal control problems can be studied as well. On the other hand, in some cases, the heuristic determination of controls can also be quite convenient. The approximate controllability is studied by using some inequalities to estimate the residue from above. Eventually, nonlinear inequality type constraints are derived for the control function. However, those constraints are merely sufficient for the approximate controllability, which means that even if those constraints are not satisfied by any choice of admissible controls, the approximate controllability may still be achieved. Also, in this case, the heuristic determination of resolving controls is quite convenient for a basic analysis.

The developed approach is also efficiently applicable for analyzing approximate controllability of some nonlinear systems. It is established that second order nonlinear ordinary differential equations of a specific form can be approximated by a type of the Green's representation for-

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mula with a Green's function satisfying the appropriate *nonlinear* equation. Moreover, these equations are reduced from second order partial differential equations which are linear in the second order time derivative, for instance, by the method of generalized separation of variables. The approach provides an easily verifiable criterion for the approximate controllability. Note also that the approach is useful in the case of exactly linearizable nonlinear differential equations, since the solution of the linearized equation can be represented by the Green's representation formula. Furthermore, it is shown that the residue between the desired and implemented states is the same for the nonlinear and linearized equations. As such, their controllability analysis is equivalent.

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The book was edited several times. Nevertheless, we anticipate that there is room for improvement and correction. We highly appreciate any correspondence, whether concerning minor typos or larger issues in the text.

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