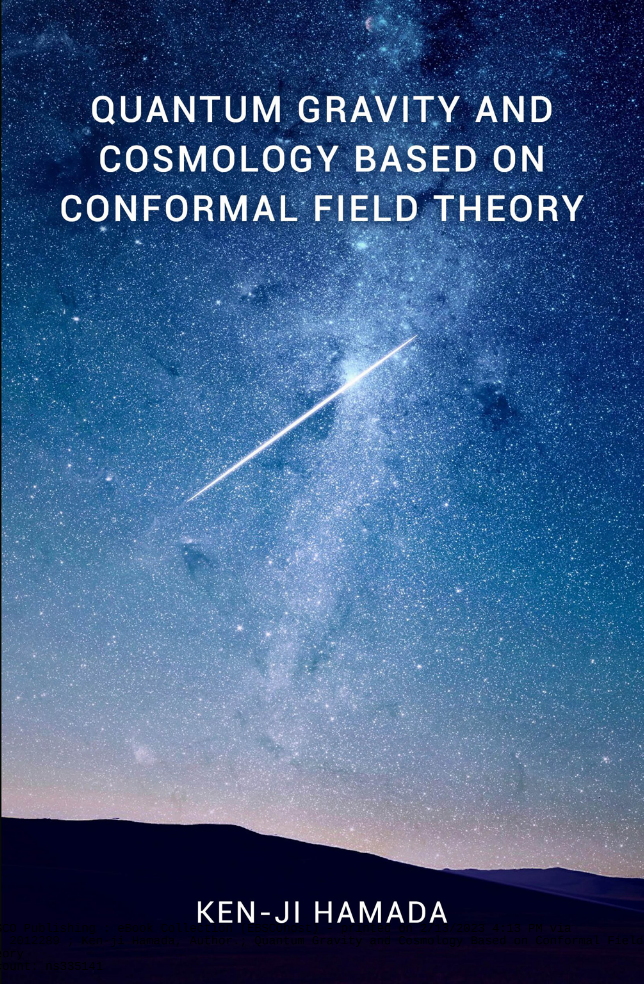
QUANTUM GRAVITY AND COSMOLOGY BASED ON CONFORMAL FIELD THEORY
KEN-JI HAMADA

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Cambridge Scholars Publishing

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PREFACE

Attempts to quantize gravity started from the 1960s and continue until now. However, in the framework of Einstein's theory, quantization of gravity did not go well because it does not become renormalizable in the standard way applied to other ordinary fields. Attempts were made to modify Einstein's theory to render it possible, but another difficulty known as ghost problem appeared, and eventually the attempt to quantize gravity with the standard field theory methods has gone away. After the 1980s, methods that do not depend on quantum field theory like string theory and loop quantum gravity have become mainstream. Many of books published in this research are about these theories. However, even though these theories have been studied for many years, realistic predictions that can explain briefly the current universe are not derived yet. Now is a good time to revisit the problem of quantization of gravity by returning to the traditional method again.

In this book, I will describe a renormalizable quantum gravity formulated with incorporating a new technique based on conformal field theory which recently has made prominent progress. Conformal invariance here appears as a gauge symmetry that gives a key property of quantum gravity known as the background-metric independence. Due to the presence of this symmetry, the theory becomes free from the problem of spacetime singularity, and thus from the information loss problem, namely the ghost problem as well. Furthermore, I will give a new scenario of the universe that evolves from quantum gravity world to the current classical world through the spacetime phase transition, including inflation driven by quantum gravity effects only.

This book also includes descriptions of recent developments in conformal field theory, renormalization theory in curved space, and conformal anomalies related thereto, almost of which do not found in other books. In addition, it includes review on evolution equations of the universe that is the foundation of modern cosmology necessary to understand results of the CMB experiments such as WMAP. Furthermore, it will be briefly shown that there is a noticeable relationship between the quantum gravity and a random lattice model that is based on the dynamical triangulation method known as another description of the background free property. I would like to describe these topics by taking enough pages as a latest advanced textbook for leading to this new area of quantum field theory that developed mainly since the beginning of this century.

Research results on the quantum gravity over the last twenty years are summarized in this book. I was helped by several collaborators in continuing this research. I am grateful to them for attending to my discussions for a long time. I especially thank Shinichi Horata and Tetsuyuki Yukawa. I could not proceed with this research without the help of two. This book is an English version of the book published in 2016 from the Pleiades Publishing in Japan, with adding a bit of new content and sentences. I appreciate the support of the Pleiades Publishing. In publishing this English version, I would like to thank Makoto Kobayashi, Kei-Ichi Kondo, and again Yukawa. I also wish to thank Cambridge Scholars Publishing for giving me the opportunity for publication. And I thank my wife Nonn and my daughter Kyouka for supporting me.

Tsukuba, Japan
June 2018

CHAPTER ONE

INTRODUCTION

The elementary particle picture represented by an ideal point without spreading is a concept incompatible with Einstein's theory of gravity.[1] Because such an object is nothing other than a black hole in terms of the theory of gravity. If its mass m is smaller than the Planck mass $m_{\rm pl}$, the Compton wavelength which gives a typical fluctuation size of particles becomes larger than the horizon size of the mass m, hence it can be approximated as a particle. However, in the world beyond the Planck scale, such an approximation does not hold because particle information is confined inside the horizon (see Fig. 1-1).

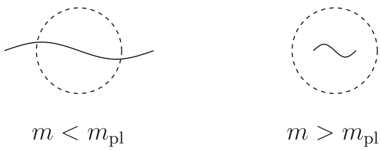

Figure 1-1: The Compton wavelength of mass m is given by $\lambda \sim 1/m$, while particle's horizon size (dotted line) is $r_g \sim m/m_{\rm pl}^2$. Therefore, $\lambda < r_g$ for $m > m_{\rm pl}$, as shown on the right, and information on such an elementary excitation is confined inside the horizon and lost. Hence, in the world beyond the Planck scale, normal particle picture is no longer established.

The goal of quantum gravity is to reveal a high energy physics beyond the Planck scale. While particles live in spacetime, gravity rules the spacetime itself, and the difference between their roles stands out there. Quantum fluctuations of gravity become large, so that the concept of time and distance will be lost. Quantization of the spacetime itself is required to describe such a world where the image of particles moving in a specific spacetime is broken.

One way to resolve the problem mentioned above is to realize such a

[1] The original paper is A. Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Phys. **49** (1916) 769.

quantum spacetime where the scale itself does not exist. It can be represented as gauge equivalence between spacetimes with different scales. This property is called the background-metric independence. In this book, as a theory with such a property, we will present a renormalizable quantum field theory, called the asymptotically background-free quantum gravity, whose ultraviolet limit is described as a special conformal field theory that has conformal invariance as a gauge symmetry.

Academic Interests

From observations of the cosmic microwave background (CMB) radiation by Wilkinson Microwave Anisotropies Probe (WMAP), which is an astronomical satellite launched from the NASA Kennedy Space Center in 2001, cosmological parameters were determined with high precision and the theory of inflation which suggests that a rapid expansion occurred in the early stage of the universe was strongly supported. On the other hand, there are still many simple and fundamental questions left, for example, why the universe is expanding or what is the source of repulsive force that ignites inflation.

Interpreting the inflation theory naturally, the universe has expanded about 10^{60} times from the birth to the present. This means that the larger size than a cluster of galaxies was within the Planck length $l_{\rm pl}$ before inflation begins. It suggests that traces of quantum fluctuations of gravity in the creation period of the universe are recorded in the CMB anisotropy spectrum observed by WMAP.

Cosmic expansion, the big bang, creation of the primordial fluctuations, and so on, it seems to be natural to consider that their origin is in quantum effects of gravity. Quantum gravity is expected as a necessary physics to understand the history of the universe from the birth of spacetime to the present. The ultimate goal of this book is to explain the spectrum of CMB using the asymptotically background-free quantum gravity. Recent studies have revealed that we can explain a number of observed facts well if considering that a spacetime phase transition suggested by this theory as the big bang occurred at 10^{17}GeV.

Historical Background

Einstein's theory of gravity has many properties unfavorable in constructing its quantum theory, for examples, the Einstein-Hilbert action given by the

Ricci scalar curvature is not positive-definite, and the Newton coupling constant has dimensions so that the theory becomes unrenormalizable. However, renormalization itself is not an idea contradictory to diffeomorphism invariance, or invariance under general coordinate transformations, which is the basis of the gravity theory.

In the early studies of the 1970s, it was considered that renormalizable quantum gravity could be obtained by simply adding fourth-order derivative gravitational actions to the Einstein-Hilbert action. It is because due to the fact that the gravitational field is dimensionless unlike other known fields, not only the coupling constant becomes dimensionless, but also the action can be made positive-definite. Furthermore, when including the Riemann curvature tensor in the action, spacetime singularities can be removed quantum mechanically because the action diverges for such field configurations.

However, with methods of treating all modes of the gravitational field perturbatively, we could not prevent undesirable gauge-invariant ghosts from appearing as asymptotic fields. It is the problem of the so-called massive graviton with negative metric.[2] Eventually, the attempt to quantize gravity with standard methods of quantum field theory had gone away, and after the 1980s, methods that do not use quantum field theory have become mainstream. Actually, there are many studies on quantum gravity, but there are few ones that have directly performed quantization of the gravitational field.

The purpose of this book is to return to the traditional method of quantum field theory again and propose a new approach to renormalizable quantum theory of gravity. In order to solve the problems, we introduce a nonperturbative technique based on conformal field theory which has recently made remarkable progress. As the result, the particle picture propagating in a specific background will be discarded.

A significant progress in methods to quantize gravity was made in the latter half of the 1980s. That is the discovery of an exact solution of two-dimensional quantum gravity. The major difference from the conventional quantum gravity mainly studied from the 1970s to the early 1980s was that it correctly took in contributions from the path integral measure and treated the conformal factor in the metric tensor field strictly. This study indicated

[2] There is a work on the unitarity issue by T. Lee and G. Wick, Nucl. Phys, **B9** (1969) 209, in which they proposed an idea that considering a full propagator including quantum corrections, a real pole representing the existence of ghosts disappears and moves to a pair of complex poles so that ghosts do not appear in the real world. For its application to quantum gravity, see E. Tomboulis, Phys. Lett. **70B** (1977) 361 and references in Bibliography. For more detailed explanations, see the end of Chapter 7. However, this idea cannot be applied to the ultraviolet limit where interactions turn off.

that diffeomorphism invariance in quantum theory should be handled more carefully than that in classical theory.

The essence of this approach is that diffeomorphism invariance involves conformal invariance, thus the quantum gravity theory is formulated as a certain conformal field theory defined on any background spacetime. The difference from normal conformal field theory is that conformal invariance is a gauge symmetry, namely BRST symmetry.[3] In normal conformal field theory, only the vacuum is conformally invariant, whereas in the quantum gravity fields must be conformally invariant as well. All of the theories with different backgrounds connected each other by conformal transformations become gauge-equivalent, thus the background-metric independence is realized. This is called the BRST conformal invariance. It represents that the so-called Wheeler-DeWitt algebra is realized at the quantum level.[4]

Developing this method in four dimensions, we have formulated a new renormalizable quantum theory of gravity. The gravitational field is then decomposed into three parts: the conformal factor defined in an exponential, the traceless tensor field, and a background metric. By quantizing the conformal factor in a non-perturbative way, the background-metric independence is strictly realized as the BRST conformal invariance in the ultraviolet limit. On the other hand, dynamics of the traceless tensor field which cannot be ignored in four dimensions is handled perturbatively by adding the fourth-order derivative Weyl action. Since the coupling constant becomes dimensionless, the theory becomes renormalizable.

In conventional quantum field theories based on Einstein's theory of gravity, the Planck scale is usually regarded as an ultraviolet cutoff. Hence, problems of spacetime singularities, ultraviolet divergences, and even the cosmological constant are substantially avoided. On the other hand, this new renormalizable quantum gravity does not require such an ultraviolet cutoff, because the beta function of the gravitational coupling constant becomes negative, like in quantum chromodynamics (QCD). Therefore, we

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[3] BRST is an abbreviation for Becchi-Rouet-Stora-Tyutin which arranged the names of four discovers. The original papers are C. Becchi, A. Rouet, and R. Stora, *Renormalization of the Abelian Higgs-Kibble Model*, Comm. Math. Phys. **42** (1975) 127; *Renormalization of Gauge Theories*, Ann. Phys. **98** (1976) 287, and I. Tyutin, Lebedev preprint FIAN, 1975. See T. Kugo and I. Ojima, *Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem*, Prog. Theor. Phys. Suppl. **66** (1979) 1 and reference books on quantum field theory in Bigliography.

[4] At the classical Poisson bracket level, the Wheeler-DeWitt algebra holds for arbitrary diffeomorphism invariant theory, but for the algebra to close at the quantum level, the theory is constrained, so that the gravitational action is determined tightly.

can describe a world beyond the Planck scale.

Furthermore, the massive graviton mode becomes unphysical in this approach, because a quadratic term of the field giving mass to this mode is not gauge invariant due to the existence of the exponential conformal factor in the Einstein-Hilbert action. Not only that, the BRST conformal symmetry shows that all modes in the fourth-order derivative gravitational field are not gauge invariant after all even in the ultraviolet limit.

As a theoretical background in which this four-dimensional quantum gravity was devised, there is a work of numerical calculations by the dynamical triangulation method.[5] It is a random lattice model in which the two-dimensional model (matrix model) is generalized to four dimensions, and the simulation result strongly suggested that scalar fluctuations are more dominant than tensor fluctuations. From this research result, we came up with this quantization method which treats only the traceless tensor field perturbatively.

After that, the first observation result of WMAP was released in 2003, and it was indicated that a scale-invariant scalar fluctuation dominates in the early universe. At the same time, the existence of a new scale close to the Planck length was suggested. At first, we could not imagine that a wavelength of observed fluctuations about 5000Mpc which corresponds to the size of the universe is related with the smallest length scale among the known ones, but it can be understood when we consider that the universe expanded about 10^{60} times from its birth to the present, including an inflationary period and the subsequent 13.7 billion years, predicted from a typical scenario of the inflation theory. From the consideration of this new scale, the idea of quantum gravity inflation was born.

Excellent Points of The Theory

A theoretical superiority of the BRST conformal field theory is that whatever background metric we choose, as far as it is conformally flat, the theory does not lose its generality. With this theory as the core, the renormalizable quantum gravity can be constructed as a quantum field theory in the flat background as usual. Dynamics that represents a deviation from the conformal invariance is controlled by only one dimensionless gravitational

[5] See S. Horata, H. Egawa, and T. Yukawa, *Clear Evidence of A Continuum Theory of 4D Euclidean Simplicial Quantum Gravity*, Nucl. Phys. B (Proc. Suppl.) **106** (2002) 971; S. Horata, H. Egawa, and T. Yukawa, *Grand Canonical Simulation of 4D Simplicial Quantum Gravity* , Nucl. Phys. B (Proc. Suppl.) **119** (2003) 921. See also the fifth section of Appendix D and the author's review article in Bibliography.

coupling constant whose beta function becomes negative.

The renormalization theory is formulated using dimensional regularization, which is a regularization method that can calculate higher loop quantum corrections while preserving diffeomorphism invariance. The long-standing problem that the form of fourth-order gravitational actions cannot be fixed from classical diffeomorphism invariance alone is settled at the quantum level, that is, it is determined by not only imposing the Wess-Zumino integrability condition but also using a certain new renormalization group equation.

The fact that the beta function is negative means that the theory can be defined correctly in the ultraviolet limit. Unlike conventional quantum field theory, however, it does not indicate that the flat spacetime in which asymptotic fields can be defined is realized. This is because the conformal factor still fluctuates non-perturbatively so that spacetime is fully quantum mechanical. Therefore, the traditional S-matrix is not defined as a physical quantity. In this book, we refer to this behavior as "asymptotic background freedom", in distinction from the conventional asymptotic freedom.

It also suggests the existence of a new dynamical infrared energy scale of quantum gravity denoted by $\Lambda_{\rm QG}$ here, like $\Lambda_{\rm QCD}$ in QCD.[6] At sufficiently high energy beyond $\Lambda_{\rm QG}$, tensor fluctuations become smaller, while scalar fluctuations by the conformal factor dominate. Below $\Lambda_{\rm QG}$, such conformal dynamics disappears. Thus, this scale divides quantum spacetime filled with conformal fluctuations of gravity from the current classical spacetime without conformal invariance. The more detailed physical implications indicated by this scale are as follows.

Inflation and spacetime phase transition If setting the magnitude relation between the Planck mass $m_{\rm pl}$ and the dynamical scale $\Lambda_{\rm QG}$ as $m_{\rm pl} > \Lambda_{\rm QG}$, there is an inflationary solution, then evolution of the early universe can be divided into three eras separated by these two scales. At high energy far beyond the Planck scale it is described as a conformally invariant spacetime where quantum scalar fluctuations of the conformal factor dominate. The conformal invariance starts breaking in the vicinity of the Planck scale, and gradually shifts to the era of inflation. The inflationary era drastically ends at $\Lambda_{\rm QG}$ where the conformal invariance loses its validity completely. At this point, the universe is expected to make a transition

[6] The existence of such a scale is a characteristic of renormalizable quantum field theory, which is a scale that does not exist in a manifestly finite continuum theory like string theory. It is also characterized by the fact that the effective action has a nonlocal form, and this point is also different from a manifestly finite theory which generally gives a local effective theory.

to the classical Friedmann spacetime in which long distance correlation has been lost. If we choose this scale as 10^{17}GeV, we can explain the CMB observation results well.

One of the excellent points in this inflationary scenario is that it can explain the evolution of the universe using the dynamics of the gravitational field alone without introducing a phenomenological scalar degree of freedom called the inflaton.[7] Interactions between the conformal-factor field and matter fields open through conformal anomaly, and become strong rapidly near $\Lambda_{\rm QG}$. The big bang is caused by that a fourth-order derivative scalar degree of freedom in the conformal factor changes to matter fields immediately at the time of the spacetime phase transition. Hence, it is suggested that quantum fluctuations of gravity are the source of everything. The origin of primordial fluctuations necessary for explaining the structure formation of the universe is given by a scale-invariant scalar spectrum predicted from conformal invariance.

Existence of physical minimum length The dynamical scale $\Lambda_{\rm QG}$ separating quantum and classical spacetimes implies that there is no concept of distance shorter than the correlation length $\xi_\Lambda = 1/\Lambda_{\rm QG}$ because spacetime totally fluctuates there. In this sense, ξ_Λ denotes a minimum length we can measure. Thus, spacetime is practically quantized by ξ_Λ, without discretizing it explicitly, that is, without breaking diffeomorphism invariance. Excitations in quantum gravity would be given by the mass of order $\Lambda_{\rm QG}$.

Although we do not know how large our universe is, at least most of the range that we are looking at today falls within the minimum length before inflation, because the present Hubble distance is given by the order of $10^{59} \times \xi_\Lambda$, as mentioned before. That is to say, we can consider that the universe we are observing now was born from a "bubble" of quantum gravity fluctuations. This is the reason why the primordial spectrum of the universe is almost scale invariant.

On the other hand, since correlations larger than ξ_Λ disappear, the sharp fall-off observed in large angular components of the CMB anisotropy spectrum can be explained by this length scale.

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[7] On the other hand, Einstein's theory of gravity is a theory that matter density determines the structure of spacetime. In other words, the current spacetime cannot be produced from the absence of matters. Therefore, the inflation model based on Einstein's theory of gravity has to introduce a scalar field as a source of all matter fields, but it is unconvincing that elementary particles with a theoretical background such as gauge principles and renormalizability are created from a scalar field that does not have these properties.

New approach to unitarity problem As stated at the beginning, in gravity theories based on the Einstein-Hilbert action, an elementary excitation that has energy beyond the Planck mass becomes a black hole, and thus unitarity is broken. On the other hand, the asymptotically background-free theory indicates that spacetime configurations where the Weyl curvature tensor disappears dominate at high energy beyond $\Lambda_{\rm QG}$. Therefore, spacetime configuration where the Riemann curvature tensor diverges like the Schwarzschild solution is excluded at the quantum level.[8] The existence of such a singular point is also denied by the realization of the BRST conformal invariance representing the background-metric independence.

Since singularities are eliminated, it is possible to discuss the problem of unitarity non-perturbatively. Algebraically, conformal invariance becomes important. The unitarity in conformal field theory is that the Hermitian nature of fields is preserved even in correlation functions. It is expressed as the conditions that not only two-point functions are positive-definite but also operator product expansion coefficients are real.[9]

The BRST conformal invariance gives far stronger constraints on the theory than conventional conformal invariance. Negative-metric ghost modes included in the fourth-order derivative gravitational field are necessary for the conformal algebra to close, but they are not gauge invariant themselves, so that they do not appear in the real world. Physical operators are given by real primary scalar composite fields with a specific conformal dimension, whereas fields with tensor indices become unphysical. Since the whole action is positive-definite, the stability of the path integral is guaranteed, and thus the Hermitian nature of the physical operators will be retained.[10]

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[8] Since the Weyl action diverges, singular configurations are obviously unphysical, whereas in Einstein's gravity theory such a singularity cannot be eliminated because the Einstein-Hilbert action given by the Ricci scalar vanishes, that is, it is physical.

[9] Since the action is often unknown in conformal field theory, such conditions will be imposed (see Chapters 2 and 3). If the action is known, it can be easily understood in statistical mechanics by considering Wick-rotated Euclidean space. If the Euclidean action I is positive-definite, the path integral with weight e^{-I} is correctly defined and thus reality of fields is preserved. If the action is not bounded from below, the path integral diverges and thus the field reality is sacrificed in order to regularize it.

[10] Each mode in the fourth-order gravitational field is not a physical quantity, thus as long as considering correlation functions of physical fields, the positivity of the whole action expressed by the original gravitational field is essential (see Footnote 9).

Outline of The Book

In Chapters 2 and 3, we explain the basis of conformal field theory and recent developments. The basis of two-dimensional conformal field theory is summarized in Chapter 4. In Chapter 5, we describe conformal anomaly involved deeply in the construction of quantum gravity. Chapter 6 is devoted to two-dimensional quantum gravity, which is the simplest theory with the BRST conformal invariance.

In Chapters 7 and 8, we formulate the BRST conformally invariant quantum gravity in four dimensions, which is one of the main subjects of this book, and construct physical field operators and physical states. As a first step to define renormalizable quantum theory of gravity by using dimensional regularization, we examine quantum field theory in curved spacetime in Chapter 9. The form of gravitational counterterms and conformal anomalies is then determined using an advanced technique of renormalization group equations applied to composite fields. Based on this result, we formulate the renormalizable asymptotically background-free quantum gravity in Chapter 10.

In the last four chapters we will discuss evolution of the universe that the quantum gravity suggests. In order to show why we can consider that its traces remains today, we first explain the Friedmann universe in Chapter 11, then present a model of inflation induced by quantum gravity effects in Chapter 12. Furthermore, in Chapter 13, we explain in detail cosmological perturbation theory describing time evolution of fluctuations. In Chapter 14, we apply it to the quantum gravity cosmology and examine time evolution of quantum gravity fluctuations in the inflationary background, then show that the amplitudes reduce during inflation. From quantum gravity spectra given before the Planck time, we derive primordial power spectra right after the spacetime phase transition, and with them as initial spectra of the Friedmann universe, the CMB anisotropy spectra are calculated and compared with experimental data.

Each chapter of the appendix supplements useful formulas for gravitational fields and also useful knowledge that will help understanding although it is slightly out of the main subject.

Finally, from the author's review article listed in Bibliography, extract the following passage:

> The wall of Planck scale reminds us the wall of sound speed. When an airplane speeds up and approaches to the sound speed, it faces violent vibrations due to the sound made by the airplane itself and sometimes breaks the airplane into pieces. People of old days thought that the sound speed is the

> unpassable wall. However, we know now once we pass the wall with durable body, a peaceful space without sounds spreads about us. Similarly, we might think that the Planck scale is the wall that we can never pass. However, once we go beyond the Planck scale, there is no singularity, but a harmonious space of conformal symmetry.

The thought of this research is summarized in this sentence.

Chapter Two

Conformal Field Theory in Minkowski Space

As places where conformal field theory appears, non-trivial fixed points in quantum field theories and critical points in statistical models are widely known. In addition, we will show that an ultraviolet limit of quantum gravity is described as a certain conformal field theory in this book.

These theories will be discussed in Minkowski space or Euclidean space, and each has advantages. First of all, the basis of conformal field theory in Minkowski space are summarized. In this case, the procedure of quantization, the Hamiltonian operator, the nature of field operators such as Hermiticity, etc. are more clear than quantum field theory in Euclidean space. Conformal field theory in Euclidean space is basically considered to be obtained by analytic continuation from Minkowski space.

On the other hand, in the case where an action or a (non-perturbative) quantization method is not clear, it is easier to discuss in Euclidean space, because we can avoid divergences specific to Minkowski space. In addition, there are advantages such as structures of correlation functions, correspondences between states and operators, and so on become clearer, and also correspondences with statistical mechanics becomes easy to understand. Conformal field theory in Euclidean space is discussed in the next chapter.

Hereinafter, when describing the basic properties of conformal field theory, we describe them in any D dimensions. When presenting specific examples, calculations are done in four dimensions for simplicity.

Conformal Transformations

Conformal transformations are coordinate transformations in which when transforming coordinates to $x^\mu \to x'^\mu$, a line element changes as

$$\eta_{\mu\nu} dx^\mu dx^\nu \to \eta_{\mu\nu} dx'^\mu dx'^\nu = \Omega^2(x) \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2-1}$$

where Ω is an arbitrary real function and the Minkowski metric is $\eta_{\mu\nu} = (-1, 1, \cdots, 1)$. Rewriting the right-hand side, the conformal transformation is expressed as

$$\eta_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)\eta_{\lambda\sigma}.$$

The $\Omega = 1$ case corresponds to the Poincaré transformation.

Conformal transformations are defined only on the background metric $\eta_{\mu\nu}$, and under the transformation this metric tensor itself does not change. On the other hand, diffeomorphism is a coordinate transformation in which the metric tensor is regarded as a field to transform together in order to preserve the line element as a scalar quantity, thus it has to be distinguished from the conformal transformation.[1] Below, all contractions of indices of tensor fields are done with the background metric $\eta_{\mu\nu}$.

Considering an infinitesimal conformal transformation $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$, we find from the above equation that ζ^μ must satisfy

$$\partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu - \frac{2}{D}\eta_{\mu\nu}\partial_\lambda\zeta^\lambda = 0. \tag{2-2}$$

This is called the conformal Killing equation, and ζ^λ is called the conformal Killing vector. The arbitrary function is then given by

$$\Omega^2 = 1 + \frac{2}{D}\partial_\lambda\zeta^\lambda. \tag{2-3}$$

Deforming the conformal Killing equation (2-2), we get

$$\left[\eta_{\mu\nu}\partial^2 + (D-2)\partial_\mu\partial_\nu\right]\partial_\lambda\zeta^\lambda = 0.$$

Furthermore, since $(D-1)\partial^2\partial_\lambda\zeta^\lambda = 0$ is obtained from the trace of this expression, we get $\partial_\mu\partial_\nu\partial_\lambda\zeta^\lambda = 0$ for $D > 2$.[2] Solving the equation with this in mind yields $(D+1)(D+2)/2$ solutions. They correspond to D translations, $D(D-1)/2$ Lorentz transformations, one dilatation, D special conformal transformations, denoted by $\zeta^\lambda_{T,L,D,S}$, respectively, which are given as follows:

$$\begin{aligned} (\zeta^\lambda_T)_\mu &= \delta^\lambda_{\ \mu}, \qquad (\zeta^\lambda_L)_{\mu\nu} = x_\mu\delta^\lambda_{\ \nu} - x_\nu\delta^\lambda_{\ \mu}, \\ \zeta^\lambda_D &= x^\lambda, \qquad (\zeta^\lambda_S)_\mu = x^2\delta^\lambda_{\ \mu} - 2x_\mu x^\lambda. \end{aligned} \tag{2-4}$$

[1] Considered the metric as a field and combined with diffeomorphism, the conformal transformation can be expressed as the Weyl rescaling of the metric field, but in this and the next two chapters it is not considered.

[2] In $D = 2$, the condition reduces to $\partial^2\partial_\lambda\zeta^\lambda = 0$, and the number of the conformal Killing vectors becomes infinite.

The indices μ, ν here represent the degrees of freedom of $\zeta^{\lambda}_{T,L,S}$. The first two correspond to isometry transformations that satisfies the Killing equation $\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = 0$, namely the Poincaré transformations.

Finite conformal transformations for dilatation and special conformal transformation are given by

$$x^\mu \to x'^\mu = \lambda x^\mu, \qquad x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2},$$

respectively. In addition to these, we introduce conformal inversion

$$x^\mu \to x'^\mu = \frac{x^\mu}{x^2}, \tag{2-5}$$

which is an important transformation that can be used in place of special conformal transformation. By combining conformal inversion and translation, special conformal transformation can be derived as

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + a^\mu \to \frac{\frac{x^\mu}{x^2} + a^\mu}{\left(\frac{x^\mu}{x^2} + a^\mu\right)^2} = \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2}.$$

Conformal Algebra and Field Transformation Law

Let P_μ, $M_{\mu\nu}$, D, and K_μ be generators of translation, Lorentz transformation, dilatation and special conformal transformation, respectively.[3] These $(D+1)(D+2)/2$ infinitesimal conformal transformation generators satisfy the following $SO(D,2)$ algebra:[4]

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\lambda] = -i\left(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu\right), \\
&[M_{\mu\nu}, M_{\lambda\sigma}] = -i\left(\eta_{\mu\lambda} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\lambda} - \eta_{\mu\sigma} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\sigma}\right), \\
&[D, P_\mu] = -iP_\mu, \quad [D, M_{\mu\nu}] = 0, \quad [D, K_\mu] = iK_\mu, \\
&[M_{\mu\nu}, K_\lambda] = -i\left(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu\right), \quad [K_\mu, K_\nu] = 0, \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D + M_{\mu\nu}\right).
\end{aligned} \tag{2-6}$$

[3] In this book, the same symbol D as spacetime dimensions is used for the generator of dilatation. They can be readily distinguished from the context.

[4] In two dimensions, the $SO(2,2)$ conformal algebra is extended to the infinite dimensional Virasoro algebra and what is called the central charge appears, but such a central extension does not exist in the conformal algebra of $D > 2$.

A subalgebra $SO(D-1,1)$ composed of the generators of translation and Lorentz transformation is the Poincaré algebra. Hermiticity of the generators is defined by

$$P_\mu^\dagger = P_\mu, \quad M_{\mu\nu}^\dagger = M_{\mu\nu}, \quad D^\dagger = D, \quad K_\mu^\dagger = K_\mu.$$

The conformal algebra can be represented collectively using the generator of $SO(D,2)$ denoted by J_{ab} as

$$[J_{ab}, J_{cd}] = -i\left(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc}\right), \qquad \text{(2-7)}$$

where the metric is set to be $\eta_{ab} = (-1, 1, \ldots, 1, -1)$, numbering as $a, b = 0, 1, 2, \ldots, D, D+1$. The generator is antisymmetric $J_{ab} = -J_{ba}$ and satisfies Hermiticity $J_{ab}^\dagger = J_{ab}$. Indeed, the conformal algebra (2-6) is obtained by choosing the spacetime indices as $\mu, \nu = 0, 1, \ldots, D-1$ and writing the generators as

$$\begin{aligned} &M_{\mu\nu} = J_{\mu\nu}, \qquad D = J_{D+1D}, \\ &P_\mu = J_{\mu D+1} - J_{\mu D}, \qquad K_\mu = J_{\mu D+1} + J_{\mu D}. \end{aligned}$$

Fields that transform regularly under conformal transformations are particularly called primary fields. We here consider a symmetric traceless tensor field $O_{\mu_1\cdots\mu_l}$ representing a field of integer spin l.[5] Let Δ be a conformal dimension and the field satisfies Hermiticity

$$O^\dagger_{\mu_1\cdots\mu_l}(x) = O_{\mu_1\cdots\mu_l}(x).$$

A primary scalar field is defined so that it transforms under conformal transformations as

$$O'(x') = \Omega^{-\Delta}(x)O(x).$$

Since $O_{\mu_1\cdots\mu_l}(x)dx^{\mu_1}\cdots dx^{\mu_l}$ transforms as a scalar quantity of conformal dimension $\Delta - l$, the transformation law of a primary tensor field is then given by

$$O'_{\mu_1\cdots\mu_l}(x') = \Omega^{l-\Delta}(x)\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x'^{\mu_l}}O_{\nu_1\cdots\nu_l}(x). \qquad \text{(2-8)}$$

[5] In $D = 4$, this is a tensor field corresponding to the $j = \tilde{j} = l/2$ case in the $(j, \tilde{j})$ representation of the Lorentz group $SO(3,1)$, which can be expressed as $O_{\mu_1\cdots\mu_l} = (\sigma_{\mu_1})^{\alpha_1\dot{\alpha}_1}\cdots(\sigma_{\mu_l})^{\alpha_l\dot{\alpha}_l}O_{\alpha_1\cdots\alpha_l\dot{\alpha}_1\cdots\dot{\alpha}_l}$. In addition, as fields with $j \neq \tilde{j}$, spinor fields of $(1/2, 0)$ and $(0, 1/2)$, Rarita-Schwinger fields of $(1, 1/2)$ and $(1/2, 1)$, antisymmetric tensor fields of $(1, 0)$ and $(0, 1)$, and so on are widely known.

Denoting a vector representation of the orthogonal group $SO(D-1,1)$ as $D_{\mu\nu}$, Jacobian of the transformation is decomposed in the form $\partial x^\nu/\partial x'^\mu = \Omega^{-1}(x) D_\mu{}^\nu(x)$. Here, a primary field of arbitrary spin is simply denoted as $O_j(x)$ and the representation matrix acting on it is written as $R[D]_{jk}$. The conformal transformation can then be expressed with a combination of scale transformations and rotations as $O'_j(x') = \Omega^{-\Delta}(x) R[D(x)]_j{}^k O_k(x)$.

If the vacuum $|0\rangle$ is conformally invariant, correlation functions of these operators satisfy

$$\langle 0|O_{j_1}(x_1)\cdots O_{j_n}(x_n)|0\rangle = \langle 0|O'_{j_1}(x_1)\cdots O'_{j_n}(x_n)|0\rangle, \qquad (2\text{-}9)$$

where note that the argument of the field on the right-hand side is x_j, which is the same as the left-hand side.

The conformal transformation law under an infinitesimal change $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$ is derived by expanding $\delta_\zeta O_j(x) \equiv O_j(x) - O'_j(x)$ by ζ^μ. Noting that $O'_j(x' = x+\zeta) = O'_j(x) + \zeta^\mu \partial_\mu O_j(x)$, $D_\nu{}^\mu = \delta_\nu{}^\mu - (\partial_\nu \zeta^\mu - \partial^\mu \zeta_\nu)/2$, and (2-3), an infinitesimal conformal transformation of primary tensor fields is given by

$$\begin{aligned}\delta_\zeta O_{\mu_1\cdots\mu_l}(x) &= \left(\zeta^\lambda \partial_\lambda + \frac{\Delta}{D}\partial_\lambda \zeta^\lambda\right) O_{\mu_1\cdots\mu_l}(x) \\ &\quad + \frac{1}{2}\sum_{j=1}^{l}\left(\partial_{\mu_j}\zeta^\lambda - \partial^\lambda \zeta_{\mu_j}\right) O_{\mu_1\cdots\mu_{j-1}\lambda\,\mu_{j+1}\cdots\mu_l}(x)\end{aligned}$$

from the transformation law (2-8).

The infinitesimal transformation is expressed as a commutator between the generator and the field operator as

$$\delta_\zeta O_{\mu_1\cdots\mu_l}(x) = i\left[Q_\zeta, O_{\mu_1\cdots\mu_l}(x)\right],$$

where Q_ζ is a generic name of $(D+1)(D+2)/2$ generators for the conformal Killing vector ζ^λ. By substituting the concrete forms of the conformal Killing vectors $\zeta^\lambda_{T,L,D,S}$ (2-4), we obtain the following transformation laws:

$$\begin{aligned} i\left[P_\mu, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \partial_\mu O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[M_{\mu\nu}, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x_\mu\partial_\nu - x_\nu\partial_\mu - i\Sigma_{\mu\nu}\right) O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[D, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x^\mu\partial_\mu + \Delta\right) O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[K_\mu, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2\Delta x_\mu + 2ix^\nu\Sigma_{\mu\nu}\right) O_{\lambda_1\cdots\lambda_l}(x), \end{aligned} \qquad (2\text{-}10)$$

where spin term is defined by

$$\Sigma_{\mu\nu} O_{\lambda_1 \cdots \lambda_l} = i \sum_{j=1}^{l} \left(\eta_{\mu\lambda_j} \delta^{\sigma}_{\ \nu} - \eta_{\nu\lambda_j} \delta^{\sigma}_{\ \mu} \right) O_{\lambda_1 \cdots \lambda_{j-1} \sigma \lambda_{j+1} \cdots \lambda_l}.$$

If defining a spin matrix as $\Sigma_{\mu\nu} O_{\lambda_1 \cdots \lambda_l} = (\Sigma_{\mu\nu})_{\lambda_1 \cdots \lambda_l}{}^{\sigma_1 \cdots \sigma_l} O_{\sigma_1 \cdots \sigma_l}$, then it satisfies the same algebra as the Lorentz generator $M_{\mu\nu}$. In the case of a vector field, it is given by $(\Sigma_{\mu\nu})_{\lambda}{}^{\sigma} = i(\eta_{\mu\lambda} \delta_{\nu}^{\ \sigma} - \eta_{\nu\lambda} \delta_{\mu}^{\ \sigma})$, and the general formula of l is represented using it as

$$(\Sigma_{\mu\nu})_{\lambda_1 \cdots \lambda_l}{}^{\sigma_1 \cdots \sigma_l} = \sum_{j=1}^{l} \delta_{\lambda_1}^{\ \sigma_1} \cdots \delta_{\lambda_{j-1}}^{\ \sigma_{j-1}} (\Sigma_{\mu\nu})_{\lambda_j}{}^{\sigma_j} \delta_{\lambda_{j+1}}^{\ \sigma_{j+1}} \cdots \delta_{\lambda_l}^{\ \sigma_l}.$$

If there is an energy-momentum tensor $\Theta_{\mu\nu}$ satisfying the traceless condition, the generators of conformal transformations can be expressed using the conformal Killing vectors as

$$Q_\zeta = \int d^{D-1}\mathbf{x}\, \zeta^\lambda \, \Theta_{\lambda 0},$$

where $d^{D-1}\mathbf{x}$ is the spatial volume element. Indeed, using the conformal Killing equation (2-2) and the conservation equation $\partial^\mu \Theta_{\mu\nu} = 0$, we can show that $\partial_\eta Q_\zeta = -(1/D) \times \int d^{D-1}\mathbf{x}\, \partial_\lambda \zeta^\lambda \Theta^\mu_{\ \mu}$, thus when the energy-momentum tensor is traceless, the time-dependence disappears and the generator is conserved. Assigning $\zeta^\lambda_{T,L,D,S}$ (2-4) to ζ^λ, we obtain the following concrete expressions:

$$\begin{aligned} P_\mu &= \int d^{D-1}\mathbf{x}\, \Theta_{\mu 0}, \quad M_{\mu\nu} = \int d^{D-1}\mathbf{x}\, (x_\mu \Theta_{\nu 0} - x_\nu \Theta_{\mu 0}), \\ D &= \int d^{D-1}\mathbf{x}\, x^\lambda \Theta_{\lambda 0}, \quad K_\mu = \int d^{D-1}\mathbf{x} \left(x^2 \Theta_{\mu 0} - 2 x_\mu x^\lambda \Theta_{\lambda 0} \right). \end{aligned} \tag{2-11}$$

As a simple example, calculations of the conformal algebra and the conformal transformation law in the case of a quantum free scalar field are given in the third section of Appendix B.

Finally, we give a differential equation that correlation functions satisfy. Conformal field theory is a theory with a conformally invariant vacuum $|0\rangle$, and such a vacuum is defined as a state that satisfies

$$Q_\zeta |0\rangle = 0, \qquad \langle 0| Q_\zeta = 0$$

for all generators $Q_\zeta\,(=Q_\zeta^\dagger)$. If any n conformal fields are simply expressed as O_{j_i} $(i=1,\ldots,n)$, correlation functions of these fields satisfy $\langle 0|\,[Q_\zeta, O_{j_1}(x_1)\cdots O_{j_n}(x_n)]\,|0\rangle = 0$. Thus,

$$\begin{aligned}&\delta_\zeta\langle 0|O_{j_1}(x_1)\cdots O_{j_n}(x_n)|0\rangle\\&= i\sum_{i=1}^n\langle 0|O_{j_1}(x_1)\cdots[Q_\zeta, O_{j_i}(x_i)]\cdots O_{j_n}(x_n)|0\rangle = 0\end{aligned}$$

holds. This is an infinitesimal version of (2-9). For example, let O_{j_i} be a primary scalar field O_i with conformal dimension Δ_i and consider the case of D and K_μ as Q_ζ, we obtain

$$\sum_{i=1}^n\left(x_i^\mu\frac{\partial}{\partial x_i^\mu}+\Delta_i\right)\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle = 0,$$

$$\sum_{i=1}^n\left(x_i^2\frac{\partial}{\partial x_i^\mu}-2x_{i\mu}x_i^\nu\frac{\partial}{\partial x_i^\nu}-2\Delta_i x_{i\mu}\right)\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle = 0,$$

respectively, from the transformation law (2-10).

Correlation Functions and Positivity

Consider two-point correlation functions of traceless symmetric primary tensor fields of integer spin l defined by

$$W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x-y) = \langle 0|O_{\mu_1\cdots\mu_l}(x)O_{\nu_1\cdots\nu_l}(y)|0\rangle. \qquad (2\text{-}12)$$

Letting Δ be conformal dimension of the field, it is generally expressed as

$$W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x) = CP_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x)\left.\frac{1}{(x^2)^\Delta}\right|_{x^0\to x^0-i\epsilon},$$

where C is a constant and ϵ is an infinitesimal ultraviolet cutoff. The function $P_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ is determined from the primary field condition.

In order to determine the form of the two-point correlation function, we use the conformal inversion (2-5), which is expressed as

$$x'_\mu = (Rx)_\mu = \frac{x_\mu}{x^2}.$$

This transformation gives $\Omega(x) = 1/x^2$. Since it returns to its original form when it is operated twice, namely $R^2 = I$, the inverse is given by $x_\mu = (Rx')_\mu$.

Primary scalar fields are transformed under the conformal inversion as

$$O'(x') = \Omega^{-\Delta}(x)O(x) = \left(x^2\right)^{\Delta} O(x).$$

It can be also written as $O'(x) = (x^2)^{-\Delta}O(Rx)$ by returning the argument to x. Here, we proceed with the discussion with the argument of O' as x'. Using this transformation law, the conformal invariance condition (2-9) expressed as $\langle 0|O'(x')O'(y')|0\rangle = \langle 0|O(x')O(y')|0\rangle$ yields

$$(x^2y^2)^{\Delta}\langle 0|O(x)O(y)|0\rangle = \langle 0|O(Rx)O(Ry)|0\rangle.$$

Noting that

$$\frac{1}{(Rx-Ry)^2} = \frac{x^2y^2}{(x-y)^2}, \tag{2-13}$$

we find that the two-point function of the primary scalar field is given by $1/(x-y)^{2\Delta}$ up to an overall coefficient. Restoring the ultraviolet cutoff, we get

$$\langle 0|O(x)O(0)|0\rangle = C\frac{1}{(x^2)^{\Delta}}\bigg|_{x^0\to x^0-i\epsilon} = C\frac{1}{(x^2+2i\epsilon x^0)^{\Delta}},$$

where $x^0 \neq 0$ and ϵ^2 is ignored. In the same way, we can determine the form of the three- and four-point functions of the primary scalar field (see the fourth section in Chapter 3).

Primary vector fields are transformed under conformal inversion as

$$O'_\mu(x') = \Omega(x)^{1-\Delta}\frac{\partial x^\nu}{\partial x'^\mu}O_\nu(x) = \left(x^2\right)^{\Delta} I_\mu{}^\nu(x)O_\nu(x),$$

where we introduce a function $I_{\mu\nu}$ of the coordinates x^μ defined by

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2},$$

which satisfies $I_\mu{}^\lambda(x)I_{\lambda\nu}(x) = \eta_{\mu\nu}$ and $I^\mu{}_\mu(x) = D-2$. Therefore, the conformal invariance condition $\langle 0|O'_\mu(x')O'_\nu(y')|0\rangle = \langle 0|O_\mu(x')O_\nu(y')|0\rangle$ is expressed as

$$(x^2y^2)^{\Delta}I_\mu{}^\lambda(x)I_\nu{}^\sigma(y)\langle 0|O_\lambda(x)O_\sigma(y)|0\rangle = \langle 0|O_\mu(Rx)O_\nu(Ry)|0\rangle.$$

Noting that

$$\begin{aligned} I_\mu{}^\lambda(x)I_\nu{}^\sigma(y)I_{\lambda\sigma}(x-y) &= I_{\mu\nu}(x-y) + 2\frac{x^2-y^2}{(x-y)^2}\left(\frac{x_\mu x_\nu}{x^2} - \frac{y_\mu y_\nu}{y^2}\right) \\ &= I_{\mu\nu}(Rx-Ry), \end{aligned}$$

we can see that the two-point function of the primary vector field is given by $I_{\mu\nu}(x-y)/(x-y)^{2\Delta}$, except for an overall coefficient. Thus, the function $P_{\mu,\nu}$ is determined to be $I_{\mu\nu}$, and we obtain

$$\langle 0|O_\mu(x)O_\nu(0)|0\rangle = CI_{\mu\nu}\frac{1}{(x^2)^\Delta}\bigg|_{x^0\to x^0-i\epsilon}.$$

The same is true for general primary tensor fields. The correlation function of a spin 2 primary tensor field is expressed as

$$\begin{aligned}&\langle 0|O_{\mu\nu}(x)O_{\lambda\sigma}(0)|0\rangle\\&=C\left(\frac{1}{2}I_{\mu\lambda}I_{\nu\sigma}+\frac{1}{2}I_{\mu\sigma}I_{\nu\lambda}-\frac{1}{D}\eta_{\mu\nu}\eta_{\lambda\sigma}\right)\frac{1}{(x^2)^\Delta}\bigg|_{x^0\to x^0-i\epsilon}.\end{aligned}$$

For general integer spin l, the function is given by

$$P_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}=\frac{1}{l!}\left(I_{\mu_1\nu_1}\cdots I_{\mu_l\nu_l}+\text{perms}\right)-\text{traces}.$$

where "perms" and "traces" reflect the symmetric and traceless properties of the tensor field.

From physical (unitary) conditions examined below, we find that the overall constant must be

$$C>0.$$

In the following, we proceed with the discussion by normalizing to $C=1$ in advance.

Using the correlation function (2-12), we define an inner product for arbitrary functions $f_{1,2}(x)$ as

$$(f_1,f_2)=\int d^Dxd^Dy\, f_1^{\mu_1\cdots\mu_l *}(x)W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x-y)f_2^{\nu_1\cdots\nu_l}(y).$$

By introducing Fourier transform of the correlation function

$$W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k)=\int d^Dx\, W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x)\, e^{-ik_\mu x^\mu},$$

the inner product is expressed in momentum space as

$$(f_1,f_2)=\int\frac{d^Dk}{(2\pi)^D}\, f_1^{\mu_1\cdots\mu_l *}(k)f_2^{\nu_1\cdots\nu_l}(k)\, W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k),$$

where $f_{1,2}(k)$ are Fourier transforms of the corresponding functions. In a physical theory that satisfies unitarity conditions, the inner product between the same functions is positive-definite as

$$(f, f) > 0.$$

This is called the Wightman positivity condition.

The positivity condition imposes restrictions on the conformal dimension Δ as follows:

$$\begin{aligned} &\Delta \geq \frac{D}{2} - 1 \quad \text{for } s = 0, \\ &\Delta \geq D - 2 + s \quad \text{for } s \neq 0, \end{aligned} \tag{2-14}$$

where s is spin of primary fields. This condition is called the unitarity bound. Below, by taking specific examples, we will see that this condition is indeed satisfied.

Specific Examples of Positivity Conditions

In the following three sections, we consider the unitarity condition at $D = 4$ for simplicity.

First, consider a primary scalar field with arbitrary conformal dimension Δ. Fourier transform of the correlation function $W(x)$ (see the first section of Appendix B) is given by

$$W(k) = (2\pi)^2 \frac{2\pi(\Delta - 1)}{4^{\Delta-1}\Gamma^2(\Delta)} \theta(k^0)\theta(-k^2)(-k^2)^{\Delta-2}.$$

Thus, the condition that the inner product $(f, f) = \int d^4k |f(k)|^2 W(k)/(2\pi)^4$ becomes positive is given by

$$\Delta \geq 1.$$

The lower bound $\Delta = 1$ is the case of a free field. From $\lim_{\Delta\to 1}(\Delta - 1)\theta(-k^2)(-k^2)^{\Delta-2} = \delta(-k^2)$, it is expressed as

$$\begin{aligned} \frac{1}{(2\pi)^2} \lim_{\Delta\to 1}(f, f) &= \int \frac{d^4k}{(2\pi)^4} |f(k)|^2 2\pi\theta(k^0)\delta(-k^2) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2|\mathbf{k}|} |f(\mathbf{k})|^2, \end{aligned}$$

which is consistent with the expression calculated directly from a canonically quantized free field.

Next, consider the positivity condition in the case of a vector field. We here consider a two-point function of more general real vector field A_μ given by

$$\begin{aligned}\langle 0|A_\mu(x)A_\nu(0)|0\rangle &= \left(\eta_{\mu\nu} - 2\alpha\frac{x_\mu x_\nu}{x^2}\right)\frac{1}{(x^2)^\Delta}\bigg|_{x^0\to x^0-i\epsilon} \\ &= \frac{1}{2\Delta}\left\{\frac{\Delta-\alpha}{2(\Delta-1)(\Delta-2)}\eta_{\mu\nu}\partial^2\right. \\ &\qquad\left. -\frac{\alpha}{\Delta-1}\partial_\mu\partial_\nu\right\}\frac{1}{(x^2)^{\Delta-1}}\bigg|_{x^0\to x^0-i\epsilon}.\end{aligned}$$

Substituting the Fourier transform of the scalar field correlation function into the last part yields Fourier transform of the vector field. Writing it as $W^{(\alpha)}_{\mu\nu}$, we get

$$\begin{aligned}W^{(\alpha)}_{\mu\nu}(k) &= (2\pi)^2\frac{2\pi(\Delta-1)}{4^{\Delta-1}\Gamma(\Delta)\Gamma(\Delta+1)}\theta(k^0)\theta(-k^2)(-k^2)^{\Delta-2} \\ &\quad\times\left\{(\Delta-\alpha)\eta_{\mu\nu} - 2\alpha(\Delta-2)\frac{k_\mu k_\nu}{k^2}\right\}.\end{aligned}$$

The correlation function of the primary vector field O_μ corresponds to $\alpha = 1$, denoted by $W^{(1)}_{\mu\nu} = W_{\mu\nu}$. On the other hand, if we choose $\alpha = \Delta$, A_μ can be considered as a descendant field $\partial_\mu O'$ discussed in the next section, where the conformal dimension of the primary scalar O' is $\Delta' = \Delta - 1$.

The Wightman positivity condition requires $f^{\mu*}f^\nu W^{(\alpha)}_{\mu\nu}$ to be positive for any function f_μ. Even if we select the center-of-mass frame $k^\mu = (k^0, k^j) = (K, 0, 0, 0)$, the generality is not lost. Then,

$$\begin{aligned}&f^{\mu*}(k)f^\nu(k)W^{(\alpha)}_{\mu\nu}(k) \\ &= C_\Delta\theta(K)\theta(K^2)\Big\{[(2\Delta-3)\alpha-\Delta]|f_0|^2 + (\Delta-\alpha)|f_j|^2\Big\}K^{2(\Delta-2)}\end{aligned}$$

is obtained, where the coefficient $C_\Delta = 4(2\pi)^3(\Delta-1)/4^\Delta\Gamma(\Delta)\Gamma(\Delta+1)$ is a positive number assuming $\Delta > 1$. The positivity condition is thus given by $(2\Delta-3)\alpha - \Delta \geq 0$ and $\Delta - \alpha \geq 0$, and solving it for Δ yields

$$\Delta \geq \frac{3\alpha}{2\alpha-1}, \qquad \Delta \geq \alpha.$$

Assigning $\alpha = 1$, the known unitarity condition

$$\Delta \geq 3$$

for the primary vector field is derived.[6] The primary vector field with the lower bound $\Delta = 3$ corresponds to a conserved current satisfying the condition $\partial^\mu O_\mu = 0$. Actually, applying differentiation to the above expression, $\partial^\mu W_{\mu\nu}(x) = 0$ is yielded at $x \neq 0$.

Descendant Fields and Positivity

Fields generated by applying the translation generator P_μ to a primary field O_j such as

$$\partial_\mu \cdots \partial_\nu O_j$$

are called descendants of the primary field O_j. In a unitary conformal field theory, not only primary fields but also their descendants must be physical. That is, two-point functions of descendant fields must satisfy the positivity condition. Here, with specific examples, we show that this condition is consistent with the unitarity bound described in the previous section.

We first consider the two-point functions of a first descendant $\partial_\mu O$ and a second descendant $\partial^2 O$ of a primary scalar field O at $D = 4$. In the latter case, we get

$$\langle 0|\partial^2 O(x)\partial^2 O(0)|0\rangle = 16\Delta^2(\Delta+1)(\Delta-1)\frac{1}{(x^2)^{\Delta+2}}\bigg|_{x^0\to x^0-i\epsilon}.$$

The descendant field $\partial^2 O$ is a scalar quantity so that the unitarity condition simply requires that the sign of the coefficient of the two-point correlation function is positive. Therefore, $\Delta \geq 1$ comes out. Here, $\Delta = 1$ is the case of a free scalar field and then the right-hand side disappears, which means that the equation of motion $\partial^2 O = 0$ holds.

In the case of the first descendant, we get

$$\langle 0|\partial_\mu O(x)\partial_\nu O(0)|0\rangle = 2\Delta\left\{\eta_{\mu\nu} - 2(\Delta+1)\frac{x_\mu x_\nu}{x^2}\right\}\frac{1}{(x^2)^{\Delta+1}}\bigg|_{x^0\to x^0-i\epsilon}.$$

Imposing the Wightman positivity condition discussed in the previous section on this expression, we obtain the condition of $\Delta \geq 1$ again.

[6] Since normal gauge field has dimension one, it does not satisfy the unitarity condition, but the gauge field itself is not a gauge invariant physical quantity so that there is no problem. On the other hand, the gauge-invariant photon field strength $F_{\mu\nu}$ satisfies the unitary condition $\Delta \geq 2$ for antisymmetric tensor fields (see the next chapter).

Next, examine the case of a primary vector field O_μ. Considering a scalar quantity $\partial^\mu O_\mu$ as a first descendant, we obtain

$$\langle 0|\partial^\mu O_\mu(x)\partial^\nu O_\nu(0)|0\rangle = 4(\Delta-1)(\Delta-3)\frac{1}{(x^2)^{\Delta+1}}\bigg|_{x^0\to x^0-i\epsilon}.$$

As before, it turns out that the coefficient becomes positive when $\Delta \geq 3$. $\Delta = 3$ is the case that O_μ is a conserved current, in which the right-hand side disappears and $\partial^\mu O_\mu = 0$ holds.

Similarly, for a primary tensor field $O_{\mu\nu}$, the two-point functions of its first descendant $\partial^\mu O_{\mu\nu}$ and a second descendant $\partial^\mu\partial^\nu O_{\mu\nu}$ that becomes a scalar quantity are given by

$$\begin{aligned}&\langle 0|\partial^\mu O_{\mu\nu}(x)\partial^\lambda O_{\lambda\sigma}(0)|0\rangle \\ &= (\Delta-4)(4\Delta-7)\left\{\eta_{\nu\sigma} - 2\frac{5\Delta-11}{4\Delta-7}\frac{x_\nu x_\sigma}{x^2}\right\}\frac{1}{(x^2)^{\Delta+1}}\bigg|_{x^0\to x^0-i\epsilon}\end{aligned}$$

and

$$\begin{aligned}&\langle 0|\partial^\mu\partial^\nu O_{\mu\nu}(x)\partial^\lambda\partial^\sigma O_{\lambda\sigma}(0)|0\rangle \\ &= 24\Delta(\Delta-1)(\Delta-3)(\Delta-4)\frac{1}{(x^2)^{\Delta+2}}\bigg|_{x^0\to x^0-i\epsilon},\end{aligned}$$

respectively. From the latter expression, we can obtain the unitarity bound $\Delta \geq 4$ easily. Also, applying the Wightman positivity condition discussed in the previous section to the former expression, we find that this condition comes out. At $\Delta = 4$, the right-hand side disappears. Thus, $O_{\mu\nu}$ with conformal dimension 4 is a conserved field satisfying $\partial^\mu O_{\mu\nu} = 0$, which is nothing but an energy-momentum tensor.

Feynman Propagators and Unitarity

Lastly, the unitarity bound for conformal dimensions discussed in the previous section is described in a slightly different way.

Feynman propagators are defined in the coordinate space by

$$\begin{aligned}\langle 0|T[O_{\mu_1\cdots\mu_l}(x)O_{\nu_1\cdots\nu_l}(0)]|0\rangle &= \theta(x^0)\langle 0|O_{\mu_1\cdots\mu_l}(x)O_{\nu_1\cdots\nu_l}(0)|0\rangle \\ &\quad +\theta(-x^0)\langle 0|O_{\nu_1\cdots\nu_l}(0)O_{\mu_1\cdots\mu_l}(x)|0\rangle.\end{aligned}$$

Fourier transform is expressed as

$$\langle 0|T[O_{\mu_1\cdots\mu_l}(x)O_{\nu_1\cdots\nu_l}(0)]|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,e^{ik_\mu x^\mu}D_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(k).$$

In the case of primary scalar fields, the propagator is given by

$$\langle 0|T[O(x)O(0)]|0\rangle = \theta(x^0)\frac{1}{(x^2+2i\epsilon x^0)^\Delta} + \theta(-x^0)\frac{1}{(x^2-2i\epsilon x^0)^\Delta}$$
$$= \frac{1}{(x^2+i\epsilon)^\Delta},$$

where $2\epsilon|x^0|$ is simply rewritten as ϵ in the last expression. Fourier transform of this equation is given by

$$D(k) = -i(2\pi)^2\frac{\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)}(k^2-i\epsilon)^{\Delta-2}.$$

Similarly, for primary vector fields, it is given by

$$\langle 0|T[O_\mu(x)O_\nu(0)]|0\rangle$$
$$= \frac{1}{2\Delta}\left\{\frac{1}{2(\Delta-2)}\eta_{\mu\nu}\partial^2 - \frac{1}{\Delta-1}\partial_\mu\partial_\nu\right\}\frac{1}{(x^2+i\epsilon)^{\Delta-1}}.$$

By substituting the expression of the scalar field propagator into the last part, its Fourier transform can be easily obtained as

$$D_{\mu,\nu}(k) = -i\frac{(2\pi)^2\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta+1)}\left\{(\Delta-1)\eta_{\mu\nu}k^2 - 2(\Delta-2)k_\mu k_\nu\right\}$$
$$\times(k^2-i\epsilon)^{\Delta-3}.$$

Let us consider an interaction between a primary scalar field O and an external field f as

$$I_{\rm int} = g\int d^4x\,(f\,O + {\rm h.c.})\,.$$

Considering the S-matrix due to this interaction and letting $S = 1 + iT$, the transition amplitude from $f^\dagger$ to f is given by

$$i\langle f|T|f\rangle = -g^2\int d^4x f^\dagger(x)\int d^4y\, f(y)\,\langle 0|T[O(x)O(y)]|0\rangle$$
$$= -g^2\int\frac{d^4k}{(2\pi)^4}\, f^\dagger(k)f(k)D(k).$$

The unitarity requires $2{\rm Im}(T) = |T|^2 \geq 0$ from $S^\dagger S = 1$ and therefore the following condition comes up:

$${\rm Im}\langle f|T|f\rangle = g^2\int\frac{d^4k}{(2\pi)^4}\,|f(k)|^2\,{\rm Im}\,\{iD(k)\} \geq 0.$$

Using $(x+i\epsilon)^\lambda - (x-i\epsilon)^\lambda = 2i\sin(\pi\lambda)\theta(-x)(-x)^\lambda$ for an infinitesimal ϵ and $\sin(\pi\lambda) = \pi/\Gamma(\lambda)\Gamma(1-\lambda)$, we obtain

$$\mathrm{Im}\left\{iD(k)\right\} = (2\pi)^2 \frac{\pi(\Delta-1)}{4^{\Delta-1}\Gamma^2(\Delta)}\theta(-k^2)(-k^2)^{\Delta-2}.$$

The right-hand side has the same form as the Fourier transform of the correlation function (there is no $\theta(k^0)$, but the whole is divided by two). Since this has to be positive, the unitary condition $\Delta \geq 1$ is obtained.

Chapter Three

Conformal Field Theory in Euclidean Space

This chapter is devoted to describing conformal field theory in Euclidean space. Since it is free from singularities originated from Lorentzian signatures, correlation functions are rather easy to handle. Moreover, states and inner products can be defined simply using field operators in coordinate space. Therefore, we can examine physical properties of them in more detail.

In Euclidean space, all spacetime indices are written with subscripts, and the same indices denote to be contracted with the Kronecker delta $\delta_{\mu\nu}$.

Critical Phenomena and Conformal Field Theory

If we take the continuum limit of a classical statistical system like the Ising model in D dimensions at a critical point, it shall be described as a conformal field theory on D dimensional Euclidean space.[1] Before discussing basic structures of Euclidean conformal field theory, we briefly describe relationships with critical phenomena.

Let T be a variable that controls critical phenomena of a statistical system such as temperature, and T_c be a critical point. In general, if it is far from the critical point, physical correlation functions decay exponentially as

$$\langle O(x)O(0)\rangle \sim \frac{1}{|x|^{2\Delta}} e^{-|x|/\xi},$$

where $|x| = \sqrt{x^2}$ and ξ is a correlation length. At the critical point $T = T_c$, the correlation length becomes $\xi \to \infty$, and the correlation function behaves in a power law like

$$\langle O(x)O(0)\rangle = \frac{1}{|x|^{2\Delta}}.$$

[1] The Euclidean path integral weight represents the Boltzmann weight in classical statistical systems. On the other hand, conformal field theory in D dimensional Minkowski spacetime corresponds to a quantum statistical system in $D-1$ dimensional space.

This indicates that conformal invariance has appeared. For convenience, an action of conformal field theory which appears exactly on the critical point is denoted by $S_{\rm CFT}$ here.

If the action is known, unitarity shall be described as a positivity of the path integral weight $e^{-S_{\rm CFT}}$, which is ensured by physical conditions that the action is real and bounded from below, as was mentioned in Footnote 9 in Chapter 1. However, the action is unknown in most cases. Hence, conformal field theory is also a study to understand critical phenomena from the conditions of conformal invariance and unitarity without relying on the action.

Critical phenomena are classified by exponents representing how to approach the critical point when a small perturbation is applied in a critical system. Consider, for example, a perturbation by a relevant operator O whose conformal dimension is $\Delta < D$. Denoting a deviation from the critical point by a dimensionless parameter t ($\ll 1$), the action is deformed as

$$S_{\rm CFT} \to S_{\rm CFT} - ta^{\Delta - D}\int d^D x\, O(x),$$

where a is a ultraviolet cutoff length, which corresponds to a lattice spacing in statistical models. If we write $ta^{\Delta-D}$ as $\xi^{\Delta-D}$ from dimensional analysis, ξ represents the correlation length and behaves as[2]

$$\xi \sim at^{-1/(D-\Delta)}.$$

For example, considering an energy operator ε as O, it represents a perturbation by the temperature $t = |T - T_c|/T_c$. Letting Δ_ε be its conformal dimension, since a corresponding critical exponent ν is defined by $\xi \sim at^{-\nu}$, a relation $\nu = 1/(D - \Delta_\varepsilon)$ is obtained. In this way, if conformal dimensions of field operators in conformal field theory are known, we can classify critical exponents, that is, critical phenomena. See the second section of Appendix B for derivations of various critical exponents.

Basic Structures

Conformal algebra in Euclidean space $\mathbb{R}^D$ is given by $SO(D+1,1)$, and it has the same form as (2-6) in Minkowski space M^D when replacing the metric from $\eta_{\mu\nu}$ to $\delta_{\mu\nu}$. Conformal transformation laws also have the same

[2] The correlation length ξ is a physical scale, which does not depend on any scale a. Thus, due to $d\xi/da = 0$, the lowest order of the beta function for the coupling constant t is obtained as $\beta = -a dt/da = -(D - \Delta)t$.

form as (2-10). A significant difference between them is that Hermiticity of the generators P_μ, K_μ, and D change as follows:

$$P_\mu^\dagger = K_\mu, \qquad D^\dagger = -D. \tag{3-1}$$

This is easy to understand by deriving the conformal algebra using the generator J_{ab} of the $SO(D,2)$ algebra given in Chapter 2 as follows. In the $D+2$ dimensional indices $a, b = 0, 1, \ldots, D, D+1$ with the metric $\eta_{ab} = (-1, 1, \ldots, 1, -1)$, we here select a D dimensional Euclidean space part as $\mu, \nu = 1, \ldots, D$. Furthermore, in order to make $SO(D+1, 1)$, we specify an imaginary unit for J_{ab} with the 0th index and identify the generators of conformal transformations as

$$\begin{aligned} &M_{\mu\nu} = J_{\mu\nu}, \qquad D = iJ_{D+1\,0}, \\ &P_\mu = J_{\mu D+1} - iJ_{\mu 0}, \qquad K_\mu = J_{\mu D+1} + iJ_{\mu 0}. \end{aligned}$$

The conformal algebra and Hermiticity in Euclidean space mentioned above can be read from the algebra (2-7) of J_{ab} and its Hermiticity.

Two-point correlation function of a symmetric traceless primary tensor field of integer spin l with conformal dimension Δ is written as

$$\langle O_{\mu_1 \cdots \mu_l}(x) O_{\nu_1 \cdots \nu_l}(0) \rangle = C P_{\mu_1 \cdots \mu_l, \nu_1 \cdots \nu_l} \frac{1}{(x^2)^\Delta},$$

where $P_{\mu_1 \cdots \mu_l, \nu_1 \cdots \nu_l}$ is a function determined from primary field conditions. As in Minkowski space M^D, using the $I_{\mu\nu}$ function in Euclidean space defined by

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2},$$

it is determined as

$$P_{\mu_1 \cdots \mu_l, \nu_1 \cdots \nu_l} = \frac{1}{l!} \left(I_{\mu_1 \nu_1} \cdots I_{\mu_l \nu_l} + \text{perms} \right) - \text{traces}.$$

In physical correlation functions, the coefficient C must be positive. We here set $C = 1$ as in the previous chapter.

Using the conformal inversion

$$x_\mu \to R x_\mu = \frac{x_\mu}{x^2}, \tag{3-2}$$

Hermiticity of a real primary tensor field in Euclidean space is defined as

$$O^\dagger_{\mu_1 \cdots \mu_l}(x) = \frac{1}{(x^2)^\Delta} I_{\mu_1 \nu_1}(x) \cdots I_{\mu_l \nu_l}(x) O_{\nu_1 \cdots \nu_l}(Rx). \tag{3-3}$$

Let us concretely see that the Hermiticity of the field is consistent with the Hermiticity of the generator (3-1). For example, considering translation of a primary scalar field $i[P_\mu, O(x)] = \partial_\mu O(x)$, its Hermitian conjugate is $i[K_\mu, O^\dagger(x)] = \partial_\mu O^\dagger(x)$. Introducing new coordinates $y_\mu = Rx_\mu = x_\mu/x^2$, Hermitian conjugate of the field can be written as $O^\dagger(x) = (y^2)^\Delta O(y)$. Hermitian conjugate of the translation is then expressed as

$$\begin{aligned} i(y^2)^\Delta \left[K_\mu, O(y)\right] &= \frac{\partial y_\nu}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\left\{(y^2)^\Delta O(y)\right\} \\ &= (y^2)^\Delta \left(y^2\partial_\mu - 2y_\mu y_\nu \partial_\nu - 2\Delta y_\mu\right) O(y). \end{aligned}$$

Apart from $(y^2)^\Delta$ on both sides, this is special conformal transformation of the primary scalar field. Similarly, considering Hermitian conjugate of dilatation $i[D, O(x)] = (x_\mu\partial_\mu + \Delta)O(x)$, we can see that it is consistent with the Hermiticity $D^\dagger = -D$.

The case of primary vector fields can be shown in the same way. Considering Hermitian conjugate of translation $i[P_\mu, O_\nu(x)] = \partial_\mu O_\nu(x)$ with attention to $I_{\mu\nu}(x) = I_{\mu\nu}(y)$, we get

$$\begin{aligned} i(y^2)^\Delta I_{\nu\lambda}[K_\mu, O_\lambda(y)] &= \frac{\partial y_\sigma}{\partial x_\mu}\frac{\partial}{\partial y_\sigma}\left\{(y^2)^\Delta I_{\nu\lambda} O_\lambda(y)\right\} \\ = (y^2)^\Delta \Bigg\{ & I_{\nu\lambda}\left(y^2\partial_\mu - 2y_\mu y_\sigma\partial_\sigma - 2\Delta y_\mu\right) O_\lambda(y) \\ & + \left(-2\delta_{\mu\nu}y_\lambda - 2\delta_{\mu\lambda}y_\nu + 4\frac{y_\mu y_\nu y_\lambda}{y^2}\right) O_\lambda(y)\Bigg\}. \end{aligned}$$

Removing extra functions on both sides with attention to $I_{\mu\lambda}I_{\nu\lambda} = \delta_{\mu\nu}$ yields special conformal transformation $i[K_\mu, O_\lambda(y)] = (y^2\partial_\mu - 2y_\mu y_\sigma\partial_\sigma - 2\Delta y_\mu + 2iy_\sigma\Sigma_{\mu\sigma})O_\lambda(y)$ for the primary vector field, where spin term is given by $i\Sigma_{\mu\sigma}O_\lambda = -\delta_{\mu\lambda}O_\sigma + \delta_{\lambda\sigma}O_\mu$.

Consider two-point correlation functions between $O_{\mu_1\cdots\mu_l}$ and its conjugate operator $O^\dagger_{\mu_1\cdots\mu_l}$ using conformal inversion. For example, in the case of primary scalar fields, it becomes

$$\langle O^\dagger(x)O(0)\rangle = \frac{1}{(x^2)^\Delta}\langle O(Rx)O(0)\rangle = 1$$

from $(Rx)^2 = 1/x^2$, which is positive-definite regardless of the coordinate x. Similarly, in the cases of primary vector and tensor fields, we get

$$\begin{aligned} \langle O^\dagger_\mu(x)O_\nu(0)\rangle &= \delta_{\mu\nu}, \\ \langle O^\dagger_{\mu\nu}(x)O_{\lambda\sigma}(0)\rangle &= \frac{1}{2}\left(\delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda} - \frac{2}{D}\delta_{\mu\nu}\delta_{\lambda\sigma}\right), \end{aligned}$$

using $I_{\mu\lambda}I_{\lambda\nu} = \delta_{\mu\nu}$. With these properties, we can define states using field operators in Euclidean space as below.

A primary state is defined as a state satisfying the following conditions for the generators of conformal transformations:

$$
\begin{aligned}
M_{\mu\nu}|\{\mu_1\cdots\mu_l\},\Delta\rangle &= (\Sigma_{\mu\nu})_{\nu_1\cdots\nu_l,\mu_1\cdots\mu_l}\,|\{\nu_1\cdots\nu_l\},\Delta\rangle,\\
iD|\{\mu_1\cdots\mu_l\},\Delta\rangle &= \Delta|\{\mu_1\cdots\mu_l\},\Delta\rangle,\\
K_\mu|\{\mu_1\cdots\mu_l\},\Delta\rangle &= 0.
\end{aligned}
$$

This state can be defined using the field operator as[3]

$$
|\{\mu_1\cdots\mu_l\},\Delta\rangle = O_{\mu_1\cdots\mu_l}(0)|0\rangle, \tag{3-4}
$$

where a conformally invariant vacuum $|0\rangle$ is defined as a state that disappears for all generators of conformal transformations. This relation is called the state-operator correspondence. States obtained by applying P_μ to the primary state (3-4) is called its descendants.

Pay attention to $I_{\mu\nu}(x) = I_{\mu\nu}(y)$ for $y_\mu = Rx_\mu$, the Hermitian conjugate of the field operator at the origin can be written as

$$
\begin{aligned}
O^\dagger_{\mu_1\cdots\mu_l}(0) &= \lim_{x^2\to 0}(x^2)^{-\Delta} I_{\mu_1\nu_1}\cdots I_{\mu_l\nu_l} O_{\nu_1\cdots\nu_l}(Rx)\\
&= \lim_{y^2\to\infty}(y^2)^{\Delta} I_{\mu_1\nu_1}\cdots I_{\mu_l\nu_l} O_{\nu_1\cdots\nu_l}(y).
\end{aligned}
$$

Therefore, Hermitian conjugate of the primary state (3-4) is defined by

$$
\begin{aligned}
\langle\{\mu_1\cdots\mu_l\},\Delta| &= \langle 0|O^\dagger_{\mu_1\cdots\mu_l}(0)\\
&= \lim_{x^2\to\infty}(x^2)^{\Delta} I_{\mu_1\nu_1}\cdots I_{\mu_l\nu_l}\langle 0|O_{\nu_1\cdots\nu_l}(x).
\end{aligned}
$$

Positive definiteness of inner products is then expressed as

$$
\begin{aligned}
(f,f) &= f^\dagger_{\mu_1\cdots\mu_l} f_{\nu_1\cdots\nu_l}\,\langle\{\mu_1\cdots\mu_l\},\Delta|\{\nu_1\cdots\nu_l\},\Delta\rangle\\
&= |f_{\mu_1\cdots\mu_l}|^2 > 0
\end{aligned}
$$

using an arbitrary symmetric traceless tensor $f_{\mu_1\cdots\mu_l}$.

[3] In Minkowski space, we cannot simply use this correspondence. In fact, considering an inner product of $|\Delta\rangle = O(0)|0\rangle$, it diverges as $\langle\Delta|\Delta\rangle = \langle 0|O^\dagger(0)O(0)|0\rangle = \langle 0|O(0)O(0)|0\rangle$ because $O^\dagger(x) = O(x)$ in M^D. On the other hand, in a cylindrical space of $\mathbb{R}\times S^{D-1}$, we can define states in the $\eta \to i\infty$ limit of time as well (see the fourth section of Appendix B and Chapters 6 and 8).

Rederivation of Two-Point Functions

Let us rederive two-point correlation function of primary scalar fields on $\mathbb{R}^D$ using conformal algebra and Hermiticity. The coordinate dependence of scalar fields is expressed using the translation generator as

$$O(x) = e^{iP_\mu x_\mu} O(0) e^{-iP_\mu x_\mu}.$$

Since $P_\mu^\dagger = K_\mu$, its Hermitian conjugate is given by

$$O^\dagger(x) = e^{iK_\mu x_\mu} O^\dagger(0) e^{-iK_\mu x_\mu}.$$

Using these expressions and Hermiticity (3-3), the two-point correlation function can be expressed as

$$\begin{aligned}\langle O(x)O(x')\rangle &= \frac{1}{(x^2)^\Delta}\langle O^\dagger(Rx)O(x')\rangle \\ &= \frac{1}{(x^2)^\Delta}\langle \Delta| e^{-iK_\mu (Rx)_\mu} e^{iP_\nu x'_\nu}|\Delta\rangle,\end{aligned}$$

where $|\Delta\rangle = O(0)|0\rangle$ and $\langle\Delta| = \langle 0|O^\dagger(0)$ are primary states. Expanding the exponential function and evaluating it, we can show that it has a value only when the numbers of K_μ and P_ν are equal. Therefore, the above expression can be expressed as

$$\langle O(x)O(x')\rangle = \frac{1}{(x^2)^\Delta}\sum_{n=0}^{\infty} C_n^\Delta(x,x')\left(\frac{x'^2}{x^2}\right)^{n/2},$$

where the expansion coefficient C_n^Δ is given by

$$C_n^\Delta = \frac{1}{(n!)^2}\frac{x_{\mu_1}\cdots x_{\mu_n} x'_{\nu_1}\cdots x'_{\nu_n}}{(x^2x'^2)^{n/2}}\langle\Delta|K_{\mu_1}\cdots K_{\mu_n}P_{\nu_1}\cdots P_{\nu_n}|\Delta\rangle.$$

Reducing the number of generators using conformal algebra yields the following recursion formula that Gegenbauer polynomials satisfy:

$$nC_n^\Delta = 2(\Delta+n-1)zC_{n-1}^\Delta - (2\Delta+n-2)C_{n-2}^\Delta,$$

where $z = x\cdot x'/\sqrt{x^2x'^2}$. Thus, the coefficients are Gegenbauer polynomials with the variable z (Legendre polynomials when $\Delta = 1/2$). Using the formula of the generating function

$$\frac{1}{(1-2zt+t^2)^\Delta} = \sum_{n=0}^{\infty} C_n^\Delta(z)t^n$$

and substituting z and $t = \sqrt{x'^2/x^2}$ into it, we obtain the known expression $\langle O(x)O(x')\rangle = 1/(x-x')^{2\Delta}$.

Three- and Four-Point Scalar Functions

Consider n-point correlation functions of primary scalar fields $\varphi_j(x_j)$ with conformal dimension Δ_j, where $j = 1, \dots, n$, denoted as

$$G_n(x_1, \dots, x_n) = \langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle.$$

Since the fields transform as $\varphi'_j(x'_j) = (x_j^2)^{\Delta_j} \varphi_j(x_j)$ under conformal inversion $x'^{\mu}_j = (Rx_j)^\mu = x^\mu_j / x^2_j$ (3-2), invariance under this transformation (see (2-9)) is expressed as

$$(x_1^2)^{\Delta_1} \cdots (x_n^2)^{\Delta_n} G_n(x_1, \dots, x_n) = G_n(Rx_1, \dots, Rx_n). \quad \text{(3-5)}$$

Moreover, from translation invariance, the correlation function is expressed as a function of the difference $|x_{ij}| = |x_i - x_j| = \sqrt{(x_i - x_j)^2}$.

Three-point functions A general form of the three-point correlation function satisfying the translation invariance is given by

$$G_3(x_1, x_2, x_3) = \sum_{a,b,c} \frac{C_{a,b,c}}{|x_{12}|^a |x_{13}|^b |x_{23}|^c}.$$

Using the relation $1/(Rx_i - Rx_j)^2 = x_i^2 x_j^2 / (x_i - x_j)^2$, it is easy to see that the conformal inversion condition (3-5) results in

$$2\Delta_1 = a + b, \qquad 2\Delta_2 = a + c, \qquad 2\Delta_3 = b + c.$$

Thus, three variables a, b, and c are fixed completely so that the three-point correlation function is determined to be

$$G_3(x_1, x_2, x_3) = \frac{C_3}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}},$$

except for a single constant C_3.

From invariance under dilatation $\varphi'_j(x'_j) = \lambda^{-\Delta_j} \varphi_j(x_j)$ where $x'^{\mu}_j = \lambda x^\mu_j$, we obtain a condition

$$\lambda^{-(\Delta_1 + \cdots + \Delta_n)} G_n(x_1, \dots, x_n) = G_n(\lambda x_1, \dots, \lambda x_n).$$

When $n = 3$, this condition gives $\Delta_1 + \Delta_2 + \Delta_3 = a + b + c$. However, since this can be derived from the conformal inversion condition, new information is not given. In general, information on dilatation is included in conformal inversion, and thus it is not considered below.

Four-point functions From the translation invariance, a general form of the four-point correlation function is given by

$$G_4(x_1, x_2, x_3, x_4) = \sum_{a,b,c,d,e,f} \frac{C_{a,b,c,d,e,f}}{|x_{12}|^a |x_{13}|^b |x_{14}|^c |x_{23}|^d |x_{24}|^e |x_{34}|^f}.$$

In the same way as in the case of the three-point function, we obtain four conditions

$$2\Delta_1 = a+b+c, \quad 2\Delta_2 = a+d+e, \quad 2\Delta_3 = b+d+f, \quad 2\Delta_4 = c+e+f$$

from the conformal inversion invariance (3-5). Since there are four conditions for six variables, two unknown variables remain. If the unknown variables are chosen as $\alpha = (b+c+d+e)/2$ and $\beta = -d$, we get

$$a = \Delta_1 + \Delta_2 - \alpha, \quad b = \Delta_{34} + \alpha + \beta, \quad c = \Delta_{12} - \Delta_{34} - \beta,$$
$$d = -\beta, \quad e = -\Delta_{12} + \alpha + \beta, \quad f = \Delta_3 + \Delta_4 - \alpha,$$

where $\Delta_{ij} = \Delta_i - \Delta_j$. Therefore, the four-point correlation function can be expressed as

$$G_4(x_1, x_2, x_3, x_4) = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}} \frac{G(u,v)}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}},$$

where the part which is not determined from conformal invariance is a function $G(u,v) = \sum_{\alpha,\beta} C_{\alpha,\beta} u^{\alpha/2} v^{\beta/2}$ of two cross ratios defined by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

Operator Product Expansions

Operator product expansions (OPE) between primary scalar fields are considered. Fields that appear on the right-hand side of the product of the same field can be expressed as $\varphi \times \varphi \sim I + \sum_{l=0,2,4,\ldots} O_{\mu_1 \cdots \mu_l}$, where I is a unit operator and $O_{\mu_1 \cdots \mu_l}$ is a primary tensor field with integer spin l. Only fields with even spin appear in the scalar field OPE, including an energy-momentum tensor that is a spin 2 primary tensor field with conformal dimension same as the spacetime dimension D. Besides these primary fields, their descendant fields (derivatives of primary fields) also appear.

Let d be a conformal dimension of the primary scalar field φ and Δ be that of a spin l primary tensor field, and these two-point correlation functions are normalized as

$$\langle \varphi(x_1)\varphi(x_2)\rangle = \frac{1}{|x_{12}|^{2d}}$$

and

$$\begin{aligned}&\langle O_{\mu_1\cdots\mu_l}(x_1)O_{\nu_1\cdots\nu_l}(x_2)\rangle \\ &= \frac{1}{|x_{12}|^{2\Delta}}\left\{\frac{1}{l!}\left[I_{\mu_1\nu_1}(x_{12})\cdots I_{\mu_l\nu_l}(x_{1,2}) + \text{perms}\right] - \text{traces}\right\},\end{aligned}$$

where $(x_{12})_\mu = x_{1\mu} - x_{2\mu}$. Moreover, the form of three-point correlation functions between them is determined from conformal invariance, apart from an overall coefficient. With a coefficient $f_{\Delta,l}$, it is normalized to

$$\begin{aligned}&\langle \varphi(x_1)\varphi(x_2)O_{\mu_1\cdots\mu_l}(x_3)\rangle \\ &= \frac{f_{\Delta,l}}{|x_{12}|^{2d-\Delta+l}|x_{13}|^{\Delta-l}|x_{23}|^{\Delta-l}}\left(Z_{\mu_1}\cdots Z_{\mu_l} - \text{traces}\right),\end{aligned} \tag{3-6}$$

where

$$Z_\mu = \frac{(x_{13})_\mu}{x_{13}^2} - \frac{(x_{23})_\mu}{x_{23}^2}.$$

The OPE between the same primary scalar fields is then expressed as

$$\begin{aligned}\varphi(x)\varphi(y) &= \frac{1}{|x-y|^{2d}} \\ &\quad + \sum_{\Delta,\, l\,(=2n)} f_{\Delta,l}\left[\frac{(x-y)_{\mu_1}\cdots(x-y)_{\mu_l}}{|x-y|^{2d-\Delta+l}}O_{\mu_1\cdots\mu_l}(y) + \cdots\right] \\ &= \frac{1}{|x-y|^{2d}} + \sum_{\Delta,\, l\,(=2n)} \frac{f_{\Delta,l}}{|x-y|^{2d-\Delta}}C_{\Delta,l}(x-y,\partial_y)O_{\Delta,l}(y),\end{aligned}$$

where in the second equality, the spin l primary tensor field is denoted simply by $O_{\Delta,l}$. The differential operator ∂_y in the coefficient $C_{\Delta,l}$ represents contributions from descendants, which is denoted by the last dots in the second line.

The constant $f_{\Delta,l}$ is called the OPE coefficient or the structure constant. As will be discussed in the penultimate section, $f_{\Delta,l}$ shall be a real number if the theory is a physical one that satisfies unitarity.

Let us derive the $l = 0$ coefficient $C_{\Delta,0}$ as an example. By applying $O = O_{\Delta,0}$ to both sides of the OPE and taking an expectation value, we obtain

$$\langle\varphi(x)\varphi(y)O(z)\rangle = \frac{f_{\Delta,0}}{|x-y|^{2d-\Delta}} C_{\Delta,0}(x-y,\partial_y)\langle O(y)O(z)\rangle.$$

From this, the following equation is derived:

$$\frac{1}{|x-z|^{\Delta}|y-z|^{\Delta}} = C_{\Delta,0}(x-y,\partial_y)\frac{1}{|y-z|^{2\Delta}}.$$

Using the Feynman parameter integral formula given in the first section of Appendix D, the left-hand side can be rewritten as

$$\begin{aligned}
&\frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta}{2})}\int_0^1 dt\,\frac{[t(1-t)]^{\frac{\Delta}{2}-1}}{[t(x-z)^2+(1-t)(y-z)^2]^{\Delta}}\\
&= \frac{1}{B(\frac{\Delta}{2},\frac{\Delta}{2})}\int_0^1 dt\,[t(1-t)]^{\frac{\Delta}{2}-1}\sum_{n=0}^{\infty}\frac{(\Delta)_n}{n!}\frac{[-t(1-t)(x-y)^2]^n}{([y-z+t(x-y)]^2)^{\Delta+n}},
\end{aligned}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Furthermore, we rewrite it to the form expanded in derivatives of $1/|y-z|^{2\Delta}$ by y using the formulas

$$\begin{aligned}
(\partial^2)^n\frac{1}{(x^2)^{\Delta}} &= 4^n(\Delta)_n(\Delta+1-D/2)_n\frac{1}{(x^2)^{\Delta+n}},\\
\frac{1}{[(y+tx)^2]^{\Delta}} &= e^{tx\cdot\partial_y}\frac{1}{(y^2)^{\Delta}}.
\end{aligned}$$

That is identified with the right-hand side, thus we obtain

$$\begin{aligned}
C_{\Delta,0}(x-y,\partial_y) = \frac{1}{B(\frac{\Delta}{2},\frac{\Delta}{2})}&\int_0^1 dt\,[t(1-t)]^{\frac{\Delta}{2}-1}\\
&\times\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n n!}\frac{[t(1-t)a^2]^n}{(\Delta+1-D/2)_n}(\partial_y^2)^n e^{ta\cdot\partial_y}\bigg|_{a=x-y}.
\end{aligned}$$

The first few terms are given as

$$\begin{aligned}
C_{\Delta,0}(x-y,\partial_y) = 1 + \frac{1}{2}(x-y)_\mu\partial_\mu^y + \frac{\Delta+2}{8(\Delta+1)}(x-y)_\mu(x-y)_\nu\partial_\mu^y\partial_\nu^y\\
-\frac{\Delta}{16(\Delta+1)(\Delta+1-D/2)}(x-y)^2\partial_y^2+\cdots.
\end{aligned}$$

Similarly, we can calculate the case of $l \neq 0$ from the three-point correlation function (3-6), though it is complicated.

Conformal Blocks

As shown in the previous section, the four-point correlation function of the primary scalar field φ_j with conformal dimension Δ_j can be simplified to the following form from conformal symmetry:

$$\begin{aligned}&\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle\\&=\left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}}\frac{G(u,v)}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}},\end{aligned} \tag{3-7}$$

where $\Delta_{ij}=\Delta_i-\Delta_j$ and the variables u and v are the cross ratios defined by

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}.$$

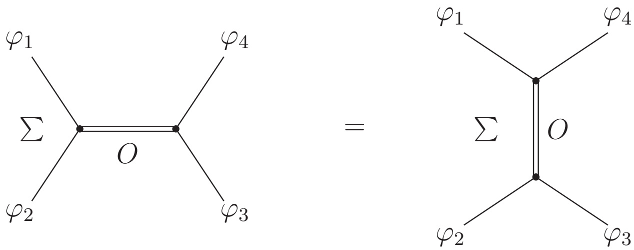

Figure 3-1: Crossing symmetry.

The right-hand side of (3-7) is in the form that OPE is taken between φ_1 and φ_2. On the other hand, the result should not be changed even when taking OPE between φ_1 and φ_4. Thus, even if (x_2,Δ_2) and (x_4,Δ_4) are exchanged on the right-hand side, the result does not change. Similarly, the exchange of (x_2,Δ_2) and (x_3,Δ_3) does not change the result. This property is called crossing symmetry. Therefore, $G(u,v)$ can also be written as a function of $G(v,u)$ or $G(1/u,v/u)$.

For simplicity, consider the case of $\Delta_1=\Delta_2=\Delta_3=\Delta_4$ in this section, while general cases will be discussed in the next section. If extracting the part proportional to the unit operator in OPE as $G(u,v)=1+\sum_{\Delta,l}f_{\Delta,l}^2g_{\Delta,l}(u,v)$, the four-point correlation function of the primary scalar field φ with conformal dimension d can be written as

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle=\frac{1}{|x_{12}|^{2d}|x_{34}|^{2d}}\Big[1+\sum_{\Delta,l}f_{\Delta,l}^2g_{\Delta,l}(u,v)\Big],$$

where $g_{\Delta,l}(u,v)$ is a function called conformal block. From the crossing relation $v^d G(u,v) = u^d G(v,u)$ coming from the exchange of x_2 and x_4, the conformal block satisfies the following relation:

$$u^d - v^d = \sum_{\Delta,l} f_{\Delta,l}^2 \left[v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u) \right]. \tag{3-8}$$

In the following, by computing the conformal block $g_{\Delta,l}$ from OPE, we will see that it can be written as the product of Gauss hypergeometric function

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

Here, the fact that the function is numerically analyzable is significant. As will be mentioned in the penultimate section, this result is a basis of studies imposing restrictions on the value of conformal dimension from the condition that the OPE coefficient $f_{\Delta,l}$ shall be real in unitary theories.

We here examine the conformal block that the $l = 0$ scalar mode propagates as an intermediate state. Using the OPE calculated in the previous section, it can be calculated as

$$g_{\Delta,0}(u,v) = |x_{12}|^\Delta |x_{43}|^\Delta C_{\Delta,0}(x_{12}, \partial_2) C_{\Delta,0}(x_{43}, \partial_3) \frac{1}{|x_{23}|^{2\Delta}}$$
$$= \frac{|x_{12}|^\Delta |x_{43}|^\Delta}{B(\frac{\Delta}{2}, \frac{\Delta}{2})^2} \int_0^1 dt \int_0^1 ds \, [t(1-t)s(1-s)]^{\frac{\Delta}{2}-1}$$
$$\times \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}}{n!m!} \frac{(\Delta)_{n+m} (\tilde{\Delta})_{n+m}}{(\tilde{\Delta})_n (\tilde{\Delta})_m} \frac{[t(1-t)x_{12}^2]^n [s(1-s)x_{43}^2]^m}{[(x_{23} + tx_{12} - sx_{43})^2]^{\Delta+n+m}},$$

where $\tilde{\Delta} = \Delta + 1 - D/2$. Letting $A^2 = t(1-t)x_{12}^2$ and $B^2 = s(1-s)x_{43}^2$ and using

$$(x_{23} + tx_{12} - sx_{43})^2 = \Lambda^2 - A^2 - B^2,$$
$$\Lambda^2 = t\left[s x_{14}^2 + (1-s)x_{13}^2\right] + (1-t)\left[s x_{24}^2 + (1-s)x_{23}^2\right],$$

the right-hand side can be rewritten as

$$\frac{|x_{12}|^\Delta |x_{43}|^\Delta}{B(\frac{\Delta}{2}, \frac{\Delta}{2})^2} \int_0^1 dt \int_0^1 ds \, \frac{[t(1-t)s(1-s)]^{\frac{\Delta}{2}-1}}{(\Lambda^2 - A^2 - B^2)^\Delta} F_4(\Delta, \tilde{\Delta}; \tilde{\Delta}, \tilde{\Delta}; X, Y),$$

where $X = -A^2/(\Lambda^2 - A^2 - B^2)$ and $Y = -B^2/(\Lambda^2 - A^2 - B^2)$. The function F_4 is a hypergeometric series with two variables (double series) called Appell function, defined as

$$F_4(a,b;c,d;x,y) = \sum_{n,m=0}^{\infty} \frac{(a)_{n+m}(b)_{n+m}}{(c)_n(d)_m} \frac{x^n}{n!} \frac{y^m}{m!}.$$

In a special case, this function is related to Gauss hypergeometric series as

$$F_4(a,b;b,b;x,y) = (1-x-y)^{-a} \, {}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; b; \frac{4xy}{(1-x-y)^2}\right).$$

Using this fact, the right-hand side can be written as

$$\frac{|x_{12}|^{\Delta}|x_{43}|^{\Delta}}{B(\frac{\Delta}{2},\frac{\Delta}{2})^2} \int_0^1 dt \int_0^1 ds \, \frac{[t(1-t)s(1-s)]^{\frac{\Delta}{2}-1}}{(\Lambda^2)^{\Delta}} \times {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \tilde{\Delta}; \frac{4A^2B^2}{\Lambda^4}\right).$$

The t and s parameter integrations are sequentially performed using integral formulas

$$\int_0^1 dt \, \frac{t^{a-1}(1-t)^{b-1}}{[t\alpha+(1-t)\beta]^{a+b}} = \frac{1}{\alpha^a\beta^b} B(a,b),$$
$$\int_0^1 ds \, \frac{s^{a-1}(1-s)^{b-1}}{(1-s\alpha)^c(1-s\beta)^d} = B(a,b)F_1(a,c,d;a+b;\alpha,\beta),$$

where F_1 is a new hypergeometric series with two variables defined by

$$F_1(a,b,c;d;x,y) = \sum_{n,m=0}^{\infty} \frac{(a)_{n+m}(b)_n(c)_m}{(d)_{n+m}} \frac{x^n}{n!} \frac{y^m}{m!}.$$

In a certain case, it can be written in terms of Gauss hypergeometric series as

$$F_1(a,b,c,b+c;x,y) = (1-y)^{-a} {}_2F_1\left(a,b;b+c;\frac{x-y}{1-y}\right).$$

Using these formulas and $4^n(\Delta/2)_n((\Delta+1)/2)_n = (\Delta)_{2n}$, we obtain

$$u^{\frac{\Delta}{2}} \sum_{n=0}^{\infty} \frac{u^n}{n!} \frac{\left(\frac{\Delta}{2}\right)_n^4}{(\Delta)_{2n}(\tilde{\Delta})_n} {}_2F_1\left(\frac{\Delta}{2}+n, \frac{\Delta}{2}+n; \Delta+2n; 1-v\right)$$

as an expression of the conformal block $g_{\Delta,0}(u,v)$. Furthermore, we introduce a new hypergeometric series with two variables defined by

$$G(a,b,c,d;x,y) = \sum_{n,m=0}^{\infty} \frac{(d-a)_n(d-b)_n}{(c)_n} \frac{(a)_{n+m}(b)_{n+m}}{(d)_{2n+m}} \frac{x^n}{n!} \frac{y^m}{m!}.$$

Rewriting the conformal block using $(n+\Delta/2)_m = (\Delta/2)_{n+m}/(\Delta/2)_n$ and $(2n+\Delta)_m = (\Delta)_{2n+m}/(\Delta)_{2n}$ finally yields the following expression:

$$g_{\Delta,0}(u,v) = u^{\frac{\Delta}{2}} G\left(\frac{\Delta}{2}, \frac{\Delta}{2}, \Delta+1-\frac{D}{2}, \Delta; u, 1-v\right).$$

Let us introduce new coordinate variables z and $\bar{z}$ defined through the relations

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}).$$

The following relation then holds:

$$\begin{aligned} &G(a,b,c-1,c;u,1-v) \\ &= \frac{1}{z-\bar{z}}\Big[z\,{}_2F_1(a,b;c;z)\,{}_2F_1(a-1,b-1;c-2;\bar{z}) \\ &\quad -\bar{z}\,{}_2F_1(a,b;c;\bar{z})\,{}_2F_1(a-1,b-1;c-2;z)\Big]. \end{aligned}$$

Thus, the conformal block in four dimensions, for example, can be written as

$$g_{\Delta,0}(u,v)|_{D=4} = \frac{z\bar{z}}{z-\bar{z}}\left[k_\Delta(z)k_{\Delta-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right],$$

where

$$k_\beta(x) = x^{\frac{\beta}{2}}\,{}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta; x\right). \tag{3-9}$$

The conformal block with $l \geq 1$ can also be obtained directly from OPE in the same way if l is small, though it is complicated. For general l, there is an approach to calculate it by deriving recursion formula about l. Writing the results only, the conformal block in four dimensions is given by

$$g_{\Delta,l}(u,v)|_{D=4} = \frac{(-1)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right]. \tag{3-10}$$

In two dimensions, it is given by

$$g_{\Delta,l}(u,v)|_{D=2} = \frac{(-1)^l}{2^l}\left[k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})\right]. \tag{3-11}$$

On the other hand, general expressions in three dimensions are still unknown except for a special case of $z = \bar{z}$.

Casimir Operator and Conformal Blocks

Consider a second order Casimir operator $C_2 = J^{ab}J_{ab}/2$ that commutes with the generator J_{ab} of the conformal algebra $SO(D+1,1)$. It can be written with the generators of conformal transformations as

$$C_2 = \frac{1}{2}M_{\mu\nu}M_{\mu\nu} - D^2 - \frac{1}{2}\left(K_\mu P_\mu + P_\mu K_\mu\right).$$

Primary states are eigenstates of this operator. In the case of spin l and conformal dimension Δ, the following holds:

$$C_2|\Delta, l\rangle = C_{\Delta,l}|\Delta, l\rangle, \qquad C_{\Delta,l} = \Delta(\Delta - D) + l(l + D - 2).$$

The four-point correlation function between different primary scalar fields is considered here. Inserting a complete set of states in between, it can be expressed as

$$\begin{aligned}&\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle \\ &= \sum_{\Delta,l,n}\langle 0|\varphi_1(x_1)\varphi_2(x_2)|n;\Delta,l\rangle\langle n;\Delta,l|\varphi_3(x_3)\varphi_4(x_4)|0\rangle,\end{aligned}$$

where $|n;\Delta,l\rangle$ denotes n th descendant states of the primary state $|\Delta, l\rangle$. Hence, consider

$$\begin{aligned}&\frac{1}{2}\langle 0|\left[J_{ab},\left[J^{ab},\varphi_1(x_1)\varphi_2(x_2)\right]\right]|n;\Delta,l\rangle \\ &= \langle 0\,|\varphi_1(x_1)\varphi_2(x_2)C_2|\,n;\Delta,l\rangle = C_{\Delta,l}\,\langle 0\,|\varphi_1(x_1)\varphi_2(x_2)|\,n;\Delta,l\rangle\,,\end{aligned}$$

where $\langle 0|J_{ab} = 0$ is used and note that the Casimir operator C_2 commutes with the translation generator P_μ which generates the descendant states. Using the conformal transformation law of the scalar field, the left-hand side can be rewritten as

$$\begin{aligned}&\Big\{x_{12}^2\partial_\mu^1\partial_\mu^2 - 2(x_{12})_\mu(x_{12})_\nu\partial_\mu^1\partial_\nu^2 - 2\Delta_1(x_{12})_\mu\partial_\mu^2 + 2\Delta_2(x_{12})_\mu\partial_\mu^1 \\ &\quad + (\Delta_1+\Delta_2)(\Delta_1+\Delta_2-D)\Big\}\langle 0|\varphi_1(x_1)\varphi_2(x_2)|n;\Delta,l\rangle.\end{aligned}$$

Parts of the four-point correlation function which have an intermediate state $O_{\Delta,l}$ can be expressed using the conformal block $g_{\Delta,l}$ as

$$\begin{aligned}&\sum_n\langle 0|\varphi_1(x_1)\varphi_2(x_2)|n;\Delta,l\rangle\langle n;\Delta,l|\varphi_3(x_3)\varphi_4(x_4)|0\rangle \\ &= \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}\frac{f_{\Delta,l}^2\,g_{\Delta,l}(u,v)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}.\end{aligned}$$

Using this fact, we find that the conformal block satisfies the following differential equation:

$$\mathcal{D} g_{\Delta,l}(u,v) = \frac{1}{2} C_{\Delta,l} g_{\Delta,l}(u,v),$$

where

$$\begin{aligned}\mathcal{D} = &(1-u+v)u\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial u}\right) + \left[(1-v)^2 - u(1+v)\right]\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}\right)\\ &-2(1+u-v)uv\frac{\partial^2}{\partial u \partial v} - Du\frac{\partial}{\partial u}\\ &+\frac{1}{2}\left(\Delta_{12}-\Delta_{34}\right)\left[(1+u-v)\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) - (1-u-v)\frac{\partial}{\partial v}\right]\\ &+\frac{1}{4}\Delta_{12}\Delta_{34}(1+u-v).\end{aligned}$$

Furthermore, if the coordinate variables are converted to z and $\bar{z}$, the differential operator can be written as

$$\begin{aligned}\mathcal{D} = &z^2(1-z)\frac{\partial^2}{\partial z^2} + \bar{z}^2(1-\bar{z})\frac{\partial^2}{\partial \bar{z}^2}\\ &+\frac{1}{2}\left(\Delta_{12}-\Delta_{34}-2\right)\left(z^2\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial \bar{z}}\right) + \frac{1}{4}\Delta_{12}\Delta_{34}(z+\bar{z})\\ &+(D-2)\frac{z\bar{z}}{z-\bar{z}}\left[(1-z)\frac{\partial}{\partial z} - (1-\bar{z})\frac{\partial}{\partial \bar{z}}\right].\end{aligned}$$

Solutions of the differential equation in $D=4$ and 2 are given by (3-10) and (3-11), respectively, with the function k_β replaced by

$$k_\beta(x) = x^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta}{2} - \frac{\Delta_{12}}{2}, \frac{\beta}{2} + \frac{\Delta_{34}}{2}, \beta; x\right).$$

Unitarity Bound Again

The unitarity condition (2-14) is considered again using primary states defined previously. Here, we consider the $D=4$ case specifically.

Consider a primary vector state $|\mu,\Delta\rangle$, for example. From the unitarity, its inner product must be positive-definite, which is normalized to $\langle\Delta',\mu|\nu,\Delta\rangle = \delta_{\Delta'\Delta}\delta_{\mu\nu}$. The unitarity further requires that inner products of its descendants are also positive-definite. Focusing first descendant states

$|\mu;\nu,\Delta\rangle = P_\mu|\nu,\Delta\rangle$, the inner product is calculated from the conformal algebra as follows:

$$\begin{aligned}\langle\mu;\lambda,\Delta'|\nu;\sigma,\Delta\rangle &= \langle\lambda,\Delta'|\,[K_\mu,P_\nu]\,|\sigma,\Delta\rangle\\ &= \langle\lambda,\Delta'|\,2i\,(D\delta_{\mu\nu}+M_{\mu\nu})\,|\sigma,\Delta\rangle\\ &= 2\delta_{\Delta'\Delta}\,(\Delta\delta_{\mu\nu}\delta_{\lambda\sigma}-\delta_{\mu\lambda}\delta_{\nu\sigma}+\delta_{\nu\lambda}\delta_{\mu\sigma})\,,\end{aligned}$$

where the Hermiticity $P_\mu^\dagger = K_\mu$, the primary state condition $K_\mu|\nu,\Delta\rangle = \langle\nu,\Delta|P_\mu = 0$, and $\langle\lambda,\Delta|M_{\mu\nu}|\sigma,\Delta\rangle = (\Sigma_{\mu\nu})_{\lambda\sigma}$ are used. When pairs of indices are expressed as $a=(\mu,\lambda)$ and $b=(\nu,\sigma)$, it is a 16×16 matrix $\langle a|b\rangle$. There are three kinds of eigenvalues, one for $2(\Delta-3)$, six for $2(\Delta-1)$ and nine for $2(\Delta+1)$. Since these all must be positive, $\Delta\geq3$ comes out.

More generally, we proceed with the discussion by denoting a representation of the rotation group $SO(4)$ as $\{r\}$. Writing a primary state that belongs to it as $|\{r\},\Delta\rangle$, the primary state condition can be expressed as

$$\begin{aligned}M_{\mu\nu}\,|\{r\},\Delta\rangle &= (\Sigma_{\mu\nu})_{\{r'\},\{r\}}\,|\{r'\},\Delta\rangle,\\ iD\,|\{r\},\Delta\rangle &= \Delta\,|\{r\},\Delta\rangle,\\ K_\mu\,|\{r\},\Delta\rangle &= 0.\end{aligned}$$

Since $SO(4)$ is expressed as $SU(2)\times SU(2)$, the representation $\{r\}$ can be denoted by a combination of (j_1,j_2) and its dimension is $(2j_1+1)(2j_2+1)$, where j_1 and j_2 are spins of the left and right $SU(2)$. For example, the integer spin l traceless symmetric tensor field $O_{\mu_1\cdots\mu_l}$ is given by $j_1 = j_2 = l/2$.

A n-th descendant states generated by applying P_μ to the primary state n times is expressed as

$$|\mu_1\cdots\mu_n;\{r\},\Delta\rangle = P_{\mu_1}\cdots P_{\mu_n}\,|\{r\},\Delta\rangle.$$

Assuming that the primary state has a positive-definite inner product normalized to $\langle\{r'\},\Delta'|\{r\},\Delta\rangle = \delta_{\{r'\}\{r\}}\delta_{\Delta'\Delta}$, the unitarity requires that all of the descendant states have positive-definite inner products.

Calculating an inner product of the first descendant state $|\mu;\{r\},\Delta\rangle = P_\mu|\{r\},\Delta\rangle$ as before, we obtain

$$\langle\mu;\{r'\},\Delta'|\nu;\{r\},\Delta\rangle = \delta_{\Delta'\Delta}\Big\{2\Delta\delta_{\{r'\}\{r\}} + 2\,\langle\{r'\},\Delta|\,iM_{\mu\nu}|\{r\},\Delta\rangle\Big\}. \tag{3-12}$$

Using the fact that Lorentz generator can be written as

$$iM_{\mu\nu} = i\frac{1}{2}\,(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})\,M_{\alpha\beta} = \frac{1}{2}\,(\Sigma_{\alpha\beta})_{\mu\nu}\,M_{\alpha\beta}$$

and denoting the matrix $\Sigma_{\alpha\beta}$ as $\langle\mu|M^{\{v\}}_{\alpha\beta}|\nu\rangle = (\Sigma_{\alpha\beta})_{\mu\nu}$ by introducing a vector state $|\mu\rangle$, the last term in (3-12) can be expressed as

$$2\langle\{r'\},\Delta|\, iM_{\mu\nu}|\{r\},\Delta\rangle = \langle\mu|\otimes\langle\{r'\},\Delta|\, M^{\{v\}}_{\alpha\beta}\cdot M^{\{r\}}_{\alpha\beta}|\{r\},\Delta\rangle\otimes|\nu\rangle. \tag{3-13}$$

This can be solved in the same way as an angular momentum coupling problem. Letting $M^{\{R\}}_{\alpha\beta} = M^{\{v\}}_{\alpha\beta} + M^{\{r\}}_{\alpha\beta}$,

$$\begin{aligned} M^{\{v\}}_{\alpha\beta}\cdot M^{\{r\}}_{\alpha\beta} &= \frac{1}{2}M^{\{R\}}_{\alpha\beta}\cdot M^{\{R\}}_{\alpha\beta} - \frac{1}{2}M^{\{v\}}_{\alpha\beta}\cdot M^{\{v\}}_{\alpha\beta} - \frac{1}{2}M^{\{r\}}_{\alpha\beta}\cdot M^{\{r\}}_{\alpha\beta} \\ &= c_2(\{R\}) - c_2(\{v\}) - c_2(\{r\}) \end{aligned}$$

holds, where c_2 is a second order Casimir operator of the $SO(4)$ rotation group. The representation $\{r\}$ is denoted as (j_1, j_2) using the expression of $SU(2)\times SU(2)$, while the vector representation $\{v\}$ is $(1/2, 1/2)$. The representation $\{R\}$ is denoted by (J_1, J_2), where, $J_{1,2}$ takes the values of $j_{1,2}\pm 1/2$. Substituting the values of the second order Casimir, it can be seen that eigenvalues of the matrix (3-13) are summarized to $2J_1(J_1+1) + 2J_2(J_2+1) - 3 - 2j_1(j_1+1) - 2j_2(j_2+1)$. Using this result, we can obtain eigenvalues of the inner product (3-12) of the first descendant state.

What we want to know here is a minimum eigenvalue of the inner product (3-12). The condition of unitarity is that it is positive. Below, we examine the condition separately. In the case of $j_1, j_2 \neq 0$, the inner product (3-12) has a minimum eigenvalue $2\Delta - 2(j_1+j_2+2)$ when $J_1 = j_1 - 1/2$ and $J_2 = j_2 - 1/2$. Therefore, the unitarity bound becomes

$$\Delta \geq j_1 + j_2 + 2 \qquad \text{for } j_1,\ j_2 \neq 0.$$

Substituting $j_1 = j_2 = l/2$, we obtain the unitarity bound $\Delta \geq l+2$ for spin l traceless symmetric primary tensor states, as is given in the previous chapter. In the case of $j_1 = 0$ and $j_2 \neq 0$, we obtain a minimum value $2\Delta - 2(j_2+1)$ when $J_1 = 1/2$ and $J_2 = j_2 - 1/2$. Thus, we get

$$\Delta \geq j_2 + 1 \qquad \text{for } j_1 = 0,\ j_2 \neq 0.$$

The result is the same even when exchanging j_1 and j_2. Substituting $j_2 = 1$, we obtain the unitarity bound $\Delta \geq 2$ for second-rank antisymmetric tensor fields.

In the case of primary scalar states with $j_1 = j_2 = 0$, we obtain a minimum eigenvalue 2Δ. This implies $\Delta \geq 0$ unlike the result in the previous chapter. Therefore, in this case, it is necessary to further consider a second descendant state such as $P^\mu P_\mu|\Delta\rangle$. The inner product is calculated as $\langle\Delta'|K^\mu K_\mu P^\nu P_\nu|\Delta\rangle = 32\Delta(\Delta-1)\delta_{\Delta'\Delta}$. The condition that it is positive gives the known

$$\Delta \geq 1 \qquad \text{for } j_1 = j_2 = 0.$$

Unitarity Bounds from Conformal Bootstrap

The unitarity bounds derived from positive definiteness of inner products gives only lower bounds of conformal dimensions. Here we introduce an approach to restrict conformal dimensions by adding a new unitarity condition to four-point correlation functions.

Consider the case of the four-point correlation function of the same scalar field with conformal dimension d discussed in the previous section. From the crossing symmetry (3-8), the conformal block $g_{\Delta,l}$ satisfies the following equation:

$$\sum_{\Delta,l} p_{\Delta,l} F_{d,\Delta,l}(z,\bar{z}) = 1,$$

$$F_{d,\Delta,l}(z,\bar{z}) = \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}, \tag{3-14}$$

where $p_{\Delta,l} = f^2_{\Delta,l}$. For example, in the case of a free scalar field with $d = 1$, $p_{\Delta,l} = \delta_{\Delta,l+2}\delta_{l,2n}2^{l+1}(l!)^2/(2l)!$ and l is a nonnegative even integer.

Since we consider correlation functions of real fields, the OPE coefficient $f_{\Delta,l}$ should be a real number unless physically dubious operations are done. Since the square is positive, that is expressed as

$$p_{\Delta,l} \geq 0. \tag{3-15}$$

If this positive-definiteness condition is newly imposed, conformal dimensions of fields appearing on the right-hand side of OPE are subject to restrictions. After some concrete results are presented below, its calculation method will be described briefly.

For example, considering OPE between scalar fields $\varphi \times \varphi \sim I + O_\Delta + \cdots$ at $D = 4$, an upper bound of the conformal dimension Δ for a lowest scalar field that appears on the right-hand side is calculated as

$$\begin{aligned}\Delta \leq\ & 2 + 0.7\,(d-1)^{1/2} + 2.1\,(d-1) + 0.43\,(d-1)^{3/2} \\ & + o((d-1)^2),\end{aligned} \tag{3-16}$$

where d is conformal dimension of φ. This condition does not say that there is no primary scalar field with conformal dimension higher than this upper limit. Although a number of scalar fields may appear in the right-hand side of the OPE regardless of whether they are present continuously or discretely, the unitarity condition (3-15) indicates that the scalar field with the lowest dimension must be within the range (3-16).

If performing the same calculation at $D = 2$, we get results that are consistent with the exact solution of two-dimensional conformal field theory. For example, in the Ising model, φ is a spin operator σ and O_Δ is an energy operator ε. Their strict conformal dimensions are known to be $d = \Delta_\sigma = 1/8$ and $\Delta = \Delta_\varepsilon = 1$, respectively. Therefore, if $d = 1/8$ is set and the upper limit of the conformal dimension of the first scalar field appearing on the right-hand side of the OPE is examined, we can see that the condition of $\Delta \leq 1$ indeed comes out. The exact solution $\Delta = 1$ appears exactly at the upper limit allowed. Furthermore, if we analyze the Ising model at $D = 3$ on the basis of this fact that it appears at an upper limit of allowed region, we obtain results consistent with Monte Carlo calculations on lattice.

The concrete calculation method will be briefly described below. Introducing new coordinates $z = 1/2 + X + iY$ and $\bar{z} = z^*$, consider the following differential operator Λ up to the Nth order for X and Y:

$$\Lambda[F] = \sum_{\substack{m,n=\text{even} \\ 2 \leq m+n \leq N}} \lambda_{m,n} \partial_X^m \partial_Y^n F|_{X=Y=0}.$$

The reason why we evaluate at $X = Y = 0$ $(z = \bar{z} = 1/2)$ is simply that convergences at this point are good when performing calculations numerically. Applying this operator to (3-14), we get

$$\sum_{\Delta,l} p_{\Delta,l} \Lambda \left[F_{d,\Delta,l} \right] = 0. \tag{3-17}$$

This equation indicates that if an inequality $\Lambda[F_{d,\Delta,l}] \geq 0$ is satisfied for all Δ and l, it is against the positive-definiteness condition, except in trivial cases.

First of all, we assume a OPE structure given as follows:

$$\varphi \times \varphi \sim I + \sum_{\Delta \geq f} O_\Delta + \sum_{\substack{l>0 \\ l=\text{even}}} \sum_{\Delta \geq D-2+l} O_{\Delta,l}.$$

Here we impose a stronger constraint $\Delta \geq f$ than the unitarity bound on the conformal dimension of the scalar field O_Δ, whereas there is no restriction more than the unitarity bound on tensor fields of $l > 0$.

Let us consider a set of inequalities $\Lambda[F_{d,\Delta,l}] \geq 0$ for all $\Delta \geq f$ $(l = 0)$ and all $\Delta \geq D-2+l$ $(l > 0)$ with fixing d and f. If there is a finite number of solutions $\lambda_{m,n}$ that satisfy this infinite number of inequality systems, it is contradicting the condition (3-17) because it means $\sum_{\Delta,l} p_{\Delta,l}\Lambda\left[F_{d,\Delta,l}\right] \neq 0$ from (3-15). Therefore, such combination of d and f is forbidden because it does not satisfy the positivity condition. If there is no solution, such combination is allowed. In this way, examine a region where the values of d and f are allowed. As d is fixed and f is gradually increased, it enters the forbidden region from the allowed region at a certain place. Letting the value be $f_c(d)$, it gives an upper limit of Δ of $l = 0$, thus the region allowed from the unitarity becomes $D/2 - 1 \leq \Delta \leq f_c(d)$.

In actual calculations, it is necessary to reduce infinite number of inequalities to finite number. Hence, we set an upper limit on l and also discretize Δ for each l. The method of determining whether or not solutions of the inequality system exist is an application of the linear programming method. The value of $\lambda_{m,n}$ itself has no physical meaning here. The expression (3-16) at $D = 4$ is obtained in this way. Numerical results at $D = 3$ are depicted in Fig. 3-2.

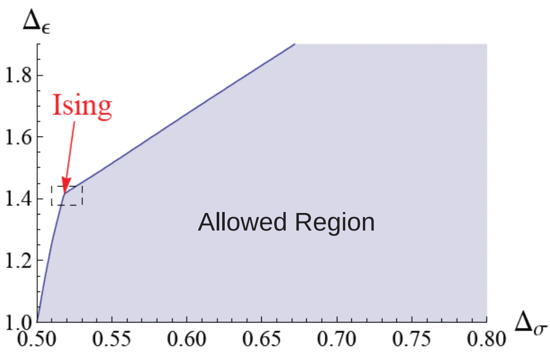

Figure 3-2: Numerical analysis of the allowed region in three dimensions, where $\Delta = \Delta_\epsilon$ and $d = \Delta_\sigma$. The Ising model appears at the upper limit of the allowed region where the boundary line breaks. [S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86** (2012) 025022.]

In order to see the OPE structure in more detail, further decompose the scalar field O_Δ by dimension by assuming that there is a discrete structure. Since the allowed region of the lowest dimensional scalar field is determined first as described above, then select, for instance, one point within that region and fix the dimension of the lowest scalar to be Δ, and for the remaining higher dimensional scalar field $O_{\Delta'}$, a new condition is imposed such as $\Delta' \geq f' \ (\geq f_c)$. That is to say that it is calculated assuming the existence of a gap between Δ and Δ'. While gradually increasing the value of f', perform the same calculation and examine an allowed region for each fixed Δ and d. In this way, in addition to restrictions on Δ', the allowed region in the $\Delta - d$ plane can be restricted further. The region is then scraped off and hollowed out so that a sharp region appears, and it is found that the upper bound $f_c(d)$ that Δ can take at a certain d emerges as a special point on a tip of the sharp region.

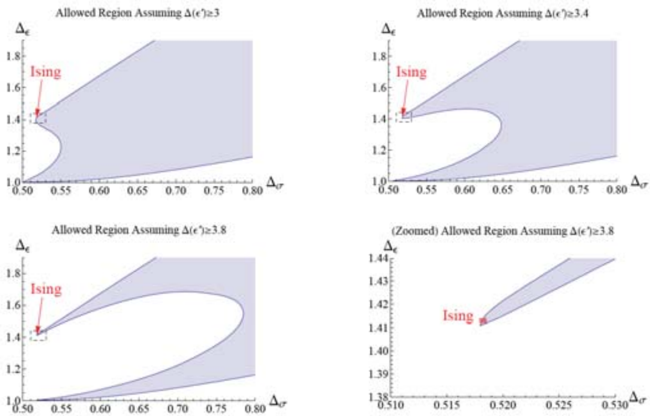

Figure 3-3: The detail analysis of the allowed region in three dimensions, where $\Delta' = \Delta(\epsilon')$. From the top, the condition is gradually strengthened as $\Delta' \geq 3$, 3.4, and 3.8. The last is the enlarged view in the vicinity of the Ising model point. [S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86** (2012) 025022.]

When this calculation is performed in three dimensions, it is found that as a special point, $d = \Delta_\sigma = 0.5182(3)$ and $f_c = \Delta_\varepsilon = 1.413(1)$ are yielded, which are consistent with Monte Carlo calculations of the three-dimensional Ising model (see Fig. 3-3). Further strengthening the restriction to higher dimensional scalar fields and tensor fields, more detailed structure can be examined and finally the allowed region becomes like an isolated island and is narrowed down to this value. Hence, a discrete OPE structure has been revealed by the (semi) analytical method also in the case of three-dimensional conformal field theory, as in two-dimensional one.

Wilson-Fisher Epsilon-Expansion

Lastly we briefly describe the Wilson-Fisher ϵ-expansion, which has been long known as a field-theoretical approach to study critical phenomena.[4] Consider a scalar field theory with a four-point interaction in $D = 4 - 2\epsilon$ dimensions (see the third section of Appendix D):[5]

$$S = \int d^D x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4 \right].$$

Calculating the beta function using dimension regularization yields

$$\beta_\lambda(\lambda) = -2\epsilon\lambda + 3\frac{\lambda^2}{(4\pi)^2} - \frac{17}{3}\frac{\lambda^3}{(4\pi)^4} + o(\lambda^4).$$

This suggests that when $\epsilon \neq 0$, there is a non-trivial fixed point

$$\frac{\lambda}{(4\pi)^2} = \frac{\lambda_*}{(4\pi)^2} = \frac{2}{3}\epsilon + \frac{68}{81}\epsilon^2 + o(\epsilon^3)$$

satisfying $\beta_\lambda(\lambda_*) = 0$. At the fixed point, it is considered that a conformally invariant quantum field theory will be realized.

Anomalous dimensions of the field φ and its normal product $[\varphi^2]$ are calculated as $\gamma = \lambda^2/12(4\pi)^4$ and $\delta = \lambda/(4\pi)^2 - 5\lambda^2/6(4\pi)^4$, respectively. Evaluating their conformal dimensions at the fixed point by taking into account a canonical dimension of the field, we obtain

$$\Delta_\varphi = \frac{D-2}{2} + \frac{1}{12}\frac{\lambda_*^2}{(4\pi)^4} = 1 - \epsilon + \frac{\epsilon^2}{27} + o(\epsilon^3),$$
$$\Delta_{[\varphi^2]} = D - 2 + \frac{\lambda_*}{(4\pi)^2} - \frac{5}{6}\frac{\lambda_*^2}{(4\pi)^4} = 2 - \frac{4}{3}\epsilon + \frac{38}{81}\epsilon^2 + o(\epsilon^3).$$

Since the beta function vanishes, these values become renormalization group invariants, namely physical values. By comparison between OPE algebras,

[4] The original paper is K. Wilson and M. Fisher, *Critical Exponents in 3.99 Dimensions*, Phys. Rev. Lett. **28** (1972) 240. For review see K. Wilson and J. Kogut, *Renormalization Group and ϵ-Expansion*, Phys. Rept. C12 (1974) 75; J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, 2002).

[5] It is common to set the dimension to $D = 4 - \epsilon$ in the ϵ-expansion.

$\sigma \times \sigma \sim \varepsilon$ in the Ising models and $\varphi \times \varphi \sim [\varphi^2]$ in the field, they are identified as $\Delta_\varphi = \Delta_\sigma$ and $\Delta_{[\varphi^2]} = \Delta_\varepsilon$. Actually, if we apply a negative perturbation by $[\varphi^2]$ to the system, the potential becomes a double well type and two minima at $\langle\varphi\rangle \neq 0$ appear. The minimum corresponds to an up or down of the Ising spin and can be regarded as a state where the spins are aligned in either. Therefore, the fixed point is considered to have the same universality as critical phenomena of the Ising models.

When taking $\epsilon \to 1/2$, we obtain $\Delta_\sigma = 0.51$ and $\Delta_\varepsilon = 1.45$ for the critical exponents of the three-dimensional Ising model.[6] These are in good agreement with the results shown in the previous section. Moreover, from (B-3), the critical exponent ν is given by

$$\nu = \frac{1}{D - \Delta_{[\varphi^2]}} = \frac{1}{2} + \frac{1}{6}\epsilon + \frac{14}{81}\epsilon^2 + o(\epsilon^3).$$

In three dimensions, $\nu = 0.63$ is obtained.

[6] The calculation has been made up to $o(\epsilon^5)$, and the result matches better and are given by $\Delta_\sigma = 0.5180$ and $\Delta_\varepsilon = 1.4102$. It is unknown how much the correctness of this method is guaranteed, but there are some achievements to resolve this doubt based on conformal invariance: S. Rychkov and Z. Tan, *The ϵ-Expansion from Conformal Field Theory*, J. Phys. **A48** (2015) 29FT01 and R. Gopakumar, A. Kaviraj, K. Sen, and A. Sinha, *Conformal Bootstrap in Mellin Space*, Phys. Rev. Lett. **118** (2017) 081601.

CHAPTER FOUR

BASIS OF TWO-DIMENSIONAL CONFORMAL FIELD THEORY

The Virasoro algebra and its representation theory are briefly summarized here, which are basic algebraic structures of two-dimensional conformal field theory. In addition, we will describe the Coulomb gas representation of the theory using a free boson field. There are many good textbooks on two-dimensional conformal field theory. For more information, please refer to them.

Virasoro Algebra and Unitary Representations

In two dimensions, there are infinitely many conformal Killing vectors, and the $SO(2,2)$ algebra extends to an infinite-dimensional conformal algebra called the Virasoro algebra. Its generator has left- and right-handed components, where one of them is denoted as L_n, then the algebra is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \qquad \text{(4-1)}$$

The last term is a central extension part of two-dimensional conformal algebra, where c is called the central charge. Hermitian conjugate of the generator is $L_n^\dagger = L_{-n}$. Let the other generator that commutes with L_n be $\tilde{L}_n$, it also satisfies the Virasoro algebra, where we consider the case that it has the same central charge c. For the six $n = 0, \pm 1$ generators, the central extension disappears and they form the subalgebra $SO(2,2) = SL(2, \mathbf{C})$. In the following, we consider only the L_n component when discussing representation theory, but similar for $\tilde{L}_n$.

A conformally invariant vacuum $|\Omega\rangle$ is defined by

$$L_n|\Omega\rangle = 0, \quad n \geq -1.$$

A primary state $|\Delta\rangle$ with conformal dimension Δ is defined by

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0, \quad n \geq 1.$$

Descendant states are generated by applying L_{-n} $(n \geq 1)$ to the primary state $|\Delta\rangle$ as

$$L_{-n_1} \cdots L_{-n_k} |\Delta\rangle, \tag{4-2}$$

which is an eigenstate of L_0. An infinite-dimensional space spanned by the descendant states is called the Verma module, and is written as V_Δ. Using $[L_0, L_{-n}] = nL_{-n}$, we can see that conformal dimension of (4-2) is given by $\Delta + \sum_j n_j$. The positive integer part $\sum_j n_j = N$ is called the level of the state. The number of independent descendant states at level N is then equal to the number $P(N)$ of partitions of the integer N. The generating function of the partition number is given by

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{N=0}^{\infty} P(N) q^N. \tag{4-3}$$

The unitarity requires that inner products of not only the primary state but also its descendant states are non-negative. Therefore, let us consider a descendant $L_{-n}|\Delta\rangle$ of the primary state $|\Delta\rangle$ and calculate its inner product as

$$\begin{aligned} \langle\Delta|L_n L_{-n}|\Delta\rangle &= \langle\Delta|\,[L_n, L_{-n}]\,|\Delta\rangle \\ &= \left[2n\Delta + \frac{c}{12}(n^3 - n)\right] \langle\Delta|\Delta\rangle, \end{aligned}$$

where the inner product of the primary state is assumed to be non-negative. Since it must be non-negative, the following conditions are derived:

$$c \geq 0, \qquad \Delta \geq 0.$$

Actually, substituting $n = 1$, the central charge term is eliminated and $\Delta \geq 0$ is obtained. Considering the case of $\Delta = 0$ and $n = 2$, only the central charge term remains and $c \geq 0$ is yielded. Here note that $L_{-2}|\Omega\rangle$ is a state obtained by applying the energy-momentum tensor to the vacuum.

Degenerate representations Degeneracy of states which is an important property widely known in two-dimensional conformal field theory is described here.[1]

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[1] See V. Kac, *Contravariant Form for Infinite-Dimensional Lie Algebras and Superalgebras*, Lecture Notes in Phys. **94** (1979) 441; B. Feigin and D. Fuks, *Invariant Skew-Symmetric Differential Operators on the Line and Verma Modules over the Virasoro Algebra*, Funct. Anal. Appl. **16** (1982) 114.

First, consider the level 2 descendant state. In general, there are two independent states, which are given by $L_{-2}|\Delta\rangle$ and $L_{-1}^2|\Delta\rangle$. They degenerate in a particular case. To see that, consider their linear combination such as

$$|\chi\rangle = \left(L_{-2} + xL_{-1}^2\right)|\Delta\rangle.$$

Let us examine conditions that this state is to be a primary state satisfying $L_n|\chi\rangle = 0$ $(n \geq 1)$. From the $n = 1$ condition, we get

$$L_1|\chi\rangle = [3 + x(4\Delta + 2)]\, L_{-1}|\Delta\rangle = 0$$

by using $[L_1, L_{-2}] = 3L_{-1}$ and $[L_1, L_{-1}^2] = L_{-1}(4L_0 + 2)$. From $n = 2$, we obtain

$$L_2|\chi\rangle = \left(4\Delta + \frac{c}{2} + 6x\Delta\right)|\Delta\rangle = 0,$$

where $[L_2, L_{-2}] = 4L_0 + c/2$ and $[L_2, L_{-1}^2] = 6L_0 + 6L_{-1}L_1$ are used. The conditions for $n \geq 3$ are trivially satisfied. Solving the conditions above simultaneously, x and Δ are expressed as functions of c, then a new primary state with conformal dimension $\Delta + 2$ can be constructed as follows:

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2\Delta+1)}L_{-1}^2\right)|\Delta\rangle,$$
$$\Delta = \frac{1}{16}\left[5 - c \pm \sqrt{(1-c)(25-c)}\right].$$

However, due to the primary condition $L_n|\chi\rangle = 0$ $(n \geq 1)$, it is obvious that the inner product of this state disappears as

$$\langle\chi|\chi\rangle = \langle\Delta|\left(L_2 - \frac{3}{2(2\Delta+1)}L_1^2\right)|\chi\rangle = 0.$$

Such states whose inner product vanishes are called null states. Since this state is orthogonal to all other states of V_Δ, it can be set as

$$|\chi\rangle = 0.$$

An irreducible representation of the Virasoro algebra obtained by eliminating such null states is called the degenerate representation.

In general, when there is a null descendant at level N, the state is called degenerate at level N. The Kac determinant is known as a formula expressing the existence of such a null descendant state. In the above $N = 2$ case, it is given by the determinant of the following 2×2 matrix:

$$M_2 = \begin{pmatrix} \langle\Delta|L_2L_{-2}|\Delta\rangle & \langle\Delta|L_1^2L_{-2}|\Delta\rangle \\ \langle\Delta|L_2L_{-1}^2|\Delta\rangle & \langle\Delta|L_1^2L_{-1}^2|\Delta\rangle \end{pmatrix} = \begin{pmatrix} 4\Delta + c/2 & 6\Delta \\ 6\Delta & 4\Delta(1+2\Delta) \end{pmatrix}.$$

The condition that degeneracy occurs is that its determinant disappears as

$$\begin{aligned}\det M_2 &= 2(16\Delta^3 - 10\Delta^2 + 2\Delta^2 c + \Delta c)\\ &= 32(\Delta - \Delta_{1,1})(\Delta - \Delta_{1,2})(\Delta - \Delta_{2,1}) = 0,\end{aligned}$$

where $\Delta_{1,1} = 0$ and $\Delta_{1,2}, \Delta_{2,1} = \left[5 - c \mp \sqrt{(1-c)(25-c)}\right]/16$. The condition $\Delta = \Delta_{1,1} = 0$ indicates that the vacuum $|\Omega\rangle$ has a null descendant at level 1 given by $L_{-1}|\Omega\rangle$, and $\Delta = \Delta_{1,2}, \Delta_{2,1}$ indicates that each primary state with these conformal dimensions has a null descendant at level 2, as mentioned above.

The determinant obtained by Kac is that extended to general N, and is given for the $P(N) \times P(N)$ matrix M_N of the Verma module (4-2) as follows:

$$\det M_N = C_N \prod_{nm \le N} (\Delta - \Delta_{n,m})^{P(N-nm)},$$

where C_N is a constant independent of c and Δ, and n, m are positive integers whose product does not exceed N. $\Delta_{n,m}$ is a function of c given by

$$\begin{aligned}\Delta_{n,m} &= \frac{c-1}{24} + \frac{1}{8}\left(n\beta_+ + m\beta_-\right)^2,\\ \beta_\pm &= \frac{1}{\sqrt{12}}\left(\sqrt{1-c} \pm \sqrt{25-c}\right).\end{aligned}$$

This is called the Kac formula.

In the following, we examine conformal field theory of $c \le 1$. The central charge is then expressed as

$$c = 1 - 12Q^2, \tag{4-4}$$

so that

$$\beta_\pm = Q \pm \sqrt{Q^2 + 2}, \tag{4-5}$$

where $\beta_+ + \beta_- = 2Q$ and $\beta_+\beta_- = -2$ hold. Introduce a pair of relatively prime positive integers p', p ($p' > p$) and consider the case that the ratio of β_+ and β_- becomes a rational number as

$$-\frac{\beta_+}{\beta_-} = \frac{p'}{p}. \tag{4-6}$$

Then, $Q^2 = (p' - p)^2/2p'p$, and thus the central charge is expressed by a rational number as

$$c = 1 - \frac{6(p'-p)^2}{p'p},$$

and the Kac formula is given by

$$\Delta_{n,m} = \frac{(np' - mp)^2 - (p'-p)^2}{4p'p}, \tag{4-7}$$

where note that $\Delta_{n,m} = \Delta_{p-n,p'-m}$ holds.

Belavin, Polyakov, and Zamolodchikov showed that operator product expansions (OPE) close between these finite number of primary states when limiting possible ranges of (n, m) to the following:

$$1 \le n \le p - 1, \quad 1 \le m \le p' - 1. \tag{4-8}$$

The series of conformal field theory whose conformal dimensions are given by (4-7) and (4-8) is called the minimal series.

Among the minimal series, especially important one is the unitary discrete series of $p' = p + 1$, which is given by

$$c = 1 - \frac{6}{p(p+1)}, \qquad \Delta_{n,m} = \frac{[n(p+1) - mp]^2 - 1}{4p(p+1)},$$

where $p = 2, 3, \ldots$ and $1 \le m \le n \le p - 1$ can be set because $\Delta_{n,m} = \Delta_{p-n,p+1-m}$ holds. Friedan, Qiu, and Shenker showed that only in this case the minimal series satisfies the unitarity condition, that is, all conformal dimensions are non-negative.

As a concrete example, the $p = 3$ unitary discrete series, which corresponds to the Ising model, is given by

$$c = \frac{1}{2}, \qquad \Delta_{1,1} = 0, \qquad \Delta_{2,1} = \frac{1}{2}, \qquad \Delta_{2,2} = \frac{1}{16}.$$

The conformal dimensions of the energy operator ε and the spin operator σ are given by combining the left and right dimensions as $\Delta_\varepsilon = \Delta_{2,1} + \tilde{\Delta}_{2,1} = 1$ and $\Delta_\sigma = \Delta_{2,2} + \tilde{\Delta}_{2,2} = 1/8$.

The unitary discrete series corresponds one-to-one with the Andrews-Baxter-Forrester (ABF) model which is a series of solvable lattice models including the Ising model.[2] The model is defined by assigning a positive

[2] It is one of integrable models given as a solution of the Yang-Baxter equation. See G. Andrews, R. Baxter, and P. Forrester, *Eight-Vertex SOS Model and Generalized Roger-Ramanujan-Type Identities*, J. Stat. Phys. **35** (1984) 193. For correspondence with the unitary discrete series, see D. Huse, *Exact Exponents for Infinitely Many New Multicritical Points*, Phys. Rev. **B30** (1984) 3908.

integer height variable $l_i = 1, \ldots, p$ to each site i on a square lattice and taking the difference of hight variables between the nearest neighbor sites (i, j) to be $|l_i - l_j| = 1$. The degenerate representation mentioned above corresponds to that the height variable is restricted to a finite range. The number of ground states of this lattice model becomes $p - 1$, and thus the Ising model that has two ground states is given by $p = 3$.

As the number of ground states increases to 3 and 4, the value of p of the corresponding unitary discrete series also increases to 4 and 5. When $p \to \infty$ there is no limit on the possible values at each site and it can be regarded as a free boson field. Such a restricted lattice model is generally called the RSOS (restricted solid on solid) model, while the case where there is no restriction is called the SOS model. The name of SOS comes from the image of stacking crystals on a solid. The six vertex model is a representative of the SOS model, which corresponds to the $c = 1$ conformal field theory.

Virasoro Character and Partition Function on Torus

Let us consider the partition function on a torus which is defined employing a cylindrical space $\mathbb{R} \times S^1$. A conformal transformation from the complex plane $(z, \bar{z})$ to a cylindrical Euclidean space $(w, \bar{w})$ is given by $w = (L/2\pi) \log z$, where L is circumference of S^1. Hamiltonian operator and momentum operator on the cylinder are given by

$$H = \frac{2\pi}{L}\left(L_0 + \tilde{L}_0\right) - \frac{\pi c}{6L}, \qquad P = \frac{2\pi}{L}\left(L_0 - \tilde{L}_0\right). \tag{4-9}$$

The shift term dependent on the central charge in the Hamiltonian operator is a Casimir term generated when the system is transformed from the plane to the cylindrical space. It is originated from the fact that energy-momentum tensors are not primary fields, so that an extra term proportional to the central charge (Schwartz derivative) appears when conformally transformed.

The partition function on the torus defined by the cylinder of the length l is given by

$$Z(l, s) = \mathrm{Tr}\, e^{-lH + isP}.$$

The translation operator e^{isP} for the S^1 direction shows that it is rotated by s when identifying the ends of the cylinder. Letting $\tau = (s + il)/L$, the partition function can be written as

$$Z(\tau) = \mathrm{Tr}\, e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \bar{\tau} (\tilde{L}_0 - c/24)}.$$

The variable τ is called the moduli of the torus. Thus, we can write the partition function on the torus using the Virasoro character $\chi_\Delta = \mathrm{Tr}_{V_\Delta} q^{L_0 - c/24}$, where $q = e^{2\pi i\tau}$

It can be found from (4-3) that the Virasoro character χ_Δ with $c = 1$ and conformal dimension Δ is given by

$$\chi_\Delta(\tau) = \mathrm{Tr}_{V_\Delta} q^{L_0 - c/24} = q^{\Delta - \frac{1}{24}} \sum_{N=0}^{\infty} P(N) q^N = \frac{q^\Delta}{\eta(\tau)},$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind η-function.

The Virasoro character for the irreducible representation of the unitary discrete series with $c = 1 - 6/p(p+1)$ is derived by subtracting null descendant states systematically. The result is given by[3]

$$\begin{aligned}\chi_{n,m}(\tau) &= \mathrm{Tr}_{V_{\Delta_{n,m}}} q^{L_0 - c/24} \\ &= \frac{q^{-c/24}}{\prod_{l=1}^{\infty}(1 - q^l)} \sum_{k \in \mathbf{Z}} \left[q^{\Delta_{n+2pk,m}} - q^{\Delta_{n+2pk,-m}} \right].\end{aligned}$$

The partition function on the torus can be obtained by combining left and right of these finite number of the Virasoro characters so that it becomes a real function. The combination is completely categorized, and the simplest one is given by[4]

$$Z_{\mathrm{uds}}(\tau) = \sum_{1 \le m \le n \le p-1} \left| \chi_{n,m}(\tau) \right|^2 .$$

Here we mention about a maniac relationship between the unitary discrete series and the ABF lattice model, which can be found when the partition function on the torus is deformed as follows. Consider the action of a

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[3] See A. Rocha-Caridi, *Vacuum Vector Representations of Virasoro Algebra*, in Vertex Operators in Mathematics and Physics, MSRI Publications 3 (Springer, 1984).

[4] Strictly, the partition function is given as an invariant function under the modular transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The combination can be classified according to the ADE type Dynkin diagram of Lie algebra, and the simplest one in the text corresponds to the A type. See A. Cappelli, C. Itzykson, and J. Zuber, *Modular Invariant Partition Functions in Two Dimensions*, Nucl. Phys. **B280** 445; A. Kato, *Classification of Modular Invariant Partition Functions in Two-Dimensions*, Mod. Phys. Lett. **A2** (1987) 585.

free boson field on the torus defined by

$$S(g,\tau) = \frac{\pi g}{4}\int_T d^2z\,\partial^\mu\varphi\partial_\mu\varphi,$$

and calculate the partition function with jumps $\delta_1\varphi(z,\bar{z}) = \varphi(z+1,\bar{z}+1) - \varphi(z,\bar{z})$ and $\delta_2\varphi(z,\bar{z}) = \varphi(z+\tau,\bar{z}+\bar{\tau}) - \varphi(z,\bar{z})$ for each cycle of the torus as

$$\begin{aligned} Z_{M,M'}(g,\tau) &= \int_{\substack{\delta_1\varphi = 2M' \\ \delta_2\varphi = 2M}} [d\varphi]\, e^{-S(g,\tau)} \\ &= \frac{1}{|\eta(\tau)|^2}\left(\frac{g}{\mathrm{Im}(\tau)}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi g}{\mathrm{Im}(\tau)}\left|M - M'\tau\right|^2\right). \end{aligned}$$

It is known that when taking $g = p/(p+1)$, the partition function of the unitary discrete series can be expressed as

$$Z_{\mathrm{uds}}(\tau) = \frac{1}{2}\sum_{M,M'\in\mathbf{Z}} Z_{M,M'}\left(g=\frac{p}{p+1},\tau\right)\sum_{j=1}^{p}\cos\left(2\pi\frac{j}{p+1}M\wedge M'\right),$$

where $M\wedge M'$ is a greatest common divisor of M and M' and $M\wedge 0 = M$. With $g = 1-\lambda/\pi$, this partition function corresponds to the ABF model of $\lambda = \pi/(p+1)$.[5] The last factor represents degeneracy or restriction in the solvable model. The partition function of $g=1$ in the absence of this factor corresponds to the six vertex model.

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[5] The Boltzmann weight of the ABF model generally given by elliptic functions reduces to trigonometric functions just at critical points. The square lattice assigned the trigonometric Boltzmann weights is tilted obliquely at 45 degrees to construct the partition function on the torus, and when lattice sites of the same height are connected by the vertical and horizontal lines, it can be rewritten to the sum of clusters of zigzag lines with branching. Furthermore, consider loops surrounding the zigzag lines, in which there are contractible and non-contractible loops. When crossing the loop, the height differs by one. Regardless of its length and its inner and outer heights, a weight $2\cos\lambda$ is assigned for each contractible loop. Each cluster can be thus expressed as $(2\cos\lambda)^n$, where n is the number of contractible loops. On the other hand, if there is a non-contractible loop that wraps around the cycle, the height increases or decreases by one when crossing it. Since the number of the non-contractible loops that appear is always even, the increase and decrease for each cycle is an even number, which is set to $2M$ and $2M'$. Letting $Z_{M,M'}$ be the sum of all configurations with such jumps, the partition function of the ABF model is given by performing the sum of M, M' with adding the factor $\sum_{j=1}^{p}\cos(2\pi jM\wedge M'/(p+1))$, while the six-vertex model is the one performing the sum without this factor. In this way, the same structure as the one of the partition function derived from conformal field theory appears. See V. Pasquier, *Lattice Derivation of Modular Invariant Partition Functions on the Torus*, J. Phys. **A20** (1987) L217; O. Foda and B. Nienhuis, *The Coulomb Gas Representation of Critical RSOS Models on the Sphere and the Torus*, Nucl.Phys. **B324** (1989) 643.

Free Boson Representation

A free boson representation of two-dimensional conformal field theory, also called the Coulomb gas representation, is briefly described here. The action is given by[6]

$$S_{\mathrm{CG}} = -\frac{1}{4\pi}\int d^2x\sqrt{-g}\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + iQR\varphi\right).$$

As a background metric $g_{\mu\nu}$, we employ a cylindrical Minkowski space $\mathbb{R}\times S^1$ with coordinates $x^\mu = (\eta, \sigma)$, where $0 < \sigma < 2\pi$. In the two-dimensional cylindrical space, the metric is given by the flat $g_{\mu\nu} = (-1, 1)$.

From the equation of motion $\partial^2\varphi = (-\partial_\eta^2 + \partial_\sigma^2)\varphi = 0$, the field is expanded in right- and left-handed oscillation modes and zero-modes as

$$\varphi(\eta,\sigma) = \hat{q} + 2\eta\hat{p} + \sum_{n\neq 0}\frac{i}{n}\left(\alpha_n e^{-in(\eta+\sigma)} + \tilde{\alpha}_n e^{-in(\eta-\sigma)}\right), \quad \text{(4-10)}$$

and its conjugate momentum is given by $\Pi = \partial_\eta\varphi/4\pi$. From the equal-time commutation relation $[\varphi(\eta,\sigma), \Pi(\eta,\sigma')] = i\delta(\sigma-\sigma')$ and an expression of the delta function $\delta(\sigma-\sigma') = \sum_{n\in\mathbf{Z}} e^{-in(\sigma-\sigma')}/2\pi$, the commutation relation of each mode is given by

$$[\hat{q},\hat{p}] = i, \qquad [\alpha_n,\alpha_m] = n\delta_{n+m,0}, \qquad [\tilde{\alpha}_n,\tilde{\alpha}_m] = n\delta_{n+m,0},$$

and $[\alpha_n, \tilde{\alpha}_m] = 0$. Since Hermiticity of the field is $\varphi^\dagger = -\varphi$ from reality of the action, it is $\hat{q}^\dagger = -\hat{q}$, $\hat{p}^\dagger = -\hat{p}$, $\alpha_n^\dagger = -\alpha_{-n}$, $\tilde{\alpha}_n^\dagger = -\tilde{\alpha}_{-n}$ for each mode. Therefore, creation modes are here expressed by negative frequency modes instead of using the symbol †.

Using the conformal Killing vector ζ^μ that satisfies the two-dimensional conformal Killing equation

$$\partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu - \eta_{\mu\nu}\partial^\lambda\zeta_\lambda = 0,$$

the generator of conformal transformation is given by

$$L_\zeta = \int_{S^1} d\sigma\,\zeta^\mu\,\Theta_{\mu 0}. \quad \text{(4-11)}$$

[6] This is an unphysical action. Actually, when written in a real function, it corresponds to the Liouville action S_{L} (5-8) introduced in the next chapter with the coefficient b_{L} of a wrong negative sign. By converting the Liouville field to $\phi \to \varphi/\sqrt{2b_{\mathrm{L}}}$ and putting $b_{\mathrm{L}} = -b$, we can obtain this action with a right kinetic term and $Q = \sqrt{b/2}$, but the imaginary unit appears. Therefore, there is no physical meaning in the boson field φ itself, and this system generally gives a non-unitary conformal field theory. It becomes unitary under the special conditions mentioned earlier.

The energy-momentum tensor is defined using a variation formula as

$$\Theta^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{CG}}}{\delta g_{\mu\nu}},$$

then taking the flat metric, we obtain

$$\Theta_{\mu\nu} = \frac{1}{4\pi} \left\{ \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu} \partial^\lambda \varphi \partial_\lambda \varphi + i2Q \left(\eta_{\mu\nu} \partial^\lambda \partial_\lambda - \partial_\mu \partial_\nu \right) \varphi \right\}.$$

The first two terms correspond to the normal energy-momentum tensor for a two-dimensional free scalar field, while the last term is derived from a variation of the $R\varphi$ term. The trace disappears according to the equation of motion of the field. Using the conformal Killing equation and the conservation equation of the energy-momentum tensor, we find that L_ζ is conserved because time derivative of the generator vanishes in proportion to the trace of the energy-momentum tensor.

In two dimensions, there are an infinite number of the conformal Killing vectors, which are expressed by arbitrary functions $\zeta^\mu = \sum_{n \in \mathbf{Z}} a_n \zeta_n^\mu$ and $\tilde{\zeta}^\mu = \sum_{n \in \mathbf{Z}} \tilde{a}_n \tilde{\zeta}_n^\mu$ with the basis

$$\zeta_n^\mu = \left(\frac{1}{2} e^{in(\eta+\sigma)}, \frac{1}{2} e^{in(\eta+\sigma)} \right), \quad \tilde{\zeta}_n^\mu = \left(\frac{1}{2} e^{in(\eta-\sigma)}, -\frac{1}{2} e^{in(\eta-\sigma)} \right), \tag{4-12}$$

where n is an integer. Substituting ζ_n^μ and $\tilde{\zeta}_n^\mu$ into the generator (4-11) and writing them L_n and $\tilde{L}_n$, respectively, the former is given by[7]

$$\begin{aligned} L_n &= e^{in\eta} \int_0^{2\pi} d\sigma \, e^{in\sigma} \frac{1}{2} : \left(\Theta_{00} + \Theta_{01} \right) : - \frac{Q^2}{2} \delta_{n,0} \\ &= \frac{1}{2} \sum_{m \in \mathbf{Z}} : \alpha_m^\pm \alpha_{n-m}^\pm : - Qn\alpha_n - \frac{Q^2}{2} \delta_{n,0}, \end{aligned} \tag{4-13}$$

where $\alpha_0 = \hat{p}$ $(= \tilde{a}_0)$ and the symbol : : represents normal ordering of free fields. Hermiticity is given by $L_n^\dagger = L_{-n}$. Likewise, $\tilde{L}_n$ is given by

[7] In two dimensions, conformal field theory is often formulated in Euclidean complex plane. If the coordinates are taken to be z and its complex conjugate $\bar{z}$, the conformal Killing vectors are given by analytic functions $\zeta(z)$ and $\tilde{\zeta}(\bar{z})$. Considering the z component, the field is expanded as $\varphi(z) = \hat{q} - i\hat{p} \log z + i \sum_{n \neq 0} \alpha_n z^{-n}/n$. The energy-momentum tensor is then given by $T(z) = -(1/2) :\partial_z \varphi \partial_z \varphi: + iQ\partial_z^2 \varphi = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}$. Writing $\alpha_0 = \hat{p}$, the Virasoro generator is $L_n = (1/2) \sum_{m \in \mathbf{Z}} : \alpha_m \alpha_{n-m} : - Q(n+1)\alpha_n$, where the constant term disappears. Hermiticity is given by $\alpha_n^\dagger = -\alpha_{-n}$ $(n \geq 1)$ and $\alpha_0^\dagger = -\alpha_0 + 2Q$. This means that a background charge $2Q$ concentrates on a conformally invariant out-vacuum at $|z| \to \infty$, while it does not exist in an in-vacuum defined at the origin.

replacing α_n in the above expression with $\tilde{\alpha}_n$. The last term $-(Q^2/2)\delta_{n,0}$ of the Virasoro generator is added here so as to satisfy the Virasoro algebra (4-1) with the central charge (4-4), which corresponds to the Casimir effect by choosing the coordinates to $\mathbb{R} \times S^1$.

Further describe this Casimir effect in more detail. As mentioned in the previous section, the Hamiltonian operator on the cylindrical space is given by $H = L_0 + \tilde{L}_0 - c/12$. On the other hand, when calculating the Hamiltonian operator in the free boson representation using a famous formula

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

as a regularization of the zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, we get

$$H = \int_0^{2\pi} d\sigma \, \Theta_{00} = L'_0 + \tilde{L}'_0 + \sum_{n=1}^{\infty} n = L'_0 + \tilde{L}'_0 - \frac{1}{12},$$

where Θ_{00} is the energy-momentum tensor without normal ordering, while L'_0 is the normal ordered part of (4-13) with the last constant term removed, and also $\tilde{L}'_0$. The shift term $-1/12$ that comes out when taking normal ordering reflects that the central charge of a free scalar boson is unity. Since the two Hamiltonian operators are the same, it is understood that the Virasoro generator satisfying (4-1) is given by $L_0 = L'_0 + (c-1)/24 = L'_0 - Q^2/2$. The same is true for $\tilde{L}_0$.

The conformally invariant vacuum is defined as a state that satisfies $L_n|\Omega\rangle = \tilde{L}_n|\Omega\rangle = 0$ $(n \geq -1)$. Since the zero-mode component of L_0 is $(1/2)(\hat{p}^2 - Q^2)$, it is given by

$$|\Omega\rangle = e^{-iQ\varphi(0)}|0\rangle = e^{-iQ\hat{q}}|0\rangle,$$

where $|0\rangle$ is not a conformally invariant vacuum, but represents the Fock vacuum which disappears for $\hat{p}$ and annihilation modes. Exponents of exponential factors are generally called charge, especially what the vacuum has is called background charge, which is $-Q$ here.[8] Noting that $\varphi^\dagger = -\varphi$, we obtain $\langle\Omega| = \langle 0|e^{-iQ\varphi(0)}$, thus the out-vacuum also has a background

[8] The background charge originates from the $R\varphi$ term in the action. It can be easily understood when considering the Euclidean path integral as follows. Extracting the part related to the zero-mode $\varphi_0 = \hat{q}$ yields the path integral weight $e^{-iQ\chi\varphi_0}$, where χ is the Euler characteristic. Since topology of states is represented by a disk of $\chi = 1$ (gluing together two disks yields inner products), we can find that an extra charge $-Q$ by the factor $e^{-iQ\varphi_0}$ is added for each state.

charge $-Q$. Consequently, the conformally invariant vacua totally have a background charge $-2Q$.

Considering a state $|\beta\rangle = e^{i\beta\varphi(0)}|\Omega\rangle$ with a charge β in addition to the background charge, it satisfies the conditions $L_n|\beta\rangle = 0$ $(n \geq 1)$ and

$$L_0|\beta\rangle = \Delta_\beta|\beta\rangle, \qquad \Delta_\beta = \frac{1}{2}\beta^2 - Q\beta.$$

Therefore, $|\beta\rangle$ is a primary state of conformal dimension Δ_β. A similar expression holds for $\tilde{L}_n$, and $\tilde{\Delta}_\beta = \Delta_\beta$ is provided. From a duality relation $\Delta_\beta = \Delta_{2Q-\beta}$, we can see that $|2Q-\beta\rangle$ is also a primary state that has the same conformal dimension. In the free boson representation, these states with the same conformal dimension are identified because the field itself does not have a physical meaning (see Footnote 6).

The primary field is given by an exponential operator as

$$V_\beta(\eta,\sigma) = :e^{i\beta\varphi(\eta,\sigma)}: = e^{i\beta\varphi_>(\eta,\sigma)}e^{i\beta\varphi_0(\eta)}e^{i\beta\varphi_<(\eta,\sigma)},$$

where $\varphi_>$ and $\varphi_<$ represent creation $(n > 0)$ and annihilation $(n < 0)$ parts of the oscillation modes α_{-n} and $\tilde{\alpha}_{-n}$ in the field expansion (4-10), respectively. The zero-mode part can be written as[9]

$$e^{i\beta\varphi_0(\eta)} = e^{i\beta\hat{q}/2}e^{2i\beta\eta\hat{p}}e^{i\beta\hat{q}/2}$$

Applying the Virasoro generator to this operator, we obtain

$$\begin{aligned}
\left[L_n, V_\beta(\eta,\sigma)\right] &= e^{in(\eta+\sigma)}\left(-i\partial_+ + \frac{n}{2}d_\beta\right)V_\beta(\eta,\sigma), \\
\left[\tilde{L}_n, V_\beta(\eta,\sigma)\right] &= e^{in(\eta-\sigma)}\left(-i\partial_- + \frac{n}{2}d_\beta\right)V_\beta(\eta,\sigma),
\end{aligned}$$

where $\partial_\pm = (\partial_\eta \pm \partial_\sigma)/2$ and d_β is the sum of left and right conformal dimensions given by

$$d_\beta = 2\Delta_\beta = \beta^2 - 2Q\beta.$$

Using the generator L_ζ (4-11) defined by the conformal Killing vector ζ^μ, the transformation law can be expressed as

$$i\left[L_\zeta, V_\beta\right] = \zeta^\mu\partial_\mu V_\beta + \frac{d_\beta}{2}\partial_\mu\zeta^\mu V_\beta.$$

[9] The Baker-Campbell-Hausdorff formula is

$$e^A e^B = \exp\left\{A + B + \frac{1}{2}[A,B] + \frac{1}{12}\Big([A,[A,B]] + [B,[B,A]]\Big) + \cdots\right\}.$$

The formula for the case that $[A,B]$ is a constant is used here.

In addition, a correspondence between the field operator V_β and the state $|\beta\rangle$ is given by

$$|\beta\rangle = \lim_{\eta \to i\infty} e^{-id_\beta \eta} V_\beta(\eta, \sigma)|\Omega\rangle.$$

In particular, the operators with $d_\beta = 2$ are called screening operators. Using (4-5), they are expressed as

$$V_\pm = :e^{i\beta_\pm \varphi}: .$$

Volume integrals of these operators both commute with the Virasoro generator L_n as

$$\left[L_n, \int d^2x V_\pm(x)\right] = -i \int d^2x\, \partial_+ \left(e^{in(\eta+\sigma)} V_\pm(x)\right) = 0.$$

Likewise, they commute with $\tilde{L}_n$.

Let us consider the minimal series defined by (4-6), then the charge β whose conformal dimension Δ_β is the Kac formula (4-7) is given by

$$\beta_{n,m} = Q - \frac{1}{2}(n\beta_+ + m\beta_-) = \frac{1}{2}\left[(1-n)\beta_+ + (1-m)\beta_-\right].$$

The conformally invariant vacua have the background charge $-2Q$ in total. From this fact, a two-point correlation function $\langle V_{2Q-\beta_{n,m}}(x) V_{\beta_{n,m}}(y)\rangle$ has a value because charges are canceled, or conserved. In general, such a charge conservation does not hold in arbitrary correlation functions. Therefore, we consider a system adding the screening operators as potential terms, which is described by the action

$$S_{\mathrm{CFT}} = S_{\mathrm{CG}} - \int d^2x\, V_+ - \int d^2x\, V_-.$$

A correlation function defined by using this action has the following structure:

$$\begin{aligned}&\langle V_{\beta_1}(x_1) \cdots V_{\beta_k}(x_k)\rangle \\ &\sim \frac{1}{n!m!} \left\langle V_{\beta_1}(x_1) \cdots V_{\beta_k}(x_k) \left(\int d^2x\, V_+\right)^n \left(\int d^2x\, V_-\right)^m \right\rangle_0,\end{aligned}$$

where $\beta_j = \beta_{n_j,m_j}$ and $\langle\cdots\rangle_0$ denotes a correlation function in the free field representation without the potential terms. The charge conservation is then given by

$$\sum_{j=1}^{k} \beta_j + n\beta_+ + m\beta_- = 2Q.$$

The correlation function has a value only when this equation holds. Paying attention to $Q = (\beta_+ + \beta_-)/2$, this condition can be expressed as

$$\frac{1}{2}\sum_{j=1}^{k}(1-n_j)+n=1, \qquad \frac{1}{2}\sum_{j=1}^{k}(1-m_j)+m=1.$$

If n and m are given by non-negative integers at this time, the calculation becomes easy. However, in general, they are given in rational numbers. In this case, in order to make them non-negative integers, the charge of one or more operators should be replaced as $\beta_j \to 2Q - \beta_j$.[10]

[10] See V. Dotsenko and V. Fateev, *Conformal Algebra and Multipoint Correlation Functions in 2D Statistical Models*, Nucl. Phys. **B240**[FS12] (1984) 312.

Chapter Five

Conformal Anomaly and Wess-Zumino Action

The Wess-Zumino action that plays a central role when we construct the background-free quantum gravity is summarized here. That is an action induced in accompany with conformal anomaly. In two-dimensional quantum gravity, it is what is called the Liouville action, or the Polyakov action. Its four-dimensional version is the Riegert action that plays an important role in this book. These actions are necessary to ensure quantum diffeomorphism invariance, namely background-metric independence, which will be mentioned in the following chapters in detail.

Wess-Zumino Integrability Condition

Conformal anomaly means that even if a classical action has conformal invariance, the invariance breaks down by quantum effects. Then, the trace of the energy-momentum tensor that vanishes classically becomes non-zero at the quantum level. In general, it arises in even dimensions in which there are dimensionless gravitational actions.

Let us consider an effective action Γ and examine $\delta_\omega \Gamma$ obtained by applying a conformal variation defined by $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$. If the effective action is derived from a theory with classical conformal invariance, its conformal variation is conformal anomaly as the name suggests. However, such a classical conformal invariance is not essential in the following discussion. Note that what we shall take care of most is diffeomorphism invariance. Conformal anomaly inevitably appears as a consequence of this invariance.

In two dimensions, we generally obtain the following formula:

$$\delta_\omega \Gamma = -\frac{b_{\rm L}}{4\pi} \int d^2x \sqrt{-g}\, \omega R. \tag{5-1}$$

The scalar curvature is the only possible gravitational term in two dimensions when there is no mass scale. When the coefficient is written by $b_{\rm L} = -c/6$, the constant c becomes the central charge of the Virasoro algebra. On the other hand, in four dimensions, there are generally four possible gravitational terms such as

$$\delta_\omega \Gamma = \frac{1}{(4\pi)^2} \int d^4x \sqrt{-g}\, \omega \left\{ \eta_1 R^2_{\mu\nu\lambda\sigma} + \eta_2 R^2_{\mu\nu} + \eta_3 R^2 + \eta_4 \nabla^2 R \right\}. \quad (5\text{-}2)$$

The form of conformal anomalies is constrained by the so-called Wess-Zumino integrability condition in which the effective action satisfies

$$[\delta_{\omega_1}, \delta_{\omega_2}]\, \Gamma = 0.$$

In two dimensions, this condition trivially holds. On the other hand, in four dimensions, with applying a conformal variation to (5-2) again, the following non-trivial condition is obtained:

$$\begin{aligned} [\delta_{\omega_1}, \delta_{\omega_2}]\, \Gamma = {} & \frac{4}{(4\pi)^2} (\eta_1 + \eta_2 + 3\eta_3) \\ & \times \int d^4x \sqrt{-g}\, R \left(\omega_1 \nabla^2 \omega_2 - \omega_2 \nabla^2 \omega_1 \right) = 0, \end{aligned} \quad (5\text{-}3)$$

thus the coefficients must satisfy $\eta_1 + \eta_2 + 3\eta_3 = 0$.[1] The combinations satisfying this condition are given by the square of the Weyl tensor

$$F_4 = C^2_{\mu\nu\lambda\sigma} = R^2_{\mu\nu\lambda\sigma} - 2R^2_{\mu\nu} + \frac{1}{3} R^2 \quad (5\text{-}4)$$

and the Euler density (Gauss-Bonnet combination)

$$G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2, \quad (5\text{-}5)$$

while $\nabla^2 R$ satisfies the condition trivially. Therefore, the conformal anomaly is classified in the following form:

$$\delta_\omega \Gamma = \frac{1}{(4\pi)^2} \int d^4x \sqrt{-g}\, \omega \left\{ \zeta_1 C^2_{\mu\nu\lambda\sigma} + \zeta_2 G_4 + \zeta_3 \nabla^2 R \right\}. \quad (5\text{-}6)$$

Note that the pure R^2 term is excluded due to the integrability condition. In the following, we will obtain the effective action Γ inversely by integrating the right-hand side of (5-6) with respect to the conformal factor.

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[1] As a quantity in the right-hand side of (5-2), we can treat $F^2_{\mu\nu}$ as well. Furthermore, gravitational quantities with a mass coefficient such as R and 1 (cosmological term) can be also considered, but they all trivially satisfy the integrability condition.

Liouville and Riegert Actions

Decompose the metric tensor field into the product of the conformal factor and others as

$$g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}, \tag{5-7}$$

and consider to carry out functional integrations with respect to the conformal-factor field ϕ. In the below, gravitational quantities with the bar are defined by using the metric tensor $\bar{g}_{\mu\nu}$.

First, consider the two-dimensional case. The scalar curvature can be decomposed as

$$\sqrt{-g}R = \sqrt{-\bar{g}}(2\bar{\Delta}_2\phi + \bar{R}),$$

where $\Delta_2 = -\nabla^2$. The differential operator $\sqrt{-g}\Delta_2$ acts conformally for a scalar A as $\sqrt{-g}\Delta_2 A = \sqrt{-\bar{g}}\bar{\Delta}_2 A$ in two dimensions. Using the above formula, we can easily carry out the integration of the scalar curvature as

$$\begin{aligned} S_{\rm L}(\phi, \bar{g}) &= -\frac{b_{\rm L}}{4\pi}\int d^2x \int_0^\phi d\phi \sqrt{-g}\, R \\ &= -\frac{b_{\rm L}}{4\pi}\int d^2x \sqrt{-\bar{g}}\left(\phi\bar{\Delta}_2\phi + \bar{R}\phi\right). \end{aligned} \tag{5-8}$$

This action is called the Liouville action. When writing $b_{\rm L} = -c/6$, for example, $c = 1$ is derived for a free scalar field in curved space (see the fourth section of Appendix D).

The reason for using the symbol S instead of Γ for the integrated quantity is because S is not diffeomorphism invariant, and to make it invariant, we need to further add a non-local term that does not depend on ϕ, as discussed later.

In four dimensions, there are three integrable quantities. Since the first term given by the square of the Weyl tensor does not depend on the conformal-factor field, we can easily integrate it and obtain the following form of the effective action:

$$\zeta_1 \int d^4x \int_0^\phi d\phi \sqrt{-g}\, C^2_{\mu\nu\lambda} = \zeta_1 \int d^4x \sqrt{-\bar{g}}\, \phi\, \bar{C}^2_{\mu\nu\lambda\sigma}.$$

Next, consider the integration of the Euler density $\sqrt{-g}G_4$. Although this quantity can be functionally integrable as it is, we here consider the combination adding a total-divergence term to it as

$$E_4 = G_4 - \frac{2}{3}\nabla^2 R. \tag{5-9}$$

This modified Euler density satisfies a decomposition formula similar to that of the two-dimensional scalar curvature as follows:

$$\sqrt{-g}E_4 = \sqrt{-\bar{g}}(4\bar{\Delta}_4\phi + \bar{E}_4),$$

where $\sqrt{-g}\Delta_4$ is a conformally invariant differential operator for a scalar satisfying $\sqrt{-g}\Delta_4 A = \sqrt{-\bar{g}}\bar{\Delta}_4 A$, which is defined by

$$\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\nabla^2 + \frac{1}{3}\nabla^\mu R\nabla_\mu. \tag{5-10}$$

This differential operator satisfies a self-adjoint condition $\int d^4x\sqrt{-g}\, A\Delta_4 B = \int d^4x\sqrt{-g}\,(\Delta_4 A)B$. Using the above formula, we can easily integrate $\sqrt{-g}E_4$ as

$$\begin{aligned} S_{\mathrm{R}}(\phi,\bar{g}) &= -\frac{b_c}{(4\pi)^2}\int d^4x\int_0^\phi d\phi\sqrt{-g}E_4 \\ &= -\frac{b_c}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\left(2\phi\bar{\Delta}_4\phi + \bar{E}_4\phi\right). \end{aligned} \tag{5-11}$$

For later convenience, we here change the sign of the coefficient as $\zeta_2 = -b_c$. The action (5-11) is called the Riegert action.

Although the Riegert action was given from the analogy with the Liouville action without offering reasons here, we will reveal in the following chapters that this action is an indispensable element to construct quantum algebra of diffeomorphism invariance together with the Weyl action. The linear term of the conformal-factor field ϕ plays an essential role to generate diffeomorphism. This is true for the Liouville action.

Lastly, we consider the last derivative term in (5-6) alone. The functional integration of the pure $\nabla^2 R$ term yields R^2. It corresponds to the inverse transformation of the four-dimensional variational formula

$$\delta_\omega\sqrt{-g}R^2 = -12\sqrt{-g}R\nabla^2\omega.$$

This indicates that local R^2 effective actions as well as local R^2 ultraviolet divergences can exist.

Here note that there is no non-local effective action that produces a pure R^2 term when the conformal variation is done because it is not integrable as discussed in previous section. On the other hand, as mentioned a bit before, there are non-local terms in the effective action associated with the F_4 and E_4 terms.

In this chapter, we considered the three terms of F_4, E_4, and $\nabla^2 R$ as a base of conformal anomalies, and obtained the effective action by integrating these terms with respect to the conformal-factor field. In these three, the last $\nabla^2 R$ term is the quantity often taken up as a problem of arbitrariness in conformal anomalies. However, if it is treated properly, it should be determined at the quantum level. Indeed, when we consider QED in curved space and determine the form of conformal anomalies using a certain renormalization group equation in Chapter 9, we can show that only the first two combinations will appear.[2]

The overall coefficients ζ_1 and b_c $(=-\zeta_2)$ are determined by calculating one-loop corrections in curved space. For instance, when considering N_S scalar fields, N_F Dirac fermions, N_A gauge fields as free fields coupled conformally with gravity, we obtain

$$\begin{aligned} \zeta_1 &= \frac{1}{120}\left(N_S + 6N_F + 12N_A\right), \\ b_c &= \frac{1}{360}\left(N_S + 11N_F + 62N_A\right). \end{aligned} \tag{5-12}$$

In quantum gravity, quantum corrections by the gravitational field itself are added to them.

Diffeomorphism Invariant Effective Actions

Let us examine the integrability condition in more detail. Since the action I for a conformally coupled field in curved space does not depend on the conformal-factor field ϕ, the relation $I(f,g) = I(f,\bar{g})$ holds, though depending on the type of the field and the spacetime dimension, it is necessary to rescale the field appropriately to exclude the ϕ-dependence. The dependence of ϕ comes from the path integral measure. That is, considering a Jacobian arising when we rewrite the diffeomorphism invariant measure defined on the full metric $g_{\mu\nu}$ to the measure defined on the metric $\bar{g}_{\mu\nu}$ such as $[df]_g = [df]_{\bar{g}}\, e^{iS(\phi,\bar{g})}$, the effective action can be written in the form

$$\begin{aligned} e^{i\Gamma(g)} &= \int [df]_g\, e^{iI(f,g)} \\ &= e^{iS(\phi,\bar{g})}\int [df]_{\bar{g}}\, e^{iI(f,\bar{g})} = e^{iS(\phi,\bar{g})} e^{i\Gamma(\bar{g})}. \end{aligned} \tag{5-13}$$

[2] We do not adopt the combination $F_4 + 2\nabla^2 R/3$ that appears in the original paper by Duff in Bibliography because it does not match with dimensional regularization at higher loops (see Footnote 13 in Chapter 10).

If we apply a simultaneous shift transformation

$$\phi \to \phi - \omega, \qquad \bar{g}_{\mu\nu} \to e^{2\omega} \bar{g}_{\mu\nu} \tag{5-14}$$

that preserves the full metric $g_{\mu\nu}$ to equation (5-13), the left-hand side is trivially invariant, while the right-hand side changes to

$$e^{iS(\phi-\omega, e^{2\omega}\bar{g})} e^{i\Gamma(e^{2\omega}\bar{g})} = e^{iS(\phi-\omega, e^{2\omega}\bar{g})} e^{iS(\omega,\bar{g})} e^{i\Gamma(\bar{g})}.$$

In order for this to return to the original $e^{i\Gamma(g)}$, the induced action S must satisfy

$$S(\phi - \omega, e^{2\omega}\bar{g}) + S(\omega, \bar{g}) = S(\phi, \bar{g}). \tag{5-15}$$

This relation expresses the Wess-Zumino integrability condition in another form. The fact that $S_{\rm L}$ (5-8) and $S_{\rm R}$ (5-11) satisfy (5-15) is obvious from the definitions when the interval of integration $[0, \phi]$ is decomposed into $[0, \omega]$ and $[\omega, \phi]$. The invariance under the simultaneous shift transformation (5-14) represents diffeomorphism invariance, hence the Liouville and Riegert actions appear exactly to ensure it.

The Liouville action $S_{\rm L}$ and the Riegert action $S_{\rm R}$ themselves are not diffeomorphism invariant, respectively. For each action, by adding a non-local term that does not depend on the ϕ field, we can obtain a diffeomorphism invariant effective action. By adding the part corresponding to $\Gamma(\bar{g})$ in (5-13), it is expressed as

$$\Gamma_{\rm L,R}(g) = S_{\rm L,R}(\phi, \bar{g}) + \Gamma_{\rm L,R}(\bar{g}),$$

and we find that each is given by

$$\begin{aligned} \Gamma_{\rm L}(g) &= -\frac{b_{\rm L}}{16\pi} \int d^2x \sqrt{-g}\, R \frac{1}{\Delta_2} R, \\ \Gamma_{\rm R}(g) &= -\frac{b_c}{8(4\pi)^2} \int d^4x \sqrt{-g}\, E_4 \frac{1}{\Delta_4} E_4, \end{aligned} \tag{5-16}$$

where $\Delta_4^{-1} E_4(x) \equiv \int d^4y \sqrt{-g}\, G(x,y) E_4(y)$ and $\Delta_4 G(x,y) = \delta^4(x - y)/\sqrt{-g}$, and a similar equation is also given in the case of two dimensions. The functions $\Gamma_{\rm L,R}(\bar{g})$ are given by these expressions defined on the metric $\bar{g}_{\mu\nu}$.

The effective action obtained by integrating the Weyl tensor squared is not diffeomorphism invariant as well. It will become a diffeomorphism invariant form by incorporating the ϕ field into a physical momentum that appears in running coupling constants. The details will be discussed in Chapter 10. Anyway, conformal anomalies are physical quantities that appear to ensure diffeomorphism invariance, unlike gauge anomalies.

Finally, we briefly mention higher order effective actions obtained by integrating the Wess-Zumino action further. They are given by

$$S_{\mathrm{F}}^{(n)} = \frac{1}{n!}\int d^4x\sqrt{-\bar{g}}\,\phi^n \bar{C}^2_{\mu\nu\lambda\sigma},$$
$$S_{\mathrm{G}}^{(n)} = \frac{1}{n!}\int d^4x\sqrt{-\bar{g}}\left\{2\phi^n\bar{\Delta}_4\phi + \bar{E}_4\phi^n\right\}$$

for the Weyl term and the modified Euler term, respectively. The $n = 1$ case is the effective action discussed above.[3] These satisfy the relations

$$\int d^4x \frac{\delta}{\delta\phi(x)} S_{\mathrm{F,G}}^{(n)} = S_{\mathrm{F,G}}^{(n-1)},$$

where we use the self-adjointness of the differential operator $\bar{\Delta}_4$ and the fact $\int d^4x\sqrt{-\bar{g}}\,\bar{\Delta}_4 A = 0$ derived from it. The effective action of $n \geq 2$ will appear in higher loop calculations of the renormalizable quantum gravity discussed in Chapter 10.

Toward BRST Conformal Symmetry

In this chapter, we discussed conformal anomalies of quantum field theory in curved space. In the following chapters, we will explain the BRST conformal symmetry as a main subject. The important thing to note in that case is a peculiar role of conformal anomalies.

As we have seen here, even when considering a field with a conformally invariant kinetic term, if we quantize it, conformal anomalies will emerge and its invariance is necessarily broken. However, if we quantize the gravitational field in the diffeomorphism invariant manner by incorporating the Wess-Zumino action for the Euler density conformal anomaly, we can show that conformal invariance is fully recovered as a gauge symmetry in a certain limit. This is the BRST conformal invariance. And then, it turns out that the kinetic terms of classical actions for mater fields and the gravitational field have to be conformal invariant after all as a condition for explicitly constructing generators of the BRST conformal algebra that close at the quantum level.

In the next chapter, as an exercise, we will explain two-dimensional quantum gravity by employing $\mathbb{R} \times S^1$ as a background spacetime. Of

[3] The expressions other than $n = 1$ do not satisfy the Wess-Zumino integrability condition (5-15), because the integrands used here are not diffeomorphism invariant.

course, it does not depend on how to choose the background, but by choosing such a cylindrical background, we will be able to describe physical states explicitly.

In Chapters 7 and 8, we will formulate the four-dimensional background-free quantum gravity by employing Minkowski background M^4 and a cylindrical background $\mathbb{R} \times S^3$, respectively. At a glance, since the number of conformal Killing vectors that is infinite in two dimensions decreases to finite 15 in four dimensions, it appears that the condition of the conformal symmetry becomes considerably weak compared with the two-dimensional case. However, since the isometry group of space is extended from the abelian group of $SO(2)$ to the non-Abelian group of $SO(4)$, this brings powerful restrictions on physical states.

Chapter Six

Two-Dimensional Quantum Gravity

Before discussing quantization of gravity in four dimensions, we briefly describe two-dimensional quantum gravity[1] whose exact solution is known and its properties are well studied. The basis of the BRST conformal symmetry is contained in this theory.

Quantization of Liouville Action

Let us decompose the gravitational field $g_{\mu\nu}$ into the conformal factor $e^{2\phi}$ and the metric $\bar{g}_{\mu\nu}$ as in (5-7). Diffeomorphism $\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu\xi^\lambda + g_{\nu\lambda}\nabla_\mu\xi^\lambda$ is then decomposed as

$$\begin{aligned} \delta_\xi\phi &= \xi^\lambda\partial_\lambda\phi + \frac{1}{2}\bar{\nabla}_\lambda\xi^\lambda, \\ \delta_\xi\bar{g}_{\mu\nu} &= \bar{g}_{\mu\lambda}\bar{\nabla}_\nu\xi^\lambda + \bar{g}_{\nu\lambda}\bar{\nabla}_\mu\xi^\lambda - \bar{g}_{\mu\nu}\bar{\nabla}_\lambda\xi^\lambda. \end{aligned} \tag{6-1}$$

The metric field with the bar can be further decomposed as $\bar{g}_{\mu\nu} = (\hat{g}e^h)_{\mu\nu}$ by introducing a non-dynamical background metric $\hat{g}_{\mu\nu}$ and the traceless tensor field satisfying $h^\mu_{\ \mu} = \hat{g}^{\mu\nu}h_{\mu\nu} = 0$. However, in two dimensions, since the number of degrees of freedom of the traceless tensor field is the same as the spacetime dimension, we can take a gauge-fixing condition called the conformal gauge defined by

$$h_{\mu\nu} = 0 \tag{6-2}$$

using two gauge degrees of freedom ξ^μ.

The partition function of two-dimensional quantum gravity under the conformal gauge is given by

$$\begin{aligned} Z &= \int [dgdf]_g \, e^{iI_{\mathrm{M}}(f,g)} \\ &= \int [d\phi dbdcdf]_{\hat{g}} \, e^{iS_{\mathrm{L}}(\phi,\hat{g}) + iI_{\mathrm{M}}(f,\hat{g}) + iI_{\mathrm{gh}}(b,c,\hat{g})}, \end{aligned}$$

[1] It is also called the non-critical string (see Footnote 10 in this chapter).

where f is a conformally invariant matter field and $I_{\rm M}$ denotes its action. Associated with the conformal gauge, what is called the bc ghost action $I_{\rm gh}$ appears, which is also conformally invariant. The $S_{\rm L}$ is the Liouville action (5-8) introduced in the previous chapter,

$$S_{\rm L}(\phi, \hat{g}) = -\frac{b_{\rm L}}{4\pi}\int d^2x\sqrt{-\hat{g}}\left(\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \hat{R}\phi\right). \tag{6-3}$$

The coefficient $b_{\rm L}$ is determined from conformal anomaly of the whole system, and letting the central charge of the matter field be $c_{\rm M}$, it is given by[2]

$$b_{\rm L} = -\frac{c_{\rm M} - 25}{6}. \tag{6-4}$$

How to determine this value will be described after defining the bc ghost action below. If the matter central charge is $c_{\rm M} < 25$, the Liouville action has a right sign such that $b_{\rm L} > 0$. We here consider the system coupled with conformal field theory (CFT) of $c_{\rm M} \leq 1$.

In two-dimensional quantum gravity, the conformal-factor field ϕ is often called the Liouville field. We call it so here as well.

From the second equation of (6-1), the transformation law of the two-dimensional traceless tensor field in the conformal gauge (6-2) is given by

$$\delta_\xi h_{\mu\nu} = \hat{\nabla}_\mu\xi_\nu + \hat{\nabla}_\nu\xi_\mu - \hat{g}_{\mu\nu}\hat{\nabla}^\lambda\xi_\lambda. \tag{6-5}$$

By replacing ξ^μ with ghost fields c^μ and introducing anti-ghost fields $b_{\mu\nu}$ according to normal gauge-fixing procedure, we obtain the ghost action as

$$I_{\rm gh} = -\frac{i}{2\pi}\int d^2x\sqrt{-\hat{g}}\, b_{\mu\nu}\delta_c h^{\mu\nu} = -\frac{i}{\pi}\int d^2x\sqrt{-\hat{g}}\, b_{\mu\nu}\hat{\nabla}^\mu c^\nu, \tag{6-6}$$

where the anti-ghost field is a symmetric traceless tensor field with two degrees of freedom.

Here, we mention what to note about the coefficient $b_{\rm L}$ (6-4) in front of the Liouville action. Unlike the path integral measure of the matter field $[df]_g$ discussed in the previous chapter, the measure of the gravitational field $[dg]_g$ gives a nested structure in which the integration over the metric field $g_{\mu\nu}$ has to be performed using the measure defined on $g_{\mu\nu}$ itself. Hence, at this time, it is an assumption that the theory can be expressed using the

[2] The field is usually redefined as $\phi \to \phi/\sqrt{2b_{\rm L}}$ so that the action density is rewritten to the form $-(1/8\pi)[(\partial\phi)^2 + 2Q\hat{R}\phi]$, where $Q = \sqrt{b_{\rm L}/2}$. In this book, to emphasize similarity with the four-dimensional quantum gravity, we proceed without redefining.

Liouville action as described above, including the measure of the gravitational field. However, once it is rewritten to a quantum field theory on the background with the action

$$S_{\rm 2DQG} = S_{\rm L} + I_{\rm M} + I_{\rm gh},$$

we can quantize it as usual, and thus it is possible to ascertain whether there is no inconsistency in the theory.

For the time being, we quantize the action $S_{\rm 2DQG}$, assuming that we do not know the value of the coefficient $b_{\rm L}$. The conformal anomaly (5-1) of this theory can be determined independent of $b_{\rm L}$, and thus writing the coefficient with a prime to distinguish it from $b_{\rm L}$, it is given by $b'_{\rm L} = -(c_{\rm M} - 25)/6$. Here, $c_{\rm M}$ is the matter central charge. The contribution -25 in the numerator of $b'_{\rm L}$ is the sum of -26 from the bc ghost field and 1 from the Liouville field. The later comes from the fact that the kinetic term of the Liouville field is that of a free scalar field.[3]

Based on this result, let us change the background metric $\hat{g}_{\mu\nu}$ conformally. The partition function then transforms as

$$\begin{aligned} Z(e^{2\omega}\hat{g}) &= \int [d\phi dbdcdf]_{e^{2\omega}\hat{g}}\, e^{iS_{\rm L}(\phi, e^{2\omega}\hat{g}) + iI_{\rm M} + iI_{\rm gh}} \\ &= \int [d\phi dbdcdf]_{\hat{g}}\, e^{iS'_{\rm L}(\omega,\hat{g})} e^{iS_{\rm L}(\phi, e^{2\omega}\hat{g}) + iI_{\rm M} + iI_{\rm gh}} \\ &= \int [d\phi dbdcdf]_{\hat{g}}\, e^{iS'_{\rm L}(\omega,\hat{g}) + iS_{\rm L}(\phi-\omega, e^{2\omega}\hat{g}) + iI_{\rm M} + iI_{\rm gh}}. \end{aligned}$$

In the first equality we use the fact that the actions of the matter field and the ghost field are conformally invariant. Moreover, the fact that the kinetic term of the Liouville field is conformally invariant is also necessary to evaluate the conformal anomaly. By rewriting the ω-dependence in the measure as a Jacobian with the Liouville action, the second equality is obtained, where $S'_{\rm L}$ is the Liouville action with the coefficient $b'_{\rm L}$. The third equality is obtained by converting the Liouville field to $\phi \to \phi - \omega$. Here, it should be noted that the measure $[d\phi]_{\hat{g}}$ defined on the background is invariant under such a shift transformation. If we put $b_{\rm L} = b'_{\rm L}$, we can show

$$\begin{aligned} Z(e^{2\omega}\hat{g}) &= \int [d\phi dbdcdf]_{\hat{g}}\, e^{iS_{\rm L}(\omega,\hat{g}) + iS_{\rm L}(\phi-\omega, e^{2\omega}\hat{g}) + iI_{\rm M} + iI_{\rm gh}} \\ &= \int [d\phi dbdcdf]_{\hat{g}}\, e^{iS_{\rm L}(\phi,\hat{g}) + iI_{\rm M} + iI_{\rm gh}} = Z(\hat{g}) \end{aligned}$$

[3] Note that the central charge of the Virasoro algebra for the Liouville field is not 1, but $1+6b_{\rm L}$ added with a contribution from the linear term of ϕ (see the next section).

using the Wess-Zumino integrability relation (5-15). In this way, the coefficient of the Liouville action is determined to be (6-4) from the condition that the theory becomes conformally invariant exactly.

As shown above, the fact that the Liouville field is an integration variable plays an essential role when showing the conformal invariance. Thus, it is important that as the gravitational field is integrated, conformal invariance is restored exactly, despite the appearance of a quantity called conformal anomaly that breaks the conformal invariance of original classical action $I_{\rm M}$. It means that this invariance is inherent to quantum gravity, and it is nothing but a realization of the so-called background-metric independence.

Virasoro Algebra and Physical States

Let us canonically quantize the action $S_{\rm 2DQG}$ and examine the background-metric independence in an algebraic manner. Here, as in the case of the free-field representation of two-dimensional conformal field theory in the third section of Chapter 4, we expand each field in the cylindrical background $\mathbb{R} \times S^1$ parametrized by the coordinates $x^\mu = (\eta, \sigma)$, where $0 < \sigma < 2\pi$. Quantization of the Liouville field can be done in the same way as in Chapter 4, but pay attention to differences in field normalization (see Footnote 2 in this chapter) and how to enter imaginary units.

From the equation of motion $\partial^2 \phi = (-\partial_\eta^2 + \partial_\sigma^2)\phi = 0$, the Liouville field is expanded in zero modes, left-handed modes, and right-handed modes as

$$\phi(\eta,\sigma) = \frac{1}{\sqrt{2b_{\rm L}}} \Big\{ \hat{q} + 2\eta\hat{p} + \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n^+ e^{-in(\eta+\sigma)} + \alpha_n^- e^{-in(\eta-\sigma)} \right) \Big\}. \tag{6-7}$$

Since ϕ is a real field, Hermitian conjugate is defined by $\alpha_n^{\pm\dagger} = \alpha_{-n}^\pm$. The conjugate momentum is given by $\Pi = (b_{\rm L}/2\pi)\partial_\eta \phi$ and the equal-time commutation relation is set as $[\phi(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma')$. Since the delta function is given by $\delta(\sigma-\sigma') = \sum_{n\in\mathbf{Z}} e^{in(\sigma-\sigma')}/2\pi$, the commutation relation of each mode is obtained as

$$[\hat{q}, \hat{p}] = i, \quad [\alpha_n^\pm, \alpha_m^\pm] = n\delta_{n+m,0}, \quad [\alpha_n^\pm, \alpha_m^\mp] = 0.$$

Residual gauge degrees of freedom after gauge-fixing to the conformal gauge (6-2) are given by the conformal Killing vectors ζ^μ satisfying the conformal Killing equation

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu - \eta_{\mu\nu}\partial^\lambda \zeta_\lambda = 0.$$

That can be seen from the fact that the gauge condition is preserved because $\delta_\zeta h_{\mu\nu} = 0$ when $\xi^\mu = \zeta^\mu$ from the transformation of the traceless tensor field (6-5). Generators of the guage transformations by the residual degrees of freedom ζ^μ that form conformal algebra are given by

$$L_\zeta = \int_{S^1} d\sigma\, \zeta^\mu\, \hat{\Theta}_{\mu 0}, \tag{6-8}$$

where $\hat{\Theta}_{\mu\nu}$ is the energy-momentum tensor defined by a variation with respect to the background metric as

$$\hat{\Theta}^{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}} \frac{\delta S_{\rm 2DQG}}{\delta \hat{g}_{\mu\nu}},$$

which satisfies the traceless condition. The indices are raised and lowered by using the background metric as $\hat{\Theta}^\mu{}_\nu = \hat{g}_{\nu\lambda}\hat{\Theta}^{\mu\lambda}$.

The energy-momentum tensor of the Liouville field is given by

$$\hat{\Theta}^{\rm L}_{\mu\nu} = \frac{b_{\rm L}}{2\pi} \left\{ \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi + \left(\eta_{\mu\nu}\partial^\lambda\partial_\lambda - \partial_\mu\partial_\nu\right)\phi \right\}.$$

The first two terms correspond to that of a normal scalar field. The last term is a term specific to the Liouville theory obtained from a variation of the $\hat{R}\phi$ term. The trace disappears according to the equation of motion of the Liouville field.

The conformal Killing vectors ζ^μ exist infinitely and are given as an arbitrary function expanded in the basis (4-12). They are here expressed as $\zeta^{+\mu}_n = (e^{in(\eta+\sigma)}/2, e^{in(\eta+\sigma)}/2)$ and $\zeta^{-\mu}_n = (e^{in(\eta-\sigma)}/2, -e^{in(\eta-\sigma)}/2)$. Substituting these into (6-8), we obtain the so-called Virasoro generators as follows:

$$\begin{aligned} L^{\rm L\pm}_n &= e^{in\eta}\int_0^{2\pi} d\sigma\, e^{\pm in\sigma}\frac{1}{2} :\!\left(\hat{\Theta}^{\rm L}_{00} \pm \hat{\Theta}^{\rm L}_{01}\right)\!: + \frac{b_{\rm L}}{4}\delta_{n,0} \\ &= \frac{1}{2}\sum_{m\in\mathbf{Z}} :\!\alpha^\pm_m \alpha^\pm_{n-m}\!: + i\sqrt{\frac{b_{\rm L}}{2}}\, n\alpha^\pm_n + \frac{b_{\rm L}}{4}\delta_{n,0}, \end{aligned}$$

where $\alpha^\pm_0 = \hat{p}$. Since the energy-momentum tensor is Hermitian, the generator satisfies the Hermiticity condition $L^{\rm L\pm\dagger}_n = L^{\rm L\pm}_{-n}$. The last term $(b_{\rm L}/4)\delta_{n,0}$ is the Casimir effect by choosing the coordinates to be $\mathbb{R}\times S^1$.

It shifts the Hamiltonian operator by $b_{\mathrm{L}}/2$ as[4]

$$H^{\mathrm{L}} = L_0^{\mathrm{L}+} + L_0^{\mathrm{L}-} = \hat{p}^2 + \frac{b_{\mathrm{L}}}{2} + \sum_{n=1}^{\infty} \left\{ \alpha_n^{+\dagger}\alpha_n^{+} + \alpha_n^{-\dagger}\alpha_n^{-} \right\}.$$

This energy shift is necessary for conformal algebra to close quantum mechanically.

Let us consider the total Virasoro generator with adding the generators of the matter field and the bc ghost field (see (6-16) in the next section) as

$$L_n^{\pm} = L_n^{\mathrm{L}\pm} + L_n^{\mathrm{M}\pm} + L_n^{\mathrm{gh}\pm}.$$

It satisfies the Virasoro algebra

$$\left[L_n^{\pm}, L_m^{\pm}\right] = (n-m)L_{n+m}^{\pm} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

and $[L_n^{+}, L_m^{-}] = 0$ whose total central charge vanishes as

$$c = 1 + 6b_{\mathrm{L}} + c_{\mathrm{M}} - 26 = 0. \tag{6-9}$$

Thus, it turns out that when the coefficient b_{L} is given by (6-4), it becomes conformally invariant at the quantum level, where c_{M} and -26 are the central charge of the matter field and the bc ghost field, respectively. The contribution from the Liouville field is $1 + 6b_{\mathrm{L}}$, of which 1 comes from the Liouville field being a scalar boson field, while $6b_{\mathrm{L}}$ comes from the fact that the Liouville action has the non-conformally invariant $\hat{R}\phi$ term. Actually, calculating the algebra of the energy-momentum tensor with Poisson brackets, a non-zero central charge $6b_{\mathrm{L}}$ is yielded. When the quantization is done, the correction 1 is added to this.[5]

The condition that the central charge vanishes indicates that diffeomorphism invariance holds in the whole system quantum mechanically. Since the conformal invariance appears as part of diffeomorphism invariance that is a gauge symmetry, all theories that connected one another by the conformal transformation are gauge equivalent. In this way, it can be shown algebraically that the theory does not depend on how to choose the background spacetime.

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[4] On the cylindrical background, $L_0^{\pm}$ corresponds to a dilatation operator that counts left/right conformal dimension, and the Hamiltonian operator $H = L_0^{+} + L_0^{-}$ is an operator that counts the sum of left and right conformal dimensions.

[5] The matter central charge c_{M} has the same structure if using the free field representation.

Next, we examine physical states of two-dimensional quantum gravity. We first consider the case where the bc ghost field is integrated out and not appear explicitly. A conformally invariant vacuum is then defined as a state that satisfies $L_n^{\pm}|\Omega\rangle = 0$ $(n \geq -1)$, which is given by

$$|\Omega\rangle = e^{-b_{\rm L}\phi_0}|0\rangle, \tag{6-10}$$

where $|0\rangle$ is a normal Fock vacuum which disappears for $\hat{p}$ and annihilation modes. The exponential factor of the zero-mode $\phi_0 = \hat{q}/\sqrt{2b_{\rm L}}$ originates from the $\hat{R}\phi$ term. Exponents in such exponential factors are generally called the Liouville charges, especially what the vacuum has is called the background charge.[6]

We also introduce another Fock vacuum $|\gamma\rangle = e^{\gamma\phi_0}|\Omega\rangle$ with the Liouville charge γ added to the conformally invariant vacuum (6-10). This state is an eigenstate of the Hamiltonian operator, which satisfies

$$H^{\rm L}|\gamma\rangle = h_\gamma|\gamma\rangle, \qquad h_\gamma = \gamma - \frac{\gamma^2}{2b_{\rm L}}. \tag{6-11}$$

Considering states in which the creation operators are applied to this vacuum as

$$|\Psi\rangle = \mathcal{O}(\alpha_n^{\pm\dagger}, \ldots)|\gamma\rangle,$$

physical states are defined by those which satisfy the Virasoro condition

$$\left(H^{\rm L} + H^{\rm M} - 2\right)|\Psi\rangle = 0, \quad \left(L_n^{\rm L\pm} + L_n^{\rm M\pm}\right)|\Psi\rangle = 0, \tag{6-12}$$

where $n \geq 1$. This is nothing but a quantum version of the Wheeler-DeWitt constraint condition that ensure diffeomorphism invariance. Here, -2 appears in the Hamiltonian condition because the bc ghost field is integrated out. This 2 comes from the spacetime dimension.

As physical states, for simplicity, we consider only the case where primary matter fields described by CFT receive corrections of quantum gravity. A real primary matter field with the same left and right conformal dimensions Δ is defined using the matter Virasoro generator as

$$L_0^{\rm M\pm}|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n^{\rm M\pm}|\Delta\rangle = 0 \quad (n \geq 1).$$

[6] Considering the Euclidean path integral and letting χ be the Euler characteristic, the weight of the path integral of the zero-mode part is given by $e^{-b_{\rm L}\chi\phi_0}$. Since topology of states is represented by a disk of $\chi = 1$, $e^{-b_{\rm L}\phi_0}$ is added to a nothing state. See Footnote 8 in Chapter 4.

The state of the matter field is symbolically expressed as $|\Delta\rangle = \Phi_\Delta^\dagger|0\rangle$ by introducing the corresponding primary field operator Φ_Δ.[7] The physical state that has undergone quantum gravity corrections, that is called a gravitationally dressed state, is given by

$$\Phi_\Delta^\dagger|\gamma_\Delta\rangle.$$

From the Hamiltonian condition, the Liouville charge γ_Δ satisfies a quadratic equation

$$h_{\gamma_\Delta} + 2\Delta = 2,$$

where h_γ is given by (6-11). Of the two solutions, choosing the one that the classical limit $b_\mathrm{L} \to \infty$ approaches a canonical value $2 - 2\Delta$, the Liouville charge is determined to be[8]

$$\begin{aligned} \gamma_\Delta &= b_\mathrm{L}\left(1 - \sqrt{1 - \frac{4-4\Delta}{b_\mathrm{L}}}\right) \\ &= 2 - 2\Delta + \frac{2(1-\Delta)^2}{b_\mathrm{L}} + \frac{4(1-\Delta)^3}{b_\mathrm{L}^2} + \cdots . \end{aligned} \tag{6-13}$$

Due to duality relation $h_\gamma = h_{2b_\mathrm{L}-\gamma}$, a state $\Phi_\Delta^\dagger|2b_\mathrm{L} - \gamma_\Delta\rangle$ also satisfies the physical state conditions (6-12). Since there is no corresponding classical gravitational state to the dual state, it is not considered as a physical object. Using the dual state, however, we can define an inner product as $\langle 2b_\mathrm{L} - \gamma_\Delta|\Phi_\Delta\Phi_\Delta^\dagger|\gamma_\Delta\rangle = \langle\Omega|e^{2b_\mathrm{L}\phi_0}|\Omega\rangle = 1$ because the Liouville charges totally cancel out. Hence, we consider the dual state to be a virtual state that appears only in intermediate states.[9]

Physical field operators corresponding to gravitationally dressed states are expressed by introducing an exponential operator with Liouville charge γ defined as

$$V_\gamma(\eta,\sigma) = :e^{\gamma\phi(\eta,\sigma)}: = e^{\gamma\phi_>(\eta,\sigma)}e^{\gamma\phi_0(\eta)}e^{\gamma\phi_<(\eta,\sigma)}, \tag{6-14}$$

[7] Strictly speaking, the correspondence between the state and the operator is given by $|\Delta\rangle = \lim_{\eta\to i\infty} e^{-i2\Delta\eta}\Phi_\Delta(\eta,\sigma)|0\rangle$. See the third section in Chapter 4 for the description of the primary field using the free boson field representation.

[8] The quantum gravity correction factor $e^{\gamma_\Delta\phi_0}$ means that eigenvalue p of the zero-mode operator $\hat{p}$ in the Liouville field is given by a pure imaginary number. If it is real as in string theory, we can normalize the state in a delta function like $\int d\phi_0\, e^{ip\phi_0}e^{ip'\phi_0} = \delta(p+p')$. However, the quantum gravity state cannot be simply normalized in this way.

[9] Unlike the free field representation of CFT discussed in Chapter 4, the Liouville field has a physical meaning as the conformal factor of the gravitational field. This fact is reflected in how correlation functions are constructed in the last section.

where ϕ_0, $\phi_>$, and $\phi_<$ correspond to the parts of the zero-modes, the creation modes, and the annihilation modes in the field (6-7), respectively. The zero-mode part in the exponential operator (6-14) can be expressed as $e^{\gamma\phi_0(\eta)} = e^{\gamma\hat{q}/2\sqrt{2b_{\rm L}}} e^{2\gamma\eta\hat{p}/\sqrt{2b_{\rm L}}} e^{\gamma\hat{q}/2\sqrt{2b_{\rm L}}}$. Applying the Virasoro generator, we find that this operator transforms as

$$\left[L_n^{\rm L\pm}, V_\gamma(\eta,\sigma)\right] = e^{in(\eta\pm\sigma)}\left(-i\partial_\pm + \frac{n}{2}h_\gamma\right)V_\gamma(\eta,\sigma), \qquad \text{(6-15)}$$

where $\partial_\pm = (\partial_\eta \pm \partial_\sigma)/2$. If we choose the $\Delta = 0$ charge in (6-13) as γ,

$$i\left[L_n^{\rm L\pm}, V_{\gamma_0}(\eta,\sigma)\right] = \partial_\pm\left\{e^{inx^\pm}V_{\gamma_0}(\eta,\sigma)\right\}$$

holds due to $h_{\gamma_0} = 2$. Therefore, it satisfies the condition of diffeomorphism invariance as follows:

$$\left[L_n^{\rm L\pm}, \int d^2x\, V_{\gamma_0}(x)\right] = 0.$$

The operator $V_{\gamma_0} =: e^{\gamma_0\phi}:$ corresponds to the cosmological term $\sqrt{-g}$. Indeed, in the classical limit $b_{\rm L} \to \infty$, it reduces to $\sqrt{-g} = e^{2\phi}$ itself.

The same can also apply to the case including the primary matter field Φ_Δ. The product of Φ_Δ and V_{γ_Δ} with the Liouville charge (6-13) satisfies $i[L_n^\pm, \Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma)] = \partial_\pm\{e^{in(\eta\pm\sigma)}\Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma)\}$ for the total Virasoro generator, and its volume integral becomes Virasoro invariant. This operator corresponds to $\sqrt{-g}\Phi_\Delta$. The correspondence between the physical state mentioned above and the operator is given by the following limit:

$$\Phi_\Delta^\dagger|\gamma_\Delta\rangle = \lim_{\eta\to i\infty} e^{-2i\eta}\Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma)|\Omega\rangle.$$

BRST Operator and Physical States

In this section, we reformulate the theory using the BRST formalism.[10] Introducing new field variables $b_{\pm\pm} = b_{00} \pm b_{01}$ and $c^\pm = c^0 \pm c^1$, the bc

[10] The original paper is M. Kato and K. Ogawa, *Covariant Quantization of String based on BRS Invariance*, Nucl. Phys. B 212 (1983) 443. See also D. Friedan, E. Martinec, and S. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Nucl. Phys. B 271 (1986) 93. The unitarity issue of string theory discussed in these papers is nothing but to show that we can introduce a Lorentzian signature into 10- or 26-dimensional target spacetime. A boson field on world sheet representing a time coordinate in the target spacetime has a wrong sign when viewed as a two-dimensional quantum field theory even considering a Euclidean world sheet, but by canceling its degrees of freedom with those of the *bc* ghost, the unitarity recovers. As the result, it can be shown that the S-matrix in the Minkowski target spacetime becomes

ghost action (6-6) can be written in the form

$$I_{\rm gh} = \frac{i}{\pi}\int d^2x \left(b_{++}\partial_- c^+ + b_{--}\partial_+ c^-\right).$$

Since the equations of motion are given by $\partial_- c^+ = \partial_+ c^- = 0$ and $\partial_- b_{++} = \partial_+ b_{--} = 0$, we can mode-expand the bc ghost fields as

$$c^\pm = \sum_{n\in\mathbf{Z}} c_n^\pm e^{-in(\eta\pm\sigma)}, \qquad b_{\pm\pm} = \sum_{n\in\mathbf{Z}} b_n^\pm e^{-in(\eta\pm\sigma)}.$$

Hermiticity of each mode is $c_n^{\pm\dagger} = c_{-n}^\pm$ and $b_n^{\pm\dagger} = b_{-n}^\pm$. The equal-time anti-commutation relations are given by $\{c^\pm(\sigma), b_{\pm\pm}(\sigma')\} = 2\pi\delta(\sigma-\sigma')$ and $\{c^\pm(\sigma), b_{\mp\mp}(\sigma')\} = 0$. The anti-commutation relation of each mode then becomes

$$\{c_n^\pm, b_m^\pm\} = \delta_{n+m,0}, \qquad \{c_n^\pm, b_m^\mp\} = 0.$$

The ghost part of the Virasoro algebra can be written in terms of the modes as

$$L_n^{\rm gh\pm} = \sum_{m\in\mathbf{Z}} (n+m) : b_{n-m}^\pm c_m^\pm : . \tag{6-16}$$

At this time, a ghost vacuum $|0\rangle_{\rm gh}$ is defined by $c_n^\pm|0\rangle_{\rm gh} = 0$ $(n \geq 2)$ and $b_n^\pm|0\rangle_{\rm gh} = 0$ $(n \geq -1)$ so as to satisfy the conformal invariance condition $L_n^{\rm gh\pm}|0\rangle_{\rm gh} = 0$ $(n \geq -1)$. The normal ordering is then defined as bringing ghost modes that disappear for this vacuum to the right.[11]

---

unitary. Two-dimensional quantum gravity may also be regarded as a string theory called the non-critical string theory that has a two-dimensional target spacetime of (t, ϕ) by introducing a time field t. The Liouville field ϕ then represents a spatial coordinate of the target spacetime called the linear dilaton background. The theory discussed here corresponds to what the time coordinate t is replaced with a free boson field of CFT introduced in the third section of Chapter 4. In any case, it is necessary to distinguish between the unitarity property on the target spacetime physics and a positive-definiteness of the field on the world sheet. From the viewpoint of quantum gravity, there is no concept of the S-matrix because the world sheet itself is fluctuating gravitationally. What is important here is that physical fields become real in the context of two-dimensional CFT.

[11] This is an ordering suitable for OPE calculations, sometimes called the conformal normal ordering. On the other hand, distinguished from this, an ordering associated with the Fock vacuum which brings $c_n^\pm$ and $b_n^\pm$ with $n > 0$ to the right is called the creation-annihilation normal ordering. If it is denoted by ‡ ‡, the Virasoro generator is written as $L_n^{\rm gh\pm} = \sum_{m\in\mathbf{Z}}(n+m) \ddagger b_{n-m}^\pm c_m^\pm \ddagger - \delta_{n,0}$.

The BRST operator is decomposed as $Q_{\rm BRST} = Q^+ + Q^-$ and each is given by

$$Q^\pm = \sum_{n\in\mathbf{Z}} c^\pm_{-n}\left(L^{\rm L\pm}_n + L^{\rm M\pm}_n\right) - \frac{1}{2}\sum_{n,m\in\mathbf{Z}}(n-m) :c^\pm_{-n}c^\pm_{-m}b^\pm_{n+m}: .$$

This can be further decomposed as

$$Q^\pm = c^\pm_0 L^\pm_0 - b^\pm_0 M^\pm + d^\pm, \qquad M^\pm = 2\sum_{n=1}^\infty n c^\pm_{-n}c^\pm_n,$$

where $L^\pm_0$ is the whole Hamiltonian operator including the ghost part, and the last term of each $Q^\pm$ that does not include the ghost zero-modes is given by

$$d^\pm = \sum_{n\neq 0} c^\pm_{-n}\left(L^{\rm L\pm}_n + L^{\rm M\pm}_n\right) - \frac{1}{2}\sum_{\substack{n,m\neq 0\\ n+m\neq 0}}(n-m) :c^\pm_{-n}c^\pm_{-m}b^\pm_{n+m}: .$$

The BRST operator satisfies nilpotency $Q^2_{\rm BRST} = 0$, which can be expressed as

$$d^{\pm 2} = L^\pm_0 M^\pm, \qquad [d^\pm, L^\pm_0] = [d^\pm, M^\pm] = [L^\pm_0, M^\pm] = 0.$$

In order to construct physical states, introduce a Fock ghost vacuum expressed by $c^+_1 c^-_1|0\rangle_{\rm gh}$, which disappears when applying annihilation modes $c^\pm_n$, $b^\pm_n$ $(n>0)$ to it.[12] Consider states obtained by applying creation modes to a whole Fock vacuum where this ghost vacuum and $|\gamma\rangle$ are combined such as

$$|\Psi\rangle = \mathcal{O}(\alpha^{\pm\dagger}_n, c^{\pm\dagger}_n, b^{\pm\dagger}_n, \cdots)|\gamma\rangle \otimes c^+_1 c^-_1|0\rangle_{\rm gh},$$

then physical states are defined by such states satisfying the BRST conformal invariance condition $Q_{\rm BRST}|\Psi\rangle = 0$.

Since the state $|\Psi\rangle$ disappears when the zero-mode $b^\pm_0$ is applied to it, we find that it is sufficient to consider the state satisfying the conditions

$$b^\pm_0|\Psi\rangle = 0, \qquad L^\pm_0|\Psi\rangle = 0 \tag{6-17}$$

as a physical state. The second condition comes from that the whole Hamiltonian operator is BRST trivial such as $L^\pm_0 = \{Q_{\rm BRST}, b^\pm_0\}$.[13] Thus, the

[12] Using the ghost fields, this vacuum is given by the limit $\lim_{\eta\to i\infty} e^{2i\eta}c^+c^-|0\rangle_{\rm gh}$.

[13] In general, the whole Virasoro generator is BRST trivial as $L^\pm_n = \{Q_{\rm BRST}, b^\pm_n\}$. Therefore, descendant states obtained by applying $L^\pm_{-n}$ to physical states become BRST trivial.

BRST conformal invariance condition for physical states composed on the subspace (6-17) reduces to

$$d^{\pm}|\Psi\rangle = 0.$$

As in the previous section, if $\mathcal{O}$ does not include the ghost mode, it is the same as the condition (6-12), where the energy shift -2 originates from $L_0^{\mathrm{gh}\pm} c_1^{\pm}|0\rangle_{\mathrm{gh}} = -c_1^{\pm}|0\rangle_{\mathrm{gh}}$.

In two-dimensional quantum gravity, there are a special physical state including derivatives (discrete state) and a state with non-trivial ghost number with a ring structure (grand ring state).[14] Moreover, by combining these states, we can construct a conserved current whose divergence becomes BRST trivial. It generates W_∞ symmetry, and as the Ward identity of this symmetry we can derive a nonlinear structure established between correlation functions, called the W and the Virasoro constraints.[15]

Because the Hamiltonian operator $L_0^{\mathrm{gh}\pm}$ does not contain $c_0^{\pm}$ and $b_0^{\pm}$, the ghost vacuum is degenerate and its inner product vanishes as ${}_{\mathrm{gh}}\langle 0|0\rangle_{\mathrm{gh}} = {}_{\mathrm{gh}}\langle 0|c_{-1}^{-}c_{-1}^{+}c_1^{+}c_1^{-}|0\rangle_{\mathrm{gh}} = 0$. This is obvious from the fact that it vanishes if $\{b_0^{\pm}, c_0^{\pm}\} = 1$ is inserted inside each inner product. One of the pair that is degenerate with the Fock ghost vacuum is $\vartheta c_1^{+}c_1^{-}|0\rangle_{\mathrm{gh}}$, where $\vartheta = ic_0^{+}c_0^{-}$. Using this fact, the ghost inner product is normalized to ${}_{\mathrm{gh}}\langle 0|c_{-1}^{-}c_{-1}^{+}\vartheta c_1^{+}c_1^{-}|0\rangle_{\mathrm{gh}} = 1$.

The BRST transformation law of the Liouville field is given by

$$i\big[Q^{\pm}, \phi(\eta,\sigma)\big] = c^{\pm}\partial_{\pm}\phi(\eta,\sigma) + \frac{1}{2}\partial_{\pm}c^{\pm}(\eta,\sigma).$$

Combining the left and right components, we obtain $i[Q_{\mathrm{BRST}}, \phi] = c^{\mu}\partial_\mu\phi + \partial_\mu c^{\mu}/2$, which is the transformation obtained by replacing the gauge transformation parameters in (6-1), that is ζ^{μ} after the gauge fixing, with the ghost field c^{μ}. The BRST transformation law of the field operator (6-14) is given by

$$i\big[Q^{\pm}, V_\gamma(\eta,\sigma)\big] = c^{\pm}\partial_{\pm}V_\gamma(\eta,\sigma) + \frac{h_\gamma}{2}\partial_{\pm}c^{\pm}V_\gamma(\eta,\sigma)$$

from (6-15). Combining the left and right components, we obtain the conformal transformation of a real scalar field with the conformal dimension

[14] See P. Bouwknegt, J. McCarthy, and K. Pilch, *BRST Analysis of Physical States for 2D Gravity Coupled to* $c \leq 1$ *Matter*, Commun. Math. Phys. **145** (1992) 541 in Bibliography.

[15] See K. Hamada, *Ward Identities of* W_∞ *Symmetry in Liouville Theory coupled to* $c_M < 1$ *Matter*, Phys. Lett. **B324** (1994) 141. As for the W and the Virasoro constraints, see M. Fukuma, H. Kawai, and R. Nakayama, *Continuum Schwinger-Dyson Equations and Universal Structures in Two-Dimensional Quantum Gravity*, Int. J. Mod. Phys. A **6** (1991) 1385.

h_γ. As shown before, the cosmological term is given as the operator of the Liouville charge γ_0 satisfying $h_{\gamma_0} = 2$, and its volume integral $\int d^2x\, V_{\gamma_0}$ becomes BRST invariant, namely diffeomorphism invariant. Generally, the volume integral of $\Phi_\Delta V_{\gamma_\Delta}$ becomes BRST invariant.

Furthermore, consider field operators multiplied by a ghost field function c^+c^-. They become locally BRST invariant operators satisfying

$$\begin{aligned} &i\big[Q_{\mathrm{BRST}}, c^+c^-\Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma)\big] \\ &= \frac{1}{2}(h_{\gamma_\Delta} + 2\Delta - 2)c^+c^- \left(\partial_+ c^+ + \partial_- c^-\right)\Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma) = 0, \end{aligned}$$

where

$$i\{Q^\pm, c^\pm\} = c^\pm \partial_\pm c^\pm$$

and $h_{\gamma_\Delta} + 2\Delta = 2$ are used. The corresponding BRST invariant physical state can be obtained by taking the limit $\lim_{\eta\to i\infty} c^+c^-\Phi_\Delta V_{\gamma_\Delta}(\eta,\sigma)|\Omega\rangle \otimes |0\rangle_{\mathrm{gh}}$ (see Footnote 12 for the Fock ghost vacuum).

On Correlation Functions

The BRST conformal invariance is the same as diffeomorphism invariance. The fact that the zero-mode p is a pure imaginary number (see Footnote 8) represents that physical fields are real composite fields. In order to calculate correlation functions between them, we have to regularize divergences resulting from an integration of the Liouville zero-mode ϕ_0. To do so, it is necessary to add a physical field with the Liouville charge as a potential term to the action.

We here consider an interacting system adding the cosmological operator $V_\alpha = \int d^2x : e^{\alpha\phi}:$ with the Liouville charge $\alpha = \gamma_0$. When considering the path integral Wick-rotated to the Euclidean space ($\tau = i\eta$), the action is expressed as $S_{\mathrm{L}} + \mu V_\alpha$. A general physical operator is denoted as $O_\gamma = \int d^2x\, \mathcal{O}_\gamma$.

Since the Euler characteristic of the space is $\int d^2x\sqrt{\hat{g}}\hat{R}/4\pi = 2$, the ϕ_0-dependence in S_{L} can be derived as $2b_{\mathrm{L}}\phi_0$, with attention to a sign in the Wick-rotated action. Since the zero-mode dependence of the field operator O_γ is given by $e^{\gamma\phi_0}$, if we introduce a variable $A = e^{\alpha\phi_0}$ and perform the zero-mode integration first, the correlation function is expressed as

$$\begin{aligned} \langle O_{\gamma_1}\cdots O_{\gamma_n}\rangle &= \frac{1}{\alpha}\int_0^\infty \frac{dA}{A} A^{-s}\langle O_{\gamma_1}\cdots O_{\gamma_n} e^{-\mu A V_\alpha}\rangle_0 \\ &= \mu^s \frac{\Gamma(-s)}{\alpha}\langle O_{\gamma_1}\cdots O_{\gamma_n}\,(V_\alpha)^s\rangle_0, \end{aligned}$$

where $\langle \cdots \rangle_0$ is a correlation function in the free theory without the potential term. The power of the cosmological term is determined to be

$$s = \frac{2b_{\mathrm{L}}}{\alpha} - \sum_{i=1}^{n} \frac{\gamma_i}{\alpha}.$$

The zero-mode then cancel out, that is, the Liouville charge is conserved and the correlation function has a value. What is important here is that the dependence of the scale μ (here, the cosmological constant) shows a power-law behavior, and it may be a negative power. Calculations of this correlation function are not easy, but a method using analytic continuation is known in two-dimensional quantum gravity.[16]

[16] See M. Goulian and M. Li, *Correlation Functions in Liouville Theory*, Phys. Rev. Lett. **66** (1991) 2051.

Chapter Seven

Background-Free Quantum Gravity

In order to construct quantum field theory of gravity, we impose the following three basic conditions:

- diffeomorphism invariance.
- finitness.
- four-dimensional spacetime.

The first condition is one of basic principles of Einstein's theory of gravity, and we think that this symmetry also holds for quantum theory. It shall be expressed as a background-metric independence in the high energy limit.

Physically meaningful quantities must be finite. When quantizing gravity, the second condition not only refers to renormalizability but also implies that there is no spacetime singularity. In addition, although several higher dimensional models have been proposed, four dimensions are dimensions that guarantee renormalizability of known quantum fields. Since there are no facts suggesting the existence of extra dimensions from observations, spacetime is assumed to be four dimensions.

These three conditions are quite realistic so that we can believe them to be true. One of the purposes of this book is to understand what we can find when thoroughly investigating these conditions. In Chapters 7-10 we will see that the action of quantum gravity is actually determined from these three conditions.

Quantum Gravity Action

When considering quantum field theories in curved spacetime, ultraviolet divergences proportional to the square of the curvature tensor necessarily occur. Thus, considering fourth-order derivative gravitational actions to renormalize such divergences is natural and becomes essential in renormalizable quantum gravity. Furthermore, since conformal invariance becomes important at high energy, we will consider quantum fields with conformally invariant couplings as matter fields.[1] Indeed, as in gauge field theories, almost of the actions of widely known quantum field theories are conformally invariant.

On the other hand, conformal invariance of classical kinetic terms will feed back as a condition necessary for constructing the BRST conformal algebra, which is a representation of the background free property. This indicates that quantum diffeomorphism invariance gives considerably stronger conditions than classical diffeomorphism invariance. Such a constraint also appears when discussing renormalization theory in Chapters 9 and 10.

Conformally invariant gravitational actions required for a gravitational system coupled with conformally invariant matter fields are the Weyl action (5-4) and the Euler term (5-5) in the fourth-order derivative actions introduced in Chapter 5. Writing the matter action density as $\mathcal{L}_{\mathrm{M}}$, the action of quantum gravity is given by

$$\frac{1}{\hbar}I = \int d^4x\sqrt{-g}\left\{-\frac{1}{t^2}C^2_{\mu\nu\lambda\sigma} - bG_4 + \frac{1}{\hbar}\left(\frac{1}{16\pi G}R - \Lambda + \mathcal{L}_{\mathrm{M}}\right)\right\}, \tag{7-1}$$

where t is a dimensionless coupling constant that controls dynamics of quantum gravity. The coefficient b is introduced for removing divergences proportional to the Euler term. However, since the Euler term does not contain a kinetic term, this constant is expanded using other coupling constants, rather than independent coupling constants (see Chapter 10). The constants G and Λ represent the Newton constant and the cosmological constant, respectively. The reduced Planck constant $\hbar$ is restored here, while the speed of light is taken to be unity.

Quantum gravity is defined by the path integral over the gravitational field with a weight $e^{iI/\hbar}$. Since the gravitational field is a dimensionless field by definition unlike gauge fields etc., its fourth-order derivative action is completely dimensionless in four dimensions. Therefore, $\hbar$ appears only before the lower derivative action such as the Einstein-Hilbert term and so on. This fact is essential and indicates that the dimensionless fourth-order gravitational action describes purely quantum mechanical dynamics.[2] This is why the Weyl action and the Riegert action induced quantum mechan-

[1] We can add interactions with mass parameters like mass terms, but they do not contribute at high energy. Since such interactions do not affect the fourth-order gravitational action, we do not consider them, except for the Einstein term and the cosmological term written by the gravitational field only.

[2] It may be regarded as part of the path integral measure, including the weight of the Weyl action.

ically can be used together as a kinetic term in later discussions. In the below, let $\hbar = 1$.

As can be seen from the action I, the conformally invariant gravitational action dominates in a high energy region beyond the Planck mass scale. Let us consider an expansion by the coupling constant t of the Weyl action in that region. It means that we consider a perturbation theory about a conformally flat configuration satisfying $C_{\mu\nu\lambda\sigma} = 0$. Therefore, the conformal factor is pulled out and the gravitational field is expanded as

$$
\begin{aligned}
g_{\mu\nu} &= e^{2\phi} \bar{g}_{\mu\nu}, \\
\bar{g}_{\mu\nu} &= (\hat{g} e^{th})_{\mu\nu} = \hat{g}_{\mu\lambda} \left(\delta^{\lambda}_{\ \nu} + t h^{\lambda}_{\ \nu} + \frac{t^2}{2} h^{\lambda}_{\ \sigma} h^{\sigma}_{\ \nu} + \cdots \right),
\end{aligned}
\tag{7-2}
$$

where $h^{\mu}_{\ \nu}$ is the traceless tensor field and $h^{\mu}_{\ \mu} = \hat{g}^{\mu\nu} h_{\mu\nu} = 0$ is satisfied. The background metric $\hat{g}_{\mu\nu}$ is an artificial non-dynamical metric introduced practically to carry out calculations. The conformal factor is given in the form of an exponential to ensure that it is positive. The important point here is that since the conformal-factor field ϕ is not subject to any restrictions from the conformally flat condition, it is necessary to handle exactly without introducing an extra coupling constant for it.

A kinetic term and interaction terms of the conformal-factor field are induced as the Wess-Zumino actions from the path integral measure, as described in Chapter 5. When rewriting the diffeomorphism invariant measure of the full metric $g_{\mu\nu}$ to a practical measure defined on the background metric $\hat{g}_{\mu\nu}$, such actions appear as Jacobians to recover diffeomorphism invariance. Thus, the path integral can be rewritten as

$$
e^{i\Gamma} = \int [dgdf]_g \, e^{iI(f,g)} = \int [d\phi dh df]_{\hat{g}} \, e^{iS(\phi,\bar{g}) + iI(f,g)}, \tag{7-3}
$$

where S is the Wess-Zumino action. The symbol f denotes matter fields with a conformally invariant kinetic term. The action S appears from zeroth order in expansions by the coupling constant t. The lowest is the Riegert action (5-11) defined before by

$$
S_{\mathrm{R}}(\phi, \bar{g}) = -\frac{b_c}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi \right), \tag{7-4}
$$

which gives the kinetic term of the conformal-factor field ϕ.

We consider quantization of the gravitational field according to the path integral expression (7-3). The coefficient b_c in (7-4) is determined from the conformal anomaly proportional to the Euler density in the whole system as

$$b_c = \frac{1}{360}\left(N_S + 11N_F + 62N_A\right) + \frac{769}{180}, \tag{7-5}$$

where the contribution from matter fields is given by (5-12). The last term is a contribution from gravitational loops, which is the sum of $-7/90$ from the conformal-factor field ϕ and $87/20$ from the traceless tensor field $h_{\mu\nu}$. Since b_c is positive, the kinetic term of the Riegert action becomes positive-definite in our signature convention.[3]

Considering the beta function of t defined by $\mu dt/d\mu = -\beta_0 t^3$, one-loop contributions from matter fields to β_0 are summarized to ζ_1 (5-12) divided by $2(4\pi)^2$. Furthermore, adding one-loop quantum corrections by the gravitational field itself, we obtain

$$\beta_0 = \frac{1}{(4\pi)^2}\left\{\frac{1}{240}\left(N_S + 6N_F + 12N_A\right) + \frac{197}{60}\right\}.$$

The last term is the sum of the contributions from the conformal-factor field $-1/15$ and from the traceless tensor field $199/30$. Since β_0 is positive, it turns out that the beta function becomes negative and the traceless tensor field $h_{\mu\nu}$ shows an asymptotically free behavior. Thus, it is justified performing the perturbation theory about a conformally flat spacetime such as (7-2) at high energy.

Here we briefly describe the meaning of the asymptotic freedom in the quantum gravity. First of all, it should be noted that this asymptotic freedom does not mean the existence of free gravitational asymptotic fields like a graviton. Fluctuations of the traceless tensor field become small, but fluctuations of the conformal-factor field that substantially defines physical distance remain large non-perturbatively. It indicates that a conformally invariant spacetime is realized at the high energy limit. As described below, this conformal invariance represents the background-metric independence which shows that the theory does not depend on how to select the background metric $\hat{g}_{\mu\nu}$.[4] Thus, we shall call this property as "asymptotic background freedom", in distinction from the conventional asymptotic freedom.

---

[3] It is easy to understand when discussing in a Wick-rotated Euclidean background space. The weight of the path integral then becomes e^{-I}, and the positive-definiteness is expressed as $I > 0$. The Weyl action also satisfies this positivity condition in the Euclidean space as $I = \int d^4x\sqrt{g}C^2_{\mu\nu\lambda\sigma}/t^2$.

[4] Since the traceless tensor field is handled perturbatively, the background-metric independence for this field is not complete, but its asymptotically free behavior means that it is not significant in the ultraviolet limit.

This also leads to the discussion of $\hbar$ earlier. Since the fourth-order gravitational action is a completely dimensionless quantum-mechanical quantity, the concept of classical asymptotic fields is not adapted here. In addition, the asymptotic background freedom indicates that spacetime singularities are eliminated, because it means that the Weyl curvature tensor including the Riemann curvature tensor disappears at a short distance, so that spacetime configurations that the Riemann curvature diverges such as the Schwarzschild solution are excluded quantum mechanically. In the first place, such a field configuration that the action diverges is unphysical. This is one of the reasons why we should consider the positive-definite action including the Riemann curvature tensor.

In this and next chapters, we consider the quantum gravity defined in the limit where the coupling constant t disappears. The action is then given by

$$S_{\rm 4DQG} = S_{\rm R}(\phi, \hat{g}) + I(g, \varphi, A, \ldots)|_{t\to 0}. \tag{7-6}$$

Since the Weyl action is divided by t^2, only the kinetic term of $h_{\mu\nu}$ remains. Furthermore, at this limit, the metric $\bar{g}_{\mu\nu}$ becomes the background metric $\hat{g}_{\mu\nu}$, thus the interaction terms between the traceless tensor field and other quantum fields disappear. Moreover, we ignore interaction terms with mass parameters such as the Einstein-Hilbert term, the cosmological term, mass terms of matter fields, and so on.

Similar to the discussion in the case of two-dimensional quantum gravity given in the first section of the previous chapter, we can show that using Wess-Zumino integrability condition (5-15), the background-metric independence of the theory $S_{\rm 4DQG}$ holds when the coefficient of the Riegert action is given by the coefficient of conformal anomaly of the whole theory (7-5). Its essence is that, since the conformal-factor field ϕ is an integration variable, the theory remains unchanged even if this field is shifted as $\phi \to \phi - \omega$. Since the theory is also invariant under a simultaneous shift $\phi \to \phi - \omega$ and $\hat{g}_{\mu\nu} \to e^{2\omega}\hat{g}_{\mu\nu}$ that preserves the full metric, it becomes invariant even if we change only the background metric as $\hat{g}_{\mu\nu} \to e^{2\omega}\hat{g}_{\mu\nu}$.

In this way, despite the involvement of the Euler-density conformal anomaly that originally breaks conformal invariance, strict conformal invariance is realized rather owing to that. In the following, we will formulate this background-metric independence as the BRST conformal invariance.

Diffeomorphism Invariance and Conformal Invariance

Diffeomorphism is defined using a vector ξ^μ by

$$\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda .$$

The transformation laws of scalar fields and gauge fields are defined respectively by[5]

$$\begin{aligned} \delta_\xi \varphi &= \xi^\lambda \partial_\lambda \varphi, \\ \delta_\xi A_\mu &= \xi^\lambda \nabla_\lambda A_\mu + A_\lambda \nabla_\mu \xi^\lambda . \end{aligned}$$

Decomposing the metric field $g_{\mu\nu}$ with the conformal factor $e^{2\phi}$ and the metric with the bar $\bar{g}_{\mu\nu}$ as in (7-2), diffeomorphism can be expressed as

$$\begin{aligned} \delta_\xi \phi &= \xi^\lambda \partial_\lambda \phi + \frac{1}{4}\hat{\nabla}_\lambda \xi^\lambda, \\ \delta_\xi \bar{g}_{\mu\nu} &= \bar{g}_{\mu\lambda}\bar{\nabla}_\nu \xi^\lambda + \bar{g}_{\nu\lambda}\bar{\nabla}_\mu \xi^\lambda - \frac{1}{2}\bar{g}_{\mu\nu}\hat{\nabla}_\lambda \xi^\lambda, \end{aligned}$$

where we use the fact that $\bar{\nabla}_\lambda \xi^\lambda = \partial_\lambda(\sqrt{-\bar{g}}\xi^\lambda)/\sqrt{-\bar{g}} = \hat{\nabla}_\lambda \xi^\lambda$ holds due to $\sqrt{-\bar{g}} = \sqrt{-\hat{g}}$. Expanding both sides of the second expression further, we obtain the transformation law of the traceless tensor field as[6]

$$\begin{aligned} \delta_\xi h_{\mu\nu} = \frac{1}{t}&\left(\hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \xi^\lambda\right) + \xi^\lambda \hat{\nabla}_\lambda h_{\mu\nu} \\ &+\frac{1}{2}h_{\mu\lambda}\left(\hat{\nabla}_\nu \xi^\lambda - \hat{\nabla}^\lambda \xi_\nu\right) + \frac{1}{2}h_{\nu\lambda}\left(\hat{\nabla}_\mu \xi^\lambda - \hat{\nabla}^\lambda \xi_\mu\right) \\ &+o(th^2), \end{aligned} \tag{7-7}$$

where gauge transformation parameters with subscript are defined by $\xi_\mu = \hat{g}_{\mu\nu}\xi^\nu$ using the background metric.

At the limit where the coupling constant t vanishes, diffeomorphism reduces to a widely known gauge transformation for the kinetic term of the Weyl action. It can be expressed by replacing ξ^μ with $\kappa^\mu = \xi^\mu/t$ and then taking the limit of $t \to 0$ as

$$\delta_\kappa h_{\mu\nu} = \hat{\nabla}_\mu \kappa_\nu + \hat{\nabla}_\nu \kappa_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \kappa^\lambda, \tag{7-8}$$

[5] Like scalar fields, the transformation law of gauge fields can also be written with normal differentiations as $\delta_\xi A_\mu = \xi^\lambda \partial_\lambda A_\mu + A_\lambda \partial_\mu \xi^\lambda$.

[6] With $\delta_\xi \bar{g}_{\mu\nu} = \delta_\xi(\hat{g}e^{th})_{\mu\nu} = t\delta_\xi h_{\mu\nu} + t^2 h_{\lambda(\mu}\delta_\xi h^\lambda{}_{\nu)} + o(t^3)$ in mind, expand both sides and determine the terms for each order.

while $\delta_\kappa \phi = 0$ and matter fields are not transformed either.

Even if the gauge transformation (7-8) is fixed as usual, 15 gauge degrees of freedom ζ^μ satisfying the conformal Killing equation

$$\hat{\nabla}_\mu \zeta_\nu + \hat{\nabla}_\nu \zeta_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \zeta^\lambda = 0 \tag{7-9}$$

still remain. For ζ^μ, the lowest term in the transformation law (7-7) disappears so that the next term becomes effective. Thus, we obtain the following transformation:

$$\begin{aligned} \delta_\zeta \phi &= \zeta^\lambda \partial_\lambda \phi + \frac{1}{4}\hat{\nabla}_\lambda \zeta^\lambda, \\ \delta_\zeta h_{\mu\nu} &= \zeta^\lambda \hat{\nabla}_\lambda h_{\mu\nu} + \frac{1}{2} h_{\mu\lambda}\left(\hat{\nabla}_\nu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\nu\right) + \frac{1}{2} h_{\nu\lambda}\left(\hat{\nabla}_\mu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\mu\right). \end{aligned} \tag{7-10}$$

The first expression represents a conformal transformation of scalar fields of conformal dimension zero with a shift term added (note that there is no field dependence in the shift term). The second is nothing but the conformal transformation of traceless and symmetric tensor fields with conformal dimension zero.

Here note that usual gauge transformations become independent of the fields as in (7-8) at the vanishing coupling limit, whereas this residual gauge transformation has field-dependences even at the limit. This means that modes in the field are mixed with each other through the gauge transformation even though there is no interaction. This fact becomes important when we show that ghost modes are unphysical.

Next, we give the transformation laws of the matter fields under this gauge fixing that leaves only the conformal Killing vectors as the residual gauge degrees of freedom. First, consider a conformally invariant scalar field φ. In this case, we can remove the conformal-factor dependence in the action explicitly by redefining the field as $\varphi = e^{-\phi}\varphi'$. Using this redefined field φ' and writing it again as φ below, diffeomorphism can be rewritten as

$$\delta_\zeta \varphi = \zeta^\mu \partial_\mu \varphi + \frac{1}{4}\varphi \hat{\nabla}_\mu \zeta^\mu. \tag{7-11}$$

The second term on the right-hand side appears to compensate for the change of the conformal-factor field. This transformation is the same as the conformal transformation of a scalar field with conformal dimension 1.[7]

[7] Specifically, looking at the invariance on the flat background, since the variable ζ^μ satisfies

Since the action of the gauge field does not depend on the conformal-factor field, it is not necessary to redefine the field as in the case of the scalar field. By restricting the gauge transformation parameter to ζ^μ and rewriting using the conformal Killing equation, diffeomorphism given before reduces to

$$\delta_\zeta A_\mu = \zeta^\nu \hat{\nabla}_\nu A_\mu + \frac{1}{4} A_\mu \hat{\nabla}_\nu \zeta^\nu + \frac{1}{2} A_\nu \left(\hat{\nabla}_\mu \zeta^\nu - \hat{\nabla}^\nu \zeta_\mu \right).$$

This means that the gauge field is transformed as a vector field of conformal dimension 1.

These conformal transformations defined on the background $\hat{g}_{\mu\nu}$ are gauge transformations, unlike those in normal conformal field theory. The conformal-factor field and the traceless tensor field are one of the gauge fields, which are not physical fields. Therefore, these fields themselves do not necessary to satisfy the unitarity bound (2-14) for conformal dimensions, as ordinary gauge fields do not so (see Footnote 6 in Chapter 2).

The background-metric independence is expressed as a symmetry in which all of theories that can be transformed one another by the conformal transformation become gauge equivalent. Although the remaining gauge degrees of freedom are only 15, this gauge symmetry gives strong constraints to physical states because the right-hand side of the transformation law (7-10) depends on the field.

On the other hand, when $t \neq 0$, diffeomorphism gradually deviates from the conformal transformation, as can be seen from the original transformation law (7-7). Dynamical processes in which conformal invariance breaks will be discussed in later chapters.

Quantization of Gravitational Field

In order to carry out quantization, we need to choose the background metric $\hat{g}_{\mu\nu}$. In the ultraviolet limit where the coupling constant t disappears, since

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the conformal Killing equation, it can be shown that the action of the scalar field is invariant as follows:

$$\begin{aligned}\delta_\zeta I_\varphi &= -\int d^4x\, \partial^\mu \varphi\, \partial_\mu \left(\zeta^\lambda \partial_\lambda \varphi + \frac{1}{4} \varphi \partial_\lambda \zeta^\lambda \right) \\ &= \int d^4x \left\{ -\frac{1}{4} \left(3\partial_\eta \zeta_0 + \partial_i \zeta^i \right) \partial_\eta \varphi \partial_\eta \varphi + \left(\partial_\eta \zeta_i + \partial_i \zeta_0 \right) \partial_\eta \varphi \partial^i \varphi \right. \\ &\quad \left. + \left[-\partial_i \zeta_j + \frac{1}{4} \delta_{ij} \left(-\partial_\eta \zeta_0 + \partial_k \zeta^k \right) \right] \partial^i \varphi \partial^j \varphi + \frac{1}{8} \left(\partial_\sigma \partial^\sigma \partial_\lambda \zeta^\lambda \right) \varphi^2 \right\} = 0.\end{aligned}$$

spacetime configurations where the Weyl tensor vanishes are realized, the background metric should be conformally flat, but as long as it is so, the background metric can be selected arbitrarily from its independence. In this chapter, we employ the Minkowski metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$ as the background metric $\hat{g}_{\mu\nu}$ and the coordinates are written as $x^\mu = (\eta, \mathbf{x})$.

In this section, we quantize the conformal-factor field and the traceless tensor field. At that time, the gauge degrees of freedom κ^μ of the transformation (7-8) are completely fixed so that only the gauge degrees of freedom ζ^μ of the conformal transformation (7-10) remain.

Conformal-factor field

First we quantize the conformal-factor field. In the Minkowski background, the Riegert action (7-4) is simply given by $-(b_c/8\pi^2)\int d^4x \phi \partial^4 \phi$, and then the linear term of ϕ disappears. The contribution from the linear term appears in the energy-momentum tensor introduced in the next section.

The gravitational field, which is a higher-order derivative field, is canonically quantized according to the Dirac quantization procedure.[8] Introducing a new variable

$$\chi = \partial_\eta \phi, \tag{7-12}$$

the action of the conformal-factor field can be written in a second-order form in time derivative as

$$S_\mathrm{R} = \int d^4x \left\{ -\frac{b_c}{8\pi^2} \left[(\partial_\eta \chi)^2 + 2\chi \partial\!\!\!/^2 \chi + \left(\partial\!\!\!/^2 \phi \right)^2 \right] + v \left(\partial_\eta \phi - \chi \right) \right\},$$

where $\partial\!\!\!/^2 = \partial^i \partial_i$ is the Laplacian operator is space. The last is the Lagrange multiplier term. Canonically conjugate momenta P_χ, P_ϕ, and P_v with respect to χ, ϕ, and v, respectively, are derived from this action, and Poisson brackets are set as

$$\begin{aligned} &\{\chi(\eta, \mathbf{x}), \mathsf{P}_\chi(\eta, \mathbf{x}')\}_\mathrm{P} = \{\phi(\eta, \mathbf{x}), \mathsf{P}_\phi(\eta, \mathbf{x}')\}_\mathrm{P} \\ &= \{v(\eta, \mathbf{x}), \mathsf{P}_v(\eta, \mathbf{x}')\}_\mathrm{P} = \delta_3(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

Since the action of the new field χ is a second-order derivative in time, it has a normal momentum variable $\mathsf{P}_\chi = -(b_c/4\pi^2)\partial_\eta \chi$, while since ϕ and

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[8] See P. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964) and the textbooks in Bibliography.

v are first order and zeroth order respectively, their momenta give constraint equations as[9]

$$\varphi_1 = \mathsf{P}_\phi - v \simeq 0, \qquad \varphi_2 = \mathsf{P}_v \simeq 0.$$

The constraint conditions constitute a subspace in the phase space spanned by six variables, ϕ, χ, v and their conjugate momenta P_ϕ, P_χ, P_v. The weak equation represents that it holds as an exact equation on the subspace.

Poisson brackets between the constraints are given by

$$C_{ab} = \{\varphi_a, \varphi_b\}_\mathrm{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where the three-dimensional delta function is expressed as 1 for simplicity. Since $\det C_{ab} \neq 0$ is satisfied, these constraints are called second class. According to the Dirac procedure, we introduce the Dirac brackets to handle the second class constraints as follows:

$$\{F, G\}_\mathrm{D} = \{F, G\}_\mathrm{P} - \{F, \varphi_a\}_\mathrm{P} C^{-1}_{ab} \{\varphi_b, G\}_\mathrm{P}.$$

The Dirac brackets satisfy the basic properties that Poisson brackets hold. Since the constraint satisfies $\{F, \varphi_a\}_\mathrm{D} = 0$ for an arbitrary function F, the Dirac brackets can be regarded as Poisson brackets in the phase subspace. Substituting Hamiltonian as F, it means that the constraint does not evolve in time, and thus if $\varphi_a = 0$ is set first, it is preserved. Therefore, we can set the constraints to zero as an exact equation using the Dirac brackets.

The Dirac brackets between the four variables in the phase subspace are given by

$$\{\chi(\eta, \mathbf{x}), \mathsf{P}_\chi(\eta, \mathbf{x}')\}_\mathrm{D} = \{\phi(\eta, \mathbf{x}), \mathsf{P}_\phi(\eta, \mathbf{x}')\}_\mathrm{D} = \delta_3(\mathbf{x} - \mathbf{x}'),$$

and the Hamiltonian is written as

$$H = \int d^3\mathbf{x} \left\{ -\frac{2\pi^2}{b_c} \mathsf{P}_\chi^2 + \mathsf{P}_\phi \chi + \frac{b_c}{8\pi^2} \left[2\chi \bar{\partial}^2 \chi + \left(\bar{\partial}^2 \phi \right)^2 \right] \right\}. \tag{7-13}$$

Equations of motion are then given by

$$\begin{aligned} \partial_\eta \phi &= \{\phi, H\}_\mathrm{D} = \chi, \\ \partial_\eta \chi &= \{\chi, H\}_\mathrm{D} = -\frac{4\pi^2}{b_c} \mathsf{P}_\chi, \\ \partial_\eta \mathsf{P}_\chi &= \{\mathsf{P}_\chi, H\}_\mathrm{D} = -\mathsf{P}_\phi - \frac{b_c}{2\pi^2} \bar{\partial}^2 \chi, \\ \partial_\eta \mathsf{P}_\phi &= \{\mathsf{P}_\phi, H\}_\mathrm{D} = -\frac{b_c}{4\pi^2} \bar{\partial}^4 \phi. \end{aligned} \tag{7-14}$$

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[9] If considering the symmetrized Lagrange multiplier term like $(v\partial_\eta \phi - \phi \partial_\eta v)/2$, the constraints become $\varphi_1 = \mathsf{P}_\phi - v/2$ and $\varphi_2 = \mathsf{P}_v + \phi/2$, but the result is the same.

Canonical quantization is performed by replacing the Dirac brackets with commutators as

$$[\phi(\eta,\mathbf{x}),\mathsf{P}_\phi(\eta,\mathbf{x}')]=[\chi(\eta,\mathbf{x}),\mathsf{P}_\chi(\eta,\mathbf{x}')]=i\delta_3(\mathbf{x}-\mathbf{x}'), \quad \text{(7-15)}$$

and other commutators disappear. From (7-14), the momentum variables are given by

$$\mathsf{P}_\chi=-\frac{b_c}{4\pi^2}\partial_\eta\chi, \qquad \mathsf{P}_\phi=-\partial_\eta\mathsf{P}_\chi-\frac{b_c}{2\pi^2}\not{\partial}^2\chi. \quad \text{(7-16)}$$

The conformal-factor field satisfies an equation of motion $\partial^4\phi=0$, which becomes $\partial_\eta\mathsf{P}_\phi=-(b_c/4\pi^2)\not{\partial}^4\phi$ when expressed using the momentum variable. The solution is given by a linear combination of $e^{ik_\mu x^\mu}$ and $\eta e^{ik_\mu x^\mu}$ and their complex conjugate, where $k_\mu x^\mu=-\omega\eta+\mathbf{k}\cdot\mathbf{x}$ and $\omega=|\mathbf{k}|$.

Decomposing the field into two parts of annihilation and creation operators as $\phi=\phi_<+\phi_>$, the annihilation operator is expanded as

$$\phi_<(x)=\frac{\pi}{\sqrt{b_c}}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\omega^{3/2}}\left[a(\mathbf{k})+i\omega\eta b(\mathbf{k})\right]e^{ik_\mu x^\mu}$$

and the creation operator is given by $\phi_>=\phi_<^\dagger$. Substituting this into the defining equations of the variables, (7-12) and (7-16), the annihilation operator part for each variable is given as follows:

$$\chi_<(x)=-i\frac{\pi}{\sqrt{b_c}}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\omega^{1/2}}\left[a(\mathbf{k})+(-1+i\omega\eta)b(\mathbf{k})\right]e^{ik_\mu x^\mu},$$
$$\mathsf{P}_{\chi<}(x)=\frac{\sqrt{b_c}}{4\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\omega^{1/2}\left[a(\mathbf{k})+(-2+i\omega\eta)b(\mathbf{k})\right]e^{ik_\mu x^\mu},$$
$$\mathsf{P}_{\phi<}(x)=-i\frac{\sqrt{b_c}}{4\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\omega^{3/2}\left[a(\mathbf{k})+(1+i\omega\eta)b(\mathbf{k})\right]e^{ik_\mu x^\mu}.$$

From the canonical commutation relations (7-15), we obtain the following commutation relation of each mode:

$$\begin{aligned}\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right]&=\delta_3(\mathbf{k}-\mathbf{k}'),\\ \left[a(\mathbf{k}),b^\dagger(\mathbf{k}')\right]&=\left[b(\mathbf{k}),a^\dagger(\mathbf{k}')\right]=\delta_3(\mathbf{k}-\mathbf{k}'),\\ \left[b(\mathbf{k}),b^\dagger(\mathbf{k}')\right]&=0.\end{aligned}$$

The Hamiltonian operator is given by doing normal ordering to (7-13), which is expressed in terms of the modes as

$$H=\int d^3\mathbf{k}\,\omega\left\{a^\dagger(\mathbf{k})b(\mathbf{k})+b^\dagger(\mathbf{k})a(\mathbf{k})-2b^\dagger(\mathbf{k})b(\mathbf{k})\right\}.$$

The two-point correlation function of the conformal-factor field is given as follows:

$$\langle 0|\phi(x)\phi(x')|0\rangle$$
$$= \frac{\pi^2}{b_c}\int_{\omega>z}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega^3}\left\{1+i\omega\left(\eta-\eta'\right)\right\}e^{-i\omega(\eta-\eta'-i\epsilon)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')},$$

where ϵ is an ultraviolet cutoff to regularize ultraviolet divergences, which is introduced only in the exponential part. This is a necessary treatment to obtain the canonical commutation relation correctly. In addition, we introduced an infinitesimal mass scale z to handle infrared divergences coming from the conformal-factor field being dimensionless. This corresponds to adding a fictitious mass term to the action. Since it breaks diffeomorphism invariance, the z-dependence shall not appear when considering diffeomorphism invariant quantities.[10] The momentum integral is evaluated under $z \ll 1$, while leaving the ultraviolet cutoff ϵ finite.

Momentum integral formulas required for calculations are given as follows:

$$\begin{aligned} I_n(\eta,\mathbf{x}) &= \int_{\omega>z}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega^n}e^{-i\omega(\eta-i\epsilon)+i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{(2\pi)^3}\int_z^\infty\omega^2 d\omega\int_{-1}^1 d\cos\theta\int_0^{2\pi}d\varphi\,\frac{1}{\omega^n}e^{i\omega|\mathbf{x}|\cos\theta}e^{-i\omega(\eta-i\epsilon)} \\ &= \frac{1}{2\pi^2}\frac{1}{|\mathbf{x}|}\int_z^\infty d\omega\,\frac{1}{\omega^{n-1}}\sin(\omega|\mathbf{x}|)e^{-i\omega(\eta-i\epsilon)}, \end{aligned} \tag{7-17}$$

where n is an integer. The infrared cutoff z is necessary when $n \geq 3$. This integral satisfies $I_n(\eta,\mathbf{x}) = i\partial_\eta I_{n+1}(\eta,\mathbf{x})$. The expressions of $n=2,3$ are obtained as

$$\begin{aligned} I_3(\eta,\mathbf{x}) &= \frac{1}{4\pi^2}\left\{-\log\left[-(\eta-i\epsilon)^2+\mathbf{x}^2\right]-\log z^2 e^{2\gamma-2}\right. \\ &\qquad\left.+\frac{\eta-i\epsilon}{|\mathbf{x}|}\log\frac{\eta-i\epsilon-|\mathbf{x}|}{\eta-i\epsilon+|\mathbf{x}|}\right\}, \\ I_2(\eta,\mathbf{x}) &= i\frac{1}{4\pi^2}\frac{1}{|\mathbf{x}|}\log\frac{\eta-i\epsilon-|\mathbf{x}|}{\eta-i\epsilon+|\mathbf{x}|}. \end{aligned}$$

[10] Note that the Einstein-Hilbert action and the cosmological term do not give normal mass terms as described here because these actions involve exponentials of the conformal-factor field.

Using these integral formulas, we obtain

$$\langle 0|\phi(x)\phi(x')|0\rangle = -\frac{1}{4b_c}\log\left\{\left[-(\eta-\eta'-i\epsilon)^2+(\mathbf{x}-\mathbf{x}')^2\right]z^2e^{2\gamma-2}\right\} -\frac{1}{4b_c}\frac{i\epsilon}{|\mathbf{x}-\mathbf{x}'|}\log\frac{\eta-\eta'-i\epsilon-|\mathbf{x}-\mathbf{x}'|}{\eta-\eta'-i\epsilon+|\mathbf{x}-\mathbf{x}'|}. \tag{7-18}$$

The last term which disappears at $\epsilon \to 0$ contributes to calculations of quantum corrections in later sections. The cutoff ϵ is taken to zero after calculating quantum corrections.

Traceless tensor field

From the Weyl action, the kinetic term of the traceless tensor field is given by

$$I = \int d^4x \left\{ -\frac{1}{2}\partial^2 h^{\mu\nu}\partial^2 h_{\mu\nu} + \partial^\mu \chi^\nu \partial_\mu \chi_\nu - \frac{1}{3}\partial_\mu \chi^\mu \partial_\nu \chi^\nu \right\},$$

where $\chi_\mu = \partial^\lambda h_{\lambda\mu}$. In order to quantize the traceless tensor field, we need to fix the gauge symmetry $\delta_\kappa h_{\mu\nu}$ (7-8). For that, we decompose the field as

$$h_{00}, \qquad h_{0i}, \qquad h_{ij} = h^{\mathrm{tr}}_{ij} + \frac{1}{3}\delta_{ij}h_{00},$$

where "tr" denotes the traceless part of the spatial component. The gauge transformation (7-8) is then decomposed as

$$\delta_\kappa h_{00} = \frac{3}{2}\partial_\eta \kappa_0 + \frac{1}{2}\partial_k \kappa^k, \qquad \delta_\kappa h_{0i} = \partial_\eta \kappa_i + \partial_i \kappa_0,$$
$$\delta_\kappa h^{\mathrm{tr}}_{ij} = \partial_i \kappa_j + \partial_j \kappa_i - \frac{2}{3}\delta_{ij}\partial_k \kappa^k.$$

First, using the four gauge degrees of freedom, we impose the transverse gauge conditions defined by

$$\partial^i h_{0i} = 0, \qquad \partial^i h^{\mathrm{tr}}_{ij} = 0.$$

At this time, the h_{00} component becomes a non-dynamical degree of freedom which does not contain time derivative in the kinetic term. Since this component can be removed further using gauge degrees of freedom that still remain after this gauge-fixing, the following additional condition is imposed:

$$h_{00} = 0.$$

The transverse gauge including this extra condition is called the radiation gauge.

When the radiation gauge is adopted, almost all degrees of freedom of κ^μ are fixed, and only the finite gauge degrees of freedom ζ^μ of the conformal transformation (7-10) remain. In fact, when solving conditions that this gauge is preserved as $\delta_\kappa(h_{00}) = (3\partial_\eta \kappa_0 + \partial_k \kappa^k)/2 = 0$, $\delta_\kappa(\partial^i h_{0i}) = \partial_\eta \partial_k \kappa^k + \not{\partial}^2 \kappa_0 = 0$, and $\delta_\kappa(\partial^i h^{\rm tr}_{ij}) = \not{\partial}^2 \kappa_j + \partial_j \partial_k \kappa^k/3 = 0$, we can see that the residual gauge degrees of freedom are $\kappa^\mu = \zeta^\mu$ only.

In this and next chapters, we will simply express the transverse (T) component of h_{0i} and the transverse-traceless (TT) component of h_{ij} in the Gothic style as

$$h^{\rm T}_{0i} = \mathsf{h}_i, \qquad h^{\rm TT}_{ij} = \mathsf{h}_{ij}.$$

Dynamical fields in the radiation gauge are given by these fields.[11]

As in the case of the conformal-factor field, in order to quantize the transverse-traceless tensor mode according to the Dirac procedure, we introduce a new variable defined by

$$\mathsf{u}_{ij} = \partial_\eta \mathsf{h}_{ij}.$$

On the other hand, since the transverse vector mode has at most second-order derivatives in time, it is not necessary to introduce a new variable. With using the variable u_{ij}, the Weyl action can be written as

$$\begin{aligned} I &= \int d^4x \left\{ -\frac{1}{2} \mathsf{h}^{ij} \left(\partial_\eta^4 - 2\not{\partial}^2 \partial_\eta^2 + \not{\partial}^4 \right) \mathsf{h}_{ij} + \mathsf{h}^j \not{\partial}^2 \left(-\partial_\eta^2 + \not{\partial}^2 \right) \mathsf{h}_j \right\} \\ &= \int d^4x \left\{ -\frac{1}{2} \partial_\eta \mathsf{u}^{ij} \partial_\eta \mathsf{u}_{ij} - \mathsf{u}^{ij} \not{\partial}^2 \mathsf{u}_{ij} - \frac{1}{2} \not{\partial}^2 \mathsf{h}^{ij} \not{\partial}^2 \mathsf{h}_{ij} \right. \\ &\qquad \left. + \partial_\eta \mathsf{h}^j \not{\partial}^2 \partial_\eta \mathsf{h}_j + \not{\partial}^2 \mathsf{h}^j \not{\partial}^2 \mathsf{h}_j + \lambda^{ij} \left(\partial_\eta \mathsf{h}_{ij} - \mathsf{u}_{ij} \right) \right\}, \end{aligned}$$

where λ^{ij} is the Lagrange multiplier.

By solving constraint equations and removing the Lagrange multiplier λ^{ij} as before, we obtain a phase subspace spanned by the canonical variables

[11] As another gauge-fixing condition, we can choose $h_{0\mu} = 0$. In this case, the spatial component can be decomposed as $h_{ij} = \mathsf{h}_{ij} + \partial_\eta^{-1}(\partial_i \mathsf{h}_j + \partial_j \mathsf{h}_i) + (\delta_{ij} - 3\partial_i \partial_j/\not{\partial}^2)\mathsf{h}$, then the h component of the Weyl action becomes $-\mathsf{h}(3\partial_\eta^2 - \not{\partial}^2)^2 \mathsf{h}/3$. On the other hand, we can see from $\delta_\kappa h_{0\mu} = 0$ that there still remains a scalar gauge degree of freedom $\psi = \partial_k \kappa^k$ satisfying $(3\partial_\eta^2 - \not{\partial}^2)\psi = 0$. Therefore, we can further impose an extra condition $\mathsf{h} = 0$ using it. In this way, the same result as the radiation gauge can be obtained.

u_{ij}, h_{ij}, h_j and their conjugate momenta

$$\begin{aligned} &\mathsf{P}_\mathsf{u}^{ij} = -\partial_\eta \mathsf{u}^{ij}, \qquad \mathsf{P}_\mathsf{h}^{ij} = -\partial_\eta \mathsf{P}_\mathsf{u}^{ij} - 2\not{\partial}^2 \mathsf{u}^{ij}, \\ &\mathsf{P}^j = 2\not{\partial}^2 \partial_\eta \mathsf{h}^j, \end{aligned}$$

respectively. By replacing the Dirac brackets with commutators, canonical commutation relations between them are provided as

$$\begin{aligned} \left[\mathsf{h}^{ij}(\eta, \mathbf{x}), \mathsf{P}_\mathsf{h}^{kl}(\eta, \mathbf{y})\right] &= \left[\mathsf{u}^{ij}(\eta, \mathbf{x}), \mathsf{P}_\mathsf{u}^{kl}(\eta, \mathbf{y})\right] = i\delta_3^{ij,kl}(\mathbf{x}-\mathbf{y}), \\ \left[\mathsf{h}^{i}(\eta, \mathbf{x}), \mathsf{P}^{j}(\eta, \mathbf{y})\right] &= i\delta_3^{ij}(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{7-19}$$

where the three-dimensional delta functions with indices are defined by $\delta_3^{ij}(\mathbf{x}) = \Delta^{ij}\delta_3(\mathbf{x})$ and $\delta_3^{ij,kl}(\mathbf{x}) = \Delta^{ij,kl}\delta_3(\mathbf{x})$ with differential operators

$$\begin{aligned} \Delta_{ij} &= \delta_{ij} - \frac{\partial_i \partial_j}{\not{\partial}^2}, \\ \Delta_{ij,kl} &= \frac{1}{2}\left(\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk} - \Delta_{ij}\Delta_{kl}\right). \end{aligned}$$

These operators satisfy the transverse and traceless conditions $\partial^i \Delta_{ij} = 0$, $\Delta^j{}_j = 2$, $\partial^i \Delta_{ij,kl} = 0$, and $\Delta^i{}_{i,kl} = 0$. They also satisfy $\Delta_{ik}\Delta^k{}_j = \Delta_{ij}$ and $\Delta_{ij,kl}\Delta^{kl,}{}_{mn} = \Delta_{ij,mn}$.

The Hamiltonian operator is given by

$$\begin{aligned} H = \int d^3\mathbf{x} : \Big\{ &-\frac{1}{2}\mathsf{P}_\mathsf{u}^{ij}\mathsf{P}^\mathsf{u}_{ij} + \mathsf{P}_\mathsf{h}^{ij}\mathsf{u}_{ij} + \mathsf{u}^{ij}\not{\partial}^2\mathsf{u}_{ij} + \frac{1}{2}\not{\partial}^2\mathsf{h}^{ij}\not{\partial}^2\mathsf{h}_{ij} \\ &+\frac{1}{4}\mathsf{P}^j\not{\partial}^{-2}\mathsf{P}_j - \not{\partial}^2\mathsf{h}^j\not{\partial}^2\mathsf{h}_j \Big\} :, \end{aligned}$$

where $\not{\partial}^{-2} = 1/\not{\partial}^2$.

The transverse-traceless tensor mode satisfies an equation of motion $\partial^4 \mathsf{h}^{ij} = 0$, which can be written as $\partial_\eta \mathsf{P}_\mathsf{h}^{ij} = -\not{\partial}^4\mathsf{h}^{ij}$ in terms of the momentum variable. As in the case of the conformal-factor field, we decompose the field into parts of annihilation and creation operators as $\mathsf{h}^{ij} = \mathsf{h}_<^{ij} + \mathsf{h}_>^{ij}$. The annihilation operator is expanded as

$$\mathsf{h}_<^{ij}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{2\omega^{3/2}} \left[\mathsf{c}^{ij}(\mathbf{k}) + i\omega\eta\, \mathsf{d}^{ij}(\mathbf{k})\right] e^{ik_\mu x^\mu}$$

and the creation operator is given by $\mathsf{h}_>^{ij} = \mathsf{h}_<^{ij\dagger}$. On the other hand, the equation of motion of the transverse vector mode is $\not{\partial}^2\partial^2\mathsf{h}^j = 0$, or $\partial_\eta \mathsf{P}^j =$

$2\not{\partial}^4 \mathsf{h}^j$ in the momentum variable. Since it is a second-order derivative in time, the annihilation operator is expanded as

$$\mathsf{h}^j_<(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{2\omega^{3/2}} \mathsf{e}_j(\mathbf{k}) e^{ik_\mu x^\mu},$$

where $\mathsf{h}^j = \mathsf{h}^j_< + \mathsf{h}^j_>$ and $\mathsf{h}^j_> = \mathsf{h}^{j\dagger}_<$ as before. Similarly, the annihilation operators of other variables are given by

$$\begin{aligned}
\mathsf{u}^{ij}_<(x) &= -i \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{2\omega^{1/2}} \left[\mathsf{c}^{ij}(\mathbf{k}) + (-1 + i\omega\eta)\mathsf{d}^{ij}(\mathbf{k})\right] e^{ik_\mu x^\mu}, \\
\mathsf{P}^{ij}_{\mathsf{u}<}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{\omega^{1/2}}{2} \left[\mathsf{c}^{ij}(\mathbf{k}) + (-2 + i\omega\eta)\mathsf{d}^{ij}(\mathbf{k})\right] e^{ik_\mu x^\mu}, \\
\mathsf{P}^{ij}_{\mathsf{h}<}(x) &= -i \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{\omega^{3/2}}{2} \left[\mathsf{c}^{ij}(\mathbf{k}) + (1 + i\omega\eta)\mathsf{d}^{ij}(\mathbf{k})\right] e^{ik_\mu x^\mu}, \\
\mathsf{P}^{j}_<(x) &= i \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \, \omega^{3/2} \, \mathsf{e}^j(\mathbf{k}) e^{ik_\mu x^\mu}.
\end{aligned}$$

Substituting these expressions into the canonical commutation relations (7-19), we obtain the following commutation relation of each mode:

$$\begin{aligned}
\left[\mathsf{c}^{ij}(\mathbf{k}), \mathsf{c}^{kl\dagger}(\mathbf{k}')\right] &= \delta_3^{ij,kl}(\mathbf{k} - \mathbf{k}'), \\
\left[\mathsf{c}^{ij}(\mathbf{k}), \mathsf{d}^{kl\dagger}(\mathbf{k}')\right] &= \left[\mathsf{d}^{ij}(\mathbf{k}), \mathsf{c}^{kl\dagger}(\mathbf{k}')\right] = \delta_3^{ij,kl}(\mathbf{k} - \mathbf{k}'), \\
\left[\mathsf{d}^{ij}(\mathbf{k}), \mathsf{d}^{kl\dagger}(\mathbf{k}')\right] &= 0, \\
\left[\mathsf{e}^{i}(\mathbf{k}), \mathsf{e}^{j\dagger}(\mathbf{k}')\right] &= -\delta_3^{ij}(\mathbf{k} - \mathbf{k}'),
\end{aligned}$$

where $\delta_3^{ij}(\mathbf{k})$ and $\delta_3^{ij,kl}(\mathbf{k})$ are the delta function $\delta_3(\mathbf{k})$ multiplied by the following functions:

$$\begin{aligned}
\tilde{\Delta}_{ij}(\mathbf{k}) &= \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}, \\
\tilde{\Delta}_{ij,kl}(\mathbf{k}) &= \frac{1}{2}\left\{\tilde{\Delta}_{ik}(\mathbf{k})\tilde{\Delta}_{jl}(\mathbf{k}) + \tilde{\Delta}_{il}(\mathbf{k})\tilde{\Delta}_{jk}(\mathbf{k}) - \tilde{\Delta}_{ij}(\mathbf{k})\tilde{\Delta}_{kl}(\mathbf{k})\right\},
\end{aligned} \quad (7\text{-}20)$$

respectively.

To further simplify the commutation relations, we introduce polarization vectors $\varepsilon^i_{(a)}$ $(a = 1, 2)$ and polarization tensors $\varepsilon^{ij}_{(a)}$ $(a = 1, 2)$. They satisfy a transverse condition $k_i \varepsilon^i_{(a)} = 0$ and transverse and traceless conditions

$k_i \varepsilon^{ij}_{(a)}(\mathbf{k}) = \varepsilon^{\ i}_{(a)i}(\mathbf{k}) = 0$, and are normalized as

$$\sum_{a=1}^{2} \varepsilon^{i}_{(a)}(\mathbf{k}) \varepsilon^{j}_{(a)}(\mathbf{k}) = \tilde{\Delta}^{ij}(\mathbf{k}), \qquad \varepsilon^{j}_{(a)}(\mathbf{k}) \varepsilon_{(b)j}(\mathbf{k}) = \delta_{ab},$$
$$\sum_{a=1}^{2} \varepsilon^{ij}_{(a)}(\mathbf{k}) \varepsilon^{kl}_{(a)}(\mathbf{k}) = \tilde{\Delta}^{ij,kl}(\mathbf{k}), \qquad \varepsilon^{ij}_{(a)}(\mathbf{k}) \varepsilon_{(b)ij}(\mathbf{k}) = \delta_{ab}.$$

Each mode can be expanded using these as

$$\mathsf{c}^{ij}(\mathbf{k}) = \sum_{a=1}^{2} \varepsilon^{ij}_{(a)}(\mathbf{k}) \mathsf{c}_{(a)}(\mathbf{k}), \quad \mathsf{d}^{ij}(\mathbf{k}) = \sum_{a=1}^{2} \varepsilon^{ij}_{(a)}(\mathbf{k}) \mathsf{d}_{(a)}(\mathbf{k}),$$
$$\mathsf{e}^{j}(\mathbf{k}) = \sum_{a=1}^{2} \varepsilon^{j}_{(a)}(\mathbf{k}) \mathsf{e}_{(a)}(\mathbf{k}),$$

then the commutation relations are given by

$$\begin{aligned}
&\left[\mathsf{c}_{(a)}(\mathbf{k}), \mathsf{c}^{\dagger}_{(b)}(\mathbf{k}')\right] = \delta_{ab}\delta_3(\mathbf{k}-\mathbf{k}'),\\
&\left[\mathsf{c}_{(a)}(\mathbf{k}), \mathsf{d}^{\dagger}_{(b)}(\mathbf{k}')\right] = \left[\mathsf{d}_{(a)}(\mathbf{k}), \mathsf{c}^{\dagger}_{(b)}(\mathbf{k}')\right] = \delta_{ab}\delta_3(\mathbf{k}-\mathbf{k}'),\\
&\left[\mathsf{d}_{(a)}(\mathbf{k}), \mathsf{d}^{\dagger}_{(b)}(\mathbf{k}')\right] = 0,\\
&\left[\mathsf{e}_{(a)}(\mathbf{k}), \mathsf{e}^{\dagger}_{(b)}(\mathbf{k}')\right] = -\delta_{ab}\delta_3(\mathbf{k}-\mathbf{k}').
\end{aligned}$$

The Hamiltonian operator is rewritten as

$$\begin{aligned}
H = \sum_{a=1}^{2} \int d^3\mathbf{k}\, \omega \Big\{ &\mathsf{c}^{\dagger}_{(a)}(\mathbf{k}) \mathsf{d}_{(a)}(\mathbf{k}) + \mathsf{d}^{\dagger}_{(a)}(\mathbf{k}) \mathsf{c}_{(a)}(\mathbf{k}) \\
&-2\mathsf{d}^{\dagger}_{(a)}(\mathbf{k}) \mathsf{d}_{(a)}(\mathbf{k}) - \mathsf{e}^{\dagger}_{(a)}(\mathbf{k}) \mathsf{e}_{(a)}(\mathbf{k}) \Big\}.
\end{aligned}$$

The two-point correlation functions of the transverse-traceless tensor mode and the transverse vector mode are calculated as follows. Let us introduce new real fields $H^{(a)}$ and $Y^{(a)}$ whose annihilation operator parts are defined as

$$H^{(a)}_{<}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{2\omega^{3/2}} \left[\mathsf{c}_{(a)}(\mathbf{k}) + i\omega\eta \mathsf{d}_{(a)}(\mathbf{k})\right] e^{ik_\mu x^\mu},$$
$$Y^{(a)}_{<}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{2\omega^{3/2}} \mathsf{e}_{(a)}(\mathbf{k}) e^{ik_\mu x^\mu}.$$

The correlation function of $H^{(a)}$ is then expressed as $\langle H^{(a)}(x)H^{(b)}(x')\rangle = \delta_{ab}\langle H(x)H(x')\rangle$ where

$$\langle H(x)H(x')\rangle = -\frac{1}{16\pi^2}\log\left\{\left[-(\eta-\eta'-i\epsilon)^2+(\mathbf{x}-\mathbf{x}')^2\right]z^2e^{2\gamma-2}\right\}$$
$$-\frac{1}{16\pi^2}\frac{i\epsilon}{|\mathbf{x}-\mathbf{x}'|}\log\frac{\eta-\eta'-i\epsilon-|\mathbf{x}-\mathbf{x}'|}{\eta-\eta'-i\epsilon+|\mathbf{x}-\mathbf{x}'|},$$

and the correlation function of $Y^{(a)}$ is also given by $\langle Y^{(a)}(x)Y^{(b)}(x')\rangle = \delta_{ab}\langle Y(x)Y(x')\rangle$ where

$$\langle Y(x)Y(x')\rangle = \frac{1}{16\pi^2}\log\left\{\left[-(\eta-\eta'-i\epsilon)^2+(\mathbf{x}-\mathbf{x}')^2\right]z^2e^{2\gamma-2}\right\}$$
$$-\frac{1}{16\pi^2}\frac{\eta-\eta'-i\epsilon}{|\mathbf{x}-\mathbf{x}'|}\log\frac{\eta-\eta'-i\epsilon-|\mathbf{x}-\mathbf{x}'|}{\eta-\eta'-i\epsilon+|\mathbf{x}-\mathbf{x}'|}.$$

Using these functions, the two-point correlation functions of the transverse-traceless tensor mode and the transverse vector mode can be expressed as

$$\begin{aligned}\langle \mathsf{h}_{ij}(x)\mathsf{h}_{kl}(x')\rangle &= \Delta_{ij,kl}(\mathbf{x})\langle H(x)H(x')\rangle,\\ \langle \mathsf{h}_i(x)\mathsf{h}_j(x')\rangle &= \Delta_{ij}(\mathbf{x})\langle Y(x)Y(x')\rangle.\end{aligned} \tag{7-21}$$

Generators of Diffeomorphism

The quantum gravity is now described as a certain quantum field theory defined on the background spacetime. Its energy-momentum tensor is thus defined by a variation of the action $S_{\rm 4DQG}$ (7-6) with respect to the background metric as

$$\hat{\Theta}^{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}}\frac{\delta S_{4DQG}}{\delta \hat{g}_{\mu\nu}}.$$

The spacetime indices are raised and lowered with using the background metric as $\hat{\Theta}_{\mu\nu} = \hat{g}_{\mu\lambda}\hat{g}_{\nu\sigma}\hat{\Theta}^{\lambda\sigma}$. After carrying out the variation, we set the background metric to the Minkowski metric. Generators of diffeomorphism δ_ζ (7-10) given in the form of the conformal transformations are then provided by

$$Q_\zeta = \int d^3\mathbf{x}\,\zeta^\lambda\,\hat{\Theta}_{\lambda 0}.$$

Conformal-factor field The energy-momentum tensor of the conformal-factor field is given by

$$\hat{\Theta}_{\mu\nu} = -\frac{b_c}{8\pi^2}\Big\{-4\partial^2\phi\partial_\mu\partial_\nu\phi + 2\partial_\mu\partial^2\phi\partial_\nu\phi + 2\partial_\nu\partial^2\phi\partial_\mu\phi$$

$$
+\frac{8}{3}\partial_\mu\partial_\lambda\phi\partial_\nu\partial^\lambda\phi - \frac{4}{3}\partial_\mu\partial_\nu\partial_\lambda\phi\partial^\lambda\phi
+\eta_{\mu\nu}\left(\partial^2\phi\partial^2\phi - \frac{2}{3}\partial^2\partial^\lambda\phi\partial_\lambda\phi - \frac{2}{3}\partial_\lambda\partial_\sigma\phi\partial^\lambda\partial^\sigma\phi\right)
-\frac{2}{3}\partial_\mu\partial_\nu\partial^2\phi + \frac{2}{3}\eta_{\mu\nu}\partial^4\phi\Big\}.
$$

The last two linear terms are derived by varying the second term of the Riegert action (7-4). It satisfies traceless and conservation conditions as $\hat{\Theta}^\lambda_{\ \lambda} = -(b_c/4\pi^2)\partial^4\phi = 0$ and $\partial^\mu\hat{\Theta}_{\mu\nu} = -(b_c/4\pi^2)\partial^4\phi\partial_\nu\phi = 0$ using the equation of motion.

With the four canonical variables, the (00) component is expressed as

$$
\begin{aligned}
\hat{\Theta}_{00} = &-\frac{2\pi^2}{b_c}\mathsf{P}_\chi^2 + \mathsf{P}_\phi\chi - \mathsf{P}_\chi\not{\partial}^2\phi - \partial_k\mathsf{P}_\chi\partial^k\phi \\
&+\frac{b_c}{8\pi^2}\left(\frac{2}{3}\chi\not{\partial}^2\chi - \frac{4}{3}\partial_k\chi\partial^k\chi + \not{\partial}^2\phi\not{\partial}^2\phi - \frac{2}{3}\partial_k\not{\partial}^2\phi\partial^k\phi \right.\\
&\left.-\frac{2}{3}\partial_k\partial_l\phi\partial^k\partial^l\phi\right) + \frac{1}{3}\not{\partial}^2\mathsf{P}_\chi + \frac{b_c}{12\pi^2}\not{\partial}^4\phi,
\end{aligned}
$$

and the $(0j)$ component is

$$
\begin{aligned}
\hat{\Theta}_{0j} = &\frac{2}{3}\mathsf{P}_\chi\partial_j\chi - \frac{1}{3}\partial_j\mathsf{P}_\chi\chi + \mathsf{P}_\phi\partial_j\phi \\
&+\frac{b_c}{8\pi^2}\left(4\partial_j\chi\not{\partial}^2\phi - \frac{8}{3}\partial_k\chi\partial_j\partial^k\phi - 2\chi\partial_j\not{\partial}^2\phi + 2\not{\partial}^2\chi\partial_j\phi \right.\\
&\left.+\frac{4}{3}\partial_j\partial_k\chi\partial^k\phi\right) - \frac{1}{3}\partial_j\mathsf{P}_\phi - \frac{b_c}{12\pi^2}\partial_j\not{\partial}^2\chi.
\end{aligned}
$$

The generator Q_ζ can be obtained by substituting these expressions into the definition above and performing normal ordering. Specifically, the generator for each conformal Killing vector (2-4) can be obtained from (2-11). Then the translation generator is given by

$$
P_0 = H = \int d^3\mathbf{x}\,\mathcal{A}, \qquad P_j = \int d^3\mathbf{x}\,\mathcal{B}_j, \tag{7-22}
$$

where the local operators $\mathcal{A}$ and $\mathcal{B}_j$ are

$$
\begin{aligned}
\mathcal{A} &= -\frac{2\pi^2}{b_c} :\mathsf{P}_\chi^2: + :\mathsf{P}_\phi\chi: + \frac{b_c}{8\pi^2}\left(2:\chi\not{\partial}^2\chi: + :\not{\partial}^2\phi\not{\partial}^2\phi:\right), \\
\mathcal{B}_j &= :\mathsf{P}_\chi\partial_j\chi: + :\mathsf{P}_\phi\partial_j\phi: .
\end{aligned} \tag{7-23}
$$

The generator of Lorentz transformation is

$$M_{0j} = \int d^3\mathbf{x} \left\{ -\eta \mathcal{B}_j - x_j \mathcal{A} - :\mathsf{P}_\chi \partial_j \phi: \right\},$$
$$M_{ij} = \int d^3\mathbf{x} \left\{ x_i \mathcal{B}_j - x_j \mathcal{B}_i \right\}. \qquad (7\text{-}24)$$

The generators of dilatation and special conformal transformation are respectively given by

$$D = \int d^3\mathbf{x} \left\{ \eta \mathcal{A} + x^k \mathcal{B}_k + :\mathsf{P}_\chi \chi: + \mathsf{P}_\phi \right\} \qquad (7\text{-}25)$$

and

$$K_0 = \int d^3\mathbf{x} \bigg\{ \left(\eta^2 + \mathbf{x}^2\right) \mathcal{A} + 2\eta x^k \mathcal{B}_k + 2\eta :\mathsf{P}_\chi \chi: + 2x^k :\mathsf{P}_\chi \partial_k \phi:$$
$$- \frac{b_c}{4\pi^2} \left(2 :\chi^2: + :\partial_k \phi \partial^k \phi: \right) + 2\eta \mathsf{P}_\phi + 2\mathsf{P}_\chi \bigg\},$$
$$K_j = \int d^3\mathbf{x} \bigg\{ \left(-\eta^2 + \mathbf{x}^2\right) \mathcal{B}_j - 2x_j x^k \mathcal{B}_k - 2\eta x_j \mathcal{A} - 2x_j :\mathsf{P}_\chi \chi:$$
$$-2\eta :\mathsf{P}_\chi \partial_j \phi: - \frac{b_c}{2\pi^2} :\chi \partial_j \phi: - 2x_j \mathsf{P}_\phi \bigg\}. \qquad (7\text{-}26)$$

Although M_{0j}, D, and K_μ include the time variable η explicitly, they are independent of the time as $\partial_\eta M_{0j} = \partial_\eta D = \partial_\eta K_\mu = 0$. The linear terms in D and K_μ generate the shift term in the transformation $\delta_\zeta \phi$ (7-10).

Traceless tensor field Similarly, we can obtain generators of the conformal transformations of the traceless tensor field in the radiation gauge by using the energy-momentum tensor derived from the Weyl action. Only the results are shown below.

The translation generator is expressed as

$$P_0 = H = \int d^3\mathbf{x}\, \mathcal{A}, \qquad P_j = \int d^3\mathbf{x}\, \mathcal{B}_j,$$

where the local operators $\mathcal{A}$ and $\mathcal{B}_j$ are given by

$$\mathcal{A} = -\frac{1}{2} :\mathsf{P}_\mathsf{u}^{kl} \mathsf{P}^\mathsf{u}_{kl}: + :\mathsf{P}_\mathsf{h}^{kl} \mathsf{u}_{kl}: + :\mathsf{u}^{kl} \partial\!\!\!/^2 \mathsf{u}_{kl}: + \frac{1}{2} :\partial\!\!\!/^2 \mathsf{h}^{kl} \partial\!\!\!/^2 \mathsf{h}_{kl}:$$
$$+ \frac{1}{4} :\mathsf{P}^k \partial\!\!\!/^{-2} \mathsf{P}_k: - :\partial\!\!\!/^2 \mathsf{h}^k \partial\!\!\!/^2 \mathsf{h}_k:,$$
$$\mathcal{B}_j = :\mathsf{P}_\mathsf{u}^{kl} \partial_j \mathsf{u}_{kl}: + :\mathsf{P}_\mathsf{h}^{kl} \partial_j \mathsf{h}_{kl}: + :\mathsf{P}^k \partial_j \mathsf{h}_k: .$$

The generator of Lorentz transformation is

$$M_{0j} = \int d^3\mathbf{x} \left\{ -\eta \mathcal{B}_j - x_j \mathcal{A} - \mathcal{C}_j \right\},$$
$$M_{ij} = \int d^3\mathbf{x} \left\{ x_i \mathcal{B}_j - x_j \mathcal{B}_i + \mathcal{C}_{ij} \right\},$$

where the local operators $\mathcal{C}_j$ and $\mathcal{C}_{ij}$ are defined by

$$\begin{aligned} \mathcal{C}_j &= :\mathsf{P}_{\mathsf{u}}^{kl} \partial_j \mathsf{h}_{kl}: + :\mathsf{P}_{\mathsf{u}\ j}^{k} \partial\!\!\!/^{-2} \mathsf{P}_k: + 2:\mathsf{P}_{\mathsf{h}\ j}^{k} \mathsf{h}_k: \\ &\quad + :\mathsf{h}^k_{\ j} \mathsf{P}_k: + 2:\mathsf{u}^k_{\ j} \partial\!\!\!/^2 \mathsf{h}_k:, \\ \mathcal{C}_{ij} &= 2\left(:\mathsf{P}_{\mathsf{u}\ i}^{k} \mathsf{u}_{kj}: - :\mathsf{P}_{\mathsf{u}\ j}^{k} \mathsf{u}_{ki}: \right) + 2\left(:\mathsf{P}_{\mathsf{h}\ i}^{k} \mathsf{h}_{kj}: - :\mathsf{P}_{\mathsf{h}\ j}^{k} \mathsf{h}_{ki}: \right) \\ &\quad + :\mathsf{P}_i \mathsf{h}_j: - :\mathsf{P}_j \mathsf{h}_i: . \end{aligned}$$

The generator of dilatation is given by

$$D = \int d^3\mathbf{x} \left\{ \eta \mathcal{A} + x^k \mathcal{B}_k + :\mathsf{P}_{\mathsf{u}}^{kl} \mathsf{u}_{kl}: \right\}.$$

The generator of special conformal transformation is

$$K_0 = -\eta^2 P_0 + 2\eta D + N_0, \qquad K_j = \eta^2 P_j + 2\eta M_{0j} + N_j,$$

where $N_0 = \int d^3\mathbf{x}\, \mathbf{x}^2 \hat{\Theta}_{00}$ and $N_j = \int d^3\mathbf{x} \left(\mathbf{x}^2 \hat{\Theta}_{0j} - 2 x_j x^k \hat{\Theta}_{0k} \right)$ are given by

$$\begin{aligned} N_0 &= \int d^3\mathbf{x} \Big\{ \mathbf{x}^2 \mathcal{A} + 2x^k \mathcal{C}_k - 2:\mathsf{u}^{kl} \mathsf{u}_{kl}: - :\partial^m \mathsf{h}^{kl} \partial_m \mathsf{h}_{kl}: \\ &\qquad\qquad - \frac{5}{4} :\partial\!\!\!/^{-2} \mathsf{P}^k \partial\!\!\!/^{-2} \mathsf{P}_k: - 4:\partial^k \mathsf{h}^l \partial_k \mathsf{h}_l: \Big\}, \\ N_j &= \int d^3\mathbf{x} \Big\{ \mathbf{x}^2 \mathcal{B}_j - 2x_j x^k \mathcal{B}_k + 2x^k \mathcal{C}_{kj} - 2x_j :\mathsf{P}_{\mathsf{u}}^{kl} \mathsf{u}_{kl}: \\ &\qquad\qquad - 2:\mathsf{u}^{kl} \partial_j \mathsf{h}_{kl}: + 2:\partial\!\!\!/^{-2} \mathsf{P}^k \partial_j \mathsf{h}_k: - 4:\mathsf{P}_{\mathsf{u}\ j}^{k} \mathsf{h}_k: \\ &\qquad\qquad - 4:\mathsf{u}^k_{\ j} \partial\!\!\!/^{-2} \mathsf{P}_k: + 4:\mathsf{h}^k_{\ j} \partial\!\!\!/^2 \mathsf{h}_k: \Big\}. \end{aligned}$$

Conformal Algebra and Primary Fields

Let us examine conformal algebra and transformation laws of various field operators using the generators obtained in the previous section. To do that, we first explain the calculation method. As a simple exercise, calculations in the case of a free scalar field are presented in the third section of Appendix B.

We here use the fact that an operator product of two Hermitian operators A and B is expressed as

$$A(x)B(y) = \langle 0|A(x)B(y)|0\rangle + :A(x)B(y): .$$

The two-point function part that diverges at a short distance can be calculated as

$$\langle 0|A(x)B(y)|0\rangle = [A_<(x), B_>(y)] ,$$

where $A_<$ is an annihilation operator part of A and $B_>$ is a creation operator part of B, as defined earlier. With care to $: A(x)B(y) :=: B(y)A(x) :$, commutation relation between the two operators can be expressed as

$$[A(x), B(y)] = \langle 0|A(x)B(y)|0\rangle - \langle 0|B(y)A(x)|0\rangle.$$

Since the second term on the right-hand side can be written using its Hermitian conjugate as $\langle 0|B(y)A(x)|0\rangle = \langle 0|A(x)B(y)|0\rangle^\dagger$, it turns out that the commutation relation disappears if the right-hand side is a real function.

The two-point correlation function of the conformal-factor field ϕ is already calculated in (7-18). Two-point correlation functions including other field variables χ, P_χ, and P_ϕ can be calculated directly using their mode expansion expressions. Alternatively, they can be obtained by differentiating the correlation function (7-18), according to the definitions of the field variables. Using these correlation functions, the equal-time commutation relation can be expressed as follows:

$$\begin{aligned}[\phi(\eta, \mathbf{x}), \mathsf{P}_\phi(\eta, \mathbf{x}')] &= \langle 0|\phi(\eta, \mathbf{x})\mathsf{P}_\phi(\eta, \mathbf{x}')|0\rangle - \mathrm{h.c.} \\ &= i\frac{1}{\pi^2}\frac{\epsilon}{[(\mathbf{x}-\mathbf{x}')^2+\epsilon^2]^2},\end{aligned}$$

where h.c. represents the Hermitian conjugate and the right-hand side is a regularized delta function[12]

$$\delta_3(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}-\epsilon\omega} = \frac{1}{\pi^2}\frac{\epsilon}{(\mathbf{x}^2+\epsilon^2)^2}. \qquad (7\text{-}27)$$

The equal-time commutation relation between χ and P_χ is the same as above. For other commutation relations, it can be shown that they all vanish

[12] It can be written as $\delta_3(\mathbf{x}) = I_0(\eta = 0, \mathbf{x})$ using the integral formula (7-17).

because corresponding two-point functions become real while keeping ϵ finite. In this way, the canonical commutation relations are derived correctly.

Next, consider equal-time commutation relations between composite operators. From Wick's operator product expansion, we obtain the following formula:

$$\begin{aligned}
\Big[:AB(x):, :\prod_k C_k(y): \Big] &= \sum_i \left[A(x), C_i(y)\right] :B(x) \prod_{k(\neq i)} C_k(y): \\
&+ \sum_i \left[B(x), C_i(y)\right] :A(x) \prod_{k(\neq i)} C_k(y): \\
&+ \sum_{i,j(i\neq j)} \big\{ \langle 0|A(x)C_i(y)|0\rangle \langle 0|B(x)C_j(y)|0\rangle - \text{h.c.} \big\} :\prod_{k(\neq i,j)} C_k(y): .
\end{aligned}$$

The last term represents a quantum correction, and it disappears if it is real.

The conformal algebra of $SO(4,2)$ (2-6) can be calculated using this formula. Compared with the case of a free scalar field in the third section of Appendix B, more complicated correction functions appear in the case of the conformal-factor field, but they all disappear and it can be shown that the conformal algebra closes at the quantum level. In the case of the traceless tensor field, unfortunately, it is not easy to show that conformal algebra closes by this method. However, on $\mathbb{R} \times S^3$ discussed in the next chapter, it can be shown that the conformal algebra for the traceless tensor field also closes successfully.

We here consider transformation laws in which a finite quantum correction term remains. First, consider conformal transformations of a composite operator $:\phi^n:$. An equal-time commutation relation with the local operator $\mathcal{A}$ (7-23) is calculated as

$$\begin{aligned}
[\mathcal{A}(\mathbf{x}), :\phi^n(\mathbf{y}):] &= -in\delta_3(\mathbf{x}-\mathbf{y}) :\chi\phi^{n-1}(\mathbf{y}): \\
&= -i\delta_3(\mathbf{x}-\mathbf{y})\, \partial_\eta :\phi^n(\mathbf{y}): . \qquad \text{(7-28)}
\end{aligned}$$

For simplicity, field operators are expressed with time dependence omitted, when considering equal-time commutation relations.

A quantum correction term appears in commutation relation with the local operator $\mathcal{B}_j$ (7-23) as

$$\begin{aligned}
[\mathcal{B}_j(\mathbf{x}), :\phi^n(\mathbf{y}):] &= -i\delta_3(\mathbf{x}-\mathbf{y})\, \partial_j :\phi^n(\mathbf{y}): \\
&+ i\frac{1}{2b_c} n(n-1) e_j(\mathbf{x}-\mathbf{y}) :\phi^{n-2}(\mathbf{x}):, \qquad \text{(7-29)}
\end{aligned}$$

where $e_j(\mathbf{x})$ representing quantum correction is given by

$$e_j(\mathbf{x}) = \frac{1}{\pi^2}\frac{\epsilon x_j[1-h(\mathbf{x})]}{\mathbf{x}^2(\mathbf{x}^2+\epsilon^2)^2}, \qquad h(\mathbf{x}) = \frac{i\epsilon}{2|\mathbf{x}|}\log\frac{i\epsilon+|\mathbf{x}|}{i\epsilon-|\mathbf{x}|}.$$

The function h satisfies $h^\dagger(\mathbf{x}) = h(\mathbf{x})$ and $\lim_{\mathbf{x}\to 0} h(\mathbf{x}) = 1$.

Since the generator of conformal transformation is conserved, or time-independent, its algebra can be calculated using equal-time commutation relations. From the commutation relations (7-28), (7-29), and $[\,:\mathsf{P}_\chi\partial_j\phi(\mathbf{x}):, :\phi^n(\mathbf{y}):] = 0$, the translation (7-22) and the Lorentz transformation (7-24) are calculated as

$$\begin{aligned} i\,[P_\mu, :\phi^n(x):] &= \partial_\mu :\phi^n(x):, \\ i\,[M_{\mu\nu}, :\phi^n(x):] &= (x_\mu\partial_\nu - x_\nu\partial_\mu) :\phi^n(x): . \end{aligned}$$

There is no quantum correction in these equations, especially in the second equation it disappears because of the antisymmetric property of the Lorentz generator and the fact that volume integrals of functions including e_j satisfy $\int d^3\mathbf{x}\, e_j(\mathbf{x}) = 0$ and

$$\int d^3\mathbf{x}\, x_i e_j(\mathbf{x}) = \frac{1}{3}\,\delta_{ij}\int_0^\infty 4\pi x^2 dx\frac{1}{\pi^2}\frac{\epsilon[1-h(x)]}{(x^2+\epsilon^2)^2} = \frac{1}{6}\delta_{ij}. \quad (7\text{-}30)$$

Similarly, the dilatation (7-25) and the special conformal transformation (7-26) are calculated as

$$\begin{aligned} i\,[D, :\phi^n(x):] &= x^\mu\partial_\mu :\phi^n(x): + n :\phi^{n-1}(x): \\ &\quad -\frac{1}{4b_c}n(n-1) :\phi^{n-2}(x):, \\ i\,[K_\mu, :\phi^n(x):] &= \left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu\right) :\phi^n(x): \\ &\quad -2x_\mu\left(n :\phi^{n-1}(x): -\frac{1}{4b_c}n(n-1) :\phi^{n-2}(x):\right). \end{aligned}$$

Each $:\phi^{n-1}:$ term is derived from commutation relations with the linear term of P_ϕ in the generator. The last term including $1/b_c$ of each transformation law is quantum correction. Those of D and K_0 are calculated using the integral formula (7-30), and that of K_j is calculated using a formula developed it as

$$\int d^3\mathbf{x}\left\{\mathbf{x}^2 e_j(\mathbf{x}-\mathbf{y}) - 2x_j x^k e_k(\mathbf{x}-\mathbf{y})\right\} = -y_j.$$

The $n = 1$ expression of the conformal transformation is nothing but the transformation law of the conformal-factor field given in (7-10). Using ζ^μ to write it all together, it is given by[13]

$$i[Q_\zeta, \phi] = \zeta^\mu \partial_\mu \phi + \frac{1}{4}\partial_\mu \zeta^\mu = \delta_\zeta \phi.$$

In this case, the quantum correction term disappears.

The simplest primary scalar field is given by

$$\mathcal{V}_\alpha(x) = :e^{\alpha\phi(x)}: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :\phi^n(x): . \tag{7-31}$$

The new exponent α is called the Riegert charge. In general, exponents in such exponential operators are called so. From the transformation laws of $:\phi^n:$ obtained above, conformal transformations of $\mathcal{V}_\alpha$ are calculated as

$$\begin{aligned}
i\left[P_\mu, \mathcal{V}_\alpha(x)\right] &= \partial_\mu \mathcal{V}_\alpha(x), \\
i[M_{\mu\nu}, \mathcal{V}_\alpha(x)] &= \left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \mathcal{V}_\alpha(x), \\
i\left[D, \mathcal{V}_\alpha(x)\right] &= \left(x^\mu \partial_\mu + h_\alpha\right) \mathcal{V}_\alpha(x), \\
i\left[K_\mu, \mathcal{V}_\alpha(x)\right] &= \left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu h_\alpha\right) \mathcal{V}_\alpha(x),
\end{aligned}$$

and its conformal dimension is obtained as

$$h_\alpha = \alpha - \frac{\alpha^2}{4b_c}. \tag{7-32}$$

The second term proportional to $1/b_c$ is quantum correction.

Next, consider composite scalar fields involving differential operators. There are two field operators containing second derivatives, which are

$$\begin{aligned}
\mathcal{R}^1_\beta &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} :\phi^n \partial^2 \phi: = :e^{\beta\phi}\left(\frac{4\pi^2}{b_c}\mathsf{P}_\chi + \hat{\partial}^2 \phi\right):, \\
\mathcal{R}^2_\beta &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} :\phi^n \partial_\lambda \phi \partial^\lambda \phi: = :e^{\beta\phi}\left(-\chi^2 + \partial_k \phi \partial^k \phi\right): .
\end{aligned}$$

[13] The transformation law of the traceless tensor field is also obtained from the commutation relation $i[Q_\zeta, h_{\mu\nu}]$, but in this case, an extra term called the Fradkin-Palchik term has to be added to $\delta_\zeta h_{\mu\nu}$ (7-10) so that the gauge-fixing conditions are preserved. See the sixth section of Appendix B or K. Hamada, Phys. Rev. D **85** (2012) 024028 in Bibliography.

These operators transform as a scalar field for the translation and the Lorentz transformation. The transformation laws for the dilatation are given by

$$i\left[D, \mathcal{R}^{1,2}_{\beta}(x)\right] = \left(x^{\mu}\partial_{\mu} + h_{\beta} + 2\right)\mathcal{R}^{1,2}_{\beta}(x),$$

and those for the special conformal transformation are given as

$$\begin{aligned} i\left[K_{\mu}, \mathcal{R}^{1}_{\beta}(x)\right] &= \left\{x^{2}\partial_{\mu} - 2x_{\mu}x^{\lambda}\partial_{\lambda} - 2x_{\mu}\left(h_{\beta}+2\right)\right\}\mathcal{R}^{1}_{\beta}(x) \\ &\quad +4 :\partial_{\mu}\phi e^{\beta\phi}(x):, \\ i\left[K_{\mu}, \mathcal{R}^{2}_{\beta}(x)\right] &= \left\{x^{2}\partial_{\mu} - 2x_{\mu}x^{\lambda}\partial_{\lambda} - 2x_{\mu}\left(h_{\beta}+2\right)\right\}\mathcal{R}^{2}_{\beta}(x) \\ &\quad -4\frac{h_{\beta}}{\beta} :\partial_{\mu}\phi e^{\beta\phi}(x):, \end{aligned}$$

respectively, where h_{β} is defined by (7-32).

From these transformation laws, considering a field operator that combine the two as

$$\mathcal{R}_{\beta} = \mathcal{R}^{1}_{\beta} + \frac{\beta}{h_{\beta}}\mathcal{R}^{2}_{\beta} = :e^{\beta\phi}\left(\partial^{2}\phi + \frac{\beta}{h_{\beta}}\partial_{\lambda}\phi\partial^{\lambda}\phi\right):, \qquad \text{(7-33)}$$

we find that $\mathcal{R}_{\beta}$ is a primary scalar field with conformal dimension $h_{\beta}+2$, which transforms under the conformal transformations as follows:

$$\begin{aligned} i\left[P_{\mu}, \mathcal{R}_{\beta}(x)\right] &= \partial_{\mu}\mathcal{R}_{\beta}(x), \\ i\left[M_{\mu\nu}, \mathcal{R}_{\beta}(x)\right] &= \left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\mathcal{R}_{\beta}(x), \\ i\left[D, \mathcal{R}_{\beta}(x)\right] &= \left(x^{\lambda}\partial_{\lambda} + h_{\beta} + 2\right)\mathcal{R}_{\beta}(x), \\ i\left[K_{\mu}, \mathcal{R}_{\beta}(x)\right] &= \left\{x^{2}\partial_{\mu} - 2x_{\mu}x^{\lambda}\partial_{\lambda} - 2x_{\mu}\left(h_{\beta}+2\right)\right\}\mathcal{R}_{\beta}(x). \end{aligned}$$

Generalizing this operator to a scalar field operator containing $2m$-th derivatives as

$$\mathcal{R}^{[m]}_{\gamma} = :e^{\gamma\phi}\left(\partial^{2}\phi + \frac{\gamma}{h_{\gamma}}\partial_{\lambda}\phi\partial^{\lambda}\phi\right)^{m}:,$$

we find that it becomes a primary scalar with conformal dimension $h_{\gamma}+2m$. The $m = 0, 1$ correspond to $\mathcal{V}_{\alpha}$ and $\mathcal{R}_{\beta}$, respectively.

Physical Field Operators

Since the conformal invariance is a gauge symmetry that occurs as part of diffeomorphism invariance, unlike normal conformal field theory, not only vacua but also field operators must be invariant under the conformal transformations. Hence, diffeomorphism invariance requires that physical fields satisfy the following conformal invariance condition:

$$\left[Q_\zeta, \int d^4x\, \mathcal{O}(x)\right] = 0. \tag{7-34}$$

The operator $\mathcal{O}$ satisfying this physical condition is given by a primary scalar field with conformal dimension 4 which is the same as the spacetime dimensions. Because it satisfies

$$i\left[Q_\zeta, \mathcal{O}(x)\right] = \partial_\mu \left\{\zeta^\mu \mathcal{O}(x)\right\}$$

for all conformal Killing vectors ζ^μ, we can see that the physical condition is satisfied. On the other hand, primary tensor fields do not satisfy this condition because of the presence of spin terms in their transformation laws.

The simplest example of such a physical operator is the operator $\mathcal{V}_\alpha$ (7-31) with $h_\alpha = 4$. Solving the condition $h_\alpha = 4$, the Riegert charge is determined as

$$\alpha = 2b_c \left(1 - \sqrt{1 - \frac{4}{b_c}}\right). \tag{7-35}$$

At this time, out of two solutions, we select this solution that α approaches the canonical value 4 in the classical limit $b_c \to \infty$ which takes the number of matter fields to infinity (large N limit). That is the solution that $\mathcal{V}_\alpha$ approaches the classical volume element $\sqrt{-g}$, and thus the operator with (7-35) is called the quantum cosmological term operator. Since α is always a real number due to $b_c > 4$ from (7-5), it becomes a real operator, as expected in gravity theories.

Likewise, the primary scalar field $\mathcal{R}_\beta$ (7-33) with $h_\beta = 2$ satisfies the physical condition. Solving the condition $h_\beta = 2$ and selecting one solution that β becomes the canonical value 2 in the classical limit $b_c \to \infty$, the Riegert charge is determined as

$$\beta = 2b_c \left(1 - \sqrt{1 - \frac{2}{b_c}}\right). \tag{7-36}$$

The operator $\mathcal{R}_\beta$ with this solution is called the quantum Ricci scalar operator. Indeed, it reduces to the Ricci scalar $\sqrt{-g}R$ in the classical limit of $\beta \to 2$.

In general, the primary scalar field $\mathcal{R}_\gamma^{[m]}$ with the Riegert charge $\gamma = 2b_c(1 - \sqrt{1 - (4 - 2m)/b_c})$ that satisfies the condition $h_\gamma + 2m = 4$ is a physical operator corresponding to the m-th power of the Ricci scalar, $\sqrt{-g}R^m$.

BRST Formulation and Physical Conditions

Lastly, we construct the BRST operator of the transformation (7-10). The BRST conformal transformation is a transformation obtained by replacing the 15 gauge transformation parameters ζ^λ with corresponding ghost fields c^λ. The ghost field can be expanded using 15 Grassmann modes, c_-^μ, $\mathrm{c}^{\mu\nu}$, c, and c_+^μ, as

$$\begin{aligned} c^\lambda &= \mathrm{c}_-^\mu \left(\zeta_T^\lambda\right)_\mu + \mathrm{c}^{\mu\nu}\left(\zeta_L^\lambda\right)_{\mu\nu} + \mathrm{c}\zeta_D^\lambda + \mathrm{c}_+^\mu\left(\zeta_S^\lambda\right)_\mu \\ &= \mathrm{c}_-^\lambda + 2x_\mu \mathrm{c}^{\mu\lambda} + x^\lambda \mathrm{c} + x^2\mathrm{c}_+^\lambda - 2x^\lambda x_\mu \mathrm{c}_+^\mu, \end{aligned}$$

where $\mathrm{c}^{\mu\nu}$ is antisymmetric. The ghost modes are Hermitian operators, and c and $\mathrm{c}^{\mu\nu}$ are dimensionless, while c_-^μ and c_+^μ have dimensions -1 and 1, respectively.

At the same time, we introduce 15 anti-ghost modes b_-^μ, $\mathrm{b}^{\mu\nu}$, b, and b_+^μ with the same properties as the ghost modes have. Take anti-commutation relations between the ghost and anti-ghost modes to

$$\begin{aligned} &\{\mathrm{c}, \mathrm{b}\} = 1, \qquad \{\mathrm{c}^{\mu\nu}, \mathrm{b}^{\lambda\sigma}\} = \eta^{\mu\lambda}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\lambda}, \\ &\{\mathrm{c}_-^\mu, \mathrm{b}_+^\nu\} = \{\mathrm{c}_+^\mu, \mathrm{b}_-^\nu\} = \eta^{\mu\nu}, \end{aligned}$$

then we obtain generators of the ghost part that satisfies the conformal algebra (2-6) as follows:

$$\begin{aligned} P_{\mathrm{gh}}^\mu &= i\left(-2\mathrm{b}\mathrm{c}_+^\mu + \mathrm{b}_+^\mu \mathrm{c} + \mathrm{b}^\mu{}_\lambda \mathrm{c}_+^\lambda + 2\mathrm{b}_+^\lambda \mathrm{c}^\mu{}_\lambda\right), \\ M_{\mathrm{gh}}^{\mu\nu} &= i\left(\mathrm{b}_+^\mu \mathrm{c}_-^\nu - \mathrm{b}_+^\nu \mathrm{c}_-^\mu + \mathrm{b}_-^\mu \mathrm{c}_+^\nu - \mathrm{b}_-^\nu \mathrm{c}_+^\mu + \mathrm{b}^{\mu\lambda}\mathrm{c}^\nu{}_\lambda - \mathrm{b}^{\nu\lambda}\mathrm{c}^\mu{}_\lambda\right), \\ D_{\mathrm{gh}} &= i\left(\mathrm{b}_-^\lambda \mathrm{c}_{+\lambda} - \mathrm{b}_+^\lambda \mathrm{c}_{-\lambda}\right), \\ K_{\mathrm{gh}}^\mu &= i\left(2\mathrm{b}\mathrm{c}_-^\mu - \mathrm{b}_-^\mu \mathrm{c} + \mathrm{b}^\mu{}_\lambda \mathrm{c}_-^\lambda + 2\mathrm{b}_-^\lambda \mathrm{c}^\mu{}_\lambda\right), \end{aligned}$$

where the suffix "gh" is applied to the ghost part.

With these generators, the BRST operator that generates the BRST conformal transformation is constructed as

$$\begin{aligned} Q_{\mathrm{BRST}} &= \mathrm{c}_-^\mu\left(P_\mu + \frac{1}{2}P_\mu^{\mathrm{gh}}\right) + \mathrm{c}^{\mu\nu}\left(M_{\mu\nu} + \frac{1}{2}M_{\mu\nu}^{\mathrm{gh}}\right) \\ &\quad + \mathrm{c}\left(D + \frac{1}{2}D^{\mathrm{gh}}\right) + \mathrm{c}_+^\mu\left(K_\mu + \frac{1}{2}K_\mu^{\mathrm{gh}}\right) \\ &= \mathrm{c}\left(D + D^{\mathrm{gh}}\right) + \mathrm{c}^{\mu\nu}\left(M_{\mu\nu} + M_{\mu\nu}^{\mathrm{gh}}\right) - \mathrm{b}\,N - \mathrm{b}^{\mu\nu}N_{\mu\nu} + \tilde{Q}, \end{aligned}$$

where P_μ, $M_{\mu\nu}$, D, and K_μ are respectively the sums of the generators of conformal transformations other than the ghost part, and other operators are defined by

$$N = 2i\mathrm{c}_+^\mu \mathrm{c}_{-\mu}, \qquad N^{\mu\nu} = \frac{i}{2}\left(\mathrm{c}_+^\mu \mathrm{c}_-^\nu + \mathrm{c}_-^\mu \mathrm{c}_+^\nu\right) + i\mathrm{c}^{\mu\lambda}\mathrm{c}^\nu{}_\lambda,$$
$$\tilde{Q} = \mathrm{c}_-^\mu P_\mu + \mathrm{c}_+^\mu K_\mu.$$

Using the conformal algebra that P_μ, $M_{\mu\nu}$, D, and K_μ satisfy, nilpotency of the BRST operator can be expressed as

$$Q_{\mathrm{BRST}}^2 = \tilde{Q}^2 - ND - 2i\mathrm{c}_+^\mu \mathrm{c}_-^\nu M_{\mu\nu} = 0.$$

Anti-commutation relations between the BRST operator and the anti-ghost modes are given by

$$\{Q_{\mathrm{BRST}}, \mathrm{b}\} = D + D_{\mathrm{gh}}, \qquad \{Q_{\mathrm{BRST}}, \mathrm{b}^{\mu\nu}\} = 2\left(M^{\mu\nu} + M_{\mathrm{gh}}^{\mu\nu}\right),$$
$$\{Q_{\mathrm{BRST}}, \mathrm{b}_-^\mu\} = K^\mu + K_{\mathrm{gh}}^\mu, \qquad \{Q_{\mathrm{BRST}}, \mathrm{b}_+^\mu\} = P^\mu + P_{\mathrm{gh}}^\mu.$$

Thus, the whole generator of conformal transformation including the ghost part becomes BRST trivial. Therefore, it is not necessary to consider the descendant field obtained by applying the whole translation generator to the physical operator given as a primary scalar field, because it becomes BRST trivial.

The BRST conformal transformation of the conformal-factor field can be directly derived from its conformal transformation law as

$$i\left[Q_{\mathrm{BRST}}, \phi(x)\right] = c^\mu \partial_\mu \phi(x) + \frac{1}{4}\partial_\mu c^\mu(x).$$

Similarly, the BRST conformal transformation of a primary scalar field $\mathcal{O}$, such as $\mathcal{V}_\alpha$ and $\mathcal{R}_\beta$ obtained in the previous section, is given by

$$i\left[Q_{\mathrm{BRST}}, \mathcal{O}(x)\right] = c^\mu \partial_\mu \mathcal{O}(x) + \frac{\Delta}{4}\partial_\mu c^\mu \mathcal{O}(x),$$

where Δ is its conformal dimension. As shown before, when $\Delta = 4$, the volume integral of $\mathcal{O}$ becomes a BRST conformal invariant as

$$i\Big[Q_{\mathrm{BRST}}, \int d^4x\, \mathcal{O}(x)\Big] = \int d^4x\, \partial_\mu\{c^\mu\, \mathcal{O}(x)\} = 0.$$

This is a rewrite of the physical condition (7-34).

Local BRST conformal invariants can be constructed by introducing a ghost field function obtained by contracting indices with a fully antisymmetric tensor as

$$\omega = \frac{1}{4!}\epsilon_{\mu\nu\lambda\sigma}c^\mu c^\nu c^\lambda c^\sigma .$$

Since the BRST conformal transformation of the ghost field is given by

$$i\{Q_{\mathrm{BRST}}, c^\mu(x)\} = c^\nu \partial_\nu c^\mu(x),$$

the function ω transforms as

$$i\left[Q_{\mathrm{BRST}}, \omega(x)\right] = c^\mu \partial_\mu \omega(x) = -\omega \partial_\mu c^\mu(x),$$

where $c^\mu \omega = 0$ is used in the second equality. Using this commutation relation, the product of ω and a primary scalar field with conformal dimension $\Delta = 4$ becomes a local BRST conformal invariant as

$$i\left[Q_{\mathrm{BRST}}, \omega\mathcal{O}(x)\right] = \frac{1}{4}\left(\Delta - 4\right)\omega\partial_\mu c^\mu \mathcal{O}(x) = 0.$$

Correlation functions of physical field operators $O_\gamma = \int d^4x\, \mathcal{O}_\gamma$ can be defined in the same way as in two-dimensional quantum gravity. However, its calculation method has not been established yet. Here, as in the last section of Chapter 6, we consider a quantum gravity system in the Wick-rotated Euclidean background, in which the cosmological term $\Lambda \int d^4x \mathcal{V}_\alpha$ is added as an interaction term and we see only the Λ-dependence of correlation functions in this system. Since the Euler characteristic of the Euclidean space is given by $\int d^4x\sqrt{\hat{g}}\hat{G}_4/32\pi^2 = 2$, we find that a correlation function $\langle\langle O_{\gamma_1} \cdots O_{\gamma_n}\rangle\rangle$ has a behavior proportional to Λ^s when the integration of the zero-mode of the conformal-factor field is performed, where the power is determined to be $s = (4b_c - \sum_{i=1}^n \gamma_i)/\alpha$.[14]

As we have seen so far, each mode itself introduced to quantize the gravitational field does not become BRST invariant. Physical quantities in the background-free quantum gravity are given as real composite scalar functions of the gravitational field. Therefore, as described in Chapter 1, the condition for ensuring that their correlation functions are real-valued is not the positive-definiteness of each mode, but the positive-definiteness of the whole action given by the Riegert action and the Weyl action written in terms of the field variables, which guarantees validity of the path integral.

[14] This equation indicates that conformally invariant vacua have a background charge $-4b_c$ in total. Such a vacuum will be discussed in detail in (8-23) in Chapter 8.

Difference from conventional thinking

Finally, we describe the difference from the unitarity argument studied in the earlier higher-order derivative quantum gravity in the 1970s.[15] In those days, the R^2 action was introduced as a kinetic term of the conformal-factor field, and all gravitational fields were treated in perturbations. Therefore, when discussing gauge invariance of the kinetic term, only the transformation of the field-independent part like (7-8) works. In this case, positive-metric and negative-metric modes are not mixed with each other by the gauge transformation. Hence, their residual modes which cannot be eliminated by gauge-fixing become gauge invariant. For that reason, it was impossible to prevent negative-metric modes from appearing as physical asymptotic states.

The idea at those days by Tomboulis who applying the work of Lee and Wick to the gravitational field is based on the fact that as the positive-metric and negative-metric modes are mixed through interactions, the negative-metric mode does not actually appear alone.[16] It can be seen from the fact that propagators acquire quantum corrections through interactions like

$$\frac{1}{p^2 M(p^2)}, \qquad M(p^2) = M_{\mathsf{P}}^2 + 4\beta_0 p^2 \log\left(\frac{p^2}{\Lambda_{\mathrm{QG}}^2}\right),$$

where $M_{\mathsf{P}} = 1/\sqrt{8\pi G}$ is the reduced Planck mass. When treating all the gravitational fields in perturbations, the Einstein-Hilbert term plays a role of mass term. The real pole $1/p^2$ represents a positive-metric mode what is called graviton. On the other hand, $1/M(p^2)$, which corresponds to a negative-metric mode called massive graviton, has no real poles as a consequence of the asymptotic freedom ($\beta_0 > 0$), and thus we can see that it does not appear in the real world.

This idea is still effective when considering contacts with the real world at low energy. However, it is clear that it cannot be avoided that the negative-metric modes appear as gauge invariant asymptotic states when coupling constants disappear at the ultraviolet limit.

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[15] In general, a free propagator of fourth-order derivative fields has the form $1/(p^4 + m^2 p^2)$. It can be decomposed as $(1/m^2)[1/p^2 - 1/(p^2 + m^2)]$, where the first term denotes a massless physical particle, while the second term is a massive ghost particle because of the wrong sign. Thus, we cannot avoid the appearance of ghosts as far as we consider such a free field.

[16] See T. Lee and G. Wick, *Negative Metric and the Unitarity of the S-Matrix*, Nucl. Phys. **B9** (1969) 209; N. Nakanishi, *Indefinite Metric Quantum Field Theory*, Prog. Theor. Phys. Suppl. **51** (1972) 1, and T. Tomboulis in Bibliography.

On the other hand, in the background-free quantum gravity, due to the non-perturbative treatment of the conformal-factor field, the BRST conformal symmetry that mixes the positive-metric and negative-metric modes still remains as part of diffeomorphism invariance even in the ultraviolet limit. It prohibits the negative-metric mode from appearing alone as a physical state. To begin with, there is no concept of asymptotic states, so that the S-matrix is not defined. Since spacetime is totally fluctuating quantum mechanically, there is no longer a picture of particles moving in the flat spacetime.

CHAPTER EIGHT

PHYSICAL STATES OF QUANTUM GRAVITY

Let us concretely construct and classify physical states in the background-free quantum gravity. For this purpose, it is convenient to employ a cylindrical spacetime $\mathbb{R} \times S^3$ as the background. By using a compact S^3 space, there is no need to worry about infrared divergences. Moreover, we can see similarities with two-dimensional quantum gravity on $\mathbb{R} \times S^1$ described in Chapter 6.

Minkowski spacetime M^4 and the cylindrical spacetime $\mathbb{R} \times S^3$ can be converted to one another by conformal transformations. Since the theory is gauge equivalent under conformal transformations (7-10), namely the background-metric independence, the result is the same regardless the background we choose.

Canonical Quantization on $\mathbb{R} \times S^3$

The metric of the background spacetime $\mathbb{R} \times S^3$ can be expressed using the Euler angles $\hat{\mathbf{x}} = (\alpha, \beta, \gamma)$, with the radius of S^3 as unity, as follows:

$$\begin{aligned} d\hat{s}^2_{\mathbb{R}\times S^3} &= \hat{g}_{\mu\nu} dx^\mu dx^\nu = -d\eta^2 + \hat{\gamma}_{ij} d\hat{x}^i d\hat{x}^j \\ &= -d\eta^2 + \frac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma), \end{aligned}$$

where possible ranges of α, β, and γ are respectively $[0, 2\pi]$, $[0, \pi]$, and $[0, 4\pi]$ (doubling the range of γ in consideration of half-integer representation). The curvatures are then given by

$$\hat{R}_{ijkl} = (\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \hat{\gamma}_{il}\hat{\gamma}_{jk}), \quad \hat{R}_{ij} = 2\hat{\gamma}_{ij}, \quad \hat{R} = 6,$$

and $\hat{C}^2_{\mu\nu\lambda\sigma} = \hat{G}_4 = 0$. The volume element of S^3 is defined by

$$d\Omega_3 = d^3\hat{x}\sqrt{\hat{\gamma}} = \frac{1}{8}\sin\beta d\alpha d\beta d\gamma,$$

and its volume is

$$\mathrm{V}_3 = \int d\Omega_3 = \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{4\pi} d\gamma \, \frac{1}{8}\sin\beta = 2\pi^2.$$

Spherical tensor harmonics on three-sphere Quantum fields are expanded in modes using spherical harmonics on S^3. A symmetric-transverse-traceless (ST^2) n-th rank tensor harmonics is classified using a representation $(J+\varepsilon_n, J-\varepsilon_n)$ of the rotation group $SO(4)=SU(2)\times SU(2)$, and it is denoted as $Y^{i_1\cdots i_n}_{J(M\varepsilon_n)}$, where $\varepsilon_n=\pm n/2$ is an index representing polarization. The spherical tensor harmonics is an eigenfunction of the Laplacian operator $\square_3=\hat{\gamma}^{ij}\hat{\nabla}_i\hat{\nabla}_j$ on S^3 satisfying

$$\square_3 Y^{i_1\cdots i_n}_{J(M\varepsilon_n)}=\{-2J(2J+2)+n\}Y^{i_1\cdots i_n}_{J(M\varepsilon_n)},$$

where $J(\geq n/2)$ is an integer or a half-integer and $M=(m,m')$ is an index representing degeneracy of the representation for each polarization and takes the following values:

$$\begin{aligned} m &= -J-\varepsilon_n,\ -J-\varepsilon_n+1,\ldots,J+\varepsilon_n-1,\ J+\varepsilon_n, \\ m' &= -J+\varepsilon_n,\ -J+\varepsilon_n+1,\ldots,J-\varepsilon_n-1,\ J-\varepsilon_n. \end{aligned}$$

From this, the degeneracy of the tensor harmonics becomes $2(2J+n+1)(2J-n+1)$ in consideration of polarization, while the degeneracy of the $n=0$ scalar harmonics is $(2J+1)^2$. Specific expressions and formulas of the spherical tensor harmonics are summarized in Appendix C.

Complex conjugate and normalization of the ST^2 spherical tensor harmonics are defined by

$$\begin{aligned} & Y^{i_1\cdots i_n *}_{J(M\varepsilon_n)}=(-1)^n\epsilon_M Y^{i_1\cdots i_n}_{J(-M\varepsilon_n)}, \\ & \int_{S^3} d\Omega_3\, Y^{i_1\cdots i_n *}_{J_1(M_1\varepsilon^1_n)} Y_{i_1\cdots i_n J_2(M_2\varepsilon^2_n)}=\delta_{J_1J_2}\delta_{M_1M_2}\delta_{\varepsilon^1_n\varepsilon^2_n}, \end{aligned}$$

where $\delta_{M_1M_2}=\delta_{m_1m_2}\delta_{m'_1m'_2}$ and ϵ_M is a sign factor defined by

$$\epsilon_M=(-1)^{m-m'},$$

which satisfies $\epsilon_M^2=1$. In the following, the polarization indices for spherical tensor harmonics where the rank n is two or less are particularly written as

$$y=\varepsilon_1=\pm\frac{1}{2}, \qquad x=\varepsilon_2=\pm 1.$$

Quantization of scalar fields The action of a conformally invariant free scalar field on $\mathbb{R}\times S^3$ is given by

$$I=\int d\eta\int_{S^3} d\Omega_3\,\frac{1}{2}\varphi\left(-\partial_\eta^2+\square_3-1\right)\varphi,$$

where missing dimensions in the action come from the fact that we set the radius of S^3 to be unity. By expanding the field as $\varphi \propto e^{-i\omega\eta} Y_{JM}$ using scalar harmonics, we obtain a dispersion relation $\omega^2 - (2J+1)^2 = 0$. Therefore, the scalar field is mode-expanded as

$$\varphi = \sum_{J\geq 0} \sum_M \frac{1}{\sqrt{2(2J+1)}} \left\{ \varphi_{JM} e^{-i(2J+1)\eta} Y_{JM} + \varphi^\dagger_{JM} e^{i(2J+1)\eta} Y^*_{JM} \right\}.$$

Quantization can be done according to normal procedures. The conjugate momentum is given by $\mathsf{P}_\varphi = \partial_\eta \varphi$ and the equal-time commutation relation is taken to be $[\varphi(\eta, \hat{\mathbf{x}}), \mathsf{P}_\varphi(\eta, \hat{\mathbf{x}}')] = i\delta_3(\hat{\mathbf{x}} - \hat{\mathbf{x}}')$, where the delta function on S^3 can be represented from completeness of the spherical harmonics as

$$\begin{aligned} \delta_3(\hat{\mathbf{x}} - \hat{\mathbf{x}}') &= \sum_{J\geq 0} \sum_M Y^*_{JM}(\hat{\mathbf{x}}) Y_{JM}(\hat{\mathbf{x}}') \\ &= 8\delta(\alpha - \alpha')\delta(\cos\beta - \cos\beta')\delta(\gamma - \gamma'). \end{aligned}$$

The commutation relations between the modes are then given by

$$[\varphi_{J_1 M_1}, \varphi^\dagger_{J_2 M_2}] = \delta_{J_1 J_2} \delta_{M_1 M_2}$$

and $[\varphi_{J_1 M_1}, \varphi_{J_2 M_2}] = [\varphi^\dagger_{J_1 M_1}, \varphi^\dagger_{J_2 M_2}] = 0$. The Hamiltonian operator can be found from the action as

$$\begin{aligned} H &= \int_{S^3} d\Omega_3 : \left\{ \frac{1}{2} \mathsf{P}^2_\varphi - \frac{1}{2} \varphi \left(\Box_3 - 1 \right) \varphi \right\} : \\ &= \sum_{J\geq 0} \sum_M (2J+1) \varphi^\dagger_{JM} \varphi_{JM}. \end{aligned} \tag{8-1}$$

Quantization of gauge fields Adopting the Coulomb gauge $\hat{\nabla}^i A_i = 0$ to quantize the gauge field, the action on $\mathbb{R} \times S^3$ is given by

$$I = \int d\eta \int_{S^3} d\Omega_3 \left\{ \frac{1}{2} A^i \left(-\partial^2_\eta + \Box_3 - 2 \right) A_i - \frac{1}{2} A_0 \Box_3 A_0 \right\},$$

where $A^i = \hat{\gamma}^{ij} A_j$. Since A_0 becomes a non-dynamical variable whose kinetic term does not contain time derivative, we further take $A_0 = 0$ using remaining gauge degrees of freedom in the Coulomb gauge. This gauge condition is called the radiation gauge.

Expanding the transverse gauge field as $A^i \propto e^{-i\omega\eta} Y^i_{J(My)}$ using the vector harmonics, the same dispersion relation $\omega^2 - (2J+1)^2 = 0$ as the scalar field has is obtained. Thus, the gauge field is mode-expanded as

$$A^i = \sum_{J\geq\frac{1}{2}} \sum_{M,y} \frac{1}{\sqrt{2(2J+1)}} \Big\{ q_{J(My)} e^{-i(2J+1)\eta} Y^i_{J(My)} \\ + q^\dagger_{J(My)} e^{i(2J+1)\eta} Y^{i*}_{J(My)} \Big\}.$$

The conjugate momentum is $\mathsf{P}^i_A = \partial_\eta A^i$ and the equal-time commutation relation is set to be $[A^i(\eta, \hat{\mathbf{x}}), \mathsf{P}^j_A(\eta, \hat{\mathbf{y}})] = i\delta^{ij}_3(\hat{\mathbf{x}} - \hat{\mathbf{y}})$, where the delta function on S^3 is given by $\delta^{ij}_3(\hat{\mathbf{x}} - \hat{\mathbf{y}}) = \sum_{J\geq\frac{1}{2}} \sum_{M,y} Y^{i*}_{J(My)}(\hat{\mathbf{x}}) Y^j_{J(My)}(\hat{\mathbf{y}})$ from completeness of the vector harmonics. The commutation relation between creation and annihilation operators becomes

$$\left[q_{J_1(M_1 y_1)}, q^\dagger_{J_2(M_2 y_2)} \right] = \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{y_1 y_2},$$

and the Hamiltonian operator is given by

$$\begin{aligned} H &= \int_{S^3} d\Omega_3 : \left\{ \frac{1}{2} \mathsf{P}^i_A \mathsf{P}^A_i - \frac{1}{2} A^i \left(\Box_3 - 2 \right) A_i \right\} : \\ &= \sum_{J\geq\frac{1}{2}} \sum_{M,y} (2J+1) q^\dagger_{J(My)} q_{J(My)}. \end{aligned} \tag{8-2}$$

Quantization of gravitational fields In order to gauge-fix the Weyl action, decompose the traceless tensor field as in the third section of Chapter 7, that is,

$$h_{00}, \qquad h_{0i}, \qquad h_{ij} = h^{\rm tr}_{ij} + \frac{1}{3} \hat{\gamma}_{ij} h_{00},$$

where $h^{\rm tr}_{ij}$ is a component satisfying the spatial traceless condition $h^{{\rm tr}\,i}{}_i = \hat{\gamma}^{ij} h^{\rm tr}_{ij} = 0$. The gauge transformation (7-8) of the traceless tensor field is then decomposed as

$$\begin{aligned} &\delta_\kappa h_{00} = \frac{3}{2} \partial_\eta \kappa_0 + \frac{1}{2} \hat{\nabla}_k \kappa^k, \qquad \delta_\kappa h_{0i} = \partial_\eta \kappa_i + \hat{\nabla}_i \kappa_0, \\ &\delta_\kappa h^{\rm tr}_{ij} = \hat{\nabla}_i \kappa_j + \hat{\nabla}_j \kappa_i - \frac{2}{3} \hat{\gamma}_{ij} \hat{\nabla}_k \kappa^k. \end{aligned}$$

As in the previous chapter, using four gauge degrees of freedom of diffeomorphism, we apply the following transverse gauge conditions:

$$\hat{\nabla}^i h_{0i} = 0, \qquad \hat{\nabla}^i h^{\rm tr}_{ij} = 0.$$

In the following, we write the transverse vector component satisfying this condition as h_i and the transverse-traceless tensor component as h_{ij}, in the Gothic style.

The four-dimensional quantum gravity action on $\mathbb{R} \times S^3$ composed of the Riegert action and the Weyl action fixed in the transverse gauge becomes

$$
\begin{aligned}
S_{\rm 4DQG} = \int d\eta \int_{S^3} d\Omega_3 \biggl\{ & -\frac{2b_c}{(4\pi)^2}\phi \left(\partial_\eta^4 - 2\Box_3\partial_\eta^2 + \Box_3^2 + 4\partial_\eta^2\right)\phi \\
& -\frac{1}{2}\mathsf{h}_{ij}\left(\partial_\eta^4 - 2\Box_3\partial_\eta^2 + \Box_3^2 + 8\partial_\eta^2 - 4\Box_3 + 4\right)\mathsf{h}^{ij} \\
& +\mathsf{h}_i\left(\Box_3 + 2\right)\left(-\partial_\eta^2 + \Box_3 - 2\right)\mathsf{h}^i \\
& -\frac{1}{27}h_{00}\left(16\Box_3 + 27\right)\Box_3 h_{00}\biggr\}. \qquad (8\text{-}3)
\end{aligned}
$$

The field h_{00} is not a dynamical field because it does not contain time derivative in the kinetic term. Therefore, using remaining gauge degrees of freedom that preserve the transverse gauge conditions, we further impose the gauge condition

$$h_{00} = 0.$$

The combination of this and the transverse conditions is called the radiation gauge.

In addition, we remove a non-dynamical transverse vector mode that satisfies $(\Box_3 + 2)\mathsf{h}_i = 0$. This mode can be written with the $J = 1/2$ vector harmonics, and thus the condition can be expressed as

$$\mathsf{h}_i|_{J=\frac{1}{2}} = 0. \qquad (8\text{-}4)$$

The radiation gauge with this condition is called the radiation$^+$ gauge. Then, residual gauge degrees of freedom of diffeomorphism become the same as degrees of freedom of the conformal Killing vectors. The details are described when constructing generators of diffeomorphism in this gauge in the next section.

As in the previous chapter, let us perform canonical quantization according to the Dirac quantization procedure. By rewriting the action of the conformal-factor field using the variable $\chi = \partial_\eta\phi$ (7-12), we obtain

$$
\begin{aligned}
S_{\rm R} = \int d\eta \int_{S^3} d\Omega_3 \biggl\{ & -\frac{b_c}{8\pi^2}\Bigl[(\partial_\eta\chi)^2 + 2\chi\Box_3\chi \\
& -4\chi^2 + (\Box_3\phi)^2\Bigr] + v(\partial_\eta\phi - \chi)\biggr\}.
\end{aligned}
$$

The Dirac brackets are set as before between the four variables in the phase subspace obtained by solving constraints. Replacing them with commutation relations yields

$$[\chi(\eta, \hat{\mathbf{x}}), \mathsf{P}_\chi(\eta, \hat{\mathbf{y}})] = [\phi(\eta, \hat{\mathbf{x}}), \mathsf{P}_\phi(\eta, \hat{\mathbf{y}})] = i\delta_3(\hat{\mathbf{x}} - \hat{\mathbf{y}}),$$

where the momentum variables on $\mathbb{R} \times S^3$ are given by

$$\mathsf{P}_\chi = -\frac{b_c}{4\pi^2}\partial_\eta \chi, \qquad \mathsf{P}_\phi = -\partial_\eta \mathsf{P}_\chi - \frac{b_c}{2\pi^2}\Box_3\chi + \frac{b_c}{\pi^2}\chi.$$

Then we obtain the Hamiltonian operator

$$H = \int_{S^3} d\Omega_3 :\left\{ -\frac{2\pi^2}{b_c}\mathsf{P}_\chi^2 + \mathsf{P}_\phi \chi + \frac{b_c}{8\pi^2}\left[2\chi\Box_3\chi - 4\chi^2 + (\Box_3\phi)^2\right]\right\}: .$$

By deriving the equation of motion of the conformal-factor field from the action (8-3) and substituting $\phi \propto e^{-i\omega\eta}Y_{JM}$ into it, we obtain a dispersion relation $\{\omega^2 - (2J)^2\}\{\omega^2 - (2J+2)^2\} = 0$. Therefore, the conformal-factor field is mode-expanded as follows:

$$\begin{aligned} \phi = \frac{\pi}{2\sqrt{b_c}} \Bigg\{ & 2(\hat{q} + \hat{p}\eta)Y_{00} \\ & + \sum_{J \geq \frac{1}{2}} \sum_M \frac{1}{\sqrt{J(2J+1)}}\left(a_{JM} e^{-i2J\eta} Y_{JM} + a^\dagger_{JM} e^{i2J\eta} Y^*_{JM}\right) \\ & + \sum_{J \geq 0} \sum_M \frac{1}{\sqrt{(J+1)(2J+1)}}\Big(b_{JM} e^{-i(2J+2)\eta} Y_{JM} \\ & \quad + b^\dagger_{JM} e^{i(2J+2)\eta} Y^*_{JM}\Big)\Bigg\}, \end{aligned}$$

where $Y_{00} = 1/\sqrt{\mathrm{V}_3} = 1/\sqrt{2}\pi$. From the canonical commutation relations, the commutation relation of each mode is given by

$$\left[\hat{q}, \hat{p}\right] = i, \quad \left[a_{J_1 M_1}, a^\dagger_{J_2 M_2}\right] = -\left[b_{J_1 M_1}, b^\dagger_{J_2 M_2}\right] = \delta_{J_1 J_2}\delta_{M_1 M_2}.$$

Thus, a_{JM} has a positive metric and b_{JM} has a negative metric.

The Hamiltonian operator is given in the modes as

$$H = \frac{1}{2}\hat{p}^2 + b_c + \sum_{J \geq 0}\sum_M \{2J a^\dagger_{JM} a_{JM} - (2J+2) b^\dagger_{JM} b_{JM}\}. \quad (8\text{-}5)$$

The energy shift b_c is a Casimir effect dependent on the coordinate system, which cannot be derived from the above normal ordered definition. For the sake of simplicity, it is here determined by requiring that conformal algebra on $\mathbb{R} \times S^3$ discussed in the next section shall close. See the last section in the next chapter as for one specific method to derive this Casimir term.

Since the transverse-traceless field h_{ij} is a fourth-order derivative field, it is quantized according to the Dirac procedure as in the conformal-factor field, whereas since the transverse vector field h_i is second order, it is quantized in a normal way. Using the tensor and the vector harmonics and expanding the fields in modes like $\mathsf{h}^{ij} \propto e^{-i\omega\eta} Y^{ij}_{J(Mx)}$ and $\mathsf{h}^i \propto e^{-i\omega\eta} Y^i_{J(My)}$, we obtain dispersion relations $\{\omega^2 - (2J)^2\}\{\omega^2 - (2J+2)^2\} = 0$ and $(2J-1)(2J+3)\{\omega^2 - (2J+1)^2\} = 0$, respectively, from the gauge-fixed action (8-3). From these, the fields are mode-expanded as[1]

$$
\begin{aligned}
\mathsf{h}^{ij} &= \frac{1}{4} \sum_{J \geq 1} \sum_{M,x} \frac{1}{\sqrt{J(2J+1)}} \Big\{ c_{J(Mx)} e^{-i2J\eta} Y^{ij}_{J(Mx)} \\
&\qquad + c^\dagger_{J(Mx)} e^{i2J\eta} Y^{ij*}_{J(Mx)} \Big\} \\
&\quad + \frac{1}{4} \sum_{J \geq 1} \sum_{M,x} \frac{1}{\sqrt{(J+1)(2J+1)}} \Big\{ d_{J(Mx)} e^{-i(2J+2)\eta} Y^{ij}_{J(Mx)} \\
&\qquad + d^\dagger_{J(Mx)} e^{i(2J+2)\eta} Y^{ij*}_{J(Mx)} \Big\}, \\
\mathsf{h}^i &= \frac{1}{2} \sum_{J \geq 1} \sum_{M,y} \frac{i}{\sqrt{(2J-1)(2J+1)(2J+3)}} \\
&\quad \times \Big\{ e_{J(My)} e^{-i(2J+1)\eta} Y^i_{J(My)} - e^\dagger_{J(My)} e^{i(2J+1)\eta} Y^{i*}_{J(My)} \Big\}. \qquad (8\text{-}6)
\end{aligned}
$$

As mentioned above, the $J = 1/2$ mode of the vector field which is the mode satisfying $(\square_3 + 2)\mathsf{h}^i|_{J=1/2} = 0$ is removed by the gauge condition.

The commutation relation of each mode is given by

$$
\begin{aligned}
&\left[c_{J_1(M_1x_1)}, c^\dagger_{J_2(M_2x_2)} \right] = -\left[d_{J_1(M_1x_1)}, d^\dagger_{J_2(M_2x_2)} \right] = \delta_{J_1J_2} \delta_{M_1M_2} \delta_{x_1x_2}, \\
&\left[e_{J_1(M_1y_1)}, e^\dagger_{J_2(M_2y_2)} \right] = -\delta_{J_1J_2} \delta_{M_1M_2} \delta_{y_1y_2}.
\end{aligned}
$$

Therefore, $c_{J(Mx)}$ is a positive-metric mode, $d_{J(Mx)}$ and $e_{J(My)}$ are negative-metric modes. The Hamiltonian operator can be written as

$$
H = \sum_{J \geq 1} \sum_{M,x} \{ 2J c^\dagger_{J(Mx)} c_{J(Mx)} - (2J+2) d^\dagger_{J(Mx)} d_{J(Mx)} \}
$$

[1] Why the expansion of h^i is unusual is merely a convention for conforming to normalization of the generator Q_M of special conformal transformation, derived in the next section.

$$-\sum_{J\geq 1}\sum_{M,y}(2J+1)e^{\dagger}_{J(My)}e_{J(My)}. \tag{8-7}$$

Generators of Conformal Transformations

Using the conformal Killing vector ζ^μ and the energy-momentum tensor as defined in the previous chapter, generators of conformal transformations can be expressed as

$$Q_\zeta = \int_{S^3} d\Omega_3\, \zeta^\mu \hat{\Theta}_{\mu 0}.$$

In fact, using the conformal Killing equation (7-9) and the conservation equation $\hat{\nabla}^\nu \hat{\Theta}_{\mu\nu} = -\partial_\eta \hat{\Theta}_{\mu 0} + \hat{\nabla}^i \hat{\Theta}_{\mu i} = 0$, we can show that the generator is preserved due to tracelessness of the energy-momentum tensor as $\partial_\eta Q_\zeta = -(1/4)\int d\Omega_3 \hat{\nabla}_\lambda \zeta^\lambda\, \hat{\Theta}^\mu_{\ \mu} = 0$.

Let us first solve the conformal Killing equation on $\mathbb{R}\times S^3$ and find 15 conformal Killing vectors. The conformal Killing equation is written for each component as

$$\begin{aligned} &3\partial_\eta\zeta_0 + \psi = 0, \qquad \partial_\eta\zeta_i + \hat{\nabla}_i\zeta_0 = 0, \\ &\hat{\nabla}_i\zeta_j + \hat{\nabla}_j\zeta_i - \frac{2}{3}\hat{\gamma}_{ij}\psi = 0, \end{aligned} \tag{8-8}$$

where $\psi = \hat{\nabla}_i\zeta^i$. Solving these equations for ψ yields $(\Box_3 + 3)\psi = 0$ and $(\partial_\eta^2 + 1)\psi = 0$. The former is obtained by applying $\hat{\nabla}^j\hat{\nabla}^i$ to the last equation in (8-8). Substituting the result into the remaining conformal Killing equations yields the latter. From this, a solution which simultaneously satisfies these two equations is expressed as $\psi = 0$ or $\psi \propto e^{\pm i\eta}Y_{\frac{1}{2}M}$.

The solution of $\psi = 0$ is that satisfying $\partial_\eta\zeta_0 = \Box_3\zeta_0 = 0$ and the Killing equation $\hat{\nabla}_i\zeta_j + \hat{\nabla}_j\zeta_i = 0$ on S^3. One of the solution is a vector for time evolution represented by $\zeta_i = 0$ as

$$\zeta^\mu_{\mathrm{T}} = (1,0,0,0).$$

Others are the six Killing vectors representing the rotation (isometry) of S^3, which satisfies $\zeta_0 = 0$ and $\partial_\eta\zeta_j = 0$ simultaneously. The Killing vector of

S^3 denoted by $\zeta_{\rm R}^\mu = (0, \zeta_{\rm R}^j)$ can be expressed using a scalar harmonics as[2]

$$(\zeta_{\rm R}^j)_{MN} = i\frac{{\rm V}_3}{4}\left\{Y^*_{\frac{1}{2}M}\hat{\nabla}^j Y_{\frac{1}{2}N} - Y_{\frac{1}{2}N}\hat{\nabla}^j Y^*_{\frac{1}{2}M}\right\},$$

where the indices M and N denote a $\mathbf{4}$-vector representation of $SU(2) \times SU(2)$. Even after this, for simplicity, when describing the $\mathbf{4}$-vector indices of the conformal Killing vectors and corresponding generators of conformal transformations, we display only the degeneracy index M with omitting $J = 1/2$.

Substituting these into the definition of the generators, we obtain the Hamiltonian operator

$$H = \int_{S^3} d\Omega_3 :\hat{\Theta}_{00}:$$

and six rotation generators of S^3 as

$$R_{MN} = \int_{S^3} d\Omega_3\, (\zeta_{\rm R}^j)_{MN} :\hat{\Theta}_{j0}:,$$

where R_{MN} satisfies

$$R_{MN} = -\epsilon_M \epsilon_N R_{-N-M}, \qquad R^\dagger_{MN} = R_{NM}.$$

Solutions of the conformal Killing equation satisfying $\psi \neq 0$ are denoted by $\zeta_{\rm S}^\mu = (\zeta_{\rm S}^0, \zeta_{\rm S}^j)$. There are eight solutions, four of which are given by

$$(\zeta_{\rm S}^0)_M = \frac{\sqrt{{\rm V}_3}}{2} e^{i\eta} Y^*_{\frac{1}{2}M}, \quad (\zeta_{\rm S}^j)_M = -i\frac{\sqrt{{\rm V}_3}}{2} e^{i\eta}\hat{\nabla}^j Y^*_{\frac{1}{2}M} \tag{8-9}$$

and the other four are complex conjugates of them. Substituting (8-9) into the definition of the generator and rewriting it using the conservation equation of the energy-momentum tensor, four generators of special conformal transformations are obtained as

$$Q_M = \sqrt{{\rm V}_3}\, P^{(+)} \int_{S^3} d\Omega_3\, Y^*_{\frac{1}{2}M} :\hat{\Theta}_{00}:, \tag{8-10}$$

where $P^{(+)} = e^{i\eta}(1 + i\partial_\eta)/2$. By performing the spatial integration over S^3, we can show that only functions of $e^{\pm i\eta}$ remain. Thus, $P^{(+)}$ is an operator that selects only the $e^{-i\eta}$ part and makes the generator time-independent.

---

[2] The Killing vector on S^3 can be expressed using the $J = 1/2$ component of the vector harmonics $Y^j_{J(My)}$ and a $SU(2) \times SU(2)$ Clebsch-Gordan coefficient of the $\mathbf{G}$ type (C-4) as $(\zeta_{\rm R}^j)_{MN} = i(\sqrt{{\rm V}_3}/2)\sum_{V,y} \mathbf{G}^{1/2M}_{1/2(Vy);1/2N} Y^{j*}_{1/2(Vy)}$.

Their Hermitian conjugates $Q^\dagger_M$ correspond to four generators of translations.

In this place, we explain the residual gauge degrees of freedom in the radiation$^+$ gauge mentioned before in detail. The residual gauge degrees of freedom preserving normal conditions of the radiation gauge defined by $h_{00} = 0$ and $\hat{\nabla}^i h_{0i} = \hat{\nabla}^i h^{\rm tr}_{ij} = 0$ are expressed by three equations $\delta_\kappa h_{00} = (3\partial_\eta \kappa_0 + \tilde{\psi})/2 = 0$, $\delta_\kappa(\hat{\nabla}^i h_{0i}) = \partial_\eta \tilde{\psi} + \Box_3 \kappa_0 = 0$, and $\delta_\kappa(\hat{\nabla}^i h^{\rm tr}_{ij}) = (\Box_3 + 2)\kappa_j + \hat{\nabla}_j \tilde{\psi}/3 = 0$, where $\tilde{\psi} = \hat{\nabla}_i \kappa^i$. These equations indicate that the residual gauge degrees of freedom are wider than 15 gauge degrees of freedom spanned by the conformal Killing vectors. That is, the second equation is weaker than the second condition of the conformal Killing equation (8-8), and thus there is a solution satisfying $\partial_\eta \kappa^i \neq 0$ as a solution of the Killing equation on S^3. Letting $f(\eta)$ be an arbitrary function of time, it means that $\kappa^\mu = (0, f(\eta) Y^i_{1/2(My)})$ is allowed as a residual degree of freedom. With this gauge degree of freedom we can remove the $J = 1/2$ mode in h_i and impose the gauge-fixing condition (8-4). Thus, the residual degrees of freedom of the transformations after fixing in the radiation$^+$ gauge is the same as the conformal Killing vectors.

The 15 generators of the conformal transformations form a conformal algebra of $SO(4,2)$ as follows:

$$\begin{aligned}
\left[Q_M, Q^\dagger_N\right] &= 2\delta_{MN} H + 2R_{MN}, \\
\left[H, Q_M\right] &= -Q_M, \qquad \left[H, Q^\dagger_M\right] = Q^\dagger_M, \\
\left[H, R_{MN}\right] &= \left[Q_M, Q_N\right] = 0, \\
\left[Q_M, R_{M_1 M_2}\right] &= \delta_{M M_2} Q_{M_1} - \epsilon_{M_1}\epsilon_{M_2}\delta_{M -M_1} Q_{-M_2}, \\
\left[R_{M_1 M_2}, R_{M_3 M_4}\right] &= \delta_{M_1 M_4} R_{M_3 M_2} - \epsilon_{M_1}\epsilon_{M_2}\delta_{-M_2 M_4} R_{M_3 -M_1} \\
&\quad -\delta_{M_2 M_3} R_{M_1 M_4} + \epsilon_{M_1}\epsilon_{M_2}\delta_{-M_1 M_3} R_{-M_2 M_4}.
\end{aligned} \tag{8-11}$$

On the cylindrical background $\mathbb{R} \times S^3$, the Hamiltonian operator is a dilatation operator that counts conformal dimensions of states. In order to see this, consider a conformal mapping $y \to r = e^y$ from the Euclidean $\mathbb{R} \times S^3$ metric $dy^2 + d\Omega_3^2$ to the $\mathbb{R}^4$ metric $dr^2 + r^2 d\Omega_3^2$. A scale transformation $r \to e^a r$ corresponds to time evolution $y \to y + a$ in the cylindrical spacetime. From this, the way to quantize fields on $\mathbb{R}^4$ is called the radial quantization.[3] Quantum field theory on the Lorentzian $\mathbb{R} \times S^3$ is obtained

[3] See S. Fubini, A. Hanson and R. Jackiw, *New Approach to Field Theory*, Phys. Rev. D **7** (1973) 1732.

by doing analytic continuation as $y = i\eta$. We can then see that each mode of fields with a time-dependence $e^{iE\eta}$ has a conformal dimension E.

Since the rotation generator R_{MN} commutes with the Hamiltonian operator, it is an operator whose conformal dimension is zero. The generator of the special conformal transformation Q_M has conformal dimension -1 and its Hermitian conjugate has 1. Therefore, the generator Q_M is represented by a proper combination of creation and annihilation operators whose conformal dimensions differ by 1.

Parameterizing a $\mathbf{4}$-vector index $\{(1/2, 1/2), (1/2, -1/2), (-1/2, 1/2), (-1/2, -1/2)\}$ by $\{1, 2, 3, 4\}$, and setting $A_+ = R_{31}$, $A_- = R_{31}^\dagger$, $A_3 = (R_{11} + R_{22})/2$, $B_+ = R_{21}$, $B_- = R_{21}^\dagger$, and $B_3 = (R_{11} - R_{22})/2$, the last rotation algebra in (8-11) can be written in a familiar form of $SU(2) \times SU(2)$ algebra as

$$
\begin{aligned}
\left[A_+, A_-\right] &= 2A_3, \qquad & \left[A_3, A_\pm\right] &= \pm A_\pm, \\
\left[B_+, B_-\right] &= 2B_3, \qquad & \left[B_3, B_\pm\right] &= \pm B_\pm,
\end{aligned}
$$

where $A_{\pm,3}$ and $B_{\pm,3}$ commute.

We here discuss the four-dimensional quantum gravity by dividing it into four sectors: the scalar field, the gauge field, the conformal-factor field, and the traceless tensor field. Full generator of the conformal transformation is given by the sum of all sectors. The following will provide the generator concretely for each field.

Scalar field The energy-momentum tensor of the conformally invariant scalar field is given by

$$
\begin{aligned}
\hat{\Theta}_{\mu\nu} = \frac{2}{3}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi - \frac{1}{3}\varphi\hat{\nabla}_\mu\hat{\nabla}_\nu\varphi + \frac{1}{6}\hat{R}_{\mu\nu}\varphi^2 \\
-\frac{1}{6}\hat{g}_{\mu\nu}\left\{\hat{\nabla}_\lambda\varphi\hat{\nabla}^\lambda\varphi + \frac{1}{6}\hat{R}\varphi^2\right\}.
\end{aligned}
$$

Since the trace vanishes proportional to the equation of motion as $\hat{\Theta}^\lambda{}_\lambda = (1/3)\,\varphi(-\hat{\nabla}^2 + \hat{R}/6)\varphi = 0$, the generator of the conformal transformation is preserved as shown before.

The generator can be obtained by substituting the energy-momentum tensor into the definition and performing the integration over S^3. The Hamiltonian operator is already given by (8-1). The rotation generator can be expressed, using the index $\{1, 2, 3, 4\}$ parameterized as the $\mathbf{4}$-vector index above, as[4]

$$R_{11}=\sum_{J>0}\sum_{M}(m+m')\varphi^{\dagger}_{JM}\varphi_{JM},$$
$$R_{22}=\sum_{J>0}\sum_{M}(m-m')\varphi^{\dagger}_{JM}\varphi_{JM},$$
$$R_{21}=\sum_{J>0}\sum_{M}\sqrt{(J+1-m')(J+m')}\varphi^{\dagger}_{JM}\varphi_{J\overline{M}},$$
$$R_{31}=\sum_{J>0}\sum_{M}\sqrt{(J+1-m)(J+m)}\varphi^{\dagger}_{JM}\varphi_{J\underline{M}},$$

where the indices with upper and lower lines are defined by $\overline{M}=(m,m'-1)$ and $\underline{M}=(m-1,m')$.

The generator of the special conformal transformation is obtained by substituting the normal ordered energy-momentum tensor into the expression (8-10) as follows:

$$\begin{aligned}Q_M=P^{(+)}\sum_{J_1,M_1}\sum_{J_2,M_2}\frac{1}{4}\sqrt{\frac{\mathrm{V}_3}{(2J_1+1)(2J_2+1)}}\int_{S^3}d\Omega_3\,Y^{*}_{\frac{1}{2}M}Y_{J_1M_1}Y_{J_2M_2}\\ \times\Bigg\{\left[-(2J_1+1)(2J_2+1)+(2J_2+1)^2-\frac{1}{2}\right]\\ \times\Big(\varphi_{J_1M_1}\varphi_{J_2M_2}e^{-i(2J_1+2J_2+2)\eta}\\ +\epsilon_{M_1}\varphi^{\dagger}_{J_1-M_1}\epsilon_{M_2}\varphi^{\dagger}_{J_2-M_2}e^{i(2J_1+2J_2+2)\eta}\Big)\\ +\left[(2J_1+1)(2J_2+1)+(2J_2+1)^2-\frac{1}{2}\right]\\ \times\Big(\varphi_{J_1M_1}\epsilon_{M_2}\varphi^{\dagger}_{J_2-M_2}e^{-i(2J_1-2J_2)\eta}\end{aligned}$$

---

[4] Using the $SU(2)\times SU(2)$ Clebsch-Gordan coefficient of the type $\mathbf{G}$ (C-4), the rotation generator can be expressed as

$$R_{MN}=\frac{1}{2}\sum_{J\geq 0}\sum_{S_1,S_2}\sum_{V,y}\epsilon_V\,\mathbf{G}^{\frac{1}{2}M}_{\frac{1}{2}(-Vy);\frac{1}{2}N}\mathbf{G}^{JS_1}_{\frac{1}{2}(Vy);JS_2}\varphi^{\dagger}_{JS_1}\varphi_{JS_2}.$$

Substituting $\mathbf{G}^{1/2M}_{J(Vy);JN}=-\sqrt{2J(2J+2)}C^{1/2m}_{J+yv,Jn}C^{1/2m'}_{J-yv',Jn'}$ and $\mathbf{G}^{JM}_{1/2(Vy);JN}=-\sqrt{2J(2J+2)}C^{Jm}_{1/2+yv,Jn}C^{Jm'}_{1/2-yv',Jn'}$ reduces this expression to the one in the text, where the coefficient $\mathbf{G}^{JM}_{J_1(M_1y_1);J_2M_2}$ has values only at $J_1=J_2$ when $J=1/2$ and only at $J=J_2$ when $J_1=1/2$.

$$+\epsilon_{M_1}\varphi^{\dagger}_{J_1 -M_1}\varphi_{J_2 M_2}e^{i(2J_1-2J_2)\eta}\Big)\Big\}$$

$$= \sum_{J\geq 0}\sum_{M_1,M_2}\mathbf{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}M_2}\sqrt{(2J+1)(2J+2)}\epsilon_{M_1}\varphi^{\dagger}_{J-M_1}\varphi_{J+\frac{1}{2}M_2}, \tag{8-12}$$

where $\mathbf{C}$ is a $SU(2)\times SU(2)$ Clebsch-Gordan coefficient obtained by integrating the product of three scalar harmonics over S^3 defined as

$$\begin{aligned}\mathbf{C}^{JM}_{J_1M_1,J_2M_2} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^*_{JM}Y_{J_1M_1}Y_{J_2M_2} \\ &= \sqrt{\frac{(2J_1+1)(2J_2+1)}{2J+1}}C^{Jm}_{J_1m_1,J_2m_2}C^{Jm'}_{J_1m'_1,J_2m'_2},\end{aligned} \tag{8-13}$$

where $C^{Jm}_{J_1m_1,J_2m_2}$ is the normal Clebsch-Gordan coefficient, from which $J+J_1+J_2$ is an integer, and a triangular inequality $|J_1-J_2|\leq J\leq J_1+J_2$ and $M=M_1+M_2$ hold. In addition, $\mathbf{C}$ is a real function, and $\mathbf{C}^{JM}_{J_1M_1,J_2M_2}=\mathbf{C}^{JM}_{J_2M_2,J_1M_1}=\mathbf{C}^{J-M}_{J_1-M_1,J_2-M_2}=\epsilon_{M_2}\mathbf{C}^{J_1M_1}_{JM,J_2-M_2}$ and $\mathbf{C}^{JM}_{00,JN}=\delta_{MN}$ are satisfied. The $\mathbf{C}$-coefficient with $J=1/2$ appears in the generator Q_M.

The free scalar field is transformed as a primary field of conformal dimension 1. Actually calculating commutation relations between the generators and the field operator, the transformation law (7-11) is obtained as

$$i[Q_\zeta,\varphi]=\zeta^\mu\hat{\nabla}_\mu\varphi+\frac{1}{4}\hat{\nabla}_\mu\zeta^\mu\varphi.$$

For example, when the conformal Killing vector is η^μ, the generator is the Hamiltonian operator, and we can show $i[H,\varphi]=\partial_\eta\varphi$ immediately. In the case of the special conformal transformation, substituting $(\zeta^\mu_{\mathrm{S}})_M$ on the right-hand side and rewriting it using a product expansion of scalar harmonics (C-6) in the third section of Appendix C, we can show that it agrees with $i[Q_M,\varphi]$.

Gauge field The energy-momentum tensor of the gauge field is given by

$$\hat{\Theta}_{\mu\nu}=F_{\mu\lambda}F_\nu^{\ \lambda}-\frac{1}{4}\hat{g}_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma},$$

where $F^\mu_{\ \nu}=\hat{g}^{\mu\lambda}F_{\lambda\nu}$. This energy-momentum tensor is obviously traceless.

The generators of the conformal transformations are given in the radiation gauge $A_0=\hat{\nabla}^iA_i=0$. The Hamiltonian operator is already given in

(8-2). The generator of the special conformal transformation is

$$Q_M = \sum_{J\geq\frac{1}{2}} \sum_{M_1,y_1} \sum_{M_2,y_2} \mathbf{D}^{\frac{1}{2}M}_{J(M_1y_1),J+\frac{1}{2}(M_2y_2)} \times\sqrt{(2J+1)(2J+2)}(-\epsilon_{M_1}) q^\dagger_{J(-M_1y_1)} q_{J+\frac{1}{2}(M_2y_2)}, \quad \text{(8-14)}$$

where we introduce a new $SU(2)\times SU(2)$ Clebsch-Gordan coefficient of the type $\mathbf{D}$ defined by

$$\begin{aligned}\mathbf{D}^{\frac{1}{2}M}_{J(M_1y_1),J+\frac{1}{2}(M_2y_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^*_{\frac{1}{2}M} Y^i_{J(M_1y_1)} Y_{iJ+\frac{1}{2}(M_2y_2)} \\ &= \sqrt{J(2J+3)}\, C^{\frac{1}{2}m}_{J+y_1m_1,J+\frac{1}{2}+y_2m_2} C^{\frac{1}{2}m'}_{J-y_1m'_1,J+\frac{1}{2}-y_2m'_2}.\end{aligned}$$

The general expression of the $\mathbf{D}$-coefficient is given by (C-2) in the second section of Appendix C.

Here and also below, the rotation generator on S^3 is omitted because its concrete expression is not essential in the following discussion.

Conformal-factor field The energy-momentum tensor of the conformal-factor field is obtained from a variation of the Riegert action with respect to the background metric as

$$\begin{aligned}\hat{\Theta}_{\mu\nu} = -\frac{b_c}{8\pi^2}\Bigg\{ &-4\hat{\nabla}^2\phi\hat{\nabla}_\mu\hat{\nabla}_\nu\phi + 2\hat{\nabla}_\mu\hat{\nabla}^2\phi\hat{\nabla}_\nu\phi + 2\hat{\nabla}_\nu\hat{\nabla}^2\phi\hat{\nabla}_\mu\phi \\ &+\frac{8}{3}\hat{\nabla}_\mu\hat{\nabla}_\lambda\phi\hat{\nabla}_\nu\hat{\nabla}^\lambda\phi - \frac{4}{3}\hat{\nabla}_\mu\hat{\nabla}_\nu\hat{\nabla}_\lambda\phi\hat{\nabla}^\lambda\phi + 4\hat{R}_{\mu\lambda\nu\sigma}\hat{\nabla}^\lambda\phi\hat{\nabla}^\sigma\phi \\ &+4\hat{R}_{\mu\lambda}\hat{\nabla}^\lambda\phi\hat{\nabla}_\nu\phi + 4\hat{R}_{\nu\lambda}\hat{\nabla}^\lambda\phi\hat{\nabla}_\mu\phi - \frac{4}{3}\hat{R}_{\mu\nu}\hat{\nabla}_\lambda\phi\hat{\nabla}^\lambda\phi \\ &-\frac{4}{3}\hat{R}\hat{\nabla}_\mu\phi\hat{\nabla}_\nu\phi - \frac{2}{3}\hat{\nabla}_\mu\hat{\nabla}_\nu\hat{\nabla}^2\phi - 4\hat{R}_{\mu\lambda\nu\sigma}\hat{\nabla}^\lambda\hat{\nabla}^\sigma\phi \\ &+\frac{14}{3}\hat{R}_{\mu\nu}\hat{\nabla}^2\phi + 2\hat{R}\hat{\nabla}_\mu\hat{\nabla}_\nu\phi - 4\hat{R}_{\mu\lambda}\hat{\nabla}^\lambda\hat{\nabla}_\nu\phi - 4\hat{R}_{\nu\lambda}\hat{\nabla}^\lambda\hat{\nabla}_\mu\phi \\ &-\frac{1}{3}\hat{\nabla}_\mu\hat{R}\hat{\nabla}_\nu\phi - \frac{1}{3}\hat{\nabla}_\nu\hat{R}\hat{\nabla}_\mu\phi + \hat{g}_{\mu\nu}\Bigg[\hat{\nabla}^2\phi\hat{\nabla}^2\phi - \frac{2}{3}\hat{\nabla}^\lambda\hat{\nabla}^2\phi\hat{\nabla}_\lambda\phi \\ &-\frac{2}{3}\hat{\nabla}^\lambda\hat{\nabla}^\sigma\phi\hat{\nabla}_\lambda\hat{\nabla}_\sigma\phi - \frac{8}{3}\hat{R}_{\lambda\sigma}\hat{\nabla}^\lambda\phi\hat{\nabla}^\sigma\phi + \frac{2}{3}\hat{R}\hat{\nabla}^\lambda\phi\hat{\nabla}_\lambda\phi \\ &+\frac{2}{3}\hat{\nabla}^4\phi + 4\hat{R}_{\lambda\sigma}\hat{\nabla}^\lambda\hat{\nabla}^\sigma\phi - 2\hat{R}\hat{\nabla}^2\phi + \frac{1}{3}\hat{\nabla}^\lambda\hat{R}\hat{\nabla}_\lambda\phi\Bigg]\Bigg\}.\end{aligned}$$

Its trace disappears in proportion to the equation of motion of the conformal-factor field on $\mathbb{R}\times S^3$ as $\hat{\Theta}^\lambda_{\ \lambda} = -(b_c/4\pi^2)\hat{\Delta}_4\phi = 0$.

The Hamiltonian operator is already given in (8-5). From (8-10), the generator of the special conformal transformation is derived as

$$\begin{aligned} Q_M = & \left(\sqrt{2b_c} - i\hat{p}\right) a_{\frac{1}{2}M} \\ & + \sum_{J\geq 0}\sum_{M_1,M2} \mathbf{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}M_2} \Big\{ \alpha(J)\epsilon_{M_1} a^\dagger_{J-M_1} a_{J+\frac{1}{2}M_2} \\ & + \beta(J)\epsilon_{M_1} b^\dagger_{J-M_1} b_{J+\frac{1}{2}M_2} + \gamma(J)\epsilon_{M_2} a^\dagger_{J+\frac{1}{2}-M_2} b_{JM_1} \Big\}, \end{aligned} \tag{8-15}$$

where the $\mathbf{C}$-coefficient is the same as (8-13) introduced in the case of the scalar field. Other coefficients are given by

$$\begin{aligned} & \alpha(J) = \sqrt{2J(2J+2)}, \qquad \beta(J) = -\sqrt{(2J+1)(2J+3)}, \\ & \gamma(J) = 1. \end{aligned} \tag{8-16}$$

As mentioned earlier, the Casimir term in the Hamiltonian operator (8-5) is necessary for the conformal algebra (8-11) to close.

In calculations of the conformal algebra, the following crossing relation that the $\mathbf{C}$-coefficient satisfies is useful:

$$\sum_{J\geq 0}\sum_M \epsilon_M \mathbf{C}^{J_1M_1}_{J_2M_2,J-M} \mathbf{C}^{J_3M_3}_{JM,J_4M_4} = \sum_{J\geq 0}\sum_M \epsilon_M \mathbf{C}^{J_1M_1}_{J_4M_4,J-M} \mathbf{C}^{J_3M_3}_{JM,J_2M_2}. \tag{8-17}$$

This relation can be derived by integrating the product of four scalar harmonics

$$\int_{S^3} d\Omega_3\, Y^*_{J_1M_1} Y_{J_2M_2} Y^*_{J_3M_3} Y_{J_4M_4}$$

in two ways using the fact that the product of two scalar harmonics can be expanded in another scalar harmonics as

$$Y_{J_1M_1}Y_{J_2M_2} = \frac{1}{\sqrt{V_3}} \sum_{J\geq 0}\sum_M \mathbf{C}^{JM}_{J_1M_1,J_2M_2} Y_{JM}.$$

The crossing relation (8-17) is quite useful, for example, when calculating the commutation relation between Q_M and $Q^\dagger_N$. It is also useful when constructing physical states in the next section.

The conformal transformation law can be expressed using a commutation relation between the generator and the field operator as

$$i\left[Q_\zeta, \phi\right] = \zeta^\mu \hat{\nabla}_\mu \phi + \frac{1}{4}\hat{\nabla}_\mu \zeta^\mu. \tag{8-18}$$

In the case of the special conformal transformation, it can be shown easily by using the product expansion of scalar harmonics (C-6) as in the case of the scalar field.

Traceless tensor field The generators of the conformal transformations for the traceless tensor field are given in the radiation$^+$ gauge. The Hamiltonian operator H is given in (8-7). The generator of the special conformal transformation, writing only the result, becomes

$$
\begin{aligned}
Q_M = &\sum_{J\geq 1}\sum_{M_1,x_1}\sum_{M_2,x_2} \mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J+\frac{1}{2}(M_2x_2)} \\
&\times\Big\{\alpha(J)\epsilon_{M_1}c^\dagger_{J(-M_1x_1)}c_{J+\frac{1}{2}(M_2x_2)} + \beta(J)\epsilon_{M_1}d^\dagger_{J(-M_1x_1)}d_{J+\frac{1}{2}(M_2x_2)} \\
&\quad +\gamma(J)\epsilon_{M_2}c^\dagger_{J+\frac{1}{2}(-M_2x_2)}d_{J(M_1x_1)}\Big\} \\
&+\sum_{J\geq 1}\sum_{M_1,x_1}\sum_{M_2,y_2} \mathbf{H}^{\frac{1}{2}M}_{J(M_1x_1);J(M_2y_2)} \\
&\quad\times\Big\{A(J)\epsilon_{M_1}c^\dagger_{J(-M_1x_1)}e_{J(M_2y_2)} + B(J)\epsilon_{M_2}e^\dagger_{J(-M_2y_2)}d_{J(M_1x_1)}\Big\} \\
&+\sum_{J\geq 1}\sum_{M_1,y_1}\sum_{M_2,y_2} \mathbf{D}^{\frac{1}{2}M}_{J(M_1y_1),J+\frac{1}{2}(M_2y_2)}C(J)\epsilon_{M_1}e^\dagger_{J(-M_1y_1)}e_{J+\frac{1}{2}(M_2y_2)}.
\end{aligned}
\tag{8-19}
$$

The coefficients $\alpha(J)$, $\beta(J)$, and $\gamma(J)$ are given by the same as (8-16) in the conformal-factor field. Other coefficients are given by

$$
A(J) = \sqrt{\frac{4J}{(2J-1)(2J+3)}}, \qquad B(J) = \sqrt{\frac{2(2J+2)}{(2J-1)(2J+3)}},
$$
$$
C(J) = \sqrt{\frac{(2J-1)(2J+1)(2J+2)(2J+4)}{2J(2J+3)}}.
$$

New $SU(2)\times SU(2)$ Clebsch-Gordan coefficients are defined by

$$
\begin{aligned}
\mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J+\frac{1}{2}(M_2x_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^*_{\frac{1}{2}M}Y^{ij}_{J(M_1x_1)}Y_{ijJ+\frac{1}{2}(M_2x_2)} \\
&= \sqrt{(2J-1)(J+2)}\, C^{\frac{1}{2}m}_{J+x_1m_1,J+\frac{1}{2}+x_2m_2}C^{\frac{1}{2}m'}_{J-x_1m_1',J+\frac{1}{2}-x_2m_2'}, \\
\mathbf{H}^{\frac{1}{2}M}_{J(M_1x_1);J(M_2y_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^*_{\frac{1}{2}M}Y^{ij}_{J(M_1x_1)}\hat{\nabla}_i Y_{jJ(M_2y_2)} \\
&= -\sqrt{(2J-1)(2J+3)}\, C^{\frac{1}{2}m}_{J+x_1m_1,J+y_2m_2}C^{\frac{1}{2}m'}_{J-x_1m_1',J-y_2m_2'}.
\end{aligned}
$$

The general expressions for these coefficients are given in (C-3) and (C-5) in the second section of Appendix C.

Here, we have found this generator by determining the six coefficients α, β, γ, A, B, and C so that the conformal algebra closes without specifying their values in advance, instead of directly deriving from the energy-momentum tensor of the Weyl action according to the definition of Q_ζ. At that time, calculations of the algebra can be simplified by using crossing relations derived from product expansions of the vector and the tensor harmonics. Each convention such as a sign of the coefficients and the mode-expansion (8-6) already shown is decided to be consistent with the conformal transformation law of the field.

The existence of cross terms between the positive-metric mode $c_{J(Mx)}$ and the negative-metric modes $d_{J(Mx)}$, $e_{J(My)}$ indicates that the conformal algebra does not close with the positive-metric tensor mode only. Hence, these negative-metric modes are indispensable modes for the conformal algebra. In this way, in order to realize the conformal invariance that represents quantum diffeomorphism invariance, the fourth-order gravitational field including the negative-metric modes is essential.

BRST Operator and Physical State Conditions

Ghost fields c^μ satisfying the conformal Killing equation $\hat{\nabla}_\mu c_\nu + \hat{\nabla}_\nu c_\mu - \hat{g}_{\mu\nu}\hat{\nabla}_\lambda c^\lambda/2 = 0$ are represented in terms of 15 Grassmann modes. Denoting these modes as c, c_{MN}, c_M, and $\mathrm{c}_M^\dagger$ in the Roman style, the ghost fields are mode-expanded as

$$c^\mu = \mathrm{c}\eta^\mu + \sum_M \left(\mathrm{c}_M^\dagger \zeta_M^\mu + \mathrm{c}_M \zeta_M^{\mu *} \right) + \sum_{M,N} \mathrm{c}_{MN} \zeta_{MN}^\mu,$$

where c is a Hermitian operator and c_{MN} are six modes that satisfy $\mathrm{c}_{MN}^\dagger = \mathrm{c}_{NM}$ and $\mathrm{c}_{MN} = -\epsilon_M \epsilon_N \mathrm{c}_{-N-M}$.[5] In addition, we introduce anti-ghost modes b, b_{MN}, b_M, and $\mathrm{b}_M^\dagger$ with the same properties as the ghost modes have and set anti-commutation relations between them as

$$\{\mathrm{b}, \mathrm{c}\} = 1, \qquad \{\mathrm{b}_{MN}, \mathrm{c}_{LK}\} = \delta_{ML}\delta_{NK} - \epsilon_M \epsilon_N \delta_{-MK}\delta_{-NL},$$
$$\{\mathrm{b}_M^\dagger, \mathrm{c}_N\} = \{\mathrm{b}_M, \mathrm{c}_N^\dagger\} = \delta_{MN}.$$

[5] The c_{MN} mode satisfies $\sum_M \mathrm{c}_{MM} = 0$, and also $\sum_M \epsilon_M \mathrm{c}_{-M}\mathrm{c}_M = 0$ holds from the Grassmannian property.

Using these Grassmann modes, we can construct 15 generators satisfying the conformal algebra (8-11) as follows:

$$
\begin{aligned}
H^{\mathrm{gh}} &= \sum_M \left(\mathrm{c}_M^\dagger \mathrm{b}_M - \mathrm{c}_M \mathrm{b}_M^\dagger \right), \\
R_{MN}^{\mathrm{gh}} &= -\mathrm{c}_M \mathrm{b}_N^\dagger + \mathrm{c}_N^\dagger \mathrm{b}_M + \epsilon_M \epsilon_N \left(\mathrm{c}_{-N} \mathrm{b}_{-M}^\dagger - \mathrm{c}_{-M}^\dagger \mathrm{b}_{-N} \right) \\
&\quad - \sum_L \left(\mathrm{c}_{LM} \mathrm{b}_{LN} - \mathrm{c}_{NL} \mathrm{b}_{ML} \right), \\
Q_M^{\mathrm{gh}} &= -2\mathrm{c}_M \mathrm{b} - \mathrm{c}\mathrm{b}_M - \sum_L \left(2\mathrm{c}_{LM} \mathrm{b}_L + \mathrm{c}_L \mathrm{b}_{ML} \right), \\
Q_M^{\mathrm{gh}\dagger} &= 2\mathrm{c}_M^\dagger \mathrm{b} + \mathrm{c}\mathrm{b}_M^\dagger + \sum_L \left(2\mathrm{c}_{ML} \mathrm{b}_L^\dagger + \mathrm{c}_L^\dagger \mathrm{b}_{LM} \right). \qquad (8\text{-}20)
\end{aligned}
$$

In the following, the whole generators of conformal transformations including these ghost parts are expressed as

$$
\mathcal{H} = H + H^{\mathrm{gh}}, \qquad \mathcal{R}_{MN} = R_{MN} + R_{MN}^{\mathrm{gh}},
$$
$$
\mathcal{Q}_M = Q_M + Q_M^{\mathrm{gh}}, \qquad \mathcal{Q}_M^\dagger = Q_M^\dagger + Q_M^{\mathrm{gh}\dagger},
$$

where H, R_{MN}, Q_M, and $Q_M^\dagger$ are respectively the sums of the generators other than the ghost sector.

The BRST operator of the transformation (7-10) defined on the background spacetime $\mathbb{R} \times S^3$ is given by

$$
\begin{aligned}
Q_{\mathrm{BRST}} &= \mathrm{c}H + \sum_M \left(\mathrm{c}_M^\dagger Q_M + \mathrm{c}_M Q_M^\dagger \right) + \sum_{M,N} \mathrm{c}_{MN} R_{MN} \\
&\quad + \frac{1}{2} \mathrm{c} H^{\mathrm{gh}} + \frac{1}{2} \sum_M \left(\mathrm{c}_M^\dagger Q_M^{\mathrm{gh}} + \mathrm{c}_M Q_M^{\mathrm{gh}\dagger} \right) + \frac{1}{2} \sum_{M,N} \mathrm{c}_{MN} R_{MN}^{\mathrm{gh}}.
\end{aligned}
$$

Further deforming it, we obtain

$$
Q_{\mathrm{BRST}} = \mathrm{c}\mathcal{H} + \sum_{M,N} \mathrm{c}_{MN} \mathcal{R}_{MN} - \mathrm{b}\mathcal{M} - \sum_{M,N} \mathrm{b}_{MN} \mathcal{Y}_{MN} + \tilde{\mathcal{Q}}, \qquad (8\text{-}21)
$$

where $\mathcal{H}$ and $\mathcal{R}_{MN}$ are the whole generators defined above. Other terms are defined by

$$
\mathcal{M} = 2 \sum_M \mathrm{c}_M^\dagger \mathrm{c}_M, \qquad \mathcal{Y}_{MN} = \mathrm{c}_M^\dagger \mathrm{c}_N + \sum_L \mathrm{c}_{ML} \mathrm{c}_{LN},
$$
$$
\tilde{\mathcal{Q}} = \sum_M \left(\mathrm{c}_M^\dagger Q_M + \mathrm{c}_M Q_M^\dagger \right).
$$

Using the expression (8-21) and the conformal algebra (8-11), we can show nilpotency as

$$
\begin{aligned}
Q_{\mathrm{BRST}}^2 &= \tilde{Q}^2 - M\mathcal{H} - 2\sum_{M,N} \mathrm{c}_M^\dagger \mathrm{c}_N \left[\mathcal{R}_{MN} + \sum_L \left(\mathrm{c}_{LM}\mathrm{b}_{LN} - \mathrm{c}_{NL}\mathrm{b}_{ML}\right)\right] \\
&= \tilde{Q}^2 - MH - 2\sum_{M,N} \mathrm{c}_M^\dagger \mathrm{c}_N R_{MN} = 0.
\end{aligned}
$$

Anti-commutation relations between the BRST operator and the anti-ghost modes are given by

$$
\begin{aligned}
\{Q_{\mathrm{BRST}}, \mathrm{b}\} &= \mathcal{H}, \qquad \{Q_{\mathrm{BRST}}, \mathrm{b}_{MN}\} = 2\mathcal{R}_{MN}, \\
\{Q_{\mathrm{BRST}}, \mathrm{b}_M\} &= \mathcal{Q}_M, \qquad \left\{Q_{\mathrm{BRST}}, \mathrm{b}_M^\dagger\right\} = \mathcal{Q}_M^\dagger.
\end{aligned}
$$

Thus, the whole generator of the conformal transformation becomes BRST trivial. Therefore, descendant states which are generated by applying $\mathcal{Q}_M^\dagger$ to a BRST invariant physical state given as a primary scalar state in the below become BRST trivial.

The BRST transformation is a diffeomorphism replacing the gauge transformation parameter ζ^μ with the ghost field c^μ. It is expressed using a commutation relation with the BRST operator as

$$
i\left[Q_{\mathrm{BRST}}, \phi\right] = c^\mu \hat{\nabla}_\mu \phi + \frac{1}{4}\hat{\nabla}_\mu c^\mu.
$$

The same applies to other fields. In the case of the ghost field, it is given with an anti-commutation relation as

$$
i\left\{Q_{\mathrm{BRST}}, c^\mu\right\} = c^\nu \hat{\nabla}_\nu c^\mu.
$$

Physical states are represented as BRST conformally invariant states. Writing them symbolically as $|\Psi\rangle$, physical state conditions are simply expressed as

$$
Q_{\mathrm{BRST}}|\Psi\rangle = 0. \tag{8-22}
$$

In the following, we will construct physical states by solving this condition.

First of all, we define several vacuum states. Separating the ghost sector and other field sectors, we write the latter Fock vacuum as $|0\rangle$, which disappears when the zero mode $\hat{p}$ of the conformal-factor field and the annihilation operators such as a_{JM} and b_{JM} are applied. Furthermore, a conformally invariant vacuum which disappears for all of the generators H, R_{MN},

$Q_M, Q_M^\dagger$ without the ghost part is expressed as

$$|\Omega\rangle = e^{-2b_c\phi_0(0)}|0\rangle, \tag{8-23}$$

where $\phi_0(0) = \hat{q}/\sqrt{2b_c}$. Both the vacuum $|\Omega\rangle$ and its Hermitian conjugate $\langle\Omega|$ have a Riegert charge $-2b_c$ as a background charge. Thus, the total background charge of the conformally invariant vacua is $-4b_c$. This charge originates from the linear term in the Riegert action.

Let us write a conformally invariant ghost vacuum as $|0\rangle_{\rm gh}$, which disappears for all of the generators (8-20) of the ghost sector. This vacuum vanishes for all of the anti-ghost modes, but it does not disappear against the ghost modes. On the other hand, a Fock ghost vacuum which disappears when the annihilation operators c_M and b_M are applied is expressed as $\prod_M \mathrm{c}_M|0\rangle_{\rm gh}$ using the conformally invariant ghost vacuum.

Since the Hamiltonian operator $H^{\rm gh}$ does not include the c and c_{MN} ghost modes and the b and b_{MN} anti-ghost modes, the ghost vacuum is degenerate. Its degenerate pair is given by the vacuum multiplied by c and $\prod \mathrm{c}_{MN}$. Inner products between them will be discussed later.

For later convenience, we introduce the whole Fock vacuum with a Riegert charge γ, including the ghost part, as

$$|\gamma\rangle = e^{\gamma\phi_0(0)}|\Omega\rangle \otimes \prod_M \mathrm{c}_M|0\rangle_{\rm gh}.$$

Since $i\hat{p}|\gamma\rangle = (\gamma/\sqrt{2b_c} - \sqrt{2b_c})|\gamma\rangle$, this state satisfies

$$\mathcal{H}|\gamma\rangle = (h_\gamma - 4)|\gamma\rangle, \qquad h_\gamma = \gamma - \frac{\gamma^2}{4b_c}, \tag{8-24}$$

where h_γ is the same as given in (7-32), and -4 comes from the ghost part.

The physical state $|\Psi\rangle$ is constructed by applying the creation operators such as $a_{JM}^\dagger$ and $b_{JM}^\dagger$ in the field sectors, the creation operators $\mathrm{c}_M^\dagger$ and $\mathrm{b}_M^\dagger$ in the ghost sector, and the zero-mode $\hat{p}$ to the whole Fock vacuum. The zero-mode $\hat{p}$ may be replaced with its eigenvalue. Here, note that the state disappears if b or b_{MN} are applied. Since these modes satisfy $\{Q_{\rm BRST}, \mathrm{b}\} = \mathcal{H}$ and $\{Q_{\rm BRST}, \mathrm{b}_{MN}\} = 2\mathcal{R}_{MN}$, we find that as physical states it is only enough to consider states in a subspace that satisfies the following conditions:

$$\mathcal{H}|\Psi\rangle = \mathcal{R}_{MN}|\Psi\rangle = 0, \qquad \mathrm{b}|\Psi\rangle = \mathrm{b}_{MN}|\Psi\rangle = 0. \tag{8-25}$$

On this subspace, from the expression of (8-21), the BRST conformally invariant state (8-22) is the same as a $\tilde{Q}$-invariant state.

For a while, as the physical state constructed on the subspace (8-25), we consider the following form:

$$|\Psi\rangle = \mathcal{A}\left(\hat{p}, a^{\dagger}_{JM}, b^{\dagger}_{JM}, \ldots\right)|\gamma\rangle, \tag{8-26}$$

where the dots represent other field creation operators except for the ghost and anti-ghost modes. The operator $\mathcal{A}$ and the Riegert charge γ are determined from the BRST conformal invariance conditions. The case where the ghost and anti-ghost modes are included in $\mathcal{A}$ will be discussed at the end of the next section.

As long as concerning the state of the type above, the $\tilde{Q}$-invariance condition reduces to

$$\tilde{Q}|\Psi\rangle = \sum_M \mathrm{c}^{\dagger}_M Q_M|\Psi\rangle = 0$$

because of $\mathrm{c}_M|\Psi\rangle = 0$. Furthermore, adding the condition of the Hamiltonian operator and the rotation invariance condition in (8-25), we obtain

$$(H-4)|\Psi\rangle = R_{MN}|\Psi\rangle = Q_M|\Psi\rangle = 0 \tag{8-27}$$

as the BRST conformal invariance conditions of the state, where the $Q^{\dagger}_M$ condition is not necessary. The condition (8-27) indicates that the BRST conformally invariant state is given by a primary scalar with conformal dimension 4.

The BRST conformal invariance condition (8-27) requires that the operator $\mathcal{A}$ satisfies the following algebra:

$$[H, \mathcal{A}] = l\mathcal{A}, \qquad [R_{MN}, \mathcal{A}] = 0, \qquad [Q_M, \mathcal{A}] = 0,$$

where l $(\geq\ 0)$ is conformal dimension of $\mathcal{A}$. The Hamiltonian operator condition then indicates that the Riegert charge γ in the whole Fock vacuum $|\gamma\rangle$ must satisfy

$$h_\gamma + l - 4 = 0.$$

This is a quadratic equation for the Riegert charge γ. If we choose a solution that approaches the canonical value $4-l$ at the classical limit $b_c \to \infty$, γ is given for each l as

$$\gamma_l = 2b_c\left(1 - \sqrt{1 - \frac{4-l}{b_c}}\right). \tag{8-28}$$

This is a real number due to $b_c > 4$ (7-5). Here, γ_0 and γ_2 correspond to the previous α (7-35) and β (7-36), respectively.

Construction of Physical States

In order to construct physical states, we have to find primary states which disappear by applying the generator Q_M of special conformal transformations. First, we look for combinations of creation operators that commute with Q_M. After finding such operators for each field sector, we combine them to be rotation-invariant, and then construct physical states so as to satisfy the Hamiltonian operator condition.

Primary states of scalar fields As a simple example, we first examine primary states of the scalar field. A commutation relation between Q_M (8-12) and the creation mode $\varphi^{\dagger}_{JM_1}$ is given by

$$[Q_M, \varphi^{\dagger}_{JM_1}] = \sqrt{2J(2J+1)} \sum_{M_2} \epsilon_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J-\frac{1}{2}-M_2} \varphi^{\dagger}_{J-\frac{1}{2}M_2}.$$

Thus, the creation mode that commutes with Q_M is only $\varphi^{\dagger}_{00}$ with conformal dimension 1. Here we will impose a Z_2 symmetry under $\varphi \leftrightarrow -\varphi$ on the scalar field and allow only even product of $\varphi^{\dagger}_{00}$.

Next, consider operators given by the product of the creation modes. Such a quadratic operator that belongs to a J representation with conformal dimension $2L+2$ is generally expressed as

$$\Phi^{[L]\dagger}_{JN} = \sum_{K=0}^{L} \sum_{M_1} \sum_{M_2} f(L,K) \mathbf{C}^{JN}_{L-KM_1, KM_2} \varphi^{\dagger}_{L-KM_1} \varphi^{\dagger}_{KM_2}.$$

Calculating commutation relation with Q_M yields

$$\begin{aligned} \left[Q_M, \Phi^{[L]\dagger}_{JN}\right] = \sum_{K=0}^{L} \sum_{M_1} \sum_{M_2} & \varphi^{\dagger}_{L-K-\frac{1}{2}M_1} \varphi^{\dagger}_{KM_2} \\ & \times \Bigg\{ \sqrt{(2L-2K)(2L-2K+1)} f(L,K) \\ & \qquad \times \sum_{S} \epsilon_S \mathbf{C}^{\frac{1}{2}M}_{L-K-\frac{1}{2}M_1, L-K-S} \mathbf{C}^{JN}_{L-KS, KM_2} \\ & \quad + \sqrt{(2K+1)(2K+2)} f\left(L, K+\frac{1}{2}\right) \\ & \qquad \times \sum_{S} \epsilon_S \mathbf{C}^{\frac{1}{2}M}_{KM_2, K+\frac{1}{2}-S} \mathbf{C}^{JN}_{K+\frac{1}{2}S, L-K-\frac{1}{2}M_1} \Bigg\}. \end{aligned}$$

Using the crossing relation (8-17), it is understood that conditions for this commutator to vanish are only if $J=L$, L is a nonnegative integer, and the coefficient f satisfies the following recursion relation:

$$f\left(L, K+\frac{1}{2}\right) = -\sqrt{\frac{(2L-2K)(2L-2K+1)}{(2K+1)(2K+2)}}\, f(L,K).$$

Solving this recursion relation, the coefficient is determined as

$$f(L,K) = \frac{(-1)^{2K}}{\sqrt{(2L-2K+1)(2K+1)}} \begin{pmatrix} 2L \\ 2K \end{pmatrix} \tag{8-29}$$

up to a normalization constant that depends only on L. In this way, we can find a quadratic creation operator that commutes with Q_M. Writing it as $\Phi^{\dagger}_{LN} = \Phi^{[L]\dagger}_{LN}$, we obtain

$$\Phi^{\dagger}_{LN} = \sum_{K=0}^{L} \sum_{M_1} \sum_{M_2} f(L,K) \mathbf{C}^{LN}_{L-KM_1, KM_2} \varphi^{\dagger}_{L-KM_1} \varphi^{\dagger}_{KM_2},$$

where L is a nonnegative integer, and the operator of $L = 0$ is the already obtained $\Phi^{\dagger}_{00} = (\varphi^{\dagger}_{00})^2$.

rank of tensor	0
creation operators conformal dimension	$\Phi^{\dagger}_{LN}$ $2L+2$

Table 8-1: Building blocks of primary states for the scalar field, where L is a nonnegative integer.

Combining the operator $\Phi^{\dagger}_{LN}$ with the $SU(2) \times SU(2)$ Clebsch-Gordan coefficient, we can construct a basis for creation operators that commute with Q_M. It is thought that any creation operator that commutes with Q_M can be represented in such a form basically due to the crossing relation of the Clebsch-Gordan coefficient and so on. Hence, we may consider the operators $\Phi^{\dagger}_{LN}$ ($L \in \mathbb{Z}_{\geq 0}$) as building blocks of primary states for the scalar field sector, which are summarized in Table 8-1.

Let us concretely see relationships between primary states obtained in this way and field operators. First, the simplest primary state $\varphi^{2n\dagger}_{00}|0\rangle$ with conformal dimension $2n$ corresponds to a field operator $:\varphi^{2n}:$ by a state-operator correspondence.[6] A primary state $\Phi^{\dagger}_{1M}|0\rangle$ of conformal dimension

[6] In the cylindrical spacetime of $\mathbb{R} \times S^3$, the state-operator correspondence is given by $|\{\mu_1 \cdots \mu_l\}; \Delta\rangle = \lim_{\eta \to i\infty} e^{-i\Delta\eta} O_{\mu_1 \cdots \mu_l}(\eta, \hat{\mathbf{x}})|0\rangle$. See also the fourth section of Appendix B.

4 with nine independent components corresponds to the traceless energy-momentum tensor $\Theta_{\mu\nu}$. Likewise, a primary state $\Phi^\dagger_{LM}|0\rangle$ corresponds to a symmetric traceless tensor field of even spin $l = 2L$ with conformal dimension $2L + 2$.

In the same way, building blocks of primary states for the gauge field can be obtained. The results are summarized in the seventh section of Appendix B.

Primary states of gravitational fields Next, we consider primary states in the conformal-factor field sector. Commutation relations between the generator Q_M (8-15) and the zero-modes of the conformal-factor field are given by

$$[Q_M, \hat{q}] = -a_{\frac{1}{2}M}, \qquad [Q_M, \hat{p}] = 0.$$

Commutation relations with $a^\dagger_{1/2M}$ and $a^\dagger_{JM}$ $(J \geq 1)$ are

$$\begin{aligned}
\left[Q_M, a^\dagger_{\frac{1}{2}M_1}\right] &= \left(\sqrt{2b_c} - i\hat{p}\right)\delta_{M,M_1}, \\
\left[Q_M, a^\dagger_{JM_1}\right] &= \alpha\left(J - \frac{1}{2}\right)\sum_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J-\frac{1}{2}M_2}\epsilon_{M_2} a^\dagger_{J-\frac{1}{2}-M_2},
\end{aligned}$$

and that with $b^\dagger_{JM}$ $(J \geq 0)$ is

$$\begin{aligned}
\left[Q_M, b^\dagger_{JM_1}\right] = &-\gamma(J)\sum_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J+\frac{1}{2}M_2}\epsilon_{M_2} a^\dagger_{J+\frac{1}{2}-M_2} \\
&-\beta\left(J - \frac{1}{2}\right)\sum_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J-\frac{1}{2}M_2}\epsilon_{M_2} b^\dagger_{J-\frac{1}{2}-M_2}.
\end{aligned}$$

As in the same way as doing for the scalar field, we can construct quadratic creation operators with conformal dimension $2L$ that commute with Q_M. Let L be a positive integer, we obtain the following two types:

$$\begin{aligned}
S^\dagger_{LN} = &\frac{\sqrt{2}(\sqrt{2b_c} - i\hat{p})}{\sqrt{(2L-1)(2L+1)}}\, a^\dagger_{LN} \\
&+ \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}}\sum_{M_1}\sum_{M_2} x(L,K)\,\mathbf{C}^{LN}_{L-KM_1, KM_2} a^\dagger_{L-KM_1} a^\dagger_{KM_2}, \\
\mathcal{S}^\dagger_{L-1N} = &-\sqrt{2}(\sqrt{2b_c} - i\hat{p})\, b^\dagger_{L-1N} \\
&+ \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}}\sum_{M_1}\sum_{M_2} x(L,K)\,\mathbf{C}^{L-1N}_{L-KM_1, KM_2} a^\dagger_{L-KM_1} a^\dagger_{KM_2}
\end{aligned}$$

$$+\sum_{K=\frac{1}{2}}^{L-1}\sum_{M_1}\sum_{M_2} y(L,K)\,\mathbf{C}^{L-1N}_{L-K-1M_1,KM_2} b^\dagger_{L-K-1M_1} a^\dagger_{KM_2},$$

where the coefficients are given by

$$x(L,K) = \frac{(-1)^{2K}}{\sqrt{(2L-2K+1)(2K+1)}}\sqrt{\begin{pmatrix} 2L \\ 2K \end{pmatrix}\begin{pmatrix} 2L-2 \\ 2K-1 \end{pmatrix}},$$
$$y(L,K) = -2\sqrt{(2L-2K-1)(2L-2K+1)}x(L,K). \tag{8-30}$$

If L is a half integer, there is no such operators. These two quadratic creation operators will be building blocks of primary states for the conformal-factor field. They are summarized in Table 8-2.

rank of tensor	0
creation operators	$S^\dagger_{LN}$
	$\mathcal{S}^\dagger_{L-1N}$
conformal dimension	$2L$

Table 8-2: Building blocks of primary states for the conformal-factor field. Each building block exists for an integer L (≥ 1).

If we do the same analysis for the traceless tensor field, we find that the only creation mode that is commutative with Q_M (8-19) is the lowest positive-metric mode $c^\dagger_{1(Mx)}$ in the transverse-traceless field h_{ij}. Moreover, as in the cases of the conformal-factor field, we can classify Q_M-invariant quadratic creation operators using triangle inequalities and crossing relations of several $SU(2)\times SU(2)$ Clebsch-Gordan coefficients even if their explicit values are unknown. In this case, quadratic operators with tensor index up to rank 4 appear. They shall be building blocks of primary states for the traceless tensor field, which are summarized in Table 8-3. For their concrete expressions, see the seventh section of Appendix B.

From the building blocks in Tables 8-2 and 8-3, we can construct gravitational primary states. For example, let us consider the lowest scalar operator of the conformal-factor field,

$$\mathcal{S}^\dagger_{00} = -\sqrt{2}(\sqrt{2b_c} - i\hat{p})b^\dagger_{00} - \frac{1}{\sqrt{2}}\sum_M \epsilon_M a^\dagger_{\frac{1}{2}-M} a^\dagger_{\frac{1}{2}M}. \tag{8-31}$$

A gravitational primary scalar state of conformal dimension 2 can be constructed by applying this operator to the vacuum (8-23) as $\mathcal{S}^\dagger_{00}|\Omega\rangle$. This state corresponds to the Ricci scalar R, apart from an exponential factor of ϕ.

0	1	2	3	4
$A^\dagger_{LN}$	$B^\dagger_{L-\frac{1}{2}(Ny)}$	$c^\dagger_{1(Nx)}$	$D^\dagger_{L-\frac{1}{2}(Nz)}$	$E^\dagger_{L(Nw)}$
$\mathcal{A}^\dagger_{L-1N}$				$\mathcal{E}^\dagger_{L-1(Nw)}$
$2L$	$2L$	2	$2L$	$2L$

Table 8-3: Building blocks for the traceless tensor field. Items are the same as Table 8-2. Each building block of $n = 0, 1, 3, 4$ exists for an integer $L(\geq 3)$. Here, z and w are the polarization indices ϵ_3 and ϵ_4, respectively.

The next building blocks, each of which has nine independent components, are given by

$$S^\dagger_{1N} = \sqrt{\frac{2}{3}}(\sqrt{2b_c} - i\hat{p})a^\dagger_{1N} - \frac{1}{\sqrt{2}}\sum_{M_1,M_2}\mathbf{C}^{1N}_{\frac{1}{2}M_1,\frac{1}{2}M_2}a^\dagger_{\frac{1}{2}M_1}a^\dagger_{\frac{1}{2}M_2},$$

$$\mathcal{S}^\dagger_{1N} = -\sqrt{2}(\sqrt{2b_c} - i\hat{p})b^\dagger_{1N} - 4b^\dagger_{00}a^\dagger_{1N} - \sqrt{2}\sum_{M_1,M_2}\mathbf{C}^{1N}_{\frac{3}{2}M_1,\frac{1}{2}M_2}a^\dagger_{\frac{3}{2}M_1}a^\dagger_{\frac{1}{2}M_2}$$
$$+\frac{2}{\sqrt{3}}\sum_{M_1,M_2}\mathbf{C}^{1N}_{1M_1,1M_2}a^\dagger_{1M_1}a^\dagger_{1M_2} + 4\sum_{M_1,M_2}\mathbf{C}^{1N}_{\frac{1}{2}M_1,\frac{1}{2}M_2}b^\dagger_{\frac{1}{2}M_1}a^\dagger_{\frac{1}{2}M_2}.$$

From these, we obtain traceless symmetric primary tensor states, of which the state $S^\dagger_{1N}|\Omega\rangle$ of conformal dimension 2 corresponds to $R_{\mu\nu} - g_{\mu\nu}R/4$ and the state $\mathcal{S}^\dagger_{1N}|\Omega\rangle$ of conformal dimension 4 corresponds to the energy-momentum tensor of the conformal-factor field.

In addition, we obtain a primary tensor state $c^\dagger_{1(Nx)}|\Omega\rangle$ of conformal dimension 2 using the lowest building block of the traceless tensor field. It has ten independent components, and corresponds to the Weyl tensor $C_{\mu\nu\lambda\sigma}$, where $x = \pm 1$ represents selfdual and anti-selfdual components. Primary states of conformal dimension 4 are $\sum_{N,x}\epsilon_N c^\dagger_{1(-Nx)}c^\dagger_{1(Nx)}|\Omega\rangle$ and $\sum_{N_1,x_1}\sum_{N_2,x_2}\mathbf{E}^{1N}_{1(N_1x_1),1(N_2x_2)}c^\dagger_{1(N_1x_1)}c^\dagger_{1(N_2x_2)}|\Omega\rangle$. These states correspond to $C^2_{\mu\nu\lambda\sigma}$ and the energy-momentum tensor of traceless tensor field, respectively.

Some of the primary states raised as examples here do not satisfy the unitarity bound (2-14). It is a characteristic feature in higher-order derivative theories, but it is not necessary to satisfy this condition because these states are still gauge dependent.[7] In fact, they do not satisfy the conditions of H

[7] As another example, the normal $U(1)$ gauge field A_μ is a primary vector field, but the

and R_{MN} yet, and thus are not gauge-invariant physical states of quantum gravity.

Physical states Physical states in the form of (8-26) are now expressed using the building blocks obtained above as

$$|\Psi\rangle = \mathcal{A}(\Phi^\dagger, S^\dagger, \mathcal{S}^\dagger, \ldots)|\gamma\rangle.$$

All of tensor indices of the building blocks are then contracted by using various $SU(2) \times SU(2)$ Clebsch-Gordan coefficients so that $\mathcal{A}$ becomes invariant under the S^3 rotation. The conformal dimension l of the operator $\mathcal{A}$ is given by an even number because all the building blocks have even dimensions. It represents the number of derivatives of corresponding field operators. Finally, by choosing the Riegert charge (8-28) so as to satisfy the Hamiltonian operator condition for l, physical states can be constructed.

As an example, let us see physical states where the conformal dimension l is four or less. A physical state that the identity operator $\mathcal{A} = I$ is dressed by quantum gravity is given by the $l = 0$ state

$$|\gamma_0\rangle.$$

This is a state corresponding to the quantum cosmological term, which reduces to $\sqrt{-g}$ in the classical limit. Physical states of $l = 2$ are given by

$$\Phi^\dagger_{00}|\gamma_2\rangle, \qquad \mathcal{S}^\dagger_{00}|\gamma_2\rangle.$$

These are quantum states corresponding to $\sqrt{-g}\varphi^2$ and the scalar curvature $\sqrt{-g}R$, respectively. Physical states of $l = 4$ are given by

$$\begin{aligned}&(\Phi^\dagger_{00})^2|\gamma_4\rangle, \qquad \Phi^\dagger_{00}\mathcal{S}^\dagger_{00}|\gamma_4\rangle, \qquad \mathcal{S}^\dagger_{00}\mathcal{S}^\dagger_{00}|\gamma_4\rangle, \\ &\sum_N \epsilon_N S^\dagger_{1-N} S^\dagger_{1N}|\gamma_4\rangle, \qquad \sum_{N,x} \epsilon_N c^\dagger_{1(-Nx)} c^\dagger_{1(Nx)}|\gamma_4\rangle.\end{aligned}$$

These are quantum states corresponding to $\sqrt{-g}\varphi^4$, $\sqrt{-g}R\varphi^2$, $\sqrt{-g}R^2$, $\sqrt{-g}(R_{\mu\nu} - g_{\mu\nu}R/4)^2$, and $\sqrt{-g}C^2_{\mu\nu\lambda\sigma}$, respectively. Since $\gamma_4 = 0$, there is no exponential factor of ϕ in these physical states even at the quantum level.

Lastly, we discuss physical states including the ghost and anti-ghost creation modes $\mathrm{c}^\dagger_M$ and $\mathrm{b}^\dagger_M$. For example, in the case of $l = 2$, there exists another BRST conformally invariant state such as

$$\Big\{ -\left(\sqrt{2b_c} - i\hat{p}\right)^2 \sum_M \epsilon_M \mathrm{b}^\dagger_{-M}\mathrm{c}^\dagger_M + \hat{h} \sum_M \epsilon_M a^\dagger_{\frac{1}{2}-M} a^\dagger_{\frac{1}{2}M} \Big\}|\gamma_2\rangle, \quad \text{(8-32)}$$

conformal dimension is 1, which does not satisfy the unitary bound. This is because the gauge field depends on the gauge. See Footnote 6 in Chapter 2.

where $\hat{h} = \hat{p}^2/2 + b_c$. However, it turns out that (8-32) is BRST equivalent with the physical state already given above. In order to show that, we introduce a new state

$$|\Upsilon\rangle = \left(\sqrt{2b_c} - i\hat{p}\right) \sum_M \epsilon_M \mathrm{b}^\dagger_{-M} a^\dagger_{\frac{1}{2}M} |\gamma_2\rangle$$

satisfying $\mathcal{H}|\Upsilon\rangle = \mathcal{R}_{MN}|\Upsilon\rangle = \mathrm{b}|\Upsilon\rangle = \mathrm{b}_{MN}|\Upsilon\rangle = 0$. Applying the BRST operator to this state yields

$$Q_{\mathrm{BRST}}|\Upsilon\rangle = \Biggl\{ -\left(\sqrt{2b_c} - i\hat{p}\right)^2 \sum_M \epsilon_M \mathrm{b}^\dagger_{-M} \mathrm{c}^\dagger_M \\ +4\left(\sqrt{2b_c} - i\hat{p}\right) b^\dagger_{00} + 2\hat{h} \sum_M \epsilon_M a^\dagger_{\frac{1}{2}-M} a^\dagger_{\frac{1}{2}M} \Biggr\} |\gamma_2\rangle.$$

From this, paying attention to $\hat{h}|\beta\rangle = 2|\beta\rangle$, the BRST invariant state (8-32) can be written as

$$\frac{1}{2\sqrt{2}} \mathcal{S}^\dagger_{00} |\gamma_2\rangle + Q_{\mathrm{BRST}}|\Upsilon\rangle.$$

Thus, we can show that it is BRST equivalent to the physical state $\mathcal{S}^\dagger_{00}|\gamma_2\rangle$.

In general, physical states including the ghost modes in $\mathcal{A}$ seem to be BRST equivalent with physical states given by the standard form (8-26). Therefore, we will consider only the standard form in this book.

Physical Field Operators

The BRST invariant physical field operators discussed in the previous chapter are considered again on $\mathbb{R} \times S^3$. As mentioned earlier, such physical fields consist of primary scalar fields. In order to obtain such operators, we first examine transformation laws of the n-th power operator of the conformal-factor field defined by

$$:\phi^n: = :(\phi_> + \phi_0 + \phi_<)^n: = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \phi_>^{n-k} \left(\phi_0 + \phi_<\right)^k,$$

where $\phi_<$ and $\phi_> (= \phi_<^\dagger)$ are the annihilation and creation operator parts of the field, respectively, and $\phi_0 = (\hat{q} + \eta\hat{p})/\sqrt{2b_c}$ is the zero-mode part.

Transformation laws of this operator for time evolution and S^3 rotations are given by

$$i\left[H, :\phi^n:\right] = \partial_\eta :\phi^n:, \qquad i\left[R_{MN}, :\phi^n:\right] = \hat{\nabla}_j \left(\zeta^j_{MN} :\phi^n:\right).$$

In these transformations, quantum corrections do not appear. On the other hand, using the fact that each part of the conformal-factor field is transformed under special conformal transformations as

$$i\left[Q_M,\phi_>\right] = \zeta_M^\mu \hat{\nabla}_\mu \phi_> + \zeta_M^0 \partial_\eta \phi_0 + \frac{1}{4}\hat{\nabla}_\mu \zeta_M^\mu,$$
$$i\left[Q_M,\phi_0+\phi_<\right] = \zeta_M^\mu \hat{\nabla}_\mu \phi_<,$$

we obtain

$$i\left[Q_M, :\phi^n:\right] = \zeta_M^\mu \hat{\nabla}_\mu :\phi^n: + \frac{n}{4}\hat{\nabla}_\mu \zeta_M^\mu :\phi^{n-1}: - \frac{1}{16b_c} n(n-1)\hat{\nabla}_\mu \zeta_M^\mu :\phi^{n-2}: .$$

The last term on the right-hand side is a quantum correction, which can be derived by using $i\zeta_M^0 = \hat{\nabla}_\mu \zeta_M^\mu/4$ and an expansion formula

$$\partial_\eta\left(\phi_0+\phi_<\right)^k = k\partial_\eta\phi_<\left(\phi_0+\phi_<\right)^{k-1} + k\partial_\eta\phi_0\left(\phi_0+\phi_<\right)^{k-1} + i\frac{1}{4b_c}k(k-1)\left(\phi_0+\phi_<\right)^{k-2}$$

yielded by paying attention to the zero-mode commutation relation $[\phi_0, \partial_\eta\phi_0] = i/2b_c$. The transformation of $n=1$ is simply the transformation (8-18).

The simplest primary scalar field with a Riegert charge α is given by

$$\mathcal{V}_\alpha = :e^{\alpha\phi}: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :\phi^n: = e^{\alpha\phi_>} e^{\alpha\phi_0} e^{\alpha\phi_<},$$

where the exponential operator of the zero-mode can be expressed in the form $e^{\alpha\phi_0} = e^{\hat{q}\alpha/\sqrt{2b_c}} e^{\eta\hat{p}\alpha/\sqrt{2b_c}} e^{-i\eta\alpha^2/4b_c}$. The transformation laws are

$$i[H,\mathcal{V}_\alpha] = \partial_\eta \mathcal{V}_\alpha, \quad i[R_{MN},\mathcal{V}_\alpha] = \hat{\nabla}_j(\zeta_{MN}^j \mathcal{V}_\alpha),$$
$$i\left[Q_M,\mathcal{V}_\alpha\right] = \zeta_M^\mu \hat{\nabla}_\mu \mathcal{V}_\alpha + \frac{h_\alpha}{4}\hat{\nabla}_\mu \zeta_M^\mu \mathcal{V}_\alpha,$$

where the conformal dimension h_α is given by (8-24). Letting the Riegert charge α be a real number so that $\mathcal{V}_\alpha$ is a Hermitian operator, the transformation law of the translation $Q_M^\dagger$ is given by the one that ζ_M^μ is replaced with its complex conjugate $\zeta_M^{\mu *}$. These transformation laws can be expressed by one equation using the BRST operator as

$$i\left[Q_{\rm BRST},\mathcal{V}_\alpha\right] = c^\mu \hat{\nabla}_\mu \mathcal{V}_\alpha + \frac{h_\alpha}{4}\hat{\nabla}_\mu c^\mu \mathcal{V}_\alpha.$$

From this, we find that a volume integral of the primary scalar operator $\mathcal{V}_\alpha$ with $h_\alpha = 4$ becomes BRST invariant as

$$i\big[Q_{\rm BRST}, \int d\Omega_4\, \mathcal{V}_\alpha\big] = \int d\Omega_4\, \hat{\nabla}_\mu\, (c^\mu \mathcal{V}_\alpha) = 0,$$

where $d\Omega_4 = d\eta\, d\Omega_3$ is the volume element of the background spacetime. This condition is equal to that the field operator commutes with all of the 15 generators. Furthermore, as in the previous chapter, if we introduce the ghost field function contracted with a fully antisymmetric tensor as $\omega = (1/4!)\, \epsilon_{\mu\nu\lambda\sigma} c^\mu c^\nu c^\lambda c^\sigma$, the product of ω and $\mathcal{V}_\alpha$ becomes a locally BRST invariant field operator as

$$i\,[Q_{\rm BRST}, \omega \mathcal{V}_\alpha] = \frac{1}{4}\,(h_\alpha - 4)\,\omega \hat{\nabla}_\mu c^\mu \mathcal{V}_\alpha = 0,$$

because ω transforms as $i[Q_{\rm BRST}, \omega] = -\omega \hat{\nabla}_\mu c^\mu$ under the BRST transformation. The Riegert charge is then given by the $l = 0$ expression of (8-28), which is a real number $\alpha = \gamma_0 = 2b_c(1 - \sqrt{1 - 4/b_c})$. This is the same as (7-35). The operator $\mathcal{V}_a$ with this value is the quantum cosmological term operator on $\mathbb{R} \times S^3$.

Next, we consider a physical field operator corresponding to the Ricci scalar curvature. Writing only the result here, a primary scalar field with second derivatives is given by

$$\begin{aligned}
\mathcal{R}_\beta &= :e^{\beta\phi} \left(\hat{\nabla}^2 \phi + \frac{\beta}{h_\beta} \hat{\nabla}_\mu \phi \hat{\nabla}^\mu \phi - \frac{h_\beta}{\beta} \right): \\
&= \mathcal{R}^1_\beta + \frac{\beta}{h_\beta} \mathcal{R}^2_\beta - \frac{h_\beta}{\beta} \mathcal{V}_\beta,
\end{aligned}$$

where $\mathcal{R}^{1,2}_\beta$ are defined by

$$\begin{aligned}
\mathcal{R}^1_\beta &= \hat{\nabla}^2 \phi_> V_\beta + V_\beta \hat{\nabla}^2 \phi_<, \\
\mathcal{R}^2_\beta &= -\frac{1}{4} \partial_\eta \phi_0 \partial_\eta \phi_0 V_\beta - \frac{1}{2} \partial_\eta \phi_0 V_\beta \partial_\eta \phi_0 - \frac{1}{4} V_\beta \partial_\eta \phi_0 \partial_\eta \phi_0 \\
&\quad - \partial_\eta \phi_0 \left(\partial_\eta \phi_> V_\beta + V_\beta \partial_\eta \phi_< \right) - \left(\partial_\eta \phi_> V_\beta + V_\beta \partial_\eta \phi_< \right) \partial_\eta \phi_0 \\
&\quad + \hat{\nabla}_\mu \phi_> \hat{\nabla}^\mu \phi_> V_\beta + 2 \hat{\nabla}_\mu \phi_> V_\beta \hat{\nabla}^\mu \phi_< + V_\beta \hat{\nabla}_\mu \phi_< \hat{\nabla}^\mu \phi_<.
\end{aligned}$$

Since the operator $\mathcal{R}_\beta$ is transformed as a primary scalar field of conformal dimension $h_\beta + 2$, we can see that it transforms under the BRST transformation as

$$i\,[Q_{\rm BRST}, \mathcal{R}_\beta] = c^\mu \hat{\nabla}_\mu \mathcal{R}_\beta + \frac{h_\beta + 2}{4} \hat{\nabla}_\mu c^\mu \mathcal{R}_\beta.$$

Therefore, when $h_\beta = 2$, the volume integral of $\mathcal{R}_\beta$ or the field product $\omega\mathcal{R}_\beta$ becomes BRST invariant. The Riegert charge is then given by the $l = 2$ expression of (8-28), which is $\beta = \gamma_2 = 2b_c(1 - \sqrt{1-2/b_c})$, the same as (7-36). The operator $\mathcal{R}_\beta$ with this value is the quantum scalar curvature, which reduces to the normal scalar curvature on $\mathbb{R} \times S^3$ given by $d^4x\sqrt{-g}R = d\Omega_4 e^{2\phi}(-6\hat{\nabla}^2\phi - 6\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\phi + 6)$ divided by -6 when the classical limit $b_c \to \infty$ is taken.

State-Operator Correspondences and Dual States

Finally, we clarify correspondences between physical field operators and physical states and examine the structure of inner products. In general, when considering a physical field operator $\mathcal{O}_\gamma$ with Riegert charge γ that satisfies the BRST conformal invariance condition $[Q_{\mathrm{BRST}}, \omega\mathcal{O}_\gamma] = 0$, a state corresponding to this operator is given by

$$\lim_{\eta \to i\infty} e^{-4i\eta}\mathcal{O}_\gamma|\Omega\rangle = |\mathcal{O}_\gamma\rangle,$$

apart from the ghost part.

For example, for the quantum cosmological term operator $\mathcal{V}_\alpha$, paying attention to $h_\alpha = 4$, it becomes

$$\begin{aligned}|\mathcal{V}_\alpha\rangle &= \lim_{\eta \to i\infty} e^{-4i\eta}\mathcal{V}_\alpha|\Omega\rangle \\ &= \lim_{\eta \to i\infty} e^{i(-4+h_\alpha)\eta} e^{\alpha\phi_>} e^{\frac{\alpha}{\sqrt{2b_c}}\hat{q}}|\Omega\rangle = e^{\alpha\phi_0(0)}|\Omega\rangle.\end{aligned}$$

For the quantum Ricci scalar operator $\mathcal{R}_\beta$ with $h_\beta = 2$, it becomes

$$\begin{aligned}|\mathcal{R}_\beta\rangle = \lim_{\eta \to i\infty} e^{-4i\eta}\mathcal{R}_\beta|\Omega\rangle = \lim_{\eta \to i\infty} e^{i(-4+h_\beta)\eta}\Big\{&\hat{\nabla}^2\phi_> - 2i\partial_\eta\phi_> \\ + \frac{\beta}{h_\beta}\hat{\nabla}_\mu\phi_>\hat{\nabla}^\mu\phi_>\Big\}& e^{\beta\phi_>} e^{\frac{\beta}{\sqrt{2b_c}}\hat{q}}|\Omega\rangle = -\frac{\beta}{2\sqrt{2}b_c}\mathcal{S}_{00}^\dagger e^{\beta\phi_0(0)}|\Omega\rangle,\end{aligned}$$

where $\mathcal{S}_{00}^\dagger$ is given by (8-31).

Since the most divergent part of the ghost function ω at the limit of $\eta \to i\infty$ behaves like $\omega \propto e^{-4i\eta}\prod_M \mathrm{c}_M$, it can be seen that the state-operator correspondence including the ghost part is given by

$$\lim_{\eta \to i\infty} \omega\mathcal{O}_\gamma|\Omega\rangle \otimes |0\rangle_{\mathrm{gh}} \;\propto\; |\mathcal{O}_\gamma\rangle \otimes \prod_M \mathrm{c}_M|0\rangle_{\mathrm{gh}}.$$

The right-hand side is the physical state discussed in the third and fourth sections in this chapter.

In order to define inner products, we examine a conjugate state of $|\mathcal{O}_\gamma\rangle \otimes \prod \mathrm{c}_M|0\rangle_{\mathrm{gh}}$. A conjugate of the state $|\mathcal{O}_\gamma\rangle$ denoted by $\langle\tilde{\mathcal{O}}_\gamma|$ is now not given by an ordinary Hermitian conjugate state $\langle\mathcal{O}_\gamma|$. It is because such an inner product $\langle\mathcal{O}_\gamma|\mathcal{O}_\gamma\rangle$ cannot be normalized as usual from the fact that the Riegert charge γ is a real number and the vacuum has the background charge $-4b_c$ and thus the total Riegert charge has a non-vanishing value $2\gamma - 4b_c \neq 0$ (the zero-mode is not canceled out).[8]

The state $\langle\tilde{O}_\gamma|$ is given by a dual state of $|\mathcal{O}_\gamma\rangle$ which is obtained through a duality relation $h_\gamma = h_{4b_c-\gamma}$. Again, considering the physical operators $\mathcal{V}_\alpha$ and $\mathcal{R}_\beta$, their dual field operators satisfying the physical field condition are given by

$$\begin{aligned}\tilde{\mathcal{V}}_\alpha &= \mathcal{V}_{4b_c-\alpha}, \\ \tilde{\mathcal{R}}_\beta &= -\frac{b_c}{4}\mathcal{R}_{4b_c-\beta} \\ &= -\frac{b_c}{4}\left(\mathcal{R}^1_{4b_c-\beta} + \frac{4b_c-\beta}{h_\beta}\mathcal{R}^2_{4b_c-\beta} - \frac{h_\beta}{4b_c-\beta}\mathcal{V}_{4b_c-\beta}\right).\end{aligned}$$

The corresponding dual states are defined by

$$\begin{aligned}\langle\tilde{\mathcal{V}}_\alpha| &= \lim_{\eta\to -i\infty} e^{4i\eta}\langle\Omega|\tilde{\mathcal{V}}_\alpha = \langle\Omega|e^{(4b_c-\alpha)\phi_0(0)}, \\ \langle\tilde{\mathcal{R}}_\beta| &= \lim_{\eta\to -i\infty} e^{4i\eta}\langle\Omega|\tilde{\mathcal{R}}_\beta = \frac{4b_c-\beta}{8\sqrt{2}}\langle\Omega|e^{(4b_c-\beta)\phi_0(0)}\mathcal{S}_{00}.\end{aligned}$$

Using these, the inner products can be defined and normalized as

$$\langle\tilde{\mathcal{V}}_\alpha|\mathcal{V}_\alpha\rangle = 1, \qquad \langle\tilde{\mathcal{R}}_\beta|\mathcal{R}_\beta\rangle = 1,$$

where $\langle\Omega|e^{4b_c\phi_0(0)}|\Omega\rangle = 1$ is used, which is derived from the fact that the sum of Riegert charges of the field operators reduces to $4b_c$, and it cancels with the background charges of the vacua, and thus the zero-mode disappears.

As in two-dimensional quantum gravity, there is no corresponding classical gravitational state in the dual state. Hence, this state only appears in an intermediate as a purely quantum virtual state.

[8] If the Riegert charge is a pure imaginary number like $\gamma = ip$ and the vacuum does not have a background charge, the state can be then normalized to its Hermitian conjugate like $\langle\mathcal{O}_{-ip}|\mathcal{O}_{ip}\rangle = 1$ as usual.

Taking an inner product of the ghost vacuum and its Hermitian conjugate, we find that it vanishes as ${}_{\mathrm{gh}}\langle 0|0\rangle_{\mathrm{gh}} = 0$ or ${}_{\mathrm{gh}}\langle 0| \prod_M \mathrm{c}_M^\dagger \prod_N \mathrm{c}_N |0\rangle_{\mathrm{gh}} = 0$. It can be easily shown by inserting the anti-commutation relation $\{\mathrm{b}, \mathrm{c}\} = 1$ or $\{\mathrm{b}_{MN}, \mathrm{c}_{LK}\} = \delta_{ML}\delta_{NK} - \epsilon_M \epsilon_N \delta_{-MK}\delta_{-NL}$ into the inner product, such as ${}_{\mathrm{gh}}\langle 0|0\rangle_{\mathrm{gh}} = {}_{\mathrm{gh}}\langle 0|\{\mathrm{b}, \mathrm{c}\}|0\rangle_{\mathrm{gh}} = 0$, because the vacuum disappears if b or b_{MN} is applied. Therefore, the inner product in ghost states is defined and normalized by inserting a Hermitian operator $\vartheta = i\mathrm{c} \prod \mathrm{c}_{MN}$ as

$$ {}_{\mathrm{gh}}\langle 0| \prod \mathrm{c}_M^\dagger \vartheta \prod \mathrm{c}_M |0\rangle_{\mathrm{gh}} = 1. $$

Thus, a conjugate of the physical state $|\mathcal{O}_\gamma\rangle \otimes \prod \mathrm{c}_M |0\rangle_{\mathrm{gh}}$ is given by $\langle \tilde{\mathcal{O}}_\gamma| \otimes {}_{\mathrm{gh}}\langle 0| \prod \mathrm{c}_M^\dagger \vartheta$.

Chapter Nine

Gravitational Counterterms and Conformal Anomalies

As a first step to quantize gravity, we discuss renormalization of quantum field theory in curved spacetime using dimensional regularization. Since ultraviolet divergences are local, it is possible to generalize the theory renormalizable in the flat spacetime to that on curved spacetime. Here we proceed with the argument assuming that there exists renormalizable quantum field theories in curved spacetime. That is, we require that all ultraviolet divergences to be removed are local. Then, we will find that the form of gravitational counterterms is strongly restricted from this renormalizability condition. It indicates that diffeomorphism invariance at the quantum level gives stronger conditions than that at the classical level, as was mentioned from an algebraic point of view in several previous chapters. From this consideration, we will show that the form of E_4 (5-9) introduced by Riegert appears as a conformal anomaly.

In this and next chapters, from the beginning, we will consider the D-dimensional Euclidean spacetime convenient for dimensional regularization.

Summary of Gravitational Counterterms

At the beginning, we briefly summarize the results on gravitational counterterms and conformal anomalies obtained in this chapter.

Dimensional regularization we employ here is the method to regularize the theory by making spacetime dimension a little smaller than four. Since ultraviolet divergences are extracted as poles of $D - 4$, renormalization calculations can be easily carried out. After removing ultraviolet divergences, the dimension is returned to four and physical quantities are obtained. This method is the only one that we can do higher loop calculations while preserving diffeomorphism invariance among several regularization methods of ultraviolet divergences. And, this fact is significant because as mentioned earlier, conformal anomalies are physical quantities that appear in order to preserve diffeomorphism invariance.

A further advantage of dimensional regularization is that it does not depend on how to choose the path integral measure. In the DeWitt-Schwinger method and so on defined in four dimensions, conformal anomalies as contributions from the measure are derived by regularizing a divergent quantity $\delta^{(4)}(0) = \langle x|x'\rangle|_{x'\to x}$.[1] On the other hand, in dimensional regularization, this quantity becomes identically zero due to $\delta^{(D)}(0) = \int d^D k = 0$, and thus there is no contribution from the measure.

In dimensional regularization, conformal anomalies are included between D and four dimensions. That is, conformal anomalies are finite quantities obtained by canceling poles of ultraviolet divergences with zeros representing deviations from four dimensions of D-dimensional counterterms as

$$\frac{1}{D-4} \times o(D-4) \to \text{finite}.$$

Conformal anomalies are generated from higher poles as well. Therefore, it is necessary to determine the D-dependence of counterterms exactly.

Four-dimensional gravitational counterterms with dimensionless coupling constants are not uniquely determined only by diffeomorphism invariance classically. They are expressed by any combinations of three fourth-order derivative actions of the square of the Riemann curvature tensor, the square of the Ricci tensor, and the square of the scalar curvature. Which combinations are chosen is directly linked to the problem of how to define coupling constants. Here, we will impose conformal invariance on actions, considering that a scale-invariant world is realized in the ultraviolet limit. At this time, gravitational counterterms are settled in two, the square of the Weyl tensor and the Euler density.

However, one problem arises here. Even if conformal invariance is imposed on actions, it is broken in quantum theory. It is obvious from the fact that new scales appear when renormalization calculations are carried out. In addition, since dimensional regularization literally shifts the dimension, conformal invariance is obviously broken. For this reason, there is no guarantee that renormalization calculations can be accomplished with only two gravitational counterterms. Nevertheless, it can be shown that the calculations using dimensional regularization goes well for various gauge theories.

In this chapter, we will see that the form of gravitational counterterms receives strong restrictions from the renormalizability condition at all loop orders. In particular, if we consider quantum field theories where classical

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[1] It can be expressed as $\delta^{(4)}(0) = \langle x|e^{-sK}|x\rangle|_{s\to 0}$ using a positive-definite regularization operator K specific to the theory. This quantity can be obtained by solving the heat equation $(\partial_s + K)\langle x|e^{-sK}|x\rangle = 0$. See the fourth section of Appendix D.

actions, apart from mass terms, are restricted to be conformally invariant at four dimensions by gauge symmetry, such as quantum electrodynamics (QED) and non-abelian gauge theories in curved space, we will find that gravitational counterterms are limited to two combinations even in D dimensions.

We here examine QED in curved space and will determine gravitational counterterms by analyzing renormalization group equations.[2] As the result, it will be shown that gravitational counterterms are given by the following two: the square of the D dimensional Weyl tensor (A-1) defined by

$$F_D = C^2_{\mu\nu\lambda\sigma} = R^2_{\mu\nu\lambda\sigma} - \frac{4}{D-2}R^2_{\mu\nu} + \frac{2}{(D-1)(D-2)}R^2 \quad \text{(9-1)}$$

and the Euler density modified by adding a correction term of $o(D-4)$ as

$$G_D = G_4 + (D-4)\chi(D)H^2, \quad \text{(9-2)}$$

where G_4 and H are the ordinary Euler density and a rescaled scalar curvature defined by

$$G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2, \qquad H = \frac{R}{D-1},$$

and $\chi(D)$ is a finite function of only D without poles. Expanding $\chi(D)$ in a non-negative power of $D-4$ and solving renormalization group equations for each order, we can determine all of its expansion coefficients. The first three terms are calculated explicitly, which are given by

$$\chi(D) = \frac{1}{2} + \frac{3}{4}(D-4) + \frac{1}{3}(D-4)^2 + \cdots. \quad \text{(9-3)}$$

The trace of the energy-momentum tensor, that is, the conformal anomaly is determined in the following form:

$$\Theta = \frac{\beta}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}\left(D-1+2\bar{\gamma}_\psi\right)\left[E_\psi\right] - \mu^{D-4}\left(\beta_a F_D + \beta_b E_D\right),$$

where β, β_a, and β_b are the beta functions defined in the following (9-8) and (9-9), and $\bar{\gamma}_\psi$ is an anomalous dimension of fermion. The quantity with [ ] denotes its normal product (to distinguish it from : : in free fields). The

[2] Similar arguments hold for quantum chromodynamics (QCD) (see Footnote 8 in fifth section in this chapter).

gravitational part of the conformal anomaly consists of two combinations of the Weyl tensor squared (9-1) and

$$E_D = G_D - 4\chi(D)\nabla^2 H \tag{9-4}$$

defined by extending the modified Euler density (9-2).[3] When $D \to 4$, the conformal anomaly E_D reduces to E_4. In this way, it will be shown that the combination E_4, which is predicted by Riegert from the analogy with two-dimensional quantum gravity, appears at the quantum level.[4]

In this chapter, we will show that the above fact strictly holds for all orders by examining special renormalization group equations by Hathrell that correlation functions of the energy-momentum tensor satisfy.

QED in Curved Space

The reason for considering massless QED in curved spacetime with classical conformal invariance is that it is one of the most fundamental renormalizable quantum field theories whose interactions between quantum fields and gravitational fields are completely fixed at the classical level by both gauge invariance and diffeomorphism invariance. Hence, an ambiguity in this theory that cannot be determined classically from these symmetries appears only in gravitational counterterms. In spite of that, it will be shown that the ambiguity can be settled at the quantum level.

The action is divided into three parts, the QED action, the gauge-fixing term, and gravitational counterterms, and is written as $S = S_{\rm QED} + S_{\rm g.f.} + S_g$. In the following, the quantities with the suffix 0 represent bare quantities before renormalized.

The QED action in D-dimensional curved space with Euclidean signature is given by

$$S_{\rm QED} = \int d^D x \sqrt{g} \left\{ \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + i\bar{\psi}_0 \not{D} \psi_0 \right\}.$$

The Dirac operator is defined by $\not{D} = \gamma^\mu D_\mu$ and $\gamma^\mu = e_a^\mu \gamma^a$, where $e_\mu^{\ a}$ is the D-dimensional vielbein field, which satisfies $e_\mu^{\ a} e_{\nu a} = g_{\mu\nu}$ and $e_a^\mu e_{\mu b} =$

[3] These functions satisfy the Wess-Zumino integrability condition in D dimensions given by (A-2) in the first section of Appendix A.

[4] The coefficient of the $\nabla^2 R$ term of the conformal anomaly can be arbitrarily changed by adding a finite R^2 term to the action. However, that is possible only when quantization of gravitational fields is not supposed. The aim here is to determine the minimum form of the gravitational counterterm excluding such a term not necessary for renormalization.

δ_{ab}. The Dirac's gamma matrix is normalized as $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$. The covariant derivative acting on fermions is defined by

$$D_\mu \psi_0 = \partial_\mu \psi_0 + \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}\psi_0 + ie_0 A_{0\mu}\psi_0,$$
$$D_\mu \bar{\psi}_0 = \partial_\mu \bar{\psi}_0 - \frac{1}{2}\omega_{\mu ab}\bar{\psi}_0\Sigma^{ab} - ie_0 A_{0\mu}\bar{\psi}_0,$$

where the spin connection and the generator of local Lorentz group are defined by $\omega_{\mu ab} = e^\nu{}_a(\partial_\mu e_{\nu b} - \Gamma^\lambda_{\mu\nu} e_{\lambda b})$ and $\Sigma^{ab} = -[\gamma^a, \gamma^b]/4$. For details, see the third section of Appendix A.

The BRST transformation is defined by replacing a gauge transformation parameter with a Grassmann ghost field η_0. We also introduce an anti-ghost field $\tilde{\eta}_0$ conjugate to the ghost field and an auxiliary field B_0, which are defined as

$$\delta_B A_{0\mu} = \nabla_\mu \eta_0, \quad \delta_B \psi_0 = -ie_0\eta_0\psi_0, \quad \delta_B \bar{\psi}_0 = ie_0\eta_0\bar{\psi}_0,$$
$$\delta_B \eta_0 = 0, \qquad \delta_B \tilde{\eta}_0 = iB_0, \qquad \delta_B B_0 = 0.$$

This transformation indicates nilpotency $\delta_B^2 = 0$. Using the BRST transformation, the gauge-fixing term can be expressed in a BRST trivial form as

$$S_{\text{g.f.}} = \int d^D x \sqrt{g}\, \delta_B \left[-i\, \tilde{\eta}_0 \left(\nabla^\mu A_{0\mu} - \frac{\xi_0}{2} B_0 \right) \right]$$
$$= \int d^D x \sqrt{g} \left\{ B_0 \nabla^\mu A_{0\mu} - \frac{\xi_0}{2} B_0^2 + i\, \tilde{\eta}_0 \nabla^2 \eta_0 \right\}.$$

The auxiliary field satisfies an equation of motion $B_0 = \nabla^\mu A_{0\mu}/\xi_0$. Solving it, the gauge-fixing term reduces to

$$S_{\text{g.f.}} = \int d^D x \sqrt{g} \left\{ \frac{1}{2\xi_0} \left(\nabla^\mu A_{0\mu} \right)^2 - i\, \nabla^\mu \tilde{\eta}_0 \nabla_\mu \eta_0 \right\}.$$

As for the counterterms to eliminate ultraviolet divergences of the gravitational field, in order to keep generality, we first consider three possible terms such as

$$S_g = \int d^D x \sqrt{g} \left\{ a_0 F_D + b_0 G_4 + c_0 H^2 \right\}. \tag{9-5}$$

In the following argument, we will show that the latter two terms are related each other through renormalization group equations and thus the independent gravitational counterterms are only two at all orders in perturbations.

Renormalized quantities of the fields and coupling constants are defined according to the ordinary procedure of introducing renormalization factors into each as

$$A_{0\mu} = Z_3^{1/2} A_\mu, \quad \psi_0 = Z_2^{1/2} \psi, \quad e_0 = \mu^{2-D/2} Z_3^{-1/2} e, \quad \xi_0 = Z_3 \xi,$$

where μ is an arbitrary mass scale to compensate for missing dimensions, and thus the renormalized coupling constant e becomes dimensionless. The Ward-Takahashi identity is used when defining the renormalization factor of the coupling constant. In the following, all the renormalization factors are expanded in powers of the fine structure constant

$$\alpha = \frac{e^2}{4\pi}.$$

The bare constants of the gravitational counterterms are expanded as

$$\begin{aligned} a_0 &= \mu^{D-4} \left(a + L_a \right), \qquad L_a = \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{(D-4)^n}, \\ b_0 &= \mu^{D-4} \left(b + L_b \right), \qquad L_b = \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{(D-4)^n}, \\ c_0 &= \mu^{D-4} \left(c + L_c \right), \qquad L_c = \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{(D-4)^n}, \end{aligned} \tag{9-6}$$

where $L_{a,b,c}$ are pure pole terms and the residues a_n, b_n, and c_n are functions of only α, not depending on D.

As apparent from the above procedure, the essence of renormalizability is that ultraviolet divergences appear in the form of poles with the local actions, but poles with nonlocal actions do not appear.

Conventional renormalization group equations Commonly known renormalization group equations are summarized here. They are equations that correctly regularized renormalizable theories shall satisfy. In dimensional regularization, bare quantities do not depend on the mass scale μ introduced arbitrarily, and thus the following equation holds:

$$\mu \frac{d}{d\mu} \left(\text{bare} \right) = 0, \qquad \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \mu \frac{d\alpha}{d\mu} \frac{\partial}{\partial \alpha} + \mu \frac{d\xi}{d\mu} \frac{\partial}{\partial \xi}.$$

Below, we will refer all equations derived through this condition as renormalization group equations.

We first consider the following renormalization group equation:

$$\mu \frac{d}{d\mu}\left(\frac{e_0^2}{4\pi}\right) = 0 = \frac{\mu^{4-D}}{Z_3}\alpha\left(4 - D - \mu\frac{d}{d\mu}\log Z_3 + \frac{\mu}{\alpha}\frac{d\alpha}{d\mu}\right). \quad (9\text{-}7)$$

Defining the beta function of the fine structure constant as

$$\beta(\alpha, D) \equiv \frac{1}{\alpha}\mu\frac{d\alpha}{d\mu} = D - 4 + \bar{\beta}(\alpha), \quad (9\text{-}8)$$

the part that depends only on α can be written as $\bar{\beta} = \mu d(\log Z_3)/d\mu$. Furthermore, expanding the renormalization factor as

$$\log Z_3 = \sum_{n=1}^{\infty} \frac{f_n(\alpha)}{(D-4)^n},$$

and expanding the right-hand side of (9-7) to extract conditions for it to disappear, we find that the residues must satisfy

$$\alpha\frac{\partial f_{n+1}}{\partial\alpha} + \bar{\beta}\alpha\frac{\partial f_n}{\partial\alpha} = 0.$$

Moreover, we can see that the beta function is written as $\bar{\beta} = \alpha\partial f_1/\partial\alpha$ using the simple pole residue.

As a point to note in later calculations, it is worth mentioning that since β has the $D-4$ dependence, its inverse $1/\beta$ has poles when expanded by the coupling constant (see (9-17) below).

Similarly, the beta function of the coupling constant a in the gravitational part (9-6) is defined as

$$\beta_a(\alpha, D) \equiv \mu\frac{da}{d\mu} = -(D-4)a + \bar{\beta}_a(\alpha). \quad (9\text{-}9)$$

Those for the coupling constants b and c are also defined in the same way. Since the bare constant a_0 does not depend on μ, we find that by solving a renormalization group equation $\mu da_0/d\mu = 0$, the residues satisfy

$$\frac{\partial}{\partial\alpha}(\alpha a_{n+1}) + \bar{\beta}\alpha\frac{\partial a_n}{\partial\alpha} = 0. \quad (9\text{-}10)$$

The beta function can be expressed as $\bar{\beta}_a = -\partial(\alpha a_1)/\partial\alpha$ using the simple pole residue. The same is true for the bare constants b_0 and c_0.

Normal Products

In this section, we introduce some normal products (see also the third section of Appendix D). They are renormalized composite fields that behave as finite operators in correlation functions of the fundamental fields.

Equation-of-motion fields As the simplest example, we first introduce what is called the equation-of-motion field. That for the gauge field is defined by

$$\begin{aligned} E_{0A} &= \frac{1}{\sqrt{g}} A_{0\mu} \frac{\delta S}{\delta A_{0\mu}} \\ &= A_{0\mu} \nabla_\nu F_0^{\mu\nu} - e_0 \bar{\psi}_0 \gamma^\mu A_{0\mu} \psi_0 - \frac{1}{\xi_0} A_{0\mu} \nabla^\mu \nabla^\nu A_{0\nu}. \end{aligned} \tag{9-11}$$

Considering correlation functions of the renormalized gauge field with N_A different points defined by

$$\left\langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \right\rangle = Z_3^{-\frac{N_A}{2}} \int dA_{0\mu} d\psi_0 d\bar{\psi}_0 \prod_{j=1}^{N_A} A_{0\mu_j}(x_j) e^{-S},$$

and inserting the equation-of-motion field into it and performing partial integration of the functional integral of $A_{0\mu}$, we obtain

$$\begin{aligned} &\left\langle E_{0A}(x) \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \right\rangle \\ &= -Z_3^{-\frac{N_A}{2}} \int dA_{0\mu} d\psi_0 d\bar{\psi}_0 \prod_{j=1}^{N_A} A_{0\mu_j}(x_j) \frac{1}{\sqrt{g}} A_{0\mu}(x) \frac{\delta}{\delta A_{0\mu}(x)} e^{-S} \\ &= \sum_{j=1}^{N_A} \frac{1}{\sqrt{g}} \delta^D(x - x_j) \left\langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \right\rangle. \end{aligned} \tag{9-12}$$

Here note that in dimensional regularization, functional derivatives at the same point vanish as $\delta A_{0\mu}(x)/\delta A_{0\nu}(x) = \delta_\mu{}^\nu \delta^D(0) = 0$.[5]

Similarly, the equation-of-motion field of the fermion field is defined by

$$E_{0\psi} = \frac{\delta S}{\delta \chi} \equiv \frac{1}{\sqrt{g}} \left(\bar{\psi}_0 \frac{\delta S}{\delta \bar{\psi}_0} + \psi_0 \frac{\delta S}{\delta \psi_0} \right) = i \bar{\psi}_0 \overset{\leftrightarrow}{\not{D}} \psi_0, \tag{9-13}$$

[5] As mentioned at the beginning of this chapter, this represents that the path integral does not depend on how to choose the measure. Therefore, we denote the measure by a concise expression without dependencies on the metric field as $dA_{0\mu} d\psi_0 d\bar{\psi}_0$.

where the Dirac operator with left and right arrows is

$$\bar{\psi}_0 \overleftrightarrow{\not{D}} \psi_0 = \bar{\psi}_0 \gamma^\mu D_\mu \psi_0 - D_\mu \bar{\psi}_0 \gamma^\mu \psi_0.$$

From this, we obtain

$$\left\langle E_{0\psi}(x) \prod_{j=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_j) \right\rangle = \sum_{j=1}^{N_\psi} \frac{1}{\sqrt{g}} \delta^D(x - x_j) \left\langle \prod_{j=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_j) \right\rangle. \tag{9-14}$$

Since each of the right-hand sides of the expressions (9-12) and (9-14) is a finite correlation function composed of the renormalized fundamental fields, it shows that the equation-of-motion field behaves as a finite operator in such a correlation function. It is nothing but a normal product. Writing it by the notation [ ], each equation-of-motion field can be expressed as

$$E_{0A} = [E_A], \qquad E_{0\psi} = [E_\psi]. \tag{9-15}$$

The volume integrals of these equation-of-motion fields can be written respectively as

$$\int d^D x \sqrt{g} E_{0A} = \int d^D x \sqrt{g} \left\{ \frac{1}{2} F_{0\mu\nu} F_0^{\mu\nu} - e_0 \bar{\psi}_0 \gamma^\mu A_{0\mu} \psi_0 + \frac{1}{\xi_0} \left(\nabla^\mu A_{0\mu}\right)^2 \right\},$$
$$\int d^D x \sqrt{g} E_{0\psi} = \int d^D x \sqrt{g}\, 2i \bar{\psi}_0 \gamma^\mu D_\mu \psi_0.$$

By performing the volume integration for (9-12) and (9-14), we find that these normal products become the number of each field, N_A and N_ψ, in the correlation function.

Normal product of gauge field squared A normal product of the gauge field squared generally has the following structure:

$$[F_{\mu\nu} F^{\mu\nu}] = \left(1 + \sum \text{poles}\right) F_{0\mu\nu} F_0^{\mu\nu} + \sum \text{poles} \times (\text{other fields}), \tag{9-16}$$

because it returns to the product of the bare field at the limit where interactions disappear. Here, we will determine the undecided part from considerations of finite quantities obtained by differentiating correlation functions of the renormalized fundamental fields with respect to renormalized variables.

Consider a finite quantity obtained by differentiating the correlation function by ξ, then find that the following holds:

$$\xi \frac{\partial}{\partial \xi} \left\langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_k) \right\rangle = \text{finite}$$

$$
\begin{aligned}
&= \left\langle \left\{ -\frac{N_\psi}{2}\xi\frac{\partial}{\partial\xi}\log Z_2 - \xi\frac{\partial S}{\partial\xi} \right\} \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} \left(\psi \text{ or } \bar\psi\right)(x_k) \right\rangle \\
&= \frac{1}{2}\left\langle \int d^D x\sqrt{g} \left\{ \frac{1}{\xi}\left(\nabla^\mu A_\mu\right)^2 - [E_\psi]\xi\frac{\partial}{\partial\xi}\log Z_2 \right\} \right. \\
&\qquad \left. \times \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} \left(\psi \text{ or } \bar\psi\right)(x_k) \right\rangle,
\end{aligned}
$$

where we use the facts that $\partial/\partial\xi$ passes through the bare fields $A_{0\mu}$, ψ_0, and $\bar\psi_0$ which are integration variables, N_ψ can be written in terms of the volume integral of $[E_\psi]$, and $\xi\partial S/\partial\xi = -(1/2\xi)\int d^D x\sqrt{g}(\nabla^\mu A_\mu)^2$. Thus, we can see that the volume integral in the braces on the right-hand side is a finite quantity. It is denoted as $\int d^D x\sqrt{g}[(\nabla^\mu A_\mu)^2]/\xi$ using the notation of normal products.

In the same way, we consider a finite quantity obtained by differentiating the correlation function with α. The α-dependence of the bare coupling constants is calculated as

$$
\begin{aligned}
&\alpha\frac{\partial e_0}{\partial\alpha} = \frac{D-4}{2\beta}e_0, \qquad \alpha\frac{\partial\xi_0}{\partial\alpha} = \frac{\bar\beta}{\beta}\xi_0, \\
&\alpha\frac{\partial a_0}{\partial\alpha} = -\frac{D-4}{\beta}\mu^{D-4}\left(L_a + \frac{\bar\beta_a}{D-4}\right).
\end{aligned}
$$

For b_0 and c_0, the same equation as a_0 holds. As for the renormalization factors, the following equations hold:

$$
\alpha\frac{\partial}{\partial\alpha}\log Z_3 = \frac{\bar\beta}{\beta}, \qquad \alpha\frac{\partial}{\partial\alpha}\log Z_2^{1/2} = \frac{1}{\beta}\left(\gamma_\psi + \bar\beta\xi\frac{\partial}{\partial\xi}\log Z_2^{1/2}\right),
$$

where $\gamma_\psi = \mu d(\log Z_2^{1/2})/d\mu$ is the anomalous dimension of the fermion. By differentiating the action S with α using these equations, we obtain

$$
\begin{aligned}
\alpha\frac{\partial S_{\mathrm{QED}}}{\partial\alpha} &= -\frac{D-4}{2\beta}\int d^D x\sqrt{g}e_0\bar\psi_0\gamma^\mu A_{0\mu}\psi_0, \\
\alpha\frac{\partial S_{\mathrm{g.f.}}}{\partial\alpha} &= -\frac{\bar\beta}{\beta}\int d^D x\sqrt{g}\,\frac{1}{2\xi_0}\left(\nabla^\mu A_{0\mu}\right)^2, \\
\alpha\frac{\partial S_g}{\partial\alpha} &= -\frac{D-4}{\beta}\mu^{D-4}\int d^D x\sqrt{g}\left[\left(L_a + \frac{\bar\beta_a}{D-4}\right)F_D\right. \\
&\qquad \left. + \left(L_b + \frac{\bar\beta_b}{D-4}\right)G_4 + \left(L_c + \frac{\bar\beta_c}{D-4}\right)H^2\right].
\end{aligned}
$$

Calculating the α-derivative of the correlation function using these formulas and disregarding obviously finite terms proportional to $[(\nabla^\mu A_\mu)^2]/\xi$ and $[E_A]$, we obtain

$$\begin{aligned}
&\left\langle \int d^D x \sqrt{g} \left\{ \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} - \frac{\bar{\gamma}_\psi}{\beta} [E_\psi] \right.\right. \\
&\quad + \frac{D-4}{\beta} \mu^{D-4} \left[\left(L_a + \frac{\bar{\beta}_a}{D-4} \right) F_D + \left(L_b + \frac{\bar{\beta}_b}{D-4} \right) G_4 \right. \\
&\quad \left.\left.\left. + \left(L_c + \frac{\bar{\beta}_c}{D-4} \right) H^2 \right] \right\} \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_k) \right\rangle \\
&= \text{finite},
\end{aligned}$$

where

$$\bar{\gamma}_\psi = \gamma_\psi - (D-4)\xi \frac{\partial}{\partial \xi} \log Z_2^{1/2}.$$

This shows that the inside the braces is expressed as a normal product except for derivative terms that vanish in the volume integral.

Noting that $(D-4)/\beta$ is expanded by the coupling constant as

$$\frac{D-4}{\beta} = \frac{1}{1 + \frac{\bar{\beta}}{D-4}} = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\beta})^n}{(D-4)^n}, \tag{9-17}$$

it can be seen that the inside the braces has the structure of the normal product (9-16). This means that it is equal to the normal product $[F_{\mu\nu}F^{\mu\nu}]/4$, except for total divergence terms. Since $\nabla^2 H$ is the only candidate for such a term from the symmetry of the theory, with this as an unknown term we obtain the following expression:

$$\begin{aligned}
\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] &= \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} - \frac{\bar{\gamma}_\psi}{\beta} [E_\psi] \\
&\quad + \frac{D-4}{\beta} \mu^{D-4} \left[\left(L_a + \frac{\bar{\beta}_a}{D-4} \right) F_D + \left(L_b + \frac{\bar{\beta}_b}{D-4} \right) G_4 \right. \\
&\quad \left. + \left(L_c + \frac{\bar{\beta}_c}{D-4} \right) H^2 - \frac{\sigma + L_\sigma}{D-4} \nabla^2 H \right],
\end{aligned} \tag{9-18}$$

where σ in the last term is a finite function of α, and L_σ is a pure pole term. In order to determine these unknown quantities, it is necessary to consider another finite condition. It will be discussed later.

Energy-momentum tensor The energy-momentum tensor is defined by a variation of the action S with respect to the metric field as

$$\Theta^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2}\frac{1}{\sqrt{g}}\left(e^{\mu}{}_{a}\frac{\delta S}{\delta e_{\nu a}} + e^{\nu}{}_{a}\frac{\delta S}{\delta e_{\mu a}} \right),$$

and its trace is expressed as

$$\Theta = \frac{\delta S}{\delta \Omega} = \frac{2}{\sqrt{g}} g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}.$$

Since the variation of a finite correlation function with respect to the metric field is finite, we obtain

$$\begin{aligned} \frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}}\left\langle \prod A_\mu \prod \psi \prod \bar{\psi} \right\rangle &= -\left\langle \Theta^{\mu\nu} \prod A_\mu \prod \psi \prod \bar{\psi} \right\rangle \\ &= \text{finite}. \end{aligned}$$

Therefore, the energy-momentum tensor $\Theta^{\mu\nu}$ is one of normal products. Like the equation-of-motion field, it is a finite operator defined as a bare quantity. Even after this, as for the energy-momentum tensor, it will be simply written as $\Theta^{\mu\nu}$ without using the suffix 0 indicating bare quantities nor the normal product symbol.

Decomposing the energy-momentum tensor as $\Theta^{\mu\nu} = \Theta^{\mu\nu}_{\mathrm{QED}} + \Theta^{\mu\nu}_{\mathrm{g.f.}} + \Theta^{\mu\nu}_{g}$, the QED part is given by

$$\begin{aligned} \Theta^{\mu\nu}_{\mathrm{QED}} &= -F_0^{\mu\lambda}F^{\nu}_{0\lambda} + \frac{1}{4}g^{\mu\nu}F_{0\lambda\sigma}F_0^{\lambda\sigma} \\ &\quad -\frac{i}{4}\left\{ \bar{\psi}_0\gamma^\mu D^\nu \psi_0 - D^\mu \bar{\psi}_0 \gamma^\nu \psi_0 + (\mu \leftrightarrow \nu) - 2g^{\mu\nu}\bar{\psi}_0 \overleftrightarrow{\not{D}}\, \psi_0 \right\}, \end{aligned}$$

and its trace is

$$\Theta_{\mathrm{QED}} = (D-4)\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}(D-1)i\bar{\psi}_0 \overleftrightarrow{\not{D}}\, \psi_0.$$

The part derived from the gauge-fixing term can be written in a BRST trivial form as

$$\begin{aligned} \Theta^{\mu\nu}_{\mathrm{g.f.}} &= \frac{1}{\xi_0}\Big[A_0^\mu \nabla^\nu \nabla^\lambda A_{0\lambda} + A_0^\nu \nabla^\mu \nabla^\lambda A_{0\lambda} - g^{\mu\nu} A_{0\lambda}\nabla^\lambda \nabla^\sigma A_{0\sigma} \\ &\qquad -\frac{1}{2}g^{\mu\nu}\left(\nabla^\lambda A_{0\lambda}\right)^2 \Big] \\ &\qquad + i\nabla^\mu \tilde{\eta}_0 \nabla^\nu \eta_0 + i\nabla^\nu \tilde{\eta}_0 \nabla^\mu \eta_0 - i g^{\mu\nu}\nabla^\lambda \tilde{\eta}_0 \nabla_\lambda \eta_0 \\ &= -i\,\delta_B \left[\nabla^\mu \tilde{\eta}_0 A_0^\nu + \nabla^\nu \tilde{\eta}_0 A_0^\mu - g^{\mu\nu}\nabla_\lambda \tilde{\eta}_0 A_0^\lambda - \frac{1}{2}g^{\mu\nu}\tilde{\eta}_0 \nabla_\lambda A_0^\lambda \right], \end{aligned}$$

where δ_B is the on-shell BRST transformation $\delta_B \tilde{\eta}_0 = i\nabla_\lambda A_0^\lambda/\xi_0$ obtained after solving the equation of motion of B_0, and then an equation of motion $\nabla^2 \eta_0 = 0$ is used to ensure nilpotency. This means that $\Theta^{\mu\nu}_{\text{g.f.}}$ disappears when inserting it into physical correlation functions which do not include the ghost fields. Hence, $\Theta^{\mu\nu}_{\text{g.f.}}$ will be ignored in the following discussion.

The gravitational part of the energy-momentum tensor is given by

$$\begin{aligned}
\Theta^{\mu\nu}_g = a_0 \Big\{ & -4R^{\mu\lambda\sigma\rho}R^\nu{}_{\lambda\sigma\rho} - \frac{8(D-4)}{D-2}R^{\mu\lambda\nu\sigma}R_{\lambda\sigma} + 8R^{\mu\lambda}R^\nu{}_\lambda \\
& - \frac{8}{(D-1)(D-2)}R^{\mu\nu}R - \frac{8(D-3)}{D-2}\nabla^2 R^{\mu\nu} + \frac{4(D-3)}{D-1}\nabla^\mu\nabla^\nu R \\
& + g^{\mu\nu}\Big[R^2_{\lambda\sigma\rho\kappa} - \frac{4}{D-2}R^2_{\lambda\sigma} + \frac{2}{(D-1)(D-2)}R^2 \\
& + \frac{4(D-3)}{(D-1)(D-2)}\nabla^2 R \Big] \Big\} + b_0 \Big\{ -4R^{\mu\lambda\sigma\rho}R^\nu{}_{\lambda\sigma\rho} + 8R^{\mu\lambda\nu\sigma}R_{\lambda\sigma} \\
& + 8R^{\mu\lambda}R^\nu{}_\lambda - 4R^{\mu\nu}R + g^{\mu\nu}G_4 \Big\} + \frac{c_0}{(D-1)^2}\Big\{ -4R^{\mu\nu}R \\
& + 4\nabla^\mu\nabla^\nu R + g^{\mu\nu}\left[R^2 - 4\nabla^2 R \right] \Big\},
\end{aligned} \tag{9-19}$$

and its trace is

$$\Theta_g = (D-4)\left[a_0 F_D + b_0 G_4 + c_0 H^2 \right] - 4c_0 \nabla^2 H.$$

Let us see that the trace of the energy-momentum tensor can actually be described in terms of the normal products given before. If the metric field is taken to be the flat one, for simplicity, and the BRST trivial term $\Theta_{\text{g.f.}}$ is ignored, we get

$$\begin{aligned}
\Theta &= \frac{D-4}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}(D-1)E_{0\psi} \\
&= \frac{\beta}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}\left(D-1+2\bar{\gamma}_\psi\right)[E_\psi].
\end{aligned}$$

The first equality is from the definition and the second equality is from the expressions of the normal products (9-18) and (9-15). The right-hand side is the so-called conformal anomaly, and this expression shows that it is proportional to the beta function, apart from the equation-of-motion field. The expression of the conformal anomaly in curved spacetime will be derived in the later section.

Restrictions from Correlation Functions

In the following sections, we will show that the form of the gravitational counterterms is restricted through new renormalization group equations by Hathrell, which will be derived from considerations of correlation functions between the normal products obtained in the previous section.

Two-point correlation functions Since a quantity obtained by performing the variation of the partition function twice with respect to the metric field is also finite, we obtain the following finiteness condition concerning the energy-momentum tensor:

$$\langle \Theta^{\mu\nu}(x)\Theta^{\lambda\sigma}(y)\rangle - \frac{2}{\sqrt{g(y)}}\left\langle \frac{\delta\Theta^{\mu\nu}(x)}{\delta g_{\lambda\sigma}(y)}\right\rangle = \text{finite}.$$

Taking the flat spacetime and Fourier transforming this expression, we obtain

$$\langle \Theta^{\mu\nu}(p)\Theta^{\lambda\sigma}(-p)\rangle_{\text{flat}} - a_0 A^{\mu\nu,\lambda\sigma}(p) - c_0 C^{\mu\nu,\lambda\sigma}(p) = \text{finite},$$

where $A^{\mu\nu,\lambda\sigma}$ and $C^{\mu\nu,\lambda\sigma}$ are derived from the Weyl term F_D and the H^2 term in the gravitational counterterm (9-5), respectively, which are given by

$$\begin{aligned} A^{\mu\nu,\lambda\sigma}(p) &= \frac{4(D-3)}{D-2}\Big[p^4\big(\delta^{\mu\lambda}\delta^{\nu\sigma}+\delta^{\mu\sigma}\delta^{\nu\lambda}\big) - p^2\big(\delta^{\mu\lambda}p^\nu p^\sigma \\ &\quad +\delta^{\mu\sigma}p^\nu p^\lambda + \delta^{\nu\lambda}p^\mu p^\sigma + \delta^{\nu\sigma}p^\mu p^\lambda\big) + 2p^\mu p^\nu p^\lambda p^\sigma\Big] \\ &\quad -\frac{8(D-3)}{(D-1)(D-2)}\Big[p^4\delta^{\mu\nu}\delta^{\lambda\sigma} - p^2\big(\delta^{\mu\nu}p^\lambda p^\sigma + \delta^{\lambda\sigma}p^\mu p^\nu\big) \\ &\quad +p^\mu p^\nu p^\lambda p^\sigma\Big], \\ C^{\mu\nu,\lambda\sigma}(p) &= \frac{8}{(D-1)^2}\Big[p^4\delta^{\mu\nu}\delta^{\lambda\sigma} - p^2\big(\delta^{\mu\nu}p^\lambda p^\sigma + \delta^{\lambda\sigma}p^\mu p^\nu\big) \\ &\quad +p^\mu p^\nu p^\lambda p^\sigma\Big]. \end{aligned} \tag{9-20}$$

Contracting the spacetime indices of the energy-momentum tensor yields

$$\langle \Theta^{\mu\nu}(p)\Theta_{\mu\nu}(-p)\rangle_{\text{flat}} - 4(D-3)(D+1)a_0 p^4 - \frac{8}{D-1}c_0 p^4 = \text{finite} \tag{9-21}$$

and

$$\langle \Theta(p)\Theta(-p)\rangle_{\text{flat}} - 8c_0 p^4 = \text{finite}. \tag{9-22}$$

Therefore, c_0 can be determined from (9-22), and then a_0 from (9-21). On the other hand, to determine b_0, we need to consider three-point functions, which will be discussed later. In the following, we will examine the condition (9-22) only.

Let us introduce a quantity in which the trace of the energy-momentum tensor is slightly modified as

$$\bar{\Theta} = \Theta - \frac{1}{2}(D-1)[E_\psi]. \tag{9-23}$$

As an important property of the equation-of-motion field (9-13), note that two-point functions involving it disappear as

$$\langle [E_\psi(x)] P(y) \rangle_{\text{flat}} = -\int P(y) \frac{\delta}{\delta\chi(x)} e^{-S} \bigg|_{\text{flat}} = \left\langle \frac{\delta P(y)}{\delta\chi(x)} \right\rangle_{\text{flat}} = 0, \tag{9-24}$$

where P denotes any fermion composite field, and in the last equality we use the fact that one-point functions disappear in the flat spacetime, which is because tadpole diagrams vanish when dimensional regularization is applied for ordinary massless fields with actions equal to or less than second derivative. By rewriting (9-22) using this property, we obtain

$$\langle \bar{\Theta}(p) \bar{\Theta}(-p) \rangle_{\text{flat}} - 8p^4 \mu^{D-4} L_c = \text{finite}.$$

Furthermore, we introduce the following composite field defined on the flat spacetime:

$$\begin{aligned} \{A^2\} &= \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} \\ &= \frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{\bar{\gamma}_\psi}{\beta}[E_\psi]. \end{aligned} \tag{9-25}$$

Ignoring the part originating from the gauge-fixing term as not contributing, this field has a relation with $\bar{\Theta}$ (9-23) as

$$\bar{\Theta}|_{\text{flat}} = \beta \{A^2\}. \tag{9-26}$$

Note that $\bar{\Theta}$ is a finite operator, but as can be seen from the fact that $1/\beta$ has poles, $\{A^2\}$ is not a finite quantity.

Consider a two-point correlation function of the composite field $\{A^2\}$ in the flat spacetime. Its Fourier transform is written as

$$\Gamma_{AA}(p^2) = \left\langle \{A^2\}(p) \{A^2\}(-p) \right\rangle_{\text{flat}}.$$

In this case, although $\{A^2\}$ itself is not a finite quantity, the term containing $1/\beta$ which breaks the finiteness is eliminated from the property of the equation-of-motion field (9-24). Thus, Γ_{AA} is expressed by a two-point function of the normal product $[F_{\mu\nu}F^{\mu\nu}]$. In general, in correlation functions between normal products, ultraviolet divergences may occur, but are local, that is, nonlocal poles such as $p^4 \log^m(p^2/\mu^2)/(D-4)^n$ do not appear. Therefore, it can be expressed in the form

$$\Gamma_{AA}(p^2) - p^4\mu^{D-4}\left(\frac{D-4}{\beta}\right)^2 L_x = \text{finite}, \tag{9-27}$$

where

$$L_x = \sum_{n=1}^{\infty} \frac{x_n(\alpha)}{(D-4)^n}.$$

The expression (9-27) is a definition of the pole term L_x, and the factor $(D-4)^2/\beta^2$ in front of L_x is introduced for the following convenience.

Since $\beta^2\Gamma_{AA} = \langle\bar{\Theta}\bar{\Theta}\rangle_{\text{flat}}$ is established from (9-26), it is found that the pole terms satisfy the following relation:

$$(D-4)^2 L_x - 8L_c = \text{finite}. \tag{9-28}$$

From this, a relation between the residues is derived as

$$c_n = \frac{1}{8}x_{n+2}. \tag{9-29}$$

That is, if x_3 is obtained, c_1 can be determined. Since the renormalization group equation for c_n has already been given as in (9-10), this means that if x_3 is obtained, all c_n will be determined. At the same time, x_n $(n \geq 4)$ are also determined.

Next, we examine a relation that holds between the residues x_n. For this purpose, we use the fact that when F is a finite quantity,

$$\frac{1}{\beta^n}\mu\frac{d}{d\mu}(\beta^n F) = \mu\frac{dF}{d\mu} + n\alpha\frac{\partial\bar{\beta}}{\partial\alpha}F = \text{finite} \tag{9-30}$$

holds for a positive integer n even if $1/\beta$ is a divergent quantity. Assigning (9-27) as a finite quantity F and letting $n=2$, we obtain

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\beta^2\Gamma_{AA}(p^2) - p^4\mu^{D-4}(D-4)^2 L_x\right\} = \text{finite}.$$

Since $\beta\{A^2\}$ is a bare quantity from the definition (9-25), $\mu d(\beta\{A^2\})/d\mu = 0$ holds. Thus, we can see that the first term on the left-hand side vanishes. Therefore, the following holds:

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\mu^{D-4}(D-4)^2 L_x\right\} = \text{finite}.$$

By Laurent expanding this expression, we derive conditions that poles disappear and the left-hand side becomes finite. For the $n(\geq 1)$-th pole, we obtain the following renormalization group equation:

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}(\alpha x_{n+1}) + \bar{\beta}\alpha\frac{\partial x_n}{\partial\alpha} \\
&+\sum_{m=1}^{n-1}(-1)^{n-m}(n-m+1)\bar{\beta}^{n-m}\left[\frac{\partial}{\partial\alpha}(\alpha x_{m+1}) + \bar{\beta}\alpha\frac{\partial x_m}{\partial\alpha}\right] \\
&+(-1)^n(n+1)\bar{\beta}^n\frac{\partial}{\partial\alpha}(\alpha x_1) = 0.
\end{aligned}$$

In particular, the equations for $n = 1, 2$ are given by

$$\frac{\partial}{\partial\alpha}(\alpha x_2) - \frac{\bar{\beta}}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha^2 x_1\right) = 0,$$

$$\frac{\partial}{\partial\alpha}(\alpha x_3) - \frac{\bar{\beta}}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha^2 x_2\right) + \frac{\bar{\beta}^2}{\alpha^2}\frac{\partial}{\partial\alpha}\left(\alpha^3 x_1\right) = 0.$$

Solving these equations, the residues x_2 and x_3 can be determined using x_1 as

$$\begin{aligned}
x_2 &= \frac{1}{\alpha}\int_0^\alpha d\alpha'\frac{\bar{\beta}(\alpha')}{\alpha'}\frac{\partial}{\partial\alpha'}\left(\alpha'^2 x_1(\alpha')\right), \\
x_3 &= -\frac{\bar{\beta}(\alpha)}{\alpha}\int_0^\alpha d\alpha'\left\{\alpha'^2 x_1(\alpha')\frac{\partial}{\partial\alpha'}\left(\frac{\bar{\beta}(\alpha')}{\alpha'}\right)\right\}.
\end{aligned} \tag{9-31}$$

On the other hand, as can be seen from the relation (9-29), the renormalization group equation of x_n above is simplified for $n \geq 3$ and reduces to the same equation as (9-10) that c_n satisfies.

Three-point correlation functions As before, performing the variations of the partition function three times with Ω, we obtain the following finiteness condition:

$$\begin{aligned}
&\langle\Theta(x)\Theta(y)\Theta(z)\rangle - \left\langle\frac{\delta\Theta(x)}{\delta\Omega(y)}\Theta(z)\right\rangle - \left\langle\frac{\delta\Theta(y)}{\delta\Omega(z)}\Theta(x)\right\rangle \\
&-\left\langle\frac{\delta\Theta(z)}{\delta\Omega(x)}\Theta(y)\right\rangle + \left\langle\frac{\delta\Theta(x)}{\delta\Omega(y)\delta\Omega(z)}\right\rangle = \text{finite}.
\end{aligned}$$

The flat spacetime limit is then taken.

Although two-point functions including $[E_\psi]$ disappear in the flat spacetime as (9-24), three-point functions including it have values and are expressed such as

$$\langle [E_\psi(x)] P(y) Q(z) \rangle_{\rm flat} = \left\langle \frac{\delta P(y)}{\delta \chi(x)} Q(z) \right\rangle_{\rm flat} + \left\langle P(y) \frac{\delta Q(z)}{\delta \chi(x)} \right\rangle_{\rm flat}.$$

Moreover, using the fact that a variantion of $[E_\psi]$ is given as

$$\frac{\delta [E_\psi(x)]}{\delta \Omega(y)} = \frac{\delta \Theta(y)}{\delta \chi(x)} - D \frac{1}{\sqrt{g}} \delta^D(x-y) [E_\psi],$$

the finiteness condition of the three-point function can be expressed in terms of $\bar{\Theta}$ (9-23) as

$$\begin{aligned} &\langle \bar{\Theta}(x) \bar{\Theta}(y) \bar{\Theta}(z) \rangle_{\rm flat} - \langle \bar{\Theta}(x) \bar{\Theta}_2(y,z) \rangle_{\rm flat} - \langle \bar{\Theta}(y) \bar{\Theta}_2(z,x) \rangle_{\rm flat} \\ &-\langle \bar{\Theta}(z) \bar{\Theta}_2(x,y) \rangle_{\rm flat} + \left\langle \frac{\delta^3 S}{\delta \Omega(x) \delta \Omega(y) \delta \Omega(z)} \right\rangle_{\rm flat} = \text{finite}, \end{aligned}$$

where

$$\bar{\Theta}_2(x,y) = \frac{\delta \bar{\Theta}(x)}{\delta \Omega(y)} - \frac{1}{2}(D-1) \frac{\delta \bar{\Theta}(x)}{\delta \chi(y)}$$

and this function satisfies $\bar{\Theta}_2(x,y) = \bar{\Theta}_2(y,x)$.

Introduce a three-point function Γ_{AAA} of the composite field $\{A^2\}$. Its Fourier transform is denoted as

$$\Gamma_{AAA}(p_x^2, p_y^2, p_z^2) = \left\langle \{A^2\}(p_x) \{A^2\}(p_y) \{A^2\}(p_z) \right\rangle_{\rm flat}.$$

From the relation (9-26) and

$$\bar{\Theta}_2(x,y)|_{\rm flat} = -4\beta \{A^2\}(x) \delta^D(x-y) + 8c_0 \partial^4 \delta^D(x-y),$$

we find that the following holds:

$$\begin{aligned} &\beta^3 \Gamma_{AAA}(p_x^2, p_y^2, p_z^2) + 4\beta^2 \left\{ \Gamma_{AA}(p_x^2) + \Gamma_{AA}(p_y^2) + \Gamma_{AA}(p_z^2) \right\} \\ &+ b_0 B(p_x^2, p_y^2, p_z^2) + c_0 C(p_x^2, p_y^2, p_z^2) = \text{finite}, \end{aligned} \qquad (9\text{-}32)$$

where the second term in $\bar{\Theta}_2|_{\rm flat}$ does not contribute here. The functions B and C are parts derived from the variations of $b_0 G_4$ and $c_0 H^2$, respectively, which are given by

$$\begin{aligned} B(p_x^2, p_y^2, p_z^2) &= -2(D-2)(D-3)(D-4) \\ &\qquad \times \Big[p_x^4 + p_y^4 + p_z^4 - 2\left(p_x^2 p_y^2 + p_y^2 p_z^2 + p_z^2 p_x^2\right)\Big], \\ C(p_x^2, p_y^2, p_z^2) &= -4\Big[(D+2)\left(p_x^4 + p_y^4 + p_z^4\right) \\ &\qquad + 4\left(p_x^2 p_y^2 + p_y^2 p_z^2 + p_z^2 p_x^2\right)\Big]. \end{aligned}$$

In the following, in order to simplify the calculation, we consider two cases in which some momenta satisfy on-shell conditions such as $p_z^2 = 0$ and $p_x^2 = p_y^2$ or $p_y^2 = p_z^2 = 0$.[6] The functions B and C are then given by

$$\begin{aligned} B(p^2, p^2, 0) &= 0, \\ C(p^2, p^2, 0) &= -8(D+4)p^4, \\ B(p^2, 0, 0) &= -2(D-2)(D-3)(D-4)p^4, \\ C(p^2, 0, 0) &= -4(D+2)p^4. \end{aligned}$$

Furthermore, since $\beta^2 \Gamma_{AA}(p^2) - 8p^4 \mu^{D-4} L_c = \mathrm{finite}$ is established from (9-27) and (9-28), if rewriting (9-32) using these, we obtain $\beta^3 \Gamma_{AAA}(p^2, p^2, 0) - 8(D-4)p^4 \mu^{D-4} L_c = \mathrm{finite}$ and

$$\begin{aligned} &\beta^3 \Gamma_{AAA}(p^2, 0, 0) - p^4 \mu^{D-4}\big[2(D-2)(D-3)(D-4)L_b \\ &+4(D-6)L_c\big] = \mathrm{finite}. \end{aligned} \tag{9-33}$$

Next, we extract information of Γ_{AAA} in a slightly different way. Consider a finite quantity obtained by applying a differentiation $\alpha\partial/\partial\alpha$ to (9-27) and rewrite it using

$$\begin{aligned} \alpha \frac{\partial S}{\partial \alpha}\bigg|_{\mathrm{flat}} &= \int d^D x \left\{ -\{A^2\} + \frac{D-4}{2\beta}[E_A] - \frac{1}{2\xi_0}\left(\partial^\mu A_{0\mu}\right)^2 \right\}, \\ \alpha \frac{\partial}{\partial \alpha}\{A^2\} &= -\frac{\alpha}{\beta}\frac{\partial \bar{\beta}}{\partial \alpha}\{A^2\}, \end{aligned}$$

so that we obtain

$$\left\langle \{A^2\}\{A^2\} \int d^D x \{A^2\} \right\rangle_{\mathrm{flat}} - \frac{D-4}{2\beta}\left\langle \{A^2\}\{A^2\} \int d^D x\, [E_A] \right\rangle_{\mathrm{flat}}$$

[6] Although we are considering in Euclidean spacetime here, we assume that considering analytic continuation the momentum itself does not vanish even if it is on-shell and the conservation law $p_x^\mu + p_y^\mu + p_z^\mu = 0$ holds. See the original paper by Hathrell for discussion in Minkowski spacetime as it is.

$$-2\frac{\alpha}{\beta}\frac{\partial\bar{\beta}}{\partial\alpha}\left\langle\{A^2\}\,\{A^2\}\right\rangle_{\rm flat}-p^4\mu^{D-4}\alpha\frac{\partial}{\partial\alpha}\left[\left(\frac{D-4}{\beta}\right)^2L_x\right]=\text{finite},$$

in which the gauge-fixing term disappears from gauge invariance. Note that with attention to $\int d^Dx\{A^2\}(x)=\{A^2\}(p=0)$, the first term on the left-hand side gives $\Gamma_{AAA}(p^2,p^2,0)$. In the second term, performing partial integration of the gauge field according to the definition (9-11) of the equation-of-motion field $[E_A]=E_{0A}$ and using

$$\int d^Dx\,A_{0\mu}(x)\frac{\delta}{\delta A_{0\mu}(x)}\{A^2\}(y)=2\{A^2\}(y),$$

we obtain the relation

$$\left\langle\{A^2\}(y)\,\{A^2\}(z)\int d^Dx[E_A(x)]\right\rangle_{\rm flat}=4\left\langle\{A^2\}(y)\,\{A^2\}(z)\right\rangle_{\rm flat}.$$

By using Fourier transform of this expression, we finally obtain the following finiteness condition:

$$\Gamma_{AAA}(p^2,p^2,0)-2\frac{\alpha^2}{\beta}\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\Gamma_{AA}(p^2)$$
$$-p^4\mu^{D-4}\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\left[\alpha^2\left(\frac{D-4}{\beta}\right)^2L_x\right]=\text{finite}.$$

Since β^3 is not multiplied to the whole, this expression gives a stronger condition than that found before.

In general, Γ_{AAA} has the following structure:

$$\Gamma_{AAA}(p_x^2,p_y^2,p_z^2)-\sum\text{poles}\times\left\{\Gamma_{AA}(p_x^2)+\Gamma_{AA}(p_y^2)\right.$$
$$\left.+\Gamma_{AA}(p_z^2)\right\}-\mu^{D-4}\sum\text{poles}\times\left\{\text{terms in }p_i^2p_j^2\right\}=\text{finite}.$$

Unlike Γ_{AA}, Γ_{AAA} has nonlocal poles because the three-point function including $[E_\psi]$ does not disappear and thus the contribution from the term $[E_\psi]/\beta$ that breaks finiteness of $\{A^2\}$ (9-25) remains. The second term involving Γ_{AA} in the above equation plays a role of canceling out such nonlocal poles.

From this consideration, it can be easily inferred that a finiteness condition for $\Gamma_{AAA}(p^2,0,0)$ is given by

$$\Gamma_{AAA}(p^2,0,0)-\frac{\alpha^2}{\beta}\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\Gamma_{AA}(p^2)$$
$$-p^4\mu^{D-4}\left(\frac{D-4}{\beta}\right)^3L_y=\text{finite}. \tag{9-34}$$

Here, it is important that the coefficient of the factor including poles in front of Γ_{AA} is half of the case of $\Gamma_{AAA}(p^2, p^2, 0)$. On the other hand, the last term is a new pole term defined through this expression and is expanded as

$$L_y = \sum_{n=1}^{\infty} \frac{y_n(\alpha)}{(D-4)^n},$$

while the factor $(D-4)^3/\beta^3$ in front of L_y is introduced for convenience.

By multiplying (9-34) by β^3 and rewriting the part that becomes $\beta^2\Gamma_{AA}$ using (9-27) and (9-28), we get

$$\beta^3\Gamma_{AAA}(p^2,0,0) - p^4\mu^{D-4}\left[8\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)L_c + (D-4)^3 L_y\right] = \text{finite}.$$

Eliminating Γ_{AAA} by using (9-33), we obtain the following relation between pole terms:

$$\begin{aligned} &2(D-2)(D-3)(D-4)L_b + 4\left[D-6-2\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\right]L_c \\ &-(D-4)^3 L_y = \text{finite}. \end{aligned} \tag{9-35}$$

Lastly, we consider conditions that the pole term L_y satisfies. As in the case of L_x, substituting (9-34) into (9-30) as a finite function F and letting $n = 3$, we obtain

$$-\alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\Gamma_{AA}(p^2) - p^4\mu^{D-4}\left(\frac{D-4}{\beta}\right)^3\left[(D-4)L_y + \beta\alpha\frac{\partial}{\partial\alpha}L_y\right] = \text{finite},$$

where we use

$$\mu\frac{d}{d\mu}\left[\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\right] = \bar{\beta}\alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}$$

and the fact that $\mu d(\beta^2\Gamma_{AA})/d\mu = \mu d(\beta^3\Gamma_{AAA})/d\mu = 0$ holds because $\beta\{A^2\}$ is a bare quantity. Furthermore, using (9-27), we get a renormalization group equation which relates L_x and L_y as

$$\left(\frac{D-4}{\beta}\right)^3\left[(D-4)L_y + \beta\alpha\frac{\partial}{\partial\alpha}L_y\right] + \alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\left(\frac{D-4}{\beta}\right)^2 L_x = \text{finite}. \tag{9-36}$$

From conditions that the n-th poles disappear when Laurent expanding this expression, we find the following relation between the residues:

$$\frac{\partial}{\partial\alpha}(\alpha y_{n+1}) + \frac{1}{2}\sum_{m=1}^{n}(-1)^m(m+1)\bar{\beta}^m\left[(m+2)\frac{\partial}{\partial\alpha}(\alpha y_{n-m+1})\right.$$

$$-m\alpha\frac{\partial}{\partial\alpha}y_{n-m+1}\Bigg] - \alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\sum_{m=1}^{n}(-1)^m m\bar{\beta}^{m-1}x_{n-m+1} = 0. \quad (9\text{-}37)$$

Since we already give the equation to calculate general x_n from x_1, this equation shows that general y_n is also determined if y_1 is given.

Residues of the pole terms Since the pole term L_y is related to L_x through the renormalization group equation (9-36) and L_x is related to L_c through (9-28), we find that the renormalization group equation (9-35) gives a relation between L_b and L_c. We here solve the renormalization group equations and derive the residues of L_x and L_y specifically. The data necessary for that is the QED beta function and the simple pole residues x_1 and y_1, which are expanded as

$$\begin{aligned}\bar{\beta} &= \beta_1\alpha + \beta_2\alpha^2 + \beta_3\alpha^3 + o(\alpha^4),\\ x_1 &= X_1 + X_2\alpha + X_3\alpha^2 + o(\alpha^3),\\ y_1 &= Y_1 + Y_2\alpha + Y_3\alpha^2 + o(\alpha^3).\end{aligned}$$

Specific values of these coefficients will be given later.

We first calculate the residues x_n. The formulas for calculating x_2 and x_3 from x_1 are already shown in the integral representations (9-31). As is seen from the relation (9-29), $x_n\,(n \geq 3)$ satisfies the same equation as (9-10) that c_n satisfies. From these equations, we calculate x_n up to $o(\alpha^{n+1})$ using the above expressions of $\bar{\beta}$ and x_1. Writing down the case of $n = 2, 3, 4$ specifically, each is given by

$$\begin{aligned}x_2 &= \beta_1 X_1\alpha + \left(\frac{2\beta_2 X_1}{3} + \beta_1 X_2\right)\alpha^2 + \left(\frac{\beta_3 X_1}{2} + \frac{3\beta_2 X_2}{4} + \beta_1 X_3\right)\alpha^3,\\ x_3 &= -\frac{\beta_1\beta_2 X_1}{12}\alpha^3 + \left(-\frac{\beta_2^2 X_1}{15} - \frac{\beta_1\beta_3 X_1}{10} - \frac{\beta_1\beta_2 X_2}{20}\right)\alpha^4,\\ x_4 &= \frac{\beta_1^2\beta_2 X_1}{20}\alpha^4 + \left(\frac{31\beta_1\beta_2^2 X_1}{360} + \frac{\beta_1^2\beta_2 X_2}{30} + \frac{\beta_1^2\beta_3 X_1}{15}\right)\alpha^5. \qquad (9\text{-}38)\end{aligned}$$

Note that the lowest order of each x_n is given by $o(\alpha^{n-1})$ for $n \leq 2$, whereas for $n \geq 3$ it becomes $o(\alpha^n)$. Moreover, there is a three-loop contribution X_3 in the $o(\alpha^3)$ term of x_2, whereas X_3 does not appear in the $o(\alpha^{n+1})$ term of x_n for $n \geq 3$.

Since c_n is given by $x_{n+2}/8$ from the relation (9-29), the lowest order of c_n becomes $o(\alpha^{n+2})$ from the expression of x_n $(n \geq 3)$.

Next, substituting the value of x_n into the renormalization group equation (9-37) and solving it, we can derive the residue y_n. Calculating up to

$o(\alpha^{n+1})$ for each y_n, we obtain

$$
\begin{aligned}
y_2 &= \frac{3\beta_1 Y_1}{2}\alpha + \left(-\frac{2\beta_2 X_1}{3} + \beta_2 Y_1 + \frac{5\beta_1 Y_2}{3}\right)\alpha^2 \\
&\quad + \left(-\frac{3\beta_3 X_1}{2} - \frac{\beta_2 X_2}{2} + \frac{3\beta_3 Y_1}{4} + \frac{5\beta_2 Y_2}{4} + \frac{7\beta_1 Y_3}{4}\right)\alpha^3, \\
y_3 &= \frac{\beta_1^2 Y_1}{2}\alpha^2 + \left(-\frac{2\beta_1\beta_2 X_1}{3} + \frac{5\beta_1\beta_2 Y_1}{8} + \frac{2\beta_1^2 Y_2}{3}\right)\alpha^3 \\
&\quad + \left(-\frac{3\beta_1\beta_3 X_1}{2} - \frac{\beta_1\beta_2 X_2}{2} - \frac{2\beta_2^2 X_1}{5} + \frac{3\beta_1^2 Y_3}{4} + \frac{59\beta_1\beta_2 Y_2}{60}\right. \\
&\qquad \left. + \frac{\beta_2^2 Y_1}{5} + \frac{9\beta_1\beta_3 Y_1}{20}\right)\alpha^4, \\
y_4 &= \frac{\beta_1^2\beta_2 Y_1}{40}\alpha^4 + \left(\frac{\beta_1^2\beta_3 Y_1}{30} + \frac{13\beta_1\beta_2^2 Y_1}{240} + \frac{\beta_1^2\beta_2 Y_2}{90} + \frac{13\beta_1\beta_2^2 X_1}{180}\right)\alpha^5, \\
y_5 &= -\frac{\beta_1^3\beta_2 Y_1}{60}\alpha^5 \\
&\quad + \left(-\frac{53\beta_1^2\beta_2^2 X_1}{1260} - \frac{\beta_1^3\beta_3 Y_1}{42} - \frac{89\beta_1^2\beta_2^2 Y_1}{1680} - \frac{\beta_1^3\beta_2 Y_2}{126}\right)\alpha^6. \qquad (9\text{-}39)
\end{aligned}
$$

Note again that the lowest order of each y_n is given by $o(\alpha^{n-1})$ for $n \leq 3$, whereas for $n \geq 4$ it starts with $o(\alpha^n)$. Also, the $o(\alpha^{n+1})$ term of y_n includes a three-loop value Y_3 for $n \leq 3$, but it does not appear for $n \geq 4$. This structure is related to the fact that the renormalization group equation (9-37) reduces to the simple form

$$
\frac{\partial}{\partial\alpha}(\alpha y_{n+1}) + \bar{\beta}\alpha\frac{\partial y_n}{\partial\alpha} = -\alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\left(x_n + \bar{\beta}x_{n-1}\right)
$$

for $n \geq 4$, as is similar to the fact that the renormalization group equation of x_n reduces to the simple form for $n \geq 3$.

The respective residues can be calculated by substituting concrete values. Each coefficient of the beta function is given by[7]

$$
\beta_1 = \frac{8}{3}\frac{1}{4\pi}, \qquad \beta_2 = 8\frac{1}{(4\pi)^2}, \qquad \beta_3 = -\frac{124}{9}\frac{1}{(4\pi)^3}. \qquad (9\text{-}40)
$$

The values of $X_{1,2}$ and $Y_{1,2}$ are yielded by calculating Γ_{AA} and Γ_{AAA} up to $o(\alpha)$, respectively.

[7] See S. Gorishny, A. Kataev, S. Larin and L. Surguladze, *The Analytical Four Loop Corrections to The QED β-Function in The MS Scheme and to The QED ψ-Function: Total Reevaluation*, Phys. Lett. **B256** (1991) 81.

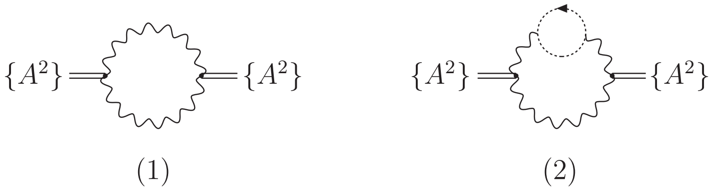


Figure 9-1: Quantum corrections to Γ_{AA}.

The contributions up to $o(\alpha)$ are given at the two-loop level. Feynman diagrams for Γ_{AA} are depicted in Fig. 9-1. Substituting Fourier transform of the composite field $\{A^2\}$, Γ_{AA} can be written as

$$\begin{aligned}&\Gamma_{AA}(p^2)(2\pi)^D\delta^D(p+q)\\&=\left(\frac{D-4}{\beta}Z_3\right)^2\frac{1}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\,K^{\mu\nu}(k,k-p)K^{\lambda\sigma}(l,l-q)\\&\quad\times\langle A_\mu(k)A_\nu(p-k)A_\lambda(l)A_\sigma(q-l)\rangle,\end{aligned}$$

where

$$K^{\mu\nu}(k,k-p)=k\cdot(k-p)\delta^{\mu\nu}-(k-p)^\mu k^\nu$$

and the Z_3 factor appears when $A_{0\mu}$ is replaced with $Z_3^{1/2}A_\mu$ in the definition of $\{A^2\}$. Calculating the four-point function of the gauge field A_μ and performing the momentum integration, we find that ultraviolet divergences are given by

$$\Gamma_{AA}(p^2)=\frac{\mu^{D-4}}{(4\pi)^2}\,p^4\left\{-\frac{1}{2}\frac{1}{D-4}+\frac{\alpha}{4\pi}\left[\frac{4}{3}\frac{1}{(D-4)^2}+\frac{5}{3}\frac{1}{D-4}\right]\right\}.$$

From this, the coefficients $X_{1,2}$ of the residue x_1 are determined to be

$$X_1=-\frac{1}{2}\frac{1}{(4\pi)^2},\qquad X_2=\frac{5}{3}\frac{1}{(4\pi)^3}. \tag{9-41}$$

Also, the lowest order term of the residue x_2 can be directly calculated as $-4\alpha/3(4\pi)^3$ from (9-27) with taking into account the $(D-4)^2/\beta^2$ factor. This value is consistent with the result (9-38) of the renormalization group equation.

Similarly, Γ_{AAA} can be calculated from the diagrams in Fig. 9-2 and we obtain

$$\Gamma_{AAA}(p^2,0,0)=\frac{\mu^{D-4}}{(4\pi)^2}p^4\left\{-\frac{1}{2}\frac{1}{D-4}+\frac{\alpha}{4\pi}\left[2\frac{1}{(D-4)^2}+\frac{11}{6}\frac{1}{D-4}\right]\right\}.$$

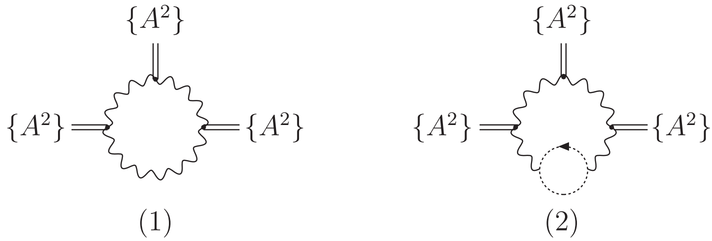


Figure 9-2: Quantum corrections to Γ_{AAA}.

From this, the coefficients $Y_{1,2}$ of the residue y_1 are determined to be

$$Y_1 = -\frac{1}{2}\frac{1}{(4\pi)^2}, \qquad Y_2 = \frac{11}{6}\frac{1}{(4\pi)^3}. \tag{9-42}$$

In addition, the lowest order term of the residue y_2 is determined to be $-2\alpha/(4\pi)^3$ from (9-34). The value is consistent with the result (9-39).

Determination of Gravitational Counterterms

As L_y has already been given in the previous section, we find that the renormalization group equation (9-35) gives a relation between L_b and L_c. It indicates that the independent gravitational counterterms are given by two of the D-dimensional Weyl term F_D (9-1) and a certain combination of G_4 and H^2. Therefore, we here introduce a new function, which is just (9-2) presented at the beginning of this chapter, as

$$G_D = G_4 + (D-4)\chi(D)H^2,$$

and define the gravitational counterterm in the following form:

$$S_g = \int d^D x\sqrt{g}\,\{a_0 F_D + b_0 G_D\},$$

where $\chi(D)$ is a finite function of only D. The modified Euler density G_D reduces to the ordinary Euler density G_4 in four dimensions. This gravitational counterterm indicates that there is a relation between L_b and L_c as follows:

$$L_c - (D-4)\chi(D)L_b = \text{finite}. \tag{9-43}$$

And also, the coupling constant c in c_0 (9-6) is removed and only b in b_0 is considered. In the following, we will see that the function $\chi(D)$ can be determined by solving the relation (9-43) and the renormalization group equation (9-35) simultaneously.

From the renormalization group equation (9-35), it turns out that the following relation holds between the residues:

$$\begin{aligned} &4b_{n+1} + 6b_{n+2} + 2b_{n+3} - 8\left[1 + \alpha^2 \frac{\partial}{\partial \alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\right] c_n \\ &+4c_{n+1} - y_{n+3} = 0. \end{aligned} \tag{9-44}$$

Since the equation of $n \geq 2$ can be derived from the $n = 1$ equation using the renormalization group equation of each residue, we will consider only the $n = 1$ equation below.

In order to solve it in perturbations, we expand $\chi(D)$ as

$$\chi(D) = \sum_{n=1}^{\infty} \chi_n (D-4)^{n-1} = \chi_1 + \chi_2(D-4) + \chi_3(D-4)^2 + \cdots$$

and determine each coefficient in order. Then, from (9-43), the following relation holds:

$$c_1 = \chi_1 b_2 + \chi_2 b_3 + \chi_3 b_4 + \cdots. \tag{9-45}$$

Since the residue b_n of $n \geq 3$ can be derived from b_2 through the renormalization group equation (9-10), c_1 can be written in terms of b_2. Expanding b_2 as

$$b_2 = B_1\alpha^3 + B_2\alpha^4 + B_3\alpha^5 + o(\alpha^6),$$

we obtain the residue b_n as

$$\begin{aligned} b_3 &= -\frac{3}{5}\beta_1 B_1 \alpha^4 - \left(\frac{1}{2}\beta_2 B_1 + \frac{2}{3}\beta_1 B_2\right)\alpha^5 + o(\alpha^6), \\ b_4 &= \frac{2}{5}\beta_1^2 B_1 \alpha^5 + o(\alpha^6), \qquad b_5 = o(\alpha^6). \end{aligned}$$

In addition, parts that depend on the coupling constant α of the lowest residue b_1 are determined to be

$$b_1 = -\frac{2B_1}{\beta_1}\alpha^2 + \left(-\frac{5B_2}{3\beta_1} + \frac{4\beta_2 B_1}{3\beta_1^2}\right)\alpha^3 + o(\alpha^4). \tag{9-46}$$

Substituting the expressions of $b_{2,3,4}$ into (9-45), the residue c_1 can be expressed as

$$c_1 = \chi_1 B_1 \alpha^3 + \left(\chi_1 B_2 - \frac{3}{5}\chi_2 \beta_1 B_1\right)\alpha^4 + \Bigg\{\chi_1 B_3 - \chi_2 \left(\frac{1}{2}\beta_2 B_1 + \frac{2}{3}\beta_1 B_2\right) + \frac{2}{5}\chi_3 \beta_1^2 B_1\Bigg\}\alpha^5 + o(\alpha^6).$$

On the other hand, the residue c_1 is given by $x_3/8$, therefore compared with the solution (9-38), we can read the coefficient B_n as

$$B_1 = -\frac{\beta_1 \beta_2 X_1}{96\chi_1},$$
$$B_2 = -\frac{\chi_2 \beta_1^2 \beta_2 X_1}{160\chi_1^2} - \frac{\beta_2^2 X_1}{120\chi_1} - \frac{\beta_1 \beta_3 X_1}{80\chi_1} - \frac{\beta_1 \beta_2 X_2}{160\chi_1}.$$

The residue c_n $(n \geq 2)$ can also be obtained from c_1 through (9-10).

Substituting the above expressions of b_n and c_n into the renormalization group equation (9-44) of $n = 1$ and ignoring $o(\alpha^6)$, we obtain

$$4(1-2\chi_1) B_1 \alpha^3 + \left\{4(1-2\chi_1) B_2 - \frac{3}{5}(6+4\chi_1-8\chi_2)\beta_1 B_1\right\}\alpha^4 + \Bigg\{4(1-2\chi_1) B_3 - 8\chi_1 \beta_2 B_1 - (6+4\chi_1-8\chi_2)\left(\frac{1}{2}\beta_2 B_1 + \frac{2}{3}\beta_1 B_2\right) + \frac{2}{5}(2+4\chi_2-8\chi_3)\beta_1^2 B_1\Bigg\}\alpha^5 - y_4(\alpha) = o(\alpha^6).$$

Noting that y_4 (9-39) starts with $o(\alpha^4)$, the coefficient χ_1 is determined to be

$$\chi_1 = \frac{1}{2}$$

in order for $o(\alpha^3)$ to disappear. Furthermore, by substituting the concrete expression of y_4 (9-39) and solving the condition that $o(\alpha^4)$ disappears, we obtain

$$\chi_2 = 1 - \frac{Y_1}{4X_1}.$$

Since the B_3-dependence disappears when assigning $\chi_1 = 1/2$, the condition of $o(\alpha^5)$ can be solved, so that

$$\chi_3 = \frac{1}{8}\left(2 - \frac{Y_1}{X_1}\right)\left(3 - \frac{Y_1}{X_1}\right) - \frac{1}{6}\frac{\beta_2}{\beta_1^2}\left(1 - \frac{Y_1}{X_1}\right) + \frac{1}{6}\frac{X_2}{\beta_1 X_1}\left(\frac{Y_2}{X_2} - \frac{3}{2}\frac{Y_1}{X_1}\right).$$

In this way, we can determine χ_n sequentially. Substituting the concrete values (9-40), (9-41), and (9-42) into them yields

$$\chi_1 = \frac{1}{2}, \qquad \chi_2 = \frac{3}{4}, \qquad \chi_3 = \frac{1}{3}. \tag{9-47}$$

Thus the expression (9-3) is derived.

Lastly, we summarize the results of the residues $b_{1,2}$, which are calculated as

$$\begin{aligned} b_1 &= \frac{73}{360}\frac{1}{(4\pi)^2} - \frac{1}{6}\frac{\alpha^2}{(4\pi)^4} + \frac{25}{108}\frac{\alpha^3}{(4\pi)^5} + o(\alpha^4), \\ b_2 &= \frac{2}{9}\frac{\alpha^3}{(4\pi)^5} + \frac{22}{135}\frac{\alpha^4}{(4\pi)^6} + o(\alpha^5). \end{aligned}$$

The residue b_1 is derived from (9-46) with the exception of the constant term. The constant term cannot be determined from the renormalization group equation. It is given by assigning $N_S = 0$, $N_F = 1$, and $N_A = 1$ to the value b_c (5-12) divided by $(4\pi)^2$ which is derived from direct one-loop calculations.

For completeness, we also present the result of the residue a_1 obtained from direct calculations of the two-point correlation function of $\Theta_{\mu\nu}$ as

$$a_1 = -\frac{3}{20}\frac{1}{(4\pi)^2} - \frac{7}{72}\frac{\alpha}{(4\pi)^3} + o(\alpha^2),$$

where the constant term is given by $-\zeta_1/(4\pi)^2$ (5-12).

If the same discussion is done in the case of QCD in curved space, we obtain renormalization group equations which have the same form as those of QED. Solving them, we find that the values of $\chi_{1,2}$ are the same as above, regardless of gauge groups and fermion representations.[8] The value of the first coefficient χ_1 is particularly important, and as is discussed soon below, the form of conformal anomaly introduced by Riegert is determined from this value.

[8] See M. Freeman, *The Renormalization of Nonabelian Gauge Theories in Curved Space-time*, Ann. Phys. **153** (1984) 339 and the appendix of K. Hamada and M. Matsuda, Phys. Rev. D **93** (2016) 064051.

Determination of Conformal Anomalies

Rewriting (9-18) based on the result in the previous section, the composite field $[F_{\mu\nu}^2]$ can be expressed as

$$\begin{aligned}\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] = \frac{D-4}{4\beta}F_{0\mu\nu}F_0^{\mu\nu} - \frac{\bar{\gamma}_\psi}{\beta}[E_\psi] \\ +\frac{D-4}{\beta}\mu^{D-4}\biggl[\left(L_a + \frac{\bar{\beta}_a}{D-4}\right)F_D + \left(L_b + \frac{\bar{\beta}_b}{D-4}\right)G_D \\ -\frac{4\chi(D)(\sigma + L_\sigma)}{D-4}\nabla^2 H\biggr],\end{aligned}$$

where the last term is multiplied by $4\chi(D)$ for convenience. Using this expression, we can rewrite the trace of the energy-momentum tensor as

$$\begin{aligned}\Theta &= (D-4)\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}(D-1)i\bar{\psi}_0\overleftrightarrow{\not{D}}\psi_0 \\ &\quad +(D-4)\left\{a_0F_D + b_0\left[G_D - 4\chi(D)\nabla^2 H\right]\right\} \\ &= \frac{\beta}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}\left(D-1+2\bar{\gamma}_\psi\right)[E_\psi] - \mu^{D-4}\left(\beta_aF_D + \beta_bG_D\right) \\ &\quad -4\mu^{D-4}\chi(D)\left[(D-4)(b+L_b) - (\sigma + L_\sigma)\right]\nabla^2 H,\end{aligned}$$

except for the gauge-fixing term. To make the last term finite, we impose a condition $(D-4)(b+L_b) - (\sigma + L_\sigma) = \text{finite} = (D-4)b - \sigma + b_1$. The residue σ_n $(n \geq 1)$ of L_σ is then expressed in terms of the residue of L_b as

$$\sigma_n = b_{n+1}.$$

Furthermore, based on this result and $\mu d\Theta/d\mu = 0$, we can solve an extra condition that $\beta^{-1}\mu d(\beta[F_{\mu\nu}^2])/d\mu$ is finite, as in (9-30). Thus, we get

$$\sigma = \bar{\beta}_b + b_1.$$

Substituting this result, the form of the conformal anomaly is determined to be

$$\Theta = \frac{\beta}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}\left(D-1+2\bar{\gamma}_\psi\right)[E_\psi] - \mu^{D-4}\left(\beta_aF_D + \beta_bE_D\right),$$

where the last term is given by

$$E_D = G_D - 4\chi(D)\nabla^2 H,$$

which is just (9-4) presented at the beginning of this chapter. When $D = 4$, this function reduces to

$$E_4 = G_4 - \frac{2}{3}\nabla^2 R$$

because of $\chi(4) = 1/2$. This is the combination (5-9) predicted by Riegert from the analogy with the gravitational effective action in two dimensions, as discussed in Chapter 5. Now we showed that the E_4 exactly appears from the renormalizability of quantum field theory in curved space.

Note that conformal variations of the volume integral of F_D and that of E_D (same as that of G_D) satisfy

$$\frac{\delta}{\delta\Omega}\int d^D x\sqrt{g}F_D = (D-4)F_D,$$
$$\frac{\delta}{\delta\Omega}\int d^D x\sqrt{g}E_D = (D-4)E_D,$$

where

$$\frac{\delta}{\delta\Omega}\int d^D x\sqrt{g}H^2 = (D-4)H^2 - 4\nabla^2 H$$

is used.

Casimir Energy

Last of all, we briefly mention the Casimir effect in curved spacetime. It is obtained from the time-time component of the energy-momentum tensor restored to the Lorentzian metric.

Consider the contribution from free fields in curved spacetime. If spacetime is restricted to be conformally flat, the Casimir effect results from the $b_0 G_D$ action only. By placing the residue of the simple pole of b_0 as $b_1 = b_c/(4\pi)^{D/2}$ and computing the quantity that remains in the four-dimensional limit, we obtain

$$\begin{aligned}\langle\Theta^{\mu\nu}\rangle|_{\mathrm{conf.}} &= -\lim_{D\to 4}\frac{b_c}{(4\pi)^{D/2}}\frac{1}{D-4}\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\int d^D x\sqrt{-g}\,G_D\right)\Big|_{\mathrm{conf.}} \\ &= -\frac{b_c}{(4\pi)^2}\Big\{2R^{\mu\lambda}R^{\nu}{}_{\lambda} - \frac{4}{3}R^{\mu\nu}R - g^{\mu\nu}\left(R^2_{\lambda\sigma} - \frac{1}{2}R^2\right) \\ &\qquad -\frac{2}{9}\left(R^{\mu\nu}R - \nabla^\mu\nabla^\nu R\right) + \frac{1}{18}g^{\mu\nu}\left(R^2 - 4\nabla^2 R\right)\Big\}.\end{aligned}$$

This is the same as the four-dimensional limit of the expression (9-19) in which c_0 is replaced with $(D-4)\chi(D)b_0$ and the Riemann curvature tensor is eliminated using the fact that the Weyl tensor (A-1) vanishes in a conformally flat spacetime. The overall negative sign is due to returning to the Lorentzian signature.

Consider the $\mathbb{R}\times S^3$ spacetime with the unit three-sphere used in Chapter 8 as space. The curvatures are then $R = 6$, $R_{0\mu} = 0$, and $R_{\mu\nu}^2 = 12$, and the volume of the unit three-sphere is given by $V_3 = \int d\Omega_3 = 2\pi^2$. Thus, the Casimir energy is calculated as

$$E_c = \int d\Omega_3 \, \langle \Theta^{00} \rangle = 8 \frac{b_c}{(4\pi)^2} V_3 = b_c,$$

in which the contribution from G_4 in G_D is $3b_c/4$ and the remaining $b_c/4$ comes from $(D-4)\chi(D)H^2$. This effect appears in the Hamiltonian operator (8-5).

CHAPTER TEN

RENORMALIZATION IN QUANTUM GRAVITY

In the first half of this book, we have shown that the background-free quantum gravity can be expressed as a special conformal field theory where conformal invariance is a gauge symmetry. It is realized by quantizing the conformal-factor field strictly, while the traceless tensor field is handled in perturbations. Renormalizable quantum gravity in which such a conformal field theory appears at very high energy beyond the Planck mass is called the asymptotically background-free quantum gravity. In this chapter, we describe renormalization of this theory using dimensional regularization.

Since it does not lose generality even if we employ the flat metric as a background metric due to the background-free property, the renormalizable quantum gravity can be formulated as usual as a quantum field theory in the flat spacetime.

D-Dimensional Action and Renormalization Procedure

Since quantum gravity includes quantum field theory in curved space as a part, we use the gravitational counterterms for dimensional regularization obtained in the previous chapter as a quantum gravity action.

As in the previous chapter, we will examine the system in which QED and gravity are coupled, as an example. The bare quantum gravity action in D-dimensional space with Euclidean signature[1] is given by[2]

$$S = \int d^D x \sqrt{g} \left\{ \frac{1}{t_0^2} F_D + b_0 G_D + \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \sum_{j=1}^{n_F} i \bar{\psi}_{0j} \not{D} \psi_{0j} \right.$$
$$\left. - \frac{M_0^2}{2} R + \Lambda_0 \right\}.$$

[1] Since the flat background can be adopted due to the background-metric independence, we can Wick-rotate to Euclidean time $\tau = i\eta$ in the flat background.

[2] This action reduces to I (7-1) at $D \to 4$, where note that the whole sign is reversed in the Euclidean space, apart from fermion. The reason why using S as a symbol instead of I is from the fact that quantum correction terms such as the Riegert action S_{R} (7-4) are included in the D-dimensional action as discussed in Chapter 9.

The first two terms are those defined in the previous chapter. Writing those again, F_D is the D-dimensional Weyl action (9-1) defined by

$$F_D = C^2_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - \frac{4}{D-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(D-1)(D-2)}R^2.$$

The bare constant t_0 is the coupling constant for the traceless tensor field, where a_0 in curved space is replaced with $1/t_0^2$. The combination G_D (9-2) is a D-dimensional generalization of the commonly used Euler density $G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2$, which is defined as

$$G_D = G_4 + (D-4)\chi(D)H^2, \qquad H = \frac{R}{D-1}. \tag{10-1}$$

It is also the bulk part of the conformal anomaly $E_D = G_D - 4\chi(D)\nabla^2 H$ (9-4). The coefficient $\chi(D)$ is a finite function of only D and is expanded as

$$\chi(D) = \sum_{n=1}^{\infty} \chi_n (D-4)^{n-1}.$$

The coefficient χ_n is a number independent of the coupling constant which can be determined for each order.

The lowest χ_1 and the second χ_2 are particularly important, and these values are considered to be universal constants. Indeed, from the calculation of QED and QCD in curved spacetime, regardless of not only gauge group but also the number or representation of fermions, they have been determined to be the values (9-47),

$$\chi_1 = \frac{1}{2}, \qquad \chi_2 = \frac{3}{4}. \tag{10-2}$$

The first value is also derived from the calculation of a scalar field in curved spacetime.[3] On the other hand, there is still a possibility that χ_3 may depend on the matter theory coupled with gravity, but from the QED calculation $\chi_3 = 1/3$ has been found. In the following, we will proceed with calculations with χ_3 as an arbitrary number. The gravitational interaction concerned with this value contributes only to quantum corrections of three or more loops.

[3] In the case of the scalar field, however, it is pointed out that we need to introduce an extra interaction term $\eta_0 R\varphi^2$ additively which represents a deviation from the conformal coupling. In this case, three counterterms are necessary: $(1/t_0^2)F_D$, $b_0 G_D$, and $\kappa_0 R^2$. The former two used here are responsible for the conformal coupling, while the last controls the deviation therefrom, which will disappear in the ultraviolet limit.

The gravitational field is decomposed into the conformal-factor field ϕ, the traceless tensor field $h_{0\mu\nu}$, and the background metric $\hat{g}_{\mu\nu}$ as before. Using the coupling constant t_0, it is expanded as

$$g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu},$$
$$\bar{g}_{\mu\nu} = (\hat{g}e^{t_0 h_0})_{\mu\nu} = \hat{g}_{\mu\nu} + t_0 h_{0\mu\nu} + \frac{t_0^2}{2} h_{0\mu}^{\lambda} h_{0\lambda\nu} + \cdots.$$

The inverse of each metric is defined by $g^{\mu\lambda}g_{\lambda\nu} = \bar{g}^{\mu\lambda}\bar{g}_{\lambda\nu} = \hat{g}^{\mu\lambda}\hat{g}_{\lambda\nu} = \delta^{\mu}_{\nu}$. The contraction of the indices of the tensor field $h_{0\mu\nu}$ is done by using the background metric such as $h_{0\mu\nu} = \hat{g}_{\mu\lambda}h_{0\nu}^{\lambda}$ and $h_{0\mu}^{\mu} = 0$.

It should be noted here that we do not introduce own coupling constant in the conformal-factor field ϕ, but treat it strictly. Since ϕ is an integration variable, even if we shift the field like $\phi \to \phi - \sigma$, it is nothing but a change of the integration variable, thus the theory remains unchanged. Since this shift is equivalent to converting the background metric as $\hat{g}_{\mu\nu} \to e^{2\sigma}\hat{g}_{\mu\nu}$, the background-metric independence is realized. Therefore, even if we employ the flat metric as the background metric, the result does not change. In the following, $\hat{g}_{\mu\nu}$ is the Euclidean flat metric $\delta_{\mu\nu}$, unless otherwise mentioned.

Renormalization factors of the traceless tensor field, gauge field, and fermion field are defined according to normal procedure as

$$A_{0\mu} = Z_3^{1/2}A_\mu, \quad \psi_{0j} = Z_2^{1/2}\psi_j, \quad h_{0\mu\nu} = Z_h^{1/2}h_{\mu\nu}. \qquad (10\text{-}3)$$

For the coupling constant of QED and that of the traceless tensor field, they are defined by

$$e_0 = \mu^{2-D/2}Z_e e, \quad t_0 = \mu^{2-D/2}Z_t t. \qquad (10\text{-}4)$$

Moreover, when the renormalization factor for the QED interaction is denoted by Z_1 as usual, the Ward-Takahashi identity $Z_1 = Z_2$ holds also in the system coupled with quantum gravity as will be shown in the later section, thus the famous relation $Z_e = Z_3^{-1/2}$ holds as well. Here, we will describe the renormalization procedure as this relation is being satisfied.

The most characteristic property of renormalization in the asymptotically background-free quantum gravity is that the conformal-factor field ϕ does not receive renormalization, which is expressed as

$$Z_\phi = 1. \qquad (10\text{-}5)$$

This property originates from not introducing a coupling constant in the conformal-factor field. In the later section, by doing concrete calculations, we will demonstrate that this non-renormalization theorem actually holds.

In dimensional regularization, ultraviolet divergences appear as negative powers of $D-4$. Therefore, the renormalization factors for removing them are defined by Laurent expansions in $D-4$ as

$$\log Z_3 = \sum_{n=1}^{\infty} \frac{f_n}{(D-4)^n}, \qquad \log Z_t^{-2} = \sum_{n=1}^{\infty} \frac{g_n}{(D-4)^n}. \tag{10-6}$$

Other renormalization factors are similarly Laurent expanded. The residues f_n and g_n are functions of the renormalized coupling constants e and t. In the following, as the coupling constants, we use

$$\alpha = \frac{e^2}{4\pi}, \qquad \alpha_t = \frac{t^2}{4\pi}.$$

On the other hand, we need to be careful for the bare constant b_0. Since the action G_D becomes topological in four dimensions, the kinetic term of the gravitational field appears from $o(D-4)$ when it is expanded around four dimensions. It means that this action does not contribute classically to dynamics of gravity. Therefore, since b_0 cannot be regarded as an independent coupling constant, it is expanded only in negative powers of $D-4$ as

$$b_0 = \frac{\mu^{D-4}}{(4\pi)^{D/2}} L_b, \qquad L_b = \sum_{n=1}^{\infty} \frac{b_n}{(D-4)^n}, \tag{10-7}$$

unlike what we did in (9-6) in curved space.

As we can see from the calculation in curved space, the residues b_n $(n \geq 2)$ are functions of only the coupling constants, whereas there is a constant term in the simple pole residue b_1 so that it can be decomposed as

$$b_1 = b + b_1', \tag{10-8}$$

where b_1' is the part that depends on the coupling constant, while b is just the number.[4]

The conformal anomaly or the corresponding Wess-Zumino action, as described in the previous chapter, is a finite quantity that poles of the renormalization factor cancel out with zeros of $D-4$ that appear when the action is expanded near four dimensions. It is a quantum mechanical quantity that appears in order to preserve diffeomorphism invariance, and hence it

[4] Note that it is different from the coupling constant b in b_0 introduced in the previous chapter, although we use the same symbol here.

is physically not an anomalous quantity. Especially important one is the Riegert action (7-4) which is derived from the cancellation of the $o(D-4)$ term in the expansion of G_D and the simple pole of b_0. It gives the kinetic term of the conformal-factor field at the zeroth order of the coupling constant. For details, see (10-20) in the next section. The fact that quantum diffeomorphism in the vanishing coupling limit is completed only when the Riegert action and the Weyl action are combined has already been shown in Chapters 7 and 8.

It should be noted here that there is a contribution to the constant b that gives the coefficient of the induced Riegert action from this action itself through loops of the conformal-factor field. Therefore, in order to resolve the nested structure and take in such loop contributions systematically in renormalization calculations, we need a manipulation described below.

First of all, we will calculate by considering the constant b as a new coupling constant for the time being. At this time, the effective action has the following structure:

$$\Gamma = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \frac{b-b_c}{D-4} \int d^D x \sqrt{\hat{g}}\, \hat{G}_4 + \Gamma_{\rm ren}(\alpha, \alpha_t, b),$$

where $\Gamma_{\rm ren}$ is a renormalized finite function. The divergence term remains only when a curved background is employed, and b_c is a constant independent of the coupling constant determined from one loop corrections of QED and the gravitational field, which is given by[5]

$$b_c = \frac{11 n_F}{360} + \frac{40}{9}.$$

After carrying out all renormalization calculations, we put $b = b_c$ and get the effective action $\Gamma_{\rm ren}(\alpha, \alpha_t, b_c)$. In this way, the renormalizable quantum gravity is constructed, in which its high energy dynamics is described by the dimensionless gravitational coupling constant α_t only.

Since b appears in front of the Riegert action, quantum loop corrections by the conformal-factor field will appear in a negative power of b (see (10-22) in the next section). That is to say that as b increases, the conformal factor becomes classical. The $1/b$ expansion corresponds to the so-called large N expansion which increases the number of matter fields.

[5] In general, it is given by $b_c = (N_S + 11N_F + 62N_A)/360 + 769/180$ (7-5), where N_S, N_F, and N_A are the number of scalars, fermions, and gauge fields with conformal couplings. The last number is the sum of $-7/90$ and $87/20$, which are the corrections from the gravitational fields ϕ and $h_{\mu\nu}$, respectively.

The beta functions for the coupling constants α and α_t are defined as

$$\beta \equiv \frac{\mu}{\alpha}\frac{d\alpha}{d\mu} = D - 4 + \bar{\beta},$$
$$\beta_t \equiv \frac{\mu}{\alpha_t}\frac{d\alpha_t}{d\mu} = D - 4 + \bar{\beta}_t, \tag{10-9}$$

where the definition of β is the same as (9-8) in the previous chapter. We have also defined β_t according to that definition.[6] On the other hand, the beta function for b is defined as

$$\mu\frac{db}{d\mu} = (D-4)\bar{\beta}_b \tag{10-10}$$

unlike the previous chapter.

Let us first consider a renormalization group equation $\mu db_0/d\mu = 0$. Rewrite the μ-differential

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \mu\frac{d\alpha}{d\mu}\frac{\partial}{\partial\alpha} + \mu\frac{d\alpha_t}{d\mu}\frac{\partial}{\partial\alpha_t} + \mu\frac{db}{d\mu}\frac{\partial}{\partial b} + \cdots$$

with (10-9) and (10-10), and Laurent-expand $\mu db_0/d\mu$ with the definition (10-7) of b_0. In order for it to disappear, all expansion coefficients must vanish, so that a renormalization group equation for the residue b_n is obtained as

$$\left(\alpha\frac{\partial}{\partial\alpha} + \alpha_t\frac{\partial}{\partial\alpha_t} + \bar{\beta}_b\frac{\partial}{\partial b} + 1\right)b_{n+1} + \left(\bar{\beta}\alpha\frac{\partial}{\partial\alpha} + \bar{\beta}_t\alpha_t\frac{\partial}{\partial\alpha_t}\right)b_n = 0.$$

Apart from the parts depending on α_t and b, this equation returns to the expression obtained by QED in curved space discussed in the previous chapter. From a finite part that must disappear, we obtain

$$\bar{\beta}_b = -\left(\frac{\partial b_1}{\partial b}\right)^{-1}\left(b_1 + \alpha\frac{\partial b_1}{\partial\alpha} + \alpha_t\frac{\partial b_1}{\partial\alpha_t}\right).$$

If the pole term L_b is decomposed as $L_b = b/(D-4) + L'_b$ according to the structure (10-8) of the simple pole b_1, it can be also expressed as $\bar{\beta}_b = -b - (D-4)L'_b - \mu dL'_b/d\mu$.

Since the function $\bar{\beta}_b$ is finite, (10-10) indicates that $\mu db/d\mu \to 0$ at the limit of $D \to 4$. Hence, it is justified that the coupling constant b is replaced with the constant b_c after finishing all renormalization calculations.

---

[6] It is slightly different from the definition when touched on the beta function of t in the first section of Chapter 7.

Parts that depend on the coupling constant α in the simple pole b_1 and the double pole b_2 have already been calculated in the previous chapter up to $o(\alpha^3)$ and $o(\alpha^4)$, respectively. Here we will use the following results:

$$b_1 = b - \frac{n_F^2}{6}\left(\frac{\alpha}{4\pi}\right)^2, \qquad b_2 = \frac{2n_F^3}{9}\left(\frac{\alpha}{4\pi}\right)^3. \tag{10-11}$$

Then we obtain

$$\bar{\beta}_b = -b + \frac{n_F^2}{2}\left(\frac{\alpha}{4\pi}\right)^2. \tag{10-12}$$

From a renormalization group equation $\mu d(e_0^2/4\pi)/d\mu = 0$, we find that the residue f_n of $\log Z_3$ satisfies

$$\left(\alpha\frac{\partial}{\partial\alpha} + \alpha_t\frac{\partial}{\partial\alpha_t} + \bar{\beta}_b\frac{\partial}{\partial b}\right)f_{n+1} + \left(\bar{\beta}\alpha\frac{\partial}{\partial\alpha} + \bar{\beta}_t\alpha_t\frac{\partial}{\partial\alpha_t}\right)f_n = 0. \tag{10-13}$$

The beta function is given by

$$\bar{\beta} = \alpha\frac{\partial f_1}{\partial\alpha} + \alpha_t\frac{\partial f_1}{\partial\alpha_t} + \bar{\beta}_b\frac{\partial f_1}{\partial b}.$$

The residue g_n of $\log Z_t^{-2}$ also satisfies similar equations. Each of the beta functions can be expressed using the renormalization factors as $\bar{\beta} = \mu d(\log Z_3)/d\mu$ and $\bar{\beta}_t = \mu d(\log Z_t^{-2})/d\mu$.

In advance, we summarize various results derived from calculations that will be performed in the later section. The residues f_1 and f_2 are given by

$$\begin{aligned} f_1 &= \frac{8n_F}{3}\frac{\alpha}{4\pi} + \left(4n_F - \frac{16n_F^2}{27b}\right)\left(\frac{\alpha}{4\pi}\right)^2, \\ f_2 &= -\frac{32n_F^2}{9}\left(\frac{\alpha}{4\pi}\right)^2 - \left(\frac{128n_F^2}{9} - \frac{160n_F^3}{81b}\right)\left(\frac{\alpha}{4\pi}\right)^3, \end{aligned} \tag{10-14}$$

which are derived from the renormalization factor (10-31) of the gauge field including ordinary QED loop corrections as well as loop corrections of the conformal-factor field given by $o(1/b)$, while there is no one-loop $o(\alpha_t)$ correction from the traceless tensor field to f_1. Paying attention to $\bar{\beta}_b = -b + o(\alpha^2)$, we can see that the residues $f_{1,2}$ satisfy the renormalization group equation (10-13). From the simple pole residue f_1, the beta function of QED is obtained as

$$\bar{\beta} = \frac{8n_F}{3}\frac{\alpha}{4\pi} + \left(8n_F - \frac{16n_F^2}{9}\frac{1}{b}\right)\left(\frac{\alpha}{4\pi}\right)^2. \tag{10-15}$$

Replacing b with the constant b_c yields the final result. Due to $b_c > 0$, the gravitational correction term of $o(1/b_c)$ gives a negative contribution.

Similarly, from the result (10-30) calculated in the later section, we obtain the beta function of the gravitational coupling constant as

$$\bar{\beta}_t = -\left(\frac{n_F}{20} + \frac{20}{3}\right)\frac{\alpha_t}{4\pi} - \frac{7n_F}{36}\frac{\alpha\alpha_t}{(4\pi)^2}. \qquad (10\text{-}16)$$

Since this becomes negative, we find that the traceless tensor field has an asymptotically free behavior. This shows that the perturbative expansion about a conformally flat spacetime is justified.

The thing to note here is that as we use the term asymptotic freedom according to historical nomenclature, this does not mean that an asymptotically free tensor field appears in the ultraviolet limit. Since no coupling constant is introduced in the conformal factor, the conformal factor remains fluctuating non-perturbatively even in the ultraviolet limit, so that the background-free world is realized. Thus this technical term simply means that fluctuations of the traceless tensor field become less significant than those of the conformal factor. For this reason, ordinary particle picture where gravitons and other particles propagate as small fluctuations in the flat spacetime is no longer established. Hence, to distinguish from the asymptotic freedom, we call this behavior as the asymptotic background freedom.

On the other hand, the negativity of the beta function indicates the existence of a new dynamical infrared energy scale Λ_{QG}, as in the case of QCD. It will be defined later when introducing the running coupling constant. Below this energy scale, the conformally invariant gravitational dynamics disappears and it becomes incorrect to treat the gravitational field separately into the conformal-factor field and the traceless tensor field. These fields are tightly coupled together at low energies to act as one gravitational field, and its dynamics will be dominated by the Einstein-Hilbert action.

Kinetic Terms and Interactions

Renormalization calculations are performed by Laurent-expanding the bare action S with renomalized quantities. Then, terms with negative power of $D - 4$ are set as counterterms for eliminating ultraviolet divergences, and terms having zero or positive power are treated as kinetic terms and interaction terms.

Let us first describe the gauge field action. When writing the Laurent expansion of the renormalization factor $Z_3 - 1$ as $\sum_{n=1} x_n/(D-4)^n$, the bare action of the gauge field is expanded as

$$\begin{aligned}
&\frac{1}{4}\int d^Dx\sqrt{g}\,F_{0\mu\nu}F_0^{\mu\nu} = \frac{1}{4}Z_3\int d^Dx\,e^{(D-4)\phi}F_{\mu\nu}F_{\lambda\sigma}\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma} \\
&= \frac{1}{4}\int d^Dx\left\{\left(1+\frac{x_1}{D-4}+\frac{x_2}{(D-4)^2}+\cdots\right)F_{\mu\nu}F_{\lambda\sigma}\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma}\right. \\
&\quad+\left(D-4+x_1+\frac{x_2}{D-4}+\cdots\right)\phi F_{\mu\nu}F_{\lambda\sigma}\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma} \\
&\quad+\frac{1}{2}\left[(D-4)^2+(D-4)x_1+x_2+\cdots\right]\phi^2F_{\mu\nu}F_{\lambda\sigma}\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma} \\
&\quad\left.+\cdots\right\},
\end{aligned} \tag{10-17}$$

where using the residues $f_{1,2}$ of $\log Z_3$ (10-6), the coefficients $x_{1,2}$ are expressed as

$$x_1 = f_1, \qquad x_2 = f_2 + \frac{1}{2}f_1^2. \tag{10-18}$$

The renormalized gauge field is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The first line on the right-hand side of (10-17) gives normal kinetic term and counterterms of the gauge field. When expanded with the coupling constant t_0 and replaced it with the renormalized one through (10-4), interaction terms with the traceless tensor field and their counterterms appear. The results of the coefficients $f_{1,2}$ are given in (10-14) in advance.

The second line gives terms which do not appear in normal quantum field theory in the flat spacetime, in which $\phi F_{\mu\nu}^2$ is a Wess-Zumino action obtained by integrating the conformal anomaly $F_{\mu\nu}^2$ with respect to the conformal-factor field. Inversely, if we perform a variation of this action by the conformal-factor field, the conformal anomaly is produced. This indicates that the conformal anomaly is related to the beta function. The third line gives terms that yield higher-order conformal anomalies.

In the same way, we can consider the Weyl action. In D dimensions, terms that depend on the conformal-factor field ϕ appear as

$$\begin{aligned}
&\frac{1}{t_0^2}\int d^Dx\sqrt{g}\,F_D = \frac{1}{t_0^2}\int d^Dx\,e^{(D-4)\phi}\,\bar{C}^2_{\mu\nu\lambda\sigma} \\
&= \int d^Dx\left[\frac{1}{t_0^2}\bar{C}^2_{\mu\nu\lambda\sigma}+\frac{D-4}{t_0^2}\phi\,\bar{C}^2_{\mu\nu\lambda\sigma}+\cdots\right].
\end{aligned} \tag{10-19}$$

When Laurent expansion is performed by replacing the bare quantities with the renormalized ones, a kinetic term, interaction terms, and counterterms are generated, in which interactions like $\phi^n \bar{C}^2_{\mu\nu\lambda\sigma}$ associated with the conformal anomalies appear.

Next, we examine the bare action for the Euler density G_D (10-1) with the coefficient (10-2). From the Laurent expansion of the bare constant b_0 (10-7) and the expansion (A-10) of the volume integral of G_D given in the fourth section of Appendix A, the bare action is expanded as follows:

$$
\begin{aligned}
b_0 \int d^D x \sqrt{g}\, G_D = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \int d^D x \Biggl\{ & \left(\frac{b_1}{D-4} + \frac{b_2}{(D-4)^2} + \cdots \right) \bar{G}_4 \\
& + \left(b_1 + \frac{b_2}{D-4} + \cdots \right) \left(2\phi \bar{\Delta}_4 \phi + \bar{G}_4 \phi - \frac{2}{3} \bar{\nabla}^2 \bar{R} \phi + \frac{1}{18} \bar{R}^2 \right) \\
& + \bigl[(D-4) b_1 + \cdots \bigr] \left(\phi^2 \bar{\Delta}_4 \phi + \frac{1}{2} \bar{G}_4 \phi^2 + 3\phi \bar{\nabla}^4 \phi \right. \\
& \quad \left. + 4\phi \bar{R}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \phi - \frac{14}{9} \phi \bar{R} \bar{\nabla}^2 \phi + \frac{10}{9} \phi \bar{\nabla}^\mu \bar{R} \bar{\nabla}_\mu \phi + \cdots \right) \\
& + \bigl[(D-4)^2 b_1 + \cdots \bigr] \left[\frac{1}{3} \phi^3 \bar{\Delta}_4 \phi + \left(4\chi_3 - \frac{1}{2} \right) \left(\bar{\nabla}^\mu \phi \bar{\nabla}_\mu \phi \right)^2 + \cdots \right] \\
& + \cdots \Biggr\},
\end{aligned}
\qquad (10\text{-}20)
$$

where $\bar{\Delta}_4$ is defined by

$$
\bar{\Delta}_4 = \bar{\nabla}^4 + 2\bar{R}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{2}{3} \bar{R} \nabla^2 + \frac{1}{3} \bar{\nabla}^\mu \bar{R} \bar{\nabla}_\mu ,
$$

which is (5-10) introduced in Chapter 5 that becomes conformally invariant for a scalar in four dimensions when multiplied by $\sqrt{\bar{g}}$. The first line of the expansion gives counterterms to eliminate ultraviolet divergences proportional to $\bar{G}_4$, which determine the residue b_n. The second line gives the Riegert action S_{R} (7-4) induced by the divergences, we write its finite part again as

$$
S_{\mathrm{R}} = \frac{\mu^{D-4}}{(4\pi)^{D/2}} b_1 \int d^D x \left(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi + \frac{1}{18} \bar{R}^2 \right), \qquad (10\text{-}21)
$$

where $\bar{E}_4 = \bar{G}_4 - 2\bar{\nabla}^2 \bar{R}/3$ (5-9) and an extra term $\bar{R}^2$ is added. This action gives the kinetic term of the conformal-factor field. The third line of the expansion (10-20) gives new self-interactions that become effective in higher loop calculations. As the coefficients $b_{1,2}$, we use the values in (10-11) here.

From the part independent of α_t and α of (10-21) when expanded with the coupling constant t_0, the kinetic term of the conformal-factor field is given as

$$\frac{\mu^{D-4}}{(4\pi)^{D/2}} 2b \int d^D x\, \phi \partial^4 \phi,$$

where $\partial^2 = \partial_\lambda \partial_\lambda$ is d'Alembertian in the flat Euclidean background. As a convention, indices in Euclidean background will be represented by subscripts and the same index will be contracted with the Kronecker delta $\delta_{\mu\nu}$. The propagator of the conformal-factor field is then given by

$$\langle \phi(k)\phi(-k) \rangle = \mu^{4-D} \frac{(4\pi)^{D/2}}{4b} \frac{1}{k^4}. \tag{10-22}$$

From this expression, it can be seen that loop quantum corrections by the conformal-factor field will appear in the order of $1/b$.

Here, we write out the gravitational interaction terms required for later calculations in advance. From the third line on the right-hand side of the expansion (10-20), we obtain a three-point self-interaction of the conformal-factor field as

$$S^{(D-4)b}_{\mathrm{G}[\phi\phi\phi]} = (D-4)b \, \frac{\mu^{D-4}}{(4\pi)^{D/2}} \int d^D x\, \phi^2 \partial^4 \phi. \tag{10-23}$$

Note that since $D-4$ is multiplied to the whole, this interaction contributes to ultraviolet divergences in more than two loops. In addition, four-point self-interactions with $(D-4)^2$ appear in the fifth line of (10-20).

Expanding the bare action (10-20) further with the coupling constant t, interaction terms between the conformal-factor field and the traceless tensor field appear. From the terms $\bar{\nabla}^2 \bar{R} \phi$ and $\bar{R}^2$ in the Riegert action S_{R} (10-21), we obtain two-point interaction terms

$$\begin{aligned} S^{bt}_{\mathrm{G}[\phi h]} &= -bt \, \frac{\mu^{D/2-2}}{(4\pi)^{D/2}} \int d^D x \, \frac{2}{3} \partial^2 \phi \partial_\mu \chi_\mu, \\ S^{bt^2}_{\mathrm{G}[hh]} &= \frac{bt^2}{(4\pi)^{D/2}} \int d^D x \, \frac{1}{18} \partial_\mu \chi_\mu \partial_\nu \chi_\nu, \end{aligned} \tag{10-24}$$

where $\chi_\mu = \partial_\nu h_{\mu\nu}$. Employing the Landau gauge defined in the next section, these two interactions will disappear and the number of Feynman diagrams can be reduced considerably.

From the $\phi\bar{\Delta}_4\phi$ term, we obtain three- and four-point interactions as follows:

$$S^{bt}_{\mathrm{G}[\phi\phi h]} = b\frac{\mu^{D-4}}{(4\pi)^{D/2}}\int d^D x\, 2\phi\bar{\Delta}_4\phi\Big|_{o(t)}$$
$$= bt\,\frac{\mu^{D/2-2}}{(4\pi)^{D/2}}\int d^D x\left\{4\partial_\mu\phi\partial_\nu\partial^2\phi + \frac{8}{3}\partial_\mu\partial_\lambda\phi\partial_\nu\partial_\lambda\phi\right.$$
$$\left. -\frac{4}{3}\partial_\lambda\phi\partial_\mu\partial_\nu\partial_\lambda\phi - 4\partial_\mu\partial_\nu\phi\partial^2\phi\right\}h_{\mu\nu} \qquad (10\text{-}25)$$

and

$$S^{bt^2}_{\mathrm{G}[\phi\phi hh]} = b\frac{\mu^{D-4}}{(4\pi)^{D/2}}\int d^D x\, 2\phi\bar{\Delta}_4\phi\Big|_{o(t^2)}$$
$$= \frac{bt^2}{(4\pi)^{D/2}}\int d^D x\left\{2\partial^2\phi\partial_\mu\partial_\nu\phi\, h_{\mu\lambda}h_{\nu\lambda}\right.$$
$$\left. +2\partial_\mu\partial_\nu\phi\partial_\lambda\partial_\sigma\phi\, h_{\mu\nu}h_{\lambda\sigma} + \text{ terms with } \partial h\right\}, \qquad (10\text{-}26)$$

where the first two terms in $S^{bt^2}_{\mathrm{G}[\phi\phi hh]}$ are from $\bar{\nabla}^2\phi\bar{\nabla}^2\phi$. The four-point interactions including derivatives of the traceless tensor field are not depicted because they give a vanishing contribution to one-loop calculations in the later section, as in a tadpole diagram (2) in Fig. 10-1.

When calculating two-loop corrections of the cosmological constant, we further need interaction terms obtained by expanding the third and fourth lines of (10-20) with t. Besides that, we cannot ignore the terms omitted in (10-26). For derivations of these interaction terms, see the expansions (A-5) and (A-6) in the first section of Appendix A.

The action of fermions is conformally invariant in any D dimensions (see the second section of Appendix A). That is to say that we can always absorb the conformal-factor field dependence into the fermion field by redefining it appropriately.[7] When calculating the effective action, it is useful to rewrite the field so that the conformal-factor field dependence disappears, because dimensional regularization does not depend on how to choose the measure. Expanding the bare fermion action up to $o(t_0^2)$ in the flat background, we obtain

$$\int d^D x\, i\bar{\psi}_0\bar{\not{D}}\psi_0$$

[7] Rewriting the action in terms of $\psi_0' = e^{(D-1)\phi/2}\psi_0$, the conformal-factor field dependence can be eliminated as $\int d^D x\sqrt{g}\bar{\psi}_0\not{D}\psi_0 = \int d^D x\sqrt{\bar{g}}\bar{\psi}_0'\bar{\not{D}}\psi_0'$. In the text, ψ_0' is written again as ψ_0.

$$= \int d^D x \left\{ i\bar{\psi}_0 \gamma_\mu \partial_\mu \psi_0 - i\frac{t_0}{4}\left(\bar{\psi}_0 \gamma_\mu \partial_\nu \psi_0 - \partial_\nu \bar{\psi}_0 \gamma_\mu \psi_0\right) h_{0\mu\nu} \right.$$
$$+ i\frac{t_0^2}{16}\left(\bar{\psi}_0 \gamma_\mu \partial_\nu \psi_0 - \partial_\nu \bar{\psi}_0 \gamma_\mu \psi_0\right) h_{0\mu\lambda} h_{0\nu\lambda}$$
$$+ i\frac{t_0^2}{16}\bar{\psi}_0 \gamma_{\mu\nu\lambda} \psi_0 h_{0\mu\sigma} \partial_\lambda h_{0\nu\sigma} - e_0 \bar{\psi}_0 \gamma_\mu \psi_0 A_{0\mu}$$
$$\left. + \frac{e_0 t_0}{2}\bar{\psi}_0 \gamma_\mu \psi_0 A_{0\nu} h_{0\mu\nu} - \frac{e_0 t_0^2}{8}\bar{\psi}_0 \gamma_\mu \psi_0 A_{0\nu} h_{0\mu\lambda} h_{0\nu\lambda} \right\}, \quad (10\text{-}27)$$

where $\gamma_{\mu\nu\lambda} = (1/3!)(\gamma_\mu \gamma_\nu \gamma_\lambda + \text{antisymmetric})$. Expanding the bare coupling constants e_0 and t_0 and the bare field ψ_0 by their renormalized quantities, we get interaction terms and counterterms.

Gauge-Fixing

From the lowest order of the expansion of the Weyl action (10-19), the kinetic term of the traceless tensor field is given by

$$\int d^D x \left\{ \frac{D-3}{D-2}\left(h_{0\mu\nu}\partial^4 h_{0\mu\nu} + 2\chi_{0\mu}\partial^2 \chi_{0\mu}\right) - \frac{D-3}{D-1}\chi_{0\mu}\partial_\mu \partial_\nu \chi_{0\nu} \right\},$$

where $\chi_{0\mu} = \partial_\lambda h_{0\lambda\mu}$.

Let us perform gauge-fixing of the kinetic terms of the traceless tensor field and the $U(1)$ gauge field. According to procedures of the BRST gauge fixing, we introduce the following gauge-fixing terms and ghost actions:

$$S_{\text{g.f.}} = \int d^D x \, \delta_{\text{B}} \left[-i\,\tilde{c}_{0\mu} N_{\mu\nu} \left(\chi_{0\nu} - \frac{\zeta_0}{2} B_{0\nu} \right) - i\,\tilde{c}_0 \left(\partial_\mu A_{0\mu} - \frac{\xi_0}{2} B_0 \right) \right],$$

where $\tilde{c}_{0\mu}$ and $\tilde{c}_0$ are anti-ghost fields, $B_{0\mu}$ and B_0 are auxiliary fields. The differential operator $N_{\mu\nu}$ is symmetric and second order, which is taken as

$$N_{\mu\nu} = \frac{2(D-3)}{D-2}\left(-2\partial^2 \delta_{\mu\nu} + \frac{D-2}{D-1}\partial_\mu \partial_\nu \right),$$

so that only the first term in the kinetic term above remains when the Feynman gauge defined later is adopted. By assigning a ghost field $c_{0\mu}$ to the diffeomorphism parameter ξ_μ / t_0 as well as assigning a ghost c_0 to the $U(1)$ gauge transformation parameter, the BRST transformations of the traceless tensor field and the gauge field are given respectively as

$$\delta_{\text{B}} h_{0\mu\nu} = \partial_\mu c_{0\nu} + \partial_\nu c_{0\mu} - \frac{2}{D}\delta_{\mu\nu}\partial_\lambda c_{0\lambda} + t_0 c_{0\lambda}\partial_\lambda h_{0\mu\nu}$$

$$
\begin{aligned}
&\quad +\frac{t_0}{2} h_{0\mu\lambda}\left(\partial_\nu c_{0\lambda}-\partial_\lambda c_{0\nu}\right)+\frac{t_0}{2} h_{0\nu\lambda}\left(\partial_\mu c_{0\lambda}-\partial_\lambda c_{0\mu}\right)+\cdots, \\
\delta_{\mathrm{B}} A_{0\mu} &= \partial_\mu c_0+t_0\left(c_{0\lambda} \partial_\lambda A_{0\mu}+A_{0\lambda} \partial_\mu c_{0\lambda}\right).
\end{aligned}
$$

The BRST transformation of the conformal-factor field is

$$
\delta_{\mathrm{B}} \phi = t_0 c_{0\lambda} \partial_\lambda \phi+\frac{t_0}{D} \partial_\lambda c_{0\lambda}.
$$

From nilpotency of the BRST transformation and Grassmannian properties of the ghost and anti-ghost fields, the BRST transformations of other fields are given by[8]

$$
\begin{aligned}
\delta_{\mathrm{B}} c_0 &= t_0 c_{0\lambda} \partial_\lambda c_0, \\
\delta_{\mathrm{B}} \tilde{c}_0 &= i B_0, \qquad \delta_{\mathrm{B}} B_0=0, \\
\delta_{\mathrm{B}} c_{0\mu} &= t_0 c_{0\lambda} \partial_\lambda c_{0\mu}, \\
\delta_{\mathrm{B}} \tilde{c}_{0\mu} &= i B_{0\mu}, \qquad \delta_{\mathrm{B}} B_{0\mu}=0.
\end{aligned}
$$

Applying the BRST transformation, the gauge-fixing term and the ghost action are written as

$$
\begin{aligned}
S_{\mathrm{g.f.}} = \int d^D x \biggl\{ & B_{0\mu} N_{\mu\nu} \chi_{0\nu}-\frac{\zeta_0}{2} B_{0\mu} N_{\mu\nu} B_{0\nu}+i \tilde{c}_{0\mu} N_{\mu\nu} \partial_\lambda(\delta_{\mathrm{B}} h_{0\nu\lambda}) \\
& +B_0 \partial_\mu A_{0\mu}-\frac{\xi_0}{2} B_0^2+i \tilde{c}_0 \partial_\mu(\delta_{\mathrm{B}} A_{0\mu}) \biggr\}.
\end{aligned}
$$

Moreover, if the auxiliary fields $B_{0\mu}$ and B_0 are integrated out, the gauge-fixing terms become[9]

$$
\int d^D x\left\{\frac{1}{2 \xi_0}\left(\partial_\mu A_{0\mu}\right)^2+\frac{1}{2 \zeta_0} \chi_{0\mu} N_{\mu\nu} \chi_{0\nu}\right\}. \tag{10-28}
$$

Note here that there is no interaction with the conformal-factor field ϕ in the gauge-fixing terms and the ghost actions.

The renormalization factors of the gauge-fixing parameters are defined as $\xi_0 = Z_3 \xi$ and $\zeta_0 = Z_h \zeta$, so that the counterterm for each kinetic term

[8] In any background $\hat{g}_{\mu\nu}$, the ghost field is defined by the superscript c_0^μ and its subscript is given by $c_{0\mu} = \hat{g}_{\mu\nu} c_0^\nu$ (see the second section of Chapter 7). The BRST transformation of the ghost field is expressed as $\delta_{\mathrm{B}} c_0^\mu = t_0 c_0^\nu \hat{\nabla}_\nu c_0^\mu = t_0 c_0^\nu \partial_\nu c_0^\mu$ from the Grassmannian property, thus it does not depend on how to choose the background metric.

[9] When $B_{0\mu}$ is integrated out, a determinant $\det^{-1/2}(N_{\mu\nu})$ appears. If considered in a curved background as in the background field method, it is necessary to evaluate this determinant.

is given in the gauge-invariant form because (10-28) is written in the renormalized quantities only. We also need to introduce renormalization factors to the ghost fields.

The equation of motion of the traceless tensor field including the gauge-fixing term (10-28) is denoted as $K^{(\zeta)}_{\mu\nu,\lambda\sigma}(k)h_{\lambda\sigma}(k) = 0$ in momentum space, where

$$K^{(\zeta)}_{\mu\nu,\lambda\sigma}(k) = \frac{2(D-3)}{D-2}\Bigg\{ I^{\mathrm{H}}_{\mu\nu,\lambda\sigma}k^4 + \frac{1-\zeta}{\zeta}\bigg[\frac{1}{2}k^2\big(\delta_{\mu\lambda}k_\nu k_\sigma + \delta_{\nu\lambda}k_\mu k_\sigma + \delta_{\mu\sigma}k_\nu k_\lambda + \delta_{\nu\sigma}k_\mu k_\lambda\big) - \frac{1}{D-1}k^2\big(\delta_{\mu\nu}k_\lambda k_\sigma + \delta_{\lambda\sigma}k_\mu k_\nu\big) + \frac{1}{D(D-1)}\delta_{\mu\nu}\delta_{\lambda\sigma}k^4 - \frac{D-2}{D-1}k_\mu k_\nu k_\lambda k_\sigma\bigg]\Bigg\}$$

and I^{H} is a projection tensor

$$I^{\mathrm{H}}_{\mu\nu,\lambda\sigma} = \frac{1}{2}\left(\delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda}\right) - \frac{1}{D}\delta_{\mu\nu}\delta_{\lambda\sigma}$$

which satisfies $(I^{\mathrm{H}})^2 = I^{\mathrm{H}}$. The propagator is given by its inverse defined as $K^{(\zeta)}_{\mu\nu,\lambda\sigma}(k)\langle h_{\lambda\sigma}(k)h_{\rho\kappa}(-k)\rangle = I^{\mathrm{H}}_{\mu\nu,\rho\kappa}$, which is expressed as

$$\langle h_{\mu\nu}(k)h_{\lambda\sigma}(-k)\rangle = \frac{D-2}{2(D-3)}\frac{1}{k^4}I^{(\zeta)}_{\mu\nu,\lambda\sigma}(k). \tag{10-29}$$

Reflecting symmetric and traceless properties of the field, the tensor in the numerator is given by

$$I^{(\zeta)}_{\mu\nu,\lambda\sigma}(k) = I^{\mathrm{H}}_{\mu\nu,\lambda\sigma} + (\zeta-1)\Bigg[\frac{1}{2}\left(\delta_{\mu\lambda}\frac{k_\nu k_\sigma}{k^2} + \delta_{\nu\sigma}\frac{k_\mu k_\lambda}{k^2} + \delta_{\mu\sigma}\frac{k_\nu k_\lambda}{k^2} + \delta_{\nu\lambda}\frac{k_\mu k_\sigma}{k^2}\right) - \frac{1}{D-1}\left(\delta_{\mu\nu}\frac{k_\lambda k_\sigma}{k^2} + \delta_{\lambda\sigma}\frac{k_\mu k_\nu}{k^2}\right) + \frac{1}{D(D-1)}\delta_{\mu\nu}\delta_{\lambda\sigma} - \frac{D-2}{D-1}\frac{k_\mu k_\nu k_\lambda k_\sigma}{k^4}\Bigg].$$

This reduces to a simple form of only the projection tensor when $\zeta = 1$. We call this choice the Feynman gauge. Also, the transverse condition is given by

$$k_\mu I^{(\zeta)}_{\mu\nu,\lambda\sigma}(k) = \zeta\left(\frac{1}{2}k_\lambda\delta_{\nu\sigma} + \frac{1}{2}k_\sigma\delta_{\nu\lambda} - \frac{1}{D}k_\nu\delta_{\lambda\sigma}\right),$$

and thus it becomes $k_\mu I^{(0)}_{\mu\nu,\lambda\sigma}(k) = 0$ when $\zeta = 0$. This choice is called the Landau gauge.

Calculations of Renormalization Factors

In this section, we will describe renormalization of ultraviolet divergences by showing various concrete calculations. As can be seen below, in the renormalizable quantum gravity, loop expansion does not become the $\hbar$-expansion. This can be seen from the fact that the Riegert action, which is the kinetic term of the conformal-factor field, appears as a quantum effect. As mentioned earlier, it also originates from the fact that the fourth-order gravitational actions in four dimensions including the Weyl action are completely dimensionless functions given by the zeroth order of $\hbar$.

In order to handle infrared divergences, we regularize them by adding an infinitesimal mass z $(\ll 1)$ to the gravitational field. In other words, the momentum dependence of the denominators of the propagators of ϕ (10-22) and $h_{\mu\nu}$ (10-29) is replaced as

$$\frac{1}{k^4} \to \frac{1}{k_z^4} = \frac{1}{(k^2+z^2)^2}.$$

Infrared divergences then appear in the form of $\log z^2$. This is the same way as introducing a photon mass when dealing with infrared divergences in QED. Since this mass term is not gauge-invariant, the infrared divergence finally shall cancel out and disappear.[10]

Here it is noted that the Einstein term and the cosmological term cannot be regarded as mass terms of the fourth-order gravitational field. In the fourth-order derivative actions, the ϕ-dependences appear as polynomials in the expansion of the coupling constant t, while these lower derivative actions have a exponential factor of ϕ even at the vanishing limit of t. Since these lower derivative actions become diffeomorphism invariant with including this conformal factor, mass terms defined by a quadratic term of the field are not gauge-invariant. In addition, it is suggested that the presence of the conformal factor renders dependency of mass parameters such as the Planck mass a power-law. Renormalization of such composite fields will be discussed in the penultimate section.

In the following, calculations will be carried out by taking the spacetime dimension as

$$D = 4 - 2\epsilon.$$

The propagators of the conformal-factor field ϕ and the traceless tensor field $h_{\mu\nu}$ are described by solid and spiral lines, respectively. The $U(1)$ gauge

[10] By checking that the infrared divergences actually cancel out, we can confirm that calculations are going successfully in the diffeomorphism invariant way.

field, namely photon is represented by a wavy line. The fermion is a dotted line with an arrow. Because loop calculations including the gravitational field are very complicated, in most cases we will write only the results.

Non-renormalization theorem (I) We first calculate corrections of the order of $\alpha_t = t^2/4\pi$ for two-point functions of the conformal-factor field given by Feynman diagrams in Fig. 10-1, and show that this field does not actually receive renormalization.

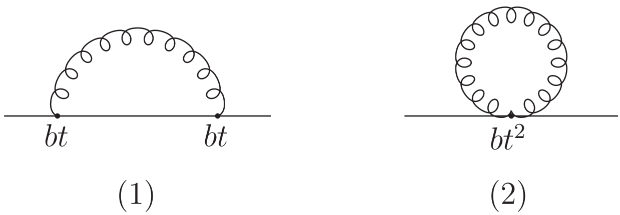


Figure 10-1: Loop corrections to the conformal-factor field at $o(\alpha_t)$.

Corrections from the diagram (1) in Fig. 10-1 are calculated using the interaction term $S^{bt}_{\mathrm{G}[\phi\phi h]}$ (10-25) in the Feynman gauge ($\zeta = 1$) as

$$\int \frac{d^D k}{(2\pi)^D}\,\phi(k)\phi(-k)\biggl\{-\frac{D-2}{2(D-3)}\frac{b}{6}\frac{t^2}{(4\pi)^{D/2}}\int\frac{d^D l}{(2\pi)^D}\frac{1}{l_z^4(l+k)_z^4}$$
$$\times\Bigl[6(l^2k^6+l^6k^2)+24l^4k^4-16(l\cdot k)(l^2k^4+l^4k^2)$$
$$-20(l\cdot k)^2l^2k^2-2(l\cdot k)^2(l^4+k^4)+8(l\cdot k)^3(l^2+k^2)+8(l\cdot k)^4$$
$$+\frac{4-D}{3D}\Bigl(-36l^4k^4+24(l\cdot k)(l^2k^4+l^4k^2)+40(l\cdot k)^2l^2k^2$$
$$-4(l\cdot k)^2(l^4+k^4)-16(l\cdot k)^3(l^2+k^2)-16(l\cdot k)^4\Bigr)\Bigr]\biggr\}.$$

Evaluating the momentum integral of l with the condition of $z \ll 1$ using integral formulas in the first section of Appendix D, the inside the braces is calculated as[11]

$$\frac{\mu^{D-4}}{(4\pi)^{D/2}}\,2bk^4\left[-3\frac{\alpha_t}{4\pi}\left(\frac{1}{\bar{\epsilon}}-\log\frac{z^2}{\mu^2}+\frac{7}{6}\right)\right],$$

[11] The infrared divergence is rewritten in the form of $\log(z^2/\mu^2)$ with attention to $1 = \mu^{D-4}\mu^{4-D} \cong \mu^{D-4}(1+\epsilon\log\mu^2)$.

where $1/\bar{\epsilon} = 1/\epsilon - \gamma + \log 4\pi$. In this case, nonlocal terms like $\log(k^2/\mu^2)$ are canceled out and do not appear.

Corrections from the tadpole diagram (2) in Fig. 10-1 can be calculated easily using the interaction $S^{bt^2}_{\mathrm{G}[\phi\phi hh]}$ (10-26). Here, note that unlike second-order derivative theories, fourth-order derivative theories have ultraviolet divergences even in such tadpole diagrams. Since contributions from interaction terms involving derivatives of $h_{\mu\nu}$ disappear at $z \to 0$ after carrying out the momentum integration, only the two terms described in (10-26) contribute. Thus, we obtain

$$\frac{\mu^{D-4}}{(4\pi)^{D/2}} 2bk^4 \left[3\frac{\alpha_t}{4\pi} \left(\frac{1}{\bar{\epsilon}} - \log\frac{z^2}{\mu^2} + \frac{7}{12} \right) \right].$$

Combining the corrections from these two Feynman diagrams, we can see that not only the ultraviolet divergences but also the infrared divergences cancel each other. Thus, $Z_\phi = 1$ (10-5) is shown at $o(\alpha_t)$.

The calculation has been done in arbitrary gauge, and we finally obtain the following finite correction:

$$\frac{\alpha_t}{4\pi} \left[-\frac{7}{4} + \frac{1}{3}(\zeta - 1) \right] \frac{\mu^{D-4}}{(4\pi)^{D/2}} 2b \int \frac{d^D k}{(2\pi)^D} \phi(k) k^4 \phi(-k).$$

Renormalization for traceless tensor field In order to determine the renormalization factor of the traceless tensor field, it is necessary to calculate two-point and three-point functions as in Fig. 10-2. In the counterterm for the Weyl action, there are two- and three-point functions of the traceless tensor field, but in the countertrem proportional to $\bar{G}_4$ that is necessary for determining the residue b_n, there are only three-point functions or more.[12]

As an example of calculations for two-point functions of the traceless tensor field, consider a one-loop correction in Fig. 10-3 in which the fermion propagates inside. From the interaction (10-27), it is calculated as[13]

$$\frac{\mu^{4-D}}{32} t^2 \int \frac{d^D k}{(2\pi)^D} h_{\mu\nu}(k) h_{\lambda\sigma}(-k) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p+k)^2}$$

---

[12] In the flat background, the $o(h^2)$ term of $\bar{G}_4$ becomes a total divergence in any dimension (see (A-4) in the first section of Appendix A).

[13] The form of conformal anomaly $F_4 + 2\nabla^2 R/3$ initially introduced by Duff is determined by performing a conformal variation to the effective action obtained when written in terms of the four-dimensional nonlocal Weyl action and the local R^2 action after eliminating the divergence from this expression. Actually, however, it is correct to include the last term in the D-dimensional Weyl action, not in R^2.

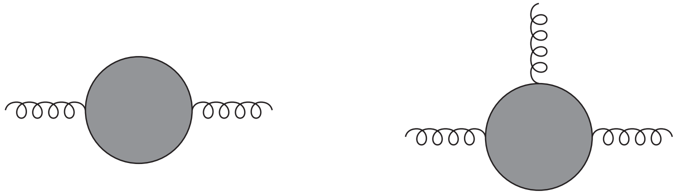


Figure 10-2: Loop corrections to two- and three-point functions by the traceless tensor field.

$$\times \mathrm{tr}(\gamma_\alpha \gamma_\mu \gamma_\beta \gamma_\lambda)\, p_\alpha (p+k)_\beta (2p+k)_\nu (2p+k)_\sigma$$
$$= \frac{\alpha_t}{4\pi} \int \frac{d^D k}{(2\pi)^D}\, h_{\mu\nu}(k) h_{\lambda\sigma}(-k) \left\{ -\frac{1}{40} \left(\frac{1}{2}\delta_{\mu\lambda}\delta_{\nu\sigma} k^4 - \delta_{\mu\lambda} k_\nu k_\sigma k^2 + \frac{1}{3} k_\mu k_\nu k_\lambda k_\sigma \right) \left(\frac{1}{\bar{\epsilon}} - \log \frac{k^2}{\mu^2} + \frac{12}{5} \right) + \frac{1}{360} k_\mu k_\nu k_\lambda k_\sigma \right\}.$$

Writing this with the D-dimensional expression of the Weyl action, we get[14]

$$\frac{\alpha_t}{4\pi} \int \frac{d^D k}{(2\pi)^D}\, h_{\mu\nu}(k) h_{\lambda\sigma}(-k) \left\{ -\frac{1}{40} \left(\frac{1}{\bar{\epsilon}} - \log \frac{k^2}{\mu^2} + \frac{17}{5} \right) \times \left[\frac{D-3}{D-2} \left(\delta_{\mu\lambda}\delta_{\nu\sigma} k^4 - 2\delta_{\mu\lambda} k_\nu k_\sigma k^2 \right) + \frac{D-3}{D-1} k_\mu k_\nu k_\lambda k_\sigma \right] \right\}.$$

In order to cancel out the ultraviolet divergence, the renormalization factor of the traceless tensor field defined by (10-3) is determined as

$$Z_h = 1 + \frac{1}{40} \frac{\alpha_t}{4\pi} \frac{1}{\epsilon}.$$

In the case of the diagram in which fermions propagate inside, since the result is independent of the gauge choice, the relation $Z_t Z_h^{1/2} = 1$ representing the gauge invariance holds. Using this relation, the fermion one-loop contribution to Z_t is determined to be $-(1/80)(\alpha_t/4\pi)(1/\epsilon)$.

Similarly, we can calculate two-point functions in which the conformal-factor field propagates inside. Feynman diagrams which may cause ultraviolet divergences are given in Fig. 10-4, in which the three- and four-point

[14] The quantity obtained by applying $h_{\mu\nu}(k)h_{\lambda\sigma}(-k)$ to the momentum function inside the square brackets is $h_{\mu\nu}(k)h_{\lambda\sigma}(-k)A_{\mu\nu,\lambda\sigma}(k)/8$ when using (9-20).

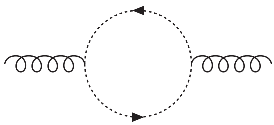


Figure 10-3: Loop corrections by the fermion.

interactions $S^{bt}_{\mathrm{G}[\phi\phi h]}$ (10-25) and $S^{bt^2}_{\mathrm{G}[\phi\phi hh]}$ (10-26) contribute. However, since there is always a derivative on the conformal-factor field in the four-point interaction, the tadpole diagram (2) disappears. Contributions from the diagram (1), when writing only the results, are expressed in the form of the D-dimensional Weyl action with

$$\frac{\alpha_t}{4\pi}\frac{1}{30}\left(\frac{1}{\bar{\epsilon}} - \log\frac{k^2}{\mu^2} + \frac{289}{60}\right),$$

as in the case of the fermion. Infrared divergences are canceled out within the calculation of the diagram (1). From this, the contribution to Z_h becomes $(-1/30)(\alpha_t/4\pi)(1/\epsilon)$. Since the gauge-invariance relation $Z_t = Z_h^{-1/2}$ is established in this case as well, the contribution to Z_t can be calculated as $(1/60)(\alpha_t/4\pi)(1/\epsilon)$.

(1) (2)

Figure 10-4: Loop corrections by the conformal-factor field.

In general, it is difficult to calculate the renormalization factor of the traceless tensor field. One-loop corrections from diagrams in which the traceless tensor field propagates inside have been calculated by using the background field method[15] which is often used in renormalization of non-Abelian gauge fields or the gravitational field. Here write only the results,

[15] For the background field method, see L. Abbott, *The Background Field Method Beyond One*

the coupling constant renormalization factor defined by (10-4) is given by

$$Z_t = 1 - \left(\frac{n_F}{80} + \frac{5}{3} \right) \frac{\alpha_t}{4\pi} \frac{1}{\epsilon} - \frac{7n_F}{288} \frac{\alpha\alpha_t}{(4\pi)^2} \frac{1}{\epsilon} + o(\alpha_t^2). \quad (10\text{-}30)$$

The contribution of $o(\alpha_t)$ is yielded by the sum of $-n_F/80$, $-1/40$, $1/60$, and $-199/120$ from the fermion, the $U(1)$ gauge field and its ghost field, the conformal-factor field, and the traceless tensor field and its ghost field, respectively. The contribution of $o(\alpha\alpha_t)$ is from two-loop Feynman diagrams in which only the fermion and the gauge field propagate inside. From this result, the beta function of the coupling constant α_t is determined as (10-16).

Here, we give a little explanation on the background field method. In this method, if defining the renormalization factor $Z_{\hat{h}}$ as $\hat{h}_{0\mu\nu} = Z_{\hat{h}}^{1/2} \hat{h}_{\mu\nu}$ for the background field $\hat{g}_{\mu\nu} = (e^{t_0 \hat{h}_0})_{\mu\nu}$, the relation $Z_t Z_{\hat{h}}^{1/2} = 1$ holds at all orders as a gauge invariance condition so that the product $t_0 \hat{h}_{0\mu\nu}$ becomes a renormalized quantity. Thus, an advantage of the background field method is that although the normal renormalization factor Z_h is gauge-dependent, the background field renormalization factor $Z_{\hat{h}}$ becomes gauge-invariant obviously. Therefore, we can obtain Z_t by calculating $Z_{\hat{h}}$. In this way, the one-loop correction in (10-30) has been calculated.

Non-renormalization theorem (II) From the calculation of the two-point function of the conformal-factor field ϕ, it has already been shown that this field does not receive renormalization at $o(\alpha_t)$. Here, using the result of the previous chapter, we demonstrate that $Z_\phi = 1$ (10-5) holds up to $o(\alpha^3)$.

First of all, we can easily find that two-point functions of ϕ at $o(\alpha)$ are obviously finite from the order of the induced interaction $\phi F_{\mu\nu}^2$ in (10-17) that contributes to the calculations.

Next, consider quantum corrections of $o(\alpha^2)$, whose Feynman diagrams are given in Fig. 10-5. The three-point interaction with $n_F e^2$ is derived from the residue $x_1 = (8n_F/3)(\alpha/4\pi)$. The part written as "2lp" in the circle denotes the ordinary two-loop photon self-energy diagram in QED. Diagrams that include counterterms as a part in order to eliminate divergences in subdiagrams are not depicted for the sake of simplicity. At $o(\alpha^2)$, there is no counterterm other than that prepared for subdiagrams. It is because since the double pole reside b_2 appears at $o(\alpha^3)$ as shown before, a simple pole coun-

Loop, Nucl. Phys. **B185** (1981) 189. For applications to the gravitational field, see E. Fradkin and A. Tseytlin, Nucl. Phys. **B201** (1982) 469; I. Antoniadis, P. Mazur and E. Mottola, Nucl. Phys. **B388** (1992) 627; K. Hamada and F. Sugino, Nucl. Phys. **B553** (1999) 283 and so on listed in Bibliography.

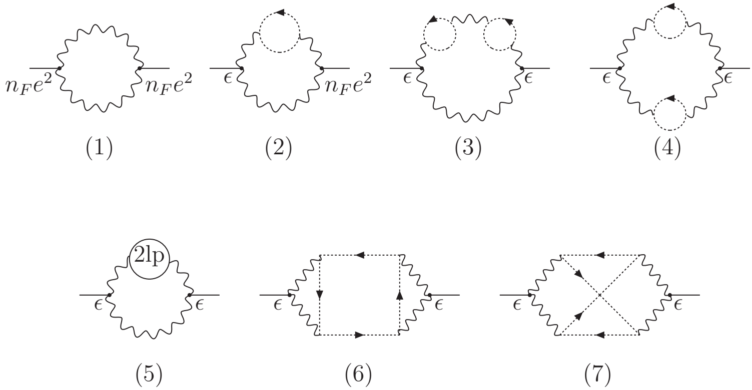


Figure 10-5: Loop corrections to the conformal-factor field at $o(\alpha^2)$.

terterm eliminating an entire ultraviolet divergence of the two-point function appears at $o(\alpha^3)$ as seen from the Laurent expansion (10-20) of G_D.

Regarding each Feynman diagram in Fig. 10-5, the diagram (5) is obviously finite because the two-loop photon self-energy diagram produces only a simple pole divergence which cancels out the ϵ present at the vertex of ϕ. Moreover, since the four-point function of photons including a fermion loop is finite, (6) and (7) also become finite. Therefore, diagrams in which a simple pole divergence occurs are from (1) to (4). Summing over these divergences yields a finite result.

In $o(\alpha^3)$ as well, we can demonstrate $Z_\phi = 1$ using the results in the previous chapter. The part to be noted in the calculation of $o(\alpha^3)$ is that, as already mentioned, since the double pole residue b_2 has a value at this order, a simple pole counterterm appears in the kinetic term of the conformal-factor field.

Ward-Takahashi identity The renormalization factor of the fermion field at $o(\alpha_t)$ is calculated from self-energy diagrams in Fig. 10-6. In the Feynman gauge, it is given by

$$Z_2 = 1 - \frac{21}{64}\frac{\alpha_t}{4\pi}\frac{1}{\epsilon}.$$

Contributions of $o(\alpha_t)$ to the vertex-function renormalization factor Z_1 are given by Fig. 10-7. By studying the relationship between them, we will show that the Ward-Takahashi identity $Z_1 = Z_2$ holds even if gravitational internal lines are included.

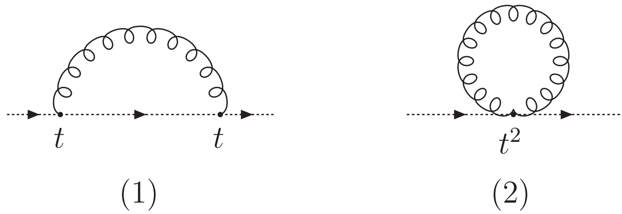


Figure 10-6: Loop corrections to Z_2 at $o(\alpha_t)$.

Let us examine the self-energy function derived from (1) in Fig. 10-6. Ignoring an overall coefficient for simplicity, it is given by

$$\Sigma^{(1)} = \gamma_\mu \frac{1}{\not{k} + \not{p}} \gamma_\lambda \frac{1}{k^4} I^H_{\mu\nu,\lambda\sigma} (k+2p)_\nu (k+2p)_\sigma,$$

where p is a momentum of the external fermion field and k is a momentum of the internal traceless tensor field. Since the momentum dependence of $k+2p$ arises from the gravitational interaction (10-27), a derivative of the self-energy function by the external momentum p is affected by this dependence, unlike the Ward-Takahashi identity in ordinary QED. Thus we obtain

$$\begin{aligned}\frac{\partial}{\partial p_\rho}\Sigma^{(1)} = &-\gamma_\mu \frac{1}{\not{k} + \not{p}} \gamma_\rho \frac{1}{\not{k} + \not{p}} \gamma_\lambda \frac{1}{k^4} I^H_{\mu\nu,\lambda\sigma} (k+2p)_\nu (k+2p)_\sigma \\ &+\gamma_\mu \frac{1}{\not{k} + \not{p}} \gamma_\lambda \frac{1}{k^4} I^H_{\mu\nu,\lambda\sigma} \big\{ 2\delta_{\nu\rho}(k+2p)_\sigma + 2\delta_{\sigma\rho}(k+2p)_\nu \big\}.\end{aligned}$$

The first term on the right-hand side corresponds to the diagram (1) in Fig. 10-7 in which momenta of external photons are taken to be zero. The second term corresponds to the sum of (2) and (3) in Fig. 10-7.

When the same operation is performed for the tadpole diagram (2) in Fig. 10-6, we can see that a term corresponding to the diagram (4) in Fig. 10-7 is produced. This reflects the fact that the interaction $\bar{\psi}\gamma_\mu\psi A_\nu (h^n)_{\mu\nu}$ in (10-27) can be generated by applying a replacement $p_\nu \to p_\nu - eA_\nu$ to the interaction $\bar{\psi}\gamma_\mu p_\nu \psi (h^n)_{\mu\nu}$. On the other hand, for other interaction terms including derivatives of the traceless tensor field in (10-27), the differential operator of p passes through them.

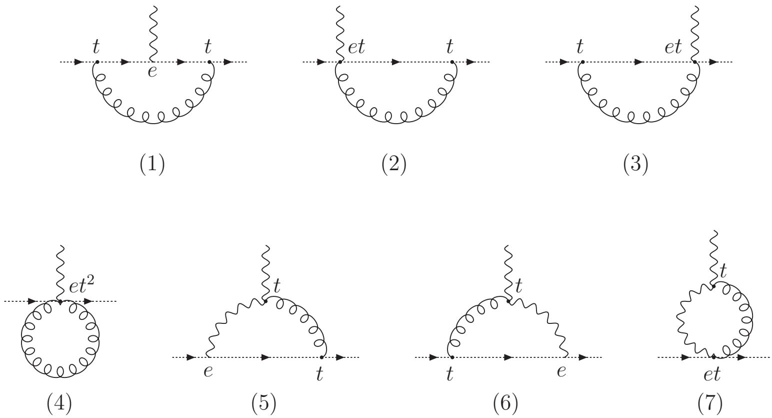


Figure 10-7: Loop correction to the vertex function renormalization factor Z_1 at $o(\alpha_t)$.

The diagrams from (5) to (7) in Fig. 10-7 which have no correspondence to those in Fig. 10-6 do not involve ultraviolet divergences. Furthermore, when momenta of external photons are taken to be zero, it can be seen that these contributions disappear due to gauge symmetry. In this way, we can show that the Ward-Takahashi identity $Z_1 = Z_2$ holds at $o(\alpha_t)$.

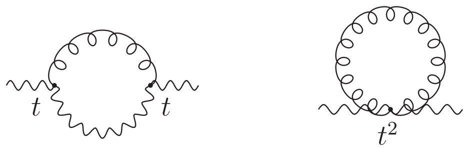
In general, if vertex function corrections have external photons coming from gravitational interactions of the type $\phi^n FF$ or $h^n FF$ such as (5) to (7) in Fig. 10-7, they disappear when external photon momenta vanish because photon fields appear in the form $F_{\mu\nu}$. Furthermore, as vertex function corrections which cannot be generated by the operation mentioned above, we need to consider diagrams such as (1) and (2) in Fig. 10-8 in which an external photon is directly connected to an internal closed fermion loop. In order that the Ward-Takahashi identity holds at all orders, they have to disappear when the external photon momentum vanishes.

The disappearance of vertex functions of the type (1) in Fig. 10-8 can be easily seen by using a generalized Furry's theorem: a fermion loop diagram with odd number of attached photons disappears regardless of the number of attached gravitational fields. It can be shown from the fact that when the charge conjugation is taken, photons change sign, but gravitational fields do not change. On the other hand, vertex functions of the type (2) in Fig. 10-8 do not vanish obviously, but the external photon field appears in the form of $F_{\mu\nu}$ after all because of gauge invariance. Therefore, it also disappears when the momentum is set to zero. In this way, $Z_1 = Z_2$ can be shown.

(1) (2)

Figure 10-8: Examples of vertex function corrections in which an external photon is attached to an internal closed fermion loop.

Renormalization of gauge field Let us calculate corrections to the renormalization factor Z_3 by the gravitational interactions. Contributions of $o(\alpha_t)$ where the traceless tensor field propagates inside are given by two Feynman diagrams in Fig. 10-9. Because ultraviolet divergences from the self-energy diagram (1) and the tadpole diagram (2) cancel each other, it is found that the correction at this order becomes finite.



(1) (2)

Figure 10-9: Loop corrections to Z_3 at $o(\alpha_t)$.

Feynman diagrams in which the conformal-factor field propagates inside appear from $o(\alpha^2/b)$. Among them, diagrams where simple poles occur are only three, given in Fig. 10-10. As mentioned earlier, Feynman diagrams including the two-loop QED photon self-energy diagram that only gives a simple pole are not depicted here because they become finite obviously. In addition, Feynman diagrams that generate a double pole at $o(\alpha^3/b)$ are given in Fig. 10-11.

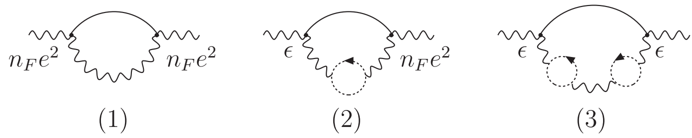


Figure 10-10: Feynman diagrams that yield a simple pole to Z_3 at $o(\alpha^2/b)$.

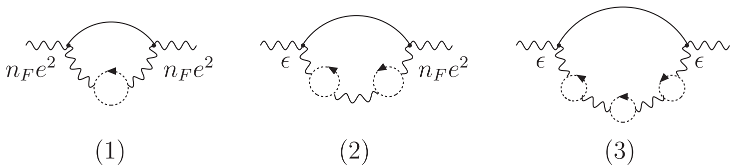


Figure 10-11: Feynman diagrams that yield a double pole to Z_3 at $o(\alpha^3/b)$.

Adding contributions from these Feynman diagrams to normal QED corrections, we obtain the following renormalization factor:

$$\begin{aligned} Z_3 = 1 &- \frac{4n_F}{3}\frac{\alpha}{4\pi}\frac{1}{\epsilon} + \left(-2n_F + \frac{8}{27}\frac{n_F^2}{b}\right)\frac{\alpha^2}{(4\pi)^2}\frac{1}{\epsilon} \\ &+ \left(-\frac{8n_F^2}{9} + \frac{8}{81}\frac{n_F^3}{b}\right)\frac{\alpha^3}{(4\pi)^3}\frac{1}{\epsilon^2} + o(\alpha\alpha_t, \alpha_t^2). \end{aligned} \tag{10-31}$$

From this result, we can read the residues $f_{1,2}$ (10-14), and the beta function is calculated as (10-15).

Non-renormalization theorem (III) [vertex function] Furthermore, we calculate vertex functions of the type $\phi F_{\mu\nu}^2$, and demonstrate that they can be renormalized by $Z_\phi = 1$, namely only information of Z_3.

Since the double pole in the renormalization factor Z_3 occurs at $o(\alpha^3)$, we find that a simple pole counterterm of the type $\phi F_{\mu\nu}^2$ is derived at $o(\alpha^3)$ from the Laurent expansion (10-17). Therefore, Feynman diagrams that produce ultraviolet divergences appear from this order.

Let us first consider the case where only the QED fields propagate inside. For simplicity, we perform the calculation by taking a momentum of the conformal-factor field to be zero. Feynman diagrams where ultraviolet divergences occur are depicted in Fig. 10-12. The diagrams (1) and (2) are derived by attaching the vertex $n_F e^2 \phi F_{\mu\nu}^2$ to the two-loop photon

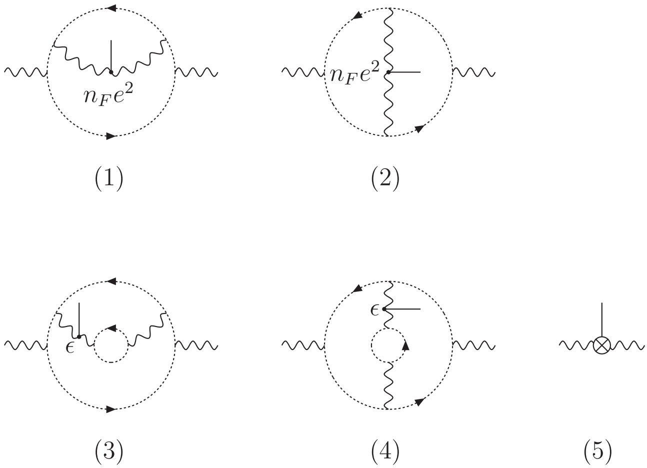


Figure 10-12: Loop corrections to the vertex function $\phi F_{\mu\nu}^2$ at $o(\alpha^3)$.

self-energy diagrams in QED. Since the two-loop photon self-energy gives a simple pole, the sum of these diagrams also gives a simple pole.

The diagrams (3) and (4) in Fig. 10-12 are derived by attaching the vertex $\epsilon\phi F_{\mu\nu}^2$ to the three-loop photon self-energy diagrams in QED. Since it is known that the three-loop self-energy diagrams that have two fermion loops are known to give at most a double pole, we find that the sum of these diagrams also gives a simple pole due to the ϵ at the vertex. The last diagram (5) is a simple pole counterterm that appears in the Laurent expansion (10-17) associated with the double pole of Z_3.

In addition, there are Feynman diagrams derived by attaching the vertex $\epsilon\phi F_{\mu\nu}^2$ to the three-loop photon self-energy diagrams with one fermion loop. However, the sum of such diagrams becomes finite obviously because the sum of the three-loop photon self-energy diagrams with one fermion loop produces at most a simple pole so that the ϵ at the vertex cancels out the pole. Therefore, such diagrams are omitted here.

Summing over ultraviolet divergences from these diagrams, we can show that the renormalized vertex function becomes finite as follows:

$$\Gamma_{\mu\nu}^{\phi AA}(0;k,-k)\big|_{o(\alpha^3)}^{\mathrm{div}} = \frac{1}{\epsilon}\left\{-\frac{8}{3}+\frac{16}{9}+\frac{8}{9}\right\} n_F^2\left(\frac{\alpha}{4\pi}\right)^3\left(\delta_{\mu\nu}k^2 - k_\mu k_\nu\right) = 0,$$

where the effective action is normalized as

$$\Gamma = \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} \, \phi(-k-l) A_\mu(k) A_\nu(l) \, \Gamma^{\phi AA}_{\mu\nu}(-k-l; k, l).$$

The first $-8/3$ is from the sum of (1) and (2) in Fig. 10-12, the second term $16/9$ is from (3) and (4), and the last $8/9$ is from the counterterm (5).

Finally, consider renormalization of the vertex function at $o(\alpha^3/b)$ where the conformal-factor field propagates inside. Feynman diagrams where ultraviolet divergences occur are drawn in Fig. 10-13. There are other Feynman diagrams including the two- and three-loop photon self-energy diagrams, but they become obviously finite, and thus they are not depicted. In the diagrams with internal lines of the conformal-factor field, there are contributions from the three-point self-interaction $S^{(D-4)b}_{\mathrm{G}[\phi\phi\phi]}$ (10-23) and the interaction of the type $\phi^2 F^2_{\mu\nu}$. Therefore, it becomes non-trivial verification of the action G_D (10-20) and the Laurent expansion of the gauge field action (10-17). Combining all of the contributions from Fig. 10-13, ultraviolet divergences cancel out and the effective action becomes finite as follows:

$$\begin{aligned}\Gamma^{\phi AA}_{\mu\nu}(0; k, -k)\big|^{\mathrm{div}}_{o(\alpha^3/b)} &= \frac{1}{\epsilon}\left\{-\frac{8}{81} + \frac{16}{81} - \frac{8}{81}\right\} \frac{n_F^3}{b} \left(\frac{\alpha}{4\pi}\right)^3 \left(\delta_{\mu\nu}k^2 - k_\mu k_\nu\right) \\ &= 0,\end{aligned}$$

where the first term is from the sum of (1) to (3) in Fig. 10-13, and the second term is from the sum of (10) to (13). The third term is the contribution from (14), which is a simple pole counterterm induced from the double pole of Z_3 at $o(\alpha^3/b)$. The sum of ultraviolet divergences from (4) to (9) becomes finite.

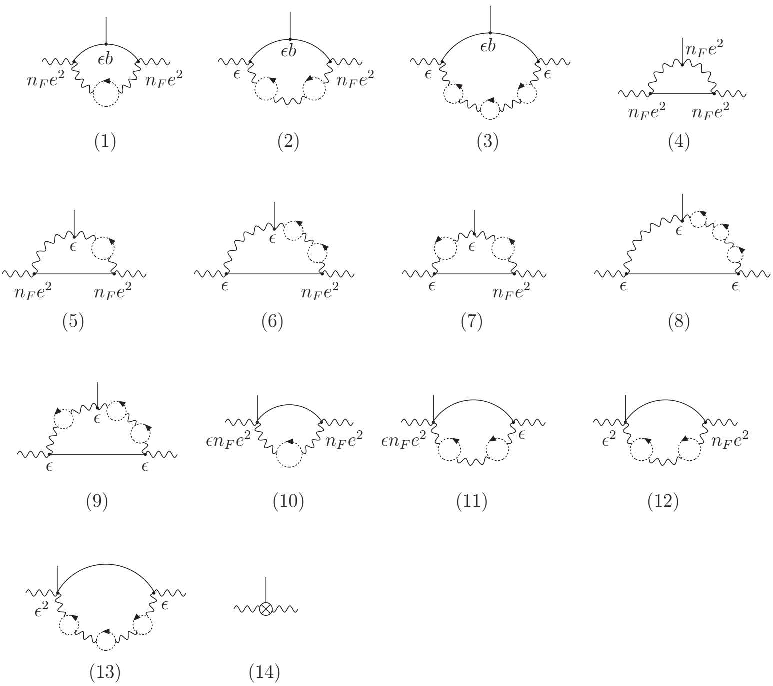


Figure 10-13: Loop corrections to the vertex function $\phi F_{\mu\nu}^2$ at $o(\alpha^3/b)$.

Background-Metric Independence Revisited

The background-metric independence is realized by treating the conformal-factor field ϕ non-perturbatively. In fact, a conformal change of the background metric $\hat{g}_{\mu\nu} \to e^{2\sigma}\hat{g}_{\mu\nu}$ is equivalent to a shift transformation of the conformal-factor field $\phi \to \phi + \sigma$. The shift is merely a change of integration variables, thus the independence is expressed as an invariance under the shift of the path integral measure. This means that the background-metric independence is a purely quantum mechanical symmetry that does not exist in the classical theory.

With this shift invariance, the background-metric independence can be expressed as

$$\int d\phi \frac{\delta}{\delta\phi(y)} \left(O(x) e^{-S} \right) = 0.$$

Given $\sqrt{g}\Theta = \delta S/\delta\phi$ as an operator O, we get

$$\langle \sqrt{g}\Theta(x)\sqrt{g}\Theta(y) \rangle - \left\langle \frac{\delta\sqrt{g}\Theta(x)}{\delta\phi(y)} \right\rangle = 0, \tag{10-32}$$

where $\Theta = \Theta_A + \Theta_\psi + \Theta_g$ is the trace of the energy-momentum tensor. Each term is given by $\Theta_A = (D-4)F_{0\mu\nu}F_0^{\mu\nu}/4$, $\Theta_\psi = (D-1)\sum_{j=1}^{n_F} i\bar{\psi}_{0j} \overset{\leftrightarrow}{\not{D}} \psi_{0j}$, and $\Theta_g = (D-4)(C^2_{\mu\nu\lambda\sigma}/t_0^2 + b_0 E_D)$. Here, it is significant that the right-hand side of (10-32) vanishes, unlike the case in curved space discussed in the previous chapter.

Let us see that the equation (10-32) actually holds with attention to the role of the conformal-factor field ϕ. First, we will confirm it directly by performing calculations on the flat background. Expanding $\sqrt{g}\Theta$ with $D-4$, the gauge field part is given by

$$\begin{aligned} \sqrt{g}\Theta_A &= (D-4)Z_3 \frac{1}{4} e^{(D-4)\phi} F_{\mu\nu}F_{\mu\nu} \\ &= \frac{D-4}{4}\left[1 + \frac{f_1}{D-4} + (D-4+f_1)\,\phi + \cdots \right] F_{\mu\nu}F_{\mu\nu}, \end{aligned} \tag{10-33}$$

and the gravitational part that depends on b is given by

$$\begin{aligned} \sqrt{g}\Theta_g = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \Big\{ & 4b\partial^4\phi + (D-4)b\left[2\phi\partial^4\phi + \partial^4(\phi^2)\right] \\ & + (D-4)^2 b \Big[\phi^2\partial^4\phi + \frac{1}{3}\partial^4(\phi^3) \\ & -2\left(8\chi_3 - 1\right)\partial_\lambda\left(\partial_\lambda\phi\partial_\sigma\phi\partial_\sigma\phi\right) \Big] + \cdots \Big\}. \end{aligned} \tag{10-34}$$

In these expansions, interactions with the traceless tensor field are omitted. The gravitational part derived from the Weyl action is also given by $(D-4)e^{(D-4)\phi}\bar{C}^2_{\mu\nu\lambda\sigma}/t_0^2$. This part is difficult to handle because it involves self-interactions of the traceless tensor field, but since it has a structure similar to (10-33), we will be able to infer the behavior of this part from the gauge field part discussed below.

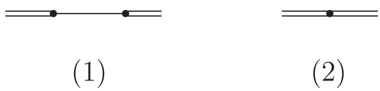


Figure 10-14: Tree diagrams indicating the background-metric independence.

For each order, we specifically demonstrate that the equation (10-32) holds. A contribution of $o(b)$ of the two-point correlation function of $\sqrt{g}\Theta$ is given by the tree diagram (1) in Fig. 10-14, where $\sqrt{g}\Theta$ is denoted by a double line. From the first term of the expansion (10-34), it is easily calculated as $4b\mu^{D-4}k^4/(4\pi)^{D/2}$ using the propagator (10-22). The second term of (10-32) represented by the diagram (2) in Fig. 10-14 can be calculated from the first term in (10-34) as well, which cancels out the contribution from the diagram (1), and thus the equation holds at $o(b)$.

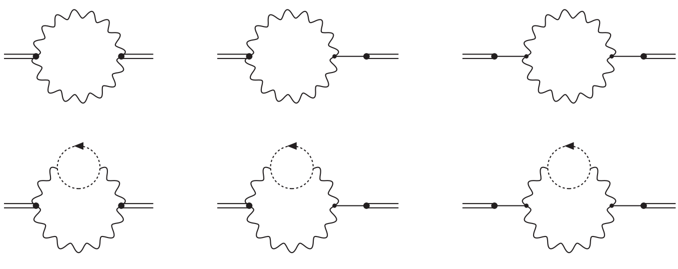


Figure 10-15: Feynman diagrams representing the background-metric independence at $o(1)$ and $o(\alpha)$ with internal lines of the QED fields.

Moreover, we can show that (10-32) holds at $o(1)$ and $o(\alpha)$ using the expansion (10-34) and the interaction (10-20), as in Figs. 10-15 and 10-16. The last tadpole diagram in Fig. 10-16 represents a contribution from the second term of (10-32), in which there is no contribution of $o(\alpha)$. This is consistent with the result (10-11) that there is no $o(\alpha)$ term in the residue b_1. For each order considered here, the equation holds regardless of the value

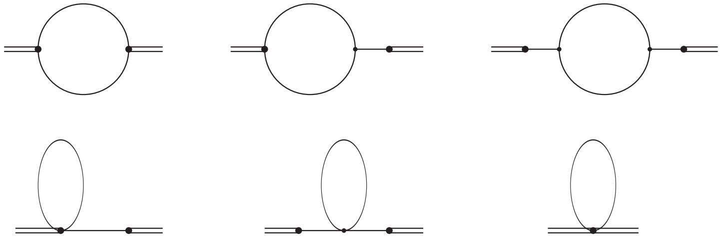


Figure 10-16: Feynman diagrams representing the background-metric independence at $o(1)$ with internal lines of the conformal-factor field.

of χ_3.

Finally, we show in a more obvious way that the equation (10-32) represents the background-metric independence. Consider the trace of another energy-momentum tensor $\sqrt{\hat{g}}\hat{\Theta} = \delta S/\delta\sigma$ which is yielded by doing a variation of the action with respect to the background metric, where $\delta/\delta\sigma = 2\hat{g}_{\mu\nu}\delta/\delta\hat{g}_{\mu\nu}$. Since the change in ϕ can be expressed as a change in σ in quantities written in terms of the original metric $g_{\mu\nu}$, the relationship $\sqrt{\hat{g}}\hat{\Theta} = \sqrt{g}\Theta$ is established, except for parts derived from the gauge-fixing term. Therefore, taking gauge invariance into account, (10-32) can be expressed in terms of the σ-variation as $\delta^2\langle 1\rangle/\delta\sigma(x)\delta\sigma(y) = 0$, because the variation by the background metric can be put outside the correlation function. In this way, we can show that the partition function does not depend on how to choose the background σ.

Diffeomorphism Invariant Effective Actions

In this section, from considerations of the effective action, we will see that the Wess-Zumino action associated with conformal anomaly comes out to ensure diffeomorphism invariance.

We first describe the relationship between conformal anomaly and the beta function in QED again. Carrying out renormalization, a scale appears in the effective action through a non-local term that takes the form of $\log(k^2/\mu^2)$ in momentum space.

The effective action of QED including a dependence of the conformal-factor field is given as

$$\Gamma_{\rm QED} = \left\{ 1 - \frac{\bar{\beta}}{2} \log\left(\frac{k^2}{\mu^2}\right) + x_1\phi + 4n_F \frac{\alpha^2}{(4\pi)^2}\phi \right\} \frac{1}{4}\bar{F}^2_{\mu\nu}(k).$$

The third term on the right-hand side is the Wess-Zumino action induced by the residue x_1. The fourth term is a finite contribution coming from diagrams in Fig. 10-17. Since the two-loop photon self-energy has only a simple pole, it becomes finite by the cancellation of the pole and the ϵ at the vertex. For simplicity, we are considering only the zero-mode part of ϕ here. Performing the ϕ-variation of the effective action, the conformal anomaly is found and its coefficient is proportional to the beta function as

$$\delta_\phi \Gamma_{\rm QED} = \left(x_1 + 4n_F \frac{\alpha^2}{(4\pi)^2} \right) \frac{1}{4}\sqrt{g}F^2_{\mu\nu} = \bar{\beta}\frac{1}{4}\sqrt{g}F^2_{\mu\nu}.$$

This is the equation of the conformal anomaly obtained in the previous chapter. Note that the coefficient of the $o(\alpha^2)$ term in $\bar{\beta}$ is twice that of the residue x_1.

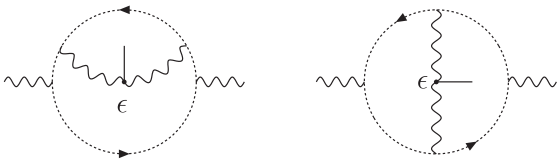


Figure 10-17: Finite loop corrections to the vertex function $\phi F^2_{\mu\nu}$ at $o(\alpha^2)$.

Recall that the momentum squared is defined by k^2 $(= k_\mu k_\nu \delta^{\mu\nu})$ in the flat background. Here we introduce a physical momentum squared defined on the full metric $g_{\mu\nu}$ $(= e^{2\phi}\delta_{\mu\nu})$ as

$$k^2_{\rm phy} = \frac{k^2}{e^{2\phi}}. \tag{10-35}$$

By using this, the effective action can be written as a diffeomorphism invariant form

$$\Gamma_{\rm QED} = \left\{ 1 - \frac{\bar{\beta}}{2} \log\left(\frac{k^2_{\rm phy}}{\mu^2}\right) \right\} \frac{1}{4}\sqrt{g}F^2_{\mu\nu}.$$

Thus, conformal anomalies are quantities in accompany with the scale that appears in the process of renormalization, and the Wess-Zumino actions occur to make non-local terms diffeomorphism invariant. Consequently, conformal anomalies are physically indispensable ingredients to preserve diffeomorphism invariance, unlike usual gauge anomalies. The interaction of the type $\phi^n F_{\mu\nu}^2$ will correspond to a nonlocal term $\log^n(k^2/\mu^2)$ associated with higher order corrections.

The same holds true for the Weyl action. The non-local term $\log(k^2/\mu^2)$ and the Wess-Zumino action $\phi C_{\mu\nu\lambda\sigma}^2$ are generated with carrying out renormalization. Letting the beta function be $\bar{\beta}_t = -8\pi\beta_0\alpha_t$ $(\beta_0 > 0)$,[16] the effective action is given by

$$\begin{aligned}\Gamma_{\mathrm{W}} &= \left\{\frac{1}{t^2} - 2\beta_0\phi + \beta_0 \log\left(\frac{k^2}{\mu^2}\right)\right\} \bar{C}_{\mu\nu\lambda\sigma}^2 \\ &= \frac{1}{\tilde{t}^2(k_{\mathrm{phy}}^2)}\sqrt{g}C_{\mu\nu\lambda\sigma}^2.\end{aligned}$$

The function $\tilde{t}(k_{\mathrm{phy}}^2)$, which puts together the inside the braces, is the running coupling constant expressed as

$$\tilde{t}^2(k_{\mathrm{phy}}^2) = \frac{1}{\beta_0 \log(k_{\mathrm{phy}}^2/\Lambda_{\mathrm{QG}}^2)}, \tag{10-36}$$

where k_{phs}^2 is the physical momentum squared (10-35). The dynamical infrared energy scale is defined as $\Lambda_{\mathrm{QG}} = \mu \exp\{-1/(2\beta_0 t^2)\}$. This is a physical scale which is renormalization group invariant as $\mu d\Lambda_{\mathrm{QG}}/d\mu = 0$. The Wess-Zumino action of the type $\phi^n C_{\mu\nu\lambda\sigma}^2$ will correspond to the nonlocal term $\log^n(k^2/\mu^2)$ as well.

Next, we derive a diffeomorphism invariant effective action which produces the conformal anomaly E_4. Consider the effective action that occurs together with the simple pole divergence of the residue b_1 proportional to the Euler density $\bar{G}_4$. Since $\bar{G}_4$ does not have quadratic terms, Feynman diagrams to determine it are given by three-point functions of the traceless tensor field, and the form of their finite parts will be

$$W_{\mathrm{G}}(\bar{g}) = \frac{b_c}{(4\pi)^2}\int d^4x \left\{\frac{1}{8}\bar{E}_4\frac{1}{\bar{\Delta}_4}\bar{E}_4 - \frac{1}{18}\bar{R}^2\right\}$$

at the lowest order of b_1. The term proportional to $\bar{R}^2$ guarantees that W_G does not have quadratic terms of the traceless tensor field when it expands in the flat background.

[16] The coefficient β_0 is that when the beta function is defined as $\mu dt/d\mu = -\beta_0 t^3 + o(t^5)$. This convention is used in Chapters 7, 12, and 14.

The effective action is given by the sum of the Riegert action $S_{\rm R}$ (10-21) and this finite part W_G, and is expressed as

$$\Gamma_{\rm R} = S_{\rm R}(\phi, \bar{g}) + W_{\rm G}(\bar{g}) = \frac{b_c}{8(4\pi)^2}\int d^4x\sqrt{g}\, E_4 \frac{1}{\Delta_4} E_4.$$

The $\bar{R}^2$ terms cancel out, thus it becomes a diffeomorphism invariant form. In this way, the scale-invariant non-local Riegert action (5-16) given in Chapter 5 is obtained.

Furthermore, consider the case where in the coefficient b_1 of the Riegert action there are higher order corrections that depend on the coupling constant t. If we replace t with the running coupling constant and expand it like $\tilde{t}^2(k^2_{\rm phy}) = t^2 + 2\beta_0 t^4\phi - \beta_0 t^4 \log(k^2/\mu^2) + \cdots$, we find from its ϕ-dependence that the $\phi^2\bar{\Delta}_4\phi$ term is generated. This suggests that the $\phi^n\bar{\Delta}_4\phi$ $(n \geq 2)$ term appears so as to ensure that the coupling constant can be replaced with the running coupling constant in the effective action.

Renormalization of Mass Parameters

Renormalization calculations for the Einstein term and the cosmological term are performed, and anomalous dimensions of the Planck mass and the cosmological constant are derived here.

Since the conformal-factor field ϕ is strictly handled without introducing its own coupling constant, the cosmological term is expressed by its exponential as

$$S_\Lambda = \Lambda_0 \int d^Dx\sqrt{g} = \Lambda_0 \int d^Dx\, e^{D\phi}.$$

The Einstein-Hilbert action is expanded up to $o(h^2)$ as follows:

$$\begin{aligned}
S_{\rm EH} &= -\frac{M_0^2}{2}\int d^Dx\sqrt{g}\, R \\
&= -\frac{M_0^2}{2}\int d^Dx\, e^{(D-2)\phi}\left\{\bar{R} - (D-1)\bar{\nabla}^2\phi\right\} \\
&= \frac{3}{2}M_0^2\int d^Dx\, e^{(D-2)\phi}\left\{\frac{D-1}{3}\partial^2\phi \right. \\
&\quad + \frac{D-2}{3}t_0 h_{0\mu\nu}\Big(-\partial_\mu\partial_\nu\phi + \partial_\mu\phi\partial_\nu\phi\Big) \\
&\quad + \frac{D-1}{6}t_0^2 h_{0\mu\lambda}h_{0\nu\lambda}\partial_\mu\partial_\nu\phi + \frac{D-1}{6}t_0^2 h_{0\mu\nu}\partial_\mu h_{0\nu\lambda}\partial_\lambda\phi
\end{aligned}$$

$$-\frac{D-3}{6}t_0^2 h_{0\mu\nu}\chi_{0\mu}\partial_\nu\phi + \frac{t_0^2}{12}\partial_\lambda h_{0\mu\nu}\partial_\lambda h_{0\mu\nu} - \frac{t_0^2}{6}\chi_{0\mu}\chi_{0\mu}\Bigg\}. \tag{10-37}$$

Recall that the conformal-factor field is not renormalized, namely $Z_\phi = 1$. Therefore, renormalization can be done by replacing the bare Planck mass and the bare cosmological constant with

$$\begin{aligned} M_0^2 &= \mu^{D-4} Z_{\mathrm{EH}} M^2, \\ \Lambda_0 &= \mu^{D-4} Z_\Lambda \left(\Lambda + L_M M^4\right), \end{aligned}$$

where M is a renormalized Planck mass with dimension 1 and Λ is a renormalized cosmological constant with dimension 4. Their renormalization factors are denoted by Z_{EH} and Z_Λ, while L_M is a term with only poles.

An anomalous dimension of the Planck mass squared is defined by

$$\gamma_{\mathrm{EH}} \equiv -\frac{\mu}{M^2}\frac{dM^2}{d\mu} = D - 4 + \bar{\gamma}_{\mathrm{EH}},$$

where $\bar{\gamma}_{\mathrm{EH}} = \mu d(\log Z_{\mathrm{EH}}) d\mu$. An anomalous dimension of the cosmological constant is defined by

$$\gamma_\Lambda \equiv -\frac{\mu}{\Lambda}\frac{d\Lambda}{d\mu} = D - 4 + \bar{\gamma}_\Lambda + \frac{M^4}{\Lambda}\bar{\delta}_\Lambda, \tag{10-38}$$

where $\bar{\gamma}_\Lambda = \mu d(\log Z_\Lambda) d\mu$ and

$$\bar{\delta}_\Lambda = \mu\frac{dL_M}{d\mu} - (D-4)L_M + (\bar{\gamma}_\Lambda - 2\bar{\gamma}_{\mathrm{EH}})L_M.$$

Consider contributions from the gravitational coupling constant only here. The renormalization factor Z_Λ is Laurent-expanded as

$$\log Z_\Lambda = \sum_{n=1}^{\infty} \frac{u_n}{(D-4)^n}.$$

Expanding $\bar{\gamma}_\Lambda$ using this and extracting finite parts, we obtain

$$\bar{\gamma}_\Lambda = \left(\bar{\beta}_b \frac{\partial}{\partial b} + \alpha_t \frac{\partial}{\partial \alpha_t}\right) u_1,$$

and also a renormalization group equation

$$\left(\bar{\beta}_b \frac{\partial}{\partial b} + \alpha_t \frac{\partial}{\partial \alpha_t}\right) u_{n+1} + \bar{\beta}_t \alpha_t \frac{\partial}{\partial \alpha_t} u_n = 0$$

from pole parts which have to vanish. The same is true for the renormalization factor Z_{EH}.

To solve the renormalization group equation, we need to decide a dependence of α_t in $\bar{\beta}_b$. It has not yet been calculated, but we have known that the dependence of the gauge coupling constant α is given by (10-12). Based on this result and from the similarity between the gauge field action and the Weyl action of the traceless tensor field, we guess that the dependence of α_t in $\bar{\beta}_b$ is given by

$$\bar{\beta}_b = -b + o(\alpha_t^2)$$

as similar to the case of α.

We first calculate the anomalous dimensions at $\alpha_t \to 0$. In this case, we can check correctness of the results by comparing with the exact solutions obtained from the BRST conformal invariance. Feynman diagrams up to the third order in the $1/b$-expansion that contribute to the simple pole residue u_1 are depicted in Fig. 10-18, in which the bottom represents ϕ^n that appears when the exponential is expanded as $e^{D\phi} = \sum_n D^n \phi^n / n!$.

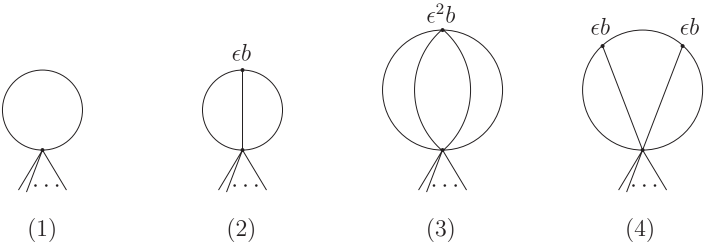


Figure 10-18: Loop corrections to the cosmological term up to $o(1/b^3)$.

Contributions to the simple pole u_1 are $4/b$ from (1), $4/b^2$ from (2), $-8/3b^3$ and $28/3b^3$ from (3) and (4), respectively, in Fig. 10-18. Although infrared divergences remain at this time, it will be shown that they all cancel out each other when constructing the effective potential in the next section. Therefore, ignoring them, we consider only ultraviolet divergences here. Recalling $\bar{\beta}_b = -b$ at $\alpha_t \to 0$, we obtain

$$\bar{\gamma}_\Lambda = -b \frac{\partial u_1}{\partial b} = \frac{4}{b} + \frac{8}{b^2} + \frac{20}{b^3} + \cdots .$$

Lastly, by replacing b with b_c, we find the anomalous dimension of the cosmological constant. Disappearance of the anomalous dimension at $b \to \infty$ is consistent with the fact that it is the classical limit where the conformal-factor field does not propagate.

Compare this result with the exact solution. The conformal-factor field dependence of the renormalized cosmological term is expressed as $\delta_\phi S_\Lambda = (4+\gamma_\Lambda)S_\Lambda$.[17] On the other hand, the quantum cosmological operator determined from the BRST conformal invariance is $\int d^4x :e^{\alpha\phi}:$ and the Riegert charge α is given by (7-35). From this, the exact solution of γ_Λ is given by

$$\gamma_\Lambda = \alpha - 4 = 2b_c\left(1-\sqrt{1-\frac{4}{b_c}}\right) - 4.$$

Expanding this expression with $1/b_c$, we can see that the first three terms agree with the calculation result.

Likewise, we can perform renormalization calculations of the Einstein term and obtain

$$Z_{\rm EH} = 1 - \left[\frac{1}{2b} + \frac{1}{4b^2}\right]\frac{1}{\epsilon}, \qquad L_M = \frac{9}{16}\frac{(4\pi)^2}{b^2}\frac{1}{\epsilon}.$$

The anomalous dimension of the Planck mass squared is then given by

$$\gamma_{\rm EH} = \frac{1}{b_c} + \frac{1}{b_c^2} + \cdots.$$

This result also agree with the $1/b_c$-expansion of the exact anomalous dimension $\gamma_{\rm EH} = \beta - 2$ derived from the Riegert charge β (7-36) of the quantum Ricci scalar. Moreover, from the pole term L_M, we obtain

$$\bar{\delta}_\Lambda = -\frac{9}{8}\frac{(4\pi)^2}{b_c^2}.$$

Loop corrections by traceless tensor field Loop corrections by the traceless tensor field are calculated in the Landau gauge. Since the two-point interaction (10-24) disappears when this gauge is employed, the number of Feynman diagrams can be reduced considerably.[18]

Feynman diagrams that may give corrections of $o(\alpha_t)$ to the anomalous dimension of the Planck mass in the Landau gauge are depicted in Fig. 10-20. However, ultraviolet divergences from (2), (3), and (4) disappear in this gauge. Moreover, (5) that contributes to L_M becomes finite. Consequently, only the contribution from (1) remains and $Z_{\rm EH} - 1 = -(5/8)(\alpha_t/4\pi\epsilon)$ is

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[17] Since the variation with respect to the conformal-factor field corresponds to a scale change in real space, δ_ϕ can be identified with $-\mu d/d\mu$.

[18] See K. Hamada and M. Matsuda, Phys. Rev. D **93** (2016) 064051 on the calculations in the Landau gauge and some issues about gauge-dependence.

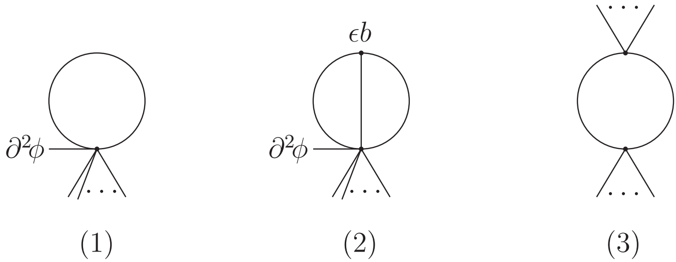

Figure 10-19: First two diagrams give corrections to $Z_{\rm EH}$ at $o(1/b)$ and $o(1/b^2)$, respectively. The last gives a correction to L_M of $o(1/b^2)$.

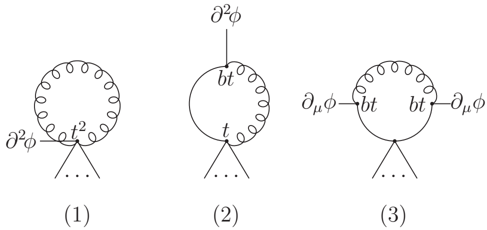

Figure 10-20: First four diagrams give corrections to $Z_{\rm EH}$ of $o(\alpha_t)$ and the last gives a correction to L_M of $o(\alpha_t/b)$ in the Landau gauge.

obtained. Thus, together with the previous result, the anomalous dimension is calculated as

$$\gamma_{\rm EH} = \frac{1}{b_c} + \frac{1}{b_c^2} + \frac{5}{4}\frac{\alpha_t}{4\pi}. \tag{10-39}$$

Loop corrections to the cosmological constant by the traceless tensor field are yielded from two-loop diagrams of $o(\alpha_t/b)$ in the Landau gauge, which are given by Feynman diagrams in Fig. 10-21 with Fig. 10-22 as subdiagrams, where (1) and (2) are the same as those in Fig. 10-1. In the subdiagram (3), there are contributions from two three-point interactions in which one is derived from the t-independent part of the Wess-Zumino action (10-19) associated with the Weyl action with a factor $D-4$ given by

$$S^{D-4}_{F[\phi hh]} = \frac{D-4}{t_0^2}\int d^D x\, \phi\, \bar{C}^2_{\mu\nu\lambda\sigma}\bigg|_{o(t^0)}$$

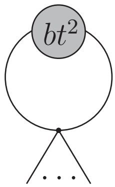

Figure 10-21: Loop corrections of $o(\alpha_t/b)$ to Z_Λ in the Landau gauge. The gray part including bt^2 is given in Fig. 10-22.

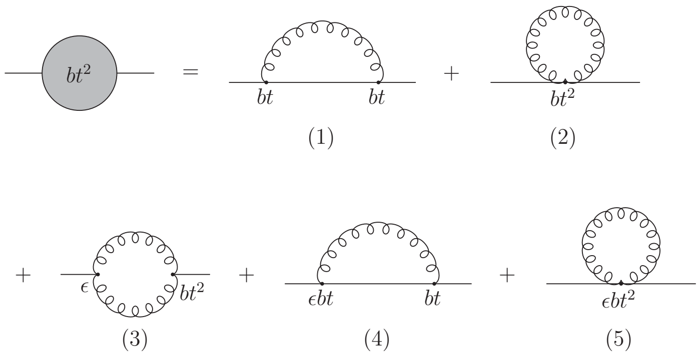

Figure 10-22: Subdiagrams necessary for two-loop corrections of $o(\alpha_t/b)$ to Z_Λ.

and the other is the $o(bt^2)$ part derived from the expansion (10-20) of the Euler term as

$$S^{bt^2}_{\mathrm{G}[\phi hh]} = b\frac{\mu^{D-4}}{(4\pi)^{D/2}}\int d^D x\, \phi \left(\bar{G}_4 - \frac{2}{3}\bar{\nabla}^2 \bar{R} \right)\bigg|_{o(t^2)}.$$

In the subdiagram (4), there are contributions from the three-point interaction used in (1) and the three-point interaction of $(D-4)\times o(bt)$ given by

$$\begin{aligned} S^{(D-4)bt}_{\mathrm{G}[\phi\phi h]} &= (D-4)b\frac{\mu^{D-4}}{(4\pi)^{D/2}}\int d^D x \bigg[\frac{1}{2}\bar{G}_4\phi^2 + 3\phi\bar{\nabla}^4\phi \\ &\qquad + 4\phi\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi - \frac{14}{9}\phi\bar{R}\bar{\nabla}^2\phi + \frac{10}{9}\phi\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi \bigg]\bigg|_{o(t)}. \end{aligned}$$

The subdiagram (5) is from the four-point interaction $S^{(D-4)bt^2}_{\mathrm{G}[\phi\phi hh]}$ which is derived by expanding the right-hand side of above up to $o(t^2)$. The two-loop computation is cumbersome. Here we write only the result of summing all the contributions from these diagrams, which is given as follows:

$$(Z_\Lambda - 1)|_{\zeta=0} = \frac{155}{18}\frac{t^2}{b(4\pi)^2}\frac{1}{\epsilon}.$$

Thus, the anomalous dimension of the cosmological constant is given by

$$\gamma_\Lambda = \frac{4}{b_c} + \frac{8}{b_c^2} + \frac{20}{b^3} - \frac{9(4\pi)^2}{8b_c^2}\frac{M^4}{\Lambda} - \frac{310}{9b_c}\frac{\alpha_t}{4\pi}, \qquad \text{(10-40)}$$

together with the previous results.

Physical Cosmological Constant

In general, a physical mass scale M_{phy} must be renormalization group invariant as

$$\mu\frac{d}{d\mu}M_{\mathrm{phy}} = 0.$$

One of such gravitational scales is Λ_{QG}, which is already introduced when we define the running coupling constant. In addition, there are the physical Planck mass and the physical cosmological constant denoted by M_{P} and $\Lambda\mathrm{cos}$, respectively.

These physical mass scales are determined through the effective action. Introducing a background σ and decomposing the conformal-factor field as $\phi = \sigma + \varphi$, the effective action is expanded in powers of σ as

$$\begin{aligned}\Gamma(\sigma) &= \sum_n \frac{1}{n!}\int d^D x_1 \cdots d^D x_n\, \Gamma^{(n)}(x_1, \ldots, x_n)\, \sigma(x_1)\cdots\sigma(x_n) \\ &= \sum_n \frac{1}{n!}\int \frac{d^D k_1}{(2\pi)^D}\cdots\frac{d^D k_n}{(2\pi)^D}(2\pi)^D\delta^{(D)}(k_1 + \cdots + k_n) \\ &\quad \times\Gamma^{(n)}(k_1, \ldots, k_n)\, \sigma(k_1)\cdots\sigma(k_n),\end{aligned}$$

where $\Gamma^{(n)}$ is a renormalized n-point Green function given by the sum of all 1PI Feynman diagrams with n external legs of σ. Renormalization group analysis for the 1PI Green function $\Gamma^{(n)}$ can be carried out as in the case of the φ^4-theory. One of the crucial differences is that the conformal-factor field is not renormalized, namely $Z_\phi = 1$, and also for the background

σ. Therefore, the renormalized $\Gamma^{(n)}$ is the same as the bare one, and thus $\mu d\Gamma^{(n)}/d\mu = 0$ holds.[19]

A physical cosmological constant is defined by an effective potential V given by the zero-momentum part of $\Gamma^{(n)}(k_1, \ldots, k_n)$ as

$$\Gamma(\sigma)|_V = \int d^D x\, V(\sigma) = \sum_n \frac{1}{n!}\,\Gamma^{(n)}(0, \ldots, 0)\int d^D x\, \sigma^n(x).$$

Diffeomorphism invariance requires that $\Gamma^{(n)}(0, \ldots, 0) = \Lambda_{\rm cos} D^n$ so that the effective potential has the form

$$V(\sigma) = \Lambda_{\rm cos}\, e^{D\sigma(x)}. \tag{10-41}$$

The renormalization group equation indicates that $\Lambda_{\rm cos}$ satisfies

$$\mu\frac{d}{d\mu}\Lambda_{\rm cos} = 0.$$

Thus, we find that the effective potential gives the physical cosmological constant, which can be observed cosmologically. Likewise, the second derivative term defines the physical Planck mass.

Let us examine the physical cosmological constant here. We calculate the one-loop effective potential within the large b approximation, where the background σ is taken to be a constant. Moreover, as lower-derivative interactions, we consider only the cosmological term for simplicity. By expanding the conformal-factor field φ up to the second order, the action becomes

$$\begin{aligned} S|_{\varphi^2} = \int d^D x\, \mu^{D-4}\Biggl\{ &\frac{1}{(4\pi)^{\frac{D}{2}}}\left[2b\,\varphi\partial^4\varphi + (D-4)b\,(2\sigma+3)\varphi\partial^4\varphi\right] \\ &+\Lambda e^{D\sigma}\left(1 + \frac{D^2}{2}\varphi^2\right) + (Z_\Lambda - 1)\Lambda e^{D\sigma}\Biggr\}, \end{aligned}$$

where the term with $D-4$ comes from the three-point interaction (10-23). Furthermore, by rescaling the field as

$$\varphi = \frac{a}{\sqrt{\mu^{D-4}}}\varphi', \quad a = \sqrt{\frac{(4\pi)^{\frac{D}{2}}}{4b[1 + (D-4)(\sigma + \frac{3}{2})]}},$$

[19] In the φ^4 theory, the field receives renormalization so that $\Gamma^{(n)}$ is not renormalization group invariant, though $\Gamma(\varphi)$ itself is invariant.

we obtain

$$S|_{\varphi^2} = \int d^D x \left\{ \frac{1}{2}\varphi' \partial^4 \varphi' + \frac{D^2 a^2}{2} \Lambda e^{D\sigma} \varphi'^2 - \frac{1}{\bar{\epsilon}} \frac{2}{b} \mu^{D-4} \Lambda e^{D\sigma} \right\}.$$

The last term is the one-loop counterterm for the cosmological constant, where the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme is adopted, that is, divergences are removed by the combination of $1/\bar{\epsilon}$.

Letting $A = D^2 a^2 \Lambda e^{D\sigma}$ and writing the kinetic term to $\mathcal{D} = \mathcal{D}_0 + A$, where $\mathcal{D}_0 = k^4$ in momentum space, the one-loop effective potential (see Fig. 10-23) can be calculated as

$$\begin{aligned} V^{\mathrm{loop}} &= -\log \left[\det \left(\mathcal{D}_0^{-1} \mathcal{D} \right) \right]^{-\frac{1}{2}} = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \log \left(1 + \frac{A}{k_z^4} \right) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} A^n I_n, \end{aligned}$$

where I_n is a tadpole-type integral defined by

$$I_n(z) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k_z^4)^n}.$$

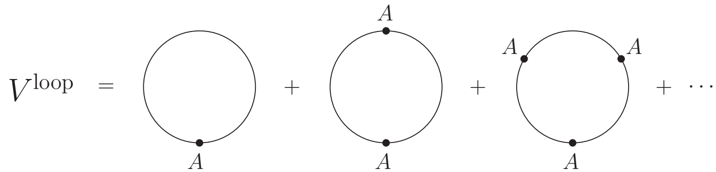

Figure 10-23: One-loop effective potential to the physical cosmological term.

The integral I_1 has divergences of both ultraviolet and infrared, while for $n \geq 2$ only infrared divergences appear, which are given by

$$I_1 = \frac{1}{(4\pi)^2} \left(\frac{1}{\bar{\epsilon}} - \log z^2 \right), \qquad I_{n\,(\geq 2)} = \frac{1}{(4\pi)^2} \frac{z^{2(2-2n)}}{(2n-1)(2n-2)}.$$

Substituting these integrated values, we obtain the following loop correction:[20]

$$V^{\rm loop} = \frac{1}{(4\pi)^2}\left\{\frac{A}{2}\left(\frac{1}{\bar\epsilon} - \log z^2\right) + z^4\sum_{n=2}^{\infty}\frac{(-1)^{n-1}A^n z^{-4n}}{2n(2n-1)(2n-2)}\right\}$$
$$= \frac{1}{(4\pi)^2}\left\{\frac{A}{2}\left(\frac{1}{\bar\epsilon} - \log z^2\right) + \frac{1}{4}\left(z^4 - A\right)\log\left(1+\frac{A}{z^4}\right)\right.$$
$$\left. - z^2\sqrt{A}\arctan\left(\frac{\sqrt{A}}{z^2}\right) + \frac{3}{4}A\right\}.$$

With attention to the D-dependence of A, we find that a finite σ-dependent term arises from the first term as

$$\frac{1}{(4\pi)^2}\frac{A}{2}\frac{1}{\bar\epsilon} = \frac{2}{b}\mu^{D-4}\Lambda e^{D\sigma}\left(\frac{1}{\bar\epsilon} + 2\sigma + 2 - \log 4\pi + \log\mu^2\right),$$

in which in order to match the form of the counterterm, the μ-dependence is recovered through the relation $\mu^{D-4}[1-(D-4)\log\mu] = 1 + o[(D-4)^2]$. Removing the ultraviolet divergence by the counterterm and taking $D=4$, the loop correction $V^{\rm loop}$ results in

$$\frac{1}{(4\pi)^2}\left\{\frac{\bar A}{2}\left(2\sigma + 2 - \log 4\pi\right) - \frac{\bar A}{2}\log\frac{z^2}{\mu^2}\right.$$
$$\left. + \frac{1}{4}\left(z^4 - \bar A\right)\log\left(1+\frac{\bar A}{z^4}\right) - z^2\sqrt{\bar A}\arctan\left(\frac{\sqrt{\bar A}}{z^2}\right) + \frac{3}{4}\bar A\right\},$$

where $\bar A = A|_{D=4} = 4(4\pi)^2\Lambda e^{4\sigma}/b$.

This expression can take the limit of $z \to 0$ and we obtain a finite result $V^{\rm loop} = \bar A\,[7 - 2\log 4\pi - \log(\bar A/\mu^4)]/4(4\pi)^2$. In this way, we can demonstrate that infrared divergences cancel out by summing up all contributions. Adding the classical (tree) part $\Lambda e^{4\sigma}$ and putting $b = b_c$, we get the following effective potential:

$$V = \Lambda\left\{1 + \frac{1}{b_c}\left[7 - 2\log 4\pi - \log\left(\frac{64\pi^2}{\mu^4}\frac{\Lambda}{b_c}\right)\right]\right\}e^{4\sigma}.$$

Note that the σ-dependence disappears except for the $e^{4\sigma}$ at last. This σ-independent part gives the physical cosmological constant $\Lambda_{\rm cos}$.

[20] To calculate the sum of series $f(x) = \sum_{n=2}^{\infty}(-1)^{n-1}x^{2n}/2n(2n-1)(2n-2)$, it is good to consider $h(x) = \partial^2/\partial x^2\{f(x)/x\}$. Since this is easily obtained as $h(x) = [\log(1+x^2) - x^2]/2x^3$, the original series can be derived as $f(x) = x\int_0^x du\int_0^u dv\, h(v)$. The series part here is then given by $z^4 f(\sqrt{A}/z^2)$.

More generally, adding the Einstein-Hilbert action and taking in loop corrections by the traceless tensor field in the Landau gauge, we obtain the following physical cosmological constant at the one-loop level:[21]

$$
\begin{aligned}
\Lambda_{\cos} = \Lambda &+ (7 - 2\log 4\pi)\frac{\Lambda}{b_c} - \left(\frac{\Lambda}{b_c} - \frac{9\pi^2 M^4}{2b_c^2}\right)\log\left(\frac{64\pi^2}{\mu^4}\frac{\Lambda}{b_c}\right) \\
&- \frac{9\pi^2}{2}\left(\frac{25}{3} - 4\log 4\pi\right)\frac{M^4}{b_c^2} \\
&- 6\pi\frac{M^2}{b_c}\sqrt{\frac{\Lambda}{b_c} - \frac{9\pi^2 M^4}{4b_c^2}}\arccos\left(\frac{3\pi M^2}{2\sqrt{b_c\Lambda}}\right) \\
&+ \frac{5}{128}\alpha_t^2 M^4\left(\log\frac{\pi^2\alpha_t^2 M^4}{\mu^4} - \frac{21}{5}\right),
\end{aligned}
$$

where we assume that the ratios Λ/b_c and M^4/b_c^2 are comparable, and $\alpha_t/4\pi$ and $1/b_c$ are also so, in the large b_c approximation.

[21] See K. Hamada and M. Matsuda, Phys. Rev. D **96** (2017) 026010.

Chapter Eleven

The Universe in Einstein Gravity

Why is quantum gravity necessary? Why is it possible to think that its traces are left in the cosmic microwave background radiation today? Before answering these questions, we first need to know about the Friedmann universe given as a solution of the Einstein equation. The purpose of this chapter is to concisely summarize what is known about the Friedmann universe and connect it to later chapters.

Instability and Evolution of Fluctuations

What we have to mention first is that the Friedmann solution of the Einstein equation is an unstable solution. Usually, such a solution is never chosen as physics. This is because if we consider a small fluctuation (perturbation) around this solution it will grow with time and deviate significantly from the solution. Nevertheless, the universe can be approximated well by the Friedmann solution. This means that initial amplitudes of the fluctuations were unnaturally small.[1] It was one of the big problems of cosmology to clarify what initial dynamics of the universe that produces such very small fluctuations is.

Inflation theory was proposed around 1980 as an idea to create very small initial fluctuations necessary for the universe to last more than 10 billion years. It is an idea that there was an era when the universe expands exponentially before the big bang where matters are created, and fluctuations become small at that time. At the same time, this idea can also explain why correlations larger than the horizon size existed in the early universe (the horizon problem).

After the big bang, fluctuations grow and structures such as stars, galaxies, and clusters of galaxies are built. Since these structure formation includes nonlinear effects, it cannot be described by a simple perturbation theory, but until the universe is neutralized, fluctuations are still small and can

[1] Although it depends on perturbation variables we employ, when considering the matter density fluctuation $\delta\rho/\rho$, it is required to be as small as 10^{-60}.

be handled within a linear perturbation. Furthermore, after neutralization, radiation fluctuations do not grow because photons become free (decoupled) from interactions with matters. The so-called Sachs-Wolfe effect expresses that.[2] Therefore, if we know spectra of fluctuations at the time of decoupling, we can roughly know the current spectra of the cosmic microwave background radiation (CMB).

The CMB discovered in 1964 by Penzias and Wilson has a Planck distribution whose temperature T is 3 degrees Kelvin.[3] The temperature fluctuation amplitude $\delta T/T$ that denotes a deviation from it is the order of 10^{-5} in which a linear perturbation method works well enough. This value was observed for the first time in the 1990's by the Cosmic Background Explorer (COBE). The method describing evolution of the universe perturbatively based on this fact is called cosmological perturbation theory.

The temperature fluctuation spectrum of CMB obtained from the observation by Wilkinson Microwave Anisotropies Probe (WMAP) depicted in Fig. 11-1, which is a successor of COBE, records the history of the universe from the radiation-dominated era to the present. It is believed that the spectrum in the early universe was almost scale-invariant. When writing its spectrum what is called the Harrison-Zel'dovich type[4] in Fig. 11-1, it becomes a horizontal straight line whose amplitude almost coincides with that of the dent in large angle components. That is to say that the deformation from the straight line represents dynamics that the fluctuation was subjected to during evolution of the universe.

Most of these deformations occur during the period between the time that the universe is moving from the radiation-dominated era to the matter-dominated era and the time it is neutralized. Since the scale-invariant spectrum is retained before that, we can go back to the past anywhere as far as the radiation-dominated era continues.[5] Therefore, as long as the Einstein's

---

[2] The Sachs-Wolfe effect is expressed as a relation that connects the current temperature fluctuation $(\delta T/T)(\eta_0)$ with a gravitational potential value $\Psi(\eta_{\rm dec})$ at the decoupling (last scattering surface). For long wavelength fluctuations (large angle components) which leaves information of the primordial spectra intact, the relation $(\delta T/T)(\eta_0, \mathbf{x}_0) \simeq \Psi(\eta_{\rm dec}, \mathbf{x}_{\rm dec})/3$ holds. See the first section of Appendix E for details. The original paper is R. Sachs and A. Wolfe, *Perturbations of a Cosmological Model and Angular Variations of the Microwave Background*, Astrophys. J. **147** (1967) 73.

[3] The original paper is A. Penzias and R. Wilson, *A Measurement of Excess Antenna Temperature at 4080-Mc/s*, Astrophys. J. **142** (1965) 419.

[4] The original papers are E. Harrison, *Fluctuations at the Threshold of Classical Cosmology*, Phys. Rev. D **1** (1970) 2726 and Ya. Zel'dovich, *A Hypothesis, Unifying the Structure and the Entropy of the Universe*, Mon. Not. R. Astron. Soc. **160** (1972) P1.

[5] In the period when the fluctuation size is larger than the horizon size, the spectrum hardly

theory of gravity is correct, from the current CMB spectrum, we can derive information on the big bang where the primordial spectrum was generated.

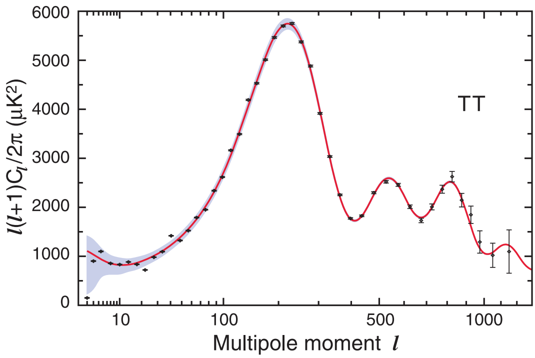

Figure 11-1: WMAP temperature fluctuation spectrum [C. Bennett, et.al., Astrophys. J. Suppl. **208** (2013) 20]. Low multipoles correspond to large angle components.

The spectrum can be roughly divided into three regions: the low multipole component region $l < 30$ in which the scale-invariant spectrum of the early universe is considered to be preserved almost as it is, the region $30 < l < 700$ where a plasma fluid oscillation of photons and baryons appears, and the Silk damping region $l > 700$ where the amplitude of photon fluctuations decreases exponentially in the process of neutralization. In this damping region, the approximation of perfect fluids breaks and an anisotropic stress appears.

The Silk damping occurs because thermal equilibrium cannot be maintained gradually as the mean free path of light becomes longer during the period from the start of the recombination process until the light is completely free from matters. If the wavelength is longer than the mean free path, the perfect fluid approximation holds, but if it is short, photon diffusions occur and the amplitude of the fluctuation decreases.[6] This effect begins to appear beyond the first acoustic peak, and becomes significant where l exceeds 700.

changes. In particular, the CMB multipole components of $l < 30$ are maintaining the primordial spectrum just after the big bang, which represent fluctuations that entered the horizon after the neutralization of the universe, or do not enter the horizon currently.

[6] The perfect fluid is a fluid with zero viscosity ($\neq$ perfect gas). Viscosity is proportional to the mean free path, and having zero viscosity means a strongly coupled system with zero mean free path. In such a frequently interacting system, heat exchange is closed in the system and thermal equilibrium can be realized.

Therefore, the cosmological perturbation theory assuming the perfect fluid is effective in the long wavelength region up to at most $l < 700$, which will be described in detail in Chapter 13. In order to analyze the Silk damping, it is necessary to solve Boltzmann equations considering the Thomson scattering, but it is not dealt with in this book.

Calculations of the CMB temperature fluctuation spectra have already been programmed, and existing calculation codes such as CMBFAST are published. One of the important things in early cosmology is to give the primordial spectrum as its initial condition. The goal of the quantum gravity theory is to give the scale-invariant primordial spectrum and the dynamics of inflation that generates it.

Friedmann Universe

The Friedmann solution of the Einstein equation is summarized, and various scales that appears there are explained. Since the spatial curvature is almost zero from the observation result, we ignore it for the sake of simplicity below. It is also assumed that matters are perfect fluids. The metric tensor and the energy-momentum tensor for each matter state α are given by

$$\begin{aligned} ds^2 &= e^{2\hat{\phi}}\left(-d\eta^2 + d\mathbf{x}^2\right), \\ T_{(\alpha)\nu}^{\ \mu} &= \mathrm{diag}(-\rho_\alpha, P_\alpha, P_\alpha, P_\alpha), \end{aligned}$$

respectively, where $\hat{\phi}(\eta)$ is a conformal-factor background field. The time variable η is called the conformal time, and $\mathbf{x} = (x^1, x^2, x^3)$ are the comoving coordinates which are coordinates like angles that will not change even if the universe expands. The physical (proper) time τ is defined by

$$d\tau = e^{\hat{\phi}} d\eta \tag{11-1}$$

and a physical distance is given by $e^{\hat{\phi}} d\mathbf{x}$. The variables ρ_α and P_α represent energy density and pressure of each state α, respectively. Usually, we consider multiple matter states. These variables depend only on the conformal time η.

The matter state α is represented by the equation-of-state parameter defined as a proportionality factor between pressure and energy density as

$$w_\alpha = \frac{P_\alpha}{\rho_\alpha}.$$

In radiation states, the equation-of-state parameter is given by $1/3$ and the energy-momentum tensor becomes traceless. Also, if temperature is sufficiently high like just after the big bang, even massive particles can be regarded as massless ones and described as a radiation state. When the universe cools and temperature becomes lower than the particle mass, pressure disappears and the equation-of-state parameter becomes 0.

The conservation law $\nabla_\mu T^{\ \mu}_{(\alpha)\nu} = 0$ in the absence of particle sources is expressed for each state as

$$\partial_\eta \rho_\alpha + 3\partial_\eta \hat{\phi} \left(\rho_\alpha + P_\alpha\right) = 0. \tag{11-2}$$

The trace part and the (00) component of the Einstein equation are respectively given by

$$\begin{aligned} M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \left(6\partial_\eta^2 \hat{\phi} + 6\partial_\eta \hat{\phi} \partial_\eta \hat{\phi}\right) - \rho + 3P - 4\Lambda_{\cos} &= 0, \\ -3M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \partial_\eta \hat{\phi} \partial_\eta \hat{\phi} + \rho + \Lambda_{\cos} &= 0, \end{aligned} \tag{11-3}$$

where $M_{\mathsf{P}} = 1/\sqrt{8\pi G}$ is the reduced Planck mass and $\Lambda_{\cos}$ is the cosmological constant. The energy density ρ and the pressure P are the sums over all states, respectively, given by

$$\rho = \sum_\alpha \rho_\alpha, \qquad P = \sum_\alpha P_\alpha. \tag{11-4}$$

Note that the conservation law (11-2) holds for each state α, whereas variables in the Einstein equation come in the form of the sum of states. Of course, the conservation law also holds for the variables ρ and P, and the equation-of-state parameter is then given by $w = P/\rho$.

Introduce a commonly used scale factor a and the Hubble variable H, which are defined with the conformal-factor field as

$$a = e^{\hat{\phi}}, \qquad \partial_\eta \hat{\phi} = \frac{\partial_\eta a}{a} = aH. \tag{11-5}$$

Rewriting the equations with the Hubble variable, the Einstein equation (11-3) can be expressed as

$$\begin{aligned} 6M_{\mathsf{P}}^2 \left(a^{-1}\partial_\eta H + 2H^2\right) &= \rho - 3P + 4\Lambda_{\cos} = (1-3w)\rho + 4\Lambda_{\cos}, \\ 3M_{\mathsf{P}}^2 H^2 &= \rho + \Lambda_{\cos}, \end{aligned} \tag{11-6}$$

and the energy conservation equation (11-2) is

$$\partial_\eta \rho_\alpha = -3aH\left(\rho_\alpha + P_\alpha\right) = -3\left(1 + w_\alpha\right) aH \rho_\alpha. \tag{11-7}$$

The Hubble constant representing the current value of the Hubble variable is one of the cosmological parameters specifying the Friedmann universe, often denoted as $H_0 = 100h$ [kms^{-1}Mpc^{-1}] using the small letter h. In natural units ($c = \hbar = 1$), it is given by

$$H_0 = \frac{h}{2997.9} = 0.00024\text{Mpc}^{-1},$$

where the value of $h = 0.72$ is adopted. Then, the Hubble distance $1/H_0$ which is often used to represent the size of the universe currently visible is 4164 Mpc.

Current density parameters are defined by using the Hubble constant as

$$\Omega_\alpha = \frac{\rho_{\alpha 0}}{3M_{\mathsf{P}}^2 H_0^2},$$

where $\rho_{\alpha 0}$ denotes current energy density for each state α. As a mater state α, cold dark matters (CDM) are denoted by c,[7] baryons are b and radiations are r. The whole massive matters combined c and b are called dusts, which are denoted by d. Also, the cosmological constant can be regarded as a matter of $w = -1$ and thus it is useful to define the quantity $\Omega_\Lambda = \Lambda_{\cos}/3M_{\mathsf{P}}^2 H_0^2$. In this book, we use the following values:

$$\begin{aligned}
&\Omega_r = \Omega_\gamma + \Omega_\nu = 4.2 \times 10^{-5}/h^2 = 8.1 \times 10^{-5}, \\
&\Omega_b = 0.042, \\
&\Omega_d = \Omega_c + \Omega_b = 0.27, \\
&\Omega_\Lambda = 0.73,
\end{aligned}$$

where Ω_r is the sum of Ω_γ from photons and Ω_ν from neutrinos. From $\rho_{\gamma 0} = 2(\pi^2/30)T_\gamma^4$ and $\rho_{\nu 0} = N_\nu 2(\pi^2/30)(7/8)T_\nu^4$, the ratio is given

$$\frac{\Omega_\nu}{\Omega_\gamma} = \frac{\rho_{\nu 0}}{\rho_{\gamma 0}} = N_\nu \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 = 0.68,$$

where N_ν is the number of generations. In the last equality, $T_\nu/T_\gamma = (4/11)^{1/3}$ and $N_\nu = 3$ are used. From this, the photon density is given by $\Omega_\gamma = 4.8 \times 10^{-5}$.

From the Einstein equation (11-6), we find that these quantities satisfy

$$\Omega_d + \Omega_\Lambda = 1,$$

[7] Dark matter is an unknown object giving only gravitational effects which hardly interact with ordinary matters. Its existence is predicted indirectly to explain observed results such as CMB spectra, galactic rotation curves, gravitational lens effects, etc.

where Ω_r is small and thus ignored here. This is an equation showing that the space is flat. If there is a curvature in the space, the right-hand side is deviated from one, but the observation suggests that it is one.

Since the equation-of-state parameters of the states r, c, and b are

$$w_r = \frac{1}{3}, \quad w_c = 0, \quad w_b = 0,$$

respectively, we obtain

$$\begin{aligned}
\rho_r &= \rho_{r0}\left(\frac{a_0}{a}\right)^4 = 3M_{\mathsf{P}}^2 H_0^2 \Omega_r \left(\frac{a_0}{a}\right)^4, \\
\rho_c &= \rho_{c0}\left(\frac{a_0}{a}\right)^3 = 3M_{\mathsf{P}}^2 H_0^2 \Omega_c \left(\frac{a_0}{a}\right)^3, \\
\rho_b &= \rho_{b0}\left(\frac{a_0}{a}\right)^3 = 3M_{\mathsf{P}}^2 H_0^2 \Omega_b \left(\frac{a_0}{a}\right)^3,
\end{aligned}$$

by solving the conservation equation (11-7), where a_0 is a current scale factor and is often normalized as $a_0 = 1$. Unless otherwise specified, $a_0 = 1$ will be taken when calculating numerical values below. Substituting this into the Einstein equation (11-6), the Hubble variable can be written as

$$H^2 = H_0^2 \left\{ \Omega_r \left(\frac{a_0}{a}\right)^4 + \Omega_d \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda \right\}. \tag{11-8}$$

As can be seen from this equation, the cosmological constant does not contribute to dynamics unless the scale factor is about $a_0/a < 2$ which is close to the current value.

The red shift z representing a wavelength elongation due to cosmic expansion is defined by

$$z + 1 = \frac{a_0}{a},$$

which is used to indicate how far back in the past since the present time. As we integrate the equation (11-8) by rewritten in the form $d\eta = \cdots$ using $H = \partial_\eta a / a^2$, the relation between the red shift and the comoving angular size distance is calculated as

$$d = \eta_0 - \eta = \frac{1}{a_0 H_0} \int_0^z \frac{dz}{\sqrt{\Omega_r (z+1)^4 + \Omega_d (z+1)^3 + \Omega_\Lambda}}.$$

Specific numerical values are as follows:

$$\begin{aligned}
z = 0.1 &\Leftrightarrow d = 408\,\text{Mpc}, \\
z = 1 &\Leftrightarrow d = 3271\,\text{Mpc}, \\
z = 5 &\Leftrightarrow d = 7822\,\text{Mpc}, \\
z_{\text{dec}} = 1100 &\Leftrightarrow d_{\text{dec}} = 13808\,\text{Mpc}\ (\simeq \text{the size of the universe}),
\end{aligned}$$

where $z_{\rm dec}$ denotes the redshift of the last scattering surface where the universe neutralized.

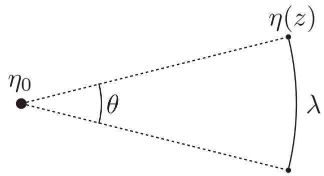

Figure 11-2: Redshift and distance.

The relationship between the distance and an angle is given by $\theta \simeq \lambda/d$ as in Fig. 11-2. We here give a relationship of multipole l and comoving wavenumber $k = \pi/\lambda$ denoting the size of fluctuations. Since the distance $d_{\rm dec}$ comes in when defining the angular power spectrum C_l through the Sachs-Wolfe effect, l is evaluated at $d = d_{\rm dec}$ so that

$$l \simeq \frac{\pi}{\theta} = k d_{\rm dec} \tag{11-9}$$

holds. Therefore,

$$\begin{aligned} k = 0.0002\,\mathrm{Mpc}^{-1} &\Leftrightarrow l \simeq 3, \\ k = 0.002\,\mathrm{Mpc}^{-1} &\Leftrightarrow l \simeq 30, \\ k = 0.005\,\mathrm{Mpc}^{-1} &\Leftrightarrow l \simeq 70, \\ k = 0.015\,\mathrm{Mpc}^{-1} &\Leftrightarrow l \simeq 210, \\ k = 0.05\,\mathrm{Mpc}^{-1} &\Leftrightarrow l \simeq 700. \end{aligned}$$

The physical wavenumber of fluctuations is given by dividing the comoving wavenumber by the scale factor as $p = k/a$. Since $a_0 = 1$ is taken here, the value of k represents the size of current physical fluctuations. The wavelength of the $l = 3$ fluctuation is $1/k \simeq 5000\,\mathrm{Mpc}$ which is about the same as the Hubble distance ($1/H_0$). The $l \simeq 30$ corresponds to the size of $1/k \simeq 500\,\mathrm{Mpc}$. Even $l \simeq 700$ is $1/k \simeq 20\,\mathrm{Mpc}$, which is the size of a super cluster of galaxies given by 10 to 30 Mpc.

Since energy density of radiations decreases with a^{-4} as the universe expands, whereas that of massive matters decreases with a^{-3}, there is a time that the universe shifts from the radiation-dominated era of $\rho_r > \rho_d$ to the matter-dominated era of $\rho_r < \rho_d$. Solving the defining equation $\rho_r = \rho_d$, the redshift value that indicates the time is given by

$$z_{\rm eq} + 1 = \frac{\Omega_d}{\Omega_r} = 3333.$$

Lastly, we briefly describe the behavior of the scale factor. In the radiation-dominated era approximated as $\rho_d = 0$, the Einstein equation can be easily solved and $a \propto \eta$ is obtained. In the mater-dominated era approximated as $\rho_r = 0$, we get $a \propto \eta^2$. In the physical time, it is expressed as $a \propto \tau^{1/2}$ and $a \propto \tau^{2/3}$, respectively.

Chapter Twelve

Quantum Gravity Cosmology

First of all, we assume that the universe is only one and we are inside it. Although we do not know how large the universe is, we think that the whole is homogeneous and the area we are looking at is part of it. The spacetime that we recognize now shall be born through a phase transition from quantum space to classical space. It is not a discussion based on a perspective as we are watching the universe from the outside, such as suddenly the time and space itself is created out of nothing.

In this chapter, we describe evolution of the initial universe based on the asymptotically background-free quantum gravity. Based on the existence of two energy scales, the Planck mass $m_{\rm pl} = 1/\sqrt{G} \simeq 10^{19}$GeV and the dynamical scale $\Lambda_{\rm QG}$, we construct a model of inflation in which our current universe comes out from a conformally invariant spacetime. Taking the value of the dynamical energy scale to be $\Lambda_{\rm QG} \simeq 10^{17}$GeV, lower than the Planck mass, the evolution can be regarded as a violating process of conformal invariance by these two scales, and eventually the spacetime changes from inflation to the Friedmann universe through the big bang described as a phase transition.

Inflation and Spacetime Phase Transition

At high energy beyond the Planck mass, higher-order derivative actions dominate, and spacetime fluctuations will be described by the background-free quantum gravity. Here, we discuss a region where energy falls to the Planck scale and the Einstein-Hilbert action becomes effective. On the other hand, the cosmological constant $\Lambda_{\rm cos}$ is here ignored as it is extremely small.

As mentioned earlier, we assume the following magnitude relationship between the two gravitational energy scales that govern dynamics:

$$m_{\rm pl} \gg \Lambda_{\rm QG}.$$

In this case,[1] an inflationary solution exists.

[1] This magnitude condition is also involved with a problem of unitarity. As mentioned in

In the chapters on cosmology, the metric tensor field is decomposed as $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$ as before, but the metric with the bar is expanded as $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \cdots$ without introducing the coupling constant t in order to consider the case where t becomes large. That is, we write $th_{\mu\nu}$ as $h_{\mu\nu}$ in the following. The coordinates of the flat background with the metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$ are $x^\mu = (\eta, x^i)$, where η is the conformal time and x^i is the comoving spatial coordinate.

Stable inflationary solution When energy is sufficiently higher than the dynamical scale $\Lambda_{\rm QG}$, the coupling constant t can be ignored due to the asymptotic background freedom. In this region, dynamics of the quantum gravity is described by the action $S_{\rm 4DQG}$ (7-6) which is a combination of the original action I (7-1) and the Riegert action $S_{\rm R}$ (5-11) induced quantum mechanically.

Consider the equation of motion of the conformal-factor field ϕ. The equation for its spatially homogeneous component $\hat{\phi}(\eta)$ is given by

$$-\frac{b_c}{4\pi^2}\partial_\eta^4\hat{\phi} + 6M_{\rm P}^2\, e^{2\hat{\phi}}\left(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right) = 0,$$

where we use the fact that matter actions do not depend on the conformal-factor field. Rewriting this equation using the Hubble variable H (11-5) introduced in the previous chapter, we obtain

$$\frac{b_c}{8\pi^2}\left(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 18H^2\dot{H} + 6H^4\right) - 3M_{\rm P}^2\left(\dot{H} + 2H^2\right) = 0,$$

where the dot denotes a derivative by the physical time τ (11-1). This equation has the following inflationary solution (de Sitter solution):

$$H = H_{\mathbf{D}}, \qquad H_{\mathbf{D}} = M_{\rm P}\sqrt{\frac{8\pi^2}{b_c}} = m_{\rm pl}\sqrt{\frac{\pi}{b_c}}. \tag{12-1}$$

The scale factor a (11-5) increases exponentially as a function of the physical time as

$$a(\tau) \propto e^{H_{\mathbf{D}}\tau}.$$

Time in the universe is a monotonically increasing variable, and the inflationary solution shows that it is nothing but a scale factor. This means

Chapter 1, assuming that gravity is classical until the Planck energy, an elementary excitation with the Planck mass becomes a black hole and its information is lost. Under this condition, however, the problem can be avoided because the effect of quantum gravity begins to work before reaching the Planck energy.

that time is dynamically generated by the Planck scale which causes the exponential expansion. Before that, it can be thought of as a world without time, where time changes are quite moderate and fluctuations dominate.

The value of the coefficient b_c (7-5) is about 10 for the Standard Model and various GUT models, and therefore the constant $H_{\mathbf{D}}$ is located between the reduced Planck mass $M_{\mathsf{P}} = 2.4 \times 10^{18}$ GeV and the normal Planck mass $m_{\rm pl} = 1.2 \times 10^{19}$ GeV. In the following, $H_{\mathbf{D}}$ is treated as one of the Planck mass scale, and

$$\tau_{\mathsf{P}} = \frac{1}{H_{\mathbf{D}}} \tag{12-2}$$

is defined as the Planck time, which is the time when the universe starts to exponentially expand.

Let us show that the inflationary solution is stable. With a deviation from the solution as δ, and assigning $H = H_{\mathbf{D}}(1+\delta)$ to the equation of motion, we get

$$\dddot{\delta} + 7H_{\mathbf{D}}\ddot{\delta} + 15H_{\mathbf{D}}^2\dot{\delta} + 12H_{\mathbf{D}}^3\delta = 0,$$

where $o(\delta^2)$ is ignored. Solving this equation by substituting $\delta = e^{\upsilon\tau}$ yields $-4H_{\mathbf{D}}$ and $(-3/2 \pm i\sqrt{3}/2)H_{\mathbf{D}}$ as solutions for υ. Since all three modes have negative real parts, it is found that the deviation decreases exponentially with time, and thus the inflationary solution is stable. As shown in the later chapter, it is also stable against spatial fluctuations (perturbations), that is, fluctuations will be found to be gradually decreasing.

Spacetime phase transition Near the Planck scale, the breaking of conformal invariance is still small and quantum correlations shall behave in powers of the scale. On the other hand, the breaking at the dynamical energy scale $\Lambda_{\rm QG}$ occurs logarithmically through the running coupling constant, thus the conformal invariance rapidly and completely breaks at this scale.

The physical correlation length of the quantum gravity is given by $\xi_\Lambda = 1/\Lambda_{\rm QG}$. Fluctuations shorter than it are quantum mechanical, while those longer than it can be considered as classical fluctuations. If energy falls below $\Lambda_{\rm QG}$, all fluctuations of spacetime will become classical. We call it the spacetime phase transition.

In considering dynamics of the phase transition, we refer to quantum chromodynamics (QCD) which is a representative of quantum field theory showing the asymptotic freedom. There is a dynamical energy scale $\Lambda_{\rm QCD}$ in QCD, and the kinetic term of the gauge field disappears at low energy below this scale. Likewise, it can be considered that the conformally invariant kinetic terms of the gravitational field disappear at the spacetime phase transition. In fact, if the coupling constant becomes infinite at the dynamical scale Λ_{QG}, we know that the Weyl action $-(1/t^2)C^2_{\mu\nu\lambda\sigma}$ disappears because the curvature should be finite.[2]

The dynamics of the conformal-factor field is considered as follows. The coefficient b_c before the Riegert action will be replaced by a function including quantum corrections as

$$b_c \to b_c\big(1 - a_1 t^2 + \cdots\big) = b_c B_0(t).$$

As its non-perturbative expression, we here use the form bravely summed up as

$$B_0(t) = \frac{1}{(1 + \frac{a_1}{\kappa} t^2)^\kappa}, \tag{12-3}$$

where κ is a phenomenological parameter that controls higher order effects and is assumed to be in the range $0 < \kappa \leq 1$.

Incorporating this effect, the equation of motion of the conformal-factor field becomes

$$-\frac{b_c}{4\pi^2} B_0 \partial_\eta^4 \hat{\phi} + M_{\mathrm{P}}^2\, e^{2\hat{\phi}} \Big\{ 6\partial_\eta^2 \hat{\phi} + 6\partial_\eta \hat{\phi} \partial_\eta \hat{\phi} \Big\} = 0. \tag{12-4}$$

In addition, the energy conservation equation is obtained from the $(0,0)$ component of the energy-momentum tensor as

$$\frac{b_c}{8\pi^2} B_0 \Big\{ 2\partial_\eta^3 \hat{\phi} \partial_\eta \hat{\phi} - \partial_\eta^2 \hat{\phi} \partial_\eta^2 \hat{\phi} \Big\} - 3M_{\mathrm{P}}^2\, e^{2\hat{\phi}} \partial_\eta \hat{\phi} \partial_\eta \hat{\phi} + e^{4\hat{\phi}} \rho = 0, \tag{12-5}$$

where ρ is a matter energy density.

By taking in the running effect of the coupling constant further, we describe time evolution in the inflationary era until the spacetime phase transition. The running coupling constant is here defined as a response to the physical time by the renormalization group equation $-\tau d\tilde{t}/d\tau = \beta(\tilde{t}) = -\beta_0 \tilde{t}^3$. The solution that becomes infinite at a dynamical time scale

$$\tau_\Lambda = \frac{1}{\Lambda_{\mathrm{QG}}}$$

is given by

$$\tilde{t}^2(\tau) = \frac{1}{\beta_0 \log(1/\tau^2 \Lambda_{\mathrm{QG}}^2)}. \tag{12-6}$$

[2] Actually, it seems natural to think that the change occurs before the coupling constant becomes infinite, but here we represent the disappearance of the kinetic term assuming that it becoms ideally infinite.

This corresponds to the running coupling constant (10-36) with the physical momentum $k_{\rm phy}$ replaced by the inverse of the physical time τ (> 0).

Replacing the coupling constant t with the time-dependent running coupling constant $\tilde{t}(\tau)$, the dynamical factor B_0 is expressed as a function decreasing with time. Rewriting it using the Hubble variable, we obtain the following equation of motion:

$$B_0(\tau)\Big(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 18H^2\dot{H} + 6H^4\Big) - 3H_{\mathbf{D}}^2\Big(\dot{H} + 2H^2\Big) = 0. \tag{12-7}$$

The equation of energy conservation is also given by

$$B_0(\tau)\Big(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} + 3H^4\Big) - 3H_{\mathbf{D}}^2H^2 + \rho = 0. \tag{12-8}$$

In the early epoch of inflation where the coupling constant is small, the solution of the equation of motion is given by $H \simeq H_{\mathbf{D}}$. Substituting this solution into the conservation equation yields $\rho \simeq 0$. Therefore, the matter energy density is generated when starting to deviate from the inflationary solution $H = H_{\mathbf{D}}$. The coupling constant increases gradually during inflation and rapidly in the neighborhood of the spacetime phase transition. Along with this, the dynamical factor B_0 decreases gradually and disappears at the phase transition point.[3]

The generation of the matter energy density can be explained from the fact that a new Wess-Zumino interaction associated with conformal anomaly such as

$$\phi\, F_{\mu\nu}^2$$

opens at $t \neq 0$. At the time of the phase transition, the strength of this interaction becomes very large, and it is considered that the big bang occurs when the conformal-factor fields changes to matter fields at once.

At the phase transition point, the fourth-order derivative terms disappear. From the conservation equation (12-8), we can see that gravitational energies that these terms have are transferred to matter energies and the density $\rho(\tau_\Lambda) = 3M_{\rm P}^2H^2(\tau_\Lambda)$ is generated. This is easy to understand by considering the following equation obtained by differentiating conservation equation with time:

$$\dot{\rho} + 4H\rho = \frac{b_c}{8\pi^2}\dot{B}_0(\tau)\Big(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} + 3H^4\Big).$$

[3] At the phase transition point, the third derivative of the Hubble variable diverges when $0 < \kappa < 1$. When $\kappa = 1$, the second derivative also diverges, but in any case $B_0\ddot{H}$ becomes finite, thus the matter energy density which is a physical quantity remains finite.

The right-hand side corresponds to a source term, which means that when the dynamical factor B_0 is greatly changed with time, the matter is generated. In this way, we can explain inflation and the big bang without introducing an artificial scalar degree of freedom like inflaton.

If the inflationary era is defined by the period from the Planck time $\tau_{\mathrm{P}}\ (= 1/H_{\mathbf{D}})$ at which the universe starts expanding rapidly to the dynamical time $\tau_\Lambda\ (= 1/\Lambda_{\mathrm{QG}})$ at which the spacetime phase transition occurs, the expansion rate of the universe called the number of e-foldings in this period is given by

$$\mathcal{N}_e = \log \frac{a(\tau_\Lambda)}{a(\tau_{\mathrm{P}})}.$$

If expanding almost exponentially as $a \simeq e^{H_{\mathbf{D}}\tau}$ until the phase transition, the number of e-foldings is given by a ratio of the two energy scales as

$$\mathcal{N}_e \simeq \frac{H_{\mathbf{D}}}{\Lambda_{\mathrm{QG}}}. \tag{12-9}$$

The actual number of e-foldings changes depending on the dynamical parameters β_0, a_1, and κ. Since these are phenomenological parameters that depend on strong-coupling dynamics of t, we do not consider them to be exact here and we will choose appropriately for convenience of the calculation. In Figs. 12-1 and 12-2, the calculation results in the case of $H_{\mathbf{D}}/\Lambda_{\mathrm{QG}} = 60$, $\beta_0/b_c = 0.06$, $a_1/b_c = 0.01$, and $\kappa = 0.5$ are shown. In these figures, the scale is normalized as $H_{\mathbf{D}} = 1$, thus the phase transition time is $\tau_\Lambda = 60$. The number of e-foldings then becomes $\mathcal{N}_e = 65.0$. The low energy region after the transition ($\tau > \tau_\Lambda$) will be described in the next section.

Actually, the total number of e-foldings in the inflationary era will be about $\mathcal{N}_e = 70$, which is given by the sum of the rapid expansion until the phase transition and the following expansion until settling to the Friedmann solution after that. This means that the scale factor expands by 10^{30} times. This value is used in the evolution scenario of the universe shown at the end of this chapter.

With $b_c = 10$ as mentioned before, $H_{\mathbf{D}} = 6.7 \times 10^{18}$ GeV is obtained since the reduced Planck mass is $M_{\mathrm{P}} = 2.4 \times 10^{18}$ GeV. From this, the dynamical energy scale deriving the above number of e-foldings is given by

$$\Lambda_{\mathrm{QG}} = 1.1 \times 10^{17}\ \mathrm{GeV}. \tag{12-10}$$

Moreover, using this value to estimate an amplitude of scalar fluctuations from dimensional analysis, it is expected to decrease in the inflationary era

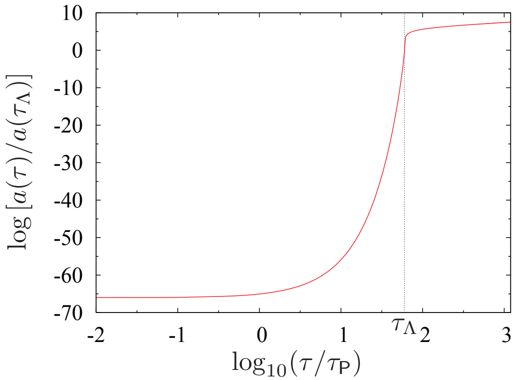

Figure 12-1: Time evolution of the scale factor $a(\tau)$. It begins to expand rapidly from the Planck time τ_{P}. The inflationary expansion terminates at the dynamical time τ_Λ $(= 60\tau_{\mathrm{P}})$, and there the universe turns to the classical Friedmann spacetime.

and become

$$\left.\frac{\delta R}{R}\right|_{\tau_\Lambda} \sim \frac{\Lambda_{\mathrm{QG}}^2}{12H_{\mathbf{D}}^2} \sim 10^{-5}$$

at the time of the spacetime phase transition. The denominator is the curvature of the inflationary (de Sitter) spacetime. This value agrees with the magnitude of the scalar amplitude required from CMB observation. In Chapter 14, we will see that the amplitude actually decreases considering time evolution of the fluctuation. Quantum gravity cosmology based on this inflationary scenario is summarized in Fig. 12-3.

Low Energy Effective Gravity Theory

The Einstein-Hilbert action dominates in the low energy region below the dynamical energy scale Λ_{QG}, and the classical spacetime where particles come and go as we normally consider appears. In this section, we discuss the low energy effective theory of quantum gravity.

In the case of QCD, dynamics at high energy beyond the dynamical scale Λ_{QCD} is described by the kinetic term of the gauge field, but below this scale the kinetic term disappears, and mesons and baryons become dynamical field variables. The situation is slightly different in the renormalizable quantum gravity. In energy regions sufficiently higher than Λ_{QG}, the conformal-factor field and the traceless tensor field, which are two modes of

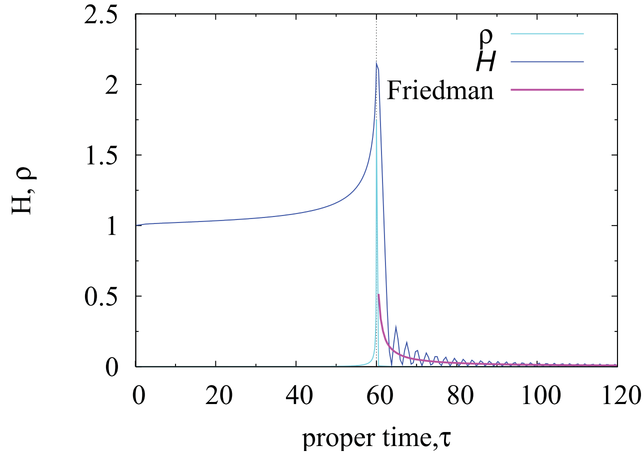

Figure 12-2: Time evolution of the Hubble variable H and the matter energy density ρ. The scale is normalized by $H_{\mathbf{D}} = 1$. After the phase transition at $\tau = 60$, the solution gradually approaches the Friedmann solution as time passes.

the gravitational field, each have the unique kinetic terms and describe conformally invariant dynamics. Below Λ_{QG}, the conformal gravity dynamics is lost with the disappearance of these kinetic terms, but the Einstein-Hilbert action remains as a kinetic term of the gravitational field. Therefore, as a composite field in which the two modes are tightly coupled, the gravitational field still remains as a dynamical variable.

The low energy effective theory of quantum gravity is given by an expansion in derivatives of the gravitational field as

$$I_{\mathrm{low}} = \int d^4x \sqrt{-g} \left\{ \mathcal{L}_2 + \mathcal{L}_4 + \cdots \right\},$$

where the number of subscripts represents the order of derivatives. The cosmological term is not considered because it does not contain derivatives and can be ignored in the early universe. The second-order derivative term that consists of the Einstein-Hilbert action and matter actions is given by

$$\mathcal{L}_2 = \frac{M_{\mathrm{P}}^2}{2} R + \mathcal{L}_{\mathrm{M}}$$

where $\mathcal{L}_{\mathrm{M}}$ is a matter action density.

The reduced Planck mass M_{P} is analogous to the pion decay constant $4\pi F_\pi$ in chiral perturbation theory. The expansion is performed by the inverse of the reduced Planck mass, which is guaranteed by the magnitude relationship of $M_{\mathsf{P}} \gg \Lambda_{\mathrm{QG}}$. In practice, we employ the terms up to $\mathcal{L}_4$ given by

$$R^2, \quad R^2_{\mu\nu}, \quad R^2_{\mu\nu\lambda\sigma}, \quad \frac{1}{M_{\mathsf{P}}^2} R_{\mu\nu} T^{\mu\nu}, \quad \frac{1}{M_{\mathsf{P}}^4} T^{\mu\nu} T_{\mu\nu},$$

where $T_{\mu\nu}$ is a matter energy-momentum tensor satisfying the traceless condition.

Since the low energy effective theory is defined as an expansion around the Einstein theory which is the lowest order, we consider that higher order terms related each other through the Einstein equation $M_P^2 R_{\mu\nu} = T_{\mu\nu}$ are not independent ones. The Einstein equation is also $R = 0$ due to the traceless condition of $T_{\mu\nu}$. Taking into account these equations and the Euler relation to remove the square of the Riemann curvature tensor, we can reduce the number of the terms in $\mathcal{L}_4$ to one as

$$\mathcal{L}_4 = \frac{\kappa}{(4\pi)^2} R^{\mu\nu} R_{\mu\nu},$$

where κ is a positive parameter which is phenomenologically determined.

The coupling constant κ undergoes loop corrections from $\mathcal{L}_2$. With introducing a cutoff E $(< \Lambda_{\mathrm{QG}})$, calculations can be done by using the background field method about a background satisfying the Einstein equation. Including quantum corrections in κ, the following function that depends on the cutoff is obtained:

$$\kappa(E) = \kappa(\Lambda_{\mathrm{QG}}) + \zeta \log(E^2/\Lambda_{\mathrm{QG}}^2). \tag{12-11}$$

For contributions from Feynman diagrams in which only matter fields propagate inside, we obtain $\zeta = (N_S + 6N_F + 12N_A)/120$, where N_S, N_F, and N_A are the numbers of scalars, Dirac fermions, and gauge fields, respectively. If $\kappa(\Lambda_{\mathrm{QG}})$ is taken to be a positive number, the phenomenological coupling constant $\kappa(E)$ indicates that since ζ is positive it decreases at low energy and the fourth order terms become irrelevant immediately.

In addition, although ghost poles arising from the higher-order derivative terms appear at the Planck scale, the low energy effective theory is defined in energy regions less than Λ_{QG}. Hence, ghosts do not appear and there is no conflict with the unitarity problem.

The equation of motion for the homogeneous component is given by

$$M_{\mathsf{P}}^2\left(\dot{H} + 2H^2\right) + \frac{\kappa}{4\pi^2}\left(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H}\right) = 0, \tag{12-12}$$

and the energy conservation equation is

$$-3M_{\mathsf{P}}^2H^2 + \rho + \frac{\kappa}{4\pi^2}\left(-6H\ddot{H} + 3\dot{H}^2 - 18H^2\dot{H}\right) = 0. \quad (12\text{-}13)$$

As in the previous section, we take in quantum effects by replacing the coupling constant κ with a time-dependent running coupling constant $\kappa(\tau)$. It is defined by replacing the cutoff with $E = 1/\tau$ as

$$\kappa(\tau) = \kappa_\Lambda + \zeta \log\left(\frac{1}{\tau^2\Lambda_{\mathrm{QG}}^2}\right) \simeq \frac{\kappa_\Lambda}{1 + \frac{\zeta}{\kappa_\Lambda}\log(\tau^2\Lambda_{\mathrm{QG}}^2)},$$

where $\kappa_\Lambda = \kappa(\Lambda_{\mathrm{QG}})$. The last rewriting assumes that the running coupling constant eventually vanishes.

In order to describe behaviors before and after the phase transition, a non-perturbative method such as lattice QCD is necessary, but here let us simply connect the inflationary solution and the solution obtained by solving the equations of motion of the low energy effective theory at the time $\tau = \tau_\Lambda$. The initial values of H, $\dot{H}$, and ρ for solving the coupled equations (12-12) and (12-13) are chosen so that the scale factor a is smoothly connected with the inflationary solution. The initial value of $\ddot{H}$ to solve (12-12) is then determined from the conservation equation (12-13). The numerical results are shown in Figs. 12-1 and 12-2, where the parameters are chosen as $\kappa_\Lambda = 1$ and $\zeta = 1$.

The equations of motion (12-12) and (12-13) include the Friedmann solution that satisfies $\dot{H} + 2H^2 = 0$ and $3M_{\mathsf{P}}^2H^2 = \rho$. At first, the value of H decreases sharply, and it gradually approaches the Friedmann solution while oscillating. The asymptotic Friedmann solution is also written in Fig. 12-2.

Order parameters The scalar curvature is a variable which greatly changes before and after the phase transition, because inflation is expressed as $R \neq 0$, whereas the Friedmann solution is $R = 0$. To see behaviors of the change, we introduce the scalar curvature $R = 6\dot{H} + 12H^2$ as a variable and rewrite (12-12) and (12-13) as

$$\ddot{R} + 3H\dot{R} + \frac{4\pi^2}{\kappa}M_{\mathsf{P}}^2R = 0,$$
$$\rho = 3M_{\mathsf{P}}^2H^2 + \frac{\kappa}{4\pi^2}\left(H\dot{R} + H^2R - \frac{1}{12}R^2\right).$$

Defining the mass scale $m_{\mathrm{rsp}} = M_{\mathsf{P}}\sqrt{8\pi^2/2\kappa}$ of the order of the Planck scale, this equation shows that the spacetime changes from $R \neq 0$ to $R = 0$ within about the Planck time of $1/m_{\mathrm{rsp}}$.

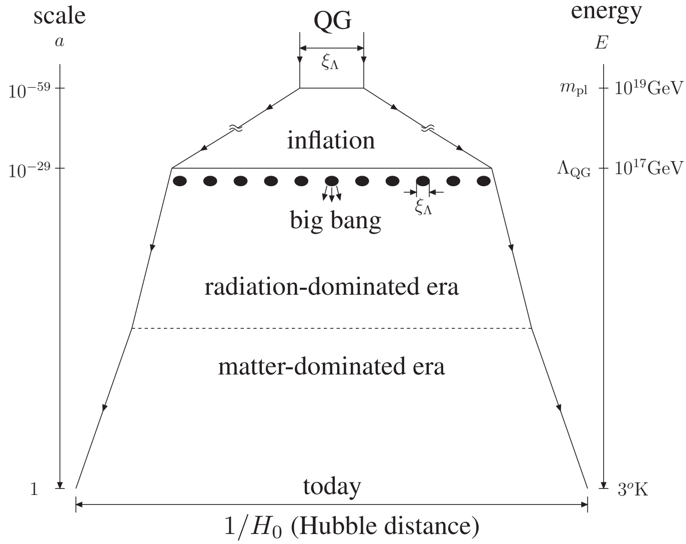

Figure 12-3: Quantum gravity cosmology. A fluctuation which was the size of the correlation length $\xi_\Lambda = 1/\Lambda_{\rm QG}$ ($\gg l_{\rm pl}$) before the Planck time expands about 10^{59} times until today to the Hubble distance $1/H_0$ ($\simeq 5000\,{\rm Mpc}$) comparable to the size of the universe, namely $1/H_0 \simeq 10^{59}\xi_\Lambda$.

Chapter Thirteen

Cosmological Perturbation Theory

Cosmological perturbation theory is a method to describe time evolution of the universe in linear approximation assuming that fluctuations (perturbations) around a certain homogeneous background are sufficiently small. In this chapter, we solve the Einstein equation around the Friedmann solution and examine time evolution of fluctuations. Once we know how small fluctuations obtained by inflation evolve after the big bang, we can get information on quantum fluctuations at the beginning of the universe from the current observations by tracing back in time.

Perturbation Variables

First, we introduce gauge invariant perturbation variables that are used in cosmological perturbation theory. The gravitational field is decomposed into the conformal-factor field ϕ and the traceless tensor field $h_{\mu\nu}$, which is expanded as

$$g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}, \quad \bar{g}_{\mu\nu} = (e^h)_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \cdots,$$

where the indices of the traceless tensor field are raised or lowered by the flat metric $\eta_{\mu\nu}$ and $h^\lambda{}_\lambda = 0$. The conformal-factor field is also decomposed into a background field $\hat{\phi}$ and perturbation φ as

$$\phi(\eta, \mathbf{x}) = \hat{\phi}(\eta) + \varphi(\eta, \mathbf{x}).$$

The gravitational field is then expanded within linear approximation as follows:[1]

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= e^{2\hat{\phi}}(1+2\varphi)\left(\eta_{\mu\nu}+h_{\mu\nu}\right)dx^\mu dx^\nu \\ &= a^2\Big\{-(1+2\varphi-h_{00})d\eta^2 + 2h_{0i}d\eta dx^i \\ &\qquad + \left(\delta_{ij}+2\varphi\delta_{ij}+h_{ij}\right)dx^i dx^j\Big\}, \end{aligned}$$

[1] The symbols commonly used in cosmological perturbation theory are A, B_i, and H_{ij}, which are defined by $ds^2 = a^2\left\{-(1+2A)d\eta^2 - 2B_i d\eta dx^i + \left(\delta_{ij}+2H_{ij}\right)dx^i dx^j\right\}$.

where $a = e^{\hat{\phi}}$ (11-5) is the scale factor and $i, j = 1, 2, 3$ represents the components of the spatial coordinates. Perturbations for energy density and pressure of matters are defined for each state α as

$$\begin{aligned}\rho_\alpha(\eta, \mathbf{x}) &= \rho_\alpha(\eta) + \delta\rho_\alpha(\eta, \mathbf{x}),\\ P_\alpha(\eta, \mathbf{x}) &= P_\alpha(\eta) + \delta P_\alpha(\eta, \mathbf{x}).\end{aligned}$$

In the following, unless otherwise noted, ρ and P represent the homogeneous parts that depend only on time.

Matters are described as almost perfect fluids. For each state α, the energy-momentum tensor is given as

$$T_{(\alpha)\nu}^{\ \mu} = \{\rho_\alpha(\eta, \mathbf{x}) + P_\alpha(\eta, \mathbf{x})\}\, u_\alpha^\mu u_\nu^\alpha + P_\alpha(\eta, \mathbf{x})\left(\delta^\mu_{\ \nu} + \Pi^{\alpha\mu}_{\ \ \ \nu}\right),$$

where $\Pi^\alpha_{\mu\nu}$ is an anisotropic stress tensor that represents deviations from perfect fluids, which has only spatial traceless components such that $\Pi^\alpha_{0\nu}$. The variable u^μ_α is a four-velocity of particles in state α satisfying

$$g_{\mu\nu} u^\mu_\alpha u^\nu_\alpha = -1. \tag{13-1}$$

The four-velocity without perturbations is $u^\mu_\alpha = (1/a, 0, 0, 0)$. By solving (13-1) to the first order of perturbations, the variables u^μ_α and $u^\alpha_\mu = g_{\mu\nu} u^\nu_\alpha$ are given by

$$\begin{aligned}u^0_\alpha &= \frac{1}{a}\left(1 - \varphi + \frac{1}{2}h_{00}\right), \quad u^i_\alpha = \frac{v^i_\alpha}{a},\\ u^\alpha_0 &= -a\left(1 + \varphi - \frac{1}{2}h_{00}\right), \quad u^\alpha_i = a\left(v^\alpha_i + h_{0i}\right),\end{aligned}$$

where v^i_α and $v^\alpha_i = \delta_{ij} v^j_\alpha$ are spatial components of the four-velocity which are perturbation variables not determined from (13-1).

Substituting the expression of the four-velocity into the matter energy-momentum tensor and expanding it up to the first order, we obtain

$$\begin{aligned}T_{(\alpha)0}^{\ 0} &= -(\rho_\alpha + \delta\rho_\alpha),\\ T_{(\alpha)0}^{\ i} &= -(\rho_\alpha + P_\alpha) v^i_\alpha,\\ T_{(\alpha)j}^{\ 0} &= (\rho_\alpha + P_\alpha)(v^\alpha_j + h_{0j}),\\ T_{(\alpha)j}^{\ i} &= (P_\alpha + \delta P_\alpha)\delta^i_{\ j} + P_\alpha \Pi^{\alpha i}_{\ \ \ j}.\end{aligned} \tag{13-2}$$

When written in terms of $T_{(\alpha)\nu}^{\ \mu}$ as in the left-hand side, the indices are raised or lowered by the physical metric $g_{\mu\nu}$, but the spatial indices of perturbation variables in the right-hand side are done with δ_{ij} such as $v_i = \delta_{ij}v^j$. That is why the second and third equations are not symmetric, namely $T^0_{\ i} = g^{0\lambda}T_{\lambda i} \neq g^{i\lambda}T_{\lambda 0} = T^i_{\ 0}$.

The velocity perturbation variable is decomposed as

$$v_i^\alpha = \partial_i v^\alpha + v_i^{\mathrm{T}\alpha},$$

where $v_i^{\mathrm{T}\alpha}$ satisfies the transverse condition. The anisotropy stress tensor satisfies the traceless condition $\Pi^{\alpha i}{}_i = 0$.

Gauge transformations In linear approximation, the perturbations of the gravitational field are transformed as

$$\delta_\xi \varphi = \xi^\lambda \partial_\lambda \hat{\phi} + \frac{1}{4}\partial_\lambda \xi^\lambda,$$
$$\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda \xi^\lambda$$

under diffeomorphism $\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda$, where the subscript of the gauge transformation parameter is defined by $\xi_\mu = \eta_{\mu\nu}\xi^\nu$ using the flat metric. The traceless tensor field is further decomposed as

$$h_{00} = h, \qquad h_{0i} = h_i^{\mathrm{T}} + \partial_i h',$$
$$h_{ij} = h_{ij}^{\mathrm{TT}} + \partial_{(i}h_{j)}^{\mathrm{T}\prime} + \frac{1}{3}\delta_{ij}h + \left(\frac{\partial_i\partial_j}{\partial\!\!\!/^2} - \frac{1}{3}\delta_{ij}\right)h'',$$

where the vectors h_i^{T} and $h_i^{\mathrm{T}\prime}$ satisfy the transverse condition and the tensor h_{ij}^{TT} satisfies the transverse and traceless conditions. The spatial Laplacian $\partial\!\!\!/^2 = \partial^i\partial_i$ is defined in the comoving coordinates. Decomposing the parameter ξ^μ into ξ^0 and $\xi_i = \xi_i^{\mathrm{T}} + \partial_i \xi^{\mathrm{S}}$, the transformation can be expressed as

$$\delta_\xi \varphi = \xi^0 \partial_\eta \hat{\phi} + \frac{1}{4}\partial_\eta \xi^0 + \frac{1}{4}\partial\!\!\!/^2 \xi^{\mathrm{S}},$$
$$\delta_\xi h = -\frac{3}{2}\partial_\eta \xi^0 + \frac{1}{2}\partial\!\!\!/^2 \xi^{\mathrm{S}}, \quad \delta_\xi h' = -\xi^0 + \partial_\eta \xi^{\mathrm{S}}, \quad \delta_\xi h'' = 2\partial\!\!\!/^2 \xi^{\mathrm{S}},$$
$$\delta_\xi h_i^{\mathrm{T}} = \partial_\eta \xi_i^{\mathrm{T}}, \quad \delta_\xi h_i^{\mathrm{T}\prime} = 2\xi_i^{\mathrm{T}}, \quad \delta_\xi h_{ij}^{\mathrm{TT}} = 0.$$

The matter energy-momentum tensor is transformed under diffeomorphism as

$$\delta_\xi T_{(\alpha)\ \nu}^{\ \ \mu} = \partial_\nu \xi^\lambda T_{(\alpha)\ \lambda}^{\ \ \mu} - \partial_\lambda \xi^\mu T_{(\alpha)\ \nu}^{\ \ \lambda} + \xi^\lambda \partial_\lambda T_{(\alpha)\ \nu}^{\ \ \mu}.$$

Therefore, the matter perturbations are transformed for each state α as

$$\delta_\xi v^\alpha = -\partial_\eta \xi^{\mathrm{S}}, \qquad \delta_\xi v_i^{\mathrm{T}\alpha} = -\partial_\eta \xi_i^{\mathrm{T}},$$
$$\delta_\xi(\delta\rho_\alpha) = \xi^0 \partial_\eta \rho_\alpha, \qquad \delta_\xi(\delta P_\alpha) = \xi^0 \partial_\eta P_\alpha, \qquad \delta_\xi \Pi^{\alpha i}{}_j = 0.$$

Gauge invariant variables Introduce gauge invariant variables within the linear approximation. As scalar variables, there are two gravitational potentials called the Bardeen potentials defined by

$$\begin{aligned} \Phi &= \varphi + \frac{1}{6}h - \frac{1}{6}h'' + \sigma\partial_\eta\hat{\phi}, \\ \Psi &= \varphi - \frac{1}{2}h + \sigma\partial_\eta\hat{\phi} + \partial_\eta\sigma, \end{aligned} \tag{13-3}$$

where

$$\sigma = h' - \frac{1}{2}\frac{\partial_\eta h''}{\partial^2}. \tag{13-4}$$

Since the σ variable transforms as $\delta_\xi\sigma = -\xi^0$, we can easily show that the Bardeen potentials are gauge invariant variables satisfying $\delta_\xi\Phi = \delta_\xi\Psi = 0$.

Adopting the $h' = h'' = 0$ gauge called the conformal Newtonian gauge or the longitudinal gauge, the Bardeen potentials are written as $\Phi = \varphi + h/6$ and $\Psi = \varphi - h/2$. Then, the scalar components of the metric are simply expressed as

$$ds^2 = a^2\left[-\left(1+2\Psi\right)d\eta^2 + \left(1+2\Phi\right)d\mathbf{x}^2\right].$$

From this expression, Ψ which appears in the time component is also called the Newton potential.

Gauge invariant vector and tensor perturbations of the gravitational field are defined by

$$\Upsilon_i = h_i^{\mathrm{T}} - \frac{1}{2}\partial_\eta h_i^{\mathrm{T}\prime}, \qquad h_{ij}^{\mathrm{TT}}.$$

The transverse-traceless tensor field becomes gauge invariant in itself.

Gauge-invariant perturbation variables commonly used for each matter state α are given by

$$\begin{aligned} V^\alpha &= v^\alpha + \frac{1}{2}\frac{\partial_\eta h''}{\partial^2}, \qquad V_i^\alpha = v_i^{\mathrm{T}\alpha} + \frac{1}{2}\partial_\eta h_i^{\mathrm{T}\prime}, \\ D^\alpha &= \frac{\delta\rho_\alpha}{\rho_\alpha} + \frac{\partial_\eta\rho_\alpha}{\rho_\alpha}\sigma - 3(1+w_\alpha)\partial_\eta\hat{\phi}V^\alpha \\ &= \frac{\delta\rho_\alpha}{\rho_\alpha} - 3(1+w_\alpha)\partial_\eta\hat{\phi}(\sigma + V^\alpha), \\ \mathcal{D}^\alpha &= \frac{\delta\rho_\alpha}{\rho_\alpha} + \frac{\partial_\eta\rho_\alpha}{\rho_\alpha}\sigma + 3(1+w_\alpha)\Phi \\ &= \frac{\delta\rho_\alpha}{\rho_\alpha} + 3(1+w_\alpha)(\Phi - \partial_\eta\hat{\phi}\sigma), \\ \Omega_i^\alpha &= v_i^{\mathrm{T}\alpha} + h_i^{\mathrm{T}}, \qquad \Pi_{ij}^\alpha. \end{aligned} \tag{13-5}$$

In the variables D^α and $\mathcal{D}^\alpha$, the $\partial_\eta\rho_\alpha$ terms are rewritten using the conservation equation $\partial_\eta\rho_\alpha = -3(1+w_\alpha)\partial_\eta\hat{\phi}\rho_\alpha$ without a source. Therefore, when there is a source, they must be defined by the first expressions. The scalar and vector variables introduced here are not independent, which satisfy $D^\alpha = \mathcal{D}^\alpha - 3(1+w_\alpha)(\Phi + \partial_\eta\hat{\phi}V^\alpha)$ and $\Upsilon_i + V_i^\alpha - \Omega_i^\alpha = 0$, respectively. As discussed in the next section, D^α is an energy density variable appearing on the right-hand side of the Poisson equation. On the other hand, $\mathcal{D}^\alpha$ is a useful density variable when considering angular power spectra of CMB.[2]

The variables ρ and P defined as the sum of states are already introduced in (11-4). For the perturbation variables, D, $\mathcal{D}$, V, and Π_{ij} are defined as

$$\begin{aligned}
\rho D &= \sum_\alpha \rho_\alpha D^\alpha, \qquad \rho\mathcal{D} = \sum_\alpha \rho_\alpha \mathcal{D}^\alpha, \\
(1+w)\rho V &= (\rho + P)V = \sum_\alpha (\rho_\alpha + P_\alpha)V^\alpha = \sum_\alpha (1+w_\alpha)\rho_\alpha V^\alpha, \\
P\Pi_{ij} &= \sum_\alpha P_\alpha \Pi_{ij}^\alpha.
\end{aligned} \tag{13-6}$$

The equation-of-state parameter is defined by

$$w = \frac{P}{\rho} = \frac{\sum_\alpha P_\alpha}{\sum_\alpha \rho_\alpha}.$$

Here, it should be noted that $D \neq \sum_\alpha D^\alpha$, $\mathcal{D} \neq \sum_\alpha \mathcal{D}^\alpha$, $V \neq \sum_\alpha V^\alpha$, and $w \neq \sum_\alpha w_\alpha$. This is because D^α and $\mathcal{D}^\alpha$ are defined by dividing by ρ_α, while quantities that can be summed must be in the form $\rho_\alpha D_\alpha \sim \delta\rho_\alpha$ which appears in the energy-momentum tensor. Similarly, since the velocity variable appears in the form of $(\rho_\alpha + P_\alpha)V^\alpha$, the sum V must be defined by the above equation.

Finally we introduce variables related to entropy. Between pressure and energy density perturbations, the thermodynamic relationship

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \delta S = c_s^2\delta\rho + T\delta S$$

holds. If considering an adiabatic fluid ($\delta S = 0$), δP becomes proportional to $\delta\rho$ and its coefficient is given by the sound speed squared, $c_s^2 = \partial P/\partial\rho$.

[2] The relationship between the photon density variable and the variable Θ often used is $\mathcal{D}^\gamma/4 = \Theta + \Phi$ in the conformal Newtonian gauge.

As a gauge invariant variable associated with the thermodynamic relationship, we introduce

$$\Gamma^\alpha = \frac{1}{P_\alpha}\left(\delta P_\alpha - c_\alpha^2 \delta\rho_\alpha\right) = \frac{\delta P_\alpha}{P_\alpha} - \frac{c_\alpha^2}{w_\alpha}\frac{\delta\rho_\alpha}{\rho_\alpha}$$

proportional to entropy of each state α.

In addition, the invariant Γ representing entropy of the whole system is defined by

$$P\Gamma = \delta P - c_s^2 \delta\rho, \tag{13-7}$$

where δP and $\delta\rho$ are the simple sum of states and the sound speed squared is defined as

$$c_s^2 = \frac{\partial_\eta P}{\partial_\eta \rho} = \frac{\sum_\alpha \partial_\eta P_\alpha}{\sum_\alpha \partial_\eta \rho_\alpha}.$$

Note here that $c_s^2 \neq \sum_\alpha c_\alpha^2$.

Evolution Equations of Fluctuations

Let us derive evolution equations of fluctuations from the Einstein equation and conservation equations for each matter state α.

Einstein Equations

In order to treat the conformal-factor field specially, a variation of the action I is defined as follows:

$$\begin{aligned}\delta I &= \frac{1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu}\delta g_{\mu\nu} \\ &= \frac{1}{2}\int d^4x \sqrt{-\bar{g}}\left\{2\bar{T}^\lambda{}_\lambda \delta\phi + \bar{T}^{\mu\nu}\delta\bar{g}_{\mu\nu}\right\} \\ &= \int d^4x \left\{\mathbf{T}^\lambda{}_\lambda \delta\phi + \frac{1}{2}\mathbf{T}^\mu{}_\nu \delta h^\nu{}_\mu\right\}.\end{aligned}$$

Here we introduce three kinds of energy-momentum tensors, $T_{\mu\nu}$, $\bar{T}_{\mu\nu}$, and $\mathbf{T}_{\mu\nu}$. These are useful when considering perturbations around a conformally flat spacetime. The second equality is shown by using an expression that the variation of the metric $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$ is decomposed in modes as

$$\delta g_{\mu\nu} = 2e^{2\phi}\bar{g}_{\mu\nu}\delta\phi + e^{2\phi}\delta\bar{g}_{\mu\nu}.$$

For each tensor, we should be aware of the metric used to contract the index. The normal energy-momentum tensor $T_{\mu\nu}(g)$ defined in the first equation is contracted with the physical metric $g_{\mu\nu}$. The tensor $\bar{T}_{\mu\nu}(\phi, \bar{g})$ on the second line is done with the metric $\bar{g}_{\mu\nu}$ excluding the conformal-factor field, and the last $\mathbf{T}_{\mu\nu}(\varphi, h)$ is with the flat metric $\eta_{\mu\nu}$.

The difference between the normal energy-momentum tensor and the tensor with the bar appears as a conformal-factor dependence, which is expressed as $T^{\mu\nu} = e^{-6\phi}\bar{T}^{\mu\nu} = e^{-6\hat{\phi}}(1-6\varphi)\bar{T}^{\mu\nu}$ or $T^{\mu}_{\ \nu} = e^{-4\phi}\bar{T}^{\mu}_{\ \nu} = e^{-4\hat{\phi}}(1-4\varphi)\bar{T}^{\mu}_{\ \nu}$. The relationship between the tensors with the bar and written in the bold is given by $\mathbf{T}_{\mu\nu} = \eta_{\lambda(\mu}\bar{T}^{\lambda}_{\ \nu)}$, symmetrized within the linear approximation, which satisfies $\mathbf{T}^{\lambda}_{\ \lambda}(= \eta^{\mu\nu}\mathbf{T}_{\mu\nu}) = \bar{T}^{\lambda}_{\ \lambda}$ by definition.

The Einstein equation can be written by the sum of the Einstein term, the cosmological term, and the matter term as

$$\mathbf{T}_{\mu\nu} = \mathbf{T}^{\mathrm{EH}}_{\mu\nu} + \mathbf{T}^{\Lambda}_{\mu\nu} + \mathbf{T}^{\mathrm{M}}_{\mu\nu} = 0,$$

where the matter energy-momentum tensor is given by the sum of all states as

$$\mathbf{T}^{\mathrm{M}}_{\mu\nu} = \sum_{\alpha} \mathbf{T}^{(\alpha)}_{\mu\nu}.$$

Rewriting (13-2) to the tensor in the bold and summing all states, it is expressed as

$$\begin{aligned}
\mathbf{T}^{\mathrm{M}\lambda}_{\ \ \ \lambda} &= e^{4\hat{\phi}}\left\{-\rho + 3P - \delta\rho + 3\delta P + 4(-\rho + 3P)\varphi\right\}, \\
\mathbf{T}^{\mathrm{M}}_{00} &= e^{4\hat{\phi}}(\rho + \delta\rho + 4\rho\varphi), \\
\mathbf{T}^{\mathrm{M}}_{0i} &= -e^{4\hat{\phi}}(\rho + P)\left(v_i + \frac{1}{2}h_{0i}\right), \\
\mathbf{T}^{\mathrm{M}}_{ij} &= e^{4\hat{\phi}}\left\{(P + \delta P + 4P\varphi)\delta_{ij} + P\Pi_{ij}\right\},
\end{aligned} \tag{13-8}$$

where $\delta\rho$ and δP are given by the sums of states as in (11-4), and Π_{ij} is defined in (13-6). The velocity variable v_i is also given by the sum of states as in the definition of V in (13-6).

Since the anisotropic stress tensor is traceless, it is decomposed as

$$\Pi_{ij} = \left(-\frac{\partial_i\partial_j}{\partial^2} + \frac{1}{3}\delta_{ij}\right)\Pi^S + \partial_{(i}\Pi^V_{j)} + \Pi^T_{ij},$$

where Π^V_i satisfies the transverse condition and Π^T_{ij} satisfies the transverse and traceless conditions. Decomposition of the variable Π^{α}_{ij} for each state is the same.

The energy-momentum tensor derived from the Einstein-Hilbert action is given, when expanded up to the first order of the traceless tensor field, by

$$
\begin{aligned}
\mathbf{T}^{\mathrm{EH}}_{\mu\nu} = M_{\mathrm{P}}^2 e^{2\phi}\Big\{ & 2\partial_\mu\partial_\nu\phi - 2\partial_\mu\phi\partial_\nu\phi + \eta_{\mu\nu}\left(-2\partial^2\phi - \partial^\lambda\phi\partial_\lambda\phi\right) \\
& -\partial_{(\mu}\chi_{\nu)} + \frac{1}{2}\partial^2 h_{\mu\nu} - 2h^\lambda_{(\mu}\partial_{\nu)}\partial_\lambda\phi + 2h^\lambda_{(\mu}\partial_{\nu)}\phi\partial_\lambda\phi \\
& -2\partial_{(\mu}h^\lambda_{\nu)}\partial_\lambda\phi + \partial^\lambda h_{\mu\nu}\partial_\lambda\phi \\
& +\eta_{\mu\nu}\left(\frac{1}{2}\partial_\lambda\chi^\lambda + 2h^{\lambda\sigma}\partial_\lambda\partial_\sigma\phi + h^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi + 2\chi^\lambda\partial_\lambda\phi\right)\Big\},
\end{aligned}
$$

where $\chi_\mu = \partial_\lambda h^\lambda{}_\mu$ and $\partial^2 = \partial^\lambda\partial_\lambda = -\partial_\eta^2 + \not\partial^2$. The conformal-factor field ϕ has not expanded yet. Taking the trace, we obtain

$$
\begin{aligned}
\mathbf{T}^{\mathrm{EH}\lambda}{}_\lambda = M_{\mathrm{P}}^2 e^{2\phi}\Big\{ & -6\partial^2\phi - 6\partial^\lambda\phi\partial_\lambda\phi + \partial_\lambda\chi^\lambda + 6h^{\lambda\sigma}\partial_\lambda\partial_\sigma\phi \\
& +6\chi^\lambda\partial_\lambda\phi + 6h^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi\Big\}.
\end{aligned}
$$

Decompose the conformal-factor field ϕ into the background $\hat{\phi}$ and the perturbation φ, and further expand the energy-momentum tensor up to the first order of φ. For the sake of simplicity, we take the conformal Newtonian gauge of $h' = h'' = 0$ and $h_i^{\mathrm{T}\prime} = 0$ by using four gauge degrees of freedom. Each component of the energy-momentum tensor for the Einstein term is expanded as follows:

$$
\begin{aligned}
\mathbf{T}^{\mathrm{EH}\lambda}{}_\lambda = M_{\mathrm{P}}^2 e^{2\hat{\phi}}\Big\{ & 6\partial_\eta^2\hat{\phi} + 6\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + 12(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi})\varphi + 6\partial_\eta^2\varphi \\
& -6\not\partial^2\varphi + 12\partial_\eta\hat{\phi}\partial_\eta\varphi + \partial_\eta^2 h + \frac{1}{3}\not\partial^2 h + 6\partial_\eta\hat{\phi}\partial_\eta h \\
& +6(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi})h\Big\}, \\
\mathbf{T}^{\mathrm{EH}}_{00} = M_{\mathrm{P}}^2 e^{2\hat{\phi}}\Big\{ & -3\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} - 6\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\varphi - 6\partial_\eta\hat{\phi}\partial_\eta\varphi + 2\not\partial^2\varphi \\
& -3\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}h - \partial_\eta\hat{\phi}\partial_\eta h + \frac{1}{3}\not\partial^2 h\Big\}, \\
\mathbf{T}^{\mathrm{EH}}_{0i} = M_{\mathrm{P}}^2 e^{2\hat{\phi}}\Big\{ & 2\partial_\eta\partial_i\varphi - 2\partial_\eta\hat{\phi}\partial_i\varphi + \frac{1}{3}\partial_\eta\partial_i h + \partial_\eta\hat{\phi}\partial_i h + \frac{1}{2}\not\partial^2 h_i^{\mathrm{T}} \\
& +(\partial_\eta^2\hat{\phi} - \partial_\eta\hat{\phi}\partial_\eta\hat{\phi})h_i^{\mathrm{T}}\Big\},
\end{aligned}
$$

$$\mathbf{T}^{\rm EH}_{ij} = M_{\sf P}^2 e^{2\hat\phi}\Big\{ 2\partial_i\partial_j\varphi + \delta_{ij}\Big[2\partial_\eta^2\hat\phi + \partial_\eta\hat\phi\partial_\eta\hat\phi + 2\partial_\eta^2\varphi - 2\partial\!\!\!/^2\varphi$$
$$+2\partial_\eta\hat\phi\partial_\eta\varphi + \left(4\partial_\eta^2\hat\phi + 2\partial_\eta\hat\phi\partial_\eta\hat\phi\right)\varphi\Big] - \frac{1}{3}\partial_i\partial_j h$$
$$+\delta_{ij}\left[\frac{1}{3}\partial_\eta^2 h + \frac{1}{3}\partial\!\!\!/^2 h + \frac{5}{3}\partial_\eta\hat\phi\partial_\eta h + \left(2\partial_\eta^2\hat\phi + \partial_\eta\hat\phi\partial_\eta\hat\phi\right)h\right]$$
$$+\partial_\eta\partial_{(i}h^{\rm T}_{j)} + 2\partial_\eta\hat\phi\partial_{(i}h^{\rm T}_{j)} - \frac{1}{2}\partial_\eta^2 h^{\rm TT}_{ij} + \frac{1}{2}\partial\!\!\!/^2 h^{\rm TT}_{ij} - \partial_\eta\hat\phi\partial_\eta h^{\rm TT}_{ij}\Big\}.$$

The energy-momentum tensor for the cosmological term is given by

$$\mathbf{T}^{\Lambda}_{\mu\nu} = -\Lambda_{\cos} e^{4\hat\phi}(1+4\varphi)\eta_{\mu\nu}.$$

Lastly, we rewrite the above expressions in terms of the gauge-invariant variables using $\varphi = (3\Phi+\Psi)/4$ and $h = 3(\Phi-\Psi)/2$ in the conformal Newtonian gauge.

Linear scalar equations Consider the following four types of equations satisfied by the scalar variables:

$$e^{-4\hat\phi}\mathbf{T}^{\lambda}{}_{\lambda} = 0, \qquad e^{-4\hat\phi}\left(\mathbf{T}^{i}{}_{i} - 3\frac{\partial^i\partial^j}{\partial\!\!\!/^2}\mathbf{T}_{ij}\right) = 0,$$
$$e^{-4\hat\phi}\left(\mathbf{T}_{00} + 3\partial_\eta\hat\phi\frac{\partial^i}{\partial\!\!\!/^2}\mathbf{T}_{i0}\right) = 0, \qquad e^{-4\hat\phi}\frac{\partial^i}{\partial\!\!\!/^2}\mathbf{T}_{i0} = 0. \quad (13\text{-}9)$$

The combinations on the left-hand side are normalized so that they become the left-hand side of equations which will be obtained below directly as it is.

The first trace equation is expressed as

$$M_{\sf P}^2 e^{-2\hat\phi}\Big\{6\partial_\eta^2\Phi + 18\partial_\eta\hat\phi\partial_\eta\Phi - 4\partial\!\!\!/^2\Phi - 6\partial_\eta\hat\phi\partial_\eta\Psi$$
$$+\left(12\partial_\eta^2\hat\phi + 12\partial_\eta\hat\phi\partial_\eta\hat\phi - 2\partial\!\!\!/^2\right)\Psi\Big\}$$
$$+(3c_s^2-1)\rho\left\{D + 3(1+w)\partial_\eta\hat\phi V\right\} + 3w\rho\Gamma$$
$$+(3w-1)\rho(3\Phi+\Psi) - 4\Lambda_{\cos}(3\Phi+\Psi) = 0, \quad (13\text{-}10)$$

where the matter term is also rewritten in terms of the gauge-invariant variables after replacing δP with $P\Gamma + c_s^2\delta\rho$ using (13-7). From the second equation, we obtain a relation between Φ and Ψ as

$$M_{\sf P}^2 e^{-2\hat\phi}(-2\partial\!\!\!/^2)(\Phi+\Psi) + 2P\Pi^S = 0. \quad (13\text{-}11)$$

From the third equation, we obtain the Poisson equation

$$M_{\mathsf{P}}^2\, e^{-2\hat{\phi}}\, 2\not\partial^2\Phi + \rho D = 0. \tag{13-12}$$

The fourth equation, which includes the velocity variable, becomes

$$M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \left\{ 2\partial_\eta \Phi - 2\partial_\eta \hat{\phi} \Psi \right\} - (1+w)\rho V = 0. \tag{13-13}$$

In order to derive the equations above, we used the Einstein equation (11-3) that the background $\hat{\phi}$ satisfies.

Let us set $\Pi^S = 0$. Actually, since the perfect fluid approximation holds for relatively large size fluctuations, it is not contradictory to the observations as for such fluctuations. In this case, these equation systems can be solved because there are four equations for four variables. It should be noted here that the variables that we can solve are D, V, Φ, and Ψ only which are defined by the sum of all states. If the universe can be approximated as one radiation state just after the big bang, there is no problem, but if various matter states coexist, it is necessary to solve conservation equations for each state, as will be shown below.

Linear vector equations As equations that the vector variables satisfy, we consider the following two types:

$$e^{-4\hat{\phi}} \frac{\partial^j}{\not\partial^2} \mathbf{T}_{ij} = 0, \qquad e^{-4\hat{\phi}} \mathbf{T}_{0i} = 0.$$

Extracting vector components from each equation, we obtain

$$M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \left\{ \frac{1}{2}\partial_\eta \Upsilon_i + \partial_\eta \hat{\phi} \Upsilon_i \right\} + \frac{1}{2} P \Pi_i^V = 0 \tag{13-14}$$

and

$$\frac{1}{2} M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \not\partial^2 \Upsilon_i - (1+w)\rho \Omega_i = 0. \tag{13-15}$$

Also when deriving these equations, the Einstein equation (11-3) for the background $\hat{\phi}$ is used. Like the scalar equation, (13-14) can be solved easily if $\Pi_i^V = 0$, and Ω_i can be obtained by substituting its solution into (13-15).

Linear tensor equations From $e^{-4\hat{\phi}} \mathbf{T}_{ij} = 0$, we obtain the following equation for the tensor variable:

$$M_{\mathsf{P}}^2\, e^{-2\hat{\phi}} \left\{ -\frac{1}{2}\partial_\eta^2 h_{ij}^{\mathrm{TT}} - \partial_\eta \hat{\phi} \partial_\eta h_{ij}^{\mathrm{TT}} + \frac{1}{2} \not\partial^2 h_{ij}^{\mathrm{TT}} \right\} + P \Pi_{ij}^T = 0. \tag{13-16}$$

This equation can be easily solved if $\Pi_{ij}^T = 0$.

Conservation Equations for Matter Fields

Since the variables in the Einstein equation are those of the sum of all states as shown above, we cannot follow changes in each state with the Einstein equation only. On the other hand, as far as there is no source term, the matter energy-momentum tensor (13-2) satisfies a conservation equation

$$\nabla_\mu T^{\ \mu}_{(\alpha)\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^{\ \mu}_{(\alpha)\nu}\right) + \frac{1}{2}\left(\partial_\nu g_{\mu\lambda}\right)g^{\lambda\sigma}T^{\ \mu}_{(\alpha)\sigma} = 0$$

for each state. If the state is not one, we have to solve the Einstein equation and the conservation equation in combination.

Let us consider the following normalized equations for each component of the conservation equation:

$$-\frac{1}{\rho_\alpha}\nabla_\mu T^{\ \mu}_{(\alpha)0} = 0, \qquad \frac{1}{(1+w_\alpha)\rho_\alpha}\nabla_\mu T^{\ \mu}_{(\alpha)i} = 0.$$

From the first equation, we obtain

$$\partial_\eta \mathcal{D}^\alpha + 3\left(c_\alpha^2 - w_\alpha\right)\partial_\eta\hat{\phi}\mathcal{D}^\alpha + (1+w_\alpha)\partial\!\!\!/^2 V^\alpha + 3w_\alpha\partial_\eta\hat{\phi}\Gamma^\alpha = 0. \quad (13\text{-}17)$$

By applying $\partial^i/\partial\!\!\!/^2$ to the second equation to remove the transverse component and extracting the scalar component, we obtain

$$\begin{aligned}&\partial_\eta V^\alpha + \left(1-3c_\alpha^2\right)\partial_\eta\hat{\phi}V^\alpha + \Psi - 3c_\alpha^2\Phi\\ &+\frac{c_\alpha^2}{1+w_\alpha}\mathcal{D}^\alpha + \frac{w_\alpha}{1+w_\alpha}\left[\Gamma^\alpha - \frac{2}{3}\Pi^{S\alpha}\right] = 0.\end{aligned} \quad (13\text{-}18)$$

An equation for the vector variables is also yielded by extracting the transverse component from the second equation as

$$\partial_\eta\Omega_i^\alpha + \left(1-3c_\alpha^2\right)\partial_\eta\hat{\phi}\Omega_i^\alpha + \frac{w_\alpha}{2(1+w_\alpha)}\partial\!\!\!/^2\Pi_i^{V\alpha} = 0. \quad (13\text{-}19)$$

In order to derive these equations, we used the conservation equation (11-2) for the background field and a differential equation for the equation-of-state parameter

$$\partial_\eta w_\alpha = \left(c_\alpha^2 - w_\alpha\right)\frac{\partial_\eta\rho_\alpha}{\rho_\alpha} = -3(1+w_\alpha)\left(c_\alpha^2 - w_\alpha\right)\partial_\eta\hat{\phi}.$$

What we note here is that when deforming the equations, we use the conservation equations of each state, but do not use the Einstein equation, so that the equations hold for each state.

These equations also hold true when the variables with α are all replaced with the variables defined by the sum of states. They can be easily derived by considering the equations $-\rho^{-1}\nabla_\mu T^{\mathrm{M}\mu}_{\ \ 0} = 0$ and $(1+w)^{-1}\rho^{-1}\nabla_\mu T^{\mathrm{M}\mu}_{\ \ i} = 0$.

Finally, we examine scalar equations that D^α, not $\mathcal{D}^\alpha$, satisfies. Using the relation between two variables

$$D^\alpha = \mathcal{D}^\alpha + 3\,(1+w_\alpha)\left(\Phi + \partial_\eta\hat{\phi}V^\alpha\right),$$

the equation (13-18) including a derivative of V^α can be rewritten as

$$\partial_\eta V^\alpha + \partial_\eta\hat{\phi}V^\alpha + \Psi + \frac{c_\alpha^2}{1+w_\alpha}D^\alpha + \frac{w_\alpha}{1+w_\alpha}\left[\Gamma^\alpha - \frac{2}{3}\Pi^{S\alpha}\right] = 0$$

for each state. On the other hand, when rewriting (13-17) to get a differential equation for D^α, we have to use the Einstein equation (13-13) to remove $\partial_\eta\Phi$ and also use (11-3) to remove $\partial_\eta^2\hat{\phi}$. Thus, the following equation is obtained:

$$\begin{aligned}&\partial_\eta D^\alpha - 3w_\alpha\partial_\eta\hat{\phi}D^\alpha + (1+w_\alpha)\partial^2 V^\alpha + 2w_\alpha\partial_\eta\hat{\phi}\Gamma^\alpha \\ &+\frac{3}{2M_{\mathrm{P}}^2}\,(1+w_\alpha)\,(1+w)\rho a^2\,(V - V^\alpha) = 0,\end{aligned}$$

in which the state sum variables ρ, w, and V appear in the last term. For this reason, we will use the perturbation variable $\mathcal{D}^\alpha$ in the following calculations.

Fourier Transform of Evolution Equations

We will solve the evolution equations in the comoving momentum space. Since the curvature of the three-dimensional space is now assumed to be zero, we can consider normal Fourier transform.[3] For the dimensionless scalar variables Ψ, Φ, D, and $\mathcal{D}$, Fourier transform is defined such as

$$\Psi(\eta,\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\Psi(\eta,\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Fourier transform for the dimensionless transverse vector variables V_i, Ω_i, Υ_i and the transverse traceless tensor variable h_{ij}^{TT} is the same. For the

[3] If the space is curved, it expands with harmonic functions on the space.

scalar variable V with dimension, it is defined as

$$V(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(-\frac{1}{k}\right) V(\eta, \mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $k = |\mathbf{k}|$.

In the following, the cosmological constant $\Lambda_{\cos}$ and the anisotropic stress tensor Π^{α}_{ij} are assumed to be zero for simplicity. In fact, the cosmological constant can be ignored in the era before the universe is neutralized. Its effect becomes significant near today, which is of increasing the amplitude of the large angle component of CMB spectrum known as the integrated Sachs-Wolfe effect, but we will not discuss it here. Also, we assume that matters are adiabatic fluids of $\Gamma^{\alpha} = 0$.

From (13-11), (13-12), (13-17), and (13-18), the scalar equations can be expressed in momentum space as

$$\begin{aligned}
&\Phi = -\Psi, \\
&k^2\Phi = \frac{a^2}{2M_{\mathsf{P}}^2} \sum_{\alpha} \rho_{\alpha} D^{\alpha} \\
&\quad\;\; = \frac{a^2}{2M_{\mathsf{P}}^2} \sum_{\alpha} \rho_{\alpha} \left\{ \mathcal{D}^{\alpha} + 3\left(1 + w_{\alpha}\right) \left(\Psi + aH\frac{V^{\alpha}}{k} \right) \right\}, \\
&\partial_{\eta}\mathcal{D}^{\alpha} + 3\left(c_{\alpha}^2 - w_{\alpha}\right) aH\mathcal{D}^{\alpha} = -\left(1 + w_{\alpha}\right) kV^{\alpha}, \\
&\partial_{\eta}V^{\alpha} + \left(1 - 3c_{\alpha}^2\right) aHV^{\alpha} = k\left(\Psi - 3c_{\alpha}^2\Phi\right) + \frac{c_{\alpha}^2}{1 + w_{\alpha}} k\mathcal{D}^{\alpha},
\end{aligned} \tag{13-20}$$

respectively. From (13-14) and (13-19), the vector equations are

$$\begin{aligned}
&\partial_{\eta}\Upsilon_i + 2aH\Upsilon_i = 0, \\
&\partial_{\eta}\Omega_i^{\alpha} + \left(1 - 3c_{\alpha}^2\right) aH\Omega_i^{\alpha} = 0.
\end{aligned} \tag{13-21}$$

From (13-16), the tensor equation is

$$\partial_{\eta}^2 h_{ij}^{\mathrm{TT}} + 2aH\partial_{\eta} h_{ij}^{\mathrm{TT}} + k^2 h_{ij}^{\mathrm{TT}} = 0. \tag{13-22}$$

Adiabatic Conditions

The initial universe is in a thermal equilibrium state and it is considered to be in an adiabatic state with no exchanges of heat from the outside because it is a closed system. It is also seen from the fact that the spectrum of cosmic microwave background radiation shows a Planck distribution of black body radiation. Hence, the adiabatic condition shall be imposed as the initial condition necessary to solve the evolution equation below.

We here give the adiabatic condition of a mixed fluid composed of radiations and dusts. Since the dust is $P_d = \delta P_d = 0$, energy density and pressure in this system are given by

$$\rho = \rho_r + \rho_d, \qquad \delta\rho = \delta\rho_r + \delta\rho_d,$$
$$P = P_r = \frac{1}{3}\rho_r, \qquad \delta P = \delta P_r = \frac{1}{3}\delta\rho_r.$$

The sound speed squared is then given by

$$c_s^2 = \frac{\partial_\eta P}{\partial_\eta \rho} = \frac{1}{3}\frac{1}{1+\frac{3}{4}\frac{\rho_d}{\rho_r}},$$

where the time derivatives are rewritten using the conservation equations. From these expressions, entropy of the system is calculated as

$$T\delta S = \delta P - c_s^2\delta\rho = \frac{1}{3}\frac{\rho_d}{1+\frac{3}{4}\frac{\rho_d}{\rho_r}}\left(\frac{3}{4}\frac{\delta\rho_r}{\rho_r} - \frac{\delta\rho_d}{\rho_d}\right).$$

Therefore, the adiabatic condition $\delta S = 0$ of the mixed fluid is given by

$$\frac{\delta\rho_r}{\rho_r} = \frac{4}{3}\frac{\delta\rho_d}{\rho_d}. \tag{13-23}$$

In terms of the gauge-invariant variables, it becomes $\mathcal{D}^r = (4/3)\mathcal{D}^d$. Considering that the adiabatic condition for the velocity variable is given by $V^r = V^d$, it is also $D^r = (4/3)D^d$.

The adiabatic condition as an initial condition of the evolution equation is set in the radiation-dominated era. Assuming fluctuations come from one scalar component, we here take the following initial condition as the adiabatic condition (13-23) separately holds for any pair of radiation and dust components:

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = \frac{\delta\rho_\nu}{\rho_\nu} = \frac{4}{3}\frac{\delta\rho_c}{\rho_c} = \frac{4}{3}\frac{\delta\rho_b}{\rho_b}\left(= \frac{\delta\rho}{\rho}\right), \tag{13-24}$$

where the last equality in the parenthesis represents that energy density of radiations is overwhelmingly larger than that of CDM and baryons in the radiation-dominated era. Also, as described later, since photons and baryons are strongly coupled until the universe is neutralized, they behave as one fluid with the adiabatic condition maintained in good approximation until then.

Solutions of Vector and Tensor Equations

We first solve the vector and tensor equations which are simpler than the scalar equation. After grasping its physical behavior from the equation written in terms of the physical (proper) time, we solve the equation with the conformal time as it is which is easy to handle.

Solutions in Physical Time

The linear equation can solve for each comoving momentum k. Therefore, the physical momentum $p = k/a$, which represents an actual fluctuation size for each k, decreases with the expansion of the universe. If we normalize the current scale factor to $a_0 = 1$, k represents a current fluctuation size. For example, a fluctuation $k = 0.0002\mathrm{Mpc}^{-1}$ corresponding to the current horizon size was $0.2\mathrm{Mpc}^{-1}$ when the universe was neutralized because of $1/a = 1 + z = 1100$.

Let us first examine the vector equations. Rewritten in the physical time τ using $d\tau = ad\eta$ (11-1), the first vector equation in (13-21) becomes

$$\dot{\Upsilon}_i + 2H\Upsilon_i = 0,$$

where the dot denotes a derivative with respect to τ. If the Hubble variable H is a positive constant, this equation indicates that the vector fluctuation reduces with time by $e^{-2H\tau}$. Actually, the reduction eventually stops because H is a positive function decreasing with time. However, even if the vector fluctuation is large in the initial universe, it will decrease soon and not be observed today.

The variable Ω_i^α also will be vanishing eventually. From the second equation of (13-21), we can see that although its amplitude in the radiation-dominated era does not change almost due to $c_\alpha^2 \simeq 1/3$, it starts to reduce when turning to $c_\alpha^2 < 1/3$. Therefore, the vector fluctuations are not usually considered in analysis of the CMB anisotropy spectrum.

The tensor equation (13-22) written in the physical time is given by

$$\ddot{h}_{ij}^{\mathrm{TT}} + 3H\dot{h}_{ij}^{\mathrm{TT}} + \frac{k^2}{a^2}h_{ij}^{\mathrm{TT}} = 0.$$

In this case, the physical momentum k/a appears in the last term. This term decreases as the scale factor increases with time. In regions where the last term is negligible compared to the second term, a solution satisfying $\dot{h}_{ij}^{\mathrm{TT}} = 0$ becomes stable and the tensor fluctuation maintains a constant value, but in regions where the last term dominates, the tensor fluctuation reduces.

Whether it decreases or not depends on whether the physical momentum k/a is larger or smaller than the Hubble variable H. Considering the size of fluctuation a/k in real space, the tensor fluctuation, which was larger than the horizon size $1/H$ in the early universe, decreases as it enters the horizon along with the expansion of the universe. That is, although both a/k and $1/H$ increase with the expansion of the universe, there is a time when the horizon size overtakes the fluctuation size on the way, after which the reduction of the tensor fluctuation starts. Here, a scale larger than the horizon size is called super-horizon scale, and a smaller one is called sub-horizon scale.

The wavelength $1/k$ observed as the CMB temperature fluctuation spectrum today has a size of 10 to 5000 Mpc, and it is from sizes comparable to the Hubble distance $1/H_0 = 4164$ Mpc to smaller ones. These fluctuations are all $a/k > 1/H$ in the past, and it indicates that the largest size fluctuation has been propagated to the present without entering the horizon from the beginning of the universe. That is, it means that if there is the tensor fluctuation in the early universe, it is left in large angular components of the CMB spectrum without reducing. Conversely, with small angular components, the tensor fluctuation begins to reduce at the stage of entering the horizon, and becomes so small that it can be hardly observed now.

Solutions in Conformal Time

Let us solve the same equations using the conformal time. In this case, it is convenient to introduce

$$x = k\eta \quad (0 < x < \infty)$$

as a time variable. Using this variable, a region where the size of fluctuation becomes a super-horizon scale in the radiation-dominated (matter-dominated) era is expressed as

$$\frac{a}{k} > \frac{1}{H} \implies x < 1 \; (x < 2)$$

due to $aH = \partial_\eta a/a = 1/\eta \; (2/\eta)$ from $a \propto \eta \; (\eta^2)$. In other words, if x is less than 1 (2), it is a super-horizon scale, and if time goes by more than 1 (2), it enters the horizon and becomes a sub-horizon scale. The distinction between 1 or 2 is whether it enters in the radiation-dominated era or in the matter-dominated era.

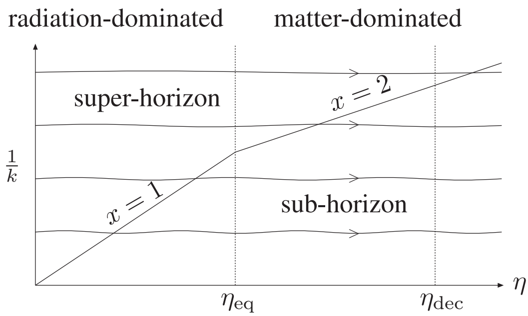

Figure 13-1: A fluctuation size $1/k$ and the time to enter the horizon. The oblique solid line represents the position of the horizon ($k = aH$).

The thing to note here is that the time variable x can still take the value of $x \ll 1$ even today for very large size fluctuations. Such fluctuations will have remained super-horizon ones from the beginning to the present. The low multipole components of $l = 2, 3$ in the CMB spectrum correspond to such fluctuations.

A typical fluctuation size $1/k$ and the time when it entered the horizon are shown in Fig. 13-1. It corresponds to $k = 0.002\text{Mpc}^{-1}$ ($l \simeq 30$), $k = 0.005\text{Mpc}^{-1}$ ($l \simeq 70$), $k = 0.015\text{Mpc}^{-1}$ ($l \simeq 210$), and $k = 0.05\text{Mpc}^{-1}$ ($l \simeq 700$) from the top, respectively, where the relationship with the multipole l is given by $l \simeq \pi/\theta = kd_{\rm dec}$ (11-9). It shows that the large angle fluctuation ($l \simeq 30$) observed today as the CMB temperature fluctuation became a sub-horizon size after the universe was neutralized. In contrast to this, we can see that the fluctuation of $l \simeq 210$ near the first acoustic peak became a sub-horizon size in the radiation-dominated era.

The vector equation in the conformal time can be written using $aH = \partial_\eta a/a$ as

$$\partial_\eta \Upsilon_i + 2aH\Upsilon_i = \partial_\eta(a^2\Upsilon_i) = 0,$$
$$\partial_\eta \Omega_i^\alpha + (1 - 3c_\alpha^2)aH\Omega_i^\alpha \simeq \partial_\eta(a^{1-3c_\alpha^2}\Omega_i^\alpha) = 0,$$

where the sound speed c_α is assumed to be a constant for simplicity. Thus, Υ_i quickly reduces with the expansion of the universe and Ω_i^α also reduces at $c_\alpha^2 < 1/3$ as follows:

$$\Upsilon_i \propto a^{-2}, \quad \Omega_i^\alpha \propto a^{3c_\alpha^2 - 1}.$$

Rewriting the tensor equation in terms of the variable x and using $aH = q/x$ in the radiation-dominated ($q = 1$) and matter-dominated ($q = 2$) eras, we obtain

$$\partial_x^2 h_{ij}^{\mathrm{TT}} + 2\frac{q}{x}\partial_x h_{ij}^{\mathrm{TT}} + h_{ij}^{\mathrm{TT}} = 0.$$

Using Bessel functions, its solution is given by $h_{ij}^{\mathrm{TT}} = e_{ij} x^{1/2-q} J_{q-1/2}(x)$, where e_{ij} is the transverse-traceless polarization tensor. It behaves as

$$h_{ij}^{\mathrm{TT}} = \begin{cases} \text{const.} & \text{for } x \ll 1 \text{ (super-horizon)} \\ \frac{1}{a} & \text{for } x > 1 \text{ (sub-horizon)}\,. \end{cases}$$

Thus, the tensor fluctuation decreases when it enters the horizon.

Solutions of Scalar Equations Without Baryons

In order to understand properties of the scalar equations, we here consider a simple system that can be easily solved, in which there are cold dark matters (CDM) and radiations only, and the anisotropic stress tensor and the cosmological constant are zero. As a time variable, we use $x = k\eta$ introduced in the previous section.

Radiation-Dominated Era

Consider a system in the radiation-dominated era where CDM and radiations exist. Since it is radiation-dominated,

$$\rho_r \gg \rho_c$$

holds. Therefore, the Friedmann equation can be approximated as $3M_{\mathsf{P}}^2 H^2 = \rho \simeq \rho_r$. Similarly, ignoring ρ_c from the right-hand side of the Poisson equation (the second equation of (13-20)), we get

$$-\Psi \simeq \frac{3}{2}\frac{1}{x^2}\left\{\mathcal{D}^r + 4\left(\Psi + \frac{1}{x}V^r\right)\right\}, \qquad \text{(13-25)}$$

where we use the first equation of (13-20), and also $aH = \partial_\eta a/a = 1/\eta$ and $a^2\rho_r/2M_{\mathsf{P}}^2 = (3/2)(aH)^2 = 3/2\eta^2$ derived from the fact that $a \propto \eta$ in the radiation-dominated era.

The conservation equations for radiations are given by

$$\partial_x \mathcal{D}^r + \frac{4}{3}V^r = 0, \qquad \partial_x V^r = 2\Psi + \frac{1}{4}\mathcal{D}^r \qquad \text{(13-26)}$$

from $w_r = c_r^2 = 1/3$. The conservation equations for CDM are

$$\partial_x \mathcal{D}^c + V^c = 0, \qquad \partial_x V^c + \frac{1}{x} V^c = \Psi \tag{13-27}$$

from $w_c = c_c^2 = 0$.

Combining the differential equations (13-25) and (13-26), we obtain

$$(x^2 + 6)\partial_x^2 \mathcal{D}^r + \frac{12}{x}\partial_x \mathcal{D}^r + \frac{1}{3}(x^2 - 6)\mathcal{D}^r = 0.$$

A general solution of this differential equation is given by

$$\begin{aligned}\mathcal{D}^r = A &\left\{\cos\left(\frac{x}{\sqrt{3}}\right) - \frac{2\sqrt{3}}{x}\sin\left(\frac{x}{\sqrt{3}}\right)\right\} \\ &+ B\left\{\sin\left(\frac{x}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{x}\cos\left(\frac{x}{\sqrt{3}}\right)\right\}.\end{aligned}$$

Imposing regularity at $x \to 0$ as an initial condition, we get $B = 0$ and thus

$$\begin{aligned}\mathcal{D}^r &= A\left\{\cos\left(\frac{x}{\sqrt{3}}\right) - \frac{2\sqrt{3}}{x}\sin\left(\frac{x}{\sqrt{3}}\right)\right\}, \\ V^r &= -\frac{3}{4}\partial_x \mathcal{D}^r = A\frac{3}{4}\left\{\frac{x^2 - 6}{\sqrt{3}x^2}\sin\left(\frac{x}{\sqrt{3}}\right) + \frac{2}{x}\cos\left(\frac{x}{\sqrt{3}}\right)\right\}, \\ \Psi &= -\frac{1}{12 + 2x^2}\left(3\mathcal{D}^r + \frac{12}{x}V^r\right).\end{aligned}$$

Each solution behaves at the super-horizon limit ($x \ll 1$) as

$$\begin{aligned}\Psi &= \Psi_{\mathrm{I}} - \frac{1}{30}\Psi_{\mathrm{I}} x^2 + \cdots, \\ \mathcal{D}^r &= -6\Psi_{\mathrm{I}} - \frac{1}{3}\Psi_{\mathrm{I}} x^2 + \cdots, \\ V^r &= \frac{1}{2}\Psi_{\mathrm{I}} x + \cdots,\end{aligned} \tag{13-28}$$

where the initial value $\Psi_{\mathrm{I}} = A/6$ of the Bardeen potential is a function of the wave number k only. In the super-horizon region, since the x^2 terms can be ignored, Ψ and $\mathcal{D}^r$ do not change almost. However, the energy density fluctuation D^r that appears on the right-hand side of the Poisson equation becomes

$$D^r = -\frac{2}{3}\Psi_{\mathrm{I}} x^2,$$

and its initial value almost vanishes. The requirement that initial fluctuations must be very small in order to solve the flatness problem means that among energy density fluctuations, this D variable is small.

The velocity fluctuation V^c of CDM can be obtained by substituting the solution of Ψ into the second equation in (13-27). Furthermore, the energy density fluctuation $\mathcal{D}^c$ can be obtained by substituting the solution of V^c into the first equation in (13-27). As initial conditions of them, the adiabatic conditions (13-24) are imposed as

$$\mathcal{D}^c(x=0) = \frac{3}{4}\mathcal{D}^r(x=0), \qquad V^c(x=0) = V^r(x=0).$$

By substituting the solution (13-28) of Ψ into (13-27) and solving under the adiabatic condition, we obtain

$$\begin{aligned} \mathcal{D}^c &= -\frac{9}{2}\Psi_{\mathrm{I}} - \frac{1}{4}\Psi_{\mathrm{I}} x^2 + \cdots, \\ V^c &= \frac{1}{2}\Psi_{\mathrm{I}} x + \cdots. \end{aligned}$$

Thus, $\mathcal{D}^c$ also does not change almost in the super-horizon region.

In the sub-horizon region ($x \gg 1$) where fluctuations enter the horizon, $\mathcal{D}^r$ and V^r begin to oscillate. On the other hand, the Bardeen potential decreases by $1/x^2$ as

$$\Psi = -\frac{3}{2x^2}\mathcal{D}^r.$$

Therefore, solving the CDM fluctuation as $\Psi \simeq 0$, we get

$$\mathcal{D}^c \propto \log x, \qquad V^c \propto -\frac{1}{x},$$

and thus the growth of $\mathcal{D}^c$ is slow even in the sub-horizon region (Meszaros effect).

Matter-Dominated Era

Since $\rho_r \ll \rho_c$ in the matter-dominated era, the Friedmann equation can be approximated by $3M_{\mathsf{P}}^2 H^2 = \rho_c$. The Poisson equation in (13-20) can be expressed as $-k^2\Psi = (a^2/2M_{\mathsf{P}}^2)\rho_c D^c = (6/\eta^2)D^c$ because of $aH = 2/\eta$ from the behavior $a \propto \eta^2$, and thus the Bardeen potential is determined from the CDM fluctuations. Substituting the parameter $w_c = c_c^2 = 0$ representing the CDM state, the differential equations that the CDM fluctuations satisfy are given by

$$-(x^2+18)\Psi = 6\mathcal{D}^c + \frac{36}{x}V^c, \quad \partial_x \mathcal{D}^c + V^c = 0, \quad \partial_x V^c + \frac{2}{x}V^c = \Psi.$$

Combining these equations, we obtain

$$(x^2+18)\partial_x^2 V^c + \left(4x + \frac{72}{x}\right)\partial_x V^c - \left(4 + \frac{72}{x^2}\right)V^c = 0.$$

A general solution of this differential equation is given by

$$V^c = V_0 x + \frac{V_1}{x},$$

and if imposing finiteness at $x \to 0$, we obtain $V_1 = 0$ as an initial condition. Substituting this solution into the above differential equations, we can calculate other fluctuations and obtain

$$\begin{aligned} \Psi &= \Psi_{\mathrm{I}}, \\ \mathcal{D}^c &= -5\Psi_{\mathrm{I}} - \frac{1}{6}\Psi_{\mathrm{I}} x^2, \\ V^c &= \frac{1}{3}\Psi_{\mathrm{I}} x, \end{aligned} \tag{13-29}$$

where $\Psi_{\mathrm{I}} = 3V_0$.

The differential equations for radiations are given by

$$\partial_x \mathcal{D}^r + \frac{4}{3}V^r = 0, \qquad \partial_x V^r = 2\Psi + \frac{1}{4}\mathcal{D}^r$$

due to $w_r = c_r^2 = 1/3$. Combining these equations yields

$$\partial_x^2 V^r + \frac{1}{3}V^r = 2\partial_x \Psi.$$

Since the Bardeen potential is a constant, the right-hand side vanishes, and thus this equation can be easily solved. Also, substituting the solution into the original equation, $\mathcal{D}^r$ can be calculated. We thus obtain the following general solutions:

$$\begin{aligned} V^r &= A\sin\left(\frac{x}{\sqrt{3}}\right) + B\cos\left(\frac{x}{\sqrt{3}}\right), \\ \mathcal{D}^r &= \frac{4}{\sqrt{3}}A\cos\left(\frac{x}{\sqrt{3}}\right) - \frac{4}{\sqrt{3}}B\sin\left(\frac{x}{\sqrt{3}}\right) - 8\Psi_{\mathrm{I}}\,. \end{aligned}$$

The coefficients A and B are determined by the adiabatic condition. Imposing that $V^r = V^c$ and $\mathcal{D}^r = (4/3)\mathcal{D}^c$ hold at the limit $x \to 0$, we get $B = 0$ and $A = \Psi_\mathrm{I}/\sqrt{3}$. Therefore, the solution is

$$\mathcal{D}^r = -8\Psi_\mathrm{I} + \frac{4}{3}\Psi_\mathrm{I}\cos\left(\frac{x}{\sqrt{3}}\right),$$
$$V^r = \frac{\Psi_\mathrm{I}}{\sqrt{3}}\sin\left(\frac{x}{\sqrt{3}}\right). \tag{13-30}$$

In the super-horizon region ($x \simeq 0$), it becomes $\mathcal{D}^r \simeq (-20/3)\Psi_\mathrm{I}$.

It turns out that the Bardeen potential does not change at all in the matter-dominated era. Also, it is already mentioned that the Bardeen potential does not change for super-horizon size fluctuations even in the radiation-dominated era. Furthermore, it is known that the amplitude of the large size CMB temperature fluctuation can be determined by the magnitude of the Bardeen potential at the decoupling time through the Sachs-Wolfe effect (E-5). Thus, it can be thought that the current observed value $\Delta T/T \simeq 10^{-5}$ directly conveys the magnitude of the Bardeen potential at the time of the big bang.

The energy density fluctuations $\mathcal{D}^c$ and $\mathcal{D}^r$ change greatly after entering the horizon ($x \gg 2$). The CDM fluctuation $\mathcal{D}^c$ rapidly increases with x^2, while the radiation fluctuation $\mathcal{D}^r$ starts to oscillate, although they are almost constants in the super-horizon region ($x \ll 2$).

The CDM velocity fluctuation V^c monotonically grows with x. On the other hand, the radiation velocity fluctuation V^r grows with x in the super-horizon region, but begins to oscillate like $\mathcal{D}^r$ after entering the horizon.

The thing to note here is that since the solution is obtained under the matter-dominated condition, it will not lead to that in the radiation-dominated era even at $x \to 0$. Since the initial condition is set in the super-horizon region ($x \simeq 0$), it is assumed that the fluctuation is not yet inside the horizon even going into the matter-dominated era. In other words, we have considered relatively large size fluctuations about $l < 200$ in the CMB multipole. In addition, it can be seen that even for sufficiently large fluctuations as always staying in a super-horizon size even in the matter-dominated era, Ψ does not change, while $\mathcal{D}^r$ will slightly increase from $\mathcal{D}^r = -6\Psi_\mathrm{I}$ to $\mathcal{D}^r = (-20/3)\Psi_\mathrm{I}$ before reaching the turning point ($\eta = \eta_\mathrm{eq}$) of the era.

Finally, we briefly describe the position of the first acoustic peak. The CMB temperature fluctuation spectrum in this region is almost determined from the amplitude of the fluctuation when the universe is neutralized. Using the scalar component of the Sachs-Wolfe effect (E-4) derived in the first section of Appendix E, the CMB temperature fluctuation today is given by

$$\frac{\Delta T}{T}(\eta_0) \simeq \frac{1}{4}\mathcal{D}^r(\eta_{\rm dec}) + 2\Psi(\eta_{\rm dec}) = \frac{1}{3}\Psi_{\rm I}\cos\left(c_s x_{\rm dec}\right), \quad (13\text{-}31)$$

where (13-29) and (13-30) are used and $c_s = c_r = 1/\sqrt{3}$ is applied as a sound speed. The extreme values are therefore given by $c_s x_{\rm dec} = c_s k\eta_{\rm dec} = 0, \pi, 2\pi, \ldots$. The first extreme is given by $k_{\rm 1peak} = \pi/r_s$ except for zero, where $r_s = c_s\eta_{\rm dec}$ is called the sound horizon at the decoupling time. The position in the multipoles can be calculated using (11-9) as

$$l_{\rm 1peak} \simeq k_{\rm 1peak} d_{\rm dec} = \frac{\pi(\eta_0 - \eta_{\rm dec})}{c_s\eta_{\rm dec}} = \frac{\pi}{c_s}\left(\sqrt{z_{\rm dec}+1} - 1\right). \quad (13\text{-}32)$$

Substituting $z_{\rm dec} = 1100$ and the value of the sound speed yields $l_{\rm 1peak} \simeq 174$. This value is smaller than the observed value because no baryon effects are included in the sound speed. As shown in the following section, a sound speed in a fluid composed of baryons and radiations becomes $c_s < 1/\sqrt{3}$, and thus the position of the peak moves to the larger of l.

Solutions of Scalar Equations With Baryons

Let us consider states including baryons before the universe is neutralized. Before the neutralization, electrons and baryons are strongly interacting and can be regarded as one. Therefore, what we call baryons usually represents a state where electrons and baryons united.

Interactions between baryons and photons are due to the Thomson scattering, and its scattering cross section is given by $\sigma_T = 8\pi\alpha^2/3m_e^2$ ($\alpha = e^2/4\pi \simeq 1/137$). In order to find evolution equations for fluctuations incorporating effects of the Thomson scattering, it is strictly necessary to handle Boltzmann equations. Here, we will give the evolution equation without showing its derivation, and only examine the behavior of the solution.

The equation-of-state parameter and the sound speed squared of baryons are w_b, $c_b^2 \ll 1$, respectively.[4] For simplicity, we make them zero here.

[4] For details, they are determined by $\rho_b = nm_p$ and $P_b = nT_b$, where $n\,(\propto 1/a^3)$ and $T_b\,(\propto 1/a)$ are the number density and temperature, respectively, and $m_p \simeq 1$ GeV is a typical mass of baryons. We then get $w_b = P_b/\rho_b = T_b/m_p$ and $c_b^2 = \partial_\eta P_b/\partial_\eta\rho_b = 4T_b/3m_p$. In the period considering now which is from around $z_{\rm eq}$ to $z_{\rm dec}$ where the universe is neutralized, baryons can be described as a sufficiently non-relativistic state of $T_b \ll m_p$, and therefore we can approximate them to $w_b = c_b^2 = 0$.

Also, as components of radiations, we consider photons and neutrinos separately as

$$w_\gamma = w_\nu = c_\gamma^2 = c_\nu^2 = \frac{1}{3},$$
$$w_c = w_b = c_c^2 = c_b^2 = 0,$$

namely $P_\gamma = \rho_\gamma/3$, $P_\nu = \rho_\nu/3$, and $P_c = P_b = 0$. The Poisson equation then becomes

$$\begin{aligned}
-2M_{\mathsf{P}}^2\frac{k^2}{a^2}\Psi = \rho_c &\left\{\mathcal{D}^c + 3\left(\Psi + aH\frac{V^c}{k}\right)\right\} \\
&+\rho_\nu\left\{\mathcal{D}^\nu + 4\left(\Psi + aH\frac{V^\nu}{k}\right)\right\} \\
&+\rho_\gamma\left\{\mathcal{D}^\gamma + 4\left(\Psi + aH\frac{V^\gamma}{k}\right)\right\} \\
&+\rho_b\left\{\mathcal{D}^b + 3\left(\Psi + aH\frac{V^b}{k}\right)\right\}. \qquad (13\text{-}33)
\end{aligned}$$

Taking into account that baryons interact with photons but not neutrinos, we obtain the following matter conservation equations:

$$\begin{aligned}
&\partial_\eta\mathcal{D}^c = -kV^c, \quad \partial_\eta V^c + aHV^c = k\Psi, \\
&\partial_\eta\mathcal{D}^\nu = -\frac{4}{3}kV^\nu, \quad \partial_\eta V^\nu = 2k\Psi + \frac{1}{4}k\mathcal{D}^\nu, \\
&\partial_\eta\mathcal{D}^\gamma = -\frac{4}{3}kV^\gamma, \quad \partial_\eta V^\gamma = 2k\Psi + \frac{1}{4}k\mathcal{D}^\gamma - \frac{1}{\eta_T}\left(V^\gamma - V^b\right), \\
&\partial_\eta\mathcal{D}^b = -kV^b, \quad \partial_\eta V^b + aHV^b = k\Psi + \frac{1}{\eta_T}\frac{4}{3}\frac{\rho_\gamma}{\rho_b}\left(V^\gamma - V^b\right). \qquad (13\text{-}34)
\end{aligned}$$

The variable on the Thomson scattering is defined by

$$\eta_T = \frac{1}{a\sigma_T n_e}.$$

It means that photons and baryons are strongly coupled when η_T is small. From the equations, we see that the $\eta_T \to 0$ limit signifies the adiabatic condition $V^\gamma = V^b$.

The initial adiabatic conditions (13-24) for solving these equations are set in the radiation-dominated era as

$$\mathcal{D}^c(0) = \mathcal{D}^b(0) = \frac{3}{4}\mathcal{D}^\gamma(0) = \frac{3}{4}\mathcal{D}^\nu(0),$$
$$V^c(0) = V^b(0) = V^\gamma(0) = V^\nu(0).$$

When photons and baryons are tightly coupled, they can be described as one plasma fluid. Therefore, the adiabatic condition between these components holds like

$$\mathcal{D}^b(x) \simeq \frac{3}{4}\mathcal{D}^\gamma(x)$$

as a good approximation until the universe is neutralized. After that, the interaction becomes ineffective, and this condition does not hold. In fact, by combining the conservation equations in (13-34), $\partial_\eta(\mathcal{D}^b - 3\mathcal{D}^\gamma/4) = -k(V^b - V^\gamma)$ can be derived. This equation means that the initial adiabatic condition is maintained long enough as long as photons and baryons are tightly coupled so that $V^b = V^\gamma$.

In order to emphasize that it is one plasma fluid, we introduce

$$\rho = \rho_\gamma + \rho_b, \qquad P = P_\gamma + P_b = \frac{1}{3}\rho_\gamma$$

as new variables representing the sum of the two states. The equation-of-state parameters and the sound speed squared of this fluid are

$$w = \frac{P}{\rho} = \frac{1}{3}\frac{1}{1+\frac{\rho_b}{\rho_\gamma}}, \qquad c_s^2 = \frac{\partial_\eta P}{\partial_\eta \rho} = \frac{1}{3}\frac{1}{1+\frac{3}{4}\frac{\rho_b}{\rho_\gamma}}. \qquad (13\text{-}35)$$

The perturbation variables are given by

$$\mathcal{D} = \frac{1}{\rho}\left(\rho_\gamma \mathcal{D}^\gamma + \rho_b \mathcal{D}^b\right),$$
$$V = \frac{1}{\rho + P}\left\{\left(\rho_\gamma + P_\gamma\right)V^\gamma + \rho_b V^b\right\}.$$

This state is denoted as $b\gamma$ in the following sentences, although it is omitted in the variables.

Substituting the three states $\alpha = c, \nu, b\gamma$ into the second equation of (13-20), the Poisson equation written with these variables is given by

$$\begin{aligned} -2M_{\mathsf{P}}^2\frac{k^2}{a^2}\Psi = \rho_c &\left\{\mathcal{D}^c + 3\left(\Psi + aH\frac{V^c}{k}\right)\right\} \\ &+\rho_\nu\left\{\mathcal{D}^\nu + 4\left(\Psi + aH\frac{V^\nu}{k}\right)\right\} \\ &+\rho\left\{\mathcal{D} + 3(1+w)\left(\Psi + aH\frac{V}{k}\right)\right\}. \qquad (13\text{-}36)\end{aligned}$$

This equation is exactly the same as (13-33). From the first, third, and fourth equations in (13-20), the conservation equations for $\alpha = b\gamma$ are given as

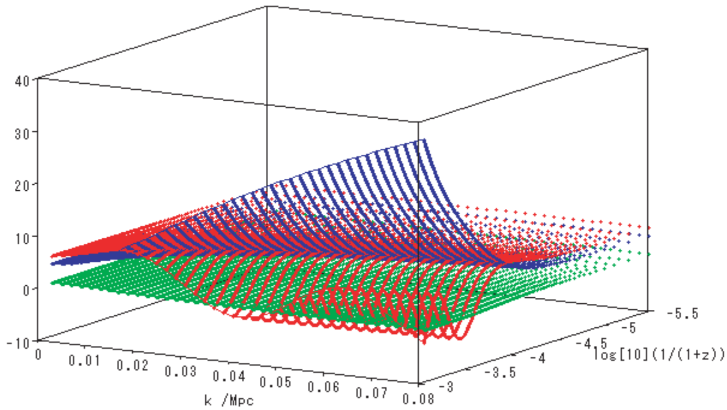

Figure 13-2: Time evolution of the perturbation variables. From the top, the CDM density perturbation $\mathcal{D}^c$ (black), the photon density perturbation $\mathcal{D}^\gamma$ (dark gray), and the Bardeen potential Φ (light gray). Time is represented using a common logarithm of the redshift z. Calculations are performed from the radiation-dominated era to the decoupling time ($z \simeq 10^3$) by taking the initial value of Φ as 1 independent of k. The density perturbation $\mathcal{D}^c$ increases monotonically from the shorter wavelength which enters the horizon first after z_{eq}, while $\mathcal{D}^\gamma$ oscillates greatly. Changes in Φ are easier to understand in Fig. 13-3.

follows:

$$\partial_\eta \mathcal{D} + 3\left(c_s^2 - w\right) aH\mathcal{D} = -(1+w)kV,$$
$$\partial_\eta V + \left(1 - 3c_s^2\right) aHV = k\left(1 + 3c_s^2\right)\Psi + \frac{c_s^2}{1+w}k\mathcal{D}. \quad (13\text{-}37)$$

These conservation equations are related to the original equations (13-34) with the Thomson scattering terms as follows. Calculating $\partial_\eta \mathcal{D}$ by combining the photon and the baryon equations in (13-34), we get

$$\rho\partial_\eta \mathcal{D} = -(1+w)\rho kV + 3waH\rho\mathcal{D} - aH\rho_r\mathcal{D}^\gamma.$$

As mentioned before, since the derivatives of the two density perturbations satisfy $\partial_\eta(\mathcal{D}^b - 3\mathcal{D}^\gamma/4) \sim 0$ when $V^b \sim V^\gamma$, the adiabatic condition is maintained with a good approximation so that $\mathcal{D}^b \simeq 3\mathcal{D}^\gamma/4$ can be always

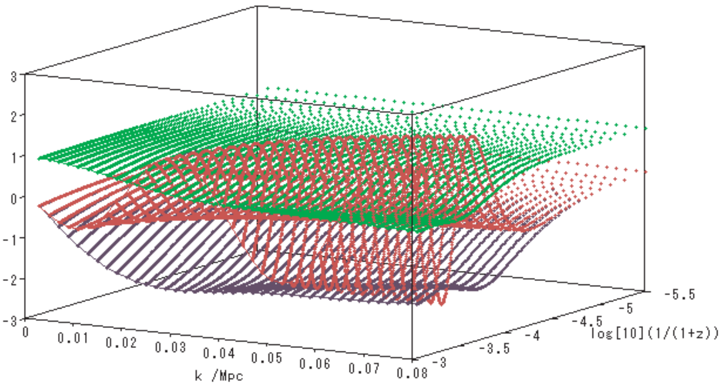

Figure 13-3: From the top, the Bardeen potential Φ (light gray), the baryon velocity perturbation V^b (dark gray) that oscillates greatly, and the CDM velocity perturbation V^c (black). Calculated under the same conditions as in Fig. 13-2. The amplitude of Φ reduces slightly in the high wavenumber region before $z_{\rm eq}$, but it does not change after $z_{\rm eq}$.

fulfilled. Therefore, applying the relation $\mathcal{D} = 3(1+w)\mathcal{D}^\gamma/4$ obtained from the adiabatic condition to the last term and rewriting $\rho^\gamma \mathcal{D}^\gamma$ to $3c_s^2\rho\mathcal{D}$, we can derive the first conservation equation in (13-37). Similarly, calculating the combination $(1+w)\rho\partial_\eta V$ from the equations including derivatives of the photon and baryon velocity perturbations in (13-34), the Thomson scattering terms cancel out and we get

$$(1+w)\rho\partial_\eta V = (1+w)\rho\left\{-\left(1-3c_s^2\right)aHV + \left(1+3c_s^2\right)k\Psi\right\} + \frac{1}{3}k\rho_r\mathcal{D}^\gamma.$$

Rewriting the last term as above, we get the second equation of (13-37).

Let us numerically examine the differential equations in the case that baryons and photons are tightly coupled. We simultaneously solve the conservation equations of the CDM variables $\mathcal{D}^c$ and V^c as well as the neutrino variables $\mathcal{D}^\nu$ and V^ν in (13-34), the equations (13-37) of the plasma fluid variables $\mathcal{D}$ and V, and the Poisson equation (13-36) which determines the Bardeen potential $\Phi\,(= -\Psi)$, together with the Friedmann equations. The cosmological parameters are taken as $\Omega_b = 0.042$, $\Omega_d = 0.27$, $\Omega_r = 8.1\times 10^{-5}$, $\Omega_\gamma = 4.8\times 10^{-5}$, $\Omega_\Lambda = 0.73$, and $h = 0.72$. The initial

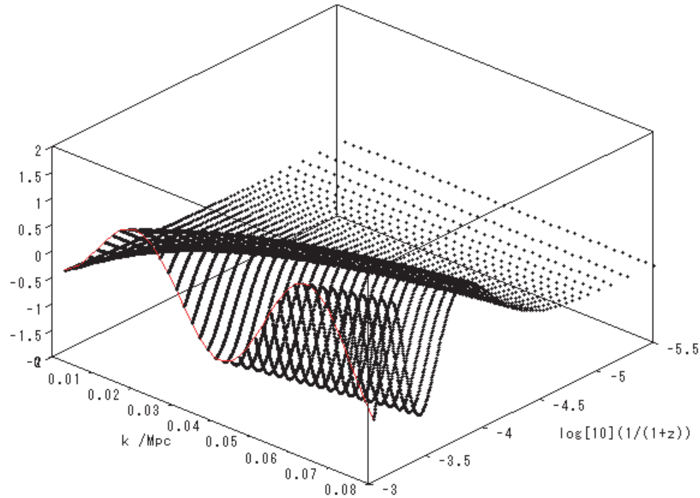

Figure 13-4: Time evolution of the combination $\mathcal{D}^\gamma/4 + 2\Psi$ that appears in the Sachs-Wolfe effect. Calculated under the same condition as in Fig. 13-2. The last solid line is the spectrum at the decoupling time ($z \simeq 10^3$), in which a cosine function appears [see (13-31)]. The first extreme value near $0.02\,\mathrm{Mpc}^{-1}$ corresponds to the first acoustic peak of CMB.

value was given at the radiation-dominated era as $\Psi_\mathrm{I} = -1$ that represents the Harrison-Zel'dovich spectrum (see the analytic solution in the radiation-dominated era).[5] The results are shown in Figs. 13-2 to 13-5, where the photon density perturbation and the baryon velocity perturbation are obtained using the adiabatic approximation relations $\mathcal{D}^\gamma = 4(1+w)\mathcal{D}/4$ and $V^b = V$. Time evolution of the combination $\mathcal{D}^\gamma/4 + 2\Psi$ that appears in the Sachs-Wolfe effect is also shown in Fig. 13-4.[6]

Here, reconsider the position of the first acoustic peak. The conservation

[5] Actually, the Harrison-Zel'dovich spectrum is the case where $k^3|\Psi|^2$ becomes a constant that does not depend on k, and therefore the initial condition should be given by $k^{3/2}\Psi_\mathrm{I} = -1$. However, within the linear approximation, even if we introduce a new dimensionless variable $\bar{\Psi} = k^{3/2}\Psi$ (similarly for other variables), this variable satisfies the same linear equation that the original variable satisfies. Thus, each figure can be regarded as the result calculated using the dimensionless variable with the initial condition $\bar{\Psi}_\mathrm{I} = -1$.

[6] This combination corresponds to $k^{3/2}(\Theta_0 + \Psi)$ in W. Hu and N. Sugiyama, Phys. Rev. D **51** (1995) 2599.

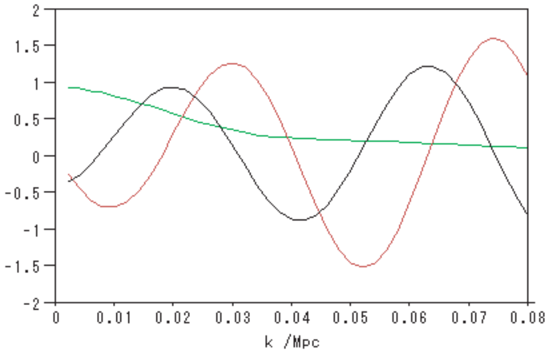

Figure 13-5: Spectra at the decoupling time ($z \simeq 10^3$) for the Bardeen potential Φ (light gray), the baryon velocity perturbation V^b (dark gray), and the combination $\mathcal{D}^\gamma/4 + 2\Psi$ (black) that appears in the Sachs-Wolfe effect.

equation (13-37) indicates that photons and baryons oscillate as one plasma fluid with the sound speed (13-35). The sound speed at the decoupling time is provided as

$$c_s(\eta_{\rm dec}) = \frac{1}{\sqrt{3\left(1 + \frac{3\Omega_b}{4\Omega_\gamma}\frac{a_{\rm dec}}{a_0}\right)}} = \frac{1}{\sqrt{3\left(1 + \frac{3\Omega_b}{4\Omega_\gamma}\frac{1}{z_{\rm dec}+1}\right)}}.$$

Substituting the numerical values of the cosmological parameters, we obtain $c_s(\eta_{\rm dec}) = 0.456$. Therefore, assigning this value into the expression (13-32) that determines the position of the first acoustic peak yields

$$l_{\rm 1peak} \simeq \frac{\pi}{c_s(\eta_{\rm dec})}\left(\sqrt{z_{\rm dec}+1} - 1\right) = 220,$$

which fits well with the observed value.

Evolution of Matter Fluctuations After Neutralization

After the universe is neutralized, photons are less affected by the evolution of the universe and its spectrum is maintained until today (Sachs-Wolfe effect). On the other hand, fluctuations of dusts like CDM and baryons continue to grow, and build structures such as galaxies and clusters of galaxies. Here we briefly describe evolution of such fluctuations after the decoupling.

Evolution equations for fluctuations of CDM and neutralized baryons are given by the same equation as

$$\partial_\eta \mathcal{D}^{c,b} = -kV^{c,b}, \qquad \partial_\eta V^{c,b} + aHV^{c,b} = k\Psi.$$

By eliminating $V^{c,b}$, we get

$$\partial_\eta^2 \mathcal{D}^{c,b} + aH\partial_\eta \mathcal{D}^{c,b} = k^2\Psi.$$

Since both satisfy the same equation, the difference between the two variables satisfies

$$\partial_\eta^2 \left(\mathcal{D}^c - \mathcal{D}^b\right) + aH\partial_\eta \left(\mathcal{D}^c - \mathcal{D}^b\right) = 0.$$

A stable solution of this equation satisfies $\partial_\eta \mathcal{D}^c = \partial_\eta \mathcal{D}^b$. The CDM density perturbation starts growing when going into the matter-dominated era, whereas the growth of the baryon perturbation is suppressed until the decoupling due to interactions with photons. The solution that their derivatives become the same means that the growth of the baryon perturbation which has been suppressed for a long time like $\partial_\eta \mathcal{D}^b(\eta_{\rm dec}) \simeq 0$ on the average is accelerated after the decoupling so as to follow the already grown CDM perturbation of $\partial_\eta \mathcal{D}^c(\eta_{\rm dec}) > 0$.

Conversely, this means that if there is no CDM, the growth of the baryon perturbation is suppressed, and the distribution of galaxies constituted by baryons will be different from that of the present. This behavior is regarded as an indirect evidence of the existence of CDM, as well as the behavior of the galactic rotation curve.

Chapter Fourteen

From Quantum Gravity to CMB

If the big bang as the spacetime phase transition occurs at the energy scale $\Lambda_{\rm QG} \simeq 10^{17}$ GeV, the universe after that expands about 10^{29} times. Since the universe expands 10^{30} times during the inflationary era, a fluctuation that was the Planck length at the Planck time expands as much as 10^{59} times so that its size is now hundreds of megaparsec (Mpc) that is bigger than the size of a cluster of galaxies, as shown in Fig. 12-3. As described in the previous chapter, the fluctuation of this size is long staying in the super-horizon region, and reaching to the present without much changing its amplitude. As the result, it can be observed through CMB. Therefore, by studying its power spectrum, we can understand phenomena at the Planck scale.

In this chapter, we consider fluctuations around the inflationary solution, and applying cosmological perturbation theory, we derive evolution equations of quantum gravity fluctuations. By solving them actually, we show that the inflationary solution is stable, that is, the fluctuations gradually decrease. We also derive spectra at the time of the phase transition by examining how conformally invariant initial spectra set before the Planck time evolve with time. By identifying them with the primordial power spectra which become initial conditions of the Friedmann universe, we calculate the CMB angular power spectra.

Brief Summary After Big Bang

We begin this chapter from summarizing the linear evolution equations in the Einstein theory when the entire universe is in thermal equilibrium just after the big bang. Since the Bardeen (gravitational) potentials have the relation

$$\Phi + \Psi = 0$$

from (13-11), the equations can be described by one Bardeen potential. By applying this relation and considering that the trace of the matter energy-momentum tensor disappears, we find that from the Einstein equation (13-

10) it satisfies a scalar equation

$$3\partial_\eta^2\Phi + 12\partial_\eta\hat{\phi}\partial_\eta\Phi - \partial\!\!\!/^2\Phi = 0,$$

where the background $\hat{\phi}$ satisfies the Friedmann equation. The tensor evolution equation (13-16) is given by

$$\partial_\eta^2 h_{ij}^{\mathrm{TT}} + 2\partial_\eta\hat{\phi}\partial_\eta h_{ij}^{\mathrm{TT}} - \partial\!\!\!/^2 h_{ij}^{\mathrm{TT}} = 0.$$

Since the vector fluctuation disappears with time, it is not usually considered.

As time goes on and temperature of the universe goes down, various matter states come away from the thermal equilibrium state. At that time, depending on the state, its energy-momentum tensor is no longer traceless, and the Einstein equation is also affected. Therefore, it is necessary to solve the Einstein equation together with the conservation equations of each state.

As mentioned in Chapter 11, the amplitude of the current CMB temperature fluctuation in the long wavelength region is given by the value of the Bardeen potential at the time when the universe was neutralized through the Sachs-Wolfe effect. Moreover, from the evolution equation, we find that its value does not change almost from after the big bang until the universe is neutralized. Hence, it can be seen that the primordial value of the Bardeen potential just after the big bang was on the order of the same 10^{-5} that the current temperature fluctuation has. One of the aims of quantum gravity cosmology is to give this value as an initial condition of the current Friedmann universe. Its rough estimation by dimensional analysis has already been given at the end of the first section in Chapter 12.

Evolution Equations in Quantum Gravity

Since scalar fluctuations are expected to be small with time, as a first step we consider linear equations of motion obtained by expanding in perturbations about the homogeneous inflationary solution. Discussion on whether or not the linear approximation can apply, in particular whether it is effective even in the initial region, will be done after solving the equations actually.

Also, in order for the linear approximation to be effective even in the vicinity of the phase transition, it is necessary that spectra do not depend on non-perturbative dynamics of the phase transition. If fluctuations under consideration have a size comparable with the dynamical correlation length $\xi_\Lambda = 1/\Lambda_{\mathrm{QG}}$ at the phase transition, detailed information on the dynamics is needed. However, fluctuations considered here have a size of about the Planck length at the Planck time, and by the time that inflation is over, its size will become much longer than the dynamical correlation length. It suggests that such fluctuations are not affected by the phase transition dynamics.

We also solve a linear equation of motion for the tensor field which is expected to be small initially due to the asymptotically free behavior, and it will be shown that its amplitude for the size considered here are maintained small until the phase transition.

Evolution equations in the renormalizable quantum gravity are expressed as equations that the sum of energy-momentum tensors from each action disappears as follows:

$$\mathbf{T}_{\mu\nu} = \mathbf{T}^{\mathrm{R}}_{\mu\nu} + \mathbf{T}^{\mathrm{W}}_{\mu\nu} + \mathbf{T}^{\mathrm{EH}}_{\mu\nu} + \mathbf{T}^{\mathrm{M}}_{\mu\nu} = 0,$$

where R, W, EH, and M denote that they are derived from the Riegert, Weyl, Einstein-Hilbert, and matter actions, respectively. The cosmological term is ignored as small. Derivation of the evolution equations is very troublesome.[1] Practically, if we take the conformal Newtonian gauge that satisfies the $\sigma = 0$ condition mentioned earlier, calculations can be reduced. The Bardeen potentials (14-1) are then simply expressed as $\Phi = \varphi + h/6$ and $\Psi = \varphi - h/2$. In the following, we examine the equations in this gauge.

Without describing the details here, we will give only the results obtained and examine their properties. Parts with the coefficients b_c, $1/\tilde{t}^2$, and M_{P}^2 are derived from the Riegert, Weyl, and Einstein-Hilbert actions, respectively, where $\tilde{t}^2$ is the time-dependent running coupling constant (12-6).

Linear scalar equations Since the matter energy-momentum tensor is traceless such that $\mathbf{T}^{\mathrm{M}\lambda}{}_{\lambda} = 0$, we obtain the following scalar evolution equation from the trace component:

$$\begin{aligned}\frac{b_c}{8\pi^2} B_0(\tau) \Big\{ & -2\partial_\eta^4 \Phi - 2\partial_\eta \hat{\phi}\, \partial_\eta^3 \Phi + \left(-8\partial_\eta^2 \hat{\phi} + \frac{10}{3} \not\partial^2 \right) \partial_\eta^2 \Phi \\ & + \left(-12\partial_\eta^3 \hat{\phi} + \frac{10}{3} \partial_\eta \hat{\phi} \not\partial^2 \right) \partial_\eta \Phi + \left(\frac{16}{3} \partial_\eta^2 \hat{\phi} - \frac{4}{3} \not\partial^2 \right) \not\partial^2 \Phi \\ & + 2\partial_\eta \hat{\phi}\, \partial_\eta^3 \Psi + \left(8\partial_\eta^2 \hat{\phi} + \frac{2}{3} \not\partial^2 \right) \partial_\eta^2 \Psi + \left(12\partial_\eta^3 \hat{\phi} - \frac{10}{3} \partial_\eta \hat{\phi} \not\partial^2 \right) \partial_\eta \Psi\end{aligned}$$

[1] In order to make the evolution equation gauge invariant, it is necessary to modify the time-dependent dynamical factor B_0 (12-3) to a function that transforms as a scalar like $\delta_\xi B = \xi^\lambda \partial_\lambda B = \xi^0 \partial_\eta B$. Using the variable σ (13-4), the modified dynamical factor can be expressed as $B = B_0 - \sigma \partial_\eta B_0$. However, if we take the conformal Newtonian gauge, we do not have to consider this modification.

$$
\begin{aligned}
&+\left(-\frac{16}{3}\partial_\eta^2\hat{\phi}-\frac{2}{3}\partial\!\!\!/^2\right)\partial\!\!\!/^2\Psi\Big\} \\
&+M_{\mathrm{P}}^2\,e^{2\hat{\phi}}\Big\{6\partial_\eta^2\Phi+18\partial_\eta\hat{\phi}\,\partial_\eta\Phi-4\partial\!\!\!/^2\Phi-6\partial_\eta\hat{\phi}\,\partial_\eta\Psi \\
&+\left(12\partial_\eta^2\hat{\phi}+12\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}-2\partial\!\!\!/^2\right)\Psi\Big\}=0,
\end{aligned} \tag{14-1}
$$

where the term including $\partial_\eta^4\hat{\phi}$ is eliminated using the equation of motion (12-4) of the background field.

From the second combination of the energy-momentum tensor in (13-9) further divided by $\partial\!\!\!/^2$, we obtain a second-order differential equation

$$
\begin{aligned}
&\frac{2}{\tilde{t}^2(\tau)}\Big\{4\partial_\eta^2\Phi-\frac{4}{3}\partial\!\!\!/^2\Phi-4\partial_\eta^2\Psi+\frac{4}{3}\partial\!\!\!/^2\Psi\Big\}+\frac{b_c}{8\pi^2}B_0(\tau)\Big\{\frac{4}{3}\partial_\eta^2\Phi \\
&\quad+4\partial_\eta\hat{\phi}\,\partial_\eta\Phi+\left(\frac{28}{3}\partial_\eta^2\hat{\phi}-\frac{8}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}-\frac{8}{9}\partial\!\!\!/^2\right)\Phi \\
&\quad-\frac{4}{3}\partial_\eta\hat{\phi}\,\partial_\eta\Psi+\left(-\frac{4}{3}\partial_\eta^2\hat{\phi}+\frac{8}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}-\frac{4}{9}\partial\!\!\!/^2\right)\Psi\Big\} \\
&-2M_{\mathrm{P}}^2\,e^{2\hat{\phi}}\{\Phi+\Psi\}=0.
\end{aligned} \tag{14-2}
$$

This equation plays a role of a constraint condition connecting from the inflation era to the Friedmann universe. In the small running coupling limit $\tilde{t}\to 0$ as in the early stage of inflation, the scalar mode h originated from the traceless tensor field disappears and the $\Phi=\Psi\ (=\varphi)$ fluctuation becomes significant. On the other hand, at the phase transition where the running coupling constant diverges, the last Einstein term dominates, indicating that the fluctuation of $\Phi=-\Psi$, which holds in the Friedmann universe, is realized.

In the vanishing coupling limit where $\Phi=\Psi=\varphi$, the left-hand side of the trace equation (14-1) can be written using only the fluctuation variable φ of the conformal-factor field as

$$
\begin{aligned}
\mathbf{T}^\mu{}_\mu|_{t\to 0}=&-\frac{b_c}{4\pi^2}\left(\partial_\eta^4\varphi-2\partial_\eta^2\partial\!\!\!/^2\varphi+\partial\!\!\!/^4\varphi\right)+M_{\mathrm{P}}^2\,e^{2\hat{\phi}}\Big\{6\partial_\eta^2\varphi-6\partial\!\!\!/^2\varphi \\
&+12\partial_\eta\hat{\phi}\,\partial_\eta\varphi+12\left(\partial_\eta^2\hat{\phi}+\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\varphi\Big\}.
\end{aligned}
$$

Linear tensor and vector equations From the equation of motion $\mathbf{T}_{ij}=0$, we obtain a linear evolution equation for the tensor fluctuation

$$
\frac{2}{\tilde{t}^2(\tau)}\left\{-\partial_\eta^4 h_{ij}^{\mathrm{TT}}+2\partial\!\!\!/^2\partial_\eta^2 h_{ij}^{\mathrm{TT}}-\partial\!\!\!/^4 h_{ij}^{\mathrm{TT}}\right\}+\frac{b_c}{8\pi^2}B_0(\tau)\Big\{
$$

$$\left(\frac{1}{3}\partial_\eta^2\hat{\phi}+\frac{4}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\partial_\eta^2 h_{ij}^{\rm TT}+\left(\frac{1}{3}\partial_\eta^3\hat{\phi}+\frac{8}{3}\partial_\eta^2\hat{\phi}\partial_\eta\hat{\phi}\right)\partial_\eta h_{ij}^{\rm TT}$$
$$+\left(-\frac{7}{3}\partial_\eta^2\hat{\phi}+\frac{2}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\not{\partial}^2 h_{ij}^{\rm TT}\bigg\}$$
$$+M_{\rm P}^2\, e^{2\hat{\phi}}\left\{-\frac{1}{2}\partial_\eta^2 h_{ij}^{\rm TT}-\partial_\eta\hat{\phi}\partial_\eta h_{ij}^{\rm TT}+\frac{1}{2}\not{\partial}^2 h_{ij}^{\rm TT}\right\}=0. \tag{14-3}$$

A linear evolution equation that the vector fluctuation satisfies is derived from $\not{\partial}^{-2}\partial^j\mathbf{T}_{ij}=0$, which is

$$\frac{2}{\tilde{t}^2(\tau)}\left\{\partial_\eta^3\Upsilon_i-\partial_\eta\not{\partial}^2\Upsilon_i\right\}-\frac{b_c}{8\pi^2}B_0(\tau)\Bigg\{\left(\frac{1}{3}\partial_\eta^2\hat{\phi}+\frac{4}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\partial_\eta\Upsilon_i$$
$$+\left(\frac{1}{3}\partial_\eta^3\hat{\phi}+\frac{8}{3}\partial_\eta^2\hat{\phi}\partial_\eta\hat{\phi}\right)\Upsilon_i\Bigg\}+M_{\rm P}^2\, e^{2\hat{\phi}}\left\{\frac{1}{2}\partial_\eta\Upsilon_i+\partial_\eta\hat{\phi}\Upsilon_i\right\}=0. \tag{14-4}$$

Conversion formulas to physical time When solving the equations of motion, we will use the physical time τ defined by $d\tau=a(\tau)d\eta$ (11-1). Below we summarize useful conversion formulas from the conformal time to the physical time. Using the scale factor $a(\tau)=e^{\hat{\phi}(\tau)}$ and the Hubble variable $H(\tau)=\dot{a}(\tau)/a(\tau)$, the differential operators can be rewritten as

$$\not{\partial}^2=a^2\left(-\frac{k^2}{a^2}\right),\qquad \partial_\eta=a\partial_\tau,\qquad \partial_\eta^2=a^2\left(\partial_\tau^2+H\partial_\tau\right),$$
$$\partial_\eta^3=a^3\left\{\partial_\tau^3+3H\partial_\tau^2+\left(\dot{H}+2H^2\right)\partial_\tau\right\},$$
$$\partial_\eta^4=a^4\left\{\partial_\tau^4+6H\partial_\tau^3+\left(4\dot{H}+11H^2\right)\partial_\tau^2+\left(\ddot{H}+7H\dot{H}+6H^3\right)\partial_\tau\right\}.$$

The derivatives of the background field are rewritten as

$$\partial_\eta\hat{\phi}=aH,\quad \partial_\eta^2\hat{\phi}=a^2\left(\dot{H}+H^2\right),\quad \partial_\eta^3\hat{\phi}=a^3\left(\ddot{H}+4H\dot{H}+2H^3\right),$$
$$\partial_\eta^4\hat{\phi}=a^4\left(\dddot{H}+7H\ddot{H}+4\dot{H}^2+18H^2\dot{H}+6H^4\right).$$

On contributions from nonlinear terms In the spectral region we are considering here, amplitudes of the scalar fluctuations decrease with time as will be shown in the later section, and the linear approximation actually becomes well. Even in the initial stage, since the coupling constant $\tilde{t}$ is small, the scalar mode h in the traceless tensor field may be handled with the linear approximation. Even if $\tilde{t}$ grows with time, the amplitude also decreases at the same time, and therefore it is considered that nonlinear terms derived from three-point interactions including h in the fourth-order gravitational action may be ignored. Likewise, nonlinear terms derived from self-interactions of the conformal-factor field φ with the coupling $\tilde{t}$ can be neglected.

Moreover, since the tensor fluctuation remains small from the beginning to the end as will be sown later, nonlinear terms including it are also negligible. Although not discussed in this book, the vector fluctuation that initially has a small amplitude grows around the phase transition, but it will have little effect.

However, since interaction terms derived from the exponential conformal factor in the Einstein-Hilbert action exist even at $\tilde{t} \to 0$, they may affect the initial stage before the amplitude becomes small yet. Actually, as we will see later, this nonlinear terms cannot be ignored in the region where the wavenumber k is larger than the Planck mass scale m (14-8) in the comoving coordinates. It means that we have to add the nonlinear terms to the evolution equation (14-1). The second and third order nonlinear terms are given as follows:

$$\begin{aligned}
\mathbf{T}^{\mathrm{EH}\lambda}_{\lambda}|_{\mathrm{NL2}} &= M_{\mathrm{P}}^2 e^{2\hat{\phi}}\Big\{12\varphi\partial_\eta^2\varphi - 12\varphi\partial^2\varphi + 24\partial_\eta\hat{\phi}\,\varphi\partial_\eta\varphi \\
&\quad + 6\partial_\eta\varphi\partial_\eta\varphi - 6\partial_i\varphi\partial^i\varphi + 12\left(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\varphi^2\Big\}, \\
\mathbf{T}^{\mathrm{EH}\lambda}_{\lambda}|_{\mathrm{NL3}} &= M_{\mathrm{P}}^2 e^{2\hat{\phi}}\Big\{12\varphi^2\partial_\eta^2\varphi - 12\varphi^2\partial^2\varphi + 24\partial_\eta\hat{\phi}\,\varphi^2\partial_\eta\varphi \\
&\quad + 12\varphi\partial_\eta\varphi\partial_\eta\varphi - 12\varphi\partial_i\varphi\partial^i\varphi + 8\left(\partial_\eta^2\hat{\phi} + \partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\varphi^3\Big\}. \qquad (14\text{-}5)
\end{aligned}$$

Similarly, the higher order nonlinear terms can be obtained.

On the other hand, in the constraint equation (14-2), when $\tilde{t}$ is small as in the initial stage, the first term dominates and the Einstein term does not contribute. Moreover, at the final stage when the Einstein term becomes dominant, the scalar fluctuations are already small. Thus, it will not be necessary to add nonlinear terms to the constraint equation.

Evolution Equations for Matter Fields

Consider equations of motion including the perturbation variables of matter fields. Although these equations are not necessary for calculations of the primordial spectrum to be done in the next section, we here present them to complete the equation system.

The matter energy-momentum tensor (13-8) is traceless and can be expressed as

$$\mathbf{T}^{\mathrm{M}}_{00} = e^{4\hat{\phi}} \left(\rho + \delta\rho + 4\rho\varphi\right), \qquad \mathbf{T}^{\mathrm{M}}_{0i} = -\frac{4}{3} e^{4\hat{\phi}} \rho \left(v_i + \frac{1}{2} h_{0i}\right),$$
$$\mathbf{T}^{\mathrm{M}}_{ij} = \frac{1}{3} e^{4\hat{\phi}} \left(\rho + \delta\rho + 4\rho\varphi\right) \delta_{ij}. \tag{14-6}$$

Similar to the previous chapter, a differential equation including the energy density perturbation D is obtained from the third combination of the energy-momentum tensors in (13-9) as

$$\begin{aligned}
&\frac{b_c}{8\pi^2} B_0(\tau) \bigg\{ \left(-2\partial_\eta^2\hat{\phi} + 2\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} - \frac{2}{3}\partial\!\!\!/^2\right)\partial_\eta^2\Phi + \left(2\partial_\eta^3\hat{\phi} - 4\partial_\eta^2\hat{\phi}\partial_\eta\hat{\phi}\right)\partial_\eta\Phi \\
&+\partial_\eta\hat{\phi}\left(-2\partial_\eta^2\hat{\phi} + 2\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} - 2\partial\!\!\!/^2\right)\partial_\eta\Phi + \left(-\frac{20}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + \frac{4}{9}\partial\!\!\!/^2\right)\partial\!\!\!/^2\Phi \\
&+\partial_\eta\hat{\phi}\left(2\partial_\eta^2\hat{\phi} - 2\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + \frac{2}{3}\partial\!\!\!/^2\right)\partial_\eta\Psi + \left(-2\partial_\eta^3\hat{\phi}\partial_\eta\hat{\phi} + 4\partial_\eta^2\hat{\phi}\partial_\eta^2\hat{\phi}\right)\Psi \\
&+\left(2\partial_\eta^2\hat{\phi} + \frac{2}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + \frac{2}{9}\partial\!\!\!/^2\right)\partial\!\!\!/^2\Psi \bigg\} \\
&+\frac{2}{\tilde{t}^2(\tau)}\left\{-\frac{4}{3}\partial\!\!\!/^4\Phi - 4\partial_\eta\hat{\phi}\,\partial\!\!\!/^2\partial_\eta\Phi + \frac{4}{3}\partial\!\!\!/^4\Psi + 4\partial_\eta\hat{\phi}\,\partial\!\!\!/^2\partial_\eta\Psi\right\} \\
&+M_{\mathsf{P}}^2\, e^{2\hat{\phi}}\, 2\partial\!\!\!/^2\Phi + e^{4\hat{\phi}}\rho D = 0.
\end{aligned}$$

Since this equation only contains at most the second-order time derivative of the Bardeen potential Φ, we can determine the value of D by substituting the values of Φ and Ψ obtained by solving the simultaneous differential equations of (14-1) and (14-2).

From the fourth expression in (13-9), a differential equation including the scalar velocity perturbation V is obtained as

$$\begin{aligned}
&\frac{b_c}{8\pi^2} B_0(\tau) \bigg\{ -\frac{2}{3}\partial_\eta^3\Phi + \left(-\frac{10}{3}\partial_\eta^2\hat{\phi} + \frac{2}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + \frac{4}{9}\partial\!\!\!/^2\right)\partial_\eta\Phi - \frac{4}{3}\partial_\eta\hat{\phi}\,\partial\!\!\!/^2\Phi \\
&+\frac{2}{3}\partial_\eta\hat{\phi}\,\partial_\eta^2\Psi + \left(2\partial_\eta^2\hat{\phi} - \frac{2}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi} + \frac{2}{9}\partial\!\!\!/^2\right)\partial_\eta\Psi + \left(2\partial_\eta^3\hat{\phi} - \frac{2}{3}\partial_\eta\hat{\phi}\,\partial\!\!\!/^2\right)\Psi \bigg\} \\
&+\frac{2}{\tilde{t}^2(\tau)}\left\{-\frac{4}{3}\partial\!\!\!/^2\partial_\eta\Phi + \frac{4}{3}\partial\!\!\!/^2\partial_\eta\Psi\right\} \\
&+M_{\mathsf{P}}^2\, e^{2\hat{\phi}}\left\{2\partial_\eta\Phi - 2\partial_\eta\hat{\phi}\,\Psi\right\} - \frac{4}{3}e^{4\hat{\phi}}\rho V = 0.
\end{aligned}$$

This equation is also third order in time derivatives of the Bardeen potential Φ, and thus we can obtain V by substituting the solution of the simultaneous differential equations of (14-1) and (14-2).

Extracting a vector component from the equation of motion $\mathbf{T}_{0i} = 0$ yields a differential equation including the vector variable Ω_i as

$$\frac{2}{\tilde{t}^2(\tau)}\left\{\partial_\eta^2 \partial\!\!\!/^2 \Upsilon_i - \partial\!\!\!/^4 \Upsilon_i\right\} - \frac{b_c}{8\pi^2}B_0(\tau)\left(\frac{1}{3}\partial_\eta^2\hat{\phi} + \frac{4}{3}\partial_\eta\hat{\phi}\partial_\eta\hat{\phi}\right)\partial\!\!\!/^2\Upsilon_i$$
$$+\frac{1}{2}M_{\mathrm{P}}^2\, e^{2\hat{\phi}}\, \partial\!\!\!/^2\Upsilon_i - \frac{4}{3}e^{4\hat{\phi}}\rho\,\Omega_i = 0.$$

Since this equation only contains at most the second-order time derivative of the vector variable Υ_i, we can obtain Ω_i by substituting the solution of the differential equation (14-4).

Initial Spectra of Quantum Gravity

As an initial condition for solving the linear evolution equations, we here consider spectra obtained from two-point functions of the gravitational field. The initial spectra are set at an appropriate physical time $\tau_{\mathrm{i}} = 1/E_{\mathrm{i}}$ before inflation starts, where $E_{\mathrm{i}} \geq H_{\mathbf{D}}$.

In the early epoch, the conformal-factor fluctuation φ represented by $\Phi = \Psi$ dominates, and its dynamics is described by the fourth-order derivative Riegert action. The correlation function is given by a logarithmic function reflecting that the field is dimensionless, which is

$$\langle\varphi(\tau_{\mathrm{i}},\mathbf{x})\varphi(\tau_{\mathrm{i}},\mathbf{x}')\rangle = -\frac{1}{4b_c}\log\left(m^2|\mathbf{x}-\mathbf{x}'|^2\right) \tag{14-7}$$

at the equal time, where b_c is the coefficient in front of the Riegert action which is positive. The mass scale m is the Planck mass in the comoving coordinates at the time τ_{i} defined by

$$m = a(\tau_{\mathrm{i}})H_{\mathbf{D}}. \tag{14-8}$$

A physical distance on the hypersurface of the time τ_{i} is $|\mathbf{r}-\mathbf{r}'| = a(\tau_{\mathrm{i}})|\mathbf{x}-\mathbf{x}'|$, and thus $H_{\mathbf{D}}^2|\mathbf{r}-\mathbf{r}'|^2 = m^2|\mathbf{x}-\mathbf{x}'|^2$. The logarithmic correlation function (14-7) indicates that there are fluctuations with correlations longer than the Planck length $L_{\mathrm{P}} = 1/H_{\mathbf{D}}$ which is the horizon distance in inflation.

Spectra are expressed using Fourier transform in the three-dimensional comoving space. For the variable $\varphi(\mathbf{x})$, it is defined as

$$\varphi(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\varphi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

and its mean square $\langle|\varphi(\mathbf{k})|^2\rangle$ is defined by

$$\langle\varphi(\mathbf{k})\varphi(\mathbf{k}')\rangle = \langle|\varphi(\mathbf{k})|^2\rangle(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}'). \tag{14-9}$$

Fourier transform of the logarithm function is given by

$$-\log\left(m^2|\mathbf{x}|^2\right) = \int_{k>\epsilon}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{4\pi^2}{k^3}e^{i\mathbf{k}\cdot\mathbf{x}} - \log\left(\frac{m^2}{\epsilon^2 e^{2\gamma-2}}\right),$$

where $k = |\mathbf{k}|$ and γ is the Euler constant. The $\epsilon(\ll 1)$ is a small infrared cutoff, and its effect will be introduced as the correlation length ξ_Λ later. Since the constant term on the right-hand side is proportional to $\delta^3(\mathbf{k})$ in Fourier space, it is ignored. Using this formula, we get

$$\langle|\varphi(\tau_\mathrm{i},\mathbf{k})|^2\rangle = \frac{\pi^2}{b_c}\frac{1}{k^3}.$$

Thus, a scale-invariant scalar power spectrum is obtained as

$$P_\varphi(\tau_\mathrm{i},k) = \frac{k^3}{2\pi^2}\langle|\varphi(\tau_\mathrm{i},\mathbf{k})|^2\rangle = \frac{1}{2b_c}. \tag{14-10}$$

Since the Riegert action is positive-definite with $b_c > 0$, the amplitude becomes a physical positive value. This scalar spectrum corresponds to a spectral index n_s being 1, called the Harrison-Zel'dovich spectrum, when it is expressed in the form of $A_s k^{n_s-1}$ as usual. Here note that how to define the exponent as $n_s - 1$ is a traditional convention used only for the scalar spectrum.

The initial tensor spectrum is obtained from the two-point correlation function of the transverse traceless tensor field h^{TT}_{ij}. The dynamics of the tensor field is described by the fourth-order derivative Weyl action. From (7-21), it is expressed as

$$\langle h^{\mathrm{TT}}_{ij}(x)h^{\mathrm{TT}}_{kl}(x')\rangle = 2\Delta_{ij,kl}(\mathbf{x})\langle h^{\mathrm{TT}}(x)h^{\mathrm{TT}}(x')\rangle,$$

where the relationship between the field notation in Chapter 7 and that here is given by $h^{\mathrm{TT}}_{ij} = t\mathsf{h}_{ij}$ and $h^{\mathrm{TT}} = tH/\sqrt{2}$. The normalization is in accordance with (E-13) in Appendix E. The two-point correlation function of h^{TT} is given by a logarithmic function, and at the equal time it is

$$\langle h^{\mathrm{TT}}(\tau_\mathrm{i},\mathbf{x})h^{\mathrm{TT}}(\tau_\mathrm{i},\mathbf{x}')\rangle = -\frac{t_\mathrm{i}^2}{32\pi^2}\log(m^2|\mathbf{x}-\mathbf{x}'|^2),$$

where t_i is an initial value of the coupling constant t.

In the same way for φ, the tensor power spectrum is defined using the correlation function of h^{TT} so that a scale-invariant spectrum is yielded in Fourier space as

$$P_h(\tau_{\mathrm{i}}, k) = \frac{k^3}{2\pi^2}\langle|h^{\mathrm{TT}}(\tau_{\mathrm{i}}, \mathbf{k})|^2\rangle = \frac{t_{\mathrm{i}}^2}{16\pi^2}.$$

This corresponds to $n_t = 0$ when the tensor spectrum is expressed using the spectral index in the form of $A_t k^{n_t}$. The initial tensor amplitude is expected to be sufficiently smaller than that of the scalar amplitude due to the asymptotically free behavior.

Finally, it should be recognized that there is room for discussion as to whether the spectra given here are physical. Although we adopt the linear approximation from practicality, since there is the BRST conformal invariance in the foundation of the theory, it seems necessary to think about physical quantities more properly. Indeed, the existence of any tensor fluctuation is denied from the discussion of physical states. Here, the tensor equation is introduced to complete the entire equation system. And also, since a small tensor fluctuation may be dynamically created, it is added to the discussion. On the other hand, it is considered that the scalar spectrum contains the essence of fluctuations.

Solutions of Evolution Equations and Stability

Let us numerically solve the linear evolution equations and see how the amplitude changes during inflation. We then obtain spectra at the spacetime phase transition point $\tau = \tau_\Lambda$ and identify them with the primordial power spectra for the structure formation of the universe after the big bang.

Since the Bardeen potentials satisfy $\Psi = \Phi = \varphi$ in the initial time τ_{i} where the running coupling constant is sufficiently small, we take the initial condition as

$$\Phi(\tau_{\mathrm{i}}, k) = \Psi(\tau_{\mathrm{i}}, k),$$

whereas it can be seen from the constraint equation (14-2) that $\Psi = -\Phi$ will be realized at the phase transition point where the running coupling diverges. Therefore, we numerically solve the simultaneous differential equations of (14-1) and (14-2) in the physical time τ as a boundary value problem by imposing the boundary condition

$$\Phi(\tau_\Lambda, k) + \Psi(\tau_\Lambda, k) = 0. \qquad (14\text{-}11)$$

When carrying out the calculation specifically, the two variables Φ and h are considered by putting $\Psi = \Phi - 2h/3$. The fourth order equation (14-1) from which $\partial_\eta^3 h$ is removed using the equation obtained by differentiating the second order equation (14-2) is solved simultaneously with (14-2) itself.[2] The initial condition is $\Phi(\tau_{\rm i}, k) = \sqrt{P_\varphi(k)}$ and $h(\tau_{\rm i}, k) = 0$, and further the derivatives of Φ with respect to the physical time up to third order are set to zero. The boundary condition is given by $\Phi(\tau_\Lambda, k) - h(\tau_\Lambda, k)/3 = 0$ from (14-11).

In the linear approximation, it can be solved by fixing the comoving wavenumber k. Factoring out the scale factor $a(\tau)$ and making the whole equation dimensionless by using $H_{\mathbf{D}}$,[3] the terms containing $-\partial\!\!\!/^2$ that depend on k are replaced with a physical wavenumber function $k^2/H_{\mathbf{D}}^2 a^2(\tau)$, where the scale factor of the denominator is given by the solution of the background field equation. When doing the calculation, the initial value of the scale factor is normalized to $a(\tau_{\rm i}) = 1$ and the Planck constant $H_{\mathbf{D}}$ is then rewritten to m.

Both at the initial time and at the phase transition point, the two-point function of Φ and that of Ψ become the same, thus we use Φ to represent the scalar spectra below. Defining the transfer function representing the temporal change of the Bardeen potential as

$$\Phi(\tau_\Lambda, k) = \mathcal{T}_\Phi(\tau_\Lambda, \tau_{\rm i})\Phi(\tau_{\rm i}, k),$$

the primordial power spectrum that is the initial condition of the Friedmann universe is given by $P_\Phi(\tau_\Lambda, k) = \mathcal{T}_\Phi^2(\tau_\Lambda, \tau_{\rm i}) P_\varphi(\tau_{\rm i}, k)$.

Since the physical wavenumber rapidly decreases as the scale factor $a(\tau)$ increases during inflation, the equations of motion no longer have the wavenumber dependence near the phase transition. Therefore, although the phenomenological parameters β_0, a_1, and κ which relate to the dynamics of the phase transition affect an amplitude of the primordial power spectrum, it is considered that they do not affect its pattern.

We here use the values adopted in the inflationary solution in Chapter 12, which are $b_c = 10$, $H_{\mathbf{D}}/\Lambda_{\rm QG} = 60$, and the phenomenological parameters

[2] It is treated as a boundary value problem in order to solve it while retaining the constraint equation (14-2). If we try to solve it as an initial value problem, it does not go well because it is difficult to maintain the constraint (14-2), which will be not satisfied gradually with time. In actual calculations, it has been solved by using Fortran software published on the web, "BVP_SOLVER", which can be applied to cases where some derivative quantities other than physical quantities diverge at the boundary. See W. Enright and P. Muir, SIAM J. Sci. Comput. **17** (1996) 479. It seems that commercially available Maple software also has this performance, and if we want to calculate a single line as shown in Fig. 14-1, we can solve it somehow. At that time, we set the boundary condition just before the phase transition time τ_Λ at which the coupling constant diverges and calculate it up to that point.

[3] It is useful to replace the physical time τ with a dimensionless time $t = H_{\mathbf{D}}\tau$.

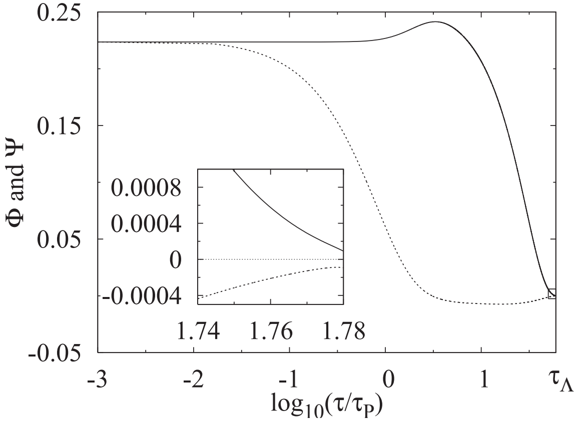

Figure 14-1: Solutions of the linear evolution equations for the Bardeen potentials Φ (solid) and Ψ (dotted) in the inflationary background. The initial value is $\Phi = \Psi\,(=\varphi) = 1/\sqrt{20}$, the comoving wavenumber is set to $k = 0.01\,\mathrm{Mpc}^{-1}$, and the comoving Planck mass is $m = 0.0156\,(= 60\lambda)\,\mathrm{Mpc}^{-1}$ (λ is defined later). The two Bardeen potentials change while decreasing the amplitude, respectively, and become $\Phi = -\Psi$ at the phase transition point τ_Λ. [K. Hamada, S. Horata, and T. Yukawa, Phys. Rev. D **81** (2010) 083533.]

$\beta_0/b_c = 0.06$, $a_1/b_c = 0.01$, and $\kappa = 0.5$. The number of e-foldings is then $\mathcal{N}_e = 65.0$. The comoving Planck mass is $m = 0.0156\,\mathrm{Mpc}^{-1}$, and the initial amplitude of the Bardeen potential is given by $\sqrt{P_\varphi} = 1/\sqrt{2b_c}$. The calculation results are shown in Figs. 14-1 and 14-2. The decreasing of the amplitude with time indicates that the inflationary solution is stable.

The linear tensor evolution equation (14-3) is solved with the initial value $\sqrt{P_h} = t_\mathrm{i}/4\pi = 10^{-5}$, and we get Fig. 14-3.[4] It can be seen that the amplitude of the tensor fluctuation is maintained small until the end.

Limit of linear approximation The wavenumber region for which the calculation is valid within the linear approximation is $k < m$. In $k > m$, the nonlinear terms derived from the conformal factor in the Einstein-Hilbert

[4] If β_0 which is one of the phenomenological parameters is decided, the running coupling constant is determined, and thus the initial value t_i will be determined in principle, but its value is not used here. This is because the phenomenological parameters have no absolute meaning since they are appropriately selected considering the convenience of the calculation. In fact, in this setting, we can reduce the initial value as much as possible by making the start time earlier.

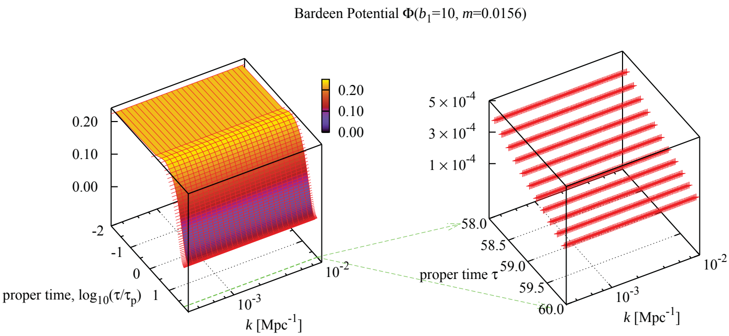

Figure 14-2: Time evolution of the Bardeen potential Φ. The line at the phase transition point $\tau = 60$ gives the primordial power spectrum.

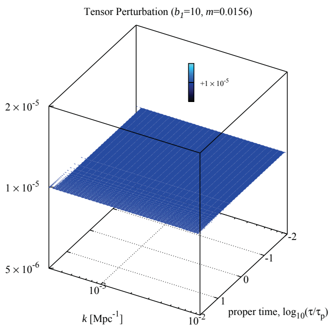

Figure 14-3: Solution of the tensor evolution equation.

action cannot be ignored anymore. Since this exponential factor is due to conformal invariance, this invariance will not be retained for $k > m$ unless the nonlinear terms are taken into account.

In fact, in the linear approximation, the Einstein term acts like a mass term, so that an exponential damping occurs in $k > m$. The nonlinear terms have effects of relaxing it to a power-law behavior. When the nonlinear effects are taken in, it can be expected that the scalar amplitude has a gentle slope to $n_s < 1$ called the red tilt for $k > m$.

Numerical calculations with the nonlinear terms are very difficult and not done yet. Therefore, we will proceed with calculations of the CMB spectra simply assuming that the scale invariant spectrum of $n_s = 1$ is retained for $k > m$.

Effects of correlation length We here consider spectra in which the dynamical correlation length given by $\xi_\Lambda = 1/\Lambda_{\rm QG}$ ($\gg L_{\rm P}$) is taken into account. This length scale indicates that there is no correlation between two points separated by a physical distance over ξ_Λ before the Planck time where spacetime has not begun to expand yet.

This effect can be expressed by adding a correction of $o(t^2)$ to the spectral index n_s that is now setting to 1. Replacing the coupling constant t^2 with the k-dependent running coupling constant $\bar{t}^2(k) = 1/\beta_0 \log(k^2/\lambda^2)$, we obtain

$$P_s(k) = A_s \left(\frac{k}{m}\right)^{v/\log(k^2/\lambda^2)}, \tag{14-12}$$

where v is a positive constant and λ is the dynamical energy scale in the comoving coordinates defined by

$$\lambda = a(\tau_{\rm i})\Lambda_{\rm QG}. \tag{14-13}$$

Between this scale and the comoving Planck mass defined before, $\lambda/m = \Lambda_{\rm QG}/H_{\mathbf{D}} \ll 1$ holds. The spectrum (14-12) sharply drops near $k = \lambda$, indicating that the correlation vanishes.

As mentioned in the caption of Fig. 12-3, when looking at the comoving wavenumber k, quantum correlations for $k < \lambda$, which did not exist before the universe began to expand, never exist during the evolution. Therefore, we adopt the spectrum (14-12) as the primordial scalar spectrum at the phase transition point.

Similarly for the tensor fluctuation, we give a spectrum at the phase transition point as

$$P_t(k) = A_t \left(\frac{k}{m}\right)^{v/\log(k^2/\lambda^2)}. \tag{14-14}$$

As shown in Fig. 14-3, the amplitude A_t remains small during the inflationary era.

The amplitude of the scalar spectrum decreases during the inflationary era, whereas the amplitude of the tensor spectrum does not change. Therefore, the tensor amplitude may become comparable to the scalar amplitude at the phase transition, if there is a small tensor fluctuation at the beginning. In this case, the tensor-to-scalar ratio

$$r = \frac{A_t}{A_s}$$

is to be an element for determining the CMB spectrum.

CMB Angular Power Spectra

We calculate the CMB angular power spectra by setting almost scale-invariant spectra P_s (14-12) and P_t (14-14) derived from the consideration of the evolution equations as the primordial power spectra of the Friedmann universe. The calculations are performed using the widely known calculation code, "CMBFAST", and the results are displayed in Figs. 14-4 and 14-5, together with the observed data of such as WMAP.

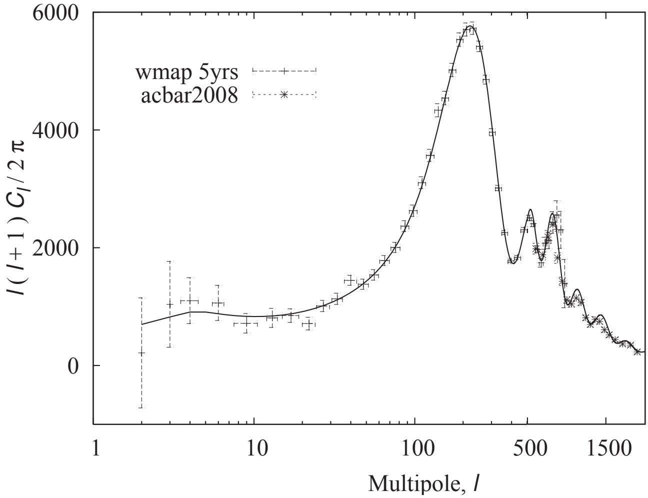

Figure 14-4: The temperature-temperature (TT) angular power spectrum of CMB. The calculation result (solid) is displayed together with the data of WMAP5 and ACBAR 2008. The tensor-to-scalar ratio is $r = 0.06$. The parameters of the dynamical damping factor are $\lambda = 0.00026\ (= m/60)\mathrm{Mpc}^{-1}$ and $v = 0.00002$. The cosmological parameters are $\tau_e = 0.08$, $\Omega_b = 0.043$, $\Omega_c = 0.20$, $\Omega_{\mathrm{vac}} = 0.757$, $H_0 = 73.1$, $T_{\mathrm{cmb}} = 2.726$, and $Y_{\mathrm{He}} = 0.24$. [K. Hamada, S. Horata, and T. Yukawa, Phys. Rev. D **81** (2010) 083533.]

The parameters A_s and v are adjusted to be consistent with the experimental data. We also add the tensor amplitude in order to compensate for the lack of the scalar amplitude at large angle components ($l < 100$). The tensor-to-scalar ratio is here set to $r = 0.06$. The values of the Planck mass and the dynamical mass scale in the comoving coordinates are $m = 0.0156$

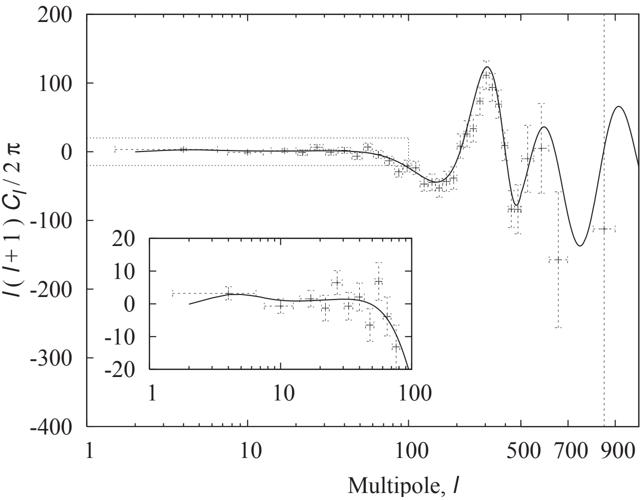

Figure 14-5: The temperature-polarization (TE) angular power spectrum of CMB together with the WMAP5 data. The parameters are the same as those in Fig. 14-4. [K. Hamada, S. Horata, and T. Yukawa, Phys. Rev. D **81** (2010) 083533.]

and

$$\lambda = \frac{m}{60} = 0.00026\,\mathrm{Mpc}^{-1}, \tag{14-15}$$

adopted in the calculation in the previous section. The ratio of these two scales is almost determined by the number of e-foldings, as shown in (12-9). Among the cosmological parameters, the optical depth is determined to be $\tau_e = 0.08$ from the polarization-polarization (EE) power spectrum (not depicted here). Other cosmological parameters are also determined to match the experimental data.

First, we pay attention to the sharp falloff at low multipole components in the WMAP data. If the falloff suggests the existence of a new physical scale, its value just becomes about $0.0002\,\mathrm{Mpc}^{-1}$ above, which is given by substituting $l = 2$ to the relation (11-9) between multipole l and comoving wavenumber k. Therefore, the primordial power spectra P_s (14-12) and P_t (14-14) with the value (14-15) can explain the observed data.[5]

[5] The deviation seen at the low multipole in Fig. 11-1 is often regarded as an error that can be explained by cosmic variance, but we here consider that it represents the existence of the scale.

Substituting λ (14-15) and $\Lambda_{\rm QG} \simeq 1.1 \times 10^{17}$ GeV (12-10) into (14-13), it turns out that the initial scale factor before inflation becomes the following order:

$$a(\tau_{\rm i}) = \frac{0.00026\,{\rm Mpc}^{-1}}{1.1 \times 10^{17}\,{\rm GeV}} \simeq 1.5 \times 10^{-59}$$

with the present as 1. In other words, it indicates that the wavelength of $1/\lambda \simeq 4000\,{\rm Mpc}$ today was a wavelength of the correlation length $\xi_\Lambda = 1/\Lambda_{\rm QG} \simeq 2 \times 10^{-31}$cm before inflation. This value well matches the inflationary scenario given in Chapter 12. In the inflationary era, the universe expands about 10^{30} times from the Planck time to the end of the phase transition process (see Figs. 12-1 and 12-2). After moving to the Friedmann spacetime, from the ratio of the dynamical energy scale $\Lambda_{\rm QG}$ and 3^oK, it can be estimated that the universe expands about 10^{29} times until today. In total, 10^{59} is derived.

From the latest CMB observations, it is suggested that the primordial power spectrum is slightly inclined to $n_s < 1$ (red tilt). We consider that this is not a feature of initial quantum spectra, but rather is a secondary one caused by dynamics during inflation. Such an effect will be expected from the nonlinear term as mentioned before.

Finally, non-Gaussianity of the primordial power spectrum is derived from diffeomorphism invariant interactions. When expressed by $f_{\rm NG}$ which is used everywhere as an indicator of its strength,[6] it becomes $o(1)$.

Other issues In the renormalizable quantum gravity it is not necessary to introduce any ultraviolet cutoff at the Planck scale. Therefore there is no fine-tuning of the 10^{120} digit known as the cosmological constant problem in the first place. The cosmological constant is given as a renormalization group invariant physical scale that can be taken small as mentioned in Chapter 10.

In this theory we can explain the initial inflation and the current de Sitter expansion separately on different scales. The former is explained by the Planck mass and the latter is by the cosmological constant. In this way, since we do not use the cosmological constant for the initial stage, there is no need to introduce a new quantity called dark energy to distinguish the current inflation mechanism from the initial one.

There is a gravitational θ-term composed of the Weyl tensor as a CP violating term at the Plank scale. If this term actually exists, how will it affect baryogenesis? Is there any possibility that a Cohen-Kaplan-type coupling

[6] See E. Komatsu and D. Spergel, *Acoustic Signatures in the Primary Microwave Background Bispectrum*, Phys. Rev. D **63** (2001) 063002.

between the Ricci scalar curvature and a divergence of the baryon number current causing baryogenesis appears in the low energy effective theory of gravity?

The true character of dark matters is not known yet. Dark matters hardly interact with ordinary matters, but are affected by gravity. It is considered to exist because if there is no dark matter, we cannot explain the current galactic distribution of the universe as well as the behavior of the rotation curve of galaxies. If there is a stable gravitational soliton based on quantum gravity, it will be a candidate for dark matters.

Moreover, when the effect of quantum gravity turns on at the center of macroscopic black holes, is there a possibility that black holes will eventually explode due to the repulsive force that ignites inflation? It is a future task.

Appendix A

Useful Gravitational Formulas

Formulas on Curvatures

The Lorentz signature employed in this book is $(-1, 1, \cdots, 1)$.[1] In Appendix A, unless otherwise noted, dimensions of spacetime are arbitrary D. Definitions of the Christoffel symbol and the Riemann curvature tensor are

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right),$$
$$R^{\lambda}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\lambda}_{\rho\sigma} \Gamma^{\rho}_{\mu\nu} - \Gamma^{\lambda}_{\rho\nu} \Gamma^{\rho}_{\mu\sigma},$$

respectively. The Ricci tensor is defined by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ and the Ricci scalar curvature is $R = R^{\mu}{}_{\mu}$. A covariant derivative is expressed using the Christoffel symbol as

$$\nabla_\mu A^{\sigma_1 \cdots \sigma_m}_{\lambda_1 \cdots \lambda_n} = \partial_\mu A^{\sigma_1 \cdots \sigma_m}_{\lambda_1 \cdots \lambda_n} - \sum_{j=1}^{n} \Gamma^{\nu_j}_{\mu\lambda_j} A^{\sigma_1 \cdots \sigma_m}_{\lambda_1 \cdots \nu_j \cdots \lambda_n} + \sum_{j=1}^{m} \Gamma^{\sigma_j}_{\mu\nu_j} A^{\sigma_1 \cdots \nu_j \cdots \sigma_m}_{\lambda_1 \cdots \lambda_n},$$

which satisfies

$$\left[\nabla_\mu, \nabla_\nu \right] A_{\lambda_1 \cdots \lambda_n} = \sum_{j=1}^{n} R_{\mu\nu\lambda_j}{}^{\sigma_j} A_{\lambda_1 \cdots \sigma_j \cdots \lambda_n}.$$

The Riemann curvature tensor satisfies an antisymmetric property $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$ and

$$R^{\mu}{}_{\nu\lambda\sigma} + R^{\mu}{}_{\lambda\sigma\nu} + R^{\mu}{}_{\sigma\nu\lambda} = 0,$$
$$\nabla_\rho R^{\mu}{}_{\nu\lambda\sigma} + \nabla_\lambda R^{\mu}{}_{\nu\sigma\rho} + \nabla_\sigma R^{\mu}{}_{\nu\rho\lambda} = 0.$$

The last is the Bianchi identity. From this, $\nabla_\mu R^{\mu}{}_{\lambda\nu\sigma} = \nabla_\nu R_{\lambda\sigma} - \nabla_\sigma R_{\lambda\nu}$ and $\nabla_\mu R^{\mu}{}_{\nu} = \nabla_\nu R/2$ are yielded.

---

[1] In Euclidean space, rewrite $\sqrt{-g}$ to $\sqrt{g}$, and also $\eta_{\mu\nu}$ to $\delta_{\mu\nu}$ for the flat metric.

The Weyl curvature tensor is defined by

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{D-2}\left(g_{\mu\lambda}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\lambda}\right)$$
$$+\frac{1}{(D-1)(D-2)}\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}\right)R. \quad \text{(A-1)}$$

This tensor vanishes no matter which index is contracted such as $C^{\mu}{}_{\mu\lambda\sigma} = C^{\mu}{}_{\nu\mu\sigma} = 0$. The number of independent components is $(D-3)D(D+1)(D+2)/12$, which is identically zero in three dimensions and ten in four dimensions.

Variational formulas Variational formulas of the curvatures are given as follows:

$$\delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\sigma}\delta g_{\lambda\sigma}, \qquad \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu},$$
$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\right),$$
$$\begin{aligned}\delta R^{\lambda}{}_{\mu\sigma\nu} &= \nabla_\sigma\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\lambda}_{\mu\sigma}\\ &= \frac{1}{2}g^{\lambda\rho}\Big[\nabla_\sigma\nabla_\mu\delta g_{\nu\rho} + \nabla_\sigma\nabla_\nu\delta g_{\mu\rho} - \nabla_\sigma\nabla_\rho\delta g_{\mu\nu} - \nabla_\nu\nabla_\mu\delta g_{\sigma\rho}\\ &\quad -\nabla_\nu\nabla_\sigma\delta g_{\mu\rho} + \nabla_\nu\nabla_\rho\delta g_{\mu\sigma}\Big],\end{aligned}$$
$$\begin{aligned}\delta R_{\mu\nu} &= \delta R^{\lambda}{}_{\mu\lambda\nu}\\ &= \frac{1}{2}\Big[\nabla_\mu\nabla^\lambda\delta g_{\lambda\nu} + \nabla_\nu\nabla^\lambda\delta g_{\lambda\mu} - \nabla_\mu\nabla_\nu\left(g^{\lambda\sigma}\delta g_{\lambda\sigma}\right) - \nabla^2\delta g_{\mu\nu}\Big]\\ &\quad -R^{\lambda}{}_{\mu}{}^{\sigma}{}_{\nu}\delta g_{\lambda\sigma} + \frac{1}{2}\left(R_\mu{}^\lambda\delta g_{\lambda\nu} + R_\nu{}^\lambda\delta g_{\lambda\mu}\right),\end{aligned}$$
$$\begin{aligned}\delta R &= \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\\ &= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla^2\left(g^{\mu\nu}\delta g_{\mu\nu}\right).\end{aligned}$$

In addition, the following variation formulas including derivatives are useful:

$$\delta(\nabla_\mu A) = \nabla_\mu\delta A,$$
$$\delta(\nabla_\mu\nabla_\nu A) = \nabla_\mu\nabla_\nu\delta A - \frac{1}{2}\nabla^\lambda A\left(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right),$$
$$\begin{aligned}\delta(\nabla^2 A) &= \nabla^2\delta A - \delta g_{\mu\nu}\nabla^\mu\nabla^\nu A - \nabla^\mu A\nabla^\nu\delta g_{\mu\nu}\\ &\quad +\frac{1}{2}\nabla^\lambda A\nabla_\lambda(g^{\mu\nu}\delta g_{\mu\nu}),\end{aligned}$$

where A is an arbitrary scalar.

Conformal variations of curvatures A conformal (Weyl) variation $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ of the scalar curvature is given from the above variational formulas as

$$\delta_\omega \sqrt{-g}R = (D-2)\omega\sqrt{-g}R - 2(D-1)\sqrt{-g}\nabla^2\omega.$$

Conformal variations of various curvatures squared are given by

$$\begin{aligned}
\delta_\omega \sqrt{-g}R^2_{\mu\nu\lambda\sigma} &= (D-4)\omega\sqrt{g}R^2_{\mu\nu\lambda\sigma} - 8\sqrt{-g}R^{\mu\nu}\nabla_\mu\nabla_\nu\omega, \\
\delta_\omega \sqrt{g}R^2_{\mu\nu} &= (D-4)\omega\sqrt{g}R^2_{\mu\nu} - 2\sqrt{-g}R\nabla^2\omega \\
&\quad -2(D-2)\sqrt{-g}R^{\mu\nu}\nabla_\mu\nabla_\nu\omega, \\
\delta_\omega \sqrt{-g}R^2 &= (D-4)\omega\sqrt{g}R^2 - 4(D-1)\sqrt{g}R\nabla^2\omega, \\
\delta_\omega \sqrt{-g}\nabla^2 R &= (D-4)\omega\sqrt{-g}\nabla^2 R + (D-6)\sqrt{-g}\nabla^\lambda R\nabla_\lambda\omega \\
&\quad -2\sqrt{-g}R\nabla^2\omega - 2(D-1)\sqrt{g}\nabla^4\omega, \\
\delta_\omega \sqrt{-g}F_{\mu\nu}F^{\mu\nu} &= (D-4)\omega\sqrt{-g}F_{\mu\nu}F^{\mu\nu},
\end{aligned}$$

where $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ is simply expressed as $R^2_{\mu\nu\lambda\sigma}$, and so on.

A generalized expression of the Wess-Zumino integrability condition (5-3) obtained by performing the conformal variations twice to an effective action Γ in D dimensions is given by

$$\begin{aligned}
[\delta_{\omega_1}, \delta_{\omega_2}]\Gamma &= \left\{4\eta_1 + D\eta_2 + 4(D-1)\eta_3 + (D-4)\eta_4\right\} \\
&\qquad \times \frac{1}{(4\pi)^2}\int d^D x\sqrt{-g}\, R\left(\omega_1\nabla^2\omega_2 - \omega_2\nabla^2\omega_1\right) \\
&= 0.
\end{aligned} \tag{A-2}$$

Three combinations that satisfy this integrability condition are the square of the Weyl curvature tensor, the Euler density (Gauss-Bonnet combination), and a function that becomes a total divergence form in four dimensions, which are given as follows:

$$\begin{aligned}
F_D &= C^2_{\mu\nu\lambda\sigma} = R^2_{\mu\nu\lambda\sigma} - \frac{4}{D-2}R^2_{\mu\nu} + \frac{2}{(D-1)(D-2)}R^2, \\
G_4 &= R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2, \\
M_D &= (D-4)H^2 - 4\nabla^2 H,
\end{aligned}$$

where $H = R/(D-1)$. The conformal anomalies introduced in Chapter 9 can be written with F_D and $E_D = G_4 + \chi(D)M_D$. The modified Euler density $G_D = G_4 + (D-4)\chi(D)H^2$ is the bulk part of E_D excluding the total divergence term.

Euler characteristics The Euler characteristic is a topological invariant existing in even-dimensional Euclidean spaces. For a compact space in $D = 2$, it is defined by

$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}R.$$

In $D = 4$, using ${}^*R_{\mu\nu\lambda\sigma} = \epsilon_{\mu\nu}{}^{\rho\kappa}R_{\rho\kappa\lambda\sigma}/2$, it is defined by[2]

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,{}^*R_{\mu\nu\lambda\sigma}{}^*R^{\mu\nu\lambda\sigma} = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,G_4.$$

Euler's relations As relations associated with the Euler characteristics, when $D = 2$,

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$$

holds, and when $D = 4$,

$$R_{\mu\lambda\sigma\rho}R_\nu{}^{\lambda\sigma\rho} - 2R_{\mu\lambda\nu\sigma}R^{\lambda\sigma} - 2R_{\mu\lambda}R_\nu{}^\lambda + R_{\mu\nu}R = \frac{1}{4}g_{\mu\nu}G_4$$

holds.

Mode expansions When the metric field is decomposed into a conformal factor and others as $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$, the curvatures are expressed as

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} &= \bar{\Gamma}^\lambda{}_{\mu\nu} + \bar{g}^\lambda{}_\mu\bar{\nabla}_\nu\phi + \bar{g}^\lambda{}_\nu\bar{\nabla}_\mu\phi - \bar{g}_{\mu\nu}\bar{\nabla}^\lambda\phi, \\
R^\lambda{}_{\mu\sigma\nu} &= \bar{R}^\lambda{}_{\mu\sigma\nu} + \bar{g}^\lambda{}_\nu\bar{\Delta}_{\mu\sigma} - \bar{g}^\lambda{}_\sigma\bar{\Delta}_{\mu\nu} + \bar{g}_{\mu\sigma}\bar{\Delta}^\lambda{}_\nu - \bar{g}_{\mu\nu}\bar{\Delta}^\lambda{}_\sigma \\
&\quad + \left(\bar{g}^\lambda{}_\nu\bar{g}_{\mu\sigma} - \bar{g}^\lambda{}_\sigma\bar{g}_{\mu\nu}\right)\bar{\nabla}_\rho\phi\bar{\nabla}^\rho\phi, \\
R_{\mu\nu} &= \bar{R}_{\mu\nu} - (D-2)\bar{\Delta}_{\mu\nu} - \bar{g}_{\mu\nu}\left[\bar{\nabla}^2\phi + (D-2)\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi\right], \\
R &= e^{-2\phi}\left[\bar{R} - 2(D-1)\bar{\nabla}^2\phi - (D-1)(D-2)\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi\right], \\
C^\lambda{}_{\mu\sigma\nu} &= \bar{C}^\lambda{}_{\mu\sigma\nu},
\end{aligned}$$

where $\bar{\Delta}_{\mu\nu} = \bar{\nabla}_\mu\bar{\nabla}_\nu\phi - \bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi$. The quantity with the bar is defined using the metric field $\bar{g}_{\mu\nu}$. The square of the Weyl tensor is then expressed as

$$\sqrt{-g}\,C^2_{\mu\nu\lambda\sigma} = \sqrt{-\bar{g}}\,e^{(D-4)\phi}\,\bar{C}^2_{\mu\nu\lambda\sigma}$$

[2] The expression ${}^*R_{\mu\nu\lambda\sigma}{}^*R^{\mu\nu\lambda\sigma}$ of the Euler density is a function defined only in four dimensions, whereas G_4 is defined in any dimension so that it can be used in renormalization theory by dimension regularization.

and the Euler density is

$$\sqrt{-g}\,G_4 = \sqrt{-\bar{g}}\,e^{(D-4)\phi}\left[\bar{G}_4 + (D-3)\bar{\nabla}_\mu J^\mu + (D-3)(D-4)K\right], \tag{A-3}$$

where

$$\begin{aligned} J^\mu &= 8\bar{R}^{\mu\nu}\bar{\nabla}_\nu\phi - 4\bar{R}\bar{\nabla}^\mu\phi + 4(D-2)\big(\bar{\nabla}^\mu\phi\bar{\nabla}^2\phi - \bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi \\ &\quad +\bar{\nabla}^\mu\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi\big), \\ K &= 4\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + 4(D-2)\bar{\nabla}^2\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi \\ &\quad +(D-1)(D-2)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2. \end{aligned}$$

Thus, the conformal-factor dependent part of the Euler density multiplied by $\sqrt{-g}$ becomes a total divergence form in four dimensions.

Furthermore, when the metric field with the bar is expanded in terms of the traceless tensor field $h_{\mu\nu}$ as $\bar{g}_{\mu\nu} = (\hat{g}e^h)_{\mu\nu}$ up to $o(h^2)$, we obtain

$$\begin{aligned} \bar{\Gamma}^\lambda_{\mu\nu} &= \hat{\Gamma}^\lambda_{\mu\nu} + \hat{\nabla}_{(\mu}h^\lambda{}_{\nu)} - \frac{1}{2}\hat{\nabla}^\lambda h_{\mu\nu} + \frac{1}{2}\hat{\nabla}_{(\mu}(h^2)^\lambda{}_{\nu)} - \frac{1}{4}\hat{\nabla}^\lambda(h^2)_{\mu\nu} \\ &\quad - h^\lambda{}_\sigma\hat{\nabla}_{(\mu}h^\sigma{}_{\nu)} + \frac{1}{2}h^\lambda{}_\sigma\hat{\nabla}^\sigma h_{\mu\nu}, \\ \bar{R} &= \hat{R} - \hat{R}_{\mu\nu}h^{\mu\nu} + \hat{\nabla}_\mu\hat{\nabla}_\nu h^{\mu\nu} + \frac{1}{2}\hat{R}^\sigma{}_{\mu\lambda\nu}h^\lambda{}_\sigma h^{\mu\nu} - \frac{1}{4}\hat{\nabla}^\lambda h^\mu{}_\nu\hat{\nabla}_\lambda h^\nu{}_\mu \\ &\quad + \frac{1}{2}\hat{\nabla}_\nu h^\nu{}_\mu\hat{\nabla}_\lambda h^{\lambda\mu} - \hat{\nabla}_\mu(h^\mu{}_\nu\hat{\nabla}^\lambda h^\nu{}_\lambda), \\ \bar{R}_{\mu\nu} &= \hat{R}_{\mu\nu} - \hat{R}^\sigma{}_{\mu\lambda\nu}h^\lambda{}_\sigma + \hat{R}^\lambda_{(\mu}h_{\nu)\lambda} + \hat{\nabla}_{(\mu}\hat{\nabla}^\lambda h_{\nu)\lambda} - \frac{1}{2}\hat{\nabla}^2 h_{\mu\nu} \\ &\quad - \frac{1}{2}h^\lambda_{(\mu}\hat{\nabla}^2 h_{\nu)\lambda} - \frac{1}{2}\hat{\nabla}^\lambda h^\sigma{}_\mu\hat{\nabla}_\sigma h_{\nu\lambda} - \frac{1}{4}\hat{\nabla}_\mu h^\lambda{}_\sigma\hat{\nabla}_\nu h^\sigma{}_\lambda \\ &\quad - \frac{1}{2}\hat{\nabla}_\lambda(h^\lambda{}_\sigma\hat{\nabla}_{(\mu}h^\sigma{}_{\nu)}) + \frac{1}{2}\hat{\nabla}_\lambda(h^\sigma_{(\mu}\hat{\nabla}_{\nu)}h^\lambda{}_\sigma) + \frac{1}{2}\hat{\nabla}_\lambda(h^\lambda{}_\sigma\hat{\nabla}^\sigma h_{\mu\nu}), \end{aligned}$$

where the raising and lowering of the index is performed using the background metric $\hat{g}_{\mu\nu}$, and the traceless condition is $h^\mu{}_\mu = \hat{g}^{\mu\nu}h_{\mu\nu} = 0$. The symmetric product is defined by $a_{(\mu}b_{\nu)} = (a_\mu b_\nu + a_\nu b_\mu)/2$. With attention to $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$ and $\bar{g}^{\mu\nu} = (\hat{g}e^{-h})^{\mu\nu} = \hat{g}^{\mu\nu} - h^{\mu\nu} + \cdots$, we can derive $\bar{R}$ from $\bar{R}_{\mu\nu}$, where $[\hat{\nabla}_\lambda, \hat{\nabla}_\nu]h^\lambda{}_\mu = h^\lambda{}_\sigma\hat{R}^\sigma{}_{\mu\nu\lambda} + h_{\mu\sigma}\hat{R}^\sigma{}_\nu$ is used.

When the flat background metric $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ is employed, the square of each curvature with the bar, and so on, is expanded up to $o(h^2)$ as

$$\bar{R}^2_{\mu\nu\lambda\sigma} = \partial_\lambda\partial_\sigma h_{\mu\nu}\partial^\lambda\partial^\sigma h^{\mu\nu} - 2\partial_\nu\partial_\lambda h_{\mu\sigma}\partial^\mu\partial^\lambda h^{\nu\sigma} + \partial_\lambda\partial_\sigma h_{\mu\nu}\partial^\mu\partial^\nu h^{\lambda\sigma},$$

$$
\begin{aligned}
\bar{R}^2_{\mu\nu} &= \frac{1}{2}\partial_\mu\chi_\nu\partial^\mu\chi^\nu - \partial^2 h_{\mu\nu}\partial^\mu\chi^\nu + \frac{1}{2}\partial_\mu\chi_\nu\partial^\nu\chi^\mu + \frac{1}{4}\partial^2 h_{\mu\nu}\partial^2 h^{\mu\nu}, \\
\bar{R}^2 &= \partial_\mu\chi^\mu\partial_\nu\chi^\nu, \\
\bar{\nabla}^2\bar{R} &= \partial^2\partial_\mu\chi^\mu - \frac{1}{4}\partial^2\left(\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}\right) + \frac{1}{2}\partial^2\left(\chi_\mu\chi^\mu\right) - \partial^2\partial^\mu\left(h_{\mu\nu}\chi^\nu\right) \\
&\quad - h_{\mu\nu}\partial^\mu\partial^\nu\partial_\lambda\chi^\lambda - \chi_\mu\partial^\mu\partial_\nu\chi^\nu,
\end{aligned}
$$

where $\chi_\mu = \partial_\nu h^\nu{}_\mu$. From these, we can see that the $o(h^2)$ term of the Euler density with the bar can be written in a total divergence form in any dimension as

$$\bar{G}_4 = \partial_\sigma L^\sigma, \tag{A-4}$$

where

$$
\begin{aligned}
L^\sigma &= \partial_\lambda h_{\mu\nu}\partial^\lambda\partial^\sigma h^{\mu\nu} - \partial^\sigma h_{\mu\nu}\partial^2 h^{\mu\nu} - 2\partial_\lambda h_{\mu\nu}\partial^\lambda\partial^\mu h^{\nu\sigma} - 2\partial_\lambda h_\nu{}^\sigma\partial^\lambda\chi^\nu \\
&\quad + 4\partial^\sigma h_{\mu\nu}\partial^\mu\chi^\nu + \partial_\lambda h_{\mu\nu}\partial^\mu\partial^\nu h^{\lambda\sigma} - \partial_\lambda h_\nu{}^\sigma\partial^\nu\chi^\lambda - \chi_\lambda\partial^\lambda\chi^\sigma \\
&\quad + \chi^\sigma\partial_\lambda\chi^\lambda.
\end{aligned}
$$

Moreover, quantities involving ϕ are expanded up to $o(h^2)$ as follows:

$$
\begin{aligned}
\bar{\nabla}^2\phi &= \partial^2\phi - \chi^\mu\partial_\mu\phi - h^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{1}{2}h^{\mu\lambda}\partial_\mu h^\nu{}_\lambda\partial_\nu\phi + \frac{1}{2}h^{\mu\nu}\chi_\mu\partial_\nu\phi \\
&\quad + \frac{1}{2}h^{\mu\lambda}h^\nu{}_\lambda\partial_\mu\partial_\nu\phi, \\
\bar{\nabla}_\mu\bar{\nabla}_\nu\phi &= \partial_\mu\partial_\nu\phi - \frac{1}{2}\left(\partial_\mu h^\lambda{}_\nu + \partial_\nu h^\lambda{}_\mu - \partial^\lambda h_{\mu\nu}\right)\partial_\lambda\phi \\
&\quad - \frac{1}{4}\Big(h^\sigma{}_\mu\partial_\nu h^\lambda{}_\sigma + h^\sigma{}_\nu\partial_\mu h^\lambda{}_\sigma - h^\lambda{}_\sigma\partial_\mu h^\sigma{}_\nu - h^\lambda{}_\sigma\partial_\nu h^\sigma{}_\mu \\
&\quad - h^\sigma{}_\mu\partial^\lambda h_{\nu\sigma} - h^\sigma{}_\nu\partial^\lambda h_{\mu\sigma} + 2h^{\lambda\sigma}\partial_\sigma h_{\mu\nu}\Big)\partial_\lambda\phi, \\
\bar{\nabla}^4\phi &= \partial^4\phi - \partial^2(h^{\mu\nu}\partial_\mu\partial_\nu\phi + \chi^\mu\partial_\mu\phi) - h^{\mu\nu}\partial_\mu\partial_\nu\partial^2\phi - \chi^\mu\partial_\mu\partial^2\phi \\
&\quad + \partial^2\left(\frac{1}{2}h^{\mu\lambda}h^\nu{}_\lambda\partial_\mu\partial_\nu\phi + \frac{1}{2}h^{\mu\nu}\partial_\mu h^\lambda{}_\nu\partial_\lambda\phi + \frac{1}{2}h^{\mu\nu}\chi_\mu\partial_\nu\phi\right) \\
&\quad + h^{\mu\nu}\bigg(\partial_\mu\partial_\nu h^{\lambda\sigma}\partial_\lambda\partial_\sigma\phi + 2\partial_\mu h^{\lambda\sigma}\partial_\nu\partial_\lambda\partial_\sigma\phi + h^{\lambda\sigma}\partial_\mu\partial_\nu\partial_\lambda\partial_\sigma\phi \\
&\quad + \frac{1}{2}\partial_\mu h^\lambda{}_\nu\partial_\lambda\partial^2\phi + \frac{1}{2}\chi_\mu\partial_\nu\partial^2\phi + \partial_\mu\partial_\nu\chi^\lambda\partial_\lambda\phi + 2\partial_\mu\chi^\lambda\partial_\nu\partial_\lambda\phi \\
&\quad + 2\chi^\lambda\partial_\mu\partial_\nu\partial_\lambda\phi\bigg) + \chi^\mu\partial_\mu\chi^\nu\partial_\nu\phi + \chi^\mu\chi^\nu\partial_\mu\partial_\nu\phi \\
&\quad + \chi^\mu\partial_\mu h^{\nu\lambda}\partial_\nu\partial_\lambda\phi + \frac{1}{2}h^{\mu\lambda}h^\nu{}_\lambda\partial_\mu\partial_\nu\partial^2\phi,
\end{aligned} \tag{A-5}
$$

and

$$\begin{aligned}
\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi = {} & \partial^\mu\chi^\nu\partial_\mu\partial_\nu\phi - \frac{1}{2}\partial^2 h^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\partial^\lambda h^{\mu\nu}\partial_\lambda\chi_\mu\partial_\nu\phi \\
& -\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial_\mu\chi^\lambda\partial_\nu\phi + \frac{1}{2}\partial^\lambda h^{\mu\nu}\partial_\mu\chi_\nu\partial_\lambda\phi \\
& +\frac{1}{2}\partial^2 h^{\mu\nu}\partial_\mu h^\lambda{}_\nu\partial_\lambda\phi - \frac{1}{4}\partial^2 h^{\mu\nu}\partial^\lambda h_{\mu\nu}\partial_\lambda\phi \\
& -h^{\mu\nu}\partial^\lambda\chi_\mu\partial_\nu\partial_\lambda\phi - h^{\mu\nu}\partial_\mu\chi^\lambda\partial_\nu\partial_\lambda\phi \\
& +\frac{1}{2}h^{\mu\nu}\partial^2 h^\lambda{}_\mu\partial_\nu\partial_\lambda\phi - \frac{1}{2}\partial_\lambda h^{\mu\nu}\partial_\mu h^{\lambda\sigma}\partial_\nu\partial_\sigma\phi \\
& -\frac{1}{4}\partial^\lambda h^{\mu\nu}\partial^\sigma h_{\mu\nu}\partial_\lambda\partial_\sigma\phi - \frac{1}{2}\partial_\lambda(h^{\lambda\sigma}\partial^\mu h^\nu{}_\sigma)\partial_\mu\partial_\nu\phi \\
& +\frac{1}{2}\partial_\lambda(h^{\mu\sigma}\partial^\nu h^\lambda{}_\sigma)\partial_\mu\partial_\nu\phi + \frac{1}{2}\partial_\lambda(h^{\lambda\sigma}\partial_\sigma h^{\mu\nu})\partial_\mu\partial_\nu\phi, \\
\bar{R}\bar{\nabla}^2\phi = {} & \partial_\mu\chi^\mu\partial^2\phi - \chi^\mu\left(\partial_\nu\chi^\nu\partial_\mu\phi + \frac{1}{2}\chi_\mu\partial^2\phi\right) \\
& -\frac{1}{4}\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu}\partial^2\phi - h^{\mu\nu}\left(\partial_\mu\chi_\nu\partial^2\phi + \partial_\lambda\chi^\lambda\partial_\mu\partial_\nu\phi\right), \\
\bar{\nabla}^\mu\bar{R}\bar{\nabla}_\mu\phi = {} & \partial^\mu\partial_\nu\chi^\nu\partial_\mu\phi - \frac{1}{2}\partial^\lambda\partial^\sigma h^{\mu\nu}\partial_\lambda h_{\mu\nu}\partial_\sigma\phi - \frac{1}{2}\partial^\mu(\chi^\nu\chi_\nu)\partial_\mu\phi \\
& -\partial^\lambda(h^{\mu\nu}\partial_\mu\chi_\nu)\partial_\lambda\phi - h^{\mu\nu}\partial_\mu\partial_\lambda\chi^\lambda\partial_\nu\phi. \qquad \text{(A-6)}
\end{aligned}$$

Scalar Fields in Curved Space

The kinetic term of a free scalar field in curved space is given by

$$S = -\frac{1}{2}\int d^D x\sqrt{-g}\,\varphi\left(-\nabla^2 + \xi R\right)\varphi.$$

Under conformal transformations, since a scalar field transforms as $\delta_\omega\varphi = A\omega\varphi$, where $A = -(D-2)/2$, the action transforms as

$$\begin{aligned}
\delta_\omega S = -\frac{1}{2}\int d^D x\sqrt{-g}\,\omega\Big\{ & (D-2+2A)\,\varphi\left(-\nabla^2+\xi R\right)\varphi \\
& + [D-2-4\xi(D-1)]\left(\nabla^\mu\varphi\nabla_\mu\varphi + \varphi\nabla^2\varphi\right)\Big\}.
\end{aligned}$$

The first term on the right-hand side disappears when introducing the value of A. The second term disappears and the action becomes conformally invariant when the parameter representing the strength of the coupling is set as

$$\xi = \frac{D-2}{4(D-1)}. \tag{A-7}$$

Thus, the action has conformal invariance when $\xi = 0$ in two dimensions and $\xi = 1/6$ in four dimensions.

The energy-momentum tensor is defined by $\Theta^{\mu\nu} = (2/\sqrt{-g})\delta S/\delta g_{\mu\nu}$, which is given by

$$\begin{aligned}\Theta^{\mu\nu} = \nabla^\mu\varphi\nabla^\nu\varphi - \frac{1}{2}g^{\mu\nu}\nabla^\lambda\varphi\nabla_\lambda\varphi + \xi\Big[R^{\mu\nu}\varphi^2 - \frac{1}{2}g^{\mu\nu}R\varphi^2 \\ - \left(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2\right)\varphi^2\Big]\end{aligned} \tag{A-8}$$

for any ξ. Its trace $\Theta = \Theta^\mu_{\ \mu}$ is

$$\begin{aligned}\Theta = -\frac{1}{2}\left[D-2-4\xi(D-1)\right]\nabla^\mu\varphi\nabla_\mu\varphi \\ -2\xi(D-1)\,\varphi\left[-\nabla^2 + \frac{D-2}{4(D-1)}R\right]\varphi.\end{aligned}$$

In the conformal coupling (A-7), the first term disappears. In addition, only then, the second term is proportional to the equation of motion $(-\nabla^2 + \xi R)\varphi = 0$. Therefore, Θ disappears in proportion to the equation of motion in the case of the conformal coupling.

Fermions in Curved Space

In terms of the vielbein e^a_μ in D dimensions,[3] the metric field can be represented as $g_{\mu\nu} = e^a_\mu e_{\nu a}$. In the following, unless otherwise noted, a, b, c, and d denote the local Lorentz indices, while μ, ν, λ, and σ are the Einstein indices. The gamma matrix that basically has the local Lorentz index is defined by an anti-commutation relation $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. The gamma matrix with the Einstein index is then expressed as $\gamma^\mu = e^\mu_a\gamma^a$ using the vielbein. The Dirac adjoint of a fermion field ψ is defined by $\bar{\psi} = \psi^\dagger\gamma^0$ using the gamma matrix γ^0 with the local Lorentz index.[4]

[3] In four dimensions, it is called the vierbein or tetrad.

[4] Hermitian conjugate of the product of two Grassmann numbers θ and ψ follows a rule of $(\theta\psi)^\dagger = \psi^\dagger\theta^\dagger$.

Covariant derivatives Introducing the vielbein, we can define a covariant derivative acting on the local Lorentz index. For a vector, by taking $V_a = e^\mu_a V_\mu$, it is represented as

$$\nabla_\mu V_a = \partial_\mu + \omega_{\mu a}{}^b V_b,$$

where $\omega_{\mu ab}$ is the spin connection. To coincide with a normal covariant derivative $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$, we require the condition that a covariant differentiation of the vielbein vanishes as $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0$. The spin connection is thus represented as

$$\omega_{\mu ab} = e^\nu_a \left(\partial_\mu e_{\nu b} - \Gamma^\lambda_{\mu\nu} e_{\lambda b} \right),$$

where an antisymmetric property $\omega_{\mu ab} = -\omega_{\mu ba}$ holds for the local Lorentz indices.

In order to apply the covariant derivative to fermions, it is generalized as

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \Sigma^{ab},$$

where Σ^{ab} is a generator of the local Lorentz group satisfying[5]

$$\left[\Sigma^{ab}, \Sigma^{cd} \right] = -\eta^{ac} \Sigma^{bd} + \eta^{ad} \Sigma^{bc} + \eta^{bc} \Sigma^{ad} - \eta^{bd} \Sigma^{ac}.$$

Using this algebra, the commutation relation between the covariant derivatives is expressed as

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] &= \frac{1}{2} \left(\partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + [\omega_\mu, \omega_\nu]_{ab} \right) \Sigma^{ab} \\ &= \frac{1}{2} R_{\mu\nu ab} \Sigma^{ab}, \end{aligned}$$

where $R_{\mu\nu ab} = e^\lambda_a e^\sigma_b R_{\mu\nu\lambda\sigma}$. The Riemann curvature tensor can be represented as above by using the spin connection.

The generator of the Lorentz group for scalar fields is $\Sigma^{ab} = 0$. When acting on vector fields, it is given by $(\Sigma^{ab})_{cd} = \delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c$. For fermions, it is given by using the gamma matrix as

$$\Sigma^{ab} = -\frac{1}{4} \left[\gamma^a, \gamma^b \right].$$

5 A slightly different convention than that in Chapters 2 and 3 is used here.

Fermion action The action for a massless free fermion is given by

$$\begin{aligned} S &= i\int d^D x\sqrt{-g}\bar{\psi}\gamma^\mu \nabla_\mu \psi \\ &= \frac{i}{2}\int d^D x\sqrt{-g}\left(\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi\right), \end{aligned} \tag{A-9}$$

where the covariant derivatives for fermions are defined by

$$\begin{aligned} \nabla_\mu \psi &= \left(\partial_\mu + \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}\right)\psi, \\ \nabla_\mu \bar{\psi} &= \bar{\psi}\overleftarrow{\nabla}_\mu = \bar{\psi}\left(\overleftarrow{\partial}_\mu - \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}\right). \end{aligned}$$

The second equality in (A-9) can be shown by performing a partial integration with attention to $\partial_\mu(\sqrt{-g}e_a^\mu) = -\sqrt{-g}\omega_{\mu a}{}^b e_b^\mu$ and rewriting the expression using $[\Sigma^{ab}, \gamma^c] = \eta^{bc}\gamma^a - \eta^{ac}\gamma^b$. This can also be shown from the fact that the covariant derivative satisfies the Leibniz rule $\nabla_\mu(\bar{\psi}\gamma^\nu\psi) = \nabla_\mu\bar{\psi}\gamma^\nu\psi + \bar{\psi}\gamma^\nu\nabla_\mu\psi$, where $\nabla_\mu\gamma^\nu = \gamma^a\nabla_\mu e_a^\nu = 0$ is used. This quantity can also be expressed as

$$\begin{aligned} \nabla_\mu(\bar{\psi}\gamma^\nu\psi) &= \partial_\mu(\bar{\psi}\gamma^\nu\psi) + \Gamma^\nu_{\mu\lambda}\bar{\psi}\gamma^\lambda\psi \\ &= \partial_\mu\bar{\psi}\gamma^\nu\psi + \bar{\psi}\gamma^\nu\partial_\mu\psi - \omega_{\mu a}{}^b e_b^\nu\bar{\psi}\gamma^a\psi. \end{aligned}$$

The action can be rewritten as

$$S = i\int d^D x\sqrt{-g}\left[\frac{1}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right) - \frac{1}{4}\omega_{\mu ab}e_c^\mu\bar{\psi}\gamma^{abc}\psi\right],$$

where $\gamma^{abc} = (\gamma^a\gamma^b\gamma^c + \text{antisymmetric})/3!$ is a completely antisymmetric product of the gamma matrix and $\gamma^c\Sigma^{ab} + \Sigma^{ab}\gamma^c = -\gamma^{abc}$ is used.

Energy-momentum tensor The variation of the metric field is expressed with the vielbein as $\delta g_{\mu\nu} = \delta e_\mu^a e_{\nu a} + e_\mu^a \delta e_{\nu a}$, and thus $\delta\sqrt{-g} = \sqrt{-g}e_a^\mu\delta e_\mu^a$ is obtained. Since a variation of δ_{ab} expressed in terms of the vielbein disappears, we get $\delta e_a^\mu = -e_b^\mu e_a^\nu \delta e_\nu^b$. Using these, a variation of the spin connection is expressed as $\delta\omega_{\mu ab} = e_a^\nu\nabla_\mu\delta e_{\nu b} - e_a^\nu e_{\lambda b}\delta\Gamma^\lambda_{\mu\nu}$ and a variation of the action is given by

$$\begin{aligned} \delta S = \frac{i}{2}\int d^D x\sqrt{-g}\,\delta e_{\mu a}\Big[& e^{\mu a}\left(\bar{\psi}\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^\lambda\psi\right) \\ & - e_\lambda^a\left(\bar{\psi}\gamma^\mu\nabla^\lambda\psi - \nabla^\lambda\bar{\psi}\gamma^\mu\psi\right) - \frac{1}{2}e_b^\mu e_c^\lambda\nabla_\lambda\left(\bar{\psi}\gamma^{abc}\psi\right)\Big]. \end{aligned}$$

Therefore, the energy-momentum tensor is yielded as

$$\Theta^{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{-g}}\left(e_a^\mu \frac{\delta S}{\delta e_{\nu a}} + e_a^\nu \frac{\delta S}{\delta e_{\mu a}}\right)$$
$$= -\frac{i}{4}\left[\bar{\psi}\gamma^\mu\nabla^\nu\psi + \bar{\psi}\gamma^\nu\nabla^\mu\psi - \nabla^\mu\bar{\psi}\gamma^\nu\psi - \nabla^\nu\bar{\psi}\gamma^\mu\psi \right.$$
$$\left. -2g^{\mu\nu}\left(\bar{\psi}\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^\lambda\psi\right)\right].$$

The trace is

$$\Theta = \Theta^\mu_{\ \mu} = i\frac{1}{2}(D-1)\left(\bar{\psi}\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^\lambda\psi\right).$$

As in scalar fields with the conformal coupling, the trace vanishes according to the equation of motion $\gamma^\mu\nabla_\mu\psi = \nabla_\mu\bar{\psi}\gamma^\mu = 0$.

Conformal invariance The massless fermion action is conformally invariant at any dimension. Under the conformal transformation $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$, the vielbein and fermions transform as

$$\delta_\omega e_a^\mu = -\omega e_a^\mu, \qquad \delta_\omega e_{\mu a} = \omega e_{\mu a},$$
$$\delta_\omega\psi = \frac{1-D}{2}\omega\psi, \qquad \delta_\omega\bar{\psi} = \frac{1-D}{2}\omega\bar{\psi}.$$

The spin connection and so on transform as

$$\delta_\omega\omega_{\mu ab} = \left(e_{\mu a}e_b^\lambda - e_{\mu b}e_a^\lambda\right)\partial_\lambda\omega, \quad \delta_\omega\left(\gamma^\mu\nabla_\mu\psi\right) = -\frac{D+1}{2}\omega\gamma^\mu\nabla_\mu\psi.$$

where $\gamma_a\Sigma^{ab} = (D-1)\gamma^b/2$ is used in the second. Thus, it can be shown that the kinetic term of fermions is conformally invariant in arbitrary D dimensions as

$$\delta_\omega\left(\sqrt{-g}\bar{\psi}\gamma^\mu\nabla_\mu\psi\right) = \left(D\omega + \frac{1-D}{2}\omega - \frac{D+1}{2}\omega\right)\sqrt{-g}\bar{\psi}\gamma^\mu\nabla_\mu\psi = 0.$$

Expansions of spin connection Here give an expansion formula of the spin connection in the flat background. Since the kinetic term of fermions is conformally invariant, it is presented excluding the conformal-factor field dependence. The vielbein with the bar is expanded with the traceless tensor field as

$$\bar{e}_{\mu a} = (e^{\frac{1}{2}h})_{\mu a} = \eta_{\mu a} + \frac{1}{2}h_{\mu a} + \frac{1}{8}(h^2)_{\mu a} + \cdots,$$
$$\bar{e}_a^\mu = (e^{-\frac{1}{2}h})_a^\mu = \delta_a^\mu - \frac{1}{2}h_a^\mu + \frac{1}{8}(h^2)_a^\mu + \cdots,$$

where $\bar{e}^a_\mu \bar{e}_{\nu a} = \bar{g}_{\mu\nu}$ and $\bar{e}^\mu_a \bar{e}_{\mu b} = \eta_{ab}$. Since it is expanded employing the flat background, $\eta_{\mu a}$ is a vielbein of the flat background metric, so that all indices that appear on the right-hand sides can be regarded as the local Lorentz indices (or the Einstein indices). By using these expansions, we obtain

$$\begin{aligned}\bar{\omega}_{\mu ab} &= \bar{e}^\nu_a \left(\partial_\mu \bar{e}_{\nu b} - \bar{\Gamma}^\lambda_{\mu\nu} \bar{e}_{\lambda b}\right) \\ &= -\frac{1}{2}\left(\partial_a h_{\mu b} - \partial_b h_{\mu a}\right) - \frac{1}{8}\left(h^\lambda_a \partial_\mu h_{\lambda b} - h^\lambda_b \partial_\mu h_{\lambda a}\right) \\ &\quad -\frac{1}{4}\left(h_{\mu\lambda}\partial_a h^\lambda_b - h_{\mu\lambda}\partial_\beta h^\lambda_a\right) + \frac{1}{4}\left(h^\lambda_a \partial_\lambda h_{\mu b} - h^\lambda_b \partial_\lambda h_{\mu a}\right).\end{aligned}$$

Expansions of Gravitational Actions Around $D = 4$

Expanding the D-dimensional Weyl action around $D = 4$ yields

$$\begin{aligned}\int d^D x \sqrt{-g}\, C^2_{\mu\nu\lambda\sigma} &= \int d^D x \sqrt{-\hat{g}}\, e^{(D-4)\phi}\, \bar{C}^2_{\mu\nu\lambda\sigma} \\ &= \sum_{n=0}^{\infty} \frac{(D-4)^n}{n!} \int d^D x \sqrt{-\hat{g}}\, \phi^n\, \bar{C}^2_{\mu\nu\lambda\sigma}.\end{aligned}$$

The action for the generalized Euler density G_D in D dimensions is expanded as follows. First, the volume integral of G_4 is expanded using (A-3) as

$$\begin{aligned}&\int d^D x \sqrt{-g}\, G_4 \\ &= \int d^D x \sqrt{-\bar{g}}\, e^{(D-4)\phi} \left[\bar{G}_4 + (D-3)\bar{\nabla}_\mu J^\mu + (D-3)(D-4)K\right] \\ &= \sum_{n=0}^{\infty} \frac{(D-4)^n}{n!} \int d^D x \sqrt{-\bar{g}} \left[\phi^n \bar{G}_4 + (D-3)\left(\phi^n \bar{\nabla}_\mu J^\mu + n\phi^{n-1} K\right)\right] \\ &= \sum_{n=0}^{\infty} \frac{(D-4)^n}{n!} \int d^D x \sqrt{-\hat{g}} \Big[\phi^n \bar{G}_4 + 4(D-3)\phi^n \bar{R}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \phi \\ &\quad -2(D-3)\phi^n \bar{R}\bar{\nabla}^2\phi - 2(D-2)(D-3)(D-4)\phi^n \bar{\nabla}^2\phi \bar{\nabla}^\lambda\phi \bar{\nabla}_\lambda \phi \\ &\quad -(D-2)(D-3)^2(D-4)\phi^n (\bar{\nabla}^\lambda \phi \bar{\nabla}_\lambda \phi)^2\Big].\end{aligned}$$

From the H squared term multiplied by $D-4$, where $H = R/(D-1)$, we obtain

$$
\begin{aligned}
&(D-4)\int d^D x\sqrt{-g}\,H^2 \\
&= \sum_{n=0}^{\infty}\frac{(D-4)^n}{n!}\int d^D x\sqrt{-\hat{g}}\,\biggl[\frac{(D-4)}{(D-1)^2}\phi^n\bar{R}^2-\frac{2(D-6)}{D-1}\phi^n\bar{R}\bar{\nabla}^2\phi \\
&\quad +\frac{2(D-2)}{D-1}\phi^n\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi+4\phi^n\bar{\nabla}^4\phi+8(D-4)\phi^n\bar{\nabla}^2\phi\bar{\nabla}^\lambda\phi\bar{\nabla}_\lambda\phi \\
&\quad +(D-2)^2(D-4)\phi^n(\bar{\nabla}^\lambda\phi\bar{\nabla}_\lambda\phi)^2\biggr].
\end{aligned}
$$

Substituting these expansions and $\chi(D)=1/2+3(D-4)/4+\chi_3(D-4)^2+\chi_4(D-4)^3+\cdots$, the G_D action is expanded as follows:[6]

$$
\begin{aligned}
&\int d^D x\sqrt{-g}\,G_D \\
&= \int d^D x\sqrt{-g}\,\left[G_4+(D-4)\chi(D)H^2\right] \\
&= \sum_{n=0}^{\infty}\frac{(D-4)^n}{n!}\int d^D x\sqrt{-\hat{g}}\,\biggl\{\phi^n\bar{G}_4+\frac{D-4}{(D-1)^2}\chi(D)\phi^n\bar{R}^2 \\
&\quad +4(D-3)\phi^n\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi-2\left[D-3+\frac{D-6}{D-1}\chi(D)\right]\phi^n\bar{R}\bar{\nabla}^2\phi \\
&\quad +\frac{2(D-2)}{D-1}\chi(D)\phi^n\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi+4\chi(D)\phi^n\bar{\nabla}^4\phi \\
&\quad +2(D-4)\left[-(D-2)(D-3)+4\chi(D)\right]\phi^n\bar{\nabla}^2\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi \\
&\quad +(D-2)(D-4)\left[-(D-3)^2+(D-2)\chi(D)\right]\phi^n(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2\biggr\} \\
&= \int d^D x\sqrt{-\hat{g}}\,\biggl\{\bar{G}_4+(D-4)\Bigl(2\phi\bar{\Delta}_4\phi+\bar{G}_4\phi-\frac{2}{3}\bar{R}\bar{\nabla}^2\phi+\frac{1}{18}\bar{R}^2\Bigr) \\
&\quad +(D-4)^2\Bigl(\phi^2\bar{\Delta}_4\phi+\frac{1}{2}\bar{G}_4\phi^2+3\phi\bar{\nabla}^4\phi+4\phi\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi
\end{aligned}
$$

[6] In two-dimensional quantum gravity, the action in dimensional regularization is given by the volume integral of the D-dimensional Ricci scalar curvature, which is expanded around $D=2$ as

$$
\int d^D x\sqrt{-g}R=\sum_{n=0}^{\infty}\frac{(D-2)^n}{n!}\int d^D x\sqrt{-\hat{g}}\left[-(D-1)\phi^n\bar{\nabla}^2\phi+\bar{R}\phi^n\right].
$$

The $n=1$ part gives the Liouville action.

$$
\begin{aligned}
&-\frac{14}{9}\phi\bar{R}\bar{\nabla}^2\phi+\frac{10}{9}\phi\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi-\frac{7}{9}\bar{R}\bar{\nabla}^2\phi+\frac{1}{18}\bar{R}^2\phi+\frac{5}{108}\bar{R}^2\Big)\\
&+(D-4)^3\left[\frac{1}{3}\phi^3\bar{\Delta}_4\phi+\frac{1}{6}\bar{G}_4\phi^3+\left(4\chi_3-\frac{1}{2}\right)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2\right.\\
&\quad+(8\chi_3-2)\,\bar{\nabla}^2\phi\bar{\nabla}^\lambda\phi\bar{\nabla}_\lambda\phi+\frac{3}{2}\phi^2\bar{\nabla}^4\phi+2\phi^2\bar{R}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi\\
&\quad-\frac{7}{9}\phi^2\bar{R}\bar{\nabla}^2\phi+\frac{5}{9}\phi^2\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi+\frac{1}{36}\bar{R}^2\phi^2+4\chi_3\phi\bar{\nabla}^4\phi\\
&+\left(\frac{4}{3}\chi_3-\frac{35}{54}\right)\phi\bar{R}\bar{\nabla}^2\phi+\left(\frac{4}{3}\chi_3+\frac{7}{54}\right)\phi\bar{\nabla}^\lambda\bar{R}\bar{\nabla}_\lambda\phi\\
&\quad\left.+\frac{5}{108}\bar{R}^2\phi+\left(-\frac{4}{3}\chi_3+\frac{7}{27}\right)\bar{R}\bar{\nabla}^2\phi+\left(\frac{1}{9}\chi_3-\frac{1}{27}\right)\bar{R}^2\right]\\
&+o((D-4)^4)\Big\}, \qquad \text{(A-10)}
\end{aligned}
$$

where $\sqrt{-g}\Delta_4$ is the differential operator defined by (5-10) in D dimensions which becomes conformally invariant at four dimensions. Note that the coefficients up to $o((D-4)^3)$ do not depend on the value of χ_4.

Appendix B

Addenda to Conformal Field Theory

Fourier Transform of Two-Point Function

In D-dimensional Euclidean space, the two-point correlation function of a scalar field with conformal dimension Δ is given by $\langle O(x)O(0)\rangle = 1/(x^2)^\Delta$ and its Fourier transform is expressed as

$$\frac{1}{(x^2)^\Delta} = \frac{(2\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}-\Delta)}{4^{\Delta-\frac{D}{4}}\Gamma(\Delta)}\int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x}\left(k^2\right)^{\Delta-\frac{D}{2}} . \tag{B-1}$$

where $\Delta < D/2$ is assumed.

Fourier transform of two-point functions in Minkowski space can be found using the expression above. The product $k \cdot x$ in Euclidean space is rewritten as $\mathbf{k}\cdot\mathbf{x} + k^D x^D$, and hereafter $k \cdot x$ represents the product in Minkowski space. When the D-th coordinate is rewritten as $x^D = ix^0 + \epsilon$, the left-hand side of (B-1) becomes a correlation function $\langle 0|O(x)O(0)|0\rangle$ in Minkowski spacetime. Rewriting the right-hand side as well, we obtain

$$\begin{aligned}\frac{1}{[-(x^0-i\epsilon)^2+\mathbf{x}^2]^\Delta} = \frac{(2\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}-\Delta)}{4^{\Delta-\frac{D}{4}}\Gamma(\Delta)}&\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\times\int \frac{dk^D}{2\pi} e^{-k^D(x^0-i\epsilon)}\left\{\mathbf{k}^2+(k^D)^2\right\}^{\Delta-\frac{D}{2}} .\end{aligned} \tag{B-2}$$

Due to the presence of a phase factor $e^{i\epsilon k^D}$, the path of the k^D-integration can be extended to the upper half of the complex plane. There are cuts on the imaginary axis from $k^D = i|\mathbf{k}|$ to $i\infty$ in the upper half and from $k^D = -i|\mathbf{k}|$ to $-i\infty$ in the lower half because there are poles at $k^D = \pm i|\mathbf{k}|$ and Δ is not an integer. Therefore, the path $-\infty < k^D < \infty$ can be deformed to a path that traces the left and right sides of the imaginary axis in the upper half avoiding the cut and the pole (in the case of a free scalar with $\Delta = D/2-1$, pick up only a residue of the pole). Letting $k^D = ik^0$, the k^D-integration can be rewritten as

$$\int_{-\infty}^{\infty} \frac{dk^D}{2\pi} e^{-k^D(x^0-i\epsilon)}\left\{\mathbf{k}^2+(k^D)^2\right\}^{\Delta-\frac{D}{2}}$$

$$= i\int_0^\infty \frac{dk^0}{2\pi} e^{-ik^0x^0-\epsilon k^0}\Big\{\big[\mathbf{k}^2-(k^0-io)^2\big]^{\Delta-\frac{D}{2}} - \big[\mathbf{k}^2-(k^0+io)^2\big]^{\Delta-\frac{D}{2}}\Big\},$$

where a new positive infinitesimal o is introduced to avoid the cuts. Furthermore, using

$$(x+io)^\lambda-(x-io)^\lambda = \begin{cases} 0 & \text{for } x>0 \\ 2i|x|^\lambda \sin\pi\lambda & \text{for } x<0 \end{cases} = 2i(-x)^\lambda\theta(-x)\sin\pi\lambda,$$

the integrand can be written as $[\mathbf{k}^2-(k^0-io)^2]^{\Delta-D/2}-[\mathbf{k}^2-(k^0+io)^2]^{\Delta-D/2} = 2i(-k^2)^{\Delta-D/2}\theta(-k^2)\sin[\pi(\Delta-D/2)]$, where $k^2=\mathbf{k}^2-(k^0)^2$. Therefore, the right-hand side of (B-2) is expressed as

$$-2\sin\left[\pi\left(\Delta-\frac{D}{2}\right)\right]\frac{(2\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}-\Delta)}{4^{\Delta-\frac{D}{4}}\Gamma(\Delta)}\int\frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}}e^{i\mathbf{k}\cdot\mathbf{x}} \times\int_0^\infty\frac{dk^0}{2\pi}e^{-ik^0x^0}(-k^2)^{\Delta-\frac{D}{2}}\theta(-k^2)$$
$$=\frac{(2\pi)^{\frac{D}{2}+1}4^{\frac{D}{4}-\Delta}}{\Gamma(\Delta)\Gamma(\Delta-\frac{D}{2}+1)}\int\frac{d^Dk}{(2\pi)^D}e^{ik\cdot x}\theta(k^0)\theta(-k^2)(-k^2)^{\Delta-\frac{D}{2}},$$

where $\Gamma(\lambda)\Gamma(1-\lambda)=\pi/\sin(\pi\lambda)$ and $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$ are used. From this, we can read the Fourier transform $W(k)$ for scalar fields introduced in Chapter 2.

Derivation of Critical Exponents

Various critical exponents are derived by adding small perturbations to conformal field theory $S_{\rm CFT}$. First, consider a perturbation by energy operator ε, which is a representative of relevant operators whose conformal dimension satisfies $\Delta<D$ in D dimensions. It corresponds to a perturbation by temperature. Letting a dimensionless temperature parameter representing a deviation from the critical point be $t=|T-T_c|/T_c$, the action of such a system is given by

$$S_t = S_{\rm CFT} - ta^{\Delta_\varepsilon-D}\int d^Dx\,\varepsilon(x),$$

where Δ_ε is conformal dimension of the energy operator. The length scale a introduced to compensate for the dimension is a ultraviolet cutoff corresponding to lattice spacings in statistical models.[1]

The correlation length ξ is then given by

$$\xi \sim a t^{-\nu}, \qquad \nu = \frac{1}{D - \Delta_\varepsilon}, \tag{B-3}$$

because $t a^{\Delta_\varepsilon - D} \sim \xi^{\Delta_\varepsilon - D}$ holds from dimensional analysis. The limit $\xi \to \infty$ is equivalent to $t \to 0$ because the exponent ν is positive from the condition $\Delta_\varepsilon < D$.

When the temperature perturbation t is added, one-point function of an operator O is defined by $\langle\langle O \rangle\rangle_t = \int O\, e^{-S_t}$, and is expanded as

$$\langle\langle O \rangle\rangle_t = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle O \left(t a^{\Delta_\varepsilon - D} \int d^D x\, \varepsilon(x) \right)^n \right\rangle,$$

where $\langle O \rangle = \int O\, e^{-S_{\mathrm{CFT}}}$ represents a normal correlation function of CFT.

In order to obtain critical exponents, we investigate behaviors near the critical point where the correlation length ξ is sufficiently large. In this limit, the ultraviolet cutoff a corresponding to lattice spacings does not affect the results, and thus $a = 1$ is taken below unless otherwise required.

Consider a spin operator σ in addition to ε as a relevant operator. To specify the statistical system, we assume that they satisfy the following OPE structure which the Ising model possesses:

$$\sigma \times \sigma \sim I + \varepsilon, \qquad \sigma \times \varepsilon \sim \varepsilon, \qquad \varepsilon \times \varepsilon \sim I.$$

First, examine a critical exponent of specific heat. Writing free energy per unit volume as f, its specific heat is given by $C = -\partial^2 f / \partial t^2 = \partial \langle\langle \varepsilon(0) \rangle\rangle_t / \partial t$. Since one-point function $\langle \varepsilon \rangle$ in CFT disappears, the term that contributes most in the large ξ (small t) limit is given by a two-point function

$$C = \left\langle \varepsilon(0) \int_{|x| \le \xi} d^D x\, \varepsilon(x) \right\rangle = \int_{|x| \le \xi} d^D x \frac{1}{|x|^{2\Delta_\varepsilon}},$$

[1] In lattice models, a parameter giving the dimension is only the lattice spacing, whereas coupling constants corresponding to a temperature variable, and so on, are dimensionless quantities. From this fact, the temperature parameter t is introduced as a dimensionless quantity, and the dimensions is made up for with a. Although this scale is necessary to make the system finite, the value of a itself is not important when finding critical exponents in the large correlation length limit, and it appears only in dimensional analysis.

where $\langle\langle\varepsilon\rangle\rangle_t$ is evaluated at the origin for convenience. Evaluating the integral inside the correlation length such as $|x| \le \xi$, it can be shown that the specific heat is given as a function of ξ as follows:

$$C \sim \xi^{D-2\Delta_\varepsilon} + \text{const.}$$

The constant term is a contribution from the ultraviolet cutoff, which is ignored as being smaller than the first term in the vicinity of the critical point where ξ is large.[2] The critical exponent is defined by $C \sim t^{-\alpha}$, thus rewriting the right-hand side to the behavior of t using $\xi \sim t^{-\nu}$ yields

$$\alpha = \nu(D - 2\Delta_\varepsilon).$$

Next, consider a critical exponent of magnetization given by a one-point function of the spin operator. In CFT, one-point functions $\langle\sigma\rangle$ and $\langle\varepsilon\rangle$ as well as a two-point function $\langle\sigma\varepsilon\rangle$ disappear from the OPE structure. Therefore, when a perturbation by temperature is applied, the behavior of magnetization in the vicinity of the critical point is given by

$$M = \langle\langle\sigma(0)\rangle\rangle_t = \frac{t^2}{2!}\int_{|x|\le\xi} d^D x \int_{|y|\le\xi} d^D y \ \langle\sigma(0)\varepsilon(x)\varepsilon(y)\rangle\,.$$

The integrals can be easily evaluated from dimensional analysis, and the most contributing part at $\xi \to \infty$ $(t \to 0)$ is given by

$$M \sim t^2 \times \xi^{2D-\Delta_\sigma-2\Delta_\varepsilon} \sim t^{\nu\Delta_\sigma},$$

where Δ_σ is conformal dimension of the spin operator. Thus, the critical exponent defined by $M \sim t^\beta$ is given by

$$\beta = \nu\Delta_\sigma.$$

In order to derive a critical exponent of magnetic susceptibility, it is necessary to add perturbations by not only temperature but also an external magnetic field h, whose action is given as follows:

$$S_{t,h} = S_{\mathrm{CFT}} - ta^{\Delta_\varepsilon - D}\int d^D x\, \varepsilon(x) - ha^{\Delta_\sigma - D}\int d^D x\, \sigma(x).$$

The magnetic susceptibility near the critical point is calculated as

$$\chi = \frac{\partial}{\partial h}\langle\langle\sigma(0)\rangle\rangle_{t,h}\Big|_{h=0} = \int_{|x|\le\xi} d^D x\ \langle\sigma(0)\sigma(x)\rangle \sim \xi^{D-2\Delta_\sigma} \sim t^{-\nu(D-2\Delta_\sigma)}.$$

[2] In the case of the $D = 2$ Ising model, the exponent becomes zero, but a logarithmic divergence appears.

Therefore, the critical exponent defined by $\chi \sim t^{-\gamma}$ is determined to be

$$\gamma = \nu(D - 2\Delta_\sigma).$$

Moreover, the h-dependence of magnetization defined by $M \sim h^{1/\delta}$ at the critical temperature $t = 0$ is obtained from the following action that has only the perturbation by the external magnetic field h:

$$S_h = S_{\mathrm{CFT}} - h a^{\Delta_\sigma - D} \int d^D x \, \sigma(x).$$

From this action, the magnetization is given by

$$M = \langle\langle \sigma(0) \rangle\rangle_h = h \int_{|x| \leq \xi} d^D x \, \langle \sigma(0)\sigma(x) \rangle .$$

Note that the correlation length in this system is represented by $\xi^{\Delta_\sigma - D} \sim h a^{\Delta_\sigma - D}$. Evaluating the term that contributes most in $\xi \to \infty$ $(h \to 0)$ from dimensional analysis yields

$$M \sim h \times \xi^{D - 2\Delta_\sigma} \sim h^{\Delta_\sigma/(D-\Delta_\sigma)}.$$

Thus, the critical exponent is expressed as

$$\delta = \frac{D - \Delta_\sigma}{\Delta_\sigma}.$$

Finally, summarize widely known scaling relations. By defining a new exponent η by a relation $2\Delta_\sigma = D - 2 + \eta$ and eliminating Δ_σ and Δ_ε using this relation and (B-3), each critical exponent can be expressed as

$$\begin{aligned}
&\alpha = 2 - \nu D \quad \text{[Josephson's law]}, \\
&\beta = \frac{1}{2}\nu(D - 2 + \eta), \\
&\gamma = \nu(2 - \eta) \quad \text{[Fisher's law]}, \\
&\delta = \frac{D + 2 - \eta}{D - 2 + \eta}.
\end{aligned}$$

In addition, the following relations hold:

$$\begin{aligned}
&\alpha + 2\beta + \gamma = 2 \quad \text{[Rushbrooke' law]}, \\
&\gamma = \beta(\delta - 1) \quad \text{[Widom's law]}.
\end{aligned}$$

For the Ising model in $D = 2$, as a concrete example, $\nu = 1$ and $\eta = 1/4$ are obtained from $\Delta_\varepsilon = 1$ and $\Delta_\sigma = 1/8$, and thus

$$\alpha = 0, \qquad \beta = \frac{1}{8}, \qquad \gamma = \frac{7}{4}, \qquad \delta = 15.$$

Conformal Algebra for Free Scalar on M^4

As a simple exercise, we derive conformal algebra and transformation laws for the conformally coupled massless free scalar field. The canonical momentum is given by $\mathsf{P}_\varphi = \partial_\eta \varphi$ and the canonical commutation relation is set to $[\varphi(\eta, \mathbf{x}), \mathsf{P}_\varphi(\eta, \mathbf{x}')] = i\delta_3(\mathbf{x} - \mathbf{x}')$. Decomposing the scalar field into annihilation and creation operator parts as $\varphi = \varphi_< + \varphi_>$, the annihilation part is expanded as

$$\varphi_<(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}}\, \varphi(\mathbf{k})\, e^{ik_\mu x^\mu},$$

where $\omega = |\mathbf{k}|$, while the creation part is given by $\varphi_> = \varphi_<^\dagger$. The mode operator then satisfies $[\varphi(\mathbf{k}), \varphi^\dagger(\mathbf{k}')] = \delta_3(\mathbf{k} - \mathbf{k}')$. The two-point correlation function is expressed as $\langle 0|\varphi(x)\varphi(0)|0\rangle = [\varphi_<(x), \varphi_>(0)]$, which is given by

$$\begin{aligned}\langle 0|\varphi(x)\varphi(0)|0\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega}\, e^{-i|\mathbf{k}|(\eta - i\epsilon) + i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{4\pi^2} \frac{1}{-(\eta - i\epsilon)^2 + \mathbf{x}^2},\end{aligned}$$

where ϵ is a ultraviolet cutoff, and see (7-17) given in Chapter 7 for the momentum integration.

The energy-momentum tensor is given by substituting $\xi = 1/6$ into (A-8) as

$$\Theta_{\mu\nu} = \frac{2}{3}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{3}\varphi\partial_\mu\partial_\nu\varphi - \frac{1}{6}\eta_{\mu\nu}\partial^\lambda\varphi\partial_\lambda\varphi.$$

The trace disappears when using the equation of motion. Each generator of conformal transformation can be expressed as follows:

$$\begin{aligned}
P_0 = H &= \int d^3\mathbf{x}\, \mathcal{A}, \qquad P_j = \int d^3\mathbf{x}\, \mathcal{B}_j, \\
M_{0j} &= \int d^3\mathbf{x}\, (-\eta\mathcal{B}_j - x_j\mathcal{A}), \qquad M_{ij} = \int d^3\mathbf{x}\, (x_i\mathcal{B}_j - x_j\mathcal{B}_i), \\
D &= \int d^3\mathbf{x}\, \left(\eta\mathcal{A} + x^k\mathcal{B}_k + :\mathsf{P}_\varphi\varphi:\right), \\
K_0 &= \int d^3\mathbf{x} \left[\left(\eta^2 + \mathbf{x}^2\right)\mathcal{A} + 2\eta x^k\mathcal{B}_k + 2\eta :\mathsf{P}_\varphi\varphi: + \frac{1}{2} :\varphi^2:\right], \\
K_j &= \int d^3\mathbf{x} \left[\left(-\eta^2 + \mathbf{x}^2\right)\mathcal{B}_j - 2x_j x^k\mathcal{B}_k - 2\eta x_j\mathcal{A} - 2x_j :\mathsf{P}_\varphi\varphi:\right],
\end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}_j$ are energy and momentum densities, respectively, defined by

$$\mathcal{A} = \frac{1}{2}{:}\mathsf{P}_\varphi^2{:} - \frac{1}{2}{:}\varphi \not{\partial}^2 \varphi{:}, \qquad \mathcal{B}_j = {:}\mathsf{P}_\varphi \partial_j \varphi{:},$$

and $\not{\partial}^2 = \partial^i \partial_i$ is the Laplacian in space.

Since the generators are time-independent operators, conformal algebra can be calculated using equal-time commutation relations. Using two-point functions at the same time

$$\langle 0|\varphi(\mathbf{x})\varphi(\mathbf{x}')|0\rangle = \frac{1}{4\pi^2}\frac{1}{(\mathbf{x}-\mathbf{x}')^2+\epsilon^2},$$
$$\langle 0|\varphi(\mathbf{x})\mathsf{P}_\varphi(\mathbf{x}')|0\rangle = i\frac{1}{2\pi^2}\frac{\epsilon}{[(\mathbf{x}-\mathbf{x}')^2+\epsilon^2]^2},$$
$$\langle 0|\mathsf{P}_\varphi(\mathbf{x})\mathsf{P}_\varphi(\mathbf{x}')|0\rangle = -\frac{1}{2\pi^2}\frac{(\mathbf{x}-\mathbf{x}')^2-3\epsilon^2}{[(\mathbf{x}-\mathbf{x}')^2+\epsilon^2]^3},$$

we can express the equal-time commutation relation between φ and P_φ as

$$\begin{aligned}[\varphi(\eta,\mathbf{x}), \mathsf{P}_\varphi(\eta,\mathbf{x}')] &= \langle 0|\varphi(\eta,\mathbf{x})\mathsf{P}_\varphi(\eta,\mathbf{x}')|0\rangle - \text{h.c.} \\ &= i\frac{1}{\pi^2}\frac{\epsilon}{[(\mathbf{x}-\mathbf{x}')^2+\epsilon^2]^2},\end{aligned}$$

where the right-hand side is a regularized three-dimensional δ-function

$$\delta_3(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}-\epsilon\omega} = \frac{1}{\pi^2}\frac{\epsilon}{(\mathbf{x}^2+\epsilon^2)^2}.$$

It is found that the equal-time commutation relation between φ's disappears because it becomes real. It is as well between P_φ's.

Similarly, the equal-time commutation relations for $\mathcal{A}$ and $\mathcal{B}_j$ are calculated as

$$\begin{aligned}
[\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})] &= \frac{1}{2} i \not{\partial}_x^2 \delta_3(\mathbf{x}-\mathbf{y}) \left({:}\mathsf{P}_\varphi(\mathbf{x})\varphi(\mathbf{y}){:} - {:}\varphi(\mathbf{x})\mathsf{P}_\varphi(\mathbf{y}){:}\right), \\
[\mathcal{B}_j(\mathbf{x}), \mathcal{B}_k(\mathbf{y})] &= i\partial_k^x \delta_3(\mathbf{x}-\mathbf{y})\ {:}\partial_j\varphi(\mathbf{x})\mathsf{P}_\varphi(\mathbf{y}){:} \\
&\quad + i\partial_j^x \delta_3(\mathbf{x}-\mathbf{y})\ {:}\mathsf{P}_\varphi(\mathbf{x})\partial_k\varphi(\mathbf{y}){:}, \\
[\mathcal{A}(\mathbf{x}), \mathcal{B}_j(\mathbf{y})] &= i\partial_j^x \delta_3(\mathbf{x}-\mathbf{y})\ {:}\mathsf{P}_\varphi(\mathbf{x})\mathsf{P}_\varphi(\mathbf{y}){:} \\
&\quad - \frac{1}{2} i \delta_3(\mathbf{x}-\mathbf{y})\ {:}\not{\partial}^2\varphi\partial_j\varphi(\mathbf{y}){:} \\
&\quad - \frac{1}{2} i \not{\partial}_x^2 \delta_3(\mathbf{x}-\mathbf{y})\ {:}\varphi(\mathbf{x})\partial_j\varphi(\mathbf{y}){:} - i\frac{2}{\pi^2} f_j(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

Other necessary equal-time commutation relations are given by

$$
\begin{aligned}
[\mathcal{A}(\mathbf{x}), :\mathsf{P}_\varphi\varphi(\mathbf{y}):] &= -i\delta_3(\mathbf{x}-\mathbf{x})\left(:\mathsf{P}_\varphi^2(\mathbf{y}): + \frac{1}{2} :\varphi\partial\!\!\!/^2\varphi(\mathbf{y}):\right) \\
&\quad -\frac{1}{2}i\partial\!\!\!/_x^2\delta_3(\mathbf{x}-\mathbf{y}) :\varphi(\mathbf{x})\varphi(\mathbf{y}): + i\frac{10}{\pi^2}f(\mathbf{x}-\mathbf{y}), \\
[\mathcal{B}_j(\mathbf{x}), :\mathsf{P}_\varphi\varphi(\mathbf{y}):] &= -i\delta_3(\mathbf{x}-\mathbf{x})\mathcal{B}_j(\mathbf{y}) + i\partial_j^x\delta(\mathbf{x}-\mathbf{y}) :\mathsf{P}_\varphi(\mathbf{x})\varphi(\mathbf{y}):,
\end{aligned}
$$

where f_j and f that represent quantum corrections are defined by

$$
f_j(\mathbf{x}) = \frac{1}{\pi^2}\frac{\epsilon x_j(\mathbf{x}^2-\epsilon^2)}{(\mathbf{x}^2+\epsilon^2)^6}, \qquad f(\mathbf{x}) = -\frac{1}{40\pi^2}\frac{\epsilon(5\mathbf{x}^2-3\epsilon^2)}{(\mathbf{x}^2+\epsilon^2)^5},
$$

which satisfy $f_j(\mathbf{x}) = \partial_j f(\mathbf{x})$. Spatial integrals for these functions satisfy

$$
\int d^3\mathbf{x}\, f_j(\mathbf{x}) = 0, \quad \int d^3\mathbf{x}\, f(\mathbf{x}) = 0, \quad \int d^3\mathbf{x}\, x^j f(\mathbf{x}) = 0,
$$

while keeping ϵ finite, whereas the integral $\int d^3\mathbf{x}\,\mathbf{x}^2 f(\mathbf{x}) = -1/160\epsilon^2$ diverges at $\epsilon \to 0$.[3]

Calculating the conformal algebra (2-6) using the equal-time commutation relations above, we can show that all the quantum corrections that diverge at $\epsilon \to 0$ cancel out and the algebra closes quantum mechanically.

Next, examine transformation laws of a composite field $:\varphi^n:$. Equal-time commutation relations between this operator and the variables that appear in the generators are calculated as

$$
\begin{aligned}
[\mathcal{A}(\mathbf{x}), :\varphi^n(\mathbf{y}):] &= -i\delta_3(\mathbf{x}-\mathbf{y})\,\partial_\eta :\varphi^n(\mathbf{y}):, \\
[\mathcal{B}_j(\mathbf{x}), :\varphi^n(\mathbf{y}):] &= -i\delta_3(\mathbf{x}-\mathbf{y})\,\partial_j :\varphi^n(\mathbf{y}): \\
&\quad + i\frac{1}{2\pi^2}n(n-1)g_j(\mathbf{x}-\mathbf{y}) :\varphi^{n-2}(\mathbf{y}):, \\
[:\mathsf{P}_\varphi\varphi(\mathbf{x}):, :\varphi^n(\mathbf{y}):] &= -in\,\delta_3(\mathbf{x}-\mathbf{y}) :\varphi^n(\mathbf{y}): \\
&\quad + i\frac{3}{2\pi^2}n(n-1)g(\mathbf{x}-\mathbf{y}) :\varphi^{n-2}(\mathbf{y}):,
\end{aligned}
$$

where quantum correction functions are defined by

$$
g_j(\mathbf{x}) = \frac{1}{\pi^2}\frac{\epsilon x_j}{(\mathbf{x}^2+\epsilon^2)^4}, \qquad g(\mathbf{x}) = -\frac{1}{6\pi^2}\frac{\epsilon}{(\mathbf{x}^2+\epsilon^2)^3},
$$

[3] The function f can be expressed as $f(\mathbf{x}) = (-1/320) \times \partial\!\!\!/^2\left(\delta_3(\mathbf{x})/\mathbf{x}^2\right)$, where another expression $\pi^2\delta_3(\mathbf{x}) = 4\epsilon^3/(\mathbf{x}^2+\epsilon^2)^3$ of the δ-function is used.

which satisfy $g_j(\mathbf{x}) = \partial_j g(\mathbf{x})$.

From these commutation relations, it can be shown that all the quantum corrections disappear and the composite operator transforms as

$$\begin{aligned} i\,[P_\mu, :\varphi^n(x):] &= \partial_\mu :\varphi^n(x):, \\ i\,[M_{\mu\nu}, :\varphi^n(x):] &= (x_\mu \partial_\nu - x_\nu \partial_\mu) :\varphi^n(x):, \\ i\,[D, :\varphi^n(x):] &= (x^\mu \partial_\mu + n) :\varphi^n(x):, \\ i\,[K_\mu, :\varphi^n(x):] &= \left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu n\right) :\varphi^n(x): . \end{aligned}$$

In this way, we can see that $:\varphi^n:$ is a primary scalar field with conformal dimension n.

Mapping to $\mathbb{R} \times S^3$ Space

First, consider a mapping from $\mathbb{R}^4$ to Euclidean $\mathbb{R} \times S^3$. Define the radial component r of the coordinates x_μ in $\mathbb{R}^4$ with $x_\mu x_\mu = r^2$, while the coordinates of a unit S^3 are expressed as $X_\mu = x_\mu / r$ satisfying $X_\mu X_\mu = 1$. The $\mathbb{R}^4$ metric $ds^2_{\mathbb{R}^4} = dx_\mu dx_\mu$ can then be written as

$$\begin{aligned} ds^2_{\mathbb{R}^4} &= dr^2 + r^2 dX_\mu dX_\mu \\ &= e^{2\tau}\left(d\tau^2 + dX_\mu dX_\mu\right) = e^{2\tau} ds^2_{\mathbb{R}\times S^3}, \end{aligned}$$

where $r = e^\tau$. Thus, $\mathbb{R}^4$ and Euclidean $\mathbb{R} \times S^3$ with the metric $ds^2_{\mathbb{R}\times S^3}$ are linked by the coordinate transformation.

Let us consider a mapping of a primary scalar field from $\mathbb{R}^4$ to $\mathbb{R} \times S^3$. Scalar fields do not change in the transformation from the coordinates x_μ to the coordinates (r, X_μ), while the mapping $r = e^\tau$ from (r, X_μ) to (τ, X_μ) is a conformal transformation. Thus, denoting a scalar field in $\mathbb{R}^4$ as $\mathcal{O}(x)$, it is expressed as

$$\mathcal{O}(x) = e^{-\Delta\tau} O(\tau, X),$$

where Δ is conformal dimension of the scalar field. The transformation law of translation in Euclidean $\mathbb{R} \times S^3$ then becomes

$$\begin{aligned} & i\,[P_\mu, O(\tau, X)] = e^{\Delta\tau} i\,[P_\mu, \mathcal{O}(x)] = e^{\Delta\tau} \partial_\mu \mathcal{O}(x) \\ & = e^{\Delta\tau}\left(\frac{\partial\tau}{\partial x_\mu}\frac{\partial}{\partial\tau} + \frac{\partial X_\nu}{\partial x_\mu}\frac{\partial}{\partial X_\nu}\right) e^{-\Delta\tau} O(\tau, X) \\ & = e^{-\tau}\left\{X_\mu \partial_\tau + (\delta_{\mu\nu} - X_\mu X_\nu)\frac{\partial}{\partial X_\nu} - \Delta X_\mu\right\} O(\tau, X). \end{aligned} \quad \text{(B-4)}$$

Similarly, special conformal transformation yields

$$i\,[K_\mu, O(\tau, X)] = e^{\Delta\tau}\left(x^2\partial_\mu - 2x_\mu x_\nu\partial_\nu - 2\Delta x_\mu\right)\mathcal{O}(x)$$
$$= e^\tau\left\{-X_\mu\partial_\tau + (\delta_{\mu\nu} - X_\mu X_\nu)\frac{\partial}{\partial X_\nu} - \Delta X_\mu\right\}O(\tau, X). \quad \text{(B-5)}$$

Dilatation and Lorentz transformation are given by

$$i\,[D, O(\tau, X)] = \partial_\tau O(\tau, X),$$
$$i\,[M_{\mu\nu}, O(\tau, X)] = \left(X_\mu\frac{\partial}{\partial X_\nu} - X_\nu\frac{\partial}{\partial X_\mu}\right)O(\tau, X). \quad \text{(B-6)}$$

In Euclidean $\mathbb{R}\times S^3$, the dilatation represents evolution of the radial direction $r = e^\tau$. Therefore, the field operator at any τ can be expressed as $O(\tau, X) = e^{i\tau D}O(0, X)\,e^{-i\tau D}$. If the field at $\tau = 0$ is identified with a Minkowski field that satisfies Hermiticity $O^\dagger(0, X) = O(0, X)$, Hermiticity in Euclidean $\mathbb{R}\times S^3$ is expressed as $O^\dagger(\tau, X) = e^{-i\tau D}O(0, X)\,e^{i\tau D} = O(-\tau, X)$ from $D^\dagger = -D$.

Further rewrite these conformal transformations using harmonic functions on S^3. Introduce the Euler angles $\hat{\mathbf{x}} = (\alpha, \beta, \gamma)$ and let their ranges be $[0, 2\pi]$, $[0, \pi]$, and $[0, 4\pi]$, respectively. The line element of the unit S^3 is expressed as

$$dX_\mu dX_\mu = \hat{\gamma}_{ij}d\hat{x}^i d\hat{x}^j = \frac{1}{4}\left(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta\, d\alpha d\gamma\right).$$

The coordinate X_μ is then expressed using the Euler angles as

$$X_0 = \cos\frac{\beta}{2}\cos\frac{1}{2}(\alpha+\gamma), \qquad X_1 = \sin\frac{\beta}{2}\sin\frac{1}{2}(\alpha-\gamma),$$
$$X_2 = -\sin\frac{\beta}{2}\cos\frac{1}{2}(\alpha-\gamma), \qquad X_3 = -\cos\frac{\beta}{2}\sin\frac{1}{2}(\alpha+\gamma),$$

and the (induced) metric $\hat{\gamma}_{ij}$ on S^3 can be written using X_μ as

$$\hat{\gamma}_{ij} = \frac{\partial X_\mu}{\partial\hat{x}^i}\frac{\partial X_\nu}{\partial\hat{x}^j}\delta_{\mu\nu}. \quad \text{(B-7)}$$

In particular, we need a scalar spherical harmonics of $J = 1/2$ to describe the generators. The $J = 1/2$ component of the Wigner D-function can be expressed using the coordinates X_μ as

$$D^{\frac{1}{2}}_{mm'} = \begin{pmatrix} X_0 + iX_3 & X_2 + iX_1 \\ -X_2 + iX_1 & X_0 - iX_3 \end{pmatrix} = \sqrt{2}\,(T_\mu)_M X_\mu,$$

where $M = (m, m')$ and T_μ is defined by this expression. Since $J = 1/2$, M denotes a **4**-vector index here. Introducing a sign factor $\epsilon_M = (-1)^{m-m'}$, complex conjugate of T_μ is expressed as $(T^*_\mu)_M = \epsilon_M (T_\mu)_{-M}$,[4] then it is found that

$$(T^*_\mu)_M (T_\mu)_N = \delta_{MN}, \qquad \sum_M (T^*_\mu)_M (T_\nu)_M = \delta_{\mu\nu},$$

hold, where $\delta_{MN} = \delta_{mn}\delta_{m'n'}$. Using this expression, the $J = 1/2$ scalar spherical harmonics $Y_{1/2M}$ can be expressed as

$$\frac{\sqrt{V_3}}{2} Y_{\frac{1}{2}M} = (T_\mu)_M X_\mu.$$

This scalar harmonics satisfies the following product formulas:

$$\frac{V_3}{4}\sum_M Y^*_{\frac{1}{2}M} Y_{\frac{1}{2}M} = 1, \qquad \frac{V_3}{4}\sum_M \hat{\nabla}_i Y^*_{\frac{1}{2}M} \hat{\nabla}_j Y_{\frac{1}{2}M} = \hat{\gamma}_{ij},$$
$$\frac{V_3}{4}\hat{\nabla}_i Y^*_{\frac{1}{2}M} \hat{\nabla}_j Y_{\frac{1}{2}N} = \delta_{MN} - \frac{V_3}{4} Y^*_{\frac{1}{2}M} Y_{\frac{1}{2}N}. \tag{B-8}$$

From the first equation, it also satisfies $\sum_M Y^*_{1/2M} \hat{\nabla}_j Y_{1/2M} = 0$. These four equations correspond to $X_\mu X_\mu = 1$, the induced metric (B-7),

$$\hat{\gamma}^{ij} \frac{\partial X_\mu}{\partial \hat{x}^i} \frac{\partial X_\nu}{\partial \hat{x}^j} = \delta_{\mu\nu} - X_\mu X_\nu, \tag{B-9}$$

and $X_\mu dX_\mu = 0$, respectively. Actually, letting T^*_μ and T_ν act on both sides of (B-9), we can derive the third equation in (B-8). Furthermore, using

$$\hat{\gamma}^{ij} \frac{\partial X_\mu}{\partial \hat{x}^j} = (\delta_{\mu\nu} - X_\mu X_\nu) \frac{\partial \hat{x}^i}{\partial X_\nu}, \tag{B-10}$$

which is a modified version of (B-9), we get

$$\frac{\sqrt{V_3}}{2} \hat{\gamma}^{ij} \frac{\partial}{\partial \hat{x}^j} Y_{\frac{1}{2}M} = (T_\mu)_M (\delta_{\mu\nu} - X_\mu X_\nu) \frac{\partial \hat{x}^i}{\partial X_\nu}.$$

Let us rewrite conformal algebra using these tools. By applying $(T_\mu)_M$, we write the generators of the conformal transformations as follows:

$$H = iD, \qquad R_{MN} = i(T^*_\mu)_M (T_\nu)_N M_{\mu\nu},$$
$$Q_M = -i(T^*_\mu)_M K_\mu, \qquad Q^\dagger_M = i(T_\mu)_M P_\mu,$$

[4] The functions T_μ can be represented as $(T_0)_M = (I)_M/\sqrt{2}$ and $(T_j)_M = i(\sigma_j)_M/\sqrt{2}$ using the identity matrix I and the Pauli matrix σ_i. Note here that T^*_μ is not Hermitian conjugate.

where H becomes a Hermitian operator. The generators Q_M and $Q_M^\dagger$ are Hermitian conjugate with each other, and the rotation generator of S^3 satisfies the relations $R_{MN}^\dagger = R_{NM}$ and $R_{MN} = -\epsilon_M\epsilon_N R_{-N-M}$. Conformal algebra can be then written as

$$\begin{aligned}
\left[Q_M, Q_N^\dagger\right] &= 2\delta_{MN}H + 2R_{MN}, \\
\left[H, Q_M\right] &= -Q_M, \qquad \left[H, Q_M^\dagger\right] = Q_M^\dagger, \qquad \left[H, R_{MN}\right] = 0, \\
\left[Q_M, Q_N\right] &= 0, \qquad \left[Q_M, R_{NL}\right] = \delta_{ML}Q_N - \epsilon_N\epsilon_L\delta_{M-N}Q_{-L}, \\
\left[R_{MN}, R_{LK}\right] &= \delta_{MK}R_{LN} - \epsilon_M\epsilon_N\delta_{-NK}R_{L-M} \\
&\quad -\delta_{NL}R_{MK} + \epsilon_M\epsilon_N\delta_{-ML}R_{-NK}.
\end{aligned} \tag{B-11}$$

Since the rotation generator R_{MN} commutes with H, it is an operator whose conformal dimension is zero. The generator of special conformal transformation Q_M has conformal dimension -1, while its Hermitian conjugate $Q_M^\dagger$ which is the translation generator has conformal dimension 1.

Using H and R_{MN}, the transformation laws (B-6) are rewritten as

$$\begin{aligned}
i\left[H, O(\tau, \hat{\mathbf{x}})\right] &= i\partial_\tau O(\tau, \hat{\mathbf{x}}), \\
i\left[R_{MN}, O(\tau, \hat{\mathbf{x}})\right] &= (\rho_{\mathrm{R}}^\mu)_{MN}\hat{\nabla}_\mu O(\tau, \hat{\mathbf{x}}),
\end{aligned}$$

where $\hat{\nabla}_\mu = (\partial_\tau, \hat{\nabla}_j)$ is a covariant derivative for the metric $\hat{g}_{\mu\nu} = (1, \hat{\gamma}_{ij})$. The first equation describes a time-translation corresponding to a Killing vector $\upsilon^\mu = (i, 0, 0, 0)$ on $\mathbb{R} \times S^3$. The vector $(\rho_{\mathrm{R}}^\mu)_{MN} = (0, \rho_{MN}^j)$ in the second equation is a Killing vector on S^3, where

$$\rho_{MN}^j = i\frac{\mathrm{V}_3}{4}\left(Y_{\frac{1}{2}M}^*\hat{\nabla}^j Y_{\frac{1}{2}N} - Y_{\frac{1}{2}N}\hat{\nabla}^j Y_{\frac{1}{2}M}^*\right). \tag{B-12}$$

Since $\rho_{MN}^{j*} = \rho_{NM}^j$ and $\rho_{MN}^j = -\epsilon_M\epsilon_N\rho_{-N-M}^j$ are satisfied for the 4-vector indices M and N, the vector has six components.

The transformation laws for the special conformal transformation and the translation are rewritten as

$$\begin{aligned}
i\left[Q_M, O(\tau, \hat{\mathbf{x}})\right] &= (\rho^\mu)_M\hat{\nabla}_\mu O(\tau, \hat{\mathbf{x}}) + \frac{\Delta}{4}\hat{\nabla}_\mu(\rho^\mu)_M O(\tau, \hat{\mathbf{x}}), \\
i\left[Q_M^\dagger, O(\tau, \hat{\mathbf{x}})\right] &= (\tilde{\rho}^{\mu*})_M\hat{\nabla}_\mu O(\tau, \hat{\mathbf{x}}) + \frac{\Delta}{4}\hat{\nabla}_\mu(\tilde{\rho}^{\mu*})_M O(\tau, \hat{\mathbf{x}})
\end{aligned}$$

from (B-5) and (B-4) of K_μ and P_μ, respectively, where ρ^μ is a conformal Killing vector on Euclidean $\mathbb{R} \times S^3$ defined by

$$(\rho^\mu)_M = \left(\rho_M^0, \rho_M^j\right) = \left(i\frac{\sqrt{\mathrm{V}_3}}{2}e^\tau Y_{\frac{1}{2}M}^*, -i\frac{\sqrt{\mathrm{V}_3}}{2}e^\tau\hat{\nabla}^j Y_{\frac{1}{2}M}^*\right),$$

and $\tilde{\rho}^\mu$ is also a conformal Killing vector that is defined by $\tilde{\rho}^0_M(\tau, \hat{\mathbf{x}}) = -\rho^{0*}_M(-\tau, \hat{\mathbf{x}})$ and $\tilde{\rho}^j_M(\tau, \hat{\mathbf{x}}) = \rho^{j*}_M(-\tau, \hat{\mathbf{x}})$.

Thus, the conformal Killing vectors on Euclidean $\mathbb{R} \times S^3$ for the dilatation H, the S^3 rotation R_{MN}, the special conformal transformation Q_M, and the translation $Q^\dagger_M$ are given by υ^μ, ρ^μ_{R}, ρ^μ, and $\tilde{\rho}^\mu$, respectively.

Conformal Killing vectors on $\mathbb{R} \times S^3$ with Lorentzian signature can be obtained by Wick rotating the above vectors with $\tau = i\eta$. The dilatation is then expressed as $i[H, O] = \partial_\eta O$, that is, H is the Hamiltonian operator on Lorentzian $\mathbb{R} \times S^3$. Moreover, the relationship between states and operators is given by

$$|\Delta\rangle = \lim_{\eta \to i\infty} e^{-i\Delta\eta} O(\eta, \hat{\mathbf{x}})|0\rangle.$$

A primary scalar state, for instance, is defined by the following conditions:

$$H|\Delta\rangle = \Delta|\Delta\rangle, \qquad R_{MN}|\Delta\rangle = 0, \qquad Q_M|\Delta\rangle = 0.$$

Descendant states are generated by applying $Q^\dagger_M$ to the primary state.

Two-Point Correlation Functions on $\mathbb{R} \times S^3$

The scalar harmonics, defined in Chapter 8 and Appendix C, satisfies the following formula:

$$\sum_M Y_{JM}(\hat{\mathbf{x}}) Y_{JM}(\hat{\mathbf{x}}') = \frac{2J+1}{V_3} \chi^J(\omega),$$

where

$$\chi^J(\omega) = \frac{\sin[(2J+1)\frac{\omega}{2}]}{\sin\frac{\omega}{2}}$$

is the character of the rank J irreducible representation of $SU(2)$. The angular variable ω is defined using the Euler angles as

$$\cos\frac{\omega}{2} = \cos\frac{\beta-\beta'}{2}\cos\frac{\alpha-\alpha'}{2}\cos\frac{\gamma-\gamma'}{2} - \cos\frac{\beta+\beta'}{2}\sin\frac{\alpha-\alpha'}{2}\sin\frac{\gamma-\gamma'}{2}.$$

Scalar fields Using this formula, the two-point correlation function of the scalar field φ defined in Chapter 8 can be written as

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \frac{1}{2V_3} \sum_{J \geq 0} e^{-i(2J+1)(\eta-\eta')} \chi^J(\omega).$$

To make the series converge, we regularize it by introducing an infinitesimal cutoff as $\eta - \eta' \to \eta - \eta' - i\epsilon$. Using a formula for the character

$$\sum_{J \geq 0} t^{2J} \chi^J(\omega) = \frac{1}{1 - 2t \cos\frac{\omega}{2} + t^2},$$

we obtain

$$\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \frac{1}{2V_3} \frac{1}{L^2(\eta - \eta', \omega)},$$

where ϵ is omitted and L^2 is a function on $\mathbb{R} \times S^3$ defined by

$$L^2(\eta - \eta', \omega) = 2 \left[\cos(\eta - \eta') - \cos\frac{\omega}{2} \right].$$

At short distance, it becomes $L^2 \approx -(\eta - \eta')^2 + (\alpha - \alpha')^2/4 + (\beta - \beta')^2/4 + (\gamma - \gamma')^2/4 + (\alpha - \alpha')(\gamma - \gamma')/2$.

Conformal-factor field In the same way, consider the two-point function of the conformal-factor field. Since the conformal-factor field has the zero-mode, we calculate the following operator product with attention to it:

$$\phi(x)\phi(x') = \frac{1}{2} [\phi_0(\eta), \phi_0(\eta')] + [\phi_<(x), \phi_>(x')] + :\phi(x)\phi(x'):.$$

The part that diverges at short distances is given by

$$\begin{aligned} [\phi_<(x), \phi_>(x')] &= \frac{1}{4b_c} \frac{\pi^2}{V_3} \sum_{J \geq 0} \frac{4}{2J+1} e^{-i(2J+1)(\eta - \eta')} \cos\left\{ (2J+1)\frac{\omega}{2} \right\} \\ &= -\frac{1}{4b_c} \log\left\{ 1 - 2e^{-i(\eta - \eta')} \cos\frac{\omega}{2} + e^{-2i(\eta - \eta')} \right\}. \end{aligned}$$

Adding the contribution from the zero-mode, we get

$$\phi(x)\phi(x') = -\frac{1}{4b_c} \log L^2(\eta - \eta', \omega) + :\phi(x)\phi(x'):.$$

Using this, we can obtain an operator product of the primary scalar field $V_\alpha =: e^{\alpha\phi}:$ as

$$V_\alpha(x) V_{\alpha'}(x') = \left(\frac{1}{L^2(\eta - \eta', \omega)} \right)^{\frac{\alpha\alpha'}{4b_c}} :V_\alpha(x) V_{\alpha'}(x'):,$$

where the normal ordering is defined by

$$:V_\alpha(x) V_{\alpha'}(x'): = e^{\alpha\phi_0(\eta) + \alpha'\phi_0(\eta')} e^{\alpha\phi_>(x)} e^{\alpha'\phi_>(x')} e^{\alpha\phi_<(x)} e^{\alpha'\phi_<(x')}.$$

Here, for the zero-mode part, we use the Baker-Campbell-Hausdorff formula when $[A, B]$ is a constant such that $e^A e^B = e^{\frac{1}{2}[A,B]} e^{A+B}$.

If one is a dual operator of the other, such as $\alpha' = 4b_c - \alpha$, its expectation value by the conformally invariant vacuum is given by

$$\langle \Omega | V_\alpha(x) V_{4b_c - \alpha}(x') | \Omega \rangle = \left(\frac{1}{L^2(\eta - \eta', \omega)} \right)^{h_\alpha},$$

where we use the facts that V_α has the conformal dimension $h_\alpha = \alpha - \alpha^2/4b_c$ (8-24), and since the conformally invariant vacuum has the background charge $-4b_c$, $\langle \Omega | e^{4b_c \phi(0)} | \Omega \rangle = 1$. When V_α is the physical cosmological operator in quantum gravity, it has $h_\alpha = 4$.

Correction Terms in Gauge-Fixed Conformal Transformations

A relationship between conformal transformations and gauge-fixing conditions is discussed here. We first describe the case of the $U(1)$ gauge field. When the radiation gauge $A_0 = \hat{\nabla}^i A_i = 0$ is adopted, the conformal transformation of the transverse component becomes

$$\delta_\zeta A_i = \zeta^0 \partial_\eta A_i + \zeta^j \hat{\nabla}_j A_i + \frac{1}{3} \hat{\nabla}_j \zeta^j A_i + \frac{1}{2} \left(\hat{\nabla}_i \zeta^j - \hat{\nabla}^j \zeta_i \right) A_j. \quad \text{(B-13)}$$

However, this transformation does not preserve the transverse condition. Moreover, it turns out that the transformation of the time component of the gauge field becomes

$$\delta_\zeta A_0 = \hat{\nabla}^i (\zeta^0 A_i), \quad \text{(B-14)}$$

hence it does not preserve the radiation gauge condition. In fact, in terms of the generator Q_ζ, the transformation law is written in the form with an extra term to preserve the gauge-fixing condition as follows:

$$\delta_\zeta A_\mu = i \left[Q_\zeta, A_\mu \right] + \hat{\nabla}_\mu \tilde{\lambda},$$

where $\tilde{\lambda}$ is given as a function depending on the field variable. The last extra term which has the form of a gauge transformation is called the Fradkin-Palchik term.

Let us specifically see the gauge field on $\mathbb{R} \times S^3$ discussed in Chapter 8 as an example. In the conformal transformations (B-13) and (B-14), the transformations that do not preserve the radiation gauge are the special conformal transformation with the conformal Killing vector ζ_{S}^{μ} and the translation with its complex conjugate, whereas it is preserved for the case of the Killing vectors ζ_{T}^{μ} and ζ_{R}^{μ}.

Assigning $(\zeta_{\mathrm{S}}^{\mu})_M$ (8-9), we examine the transformation law of the transverse component. Expanding the right-hand side of (B-13) using the harmonics product expansion (C-7), an extra term appears in addition to the commutation relation between the special conformal transformation generator Q_M and the field operator as follows:

$$(\delta_{\zeta_{\mathrm{S}}} A_i)_M = i[Q_M, A_i] + \hat{\nabla}_i(\tilde{\lambda})_M, \tag{B-15}$$

where a scalar function $\tilde{\lambda}$ is given by

$$\begin{aligned}(\tilde{\lambda})_M = \frac{i}{2} \sum_{J \geq \frac{1}{2}} \frac{1}{\sqrt{2(2J+1)}} \sum_{N,y} \sum_{S} \biggl\{ &-\frac{1}{2J} q_{J(Ny)} e^{-i2J\eta} \mathbf{G}^{\frac{1}{2}M}_{J(Ny);JS} \\ &+ \frac{1}{2J+2} q^{\dagger}_{J(Ny)} e^{i(2J+2)\eta} (-\epsilon_N) \mathbf{G}^{\frac{1}{2}M}_{J(-Ny);JS} \biggr\} Y^{*}_{JS}.\end{aligned}$$

Since the extra term is in the form of a gauge transformation, (B-15) can be written as $(\delta_{\zeta_{\mathrm{S}}} A_i - \delta_{\tilde{\lambda}} A_i)_M = i[Q_M, A_i]$ when $(\delta_{\tilde{\lambda}} A_\mu)_M = \hat{\nabla}_\mu(\tilde{\lambda})_M$ is considered as a gauge transformation following the special conformal transformation. Furthermore, it can be seen that the transformation of the time component is calculated as

$$(\delta_{\zeta_{\mathrm{S}}} A_0 - \delta_{\tilde{\lambda}} A_0)_M = (\hat{\nabla}^i(\zeta_{\mathrm{S}}^0 A_i) - \partial_\eta \tilde{\lambda})_M = 0.$$

Thus, the transformation generated by Q_ζ, which forms the closed conformal algebra, can be expressed by $\delta_\zeta^{\mathrm{T}} = \delta_\zeta - \delta_{\tilde{\lambda}}$ as a combination of the normal conformal transformation δ_ζ and the associated mode-dependent gauge transformation $\delta_{\tilde{\lambda}}$, where $(\tilde{\lambda})_M$ and its Hermitian conjugate are assigned for the special conformal transformation Q_M and the translation $Q_M^{\dagger}$, respectively, while for others it should be zero. This conformal transformation which preserves the radiation gauge conditions can be summarized as follows:

$$\delta_\zeta^{\mathrm{T}} A_i = i[Q_\zeta, A_i], \qquad \delta_\zeta^{\mathrm{T}} A_0 = 0.$$

The same holds true for the conformal transformation of the traceless tensor field. The conformal transformation in the radiation$^+$ gauge adopted in Chapter 8 can be written as

$$\begin{aligned}\delta_\zeta \mathsf{h}_{ij} &= \zeta^0 \partial_\eta \mathsf{h}_{ij} + \zeta^k \hat\nabla_k \mathsf{h}_{ij} + \frac{1}{2}\left(\hat\nabla_i \zeta^k - \hat\nabla^k \zeta_i\right) \mathsf{h}_{kj} \\ &\quad + \frac{1}{2}\left(\hat\nabla_j \zeta^k - \hat\nabla^k \zeta_j\right) \mathsf{h}_{ki} + \mathsf{h}_i \hat\nabla_j \zeta^0 + \mathsf{h}_j \hat\nabla_i \zeta^0 - \frac{2}{3}\hat\gamma_{ij} \hat\nabla_k \left(\zeta^0 \mathsf{h}^k\right), \\ \delta_\zeta \mathsf{h}_i &= \zeta^0 \partial_\eta \mathsf{h}_i + \zeta^k \hat\nabla_k \mathsf{h}_i + \frac{1}{2}\left(\hat\nabla_i \zeta^k - \hat\nabla^k \zeta_i\right) \mathsf{h}_k + \hat\nabla^k \left(\zeta^0 \mathsf{h}_{ik}\right), \\ \delta_\zeta h_{00} &= 2\hat\nabla^k \left(\zeta^0 \mathsf{h}_k\right).\end{aligned}$$

The transformations do not preserve the radiation$^+$ gauge obviously. Therefore, as in the case of the gauge field, consider a transformation $\delta^{\mathrm{T}}_\zeta = \delta_\zeta - \delta_{\tilde\kappa}$ by introducing a gauge transformation of the form (7-8) with a mode-dependent parameter $\tilde\kappa^\mu$, then the conformal transformation that preserves the radiation$^+$ gauge can be expressed by the commutation relation with the generator as

$$\delta^{\mathrm{T}}_\zeta \mathsf{h}_{ij} = i[Q_\zeta, \mathsf{h}_{ij}], \qquad \delta^{\mathrm{T}}_\zeta \mathsf{h}_i = i[Q_\zeta, \mathsf{h}_i], \qquad \delta^{\mathrm{T}}_\zeta h_{00} = 0.$$

In the cases of the special conformal transformation and the translation, $\tilde\kappa^\mu$ has a value. Its expression is a little complicated, and therefore not shown here, though we can get it as in the case of the gauge field.

Building Blocks for Vector and Tensor Fields

Building blocks of primary states for the gauge field sector and the traceless tensor field sector, which will give a basis of Q_M-invariant creation operators, are summarized here.[5]

The commutation relation between a creation mode $q^\dagger_{J(My)}$ in the gauge field and the special conformal transformation generator Q_M is given by

$$\begin{aligned}&\left[Q_M, q^\dagger_{J(M_1 y_1)}\right] \\ &= -\sqrt{2J(2J+1)} \sum_{M_2, y_2} \epsilon_{M_2} \mathbf{D}^{\frac{1}{2}M}_{J(M_1 y_1), J-\frac{1}{2}(-M_2 y_2)} q^\dagger_{J-\frac{1}{2}(M_2 y_2)}.\end{aligned}$$

From this, we find that only the lowest $J = 1/2$ creation mode $q^\dagger_{1/2(My)}$ commutes with Q_M. Since other creation modes do not commute alone,

[5] See K. Hamada, Int. J. Mod. Phys. A **20** (2005) 5353 in Bibliography.

consider a quadratic form of them, then we obtain the following Q_M-invariant creation operators:

$$\Psi^\dagger_{LN} = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{\substack{M_1,y_1;\\ M_2,y_2}} f(L,K)\, \mathbf{D}^{LN}_{L-K(M_1y_1),K(M_2y_2)} q^\dagger_{L-K(M_1y_1)} q^\dagger_{K(M_2y_2)},$$

$$\Upsilon^\dagger_{L(Nx)} = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{\substack{M_1,y_1;\\ M_2,y_2}} f(L,K)\, \mathbf{F}^{L(Nx)}_{L-K(M_1y_1),K(M_2y_2)} q^\dagger_{L-K(M_1y_1)} q^\dagger_{K(M_2y_2)},$$

where $f(L,K)$ is given by the same as (8-29) for the scalar field, and L is a positive integer, whereas these operators do not exist in the case of half-integers. A new $SU(2)\times SU(2)$ Clebsch-Gordan coefficient is defined by

$$\mathbf{F}^{J(Mx)}_{J_1(M_1y_1),J_2(M_2y_2)} = \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^{ij*}_{J(Mx)} Y_{iJ_1(M_1y_1)} Y_{jJ_2(M_2y_2)}.$$

Since $L=1$ is expressed using $q^\dagger_{1/2(My)}$ only, $L\geq 2$ gives new invariant operators. Thus, the building blocks are summarized in Table B-4.

rank of tensor	0	1	2
creation operators	$\Psi^\dagger_{LN}$	$q^\dagger_{\frac{1}{2}(Ny)}$	$\Upsilon^\dagger_{L(Nx)}$
conformal dim. ($L\in\mathbb{Z}_{\geq 2}$)	$2L+2$	2	$2L+2$

Table B-4: Building blocks of primary states for the gauge field.

The lowest primary state is given by $q^\dagger_{1/2(Ny)}|0\rangle$ of conformal dimension 2 with 6 independent components. It corresponds to the field strength $F_{\mu\nu}$, and the polarization $y=\pm 1/2$ represents the selfdual and anti-selfdual components.

The results for the traceless tensor field shown in Table 8-3 are summarized below. The commutation relation between Q_M and each creation mode of $c^\dagger_{J(Mx)}$, $d^\dagger_{J(Mx)}$, and $e^\dagger_{J(My)}$ is given as follows:

$$\left[Q_M, c^\dagger_{J(M_1x_1)}\right] = \alpha\left(J-\frac{1}{2}\right)\sum_{M_2,x_2}\epsilon_{M_2}\mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J-\frac{1}{2}(-M_2x_2)} c^\dagger_{J-\frac{1}{2}(M_2x_2)},$$

$$\left[Q_M, d^\dagger_{J(M_1x_1)}\right] = -\gamma(J)\sum_{M_2,x_2}\epsilon_{M_2}\mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J+\frac{1}{2}(-M_2x_2)} c^\dagger_{J+\frac{1}{2}(M_2x_2)}$$

$$
\begin{aligned}
&\qquad -\beta\left(J-\frac{1}{2}\right)\sum_{M_2,x_2}\epsilon_{M_2}\mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J-\frac{1}{2}(-M_2x_2)}d^\dagger_{J-\frac{1}{2}(M_2x_2)} \\
&\qquad -B(J)\sum_{M_2,y_2}\epsilon_{M_2}\mathbf{H}^{\frac{1}{2}M}_{J(M_1x_1);J(-M_2y_2)}e^\dagger_{J(M_2y_2)}, \\
\left[Q_M, e^\dagger_{J(M_1y_1)}\right] &= -A(J)\sum_{M_2,x_2}\epsilon_{M_2}\mathbf{H}^{\frac{1}{2}M}_{J(-M_2x_2);J(M_1y_1)}c^\dagger_{J(M_2x_2)} \\
&\qquad -C\left(J-\frac{1}{2}\right)\sum_{M_2,y_2}\epsilon_{M_2}\mathbf{D}^{\frac{1}{2}M}_{J(M_1y_1),J-\frac{1}{2}(-M_2y_2)}e^\dagger_{J-\frac{1}{2}(M_2y_2)}.
\end{aligned}
$$

Therefore, only the lowest positive-metric creation mode $c^\dagger_{1(Mx)}$ commutes with Q_M. Moreover, as shown in Table 8-3, this is the only one with an index of second-rank tensor harmonics.

Building blocks given in a quadratic form with a scalar harmonics index are given by the following two with L as a positive integer:

$$
A^\dagger_{LN} = \sum_{K=1}^{L-1}\sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\,\mathbf{E}^{LN}_{L-K(M_1x_1),K(M_2,x_2)}c^\dagger_{L-K(M_1x_1)}c^\dagger_{K(M_2x_2)},
$$

$$
\begin{aligned}
\mathcal{A}^\dagger_{L-1N} &= \sum_{K=1}^{L-1}\sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\,\mathbf{E}^{L-1N}_{L-K(M_1x_1),K(M_2,x_2)}c^\dagger_{L-K(M_1x_1)}c^\dagger_{K(M_2x_2)} \\
&+\sum_{K=1}^{L-2}\sum_{\substack{M_1,x_1;\\ M_2,x_2}} y(L,K)\,\mathbf{E}^{L-1N}_{L-K-1(M_1x_1),K(M_2,x_2)}d^\dagger_{L-K-1(M_1x_1)}c^\dagger_{K(M_2x_2)} \\
&+\sum_{K=1}^{L-\frac{3}{2}}\sum_{\substack{M_1,x_1;\\ M_2,y_2}} w(L,K)\,\mathbf{H}^{L-1N}_{L-K-\frac{1}{2}(M_1x_1);K(M_2,y_2)}c^\dagger_{L-K-\frac{1}{2}(M_1x_1)}e^\dagger_{K(M_2y_2)} \\
&+\sum_{K=1}^{L-2}\sum_{\substack{M_1,y_1;\\ M_2,y_2}} v(L,K)\,\mathbf{D}^{L-1N}_{L-K-1(M_1y_1),K(M_2y_2)}e^\dagger_{L-K-1(M_1y_1)}e^\dagger_{K(M_2y_2)},
\end{aligned}
$$

where $x(L,K)$ and $y(L,K)$ are the same as (8-30) for the conformal-factor field, and other new coefficients are given by

$$
w(L,K) = 2\sqrt{2}\sqrt{\frac{(2L-2K-1)(2L-2K+1)}{2K(2K-1)(2K+3)}}\,x(L,K),
$$

$$v(L,K) = -\sqrt{\frac{(2K-1)(2K+2)(2L-2K-3)(2L-2K)}{(2K+3)(2L-2K+1)}}\, x\left(L, K+\frac{1}{2}\right).$$

These operators do not exist if L is a half integer. Also, it is obvious that $L = 1$ does not exist. Although $L = 2$ exists, it becomes Q_M-invariant obviously because it is represented by $c^\dagger_{1(Mx)}$ only. Therefore, $L \geq 3$ gives new Q_M-invariant creation operators.

Furthermore, there are building blocks with indices of first, third, and fourth rank tensor harmonics. To describe them, introduce new $SU(2) \times SU(2)$ Clebsch-Gordan coefficients of the types ${}^n\mathbf{E}$ and ${}^n\mathbf{H}$, which include at least one tensor harmonics of the rank n. The coefficients of $n = 2$ are defined by

$$
\begin{aligned}
{}^2\mathbf{E}^{J(Mx)}_{J_1(M_1x_1),J_2(M_2x_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^{ij*}_{J(Mx)} Y_i{}^k{}_{J_1(M_1x_1)} Y_{jkJ_2(M_2x_2)}, \\
{}^2\mathbf{H}^{J(Mx)}_{J_1(M_1x_1);J_2(M_2y_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^{ij*}_{J(Mx)} Y_i{}^k{}_{J_1(M_1x_1)} \hat{\nabla}_{(j} Y_{k)J_2(M_2y_2)},
\end{aligned}
$$

while the $n = 1$ coefficients ${}^1\mathbf{E}^{J(My)}_{J_1(M_1x_1),J_2(M_2x_2)}$ and ${}^1\mathbf{H}^{J(My)}_{J_1(M_1x_1);J_2(M_2y_2)}$ are defined by replacing the first $Y^{ij}_{J(Mx)}$ with $\hat{\nabla}^{(i}Y^{j)}_{J(My)}$ in the above expressions. The $n = 4$ coefficients are

$$
\begin{aligned}
{}^4\mathbf{E}^{J(Mw)}_{J_1(M_1x_1),J_2(M_2x_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^{ijkl*}_{J(Mw)} Y_{ijJ_1(M_1x_1)} Y_{klJ_2(M_2x_2)}, \\
{}^4\mathbf{H}^{J(Mw)}_{J_1(M_1x_1);J_2(M_2y_2)} &= \sqrt{\mathrm{V}_3}\int_{S^3} d\Omega_3\, Y^{ijkl*}_{J(Mw)} Y_{ijJ_1(M_1x_1)} \hat{\nabla}_{(k} Y_{l)J_2(M_2y_2)},
\end{aligned}
$$

while the $n = 3$ coefficients ${}^3\mathbf{E}^{J(Mz)}_{J_1(M_1x_1),J_2(M_2x_2)}$ and ${}^3\mathbf{H}^{J(Mz)}_{J_1(M_1x_1);J_2(M_2y_2)}$ are defined by replacing the first $Y^{ijkl}_{J(Mw)}$ with $\hat{\nabla}^{(i}Y^{jkl)}_{J(Mz)}$.

Building blocks with a vector harmonics index are given by the following with an integer $L\,(\geq 3)$:

$$
\begin{aligned}
B^\dagger_{L-\frac{1}{2}(Ny)} =& \sum_{K=1}^{L-1} \sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\, {}^1\mathbf{E}^{L-\frac{1}{2}(Ny)}_{L-K(M_1x_1),K(M_2x_2)} c^\dagger_{L-K(M_1x_1)} c^\dagger_{K(M_2x_2)} \\
&+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{\substack{M_1,x_1;\\ M_2,y_2}} w(L,K)\, {}^1\mathbf{H}^{L-\frac{1}{2}(Ny)}_{L-K-\frac{1}{2}(M_1x_1);K(M_2y_2)} c^\dagger_{L-K-\frac{1}{2}(M_1x_1)} e^\dagger_{K(M_2y_2)}.
\end{aligned}
$$

Building blocks with an index of third-rank tensor harmonics are given by the following with an integer $L\,(\geq 3)$:

$$D^{\dagger}_{L-\frac{1}{2}(Nz)} = \sum_{K=1}^{L-1} \sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\,{}^{3}\mathbf{E}^{L-\frac{1}{2}(Nz)}_{L-K(M_1x_1),K(M_2x_2)} c^{\dagger}_{L-K(M_1x_1)} c^{\dagger}_{K(M_2x_2)}$$

$$+\sum_{K=1}^{L-\frac{3}{2}} \sum_{\substack{M_1,x_1;\\ M_2,y_2}} w(L,K)\,{}^{3}\mathbf{H}^{L-\frac{1}{2}(Nz)}_{L-K-\frac{1}{2}(M_1x_1);K(M_2y_2)} c^{\dagger}_{L-K-\frac{1}{2}(M_1x_1)} e^{\dagger}_{K(M_2y_2)}.$$

Building blocks with an index of fourth-rank tensor harmonics are given by the following two with an integer $L\,(\geq 3)$:

$$E^{\dagger}_{L(Nw)} = \sum_{K=1}^{L-1} \sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\,{}^{4}\mathbf{E}^{L(Nw)}_{L-K(M_1x_1),K(M_2x_2)} c^{\dagger}_{L-K(M_1x_1)} c^{\dagger}_{K(M_2x_2)},$$

$$\mathcal{E}^{\dagger}_{L-1(Nw)} = \sum_{K=1}^{L-1} \sum_{\substack{M_1,x_1;\\ M_2,x_2}} x(L,K)\,{}^{4}\mathbf{E}^{L-1(Nw)}_{L-K(M_1x_1),K(M_2x_2)} c^{\dagger}_{L-K(M_1x_1)} c^{\dagger}_{K(M_2x_2)}$$

$$+\sum_{K=1}^{L-2} \sum_{\substack{M_1,x_1;\\ M_2,x_2}} y(L,K)\,{}^{4}\mathbf{E}^{L-1(Nw)}_{L-K-1(M_1x_1),K(M_2x_2)} d^{\dagger}_{L-K-1(M_1x_1)} c^{\dagger}_{K(M_2x_2)}$$

$$+\sum_{K=1}^{L-\frac{3}{2}} \sum_{\substack{M_1,x_1;\\ M_2,y_2}} w(L,K)\,{}^{4}\mathbf{H}^{L-1(Nw)}_{L-K-\frac{1}{2}(M_1x_1);K(M_2y_2)} c^{\dagger}_{L-K-\frac{1}{2}(M_1x_1)} e^{\dagger}_{K(M_2y_2)}.$$

Appendix C

Useful Functions on Three-Sphere

Spherical Tensor Harmonics on S^3

Let us define symmetric-transverse-traceless tensor harmonics (ST^2 tensor harmonics) on S^3.[1] First of all, we introduce two coordinate systems representing $\mathbb{R}^4$. One is the Cartesian coordinate system expressed by $x^{\bar{\mu}}$ ($\bar{\mu} = \bar{0}, \bar{1}, \bar{2}, \bar{3}$), and the other is a spherical coordinate system expressed by $\hat{x}^\mu = (\hat{x}^0, \hat{x}^i)$, where $i = 1, 2, 3$ and $\hat{x}^0 = r = (x^{\bar{\mu}} x_{\bar{\mu}})^{1/2}$. In order to distinguish the indices, the bars are attached to the indices of the Cartesian coordinates. The $\mathbb{R}^4$ space is expressed using the metric of each coordinate system as follows:

$$ds^2_{\mathbb{R}^4} = \delta_{\bar{\mu}\bar{\nu}} dx^{\bar{\mu}} dx^{\bar{\nu}} = dr^2 + r^2 \hat{\gamma}_{ij} d\hat{x}^i d\hat{x}^j,$$

where $\hat{\gamma}_{ij}$ is the metric of the unit S^3. If we use the Euler angles to represent the coordinates of S^3 as $\hat{x}^i = (\alpha, \beta, \gamma)$, the relation connecting the two coordinate systems is given by

$$x^{\bar{0}} = r \cos\frac{\beta}{2} \cos\frac{1}{2}(\alpha + \gamma), \qquad x^{\bar{1}} = r \sin\frac{\beta}{2} \sin\frac{1}{2}(\alpha - \gamma),$$
$$x^{\bar{2}} = -r \sin\frac{\beta}{2} \cos\frac{1}{2}(\alpha - \gamma), \qquad x^{\bar{3}} = -r \cos\frac{\beta}{2} \sin\frac{1}{2}(\alpha + \gamma).$$

Spherical ST^2 tensor harmonics Spherical tensor harmonics can be defined using the Clebsch-Gordan coefficient and the Wigner D-function. In general, the D-function can be expressed using a symmetric traceless tensor $\tau_{\bar{\mu}_1 \cdots \bar{\mu}_n}$ as

$$D^J_{mm'} = \frac{1}{r^{2J}} x^{\bar{\mu}_1} \cdots x^{\bar{\mu}_{2J}} (\tau_{\bar{\mu}_1 \cdots \bar{\mu}_{2J}})_{mm'}.$$

Complex conjugate of the symmetric traceless tensor part is defined by $(\tau_{\bar{\mu}_1 \cdots \bar{\mu}_n})^*_{mm'} = (-1)^{m-m'} (\tau_{\bar{\mu}_1 \cdots \bar{\mu}_n})_{-m-m'}$.

---

[1] See M. Rubin and C. Ordóñez, *Eigenvalues and Degeneracies for n- Dimensional Tensor Spherical Harmonics*, J. Math. Phys. 25 (1984) 2888.

Scalar harmonics which belongs to the (J,J) representation of the rotation (isometry) group $SU(2) \times SU(2)$ $(= SO(4))$ of S^3 can be expressed using the Wigner D-function as

$$Y_{JM} = \sqrt{\frac{2J+1}{\mathrm{V}_3}} D^J_{mm'},$$

where $M = (m, m')$. It is normalized to

$$\int_{S^3} d\Omega_3 \, Y^*_{J_1 M_1} Y_{J_2 M_2} = \delta_{J_1 J_2} \delta_{M_1 M_2}, \qquad Y^*_{JM} = \epsilon_M Y_{J-M},$$

where the second Kronecker delta is defined by $\delta_{M_1 M_2} = \delta_{m_1 m_2} \delta_{m'_1 m'_2}$ and $\epsilon_M = (-1)^{m-m'}$ is a sign factor.

Next, consider spherical harmonics with spacetime indices. We first express it in the Cartesian coordinate system of $\mathbb{R}^4$. Vector harmonics which belongs to the $(J+y, J-y)$ representation of $SU(2) \times SU(2)$ with a polarization parameter $y = \pm 1/2$ is expressed by

$$Y^{\bar{\mu}}_{J(My)} = \frac{1}{\sqrt{2}} \frac{1}{r} \sum_{S,T} C^{J+ym}_{Js, \frac{1}{2}t} C^{J-ym'}_{Js', \frac{1}{2}t'} Y_{JS} (\tau^{\bar{\mu}})_{tt'},$$

and tensor harmonics which belongs to the $(J+x, J-x)$ representation with a polarization $x = \pm 1$ is

$$Y^{\bar{\mu}\bar{\nu}}_{J(Mx)} = \frac{1}{2} \frac{1}{r^2} \sum_{S,T} C^{J+xm}_{Js,1t} C^{J-xm'}_{Js',1t'} Y_{JS} (\tau^{\bar{\mu}\bar{\nu}})_{tt'},$$

where $\tau_{\bar{\mu}}$ and $\tau_{\bar{\mu}\bar{\nu}}$ are those in the expression of the D-function above, which are normalized to $(\tau^{\bar{\mu}})^*_{mm'} (\tau_{\bar{\mu}})_{nn'} = 2\delta_{MN}$ and $(\tau^{\bar{\mu}\bar{\nu}})^*_{mm'} (\tau_{\bar{\mu}\bar{\nu}})_{nn'} = 4\delta_{MN}$, respectively. Each complex conjugate is given by

$$Y^{\bar{\mu}*}_{J(My)} = -\epsilon_M Y^{\bar{\mu}}_{J(-My)}, \qquad Y^{\bar{\mu}\bar{\nu}*}_{J(Mx)} = \epsilon_M Y^{\bar{\mu}\bar{\nu}}_{J(-Mx)},$$

and each overall coefficient is normalized to

$$\int_{S^3} d\Omega_3 \, Y^{\bar{\mu}*}_{J_1(M_1 y_1)} Y_{\bar{\mu} J_2 (M_2 y_2)} = \frac{1}{r^2} \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{y_1 y_2},$$
$$\int_{S^3} d\Omega_3 \, Y^{\bar{\mu}\bar{\nu}*}_{J_1(M_1 x_1)} Y_{\bar{\mu}\bar{\nu} J_2 (M_2 x_2)} = \frac{1}{r^4} \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{x_1 x_2}.$$

These spherical harmonics satisfy

$$x_{\bar{\mu}} Y^{\bar{\mu}}_{J(My)} = x_{\bar{\mu}} Y^{\bar{\mu}\bar{\nu}}_{J(Mx)} = 0. \tag{C-1}$$

The vector and tensor harmonics in the spherical coordinate system are obtained by performing coordinate transformations as

$$Y_{\mu J(My)} = \frac{\partial x^{\bar{\mu}}}{\partial \hat{x}^{\mu}} Y_{\bar{\mu} J(My)}, \qquad Y_{\mu\nu J(Mx)} = \frac{\partial x^{\bar{\mu}}}{\partial \hat{x}^{\mu}} \frac{\partial x^{\bar{\nu}}}{\partial \hat{x}^{\nu}} Y_{\bar{\mu}\bar{\nu} J(Mx)}.$$

The equation (C-1) represents that all components with the $r(=\hat{x}^0)$ coordinate disappear like $0 = x^{\bar{\mu}} Y_{\bar{\mu}} = x^{\bar{\mu}} (\partial \hat{x}^{\mu}/\partial x^{\bar{\mu}}) Y_{\mu} = r(\partial \hat{x}^{\mu}/\partial r) Y_{\mu} = Y_r$ when transforming to the spherical coordinates. That is, in the spherical coordinates, they become

$$Y^r_{J(My)} = 0, \qquad Y^{rr}_{J(Mx)} = Y^{ri}_{J(Mx)} = 0,$$

and only the S^3 coordinate components survive. Therefore, we can calculate scalar quantities composed of tensor harmonics using the expressions in the $\mathbb{R}^4$ coordinates such as

$$Y^{\bar{\mu}} Y_{\bar{\mu}} = \left(\frac{1}{r^2}\right) Y^i Y_i, \qquad Y^{\bar{\mu}\bar{\nu}} Y_{\bar{\mu}} Y_{\bar{\nu}} = \left(\frac{1}{r^4}\right) Y^{ij} Y_i Y_j,$$

even if we do not know each specific expression in the spherical coordinates. This fact is used for calculating their normalizations and the $SU(2)\times SU(2)$ Clebsch-Gordan coefficients.

In general, spherical tensor harmonics of rank n belonging to the $(J+\varepsilon_n, J-\varepsilon_n)$ representation with a polarization parameter $\varepsilon_n = \pm n/2$ can be expressed as

$$Y^{\bar{\mu}_1\cdots\bar{\mu}_n}_{J(M\varepsilon_n)} \propto \sum_{S,T} C^{J+\varepsilon_n m}_{Js,\frac{n}{2}t} C^{J-\varepsilon_n m'}_{Js',\frac{n}{2}t'} Y_{JS} (\tau^{\bar{\mu}_1\cdots\bar{\mu}_n})_{tt'},$$

and its complex conjugate is given by $Y^{\bar{\mu}_1\cdots\bar{\mu}_n *}_{J(M\varepsilon_n)} = (-1)^n \epsilon_M Y^{\bar{\mu}_1\cdots\bar{\mu}_n}_{J(-M\varepsilon_n)}$.

Finally, the Euler angle expression of the vector harmonics obtained by the above method is displayed as an example. For $y=1/2$, it is given by

$$Y_{\alpha J(M\frac{1}{2})} = \frac{i}{2\sqrt{2}} \sqrt{\frac{(2J+2m+1)(2J-2m+1)}{(2J+1)\mathrm{V}_3}} D^{J-\frac{1}{2}}_{mm'},$$

$$Y_{\beta J(M\frac{1}{2})} = \frac{1}{\sqrt{2}(2J+1)} \frac{1}{\sin\beta} \left\{ m \sqrt{\frac{(2J+2m'+1)(2J-2m'+1)}{(2J+1)\mathrm{V}_3}} D^{J+\frac{1}{2}}_{mm'} \right.$$
$$\left. - m' \sqrt{\frac{(2J+2m+1)(2J-2m+1)}{(2J+1)\mathrm{V}_3}} D^{J-\frac{1}{2}}_{mm'} \right\},$$

$$Y_{\gamma J(M\frac{1}{2})} = \frac{i}{2\sqrt{2}} \sqrt{\frac{(2J+2m'+1)(2J-2m'+1)}{(2J+1)\mathrm{V}_3}} D^{J+\frac{1}{2}}_{mm'},$$

and for $y=-1/2$, it is

$$Y_{\alpha J(M-\frac{1}{2})}=\frac{i}{2\sqrt{2}}\sqrt{\frac{(2J+2m+1)(2J-2m+1)}{(2J+1)\mathrm{V}_3}}D^{J+\frac{1}{2}}_{mm'},$$

$$Y_{\beta J(M-\frac{1}{2})}=\frac{1}{\sqrt{2}(2J+1)}\frac{1}{\sin\beta}\left\{m'\sqrt{\frac{(2J+2m+1)(2J-2m+1)}{(2J+1)\mathrm{V}_3}}D^{J+\frac{1}{2}}_{mm'}\right.$$
$$\left.-m\sqrt{\frac{(2J+2m'+1)(2J-2m'+1)}{(2J+1)\mathrm{V}_3}}D^{J-\frac{1}{2}}_{mm'}\right\},$$

$$Y_{\gamma J(M-\frac{1}{2})}=\frac{i}{2\sqrt{2}}\sqrt{\frac{(2J+2m'+1)(2J-2m'+1)}{(2J+1)\mathrm{V}_3}}D^{J-\frac{1}{2}}_{mm'}.$$

$SU(2)\times SU(2)$ Clebsch-Gordan Coefficients

The $SU(2)\times SU(2)$ Clebsch-Gordan coefficients are defined by integrals over S^3 of the three products of the ST^2 tensor harmonics. The general expressions of the coefficients other than the type **C** (8-13) defined in Chapter 8 are summarized below.[2]

Type D

$$\mathbf{D}^{JM}_{J_1(M_1y_1),J_2(M_2y_2)}=\sqrt{\mathrm{V}_3}\int_{S^3}d\Omega_3\,Y^*_{JM}Y^i_{J_1(M_1y_1)}Y_{iJ_2(M_2y_2)}$$
$$=-\sqrt{\frac{2J_1(2J_1+1)(2J_1+2)2J_2(2J_2+1)(2J_2+2)}{2J+1}}$$
$$\times\left\{\begin{array}{ccc} J & J_1 & J_2 \\ \frac{1}{2} & J_2+y_2 & J_1+y_1 \end{array}\right\}\left\{\begin{array}{ccc} J & J_1 & J_2 \\ \frac{1}{2} & J_2-y_2 & J_1-y_1 \end{array}\right\}$$
$$\times C^{Jm}_{J_1+y_1m_1,J_2+y_2m_2}C^{Jm'}_{J_1-y_1m'_1,J_2-y_2m'_2}, \tag{C-2}$$

which satisfies $M=M_1+M_2$ and triangle inequality $|J_1-J_2|\le J\le J_1+J_2$, where $J+J_1+J_2$ is an integer, and the equality on the lower side (higher side) of the inequality holds in the case of $y_1=y_2$ ($y_1\neq y_2$).

Type E

$$\mathbf{E}^{JM}_{J_1(M_1x_1),J_2(M_2x_2)}=\sqrt{\mathrm{V}_3}\int_{S^3}d\Omega_3\,Y^*_{JM}Y^{ij}_{J_1(M_1x_1)}Y_{ijJ_2(M_2x_2)}$$

[2] See K. Hamada and S. Horata, Prog. Theor. Phys. **110** (2003) 1169 in Bibliography.

$$= \sqrt{\frac{(2J_1-1)(2J_1+1)(2J_1+3)(2J_2-1)(2J_2+1)(2J_2+3)}{2J+1}}$$
$$\times \begin{Bmatrix} J & J_1 & J_2 \\ 1 & J_2+x_2 & J_1+x_1 \end{Bmatrix} \begin{Bmatrix} J & J_1 & J_2 \\ 1 & J_2-x_2 & J_1-x_1 \end{Bmatrix}$$
$$\times C^{Jm}_{J_1+x_1 m_1, J_2+x_2 m_2} C^{Jm'}_{J_1-x_1 m'_1, J_2-x_2 m'_2}, \tag{C-3}$$

which satisfies $M = M_1 + M_2$ and triangular inequality $|J_1 - J_2| \le J \le J_1 + J_2$, where $J + J_1 + J_2$ is an integer, and the equality on the lower side (higher side) of the inequality holds in the case of $x_1 = x_2$ $(x_1 \neq x_2)$.

Type G

$$\mathbf{G}^{JM}_{J_1(M_1 y_1); J_2 M_2} = \sqrt{\mathrm{V}_3} \int_{S^3} d\Omega_3 \, Y^*_{JM} Y^i_{J_1(M_1 y_1)} \hat{\nabla}_i Y_{J_2 M_2}$$
$$= -\frac{1}{2\sqrt{2}} \sqrt{\frac{2J_1(2J_1+1)(2J_1+2)(2J_2+1)}{2J+1}} \sum_{K=J_2 \pm \frac{1}{2}} 2K(2K+1)$$
$$\times (2K+2) \begin{Bmatrix} J & J_1 & K \\ \frac{1}{2} & J_2 & J_1+\frac{1}{2} \end{Bmatrix} \begin{Bmatrix} J & J_1 & K \\ \frac{1}{2} & J_2 & J_1-\frac{1}{2} \end{Bmatrix}$$
$$\times C^{Jm}_{J_1+y_1 m_1, J_2 m_2} C^{Jm'}_{J_1-y_1 m'_1, J_2 m'_2}, \tag{C-4}$$

which satisfies $M = M_1 + M_2$ and triangular inequality $|J_1 - J_2| + \frac{1}{2} \le J \le J_1 + J_2 - \frac{1}{2}$, where $J + J_1 + J_2$ is a half integer.

Type H

$$\mathbf{H}^{JM}_{J_1(M_1 x_1); J_2(M_2 y_2)} = \sqrt{\mathrm{V}_3} \int_{S^3} d\Omega_3 \, Y^*_{JM} Y^{ij}_{J_1(M_1 x_1)} \hat{\nabla}_i Y_{j J_2(M_2 y_2)}$$
$$= -\frac{3}{2\sqrt{2}} \sqrt{\frac{(2J_1-1)(2J_1+1)(2J_1+3)2J_2(2J_2+1)(2J_2+2)}{2J+1}}$$
$$\times \sum_{K=J_2 \pm \frac{1}{2}} 2K(2K+1)(2K+2)$$
$$\times \begin{Bmatrix} K & 1 & J_2+y_2 \\ \frac{1}{2} & J_2 & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} K & 1 & J_2-y_2 \\ \frac{1}{2} & J_2 & \frac{1}{2} \end{Bmatrix}$$
$$\times \begin{Bmatrix} J & J_1+x_1 & J_2+y_2 \\ 1 & K & J_1 \end{Bmatrix} \begin{Bmatrix} J & J_1-x_1 & J_2-y_2 \\ 1 & K & J_1 \end{Bmatrix}$$
$$\times C^{Jm}_{J_1+x_1 m_1, J_2+y_2 m_2} C^{Jm'}_{J_1-x_1 m'_1, J_2-y_2 m'_2}, \tag{C-5}$$

which satisfies $M = M_1 + M_2$ and triangular inequality $|J_1 - J_2| + \frac{1}{2} \le J \le J_1 + J_2 - \frac{1}{2}$, where $J + J_1 + J_2$ is a half integer, and the equality on the lower side (higher side) of the inequality holds in the case of $x_1 = 2y_2$ $(x_1 \neq 2y_2)$.

Product Formulas for Spherical Harmonics on S^3

As product formulas involving the $J = 1/2$ scalar harmonics, the following are useful:

$$
\begin{aligned}
Y^*_{\frac{1}{2}M} Y_{JN} &= \frac{1}{\sqrt{\mathrm{V}_3}} \Big\{ \sum_S \mathbf{C}^{\frac{1}{2}M}_{JN,J+\frac{1}{2}S} Y^*_{J+\frac{1}{2}S} + \sum_S \mathbf{C}^{\frac{1}{2}M}_{JN,J-\frac{1}{2}S} Y^*_{J-\frac{1}{2}S} \Big\}, \\
\hat{\nabla}^i Y^*_{\frac{1}{2}M} \hat{\nabla}_i Y_{JN} &= \frac{1}{\sqrt{\mathrm{V}_3}} \Big\{ -2J \sum_S \mathbf{C}^{\frac{1}{2}M}_{JN,J+\frac{1}{2}S} Y^*_{J+\frac{1}{2}S} \\
&\qquad + (2J+2) \sum_S \mathbf{C}^{\frac{1}{2}M}_{JN,J-\frac{1}{2}S} Y^*_{J-\frac{1}{2}S} \Big\}, \qquad \text{(C-6)}
\end{aligned}
$$

and

$$
\begin{aligned}
Y^*_{\frac{1}{2}M} Y^j_{J(Ny)} &= \frac{1}{\sqrt{\mathrm{V}_3}} \Big\{ \sum_{V,y'} \mathbf{D}^{\frac{1}{2}M}_{J(Ny),J+\frac{1}{2}(Vy')} Y^{j*}_{J+\frac{1}{2}(Vy')} \\
&\qquad + \sum_{V,y'} \mathbf{D}^{\frac{1}{2}M}_{J(Ny),J-\frac{1}{2}(Vy')} Y^{j*}_{J-\frac{1}{2}(Vy')} \\
&\qquad + \frac{1}{2J(2J+2)} \sum_S \mathbf{G}^{\frac{1}{2}M}_{J(Ny);JS} \hat{\nabla}^j Y^*_{JS} \Big\}, \\
\hat{\nabla}^i Y^*_{\frac{1}{2}M} \hat{\nabla}_i Y^j_{J(Ny)} &= \frac{1}{\sqrt{\mathrm{V}_3}} \Big\{ -2J \sum_{V,y'} \mathbf{D}^{\frac{1}{2}M}_{J(Ny),J+\frac{1}{2}(Vy')} Y^{j*}_{J+\frac{1}{2}(Vy')} \\
&\qquad + (2J+2) \sum_{V,y'} \mathbf{D}^{\frac{1}{2}M}_{J(Ny),J-\frac{1}{2}(Vy')} Y^{j*}_{J-\frac{1}{2}(Vy')} \\
&\qquad + \frac{2}{2J(2J+2)} \sum_S \mathbf{G}^{\frac{1}{2}M}_{J(Ny);JS} \hat{\nabla}^j Y^*_{JS} \Big\}. \qquad \text{(C-7)}
\end{aligned}
$$

The product with the tensor harmonics can be expanded similarly using the $\mathbf{E}$ and $\mathbf{H}$ coefficients.

Formulas containing Clebsch-Gordan Coefficients and Wigner D-Functions

The normal Clebsch-Gordan coefficient $C^{c\gamma}_{a\alpha,b\beta}$ has a value when satisfying the triangular inequality $|a-b| \leq c \leq a+b$ and $\alpha + \beta = \gamma$, where a, b, and c are non-negative integers or half integers, while $a+b+c$, $a+\alpha$,

$b+\beta$, and $c+\gamma$ are nonnegative integers. This coefficient is normalized to $C^{a\alpha}_{a\alpha,00} = C^{a+ba+b}_{aa,bb} = 1$ and satisfies the following relations:[3]

$$C^{c\gamma}_{a\alpha,b\beta} = (-1)^{a+b-c}\, C^{c-\gamma}_{a-\alpha,b-\beta} = (-1)^{a+b-c}\, C^{c\gamma}_{b\beta,a\alpha}$$
$$= (-1)^{b+\beta}\sqrt{\frac{2c+1}{2a+1}}\, C^{a-\alpha}_{c-\gamma,b\beta}.$$

Various formulas containing the Clebsch-Gordan coefficients and the $6j$ symbols are

$$\sum_{\alpha,\beta} C^{c\gamma}_{a\alpha,b\beta}\, C^{c'\gamma'}_{a\alpha,b\beta} = \delta_{cc'}\delta_{\gamma\gamma'},$$

$$\sum_{\alpha,\beta,\delta} (-1)^{a-\alpha}\, C^{c\gamma}_{b\beta,a\alpha}\, C^{e\epsilon}_{b\beta,d\delta}\, C^{f\varphi}_{d\delta,a-\alpha} = (-1)^{a+b+e+f}\sqrt{(2c+1)(2f+1)}$$
$$\times C^{e\epsilon}_{c\gamma,f\varphi} \left\{ \begin{array}{ccc} a & b & c \\ e & f & d \end{array} \right\},$$

$$\sum_{\psi,\kappa,\rho,\sigma,\tau} (-1)^{\psi+\kappa+\rho+\sigma+\tau}\, C^{a\alpha}_{p\psi,q\kappa}\, C^{b\beta}_{q\kappa,r\rho}\, C^{c\gamma}_{r\rho,s\sigma}\, C^{d\delta}_{s\sigma,t\tau}\, C^{e\varepsilon}_{t\tau,p-\psi}$$
$$= (-1)^{-a-b-2c-2p-2r-t+\alpha+\delta}\sqrt{(2a+1)(2d+1)}$$
$$\times \sum_{x,y}\sum_{\xi,\eta}(-1)^{\xi+\eta}(2x+1)(2y+1)$$
$$\times C^{b\beta}_{a\alpha,x\xi}\, C^{e\varepsilon}_{x\xi,y\eta}\, C^{c-\gamma}_{y\eta,d-\delta} \left\{ \begin{array}{ccc} a & b & x \\ r & p & q \end{array} \right\} \left\{ \begin{array}{ccc} x & e & y \\ t & r & p \end{array} \right\} \left\{ \begin{array}{ccc} y & c & d \\ s & t & r \end{array} \right\},$$

$$\sum_{x}(-1)^{p+q+x}(2x+1) \left\{ \begin{array}{ccc} a & b & x \\ c & d & p \end{array} \right\} \left\{ \begin{array}{ccc} a & b & x \\ d & c & q \end{array} \right\} = \left\{ \begin{array}{ccc} a & c & q \\ b & d & p \end{array} \right\},$$

and so on.

As formulas containing the Wigner D-function, there are the following:

$$D^{J*}_{mm'} = (-1)^{m-m'}\, D^{J}_{-m-m'},$$

$$\sum_{m'=-J}^{J} D^{J*}_{m_1 m'} D^{J}_{m_2 m'} = \delta_{m_1 m_2},$$

$$\int_{S^3} d\Omega_3\, D^{J_1 *}_{m_1 m_1'}\, D^{J_2}_{m_2 m_2'} = \frac{\mathrm{V}_3}{2J_1+1}\delta_{J_1 J_2}\delta_{m_1 m_2}\delta_{m_1' m_2'},$$

[3] For the following formulas, see D. Varshalovich, A. Moskalev and V. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).

$$D^{J_1}_{m_1m'_1}D^{J_2}_{m_2m'_2} = \sum_{J=|J_1-J_2|}^{J_1+J_2}\sum_{m,m'} C^{Jm}_{J_1m_1,J_2m_2}C^{Jm'}_{J_1m'_1,J_2m'_2}\,D^J_{mm'},$$

$$\begin{aligned}\Box_3 D^J_{mm'} &= 4\left\{\partial_\beta^2 + \cot\beta\partial_\beta + \frac{1}{\sin^2\beta}\left(\partial_\alpha^2 - 2\cos\beta\partial_\alpha\partial_\gamma + \partial_\gamma^2\right)\right\}D^J_{mm'}\\ &= -4J(J+1)D^J_{mm'},\end{aligned}$$

$$\sum_{m_1,m'_1}\sum_{m_2,m'_2} C^{Jm}_{J_1m_1,J_2m_2}C^{J'm'}_{J_1m'_1,J_2m'_2}D^{J_1}_{m_1m'_1}D^{J_2}_{m_2m'_2} = \delta_{JJ'}\{J_1J_2J\}D^J_{mm'},$$

$$\begin{aligned}&\sum_{m_1,m'_1}\sum_{m_2,m'_2}\sum_{m_3,m'_3} C^{Jm}_{Kn,J_3m_3}C^{Kn}_{J_1m_1,J_2m_2}C^{J'm'}_{K'n',J_3m'_3}C^{K'n'}_{J_1m'_1,J_2m'_2}\\ &\quad\times D^{J_1}_{m_1m'_1}D^{J_2}_{m_2m'_2}D^{J_3}_{m_3m'_3} = \delta_{JJ'}\delta_{KK'}\{J_1J_2K\}\{KJ_3J\}D^J_{mm'},\end{aligned}$$

where all the D-functions have the same arguments α, β, and γ. The $3j$-symbol $\{J_1J_2J_3\}$ is 1 if $J_1+J_2+J_3$ is an integer and $|J_1-J_2| \leq J_3 \leq J_1+J_2$, otherwise disappears. In addition, it is invariant with respect to permutations of J_1, J_2, and J_3.

The specific expressions of the Wigner D-function of $J = 1/2$ and $J = 1$ in terms of the Euler angles are given as follows:

$$D^{\frac{1}{2}}_{mm'} = \begin{pmatrix} \cos\frac{\beta}{2}e^{-\frac{i}{2}(\alpha+\gamma)} & -\sin\frac{\beta}{2}e^{-\frac{i}{2}(\alpha-\gamma)} \\ \sin\frac{\beta}{2}e^{\frac{i}{2}(\alpha-\gamma)} & \cos\frac{\beta}{2}e^{\frac{i}{2}(\alpha+\gamma)} \end{pmatrix},$$

$$D^1_{mm'} = \begin{pmatrix} \frac{1+\cos\beta}{2}e^{-i(\alpha+\gamma)} & -\frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} & \frac{1-\cos\beta}{2}e^{-i(\alpha-\gamma)} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}}e^{i\gamma} \\ \frac{1-\cos\beta}{2}e^{i(\alpha-\gamma)} & \frac{\sin\beta}{\sqrt{2}}e^{i\alpha} & \frac{1+\cos\beta}{2}e^{i(\alpha+\gamma)} \end{pmatrix}.$$

Appendix D

Addenda to Renormalization Theory

Useful Formulas in Dimensional Regularization

Renormalization calculations are done after Wick-rotating to Euclidean space in advance. In that case, all spacetime indices are written with subscripts, and the same index is contracted by the Euclidean metric $\delta_{\mu\nu}$.

The integral in D-dimensional Euclidean momentum space is given by

$$\int d^D p = \int p^{D-1} dp \int d\Omega_D, \quad (p^2 = p_\mu p_\mu)$$

$$\int d\Omega_D = \int \prod_{l=1}^{D-1} \sin^{D-1-l}\theta_l d\theta_l = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}.$$

Basic integral formulas The basic form of the momentum integral appearing in dimensional regularization is

$$\int \frac{d^D p}{(2\pi)^D} \frac{p^{2n}}{(p^2+L)^\alpha} = \frac{\Gamma\left(n+\frac{D}{2}\right)\Gamma\left(\alpha-n-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)\Gamma(\alpha)} L^{\frac{D}{2}+n-\alpha},$$

whose integrand is a function of p^2 only. If p_μ is included in the integrand, rewrite the integral to the basic form such as

$$\int \frac{d^D p}{(2\pi)^D} p_\mu p_\nu f(p^2) = \frac{1}{D}\delta_{\mu\nu} \int \frac{d^D p}{(2\pi)^D} p^2 f(p^2),$$

$$\int \frac{d^D p}{(2\pi)^D} p_\mu p_\nu p_\lambda p_\sigma f(p^2) = \frac{1}{D(D+2)} (\delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda}) \times \int \frac{d^D p}{(2\pi)^D} p^4 f(p^2).$$

If the integrand has an odd number of p_μ's, it vanishes. In general, we obtain

$$\int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^n (p\cdot l)^{2m}}{(p^2+L)^\alpha} = (l^2)^m \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{D}{2}\right)\Gamma\left(\alpha-n-m-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{D}{2}\right)\Gamma(\alpha)} L^{\frac{D}{2}+n+m-\alpha}.$$

Feynman parameter integrals More complex integrals that appear in renormalization calculations of self-energy diagrams and vertex functions are evaluated by rewriting them using the Feynman parametrization

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{(1-x)^{\alpha-1}x^{\beta-1}}{[(1-x)A+xB]^{\alpha+\beta}}$$

into a form in which the basic integral formula can apply.

For example, consider a combination of $A = p^2+z^2$ and $B = (p+q)^2+z^2$, which appear in various self-energy calculations, where z^2 corresponds to a mass term. In this case, we obtain the following:

$$\begin{aligned}&\int \frac{d^Dp}{(2\pi)^D} \frac{f(p_\mu, q_\nu)}{(p^2+z^2)^\alpha((p+q)^2+z^2)^\beta} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, (1-x)^{\alpha-1}x^{\beta-1} \int \frac{d^Dp'}{(2\pi)^D} \frac{f(p'_\mu - xq_\mu, q_\nu)}{[p'^2+z^2+x(1-x)q^2]^{\alpha+\beta}}.\end{aligned}$$

In vertex-function calculations, we repeat this procedure.

Evaluation of divergences In dimensional regularization, we take $D = 4-2\epsilon$, then ultraviolet divergences are extracted as poles of ϵ. At that time,

$$\begin{aligned}\Gamma(\epsilon) &= \frac{1}{\epsilon} - \gamma + \frac{\epsilon}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + o(\epsilon^2), \\ a^\epsilon &= e^{\epsilon \ln a} = 1 + \epsilon \ln a + o(\epsilon^2),\end{aligned}$$

are used, where as examples of a, there will appear a square of momentum p^2, an infinitesimal z^2 for handling infrared divergences, and so on.

Gamma matrices formulas Define the Dirac's gamma matrix in the D-dimensional flat Euclidean space as

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \qquad \mathrm{tr}(1) = 4.$$

Various formulas of the gamma matrix used in dimensional regularization are as follows:

$$\begin{aligned}\gamma_\lambda\gamma_\lambda &= -D, \qquad \gamma_\lambda\gamma_\mu\gamma_\lambda = (D-2)\gamma_\mu, \\ \gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\lambda &= -(D-4)\gamma_\mu\gamma_\nu + 4\delta_{\mu\nu}, \\ \gamma_\mu\gamma_{\nu\lambda} &= \gamma_{\mu\nu\lambda} - \delta_{\mu\nu}\gamma_\lambda + \delta_{\mu\lambda}\gamma_\nu, \\ \gamma_{\nu\lambda}\gamma_\mu &= \gamma_{\nu\lambda\mu} + \delta_{\mu\nu}\gamma_\lambda - \delta_{\mu\lambda}\gamma_\nu, \\ \gamma_\mu\gamma_{\nu\lambda\sigma} &= \gamma_{\mu\nu\lambda\sigma} - \delta_{\mu\nu}\gamma_{\lambda\sigma} + \delta_{\mu\lambda}\gamma_{\nu\sigma} - \delta_{\mu\sigma}\gamma_{\nu\lambda}, \\ \gamma_{\nu\lambda\sigma}\gamma_\mu &= \gamma_{\nu\lambda\sigma\mu} - \delta_{\mu\nu}\gamma_{\lambda\sigma} + \delta_{\mu\lambda}\gamma_{\nu\sigma} - \delta_{\mu\sigma}\gamma_{\nu\lambda},\end{aligned}$$

where completely antisymmetric products of the gamma matrices are defined as $\gamma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2$, $\gamma_{\mu\nu\lambda} = (\gamma_\mu\gamma_\nu\gamma_\lambda + \text{antisymmetric})/3!$, and $\gamma_{\mu\nu\lambda\sigma} = (\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma + \text{antisymmetric})/4!$.

Examples of Renormalization Calculations in QED

Renormalization calculations are shortly described by taking the self-energy function of the $U(1)$ gauge field as an example. Strict discussions on the effective action using the Legendre transformation refer to field theory textbooks. Here we perform loop calculations in Euclidean space. One of the advantages of doing so is that troublesome imaginary units do not appear, and thus it is only necessary to pay attention to signs and coefficients, and integrations can be performed immediately. The partition function is given by the path integral $Z = \int e^{-S_{\rm QED}}$, and its effective action $\Gamma = -\log Z$ is simply expressed as follows:

$$\Gamma = -e^{-S_{\rm int}}\big|_{\rm 1PI} = -\sum_{n=0}^{\infty} \frac{1}{n!}\left(-S_{\rm int}\right)^n\big|_{\rm 1PI},$$

where $S_{\rm int}$ denotes interaction terms, including counterterms. The 1PI restriction which corresponds to taking the logarithm of the partition function is that for all possible one-particle irreducible diagrams, Wick contractions are taken, that is, connected by propagators. The 1PI connections are represented by Feynman diagrams, but Feynman rule is not introduced here because it differs from person to person. Moreover, since the symmetric factor is included as long as all possible Wick contractions are taken into account, we do not have to worry about it.

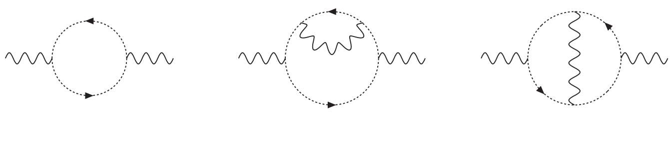

Figure D-1: One-loop and two-loop self-energy diagrams of the $U(1)$ gauge field.

First of all, we write the QED action $S_{\rm QED}$ in momentum space. Define Fourier transform of the field as $f(x) = \int [dk] f(k) e^{ik\cdot x}$, where $[dk] = d^D k/(2\pi)^D$, then the kinetic terms are given by

$$S^A_{\rm kin} = \int [dk]\, \frac{1}{2} A_\mu(k)\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) A_\nu(-k),$$

$$S_{\rm kin}^{\psi} = \int [dk]\, \bar{\psi}(k) \not{k} \psi(-k),$$

where $\not{k} = \gamma_\mu k_\mu$. The interaction term is

$$S_{\rm v} = -\mu^{2-\frac{D}{2}} e \int [dkdldp]\, \bar{\delta}(k+l+p)\, \bar{\psi}(k)\gamma_\mu \psi(l) A_\mu(p),$$

where $\bar{\delta}(k) = (2\pi)^D \delta^D(k)$. The counterterms $S_{\rm kin}^c = (Z_3 - 1) S_{\rm kin}^A + (Z_2 - 1) S_{\rm kin}^{\psi}$ and $S_{\rm v}^c = (Z_1 - 1) S_{\rm v}$ are also added to $S_{\rm int}$.

From the kinetic terms, the propagators are given by

$$\begin{aligned}\langle A_\mu(k') A_\nu(k) \rangle &= \frac{1}{k^2} \left(\delta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \bar{\delta}(k'+k), \\ \langle \psi(k') \bar{\psi}(k) \rangle &= \frac{1}{\not{k}} \bar{\delta}(k'+k) = -\frac{\not{k}}{k^2} \bar{\delta}(k'+k),\end{aligned}$$

where ξ is a parameter associate with the usual gauge-fixing. In order to avoid mistakes in coeffcients, the delta functions are left.

The effective action for the one-loop self-energy correction in Fig. D-1 is obtained by Wick-contracting all of the others with leaving only two gauge fields as follows:

$$\begin{aligned}\Gamma^{(1)} &= -\frac{1}{2!} \left(-S_{\rm v}\right)^2 \Big|_{\rm 1PI} \\ &= -\frac{1}{2!} \mu^{4-D} e^2 \int [dk' dl' dp'][dkdldp]\, \bar{\delta}(k'+l'+p')\, \bar{\delta}(k+l+p) \\ &\quad \times (-1)\, {\rm tr}\left[\langle \psi(l) \bar{\psi}(k') \rangle \gamma_\mu \langle \psi(l') \bar{\psi}(k) \rangle \gamma_\nu \right] A_\mu(p') A_\nu(p).\end{aligned}$$

In this example, there is only one type of Wick contraction, thus the overall factor remains $1/2!$. The operation of $(-1)\,{\rm tr}$ is a loop factor caused by moving $\psi(l)$ from the right end to the left end. Assigning the fermion propagator yields

$$\Gamma^{(1)} = \frac{\mu^{4-D} e^2}{2!} \int [dp]\, A_\mu(-p) A_\nu(p) \int [dl]\, \frac{l_\lambda (l+p)_\sigma}{l^2 (l+p)^2}\, {\rm tr}[\gamma_\lambda \gamma_\mu \gamma_\sigma \gamma_\nu].$$

Expressing the effective action as[1]

$$\Gamma(A) = \int [dp] \frac{1}{2} A_\mu(-p) A_\nu(p) \Gamma_{\mu\nu}(p),$$

and expanded with ϵ after setting $D = 4 - 2\epsilon$, the one-loop contribution is obtained as follows:

$$\begin{aligned}
\Gamma^{(1)}_{\mu\nu}(p) &= \frac{8e^2}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2 - \frac{D}{2}\right) B\left(\frac{D}{2}, \frac{D}{2}\right) \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \left(\frac{p^2}{\mu^2}\right)^{\frac{D}{2}-2} \\
&\overset{\epsilon\to 0}{=} \frac{\alpha}{4\pi} \left\{ \frac{4}{3}\frac{1}{\bar{\epsilon}} - \frac{4}{3}\log\left(\frac{p^2}{\mu^2}\right) + \frac{20}{9} \right\} \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right),
\end{aligned}$$

where $\alpha = e^2/4\pi$ and $1/\bar{\epsilon} = 1/\epsilon - \gamma + \log 4\pi$. Thus, in order to remove the ultraviolet divergence, we find that the renormalization factor should be $Z_3 - 1 = -(\alpha/4\pi)(4/3\epsilon)$, where the minimal subtraction (MS) scheme (subtracting only poles) is adopted.

The contribution from the two-loop diagram in Fig. D-1 is given by $-S_{\mathrm{v}}^4/4!$. In addition to that, there are contributions from $(S_{\mathrm{v}} + S^c_{\mathrm{kin}})^3/3!$ and $-(S_{\mathrm{v}} + S^c_{\mathrm{v}})^2/2!$ including one-loop counterterms. Diagrams including the counterterms are not depicted here. Writing only the result obtained by adding these contributions, it is given by[2]

$$\Gamma^{(2)}_{\mu\nu}(p) = \frac{\alpha^2}{(4\pi)^2} \left\{ \frac{2}{\bar{\epsilon}} - 4\log\left(\frac{p^2}{\mu^2}\right) + \text{const.} \right\} \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right).$$

Note that double poles disappear in this case. Therefore, the renormalization factor up to $o(\alpha^2)$ is determined as follows:

$$Z_3 = 1 - \frac{4}{3}\frac{\alpha}{4\pi}\frac{1}{\epsilon} - 2\frac{\alpha^2}{(4\pi)^2}\frac{1}{\epsilon}.$$

---

[1] A two-point correlation function of the gauge field is given by

$$\begin{aligned}
\langle A_\mu(p) A_\nu(q) \rangle &= A_\mu(p) A_\nu(q) e^{-S_{\mathrm{int}}} |_{\text{Wick contraction}} \\
&= A_\mu(p) A_\nu(q) \left[-\Gamma(A)\right] |_{\text{Wick contraction}} = \bar{\delta}(p+q) \frac{\delta_{\mu\lambda}}{p^2} \frac{\delta_{\nu\sigma}}{q^2} \left[-\Gamma_{\lambda\sigma}(p)\right].
\end{aligned}$$

Therefore, $-\Gamma_{\mu\nu}(p)$ corresponds to a correlation function (truncated Green's function) with propagators from external lines removed. The same is true for multipoint functions.

[2] For details, see C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill Inc, 1980), Chap.8-4-4.

The renormalized effective action is then given by

$$\Gamma^{\text{ren}}_{\mu\nu}(p) = \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left\{1 - \left[\frac{4}{3}\frac{\alpha}{4\pi} + 4\frac{\alpha^2}{(4\pi)^2}\right]\log\left(\frac{p^2}{\mu^2}\right)\right\},$$

where the first term is a contribution from S^A_{kin}, and local terms that depend on α are disregarded.

Renormalization of Composite Scalar Fields

Renormalization of composite fields using dimensional regularization is described by taking the $\lambda\varphi^4$-theory as an example.[3] Its Euclidean action is given by

$$S = \int d^D x \left(\frac{1}{2}\partial_\mu\varphi_0\partial_\mu\varphi_0 + \frac{1}{2}m_0^2\varphi_0^2 + \frac{\lambda_0}{4!}\varphi_0^4\right).$$

Ordinary renormalization factors are defined by

$$\varphi_0 = Z_\varphi^{1/2}\varphi, \qquad m_0^2 = Z_m m^2, \qquad \lambda_0 = \mu^{4-D}Z_\lambda\lambda, \qquad \text{(D-1)}$$

where μ is an arbitrary mass scale to compensate for missing dimensions. The beta function and the mass anomalous dimension are respectively defined by

$$\beta_\lambda \equiv \mu\frac{d\lambda}{d\mu} = (D-4)\lambda + \bar{\beta}_\lambda, \qquad \gamma_m \equiv -\frac{\mu}{m^2}\frac{dm^2}{d\mu},$$

where $\bar{\beta}_\lambda$ and γ_m are functions of λ only.

Letting $\mu d/d\mu$ act on the definitions (D-1) while be aware that bare quantities do not depend on the arbitrary scale μ, the renormalization group functions above can be expressed using the renormalization factors as $\bar{\beta}_\lambda = -\lambda\mu d(\log Z_\lambda)/d\mu$ and $\gamma_m = \mu d(\log Z_m)/d\mu$.[4] The anomalous dimension of the field is also defined as $\gamma = \mu d(\log Z_\varphi^{1/2})/d\mu$, where φ satisfies

[3] See L. Brown, *Dimensional Regularization of Composite Operators in Scalar Field Theory*, Ann. Phys. 126 (1980) 135.

[4] Expanding as $\log Z_\lambda = \sum_{n=1}^{\infty} f_n(\lambda)/(D-4)^n$ yields $\bar{\beta}_\lambda = -\lambda^2\partial f_1/\partial\lambda$ and $\lambda\partial f_{n+1}/\partial\lambda + \bar{\beta}_\lambda\partial f_n/\partial\lambda = 0$.

$\mu d\varphi/d\mu = -\gamma\varphi$ due to $\mu d\varphi_0/d\mu = 0$.[5] Each value is calculated as

$$\bar{\beta}_\lambda = 3\frac{\lambda^2}{(4\pi)^2} - \frac{17}{3}\frac{\lambda^3}{(4\pi)^4} + o(\lambda^4),$$
$$\gamma = \frac{1}{12}\frac{\lambda^2}{(4\pi)^4} + o(\lambda^3), \qquad \gamma_m = -\frac{\lambda}{(4\pi)^2} + \frac{5}{6}\frac{\lambda^2}{(4\pi)^4} + o(\lambda^3).$$

Note here that $\bar{\beta}_\lambda$, γ_m, and γ are finite functions of λ only, whereas β_λ has a $D-4$ dependence, and thus its inverse $1/\beta_\lambda$ has poles when expanded in the coupling constant (see the expansion (D-5)).

A N-point correlation function of the renormalized field φ is defined using the path integral as

$$\left\langle \prod_{j=1}^{N} \varphi(x_j) \right\rangle = Z_\varphi^{-\frac{N}{2}} \int d\varphi_0 \prod_{j=1}^{N} \varphi_0(x_j)\, e^{-S}. \qquad \text{(D-2)}$$

Normal product $[\varphi^2]$ First, consider a normal product of the field squared. This finite operator is related to the square of the bare field as follows:

$$\varphi_0^2 = Z_2[\varphi^2], \qquad \text{(D-3)}$$

where Z_2 is a new renormalization factor. Once the factor is known, it is possible to calculate the anomalous dimension of $[\varphi^2]$ as

$$\delta = \mu\frac{d}{d\mu}\log Z_2,$$

and thus the composite field satisfies $\mu d[\varphi^2]/d\mu = -\delta[\varphi^2]$.

The renormalization factor Z_2 can be determined from the condition that $[\varphi^2]$ is a finite operator. For example, consider the condition that correlation functions of the renormalized field including one normal product is finite such as[6]

$$\langle [\varphi^2]\varphi\varphi \rangle = \text{finite}.$$

Concretely, expressing $[\varphi^2]$ by the product of φ using (D-3) and (D-1) yields

$$[\varphi^2(k)] = Z_2^{-1} Z_\varphi \int \frac{d^D l}{(2\pi)^D}\, \varphi(l)\varphi(k-l)$$

[5] The dimension of the field φ, including its canonical dimension, is given by $\Delta_\varphi = (D-2)/2 + \gamma$. The behavior in correlation functions due to the anomalous dimension becomes $(p^2/\mu^2)^{\gamma/2}$ in each field.

[6] Note that correlation functions between normal products are not finite in general.

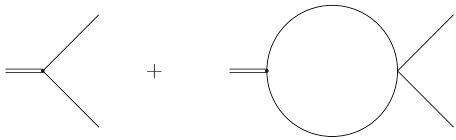

Figure D-2: Three-point functions including $[\varphi^2]$, where the right is a quantum correction.

in momentum space, then the correlation function $\langle[\varphi^2(k)]\varphi(p)\varphi(q)\rangle$ is calculated as

$$\begin{aligned}&Z_2^{-1}Z_\varphi\int\frac{d^Dl}{(2\pi)^D}\langle\varphi(l)\varphi(k-l)\varphi(p)\varphi(q)\rangle\\&=(2\pi)^D\delta^D(k+p+q)\frac{2}{(p^2+m^2)(q^2+m^2)}Z_2^{-1}Z_\varphi\left\{1+\frac{\lambda}{(4\pi)^2}\frac{1}{D-4}\right\}.\end{aligned}$$

The four-point correlation function is calculated as in Fig. D-2, where a double line represents the composite field. In order for poles to cancel out so that the right-hand side becomes finite, with attention to $Z_\varphi = 1+o(\lambda^2)$, the renormalization factor Z_2 has to be

$$Z_2 = 1+\frac{\lambda}{(4\pi)^2}\frac{1}{D-4}+o(\lambda^2).$$

In this way, Z_2 can be obtained for each order.

On the other hand, we can determine Z_2 without doing such calculations in the case of the $\lambda\varphi^4$-theory as follows. First, note that

$$m^2\frac{\partial F}{\partial m^2} = m^2\frac{\partial\lambda_0}{\partial m^2}\frac{\partial F}{\partial\lambda_0}+m^2\frac{\partial m_0^2}{\partial m^2}\frac{\partial F}{\partial m_0^2} = m_0^2\frac{\partial F}{\partial m_0^2}$$

holds for any function F, where $\partial\lambda_0/\partial m^2 = 0$ and $\partial m_0^2/\partial m^2 = m_0^2/m^2$ are used. Therefore, considering the finite correlation function $\langle\prod\varphi\rangle$ as F, we get

$$m^2\frac{\partial}{\partial m^2}\left\langle\prod\varphi\right\rangle = m_0^2\frac{\partial}{\partial m_0^2}\left\langle\prod\varphi\right\rangle = \int d^Dx\left\langle -\frac{1}{2}m_0^2\,\varphi_0^2(x)\prod\varphi\right\rangle.$$

Since the left-hand side is obviously a finite quantity, the right-hand side is also finite. This indicates that $m_0^2\varphi_0^2$ is a finite operator. In other words, noting that $m_0^2\varphi_0^2$ can be generally expressed as $m^2[\varphi^2](1+\sum \mathrm{poles})$, the finiteness shows that the poles strictly disappear. Thus,

$$m_0^2\,\varphi_0^2 = m^2[\varphi^2]$$

holds. From this relation, Z_2 can be determined as

$$Z_2 = \frac{m^2}{m_0^2} = Z_m^{-1}.$$

Therefore, we find that the anomalous dimension of the composite field $[\varphi^2]$ is given by

$$\delta = -\gamma_m = \frac{\lambda}{(4\pi)^2} - \frac{5}{6}\frac{\lambda^2}{(4\pi)^4} + o(\lambda^3).$$

Normal product $[\varphi^4]$ Next consider a more complicated normal product $[\varphi^4]$. In this case, unlike $[\varphi^2]$, we have to consider mixing with other operators with the same dimensions.

One of the field operators to be mixed is the equation-of-motion field, which is defined by

$$E_{0\varphi} \equiv \varphi_0 \frac{\delta S}{\delta\varphi_0} = \varphi_0 \left\{ \left(-\partial^2 + m_0^2\right)\varphi_0 + \frac{\lambda_0}{3!}\varphi_0^3 \right\}.$$

Inserting $E_{0\varphi}$ into the correlation function yields

$$\begin{aligned}\left\langle E_{0\varphi}(x)\prod_{j=1}^{N}\varphi(x_j)\right\rangle &= -Z_\varphi^{-\frac{N}{2}}\int d\varphi_0 \prod_{j=1}^{N}\varphi_0(x_j)\varphi_0(x)\frac{\delta}{\delta\varphi_0(x)}e^{-S}\\ &= \sum_{j=1}^{N}\delta^D(x-x_j)\left\langle \prod_{j=1}^{N}\varphi(x_j)\right\rangle.\end{aligned}$$

The first equality can be found immediately by assigning the definition of $E_{0\varphi}$ to the correlation function (D-2). The second equality is derived by moving $\delta/\delta\varphi_0$ by performing partial integration and using $\delta\varphi_0(x_j)/\delta\varphi_0(x)$ $= \delta^D(x-x_j)$. In doing so, note that as a significant feature of dimensional regularization, contributions from the derivative at the same point disappear such as $\delta^D(0) = \int d^Dk = 0$. As mentioned in Chapter 9, this property indicates that dimensional regularization does not depend on how to choose the measure of the path integral. Since the right-hand side of the above equation is a finite quantity obviously, the left-hand side is also finite. Therefore, $E_{0\varphi}$ is one of normal products and can be denoted as

$$E_{0\varphi} = [E_\varphi]\,.$$

Moreover, we can see that its volume integral $\int d^D x\,[E_\varphi]$ becomes the number of the fields, N, in the correlation function.

The normal product $[\varphi^4]$ generally has the following structure in consideration of mixing with other possible operators:

$$\frac{\mu^{4-D}}{4!}[\varphi^4] = \left(1 + \sum \text{poles}\right)\frac{1}{4!}\varphi_0^4 + \sum \text{poles}\ \frac{1}{2}m_0^2\varphi_0^2 + \sum \text{poles}\ E_{0\varphi} + \sum \text{poles}\ \partial^2\varphi_0^2. \tag{D-4}$$

In order to determine the unknown pole parts, consider a quantity obtained by differentiating the correlation function by λ below. From renormalization group equations $\mu d\lambda_0/d\mu = (4-D)\lambda_0 + \beta_\lambda \partial\lambda_0/\partial\lambda = 0$ and $\mu dm_0^2/d\mu = \beta_\lambda \partial m_0^2/\partial\lambda - \gamma_m m^2 \partial m_0^2/\partial m^2 = 0$, we find that quantities obtained by differentiating the bare constants by λ are

$$\frac{\partial\lambda_0}{\partial\lambda} = \frac{D-4}{\beta_\lambda}\lambda_0, \qquad \frac{\partial m_0^2}{\partial\lambda} = \frac{\gamma_m}{\beta_\lambda}m_0^2.$$

From these, the derivative of the action S by λ is given by

$$\frac{\partial S}{\partial\lambda} = \int d^D x \left\{\frac{\gamma_m}{\beta_\lambda}\frac{1}{2}m_0^2\varphi_0^2 + \frac{D-4}{\beta_\lambda}\frac{1}{4!}\lambda_0\varphi_0^4\right\}.$$

Here, since the bare field φ_0 is an integral variable, the differentiation by λ passes through it. With this and $\partial(\log Z_\varphi^{1/2})/\partial\lambda = \gamma/\beta_\lambda$, we can calculate $\partial(\langle\prod_{j=1}^N \varphi(x_j)\rangle)/\partial\lambda$ as

$$\left\langle -\int d^D x \left\{\frac{D-4}{\beta_\lambda}\frac{1}{4!}\lambda_0\varphi_0^4 + \frac{\gamma_m}{\beta_\lambda}\frac{1}{2}m_0^2\varphi_0^2 + \frac{\gamma}{\beta_\lambda}E_{0\varphi}\right\}\prod_{j=1}^N \varphi(x_j)\right\rangle.$$

The volume integral of the last $E_{0\varphi}$ is a replacement of N. Since the above quantity is finite, the inside the braces is also a finite quantity up to derivative terms that vanish in the volume integral. Noting that

$$\frac{D-4}{\beta_\lambda} = \frac{1}{\lambda}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(D-4)^n}\left(\frac{\bar\beta_\lambda}{\lambda}\right)^n\right], \tag{D-5}$$

we can see that the inside the braces has the structure (D-4) of the normal product $[\varphi^4]$. Therefore, identifying it with $\mu^{4-D}[\varphi^4]/4!$ except a total divergence term, we obtain

$$\frac{\mu^{4-D}}{4!}[\varphi^4] = \frac{D-4}{\beta_\lambda}\frac{1}{4!}\lambda_0\varphi_0^4 + \frac{\gamma_m}{\beta_\lambda}\frac{1}{2}m_0^2\varphi_0^2 + \frac{\gamma}{\beta_\lambda}E_{0\varphi} - \frac{Z_2^{-1}d}{\beta_\lambda}\,\partial^2\varphi_0^2,$$

where d in the last term is a function including poles, and Z_2^{-1} is introduced for later convenience. This d is a function not determined by the above method.

Differentiating the normal product $[\varphi^4]$ by μ, we get

$$\mu\frac{d}{d\mu}[\varphi^4] = -\frac{\partial\bar{\beta}_\lambda}{\partial\lambda}[\varphi^4] + 4!\mu^{D-4}\left\{\frac{\partial\gamma_m}{\partial\lambda}\frac{1}{2}m^2[\varphi^2]\right.$$
$$\left.+\frac{\partial\gamma}{\partial\lambda}[E_\varphi] - \frac{\partial(Z_2^{-1}d)}{\partial\lambda}Z_2\,\partial^2[\varphi^2]\right\}, \qquad \text{(D-6)}$$

where $\mu d(1/\beta_\lambda)/d\mu = -(1/\beta_\lambda)(\partial\beta_\lambda/\partial\lambda)$, $\mu d\gamma_m/d\mu = \beta_\lambda(\partial\gamma_m/\partial\lambda)$, and $\mu d\gamma/d\mu = \beta_\lambda(\partial\gamma/\partial\lambda)$ are used. For the right-hand side to be finite, the coefficient before the last $\partial^2[\varphi^2]$ must be finite. If it is expressed as $Z_2\partial(Z_2^{-1}d)/\partial\lambda = \zeta/\lambda$, d can be written as

$$d = Z_2\int_0^\lambda \frac{d\lambda}{\lambda}\frac{\zeta}{Z_2}.$$

Noting also that $\partial(\log Z_2)/\partial\lambda = \delta/\beta_\lambda = -\gamma_m/\beta_\lambda$, we find that d satisfies a differential equation $(\partial/\partial\lambda + \gamma_m/\beta_\lambda)d = \zeta/\lambda$. The last remaining ζ can only be determined by directly computing correlation functions including $[\varphi^4]$.

DeWitt-Schwinger Method

The DeWitt-Schwinger method is a widely known technique used when calculating one-loop effective actions in curved spacetime.[7] It is briefly described here using the kinetic term of a scalar field in curved spacetime as an example. Consider Euclidean space, and let the action be $I = (1/2)\int d^dx\sqrt{g}\varphi K\varphi$, where the dimension d is an even number and the differential operator is defined by

$$K = -\nabla^2 + \xi R.$$

The conformal coupling is given by $\xi = 0$ in two dimensions and $\xi = 1/6$ in four dimensions.

[7] In addition to books of quantum field theory in curved spacetime listed in Bibliography, see also, in two dimensions, O. Alvarez, *Theory of Strings with Boundries: Fluctuations, Topology and Quantum Geometry*, Nucl. Phys. B216 (1983) 125. For studies containing higher-order derivative terms, see A. Barvinsky and G. Vilkovisky, *The Generalized Schwinger-DeWitt Technique in Gauge Theories and Quantum Gravity*, Phys. Rep. 119 (1985) 1.

The effective action is given by

$$\Gamma = -\log \int [d\varphi]_g e^{-I} = -\log (\det K)^{-1/2} .$$

This can be rewritten in an integral representation as

$$-\log (\det K) = \int_{\varepsilon}^{\infty} \frac{ds}{s} \operatorname{Tr}\left(e^{-sK}\right),$$

where ε is a ultraviolet cutoff, and $\operatorname{Tr}(A) = \int d^d x \sqrt{g} \langle x|A|x\rangle$, $\langle x|y\rangle = \delta^d(x-y)/\sqrt{g}$. Therefore, a conformal variation of the effective action is given by

$$\delta_\omega \Gamma = \frac{1}{2} \int_{\varepsilon}^{\infty} ds \operatorname{Tr}\left(\delta_\omega K e^{-sK}\right).$$

The conformal variation of the differential operator K can be written as

$$\delta_\omega K = -2\omega K + \delta L,$$

where

$$\delta L = -(d-2)\nabla^\mu \omega \nabla_\mu - 2\xi(d-1)\nabla^2 \omega.$$

Using this expression, we obtain

$$\begin{aligned} \delta_\omega \Gamma &= \int_{\varepsilon}^{\infty} ds \frac{\partial}{\partial s} \left[\operatorname{Tr}\left(\omega\, e^{-sK}\right)\right] + \frac{1}{2} \int_{\varepsilon}^{\infty} ds \operatorname{Tr}\left(\delta L\, e^{-sK}\right) \\ &= -\operatorname{Tr}\left(\omega\, e^{-\varepsilon K}\right) + \frac{1}{2} \operatorname{Tr}\left(\delta L \frac{1}{K}\right). \end{aligned} \tag{D-7}$$

The first term of (D-7) can be written as $-\int d^d x \sqrt{g}\omega \langle x|e^{-\varepsilon K}|x\rangle$. The integrand is obtained from $G^{(s)}(x,y) = \langle x|e^{-sK}|y\rangle$ called the heat kernel, which is a solution of the following heat conduction equation:

$$\left(\frac{\partial}{\partial s} + K\right) G^{(s)}(x,y) = 0,$$

which satisfies an initial condition $\lim_{s\to 0} G^{(s)}(x,y) = \delta^d(x-y)/\sqrt{g}$. From this, the DeWitt-Schwinger method is also called the heat kernel method. Assuming that the parameter s is small, the heat kernel at the same point is expanded as

$$G^{(s)}(x,x) = \frac{1}{(4\pi)^{d/2}} \frac{1}{s^{d/2}} \left(1 + s a_1 + s^2 a_2 + \cdots\right).$$

The contribution from the second term of (D-7) disappears when $\delta L = 0$ in the case of the conformal coupling in two dimensions. On the other hand, when the conformal coupling in four-dimensions ($\xi = 1/6$), terms including differentiations of the two-point function $\langle x|K^{-1}|y\rangle$ at the same point appear, such as $(1/2)\int d^4x\sqrt{g}\omega\nabla^2\langle x|K^{-1}|x\rangle$. It seems that this term is in the form $\nabla^2 R$, but its regularization is difficult, and it will remain arbitrary in this method.

First, we find the coefficient a_1 of the the heat kernel when $d = 2$ and $\xi = 0$. In two dimensions, since the traceless tensor field can be eliminated by using gauge degrees of freedom, we can consider only the conformal-factor field ϕ, and thus the differential operator can be written as $K = -\nabla^2 = -e^{-2\phi}\partial^2$. Let us take coordinates that are locally flat at the origin and put $\phi(0) = \partial_a\phi(0) = 0$ $(a = 1, 2)$. Expand the differential operator around the origin and decompose it into the flat space Laplacian operator and a potential term $V = -x^a x^b \partial_a\partial_b\phi(0)\partial^2 + o(x^3)$ so that $K = -\partial^2 - V$. The heat kernel in the flat space is given by

$$G_0^{(s)}(x, y) = \frac{1}{4\pi s} e^{-(x-y)^2/4s},$$

and the heat conduction equation is solved by the integral equation $G = G_0 + G_0 V G$. Solving it sequentially, the first two terms of $G^{(s)}(0, 0)$ are given as follows:

$$\begin{aligned}
&G_0^{(s)}(0,0) + \int d^2x \int_0^s ds'\, G_0^{(s-s')}(0,x)V(x)G_0^{(s')}(x,0) \\
&= \frac{1}{4\pi s} - \partial_a\partial_b\phi(0)\int_0^s ds' \int d^2x\, \frac{e^{-x^2/4(s-s')}}{4\pi(s-s')} x^a x^b \partial^2 \frac{e^{-x^2/4s'}}{4\pi s'} \\
&= \frac{1}{4\pi s} - \frac{1}{12\pi}\partial^2\phi(0).
\end{aligned}$$

Here, the volume integral of the coordinates x^a is concentrated on the origin at $s \to 0$ due to the structure of the heat kernel. Evaluating the volume integral and the s'-integral produces a finite value independent of s. Since the curvature at the origin can be written as $R(0) = -2\partial^2\phi(0)$, the heat kernel coefficient a_1 in the case of the two-dimensional conformal coupling is given by

$$a_1 = \frac{1}{6}R.$$

Because of $\delta_\omega\Gamma = -\int d^2x\sqrt{g}\,\omega a_1/4\pi$, we find that the coefficient of the conformal anomaly defined in (5-1), namely, the central charge of a scalar field is given by $c = 1$ ($b_{\rm L} = -1/6$). Note here that the sign of the whole action is reversed in Euclidean space.

Similar calculations in four dimensions are difficult because it is impossible to classify necessary terms merely by the dependence of the conformal-factor field. Here we only present the results of a_1 and a_2 calculated with any ξ, which are given by

$$a_1 = \left(\frac{1}{6} - \xi\right) R,$$
$$a_2 = \frac{1}{180} R^2_{\mu\nu\lambda\sigma} - \frac{1}{180} R^2_{\mu\nu} + \frac{1}{2}\left(\frac{1}{6} - \xi\right) R^2 + \frac{1}{6}\left(\frac{1}{5} - \xi\right) \nabla^2 R.$$

The finite part of the first term in (D-7) at $d = 4$ is given by $\delta_\omega \Gamma = -\int d^4x \sqrt{g}\, \omega a_2/(4\pi)^2$, and thus the conformal anomaly coefficients (5-6) derived from a scalar field with the conformal coupling $\xi = 1/6$ are found to be $\zeta_1 = 1/120$ and ζ_2 ($= -b_c$) $= -1/360$. On the other hand, the derivative term ζ_3 cannot be determined by this method because of the unknown part already mentioned.

Dynamical Triangulation Method and Quantum Gravity

The relationship between a simplicial gravity based on dynamical triangulation and quantum gravity is briefly mentioned here. The former method is defined by summing over all possible lattice manifolds made by bonding simplices.

In two dimensions, we consider random surfaces made by gluing polygons (mainly triangles and quadrangles). Let the number of polygons be N_2 and denote possible ways of the gluing as $\mathcal{T}_2$. Then we examine the following partition function:

$$Z_{\rm SG2}(\lambda) = \sum_{N_2=0}^{\infty} \sum_{\mathcal{T}_2} e^{-\lambda N_2} = \sum_{N_2=0}^{\infty} \Omega(N_2) e^{-\lambda N_2},$$

where $\Omega(N_2)$ is the partition number of possible surfaces with N_2 polygons. This is known as a statistical model in which a second-order phase transition occurs. Denoting the critical point by $\lambda = \lambda^c$, the partition number behaves in the large N_2 as

$$\Omega(N_2) \sim N_2^{\gamma_{\rm st}^{(2)} - 3} e^{\lambda^c N_2},$$

where $\gamma_{\rm st}^{(2)}$ is a constant called the string susceptibility. Assuming that an area of each polygon is one, N_2 is an area of the surface made by gluing, and therefore λ acts as a cosmological constant to control it.

Introducing a scale a corresponding to lattice spacings, two-dimensional quantum gravity is then derived by taking a continuum limit that $N_2 \to \infty$ at the same time as $a \to 0$ while retaining $(\lambda - \lambda^c)/a^2 \to \mu$ at the phase transition point, where μ is a physical cosmological constant.

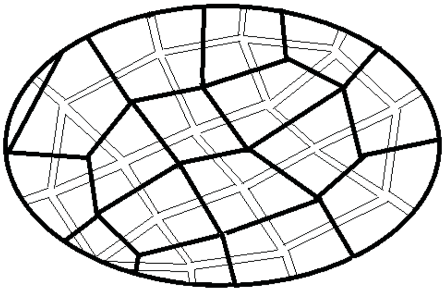

Figure D-3: A random surface constructed by gluing quadrangles. Its dual is given by a vacuum Feynman diagram of the matrix model with a four-point interaction, where a double line denotes a propagator of the matrix.

In two dimensions, the above partition function can be analytically calculated using matrix models. Consider, for example, a zero-dimensional path integral over a $N \times N$ Hermitian matrix M with a potential term $g\mathrm{tr}(M^4)$ as well as a kinetic term $\mathrm{tr}(M^2)/2$,

$$F_N(g) = \frac{1}{N^2} \log \int d^{N^2}M \, e^{-\mathrm{tr}\left(\frac{1}{2}M^2 + gM^4\right)}.$$

Each four-point interaction in Feynman diagrams corresponds to a quadrangle, and a vacuum diagram with N_2 interactions is identified with a compact random surface with area N_2 (see Fig. D-3). Various vacuum diagrams represent how quadrangles are connected, that is $\mathcal{T}_2$.[8] Topologies are classified

[8] An odd-point interaction $g\mathrm{tr}(M^3)$ which corresponds to a triangle can be considered as well. Although it is generally ill-defined because it is not bounded below, the path integral can be actually defined at $N \to \infty$ for the reason mentioned soon below. At the critical point, it belongs to the same universality class as the four-point interaction.

with $1/N^2$ expansion, and the lowest is planar Feynman diagrams that give random surfaces of S^2 we consider here.

Since the path integral is zero-dimensional, it can be evaluated exactly and vacuum Feynman diagrams can be summed over at all orders. The second-order phase transition point g_c is defined as a point where analyticity of g is broken. In general, for finite N, it is analytic for $g \geq 0$ where the potential term is bounded from below. However, taking the limit $N \to \infty$, contributions from the $N \times N$ matrix integration measure dominate, so that the analyticity extends to the negative region and the breaking point appears at a value $g_c = -e^{\lambda^c}$.[9] This negativity is essential when identifying $\lim_{N\to\infty} F_N(g = -e^{\lambda})$ with the simplicial gravity partition function above, and its continuum limit can be taken at g_c.[10]

On the other hand, in two-dimensional quantum gravity, we can calculate the behavior of the partition function in a large surface area by focusing the zero-mode ϕ_0 of the Liouville field. The ϕ_0-dependence of the Liouville action except for the cosmological term is given by $S_{\mathrm{L}} = b_{\mathrm{L}}\chi\phi_0$, and when fixed the value of $v_2 = e^{\alpha\phi_0}$, the partition function has the following v_2-dependence:

$$\int d\phi_0 \, e^{-b_{\mathrm{L}}\chi\phi_0}\delta(e^{\alpha\phi_0} - v_2) \sim v_2^{-1-b_{\mathrm{L}}\chi/\alpha},$$

where $\alpha = \gamma_0$ is the Liouville charge (6-13) with $\Delta = 0$ and χ is the Euler characteristic. Noting that $v_2 \sim N_2$, we can see that the string susceptibility is expressed as $\gamma_{\mathrm{st}}^{(2)} = 2 - b_{\mathrm{L}}\chi/\alpha$. Considering $\chi = 2$ of spherical topology and letting the central charge of CFT coupled with gravity be c_{M} yields

$$\gamma_{\mathrm{st}}^{(2)} = 2 + \frac{1}{12}\left\{c_{\mathrm{M}} - 25 - \sqrt{(25 - c_{\mathrm{M}})(1 - c_{\mathrm{M}})}\right\}.$$

This is consistent with the analytic solution of the matrix model. Here, matrix models with the central charge like the Ising model are constructed

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[9] The path integral can be rewritten in terms of N real eigenvalues λ_i of M. The measure is then expressed as $\prod_i d\lambda_i \Delta^2(\lambda)$, where $\Delta(\lambda) = \prod_{1\leq i<j\leq N}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. Exponentiating the determinant factor, logarithmic interactions are generated. See E. Brezin, C. Itzykson, G. Parisi, and J. Zuber, *Planar Diagrams*, Commun. Math. Phys. **59** (1978) 35; D. Bessis, C. Itzykson, and J. Zuber, *Quantum Field Theory Techniques in Graphical Enumeration*, Adv. Appl. Math. **1** (1980) 109.

[10] See Footnote 9. For the continuum limit, see E. Brezin and V. Kazakov, *Exactly Solvable Field Theories of Closed Strings*, Phys. Lett. **B236** (1990) 144; M. Dougls and S. Shenker, *Strings in Less Than One Dimension*, Nucl. Phys. **B335** (1990) 635; D. Gross and A. Migdal, *Nonperturbative Two-Dimensional Quantum Gravity*, Phys. Rev. Lett. **64** (1990) 127.

by increasing the number of matrices to introduce spin degrees of freedom on random lattices and considering couplings between them.[11]

This method can be generalized to any integer dimension. In four dimensions, we will make a four-dimensional manifold made by bonding 4-simplices. Unlike the two-dimensional case, it is difficult to draw it in diagrams, but it is possible algebraically. Let N_4 be the number of 4-simplices representing the total volume and N_2 be the number of 2-simplices contained therein, and denote possible ways of the bonding of N_4 4-simplices as $\mathcal{T}_4$. The partition function is then defined by

$$Z_{\mathrm{SG4}}(\kappa_2,\kappa_4)=\sum_{N_4}\sum_{\mathcal{T}_4}e^{\kappa_2 N_2-\kappa_4 N_4}=\sum_{N_4}\Omega(\kappa_2,N_4)e^{-\kappa_4 N_4},$$

where κ_2 corresponds to the square of the Planck mass and κ_4 is the cosmological constant. Assuming that the second-order phase transition occurs at $\kappa_4=\kappa_4^c(\kappa_2)$, the partition number will behave as

$$\Omega(\kappa_2,N_4)\sim N_4^{\gamma_{\mathrm{st}}^{(4)}-3}e^{\kappa_4^c(\kappa_2)N_4},$$

where $\gamma_{\mathrm{st}}^{(4)}$ is a four-dimensional version of the string susceptibility. Although the existence of the second-order phase transition point is non-trivial in four dimensions, its existence is numerically suggested in a system in which matter fields are added (see Fig. D-4).[12]

A prediction from the asymptotically background-free quantum gravity in four dimensions can be derived from the behavior of the zero-mode ϕ_0 of the conformal-factor field as in the case of two dimensions. The zero-mode dependence of the Euclidean action at the ultraviolet limit where the coupling constant t disappears is given by $S_{\mathrm{4DQG}}=2b_c\chi\phi_0-\kappa_2 e^{\beta\phi_0}\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar part other than the zero-mode and β is the Riegert charge (7-36) of its operator. Therefore, the $v_4=e^{\alpha\phi_0}$ dependency of the partition function is found as follows:

$$\begin{aligned}&\int d\phi_0\exp\left(-2b_c\chi\phi_0+\kappa_2 e^{\beta\phi_0}\mathcal{R}\right)\delta(e^{\alpha\phi_0}-v_4)\\&\sim v_4^{-1-2b_c\chi/\alpha}f(\kappa_2 v_4^{\beta/\alpha}),\end{aligned}$$

[11] For the Ising model, see V. Kazakov, *Ising Model on a Dynamical Planar Random Lattice: Exact Solution*, Phys. Lett. **A119** (1986) 140; D. Boulatov and V. Kazakov, *The Ising Model on Random Planar Lattice*, Phys. Lett. **B186** (1987) 379.

[12] See S. Horata, H. Egawa, and T. Yukawa, *Grand Canonical Simulation of 4D Simplicial Quantum Gravity* , Nucl. Phys. B (Proc. Suppl.) **119** (2003) 921.

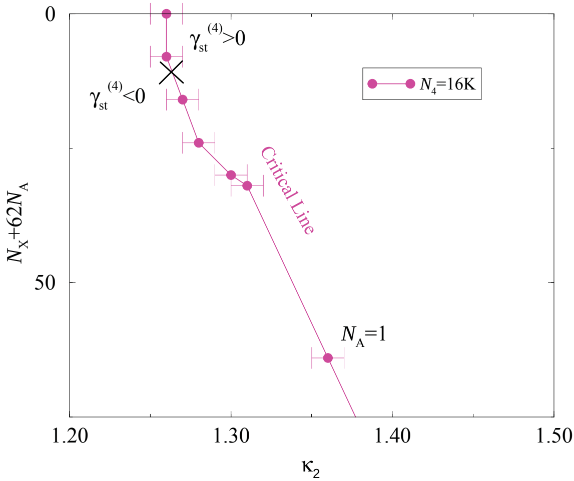

Figure D-4: The phase transition point κ_4^c and its κ_2-dependence in numerical calculations when the number of matter fields is $N_X + 62N_A$. Below the X mark, the susceptibility becomes negative and a second-order phase transition point appears. [K. Hamada, S. Horata, and T. Yukawa, *Focus on Quantum Gravity Research* (Nova Science Publisher, NY, 2006), Chap. 1.]

where α is the Riegert charge (7-35) of the cosmological term operator. From this, the susceptibility is expressed as $\gamma_{\rm st}^{(4)} = 2 - 2b_c\chi/\alpha$. Considering topology of four-sphere S^4 with $\chi = 2$ yields

$$\gamma_{\rm st}^{(4)} = 2 - \frac{1}{2}\left(b_c + \sqrt{b_c^2 - 4b_c}\right). \qquad \text{(D-8)}$$

In Fig. D-5, the numerical result for $\gamma_{\rm st}^{(4)}$ that has been calculated with sixteen thousand 4-simplices is compared with the prediction from the continuum theory. The values of b_1's shown in the figure are calculated using the measured $\gamma_{\rm st}^{(4)}$'s through the expression (D-8) in which b_c is replaced with b_1. It indicates that the results are in good agreement. Since the value of b_c in the continuum theory (7-5) is given by $b_c = 0.00278(N_X + 62N_A) + 4.27$, the b_1 can be thought of as b_c with a small negative correction of $o(t^2)$ added, where N_X and N_A are the numbers of scalar and gauge fields, respectively. The point to note here is that the matter field dependence appears correctly with the non-trivial combination of $N_X + 62N_A$ predicted from the conformal anomaly.

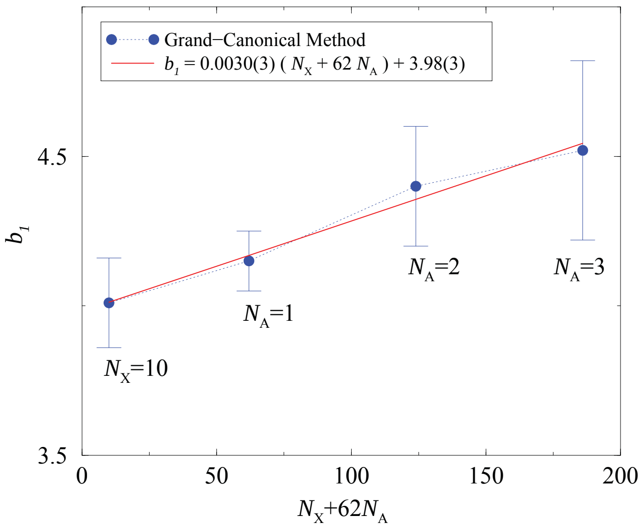

Figure D-5: Numerical results of the susceptibility $\gamma_{\rm st}^{(4)}$. The values of b_1's are calculated using the measured $\gamma_{\rm st}^{(4)}$'s through the expression (D-8) predicted from the continuum theory in which b_c is replaced with b_1. The solid line is the best fit of b_1 as a function of $N_X + 62N_A$. [K. Hamada, S. Horata, and T. Yukawa, *Focus on Quantum Gravity Research* (Nova Science Publisher, NY, 2006), Chap. 1.]

Appendix E

Addenda to Cosmology

Sachs-Wolfe Effect

Calculate an energy shift by gravity of light which is emitted from sender i and observed by receiver f. Electromagnetic waves are expressed as $F_{\mu\nu} = \mathrm{Re}(A_{\mu\nu} e^{i\psi})$, in which the amplitude $A_{\mu\nu}$ changes slowly compared to the phase part $e^{i\psi}$. If waves under consideration have a wavelength shorter than length of gravitational changes, we can do such a separation approximately. In this case, moving of planes with a constant phase describes the propagation of light.

Since the geodesic of light is conformally invariant, the square of the scale factor $a = e^{\hat{\phi}}$ is factored out and the line element is expressed by $ds^2 = a^2 d\sigma^2$ as in Chapter 13. Linear perturbations are here expressed as[1]

$$d\sigma^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + \mathcal{H}_{\mu\nu})\, dx^\mu dx^\nu.$$

Letting λ be an affine parameter for the line element $d\sigma^2$, a null vector in

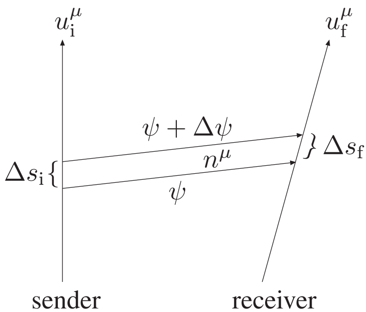

Figure E-1: Light path from sender i to receiver f.

[1] Unlike the metric $\bar{g}_{\mu\nu}$ for the traceless tensor field, $\mathcal{G}_{\mu\nu}$ includes the perturbation φ of the conformal-factor field as well ($g_{\mu\nu} = a^2 \mathcal{G}_{\mu\nu}$).

the propagation direction and its geodesic equation are given by

$$n^\mu = \frac{dx^\mu}{d\lambda}, \qquad \mathcal{G}_{\mu\nu} n^\mu n^\nu = 0,$$
$$\frac{dn^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}(\mathcal{G}) n^\alpha n^\beta = 0.$$

Then, planes with a constant phase can be described as

$$\psi\left(x^\mu + n^\mu \Delta\lambda\right) = \psi\left(x^\mu\right) \quad \Rightarrow \quad n^\mu \frac{\partial \psi}{\partial x^\mu} = 0.$$

Moreover, from the null condition, we can describe it as

$$\frac{\partial \psi}{\partial x^\mu} = K \mathcal{G}_{\mu\nu} n^\nu$$

where K is a proportionality constant.

Consider another plane with a constant phase that is transmitted with a slightly delay as shown in Fig. E-1. Writing the proper time as s, a phase difference is expressed as

$$\Delta\psi = \frac{\partial \psi}{\partial x^\mu} \frac{dx^\mu}{ds} \Delta s = \frac{\partial \psi}{\partial x^\mu} u^\mu \Delta s,$$

where $u^\mu = dx^\mu/ds$ is a four-velocity of the sender or receiver satisfying $g_{\mu\nu} u^\mu u^\nu = -1$ (13-1). Therefore, denoting a frequency of light as ν, an energy shift between the sender and the receiver is given by

$$\frac{E_\mathrm{f}}{E_\mathrm{i}} = \frac{\nu_\mathrm{f}}{\nu_\mathrm{i}} = \frac{\frac{\Delta\psi}{\Delta s_\mathrm{f}}}{\frac{\Delta\psi}{\Delta s_\mathrm{i}}} = \frac{(\mathcal{G}_{\mu\nu} n^\mu u^\nu)_\mathrm{f}}{(\mathcal{G}_{\mu\nu} n^\mu u^\nu)_\mathrm{i}}. \tag{E-1}$$

In the absence of the perturbation, since the null vector is $n^0 = 1$, $n^i n_i = 1$ and the four-velocity is $u^\mu = (1/a, 0, 0, 0)$, this results in $E_\mathrm{f}/E_\mathrm{i} = a_\mathrm{i}/a_\mathrm{f}$ that represents the red shift.

Let us calculate the energy shift in the presence of the linear perturbations. From the geodesic, writing the null vector containing the perturbation as $n^\mu = (1, \mathbf{n}) + \delta n^\mu$ yields

$$\frac{d\delta n^\mu}{d\lambda} = \left(-\partial_{(\alpha} \mathcal{H}^\mu_{\beta)} + \frac{1}{2} \partial^\mu \mathcal{H}_{\alpha\beta}\right) n^\alpha n^\beta.$$

The first term of the right-hand side can be easily integrated using

$$\frac{d(\mathcal{H}^\mu{}_\beta n^\beta)}{d\lambda} = \frac{d\mathcal{H}^\mu{}_\beta}{d\lambda} n^\beta = \frac{dx^\alpha}{d\lambda} \partial_\alpha \mathcal{H}^\mu{}_\beta n^\beta = (\partial_\alpha \mathcal{H}^\mu{}_\beta) n^\alpha n^\beta,$$

which holds within the linear approximation. Then, we get the following expression:

$$\delta n^\mu\big|_{\rm i}^{\rm f} = -\mathcal{H}^\mu{}_\beta\, n^\beta\Big|_{\rm i}^{\rm f} + \frac{1}{2}\int_{\rm i}^{\rm f} d\lambda\,(\partial^\mu \mathcal{H}_{\alpha\beta})\, n^\alpha n^\beta. \tag{E-2}$$

Letting the current background temperature be T_0 and writing an observable temperature as $T = T_0/a + \delta T$, the following relation holds:

$$\frac{T_{\rm f}}{T_{\rm i}} = \frac{a_{\rm i}}{a_{\rm f}}\left(1 + \frac{\delta T_{\rm f}}{T_{\rm f}} - \frac{\delta T_{\rm i}}{T_{\rm i}}\right) = \frac{a_{\rm i}}{a_{\rm f}}\left(1 + \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}\bigg|_{\rm i}^{\rm f}\right), \tag{E-3}$$

where $\rho_\gamma \propto T^4$ is used in the last equality. Assigning the 0-component of (E-2) and the relation (E-3) into the energy shift equation (E-1), we obtain

$$\begin{aligned}
\frac{E_{\rm f}}{E_{\rm i}} &= \frac{a_{\rm i}}{a_{\rm f}}\left\{1 + \left[\varphi - \frac{1}{2}h_{00} - (v_i + h_{0i})\, n^i + \delta n^0\right]_{\rm i}^{\rm f}\right\} \\
&= \frac{T_{\rm f}}{T_{\rm i}}\left\{1 - \left[\frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} - \varphi + \frac{1}{2}h_{00} + (v_i + h_{0i})\, n^i - \delta n^0\right]_{\rm i}^{\rm f}\right\} \\
&= \frac{T_{\rm f}}{T_{\rm i}}\Bigg\{1 - \left[\frac{1}{4}\mathcal{D}^\gamma + \partial_i V^b n^i + \Psi - \Phi + \Omega^b_i n^i\right]_{\rm i}^{\rm f} \\
&\qquad + \int_{\rm i}^{\rm f} d\lambda\left(\partial_\eta\Psi - \partial_\eta\Phi + \partial_\eta\Upsilon_i n^i - \partial_\eta h^{\rm TT}_{ij} n^i n^j\right)\Bigg\},
\end{aligned}$$

where V^b is a velocity of baryons as the sender or receiver. Because of $E_{\rm f}/E_{\rm i} = T_{\rm f}/T_{\rm i}$, the inside the braces becomes unity. Especially in a scalar component, if the Bardeen potential does not change with time, the quantity in the square brackets is preserved and its initial and the final values become the same.

Let the initial value set at the decoupling time $\rm i = \eta_{dec}$, and the final value is taken at the present $\rm f = \eta_0$. The temperature fluctuation of CMB is given by $(\Delta T/T)(\eta_0) = (1/4)\mathcal{D}^\gamma(\eta_0)$, except for $\Psi(\eta_0)$ and $n^i\partial_i V^b(\eta_0)$ contributing to monopole and dipole components which are excluded in observations.[2] Decomposing it into contributions from scalar, vector, and ten-

[2] A relative velocity of the observer to CMB gives a dipole. Since power spectra are considered in a rest frame of the CMB in which the dipole component is removed, intrinsic anisotropies are represented by multipoles with $l \geq 2$.

sor fluctuations, it can be described as follows:

$$\begin{aligned}
\frac{\Delta T}{T}(\eta_0, \mathbf{x}_0) &= \left(\frac{\Delta T}{T}\right)^S + \left(\frac{\Delta T}{T}\right)^V + \left(\frac{\Delta T}{T}\right)^T , \\
\left(\frac{\Delta T}{T}\right)^S &= \left\{\frac{1}{4}\mathcal{D}^\gamma + \partial_i V^b n^i + \Psi - \Phi\right\}(\eta_{\mathrm{dec}}, \mathbf{x}_{\mathrm{dec}}) \\
&\quad + \int_{\eta_{\mathrm{dec}}}^{\eta_0} d\eta \left(\partial_\eta \Psi - \partial_\eta \Phi\right)(\eta, \mathbf{x}(\eta)) , \\
\left(\frac{\Delta T}{T}\right)^V &= \Omega_i^b\left(\eta_{\mathrm{dec}}, \mathbf{x}_{\mathrm{dec}}\right) n^i + \int_{\eta_{\mathrm{dec}}}^{\eta_0} d\eta \, \partial_\eta \Upsilon_i\left(\eta, \mathbf{x}(\eta)\right) n^i , \\
\left(\frac{\Delta T}{T}\right)^T &= -\int_{\eta_{\mathrm{dec}}}^{\eta_0} d\eta \, \partial_\eta h_{ij}^{\mathrm{TT}}\left(\eta, \mathbf{x}(\eta)\right) n^i n^j ,
\end{aligned} \tag{E-4}$$

where the path of light is given by $\mathbf{x}(\eta) = \mathbf{x}_0 + (\eta - \eta_0)\mathbf{n}$. This relation is called the Sachs-Wolfe effect.

Finally, we write a famous formula for scalar fluctuations that holds only for large angle components. The multipole component of $l < 10$ represents a large size fluctuation that does not enter the horizon until today. The solution to such a fluctuation is given by the super-horizon limit $x\,(= k\eta) \ll 1$ of (13-30). That is, $\mathcal{D}^\gamma = (-20/3)\Psi_{\mathrm{I}}$ and $V^\gamma = (1/3)x\Psi_{\mathrm{I}} \simeq 0$, where Ψ_{I} is an initial value given after the big bang. Since at the decoupling time still $V^b \simeq V^\gamma$ and the Bardeen potential $\Psi(= -\Phi)$ is almost constant such as $\Psi(\eta_{\mathrm{dec}}) = \Psi_{\mathrm{I}}$, we obtain

$$\left(\frac{\Delta T}{T}\right)^S(\eta_0, \mathbf{x}_0) \simeq \frac{1}{3}\Psi(\eta_{\mathrm{dec}}, \mathbf{x}_{\mathrm{dec}}). \tag{E-5}$$

This relation which has been derived first is called the ordinary Sachs-Wolfe (OSW) effect, while the integral part of the scalar fluctuations in (E-4) is called the integrated Sachs-Wolfe (ISW) effect.

CMB Angular Power Spectra

Expand the CMB temperature fluctuation using spherical harmonics as

$$\frac{\Delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{n}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}(\mathbf{x}_0) Y_{lm}(\mathbf{n}).$$

Considering an ensemble average of the multipole a_{lm} satisfying $\langle a_{lm} \rangle = 0$, the mean square $C_l = \langle |a_{lm}|^2 \rangle$ is given by

$$\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}.$$

Then, using the formula of Legendre polynomials

$$\sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}') = \frac{1}{4\pi}(2l+1) P_l(\mathbf{n} \cdot \mathbf{n}') \tag{E-6}$$

and $\mathbf{n} \cdot \mathbf{n}' = \cos\theta$, we can write a two-point correlation function of the temperature fluctuation as

$$\begin{aligned} C(\theta) &= \left\langle \frac{\Delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{n}) \frac{\Delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{n}') \right\rangle \\ &= \sum_{l,l',m,m'} \langle a_{lm} a^*_{l'm'} \rangle Y_{lm}(\mathbf{n}) Y^*_{l'm'}(\mathbf{n}') \\ &= \frac{1}{4\pi} \sum_{l} (2l+1) C_l P_l(\cos\theta). \end{aligned} \tag{E-7}$$

In the following, we calculate the multipole components of the angular power spectrum C_l of unpolarized CMB temperature fluctuations, usually called the temperature-temperature (TT) spectrum. There are contributions from both scalar and tensor fluctuations, and their sum is caught by observations. The scalar fluctuation contributes to the entire TT spectrum, whereas the tensor fluctuation mainly contributes only to low multipoles (large angle components) of $l < 50$.

Scalar fluctuations For simplicity, we here consider the terms other than the integral part in the scalar component of (E-4), which give main contributions to the CMB temperature anisotropy spectrum. That is, consider

$$\frac{\Delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{n}) = \left\{ \frac{1}{4}\mathcal{D}^\gamma + \partial_i V^b n^i + 2\Psi \right\}(\eta_{\rm dec}, \mathbf{x}_{\rm dec}),$$

where $\Phi = -\Psi$ is used. From $\mathbf{x}_{\rm dec} = \mathbf{x}_0 - (\eta_0 - \eta_{\rm dec})\mathbf{n}$, Fourier transform of this equation is given by

$$\begin{aligned} \frac{\Delta T}{T}(\eta_0, \mathbf{k}, \mathbf{n}) &= \left\{ \frac{1}{4}\mathcal{D}^\gamma - i\hat{\mathbf{k}} \cdot \mathbf{n} V^b + 2\Psi \right\}(\eta_{\rm dec}, \mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{n}(\eta_0 - \eta_{\rm dec})} \\ &= \left\{ \frac{1}{4}\mathcal{D}^\gamma + 2\Psi + \frac{V^b}{k}\partial_\eta \right\}(\eta_{\rm dec}, \mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{n}\eta)}|_{\eta=\eta_0-\eta_{\rm dec}}, \end{aligned} \tag{E-8}$$

where $k = |\mathbf{k}|$ and $\hat{\mathbf{k}} = \mathbf{k}/k = (\theta_k, \varphi_k)$.

First, consider relatively large size fluctuations ($l < 30$) where the ordinary Sachs-Wolfe effect holds. In this case, the CMB temperature fluctuation is given by the Bardeen potential at the decoupling as shown in (E-5), and the Fourier transform (E-8) is simplified as

$$\frac{\Delta T}{T}(\eta_0, \mathbf{k}, \mathbf{n}) \simeq \frac{1}{3}\Psi(\eta_{\rm dec}, \mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{n}(\eta_0 - \eta_{\rm dec})}.$$

Describing $C(\theta)$ using this relation yields

$$\begin{aligned} C(\theta) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \left\langle \frac{\Delta T}{T}(\eta_0, \mathbf{k}, \mathbf{n}) \frac{\Delta T}{T}(\eta_0, \mathbf{k}', \mathbf{n}') \right\rangle e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_0} \\ &\simeq \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3}\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_0}\, \frac{1}{9}\langle \Psi(\eta_{\rm dec}, \mathbf{k})\Psi(\eta_{\rm dec}, \mathbf{k}')\rangle \\ &\quad \times e^{-i\mathbf{k}\cdot\mathbf{n}(\eta_0 - \eta_{\rm dec})} e^{-i\mathbf{k}'\cdot\mathbf{n}'(\eta_0 - \eta_{\rm dec})} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{9}\langle |\Psi(\eta_{\rm dec}, \mathbf{k})|^2\rangle\, e^{-i\mathbf{k}\cdot\mathbf{n}(\eta_0 - \eta_{\rm dec})} e^{i\mathbf{k}\cdot\mathbf{n}'(\eta_0 - \eta_{\rm dec})}, \end{aligned}$$

where the two-point function is expressed as

$$\langle \Psi(\eta, \mathbf{k})\Psi(\eta, \mathbf{k}')\rangle = \langle |\Psi(\eta, \mathbf{k})|^2\rangle (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}'). \tag{E-9}$$

Expand the phase terms on the right-hand side using the following expansion formula by spherical Bessel functions:

$$e^{i\mathbf{k}\cdot\mathbf{y}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(ky) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{y}}), \tag{E-10}$$

where $y = |\mathbf{y}|$ and $\hat{\mathbf{y}}$ is defined like $\hat{\mathbf{k}}$. Denoting the distance to the last scattering surface as $d_{\rm dec} = \eta_0 - \eta_{\rm dec}$, $C(\theta)$ is then expressed as

$$\begin{aligned} &(4\pi)^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\langle \left| \frac{1}{3}\Psi(\eta_{\rm dec}, \mathbf{k}) \right|^2 \right\rangle j_l(kd_{\rm dec}) j_{l'}(kd_{\rm dec}) \\ &\times \sum_{l,l'=0}^{\infty} i^{l'-l} \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{k}}) Y^*_{lm}(\mathbf{n}) \sum_{m'=-l'}^{l'} Y^*_{l'm'}(\hat{\mathbf{k}}) Y_{l'm'}(\mathbf{n}'). \end{aligned}$$

Divide the integral over the comoving momentum $\mathbf{k}$ into radial and angular components as $d^3\mathbf{k} = k^2 dk d\Omega_k$, where $d\Omega_k = d\cos\theta_k d\varphi_k$. Using orthogonality of spherical harmonics

$$\int d\Omega_k\, Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}) = \delta_{ll'}\delta_{mm'} \tag{E-11}$$

and the formula (E-6) for Legendre polynomials, we get

$$C(\theta) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \, \frac{2}{\pi} \int k^2 dk \, \frac{1}{9} \langle |\Psi(\eta_{\rm dec}, \mathbf{k})|^2 \rangle \, j_l^2(k d_{\rm dec}).$$

By comparing this expression with (E-7), C_l can be found. From the shape of spherical Bessel functions, the relation $l \simeq k d_{\rm dec}$ (11-9) comes out as the area most contributing to the integration.

For super-horizon fluctuations where the ordinary Sachs-Wolfe effect holds, the initial value $\Psi_{\rm I}$ of the Bardeen potential is almost maintained, that is, the transfer function is $\mathcal{T}_\Psi(\eta_{\rm dec}, \eta_{\rm I}, k) \simeq 1$. Thus, $k^3 \langle |\Psi(\eta_{\rm dec})|^2 \rangle / 2\pi^2$ can be identified with the primordial scalar power spectrum defined by $P_s = k^3 \langle |\Psi_{\rm I}|^2 \rangle / 2\pi^2$. Assuming that the primordial scalar spectrum is expressed in a power of the spectral index n_s as

$$P_s(k) = A_s \left(\frac{k}{m} \right)^{n_s - 1},$$

the angular power spectrum is calculated as

$$\begin{aligned} C_l^{\rm osw} &= 4\pi \int_0^\infty \frac{dk}{k} \frac{1}{9} P_s(k) \, j_l^2(k d_{\rm dec}) \\ &= \frac{2\pi^2 A_s}{(m\, d_{\rm dec})^{n_s-1}} \frac{1}{9} \frac{\Gamma(3-n_s)\, \Gamma\left(l - \frac{1}{2} + \frac{n_s}{2}\right)}{2^{3-n_s} \Gamma^2\left(2 - \frac{n_s}{2}\right) \Gamma\left(l + \frac{5}{2} - \frac{n_s}{2}\right)}. \end{aligned}$$

In the case of a scale-invariant spectrum ($n_s = 1$) called the Harrison-Zel'dovich spectrum, it becomes

$$\frac{l(l+1) C_l^{\rm osw}}{2\pi} = \frac{A_s}{9}.$$

Thus, in the low multipole region, it is believed that the primordial power spectrum $P_s(k)$ in the early universe has been retained until today without being deformed.

In the case to examine fluctuations of a little smaller size where acoustic oscillations can be seen, we need to take into account contributions from all the terms in (E-8). Calculating in the same way as above, we obtain

$$\begin{aligned} C(\theta) = &\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \, e^{i(\mathbf{k}+\mathbf{k}')\cdot \mathbf{x}_0} \\ &\times \left\langle \left\{ \left(\frac{\mathcal{D}^\gamma}{4} + 2\Psi \right)(\eta_{\rm dec}, \mathbf{k}) + \frac{V^b(\eta_{\rm dec}, \mathbf{k})}{k} \partial_\eta \right\} \right. \end{aligned}$$

$$\left\{\left(\frac{\mathcal{D}^\gamma}{4}+2\Psi\right)(\eta_{\rm dec},\mathbf{k}')+\frac{V^b(\eta_{\rm dec},\mathbf{k}')}{k'}\partial_{\eta'}\right\}\right\rangle$$
$$\times e^{-i\mathbf{k}\cdot\mathbf{n}\eta}\,e^{-i\mathbf{k}'\cdot\mathbf{n}'\eta'}\Big|_{\eta=\eta'=d_{\rm dec}}.$$

Performing the integration over $\mathbf{k}'$ using expressions of two-point functions such as (E-9) and expanding the remaining phase terms using (E-10), it is expressed as

$$(4\pi)^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{l,l'=0}^{\infty}i^{l'-l}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{k}})Y^*_{lm}(\mathbf{n})\sum_{m'=-l'}^{l'}Y^*_{l'm'}(\hat{\mathbf{k}})Y_{l'm'}(\mathbf{n}')$$
$$\times\left\langle\left\{\left(\frac{\mathcal{D}^\gamma}{4}+2\Psi\right)(\eta_{\rm dec},\mathbf{k})\,j_l(k\eta)+\frac{V^b(\eta_{\rm dec},\mathbf{k})}{k}\partial_\eta j_l(k\eta)\right\}\right.$$
$$\left.\left\{\left(\frac{\mathcal{D}^\gamma}{4}+2\Psi\right)(\eta_{\rm dec},\mathbf{k})\,j_{l'}(k\eta)+\frac{V^b(\eta_{\rm dec},\mathbf{k})}{k}\partial_\eta j_{l'}(k\eta)\right\}^*\right\rangle\Big|_{\eta=d_{\rm dec}}.$$

Integrating the angular variables of $\mathbf{k}$ using (E-11) and (E-6) yields

$$C_l=\frac{2}{\pi}\int k^2dk\left\langle\left|\left\{\left(\frac{\mathcal{D}^\gamma}{4}+2\Psi\right)(\eta_{\rm dec},\mathbf{k})\,j_l(kd_{\rm dec})\right.\right.\right.$$
$$\left.\left.\left.+V^b(\eta_{\rm dec},\mathbf{k})\,j'_l(kd_{\rm dec})\right\}\right|^2\right\rangle,$$

where $j'_l(x)=\partial_x j_l(x)$.

The fluctuations under consideration are still super-horizon size at the initial time $\eta_{\rm I}$ set in the radiation-dominated era. Letting $\mathcal{T}_{\gamma,b,\Psi}(\eta_{\rm dec},\eta_{\rm I},k)$ be the transfer functions until decoupling at $\eta_{\rm dec}$, the value at the decoupling time can be written as

$$\left(\frac{\mathcal{D}^\gamma}{4}+2\Psi\right)(\eta_{\rm dec},\mathbf{k})=\frac{1}{4}\mathcal{T}_\gamma\,\mathcal{D}^\gamma(\eta_{\rm I},\mathbf{k})+2\mathcal{T}_\Psi\,\Psi(\eta_{\rm I},\mathbf{k})$$
$$=\left(-\frac{3}{2}\mathcal{T}_\gamma+2\mathcal{T}_\Psi\right)\Psi_{\rm I}(k),$$
$$V^b(\eta_{\rm dec},\mathbf{k})=\mathcal{T}_b\,V^b(\eta_{\rm I},\mathbf{k})=\mathcal{T}_b\frac{1}{2}k\eta_{\rm I}\Psi_{\rm I}(k)$$

using the initial value $\Psi_{\rm I}$, where we use the fact that $\mathcal{D}^\gamma\to-6\Psi_{\rm I}$, $\Psi\to\Psi_{\rm I}$, and $V^b(=V^\gamma)\to k\eta_{\rm I}\Psi_{\rm I}/2$ in the super-horizon limit $x=k\eta\to 0$ from the solution in the radiation-dominated era given in Chapter 13. Therefore, given the primordial power spectrum P_s determined by the initial value $\Psi_{\rm I}$, the CMB angular power spectrum can be calculated as

$$C_l = 4\pi \int \frac{dk}{k} \left\{ \left(-\frac{3}{2}\mathcal{T}_\gamma + 2\mathcal{T}_\Psi \right) j_l(kd_{\rm dec}) + \mathcal{T}_b \frac{1}{2} k\eta_{\rm I} j_l'(kd_{\rm dec}) \right\}^2 P_s(k).$$

Lastly, we present the TT power spectrum calculated using the CMB-FAST code in Fig. E-2, in which the cosmological parameter dependence is shown.

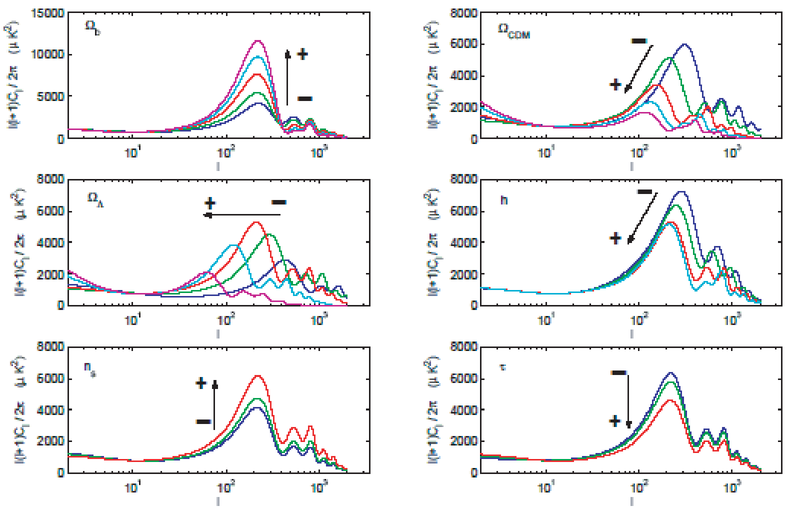

Figure E-2: Parameter dependence of the TT power spectrum. It shows changes of the spectrum when varying the density parameters Ω_b, Ω_c, Ω_Λ, the Hubble constant h, the spectral index n_s, and the optical depth τ_e from the top. [E. Martinez-Gonzalez, Lect. Note Phys. **665** (2009) 79.]

Tensor fluctuations The angular power spectrum of tensor fluctuations are calculated using the tensor part in (E-4). In the same way as for the scalar fluctuation, we obtain

$$C(\theta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{\eta_{\rm dec}}^{\eta_0} d\eta \int_{\eta_{\rm dec}}^{\eta_0} d\eta' \, e^{i\mathbf{k}\cdot\mathbf{n}(\eta_0-\eta)} e^{-i\mathbf{k}\cdot\mathbf{n}'(\eta_0-\eta')}$$
$$\times \left\langle \partial_\eta h_{ij}^{\rm TT}(\eta, -\mathbf{k}) \partial_{\eta'} h_{kl}^{\rm TT}(\eta', \mathbf{k}) \right\rangle n^i n^j n'^k n'^l, \qquad \text{(E-12)}$$

where two-point correlation function with different time appears. From the transverse and traceless conditions, the correlation function is normalized as[3]

$$\left\langle h_{ij}^{\rm TT}(\eta,-\mathbf{k})h_{kl}^{\rm TT}(\eta',\mathbf{k})\right\rangle = 2\tilde{\Delta}_{ij,kl}(\mathbf{k})\left\langle h^{\rm TT}(\eta,-\mathbf{k})h^{\rm TT}(\eta',\mathbf{k})\right\rangle, \quad \text{(E-13)}$$

where $\tilde{\Delta}_{ij,lm}(\mathbf{k})$ is defined in (7-20) as

$$\tilde{\Delta}_{ij,kl}(\mathbf{k}) = \frac{1}{2}\Bigg\{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}+\frac{1}{k^2}\big(\delta_{ij}k_kk_l+\delta_{kl}k_ik_j$$
$$-\delta_{ik}k_jk_l-\delta_{il}k_kk_j-\delta_{jk}k_ik_l-\delta_{jl}k_ik_k\big)+\frac{1}{k^4}k_ik_jk_kk_l\Bigg\}.$$

Introducing new variables $\mu = \mathbf{n}\cdot\hat{\mathbf{k}}$ and $\mu' = \mathbf{n}'\cdot\hat{\mathbf{k}}$, and noting $\mathbf{n}^2 = \mathbf{n}'^2 = 1$, we get

$$2\tilde{\Delta}_{ij,kl}(\mathbf{k})n^in^jn'^kn'^l = 2(\mathbf{n}\cdot\mathbf{n}')^2-1+\mu^2+\mu'^2-4\mu\mu'\mathbf{n}\cdot\mathbf{n}'+\mu^2\mu'^2.$$

Moreover, using

$$e^{i\mathbf{k}\cdot\mathbf{n}(\eta_0-\eta)} = e^{ik\mu(\eta_0-\eta)} = \sum_{r=0}^{\infty}(2r+1)i^rj_r(k(\eta_0-\eta))P_r(\mu),$$

we can express (E-12) as follows:

$$\sum_{r=0}^{\infty}\sum_{r'=0}^{\infty}(2r+1)(2r'+1)\,i^{r-r'}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,P_r(\mu)P_{r'}(\mu')\int_{\eta_{\rm dec}}^{\eta_0}d\eta\int_{\eta_{\rm dec}}^{\eta_0}d\eta'$$
$$\times\Bigg\{[2(\mathbf{n}\cdot\mathbf{n}')^2-1]j_r(k(\eta_0-\eta))j_{r'}(k(\eta_0-\eta'))$$
$$-j_r''(k(\eta_0-\eta))j_{r'}(k(\eta_0-\eta'))-j_r(k(\eta_0-\eta))j_{r'}''(k(\eta_0-\eta'))$$
$$+j_r''(k(\eta_0-\eta))j_{r'}''(k(\eta_0-\eta'))-4\mathbf{n}\cdot\mathbf{n}'j_r'(k(\eta_0-\eta))j_{r'}'(k(\eta_0-\eta'))\Bigg\}$$
$$\times\left\langle\partial_\eta h^{\rm TT}(\eta,-\mathbf{k})\partial_{\eta'}h^{\rm TT}(\eta',\mathbf{k})\right\rangle,$$

where $j_r'(x) = \partial_xj_r(x)$. This expression can be rewritten to a form proportional to Legendre polynomials by performing the angle integration $d\Omega_k$ using

$$\int d\Omega_k\,P_r(\mu)P_{r'}(\mu') = \frac{4\pi}{2r+1}\delta_{rr'}P_r(\mathbf{n}\cdot\mathbf{n}').$$

[3] In this normalization, $\langle h_{\rm TT}^{ij}(\eta,\mathbf{k})h_{ij}^{\rm TT}(\eta',-\mathbf{k})\rangle = 4\langle h^{\rm TT}(\eta,\mathbf{k})h^{\rm TT}(\eta',-\mathbf{k})\rangle$. If two polarization components of $h_{ij}^{\rm TT}$ commonly used are chosen as $h_{11}^{\rm TT} = -h_{22}^{\rm TT} = h_+$ and $h_{12}^{\rm TT} = h_{21}^{\rm TT} = h_\times$ on the x^1-x^2 plane, their correlation functions are given by $\langle h_+(x)h_+(x')\rangle = \langle h_\times(x)h_\times(x')\rangle = \langle h^{\rm TT}(x)h^{\rm TT}(x')\rangle$.

Moreover, $\mathbf{n}\cdot\mathbf{n}'$'s in the brackets are absorbed in the argument of the polynomial by using a recursion formula

$$xP_r(x) = \frac{r+1}{2r+1}P_{r+1}(x) + \frac{r}{2r+1}P_{r-1}(x).$$

Furthermore, rewrite it to be the form of the definition (E-7) using recursion formulas for spherical Bessel functions

$$j_r'(x) = -\frac{r+1}{2r+1}j_{r+1}(x) + \frac{r}{2r+1}j_{r-1}(x),$$
$$j_{r+1}(x) + j_{r-1}(x) = \frac{2r+1}{x}j_r(x).$$

Eventually it is put together into the following simple form:

$$\sum_{l=0}^{\infty}\frac{1}{2\pi^2}(2l+1)P_l(\mathbf{n}\cdot\mathbf{n}')\int k^2dk\int_{\eta_{\rm dec}}^{\eta_0}d\eta\int_{\eta_{\rm dec}}^{\eta_0}d\eta'\,\frac{(l+2)!}{(l-2)!}$$
$$\times\frac{j_l(k(\eta_0-\eta))}{[k(\eta_0-\eta)]^2}\frac{j_l(k(\eta_0-\eta'))}{[k(\eta_0-\eta')]^2}\left\langle\partial_\eta h^{\rm TT}(\eta,-\mathbf{k})\partial_{\eta'}h^{\rm TT}(\eta',\mathbf{k})\right\rangle.$$

Compared with (E-7) with attention to $\mathbf{n}\cdot\mathbf{n}'=\cos\theta$, we finally obtain

$$C_l = \frac{2}{\pi}\int k^2dk\left\langle\left|\int_{\eta_{\rm dec}}^{\eta_0}d\eta\,\partial_\eta h^{\rm TT}(\eta,\mathbf{k})\frac{j_l(k(\eta_0-\eta))}{k^2(\eta_0-\eta)^2}\right|^2\right\rangle\frac{(l+2)!}{(l-2)!}.$$

Let us roughly evaluate this equation. As far as the fluctuation has a super-horizon size, the solution of the tensor equation (13-22) is almost constant so that $\partial_\eta h^{\rm TT}=0$, and thus C_l will not be generated. The tensor fluctuation begins to change after entering the horizon. Therefore, the integral has a value only for large size fluctuations that enter the horizon during from the decoupling time $\eta_{\rm dec}$ to the present η_0, which correspond to low multipole components ($l<50$).

Since we know that the solution attenuates with $H\propto 1/a$ after entering the region of $x=k\eta\geq 1$, it can be written as $\partial_\eta h^{\rm TT}\simeq -aHh^{\rm TT} = (-2/\eta)h^{\rm TT}$ in this region. The last equality shows that the epoch we are considering is the matter-dominated era with $a\propto\eta^2$. From this, the integral can be evaluated as

$$\int_{\eta_{\rm dec}}^{\eta_0}d\eta\,\partial_\eta h^{\rm TT}\,\frac{j_l(k(\eta_0-\eta))}{k^2(\eta_0-\eta)^2}\simeq\frac{j_l(k\eta_0))}{k^2\eta_0^2}\int_{\eta=2/k}^{\eta_0}\frac{-2d\eta}{\eta}\,h^{\rm TT}.$$

Furthermore, considering $h^{\mathrm{TT}} \propto 1/\eta^2$, it is calculated as $\int(-2d\eta/\eta)\, h^{\mathrm{TT}} = h^{\mathrm{TT}}(\eta_0) - h^{\mathrm{TT}}(\eta = 1/k)$, where $h^{\mathrm{TT}}(\eta_0)$ is negligibly small. Since η_0 can be considered to be the same as $d_{\mathrm{dec}} = \eta_0 - \eta_{\mathrm{dec}}$, the angular power spectrum can be written for $l < 50$ as

$$C_l\,|_{l<50} \simeq \frac{2}{\pi}\int k^2 dk \left\langle \left| h^{\mathrm{TT}}\left(\eta = \frac{1}{k}, \mathbf{k}\right)\right|^2 \right\rangle \frac{j_l^2(kd_{\mathrm{dec}})}{k^4 d_{\mathrm{dec}}^4}\frac{(l+2)!}{(l-2)!}.$$

Since the tensor fluctuation hardly changes until entering the horizon, that is, the transfer function is unity until then, we can consider that the spectrum just before entering the horizon ($\eta = 1/k$) is the same as the primordial spectrum so that $k^3\langle|h^{\mathrm{TT}}(\eta = 1/k, \mathbf{k})|^2\rangle = 2\pi^2 P_t(k)$. If the primordial tensor spectrum is denoted as[4]

$$P_t(k) = A_t\left(\frac{k}{m}\right)^{n_t},$$

we get

$$\begin{aligned} C_l\,|_{l<50} &\simeq 4\pi\frac{(l+2)!}{(l-2)!}\int_0^\infty \frac{dk}{k}\frac{j_l^2(kd_{\mathrm{dec}})}{k^4 d_{\mathrm{dec}}^4} P_t(k) \\ &= \frac{2\pi^2 A_t}{(md_{\mathrm{dec}})^{n_t}}\frac{(l+2)!}{(l-2)!}\frac{\Gamma(6-n_t)\Gamma\left(l-2+\frac{n_t}{2}\right)}{2^{6-n_t}\Gamma^2\left(\frac{7}{2}-n_t\right)\Gamma\left(l+4-\frac{n_t}{2}\right)}. \end{aligned}$$

For a scale-invariant spectrum of $n_t = 0$, we obtain $l(l+1)C_l/2\pi \simeq A_t \times 8l(l+1)/15(l-2)(l+3)$. There is a divergence at $l = 2$, but it is because the approximation is rough.

Thus, the contribution from the tensor fluctuation may be included in the large angle components of the TT spectrum of CMB. However, the tensor-to-scalar ratio $r = A_t/A_s$, which shows how much the tensor fluctuation is included, cannot be determined exactly by the observed TT spectrum alone.

On polarization spectra Finally, we briefly describe other angular power spectra without touching the details. The CMB is polarized, the main cause of which is due to the Thomson scattering in the process of the universe becoming neutral. The spectra for polarizations called the E and B modes are shown in Fig. E-3. The top is the TT spectrum. The second is the TE, the third is the EE, and the bottom is the BB spectrum. The BB spectrum is generated only from the tensor fluctuation.

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[4] The convention of the tensor spectral index is different from the scalar one defined before. It comes from the historical background.

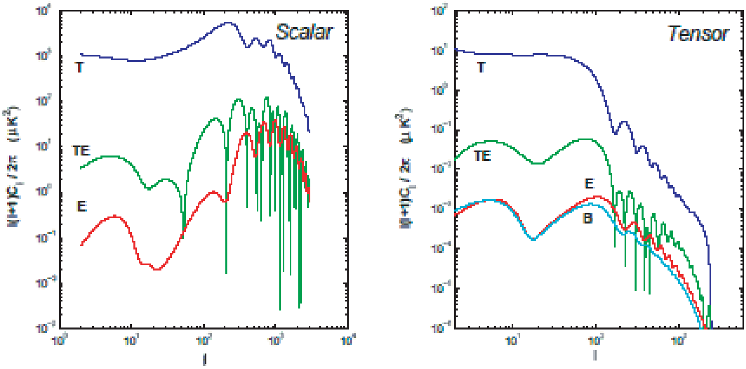

Figure E-3: Power spectrum of each mode of scalar and tensor fluctuations ($r = 0.01$). From the top, the TT, TE, EE, and BB (right only) spectra. In each mode except for the B, the CMB spectrum we can observe is only the sum of contributions from the two fluctuations. [E. Martinez-Gonzalez, Lect. Note Phys. **665** (2009) 79.]

The optical depth τ_e is determined from the behavior of the low multipoles ($l < 10$) in the EE spectrum and $\tau_e \simeq 0.1$ is obtained. It represents the degree to which the universe became a little opaque due to the reionization of particles by lights emitted at the time first stars were born. Therefore, the optical depth is a cosmological parameter that determines when the first star was born.

The tensor-to-scalar ratio r can be determined by the spectrum of the B mode originating from the tensor fluctuation, which appears at the bottom in Fig. E-3 (right).

Analytical Examinations of Evolution Equations

The solution of the simultaneous linear scalar evolution equations (14-1) and (14-2) is analytically examined here. In order to do that, we make the following simplification. First of all, let the coupling constant t be a sufficiently small constant. In this case, the Hubble variable can be regarded as a constant and it is normalized to $H = H_{\mathbf{D}}/\sqrt{B_0} = 1$. We also introduce a constant $T = b_c B_0 t^2/8\pi^2 (\ll 1)$ proportional to the square of the coupling constant. Furthermore, ignore the momentum dependence of the equation. Since it becomes k^2/a^2 when expressed using the physical time τ, such a situation will be realized at the time when the scale factor a is sufficiently increased after passed a little since inflation began. The evolution equations of the scalar fluctuations are then expressed as

$$
\begin{aligned}
&-2\ddddot{\Phi}-14\dddot{\Phi}-36\ddot{\Phi}-48\dot{\Phi}+2\dddot{\Psi}+14\ddot{\Psi}+36\dot{\Psi}+48\Psi\\
&+6\left(\ddot{\Phi}+4\dot{\Phi}-\dot{\Psi}-4\Psi\right)=0 \qquad \text{(E-14)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{4}{3}\ddot{\Phi}+\frac{16}{3}\dot{\Phi}+\frac{20}{3}\Phi-\frac{4}{3}\dot{\Psi}+\frac{4}{3}\Psi+\frac{8}{T}\left(\ddot{\Phi}+\dot{\Phi}-\ddot{\Psi}-\dot{\Psi}\right)\\
&-2(\Phi+\Psi)=0, \qquad \text{(E-15)}
\end{aligned}
$$

where the dot denotes the derivative by τ.

Introducing a new variable $f=\Psi-\dot{\Phi}$, these equations can be rewritten as

$$
\begin{aligned}
&\dddot{f}+7\ddot{f}+15\dot{f}+12f=0,\\
&\dddot{\Phi}-\left(1+\frac{7}{12}T\right)\dot{\Phi}-\frac{7}{12}T\Phi=-\ddot{f}-\left(1+\frac{1}{6}T\right)\dot{f}-\frac{1}{12}Tf.
\end{aligned}
$$

The first equation can be easily solved, and the following general solution is obtained:

$$
f=c_1e^{-4\tau}+c_2\,e^{-\frac{3}{2}\tau}\sin\left(\frac{\sqrt{3}}{2}\tau\right)+c_3\,e^{-\frac{3}{2}\tau}\cos\left(\frac{\sqrt{3}}{2}\tau\right).
$$

Substituting this solution into the second equation, we get

$$
\begin{aligned}
\Phi=&\,(a_1+c_1)\,e^{-\tau}+(a_2+c_2)\left(1-\frac{7}{12}T\tau\right)+(a_3+c_3)\left(1+\frac{7}{12}T\tau\right)e^{\tau}\\
&+c_1\frac{360-7T}{1800}e^{-4\tau}+\frac{\sqrt{3}c_2+5c_3}{14}e^{-\frac{3}{2}\tau}\cos\left(\frac{\sqrt{3}}{2}\tau\right)\\
&+\frac{5c_2-\sqrt{3}c_3}{14}e^{-\frac{3}{2}\tau}\sin\left(\frac{\sqrt{3}}{2}\tau\right), \qquad \text{(E-16)}
\end{aligned}
$$

where the solution is considered up to the first order of T.

Since we calculate it ignoring the momentum dependence here, this solution includes a vacuum solution which gives the background besides a solution of the fluctuation we are looking for. It can be seen from (E-14) that when $T=0$ the vacuum mode $\Phi=\Psi=\omega$ satisfies

$$
\ddddot{\omega}+6\dddot{\omega}+8\ddot{\omega}-3\dot{\omega}-12\omega=0,
$$

while (E-15) becomes trivial in this case. This equation is nothing but the equation of motion of the background $\hat{\phi}$ discussed in the first section of Chapter 12, which has an inflationary solution e^{τ} and three decaying solutions $e^{-4\tau}$, $e^{-3\tau/2}\sin(\sqrt{3}\tau/2)$, and $e^{-3\tau/2}\cos(\sqrt{3}\tau/2)$. What we are looking for is a solution for evolution of the fluctuation, and therefore we need to exclude solutions which become these vacuum ones at $T=0$ from the general solution (E-16). Indeed, these solutions will cease to satisfy the constraint equation (14-2) as time goes on. In this way, ignoring exponentially decaying solutions, it is found that the behavior of the fluctuation Φ with $T \ll 1$ is given by

$$\Phi \sim 1 - \frac{7}{12} T\tau.$$

This behavior appears in the attenuating part of the numerical solution given in Figs. 14-1 and 14-2.

Scattering Cross Sections in Einstein Gravity

In the energy region less than Λ_{QG}, Einstein's theory of gravity is effective and the description of the so-called graviton propagating in a classical spacetime becomes adequate. Here, we introduce the results of differential cross sections for the Rutherford scattering and the Compton scattering of scalar particles involving gravitons. For the sake of simplicity, we consider a massive scalar field which is minimally coupled with gravity ($\xi = 0$ in Appendix A).

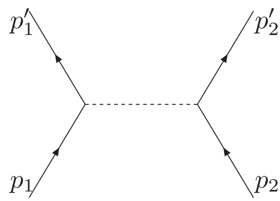

Figure E-4: Scattering of scalar particles exchanging a graviton.

Rutherford scattering Let us consider a process in which a particle of mass m with momentum p_1 and another particle of mass M with momentum p_2 are scattered to p_1' and p_2' by exchanging a graviton, as shown in Fig. E-4. Here we employ a laboratory system in which the particle p_2 is stationary as in Fig. E-5, and let the incident particle be $p_1^\mu = (\omega_1, 0, 0, q)$ whose dispersion relation is $\omega_1^2 = q^2 + m^2$. Furthermore, considering $M \gg m$, the particle p_2 remains stationary, and only the angle in the scattered particle p_1' changes, and p_1' has the same dispersion relation as the incident particle. Letting the velocity of the incident particle be v, the differential cross section is then given by

$$\frac{d\sigma_L}{d\Omega} = \frac{G^2}{4}\frac{M^2(m^2+2q^2)^2}{vq^3\sqrt{m^2+q^2}}\frac{1}{\sin^4\frac{\theta}{2}}.$$

In a non-relativistic limit $m \gg q$, putting $q = mv$ and $E = mv^2/2$, this reduces to

$$\frac{d\sigma_L}{d\Omega} = \frac{1}{4}\left(\frac{GMm}{2E}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}.$$

This agrees with the Rutherford scattering formula when the central force is given by GMm/r^2.

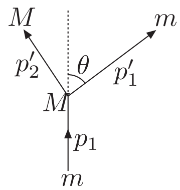

Figure E-5: Parameters in the laboratory system.

Gravitational Compton scattering Feynman diagrams of the Compton scattering between graviton and scalar particle of mass m is depicted in Fig. E-6. Differential cross section in the laboratory system where the particle p is stationary is given by[5]

$$\frac{d\sigma_L}{d\Omega} = \frac{G^2m^4}{(m+2E\sin^2\frac{\theta}{2})^2}\frac{\cos^8\frac{\theta}{2}+\sin^8\frac{\theta}{2}}{\sin^4\frac{\theta}{2}},$$

where $E = |\mathbf{k}|$ is a graviton energy.

It is thought that these scattering processes were active right after the big bang. In particular, we think that they contributed to the scattering of scalar-like dark matter which interact only with the gravitational field. However, unfortunately, it is difficult to find traces of them at present.

[5] For details, see F. Berends and R. Gastmans, *On the High-Energy Behaviour of Born Cross Sections in Quantum Gravity*, Nucl. Phys. **B88** (1975) 99.

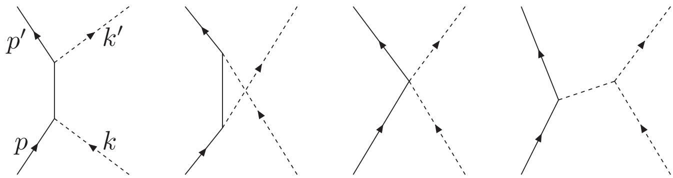

Figure E-6: The Compton scattering between graviton and scalar particle.

Fundamental Constants

Reduced Planck constant	$\hbar$	$=$	1.055×10^{-27} cm^2 g s^{-1}
Speed of light	c	$=$	2.998×10^{10} cm s^{-1}
Newton's constant	G	$=$	6.672×10^{-8} cm^3 g^{-1} s^{-2}
Planck mass	$m_{\rm pl}$	$=$	2.177×10^{-5} g
		$=$	1.221×10^{19} GeV/c^2
Reduced Planck mass	$M_{\rm P}$	$=$	2.436×10^{18} GeV/c^2
Planck length	$l_{\rm pl}$	$=$	1.616×10^{-33} cm
Planck time	$t_{\rm pl}$	$=$	5.390×10^{-44} s
Boltzmann constant	$k_{\rm B}$	$=$	1.381×10^{-16} erg K^{-1}
Megaparsec	1Mpc	$=$	3.086×10^{24} cm
Hubble constant	H_0	$=$	$100h$ km s^{-1} Mpc^{-1}
Hubble distance	c/H_0	$=$	$2998h^{-1}$ Mpc
			(current observation: $h \simeq 0.7$)

Useful constants for converting to natural units ($c = \hbar = k_{\rm B} = 1$)

1 cm	$=$	5.068×10^{13} $\hbar$/GeV
1 s	$=$	1.519×10^{24} $\hbar$/GeV/c
1 g	$=$	5.608×10^{23} GeV/c^2
1 erg	$=$	6.242×10^{2} GeV
1 K	$=$	8.618×10^{-14} GeV/$k_{\rm B}$

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