Four-Fermion Models in the Theory of Electro-Weak and Strong Interactions

S. I. Kruglov

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To Ludmila and Dzmitry

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PREFACE

There is a theoretical reason to consider a Higgs boson to be a composite particle. Thus, scalar particles such as a pion and kaon consist of quarks. In this monograph we explore the idea that a Higgs boson can be a composite particle and consists of quarks. In part I, the theoretical basis of four-fermion models is introduced. Then with the help of the path integration method, the dynamical mass generation is investigated in different four-fermion models including models with the internal symmetry groups SU(2), SU(3), SU(5), and with CP-violation. Ward-Takahashi identities and Schwinger-Dyson equations are obtained. The local SU(2)X U(1) four-fermion model with the composite Higgs boson is considered. The lepton masses and masses of W, Z bosons are formed due to the quark condensates. The Higgs boson is considered as the collective state of quarks and leptons in the model suggested. New experiments can verify the composite nature of a Higgs boson.

In part II the non-perturbative effects in strong interactions are considered. It is shown that the four-quark interaction appears naturally with the help of the gluon propagator in the infrared region. The mass formula for the sigma-meson, the Goldberger-Treiman relation and values of quark condensates are obtained. The four-quark model induced by instantons is investigated and it was proven that the current algebra is satisfied. It is shown that the four-quark models describe the region between the asymptotic freedom and quark confinement. The charge radii and electromagnetic polarizabilities of pions and nucleons are obtained within the instanton vacuum theory in good agreement with experimental data. Some quantum processes are considered in the framework of effective chiral Lagrangians. The decay of a pion into antineutrino and muon in the field of the electromagnetic wave is studied taking into account pion polarizabilities. Further developments of ideas considered may figure out the nature of quark confinement.

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Chapter 1

1.1 Introduction

In recent years, the field theories of elementary particles with four-fermion interactions have been of great interest. This is due to the fact that in the four-fermion models the internal symmetry is violated through the self-interaction of fields. This mechanism is called dynamic symmetry breaking (DSB). The first proof of the chiral γ_5-symmetry breaking in the four-fermion models was made independently by Nambu, Jona-Lasinio [1], Vaks, Larkin [2], and Arbuzov, Tavkhelidze, Faustov [3]. They started from the Lagrangian of the form

$$\mathcal{L} = -\bar{\psi}\gamma_\mu \partial_\mu \psi + g_0 \left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5\psi\right)^2 \right], \tag{1.1}$$

where γ_μ are Dirac matrices, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, g_0 possesses the dimension of $(\text{mass})^{-2}$. Authors have shown that due to the restructuring of the physical vacuum in this field model, one-parameter group of γ_5-symmetry is broken. This is due to a condensate, which is connected with the appearance of a non-zero vacuum average $\langle\bar{\psi}\psi\rangle \neq 0$, and as a result, initially massless fermions acquire a mass $m = -g_0\langle\bar{\psi}\psi\rangle$. A similar phenomenon is known to occur in the theory of superconductivity, when due to the pairing of electrons, as a result of the phase transition, an energy gap appears. Such a theory in statistical physics, with degeneracy of the vacuum state, was developed by Bogolyubov in 1960 (see, e.g., [4], [5]).

Models with DSB are well-known in connection with the nature and existence of a massive scalar Higgs boson. The theory of Glashow−Weinberg−Salam (GWS) [6] - [8]), successfully predicted the gauge vector bosons $W^{\pm}$, Z, discovered by experiments [9] - [12], and does not give the exact numerical values of the mass of the Higgs particle. The known Higgs procedure [13], [14] implies the existence of the field Φ (the weak iso-doublet) of point particles with the Lagrangian of the form

$$\mathcal{L} = -\left(D_\mu \Phi\right)^{+}\left(D_\mu \Phi\right) + \mu^2 \Phi^{+}\Phi - \lambda\left(\Phi^{+}\Phi\right)^2, \qquad (1.2)$$

where $D_\mu = \partial_\mu - i\left(g b_\mu^a(x) t^a - (1/2) g' Y a_\mu(x)\right)$, g_0, g_0' are constants of self-interaction of vector fields, Y is the weak hypercharge, and the fields of real vector bosons $W_\mu^{\pm}$, Z_μ, A_μ are defined through potentials of gauge fields b_μ^a ($a = 1, 2, 3$) and a_μ in the following way

$$W_\mu^{\pm} = \frac{1}{\sqrt{2}}\left(b_\mu^1 \pm i b_\mu^2\right), \qquad A_\mu = \frac{g a_\mu - g' b_\mu^3}{\sqrt{g^2 + g'^2}},$$

$$Z_\mu = \frac{g' a_\mu + g b_\mu^3}{\sqrt{g^2 + g'^2}}. \qquad (1.3)$$

Scalar fields Φ also participate in the Yukawa interactions of leptons and quarks. If the parameters μ^2 and λ in Lagrangian (1.2) are positive, there is spontaneous symmetry breaking with the non-zero vacuum expectation value of the scalar iso-doublet:

$$\langle\Phi\rangle_0 = \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix}, \qquad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{1.4}$$

In the unitary gauge, we have

$$\Phi(x) = \begin{bmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{bmatrix}, \tag{1.5}$$

and the mass of the field $\eta(x)$ of the physical Higgs particle is $m = \sqrt{2}\mu$. Since the parameter μ in Lagrangian (1.2) is not fixed, then the mass of the Higgs field is uncertain, and as we know, there is the main drawback of the theory of GWS. According to current estimates the mass of the Higgs particle should be $>$ 60 GeV [15], [16]. Recent LHC results [17], [18] (see also [19], [20]) on the observation of a Higgs-like particle give the mass of 125 CeV. But the existence of an unstructured scalar particle of such a large mass can be called into a question, at least in view of the fact that all known particles of spin 0 ($\pi^{\pm}$, π^0, K, etc. mesons) have a composite quark structure. Therefore, the development of alternative theories with composite Higgs fields is of particular interest. Studying various possibilities of particle mass generation, based on

DSB without the involvement of fundamental scalar fields, remains one of the most important tasks of the field theory of electro-weak interactions. There are several approaches that lead to the spontaneous symmetry breaking of the vacuum state, other than the Higgs procedure.

In one of the first of these approaches, the Schwinger mechanism [21] is used. Schwinger showed the possibility of mass generation for the vector gauge field in two-dimension electrodynamics by shifting the poles of the transverse part of the photon propagator. This scheme has been further developed in [22] - [26] by the extension to the case of 4-dimensional space-time. In subsequent studies [27] - [31], it was attempted to use the Schwinger mechanism in electro-weak theory without the Higgs Lagrangian. It should be noted, however, that the proof of the mass generation in such schemes was based on approximate nonperturbative solutions of the Dyson−Schwinger (DS) equation. A rigorous proof of the shift of the pole of the propagators of the vector particles in such models is still lacking.

The second approach, which leads to DSB of the vacuum, is based on the assumption of the composite nature of the scalar fields, built of fermion fields. In this approach, the initial Lagrangian for the scalar Higgs particle is absent. Instead, we introduce an additional interaction of fermions, for example, by (1.1), through which the vacuum becomes unstable and the condensate appeases, $\langle\bar{\psi}\psi\rangle \neq 0$, leading to a violation of the original symmetry. Scalar (composite) fields arise in this case as collective excitations of fermion fields.

Fermions, forming the scalar fields, are treated as new particles - techniquarks. This approach is called technicolor (TC). Interaction between techniquarks can occur not only by self-interaction, but also through the interaction with technigluons, as new particles. Techniquarks and technigluons are unobserved particles with the confinement radius 10^{-47} cm [32]. The authors of this approach are Weinberg [33] and Susskind [34]. Their model leads to the fact that the $W^{\pm}$- and Z-bosons become massive, however, there is no natural mechanism of a fermion mass formation [35], [36]. Developing this direction, the authors of [37], [38], in order to overcome this difficulty, have introduced extended technicolor (ETC), which led to the unification of fermions and technifermions in one multiplet and yet - to the emergence of new difficulties (see [39] - [45]). Suffice to say that there is no evidence of the existence of hadrons consisting of techniquarks.

Therefore, the original line of research based on the introduction of additional four-fermion interaction of the type (1.1) for ordinary quarks or leptons (not techniquarks), has been further developed. Thus, in [46] - [49], a single theory was constructed of the strong and electro-weak interactions using the four-fermion Lagrangian which is invariant under a global, rather than local $SU(3)_c \otimes SU(2)_L \otimes U(1)$-group. In this scheme, not only the Higgs fields were built of fermions and anti-fermions but also gauge vector bosons $W^{\pm}_{\mu}$, Z_{μ}, A_{μ} and gluons. But as we know, there are no indications of the composite nature of the well-studied photonic A_{μ}, weak intermediate $W^{\pm}_{\mu}$, Z_{μ} and gluon fields. Moreover, there are additional difficulties in the way of the principal consideration of massless photons and gluons, as composite vector particles [50]. The possibility of building massive composite vector fields was investigated in a number of subsequent papers [51] - [58].

The development of schemes with composite electroweak and Higgs bosons, despite these difficulties, continues in recent years.

Thus, in Novozhilov's work [59] composite $W^{\pm}$, Z and Higgs fields are constructed from "pre-gluons" and "pre-fermions" that are subject to the dynamics such as quantum chromo-dynamics (QCD). The "pre-fermions" are massless, and they satisfy the property of confinement. This approach can be seen as a kind of variation of the TC approach.

The electro-weak gauge-invariant model was considered without the Higgs field in [60], but with an additional "Abelian" vector boson C_{μ} with a mass M. This uses the ability to generate fermion masses based on solutions of

DS equations. Neutrinos are massless in this approach, and the intermediate bosons $W^{\pm}_{\mu}$, Z_{μ} acquire the finite masses. With some attractive features, the model still does not seem "aesthetically better" than the standard theory of GWS, because it does not reduce the number of independent degrees of freedom (instead of the scalar Higgs field, we introduce an additional vector field C_{μ}), and there are the difficulties of the interpretation and the possible existence of the field C_{μ}. A similar approach was also proposed in [61].

In the paper [62] Higgs particles are constructed from $\bar{t}$, t - quarks. To implement the DSB mechanism, the four-fermion interaction is introduced, including t-quarks, and masses of composite Higgs scalars are obtained, $m_H = 2m_t$. It is known that the analysis of experimental data on a large $\bar{B}^0_d - B^0_d$ - mixing and CP violation (see [63]) gives the lower bound for the t quark as follows: 78 GeV. In the work [62] t quark mass is obtained from the solution of an equation, $m_t = 84$ GeV. According to the current estimates, the mass of the t quark is $m_t = 174$ GeV, so that the mass of the composite Higgs particle is predicted to be $2m_t = 348$ GeV. In this approach, however, W-bosons are also composite. Their mass is generated by the Schwinger mechanism [21], $m_W = 80$ GeV. The realism of the model will show the further checking of the composite nature of the W-boson and the discovery of the Higgs boson of such mass.

A similar approach is also developed in the works [64], [65], where the masses of W and Z bosons are generated by condensation of t-quarks. This uses the four-fermion interaction in the quark sector and it is shown (see [64]), that the theory is asymptotically free, and the composite operator $\bar{\psi}\psi$ has a large anomalous dimension $\gamma_m = 2$. The importance was specified [64] of checking the phase diagrams for the four-fermion schemes in the lattice approximation using computers.

There are other papers [66] - [71], which also develop an approach that is associated with the composite nature of the Higgs particle.

Note that in the earlier work [72] it is indicated that the mass of the weak bosons W, Z can be obtained by the dynamic Higgs mechanism (without fundamental scalars) if there are heavy quarks or leptons with masses of the order of $30 - 100$ GeV. A similar result was obtained in [23].

Four-fermion field models, along with their use in the theory of electro-weak interactions, in recent years, are also used in the study of QCD at its low energy limit. We can say that a new level revives Heisenberg's ideas, but instead of "fore-matter," the quark fields are introduced.

Heisenberg (see [73]) made the first attempt to use a four-fermion model for a unified theory of elementary particles. The underlying approach to the concept of "forematter" is described by the nonlinear spinor equation. The excited states of a nonlinear spinor field were treated with strongly interacting particles - hadrons. This approach, as we know, has been studied initially in the development of the nonlinear meson theory and the theory of nuclear forces [74].

Recently, the attention of physicists was attracted by various nonlinear equations admitting particle-type solutions - solitons. This has still not diminished interest in obtaining exact solutions of nonlinear spinor equations [75], [76]. Baryons were considered as non-topological chiral solitons in the framework of the NJL model in [77].

The approach of Volkov with colleagues [78] - [91] (see also [92]) is based on a four-quark interaction Lagrangian of the type (1.1), but with given properties of the internal symmetry. On the basis of that postulated and used to describe the low-energy hadron physics Lagrangians, many characteristics were calculated: meson masses, decay widths, the scattering cross-sections and so on, which turned out to be in good agreement with the experimental values. In addition, this approach produces the vector dominance model, well-proven in practice.

A similar approach is used in [93] - [96], which also postulates a four-fermion Lagrangian, from which the effective chiral Lagrangian is derived. The latter, in particular, contains a topological Wess−Zumino [97] and Skyrme's terms [98], [99]. The corresponding coefficients of the Lagrangian calculated lead to the observed values, which are in good agreement with the phenomenology of [100] - [105]. The resulting effective Lagrangian reproduces the soft-pion theorem, the Goldberger−Treiman relation, PCAC and other well-tested relations [106] - [108]. The properties of dense and hot baryonic matter within the NJL model were investigated in [109] - [118]. We note the works [119], [120], which dealt with four-fermion Lagrangians to describe non-leptonic kaon decays $K \to 2\pi$, $K \to 3\pi$ strangeness-changing $|\Delta S| = 1$ and the change in isospin $|\Delta T| = 1/2, 3/2$. Along the way, however, we note that in the theory there are still some difficulties in explaining the increase of transitions with $|\Delta T| = 1/2$ [121] - [125]. An approach based on the Wilson lattice action for gauge fields and fermions in QCD is considered in [126] - [128]. When some assumptions in the low-energy physics were made, the authors come to the contact interaction of the four-fermion interaction with the effective Lagrangian containing Wess−Zumino's and generalized Skyrme's terms.

The effective Lagrangian directly derived from the fundamental QCD Lagrangian is very important, as it is now generally recognized that the QCD is a theory of the strong interactions of quarks and gluons. However, as you know, a reformulation of QCD in terms of hadrons as bound states of quarks possesses serious mathematical difficulties. One reason is the impossibility of functional integration over the gluon fields in the generating function for the Green functions as the corresponding path integral is not Gaussian because of the self-interaction of gluons.

Currently, intensive development of this field is running. In the papers [129], [130] (see also [131], [132]), based on simplifying assumptions of QCD at low energies, we obtain the effective Lagrangian for pseudoscalar meson nonet, comprising Wess−Zumino's interaction. The central point of this work is the consideration of the axial anomaly [133] - [148]. The integration of the chiral anomaly using methods of differential geometry was considered in [140] - [143].

Accounting anomalies lead to violation of U_1-symmetry [144], [145], which can help to solve the "old" U_1-problem formulated by Weinberg in 1974, the great mass difference of $\eta(549)$ and $\eta'(958)$ mesons (see reviews [146] - [149]).

In some other way, the chiral Lagrangian was obtained by Andrianov and Novozhilov [150] - [154]. This Lagrangian describes correctly the $\pi\pi$-scattering [130], [150]. Important components of the chiral Lagrangian are the term Wess−Zumino [97] (see also [155]), which describes the decay of $\pi^0 \to 2\gamma$ and is associated with the Adler anomaly.

In the Karchev and Slavnov work [156] the nonlinear chiral Lagrangian is obtained as a low-energy approximation of QCD at large N_c (N_c is the number of quark colors) under the assumption of chiral symmetry breaking. One of the first studies where a computational scheme has been developed for large N_c is the work of ‘t Hooft [157].

Note also the works of [158] - [163], where using a variety of mathematical techniques, the authors derive the effective chiral Lagrangians directly from the fundamental QCD Lagrangian.

Thus, the effective chiral Lagrangians describe well the low-energy hadron physics and the ability to dynamically implement the current algebra [164] - [168]. Under this approach, the nonlinear chiral Lagrangian containing the pion field, leads to the existence of stable soliton solutions, which describe the nucleon. By the way, the chiral Lagrangian, which leads to a stable soliton, interpreted as a nucleon, was proposed long ago by Skyrme [98], [99]. This ideology was then developed in [169], [170] (see also reviews [171] - [173]). Today, we know that the quantum numbers of the chiral soliton determine the above Wess−Zumino term. In this case, the baryon charge of the nucleon is treated as a topological charge of the soliton, and Δ-resonances arise here as rotational states of the quantum soliton.

It should be noted that the Skyrme model was postulated independently of the QCD, and is therefore an approximation to the true theory. Chiral Lagrangians, obtained from QCD besides Skyrme members also contain higher derivatives of the chiral pion field. Therefore, the observed values, calculated from a consistent theory, will, in general, differ from the values obtained from the Skyrme model.

In the papers [100] - [102], [174] - [185] the chiral Lagrangians of the general form are constructed having a large number of parameters, which are then determined from a comparison with the experimental values of the decay widths and the scattering cross-sections of mesons. This approach should also recognize the model, which does not follow directly from QCD.

Instantons play a very important role in the nonperturbative QCD, i.e., at low energies, when not to use perturbation theory [186]. Instantons are associated with an infinite number of sub-barrier transition oscillators from one state to another and can be obtained in pure gluodynamics as solutions of equations of motion in Euclidean space-time. It was shown in papers of Shuryak [187] - [189], Diakonov and Petrov [190] - [192] that the true vacuum of QCD can be considered as instantons and anti-instantons gas. This gas is in equilibrium with the average size of instantons $\bar{\rho} \simeq 0.3$ fm and the average distance between the centers of instantons is $\bar{R} \simeq 1$ fm.

The introduction of instantons leads to the challenging problem of the origin of the gluon condensate, the formation of which is associated with the finiteness of the action that is functional for the instanton (anti-instanton) and short-wavelength fluctuations of the gluon field. Instantons at large distances attract and repel each other at short distances, so that the gas-phase environment is stable [190]. When the quarks are present in the instanton medium, the chiral pairs $\bar{\psi}_R\psi_L$ and $\bar{\psi}_L\psi_R$ ($\psi_L = \frac{1}{2}(1+\gamma_5)\psi$, $\psi_R = \frac{1}{2}(1-\gamma_5)\psi$) are created. As a result, at the finite density of the instanton gas, the chiral condensate $\langle\bar{\psi}\psi\rangle \neq 0$ will occur, which leads to the DSB.

Thus, instantons lead to the axial anomaly needed to solve the U_1-problem, and break chiral symmetry.

Note that the concept of instantons still does not provide an explanation of quark confinement, as the latter is not associated with short- and long-wavelength fluctuations. These types of fluctuations are usually taken into account, attracting the bag model [193]. However, it is well known that the confinement problem of the quarks is still not solved in the framework of nonperturbative QCD.

The complicated structure of the QCD vacuum and the vacuum fluctuations are pointed out in the Novikov, Shifman, Vainshtein and Zakharov works [194] - [196].

In the papers [197], [198] it is shown that the inclusion of the small size instantons leads to the quark interaction which is described by the Lagrangian of the form

$$\mathcal{L}_{det} = \lambda \det \left(\bar{\psi}_R^i \psi_L^j \right) + c.c., \tag{1.6}$$

where ψ^i is the quark field of flavor i ($i = 1, 2, ...N$), c.c. means the complex conjugate of the expression, λ is a constant, which is related to the density of instantons.

The role of interactions (1.6) for the low-energy hadron physics is noted in [199] - [207].

Considering only the u, d-quarks ($\psi^1 = u$, $\psi^2 = d$), the Lagrangian (1.6) can be written as [208], [209]

$$\mathcal{L}_{det} = \frac{\lambda}{2} \left[\left(\bar{\psi}\psi \right)^2 + \left(\bar{\psi}\gamma_5\psi \right)^2 - \left(\bar{\psi}\tau^a\psi \right)^2 - \left(\bar{\psi}\gamma_5\tau^a\psi \right)^2 \right], \tag{1.7}$$

where τ^a are Pauli matrices. In (1.6), (1.7), a summation over the color degrees of freedom of quarks ($N_c = 3$) is implied.

Thus, we arrive again at the four-fermion interaction. It is important to note that the Lagrangian $\mathcal{L}_{det}$ in the general case is invariant under the group $SU(N_f) \otimes SU(N_f)$, but breaks the $U_A(1)$ symmetry, in contrast to the NJL-type of Lagrangians (1.1). This is a consequence of the axial anomaly.

It is supposed in [210] - [217] that quark flavor dynamics are described at medium energies by the interaction

Lagrangian

$$\mathcal{L}_{int} = \mathcal{L}_{NJL} + \mathcal{L}_{det}, \tag{1.8}$$

where

$$\begin{aligned}\mathcal{L}_{NJL} = G_1 \sum_{\alpha=0}^{N_f-1} \left[\left(\bar{\psi}\frac{1}{2}\lambda^\alpha \psi \right)^2 + \left(\bar{\psi}\frac{1}{2}\lambda^\alpha i\gamma_5 \psi \right)^2 \right] \\ + G_2 \sum_{\alpha=0}^{N_f-1} \left[\left(\bar{\psi}\frac{1}{2}\lambda^\alpha \gamma_\mu \psi \right)^2 + \left(\bar{\psi}\frac{1}{2}\lambda^\alpha i\gamma_5\gamma_\mu \psi \right)^2 \right].\end{aligned} \tag{1.9}$$

Here λ^a are the Gell-Mann matrices acting in the color space.

Authors of papers [210], [211] suggest that the introduction of the interaction with the Lagrangian of NJL type (1.9) is necessary to account for the exchange of heavy gluons, and this interaction is expected to be dominant. In these works, bosonization of the Lagrangian (1.8) is carried in the general case of an arbitrary number of flavors.

Calculated on the basis of (1.8) the mass spectrum of light mesons and some structural properties of mesons received [210] - [218] are in good agreement with the experiment. This suggests, that both, the instanton effect (taking into account the axial anomaly of QCD) and the four-fermion interaction (due to gluon exchange, which has a larger symmetry group $U(N_f) \otimes U(N_f)$) contribute to the low-energy physics of mesons.

The properties of hadrons were discussed in the literature, based only on the NJL Lagrangian (see [78] - [237]), which gave good description of the low-energy physics of mesons. Solitons arising in the NJL model (1.9) were studied by Alkofer and Reinhardt [238] - [243]. It has been shown that this approach approximately mimics the Skyrme model.

There are also investigations [244] - [255], to study models of type (1.6), (1.9) at the finite density of matter and temperatures. The chiral symmetry restoration at a critical temperature, the density of the matter and the external electromagnetic field were found. The results obtained can be used in the study of the theory of nuclear matter.

An important role in physics of electro-weak interactions is played by weak instantons (relevant fields are $W^{\pm}$ bosons). As shown by ‘t Hooft (see [256] and references therein), in GWS theory, there are processes with non-conservation of baryon and lepton charges. At energies of 15 TeV, cross-sections (due to many-fermion interactions (lepton-quark)), due to weak instantons, can reach the observed values. Now there is intensive work on the calculation of the function $F(\varepsilon)$, which determines the dependence of the total cross-sections in violation of the baryon number (VBN) on the energy. The knowledge of this function would appreciate the value of the energy at which processes can be observed with the VBN on the accelerator SSC ($\sqrt{s} = 40$ TeV).

Various aspects of the models with four-fermion interactions are considered also in [257] - [297]. In [298] Fukushima included the Polyakov loop in the NJL model. In this model (PNJL) the confinement of quarks holds in the NJL model via the Polyakov loop. The phase diagrams of strongly interacting matter were obtained [299]-[301]. Recent studies within the PNJL model are [302]-[303].

Thus, the four-fermion interactions play an important role in the theory of electro-weak and strong interactions of elementary particles. Hence, in particular, the further development of models of the NJL type is important. In the way of learning and using the NJL model, despite the progress and above-mentioned success, there are still many unsolved problems.

In the area of electro-weak interactions, first of all, the construction of such a model, while maintaining the fundamental property of local gauge invariance, would disclose a mechanism of mass generation of leptons and quarks, and intermediate vector bosons. This scheme, in contrast to the theory of GWS should not contain as source fields, fundamental scalar fields. This is due to natural reasons of simplicity and a minimal number of inputs to the theory of the fundamental fields. But here the scalar Higgs field can also occur, but as a collective excitation of the initial fields of quarks and leptons.

The above brief analysis of the situation in the field of non-perturbative QCD shows that there is an actual derivation of the effective low-energy Lagrangian, which would be based on the use of known solutions of the DS equations for the propagator of the gluon fields. The proof of the chiral symmetry breaking is very important. Of great interest is the study of the pseudoscalar bosons in the framework of this approach.

In addition to solving these global problems, it is obviously necessary, along with the study and calculation of the nonperturbative properties of hadrons in the well-known model of the instanton vacuum, to perform calculations for specific processes on the basis of phenomenological chiral Lagrangians by the known limitations of the application of the concept of the instanton vacuum in the range of energy above 600 MeV. Only within the framework of nonperturbative QCD, can we calculate the following characteristics of hadrons: form factors and polarizabilities.

The main objectives of this work are as follows:

1) to study the general properties of quantum field theory, taking into account the four-fermion interaction of different groups of internal symmetry;

2) the construction and study of the local $SU(2)_L \otimes U(1)$-invariant model of the electro-weak interaction with the four-fermion link without fundamental Higgs bosons;

3) to study the low-energy limit of QCD as a model with four-fermion interactions, and to solve specific problems in the physics of hadrons in the framework of non-perturbative QCD (for the calculation of the mass spectrum of light mesons and polarizabilities of hadrons and the study of some of the pion decays, taking into account their polarizabilities, in external strong electromagnetic fields, and the consideration of the possible processes of strong interactions involving the Wess−Zumino action).

In Chapter 2, using the method of functional integration in collective variables, the dynamic mass formation was studied in the four-fermion models with scalar-scalar, pseudoscalar-pseudoscalar and vector-vector interactions. Ward−Takahashi identities and the DS equation for fermions and bosons Green's functions are obtained. The dynamic symmetry breaking and the mass spectrum of bound states in the $SU(n) \otimes U(1)$-four-fermion models with two coupling constants for $n = 2,\ 3,\ 5$ are investigated. Using the method of the effective potential, discussed in detail, CP-odd, chiral-violating the four-fermion model with three coupling constants is considered. It is shown that in the one-loop approximation (mean-field) effective action has a full view of the linear σ-model. Using the method of Gell-Mann−Levy, the axial current is found and the performance of PCAC is demonstrated. Thus, it is shown that the model reproduces the characteristic properties of the non-perturbative QCD - chiral symmetry and CP-parity violation.

Chapters 3 and 4 offered locally $SU(2)_L \otimes U(1)$-invariant GWS-type model, where instead of the Higgs Lagrangian, the Lagrangian of four-fermion interactions is introduced and used. The different ways of including the four-fermion interaction of different generations are considered. It is shown that as a result of dynamic symmetry breaking "top" fermions (neutrinos) of different generations remain massless, and the "lower" fermions (leptons e, μ, τ) acquire different masses. With the use of an expansion in loops, the effective interaction Lagrangian of the gauge vector, fermion (lepton and quark) and collective scalar fields in the one-loop approximation is calculated, which is the same form as the corresponding expression in the theory of GWS. Mass formulas for the scalar Higgs field, to gauge vector $W^{\pm}$, Z-fields are found. In this case, the scalar field is considered as a superposition of quark-antiquark and lepton-antilepton states. An approximate value for the mass of the composite scalar Higgs is $m_H \simeq 2m_t$ (where m_t is a mass of t-quark).

In Chapter 5, we study the low-energy limit of QCD. It is shown that in the model of Abelian QCD, where the gluon propagator as a solution of the DS equation in the infrared region is used, we have chiral symmetry breaking, and the effective interaction is a four-quark interaction.

This approach is compared with the notion of the QCD vacuum as a gas of instantons and anti-instantons, which also have the multi-quark ‘t Hooft interaction (1.6). In this model the spectrum of masses of light mesons shows that the pion is a pseudo-Goldstone boson. The infrared asymptotic behavior of Green's functions in the four-fermion model is investigated. It is concluded that the four-quark model can be used to describe the intermediate region between the region of asymptotic freedom and quark confinement.

In Chapter 6, based on the formulation of low-energy QCD, non-perturbative characteristics of hadrons and some questions of strong and electro-weak interactions are studied. We calculate the electromagnetic polarizabilities of nucleons and pions in the concept of the instanton vacuum. We study the effective chiral Lagrangian (ECL), which includes as a normal part the kinetic term, and the terms of the third-order derivatives, and as the anomalous part - the Lagrangian of Wess−Zumino. The coefficients in the non-minimal terms of ECL are determined by comparing the values of the mass and width of the A_1-meson with the known values of the experiment, resulting in the Lagrangian which has no free parameters. Based on ECL, interaction Lagrangians of vector and pseudoscalar and axial-vector mesons, which allow the counting of the cross-sections in e^+e^-collisions, are discussed. Also the pion decay is studied ($\pi^+ \to \mu^+ + \overline{\nu}_\mu$), taking into consideration its polarizabilities in a strong electromagnetic field of a plane electromagnetic wave.

In the appendix, the exact solutions are obtained of the wave equation for the pion in constant electromagnetic fields and in the field of a plane electromagnetic wave.

Chapter 2

2.1 Dynamical symmetry breaking

The model with the most general four-fermion interaction is introduced using the method of functional integration in collective variables. Ward−Takahashi identities and Dyson−Schwinger equations are obtained. The finite renormalization procedure is performed and it is shown that the matrix elements of the interaction of fermions with their bound states are independent of the renormalization constants. The spectrum of masses in models with different internal symmetry groups $SU(n) \otimes U(1)$ (for n = 2, 3, 5), and in models with CP-parity violation, which reproduces the main features of the non-perturbative QCD, is investigated. The dependencies of the fermion masses and their bound states (collective fields) on the momentum cutoff Λ are obtained, and the Λ acquires a physical meaning. The material in this chapter is auxiliary and not associated with realistic models. Mathematical techniques used here and the results are the basis for the following chapters where realistic models of the theory of electroweak and strong interactions are investigated.
The content of the chapter is based on the results of [304] - [311].

2.1.1 Collective variables and perturbation theory in a nonlinear spinor model

Let us consider a model based on the Lagrangian with the four-fermion interaction of a general type:

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + \frac{\alpha}{2}\left(\bar{\psi}\psi\right)^2 - \frac{\beta}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$
$$-\frac{\delta}{16}\left(\bar{\psi}\gamma_{[\mu}\gamma_{\nu]}\psi\right)^2 - \frac{\sigma}{2}\left(\bar{\psi}\gamma_\mu\gamma_5\psi\right)^2 - \frac{\xi}{2}\left(\bar{\psi}\gamma_5\psi\right)^2, \qquad (2.1)$$

where α, β, δ, σ and ξ are coupling constants of the dimension m^{-2}, $\gamma_{[\mu}\gamma_{\nu]} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$. Many are considered in the literature [257] - [269] cases and, in particular (1.1), follow from (2.1) for certain restrictions on the constants. Using Fierz transformations (2.1) can be transformed to the form

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + \frac{\kappa}{2}\left(\bar{\psi}\psi\right)^2 - \frac{\rho}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2 - \frac{\lambda}{2}\left(\bar{\psi}\gamma_5\psi\right)^2. \qquad (2.2)$$

Here $\kappa = \alpha + \delta + 2\sigma$, $\rho = \beta + \sigma$, $\lambda = \xi + 2\sigma - \delta$.

We will investigate the dynamic mass formation in the model, using the method of functional integration in collective variables [312].

Generating functional for the Green function

$$Z[\bar{\eta}, \eta] = N_0 \int D\bar{\psi}D\psi \exp\left\{i\int d^4x\left(\mathcal{L} + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\}, \qquad (2.3)$$

where $\bar{\eta}$, η - external sources, by redefining the constant N_0, with the multiplication by the constant

$$\int D\varphi D\widetilde{\varphi} DA_\mu \exp\left\{-i\int d^4x\left[\frac{\mu_0^2}{2}\left(\varphi-\frac{g_0\bar{\psi}\psi}{\mu_0^2}\right)^2\right.\right.$$

$$+\frac{\mu_0^{\prime 2}}{2}\left(\widetilde{\varphi}-\frac{ig_0^\prime\bar{\psi}\gamma_5\psi}{\mu_0^{\prime 2}}\right)^2$$

$$\left.\left.+\frac{M_0^2}{2}\left(A_\mu-i\frac{e_0\bar{\psi}\gamma_\mu\psi}{M_0^2}\right)^2\right]\right\}\prod_x\delta\left(\partial_\mu A_\mu\right),$$

is written in the form

$$Z[\bar{\eta},\eta]=N\int D\varphi D\widetilde{\varphi} DA_\mu D\bar{\psi} D\psi \exp\left\{i\int d^4x\left[-\bar{\psi}\gamma_\mu\partial_\mu\psi\right.\right.$$

$$+\bar{\psi}\left(g_0\varphi+ig_0^\prime\widetilde{\varphi}\gamma_5+ie_0A_\mu\gamma_\mu\right)\psi-\frac{\mu_0^2}{2}\varphi^2-\frac{\mu_0^{\prime 2}}{2}\widetilde{\varphi}^2 \quad (2.4)$$

$$\left.\left.-\frac{M_0^2}{2}A_\mu^2+\bar{\psi}\eta+\bar{\eta}\psi+j\varphi+\widetilde{j}\widetilde{\varphi}+j_\mu A_\mu\right]\right\}\prod_x\delta\left(\partial_\mu A_\mu\right).$$

Here we introduce the collective scalar φ, pseudoscalar $\widetilde{\varphi}$ and vector A_μ neutral fields, $\kappa = g_0^2/\mu_0^2$, $\lambda = g_0^{\prime 2}/\mu_0^{\prime 2}$, $\rho = e_0^2/M_0^2$; g_0, $g_0^\prime$, e_0 are dimensionless constants, and the constants μ_0, $\mu_0^\prime$ and M_0 have the dimensions of the mass;

j, $\widetilde{j}$, j_μ - external sources of collective fields, $D\varphi = \prod_x d\varphi$. The factor $\prod_x \delta\left(\partial_\mu A_\mu\right)$ in the functional integral (2.4) takes into account the transversality of the vector field A_μ [313].

Substituting the representation of the continuum δ-function in the form

$$\prod_x \delta\left(\partial_\mu A_\mu\right) = \lim_{\alpha\to 0} \exp\left\{-\frac{i}{2\alpha}\int d^4x \left(\partial_\mu A_\mu\right)^2\right\} \tag{2.5}$$

in (2.4) and integrating over the Fermi fields $\bar{\psi}$, ψ, we obtain

$$Z[\bar{\eta}, \eta, j] = \lim_{\alpha\to 0} N \int D\Phi \exp\left(iW_0\right),$$

$$W_0 = \int d^4x d^4y \Big\{\bar{\eta}(x)S(x,y)\eta(y) - \delta(x-y)\Big[\frac{\mu_0^2}{2}\varphi^2 + \frac{\mu_0^{\prime 2}}{2}\widetilde{\varphi}^2 \tag{2.6}$$
$$+\frac{M_0^2}{2}A_\mu^2 + \frac{1}{2\alpha}\left(\partial_\mu A_\mu\right)^2 - j_A\Phi_A\Big]\Big\} - i\mathrm{Tr}\,\ln\left(1 + \widehat{G}_0 g_A \Phi_A \gamma_A\right).$$

Here the notations $\Phi_A = (A_\mu, \varphi, \widetilde{\varphi})$, $g_A = (e_0, g_0, g_0')$, $\gamma_A = (i\gamma_\mu, I, i\gamma_5)$ (I is a unit 4×4-matrix), $j_A = (j_\mu, j, \widetilde{j})$, $g_A\Phi_A\gamma_A = ie_0A_\mu\gamma_\mu + g_0\varphi + ig_0'\widetilde{\varphi}\gamma_5$, $D\Phi = D\varphi D\widetilde{\varphi} DA_\mu$ are used. The operator Tr includes the trace in matrices and space-time variables, and it is taken into account that the fermion Green's functions $S(x,y)$ and $G_0(x,y)$ satisfy the respective equation

$$\Big(\gamma_\mu \partial_\mu - g_A \Phi_A \gamma_A\Big) S(x,y) = \delta(x-y) \tag{2.7}$$

in the presence of external collective fields, and the equation

$$\gamma_\mu \partial_\mu \widehat{G}_0(x,y) = -\delta(x-y) \tag{2.8}$$

matches the free massless fermions. The limit of $\alpha \to 0$ corresponds to the Lorentz gauge where the field A_μ describes quanta with a "pure" spin 1.

As seen from (2.6), the dimensional constants μ_0, μ_0', M_0 logged in mass terms of the Lagrangian, and the dimensionless constants g_A: g_0, g_0', e_0 play the role of the coupling constants of collective fields φ, $\widetilde{\varphi}$, A_μ with spinors. Thus, the functions of these constants (dimensional and dimensionless), originally included in a single constant of four-fermion interactions, are divided. In the one-loop approximation (mean-field), where the collective fields φ, $\widetilde{\varphi}$, A_μ are constants (Φ_A = const.), the solution of (2.7) in the momentum space is

$$S(p) = \frac{-i\widehat{p} - g_0\varphi + ig_0'\widetilde{\varphi}\gamma_5}{p^2+m^2}, \tag{2.9}$$

where $\widehat{p} = p_\mu\gamma_\mu$, $m^2 = g_0^2\varphi^2 + g_0'^2\widetilde{\varphi}^2$. The equations of motion for the collective fields $\delta W_0/\delta\Phi_A = 0$, with $\eta =$

$\bar{\eta} = j = \widetilde{j} = j_\mu = 0$, have the form

$$\varphi = -\frac{ig_0^2}{4\pi^4\mu_0^2}\int\frac{d^4p\ \varphi}{p^2+m^2}, \qquad \widetilde{\varphi} = -\frac{ig_0^{\prime 2}}{4\pi^4\mu_0^{\prime 2}}\int\frac{d^4p\ \widetilde{\varphi}}{p^2+m^2}. \tag{2.10}$$

The equation for the collective vector field A_μ has only the trivial solution $A_{0\mu} = 0$. Nontrivial nonanalytic solutions $\varphi_0 \neq 0$, $\widetilde{\varphi}_0 \neq 0$ of equations (2.10) exist only under conditions $g_0 = g_0'$, $\mu_0 = \mu_0'$, $\Lambda^2\mu_0^2 > 4\pi^2$, where Λ is a cutoff momentum [1], [2]. If these conditions are satisfied, fermions acquire masses $m_0^2 = g_0^2(\varphi_0^2 + \widetilde{\varphi}_0^2)$ (see (2.9)), and a phase transition to an asymmetric phase takes place.

A similar situation is known to be the case in the theory of superconductivity, because (2.10) with conditions $g_0 = g_0'$, $\mu_0 = \mu_0'$ is similar to the equation for the energy gap [4], [5]. Here, due to the phase transition initially massless fermions also become massive. This massive fermion state is energetically more favorable, since it corresponds to the minimum of the effective potential (see [314]), [315]).

$$\begin{aligned} V(\Phi) = {} & \frac{\mu_0^2}{2}\left(\varphi_0^2 + \widetilde{\varphi}_0^2\right) \\ & + \frac{i}{(2\pi)^4}\int d^4p \ \ln\det\left[1 + \widehat{G}_0(p) g_0\left(\varphi_0 + i\widetilde{\varphi}_0\gamma_5\right)\right]. \end{aligned} \tag{2.11}$$

Indeed, the potential extremum conditions of (2.11) $\partial V/\partial\varphi_0 = \partial V/\partial\widetilde{\varphi}_0 = 0$ yield gap equations (2.10) (with

$g_0 = g_0'$, $\mu_0 = \mu_0'$), and the one-loop correction to the effective potential corresponding to the second term in (2.11), after the evaluation of the determinant is negative

$$V_{one\ loop} = \frac{i}{8\pi^4} \int d^4p \ln\left(1 + \frac{m^2}{p^2}\right) < 0 \qquad (2.12)$$

and, therefore, reduces the energy of the vacuum, which points to the implementation of the minimum of the effective potential (2.11).

It should be noted that the condition for the constants $\kappa = \lambda$ ($g_0 = g_0'$, $\mu_0 = \mu_0'$) results in the case of the chiral symmetric Lagrangian (see (1.1)). At the same time, the non-zero vacuum fields $\varphi_0 \neq 0$, $\widetilde{\varphi}_0 \neq 0$ violate the chiral symmetry. Finally, we come to the typical situation of spontaneous symmetry breaking, when the Lagrangian is invariant under the transformations of a group symmetry, but the ground state of the vacuum is not invariant under these transformations.

It is also possible that choices for solving equations (2.10) are $\varphi_0 \neq 0$, $\widetilde{\varphi}_0 = 0$, which correspond to a specific gauge. One cannot require equality of constants $\kappa = \lambda$.

Expanding the field Φ_A in equation (2.6) around the static solutions $A_\mu(x) \to A_\mu(x)$, $\varphi(x) \to \varphi_0 + \varphi(x)$, $\widetilde{\varphi}(x) \to \widetilde{\varphi}_0 + \widetilde{\varphi}(x)$, where φ_0, $\widetilde{\varphi}_0$ are governed by the equations (2.10), the expression (2.6) can be represented as a series of perturbation theories in powers of g_A:

$$W_0 = \int d^4x d^4y \Big\{ \bar{\eta}(x) S(x,y) \eta(y) + \delta(x-y) j_A(x) \Phi_A(x)$$
$$- \frac{1}{2} \Phi_A(x) \Delta^{-1}_{AB}(x,y) \Phi_B(y) \Big\} + \sum_{n=3}^{\infty} \frac{i}{n} \mathrm{Tr} \left(S_0 g_A \Phi_A \gamma_A \right)^n . \tag{2.13}$$

Here $S_0 = S(\Phi_0)$ defined by expression (2.9) with $\varphi = \varphi_0$, $\widetilde{\varphi} = \widetilde{\varphi}_0$, $g_0 = g_0'$, $\mu_0 = \mu_0'$, and the propagator for the collective fields in momentum space is given by

$$\Delta^{-1}_{AB}(p) = -i g_A g_B \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} S_0(p+k) \gamma_A S_0(k) \gamma_B$$
$$+ \delta_{AB} M_A + \frac{1}{\alpha} p_\mu p_\nu \delta_{\mu A} \delta_{\nu B}, \tag{2.14}$$

where $M_A = (M_0, \mu_0, \mu_0)$, and the tr means the trace only in matrices. In (2.14) the summation over repeated indices is implied. Substituting the expression (2.9) in (2.14), and calculating the traces of the matrices, we obtain

$$\Delta^{-1}_{\mu\nu}(p) = \delta_{\mu\nu} M_0^2 + \frac{1}{\alpha} p_\mu p_\nu$$

$$+ \frac{i e_0^2}{4\pi^4} \int \frac{d^4q \left\{ \delta_{\mu\nu} \left[q(q-p) + m_0^2 \right] + p_\mu q_\nu + p_\nu q_\mu - 2 q_\mu q_\nu \right\}}{\left[(q-p)^2 + m_0^2 \right] \left(q^2 + m_0^2 \right)},$$

$$
\begin{aligned}
\Delta_{55}^{-1}(p) &= \mu_0^2 - \frac{ig_0^2}{4\pi^4}\int \frac{d^4q\left[g_0^2\left(\varphi_0^2-\widetilde{\varphi}_0^2\right)+q(p-q)\right]}{\left[(q-p)^2+m_0^2\right]\left(q^2+m_0^2\right)}, \\
\Delta_{66}^{-1}(p) &= \mu_0^2 - \frac{ig_0^2}{4\pi^4}\int \frac{d^4q\left[g_0^2\left(\widetilde{\varphi}_0^2-\varphi_0^2\right)+q(p-q)\right]}{\left[(q-p)^2+m_0^2\right]\left(q^2+m_0^2\right)}, \\
\Delta_{5\mu}^{-1}(p) &= \Delta_{\mu 5}^{-1}(p) = \Delta_{6\mu}^{-1}(p) = \Delta_{\mu 6}^{-1}(p) = 0, \\
\Delta_{56}^{-1}(p) &= \Delta_{65}^{-1}(p) = -\frac{ig_0^2}{4\pi^4}\int \frac{d^4q 2g_0^2\varphi_0\widetilde{\varphi}_0}{\left[(q-p)^2+m_0^2\right]\left(q^2+m_0^2\right)}.
\end{aligned} \tag{2.15}
$$

Evaluating the integrals in (2.15) using (2.10), we find

$$
\begin{aligned}
\Delta_{55}^{-1}(p) &= \left(p^2+4g_0^2\varphi_0^2\right)\left(Z_3^{-1}-\frac{g_0^2}{8\pi^2}J_1(p)\right), \\
\Delta_{66}^{-1}(p) &= \left(p^2+4g_0^2\widetilde{\varphi}_0^2\right)\left(Z_3^{-1}-\frac{g_0^2}{8\pi^2}J_1(p)\right), \\
\Delta_{56}^{-1}(p) &= 4g_0^2\varphi_0\widetilde{\varphi}_0\left(Z_3^{-1}-\frac{g_0^2}{8\pi^2}J_1(p)\right), \\
\Delta_{\mu\nu}^{-1}(p) &= \left(M_0^2-\frac{\Lambda^2-m_0^2}{2}\frac{e_0^2}{4\pi^2}\right)\delta_{\mu\nu}+\frac{1}{\alpha}p_\mu p_\nu \\
&+\frac{2}{3}\frac{e_0^2}{g_0^2}Z_3^{-1}\left(\delta_{\mu\nu}p^2-p_\mu p_\nu\right)-\frac{e_0^2}{4\pi^2}J_{\mu\nu}(p), \\
J_{\mu\nu}(p) &= \frac{2}{9}p^2\delta_{\mu\nu}+\frac{1}{18}p_\mu p_\nu+\left(\delta_{\mu\nu}p^2-p_\mu p_\nu\right)
\end{aligned} \tag{2.16}
$$

$$\times \int_0^1 dx\ 2x(1-x)\ln\left[1+\frac{p^2}{m_0^2}(1-x)x\right],$$

$$J_1(p) = \int_0^1 dx\ \ln\left[1+\frac{p^2}{m_0^2}(1-x)x\right],$$

$$Z_3^{-1} = \frac{g_0^2}{8\pi^2}\left(\ln\frac{\Lambda^2}{m_0^2}-1\right).$$

We now introduce the renormalized quantities

$$\Delta'_{55}(p) = Z_3^{-1}\Delta_{55}(p), \quad \Delta'_{66}(p) = Z_3^{-1}\Delta_{66}(p),$$
$$\Delta'_{56}(p) = Z_3^{-1}\Delta_{56}(p), \quad {g_0'}^2 = g_0^2 Z_3,\ {e_0'}^2 = e_0^2 Z_3, \tag{2.17}$$
$${\varphi_0'}^2 = \varphi_0^2 Z_3^{-1}, \quad {\widetilde{\varphi}_0'}^2 = \widetilde{\varphi}_0^2 Z_3^{-1}, \quad A'_\mu = A_\mu Z_v^{-1/2},$$
$$\Delta'_{\mu\nu}(p) = Z_v^{-1}\Delta_{\mu\nu}(p), \quad {\varphi'}^2 = \varphi^2 Z_3^{-1},\ {\widetilde{\varphi}'}^2 = \widetilde{\varphi}^2 Z_3^{-1},$$

where Z_3, $Z_v = (3g_0^2/2e_0^2)Z_3$ are renormalization constants of fields φ, $\widetilde{\varphi}$ and A_μ. It follows from (2.16) that

$$\Delta_{55}^{-1}(p^2 = -4g_0^2\varphi_0^2) = 0, \quad \Delta_{66}^{-1}(p^2 = -4g_0^2\widetilde{\varphi}_0^2) = 0, \tag{2.18}$$

and masses of collective fields φ, $\widetilde{\varphi}$ are

$$m_\varphi^2 = 4g_0^2\varphi_0^2, \qquad m_{\widetilde{\varphi}}^2 = 4g_0^2\widetilde{\varphi}_0^2. \tag{2.19}$$

From (2.19) we obtain $4m_0^2 = m_\varphi^2 + m_{\widetilde{\varphi}}^2$, where m_0 is a fermion mass. Along with the renormalization of the fields and coupling constants (2.19), we define the renormalized mass of the neutral vector collective field A_μ

$$M^2 = \left[M_0^2 - \frac{e_0^2}{4\pi^2} \frac{\Lambda^2 - m_0^2}{2} \right] Z_v. \tag{2.20}$$

Finally, using (2.17), (2.19), from (2.16) up to terms of orders $g_0^2/(4\pi^2)$, $e_0^2/(4\pi^2)$, determining the radiative corrections, we find the Lagrangian, which is bilinear on composite (collective) fields

$$\begin{aligned} \mathcal{L}^{(2)} = -\frac{1}{2}\left[(\partial_\mu \varphi)^2 + (\partial_\mu \widetilde{\varphi})^2 \right] + 4g_0^2 \left(\varphi_0 \varphi + \widetilde{\varphi}_0 \widetilde{\varphi} \right)^2 \\ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} M^2 A_\mu^2 - \frac{1}{2a} \left(\partial_\mu A_\mu \right)^2 , \end{aligned} \tag{2.21}$$

where $a = \alpha Z_v^{-1}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the most convenient is the gauge in which, for example, $\widetilde{\varphi}_0 = 0$, $\varphi_0 \neq 0$. In this case, as follows from (2.21), the field $\widetilde{\varphi}$ corresponds to a massless (Goldstone) particle and the field φ describes a scalar particle with the mass $2m_0$ (an analogue of the Higgs particle.) In writing (2.21) we allowed ambiguity in the choice of the Lagrangian up to terms of the divergence-type.

Note that in the papers [265] - [267] a model was investigated, which was a particular case of (2.2) with $\kappa = \lambda$

and $\rho \to \infty$. In this model the equivalence with quantum electrodynamics was established. This occurs when a massless vector field is identified with the photon field. In our notation, this limiting case corresponds to the choice of $\kappa = \lambda$, $M^2 \to 0$. We shall return to this subject below in the discussion of the Ward identities.

From the Lagrangian (2.21) (or directly from the equation (2.16)), we find the expression for the renormalized propagator of A_μ in the lowest order of the perturbation theory

$$\Delta'_{\mu\nu}(p) = \frac{1}{p^2 + M^2}\left(\delta_{\mu\nu} + p_\mu p_\nu \frac{a-1}{aM^2 + p^2}\right). \qquad (2.22)$$

Hence, in the limit $\alpha \to 0$ ($a \to 0$), to be considered in accordance with (2.6), we obtain the propagator for the vector field, which behaves like $O(1/p^2)$.

So, we have renormalized single-particle Green's functions in the lowest order of the perturbation theory.

We now consider the divergence (with $\Lambda \to \infty$) of the highest (multi-particle) Green's functions. First of all, taking into account that $\Delta_{AB}(p) \sim O(1/p^2)$, $S_0(p^2) \sim O(1/p)$ at $p^2 \to \infty$ in the usual way it is possible to derive (see [316]) a formula for the degree of divergence of the diagram (the number of degrees of internal 4-momenta in the numerator minus the number of degrees in the denominator)

$$D = 4 - \frac{3}{2}F - B_{\varphi} - B_{\widetilde{\varphi}} - B_A, \qquad (2.23)$$

where B_{φ}, $B_{\widetilde{\varphi}}$, B_A are the numbers of external lines of boson fields φ, $\widetilde{\varphi}$, A_{μ}, respectively, and F is the number of the external fermion lines of the diagrams when $D < 0$ integrals converge and diagrams have finite values. It is important to emphasize that the value of D does not depend on the nature and number of internal lines of diagrams, but only depends on the external particles. Diagrams will converge if the integrals corresponding to any of the internal blocks are finite. Since D does not depend on the order of perturbation theory, this model contains a finite number of types of divergent diagrams [317]. In (2.17) we had a renormalization of the propagators of the collective fields, which are determined by the diagram of Figure 2.1. According to (2.23) the three- and four-point diagrams will diverge, represented by Fig. 2.2 and Fig. 2.3. Here the line $- - --$ is the set of fields Φ_A. We find from (2.13) the vertex function in the lowest order of the perturbation theory, corresponding to Fig. 2.2

$$\Gamma^0_{ABC}(x, y, z) = \frac{\delta^3 W_0}{\delta\Phi_A(x)\delta\Phi_B(y)\delta\Phi_C(z)},$$

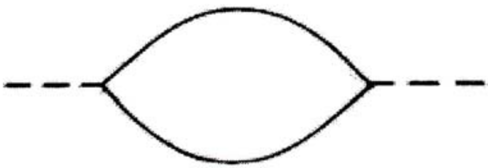

Figure 2.1: Two-point diagram.

$$\Gamma^0_{ABC}(x,y,z) = ig_A g_B g_C \mathrm{tr}\Big[S_0(z,x)\gamma_A S_0(x,y)\gamma_B S_0(y,z)\gamma_C \tag{2.24}$$

$$+S_0(z,y)\gamma_B S_0(y,x)\gamma_A S_0(x,z)\gamma_C\Big].$$

Going into the momentum space

$$\Gamma^0_{ABC}(x,y,z) = \int \frac{d^4p d^4q}{(2\pi)^8}\Gamma^0_{ABC}(p,q)e^{ip(y-x)}e^{iq(x-z)}, \tag{2.25}$$

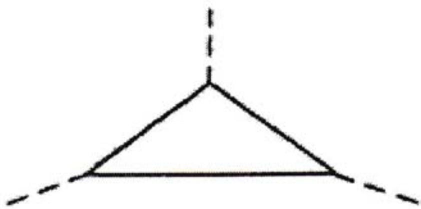

Figure 2.2: Three-point diagram.

from (2.24), we obtain the following expression

$$\Gamma^0_{ABC}(p,q) = ig_A g_B g_C \mathrm{tr}\Big\{\int \frac{d^4k}{(2\pi)^4}\Big[S_0(k+p-q)\gamma_A \times S_0(k)\gamma_B S_0(k+p) + S_0(k-p)\gamma_B S_0(k)\gamma_A S_0(k+q-p)\Big]\gamma_C\Big\}. \tag{2.26}$$

Taking into account the linear divergence of the integrals (2.26), we can regularize the vertex function (2.26) as follows:

$$\Gamma^0_{ABC}(p,q) = \lambda_{ABC} Z_A^{-1/2} Z_B^{-1/2} Z_C^{-1/2} + ig_A g_B g_C \mathrm{reg}\Gamma^0_{ABC}(p,q), \tag{2.27}$$

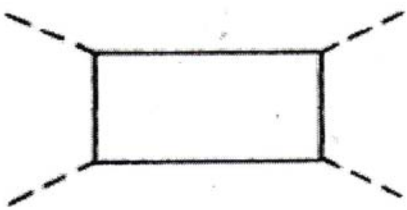

Figure 2.3: Four-point diagram.

where

$$\Gamma^0_{ABC}(0,0) = \lambda_{ABC} Z_A^{-1/2} Z_B^{-1/2} Z_C^{-1/2},$$

$$Z_B = \begin{cases} Z_v, \ B = \mu \\ Z_3, \ B = 5,6 \end{cases},$$

so that the renormalized vertex function Γ'_{ABC} will be determined by the relation

$$\Gamma^{0\prime}_{ABC} = \Gamma^0_{ABC} Z_A^{1/2} Z_B^{1/2} Z_C^{1/2}. \qquad (2.28)$$

Using (2.26), we can calculate the values of λ_{ABC} in (2.27).

Similarly (see [259]), the four-point Green's function, which is described by the diagrams in Fig. 2.3, is regularized.

By definition we have

$$\Gamma^0_{ABCD}(x,y,z,t) = \frac{\delta^4 W_0}{\delta\Phi_A(x)\delta\Phi_B(y)\delta\Phi_C(z)\delta\Phi_D(t)}, \quad (2.29)$$

and in the momentum space it is

$$\begin{gathered}\Gamma^0_{ABCD}(k_1,k_2,k_3) = ig_A g_B g_C g_D \mathrm{tr}\Bigg\{\int \frac{d^4p}{(2\pi)^4} \\ \times\Bigg\{S_0(p)\gamma_D S_0(p+k_2)\Bigg[\gamma_C S_0(p+k_2-k_3) \\ \times\gamma_B S_0(p-k_1)\gamma_A+\gamma_C S_0(p+k_2-k_3)\gamma_A S_0(p+k_1+k_2-k_3)\gamma_B \\ +\gamma_B S_0(p-k_1+k_3)\gamma_C S_0(p-k_1)\gamma_A+\gamma_A S_0(p+k_1+k_2)\gamma_C \\ \times S_0(p+k_1+k_2-k_3)\gamma_B+\gamma_A S_0(p+k_1+k_2)\gamma_B S_0(p+k_3)\gamma_C \\ +\gamma_B S_0(p-k_1+k_3)\gamma_A S_0(p+k_3)\gamma_C\Bigg]\Bigg\}\Bigg\},\end{gathered} \quad (2.30)$$

where

$$\begin{gathered}\Gamma^0_{ABCD}(x,y,z,t) = \int \frac{d^4k d^4q d^4p}{(2\pi)^{12}} \Gamma^0_{ABCD}(k,q,p) \\ \times \exp\left[ik(x-y)+iq(t-y)+ip(y-z)\right].\end{gathered}$$

Note that for the field A_μ, the Furry theorem holds: the total matrix element corresponding to the diagrams with closed fermion loops with an odd number of external vector lines is zero.

In contrast to renormalizable theories, in this model the regularization parameter Λ cannot be removed from the equations (2.10), which define the mass spectrum. Therefore, one should speak of the finite renormalization, because, generally speaking, all diagrams are finite by the finiteness of Λ.

2.1.2 The Ward–Takahashi identities and Dyson–Schwinger equations

We now derive the Ward–Takahashi identities and the DS equation for the total propagators and vertex functions. We introduce in the usual way [314] - [318] the generating functional of connected Green's functions

$$W[\bar{\eta}, \eta, j] = -i\ln Z[\bar{\eta}, \eta, j]. \tag{2.31}$$

To uniquely determine Green's functions we need to go into the Euclidean space-time and to return to the Minkowski space after the functional integration [319]. We will formally work in the Minkowski space, suggesting the need for this procedure.

We use the following definitions of [315]

$$\langle \Phi_A \rangle = \frac{\delta W[J]}{\delta j_A(x)}, \quad \langle \psi(x) \rangle = \frac{\delta W[J]}{\delta \bar{\eta}(x)}, \quad \langle \bar{\psi}(x) \rangle = \frac{\delta W[J]}{\delta \eta(x)}, \tag{2.32}$$

$$\frac{\delta \Gamma[\Phi]}{\delta \langle \Phi_A(x) \rangle} = -j_A(x), \;\; \frac{\delta \Gamma[\Phi]}{\delta \langle \psi(x) \rangle} = -\bar{\eta}(x), \;\; \frac{\delta \Gamma[\Phi]}{\delta \langle \bar{\psi}(x) \rangle} = -\eta(x),$$

where

$$\Gamma[\Phi] = W[J] - \int d^4x \left(j_A \langle \Phi_A \rangle + \bar{\eta} \langle \psi \rangle + \bar{\psi} \langle \eta \rangle \right)$$

is the effective action. The generating functional (2.31) (with $g_0 = g_0'$, $\mu_0 = \mu_0'$, see (2.4)), is invariant under γ_5-transformations of the sources

$$\eta' = e^{-i\alpha\gamma_5}\eta, \qquad \bar{\eta}' = \bar{\eta}e^{-i\alpha\gamma_5},$$
$$j' = j\cos 2\alpha + \widetilde{j}\sin 2\alpha, \quad \widetilde{j}' = -j\sin 2\alpha + \widetilde{j}\cos 2\alpha, \tag{2.33}$$

which is easily verified by a change of variables

$$\psi' = e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi}' = \bar{\psi}e^{i\alpha\gamma_5},$$
$$\varphi' = \varphi\cos 2\alpha + \widetilde{\varphi}\sin 2\alpha, \quad \widetilde{\varphi}' = -\varphi\sin 2\alpha + \widetilde{\varphi}\cos 2\alpha. \tag{2.34}$$

Given this, the condition of independence of the generating functional of the parameters of the transformations α is $dW/d\alpha = 0$, we find

$$\int d^4x \Big\{ \frac{\delta W[J]}{\delta\eta'(x)}\frac{\delta\eta'(x)}{\delta\alpha} + \frac{\delta W[J]}{\delta\bar{\eta}'(x)}\frac{\delta\bar{\eta}'(x)}{\delta\alpha}$$
$$+ \frac{\delta W[J]}{\delta j'(x)}\frac{\delta j'(x)}{\delta\alpha} + \frac{\delta W[J]}{\delta\widetilde{j}'(x)}\frac{\delta\widetilde{j}'(x)}{\delta\alpha}\Big\} = 0, \tag{2.35}$$

where $J = (j_A, \bar{\eta}, \eta)$.

From (2.35) with (2.34) and definitions (2.32), by taking into account $\eta = \bar{\eta} = 0$, we obtain the Ward−Takahashi identity

$$\int d^4x \left\{ \langle \varphi(x) \rangle \frac{\delta \Gamma[\Phi]}{\delta \langle \widetilde{\varphi}(x) \rangle} - \langle \widetilde{\varphi}(x) \rangle \frac{\delta \Gamma[\Phi]}{\delta \langle \varphi(x) \rangle} \right\} = 0. \tag{2.36}$$

A similar derivation was used in the σ-model [319]. Taking the functional derivative of the average fields $\langle \varphi \rangle$, $\langle \widetilde{\varphi} \rangle$ and going to the momentum space, from (2.36) we obtain the relation between the propagators and vertex functions

$$\begin{aligned}
\Delta_{66}^{-1}(x,y) - \Delta_{55}^{-1}(x,y) &= \int d^4z \Big\{ \langle \varphi(z) \rangle \Gamma_{566}(x,z,y) - \langle \widetilde{\varphi}(z) \rangle \\
&\quad \times \Gamma_{556}(x,z,y) \Big\} = \int d^4z \Big\{ \langle \varphi(z) \rangle \Gamma_{665}(x,z,y) \\
&\quad - \langle \widetilde{\varphi}(z) \rangle \Gamma_{655}(x,z,y) \Big\}, \\
\Delta_{56}^{-1}(x,y) + \Delta_{65}^{-1}(x,y) &= \int d^4z \Big\{ \langle \widetilde{\varphi}(z) \rangle \Gamma_{656}(x,z,y) - \langle \varphi(z) \rangle \\
&\quad \times \Gamma_{666}(x,z,y) \Big\} = \int d^4z \Big\{ \langle \varphi(z) \rangle \Gamma_{565}(x,z,y) \\
&\quad - \langle \widetilde{\varphi}(z) \rangle \Gamma_{555}(x,z,y) \Big\},
\end{aligned} \tag{2.37}$$

where

$$\Delta_{AB}^{-1}(x,y) = -\frac{\delta^2\Gamma[\Phi]}{\delta\langle\Phi_A(x)\rangle\delta\langle\Phi_B(y)\rangle},$$
$$\Gamma_{ABC}(x,y,z) = \frac{\delta^3\Gamma[\Phi]}{\delta\langle\Phi_A(x)\rangle\delta\langle\Phi_B(y)\rangle\delta\langle\Phi_C(z)\rangle}.$$

Equations (2.37) are the exact relations for the full Green functions. Hence it is easy to conclude that the validity of relations analogous to (2.37) holds, and for any n-loop approximation if we use an expansion in loops and note that this expansion preserves the symmetry properties of the generating functional. In particular, in the one-loop approximation, where $\langle\Phi_A(x)\rangle = \Phi_A$ is the constant and vertex functions are given by (2.24), (2.26), and (2.37), we obtain in the momentum space

$$\begin{aligned}
\Delta_{66}^{-1}(p) - \Delta_{55}^{-1}(p) &= \varphi_0\Gamma_{566}^0(0,p) - \widetilde{\varphi}_0\Gamma_{556}^0(0,p)\\
&= \varphi_0\Gamma_{665}^0(0,p) - \widetilde{\varphi}_0\Gamma_{655}^0(0,p),\\
\Delta_{56}^{-1}(p) + \Delta_{65}^{-1}(p) &= \widetilde{\varphi}_0\Gamma_{656}^0(0,p) - \varphi_0\Gamma_{666}^0(0,p)\\
&= \varphi_0\Gamma_{565}^0(0,p) - \widetilde{\varphi}_0\Gamma_{555}^0(0,p).
\end{aligned} \tag{2.38}$$

From equation (2.24) by taking into consideration (2.9), after the calculations, we find the following vertex functions (for $A, B, C = 5, 6$)

$$\Gamma_{555}^0(p,q) = \frac{ig_0^4\varphi_0}{4\pi^4}\int\frac{d^4k}{F}\left\{\frac{1}{R}\left[g_0^2\left(3\widetilde{\varphi}_0^2 - \varphi_0^2\right)\right.\right.$$

$$
\begin{aligned}
&+2pk - pq + 3k^2 - 4kq + q^2\Big] \\
&+\frac{1}{H}\left[g_0^2\left(3\widetilde{\varphi}_0^2 - \varphi_0^2\right) - 2pk - 2kq + pq + 3k^2\right]\Big\}, \\
\Gamma^0_{655}(p,q) &= \frac{ig_0^4\widetilde{\varphi}_0}{4\pi^4}\int\frac{d^4k}{F}\Big\{\frac{1}{R}\Big[g_0^2\left(\widetilde{\varphi}_0^2 - 3\varphi_0^2\right) \\
&-2qk - pq + k^2 + q^2\Big] \\
&+\frac{1}{H}\left[g_0^2\left(\widetilde{\varphi}_0^2 - 3\varphi_0^2\right) - pq + k^2\right]\Big\}, \\
\Gamma^0_{666}(p,q) &= \frac{ig_0^4\widetilde{\varphi}_0}{4\pi^4}\int\frac{d^4k}{F}\Big\{\frac{1}{R}\Big[g_0^2\left(3\varphi_0^2 - \widetilde{\varphi}_0^2\right) \\
&+2pk - pq + 3k^2 - 4kq + q^2\Big] \\
&+\frac{1}{H}\left[g_0^2\left(3\varphi_0^2 - \widetilde{\varphi}_0^2\right) - 2pk - 2kq + pq + 3k^2\right]\Big\}, \\
\Gamma^0_{566}(p,q) &= \frac{ig_0^4\varphi_0}{4\pi^4}\int\frac{d^4k}{F}\Big\{\frac{1}{R}\Big[g_0^2\left(\varphi_0^2 - 3\widetilde{\varphi}_0^2\right) \\
&-2qk - pq + k^2 + q^2\Big] \\
&+\frac{1}{H}\left[g_0^2\left(\varphi_0^2 - 3\widetilde{\varphi}_0^2\right) - pq + k^2\right]\Big\},
\end{aligned}
\tag{2.39}
$$

$$\Gamma^0_{656}(p,q) = \frac{ig_0^4\varphi_0}{4\pi^4}\int\frac{d^4k}{F}\left\{\frac{1}{R}\left[g_0^2\left(\varphi_0^2-3\widetilde{\varphi}_0^2\right)+pq+k^2-q^2\right]\right.$$

$$\left.+\frac{1}{H}\left[g_0^2\left(\varphi_0^2-3\widetilde{\varphi}_0^2\right)-2qk+pq+k^2\right]\right\},$$

$$\Gamma^0_{556}(p,q) = \frac{ig_0^4\varphi_0}{4\pi^4}\int\frac{d^4k}{F}\left\{\frac{1}{R}\left[g_0^2\left(\widetilde{\varphi}_0^2-3\varphi_0^2\right)\right.\right.$$

$$\left.-pq+k^2+q^2+2pk-2kq\right]$$

$$\left.+\frac{1}{H}\left[g_0^2\left(\widetilde{\varphi}_0^2-3\varphi_0^2\right)+pq+k^2-2kp\right]\right\},$$

$$\Gamma^0_{665}(p,q) = \frac{ig_0^4\varphi_0}{4\pi^4}\int\frac{d^4k}{F}\left\{\frac{1}{R}\left[g_0^2\left(\varphi_0^2-3\widetilde{\varphi}_0^2\right)\right.\right.$$

$$\left.+2pk+(k-q)^2-qp\right]$$

$$\left.+\frac{1}{H}\left[g_0^2\left(\varphi_0^2-3\widetilde{\varphi}_0^2\right)-2pk+pq+k^2\right]\right\},$$

$$\Gamma^0_{565}(p,q) = \frac{ig_0^4\widetilde{\varphi}_0}{4\pi^4}\int\frac{d^4k}{F}\left\{\frac{1}{R}\left[g_0^2\left(\widetilde{\varphi}_0^2-3\varphi_0^2\right)\right.\right.$$

$$\left.+pq+k^2-q^2\right]$$

$$\left.+\frac{1}{H}\left[g_0^2\left(\widetilde{\varphi}_0^2-3\varphi_0^2\right)+pq+k^2-2kq\right]\right\},$$

where the notations $F = [(k-q)^2 + m_0^2](k^2 + m_0^2)$, $R = (p+k-q)^2 + m_0^2$, $H = (k-p)^2 + m_0^2$ are introduced.

Using (2.16) and (2.39), it is easy to verify that the naive Ward–Takahashi identities (2.38) for the unrenormalized quantities are satisfied. We illustrate the renormalization of the vertex functions in the example of the quantities in (2.38). Putting in the expression (2.39) $p = 0$, and integrating, using the notation (2.16), we obtain

$$\Gamma^0_{566}(0,q) = \Gamma^0_{665}(0,q) = -4g_0^2\varphi_0 Z_3^{-1} + \frac{g_0^4\varphi_0}{2\pi^2}J_1(q)$$
$$-g_0^6\varphi_0\widetilde{\varphi}_0^2 J_2(q),$$
$$\tag{2.40}$$
$$\Gamma^0_{556}(0,q) = \Gamma^0_{655}(0,q) = -4g_0^2\widetilde{\varphi}_0 Z_3^{-1} + \frac{g_0^4\widetilde{\varphi}_0}{2\pi^2}J_1(q)$$
$$-g_0^6\varphi_0^2\widetilde{\varphi}_0 J_2(q),$$

where

$$J_2(q) = i\int \frac{d^4k}{\pi^4}\Big[\frac{1}{[(k-q)^2+m_0^2]^2(k^2+m_0^2)}$$
$$+\frac{1}{[(k-q)^2+m_0^2](k^2+m_0^2)^2}\Big]$$

is a finite integral.

By taking into account (2.17) and (2.28), we see that the Ward–Takahashi identities (2.38) for the renormalized quantities are satisfied.

We now derive the Ward–Takahashi identities connecting the vertex function

$$\Gamma_\mu(z,x,y) = \frac{\delta\langle S(x,y)\rangle^{-1}}{e_0\delta\langle A_\mu(z)\rangle} \qquad \left(\langle A_\mu(z)\rangle = \frac{\delta W[J]}{\delta j_\mu(z)}\right), \tag{2.41}$$

with the complete reverse fermion propagator $\langle S(x,y)\rangle^{-1}$, where

$$\langle S(x,y)\rangle = \frac{\delta^2 W[J]}{\delta\bar\eta(x)\delta\eta(y)}.$$

To do this, we represent (2.41) in the form

$$\Gamma_\mu(z,x,\xi) = -\int d^4y d^4t \langle S(x,y)\rangle^{-1}\frac{\delta\langle S(y,t)\rangle}{e_0\delta\langle A_\mu(z)\rangle}\langle S(t,\xi)\rangle^{-1}. \tag{2.42}$$

Taking into account the equation of motion for the field $A_\mu(x)$

$$\frac{\delta W_0}{\delta A_\mu(x)} = 0 \tag{2.43}$$

and $\delta\langle A_\mu(x)\rangle/\delta A_\nu(y) = \delta_{\mu\nu}\delta(x-y)$, we get the equality

$$\frac{\delta\langle S(x,t)\rangle}{e_0\delta\langle A_\mu(z)\rangle} = \frac{\int D\Phi S(x,z) i\gamma_\mu S(z,t)\exp(iW_0)}{\int D\Phi\exp(iW_0)}. \tag{2.44}$$

Differentiating (2.44) by z_μ, we have

$$\frac{\partial}{\partial z_\mu}\frac{\delta\langle S(y,t)\rangle}{e_0\delta\langle A_\mu(z)\rangle} = i\left(\langle S(y,z)\rangle\delta(z-t) - \langle S(z,t)\rangle\delta(y-z)\right). \tag{2.45}$$

Differentiating (2.42) by z_μ and using (2.45), we obtain the Ward−Takahashi identity

$$\frac{\partial\Gamma_\mu(z,x,y)}{\partial z_\mu} = i\left[\langle S(x,z)\rangle^{-1}\delta(z-y) - \langle S(z,y)\rangle^{-1}\delta(x-z)\right]. \tag{2.46}$$

Passing in (2.46) into momentum space according to formula (2.25), we find

$$\left(p'_\mu - p_\mu\right)\Gamma_\mu(p,p') = \langle S(p)\rangle^{-1} - \langle S(p')\rangle^{-1}. \tag{2.47}$$

We now obtain the DS equations in the model. First of all, the generating functional (2.31) in the Minkowski space reads in the form

$$W[\overline{\eta},\eta,j] = -i\varepsilon\ln\int D\Phi\exp\left(\frac{iW_0}{\varepsilon}\right), \tag{2.48}$$

conveniently for the subsequent expansion of the loops. The expansion in the input parameter ε, which is assumed to be 1 at the end of the calculations [319], corresponds precisely to the expansion in loops.

Now, using the definitions (2.35), (2.48), (2.6), and (2.7), we can derive the following DS equation

$$\left(\gamma_\mu \partial_\mu - g_A \langle \Phi_A(x) \rangle \gamma_A + i\varepsilon g_A \gamma_A \frac{\delta}{\delta j_A}\right) \langle \psi(x) \rangle = \eta(x). \tag{2.49}$$

Then taking the functional derivative $\delta/\delta\eta$ of both sides of (2.49) and assuming $\eta = \bar{\eta} = 0$, we obtain the DS equation for the one-particle Green function

$$\left(\gamma_\mu \partial_\mu - g_A \langle \Phi_A(x) \rangle \gamma_A + i\varepsilon g_A \gamma_A \frac{\delta}{\delta j_A}\right) \langle S(x,y) \rangle = \delta(x-y). \tag{2.50}$$

Similarly, from (2.49) we can obtain equations for the many-particle Green function.

Starting from (2.50) and using the definitions

$$\Delta_{AB}(x,y) = \frac{\delta \langle \Phi_A(x) \rangle}{\delta j_B(y)}, \quad \Gamma_A(z,x,y) = \frac{\delta \langle S(x,y) \rangle^{-1}}{g_A \delta \langle \Phi_A(z) \rangle}, \tag{2.51}$$

we arrive at the integral form of DS equations

$$\langle S(x,y) \rangle^{-1} - (\gamma_\mu \partial_\mu - g_A \langle \Phi_A(x) \rangle \gamma_A)\, \delta(x-y) = \Sigma(x,y), \tag{2.52}$$

where

$$\Sigma(x,y) = -\varepsilon g_A^2 \gamma_A \int d^4t d^4z \langle S(x,t) \rangle \Delta_{AB}(x,z) \Gamma_B(z,t,y)$$

is the mass operator.

The equations of motion for the collective fields $\langle\Phi(x)\rangle$ follow from the condition $\delta W_0/\delta\Phi_A = 0$ (see (2.48) and (2.6)):

$$M_A^2\langle\Phi_A(x)\rangle = j_A + ig_A \mathrm{Tr}\gamma_A\langle S(x,x)\rangle. \tag{2.53}$$

If we now consider that $\delta j_A(x)/\delta\langle\Phi_B(y)\rangle = \Delta^{-1}_{AB}(x,y)$, and making use of the definition (2.51) and the equality

$$\frac{\delta\langle S(x,x)\rangle}{\delta\langle\Phi_A(y)\rangle} = -g_A\int d^4zd^4t\langle S(x,t)\rangle\Gamma_A(y,t,z)\langle S(z,x)\rangle,$$

from (2.53), we find that

$$\begin{aligned} M_A^2\delta_{AB}\delta(x-y) &= \Delta^{-1}_{AB}(x,y) \\ &-ig_A^2\mathrm{tr}\gamma_A\int d^4zd^4t\langle S(x,t)\rangle\Gamma_B(y,t,z)\langle S(z,x)\rangle. \end{aligned} \tag{2.54}$$

From the definitions (2.51), we obtain an expression for the vertex function

$$\Gamma_A(z,x,y) = -\gamma_A\delta(x-y)\delta(x-z) - \frac{\delta\Sigma(x,y)}{g_A\delta\langle\Phi_A(z)\rangle}. \tag{2.55}$$

Similarly, defining the following vertex functions

$$\Gamma_{ABC}(z,x,y) = -\frac{\delta\Delta^{-1}_{BC}(x,y)}{\delta\langle\Phi_A(z)\rangle},$$

$$\Gamma_{ABCD}(t,z,x,y) = -\frac{\delta^2 \Delta^{-1}_{CD}(x,y)}{\delta\langle\Phi_A(t)\rangle\delta\langle\Phi_B(z)\rangle}. \tag{2.56}$$

From (2.54), in particular, it follows, in view of the Ward −Takahashi identities (2.46), that

$$\frac{\partial \Delta^{-1}_{\mu\nu}(x,y)}{\partial y_\nu} = M_0^2 \frac{\partial \delta(x-y)}{\partial y_\mu}, \tag{2.57}$$

i.e. the inverse propagator for the collective fields $\Delta^{-1}_{\mu\nu}$ does not satisfy the transversality, which is associated with a massive field A_μ. At the same time, $M_0 = 0$, $\Delta^{-1}_{\mu\nu}$ as it should be, satisfies the transversality condition, which is consistent with other authors' results [265] - [267].

Consider now the renormalization. Going to the momentum space, from (2.52), (2.54) with (2.25), we obtain

$$\langle S(p)\rangle^{-1} = i\widehat{p} - g_A\langle\Phi_A\rangle\gamma_A - \Sigma(p),$$

$$\Sigma(p) = i\varepsilon\gamma_A \int \frac{d^4k}{(2\pi)^4}\langle S(p-k)\rangle\Delta_{AB}(k)\Gamma_B(p-k,p),$$

$$\Delta^{-1}_{AB}(p) = M_A^2\delta_{AB} + i\mathrm{tr}\gamma_A \tag{2.58}$$

$$\times \int \frac{d^4k}{(2\pi)^4}\langle S(p+k)\rangle\Gamma_B(p+k,k)\langle S(k)\rangle.$$

The renormalization procedure, based on DS equations (2.54) and (2.58) and relations (2.56), which is not associated with the perturbation theory, is carried out in the original model similar to the case of the scalar-scalar interaction considered in [259]. In this case, the basic relations in the momentum space will take the form

$$\Delta'_{AB}(p) = \Delta_{AB}(p) Z_A^{-1}, \quad \Gamma'_A(p,q) = \Gamma_A(p,q) Z_1,$$

$$\psi' = \psi Z_2^{-1/2}, \langle S(p)\rangle' = \langle S(p)\rangle Z_2^{-1},$$

$$\langle \Phi_A \rangle' = \langle \Phi_A \rangle Z_A^{-1/2}, \quad g_A'^2 = g_A^2 Z_A \left(\frac{Z_2}{Z_1}\right)^2, \tag{2.59}$$

$$\Delta_{AB}^{-1}(0) = M_A^2 \delta_{AB} Z_A^{-1}, \qquad \langle S(0)\rangle^{-1} = m Z_2^{-1}.$$

Similar relations can be written for the three- and four-vertex functions.

We now recognize that the renormalized matrix element corresponding to n-vertex diagrams involving m vector-fermion vertices can be schematically written as

$$\mathcal{M} \sim e_0^m g_0^{n-m} \int (\Gamma_A)^n \langle S \rangle^{F_i} \Delta^{B_i^A} \Delta^{B_i^\varphi} \psi^{F_i} \langle A \rangle^{B_e^A} \langle \varphi \rangle^{B_e^\varphi} \psi^{F_e}, \tag{2.60}$$

where F_i is the number of internal fermion lines, F_e - external fermion lines, B_i^A - internal vector lines, B_e^φ - external scalar and pseudoscalar lines, B_e^A - external vector lines,

B_i^{φ} - internal scalar and pseudoscalar lines. Then substituting in (2.60) the renormalized quantities (2.59) and taking into account that $n = F_i + (1/2)F_e$, $m = B_e^A + 2B_i^A$, $n - m = B_e^{\varphi} + 2B_i^{\varphi}$, we find a regularized matrix element

$$\mathcal{M}_R \sim (e_0')^m (g_0')^{n-m} \int (\Gamma'_A)^n (\langle S\rangle')^{F_i} (\Delta')^{B_i^A} (\Delta')^{B_i^{\varphi}}$$

$$\times (\psi')^{F_i} (\langle A\rangle')^{B_e^A} (\langle\varphi\rangle')^{B_e^{\varphi}} (\psi')^{F_e}, \qquad (2.61)$$

Hence one can see that the matrix element of the process does not contain any of the renormalization constants and depends only on the renormalized quantities. It should be noted here that the renormalization procedure is performed just as well as in quantum electrodynamics. This has been possible due to the dependence of the propagators of the collective scalar and vector fields on the momentum: $\Delta_{AB}(p) \sim 1/p^2$ (see (2.22)). Therefore, we can conclude that in this model the renormalization of the charge, fermion masses and collective fields leads to the elimination of their dependence on the cutoff momentum in all orders of perturbation theory. As noted above, the momentum cutoff Λ is present in the theory, and possesses the physical meaning, as it is included in the solution of the gap equation.

2.1.3 Four-fermion interaction with the violation of CP-parity

It is easy to see that Lagrangian (2.1), which was investigated in the preceding sections, is invariant under CP-transformations. It is known, however, that a realistic theory of electro-weak interactions is permitting the violation of CP-parity, and in the theory of strong interactions CP-parity is violated by the topological structure of the QCD vacuum (with a very small parameter of the violation).

Therefore, it is interesting to generalize the Lagrangian (2.1), supplemented by a four-fermion term, violating the conservation of CP-parity. Consider a nonlinear fermion model in four-dimensional space-time with the Lagrangian of the form

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + \frac{\kappa}{2}\left(\bar{\psi}\psi\right)^2 - \frac{\lambda}{2}\left(\bar{\psi}\gamma_5\psi\right)^2 + i\frac{\gamma}{2}\left(\bar{\psi}\psi\right)\left(\bar{\psi}\gamma_5\psi\right). \tag{2.62}$$

It contains an addition, compared to (2.2), the last term, which is the P- and T-odd, and, therefore, results in a strong violation of CP-parity, as the θ-term in QCD. At the same time here the four-fermion vector-vector interaction, contained in (2.2), is not taken into account.

We note that the model in two-dimensional space-time with a structure similar to the nonlinearity in the Lagrangian was considered in [321].

We will investigate the model using the effective potential discussed in the previous sections. First, let us diagonalize the quadratic form in the interaction Lagrangian (2.62):

$$\frac{\kappa}{2}J_0^2 + \frac{\lambda}{2}J_1^2 + \frac{\gamma}{2}J_0 J_1 = \frac{\kappa'}{2}\widehat{J}_0^2 + \frac{\lambda'}{2}\widehat{J}_1^2,$$

$$\widehat{J}_0 = J_0\cos\vartheta - J_1\sin\vartheta, \qquad \widehat{J}_1 = J_0\sin\vartheta + J_1\cos\vartheta, \tag{2.63}$$
$$\kappa' = \frac{1}{2}(\kappa+\lambda) + \frac{1}{2}\sqrt{(\kappa-\lambda)^2+\gamma^2},$$
$$\lambda' = \frac{1}{2}(\kappa+\lambda) - \frac{1}{2}\sqrt{(\kappa-\lambda)^2+\gamma^2},$$

where $J_0 = \bar{\psi}\psi$, $J_1 = i\bar{\psi}\gamma_5\psi$, $\tan 2\vartheta = \gamma/(\lambda-\kappa)$. Then, by introducing collective fields φ, $\widetilde{\varphi}$ and using the notations

$$\Phi = g_0\varphi\cos\vartheta + \widetilde{g}_0\widetilde{\varphi}\sin\vartheta,\ \ \widetilde{\Phi} = -g_0\varphi\sin\vartheta + \widetilde{g}_0\widetilde{\varphi}\cos\vartheta, \tag{2.64}$$

for the generating functional (see (2.3)) after the path integrating over the fields $\bar{\psi}$, ψ, we get

$$Z[\bar{\eta},\eta] = N\int D\varphi D\widetilde{\varphi}\ \det\left(-\gamma_\mu\partial_\mu + \Phi + i\widetilde{\Phi}\gamma_5\right) \tag{2.65}$$
$$\times\exp\Big\{i\int d^4x d^4y\Big[-\frac{1}{2}\left(\mu_0^2\varphi^2 + \widetilde{\mu}_0^2\widetilde{\varphi}^2\right)\delta(x-y)$$

$$+\bar{\eta}(x)S_\theta(x,y)\eta(y)\Big]\Big\}.$$

Here $\kappa' = g_0^2/\mu_0^2$, $\lambda' = \widetilde{g}_0^2/\widetilde{\mu}_0^2$. The fermion Green function $S_\theta(x,y)$ satisfies the equation (see (2.7))

$$\left(\gamma_\mu\partial_\mu - \Phi - i\widetilde{\Phi}\gamma_5\right) S_\theta(x,y) = \delta(x-y). \tag{2.66}$$

After introducing the condensate, according to "shifts" $\varphi(x) = \varphi_0 + \varphi'(x)$, $\widetilde{\varphi}(x) = \widetilde{\varphi}_0 + \widetilde{\varphi}'(x)$, we use the gauge $\widetilde{\varphi}_0 = 0$, $\varphi_0 \neq 0$. Then the equation of the gap (see (2.10)) takes the form here

$$1 = -\frac{ig_0^2}{4\pi^4\mu_0^2}\int \frac{d^4p}{p^2+m^2}, \tag{2.67}$$

where $m = -g_0\varphi_0$.

The solution of (2.66) in the one-loop approximation for the fermion Green function in the momentum space will be expressed as follows:

$$S_\theta = \frac{-i\widehat{p} + m\exp\left(i\theta\gamma_5\right)}{p^2+m^2}. \tag{2.68}$$

Note that the fermion Green's function (2.68), corresponding to the one-loop approximation, breaks CP-parity. There is a similar solution in the infrared region of QCD obtained in [322].

We now use the expression for the effective action of [318]

$$S_{eff} = -\frac{1}{2}\int d^4x d^4y\ \varphi'_A(x)\Delta^{-1}_{AB}(x,y)\varphi'_B(y)$$
$$+\frac{1}{3!}\int d^4x d^4y d^4z\ \varphi'_A(x)\varphi'_B(y)\varphi'_C(z)\Gamma_{ABC}(x,y,z) \quad (2.69)$$
$$+\frac{1}{4!}\int d^4x d^4y d^4z d^4t\ \varphi'_A(x)\varphi'_B(y)\varphi'_C(z)\varphi'_D(t)$$
$$\times\Gamma_{ABCD}(x,y,z,t),$$

where the inverse propagators Δ^{-1}_{AB}, and 3- and 4-point Green's functions Γ_{ABC}, Γ_{ABCD} are given by formulas (2.14), (2.26) and (2.30).

Calculating Green's functions from (2.14), (2.26), and (2.30) with the replacement

$$\gamma_A \to \Gamma_A = (g_0\cos\vartheta - i\gamma_5 g_0\sin\vartheta, \quad \widetilde{g}_0\sin\vartheta + i\gamma_5\widetilde{g}_0\cos\vartheta),$$

$$M_A \to \mu_A = (\mu_0, \widetilde{\mu}_0)$$

and substituting in (2.69), after renormalization of fields and the introduction of the renormalized constant $g^2 = Z_3$, up to $\mathcal{O}(g_0^2)$, one finds

$$S_{eff} = \int d^4x\Big\{-\frac{1}{2}\left[(\partial_\mu\varphi)^2 + (\partial_\mu\widetilde{\varphi})^2 + 4m^2\varphi^2 + M^2\widetilde{\varphi}^2\right]$$

$$+2mg\varphi\left(\varphi^2+\widetilde{\varphi}^2\right)-\frac{1}{2}g^2\left(\varphi^2+\widetilde{\varphi}^2\right)^2\Big\}, \tag{2.70}$$

where

$$M=\sqrt{\widetilde{\mu}^2-\mu^2},\qquad \mu^2=\frac{Z_3}{\kappa'},\qquad \widetilde{\mu}^2=\frac{Z_3}{\lambda'},$$

$$Z_3^{-1}=-\frac{i}{8\pi^4}\int\frac{d^4q}{\left(q^2+m^2\right)^2},$$

$$\varphi=g_0Z_3^{-1/2}\varphi',\quad \widetilde{\varphi}=\widetilde{g}_0Z_3^{-1/2}\widetilde{\varphi}'.$$

The requirement of the absence of tachyons in the spectrum, that is $\widetilde{\mu}^2-\mu^2>0$, leads to the condition $\gamma<2\sqrt{\kappa\lambda}$. Also in this case $\lambda'>0$ (see (2.63)).

We leave in (2.69) only terms having the maximum fourth degree in the fields. Here n-point Green's functions for $n\geq 5$ are ignored because they are convergent and define higher (radiation) corrections in constants g_0, $\widetilde{g}_0$ [259].

Thus, the effective action for the collective fields φ, $\widetilde{\varphi}$ takes the form of the well-known renormalizable σ-model with a single dimensionless coupling constant g [108].

Chiral invariance violation (2.33) of this model is due to the inequalities $\kappa-\lambda\neq 0$, $\gamma\neq 0$, and by (even if $\kappa=\lambda$, $\gamma=0$) the restructuring of the physical vacuum.

It follows from (2.70) that the field φ has the mass of $2m$ - twice the mass of the fermion, and the field $\widetilde{\varphi}$ has the mass $M = \sqrt{\widetilde{\mu}^2 - \mu^2}$. However, we note that the fields φ, $\widetilde{\varphi}$ do not constitute a definite parity in view of (2.64), since there is a mix of a scalar field Φ and a pseudoscalar field $\widetilde{\Phi}$. Lagrangian (2.62) takes the form of chirally-invariant Lagrangian (1.1) [1] - [3] in the case when the angle of the mixing $\theta = 0$, corresponding to $\gamma = 0$, $\kappa = \lambda$. Then $\widetilde{\mu}^2 - \mu^2 = 0$, and the field $\widetilde{\varphi}$ is the massless Goldstone field (see [318]), arising due to the dynamic violation of chiral invariance of the Lagrangian under the transformations (2.33). But since, in general, the Lagrangian (2.62) is not invariant under the transformations (2.33), the corresponding axial-vector current is only partially conserved.

This can be verified directly using, for obtaining the axial-vector current and its divergence, the variational method of Gell-Mann$-$Levi [108]. At the same time, we note that the chiral transformations of fermion fields (2.33) generate the corresponding transformations of boson fields

$$\begin{aligned} \varphi - \frac{m}{g} &\to \left(\varphi - \frac{m}{g}\right)\cos\alpha + \widetilde{\varphi}\sin\alpha, \\ \widetilde{\varphi} &\to -\left(\varphi - \frac{m}{g}\right)\sin\alpha + \widetilde{\varphi}\cos\alpha, \end{aligned} \tag{2.71}$$

so that $\delta\varphi = \widetilde{\varphi}\alpha$, $\delta\widetilde{\varphi} = -(\varphi - m/g)\,\alpha$ (with $\alpha \ll 1$). Equation (2.71) shows that due to the "shift" of the field here, a dynamic violation of the $SO(2)$-transformations of fields φ, $\widetilde{\varphi}$ occurs. Applying the method of Gell-Mann−Levi [108], from (2.70) by taking into account (2.71), we find the following expression for the axial current:

$$A_\mu = \frac{\partial\delta\mathcal{L}_{eff}}{\partial\left(\partial_\mu\alpha(x)\right)} = \varphi\partial_\mu\widetilde{\varphi} - \widetilde{\varphi}\partial_\mu\varphi - \frac{m}{g}\partial_\mu\widetilde{\varphi}, \qquad (2.72)$$

in which the last extra term is precisely due to the Goldstone mechanism of symmetry breaking. According to the equation of Gell-Mann−Levy $\partial_\mu A_\mu = \partial\delta\mathcal{L}_{eff}/\partial\alpha$, we find the divergence of axial current (2.72)

$$\partial_\mu A_\mu = M^2\left(\varphi - \frac{m}{g}\right)\widetilde{\varphi}, \qquad (2.73)$$

which is in agreement with PCAC. Expression (2.73) can also be obtained directly by taking the divergence from (2.72) and using the equations of motion of the fields φ, $\widetilde{\varphi}$. Zero inequality of $\partial_\mu A_\mu$ represents a clear violation of the invariance of the Lagrangian (2.62) with respect to chiral transformations. In the chiral limit ($\alpha = \beta$, $\gamma = 0$), we have $\widetilde{\mu}^2 - \mu^2 = (1/\kappa' - 1/\lambda')Z_3 = 0$ and obtain the condition $\partial_\mu A_\mu = 0$.

2.1.4 Dynamical mass formation and symmetry breaking in $SU(n) \otimes U(1)$ four-fermion models

In the previous sections we have considered the four-fermion models that allow only one-parameter chiral (γ_5) symmetry. Since the realistic theories of electro-weak and strong interactions have multi-parameter internal symmetry then it is of interest to study four-fermion models with other, broader groups of the symmetry.

Consider a model based on the Lagrangian with the internal symmetry group $SU(n) \otimes U(1)$ and two coupling constants

$$\mathcal{L} = -\bar{\psi}\left(\gamma_\mu \partial_\mu + m\right)\psi + \frac{F}{2}\left(\bar{\psi}\psi\right)^2 + \frac{G}{2}\left(\bar{\psi}T^a\psi\right)^2, \quad (2.74)$$

where T^a $(a = 1, 2, ...n^2 - 1)$ are generators of the $SU(n)$ group, m is a bare mass of the fermions. Moreover, since the four-fermion scalar-scalar interactions result in the dynamic symmetry breaking we take into account only such interaction in the Lagrangian (2.74). In this case, the generating functional for the Green function can be represented by introducing collective Bose fields in the form (see Sec. 1)

$$Z[\bar{\eta}, \eta] = N_1 \int D\Phi_A \exp\{iS[\Phi]$$

$$+i \int d^4x d^4y \bar{\eta}(x) S_f(x,y) \eta(y)\}, \tag{2.75}$$

where

$$S[\Phi] = -\frac{1}{2} \int d^4x \left(M^2 \Phi_0^2 + \mu^2 \Phi_a^2 \right)$$
$$-i\mathrm{Tr}\ \ln \left[1 + \widehat{G} \left(f\Phi_0 + g\Phi_a T^a \right) \right]$$

is the effective action of collective fields, $F = f^2/M^2$, $G = g^2/\mu^2$, $\widehat{G}$ is Green's function for the free Dirac equation

$$\left(\gamma_\mu \partial_\mu + m \right) \widehat{G}(x,y) = -\delta(x-y), \tag{2.76}$$

and $S_f(x,y)$ being Green's function of fermions in external fields Φ_A:

$$\left(\gamma_\mu \partial_\mu + m - f\Phi_0 - g\Phi_a T^a \right) S_f(x,y) = \delta(x-y). \tag{2.77}$$

We seek a solution of equation (2.77) in the case of the fields Φ_A being constants, independent of the coordinates (one-loop approximation). We write (2.77) in the momentum space

$$\left(i\hat{p} - A \right) S_f(p) = 1, \tag{2.78}$$

where $\hat{p} = p_\mu \gamma_\mu$, $A = -m + f\Phi_0 + g\Phi_a T^a$.

To find $S_f(p)$ the method proposed in [324] is used here. The matrix A, according to the Hamilton$-$Cayley theorem (see e.g. [325]), satisfies its characteristic equation

$$A^2 - Ab_1 + |A| = 0 \quad \text{(for SU(2))},$$

$$A^3 - A^2 b_1 + A b_2 - |A| = 0 \quad \text{(for SU(3))}, \tag{2.79}$$

$$A^5 - A^4 b_1 + A^3 b_2 - A^2 b_3 + A b_4 - |A| = 0 \quad \text{(for SU(5))},$$

where

$$b_1 = \mathrm{tr} A \equiv A_t,\ |A| \equiv \det A,\ b_2 = \frac{1}{2}\left[(A_t)^2 - \left(A^2\right)_t\right],$$

$$b_3 = \frac{1}{3}\left[\left(A^3\right)_t - \frac{3}{2} A_t \left(A^2\right)_t + \frac{1}{2}(A_t)^3\right],$$

$$b_4 = -\frac{1}{4}\Bigg[\left(A^4\right)_t - \frac{4}{3} A_t \left(A^3\right)_t + (A_t)^2 \left(A^2\right)_t$$

$$-\frac{1}{2}\left(\left(A^2\right)_t\right)^2 - \frac{1}{6}(A_t)^4\Bigg].$$

We look for a solution to equation (2.78) in the form

$$S_f(p) = a + b\hat{p} + c_n A^n + d_n \hat{p} A^n. \tag{2.80}$$

Here, the A^n means the n-degree power of the matrix A and a summation on n is implied. Substituting (2.80) into (2.78), using (2.79) and the linear independence of the matrices 1, $\hat{p}$, A^n, $\hat{p}A^n$ ($\hat{p}A^n$ makes sense to the direct product of matrices $\hat{p}$ and A^n), we obtain a system of equations for the unknown coefficients, which are found by solving

$$a = -\frac{b_1 |A|}{\Delta_1}, \quad b = -\frac{i}{\Delta_1}\left(p^2 - |A| + b_1^2\right), \quad d_1 = \frac{i}{\Delta_1} b_1,$$

$$c_1 = -\frac{1}{\Delta_1}\left(p^2 - |A|\right), \quad \Delta_1 = \left(p^2 + m_1^2\right)\left(p^2 + m_2^2\right),$$

$$m_1 = f\Phi_0 + g\sqrt{\Phi_a^2} - m, \quad m_2 = f\Phi_0 - g\sqrt{\Phi_a^2} - m,$$

$$\text{(for SU(2))}$$

$$a = \frac{i|A|}{\Delta_2}\left(p^2 - b_2\right),$$

$$b = -\frac{i}{\Delta_2}\left[\left(p^2 - b_2\right)^2 + b_1\left(p^2 b_1 - |A|\right)\right],$$

$$c_1 = -\frac{1}{\Delta_2}\left([p^4 - \left(b_2 + b_1^2\right)p^2 - b_1|A|\right],$$

$$c_2 = \frac{1}{\Delta_2}\left(b_1 p^2 - |A|\right),$$

$$d_1 = -\frac{i}{\Delta_2}\left(|A| - b_1 b_2\right), \ d_2 = \frac{i}{\Delta_2}\left(p^2 - b_2\right),$$

$$\Delta_2 = p^2\left(p^2 - b_2\right)^2 + \left(|A| - p^2 b_1\right)^2, \ \text{(for SU(3))} \tag{2.81}$$

$$a = -\frac{|A|}{\Delta}\left(p^4 - p^2 b_2 + b_4\right),$$

$$b = \frac{i}{\Delta}\Big[-p^8 + p^6\left(2b_2 - b_1^2\right) + p^4\left(2b_1 b_3 - 2b_4 - b_2^2\right)$$

$$+p^2\left(2b_2 b_4 - b_1|A| - b_3^2\right) + b_3|A| - b_4^2\Big],$$

$$c_1 = \frac{1}{\Delta}\Big[-p^8 + p^6\left(2b_2 - b_1^2\right) + p^4\left(2b_1b_3 - b_4 - b_2^2\right)$$

$$+p^2\left(b_2b_4 - b_1|A| - b_3^2\right) + b_3|A|\Big],$$

$$c_2 = \frac{1}{\Delta}\left[-p^4b_3 + p^2\left(-b_1b_4 + b_2b_3 + |A|\right) - b_2|A|\right],$$

$$c_3 = \frac{1}{\Delta}\left[p^6 + p^4\left(b_1^2 - b_2\right) + p^2\left(b_4 - b_1b_3\right) + b_1|A|\right],$$

$$c_4 = \frac{1}{\Delta}\left[-p^4b_1 + p^2b_3 - |A|\right],$$

$$d_1 = \frac{i}{\Delta}\left[p^2\left(-b_1b_4 + |A|\right) - b_2|A| + b_3b_4\right],$$

$$d_2 = \frac{i}{\Delta}\Big[p^6 + p^4\left(b_1^2 - 2b_2\right)$$

$$+p^2\left(b_4 + b_2^2 - b_1b_3\right) + b_1|A| - b_2b_4\Big],$$

$$d_3 = \frac{i}{\Delta}\left[p^2\left(-b_1b_2 + b_3\right) - |A| + b_1b_4\right],$$

$$d_4 = \frac{i}{\Delta}\left[-p^4 + p^2b_2 - b_4\right],$$

$$\Delta = p^{10} + p^8\left(b_1^2 - 2b_2\right) + p^6\left(2b_4 + b_2^2 - 2b_1b_3\right)$$

$$+p^4\left(b_3^2 + 2b_1|A| - 2b_2b_4\right) + p^2\left(b_4^2 - 2b_3|A|\right) + |A|^2$$

(for SU(5)).

The poles of the Green function $S_f(p)$ define the fermion masses of the field ψ. You can make sure that we have the expansions (see [324])

$$\Delta_2 = \left(p^2 + m_1^2\right)\left(p^2 + m_2^2\right)\left(p^2 + m_3^2\right) = \det|p^2 + A^2|$$

(for SU(3)),

$$\Delta = \left(p^2 + m_1^2\right)\left(p^2 + m_2^2\right)\left(p^2 + m_3^2\right) \tag{2.82}$$

$$\times\left(p^2 + m_4^2\right)\left(p^2 + m_5^2\right) = \det|p^2 + A^2| \quad \text{(for SU(5))},$$

where m_i are eigenvalues of the matrix $(-A)$.

Thus, the eigenvalues of $(-A)$ identify the dynamic mass of the fermions. Expressions (2.80) - (2.82) define the Green function of fermions in the general covariant form, since all the coefficients are expressed in terms of invariants of the $U(n)$ group. The convenient gauge is that in which the matrix A is diagonal. In this case we choose $\Phi_0 \neq 0$, $\Phi_3 \neq 0$, $\Phi_{15} \neq 0$, $\Phi_{24} \neq 0$, and the remaining $\Phi_A = 0$. Then (2.80) takes the quasi-diagonal form

$$S_f(p) = \begin{pmatrix} . & 0 & 0 \\ 0 & \frac{-i\hat{p}+m_j}{p^2+m_j^2} & 0 \\ 0 & 0 & . \end{pmatrix}, \tag{2.83}$$

where

$$m_1 = m_0 - g\Phi_3, \quad m_2 = m_0 + g\Phi_3 \quad (\text{for SU}(2)),$$

$$\begin{aligned}
m_1 &= m_0 - g\Phi_3 - \frac{g}{\sqrt{3}}\Phi_8, \quad m_2 = m_0 + g\Phi_3 - \frac{g}{\sqrt{3}}\Phi_8, \\
m_3 &= m_0 + \frac{2g}{\sqrt{3}}\Phi_8 \quad (\text{for SU}(3)), \\
m_1 &= m_0 - g\left(\Phi_3 + \frac{1}{\sqrt{3}}\Phi_8 + \frac{2}{\sqrt{15}}\Phi_{24}\right), \\
m_2 &= m_0 - g\left(-\Phi_3 + \frac{1}{\sqrt{3}}\Phi_8 + \frac{2}{\sqrt{15}}\Phi_{24}\right), \\
m_3 &= m_0 + g\left(\frac{2}{\sqrt{3}}\Phi_8 + \frac{2}{\sqrt{15}}\Phi_{24}\right), \\
m_4 &= m_0 - g\left(\Phi_{15} - \frac{3}{\sqrt{15}}\Phi_{24}\right), \\
m_5 &= m_0 + g\left(\Phi_{15} + \frac{3}{\sqrt{15}}\Phi_{24}\right) \quad (\text{for SU}(5)).
\end{aligned} \tag{2.84}$$

Since there is $m_0 = m - f\Phi_0$, then it follows from (2.84) that, even if the bare mass of fermions is $m = 0$, then the fermions still acquire non-zero dynamic masses.

Receiving, from (2.75), the equations for the fields $\Phi_A(x)$ are

$$\begin{aligned}\frac{\delta S[\Phi]}{\delta \Phi_0(x)} &= -M^2\Phi_0(x) + if\mathrm{Tr}S_f(x,x) = 0,\\ \frac{\delta S[\Phi]}{\delta \Phi_a(x)} &= -\mu^2\Phi_a(x) + ig\mathrm{Tr}\left[S_f(x,x)T^a\right] = 0,\end{aligned} \tag{2.85}$$

and then inserting them into (2.83), we obtain a system of equations for the vacuum expectation values of Φ_A (for the group $SU(5)$):

$$\begin{gathered} M^2\Phi_0 = f(I_1m_1 + I_2m_2 + I_3m_3 + I_4m_4 + I_5m_5),\\ \mu^2\Phi_3 = g(I_1m_1 - I_2m_2),\\ \sqrt{3}\mu^2\Phi_8 = g(I_1m_1 + I_2m_2 - 2I_3m_3),\\ \mu^2\Phi_{15} = g(I_4m_4 - I_5m_5),\\ \sqrt{15}\mu^2\Phi_{24} = g(2I_1m_1 + 2I_2m_2 + 2I_3m_3 - 3I_4m_4 - 3I_5m_5), \end{gathered} \tag{2.86}$$

where

$$I_j = \frac{i}{4\pi^4}\int \frac{d^4p}{p^2 + m_j^2} \quad (j = 1, 2, ...5), \tag{2.87}$$

where $d^4p = id^3pdp_0$. These equations can be seen as a self-consistency condition, and hence they can be used to obtain mass formulas for fermions. When considering the group $SU(3)$, one has to put $m_4 = m_5 = 0$ in equation (2.86), and for a group $SU(2)$, additionally, it requires $m_3 = 0$. Indeed, given that the integrals I_j (2.87), appearing in (2.86) are quadratically divergent and make use of eliminating infinities by dimensional regularization [326] (see also [327]), which is most appropriate way to preserve the symmetry properties of the model, we have [327]

$$I = \frac{i}{4\pi^4} \int \frac{d^4p}{p^2 + m^2} = -\frac{m^2}{(2\pi)^2}\Gamma(-1),$$

where the Gamma function is given by

$$\Gamma(-1) = -\left(\frac{1}{\varepsilon} + \psi(2)\right),$$

and $\psi(2) = 1.5 - \gamma$, $\gamma = 0.5772...$ is Euler's constant, $\varepsilon \to 0$. From these relations we arrive at the connection between integrals:

$$I_i = \left(\frac{m_i}{m_j}\right)^2 I_j \quad (i, j = 1, 2, ...5). \tag{2.88}$$

In the end, allowing the self-consistency conditions (2.86) with (2.88), we arrive at the following formulas for the fermions:

$$m_1 (m_1 + m_2) = m_3 (m_2 + m_3),$$

$$m_2\,(m_3+m_2)=m_4\,(m_4+m_3)\,,$$
$$m_3\,(m_3+m_4)=m_5\,(m_4+m_5)\,. \tag{2.89}$$

It follows that in the case of the $SU(5)$ group that from five fermion masses (or vacuum expectation Φ_0, Φ_3, Φ_8, Φ_{15}, Φ_{24}) the only two are independent (e.g. m_1, m_2). The rest of the masses can be expressed in terms of these two quantities. The equation for these mass values can be obtained from (2.86) taking into account (2.88) and (2.89).

We now calculate the "free" effective action which is quadratic in the collective fields (with $n = 2, 3, 5$). Let us expand the fields in equation (2.75) in the vicinity of the vacuum expectation values which are the solutions of (2.86):

$$\Phi_0(x)=\Phi_0+\Phi_0'(x),\ \Phi_3(x)=\Phi_3+\Phi_3'(x),$$
$$\Phi_8(x)=\Phi_8+\Phi_8'(x),\quad \Phi_{15}(x)=\Phi_{15}+\Phi_{15}'(x), \tag{2.90}$$
$$\Phi_{24}(x)=\Phi_{24}+\Phi_{24}'(x),\ \Phi_b(x)=\Phi_b'(x),$$

($b \neq 0, 3, 8, 15, 24$). Inverse propagators calculated by a formula similar to (2.14) read

$$\Delta^{-1}_{AB}(p)=-ig_Ag_B\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S_f(k)T^AS_f(k-p)T^B+\delta_{AB}M_A^2, \tag{2.91}$$

where $g_A = (f, g)$, $M_A = (M, \mu)$, $T^A = (1, T^a)$. Generators of the group $SU(2)$ are the Pauli matrices τ^a, generators of the $SU(3)$ group are Gell-Mann matrices λ^a, and we use for the $SU(5)$ group the following expressions, normalized by the condition $\mathrm{tr} T^a T^b = 2\delta_{ab}$: the diagonal matrices

$$T^3 = \mathrm{diag}(1, -1, 0, 0, 0),\ T^8 = \frac{1}{\sqrt{3}}\mathrm{diag}(1, 1, -2, 0, 0),$$

$$T^{15} = \mathrm{diag}(0, 0, 0, 1, -1),\ T^{24} = \frac{2}{\sqrt{15}}\mathrm{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}),$$

and non-diagonal generators of the form

$$T^{(mn)} = \varepsilon^{m,n} + \varepsilon^{n,m},\ T^{[mn]} = i\left(\varepsilon^{m,n} - \varepsilon^{n,m}\right),$$

$m, n = 1, 2, 3, 4, 5$ $(m \neq n)$, where $\varepsilon^{m,n}$ are the elements of the entire matrix algebra with the properties [328], [329]

$$\varepsilon^{m,n}\varepsilon^{k,p} = \delta_{nk}\varepsilon^{m,p}, \quad \left(\varepsilon^{m,n}\right)_{ab} = \delta_{ma}\delta_{nb}.$$

We introduce renormalized fields $\Phi_a = Z_3^{-1/2}\Phi'_a$, $\Phi_0 = \sqrt{\frac{n}{2}}\frac{f}{g}Z_3^{-1/2}\Phi'_0$ and constants $g'^2 = Z_3 g^2$, $f'^2 = Z_3 f^2$. Consider that in the scheme of the dimensional regularization there is a connection between quadratic and logarithmically divergent integrals [327], [330], [331]. Indeed, we have

[327]

$$Z_3^{-1} = -\frac{ig^2}{4\pi^4}\int \frac{d^4p}{\left(p^2+m_1^2\right)^2} = \frac{\Gamma(0)}{(2\pi)^2)\Gamma(2)},$$

$$\Gamma(0) = \frac{1}{\varepsilon} + 1 - \gamma,$$

so that

$$Z_3^{-1} = \frac{g^2}{m_1^2}I_1 - \frac{g^2}{8\pi^2}. \tag{2.92}$$

Then, after the computations of (2.91) and taking into consideration (2.92), up to radiative corrections $\mathcal{O}(g^2)$, $\mathcal{O}(f^2)$, $\mathcal{O}(fg)$, we find that the renormalized effective action is

$$S_{free} = -\frac{1}{2}\int d^4x \left[(\partial_\mu \Phi_A)^2 + m_{AB}^2 \Phi_A \Phi_B\right], \tag{2.93}$$

where the elements of the mass matrices have the form

$$m_{00}^2 = 3\left(m_1^2 + m_2^2\right) + \frac{2\left(m_1^3 + m_2^3\right)}{2m - m_1 - m_2},$$

$$m_{03}^2 = 3\left(m_1^2 - m_2^2\right), \quad m_{33}^2 = (m_1 - m_2)^2 \quad \text{(for SU(2))},$$

$$m_{00}^2 = 2\left(m_1^2 + m_2^2 + m_3^2\right) + \frac{2\left(m_1^3 + m_2^3 + m_3^3\right)}{3m - m_1 - m_2 - m_3},$$

$$m_{03}^2 = \sqrt{6}\left(m_1^2 - m_2^2\right), \quad m_{33}^2 = (m_1 - m_2)^2,$$

$$
\begin{gathered}
m_{88}^2 = 4m_3^2 - (m_1+m_2)^2\,, \quad m_{38}^2 = \sqrt{3}\left(m_1^2 - m_2^2\right), \\
m_{08}^2 = \sqrt{2}\left(m_1^2 + m_2^2 - 2m_3^2\right) \quad \text{(for SU(3))},
\end{gathered}
\tag{2.94}
$$

$$
\begin{gathered}
m_{00}^2 = \frac{6}{5}\left(m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2\right) \\
+ \frac{2\left(m_1^3 + m_2^3 + m_3^3 + m_4^3 + m_5^3\right)}{5m - m_1 - m_2 - m_3 - m_4 - m_5}, \\
m_{03}^2 = \frac{6\sqrt{2}}{\sqrt{5}}\left(m_1^2 - m_2^2\right), \quad m_{33}^2 = (m_1 - m_2)^2\,, \\
m_{88}^2 = 4m_3^2 - (m_1+m_2)^2\,, \quad m_{38}^2 = 2\sqrt{3}\left(m_1^2 - m_2^2\right), \\
m_{08}^2 = 2\sqrt{\frac{6}{5}}\left(m_1^2 + m_2^2 - 2m_3^2\right), \\
m_{15,15}^2 = (m_4 - m_5)^2\,, \quad m_{0,15}^2 = \frac{6\sqrt{2}}{5}\left(m_4^2 - m_5^2\right), \\
m_{24,24}^2 = \frac{12}{15}\left(m_1^2 + m_2^2 + m_3^2\right) + \frac{27}{15}\left(m_4^2 + m_5^2\right) \\
-2\left(m_1^2 + m_1 m_2 + m_2^2\right), \\
m_{0,24}^2 = \frac{12}{5}\sqrt{\frac{2}{3}}\left(m_1^2 + m_2^2 + m_3^2 - \frac{3}{2}m_4^2 - \frac{3}{2}m_5^2\right), \\
m_{3,24}^2 = \frac{12}{\sqrt{15}}\left(m_1^2 - m_2^2\right),
\end{gathered}
$$

$$m_{15,24}^2 = \frac{18}{\sqrt{15}} \left(m_5^2 - m_4^2 \right) \quad \text{(for SU(5))}.$$

To obtain the mass spectrum of collective fields Φ_A the mass matrix must be brought to the diagonal form.

It follows from (2.93) and (2.94) that the masses of the fields Φ_A $(A \neq 0, 3, 8, 15, 24)$ are equal to zero, which is consistent with the Goldstone theorem [318] of spontaneous (or dynamic) symmetry breaking; the rest of the fields Φ_A acquire non-zero masses.

Let us diagonalize the mass matrix (2.94) for the consideration of the $SU(2)$ group. We make the transformation from the $SO(2)$ group

$$\Phi_0' = \Phi_0 \cos\alpha - \Phi_3 \sin\alpha, \quad \Phi_3' = \Phi_0 \sin\alpha + \Phi_3 \cos\alpha, \quad (2.95)$$

where

$$\tan 2\alpha = \frac{3 \left(m_2^2 - m_1^2 \right) \left(2m - m_1 - m_2 \right)}{2 \left[m \left(m_1^2 + m_1 m_2 + m_2^2 \right) - m_1 m_2 \left(m_1 + m_2 \right) \right]}.$$

From the above way to diagonalize the mass matrix, we find the following expressions for the masses of collective fields Φ_0', Φ_3' (fields Φ_1, Φ_2 remain massless):

$$m_{00}'^2 = \frac{1}{2} \left[m_{00}^2 + \left(m_1 - m_2 \right)^2 \right]$$

$$+ \frac{1}{2} \sqrt{\left[m_{00}^2 - \left(m_1 - m_2 \right)^2 \right]^2 + 36 \left(m_1^2 - m_2^2 \right)^2},$$

$$m_{33}^{\prime 2} = \frac{1}{2}\left[m_{00}^2 + (m_1 - m_2)^2\right]$$

$$-\frac{1}{2}\sqrt{\left[m_{00}^2 - (m_1 - m_2)^2\right]^2 + 36\left(m_1^2 - m_2^2\right)^2}.$$

The computation using formulas similar to (2.26), (2.30), and (2.69), gives the following effective Lagrangian of self-interaction of collective fields for the $SU(2)$ group:

$$\mathcal{L}_{int}[\Phi] = -3\left(m_1 + m_2\right)\Phi_0\Phi_a^2 - \left(m_1 + m_2\right)\Phi_0^3$$
$$-3\left(m_1 - m_2\right)\Phi_3\Phi_0^2 - \left(m_1 - m_2\right)\Phi_3\Phi_a^2 - \frac{1}{4}\mathrm{tr}\Phi^4, \tag{2.96}$$

where $\Phi = \Phi_0 + \tau^a\Phi_a$.

Note that the field Φ_0 is a singlet, and the fields Φ_a transform according to the adjoint representation of the $SU(2)$ group. If the vacuum field $\Phi_3 = 0$, then it follows from (2.84) that $m_1 = m_2$. Then (2.94) and (2.96) imply that the initial symmetry is restored, and all the fields Φ_a become massless (Φ_0 is a massive field).

Thus, considering this section the four-fermion model allows you to set, when using dimensional regularization, on the one hand, the relation (2.89) between the dynamical masses of the fermion multiplet, and to enter on the other hand, the relation (2.94) for fermion masses and the masses of collective fields Φ_A. In this case there are only two independent parameters used to express the masses of all particles.

The self-consistent consideration of four-fermion models, using it as the basis for the dimensional regularization of divergent integrals (2.92), leads to the establishment of a rigid connection between the fermion masses and the masses of their bound states, which are described by the collective fields.

So, discussed in this chapter the four-fermion models including scalar-scalar, pseudoscalar-pseudoscalar and vector-vector interaction, are reformulated using the method of functional integration in terms of the interaction of fermions with the collective scalar, pseudoscalar and vector fields. In this case, the kinetic terms, and terms of self-interaction of collective fields appear from the vacuum polarization diagrams. The theory of perturbations, corresponding to the expansion in loops, leads to a finite number of divergent diagrams. However, the cut-off momentum (or parameter ε in the scheme with dimensional regularization) enters a gap equation that determines the mass formulas, and therefore must be finite and have physical meaning. Therefore, one should comment here on the finite renormalization. Some variants of the renormalization procedure in four-fermion models were considered in [332], [333].

Chapter 3

3.1 Global $SU(2)\otimes U(1)$-invariant models

We investigate the $SU(2)_L \otimes U(1)$-invariant models with various schemes including four-fermion interactions of different generations of leptons and quarks. It is shown that as a result of dynamic symmetry breaking "top" fermions (neutrinos) of different generations remain massless, and the "lower" fermions (leptons e, μ, τ) acquire different masses. For each generation, the spectrum of collective excitations is obtained, which includes the Goldstone fields and massive scalar particles, the Higgs particle analogues, which are fermion-antifermion bound states. The form of effective action calculated is similar to the Higgs action, with distinction in the composite nature of the Higgs fields. The possibility of the generation of the current quark masses is shown. For this in the $SU(2)_L \otimes U(1)$-invariant model the four-fermion interaction is included, containing in right-handed singlets both the lower and upper fermions. As the part of the one-loop approximation, propagators are obtained and the spectrum of fermion masses is established as a result of dynamic symmetry breaking collective scalar fields. The content of this chapter is based on the results of [334] - [345].

3.1.1 The initial model

Usually, to get the masses of leptons, quarks and bosons in the GWS theory, the Higgs−Kibble mechanism (see [13], [14]) of spontaneous symmetry breaking is used. At the same time, we know that the introduction of masses can be explained by dynamics, due to the nonlinear interaction of the fields (see the Introduction). This possibility is examined in this chapter. The role of the Higgs Lagrangian here will be performed by a nonlinear Lagrangian which is built from the original functions of the fermion fields.

Following the work of [13], [14], we enter the doublet L^i and singlet R^i of basic (massless) lepton fields ν^i, ψ^i

$$L^i = \frac{1}{2}\,(1+\gamma_5)\begin{pmatrix} \nu^i \\ \psi^i \end{pmatrix} = \begin{pmatrix} \nu^i \\ \psi^i_L \end{pmatrix},$$

$$R^i = \frac{1}{2}\,(1-\gamma_5)\,\psi^i \equiv \psi^i_R, \tag{3.1}$$

where ψ^i is a charged 4-component fermion field, and the field ν^i is a neutral field and has only left-handed components $\nu^i = \frac{1}{2}\,(1+\gamma_5)\,\nu^i$, i is the index of generations of fermions, $\nu^i = (\nu^e, \nu^\mu, \nu^\tau)$, $\psi^i = (e, \mu, \tau)$.

Multiplets of L and R, (3.1), possess the usual transformation properties under global transformations of the

$SU(2)\otimes U(1)$ group

$$L'(x) = \exp\left(-i\frac{\tau^a}{2}\xi^a - i\frac{\eta}{2}\right)L(x),$$
$$R'(x) = \exp\left(-i\eta\right)R(x), \tag{3.2}$$

where ξ^a and η are parameters of the $SU(2)$ and $U(1)$ groups, respectively. One can build values $\bar{\psi}^i\nu^i$ and $\bar{\psi}^i\psi^i_L$, from the original functions of the fundamental fields, which are scalars under the transformations of the proper Lorentz group and it is easy to verify directly, automatically forming a doublet

$$\varphi^i = \begin{pmatrix} \varphi^i_1 \\ \varphi^i_2 \end{pmatrix} = \begin{pmatrix} \bar{\psi}^i\nu^i \\ \bar{\psi}^i\psi^i_L \end{pmatrix}, \quad \bar{\psi} = \psi^+\gamma_4, \tag{3.3}$$

i.e. it is transformed by the fundamental representation of the $SU(2)$ group:

$$\varphi'(x) = \exp\left(-i\frac{\tau^a}{2}\xi^a\right)\varphi(x).$$

Let us consider the Lagrangian for the fundamental fields ν^i, ψ^i (which is invariant under the global transformations (3.2)) introducing the self-interaction of fields φ^i (3.3), which are considered as an analogue of the Higgs doublet of scalar fields

$$\mathcal{L} = -\bar{L}^i\gamma_\mu\partial_\mu L^i - \bar{R}^i\gamma_\mu\partial_\mu R^i + \lambda_{ij}\varphi^i\varphi^{j+}, \tag{3.4}$$

where the summation on the indices $i, j = 1, 2, 3$ is implied.

From the requirement of reality of the Lagrangian (3.4) ($\mathcal{L}^* = \mathcal{L}$) it follows that λ_{ij} is the Hermitian matrix, i.e. $\|\lambda\|^+ = \|\lambda\|$. Elements of the matrix λ_{ij} have the dimension of $(m)^{-2}$.

If we substitute the expression for the doublet φ^i (3.3) to (3.4), we obtain a four-fermion interaction. The matrix λ_{ij} defines a lepton mixing matrix of fields of different generations (see also [346]). It can be considered a different mix of generations of fermions, setting various types of matrices λ_{ij}. We will analyze two distinctly different types of the matrix λ_{ij}.

The transition to full local invariant Lagrangian, describing the interaction of leptons with $W^{\pm}$, Z, A bosons, can be done by replacing

$$\partial_\mu \to D_\mu = \partial_\mu - i\left[g b^a_\mu(x) t^a - \frac{1}{2} g' Y a_\mu(x)\right]$$

and adding the free Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F^2_{\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad G^a_{\mu\nu} = \partial_\mu b^a_\nu - \partial_\nu b^a_\mu + g\varepsilon^{abc} b^b_\mu b^c_\nu,$$

and the fields of the observed bosons are constructed in the usual way by the rule (1.3) (see e.g. [347]).

In this section we investigate the dynamical breakdown of the $SU(2)_L \otimes U(1)$ symmetry of the model with the Lagrangian (3.4), and the formation of the masses of the leptons, and we obtain the spectrum of collective excitations (composite fields).

We start from the fact that according to the definition (see Chapter 1), the generating functional for the Green functions of the fields can be written as

$$\begin{aligned} Z[\bar{\eta}, \eta] = N_0 \int D\bar{\psi} D\psi D\bar{\nu} D\nu \exp\Big[i \int d^4x \Big(\mathcal{L} \\ + \bar{L}^i \eta_L^i + \bar{\eta}_L^i L^i + \bar{R}^i \eta_R^i + \bar{\eta}_R^i R^i \Big) \Big], \end{aligned} \tag{3.5}$$

where η_L^i, η_R^i are the external sources, and the Lagrangian $\mathcal{L}$ is defined by (3.4). To linearize the four-fermion interaction, which is part of (3.5), we use the formula

$$\begin{aligned} \int D\Phi \exp\left\{ i \left[\Phi_a^i \varphi_a^{i*} + \Phi_a^{i*} \varphi_a^i - g_{ij} \Phi_a^i \Phi_a^{j*} \right] \right\} \\ = (\det \|g\|)^{-1} \exp\left(i \lambda_{ij} \varphi_a^i \varphi_a^{j*} \right), \end{aligned} \tag{3.6}$$

where the summation on the indices $i, j = 1, 2, 3$ is implied, $a = 1, 2$, and the matrix g_{ij} is the inverse of λ_{ij}, i.e. $\|g\| = \|\lambda\|^{-1}$. Here Φ_a^i are collective scalar fields. For convenience we introduce the 6-dimensional functions and matrices

$$\xi^i = \begin{pmatrix} \nu_L^i \\ \psi^i \end{pmatrix}, \quad \hat{\partial} = \begin{pmatrix} -i\bar{\tau}_\mu \partial_\mu & 0 \\ 0 & \gamma_\mu \partial_\mu \end{pmatrix}, \quad \bar{\tau}_\mu = (-\tau^a, iI_2)\,,$$

$$M_1 = \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_2 & 0 & 0 \end{pmatrix}, \qquad (3.7)$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix},$$

$$\gamma_a = \begin{pmatrix} 0 & i\tau^a \\ -i\tau^a & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where I_2 is the unit 2×2 matrix. From the notations (3.7) and (3.6), the expression (3.5) is rewritten as follows:

$$Z[\bar{\eta},\eta] = N \int D\bar{\xi} D\xi D\Phi \exp\Big\{ i \int d^4x \Big[-\bar{\xi}^i \hat{\partial} \xi^i$$
$$+\bar{\xi}^i \left(\Phi_a^i M_a + \Phi_a^{i*} M_a' \right) \xi^i - g_{ij} \Phi_a^i \Phi_a^{j*} + \bar{\xi}^i \eta^i + \bar{\eta}^i \xi^i \Big] \Big\}. \qquad (3.8)$$

For the existing functional integral (3.8) in Euclidean space-time, you need to decrease the integrand at large

Φ_a^i. Therefore, it is necessary to impose the requirement of positive definite of the matrices g_{ij}. Due to the fact that the matrix $\|g\|$ as well as the matrix $\|\lambda\|$ is Hermitian, its eigenvalues are real and need to be positive.

The collective Bose fields Φ_a^i introduced are collective variables. Since the path integral (3.8) is a Gaussian with respect to fields $\bar{\xi}^i$, ξ^i, then we can perform the integration, here the result is

$$Z[\overline{\eta},\eta] = N\int D\Phi \det\left[\left(-\hat{\partial} + \Phi_a^i M_a + \Phi_a^{i*} M_a'\right)(\delta_{ij})\right]$$
$$\times \exp\left\{i\int d^4x d^4y\left[-g_{ij}\Phi_a^{i*}(x)\Phi_a^j(y)\delta(x-y)\right.\right. \quad (3.9)$$
$$\left.\left.+\overline{\eta}^i(x)K^{ij}(x,y)\eta^j(y)\right]\right\},$$

where (δ_{ij}) is the identity 3×3-matrix acting in the space of generations and $K^{ij}(x,y)$ is the Green's function of fermions in external collective fields Φ_a^i, satisfying

$$\left(-\hat{\partial} + \Phi_a^i M_a + \Phi_a^{i*} M_a'\right) K^i(x,y) = -\delta(x-y). \quad (3.10)$$

Here we have that $K^{ij}(x,y)$ is the diagonal matrix, i.e. $K^{ij}(x,y) = \delta_{ij}K^i(x,y)$ and in (3.10) there is no summation over the index i.

Suppose that in the model with the generating functional (3.5) there is a dynamic breaking $SU(2)\otimes U(1)$

symmetry. In this case, the collective fields Φ_a^i acquire non-zero vacuum expectation values $\langle\Phi_a^i\rangle = \Phi_{0a}^i$. It will be shown that this corresponds to the minimum of the effective potential. Separating the condensate and making the shift $\Phi_a^i(x) \to \Phi_{0a}^i + \Phi_a^i(x)$ (Φ_{0a}^i = const) using the formula det = exp Tr ln, we represent (3.9) in the form

$$Z[\bar\eta,\eta] = N\int D\Phi \exp\Big[iS_{eff}$$

$$+i\int d^4x d^4y\ \bar\eta^i(x)K^i(x,y)\eta^i(y)\Big], \qquad (3.11)$$

where the effective action is

$$S_{eff} = \int d^4x \left[-g_{ij}\left(\Phi_{0a}^{i*} + \Phi_a^{i*}(x)\right)\left(\Phi_{0a}^{j} + \Phi_a^{j}(x)\right)\right]$$

$$-i\mathrm{Tr}\ \ln\Big[\Big(-\hat\partial + \left(\Phi_{0a}^i + \Phi_a^i(x)\right)M_a \qquad (3.12)$$

$$+\left(\Phi_{0a}^{i*} + \Phi_a^{i*}(x)\right)M_a'\Big)\left(\delta_{ij}\right)\Big].$$

Note that this shift of collective fields must also be made in the equation (3.10).

In the action (3.12) $\Phi_a^i(x)$ are the physical fields, which are the quantum excitations of the physical vacuum. We find equations of motion of the fields $\Phi_a^i(x)$ by varying effective action (3.12) (and putting $\Phi_a^i(x) = 0$):

$$\frac{\delta S_{eff}}{\delta \Phi_a^{i*}(x)} = -g_{ij}\Phi_{0a}^i + i\mathrm{Tr}K_0^i M_a' = 0, \tag{3.13}$$

and adding the complex conjugated equation. Here K_0^i is the free fermion Green's function, which satisfies the equation (see (3.10))

$$\left(-\hat{\partial} + \Phi_{0a}^i M_a + \Phi_a^{i*} M_a'\right) K_0^i(x,y) = -\delta(x-y). \tag{3.14}$$

Going to the momentum space, we can verify that the solution of (3.14) is the matrix

$$K_0^i(p) = \frac{1}{p^2\left(p^2+m_i^2\right)}$$

$$\times \begin{pmatrix} -\left(p^2+|\Phi_{02}^i|^2\right)p & \Phi_{01}^i\Phi_{02}^{i*}p & -p^2\Phi_{01}^i \\ -p^2\Phi_{01}^{i*} & -p^2\Phi_{02}^{i*} & -p^2\bar{p} \\ \Phi_{02}^i\Phi_{01}^{i*}p & -\left(p^2+|\Phi_{01}^i|^2\right)p & -p^2\Phi_{02}^i \end{pmatrix}, \tag{3.15}$$

where

$$\bar{p} = p_\mu\bar{\tau}_\mu, \quad m_i^2 = |\Phi_{01}^i|^2 + |\Phi_{02}^i|^2, \quad p = p_\mu\tau_\mu,$$

$$\tau_\mu = \left(\tau^a, iI_2\right), \quad \bar{\tau}_\mu = \left(-\tau^a, iI_2\right), \quad p^2 = \mathbf{p}^2 - p_0^2.$$

Substituting (3.15) into (3.13) and calculating the traces, we obtain the equations

$$g_{ij}\Phi^i_{0a} = -\frac{i}{8\pi^4}\int \frac{d^4p\Phi^i_{0a}}{p^2+m_i^2} \quad (i=1,2,3, \quad a=1,2). \quad (3.16)$$

Equations (3.16) are called the gap equations, because they have the same form as the equation for the energy gap in the theory of superconductivity; non-trivial solutions correspond to the super-conducting state, and the trivial solutions correspond to the normal state.

3.1.2 Independent generations of leptons

In this section we consider the special case where the matrix λ_{ij}, appearing in (3.4), is chosen in the diagonal form

$$\lambda_{ij} = \lambda^i \delta_{ij}, \quad g_{ij} = g^i \delta_{ij}, \quad g^i = \frac{1}{\lambda^i} \tag{3.17}$$

(the summation in the index i is not assumed). This choice of the matrix λ_{ij} leads to the fact that there is no interaction between the lepton fields of different generations. The values g_i are constants of self-interaction of leptons of i-generation. Then the gap equation (3.16) can be rewritten as follows:

$$g_i \Phi^i_{0a} = -\frac{i}{8\pi^4} \int \frac{d^4 p \Phi^i_{0a}}{p^2 + m_i^2}. \tag{3.18}$$

Equation (3.18) has the trivial solution $\Phi^i_{0a} = 0$ ($m_i = 0$), corresponding to an unbroken $SU(2)_L \otimes U(1)$-symmetry, and also nontrivial nonanalytic solutions ($\Phi^i_{0a} \neq 0$) for $0 < 8\pi^2 g^i / \Lambda^2 < 1$, where Λ is the cutoff momentum [1]. This means that at $\Lambda^2 > 8\pi^2 g^i$ a phase transition holds to a state with the mass $m_i^2 = \mid \Phi^i_{01} \mid^2 + \mid \Phi^i_{02} \mid^2 \neq 0$.

To determine the mass spectrum of fermions, it is necessary to bring the Green function (3.15) for such a quasi-diagonal form in which non-zero matrix elements are related to the neutrino and "lower" leptons would be absent simultaneously. This can be done in two ways. One of them involves a move to new physical fields which are a superposition of the "upper" and "lower" leptons. After this substitution, we can ensure that the appropriate Green's function takes the quasi-diagonal form. The second method is more simple and involves the use of the unitary gauge, in which $\Phi^i_{01} = 0$, $\Phi^i_{02} = \Phi^{i*}_{02} = -m_i \neq 0$. In this case, the fermion Green's function (3.15) takes the form

$$K^i_0(p) = \begin{pmatrix} -\frac{p}{p^2} & 0 \\ 0 & \frac{m_i - i\hat{p}}{p^2+m_i^2} \end{pmatrix}. \qquad (3.19)$$

Here $p = p_\mu\tau_\mu$ and $\hat{p} = p_\mu\gamma_\mu$ refer to the states of fermions ν and ψ. It follows from the form (3.19) (see (3.7)) that as a result of dynamic symmetry breaking neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ remain massless, and fermions (e, μ, τ) gain non-zero masses m_i $(i = 1, 2, 3)$.

To prove the stability of the vacuum at $\Phi^i_{0a} \neq 0$, we calculate the effective potential. In the one-loop approximation, we limit ourselves to the constant fields Φ_{0a}, and the effective action associated with the effective potential by

$$S^0_{eff} = -\int d^4x\ V^0,$$

has the form

$$S_{eff} = \sum_i \Big\{ -\int d^4x g_i \Phi^i_{0a}\Phi^{i*}_{0a}$$

$$-i\mathrm{Tr}\ \ln\left[\left(-\hat{\partial}+\Phi_{0a}^{i}M_{a}+\Phi_{0a}^{i*}M_{a}'\right)(\delta_{ij})\right]\Bigg\}. \tag{3.20}$$

Subtracting from (3.20) the action corresponding to the unbroken symmetry $S^0 = -i\mathrm{tr}\ \ln\left(-\hat{\partial}\right)$, and using the property Tr ln = ln det, we find

$$S_{eff}-S^{0}=-\sum_{i}\int d^{4}x\Bigg\{g_{i}\Phi_{0a}^{i}\Phi_{0a}^{i*}$$
$$+i\int\frac{d^{4}p}{(2\pi)^{4}}\ln\ \det\left[\left(1+\hat{G}_{0}(p)\left(\Phi_{0a}^{i}M_{a}+\Phi_{0a}^{i*}M_{a}'\right)\right)(\delta_{ij})\right]\Bigg\}, \tag{3.21}$$

where

$$\hat{G}_{0}(p)=\frac{1}{p^{2}}\begin{pmatrix} p & 0\\ 0 & i\hat{p}\end{pmatrix}. \tag{3.22}$$

Calculating the determinant appearing in (3.21), we obtain

$$V_{eff}^{0}=\sum_{i}\left\{g_{i}\Phi_{0a}^{i}\Phi_{0a}^{i*}+\frac{i}{8\pi^{4}}\int d^{4}p\ \ln\left(1+\frac{m_{i}^{2}}{p^{2}}\right)\right\}. \tag{3.23}$$

The second term in (3.23) defines a one-loop correction to the effective potential. The extremum of the effective potential (3.23) is determined from $\partial V_{eff}^{0}/\partial\Phi_{0a}^{i*}=0$ and its complex conjugate. It is easy to check that this condition gives exactly the self-consistency equation (3.18). You can also verify directly that the condition $\partial^2 V^0_{eff}/(\partial\Phi^i_{02})^2 > 0$ (in the gauge of $\Phi^i_{01} = 0$, $\Phi^i_{02} = \Phi^{i*}_{02} \neq 0$), for the minimum of the potential, holds. Thus, non-trivial solutions of (3.16) actually correspond to the minimum of the effective potential.

We now transform the effective action (3.21), expanding it to small fields (excitations) $\Phi^i_0(x)$. The constant S^0_{eff}, appearing in (3.9), determines the vacuum energy and is removed by the redefinition of N in equation (3.11). For further discussion, this constant is negligible and can be omitted. One takes into account the equality

$$\mathrm{Tr}\ \ln\Bigg[\bigg(-\hat{\partial} + \Big(\Phi^i_{0a} + \Phi^i_{0a}(x)\Big)\, M_a$$

$$+\Big(\Phi^{i*}_{0a} + \Phi^{i*}_{0a}(x)\Big)\, M'_a\bigg)\,(\delta_{ij})\Bigg]$$

$$= \mathrm{Tr}\ \ln\left[\left(-\hat{\partial} + \Phi^i_{0a}M_a + \Phi^{i*}_{0a}M'_a\right)(\delta_{ij})\right]$$

$$+\mathrm{Tr}\ \ln\left[\left(1 - K^i_0\left(\Phi^i_a(x)M_a + \Phi^{i*}_a(x)M'_a\right)\right)(\delta_{ij})\right].$$

We use the fact that in (3.21) the linear terms in the fields $\Phi^i_a(x)$ are absent in view of (3.18). With that said, we expand the action (3.21) around the static solutions Φ^i_{0a}:

$$S_{eff} = S^{(2)}_{eff} + \sum_{n=3}^{\infty} L_n,$$

$$\begin{aligned} S^{(2)}_{eff} = &-\sum_i \int d^4x \Big\{ g_i \Phi^i_a \Phi^{i*}_a \\ &+ \frac{i}{2} \mathrm{Tr} \left[K^i_0 \left(\Phi^i_a M_a + \Phi^{i*}_a M'_a \right) \right]^2 \Big\}, \end{aligned} \qquad (3.24)$$

$$L_n = \frac{i}{n} \sum_i \mathrm{Tr} \left[K^i_0 \left(\Phi^i_a M_a + \Phi^{i*}_a M'_a \right) \right]^n .$$

The quadratic term in the fields (3.24), which determines the propagation of the fields $\Phi^i_a(x)$, can be written as

$$S^{(2)}_{eff} = -\frac{1}{2} \int d^4x d^4y \; \Phi^i(x) \left(T^{ij}(x,y) \right)^{-1} \Phi^j(y). \qquad (3.25)$$

Here we introduced the four-component wave function $\Phi^i = (\Phi^i_1, \Phi^{i*}_1, \Phi^i_2, \Phi^{i*}_2)$, and the propagator $T^{ij}(x,y)$ in the momentum space is determined by the relations

$$\begin{aligned} \left(T^{ij}_{AB}(p) \right)^{-1} = &\; g_i \delta_{ij} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{AB} \\ &- i \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} K^i(k) \Gamma_A K^i(k-p) \Gamma_B \delta_{ij}, \end{aligned} \qquad (3.26)$$

where (see (3.7)) $\Gamma_A = (M_1, M'_1, M_2, M'_2)$. Calculating the inverse propagator (3.26) using (3.18), we find its non-zero elements

$$\left(T^i_{12}(p) \right)^{-1} = \left(T^i_{21}(p) \right)^{-1}$$

$$= g_i + \frac{i}{8\pi^4}\int \frac{d^4q\ q(q-p)}{(q-p)^2(q^2+m_i^2)} = p^2 Z_3^{-1} + \mathcal{O}_R,$$

$$\left(T_{33}^i(p)\right)^{-1} = \left(T_{44}^i(p)\right)^{-1}$$

$$= -\frac{i}{8\pi^4}\int \frac{d^4q\ m_i^2}{[(q-p)^2+m_i^2](q^2+m_i^2)} = 2m_i^2 Z_3^{-1} + \mathcal{O}_R, \tag{3.27}$$

$$\left(T_{34}^i(p)\right)^{-1} = \left(T_{43}^i(p)\right)^{-1}$$

$$= g_i + \frac{i}{8\pi^4}\int \frac{d^4q\ q(q-p)}{[(q-p)^2+m_i^2](q^2+m_i^2)}$$

$$= \left(p^2 + 2m_i^2\right) Z_3^{-1} + \mathcal{O}_R.$$

Here we take into account that

$$\left(T^{ij}(p)\right)^{-1} = \left(T^i(p)\right)^{-1}\delta_{ij},\ \ Z_3^{-1} = \frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2}{m^2} - 1\right),$$

where Λ is the cut-off momentum, m is the normalization point (e.g. $m = m_1$), $\mathcal{O}_R$ represents the finite part, which is independent of the cut-off momentum, and determines the radiative corrections. Substituting (3.27) into (3.25) in the momentum space and performing the renormalization of the fields $\Phi^i \to \Phi^i Z_3^{1/2}$, we find the quadratic action in these fields up to the higher radiative corrections

$$S_{eff}^{(2)} = \frac{1}{2}\int d^4x\ \Big[\Phi_1^i \partial_\mu^2 \Phi_1^{i*} + \Phi_1^{i*}\partial_\mu^2 \Phi_1^i$$

$$+\Phi_2^i\partial_\mu^2\Phi_2^{i*} + \Phi_2^{i*}\partial_\mu^2\Phi_2^i - 2m_i^2\left(\Phi_2^i + \Phi_2^{i*}\right)^2\Big]. \tag{3.28}$$

In (3.28) the summation in index i is implied. Denoting $\Phi_2^i = \kappa^i + i\chi^i$, we rewrite (3.28) in the form

$$S_{eff}^{(2)} = \int d^4x\, \left[\frac{1}{2}\left(\Phi_1^i\partial_\mu^2\Phi_1^{i*} + \Phi_1^{i*}\partial_\mu^2\Phi_1^i\right)\right. \tag{3.29}$$
$$\left. +\chi^i\partial_\mu^2\chi^i + \kappa^i\left(\partial_\mu^2 - 4m_i^2\right)\kappa^i\right].$$

It follows from (3.29) that we have in each generation, three massless fields $\mathrm{Re}\Phi_1^i$, $\mathrm{Im}\Phi_1^i$, χ^i and one massive field κ^i with the mass $2m_i$. In this case, Φ_1^i are charged massless fields, and χ^i and κ^i are neutral massless and massive fields, respectively.

Thus, as a result of the dynamical breaking of $SU(2)_L\otimes U(1)$-symmetry in every generation we have three Goldstone fields and one massive field - an analogue of the Higgs field. This is in agreement with the general Goldstone theorem [318]. It is recalled that the fields Φ_a^i are composite and are bound fermion-antifermion states. The fact that the mass of the composite boson field is twice the mass of the fermion, was originally discovered in a simple four-fermion model (1.1). This phenomenon is analogous to the formation of Cooper pairs in the theory of superconductivity.

Three- and four-point Green's functions give a divergent contribution in the effective action (3.24), which follows from (3.24) for $n = 3, 4$. At the same time, the terms in (3.24) with the number $n = 5$ are convergent, and determine the higher-order corrections and in the future will not be counted. Therefore, calculating formulas similar to (2.26), (2.30) with (3.19), after renormalization of fields and substituting into (2.69), we finally have the following expression for the effective action of fields Φ_1^i, Φ_2^i:

$$S_{eff} = \int d^4x \left[\frac{1}{2}\left(\Phi_1^i \partial_\mu^2 \Phi_1^{i*} + \Phi_1^{i*}\partial_\mu^2 \Phi_1^i\right)\right.$$

$$\left. +\chi^i \partial_\mu^2 \chi^i + \kappa^i\left(\partial_\mu^2 - 4m_i^2\right)\kappa^i\right] \qquad (3.30)$$

$$+4m_i\lambda\kappa^i\left(|\Phi_1^i|^2 + |\Phi_2^i|^2\right) - \lambda^2\left(|\Phi_1^i|^2 + |\Phi_2^i|^2\right)^2.$$

Here we introduced the dimensionless coupling constant $\lambda^2 = Z_3$. We note that the action (3.30) is similar to the Higgs sector (after the shift of fields) of the standard theory of electro-weak interactions, with the only difference that the Higgs fields are introduced for each generation of fermions. A mass of the fields χ^i is equal to twice the mass of the fermion $(2m_i)$ corresponding generation.

As the light scalar particles with masses $2m_e$, $2m_\mu$, $2m_\tau$, are not detected, it is obviously necessary to consider a more general model, except when the leptons are included in the review as heavy quarks. Note that other mixed inclusions of lepton interaction are also possible in choosing the matrix constants in the form of the matrix-dyad $g_{ij} = g_i g_j$. In this case, as will be shown below (see section 10), there is one collective field that is common to all generations of leptons.

3.1.3 Dynamical symmetry breaking in $SU(2)_L \otimes U(1)$-model with two right singlets and quark mass generation

In order to allow the formation of dynamic masses of the bottom and top quarks, it is obviously necessary to build a four-fermion interaction with two right fermion singlets. In the beginning, for the sake of simplicity, we consider in detail only one generation of quarks.

So let

$$\mathcal{L} = -\bar{q}\gamma_\mu \partial_\mu q + \rho_0^2 \left(\bar{q}_L u_R\right)\left(\bar{u}_R q_L\right) + \lambda_0^2 \left(\bar{q}_L d_R\right)\left(\bar{d}_R q_L\right) \tag{3.31}$$

be the $SU(2) \otimes U(1)$-invariant Lagrangian for the quark doublet

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

with the four-fermion interaction, including two right singlets

$$u_R = \frac{1}{2}\left(1-\gamma_5\right) u, \quad d_R = \frac{1}{2}\left(1-\gamma_5\right) d,$$

where

$$q_L = \frac{1}{2}\left(1+\gamma_5\right) q.$$

The introduction of the quark interaction with intermediate $W^{\pm}$, Z bosons is the usual substitution in (3.31)

$$\partial_\mu \to D_\mu = \partial_\mu - i\left(g b_\mu^a(x)\frac{\tau^a}{2} - \frac{1}{2}g' Y a_\mu(x)\right)$$

and the usage of definitions (1.3). In (3.31) ρ_0 and λ_0 are bare constants (of the dimension m^{-1}) of the four-quark interaction.

Introducing the 8×8 matrices:

$$B_1 = B_3' = \lambda_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \end{pmatrix},$$

$$B_2 = B_4' = \lambda_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix},$$

$$B_3 = B_1' = \lambda_0 \begin{pmatrix} 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_4 = B_2' = \lambda_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_5 = B_7' = \rho_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_6 = B_8' = \rho_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_7 = B_5' = \rho_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_8 = B_6' = \rho_0 \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.32}$$

The Lagrangian (3.31), after the inclusion of the Schwinger sources J_A ($A = 1, 2, ...8$ and $J_3 = J_1^*,\ J_4 = J_2^*,\ J_7 = J_5^*,\ J_8 = J_6^*$), can be written in the following form:

$$\mathcal{L} = -\bar{q}\gamma_\mu \partial_\mu q + \frac{1}{2} \left(\bar{q} B_A q\right) \left(\bar{q} B_A' q\right) + \left(\bar{q} B_A' q\right) J_A. \tag{3.33}$$

From the ensuing equations of motion for the quark fields

$$\gamma_\mu \partial_\mu q - B_A q \left(\bar{q} B_A' q\right) - B_A' J_A q = 0, \tag{3.34}$$

using the equation

$$\frac{\delta}{\delta J_A}\langle Tq(x)\bar{q}(y)\rangle = i\langle Tq(x)\bar{q}(y)\bar{q}(x)B'_A q(x)\rangle$$
$$-i\langle T\bar{q}(x)B'_A q(x)\rangle\langle Tq(x)\bar{q}(y)\rangle, \quad (3.35)$$

and anti-commutation relations

$$\{q_A(x), q^+_B(y)\}|_{t=t'} = \delta_{AB}\delta\left(\mathbf{x}-\mathbf{y}\right),$$

$$T\partial_t\left(q(x)q^+(y)\right) = \partial_t\left(Tq(x)q^+(y)\right) - \delta(x-y),$$

where T is the operator of chronological ordering, for the fermion Green's function [317] $G(x,y) = i\langle Tq(x)\bar{q}(y)\rangle$ the following DS equation can be obtained:

$$\left(\gamma_\mu\partial_\mu + iB_A\frac{\delta}{\delta J_A} - B'_A\langle\Phi_A\rangle\right)G(x,y) = \delta(x-y). \quad (3.36)$$

Here

$$\langle\Phi_A\rangle = \langle T\bar{q}(x)B_A q(x)\rangle + J_A(x) \quad (3.37)$$

are eight scalar collective fields, constructed from the original quark fields and satisfying

$$\langle\Phi_3\rangle = \langle\Phi^*_1\rangle, \quad \langle\Phi_4\rangle = \langle\Phi^*_2\rangle, \quad \langle\Phi_5\rangle = \langle\Phi^*_7\rangle, \quad \langle\Phi_6\rangle = \langle\Phi^*_8\rangle,$$

$$\langle\Phi_1\rangle = \lambda_0\langle\bar{d}_R u_L\rangle + J_1, \quad \langle\Phi_2\rangle = \lambda_0\langle\bar{d}_R d_L\rangle + J_2,$$

$$\langle\Phi_3\rangle = \lambda_0\langle\bar{u}_L d_R\rangle + J_3,$$
$$\langle\Phi_4\rangle = \lambda_0\langle\bar{d}_L d_R\rangle + J_4, \quad \langle\Phi_5\rangle = \rho_0\langle\bar{u}_R d_L\rangle + J_5, \tag{3.38}$$
$$\langle\Phi_6\rangle = \rho_0\langle\bar{u}_R u_L\rangle + J_6,$$
$$\langle\Phi_7\rangle = \rho_0\langle\bar{d}_L u_R\rangle + J_7, \quad \langle\Phi_8\rangle = \rho_0\langle\bar{u}_L u_R\rangle + J_8.$$

We solve equation (3.36) by perturbation theory using an expansion in loops [348], [349]

$$G(x,y) = G^{(0)} + \varepsilon G^{(1)} + \varepsilon^2 G^{(2)} + ...,$$
$$\langle\Phi_A\rangle = \Phi_A^{(0)} + \varepsilon\Phi_A^{(1)} + \varepsilon^2\Phi_A^{(2)} + ... \tag{3.39}$$

Parameter ε relates at the end of the computation to one. Substituting (3.39) into (3.36), considering that the functional derivative of the source takes the n-loop value in the $(n+1)$-loop, i.e., $\delta/\delta J_A \to \varepsilon\delta/\delta J_A$ and equating terms of the same order with respect to ε, we obtain the equations for the Green function - propagators of fermion fields in the zero approximation ($G^{(0)}$) and the equations for ($G^{(1)}$) in the one-loop correction

$$\left(\gamma_\mu\partial_\mu - B'_A\Phi_A\right) G^{(0)}(x,y) = \delta(x-y), \tag{3.40}$$

$$\int d^4z\ G^{(0)-1}(x,z)G^{(1)}(z,y)$$

$$+\left(iB_A\frac{\delta}{\delta J_A(x)} - B'_A\Phi_A^{(1)}(x)\right)G^{(0)}(x,y) = 0. \tag{3.41}$$

Removing in equation (3.40) Schwinger's external sources, leads to the condition $\Phi_A^{(0)}(x) = const$, then selecting the gauge:

$$\Phi_1^{(0)} = \Phi_5^{(0)} = 0, \quad m_1 = -\Phi_6^{(0)}\rho_0 = -\Phi_6^{(0)*}\rho_0,$$

$$m_2 = -\Phi_2^{(0)}\lambda_0 = -\Phi_2^{(0)*}\lambda_0, \tag{3.42}$$

in the momentum representation, after diagonalization, we obtain the following expression for the fermion propagator:

$$G^{(0)}(p) = \mathrm{diag}\left(\frac{-i\hat{p}+m_1}{p^2+m_1^2},\ \frac{-i\hat{p}+m_2}{p^2+m_2^2}\right). \tag{3.43}$$

It follows that the fields of u, d-quarks describe two Dirac particles with masses m_1 and m_2, respectively.

The equations of motion for the collective fields $\Phi_A(x)$ can be easily obtained from the following relation (see (3.37)):

$$\Phi_A^{(0)}(x) = i\mathrm{Tr}\ B_A G^{(0)}(x,x) + J_A(x). \tag{3.44}$$

Putting $J_A = 0$ in (3.44), which, due to translational invariance leads to the condition $\Phi_A^{(0)}(x) = const$, and calculating the traces, we obtain

$$1 = -\frac{i\rho_0^2}{8\pi^4}\int\frac{d^4p}{p^2+m_1^2}, \quad 1 = -\frac{i\lambda_0^2}{8\pi^4}\int\frac{d^4p}{p^2+m_2^2}. \tag{3.45}$$

These are the gap equations that bind constants ρ_0, λ_0 with the momentum cut-off Λ and the masses m_1, m_2.

Differentiating $\Phi_A(x)$, (3.44), with respect to $J_B(y)$, by definition, taking

$$\Delta_{AB}(x,y) = \frac{\delta\Phi_A(x)}{\delta J_B(y)}, \tag{3.46}$$

for the propagator $\Delta_{AB}(x,y)$ of the field $\Phi_A(x)$ in the one-loop approximation, we obtain the equation:

$$\left[\delta_{AN} - \frac{i}{(2\pi)^4}\mathrm{tr}\left\{B_A\int d^4p\; G^{(0)}(p)B'_N G^{(0)}(p-k)\right\}\right]$$

$$\times\Delta_{NB}^{(1)}(k) = \delta_{AB}. \tag{3.47}$$

This equation can be written as 8×8-matrix:

$$\Delta^{(1)}(k) = \left(\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\right)^{-1}\Delta(k), \tag{3.48}$$

(Λ is the parameter of the cut-off, m being the normalization point) with the following non-zero elements:

$$\Delta_{11} = \Delta_{33} = \frac{k^2 + 2m_2^2}{\lambda_0^2 k^2 \left(k^2 + 2m_1^2 + 2m_2^2\right)},$$

$$\Delta_{22} = \Delta_{44} = \frac{k^2 + 2m_2^2}{\lambda_0^2 k^2 \left(k^2 + 4m_2^2\right)},$$

$$\Delta_{17} = \Delta_{71} = \Delta_{35} = \Delta_{53} = \frac{-2m_1 m_2}{\rho_0 \lambda_0 k^2 \left(k^2 + 2m_1^2 + 2m_2^2\right)},$$

$$\Delta_{24} = \Delta_{42} = \frac{-2m_2^2}{\lambda_0^2 k^2 \left(k^2 + 4m_2^2\right)},$$

$$\Delta_{55} = \Delta_{77} = \frac{k^2 + 2m_1^2}{\rho_0^2 k^2 \left(k^2 + 2m_1^2 + 2m_2^2\right)},$$

$$\Delta_{66} = \Delta_{88} = \frac{k^2 + 2m_1^2}{\rho_0^2 k^2 \left(k^2 + 4m_1^2\right)},$$

$$\Delta_{68} = \Delta_{86} = \frac{-2m_1^2}{\rho_0^2 k^2 \left(k^2 + 4m_1^2\right)}.$$

Making over the propagator $\Delta^{(1)}(k)$ (3.48) the transform V:

$$\Delta^{(1)'}(k) = V \Delta^{(1)}(k) V, \qquad V = \begin{pmatrix} \lambda_0 I_4 & 0 \\ 0 & \rho_0 I_4 \end{pmatrix} \tag{3.49}$$

(the transformation of V is not unitary, the propagator $\Delta^{(1)}(k)$ and Φ_A do not belong to the field of a single particle) and then convert U:

$$\Delta^{(1)''}(k) = U\Delta^{(1)'}(k)U^{-1},$$

$$U = \begin{pmatrix} -\frac{m_2}{M} & 0 & 0 & 0 & 0 & 0 & \frac{m_1}{M} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_2}{M} & 0 & \frac{m_1}{M} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_1}{M} & 0 & \frac{m_2}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{m_1}{M} & 0 & 0 & 0 & 0 & 0 & \frac{m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.50}$$

$\left(M = \sqrt{m_1^2 + m_2^2}\right)$, when overridden

$$\bar{\Delta}^{(1)''}(k) = Z_3^{-1}\Delta^{(1)''}(k), \qquad Z_3^{-1} = \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}, \tag{3.51}$$

we obtain the final expression for the desired propagator $\bar{\Delta}^{(1)''}(k)$ in the diagonal form

$$\bar{\Delta}^{(1)''}(k) = \mathrm{diag}\Bigg(\frac{1}{k^2}, \frac{1}{k^2+4m_2^2}, \frac{1}{k^2}, \frac{1}{k^2}, \frac{1}{k^2+2m_1^2+2m_2^2}, \tag{3.52}$$

$$\frac{1}{k^2+4m_1^2}, \frac{1}{k^2+2m_1^2+2m_2^2}, \frac{1}{k^2}\Big).$$

The field $\Phi''(x) = U\Phi'(x) = UV\Phi(x)$, corresponding to (3.52), becomes

$$\Phi'' = \frac{1}{M}\begin{pmatrix} -m_2\lambda_0^2\langle \bar{d}_R u_L\rangle + m_1\rho_0^2\langle \bar{d}_L u_R\rangle \\ \frac{M}{\sqrt{2}}\lambda_0^2\langle \bar{d}d\rangle \\ -m_2\lambda_0^2\langle \bar{u}_L d_R\rangle + m_1\rho_0^2\langle \bar{u}_R d_L\rangle \\ \frac{M}{\sqrt{2}}\lambda_0^2\langle \bar{d}\gamma_5 d\rangle \\ m_1\lambda_0^2\langle \bar{u}_L d_R\rangle + m_2\rho_0^2\langle \bar{u}_R d_L\rangle \\ \frac{M}{\sqrt{2}}\rho_0^2\langle \bar{u}u\rangle \\ m_1\lambda_0^2\langle \bar{d}_R u_L\rangle + m_2\rho_0^2\langle \bar{d}_L u_R\rangle \\ \frac{M}{\sqrt{2}}\rho_0^2\langle \bar{u}\gamma_5 u\rangle \end{pmatrix}. \qquad (3.53)$$

We present here also formulas to find the corrections to the fermion Green function and the propagator of the collective fields. From (3.41) and the fact that

$$\frac{\delta G^{(0)}(x,y)}{\delta J_A(x)} = -\int d^4t d^4z G^{(0)}(x,t)$$

$$\times\left(\frac{\delta G^{(0)-1}(t,z)}{\delta J_A(x)}\right) G^{(0)}(z,y), \qquad (3.54)$$

$$\frac{\delta G^{(0)-1}(t,z)}{\delta J_A(x)} = -B_0'\frac{\delta \Phi_0(t)}{\delta J_A(x)}\delta(t-z),$$

we find the one-loop correction to $G^{(0)}$

$$G^{(1)}(x,y)=\int d^4zG^{(0)}(x,z)B'_A\Phi_A^{(1)}(z)G^{(0)}(z,y)$$
$$+\int d^4zd^4tG^{(0)}(x,z)\Sigma^{(1)}(z,t)G^{(0)}(t,y), \tag{3.55}$$

$$\Sigma^{(1)}(z,t)=-iB_AG^{(0)}(z,t)B'_D\Delta_{DA}^{(0)}(t,z), \tag{3.56}$$

where $\Sigma^{(1)}(z,t)$ is the lowest approximation to the mass operator. Using (3.39), from (3.36) we can find the n-loop corrections to the one-particle Green function of fermions. To find the n-loop corrections to the propagator of the collective fields it is necessary to use the formula [349]

$$\Delta_{DA}^{(n)}(x,y)=\frac{\delta\Phi_D^{(n)}(x)}{\delta J_A(y)}. \tag{3.57}$$

Similarly, we can calculate the multi-particle Green function, and apply the reduction formulas [317], to find the amplitude of the different processes.

Thus, as a result of dynamical breaking of $SU(2)_L\otimes U(1)$-symmetry u, d-quarks acquired (dynamic) masses m_1, m_2. In the spectrum of collective excitations there are 4 massless (Goldstone) fields and 4 massive states (see (3.52)). Quark masses are given by the corresponding values of the vacuum expectation values (3.42). Similarly for the other masses of the quarks one can organize the four-fermion quark interaction of other generations

$$\begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}.$$

Chapter 4

4.1 Local $SU(2)_L \otimes U(1)$-invariant model

In the original local $SU(2)_L \otimes U(1)$-invariant model, an independent as well as a mixed inclusion of different generations of fermions in the four-fermion interaction are considered. This leads to the fact that as a result of dynamic symmetry breaking the effective interaction Lagrangian of the gauge vector, fermion and collective scalar fields in the one-loop approximation is of the same form as in the theory of GWS. If the inclusion of fermions of different generations is in an independent manner, the role of massive scalar fields in each sector plays fermion-antifermion bound states. In the model with the mixed inclusion of fermions, the role of the Higgs boson plays one collective scalar field, which is the sum of fermion-antifermion pairs of all generations.

Mass formulas for the scalar field and the vector gauge fields are found. It is noted that the mass of the composite Higgs particle can be assumed to be $m_H \approx 2m_t$ (m_t is the mass of t-quark). The content of this chapter is based on the works of [341] - [345].

4.1.1 $SU(2)_L \otimes U(1)$-invariant four-fermion model with independent fermion generations

The locally $SU(2)_L \otimes U(1)$-invariant four-fermion Lagrangian with (n) generations of fermions is chosen in the form (using the diagonal matrix of the coupling constants $\lambda_{ij} = \lambda_0^i \delta_{ij}$ (3.17))

$$\mathcal{L} = -\bar{L}^n \gamma_\mu D_\mu L^n - \bar{R}^n \gamma_\mu D_\mu R^n - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \lambda_0^n \bar{L}^n R^n \bar{R}^n L^n, \tag{4.1}$$

where

$$D_\mu(x) = \partial_\mu - i \left[g b^a_\mu(x) T^a - \frac{1}{2} g' Y a_\mu(x) \right],$$

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

$$G^a_{\mu\nu} = \partial_\mu b^a_\nu - \partial_\nu b^a_\mu + g f^{abc} b^b_\mu b^c_\nu \quad (a, b, c = 1, 2, 3).$$

Combinations of potentials b^a_μ and a_μ of gauge fields, as usual, form known potentials of $W^\pm$, Z, A-fields, (1.3).

Introducing the notations ($\beta = 1/\sqrt{g^2 + g'^2}$):

$$\xi = \begin{pmatrix} \nu_L \\ \psi \end{pmatrix}, \quad C_\mu = \begin{pmatrix} W^+_\mu \\ W^-_\mu \\ A_\mu \\ Z_\mu \end{pmatrix}, \quad R = \beta \begin{pmatrix} 0 \\ 0 \\ -g' \\ g \end{pmatrix},$$

$$V = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and Schwinger external sources of scalar J_A $(A = 1, 2, 3, 4)$ and vector $J_\mu^{A+} = (J_\mu^{W^-}, J_\mu^{W^+}, J_\mu^A, J_\mu^Z)$ types, the Lagrangian (4.1) is rewritten in the form

$$\begin{aligned}
\mathcal{L} = -\bar{\xi}^n \hat{\partial} \xi^n + \frac{\lambda_0^n}{2} \left(\bar{\xi}^n M'_A \xi^n\right) \left(\bar{\xi}^n M_A \xi^n\right) + \bar{\xi}^n B_\mu^A \xi^n C_\mu^{A+} \\
-\frac{1}{4} C_{[\mu\nu]}^{A+} C_{[\mu\nu]}^A - ig C_\nu^{A+} C_{[\mu\nu]}^{F+} C_\mu^{L+} \\
\times \left(R^A V^F H^L - R^A V^L H^F + R^F V^L H^A\right) \qquad (4.2) \\
+g^2 \left(\delta_{\rho\nu}\delta_{\beta\mu} - \delta_{\rho\beta}\delta_{\nu\mu}\right) \\
\times C_\nu^{A+} C_\rho^{F+} C_\beta^{N+} C_\mu^{L+} V^A \left(\frac{1}{2} V^F H^N + R^F R^N\right) H^L \\
+\bar{\xi}^n M_A \xi^n J_A + C_\mu^{A+} J_\mu^A.
\end{aligned}$$

Here $M'_1 = M_3$, $M'_3 = M_1$, $M'_2 = M_4$, $M'_4 = M_2$, (see (3.7)),

$$B_\mu^1 = B_\mu^{\prime 2} = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \bar{\tau}_\mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_\mu^2 = B_\mu^{'1} = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ \bar{\tau}_\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_\mu^3 = B_\mu^{'3} = e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\tau}_\mu \\ 0 & \tau_\mu & 0 \end{pmatrix},$$

$$B_\mu^4 = B_\mu^{'4} = \begin{pmatrix} \frac{1}{2\beta}\bar{\tau}_\mu & 0 & 0 \\ 0 & 0 & \frac{g^{'2}-g^2}{2}\beta\bar{\tau}_\mu \\ 0 & g^{'2}\beta\tau_\mu & 0 \end{pmatrix}.$$

In order to allow the introduction of the Lorentz gauge of potentials of vector fields, the Lagrangian (4.2) is augmented by the Lagrangian of “ghost” fields [350]

$$\mathcal{L}_{ghost} = \bar{C}^+ \partial_\mu^2 C - igC_\mu^{A+} \left(\partial_\mu \bar{C}^{B+}\right) C^{D+} K^{ABD}$$
$$+\bar{C}^{D+}\eta^D + \bar{\eta}^D C^{D+}, \tag{4.3}$$

where

$$C = \begin{pmatrix} C^+ \\ C^- \\ C^A \\ C^Z \end{pmatrix}, \quad K^{ABD} = H^A H^{[B} L^{D]} - V^A V^{[B} L^{D]}$$
$$+R^A V^{[B} H^{D]},$$

$$V^{[B}L^{D]} = V^B L^D - V^D L^B, \qquad L = \beta \begin{pmatrix} 0 \\ 0 \\ g' \\ g \end{pmatrix},$$

η^D are Schwinger's external sources.

The notations introduced allow us to write initial and intermediate equations of the model in a more compact form, but do not interfere to formulate definitive solutions to these equations in the usual way.

Based on the sum of the Lagrangians (4.2) and (4.3), using conventional variational methods, equations of motion for the fields $\xi(x)$, $C_\mu^A(x)$, $\bar{C}_\mu^D(x)$ are as follows:

$$\begin{aligned}\Big(\hat{\partial} - \lambda_0^n M'_A \bar{\xi}^n(x) M_A \xi^n(x) - M_A J_A(x) \\ - B'^A_\mu C_\mu^A(x)\Big)\xi^k(x) = 0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}-\partial_\nu^2 C_\mu^L + \partial_\nu \partial_\mu C_\nu^L - ig\Big[\partial_\nu \left(C_\mu^{F+} C_\nu^{B+}\right) \\ + C_{[\nu\mu]}^{F+} C_\nu^{B+}\Big] m^{LFB} \\ - g^2 C_\mu^{F+} C_\nu^{N+} C_\nu^{B+} m^{LFNB} = J_\mu^L + \bar{\xi}^n B_\mu^L \xi^n \\ + ig\left(\partial_\mu \bar{C}^{B+}\right) C^{D+} K^{LBD},\end{aligned} \tag{4.5}$$

$$-\partial_\nu^2 \bar{C}^D - igC_\mu^{A+} \left(\partial_\mu \bar{C}^{B+}\right) K^{ABD} = \bar{\eta}^D, \tag{4.6}$$

where we use the notations

$$m^{LFB} = H^L V^{[F} R^{B]} + V^L R^{[F} H^{B]} + R^L H^{[F} V^{B]},$$

$$m^{LFNB} = H^L \left(H^{[F} V^{N]} V^B + R^N R^{[F} V^{B]}\right)$$

$$+V^L \left(V^{[F} H^{N]} H^B + R^N R^{[F} H^{B]}\right)$$

$$+R^L \left(V^{[F} R^{N]} H^B + H^{[F} R^{N]} V^B\right).$$

We take into account the definition

$$\frac{\delta S}{\delta J_\mu^{A+}(x)} = i \left(TC_\mu^A(x) S\right), \tag{4.7}$$

where the scattering matrix: $S = T \exp\{i \int d^4x \; \mathcal{L}(x)\}$. In the Heisenberg representation, we write the following relation

$$\frac{\delta}{\delta J_\mu^{A+}(x)} \langle T\xi^n(x)\bar{\xi}^n(y)\rangle = i\langle T\xi^n(x)\bar{\xi}^n(y)C_\mu^A(x)\rangle \tag{4.8}$$

$$-i\langle C_\mu^A(x)\rangle\langle T\xi^n(x)\bar{\xi}^n(y)\rangle,$$

by which, with the help of (4.4), the DS equation for the propagator $G^n(x,y) = i\langle T\xi^n(x)\bar{\xi}^n(y)\rangle$ of each generation becomes

$$\begin{aligned}\Big(\hat{\partial} + i\lambda_0^n M'_A \frac{\delta}{\delta J_A(x)} - M_A\langle\Phi_A^n(x)\rangle + iB_\mu^A \frac{\delta}{\delta J_\mu^A(x)} \\ -B_\mu^{\prime A}\langle C_\mu^A(x)\rangle\Big)G^n(x,y) = \delta(x-y).\end{aligned} \tag{4.9}$$

Here

$$\langle\Phi_A^n(x)\rangle = \lambda_0^n\langle T\bar{\xi}^n(x)M'_A\xi^n(y)\rangle + J_A(x) \tag{4.10}$$

are collective scalar fields, $\langle C_\mu^A(x)\rangle$ is the vacuum expectation value of gauge vector fields.

We now use the definition for the propagators of gauge vector fields

$$D_{\mu\nu}^{AB}(x,y) = \frac{\delta\langle C_\mu^A(x)\rangle}{\delta J_\nu^B(y)} \tag{4.11}$$

and scalar fields

$$\Delta_{AB}^n(x,y) = \frac{\delta\langle\Phi_A^n(x)\rangle}{\lambda_0^n\delta J_B(y)}. \tag{4.12}$$

With (4.11), (4.12) and the relations

$$\frac{\delta}{\delta J_\mu^A(x)} = \int d^4z D_{\nu\mu}^{BA}(z,x)\frac{\delta}{\delta\langle C_\nu^B(z)\rangle}, \tag{4.13}$$

$$\frac{\delta}{\delta J_A(x)} = \lambda_0^i \int d^4z \Delta^i_{BA}(z,x) \frac{\delta}{\delta\langle\Phi^i_B(z)\rangle}, \tag{4.14}$$

we write (4.9) for the propagator of the fermion fields in the integral form:

$$\begin{gathered}\left(\hat{\partial} - M_A\langle\Phi^n_A(x)\rangle - B'^A_\mu\langle C^A_\mu(x)\rangle\right)G^n(x,y) \\ - \int d^4t\ \Sigma(x,t)G^n(t,y) = \delta(x-y).\end{gathered} \tag{4.15}$$

Here

$$\Sigma(x,t) = \Sigma_1(x,t) + \Sigma_2(x,t) \tag{4.16}$$

is the mass operator, where

$$\begin{gathered}\Sigma_1(x,t) = i\left(\lambda_0^n\right)^2 M'_A \int d^4w d^4v G^n(x,v) \\ \times\Gamma^n_D(w,v,t)\Delta^n_{DA}(w,x),\end{gathered} \tag{4.17}$$

$$\Sigma_2(x,t) = iB^A_\mu \int d^4w d^4v G^n(x,v)\Gamma^{nN}_\nu(w,v,t)D^{nA}_{\nu\mu}(w,x), \tag{4.18}$$

and the vertex operators are defined as follows:

$$\Gamma^n_D(w,v,t) = \frac{\delta\left(G^n(v,t)\right)^{-1}}{\delta\langle\Phi^n_D(w)\rangle},\ \ \Gamma^{nN}_\nu(w,v,t) = \frac{\delta\left(G^n(v,t)\right)^{-1}}{\delta\langle C^N_\nu(w)\rangle}. \tag{4.19}$$

We obtain the equation for the vacuum expectation $\langle C_\mu^A(x)\rangle$ by using the obvious relations

$$\begin{aligned}\frac{\delta\langle C_\mu^N\rangle}{\delta J_\nu^L} &= i\langle C_\mu^N C_\nu^{L+}\rangle - i\langle C_\mu^N\rangle\langle C_\nu^{L+}\rangle,\\ \frac{\delta\langle C_{[\mu\sigma]}^N\rangle}{\delta J_\nu^L} &= i\langle C_{[\mu\sigma]}^N C_\nu^{L+}\rangle - i\langle C_{[\mu\sigma]}^N\rangle\langle C_\nu^{L+}\rangle,\end{aligned} \tag{4.20}$$

and averaging (4.5) over the vacuum:

$$\begin{gathered}-\partial_\nu^2\langle C_\mu^L(x)\rangle + \partial_\nu\partial_\mu\langle C_\nu^L(x)\rangle\\ -ig\left(2\delta_{\sigma\alpha}\delta_{\mu\rho} - \delta_{\mu\alpha}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\rho\alpha}\right)\\ \times\left(-i\frac{\delta\langle T\partial_\alpha C_\rho^F(x)\rangle}{\delta J_\sigma^B(x)} + \langle\partial_\alpha C_\rho^F(x)\rangle\langle C_\sigma^B(x)\rangle\right)m^{LF^+B}\\ -ig^2\left(2\delta_{\sigma\alpha}\delta_{\mu\rho} - \delta_{\mu\alpha}\delta_{\sigma\rho}\right)D_{\rho\sigma}^{MK}(x,x)\langle C_\alpha^P(x)\rangle M^{LP^+M^+K}\\ +g^2\left(\frac{\delta}{\delta J_\nu^B(x)}D_{\mu\nu}^{F^+N}(x,x)\right.\\ \left.-\langle C_\nu^{B^+}(x)\rangle\langle C_\nu^{N^+}(x)\rangle\langle C_\mu^{F^+}(x)\rangle\right)m^{LFNB}\\ = J_\mu^L(x) + i\mathrm{Tr}\left[B_\mu^L G(x,x)\right]\\ +\left(g\frac{\delta\langle T\partial_\mu\bar{C}^{B+}(x)\rangle}{\delta\bar{\eta}^D(x)} + ig\langle\partial_\mu\bar{C}^{B+}(x)\rangle\langle C^{D+}(x)\rangle\right)K^{LBD}.\end{gathered} \tag{4.21}$$

After the variation of (4.21) by $J^B_\rho(y)$, we obtain the DS equation for the propagator of gauge vector fields

$$
\begin{aligned}
&-\partial^2_\nu D^{LA}_{\mu\beta}(x,y) - ig\left(2\delta_{\sigma\alpha}\delta_{\mu\rho} - \delta_{\mu\alpha}\delta_{\sigma\rho}\right)\\
&\times\Bigg\{\Bigg[\int d^4v\Bigg(\partial^x_\alpha\delta(x-v)\delta_{i\rho}\langle C^B_\sigma(x)\rangle\\
&-\langle\partial^x_\alpha C^B_\rho(x)\rangle\delta(x-v)\delta_{i\sigma}\Bigg)m^{LC^+B^+}\\
&+g\left(D^{MK}_{\rho\sigma}(x,x) + i\langle C^M_\rho(x)\rangle\langle C^{K+}_\sigma(x)\rangle\right)\\
&\times\delta_{\alpha i}m^{LC^+M^+K}\int d^4v\delta(x-v)\Bigg]D^{CA}_{i\beta}(v,y)\\
&+g\langle C^P_\alpha(x)\rangle m^{LP^+F^+B}\frac{\delta}{\delta J^A_\beta(y)}D^{FB}_{\rho\sigma}(x,x)\\
&-m^{LF^+B}i\frac{\delta}{\delta J^B_\sigma(x)}\frac{\delta\langle T\partial_\alpha C^F_\rho(x)\rangle}{\delta J^A_\beta(y)}\Bigg\}\\
&+g^2\frac{\delta}{\delta J^A_\beta(y)}\frac{\delta}{\delta J^B_\nu(x)}D^{FN}_{\mu\nu}(x,x)m^{LF^+NB}\\
&=\delta_{\mu\beta}\delta_{LA}\delta(x-y) + i\mathrm{Tr}\left[B^L_\mu\frac{\delta}{J^A_\beta(y)}G(x,x)\right]\\
&+g\frac{\delta}{J^A_\beta(y)}\Bigg[\frac{\delta\langle\partial_\mu\bar{C}^{B+}(x)\rangle}{\delta\bar{\eta}^D(x)}
\end{aligned}
\tag{4.22}
$$

$$+i\langle\partial_\mu\bar{C}^{B+}(x)\rangle\langle C^{D+}(x)\rangle\Big]K^{LBD},$$

where $\partial^x_\alpha = \partial/\partial x_\alpha$,

$$m^{LF^+B} = m^{LFB}\left(V^F \leftrightarrow H^F\right),$$

$$m^{LP^+M^+K} = m^{LPMK}\left(V^P \leftrightarrow H^P,\ V^M \leftrightarrow H^M\right).$$

Determining the three-point and four-point Green's functions for vector gauge fields as follows:

$$\Gamma^{HLC}_{\alpha\mu\sigma}(z,x,y) = \frac{\delta\left(D^{LC}_{\mu\sigma}(x,y)\right)^{-1}}{\delta\langle C^H_\alpha(z)\rangle},$$
$$\Gamma^{MHLC}_{\beta\alpha\mu\sigma}(t,z,x,y) = \frac{\Gamma^{HLC}_{\alpha\mu\sigma}(z,x,y)}{\delta\langle C^M_\beta(t)\rangle}, \tag{4.23}$$

we rewrite (4.22) by taking into account equation (4.13), as

$$\int d^4v\Big\{-\partial^2_\nu\delta_{\mu i}\delta_{LC}\delta(x-v)$$

$$-ig\left(2\delta_{\sigma\alpha}\delta_{\mu\rho}-\delta_{\mu\alpha}\delta_{\sigma\rho}\right)\Big[\int d^4z\Big(\partial^x_\alpha\delta(x-z)\delta(z-v)$$

$$\times\delta_{i\rho}\langle C^B_\sigma(x)\rangle - \langle C^B_\rho(z)\rangle\delta(x-v)\delta_{i\sigma}\Big)m^{LC^+B^+}$$

$$
\begin{aligned}
&+g\bigg(D^{MK}_{\rho\sigma}(x,x)+i\langle C^{M}_{\rho}(x)\rangle \\
&\times\langle C^{K+}_{\sigma}(x)\rangle\bigg)\delta_{\alpha i}m^{LC^{+}MK}\delta(x-v) \\
&+i\int d^4zd^4td^4w\,(\partial^x_\alpha\delta(x-z))\,m^{LF^{+}B}+ig\langle C^{P}_{\alpha}(x)\rangle \\
&\times\delta(x-z)m^{LP^{+}F^{+}B}\Big]D^{FM}_{\rho\eta}(z,t)D^{NB}_{\gamma\sigma}(w,x)\Gamma^{CMN}_{i\eta\gamma}(v,t,w) \\
&+i\mathrm{Tr}\left\{B^{L}_{\mu}\int d^4td^4w\;G(x,t)\Gamma^{C}_{i}(v,t,w)G(w,x)\right\} \\
&+g\int d^4zd^4td^4w\Big(\partial^x_\mu\delta(x-z)\Big) \qquad (4.24)\\
&\times\bar{G}^{BD}(z,t)\bar{K}^{CQN}_{i}(v,t,w)\bar{G}^{ND}(w,x)K^{LB^{+}D} \\
&-ig\int d^4z\Big(\partial^x_\mu\delta(x-z) \\
&\times\left(\bar{K}^{CB}(v,z)\langle C^{D^{+}}(x)\rangle+\langle\bar{C}^{B}(z)\rangle K^{CD^{+}}_{i}(v,x)\right)K^{LB^{+}D} \\
&+g^2\int d^4zd^4td^4w\bigg(\int d^4md^4n\bigg(D^{KQ}_{\alpha j}(z,m)D^{DB}_{\tau\nu}(n,x) \\
&\times D^{FM}_{\mu\rho}(x,t)D^{PN}_{\sigma\nu}(w,x)+D^{KB}_{\alpha\nu}(z,x) \\
&\times D^{FQ}_{\mu j}(x,m)D^{DM}_{\tau\rho}(n,t)D^{PN}_{\sigma\nu}(w,x) \\
&+D^{KB}_{\alpha\nu}(z,x)D^{FM}_{\mu\rho}(x,t)D^{PQ}_{\sigma j}(w,m)D^{DN}_{\tau\nu}(n,x)
\end{aligned}
$$

$$\times\Gamma^{CQD}_{ij\tau}(v,m,n)\Gamma^{KMP}_{\alpha\rho\sigma}(z,t,w)\Big)$$

$$-D^{KB}_{\alpha\nu}(z,x)D^{FM}_{\mu\rho}(x,t)D^{PN}_{\sigma\nu}(w,x)$$

$$\times\Gamma^{CKMP}_{i\alpha\rho\sigma}(v,z,t,w)\Big)m^{LF^+NB}\Big\}D^{CA}_{i\beta}(v,y)$$

$$=\delta_{\mu\beta}\delta_{LA}\delta(x-y),$$

where

$$\bar{K}^{CQN}_i(v,t,w)=\frac{\delta\left(\bar{G}^{QN}(t,w)\right)^{-1}}{\delta\langle C^C_i(v)\rangle},$$

$$\bar{K}^{CB}_i(v,z)=\frac{\delta\langle\bar{C}^B(z)\rangle}{\delta\langle C^C_i(v)\rangle},$$

$$K^{CB^+}_i(v,z)=\frac{\delta\langle C^{B^+}(z)\rangle}{\delta\langle C^C_i(v)\rangle},\quad \bar{G}^{BD}(z,x)=\frac{\delta\langle\bar{C}^B(x)\rangle}{\delta\bar{\eta}^D(z)}.$$

We now introduce the DS equation for the vertex operator $\Gamma^B_\nu(z,x,y)$. Using (4.19), from (4.15), we obtain the equation

$$\Gamma^{nB}_\nu(z,x,y)=-B'^B_\nu\delta(z-x)\delta(x-y)$$

$$-M_A\frac{\delta\langle\Phi^n_A(x)\rangle}{\delta\langle C^B_\nu(z)\rangle}\delta(x-y)-\Lambda^B_\nu(z,x,y), \tag{4.25}$$

where

$$\Lambda_\nu^B(z,x,y) = \frac{\delta\Sigma(x,y)}{\delta\langle C_\nu^B(z)\rangle} \tag{4.26}$$

is the vertex function.

Taking the variational derivative of both sides of (4.10), we find

$$\begin{aligned}\frac{\delta\langle\Phi_A^n(x)\rangle}{\delta\langle C_\nu^B(z)\rangle} &= i(\lambda_0^n)^2 \int d^4t d^4w d^4v \Delta'^n_{AN}(x,t) \\ &\times \mathrm{Tr}\Big\{ M'_N G^n(t,w) \Big[B'^B_\nu \delta(w-v)\delta(z-w) \\ &+ \Lambda_\nu^B(z,w,v)\Big] G^n(v,z)\Big\}.\end{aligned} \tag{4.27}$$

Here we have introduced the Green function $\Delta'^n_{AN}(x,t)$, satisfying

$$\begin{aligned}\int d^4t \Big[&(\lambda_0^n)^2 \delta_{AN}\delta(x-t) - i(\lambda_0^n)^2 \\ &\times \mathrm{Tr}\Big\{ M'_N G^n(x,t) M_N G^n(t,x)\Big\}\Big] \Delta'^n_{NB}(t,y) \\ &= \delta_{AB}\delta(x-y).\end{aligned} \tag{4.28}$$

As a result, we finally obtain the equation

$$\Gamma_\nu^{nB}(z,x,y) = -B'^B_\nu \delta(z-x)\delta(x-y)$$

$$
\begin{gathered}
-M_A i(\lambda_0^n)^2 \int d^4t d^4w d^4v \Delta'^n_{AN}(x,t) \mathrm{Tr}\Big\{ M'_N \\
\times G^n(t,w) \left[B'^B_\nu \delta(z-w)\delta(w-v) + \Lambda^B_\nu(z,w,v) \right] G^n(v,t) \Big\} \\
\times \delta(x-y) - \Lambda^B_\nu(z,x,y).
\end{gathered} \tag{4.29}
$$

Taking the variation with respect to $J_B(y)$ from (4.10), we obtain the DS equation, which is subject to the propagator (4.12) of collective scalar fields:

$$
\int d^4t \left[\lambda_0^n \delta(x-t)\delta_{AN} + P^n_{NB}(x,t) \right] \Delta^n_{NB}(t,y) = \delta_{AB}\delta(x-y), \tag{4.30}
$$

where

$$
\begin{gathered}
P^n_{AN}(x,t) = i(\lambda_0^n)^2 \mathrm{Tr}\Big\{ M'_A \int d^4w d^4v G^n(x,w) \\
\times \Gamma^n_N(t,w,v) G^n(v,x) \Big\}
\end{gathered} \tag{4.31}
$$

is the polarization operator.

Proceeding in a similar way, you can get other DS equations, in particular, for the propagator of the “ghost” fields.

4.1.2 Perturbation expansion and the effective Lagrangian

To solve the DS equations (4.15), (4.24) and (4.30), we use an expansion in loops of the form (3.39) (see [348])

$$\begin{aligned}
\langle \Phi_A^n(x) \rangle &= \Phi_{0A}^n + \varepsilon \Phi_{1A}^n + \varepsilon^2 \Phi_{2A}^n + ..., \\
G^n(x,y)) &= G_0^n + \varepsilon G_1^n + \varepsilon^2 G_2^n + ..., \\
\langle C_\mu^A(x) \rangle &= C_{0\mu}^A + \varepsilon C_{1\mu}^A + \varepsilon^2 C_{2\mu}^A + ..., \\
D_{\mu\nu}^{AB}(x,y) &= D_{0\mu\nu}^{AB} + \varepsilon D_{1\mu\nu}^{AB} + \varepsilon^2 D_{2\mu\nu}^{AB} + ..., \\
\Delta_{AB}^n(x,y) &= \Delta_{0AB}^n + \varepsilon \Delta_{1AB}^n + \varepsilon^2 \Delta_{2AB}^n + ... \; .
\end{aligned} \tag{4.32}$$

After substituting (4.32) into (4.15), (4.24), and (4.30), it is necessary to equate the terms containing the same powers of ε (indicating the order of the loop expansion). It should be noted that the functional derivative of the source transfers the n-loop value into the $(n+1)$-loop value.

As a result of the first order of the expansion in loops, we obtain the following equation for the fermion propagator:

$$\left(\hat{\partial} - M_A \Phi_{0A}^n - B_\mu^{\prime A} C_{0\mu}^A\right) G_0^n(x,y) = \delta(x-y). \tag{4.33}$$

Putting $J_\mu^A(x) = 0$, we come to the condition $\langle C_\mu^A(x) \rangle = 0$ and $C_{0\mu}^A = 0$. This is a consequence of the fact that the Lorentz symmetry is not broken spontaneously. Then (4.33) takes the form (3.14), and the corresponding solution in momentum space, coincides with (3.19). Recall that it follows from the form of the Green function (3.19) that all generations of neutrinos are massless and leptons ψ (e, μ, τ) as a result of dynamic symmetry breaking acquire masses $m_n^2 = |\Phi_{01}^n|^2 + |\Phi_{02}^n|^2$.

To obtain the mass spectrum of $W_\mu^\pm$, Z_μ, and A_μ fields, we consider the DS equations for the propagators of the gauge vector fields (4.22). Using (4.32) and restricting two-loop approximation, from which we retain only the diagrams shown in Fig. 4.1, where the wavy lines correspond to the gauge vector fields, dot-dash lines correspond to "ghosts", solid lines correspond to fermions and intermittent lines correspond to collective scalar fields. Going to the momentum space, the DS equation can be written as

$$\left(k^2\delta_{\mu i}\delta_{LC} + \Pi_{1\mu i}^{LC}(k) + \Pi_{2\mu i}^{LC}(k)\right) D_{2i\beta}^{CA}(k) = \delta_{\mu\beta}\delta_{LA}. \tag{4.34}$$

Here the polarization operator $\Pi_{1\mu i}^{LC}(k)$ is defined as follows:

$$\Pi_{1\mu i}^{LC}(k) = R_{1\mu i}^{LC} + \tilde{\Pi}_{1\mu i}^{LC}(k) + \hat{\Pi}_{1\mu i}^{LC}(k) + \bar{\Pi}_{1\mu i}^{LC}(k), \tag{4.35}$$

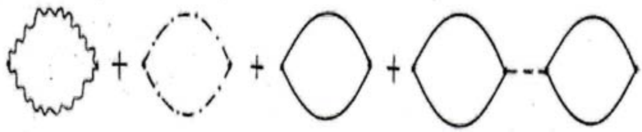

Figure 4.1: Loop diagrams.

where

$$R^{LC}_{1\mu i} = \frac{ig^2}{(2\pi)^4}\left(2\delta_{\mu\sigma}\delta_{\nu i} - \delta_{\mu i}\delta_{\nu\sigma}\right)\int d^4p D^{DK}_{0\sigma\nu}(p) m^{LD^+C^+K}, \tag{4.36}$$

$$R^{LC}_{1\mu i} \equiv \left(2\delta_{\mu\sigma}\delta_{\nu i} - \delta_{\mu i}\delta_{\nu\sigma}\right) R^{LC}_{\sigma\nu}, \tag{4.37}$$

$$\begin{aligned}\tilde{\Pi}^{LC}_{1\mu i}(k) = &\frac{ig^2}{(2\pi)^4} m^{LF^+B}\left(2\delta_{\sigma\alpha}\delta_{\mu\rho} - \delta_{\mu\alpha}\delta_{\sigma\rho}\right)\\ &\times \int d^4p p_\alpha D^{FM}_{0\rho\eta}(p) D^{NB}_{0\gamma\sigma}(p-k)\Gamma^{CMN}_{0i\eta\gamma}(p, p-k),\end{aligned} \tag{4.38}$$

$$\hat{\Pi}^{LC}_{1\mu i}(k) = \frac{ig^2}{(2\pi)^4} K^{A^+N^+Q} K^{LB^+D}$$

$$\times \int d^4 p p_\mu \bar{G}^{BQ}(p)(p_i - k_i)\bar{G}^{ND}(p-k), \tag{4.39}$$

$$\bar{\Pi}^{LC}_{1\mu i}(k) = -\frac{i}{(2\pi)^4}\mathrm{tr}\left\{B^L_\mu \int d^4 p G^n_0(p) B'^C_i G^n_0(p-k)\right\}. \tag{4.40}$$

The analytical expression corresponding to the fourth pole diagram in Fig. 4.1 has the form

$$\Pi^{LC}_{2\mu i}(k) = \frac{(\lambda^n_0)^2}{(2\pi)^4}\mathrm{tr}\left\{B^L_\mu \int d^4 p G^n_0(p) M_M G^n_0(p-k)\right\} \tag{4.41}$$

$$\times \Delta^n_{0MN}(k)\mathrm{tr}\left\{M'_N \int d^4 q G^n_0(q) B'^C_i G^n_0(q-k)\right\}.$$

Note that the propagator of the collective fields, defined by (4.12), is associated with the propagator (3.26) by a transformation of a basis in the space of functions $\Phi_A(x)$:

$$\left(T^n_{AB}(x,y)\right)^{-1} = \left[D_1 \left(\Delta^n(x,y)\right)^{-1} D_2\right]_{AB}, \tag{4.42}$$

where

$$D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.43}$$

The relationship (4.42) is due to the definition of the propagator

$$T^n_{AB}(x,y) = \frac{\delta\langle \Phi^n_A(x)\rangle}{\lambda^n_0 \delta J^*_B(y)}. \tag{4.44}$$

Using the values (3.27) and the relation (4.42), after the renormalization

$$(\lambda^n_0)^2\, \Delta^n_0(k) = \lambda^2 \bar{\Delta}^n_0(k), \tag{4.45}$$

where $\lambda^2 \equiv Z_3$, we find

$$\bar{\Delta}^n_0(k) = \frac{1}{k^2} - \frac{2m_n^2}{k^2(k^2+4m_n^2)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{4.46}$$

Evaluating (4.41) with (4.46), we obtain non-zero elements of $\Pi^{LC}_{2\mu i}$ (up to $\mathcal{O}(\lambda^2)$):

$$\Pi^{11}_{2\mu i}(k) = \Pi^{22}_{2\mu i}(k) = -\frac{\lambda^2 g^2}{2} \sum_n m_n^2 \frac{k_\mu k_\nu}{k^2} \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2, \tag{4.47}$$

$$\Pi^{44}_{2\mu i}(k) = -\lambda^2 \frac{g^2 + g'^2}{2} \sum_n m_n^2 \frac{k_\mu k_\nu}{k^2} \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2.$$

We choose the constant λ to be the same for all generations of fermions.

By imposing the requirement

$$k_\mu k_\nu \Pi_{\mu i}(k) = 0, \tag{4.48}$$

with the one-loop approximation, we obtain

$$\Pi_{1\mu i}^{LC}(k)+\Pi_{2\mu i}^{LC}(k) = \left(\delta_{\mu i} - \frac{k_\mu k_i}{k^2}\right)\sum_n \left(\Pi_1^{LC}(k) + \Pi_2^{LC}(k)\right), \tag{4.49}$$

where

$$\begin{aligned}
\Pi_1^{11}(k) = \Pi_1^{22}(k) &= \frac{g^2}{3\pi^2}k^2 \ln\frac{\Lambda^2}{m^2},\\
\Pi_1^{33}(k) &= \frac{e^2}{48\pi^2}19k^2 \ln\frac{\Lambda^2}{m^2},\\
\Pi_1^{34}(k) = \Pi_1^{43}(k) &= \frac{-16g^2+3g'^2}{48\pi^2}e\beta k^2 \ln\frac{\Lambda^2}{m^2},\\
\Pi_1^{44}(k) &= \frac{-16g^4+3g'^4}{48\pi^2}\beta^2 k^2 \ln\frac{\Lambda^2}{m^2},
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\Pi_2^{11}(k) = \Pi_2^{22}(k) &= \frac{\lambda^2 g^2}{2}\sum_n m_n^2 \left(\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\right)^2,\\
\Pi_2^{44}(k) &= \lambda^2 \frac{g^2+g'^2}{2}\sum_n m_n^2\left(\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\right)^2.
\end{aligned} \tag{4.51}$$

As seen from (4.50), one-loop approximation, $\Pi_1^{LC}(k)$, does not cause the masses of gauge vector bosons, and two-loop approximation of $\Pi_2^{LC}(k)$, as it follows from the form of equations (4.51), leads to the values $m_W \neq$, $m_Z \neq 0$. Obviously, the appearance of the masses of the intermediate vector bosons is associated with the formation of the collective scalar fields $\Phi_A(x)$; the propagator behaves as $1/k^2$ (see also [351]).

From (4.51) we find the following expression for the squared mass of gauge bosons $W^{\pm}$, Z:

$$m_W^2 = \frac{g^2}{2\lambda^2} \sum_n m_n^2, \quad m_Z^2 = \frac{g^2 + g'^2}{2\lambda^2} \sum_n m_n^2. \tag{4.52}$$

It follows from (4.50) and (4.51) that the mass of the photon field A_μ is equal to zero.

After the regularization procedure, we note that the propagator of vector gauge fields in this model can be written as (in the Landau gauge):

$$\bar{D}^C_{2\mu i}(k) = \frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + \bar{m}_C^2 + \mathcal{O}(g^4)}. \tag{4.53}$$

Here, $\bar{m}_C^2$ (at $C = W^{\pm}, Z$) are renormalized masses squared of $W^{\pm}$, Z-bosons (see (4.52)):

$$\bar{m}_W^2 = \frac{\bar{g}^2}{2\lambda^2} \sum_n m_n^2, \quad \bar{m}_Z^2 = \frac{\bar{g}^2 + \bar{g}'^2}{2\lambda^2} \sum_n m_n^2, \tag{4.54}$$

where $\bar{g}$, $\bar{g}'$ are renormalized coupling constants.

Thus, due to the four-fermion interaction, the generation of the lepton masses as e, μ, τ, and the intermediate vector bosons occurred. A similar situation exists in the quark sector.

You can go to such a basis in the space of functions Φ_A, in which the propagator of the collective fields (4.46) is diagonalized and takes the form (see Sec. 3.2)

$$\bar{\Delta}_2^n(q) = \mathrm{diag}\left(\frac{1}{q^2}, \frac{1}{q^2}, \frac{1}{q^2}, \frac{1}{q^2 + 4m_n^2}\right), \; \Phi_A = \begin{pmatrix} \Phi_1^n \\ \Phi_1^{*n} \\ \chi^n \\ \kappa^n \end{pmatrix}, \tag{4.55}$$

$$\Phi_2^n = \kappa^n + i\chi^n, \qquad \kappa^n = \frac{1}{2}\lambda_0^n \langle \bar{\psi}^n \psi^n \rangle.$$

Thus, for each generation of fermions, subject to the inclusion of each of them, we have three scalar massless (Goldstone) fields and one massive field (κ^n) with the mass $2m_n$. It will be shown that the Goldstone fields are "eaten" in the same way as in the standard model. In the next section it will be given a more realistic scheme of mixing lepton and quark generations, which will determine the existence of a heavy composite scalar (Higgs) boson. Now we calculate the total effective action of the model.

In Sec. 3.1.2 for the global $SU(2)_L \otimes U(1)$-invariant model the Lagrangian was built by taking into account only the interaction of collective scalar fields with each other (3.30). We now construct the effective Lagrangian corresponding to the interaction of fermions and vector gauge fields with the scalar collective fields. We start from the integral of the action which has the following structure (see Fig. 4.2):

$$
\begin{aligned}
S = -\Bigg[\int d^4x d^4y d^4z \Bigg\{ & \bar{\xi}^n(x)\Gamma^{(n)A}(z,x,y)\xi^n(y)\langle\Phi^n_A(z)\rangle \\
& + \frac{1}{2(\lambda^n_0)^2}\langle\Phi^{n*}_A(x)\rangle\Gamma^{(n)NAB}_{\mu}(z,x,y)\langle\Phi^n_B(y)\rangle\langle C^N_\mu(z)\rangle \\
& + \frac{1}{2(\lambda^n_0)^2}\langle C^{A*}_\mu(x)\rangle\Gamma^{(n)NAB}_{\mu\nu}(z,x,y)\langle C^B_\nu(y)\rangle\langle\Phi^n_N(z)\rangle \Bigg\} \\
& + \frac{1}{(\lambda^n_0)^2}\int d^4x d^4z \langle\Phi^{n*}_A(x)\rangle\Gamma^{(n)DA}_{\mu}(z,x)\langle C^D_\mu(z)\rangle \\
& + \frac{1}{4(\lambda^n_0)^2}\int d^4x d^4y d^4z d^4t \langle\Phi^{n*}_A(x)\rangle\Gamma^{(n)DNAB}_{\mu\nu}(t,z,x,y) \\
& \times\langle\Phi^n_B(y)\rangle\langle C^N_\nu(z)\rangle\langle C^D_\mu(t)\rangle\Bigg].
\end{aligned}
\tag{4.56}
$$

The problem reduces to finding the Green functions introduced:

$$
\Gamma^{(n)A}(z,x,y) = \lambda^n_0 \frac{\delta\left(G^n(x,y)\right)^{-1}}{\delta\langle\Phi^n_A(z)\rangle},
$$

$$\Gamma_{\mu}^{(n)DA}(z,x) = \frac{1}{\lambda_0^n}\int d^4t \frac{\delta\langle\Phi_N^n(t)\rangle}{\delta\langle C_\mu^D(z)\rangle}\left(\Delta_{NA}^{(n)}(t,x)\right)^{-1}, \tag{4.57}$$

$$\Gamma_{\mu}^{(n)NAB}(z,x,y) = \frac{\delta\left(\Delta_{AB}^n(x,y)\right)^{-1}}{\delta\langle C_\mu^N(z)\rangle}, \quad \Gamma_{\mu\nu}^{(n)NAB}(z,x,y)$$

$$= \lambda_0^n \frac{\delta D_{\mu\nu}^{AB-1}(x,y)}{\delta\langle\Phi_N^n(z)\rangle},$$

$$\Gamma_{\mu\nu}^{(n)DNAB}(t,z,x,y) = \frac{\delta\left(\Delta_{AB}^n(x,y)\right)^{-1}}{\delta\langle C_\mu^D(t)\rangle\delta\langle C_\nu^N(z)\rangle}$$

$$= \frac{(\lambda_0^n)^2\delta D_{\mu\nu}^{AB-1}(x,y)}{\delta\langle\Phi_D^n(t)\rangle\delta\langle\Phi_N^n(z)\rangle}.$$

Restricting one-loop approximation in the momentum representation, we have:

$$\Gamma_{1\mu}^{(n)DA}(p) = \frac{i\lambda_0^n}{(2\pi)^4}\int d^4k \mathrm{tr}\left\{M_A' G_0^n(k+p) B_\mu'^D G_0^n(k)\right\}. \tag{4.58}$$

After a direct computation of (4.58), we find the non-zero elements of $\Gamma_{1\mu}^{(n)DA}(p)$ (up to $\mathcal{O}(\lambda^2)$):

$$\Gamma_\mu^{(n)21}(p) = -\Gamma_\mu^{(n)13}(p) = -\frac{g}{\sqrt{2}}\frac{m_n}{\lambda^2}p_\mu,$$

$$\Gamma_\mu^{(n)42}(p) = -\Gamma_\mu^{(n)44}(p) = -\frac{m_n}{\lambda^2}p_\mu\frac{g^2+g'^2}{2}\beta. \tag{4.59}$$

Accordingly for other Green's functions, we obtain the relations of the type:

$$\Gamma_{\mu}^{(n)NAB}(p,q) = -\frac{i(\lambda_0^n)^2}{(2\pi)^4}\mathrm{tr}\Big\{M'_A\int d^4k\Big[G_0^n(k+p)B_{\mu}^{'N} \tag{4.60}$$
$$\times G_0^n(q+k)M_B G_0^n(k) + G_0^n M_B G_0^n(k-q)B_{\mu}^{'N}G_0^n(k-p)\Big]\Big\}.$$

After the integration, up to $\mathcal{O}(\lambda^2)$, we have

$$\Gamma_{\mu}^{(n)NAB}(p,q) = (p_\mu + q_\mu)Z_3^{-1}\Gamma_{NAB}^{(n)}, \tag{4.61}$$

where

$$\begin{gathered}\Gamma_{121}^{(n)} = \Gamma_{212}^{(n)} = -\Gamma_{134}^{(n)} = -\Gamma_{243}^{(n)} = -\frac{g}{\sqrt{2}},\\ \Gamma_{311}^{(n)} = -\Gamma_{333}^{(n)} = e,\\ \Gamma_{411}^{(n)} = -\Gamma_{433}^{(n)} = \frac{g^{'2} - g^2}{2}\beta,\\ \Gamma_{422}^{(n)} = -\Gamma_{444}^{(n)} = \frac{g^{'2} + g^2}{2}\beta.\end{gathered} \tag{4.62}$$

Similarly, for the function $\Gamma_{1\mu\nu}^{(n)NAB}$, we get

$$\Gamma_{1\mu\nu}^{(n)NAB}(p,q) = -\frac{i\lambda_0^n}{(2\pi)^4}\mathrm{tr}\Big\{B_\mu^A\int d^4k\Big[G_0^n(k+p)M_N$$

$$\times G_0^n(q+k)B_\nu^{'B}G_0(k)+G_0^n(k)B_\nu^{'B}G_0^n(k-q)M_N G_0^n(k-p)\Big]\Big\}, \tag{4.63}$$

and up to $\mathcal{O}(\lambda^2)$ from (4.63), we find that

$$\begin{gathered}
\Gamma_{1\mu\nu}^{(n)NAB} = \delta_{\mu\nu}\frac{m_n}{\lambda^2}\Gamma_{NAB}, \\
\Gamma_{123} = \Gamma_{131} = \Gamma_{313} = \Gamma_{332} = \frac{eg}{\sqrt{2}}, \\
\Gamma_{124} = \Gamma_{314} = \Gamma_{342} = \Gamma_{141} = \frac{eg'}{\sqrt{2}}, \\
\Gamma_{211} = \Gamma_{222} = \Gamma_{411} = \Gamma_{422} = -\frac{g^2}{2}, \\
\Gamma_{244} = \Gamma_{444} = -\frac{g'^2 + g^2}{2}.
\end{gathered} \tag{4.64}$$

The function $\Gamma_{1\mu\nu}^{(n)DNAB}(p,q,k)$ in the first approximation, using the expansion in loops, has the final form:

$$\Gamma_{1\mu\nu}^{(n)DNAB}(p,q,k) = -\frac{i(\lambda_0^n)^2}{(2\pi)^4}\mathrm{tr}\int d^4t M_A' G_0^n(p+t)B_\mu^{'D}$$
$$\times G_0^n(q+t)B_\nu^{'N}G_0^n(q+t)M_B G_0^n(t) + \ldots\ , \tag{4.65}$$

and after the direct computation leads to

$$\Gamma_{1\mu\nu}^{(n)DNAB}(p,q,k) = \delta_{\mu\nu}Z_3^{-1}\Gamma_{DNAB}, \tag{4.66}$$

where

$$
\begin{aligned}
&\Gamma_{1211} = \Gamma_{1222} = \Gamma_{1233} = \Gamma_{1244} = \Gamma_{2111} \\
&= \Gamma_{2122} = \Gamma_{2144} = \Gamma_{2133} = \frac{g^2}{2}, \\
&\Gamma_{1321} = \Gamma_{1334} = \Gamma_{2312} = \Gamma_{2343} = \Gamma_{3121} \\
&= \Gamma_{3134} = \Gamma_{3243} = \Gamma_{3212} = -\frac{eg}{\sqrt{2}}, \\
&\Gamma_{1421} = \Gamma_{1434} = \Gamma_{2412} = \Gamma_{2443} = \Gamma_{4121} \\
&= \Gamma_{4134} = \Gamma_{4212} = \Gamma_{4243} = -\frac{eg'}{\sqrt{2}}, \\
&\Gamma_{3411} = \Gamma_{3433} = \Gamma_{4311} = \Gamma_{4333} = \beta e \left(g'^2 - g^2 \right), \\
&\Gamma_{4422} = \Gamma_{4444} = \frac{g'^2 + g^2}{2}, \\
&\Gamma_{3311} = \Gamma_{3333} = 2e^2, \quad \Gamma_{4411} = \Gamma_{4433} = \frac{\left(g'^2 - g^2 \right)^2}{2} \beta^2.
\end{aligned}
\tag{4.67}
$$

Thus, the effective interaction Lagrangian corresponding to the action (4.56) by taking into consideration (4.59)-(4.67) takes the form ($\Gamma_{0A}^{(n)} = -M'_A$):

$$
\mathcal{L} = -\bar{\psi}_R^n \nu_L^n \Phi_1^{n*} - \bar{\nu}_L^n \psi_R^n \Phi_1^n - \bar{\psi}^n \psi^n \kappa^n + i\bar{\psi}^n \gamma_5 \psi^n \chi^n
$$

$$
\begin{aligned}
&-\frac{im_n}{\lambda^2}\left[\frac{g}{\sqrt{2}}\left(\partial_\mu\Phi_1^n W_\mu^+ - \partial_\mu\Phi_1^{n*}W_\mu^-\right) - \frac{i}{\beta}\partial_\mu\chi^n Z_\mu\right] \\
&+\frac{i}{\lambda^2}\Bigg[\frac{g}{\sqrt{2}}\left(W_\mu^+\Phi_2^{n*}\partial_\mu^{\leftrightarrow}\Phi_1^n + W_\mu^-\Phi_1^{n*}\partial_\mu^{\leftrightarrow}\Phi_2^n\right) - eA_\mu\Phi_1^{n*}\partial_\mu^{\leftrightarrow}\Phi_1^n \\
&-\frac{g'^2-g^2}{2}\beta Z_\mu\Phi_1^{n*}\partial_\mu^{\leftrightarrow}\Phi_1^n - \frac{1}{2\beta}Z_\mu\Phi_2^{n*}\partial_\mu^{\leftrightarrow}\Phi_2^n\Bigg] \\
&+\frac{m_n}{2}\Bigg[\frac{eg}{\sqrt{2}}\left(W_\mu^+A_\mu\Phi_1^n + W_\mu^-A_\mu\Phi_1^{n*}\right) \\
&+\frac{eg'}{\sqrt{2}}\left(W_\mu^+Z_\mu\Phi_1^n + W_\mu^-Z_\mu\Phi_1^{n*}\right) \qquad (4.68) \\
&-g^2\kappa^n W_\mu^+W_\mu^- - \frac{g'^2+g^2}{2}\kappa^n Z_\mu^2\Bigg] \\
&-\frac{1}{\lambda^2}\Bigg[\Phi^{n*}\Phi^n\left(\frac{g^2}{2}W_\mu^+W_\mu^- + \frac{g'^2+g^2}{2}Z_\mu^2\right) \\
&-\frac{e}{\sqrt{2}}\left(\Phi_2^{n*}\Phi_1^nW_\mu^+ + \Phi_2^n\Phi_1^{n*}W_\mu^-\right)\left(gA_\mu + g'Z_\mu\right) \\
&+|\Phi_1^n|^2\Bigg(\beta e\left(g'^2-g^2\right)A_\mu Z_\mu \\
&+e^2\left(A_\mu^2 - Z_\mu^2\right)\Bigg)\Bigg] \\
&-\frac{1}{\lambda^2}\left[4m_n\kappa^n\Phi^{n+}\Phi^n + \left(\Phi^{n+}\Phi^n\right)^2\right],
\end{aligned}
$$

where $\Phi^* \overleftrightarrow{\partial_\mu} \Phi = \Phi^* \partial_\mu \Phi - (\partial_\mu \Phi^*)\Phi$, $\Phi^+ \Phi = \Phi_1^* \Phi_1 + \Phi_2^* \Phi_2$.

As seen from (4.68), the resulting interaction Lagrangian has the same form with the corresponding expression of the GWS theory of electro-weak interactions, with the only difference that each generation of leptons corresponds to a variety of scalar fields.

Goldstone fields can be removed using the following conversion of fermion operators

$$L = \frac{1}{\Pi} \begin{pmatrix} \Phi_2' & -\Phi_1' \\ \Phi_1'^* & \Phi_2'^* \end{pmatrix} L', \quad R = R', \quad \Pi = \sqrt{\Phi' \Phi'^+}; \quad (4.69)$$

then

$$\Phi_1 = 0, \quad \Phi_2 = \Pi, \quad \Pi^+ = \Pi = \kappa. \quad (4.70)$$

As a result, the interaction Lagrangian of fermion, gauge vector, and massive scalar fields can be written as:

$$\mathcal{L} = -\frac{\lambda}{\sqrt{2}} \bar{\psi}^n \psi^n \bar{\kappa}^n - \sqrt{2} m_n \lambda (\bar{\kappa}^n)^3 - \frac{\lambda^4}{4} (\bar{\kappa}^n)^4$$
$$(4.71)$$
$$- \left(\sqrt{2} \frac{m_n}{\lambda} \bar{\kappa}^n + \frac{(\bar{\kappa}^n)^2}{2} \right) \left(\frac{g^2}{2} W_\mu^+ W_\mu^- + \frac{g^2 + g'^2}{2} Z_\mu^2 \right).$$

Here

$$\bar{\kappa}^n = -\frac{\sqrt{2} \lambda_0^n}{2\lambda} \langle \bar{\psi}^n \psi^n \rangle$$

is the renormalized collective scalar field playing the role of a massive Higgs field for each generation of fermions. In the Lagrangian (4.71) the self-interaction of collective fields is added, taken from (3.30).

The considered model is a modification of the standard theory of electro-weak interactions of leptons and is built by substituting the Higgs Lagrangian by the self-interaction of leptons. It is shown that, in the two-loop approximation, in this case, the formation of masses of intermediate $W^{\pm}$ and Z bosons and e, μ, τ leptons takes place. However, in this scheme, including the four-fermion interaction, the composite scalar particles - the Higgs particle counterparts, are lightweight and have the masses $2m_e$, $2m_\mu$, $2m_\tau$.

4.1.3 Mixed interactions between leptons of different generations

Let us first consider the interaction of only leptons (3.4) with the choice of the coupling constants as follows:

$$\lambda_{ij} = \lambda_{0i}\lambda_{0j}. \tag{4.72}$$

We start from the local $SU(2)_L \otimes U(1)$-invariant Lagrangian which has a form similar to (4.1), where, however, instead of the last term the four-fermion interaction is taken, and is clearly written in the following expanded form (see (3.1), (3.4)):

$$\mathcal{L}_{int} = \sum_{i,j} \left(\lambda_{0i}\bar{L}^i R^i\right) \left(\lambda_{0j}\bar{L}^j R^j\right). \tag{4.73}$$

Based on this Lagrangian, we formulate DS equations for various Green's functions of the fields, find the solutions of these equations in this approximation, and construct an effective interaction Lagrangian for the phase with broken symmetry of the vacuum.

For the fermion Green function $G^k(x,y) = i\langle\xi^k(x)\bar{\xi}^k(y)\rangle$, we have the following equation (see (4.9)):

$$\begin{aligned}\Big(\hat{\partial} + i\lambda_{0k}M'_A\frac{\delta}{\delta J_A(x)} - \lambda_{0k}M_A\langle\Phi_A(x)\rangle + iB^A_\mu\frac{\delta}{\delta J^A_\mu(x)} \\ -B'^A_\mu\langle C^A_\mu(x)\rangle\Big)G^k(x,y) = \delta(x-y),\end{aligned} \tag{4.74}$$

where, in contrast to (4.10) of Sec. 6, the relation

$$\langle\Phi_A(x)\rangle = \sum_k \lambda_{0k}\langle T\bar{\xi}^k(x)M'_A\xi^k(x)\rangle + J_A(x), \qquad (4.75)$$

defining composite scalar fields, includes the amount of the contributions of various generations of fermion-antifermion pairs.

Rewriting (4.75) as

$$\langle\Phi_A(x)\rangle = i\sum_k \lambda_{0k}\mathrm{Tr}\left\{M'_A G^k(x,x))\right\} + J_A(x), \quad (4.76)$$

for the propagator $\Delta_{AB}(x) = \langle\delta\Phi_A(x)\rangle/\delta J_B(y)$ of scalar fields $\langle\Phi_A(x)\rangle$, we obtain the equation

$$\int d^4w\Big[\delta_{AN}\delta(x-w)$$

$$+i\sum_k \lambda_{0k}\mathrm{Tr}\Big\{M'_A\int d^4td^4zG^k(x,t)\Gamma^k_N(w,t,z) \qquad (4.77)$$

$$\times G^k(z,x)\Big\}\Big]\Delta_{NB}(w,y) = \delta_{AB}\delta(x-y),$$

which differs from (4.30) by the presence of summation by generations of leptons.

The DS equation for the propagator of gauge vector fields

$$D^{AB}_{\mu\nu}(x,y) = \frac{\delta\langle C^A_\mu(x)\rangle}{\delta J^B_\nu(y)}$$

is formally identical to the same equation (4.24) of Sec. 8; only one needs to take into account in addition to this that the incoming Green's function $G^k(x,y)$ satisfies the equation (4.74).

The solution of (4.74) in one-loop approximation is similar to (3.19), but the masses of the "bottom" leptons are now determined by the formula

$$m_k^2 = \lambda_{0k}^2 \left(|\Phi_{01}|^2 + |\Phi_{02}|^2\right), \tag{4.78}$$

from which the ratio of the masses of the leptons of different generations is

$$\frac{m_k}{m_n} = \frac{\lambda_{0k}}{\lambda_{0n}}. \tag{4.79}$$

Substituting (3.19) in (4.76) and using (4.78) with $\langle \Phi_A \rangle = const$, we obtain the following gap equation connecting the λ_{0k}, Φ_{0A} and the cut-off parameter Λ:

$$1 = -\frac{i}{8\pi^4} \sum_k \int d^4p \frac{\lambda_{0k}\lambda_{0k}}{p^2 + m_k^2}. \tag{4.80}$$

The propagator of scalar fields satisfying the equation (4.77) has, in a one-loop approximation (on a certain basis), the type (4.55), but the mass of the scalar field κ (one for all generations) is expressed as follows:

$$m_\kappa^2 = 4 \frac{\sum m_n^4}{\sum m_k^4}. \tag{4.81}$$

The renormalization constant is given by

$$Z_3^{-1} = \frac{(\lambda_{0k})^2}{16\pi^2} \ln \frac{\Lambda^2}{m^2}.$$

The existence of poles in the propagators of the fields $\Phi_A(x)$ (4.55) indicates the formation of bound states, corresponding to the three massless and one massive scalar particles, which can be regarded as analogues of three Goldstone and one Higgs boson. We emphasize once again that in this case of the inclusion of mixed four-fermion interaction (see (4.73)) the role of a scalar Higgs boson plays the composite scalar field formed by fermion-antifermion pairs of all generations (see (4.81)).

The DS equation for the propagator of gauge vector fields, written in the two-loop approximation in the momentum space, looks formally as (4.34). The polarization operator $\Pi^{LC}_{1\mu\nu}(k)$ is exactly the same expression (4.35), and $\Pi^{LC}_{2\mu\nu}(k)$ now has the following (analytical) form:

$$\Pi^{LC}_{2\mu\nu}(k) = \sum_{k,n} \frac{\lambda_{0k}\lambda_{0n}}{(2\pi)^8} \mathrm{tr}\Big\{ B^L_\mu \int d^4p G^k_0(p) M_M G^k_0(p-k)$$
$$\times \Delta_{1MN}(k) \mathrm{tr}\Big\{ M'_N \int d^4q G^k_0(q) B'^C_\nu G^n_0(q-k), \tag{4.82}$$

which takes into account only the term of the two-loop approximation, which is just responsible for the appearance of the masses of the gauge fields, $W^\pm_\mu$ and Z_μ.

After direct integration in (4.82), we find that, up to $\mathcal{O}(\lambda^2)$ non-zero elements of $\Pi^{LC}_{2\mu\nu}(k)$ are as follows :

$$\Pi^{11}_{2\mu\nu}(k) = \Pi^{22}_{2\mu\nu}(k) = -Z_3 \sum_k \frac{g^2(\lambda_{0k} m_k)^2}{2} \times \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2 \frac{k_\mu k_\nu}{k^2}, \tag{4.83}$$

$$\Pi^{44}_{2\mu\nu}(k) = -Z_3 \frac{g^2 + g^{'2}}{2} \sum_k (\lambda_{0k} m_k)^2 \times \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2 \frac{k_\mu k_\nu}{k^2}.$$

Taking into account that, if the Lorentz gauge of vector fields potentials is imposed, the relation (4.48) holds, and we obtain the masses of the intermediate vector fields

$$m_W^2 = Z_3 \sum_k \frac{g^2(\lambda_{0k} m_k)^2}{2} \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2 = \frac{g^2 \sum_k m_k^2}{2\lambda^2},$$

$$m_Z^2 = Z_3 \frac{g^2 + g^{'2}}{2} \sum_k (\lambda_{0k} m_k)^2 \left(\frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \right)^2 = \frac{g^2 + g^{'2}}{2} \frac{\sum_k m_k^2}{\lambda^2}, \quad m_A = 0. \tag{4.84}$$

Here we used the relation

$$\sum_{k,n} \frac{(\lambda_{0k} m_k)^2}{\lambda_{0n}\lambda_{0n}} = \sum_k m_k^2. \tag{4.85}$$

The value of

$$\lambda^{-2} = \frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \tag{4.86}$$

plays the role of a free parameter of the theory, including the dependence on the cut-off.

It is important to note that the formulas (4.84) for the masses of gauge vector fields, although defined in this case by (4.85), based on the assumption of a mixed interaction of generations of fermions, coincide with similar expressions obtained in Sec. 4.1.2 for the case of independent incorporated different generations of fermions in the four-fermion interaction.

We now construct the complete Lagrangian of interacting fermions, collective, scalar and vector gauge fields. Acting in the same way as it was done in Sec. 4.1.2, we find the effective interaction Lagrangian in the one-loop approximation:

$$\mathcal{L} = \sum_k \Bigg\{ \lambda_{0k} \left[\bar{\psi}_R^k \nu_L^k \Phi_1^* + \bar{\nu}_L^k \psi_R^k \Phi_1 + \bar{\psi}^k \psi^k \kappa - i\bar{\psi}^k \gamma_5 \psi^k \chi \right]$$

$$+ \frac{i\lambda_{0k} m_k}{\lambda^2} \left[\frac{g^2}{\sqrt{2}} \left(\partial_\mu \Phi_1 W_\mu^+ - \partial_\mu \Phi_1^* W_\mu^- \right) - \frac{i}{\beta} \partial_\mu \chi Z_\mu \right]$$

$$+\frac{i\lambda_{0k}\lambda_{0k}}{\lambda^2}\Bigg[\frac{g^2}{\sqrt{2}}\left(W_\mu^+\Phi_2^*\partial_\mu^{\leftrightarrow}\Phi_1 + W_\mu^-\Phi_1^*\partial_\mu^{\leftrightarrow}\Phi_2\right)$$

$$-eA_\mu\Phi_1^*\partial^{\leftrightarrow}\Phi_1$$

$$-\frac{g'^2-g^2}{2}\beta Z_\mu\Phi_1^*\partial_\mu^{\leftrightarrow}\Phi_1 - \frac{1}{2\beta}Z_\mu\Phi_2^*\partial_\mu^{\leftrightarrow}\Phi_2\Bigg]$$

$$+\frac{\lambda_{0k}m_k}{\lambda^2}\Bigg[\frac{eg}{\sqrt{2}}\left(W_\mu^+A_\mu\Phi_1 + W_\mu^-A_\mu\Phi_1^*\right) \tag{4.87}$$

$$+\frac{eg'}{\sqrt{2}}\left(W_\mu^+Z_\mu\Phi_1 + W_\mu^-Z_\mu\Phi_1^*\right) - g^2\kappa W_\mu^+W_\mu^- - \frac{g'^2+g^2}{2}\kappa Z_\mu^2\Bigg]$$

$$-\frac{\lambda_{0k}\lambda_{0k}}{\lambda^2}\Bigg[\Phi^+\Phi\left(\frac{g^2}{2}W_\mu^+W_\mu^- + \frac{g'^2+g^2}{4}Z_\mu^2\right)$$

$$-\frac{e}{\sqrt{2}}\left(\Phi_2^*\Phi_1W_\mu^+ + \Phi_2\Phi_1^*W_\mu^-\right)$$

$$\times\left(gA_\mu + g'Z_\mu\right) + |\Phi_1|^2\left(A_\mu Z_\mu e\beta(g'^2-g^2) + e^2(A_\mu^2 - Z_\mu^2)\right)\Bigg]$$

$$-2\sqrt{2}\frac{m_k\lambda_{0k}^3}{\lambda^2}\kappa\Phi^+\Phi + \frac{\lambda_{0k}^4}{\lambda^2}(\Phi^+\Phi)^2\Bigg\},$$

which corresponds to a set of diagrams shown in Fig. 4.1.

Removing the Goldstone fields (see (4.69)) and introducing the notations

$$\kappa \to -\frac{\kappa}{\sqrt{2}}, \quad \bar{\kappa} = Z_3^{-1/2}\kappa, \quad G = \frac{\lambda}{(\sum m_k^2)^{1/2}}, \tag{4.88}$$

The Lagrangian (4.87) can be represented as follows:

$$\mathcal{L} = -\sum_k \frac{G}{\sqrt{2}} m_k \bar{\psi}^k \psi^k \bar{\kappa} - \bar{\kappa}\left(\frac{1}{\sqrt{2}G} + \frac{\bar{\kappa}}{4}\right) \times \left(g^2 W_\mu^+ W_\mu^- + \frac{g^2 + g^{'2}}{2} Z_\mu^2\right) - \frac{G}{4} m_\kappa^2 \bar{\kappa}^3 - \frac{G^2}{16} m_\kappa^2 \bar{\kappa}^4. \tag{4.89}$$

Thus, there is a formal agreement between the obtained Lagrangian (4.89) and the corresponding expression of the standard theory of electro-weak interactions of GWS. In this case, however, unlike the case of independent switching of different generations of fermions in the nonlinear spinor interaction (see Sec. 4.2), when there are several varieties of scalar fields, the role of the massive scalar field (the analogue of the Higgs field) has been one bound state

$$\bar{\kappa} = -\frac{1}{\sqrt{2}} \sum_{k,n} \frac{\sqrt{\lambda_{0k}\lambda_{0k}}}{\lambda} \lambda_{0n} \langle \bar{\psi}^n \psi^n \rangle, \tag{4.90}$$

representing the sum of the contributions of lepton-antilepton couples of all generations.

Since the experimental values of the masses of fields $W^{\pm}$, Z known ($m_W \simeq 80.6$ GeV, $m_Z \simeq 91$ GeV), then using the relation

$$\sum_A m_A \simeq m_\tau^2$$

and

$$\sin^2\theta_W \simeq 0.226, \quad g^2 = \frac{8m_W^2 G_F}{\sqrt{2}}, \quad G_F \simeq \frac{10^{-5}}{m_p^2}$$

from (4.81), we find (see (4.84)):

$$\lambda^2 \simeq \frac{g^2 m_\tau^2}{2m_W^2} = 2\sqrt{2}G_F m_\tau^2 \simeq 10^{-4}. \tag{4.91}$$

Therefore the use of perturbation theory in the small value of $\lambda^2/8\pi \ll 1$ is justified. It follows from (4.86) that the momentum cut-off Λ, with the values found for λ^2, (4.91), gives a very large value: $\Lambda \simeq m\exp(8\times 10^5)$ and the approach should work for large momentum transfer $q < \Lambda$.

In this case, however, it must be borne in mind that the situation may change, given the contribution of light scalar particles with the mass $2m_\tau$ and the coupling constant (4.91) in the other observables. As calculations show [352], the contribution of light scalar particles (with $m_0 \to 0$), for example, in the anomalous magnetic moment of the muon within the experimental limit gives

$$g_0 = \frac{\lambda^2}{8\pi} < 10^{-8}. \tag{4.92}$$

Obviously, the value of (4.91) is not consistent with the experimental limit (4.92). In this connection it is natural to consider the generalization of the interaction (4.73) for the case when quarks are included.

4.1.4 Quark sector

Let us consider one generation of quarks (see Sec. 7). We obtain the values of the masses of the intermediate vector bosons in the scheme with the Lagrangian (3.31) using the appropriate substitution $\partial_\mu \to D_\mu$. The DS equation here will be of the form (4.34), where we must make a change in matrices $M_A \to B'_A$, $M'_A \to B_A$, $B^L_\mu \to T^L_\mu$ (see (3.7), (3.32)). Given that

$$T^1_\mu = T^{2'}_\mu = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \bar{\tau}_\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T^2_\mu = T^{1'}_\mu = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{\tau}_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T^3_\mu = T^{3'}_\mu = e \begin{pmatrix} 0 & -\frac{2}{3}\bar{\tau}_\mu & 0 & 0 \\ -\frac{2}{3}\tau_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\bar{\tau}_\mu \\ 0 & 0 & \frac{1}{3}\tau_\mu & 0 \end{pmatrix}, \qquad (4.93)$$

$$T^4_\mu = T^{4'}_\mu = \beta$$

$$\times \begin{pmatrix} 0 & \left(-\frac{2}{3}g'^2 + \frac{1}{2\beta^2}\right)\bar{\tau}_\mu & 0 & 0 \\ -\frac{2}{3}g'^2\tau_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{3}g'^2 - \frac{1}{2\beta^2}\right)\bar{\tau}_\mu \\ 0 & 0 & \frac{1}{3}g'^2\tau_\mu & 0 \end{pmatrix},$$

the Lagrangian (3.33) by replacing $\partial_\mu \to D_\mu$ can be written as

$$\mathcal{L} = -\bar{q}\hat{\partial}q - \frac{1}{4}C^{A+}_{[\mu\nu]}C^{A}_{[\mu\nu]} + \frac{1}{2}\left(\bar{q}B_A q\right)\left(\bar{q}B'_A q\right) + \bar{q}T^A_\mu q C^A_\mu \tag{4.94}$$

$$+\mathcal{L}_C + J^A_\mu C^{A+}_\mu + \bar{q}B'_A q J_A;$$

$$C^A_{[\mu\nu]} = \partial_\mu C^A_\nu - \partial_\nu C^A_\mu, \qquad C_\mu = (W_\mu, Z_\mu, A_\mu),$$

$\mathcal{L}_C$ is the interaction Lagrangian of vector gauge fields with each other (see Chapter 3). The polarization operators $\Pi_{\mu\nu}$ are also given by (4.35) - (4.41) for the corresponding changes of the matrices.

Substituting in $\Pi_{\mu\nu}$ the expression for the propagator of collective scalar fields (3.43), after direct integration we see that the non-zero elements of the matrix $\Pi^{LC}_{2\mu\nu}(k)$ are

$$\Pi^{11}_{2\mu\nu}(k) = \Pi^{22}_{2\mu\nu}(k) = -\frac{g^2}{2}\frac{k_\mu k_\nu}{k^2}\left(m_1^2 + m_2^2\right)\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}, \tag{4.95}$$

$$\Pi^{44}_{2\mu\nu}(k) = -\frac{g'^2 + g^2}{2}\frac{k_\mu k_\nu}{k^2}\left(m_1^2 + m_2^2\right)\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}.$$

Defining

$$\lambda^{-2} = \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}$$

as a free parameter and considering the natural demand $k_\mu k_\nu \Pi_{\mu\nu}(k) = 0$, we find that the masses of $W^\pm$ and Z bosons are given by:

$$m_W^2 = \frac{g^2}{2\lambda^2}\left(m_1^2 + m_2^2\right), \quad m_Z^2 = \frac{g'^2 + g^2}{2\lambda^2}\left(m_1^2 + m_2^2\right), \tag{4.96}$$

where m_1, m_2 are masses of u and d-quarks. In this case, the propagator of gauge fields within this approximation after regularization is written, as usual, in the form (4.53). Formulas (4.96) here have the same form as in the case of the lepton sector (4.54).

Thus, in the formation of collective scalar fields, due to the mechanism of dynamic symmetry breaking, quarks make additive contributions to the masses of $W^\pm$, Z bosons.

When considering all generations of quarks for violating CP-invariance it is necessary to introduce the Kobayashi $-$Maskawa matrix.

Consider the possible generalization of the interactions (3.4) and (3.31). Suppose there are three generations of leptons and three generations of quarks:

$$l_{AL} = \begin{pmatrix} \nu'_A \\ e'_A \end{pmatrix}_L, \quad q_A = \begin{pmatrix} p'_A \\ n'_A \end{pmatrix},$$
$$e'_A = (e', \mu', \tau'), \quad \nu'_A = (\nu'_e, \nu'_\mu, \nu'_\tau), \tag{4.97}$$
$$p'_A = (u', c', t'), \quad n'_A = (d', s', b'),$$

where the prime fields indicate that they are eigenstates of the gauge states, but not eigenstates of mass matrices. The key point for future constructions is the possibility to introduce a single fermion-antifermion field of the form (see also [51]):

$$\Phi' = f^{(e)}_{AB} \bar{l}_{AL} e'_{BR} + f^{(p)}_{AB} \bar{q}^G_{AL} p^{c'}_{BR} + f^{(n)}_{AB} \bar{q}_{AL} n'_{BR}, \tag{4.98}$$

where $q^G = i\tau_2 q^c$, q^c is the charge-conjugated field.

Then we can consider a model in which the Lagrangian is standard (GWS), but instead of using the Higgs Lagrangian we consider the Lagrangian

$$\mathcal{L}_4 = \Phi' \Phi'^{+}. \tag{4.99}$$

In this case, a single collective state $\langle \Phi \rangle$, an analogue of the Higgs fields, is an excited state of leptons and quarks. After the transition to the phase with a broken ground state, we arrive at a model in a form coinciding with the

GWS, but with a composite Higgs particle. After that, the transition to the eigenstates of the mass matrix with a double unitary transformation and introduction phases violating CP-invariance can be done in the standard way (see [353]).

The mass formula for the composite Higgs particle will be here as the type (4.81) (see [51]), but the amount will be composed of masses of all fermions including quarks. Then, for the presence of such a heavy t-quark, where its mass is much greater than the masses of all other quarks, from (4.81) we obtain the following approximate equation: $m_\kappa \simeq 2m_t$. This means that if we know the mass of t-quark ($m_t = 174$ GeV) the scalar boson mass can be estimated ($m_\kappa \simeq 348$ GeV) (of course, quite rudely) in this scheme (4.98), (4.99). In turn, the mass formula (4.84) for the intermediate vector bosons in this scheme remains, and the amount will be included in the contributions of all fermions (leptons and quarks with their colors). From (4.84), approximately leaving in the sum only one term m_t^2, we obtain

$$m_W^2 \simeq \frac{N_c g^2 m_t^2}{2\lambda^2} \to \lambda^2 \simeq 0.8\frac{m_t}{m_W}, \tag{4.100}$$

where N_c is the number of colors ($N_c = 3$), so that $\lambda^2 \simeq 1.727$. Thus (see (4.92) $\lambda^2/(8\pi) \simeq 0.07 < 1$, and we can use the perturbation theory.

The approach with composite Higgs fields is more cost-effective and natural, since there is no need to introduce by "hands" a more fundamental Higgs boson. Through the introduction of four-fermion interactions (4.98) (4.99) in this model, as in the standard model of GWS, there is an analogue of the Higgs particle, but this is not an additional fundamental field, but a natural product of the dynamic symmetry breaking - some bound fermion-antifermion state formed from the fundamental fields of leptons and quarks of all generations. It should be stressed that the model will be "working" only under certain restrictions on the amount of momentum transfer q (and therefore energy). Indeed, if $\lambda^2 = 1.727$, it follows from (4.86) that the value of the momentum cut-off is

$$\Lambda = m \exp(\frac{8\pi^2}{\lambda^2}) \simeq 7m \times 10^{19}. \tag{4.101}$$

It follows, that this scheme is applicable to very high energies.

Note also that, in turn, demand $\lambda^2/(8\pi) < 1$ is not generally required. Condition $\lambda^2/(8\pi) > 1$ means the transition to a strong-coupling theory, when one can no longer use approximate calculations based on an expansion in powers of $\lambda^2/(8\pi)$, and it is necessary to make accurate calculations.

The main prediction of this approach is the mass of the composite scalar particle $m_\kappa \simeq 2m_t$. Note that in papers [68], [72] authors also make the prediction value of the mass of the Higgs particle equal to $2m_t$ assuming a condensation of t-quarks. However, the assumptions in the approaches of the authors [68], [72] differ from our consideration (see the Introduction).

Thus, we consider the original model, as mentioned above, which allows us to give, even rudely, the predictions for the value of the mass of the composite Higgs boson, m_H. In the standard model under the assumption about the value of the constant λ ($\lambda < 1$) in (1.2), one also receives, at a standard value of $v \simeq 250\ GeV$ (see (1.4)), the limit $m_H < 350\ GeV$ [353]. This restriction is obviously not inconsistent with the restriction placed above in the framework of this approach. However, in the standard model, in contrast to our scheme, there is no evidence of the mass connection of m_t and m_H. Upon the detection of the heavier Higgs boson (not 125 GeV boson), the relation $m_H \simeq 2m_t$ can be checked, and in this sense can be considered as a prediction.

Note also that when considering the phenomenological consequences of the model, one must take into account the radiative corrections. Recall that the momentum cut-off can be fixed.

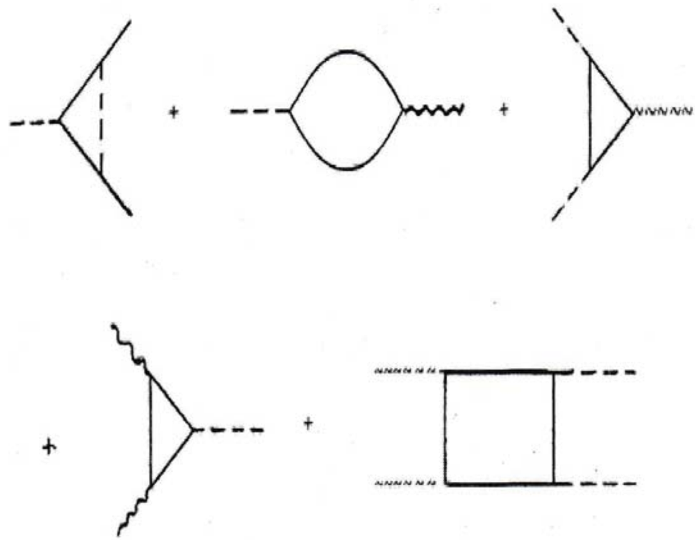

Figure 4.2: One-loop diagrams.

Chapter 5

5.1 Non-perturbative effects

This chapter explores spontaneous chiral symmetry breaking in low-energy QCD. It is shown that in the Abelian approximation of QCD, when the self-interaction of gluons is taken into account, using the gluon propagator in the infrared region ($D_{\mu\nu} \sim k^{-4}$), the chiral symmetry is spontaneously broken, and π-mesons play the role of pseudo-Goldstone bosons. It is found that in the low-energy limit, QCD provides an effective four-quark interaction. We calculate the total effective action, the mass spectrum of light mesons (π, σ) and the quark condensate in this model.

We also consider a different approach to low-energy QCD, based on the concept of the instanton vacuum. Four-quark 't Hooft's interaction (1.6) is investigated in the case of two flavors. Chiral symmetry breaking, and the spectrum of masses of light mesons are considered.

It is shown that the two approaches give, with some approximation, the same masses of π, σ mesons and the value of the quark condensate. In order to study the confinement, the infrared asymptotic behavior of the Green function is investigated in a simple four-fermion model. Since the single-fermion Green function possesses a simple pole in the infrared region, then this indicates that the four-quark model does not describe confinement. Thus, we can say that the model with four-quark interaction gives only a description of a kind of staging area of strong interactions, which lies between the regions of asymptotic freedom and quark confinement.

The content of this chapter is based on the results of [354] - [358].

5.1.1 Chiral symmetry breaking and effective action in Abelian chromodynamics

QCD describes the interactions of quarks and gluons at high energies. In the region of asymptotic freedom, perturbation theory is applicable, and the Feynman diagram technique “works”. In the region of low energy, nonperturbative effects occur for which the perturbation theory calculations are not possible due to the large value of the coupling constant.

To justify the universality of QCD, along with the ability to perform calculations at low energies, it is necessary to describe and explain the nonperturbative effects - chiral symmetry breaking, confinement of quarks, etc. That is necessary, in particular, to build effective chiral Lagrangians, which well describe hadron physics at low energy. This requires, in turn, the reformulation of QCD in terms of hadronic degrees of freedom.

An important step in this direction was made in [359], where the computational scheme for large N_c (N_c is the number of quark colors) was developed at the assumption of chiral symmetry breaking. There are other approaches to the problem [126] - [130], [156] - [162], [359].

First, the chiral symmetry breaking in the field theory, on the model level, has been demonstrated in the works [1] - [3]. In the framework of QCD, the explanation of chiral symmetry breaking is associated with the condensation of quarks. Phenomenology allows for average vacuum quark fields of the values $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = -(240 - 250\ \text{MeV})^3$. We also know that if the current quark masses are equal to zero, the Goldstone bosons are the pions.

To simplify the examination, we shall proceed from Abelian chromodynamics [360]. The non-Abelian nature of QCD and the self-interaction of gluons are here taken into account by using a singular infrared asymptotic behavior of the gluon propagator [361], [362]. The considered gluon propagator is a nonperturbative solution of the corresponding DS equation. With this simplification of the usual QCD there are no difficulties caused by the self-interaction of gluons and the presence of "ghosts". At the same time, the Ward−Slavnov−Taylor identities have the same form as in QCD [360].

This approach differs from the approach developed in [161], [359], which used the conventional gluon propagator and not the "infrared" one.

We start from the model that is QCD-inspired, with the action

$$\begin{aligned} S_0 = \int d^4x d^4y \Big\{ -\bar{\psi}^i(x) \Big[\gamma_\mu \left(\partial_\mu - i g_0 \lambda^a A^a_\mu(x) \right) + m_{0i} \Big] \\ \times \psi^i(x) \delta(x-y) - \frac{1}{2} A^a_\mu(x) \left[D^{ab}_{\mu\nu}(x,y) \right]^{-1} A^b_\nu(y) \Big\}, \end{aligned} \tag{5.1}$$

where m_{0i} are the current masses of the quarks, $A^a_\mu(x)$ are gluon fields, and λ^a are Gell-Mann matrices acting in the space of colors, $N_c = 3$. In (5.1) there is a summation over the quark flavors $i = 1, 2, ...N_f$. As used herein the gluon propagator in the infrared region is given by [361], [362]:

$$D^{ab}_{\mu\nu}(x, y) = \delta_{ab} D_{\mu\nu}(x - y)$$

$$= \delta_{ab}\kappa^2 \left[\frac{(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x_\alpha - y_\alpha)^2} - \beta\delta_{\mu\nu} \right], \qquad (5.2)$$

where κ^2 is a parameter with the dimensions of a square of a mass, and β is a dimensionless constant. Through this, as already noted, the nonperturbative effect of gluon self-interaction is automatically taken into account. In the momentum space the gluon propagator (5.2) gives the singular infrared asymptotic behavior $(k^2)^{-2}$.

Note that in this approximation for quarks in (5.1) we use the free propagator with the bare quark masses.

It will be shown below that in the restructuring of the physical vacuum, quarks acquire dynamical masses, which lead to the breaking of chiral symmetry. However, even in the case where the current quark masses are zero ($m_{0i} = 0$), the state with massive quarks is energetically more favorable, i.e. a violation of the chiral symmetry also occurs.

Assuming that the coupling constant g_0 at low energies, which we consider, is a constant (see [363]), we thus eliminate the infrared pole in the coupling constant.

We investigate in more detail the ability of chiral symmetry breaking (with $m_{0i} = 0$) in the model based on the action (5.1), motivated by QCD, and find an effective action in terms of the meson degrees of freedom.

Let us use the representation of the generating functional for Green's functions in the form of a path integral. Given that the action (5.1) is quadratic in the gluon fields, you can make a Gaussian integration; as a result we find

$$Z[\bar{\eta}, \eta] = N_0 \int D\bar{\psi} D\psi \exp\left\{i\left(S + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\},$$

$$S = \int d^4x d^4y \Big\{ -\bar{\psi}^i(x)\Big(\gamma_\mu \partial_\mu + m_{0i}\Big)\psi^i(x)\delta(x-y) \tag{5.3}$$

$$-\frac{g_0^2}{2}\bar{\psi}^i(x)\gamma_\mu\lambda^a\psi^i(x)D_{\mu\nu}(x,y)\bar{\psi}^j(y)\gamma_\nu\lambda^a\psi^j(y)\Big\}.$$

Note that the integral in the form (5.3) was studied for other types of gluon propagator in the papers [161], [359].

Now it is convenient to introduce bilocal fields [312], [364], [365]. For this, we use the Fierz transformations. First of all, we note that the gluon propagator $D^{ab}_{\mu\nu}$ (5.2) must be symmetric in its Lorentz μ, ν and unitary a, b indices. First, we find the symmetric part (indicated by parentheses) for the direct product of Dirac matrices (see [366]):

$$\begin{aligned}\gamma_{(i}\otimes\gamma_{k)} \equiv \frac{1}{2}\left(\gamma_i\otimes\gamma_k+\gamma_k\otimes\gamma_i\right) = \frac{1}{4}\delta_{ik}I_4\otimes I_4+\frac{1}{2}\gamma_{(i}\otimes\gamma_{k)}\\ -\frac{1}{4}\delta_{ik}\gamma_j\otimes\gamma_j-\frac{1}{4}\delta_{ik}\gamma_5\otimes\gamma_5+\frac{1}{2}\gamma_{(i}\gamma_5\otimes\gamma_{k)}\gamma_5 \qquad (5.4)\\ -\frac{1}{4}\delta_{ik}\gamma_l\gamma_5\otimes\gamma_l\gamma_5-2\sigma_{(i|\lambda}\otimes\sigma_{k)\lambda}+\frac{1}{2}\delta_{ik}\sigma_{\mu\nu}\otimes\sigma_{\mu\nu},\end{aligned}$$

where, as usual, repeated indices imply the summation, I_4 is the unit 4×4-matrix,

$$\sigma_{\mu\nu}=\frac{1}{2i}\left(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\right).$$

We also need the direct matrix product in the space of colors and flavors [160] - [162]:

$$\begin{aligned}\lambda^a\otimes\lambda^a &= \frac{16}{9}I_8\otimes I_8-\frac{1}{3}\lambda^a\otimes\lambda^a,\\ I_f\otimes I_f &= \frac{1}{N_F}I_f\otimes I_f+2T^a\otimes T^a,\end{aligned} \qquad (5.5)$$

where T^a are generators of the $SU(N_f)$ group, acting in the flavor space.

In the following we restrict ourselves to the light u, d-quarks with equal masses $m_{0i}=m_0$. In this case, $T^a=\tau^a/2$.

Now, using the Fierz transformations, we rewrite (5.3) with the help of (5.4), (5.5), and (5.2) in the form

$$
\begin{aligned}
Z[\bar{\eta},\eta] = N_0 \int D\bar{\psi}D\psi \exp\Big\{ i\int d^4x d^4y \Big[-\bar{\psi}(x)\Big(\gamma_\mu\partial_\mu + m_0\Big) \\
\times\psi(x)\delta(x-y) + \frac{M^2}{2}\Big[\bar{\psi}(x)\psi(y)\bar{\psi}(y)\psi(x) \\
-\bar{\psi}(x)\gamma_5\psi(y)\bar{\psi}(y)\gamma_5\psi(x) \\
+\bar{\psi}(x)\tau^a\psi(y)\bar{\psi}(y)\tau^a\psi(x) \\
-\bar{\psi}(x)\gamma_5\tau^a\psi(y)\bar{\psi}(y)\gamma_5\tau^a\psi(x) + \bar{\psi}\eta + \bar{\eta}\psi\Big]\Big]\Big\},
\end{aligned}
\tag{5.6}
$$

where $M^2 = 8g_0\kappa^2(4\beta - 1)/9$.

Only the part of the four-fermion interaction is recorded in (5.6), which can lead to spontaneous chiral symmetry breaking. This is due to the fact that the Lorentz and color symmetries are natural to consider intact. Therefore, non-zero can be only vacuum expectations: $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\bar{\psi}\gamma_5\psi\rangle \neq 0$, $\langle\bar{\psi}\gamma_5\tau^a\psi\rangle \neq 0$, $\langle\bar{\psi}\tau^a\psi\rangle \neq 0$.

When introducing bilocal boson fields $\varphi_0(x,y)$, $\tilde{\varphi}_0(x,y)$, $\varphi_a(x,y)$ $\tilde{\varphi}_a(x,y)$ it is necessary, first of all, to redefine the normalization constant N_0. To do this, we multiply (5.6) by the constant

$$
\int D\mu \exp\Big\{ -i\int d^4x d^4y \frac{1}{2M^2}\Big[\Big(\varphi_0(x,y) - M^2\bar{\psi}(x)\psi(y)\Big)
$$

$$
\begin{aligned}
&\times\Big(\varphi_0^*(y,x)-M^2\bar\psi(y)\psi(x)\Big)+\Big(\tilde\varphi_0(x,y)-iM^2\bar\psi(x)\gamma_5\psi(y)\Big)\\
&\times\Big(\tilde\varphi_0^*(y,x)-iM^2\bar\psi(y)\gamma_5\psi(x)\Big)+\Big(\varphi_a(x,y)-M^2\bar\psi(x)\tau^a\psi(y)\Big)\\
&\times\Big(\varphi_a^*(y,x)-M^2\bar\psi(y)\tau^a\psi(x)\Big)+\Big(\tilde\varphi_a(x,y)-iM^2\bar\psi(x)\gamma_5\tau^a\psi(y)\Big)\\
&\qquad\qquad\times\Big(\tilde\varphi_a^*(y,x)-iM^2\bar\psi(y)\gamma_5\tau^a\psi(x)\Big)\Big]\Big\},
\end{aligned}
\tag{5.7}
$$

where the measure being $D\mu = D\varphi_0 D\varphi_0^* D\varphi_a D\varphi_a^* D\tilde\varphi_0 D\tilde\varphi_0^* D\tilde\varphi_a \times D\tilde\varphi_a^*$. As a result, we obtain the functional

$$
\begin{aligned}
Z[\bar\eta,\eta] &= N\int D\bar\psi D\psi D\mu\exp\Big\{i\int d^4xd^4y\Big[-\bar\psi(x)\Big(\gamma_\mu\partial_\mu+m_0\Big)\\
&\times\psi(x)\delta(x-y)+\frac{1}{2}\varphi_0(x,y)\bar\psi(y)\psi(x)+\frac{1}{2}\varphi_0^*(y,x)\bar\psi(x)\psi(y)\\
&\quad+\frac{i}{2}\tilde\varphi_0(x,y)\bar\psi(y)\gamma_5\psi(x)+\frac{i}{2}\tilde\varphi_0^*(y,x)\bar\psi(x)\gamma_5\psi(y)\\
&\quad+\frac{1}{2}\varphi_a(x,y)\bar\psi(y)\tau^a\psi(x)+\frac{1}{2}\varphi_a^*(y,x)\bar\psi(x)\tau^a\psi(y)\\
&\quad+\frac{i}{2}\tilde\varphi_a(x,y)\bar\psi(y)\gamma_5\tau^a\psi(x)+\frac{i}{2}\tilde\varphi_a^*(y,x)\bar\psi(x)\gamma_5\tau^a\psi(y)\\
&-\frac{1}{2M^2}\Big(\varphi_0(x,y)\varphi_0^*(y,x)+\tilde\varphi_0(x,y)\tilde\varphi_0^*(y,x)+\varphi_a(x,y)\varphi_a^*(y,x)\\
&\qquad+\tilde\varphi_a(x,y)\tilde\varphi_a^*(y,x)\Big)+\bar\psi\eta+\bar\eta\psi\Big]\Big\}.
\end{aligned}
\tag{5.8}
$$

Integrating the expression (5.8) over the Fermi fields ψ, $\bar{\psi}$, we find

$$Z[\bar{\eta},\eta] = N \int D\mu \, \det\Big[(-\gamma_\mu \partial_\mu - m_0)\,\delta(x-y) + \frac{1}{2}\,(\varphi_A(x,y) + \varphi_A^*(x,y))\,\Gamma_A\Big] \tag{5.9}$$

$$\times \exp\Big\{i \int d^4x d^4y \Big[-\frac{1}{2M^2}\varphi_A(x,y)\varphi_A^*(y,x) + \bar{\eta}(x) S_f(x,y)\eta(y)\Big]\Big\}.$$

Here $\varphi_A(x,y) = (\varphi_0(x,y),\ \tilde{\varphi}_0(x,y),\ \varphi_a(x,y),\ \tilde{\varphi}_a(x,y))$, $\Gamma_A = (I_2,\ i\gamma_5,\ \tau^a,\ i\gamma_5\tau^a)$, and

$$S_f(x,y) = \Big[(\gamma_\mu \partial_\mu + m_0)\,\delta(x-y) - \frac{1}{2}\,(\varphi_A(x,y) + \varphi_A^*(x,y))\,\Gamma_A\Big]^{-1}$$

is Green's function of quarks in external fields.

The generating functional (5.9) can be written in the form

$$Z[\bar{\eta},\eta] = N \int D\mu \exp\Big\{i\Big[S_{eff} + \int d^4x d^4y \ \bar{\eta}(x) S_f(x,y)\eta(y)\Big]\Big\},$$

$$S_f(x,y) = -\frac{1}{2M^2}\int d^4x d^4y \ \varphi_A(x,y)\varphi_A^*(y,x)$$

$$-i\mathrm{Tr}\ \ln\Big[(-\gamma_\mu \partial_\mu - m_0)\,\delta(x-y) \tag{5.10}$$

$$+\frac{1}{2}\left(\varphi_A(x,y)+\varphi_A^*(x,y)\right)\Gamma_A\Big].$$

Now assume that there is the condensation, $\langle\bar{\psi}\psi\rangle \neq 0$. To take this into account, it is necessary to make the appropriate "shift" in the function of the bilocal field $\varphi_0(x,y)$. Assuming that the breaking of translational invariance does not occur, we must assume that the vacuum field depends only on the difference of coordinates $x-y$. Let us also assume that the Lorentz invariance, P and CP-invariance are also not violated. Using these assumptions, we make the substitution (see [161], [162]):

$$\varphi_0(x,y) = c\delta(x-y) + \Phi_0\left(\frac{x+y}{2}\right)B(x-y),$$
$$\varphi_A(x,y) = \Phi_A\left(\frac{x+y}{2}\right)B(x-y) \qquad (A\neq 0). \tag{5.11}$$

Here, $c = const.$, $t=(x+y)/2$ is the center of mass of the quark-antiquark pairs, $\Phi_A\left((x+y)/2\right)$ are real fields, which describe mesons, and $B(x-y)$ are meson form factors. Thus, we come to the field theory with nonlocal interaction [367].

The field Φ_0 matches σ-particle, the fields $\tilde{\Phi}_0$ - η-meson, $\tilde{\Phi}_a$ are identified with three-plet of pions and Φ_a with δ_a-mesons. We want to focus on the fundamental possibility of chiral symmetry breaking and correctly describe π-mesons. So here the s-quark is ignored, resulting in the identification of the fields with σ, η, δ_a-mesons to be a rough approximation. The generalization as to the $SU(3)_f \otimes SU(3)_f$-symmetry, which includes the s-quark, is not a fundamental difficulty.

Given the redefinition (5.11) and expanding the logarithm in (5.10) in powers of the meson fields Φ_A, we represent the effective action in the form:

$$
\begin{aligned}
S_f(x,y) = -\frac{1}{2M^2}\int d^4x d^4y \Big[c^2 \left(\delta(x-y)\right)^2 \\
+2c\Phi_0\left(\frac{x+y}{2}\right) B(x-y)\delta(x-y) \\
+\Phi_A^2\left(\frac{x+y}{2}\right) B^2(x-y)\Big] - i\mathrm{Tr}\,\ln\left(-\gamma_\mu\partial_\mu - m_0 + c\right) \\
+\sum_{n=1}^{\infty}\frac{i}{n}\mathrm{Tr}\Big[\int d^4x_1 ... d^4x_n \\
\times S_{0f}(x_n - x_1)\Phi_{A1}\left(\frac{x_1+x_2}{2}\right) B(x_1 - x_2) \\
\times \Gamma_{A1} S_{0f}(x_1 - x_2)\Phi_{A2}\left(\frac{x_2+x_3}{2}\right) \\
\times B(x_2 - x_3)\Gamma_{A2} \times ... S_{0f}(x_{n-1} - x_n)
\end{aligned}
\tag{5.12}
$$

$$\times \Phi_{An}\left(\frac{x_n+x_1}{2}\right) B(x_n-x_1)\Gamma_{An}\Big],$$

where we have introduced the Green function of the free quarks, satisfying

$$(\gamma_\mu \partial_\mu + m_0 - c)\, S_{0f}(x,y) = \delta(x-y). \tag{5.13}$$

As follows from (5.13), as a result of the restructuring of the physical vacuum, quarks acquire masses $m = m_0 - c$. Here we have restricted ourselves to the case of equal masses of u, d-quarks. If we also take into account the condensation $\langle \bar\psi \tau^3 \psi \rangle \neq 0$, then we come to the mass splitting of u and d quarks. Finally, assuming that the conditions $\langle \bar\psi \gamma_5 \psi \rangle \neq 0$, $\langle \bar\psi \gamma_5 \tau^3 \psi \rangle \neq 0$ hold, we can consider the violation of CP-parity. However, we shall neglect this opportunity.

The sum in (5.12) is an expansion in the loops (see Figure 1). The first term in (5.12) is an unimportant constant (infinite), which determines the energy of the vacuum.

The minimum condition of the action (5.12) is written as

$$\begin{gathered}\frac{\delta S_{eff}}{\delta \Phi_0}|_{\Phi_A=0} = \frac{ic}{M^2}\int \frac{d^4p\ B(p)}{(2\pi)^4} + 2mI = 0,\\ I = \frac{iN_c}{4\pi^4}\int \frac{d^4p\ B(p)}{p^2+m^2},\end{gathered} \tag{5.14}$$

where $m = m_0 - c$ are dynamical (constituent) quark masses, $B(p)$ is the Fourier transform of the form factor, $p^2 = \mathbf{p}^2 - p_0^2$, $d^4p = id^3pdp_0$.

It is convenient to choose the form factor of mesons in the momentum space in the form

$$B(p) = \vartheta\left(p^2 + \Lambda^2\right), \tag{5.15}$$

where $\vartheta(u) = 1$ for $u \geq 0$ and $\vartheta(u) = 0$ for $u < 0$, and the normalization condition is $\int B(x)d^4x = 1$.

The value of Λ^{-1} has the dimension of length and is a fundamental constant that characterizes the area of non-local interactions of quarks [367]. The choice of form factors of mesons in the form (5.15) leads to the convergence of the integrals with momentum cut-off Λ, which has the physical meaning here.

The transition to local fields occurs with the form factors in the form of δ-function, i.e., $B(x) \sim \delta(x)$. However, in this case, there will be difficulties associated with the divergence of integrals. It is possible, of course, to have a different choice of form factors (different from (5.15), see [367]). Note that the advantage of (5.15) is due to the fact that the relation $B^n(p) = B(p)$ holds, which will be further used in the calculation of the series (5.12) in the momentum space.

Equation (5.14) is the equation of the gap [1] - [3], which is nonanalytic in the coupling constant $M^2/B(x =$

0), and the solution determines the constant c defining a dynamical quark mass and condensate $\langle\bar{\psi}\psi\rangle$.

Due to equation (5.14) linear terms in the fields Φ_0 drop out of the equation (5.12). The quadratic terms in Φ_A in (5.12) define the propagators of mesons. From (5.12) we find the inverse propagators in the momentum space

$$\Delta^{-1}_{AB}(p) = -\delta_{AB}\frac{i}{M^2}\int\frac{d^4k\ B(k)}{(2\pi)^4}$$
$$-iN_c\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S_{0f}(k)\Gamma_A S_{0f}(k-p)\Gamma_B B^2(k). \tag{5.16}$$

Given a solution of (5.13), from (5.16) using (5.14), we obtain

$$\Delta^{-1}_{00}(p) = \Delta^{-1}_{11}(p) = \Delta^{-1}_{22}(p)$$
$$= \Delta^{-1}_{33}(p) = \frac{2m_0 I}{m_0 - m} + Z_3^{-1}\left(p^2 + 4m^2\right)(1-J),$$
$$\Delta^{-1}_{\tilde{0}\tilde{0}}(p) = \Delta^{-1}_{\tilde{1}\tilde{1}}(p) \tag{5.17}$$
$$= \Delta^{-1}_{\tilde{2}\tilde{2}}(p) = \Delta^{-1}_{\tilde{3}\tilde{3}}(p) = \frac{2m_0 I}{m_0 - m} + Z_3^{-1}p^2(1-J),$$
$$J = \frac{N_c}{4\pi^2}\int_0^1 \ln\left[1 + \frac{p^2}{m^2}x(1-x)\right].$$

Here we have introduced the renormalization constant

$$Z_3^{-1} = -\frac{iN_c}{4\pi^4} \int \frac{d^4p \vartheta(p^2+\Lambda^2)}{(p^2+m^2)^2}, \tag{5.18}$$

so that the fields are redefined, $\Phi_A Z_3^{-1/2} \to \Phi_A$, and there is a constant

$$g^2 \equiv Z_3 = \frac{4\pi^2}{N_c} \left[\ln\left(\frac{\Lambda^2}{m^2}+1\right) - \frac{\Lambda^2}{\Lambda^2+m^2} \right]^{-1}. \tag{5.19}$$

As follows from (5.19), the expansion in $g^2/4\pi^2$ (in equation (5.12)) corresponds to N_c^{-1}-expansion at the condition

$$\left[\ln\left(\frac{\Lambda^2}{m^2}+1\right) - \frac{\Lambda^2}{\Lambda^2+m^2} \right]^{-1} < 1.$$

For the parameters Λ^2, m^2, used below, this condition will be satisfied.

From (5.17) we find the masses (up to $\mathcal{O}(g^2)$)

$$m_\sigma = m_\delta = m_\pi^2 + 4m^2, \quad m_\pi^2 = m_\eta^2 = \frac{2m_0}{m_0 - m} I Z_3. \tag{5.20}$$

In the chiral limit, where the bare quark masses $m_0 = 0$, masses of π and η-mesons are zero, i.e. these mesons are Goldstone bosons in the broken $SU(2)_f \otimes SU(2)_f$-symmetry. Then, according to (5.20), we get $m_\sigma = 2m$, as in the original papers [1] - [3]. To determine the mass of π-mesons from (5.20) it is necessary to set the cut-off momentum Λ and the values of m_0, m, and to take into consideration the equation (5.14).

We will now relate these parameters to the pion decay constant $f_\pi = 93$ MeV. For this, note that the action (5.12), after calculating the loops, is similar to σ-model [108]. In this case equation (5.14) is the condition for the absence of tadpoles. The field shift (5.11) takes into account the spontaneous chiral symmetry breaking. After renormalization of fields, according to [108], we have

$$cZ_3^{-1/2} = -f_\pi. \tag{5.21}$$

Hence we find the relationship of the dynamic quark mass m, with a current mass of m_0, the renormalized constant (5.19) and f_π:

$$m - m_0 = g f_\pi. \tag{5.22}$$

This is the Goldberger−Treiman relation. If you specify $m_0 = 5$ MeV, $\Lambda = 1$ GeV, $N_c = 3$ (see [81], [82]), we arrive at (5.19), (5.20) and (5.22) to the values of $m_\pi = 140$ MeV, $m = 241$ MeV, in agreement with the experiment. For these values, according to (5.19), we find the parameter of the expansion in (5.12): $g^2/4\pi^2 \simeq 1/6 < 1$.

We now calculate the value of the quark condensate

$$\langle\bar{\psi}\psi\rangle = i\mathrm{Tr}S_{0f}(x,x) = -\frac{mN_c}{2\pi^2}\left[\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}+1\right)\right]. \tag{5.23}$$

When $\Lambda = 1$ GeV expression (5.23) yields

$$\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = (-248\ \mathrm{MeV})^3,$$

which is consistent with phenomenology.

The relations (5.20), (5.22) and (5.23) give the approximate equality

$$f_\pi^2 m_\pi^2 = -m_0\langle\bar{\psi}\psi\rangle, \tag{5.24}$$

that is obtained also in the framework of current algebra [368], [369].

The gap equation (5.14) determines the dimensional constant M^2 associated with a parameter κ^2 which is connected with the topological susceptibility of gluons. When $\Lambda = 1$ GeV we obtain the value $M = 0.02$ GeV.

Calculating the remaining diagrams in Fig. 1 up to the terms with $n = 4$, inclusive, according to (5.12), we find

$$S_{eff} = -\frac{1}{2M^2}\int d^4x d^4y\, c^2\,(\delta(x-y))^2 - i\mathrm{Tr}\,\ln\left(-\gamma_\mu\partial_\mu - m\right)$$

$$-\frac{1}{2}\int d^4x\Big\{\Phi_0\left(\partial_\mu^2 - m_0^2\right)\Phi_0 + \tilde{\Phi}_0\left(\partial_\mu^2 - m_\eta^2\right)\tilde{\Phi}_0$$

$$
\begin{aligned}
&+\tilde{\Phi}_a\left(\partial_\mu^2-m_\pi^2\right)\tilde{\Phi}_a+\Phi_a\left(\partial_\mu^2-m_\delta^2\right)\Phi_a \\
&-4gm\left[\Phi_0^3+3\Phi_0\Phi_a^2+\Phi_0\left(\tilde{\Phi}_0^2+\tilde{\Phi}_a^2\right)+2\tilde{\Phi}_0(\Phi_a\tilde{\Phi}_a)\right] \\
&+g^2\Bigg[\Phi_0^4+(\Phi_a^2)^2+6\Phi_0^2\Phi_a^2+\tilde{\Phi}_0^4+(\tilde{\Phi}_a^2)^2+6\tilde{\Phi}_0^2\tilde{\Phi}_a^2 \\
&+6\Bigg(\Phi_0^2\tilde{\Phi}_0^2+\Phi_a^2\tilde{\Phi}_0^2+\Phi_0^2\tilde{\Phi}_a^2\Bigg)+24\Phi_0\tilde{\Phi}_0(\Phi_a\tilde{\Phi}_a) \\
&+2\Phi_a^2\tilde{\Phi}_b^2+4(\Phi_a\tilde{\Phi}_a)^2\Bigg]\Bigg\}+\mathcal{O}(g^2).
\end{aligned}
\tag{5.25}
$$

Terms with the number $n > 4$ in (5.12) will give the order of $\mathcal{O}(g^2)$, since the corresponding integrals are independent of Λ. The action (5.25) involves the interaction of scalar and pseudoscalar fields with each other and can be used to calculate the decay widths and cross-sections of corresponding mesons [81], [82]. The sum of the first two terms in (5.25) includes vacuum fields and unimportant constants.

The part of the action (5.25), containing the interacting fields can be written in the algebraic form, which is clearly invariant under the group $SU(2)_f \otimes SU(2)_f$, if we use the matrix

$$
\Phi = \Phi_0 I_2 + \Phi_a\tau^a, \qquad \tilde{\Phi} = \tilde{\Phi}_0 I_2 + \tilde{\Phi}_a\tau^a. \tag{5.26}
$$

Then, taking

$$
\begin{aligned}
\mathrm{tr}\Phi^3 &= 2\left(\Phi_0^3 + 3\Phi_0\Phi_a^2\right), \\
\mathrm{tr}\Phi\tilde{\Phi}^2 &= 2\left[\Phi_0\left(\tilde{\Phi}_0^2 + \tilde{\Phi}_a^2\right) + 2\Phi_0\Phi_a\tilde{\Phi}_a\right], \\
\mathrm{tr}\Phi^4 &= 2\left[\Phi_0^4 + \left(\Phi_a^2\right)^2 + 6\Phi_0^2\Phi_a^2\right], \\
\mathrm{tr}\Phi^2\tilde{\Phi}^2 &= 2\left(\Phi_0^2 + \Phi_a^2\right)\left(\tilde{\Phi}_0^2 + \tilde{\Phi}_a^2\right) + 8\Phi_0\tilde{\Phi}_0(\Phi_a\tilde{\Phi}_a), \\
\mathrm{tr}\Phi\tilde{\Phi}\Phi\tilde{\Phi} &= 2\Phi_0^2\left(\tilde{\Phi}_a^2 + \tilde{\Phi}_0^2\right) + 2\tilde{\Phi}_0^2\Phi_a^2 \\
&\quad + 8\Phi_0\tilde{\Phi}_0(\Phi_a\tilde{\Phi}_a) + 4(\Phi_a\tilde{\Phi}_a)^2 - 2\Phi_a^2\tilde{\Phi}_b^2,
\end{aligned}
\tag{5.27}
$$

we find the following compact representation for the Lagrangian of interacting fields:

$$
\begin{aligned}
\mathcal{L}_{eff}^{int} &= gm\mathrm{tr}\left[\Phi\left(\Phi^2 + \tilde{\Phi}^2\right)\right] \\
&- \frac{g^2}{4}\mathrm{tr}\left[\Phi^4 + \tilde{\Phi}^4 + 4\Phi^2\tilde{\Phi}^2 + 2(\Phi\tilde{\Phi})^2\right].
\end{aligned}
\tag{5.28}
$$

Note that in the papers [370], [371], from phenomenological considerations, the Lagrangian for QCD at low energies was postulated in the algebraic form similar to (5.28).

The main result of this section is the proof of dynamical chiral symmetry breaking at low energies in QCD. We have used the infrared singular gluon propagator in the Abelian model of QCD. The effective chiral Lagrangian for scalar and pseudoscalar meson fields was also obtained, which is similar to the Lagrangian of the σ-model.

The main dimensional parameter in this approach is the cut-off momentum, the value of which was chosen to be $\Lambda = 1$ GeV. When setting the bare quark masses $m_0 = 5$ MeV and using the decay constant $f_\pi = 93$ MeV, the experimental values obtained: $m_\pi = 140$ MeV, dynamical quark mass $m = 241$ MeV and the quark condensate value $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = (-248 \text{ MeV})^3$, which are in good agreement with phenomenology. Similar values were obtained in [81], [81] by postulating the initial four-quark Lagrangian.

To characterize the 0^- and 0^+ meson nonet in this approach, it is necessary to generalize the above discussion only on the $SU(3)_f \otimes SU(3)_f$-symmetry.

5.1.2 On four-quark interaction induced by instantons

In the study of low-energy QCD instantons cannot be ignored. This is due to the fact that the only known significant nonperturbative solutions of nonlinear field equations in QCD are instanton solutions [186]. It is therefore natural to expect their appearance in the low-energy region.

We start from the Lagrangian (1.7) (with the addition of the free Lagrangian with the bare quark mass matrix $m_0 = \mathrm{diag}(m_{01}, m_{02})$), which describes the interaction of two quark flavors, due to the presence of the instanton vacuum.

Using the methods of Chapter 2, by introducing collective fields after the Gaussian integration over the quark fields, we can write the generating functional for Green's functions in the following form:

$$Z[\bar{\eta}, \eta] = N_0 \int D\Phi_A \exp\Big\{ i \Big[S_{eff} + \int d^4x d^4y \; \bar{\eta}(x) S_{eff}(x,y) \eta(y) \Big] \Big\}, \qquad (5.29)$$

where

$$S_{eff} = -\frac{\mu^2}{2} \int d^4x \Big[(\Phi_0 + \sigma_0)^2 - \Big(\tilde{\Phi}_0 + \tilde{\sigma}_0 \Big)^2$$

$$-\Phi_1^2 - \Phi_2^2 - (\Phi_3 + \sigma_3)^2 + \tilde{\Phi}_1^2 + \tilde{\Phi}_2^2 + \left(\tilde{\Phi}_3 + \tilde{\sigma}_3\right)^2 \Big] \quad (5.30)$$

$$-i\mathrm{Tr}\ \ln\left[-\gamma_\mu \partial_\mu - m + i\tilde{m}\gamma_5 + \Phi_A \Gamma_A\right].$$

Here

$$m = \mathrm{diag}(m_1, m_2), \quad m_1 = m_{01} - (\sigma_0 + \sigma_3),$$

$$m_2 = m_{02} - (\sigma_0 - \sigma_3), \quad \tilde{m} = \mathrm{diag}(\tilde{m}_1, \tilde{m}_2),$$

$$\tilde{m}_1 = \tilde{\sigma}_0 + \tilde{\sigma}_3, \quad \tilde{m}_2), \quad \tilde{m}_2 = \tilde{\sigma}_0 - \tilde{\sigma}_3,$$

$$\lambda = \frac{g_0^2}{\mu^2}, \quad g_0 = 1, \quad \Gamma_A = (I,\ i\gamma_5,\ \tau^a,\ i\gamma_5 \tau^a).$$

Green's function $S_f(x, y)$ of quarks in external fields obeys the equation

$$(\gamma_\mu \partial_\mu + m_0 - \Phi_A \Gamma_A)\, S_f(x, y) = \delta(x - y). \qquad (5.31)$$

We assumed that the most general conditions for vacuum expectations are $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\bar{\psi}\gamma_5\psi\rangle \neq 0$, $\langle\bar{\psi}\tau^3\psi\rangle \neq 0$, $\langle\bar{\psi}\gamma_5\tau^3\psi\rangle \neq 0$. Note that in [206] only the case with the condition $\langle\bar{\psi}\psi\rangle \neq 0$ was considered. The presence of this quark condensate is taken into account by the "shift" of the fields

$$\Phi_A \to \Big(\Phi_0 + \sigma_0,\ \tilde{\Phi}_0 + \tilde{\sigma}_0,\ \Phi_1,\ \Phi_2,\ \Phi_3 + \sigma_3,\ \tilde{\Phi}_1,\ \tilde{\Phi}_2,$$

$$\tilde{\Phi}_3 + \tilde{\sigma}_3\Big), \tag{5.32}$$

where σ_0, $\tilde{\sigma}_0$, σ_3, $\tilde{\sigma}_3$ are constants which are independent of the coordinates. The equations for the constants σ_0, $\tilde{\sigma}_0$, σ_3, $\tilde{\sigma}_3$ are obtained from the requirement of the absence of linear terms in fields Φ_A in the effective action (5.30):

$$\frac{\delta S_{eff}}{\delta \Phi_A}|_{\Phi_A=0} = -\mu^2 \sigma_A \varepsilon_A + i\mathrm{Tr}\ \Gamma_A S_{0f} = 0, \tag{5.33}$$

where $\varepsilon_A = (1,\ -1,\ -1,\ 1)$, and the free Green function S_{0f} of quarks obeys the equation (see Sec. 2.3)

$$(\gamma_\mu \partial_\mu + m - i\tilde{m}\gamma_5)\, S_{0f}(x,y) = \delta(x-y). \tag{5.34}$$

Solution of (5.34) in the momentum space is the matrix

$$S_{0f}(p) \begin{pmatrix} \frac{-i\hat{p}+m_1+i\tilde{m}_1\gamma_5}{p^2+M_1^2} & 0 \\ 0 & \frac{-i\hat{p}+m_2+i\tilde{m}_2\gamma_5}{p^2+M_2^2} \end{pmatrix}. \tag{5.35}$$

Here $M_1^2 = m_1^2 + \tilde{m}_1^2$, $M_2^2 = m_2^2 + \tilde{m}_2^2$, $\hat{p} = p_\mu \gamma_\mu$.

As seen from (5.35), by restructuring the physical vacuum, u, d-quarks acquire dynamic (constituent) masses M_1, M_2, respectively. The presence of non-zero values of $\tilde{m}_1$, $\tilde{m}_2$ (or $\tilde{\sigma}_0$, $\tilde{\sigma}_3$) leads to a dynamic violation of CP-parity. Note that the interaction Lagrangian (1.6) (and (1.7)) is not invariant under γ_5-transformations (2.35), i.e.

here, in contrast to the Lagrangian (1.1) $U_A(1)$ symmetry is violated. Therefore, the choice of gauge conditions cannot be achieved by $\langle\bar{\psi}\gamma_5\psi\rangle = 0$, $\langle\bar{\psi}\gamma_5\tau^3\psi\rangle = 0$. As it is known, instantons in the presence of quarks lead to the need to consider θ-term in the QCD Lagrangian and cause a complicated topological structure of the vacuum, which violates conservation of CP-parity [320]. In this approach, it is shown in the presence of condensates $\tilde{\sigma}_0 \neq 0$, $\tilde{\sigma}_3 \neq 0$.

With the help of (5.35), equations (5.33) become

$$\begin{aligned} \mu^2\sigma_0 &= m_1 I_1 + m_2 I_2, \quad \mu^2\sigma_3 = -m_1 I_1 + m_2 I_2, \\ \mu^2\tilde{\sigma}_0 &= \tilde{m}_1 I_1 + \tilde{m}_2 I_2, \quad \mu^2\tilde{\sigma}_3 = -\tilde{m}_1 I_1 + \tilde{m}_2 I_2, \end{aligned} \tag{5.36}$$

where

$$I_j = \frac{iN_c}{4\pi^4}\int \frac{d^4p}{p^2 + M_j^2} \qquad (j = 1, 2).$$

Equations (5.36) are the equations for the gap. As noted in Sec. 2.4, instead of cut-off regularization in the four-fermion models it is convenient to use the dimensional regularization [326]. It preserves the symmetry properties of the theory. With this regularization the link takes place (see (2.88))

$$I_2 = \left(\frac{M_2}{M_1}\right)^2 I_1. \tag{5.37}$$

Equations (5.36) can be conveniently written as

$$\mu^2 (m_{01} - m_1) = 2 m_2 I_2, \quad \mu^2 (m_{02} - m_2) = 2 m_1 I_1,$$

$$\mu^2 \tilde{m}_1 = 2 \tilde{m}_2 I_2, \quad \mu^2 \tilde{m}_2 = 2 \tilde{m}_1 I_1.$$

When taking into account the conditions (5.37), this system of equations leads to dependency

$$\frac{(m_{01} - m_1) M_1^2}{m_2 M_2^2} = \frac{(m_{02} - m_2)}{m_1} = \frac{\tilde{m}_1 M_1^2}{\tilde{m}_2 M_2^2} = \frac{\tilde{m}_2}{\tilde{m}_1}. \quad (5.38)$$

Thus, the 6 independent parameters m_{01}, m_{02}, m_1, m_2, $\tilde{m}_1$, $\tilde{m}_2$ are connected by three equations (5.38). That leaves only three independent variables. If we use the cut-off regularization, it is easy to show that equations (5.36) have the solutions for $m_1 > m_{01}$, $m_2 > m_{02}$ and constants $\tilde{m}_1$, $\tilde{m}_2$ have different signs. In this case, relations (5.37) and (5.38) are not satisfied.

By taking into consideration the conditions (5.36), the effective action (5.30) can be transformed to

$$S_{eff} = -\frac{\mu^2}{2} \int d^4x \left(\sigma_0^2 - \tilde{\sigma}_0^2 - \sigma_3^2 + \tilde{\sigma}_3^2 \right) -$$

$$i \mathrm{Tr} \ln \left(-\gamma_\mu \partial_\mu - m + i \tilde{m} \gamma_5 \right) \quad (5.39)$$

$$-\frac{\mu^2}{2} \int d^4x \left(\Phi_0^2 - \tilde{\Phi}_0^2 - \Phi_a^2 + \tilde{\Phi}_a^2 \right) + \sum_{n=2}^{\infty} \frac{i}{n} \mathrm{Tr} \, \left[S_{0f} \Phi_A \Gamma_A \right]^n .$$

We now show that equations (5.36) can be obtained from the minimum of the effective potential. Combining the constant terms in (5.39), we write

$$S_{eff}^{const} = -\frac{\mu^2}{2}\int d^4x\left(\sigma_0^2 - \tilde{\sigma}_0^2 - \sigma_3^2 + \tilde{\sigma}_3^2\right) \tag{5.40}$$
$$-i\mathrm{Tr}\,\ln\left(-\gamma_\mu\partial_\mu - m + i\tilde{m}\gamma_5\right).$$

Given that there is a relationship of constant fields (see Sec. 2.1) $S_{eff}^{const} = -\int d^4x V_{eff}$, we find from (5.40) the effective potential

$$V_{eff} = \frac{\mu^2}{2}\left[(m_1 - m_0)(m_2 - m_0) - \tilde{m}_1\tilde{m}_2\right] \tag{5.41}$$
$$+\frac{iN_c}{8\pi^4}\int d^4p\,\ln\left(p^2 + M_1^2\right)\left(p^2 + M_2^2\right).$$

Equations (5.36) follow from the minimum of the potential (5.41):

$$\frac{\partial V_{eff}}{\partial m_1} = \frac{\partial V_{eff}}{\partial m_2} = \frac{\partial V_{eff}}{\partial \tilde{m}_1} = \frac{\partial V_{eff}}{\partial \tilde{m}_2} = 0.$$

To determine the mass spectrum of mesons, Φ_A, we select from (5.39) the quadratic part

$$S_{eff}^{(2)} = -\frac{\mu^2}{2}\int d^4x\left(\Phi_0^2 - \tilde{\Phi}_0^2 - \Phi_a^2 + \tilde{\Phi}_a^2\right) + \frac{i}{2}\mathrm{Tr}\,\left[S_{0f}\Phi_A\Gamma_A\right]^2$$

$$= -\frac{1}{2}\int d^4xd^4y\ \Phi_A(x)\Delta^{-1}_{AB}(x,y)\Phi_B(y), \tag{5.42}$$

where the propagator Δ^{-1}_{AB} in the momentum space can be written as

$$\Delta^{-1}_{AB}(p) = -i\mathrm{tr}\int \frac{d^4k}{(2\pi)^4} S_{0f}(k)\Gamma_A S_{0f}(k-p)\Gamma_B + \delta_{AB}\mu^2\varepsilon_A. \tag{5.43}$$

Using the dimensional regularization, we can obtain the relationship between the logarithmically and quadratically divergent integrals [327] (see (2.92))

$$M_1^2 Z_3^{-1} \approx I_1, \tag{5.44}$$

where

$$Z_3^{-1} = -\frac{iN_c}{4\pi^4}\int \frac{d^4p}{(p^2+M_1^2)^2} \tag{5.45}$$

is the quadratically divergent integral.

When taking into account (5.44), from (5.36), one obtains the relation

$$\mu^2 \approx -2M_1M_2Z_3^{-1}. \tag{5.46}$$

After redefining the fields $\Phi_A \to Z_3^{-1/2}\Phi_A$, from (5.42) by taking into consideration (5.46), we find the elements of the mass matrix

$$m^2_{00} = (M_1 - M_2)^2 + 2(m_1^2 + m_2^2),$$

$$
\begin{gathered}
m^2_{\tilde{0}\tilde{0}} = (M_1 + M_2)^2 + 2(\tilde{m}_1^2 + \tilde{m}_2^2), \\
m^2_{0\tilde{0}} = -4(m_1\tilde{m}_1 + m_2\tilde{m}_2), \\
m^2_{11} = m^2_{22} = 2(M_1^2 + M_2^2 + M_1 M_2 + m_1 m_2 - \tilde{m}_1\tilde{m}_2), \\
m^2_{33} = (M_1 + M_2)^2 + 2(m_1^2 + m_2^2), \\
m^2_{3\tilde{3}} = -4(m_1\tilde{m}_1 + m_2\tilde{m}_2), \\
m^2_{03} = 2(M_1^2 - M_2^2 + 2m_1^2 - 2m_2^2), \\
m^2_{\tilde{0}3} = m^2_{3\tilde{0}} = 4(m_2\tilde{m}_2 - m_1\tilde{m}_1), \qquad (5.47) \\
m^2_{\tilde{0}\tilde{3}} = 2(M_1^2 - M_2^2 + 2m_2^2 - 2m_1^2), \\
m^2_{\tilde{3}\tilde{3}} = (M_1 - M_2)^2 + 2(\tilde{m}_1^2 + \tilde{m}_2^2), \\
m^2_{\tilde{1}\tilde{1}} = m^2_{\tilde{2}\tilde{2}} = 2(M_1^2 + M_2^2 - M_1 M_2 - m_1 m_2 + \tilde{m}_1\tilde{m}_2), \\
m^2_{\tilde{1}1} = m^2_{2\tilde{2}} = -4(m_1\tilde{m}_2 + \tilde{m}_1 m_2).
\end{gathered}
$$

As can be seen from (5.47), there is strong mixing of meson fields. To determine the masses of the fields Φ_A it is necessary to diagonalize the mass matrix m^2_{AB}.

Note that the field Φ_0 can be identified with σ-meson and $\tilde{\Phi}_a$ with π_a-mesons.

The quantities in (5.47) determine the degree of CP-violation. However, we know that the strong interactions with a good accuracy are CP-invariant, i.e. the parameter to CP-violation is very small. Therefore, let us consider in detail only the case of $\tilde{m}_1 = \tilde{m}_2 = 0$, which corresponds to the equalities $\langle\bar{\psi}\gamma_5\psi\rangle = 0$, $\langle\bar{\psi}\gamma_5\tau^a\psi\rangle = 0$. We must, however, bear in mind that such a transition cannot be obtained directly from (5.47). This is associated with a decrease in the number of compensation equations (5.36) in this case and with the inaccuracy of (5.38) with $\tilde{m}_1 = \tilde{m}_2 = 0$. As a result, at $\tilde{m}_1 = \tilde{m}_2 = 0$ the relation (5.46) is not satisfied. In this case, you must use only the first two equations in (5.36).

Note that it is found in [322] the nonperturbative solution for the quark propagator directly from QCD, which also violates the conservation of CP-parity. Authors of [322] also discuss the possible mechanism of reduction of CP-violating terms.

We neglect also the mass splitting of u and d-quarks, which requires the conditions $\langle\bar{\psi}\tau^3\psi\rangle = 0$, $\sigma_3 = 0$, and $m_{01} = m_{02}$ ($m_1 = m_2$). As a result of this restriction of the equations (5.36) only one equation "survives", which, after evaluation of the integral (in the scheme with a cut-off) $I_1 = I_2 = I$, takes the form

$$\mu^2 \frac{m_0 - m}{m} = \frac{N_c}{2\pi^2}\left[m^2 \ln\left(\frac{\Lambda^2}{m^2} + 1\right) - \Lambda^2\right]. \qquad (5.48)$$

Equation (5.48) is nontrivial nonanalytic in the constant $\lambda = 1/\mu^2$ solutions, provided $\alpha = 2\pi^2/(\lambda\Lambda^2 N_c) < 1$ [1].

From (5.48) the dynamical quark mass m is determined in the set-up of the constant λ and the momentum cut-off Λ (with $\alpha < 1$).

Note that the four-quark interaction, considered in this section (which is local), (1.6), differs from (5.6) (which is not local), by signs in front of the terms $(\bar{\psi}\gamma_5\psi)^2$, $(\bar{\psi}\tau^a\psi)^2$ and leads to what is already broken $U_A(1)$-symmetry, in contrast to the approach of Sec. 5.1. This difference in the signs is taken into account in the expression for the propagator of the collective fields (5.43) by the signature symbol ε_A. As already noted in the previous section, the choice of the form factor of mesons in the form (5.15) "leaves" the non-local four-fermion interaction (5.6) to the scheme of the local interaction, but with the momentum cut-off Λ. Therefore, so many values to be considered in this section, will coincide with the results of previous calculations. The difference is in the expressions for the propagators $\Delta^{-1}_{\tilde{0}\tilde{0}}$ and Δ^{-1}_{aa} which is due to the factor ε_A. The computer calculations show that equations determining the mass of the relevant collective fields

$$\Delta^{-1}_{\tilde{0}\tilde{0}} = 0, \qquad \Delta^{-1}_{aa} = 0 \tag{5.49}$$

do not have solutions at $p^2 > 0$, and at $p^2 < 0$. This suggests that the propagators of $\Delta_{\tilde{0}\tilde{0}}$, Δ_{aa} and fields $\tilde{\Phi}_0$, Φ_a do not correspond to the real particles. We noted in Sec.

5.1 that the identification of the fields $\tilde{\Phi}_0$, Φ_a with η and δ-particles there was arbitrary, since there is a contribution of the s-quark into real particles, which was not taken into account. To obtain the masses of m_δ and m_η in this approach one should generalize the analysis on $SU(3)_f \otimes SU(3)_f$-symmetry. The formula for the masses of π and σ mesons here have the same form as in the approach of Sec. 5.1.

To fix the Λ and calculate m, we find the relation of these variables with a constant f_π [78] - [82]. Let us construct an axial (not saved) current corresponding to $SU(2)_A$-transformations:

$$\psi' = \exp\left(i\alpha\gamma_5 \frac{\vec{n}\vec{\tau}}{2}\right)\psi, \quad \bar{\psi}' = \bar{\psi}\exp\left(i\alpha\gamma_5 \frac{\vec{n}\vec{\tau}}{2}\right), \quad (5.50)$$

where $\vec{n}$ is a unit vector, i.e., $\vec{n}^2 = 1$.

Transformations of the quark fields (5.50) generate infinitesimal transformations of meson fields

$$\Phi_0 + \sigma_0 \to \Phi_0 + \sigma_0 + \alpha n_a \tilde{\Phi}_a, \quad \tilde{\Phi}_0 \to \tilde{\Phi}_0 - \alpha n_a \Phi_a,$$
$$(5.51)$$
$$\tilde{\Phi}_a \to \tilde{\Phi}_a - \alpha n_a(\Phi_0 + \sigma_0), \quad \Phi_a \to \Phi_a + \alpha n_a \tilde{\Phi}_0.$$

After the renormalization of fields, we have

$$\sigma_0 = \frac{(m_0 - m)}{\sqrt{Z_3}} = \frac{(m_0 - m)}{g} \qquad (g \equiv \sqrt{Z_3}).$$

Applying the Gell-Mann−Levi method [108], we find an expression for the axial current

$$A_\mu = \frac{\partial \delta \mathcal{L}_{eff}}{\partial \partial_\mu \alpha(x)} = n_a \Big(\Phi_a \partial_\mu \tilde{\Phi}_0 - \tilde{\Phi}_0 \partial_\mu \Phi_a + \Phi_0 \partial_\mu \tilde{\Phi}_a - \tilde{\Phi}_a \partial_\mu \Phi_0 - \frac{(m - m_0)}{g} \partial_\mu \tilde{\Phi}_a \Big). \tag{5.52}$$

If we use (5.52) to describe the decay of $\pi^\pm \to \mu^\pm \nu$ (see [78] - [82]), we arrive at the identity of Goldberger−Treiman (5.22) (see also [216], [82], [372]). Substituting in (5.22) the value of g^2 (5.19), we come to the transcendental equation relating m, m_0 and the momentum cut-off Λ

$$\left(\frac{f_\pi}{m - m_0} \right)^2 = \frac{N_c}{4\pi^2} \left[\ln \left(\frac{\Lambda^2}{m^2} + 1 \right) - \frac{\Lambda^2}{\Lambda^2 + m^2} \right]. \tag{5.53}$$

In Fig. 5.1 there is a graphical relationship between the dynamic quark mass m and the momentum cut-off at $N_c = 3$, $m_0 = 0$ (see also [206], [216]). Choosing the value of $\Lambda = 1$ GeV, we obtain the value $m = 241$ MeV. From (5.48) we can find a constant λ:

$$\lambda = \left(\frac{N_c \Lambda^4}{2\pi^2 (\Lambda^2 + m^2)} - 2 f_\pi^2 \frac{m^2}{(m - m_0)^2} \right)^{-1}. \tag{5.54}$$

The graph of the value of the constant λ versus the cut-off momentum is shown in Fig. 5.2. When $\Lambda = 1$ GeV, we obtain the value $\lambda = 7.9\ (\mathrm{GeV})^{-2}$. Finally, we calculate the value of the quark condensate from (5.23). In Fig. 5.3 the dependence of the condensate $\langle \bar{d}d \rangle^{1/3} = \langle \bar{u}u \rangle^{1/3} = (\langle \bar{\psi}\psi \rangle /2)^{1/3}$ on the momentum cut-off is given.

Here, too, are the relations of current algebra (5.24), and the Lagrangian of self-interaction of collective fields is given by (5.28).

For more accurate calculations of the masses of π and σ mesons it is necessary to solve the equations

$$\Delta^{-1}_{\tilde{a}\tilde{a}}(p) = 0, \quad \Delta^{-1}_{aa}(p) = 0. \tag{5.55}$$

The numerical solutions of equations (5.55) in the argument $p^2(= -m^2)$ for $\Lambda = 1$ GeV, $m_0 = 5$ MeV, $m = 241$ MeV give the following masses of π and σ mesons

$$m_\pi = 140 \text{ MeV}, \quad m_\sigma = 500 \text{ MeV}. \tag{5.56}$$

These values do not differ from the approximate values obtained from (5.20), due to the smallness of the expansion parameter $g^2/(4\pi^2)$ (see (5.17)). Masses (5.56) agree well with the phenomenology of strong interactions [373]. It is known that σ-boson (or, as it is meant, ε-meson), along with the π, ρ and ω bosons contributes to the mechanism of exchange in nucleon-nucleon interaction [373], and it cannot be ignored. At the same time σ-boson can be treated as a "cluster" of pions with the corresponding quantum numbers. So, for example, the scalar isoscalar of two-pion exchange corresponds to the exchange of σ-boson. In addition, to obtain the correct values of the static characteristics of hadrons, such as polarizabilities (see [374]), it is necessary to consider the Feynman diagrams with internal lines corresponding to σ-boson with a mass of (5.56).

So both approaches, considered in Sec. 5.1 and Sec. 5.2, give the same values (in choosing the form factor of mesons in the form (5.15)) of the masses of π and σ mesons. The difference is that the instanton approach considers a breach of $U_A(1)$-symmetry and overcomes the U_1-problem of the mass difference between the η and η' (see the Introduction) in the $SU(3)_f \otimes SU(3)_f$-symmetric scheme. This suggests that without the contribution of instantons in the QCD vacuum, apparently, the strong interactions of hadrons cannot be correctly described. The confinement of quarks remains open, however.

5.1.3 Investigation of infrared asymptotic of Green's function in the four-fermion model

In addressing the issue of quark confinement in QCD, one can study the infrared singularities of the gluon propagator [375] - [380] and the behavior of the quark Green function near the mass shell [381] - [384]. In the previous sections we have shown that at low energy QCD turns into the theory with the effective four-quark interaction. As part of this approach the experimental situation of low energy meson physics [78] - [82] is well described. Therefore, the study of four-fermion schemes in the infrared energy is yielding obvious interest. In this connection, the question arises, is it realized or is there no confinement in the four-fermion models? To do this, one can use the criteria by which a confinement is understood as the disappearance of a simple pole of the fermion Green function at the point $p^2 = -m^2$. To this end, in this section, we examine infrared behavior of the fermionic Green's function in a simple four-fermion massless model with the Lagrangian

$$\mathcal{L} = -\bar{\psi}\gamma_\mu \partial_\mu \psi + \frac{\kappa}{2}\left(\bar{\psi}\psi\right)^2 , \tag{5.57}$$

where $\kappa = g_0^2/\mu_0^2$ is the constant of self-interaction of the fermion field with dimensionality m^{-2}. Using the results of Sec. 2.1, we can write the Green function of a fermion in the external collective field (see (2.7))

$$(\gamma_\mu \partial_\mu + m - g\Phi)\, K(x,y|\Phi) = \delta(x-y). \tag{5.58}$$

Here $m = -g_0 \Phi_0$ is the dynamical mass of the fermion, $g = g_0 Z_3^{1/2}$ is the renormalized coupling constant, and $\Phi = \Phi' Z_3^{-1/2}$ is the renormalized collective field. In the result of vacuum polarization (fermion determinant) of the field $\Phi(x)$ in the effective action the kinetic term $\Phi \partial_\mu^2 \Phi/2$ and the mass term $m^2\Phi^2/2$ are appearing, where $M = 2m$ is the mass of the collective field Φ. We now use the approximation, in which we will consider only the quadratic terms in fields Φ in the effective action, i.e. we consider only the expression of $S^{(2)}_{eff}$. Terms of the self-interaction of fields Φ^3 and Φ^4 (see diagrams in Fig. 1) are neglected. The treatment due to such an approximation has mathematical difficulties associated with the inability to accurately assess functional integration of non-Gaussian integrals.

Given the approximations, we write the expression for the quantum
Green's function

$$G(x,y) = \frac{\delta^2 Z[\bar\eta, \eta]}{\delta\bar\eta(x)\delta\eta(y)}|_{\bar\eta=\eta=0} \tag{5.59}$$

$$= N \int D\Phi K(x, y|\Phi) \exp\left\{i \int d^4x \frac{1}{2} \Phi \left(\partial_\mu^2 - M^2\right) \Phi\right\}.$$

This Green's function corresponds to the effective Yukawa interaction $g\bar{\psi}\psi\Phi$ neglecting the vacuum polarization. The infrared asymptotic of the Green's function in the model of a spinor field with Yukawa interaction was studied in [385]. To study the function (5.59), we apply another method associated with the solution of (5.58) as a functional integral [386]. For this we transform equation (5.58):

$$\left[\partial_\mu^2 - (m - g\Phi)^2 + g\gamma_\mu \partial_\mu \Phi\right] D(x, y|\Phi) = \delta(x - y), \quad (5.60)$$

where

$$K(x, y|\Phi) = (\gamma_\mu \partial_\mu - m + g\Phi(x)) D(x, y|\Phi).$$

We write the solution of (5.60) in the form [386]

$$\begin{aligned} D(x, y|\Phi) = -i \int_0^\infty ds C \int D\nu \exp\Bigg\{ i \int_0^s d\xi \Bigg[\nu_\mu^2(\xi) \\ -\left(m - g\Phi\left(x - 2\int_\xi^s d\eta\ \nu(\eta)\right)\right)^2 \quad (5.61) \\ + g\gamma_\mu(\xi)\partial_\mu \Phi\left(x - 2\int_\xi^s d\eta\ \nu(\eta)\right)\Bigg]\Bigg\} \\ \times \delta\left(x - y - 2\int_0^s d\xi\ \nu(\xi)\right). \end{aligned}$$

Here we use a functional integration over the auxiliary field ν_μ, and the constant C satisfies the condition

$$C \int D\nu \exp\left\{ i \int_0^s d\xi \ \nu_\mu^2(\xi) \right\} = 1. \tag{5.62}$$

We introduce the following notations:

$$\begin{aligned} j(z) &= \int_0^s d\xi \ \delta\left(z - x + 2\int_\xi^s d\eta \ \nu(\eta) \right), \\ [D\nu]_0^s &= C D\nu \exp\left\{ i \int_0^s d\xi \ \nu_\mu^2(\xi) \right\}, \end{aligned} \tag{5.63}$$

considering that the solution (5.61) takes the more compact form

$$\begin{aligned} D(x, y|\Phi) = -i \int_0^\infty ds \int [D\nu]_0^s \exp\Big\{ -i \int d^4z \ j(z) \\ \times \left[(m - g\Phi(z))^2 - g\gamma_\mu \partial_\mu \Phi(z) \right] \Big\} \delta\left(x - y - 2\int_0^s d\xi \ \nu(\xi) \right). \end{aligned} \tag{5.64}$$

This leads to the solution of (5.58)

$$\begin{aligned} K(x, y|\Phi) = \left(\gamma_\mu \partial_\mu - m + ig \frac{\delta}{\delta J(z)} \right) (-i) \int_0^\infty ds \int [D\nu]_0^s \\ \times \exp\Big\{ -i \int d^4z \Big[j(z) \left((m - g\Phi(z))^2 - g\gamma_\mu \partial_\mu \Phi(z) \right) \end{aligned} \tag{5.65}$$

$$+J(z)\Phi(z)\Big]\Big\}\delta\left(x-y-2\int_0^s d\xi\ \nu(\xi)\right),$$

where for convenience, the Schwinger source $J(z)$ is introduced.

In what follows we neglect the term $g\gamma_\mu\partial_\mu\Phi(z)$ in the exponent (5.65), which is responsible for the spin effects, as it contains the matrix γ_μ. It is justified in the infrared region of the energy (see [386]). Substituting (5.65) in (5.59) the functional integral of the Gaussian type arises

$$\int D\Phi\exp\Big\{-i\int d^4z\Big[j(z)\left(m-g\Phi(z)\right)^2+J(z)\Phi(z)$$

$$-\frac{1}{2}\Phi\left(\partial_\mu^2-M^2\right)\Phi\Big]\Big\}$$

$$=\exp(-ism^2)\mathrm{det}^{-1/2}\left(-\partial_\mu^2+M^2+2g^2j_E(z)\right) \quad (5.66)$$

$$\times\exp\Big[\frac{1}{2}\int d^4z_Ed^4y_E\Big(J_E(z)$$

$$-2mgj_E(z)\Big)\Delta(z-y)\left(J_E(y)-2mgj_E(y)\right)\Big].$$

Here $\Delta(z-y)$ satisfies the equation

$$\left(-\partial_\mu^2+M^2+2g^2j_E(z)\right)\Delta(z-y)=\delta_E(z-y), \quad (5.67)$$

and we made a transition to Euclidean space-time. We will assume that $g^2 < 1$. This condition is satisfied in the four-quark schemes describing the low-energy meson physics [78] - [82]. In this case the solution of (5.67) can be found by the iteration method. Zeroth and the first terms of Δ in the coupling constant g are in the momentum space as follows:

$$\Delta^0(p) = \frac{1}{p^2 + M^2},$$
$$\Delta^1(p) = -\frac{2g^2 j_E(p)}{p^2 + M^2} \int \frac{d^4 k_E}{(2\pi)^4} \frac{\exp\left[ik\left(x - 2\int_\xi^s d\eta\ \nu(\eta)\right)\right]}{k^2 + M^2}. \tag{5.68}$$

We shall only use the $\Delta^0(p)$, and therefore believe that the constant N is chosen from the condition

$$N\mathrm{det}^{-1/2}\left(-\partial_\mu^2 + M^2 + 2g^2 j_E(z)\right) = 1.$$

This requirement is consistent with the fact that the free (with $g = 0$) Green's function has the standard form. Using (5.65), (5.66) and the approximations made, we obtain the total Green's function in the infrared region

$$G(x,y) = -i\int_0^s ds \exp(-ism^2) \int [D\nu]_0^s \Bigg(\gamma_\mu \partial_\mu - m$$
$$-ig\frac{\delta}{\delta J_E(x)}\Bigg)\delta\left(x - y - 2\int_0^s d\xi\ \nu(\xi)\right)$$

$$\times \exp\Big[-2mg \int d^4 z_E J_E(z)$$
$$\times \int_0^s d\xi \Delta^0 \left(z - x + 2\int_\xi^s d\eta\ \nu(\eta)\right) \tag{5.69}$$
$$+2m^2 g^2 \int_0^s d\xi \int_0^s d\xi_1 \Delta^0 \left(2\int_{\xi_1}^{\xi} d\eta\ \nu(\eta)\right)$$
$$+\frac{1}{2}\int d^4 z_E d^4 y_E J_E(z) \Delta^0(z-y) J_E(y)\Big].$$

Taking the variational derivative in (5.69), then putting $J_E = 0$, and going to the momentum space, we find

$$G(p) = -i\int_0^s ds \exp(-ism^2) \int [D\nu]_0^s \Big(i\hat{p} - m$$
$$+2mg^2 \int_0^s d\xi \Delta^0 \left(2\int_\xi^s d\eta\ \nu(\eta)\right)\Big) \tag{5.70}$$
$$\times \exp\left[-2ip\int_0^s d\xi\ \nu(\xi) + I(s)\right].$$

Here

$$I(s) = 2m^2 g^2 \int_0^s d\xi \int_0^s d\xi_1 \Delta^0 \left(2\int_{\xi_1}^{\xi} d\eta\ \nu(\eta)\right). \tag{5.71}$$

If in (5.70) $g = 0$, then we arrive at the free Green function

$$G_0(p) = \frac{-i\hat{p} + m}{p^2 + m^2},$$

where $\hat{p} = p_\mu \gamma_\mu$.

To calculate the functional integral (5.70) we use the approximate formula [386] - [388]

$$\int [D\nu] F_1(\nu) \exp F_2(\nu) \approx \langle F_1 \rangle \exp \langle F_2 \rangle, \tag{5.72}$$

where $\langle F_i \rangle = \int [D\nu] F_i(\nu) \quad (i = 1, 2)$.

Note that the approximation (5.72) describes not only the infrared region, but also the region of high energy [388] - [390]. Applying (5.72) to (5.70), throwing the last term in parentheses in (5.70), which gives the correction to the mass of the fermion, and making the shift of the integration variable $\nu_\mu(\xi) - p_\mu = \omega_\mu(\xi)$, we obtain

$$G_0(p) = -i \int_0^s ds \exp\Big(-is(p^2 + m^2)\Big) (i\hat{p} - m) \exp\left(F(s)\right), \tag{5.73}$$

$$F(s) = 2m^2 g^2 \int_0^s d\xi \int_0^s d\xi_1$$

$$\times \int [D\omega]_0^s \Delta^0 \left(2 \int_{\xi_1}^{\xi} d\eta\ \omega(\eta) + 2p|\xi - \xi_1| \right) \tag{5.74}$$

$$= 2m^2 g^2 \int_0^s d\xi \int_0^s d\xi_1 \int \frac{d^4 k_E}{(2\pi)^4} \frac{\exp\left\{-i|\xi - \xi_1|(k^2 - 2pk)\right\}}{k^2 + M^2}.$$

We imply that there is a negative imaginary part in the exponent of (5.73) ($m^2 \to m^2 - i\varepsilon$). Expressions (5.73) and

(5.74) generalize the formulas obtained in [385], as we take into account the values which are quadratic in the boson momentum k. It is important to make the corrections to g^2 [386]. To evaluate the integral (5.74) we proceed in exactly the same way as in [385]. After replacing the variable $|\mathbf{k}| = M \sinh u$, $k_0 = M \cosh u$, $d|\mathbf{k}|/k_0 = du$, going to the rest frame of the fermion and the corresponding integration, one obtains

$$
\begin{aligned}
F(s) = \frac{sg^2 m\Lambda}{4\pi^2} + \frac{ig^2(sM^2+1)}{8\pi^2} \ln \frac{2\Lambda}{M} \\
- \frac{g^2}{16\pi} \exp(isM^2) H_0^{(2)}(2mMs) \qquad (5.75) \\
- \frac{ig^2 M^2}{8\pi} \int_0^s ds_1 \exp(is_1 M^2) H_0^{(2)}(2mMs_1),
\end{aligned}
$$

where the Hankel function is given by

$$
H_0^{(2)}(x) = \frac{2i}{\pi} \int_0^\infty du \exp(-ix \cosh u).
$$

In obtaining (5.75), we used the relation: $M = 2m$, $p^2 \approx -m^2$. The first three terms in (5.75) are eliminated by the renormalization of the fermion mass m and related fields.

At low energies, where $p^2 = -m^2$, i.e. in the infrared region, the main contribution to the integral (5.73) "accumulates" from large s. Therefore, we can replace the function $F(s)$ by its asymptotic value $\lim_{s\to\infty} F(s)$.

Considering further that $\lim_{x\to\infty} H_0^{(2)}(x) = 0$, and arguing like [385] for the integral (5.73), we obtain

$$G(p) = \frac{-i\hat{p} + m}{p^2 + m^2} + \mathrm{C}(-i\hat{p} + m), \tag{5.76}$$

where C is a constant. Thus, the Green function of fermions in the model considered has a simple pole. In this approximation, there is no gain or attenuation pole. This conclusion is consistent with the results of [385], which deals with infrared asymptotic of the fermion Green's function in scalar meson physics.

In [391] there is an indication that the consistent consideration of the propagator in quantum electrodynamics also leads to a singularity in the form of a simple pole in the infrared region.

In turn, established in [392] (see also [393]) in principle the possibility of observing the colors in QCD means none other than the existence of the pole in the asymptotic state of the quark propagator.

So, in the initial four-fermion simple model described by the Lagrangian (5.57), confinement is not realized in the infrared energies. It can be concluded that the four-quark schemes describe the intermediate region between the asymptotic freedom and confinement of quarks. In order to take into account the area of confinement one can bring the bag model (see e.g. [193]).

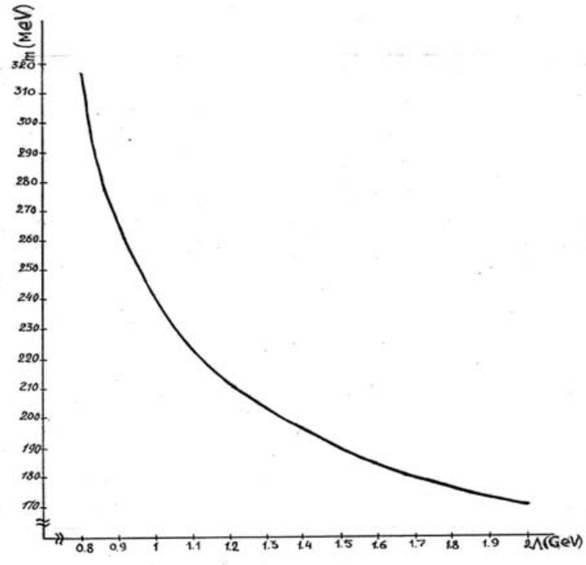

Figure 5.1: The dynamic quark mass m vs. the momentum cut-off Λ.

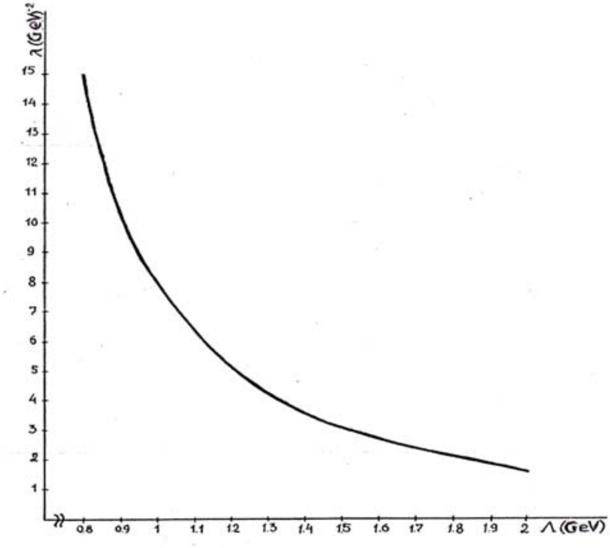

Figure 5.2: The constant λ vs. the cut-off momentum Λ.

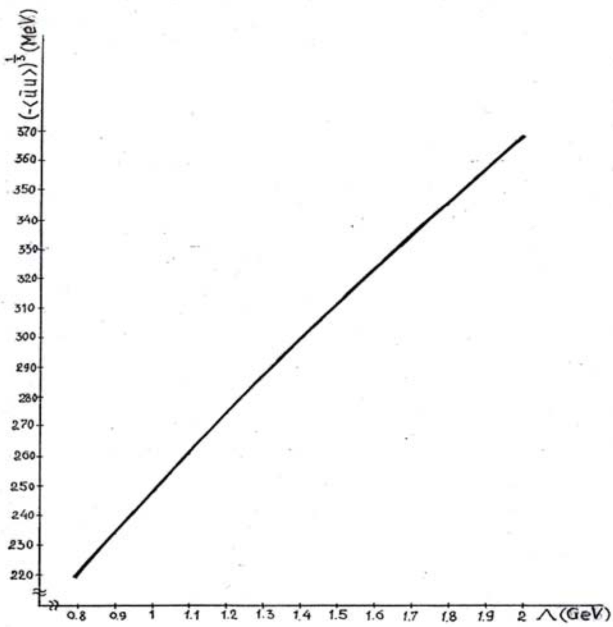

Figure 5.3: The condensate $\langle \bar{d}d \rangle^{1/3} = \langle \bar{u}u \rangle^{1/3} = (\langle \bar{\psi}\psi \rangle / 2)^{1/3}$ vs. the momentum cut-off Λ.

Chapter 6

6.1 Low-energy characteristics

Based on the effective chiral action received by Dyakonov and Petrov, obtained on the concept of the instanton vacuum, the electromagnetic polarizabilities of pions and nucleons are evaluated. The evaluated numerical values of the polarizabilities agree well with the experiment. For some long-wave approximation (when only the first three terms in the expansion of Seeley are used), the values of the polarizabilities of pions are consistent with the results of other authors obtained in the framework of the vector dominance model. Masses of nonets of scalar and pseudoscalar mesons in the framework of the $SU(3)$ Nambu−Jona-Lasinio model are calculated. The possibility of CP-violation in the model is demonstrated. We have considered problems of hadron physics at low energies on the basis of the effective chiral Lagrangian (ECL), including, in addition to the usual (normal), as well as the anomalous part of the Wess−Zumino Lagrangian. The processes predicted by ECL, that can be tested, were specified.
Analytical expressions for the probability of the decay $\pi^{\pm} \to \mu^{+}\bar{\nu}_{\mu}$ in the field of a plane electromagnetic wave, by taking into account the polarizability of the pion, are obtained.
The main results presented in this chapter were published in [394] - [407].

6.1.1 Electromagnetic polarizabilities of pions

The important characteristics of hadrons associated with a complex internal structure of particles are the electric and magnetic polarizabilities. These values show the measure of the dipole electrical $\mathbf{d} = \alpha\mathbf{E}$, and the magnetic $\mathbf{m} = \beta\mathbf{H}$ moments of the particles induced by an external electromagnetic field ($\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic fields, respectively). In other words, the electromagnetic polarizabilities α, β characterize the properties of the particles to be deformed in external fields. The potential energy of the corresponding particle interaction is given by

$$U = -\frac{1}{2}\alpha\mathbf{E}^2 - \frac{1}{2}\beta\mathbf{H}^2. \tag{6.1}$$

Since this expression (6.1) is quadratic in $\mathbf{E}$ and $\mathbf{H}$, then the polarizabilities of hadrons appear at the two-photon interactions [408]. Therefore, the additional interaction (6.1), as related to two-current effects, gives a correction to the amplitude of the photon-hadron interaction at small momenta. The experimental determination of the values of α, β is possible by studying the Compton effect [408], measuring the level shifts in meson atoms [409] and the observation of the Primakoff effect [410] (the emission of a photon by a particle in the atomic nucleus).

Thus, the electromagnetic polarizabilities can be considered as the fundamental low-energy characteristics of strong interactions of hadrons. Therefore, the calculation of these quantities is possible within nonperturbative QCD. The electric and magnetic polarizabilities of pions were calculated in different models: linear σ-model [374], [411], [412], using the current algebra [413], dispersion relations [414], chiral Lagrangians [415], the vector dominance model [412], in the lattice approach [416], in the nonlocal quark model [417], QCD-string theory [418], [419] and other models [420]. The main disadvantage of these calculations is the model-dependent variables of the discussed α and β. In this regard, of particular interest is the calculation of the electric and magnetic polarizabilities of pions within the low-energy formulation of QCD.

As noted above, in the nonperturbative QCD instantons are important. Representation of the QCD vacuum as a rather rarefied gas of instantons is a good approximation [190] - [192], [421], [422]. The presence of instantons leads to the interaction of quarks, described by the Lagrangian (1.6) [197], [198], [423]. In the region of low energies (< 1 GeV) substantial degrees of freedom belong to constituent quarks and pions, which are acting as pseudo-Goldstone bosons (see Sec. 5.1.2) [424].

Consider the problem of the effective action for pions. Note that appearing in the (5.31) matrix, composed of collective fields Φ_A, can be parameterized as follows (see, e.g. [126] - [128]):

$$m_0 - \Phi_A \Gamma_A = H U^{\gamma_5}. \tag{6.2}$$

Here H is the Hermitian matrix, $\Gamma_A = (I,\ i\gamma_5,\ \tau^a,\ i\gamma_5\tau^a)$, and the unitary matrix U^{γ_5} of the chiral field is chosen as

$$U^{\gamma_5} = \exp\left(\frac{i\pi_a \tau^a \gamma_5}{f_\pi}\right), \tag{6.3}$$

where π_a are pion fields, $f_\pi = 93$ MeV.

Configurations that minimize the effective action in the limit of large number of colors ($N_c \to \infty$), will give the value of $H = M = const$ (see [128]). As a result, in the functional integral (5.29) there will be only the integration on the fields of pions $\pi_a \equiv \tilde{\Phi}_a$. In this way you can get the appropriate effective action for pions. First the ECL for pions was obtained by Dyakonov and Eides [129] from the QCD Lagrangian by integration over the quark fields at $N_c \to \infty$. A similar effect was obtained from the theory of the instanton vacuum theory in [191]. It has been shown that (see [191]) ECL is applicable at energies up to 600 MeV ($\Lambda = 600$ MeV) in Euclidean space and corresponds to the action

$$S_{eff}[\pi] = -N_c \ln \det\left(\frac{D}{D_0}\right), \tag{6.4}$$

where $D_0 = i\hat{\partial} + iM$, $D = i\hat{\partial} + iMU^{\gamma_5}$, $\hat{\partial} = \gamma_\mu \partial_\mu$.

Here the constituent mass of light (u, d) quarks is taken to be $M = 345$ MeV.

Note, it follows from the graph of Fig. 4 that the extrapolation to the value $\Lambda = 600$ MeV gets close to this value of the dynamic mass m. It should also be noted that the transition from collective fields Φ_A to the chiral unitary field U^{γ_5}, according to (6.2) with the condition $H = M$, corresponds to a transition from the linear to the nonlinear σ- model (see [108]).

Since appearing in (6.4) the operator $D = i\hat{\partial} + iMU^{\gamma_5}$ is not Hermitian, the ECL has both real and imaginary parts. The imaginary part of ECL includes the known Wess−Zumino term [424] (see the Introduction).

Consider here only the real part of the ECL. Introducing the interaction of quarks with the electromagnetic field, according to the substitution $\partial_\mu \to \partial_\mu - iQA_\mu$, where $Q = \mathrm{diag}\,(2/3, \ -1/3)\, e$ (e is the electron charge), after squaring the Dirac operator, the authors of [424] got a real part of the effective chiral pion field in this energy range

$$\mathrm{Re}S_{eff}[\pi] = -\frac{1}{2} N_c \mathrm{Tr}\,\ln\left(\frac{DD^+}{D_0 D_0^+}\right), \qquad (6.5)$$

where $D = i\hat{\partial} + iMU^{\gamma_5} + im_0 + Q\hat{A}$, $D_0 = i\hat{\partial} + iM + im_0$, $\hat{A} = \gamma_\mu A_\mu$, m_0 is the current mass of u, d-quarks.

The use of the proper time method of Fock−Schwinger and the calculation of the functional trace (Tr) in the plane wave basis [424], [425] leads from (6.5) to the expression

$$\begin{aligned}\mathrm{Re}S_{eff}[\pi] = -\frac{N_c}{2}\int d^4x\int_0^\infty \frac{dt}{t}\varphi(t)\\ \times\int\frac{d^4p e^{-p^2t}}{(2\pi)^4}\mathrm{tr}\Big[\exp\left[-t\left(DD^+ - 2ip_\mu(\partial_\mu - iQA_\mu)\right)\right]\\ -\exp\left[-t\left(D_0D_0^+ - 2ip_\mu\partial_\mu\right)\right]\Big]\times 1,\end{aligned} \tag{6.6}$$

where the regularizing function $\varphi(t)$ cuts the integrals at small t, and the trace tr is taken only on the matrices γ_μ, τ^a.

We use the following decomposition of the chiral field U^{γ_5}, (6.3):

$$U^{\gamma_5} \simeq 1 + i\frac{\pi_a\tau^a\gamma_5}{f_\pi} - \frac{\pi_a^2}{2f_\pi^2}. \tag{6.7}$$

The obvious relation

$$[\pi_a\tau^a, \partial_\mu - iQA_\mu] = -\left[\begin{array}{cc} \partial_\mu\pi_0 & \sqrt{2}D_\mu\pi^- \\ \sqrt{2}D_\mu^+\pi^+ & -\partial_\mu\pi_0 \end{array}\right] \equiv -\Pi_\mu, \tag{6.8}$$

holds, where $D_\mu = \partial_\mu - ieA_\mu$, $D_\mu^+ = \partial_\mu + ieA_\mu$, $\pi^- = (\pi_1 - i\pi_2)/\sqrt{2}$, $\pi^+ = (\pi_1 + i\pi_2)/\sqrt{2}$, $\pi^0 = \pi_3$.

Hence we find the expression for the squared Dirac operator in an external field (see (6.5))

$$DD^{+} = -\left[\gamma_\mu \left(\partial_\mu - iQA_\mu\right)\right]^2 + \left(M + m_0\right)^2$$
$$+i\frac{M}{f_\pi}\gamma_5\hat{\Pi} - \frac{m_0 M}{f_\pi^2}\pi_a^2 - \frac{M}{f_\pi^2}\pi_a(\hat{\partial}\pi_a), \tag{6.9}$$

where $\hat{\Pi} = \gamma_\mu \Pi_\mu$.

We now use the quasi-classical approximation [425], [426], expanding the exponents in (6.6) as power series with respect to the proper time t (the Seeley decomposition [427]).

The first two terms of the expansion of (6.6) contain the mass and kinetic terms:

$$\mathrm{Re}S_{eff}^{(1)} = -\frac{N_c m_0 M}{4\pi^2 f_\pi^2}\int d^4x \int_0^\infty \frac{dt}{t^2}\varphi(t)e^{-t(M+m_0)^2}\pi_a^2$$
$$= -\frac{1}{2}m_\pi^2 \int d^4x\pi_a^2, \tag{6.10}$$
$$\mathrm{Re}S_{eff}^{(2)} = -\frac{1}{2}\int d^4x \left[2\left(D_\mu^{+}\pi^{+}\right)\left(D_\mu\pi^{-}\right) + \left(\partial_\mu\pi^0\right)^2\right] + \ldots .$$

The third term of this expansion for (6.6) includes terms with the electromagnetic field strength

$$\mathrm{Re}S_{eff}^{(3)} = \frac{N_c J_1}{3!16\pi^2(M+m_0)^2 f_\pi^2}$$

$$\times \int d^4x \Big[12iM^2 eF_{\mu\nu} \left(D_\mu^+ \pi^+ \right) \left(D_\nu \pi^- \right) \tag{6.11}$$
$$- \frac{5}{3} m_0 M e^2 F_{\mu\nu}^2 \pi_a^2 \Big] + \ldots \, ,$$

where we use the value of the integral $J_1 = \int_0^\infty d\tau \varphi(\tau) e^{-\tau}$ $= 0.682$ [425]. In (6.11) we have omitted the higher derivatives of the pion field, which do not contain the tensor $F_{\mu\nu}$. These terms do not contribute to the polarizability of pions.

Linear in $F_{\mu\nu}$ terms in (6.11) determine the charge radius r_π of a pion. To find them we use the effective Lagrangian [428]

$$\mathcal{L}_{eff}^{(1)} = 2ieF_{\mu\nu} \left(D_\mu^+ \pi^+ \right) \left(D_\nu \pi^- \right) \frac{1}{6} \langle r_\pi^2 \rangle . \tag{6.12}$$

Comparing equations (6.11) and (6.12) we find the value of the squared charge radius

$$\langle r_\pi^2 \rangle = \frac{3N_c J_1}{8\pi^2 f_\pi^2} \simeq \frac{3}{4\pi^2 f_\pi^2} \simeq 0.34 \text{ fm}^2 . \tag{6.13}$$

Value (6.13) coincides with the result of a calculation [421], obtained by other means, as well as to the value of the linear σ-model [412]. The experimental value of $\langle r_\pi^2 \rangle$ is [429]:

$$\langle r_\pi^2 \rangle = (0.44 \pm 0.02) \text{ fm}^2 .$$

Note that the quantity (6.13) is obtained only by the leading term of $1/N_c$-decomposition. The theory developed in [421] - [424] allows us also to consider the following corrections.

The fourth term of the expansion (6.6) to the proper time contains terms that are quadratic in the electromagnetic field. The general form of the Lagrangian, which is quadratic in the electromagnetic field, describes the polarizability and is given by [428], [430], [431]

$$\mathcal{L}^{(2)}_{eff} = \left(D^+_\mu \pi^+\right)\left(D_\nu \pi^-\right)\left(\frac{a+b}{m_\pi} F_{\mu\alpha}F_{\nu\alpha} - \frac{b}{2m_\pi} F^2_{\alpha\beta}\delta_{\mu\nu}\right) \tag{6.14}$$
$$+\frac{dm_\pi}{2} F^2_{\alpha\beta}\pi^+\pi^-,$$

where the constants a, b, d are related to the pion polarizabilities.

We note (see [428]) that the Lagrangian (6.12) can be rewritten up to a four-divergence as follows:

$$\mathcal{L}^{(1)}_{eff} = -\frac{e^2\langle r^2_\pi\rangle}{6} F^2_{\mu\nu}\pi^+\pi^- - ie(\partial_\mu F_{\mu\nu})(\pi^+ D^{\leftrightarrow}_\nu \pi^-)\frac{1}{6}\langle r^2_\pi\rangle, \tag{6.15}$$

$\pi^+ D^{\leftrightarrow}_\nu \pi^- \equiv \pi^+ D_\nu \pi^- - (D_\nu \pi^+)\pi^-$. Given that the first term in (6.15) is a contribution to the polarizability of the charged pions, and the second term in (6.11) will contribute to the polarizability of both charged and neutral pions. Comparing (6.11), (6.15) with (6.14), we find the following values of parameters:

$$d_{\pi^\pm} = -\frac{e^2}{3m_\pi}\langle r_\pi^2\rangle\left(1+\frac{5m_0}{9M}\right) \simeq -11.76\times 10^{-4}\ \text{fm}^3, \tag{6.16}$$

$$d_{\pi^0} = -\frac{5N_c J_1 e^2 m_0}{72\pi^2 f_\pi^2 m_\pi M} \simeq -0.1\times 10^{-4}\ \text{fm}^3. \tag{6.17}$$

Here we take the values $m_\pi = 140$ MeV, $m_0 = 5$ MeV. If one takes into consideration only Re $S_{eff}^{(3)}$ (6.11) and does not consider Re $S_{eff}^{(4)}$, in which case there will be no structure in the coefficients a and b individually. They contribute to the sum of the polarizabilities $a+b=\alpha+\beta$ (see (6.18)). The next term of the expansion Re $S_{eff}^{(4)}$ will have to give the value of $\alpha+\beta\neq 0$.

Here we restrict ourselves to the third-order expansion of the effective action on t. In this approximation, the main contribution to the polarizability of the charged pion yields

$$\alpha^{cl} = \frac{e^2\langle r_\pi^2\rangle}{3m_\pi}$$

as in the vector dominance model [412].

To establish the connection of parameters a, b, d with polarizabilities of pions, we consider the non-relativistic limit of (6.14). Taking the field of dormant pions in the form

$$\pi^- \sim \exp(-im_\pi t), \quad \pi^+ \sim \exp(im_\pi t)$$

with the normalization and the link $\mathcal{L}_{int} = -U$ (see (6.1)), we find

$$\alpha = a - d, \quad \beta = b + d,$$

$$\alpha_\pi \simeq -\beta_\pi \simeq -d_\pi = 11.76 \times 10^{-4} \text{ fm}^3, \tag{6.18}$$

$$\alpha_{\pi^0} \simeq -\beta_{\pi^0} \simeq -d_{\pi^0} = 0.1 \times 10^{-4} \text{ fm}^3.$$

The calculated values of (6.18) are in agreement with the experimental values of [432]

$$\alpha_\pi + \beta_\pi = (1.4 \pm 5.5) \times 10^{-4} \text{ fm}^3, \;\; \beta_\pi = (-7.1 \pm 4.5) \times 10^{-4} \text{ fm}^3, \tag{6.19}$$

which are not very accurate.

The experiment for the polarizabilities of neutral pions gives only the constraint $|\alpha_{\pi^0}| < 35 \times 10^{-4}$ fm^3 [429].

Thus, in the third-order expansion of the chiral effective action to t we obtain the value of the polarizabilities of the charged pions, which coincides with the calculations in the vector dominance model. The calculation of the next term in the expansion will give some corrections to the values (6.18). Since the accuracy of experimental values of α, β (6.19) is not high, the calculation of these corrections is not very important. It should be noted that the real part Re $S_{eff}^{(4)}$, and also the imaginary part Im S_{eff}, which defines the Wess−Zumino action, will contribute to the sum of the polarizabilities. This follows from the fact that the relevant Feynman diagrams contain vertices that define the anomalous part of the action.

In Appendixes A, B, and C, exact solutions of the wave equation for the charged pions, produced from the Lagrangian (6.14), are obtained for external electromagnetic fields of the following configurations: a constant magnetic field, the field of a plane wave, a constant and uniform electric field. The exact solution of the wave equation in the external plane wave is used to take into account the polarizabilities of pions in the process $\pi^+ \to \mu^+ \bar{\nu}_\mu$.

6.1.2 Electromagnetic polarizabilities of nucleons

There are experimental data on neutron polarizability [433] and proton polarizability [434]. This is therefore an interesting theoretical study of these characteristics within nonperturbative QCD. Earlier electromagnetic polarizabilities were calculated in various models of hadrons: the nonrelativistic quark model [408], the bag model [435], the Skyrme model [436], the Skyrme model with ρ-mesons [437] - [439], and the QCD-string theory [440] - [442]. Here we calculate the electric and magnetic polarizabilities of the nucleon in the chiral theory [424], based on the concept of the instanton vacuum [188], [189], [421] - [423].

Static electric and magnetic polarizabilities are associated with a shift of the nucleon mass in the electric and magnetic fields, respectively (see (6.1)):

$$\delta M_N = -\frac{1}{2}\alpha \mathbf{E}^2 - \frac{1}{2}\beta \mathbf{H}^2. \tag{6.20}$$

Low-energy formulation of QCD follows from the theory of the instanton vacuum, and the corresponding partition function is given by [421] - [423]:

$$Z = \int D\psi^+ D\psi DU \exp\left[\int d^4x\ \psi^+ \left(i\hat{\partial} + iMU^{\gamma_5} + im_0\right)\psi\right]. \tag{6.21}$$

Note that the effective Lagrangian for pions (6.4) follows from (6.21) after integration over the quark fields (at $m_0 = 0$). Nucleon mass can be calculated by the asymptotic behavior of the correlator of the nucleon currents in the Euclidean space:

$$\Pi(T) \equiv \langle J_N(T) J_N^+(0) \rangle_{T\to\infty} \sim \exp(-M_N T), \qquad (6.22)$$

where $J_N(x)$ is the current constructed from quark fields with the quantum numbers of the nucleon.

To calculate the mass shift of the nucleon in an external electromagnetic field it is necessary to study the asymptotic behavior of the correlation function (6.22) at large Euclidean times in an external electromagnetic field. Using (6.21) with the introduction of the electromagnetic field by replacing $\partial_\mu \to \partial_\mu - iQA_\mu$, the correlator of the nucleon current in the external field can be written as a path integral:

$$\Pi(T) = \frac{1}{Z} \int D\psi^+ D\psi DU \, J_N(T) J_N^+(0)$$
$$\times \exp\left[\int d^4x \; \psi^+ \left(i\hat{\partial} + QA + iMU^{\gamma_5} + im_0\right) \psi\right], \qquad (6.23)$$

where $A = A_\mu \gamma_\mu$. The integration of (6.23) on the quark fields can be done exactly as integral (6.23) is Gaussian. To calculate the integral over the field U, you can use the saddle-point method, which is justified in the limit $N_c \to \infty$. The saddle-point value of the chiral fields is given by

$$U = \exp\left(i\frac{x^a}{r}\tau^a P(r)\right). \quad (6.24)$$

Authors of the papers [424], [425] used for the profile function $P(r)$ a simple one-parameter ansatz

$$P(r) = 2\arctan\left(\frac{r_0^2}{r^2}\right) \quad (6.25)$$

when the parameter $r_0 = 0.98/M$ ($M = 375$ MeV). Integrals associated with the rotational and translational zero modes were calculated exactly [400].

To estimate the polarizabilities, the functional decomposition of traces in powers of the chiral field U can be used [400]. A similar approach was used in [424] to estimate the moment of inertia, and it was shown that the accuracy of this approximation is of the order of 10 per cent. There was obtained in [400] a relation of the electric polarizability with iso-vector charge radius $\langle r^2\rangle_{T=1}$:

$$\alpha = \frac{e^2}{2(M_\Delta - M_N)}\langle r^2\rangle_{T=1} + F(U),$$

where the first term of the functional $F(U)$ in powers of the derivative of the chiral field U is of the form:

$$F(U) = \frac{e^2}{9}\frac{N_c}{16\pi^2}\int d^3x \sin^2 P(r).$$

Note that the connection of the electric polarizability with the iso-vector charge radius follows from the general principles of quantum field theory [408], and such a link is not for the magnetic polarizability. In our present approach, in contrast to [436], there is no clear link between the magnetic polarizability and the charge radius.

To estimate the electric polarizability we can use either the experimental values of $\langle r^2\rangle_{T=1}$ and $M_\Delta - M_N$, or the values of these quantities calculated in this approach. The value of $M_\Delta - M_N$ was computed in [424], and its value is very close to the experimental one. The charge radius was estimated at [400] where in the case of a non-zero quark mass to estimate the integrals, profile function was used, which has the correct asymptotic behavior at large r:

$$P(r) = 2\arctan\left(\frac{r_0^2}{r^2}(1+m_\pi r)\exp(-m_\pi^2 r)\right). \qquad (6.26)$$

where $m_\pi = 138$ MeV is the mass of π-meson.

The computation of $\langle r^2\rangle_{T=1}$ with the help of (6.26) was given in [400], and the experimental value of $\langle r^2\rangle_{T=1} = 0.82\ \text{fm}^2$ is obtained with $r_0 = 0.89/M$. Thus, we consider a chiral theory which reproduces the value of the charge radius of the nucleon. On this basis, to evaluate the electric polarizability one can use the experimental value of the charge radius.

Magnetic polarizability can be split into two terms $\beta = \beta_{para} + \beta_{dia}$ (β_{para} is paramagnetic and β_{dia} is diamagnetic polarizabilities), where

$$\beta_{para} = 2 \sum_n \frac{|\langle 0|M_z|n\rangle|^2}{E_n - E_0}. \tag{6.27}$$

In this approach, as well as in the Skyrme model [436], [437], the sum in (6.27) is saturated by the contribution of Δ-isobar. This contribution can be calculated using the ratio between the magnetic moment of the transition $N \to \Delta$ and the magnetic moment of the nucleon $\mu_{N\Delta} = (\mu_p - \mu_n)/\sqrt{2}$ [424], which is well satisfied in the experiment. The result is:

$$\beta_{para} = \frac{e^2}{M_\Delta - M_N} \left(\frac{\mu_p - \mu_n}{2M_N} \right)^2. \tag{6.28}$$

To estimate β_{dia} it is convenient to study the amount of $\alpha + 2\beta_{dia}$, which contains no divergences associated with the slow decay of the profile function in the chiral limit, and therefore does not depend much on the choice of the variational ansatz (6.26). The value of β_{dia} has been calculated in [400].

Experimental values for proton electromagnetic polarizabilities [443] (see also [408]) are

$$\bar{\alpha} = (12 \pm 0.6) \times 10^{-4} \text{ fm}^3,$$
$$\bar{\beta} = (1.9 \pm 0.5) \times 10^{-4} \text{ fm}^3. \tag{6.29}$$

When comparing the static polarizabilities α, β, defined by (6.20), and the values of $\bar{\alpha}$, $\bar{\beta}$, taken from a review of Compton scattering, one has to consider the link

$$\bar{\alpha} = \alpha + \frac{e^2 \langle r_E^2 \rangle}{3M_N}, \quad \bar{\beta} = \beta, \tag{6.30}$$

and

$$\frac{e^2 \langle r_E^2 \rangle}{3M_N} \simeq 3.8 \times 10^{-4} \text{ fm}^3.$$

Calculations of electromagnetic polarizabilities in this approach gave values in agreement with the experimental values (6.29) [400]. Improving the accuracy of the computation of electromagnetic polarizabilities is possible by clarifying the terms in the expansion of the functional $F(U)$ in powers of the derivatives of the chiral field U.

So the considered approach gives reasonable values for the charge radii and electromagnetic polarizabilities of nucleons. Note that the chiral theory of nucleons gives an opportunity to improve the accuracy of calculations by expanding the unknown quantities in the powers of $1/N_c$ and the derivatives of the chiral field U, or using computer calculations of the functional traces.

6.1.3 The spontaneous CP violation in the $SU(3)$ NJL model

It is possible nowadays to derive the effective quark-meson Lagrangians from the fundamental QCD Lagrangian. The reformulation of QCD in terms of hadrons has not been completed yet. Therefore, in the domain of low energy, some phenomenological models are introduced. Local ECL [444], [102], [445] can describe low energy physics of hadrons with good accuracy. The instanton vacuum theory [188], [189], [421], [191] explains the appearance of the chiral condensate which leads to the dynamical symmetry breaking and to the effective four-quark interaction (for two flavors) [197], [198] (see Sec. 5.2 and also [217], [356]). So, contact four-fermion interaction modeling quark interactions, take into account both quarks and mesons [1], [257], [81], [446], [447], [448]. In such models, the gluon interactions are neglected and there is no confinement of quarks. Therefore NJL models are QCD motivated effective models with some shortcomings. In particular, NJL models make it possible to decay the scalar mesons into $q\bar{q}$.

Our goal here is to study the possibility of spontaneous CP symmetry violation in the $SU(3)$ NJL model. The electric dipole moments of particles violate CP-invariance and, in the framework of QCD, can be explained with the help of the θ-term. The effect of CP breaking in strong interactions is small, but the investigation of such a phenomenon is important. It should be noted that the θ-term is important for the solution of the $U_A(1)$ problem. The axial symmetry, $U_A(1)$, is broken by the QCD anomaly. This may be explained by the interactions of light quarks and instantons which violate the $U_A(1)$-symmetry. There is a region of quark masses [449], where CP symmetry is spontaneously broken. The CP violation leads to the exotic phenomena, the possibility of η decaying into two pions.

6.1.4 The model and perturbation expansion

We start with an NJL model possessing the internal symmetry group $SU(3) \otimes SU(3)$ in the chiral limit:

$$\mathcal{L}(x) = -\bar{\psi}(x)(\gamma_\mu \partial_\mu + \widehat{m}_0)\psi(x)$$
$$+\frac{G}{2}\left\{\left[\bar{\psi}(x)\lambda^a\psi(x)\right]^2 + \left[i\bar{\psi}(x)\gamma_5\lambda^a\psi(x)\right]^2\right\}, \tag{6.31}$$

where λ^a $(a = 0, 1, ..., 8)$ are the Gell-Mann matrices, $\lambda_0 = \sqrt{2/3}I_3$ (I_3 is the unit 3×3-matrix), $\partial_\mu = (\partial/\partial x_i, -i\partial/\partial x_0)$ (x_0 is the time), γ_μ are the Dirac matrices, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The $\widehat{m}_0$ is the matrix of bare masses of the quark triplet $\psi(x)$:

$$\psi(x) = \mathrm{diag}\left[u(x), d(x), s(x)\right], \quad \widehat{m}_0 = \mathrm{diag}\left(m_u, m_d, m_s\right).$$

The summation over color quark degrees of freedom $n = 1, 2, ..., N_C$ is implied here. The chiral symmetry is broken by quark masses and dynamically by the appearance of condensates. Therefore, for simplicity, we consider only the formation of the nonet of scalar mesons and the nonet of pseudoscalar mesons π, K, η, η'. The octets of vector and pseudovector mesons are ignored here. The $U_A(1)$ symmetry is not broken here as the Lagrangian (1) is invariant under γ_5-chiral transformations. To violate this symmetry “by hand”, one can add to the Lagrangian (1) the six-quark interaction due to instantons [217]. On the other hand the $U_A(1)$ anomaly appears because of the non-invariance of the fermion measure in the functional integral [450]. It should be noted that the QCD anomaly, $U_A(1)$, results in the existence of a ninth Goldstone boson η' with the greater mass compared to η.

Using the functional integration method [312] (see chapter 2), the generating functional for Green's functions

$$Z[\bar{\eta},\eta] = N_0 \int D\bar{\psi}D\psi \exp\Big\{ i \int d^4x \Big[\mathcal{L}(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \Big] \Big\}, \tag{6.32}$$

where $\bar{\eta}$, η are external sources, with the help of the replacement

$$N_0 = N \int D\phi_a D\widetilde{\phi}_a \exp\Big\{ -i\frac{M^2}{2} \times \int d^4x \Big[\Big(\widetilde{\phi}_a(x) - i\frac{g_0}{M^2}\bar{\psi}(x)\gamma_5\lambda^a\psi(x) \Big)^2 + \Big(\phi_a(x) - \frac{g_0}{M^2}\bar{\psi}(x)\lambda^a\psi(x) \Big)^2 \Big] \Big\},$$

can be cast into

$$Z[\bar{\eta},\eta] = N \int D\bar{\psi}D\psi D\phi_a D\widetilde{\phi}_a$$

$$\times \exp\Big\{ i \int d^4x \Big[-\bar{\psi}^n(x) \Big[\gamma_\mu \partial_\mu + \widehat{m}_0 - g_0 \Big(\phi_a(x) + i\gamma_5 \widetilde{\phi}_a(x) \Big) \lambda^a \Big] \psi(x) \tag{6.33}$$

$$- \frac{M^2}{2} \left(\widetilde{\phi}_a^2(x) + \phi_a^2(x) \right) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \Big] \Big\},$$

where $G = g_0^2/M^2$; g_0 is the dimensionless bare coupling constant and M is the dimensional constant. Equation (6.33) may be integrated over the $\bar{\psi}$, ψ, and as a result equation

$$Z[\bar{\eta}, \eta] = N \int D\phi_a D\widetilde{\phi}_a \exp\Big\{ iS[\Phi] + i \int d^4x \; d^4y \; \bar{\eta}(x) S_f(x,y) \eta(y) \Big\}, \tag{6.34}$$

where the effective action for bosonic collective fields $\Phi_a(x) = \phi_a(x) + i\gamma_5 \widetilde{\phi}_a(x)$ is given by

$$S[\Phi] = -\frac{M^2}{2} \int d^4x \left[\phi_a^2(x) + \widetilde{\phi}_a^2(x) \right] \tag{6.35}$$

$$-i\mathrm{Tr}\ln\left[-\gamma_\mu\partial_\mu - \widehat{m}_0 + g_0\Phi_a(x)\lambda^a\right].$$

We have used here the relation $\det Q = \exp\ \mathrm{Tr}\ \ln Q$ (the Q is an operator). The operator Tr in equation (6.35) includes the tracing in matrix and space-time variables. The Green function of quarks, $S_f(x,y)$, obeys the equations

$$\left[\gamma_\mu\partial_\mu + \widehat{m}_0 - g_0\Phi_a(x)\lambda^a\right] S_f(x,y) = \delta(x-y). \qquad (6.36)$$

The fields $\phi_a(x)$ and $\widetilde{\phi}_a(x)$ form nonets of scalar and pseudoscalar ($\pi^{\pm}$, π^0, $K^{\pm}$, K^0, $\bar{K}^0$, η, η') mesons.

The symmetric vacuum in the NJL models is not stable [257], [446], [447], [448]. The physical vacuum is reconstructed which results in the appearance of condensates and the dynamical breaking of $SU(3)\otimes SU(3)$ symmetry. We imply here that the condensates are formed as follows:

$$\langle\bar{\psi}\psi\rangle \neq 0,\ \langle\bar{\psi}\lambda^3\psi\rangle \neq 0,\ \langle\bar{\psi}\lambda^8\psi\rangle \neq 0,$$
$$\langle\bar{\psi}\gamma_5\psi\rangle \neq 0,\ \langle\bar{\psi}\gamma_5\lambda^3\psi\rangle \neq 0,\ \langle\bar{\psi}\gamma_5\lambda^8\psi\rangle \neq 0. \qquad (6.37)$$

The vacuum expectation values containing the γ_5 matrix are parity and time reversal odd values, and as a result, they violate CP symmetry. To take into consideration and to determine condensates, the fields have to be "shifted" by the constants. Therefore, we make the substitution in equations (6.35), (6.36)

$$\phi_0(x) = \phi_0'(x) + \sigma_0, \quad \phi_3(x) = \phi_3'(x) + \sigma_3,$$

$$
\begin{gathered}
\phi_8(x) = \phi_8'(x) + \sigma_8, \quad \phi_i(x) = \phi_i'(x), \\
\widetilde{\phi}_0(x) = \widetilde{\phi}_0'(x) + \widetilde{\sigma}_0, \quad \widetilde{\phi}_3(x) = \widetilde{\phi}_3'(x) + \widetilde{\sigma}_3, \\
\widetilde{\phi}_8(x) = \widetilde{\phi}_8'(x) + \widetilde{\sigma}_8, \quad \widetilde{\phi}_i(x) = \widetilde{\phi}_i'(x),
\end{gathered}
\tag{6.38}
$$

where $i = 1, 2, 4, 5, 6, 7$; σ_0, σ_3, σ_8, $\widetilde{\sigma}_0$, $\widetilde{\sigma}_3$, $\widetilde{\sigma}_8$ are coordinate-independent and Lorentz-invariant constants. The fields $\phi_a'(x)$, $\widetilde{\phi}_a'(x)$ in equations (6.38) represent quantum excitations over vacuum and are assumed to be small. The vacuum expectation values (condensates), $\widetilde{\sigma}_0$, $\widetilde{\sigma}_3$ and $\widetilde{\sigma}_8$, break CP symmetry. Below, the condensates σ_j and $\widetilde{\sigma}_j$ for $j = 0, 3, 8$ will be obtained from the minimum of the effective potential defining the energy density of the vacuum. To formulate the perturbation theory [312], we use the saddle-point method. Taking into consideration equations (6.38), one may rewrite equation (6.35) as follows:

$$
\begin{aligned}
S[\Phi'] = -\frac{M^2}{2} \int d^4x \Big[& \Big(\phi_j'(x) + \sigma_j \Big)^2 \\
& + \Big(\widetilde{\phi}_j'(x) + \widetilde{\sigma}_j \Big)^2 + \phi_i'^2 + \widetilde{\phi}_i'^2 \Big] \\
-i \mathrm{Tr} \ln & \left[-\gamma_\mu \partial_\mu - \widehat{m} + i \widehat{\widetilde{m}} \gamma_5 + g_0 \Phi_a'(x) \lambda^a \right],
\end{aligned}
\tag{6.39}
$$

where $\Phi_a'(x) = \phi_a'(x) + i\gamma_5 \widetilde{\phi}_a'(x)$, $i = 1, 2, 4, 5, 6, 7$; $j = 0, 3, 8$,

$$
\widehat{m} = \mathrm{diag}\left(m_{01}, m_{02}, m_{03} \right),
$$

$$\widehat{\widetilde{m}} = \mathrm{diag}\,(\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3)\,,$$
$$m_{01} = m_u - g_0\left(\sqrt{\frac{2}{3}}\sigma_0 + \sigma_3 + \frac{\sigma_8}{\sqrt{3}}\right),$$
$$m_{02} = m_d - g_0\left(\sqrt{\frac{2}{3}}\sigma_0 - \sigma_3 + \frac{\sigma_8}{\sqrt{3}}\right),$$
$$m_{03} = m_s - g_0\left(\sqrt{\frac{2}{3}}\sigma_0 - \frac{2\sigma_8}{\sqrt{3}}\right), \tag{6.40}$$
$$\widetilde{m}_1 = g_0\left(\sqrt{\frac{2}{3}}\widetilde{\sigma}_0 + \widetilde{\sigma}_3 + \frac{\widetilde{\sigma}_8}{\sqrt{3}}\right),$$
$$\widetilde{m}_2 = g_0\left(\sqrt{\frac{2}{3}}\widetilde{\sigma}_0 - \widetilde{\sigma}_3 + \frac{\widetilde{\sigma}_8}{\sqrt{3}}\right), \quad \widetilde{m}_3 = g_0\left(\sqrt{\frac{2}{3}}\widetilde{\sigma}_0 - \frac{2\widetilde{\sigma}_8}{\sqrt{3}}\right).$$

Let us consider the equality

$$\mathrm{Tr}\ln\left[-\gamma_\mu\partial_\mu - \widehat{m} + i\widehat{\widetilde{m}}\gamma_5 + g_0\Phi'_a(x)\lambda^a\right]$$
$$= \mathrm{Tr}\ln\left(-\gamma_\mu\partial_\mu - \widehat{m} + i\widehat{\widetilde{m}}\gamma_5\right)$$
$$+\mathrm{Tr}\ln\left[1 - g_0 S_{0f}(x,y)\Phi'_a(x)\lambda^a\right],$$

where the Green function $S_{0f}(x,y)$ obeys the equation

$$\left[\gamma_\mu\partial_\mu + \widehat{m} - i\widehat{\widetilde{m}}\gamma_5\right] S_{0f}(x,y) = \delta(x-y). \tag{6.41}$$

Then expanding the logarithm in small fluctuations $\Phi'_a(x)$, the effective action (6.35) takes the form

$$S[\Phi'] = -\frac{M^2}{2}\int d^4x\Big[\Big(\phi'_j(x)+\sigma_j\Big)^2$$

$$+\Big(\widetilde{\phi}'_j(x)+\widetilde{\sigma}_j\Big)^2+\phi_i'^2+\widetilde{\phi}_i'^2\Big] \qquad (6.42)$$

$$-i\mathrm{Tr}\ln\Big(-\gamma_\mu\partial_\mu-\widehat{m}+i\widehat{\widetilde{m}}\gamma_5\Big)+\sum_{n=1}^{\infty}\frac{i}{n}\mathrm{Tr}\left(g_0S_{0f}\Phi'_a\lambda^a\right)^n,$$

where

$$\mathrm{Tr}\left(g_0S_{0f}\Phi'_a\lambda^a\right)^n$$

$$=\mathrm{tr}\Big[g_0^n\int\ d^4x_1...d^4x_n\ S_{0f}(x_n-x_1)\Phi'_{a_1}\lambda^{a_1} \qquad (6.43)$$

$$\times S_{0f}(x_1-x_2)\Phi'_{a_2}\lambda^{a_2}...S_{0f}(x_{n-1}-x_n)\Phi'_{a_n}\lambda^{a_n}\Big],$$

where the tr[...] means the tracing in matrices. The terms with $n=3,4$ in equation (6.42) define the decaying and the scattering of mesons. The fields ϕ'_a and $\widetilde{\phi}'_a$ in the effective action (6.42), after renormalization, describe the physical scalar and pseudoscalar mesons.

6.1.5 Propagators of quarks and mesons

To calculate the masses of quarks and mesons it is necessary to find the propagators of quarks and mesons. The condensates σ_j and $\widetilde{\sigma}_j$ for $j = 0, 3, 8$ can be obtained from the requirement that terms linear in fields $\phi'_a(x)$, $\widetilde{\phi}'_a(x)$, which correspond to the "tadpole" diagrams, and are absent in the effective action (6.42). This leads to the gap equations

$$\frac{\delta S[\Phi']}{\delta \phi'_j(x)}|_{\phi'_j=0} = -M^2\sigma_j + ig_0 \mathrm{Tr}\left[S_{0f}(x,x)\lambda^j\right] = 0,$$
$$\frac{\delta S[\Phi']}{\delta \widetilde{\phi}'_j(x)}|_{\widetilde{\phi}'=0} = -M^2\widetilde{\sigma}_j(x) - g_0 \mathrm{Tr}\left[S_{0f}(x,x)\gamma_5\lambda^j\right] = 0. \tag{6.44}$$

To find a solution of equation (6.41), we write it down in the momentum space:

$$\left[i\widehat{p} + \widehat{m} - i\widehat{\widetilde{m}}\gamma_5\right] S_{0f}(p) = 1, \tag{6.45}$$

where $\widehat{p} = p_\mu\gamma_\mu$, $p_\mu = (\mathbf{p}, ip_0)$. It is easy to verify that the solution to equation (6.45) for the Green function is given by

$$S_{0f}(p) = \mathrm{diag}\Big(\frac{-i\widehat{p} + m_{01} + i\widetilde{m}_1\gamma_5}{p^2 + m_1^2},\ \frac{-i\widehat{p} + m_{02} + i\widetilde{m}_2\gamma_5}{p^2 + m_2^2},$$

$$\left.\frac{-i\hat{p} + m_{03} + i\widetilde{m}_3\gamma_5}{p^2 + m_3^2}\right), \tag{6.46}$$

where

$$m_1^2 = m_{01}^2 + \widetilde{m}_1^2,\ m_2^2 = m_{02}^2 + \widetilde{m}_2^2,\ m_3^2 = m_{03}^2 + \widetilde{m}_3^2. \tag{6.47}$$

The poles of the Green function (6.46) define the dynamical (constituent) masses of u, d and s quarks: m_1, m_2, m_3. The scalar (σ_j) and pseudoscalar ($\widetilde{\sigma}_j$) condensates contribute to the constituent masses of all quarks. The terms containing $\widetilde{m}_j$ in Eq.(6.46) violate CP-symmetry. Substituting equation (6.46) into equations (6.44), one obtains a system of gap equations:

$$\begin{gathered}
M^2\sigma_0 = g_0\sqrt{\frac{2}{3}}\left(I_1 m_{01} + I_2 m_{02} + I_3 m_{03}\right), \\
M^2\sigma_3 = g_0\left(I_1 m_{01} - I_2 m_{02}\right), \\
M^2\sigma_8 = \frac{g_0}{\sqrt{3}}\left(I_1 m_{01} + I_2 m_{02} - 2I_3 m_{03}\right), \\
M^2\widetilde{\sigma}_0 = -g_0\sqrt{\frac{2}{3}}\left(\widetilde{m}_1 I_1 + \widetilde{m}_2 I_2 + \widetilde{m}_3 I_3\right), \qquad (6.48) \\
M^2\widetilde{\sigma}_3 = -g_0\left(\widetilde{m}_1 I_1 - \widetilde{m}_2 I_2\right), \\
M^2\widetilde{\sigma}_8 = -\frac{g_0}{\sqrt{3}}\left(\widetilde{m}_1 I_1 + \widetilde{m}_2 I_2 - 2\widetilde{m}_3 I_3\right),
\end{gathered}$$

where quadratic diverging integrals are given by

$$I_j = \frac{iN_C}{4\pi^4}\int \frac{d^4p}{p^2+m_j^2} = \frac{N_C}{4\pi^2}\left[m_j^2 \ln\left(\frac{\Lambda^2}{m_j^2}+1\right) - \Lambda^2\right], \tag{6.49}$$

where $d^4p = id^3pdp_0$, the Λ is a cut-off and there is no summation in the index j $(j = 1, 2, 3)$ in equation (6.49). The self-consistent equations (6.48) connect such parameters of a model as the dimensional constant G, condensates σ_j (or dynamical masses of quarks) and a cut-off. The system of six gap equations (6.48), defining the vacuum expectations σ_j, $\widetilde{\sigma}_j$, with the help of equations (6.40) can be rewritten as

$$(m_u - m_{01}) = 2m_{01}GI_1, \quad (m_d - m_{02}) = 2m_{02}GI_2, \tag{6.50}$$

$$(m_s - m_{03}) = 2m_{03}GI_3,$$

$$-\widetilde{m}_1 = 2\widetilde{m}_1 GI_1, \qquad -\widetilde{m}_2 = 2\widetilde{m}_2 GI_2, \qquad -\widetilde{m}_3 = 2\widetilde{m}_3 GI_3. \tag{6.51}$$

There are different solutions of gap equations (6.50), (6.51). We are interested here in the possibilities of CP violation $(\widetilde{m}_j \neq 0)$. Therefore, consider the case when equations (6.51) have non-trivial solutions. It follows from equations (6.51) that if three vacuum expectations $\widetilde{m}_j$ $(j = 1, 2, 3)$ do not equal zero, $\widetilde{m}_j \neq 0$, then $I_1 = I_2 = I_3$, and therefore

$m_1 = m_2 = m_3$. This is not an interesting case because the strange quark s is much heavier than the u and d quarks. Another solution is $\widetilde{m}_3 = 0$, $\widetilde{m}_1 \neq 0$, $\widetilde{m}_2 \neq 0$. Then from equations (6.51), we arrive at the case $m_1 = m_2$, $\widetilde{m}_1 = \widetilde{m}_2$, i.e. isotopic symmetry is not broken, and the gap equation becomes $2g_0^2 I_1 = -M^2$ $(I_1 = I_2)$. Comparing this equation with equations (6.50), one makes a conclusion that $m_u = m_d = 0$, i.e. the chiral limit for the light quarks is realized. We expect that pions ($\pi^{\pm}$, π^0) will be massless Goldstone particles in this case. Requiring $m_s \neq 0$, one arrives from equations (6.50) at two gap equations

$$(m_s - m_{03}) = 2m_{03} G I_3, \qquad -1 = 2GI_1. \tag{6.52}$$

At the same time, if there is no CP violation, $\widetilde{m}_1 = \widetilde{m}_2 = \widetilde{m}_3 = 0$, we can analyze the case $m_u \neq 0$, $m_d \neq 0$, $m_s \neq 0$, and gap equations (6.50) are valid (see [217], [257], [446], [447], [448] for other studies). We pay attention here to the case $\widetilde{m}_1 = \widetilde{m}_2 \neq 0$, $\widetilde{m}_3 = 0$, $m_u = m_d = 0$, which requires (see equations (6.50))

$$\sigma_3 = \widetilde{\sigma}_3 = 0, \qquad \widetilde{\sigma}_0 = \sqrt{2}\widetilde{\sigma}_8. \tag{6.53}$$

The independent parameters here are the current quark mass m_s, the cut-off Λ, and the dimensional constant G.

From equation (6.52), one may obtain the part of effective action which does not depend on coordinates:

$$S[\sigma,\widetilde{\sigma}] = -\frac{M^2}{2}\int d^4x \left[(\sigma_j)^2 + (\widetilde{\sigma}_j)^2\right]$$

$$-i\mathrm{Tr}\ln\left(-\gamma_\mu\partial_\mu - \widehat{m} + i\widehat{\widetilde{m}}\gamma_5\right). \tag{6.54}$$

We can use the relation $S[\sigma,\widetilde{\sigma}] = -\int d^4x\ V_{eff}$ [314], [315] for the constant fields. As a result, one may get from equation (6.54), with the help of equation (6.40), the effective potential

$$V_{eff} = \frac{M^2}{4g_0^2}\left[(m_{01}-m_u)^2 + (m_{02}-m_d)^2 + (m_{03}-m_s)^2\right]$$
$$+\frac{iN_C}{8\pi^4}\int d^4p\ \ln\left(p^2+m_1^2\right)\left(p^2+m_2^2\right)\left(p^2+m_3^2\right). \tag{6.55}$$

We will keep all parameters as non-zero for the possibility to study as the case with CP violation as well as the case without CP breaking. Equations (6.48) or (6.50), (6.51) may be obtained from the condition of the effective potential (6.55) to realize the minimum:

$$\frac{\partial V_{eff}}{\partial m_{0j}} = \frac{\partial V_{eff}}{\partial \widetilde{m}_j} = 0 \qquad (j=1,2,3). \tag{6.56}$$

To obtain the mass spectrum of mesons, one needs to evaluate the terms in equation (6.42), quadratic in fields ϕ'_a,

$\widetilde{\phi}'_a$. From equation (6.42) we find

$$S^{(2)}[\Phi'] = -\frac{M^2}{2}\int d^4x \left[\phi'^2_a + \widetilde{\phi}'^2_a\right] + \frac{i}{2}\mathrm{Tr}\left(g_0 S_{0f}\Phi'_a\lambda^a\right)^2$$

$$\equiv -\frac{1}{2}\int d^4x\ d^4y\ \phi'_A(x)\Delta^{-1}_{AB}(x,y)\phi'_B(y). \tag{6.57}$$

In the momentum space the inverse meson symmetric propagator is given by

$$\Delta^{-1}_{AB}(p) = -ig_0^2 \mathrm{tr}\left[\int \frac{d^4k}{(2\pi^4)} S_{0f}(k)T_A S_{0f}(k-p)T_B\right] + \delta_{AB}M^2_A, \tag{6.58}$$

where $T_A = (\lambda^a, i\gamma_5\lambda^a)$, $\phi'_A = (\phi'_a, \widetilde{\phi}'_a)$, and we use the notation $A = (a, \widetilde{a})$.

Evaluating the traces in equations (6.58), we obtain the non-zero elements of the inverse propagators of the scalar $(\Phi'_a(x))$ mesons:

$$\Delta^{-1}_{00}(p) = M^2 + g_0^2\frac{2}{3}\left(I_1 + I_2 + I_3\right) + \frac{1}{3}\left(p^2 + 4m^2_{01}\right)I_{11}(p)$$

$$+\frac{1}{3}\left(p^2 + 4m^2_{02}\right)I_{22}(p) + \frac{1}{3}\left(p^2 + 4m^2_{03}\right)I_{33}(p),$$

$$\Delta^{-1}_{11}(p) = \Delta^{-1}_{22}(p) = M^2 + g_0^2\left(I_1 + I_2\right)$$

$$+\left[p^2 + (m_{02} + m_{01})^2 + (\widetilde{m}_1 - \widetilde{m}_2)^2\right] I_{12}(p),$$

$$\Delta_{33}^{-1}(p) = M^2 + g_0^2\,(I_1 + I_2) + \frac{1}{2}\left(p^2 + 4m_{01}^2\right) I_{11}(p)$$

$$+\frac{1}{2}\left(p^2 + 4m_{02}^2\right) I_{22}(p),$$

$$\Delta_{44}^{-1}(p) = \Delta_{55}^{-1}(p) = M^2 + g_0^2\,(I_1 + I_3)$$

$$+\left[p^2 + (m_{03} + m_{01})^2 + (\widetilde{m}_1 - \widetilde{m}_3)^2\right] I_{13}(p),$$

$$\Delta_{66}^{-1}(p) = \Delta_{77}^{-1}(p) = M^2 + g_0^2\,(I_2 + I_3)$$

$$+\left[p^2 + (m_{03} + m_{02})^2 + (\widetilde{m}_2 - \widetilde{m}_3)^2\right] I_{23}(p), \qquad (6.59)$$

$$\Delta_{88}^{-1}(p) = M^2 + \frac{g_0^2}{3}\,(I_1 + I_2 + 4I_3) + \frac{1}{6}\left(p^2 + 4m_{01}^2\right) I_{11}(p)$$

$$+\frac{1}{6}\left(p^2 + 4m_{02}^2\right) I_{22}(p) + \frac{2}{3}\left(p^2 + 4m_{03}^2\right) I_{33}(p),$$

$$\Delta_{03}^{-1}(p) = g_0^2\sqrt{\frac{2}{3}}\,(I_1 - I_2) + \frac{1}{\sqrt{6}}\left(p^2 + 4m_{01}^2\right) I_{11}(p)$$

$$-\frac{1}{\sqrt{6}}\left(p^2 + 4m_{02}^2\right) I_{22}(p),$$

$$\Delta_{08}^{-1}(p) = \frac{g_0^2\sqrt{2}}{3}\,(I_1 + I_2 - 2I_3) + \frac{\sqrt{2}}{6}\left(p^2 + 4m_{01}^2\right) I_{11}(p)$$

$$+\frac{\sqrt{2}}{6}\left(p^2 + 4m_{02}^2\right) I_{22}(p) - \frac{\sqrt{2}}{3}\left(p^2 + 4m_{03}^2\right) I_{33}(p),$$

$$\sqrt{2}\Delta^{-1}_{38}(p) = \Delta^{-1}_{03}(p).$$

One can get from equation (6.58) the inverse propagators of pseudoscalar ($\widetilde{\Phi}'_a(x)$) mesons:

$$\Delta^{-1}_{\tilde{0}\tilde{0}}(p) = M^2 + g_0^2 \frac{2}{3}\left(I_1 + I_2 + I_3\right) + \frac{1}{3}\left(p^2 + 4\widetilde{m}_1^2\right) I_{11}(p)$$

$$+\frac{1}{3}\left(p^2 + 4\widetilde{m}_2^2\right) I_{22}(p) + \frac{1}{3}\left(p^2 + 4\widetilde{m}_3^2\right) I_{33}(p),$$

$$\Delta^{-1}_{\tilde{1}\tilde{1}}(p) = \Delta^{-1}_{\tilde{2}\tilde{2}}(p) = M^2 + g_0^2\left(I_1 + I_2\right)$$

$$+\left[p^2 + \left(m_{02} - m_{01}\right)^2 + \left(\widetilde{m}_1 + \widetilde{m}_2\right)^2\right] I_{11}(p),$$

$$\Delta^{-1}_{\tilde{3}\tilde{3}}(p) = M^2 + g_0^2\left(I_1 + I_2\right)$$

$$+\frac{1}{2}\left(p^2 + 4\widetilde{m}_1^2\right) I_{11}(p) + \frac{1}{2}\left(p^2 + 4\widetilde{m}_2^2\right) I_{22}(p),$$

$$\Delta^{-1}_{\tilde{4}\tilde{4}}(p) = \Delta^{-1}_{\tilde{5}\tilde{5}}(p) = M^2 + g_0^2\left(I_1 + I_3\right)$$

$$+\left[p^2 + \left(m_{03} - m_{01}\right)^2 + \left(\widetilde{m}_1 + \widetilde{m}_3\right)^2\right] I_{13}(p),$$

$$\Delta^{-1}_{\tilde{6}\tilde{6}}(p) = \Delta^{-1}_{\tilde{7}\tilde{7}}(p) = M^2 + g_0^2\left(I_2 + I_3\right)$$

$$+\left[p^2 + \left(m_{03} - m_{02}\right)^2 + \left(\widetilde{m}_2 + \widetilde{m}_3\right)^2\right] I_{23}(p), \qquad (6.60)$$

$$\Delta^{-1}_{\tilde{8}\tilde{8}}(p) = M^2 + \frac{g_0^2}{3}\left(I_1 + I_2 + 4I_3\right) + \frac{1}{6}\left(p^2 + 4\widetilde{m}_1^2\right) I_{11}(p)$$

$$+\frac{1}{6}\left(p^2+4\widetilde{m}_2^2\right)I_{22}(p)+\frac{2}{3}\left(p^2+4\widetilde{m}_3^2\right)I_{33}(p),$$

$$\Delta^{-1}_{\tilde{0}\tilde{3}}(p)=g_0^2\sqrt{\frac{2}{3}}\left(I_1-I_2\right)$$

$$+\frac{1}{\sqrt{6}}\left(p^2+4\widetilde{m}_1^2\right)I_{11}(p)-\frac{1}{\sqrt{6}}\left(p^2+4\widetilde{m}_2^2\right)I_{22}(p),$$

$$\Delta^{-1}_{\tilde{0}\tilde{8}}(p)=\frac{g_0^2\sqrt{2}}{3}\left(I_1+I_2-2I_3\right)+\frac{\sqrt{2}}{6}\left(p^2+4\widetilde{m}_1^2\right)I_{11}(p)$$

$$+\frac{\sqrt{2}}{6}\left(p^2+4\widetilde{m}_2^2\right)I_{22}(p)-\frac{\sqrt{2}}{3}\left(p^2+4\widetilde{m}_3^2\right)I_{33}(p),$$

$$\Delta^{-1}_{\tilde{0}\tilde{3}}(p)=\sqrt{2}\Delta^{-1}_{\tilde{3}\tilde{8}}(p).$$

Non-diagonal scalar-pseudoscalar elements of inverse propagators are given by

$$\Delta^{-1}_{\tilde{0}0}(p)=-\frac{4}{3}\left[m_{01}\widetilde{m}_1I_{11}(p)+m_{02}\widetilde{m}_2I_{22}(p)+m_{03}\widetilde{m}_3I_{33}(p)\right],$$

$$\Delta^{-1}_{8\tilde{0}}(p)=\Delta^{-1}_{0\tilde{8}}(p)=\frac{2\sqrt{2}}{3}\Big[2m_{03}\widetilde{m}_3I_{33}(p)$$

$$-m_{01}\widetilde{m}_1I_{11}(p)-m_{02}\widetilde{m}_2I_{22}(p)\Big], \qquad (6.61)$$

$$\Delta^{-1}_{3\tilde{0}}(p)=\Delta^{-1}_{0\tilde{3}}(p)=\sqrt{2}\Delta^{-1}_{8\tilde{3}}(p)=\sqrt{2}\Delta^{-1}_{3\tilde{8}}(p)$$

$$=2\sqrt{\frac{2}{3}}\left[m_{02}\widetilde{m}_2I_{22}(p)-m_{01}\widetilde{m}_1I_{11}(p)\right],$$

where the quadratic diverging integrals read

$$I_{ij}(p) = -\frac{ig_0^2 N_C}{4\pi^4} \int \frac{d^4k}{(k^2 + m_i^2)\left[(k-p)^2 + m_j^2\right]}$$

$$= \frac{g_0^2 N_C}{4\pi^2} \left[\ln\left(\frac{\Lambda^2}{m_i^2}\right) - 1 \right. \tag{6.62}$$

$$\left. - \int_0^1 dx \;\ln \frac{m_j^2 + x\left(m_i^2 - m_j^2\right) + p^2 x(1-x)}{m_i^2}\right],$$

and there is no summation in indexes i, j. Inverse propagators (6.59)-(6.61) define the spectrum of mass for the general case including CP violation.

6.1.6 The effective action and mass spectrum of mesons

Poles of the propagators (6.58) give the masses of mesons which can be estimated by numerical calculations. Here we make some evaluations of meson masses. From equations (6.59)-(6.61), one can find the effective action of the mesonic "free" fields

$$S_{free}[\Phi] = -\frac{1}{2}\int d^4x \left[(\partial_\mu \Phi_A(x))^2 + m^2_{AB}\phi_A(x)\phi_B(x)\right], \tag{6.63}$$

where $A = (a, \tilde{a})$, $\phi_{\tilde{a}} \equiv \widetilde{\phi}_a$. The eigenvalues of the symmetric mass matrix m^2_{AB} define the mass spectrum. To obtain the mass matrix, we renormalize fields $\widetilde{\phi}_a(x) = Z^{-1/2}\widetilde{\phi}'_a(x)$, $\phi_a(x) = Z^{-1/2}\phi'_a(x)$, and the constant $g^2 = Zg_0^2$, so that the variables $g\phi_a$, $g\widetilde{\phi}_a$ are the renormalization-invariant values. It follows from equation (6.62) that the renormalization constant can be defined as follows

$$Z^{-1} = \frac{g_0^2 N_C}{4\pi^2}\left[\ln\left(\frac{\Lambda^2}{m_1^2}\right) - 1\right]. \tag{6.64}$$

It is seen from equation (6.64) that the expansion in $g^2/4\pi^2$, corresponds to the $1/N_C$ expansion. We imply here that the cut-off Λ is chosen in such a way that $g^2/4\pi^2 < 1$. Using the gap equations (6.50), (6.51), in the leading order, we find from equations (6.59) the elements of the mass matrix for scalar mesons:

$$m_{00}^2 = g^2 \frac{2}{3} \left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} + \frac{m_s I_3}{m_s - m_{03}} \right)$$
$$+ \frac{4}{3} \left(m_{01}^2 + m_{02}^2 + m_{03}^2 \right),$$
$$m_{11}^2 = m_{22}^2 = g^2 \left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} \right)$$
$$+ (m_{02} + m_{01})^2 + (\widetilde{m}_1 - \widetilde{m}_2)^2,$$
$$m_{33}^2 = g^2 \left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} \right) + 2 \left(m_{01}^2 + m_{02}^2 \right),$$
$$m_{44}^2 = m_{55}^2 = g^2 \left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_s I_3}{m_s - m_{03}} \right)$$
$$+ (m_{03} + m_{01})^2 + (\widetilde{m}_1 - \widetilde{m}_3)^2,$$
$$m_{66}^2 = m_{77}^2 = g^2 \left(\frac{m_d I_2}{m_d - m_{02}} + \frac{m_s I_3}{m_s - m_{03}} \right)$$
$$+ (m_{03} + m_{02})^2 + (\widetilde{m}_2 - \widetilde{m}_3)^2, \tag{6.65}$$
$$m_{88}^2 = \frac{g^2}{3} \left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} + \frac{4 m_s I_3}{m_s - m_{03}} \right)$$
$$+ \frac{2}{3} \left(m_{01}^2 + m_{02}^2 + 4 m_{03}^2 \right),$$

$$m_{08}^2 = \frac{g^2\sqrt{2}}{3}\left(I_1 + I_2 - 2I_3\right)$$
$$+\frac{2\sqrt{2}}{3}\left(m_{01}^2 + m_{02}^2 - 2m_{03}^2\right),$$
$$m_{03}^2 = \sqrt{2}m_{38}^2 = g^2\sqrt{\frac{2}{3}}\left(I_1 - I_2\right) + 2\sqrt{\frac{2}{3}}\left(m_{01}^2 - m_{02}^2\right).$$

One can obtain from equations (6.60) the elements of the mass matrix for pseudoscalar mesons:

$$m_{\tilde{0}\tilde{0}}^2 = g^2\frac{2}{3}\left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} + \frac{m_s I_3}{m_s - m_{03}}\right)$$
$$+\frac{4}{3}\left(\widetilde{m}_1^2 + \widetilde{m}_2^2 + \widetilde{m}_3^2\right),$$
$$m_{\tilde{0}\tilde{8}}^2 = \frac{g^2\sqrt{2}}{3}\left(I_1 + I_2 - 2I_3\right) + \frac{2\sqrt{2}}{3}\left(\widetilde{m}_1^2 + \widetilde{m}_2^2 - 2\widetilde{m}_3^2\right),$$
$$m_{\tilde{0}\tilde{3}}^2 = \sqrt{2}m_{\tilde{3}\tilde{8}}^2 = g^2\sqrt{\frac{2}{3}}\left(I_1 - I_2\right) + 2\sqrt{\frac{2}{3}}\left(\widetilde{m}_1^2 - \widetilde{m}_2^2\right),$$
$$m_{\tilde{1}\tilde{1}}^2 = m_{\tilde{2}\tilde{2}}^2 = g^2\left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}}\right)$$
$$+\left(m_{02} - m_{01}\right)^2 + \left(\widetilde{m}_1 + \widetilde{m}_2\right)^2,$$
$$m_{\tilde{4}\tilde{4}}^2 = m_{\tilde{5}\tilde{5}}^2 = g^2\left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_s I_3}{m_s - m_{03}}\right)$$
$$+\left(m_{03} - m_{01}\right)^2 + \left(\widetilde{m}_1 + \widetilde{m}_3\right)^2, \qquad (6.66)$$

$$m^2_{\tilde{6}\tilde{6}} = m^2_{\tilde{7}\tilde{7}} = g^2\left(\frac{m_d I_2}{m_d - m_{02}} + \frac{m_s I_3}{m_s - m_{03}}\right)$$
$$+ (m_{03} - m_{02})^2 + (\widetilde{m}_2 + \widetilde{m}_3)^2 ,$$
$$m^2_{\tilde{3}\tilde{3}} = g^2\left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}}\right) + 2\left(\widetilde{m}_1^2 + \widetilde{m}_2^2\right),$$
$$m^2_{\tilde{8}\tilde{8}} = \frac{g^2}{3}\left(\frac{m_u I_1}{m_u - m_{01}} + \frac{m_d I_2}{m_d - m_{02}} + \frac{4 m_s I_3}{m_s - m_{03}}\right)$$
$$+\frac{2}{3}\left(\widetilde{m}_1^2 + \widetilde{m}_2^2 + 4\widetilde{m}_3^2\right)^2 .$$

Using equations (6.61), we find non-diagonal scalar-pseudoscalar elements of the mass matrix

$$m^2_{\tilde{0}0} = -\frac{4}{3}\left(m_{01}\widetilde{m}_1 + m_{02}\widetilde{m}_2 + m_{03}\widetilde{m}_3\right),$$

$$m^2_{8\tilde{0}} = m^2_{0\tilde{8}} = \frac{2\sqrt{2}}{3}\left(2m_{03}\widetilde{m}_3 - m_{01}\widetilde{m}_1 - m_{02}\widetilde{m}_2\right), \quad (6.67)$$
$$m^2_{3\tilde{0}} = m^2_{0\tilde{3}} = \sqrt{2}m^2_{8\tilde{3}} = \sqrt{2}m^2_{3\tilde{8}}$$
$$= 2\sqrt{\frac{2}{3}}\left(m_{02}\widetilde{m}_2 - m_{01}\widetilde{m}_1\right).$$

From equations (6.65)-(6.67) is seen the Goldstone nature of pseudoscalar mesons: if bare masses of quarks are zero, $\sigma_0 = \sigma_3 = \sigma_8 = 0$, $\widetilde{m}_j = 0$, all pseudoscalar mesons are massless. We recall that if $\widetilde{m}_j \neq 0$ $(j = 1, 2)$, gap equations require the chiral limit: $m_u = m_d = 0$. If there is no CP violation ($\widetilde{m}_j = 0$), one can consider the case $m_u \neq 0$, $m_d \neq 0$ to have non-zero pion masses. It follows from equations (6.67) that there is mixing of the scalar $\phi_a(x)$ and pseudoscalar $\widetilde{\phi}_a(x)$ fields due to the CP-violating condensates $\widetilde{m}_j$. The fields $\widetilde{\phi}_0$ and $\widetilde{\phi}_8$ are also mixed corresponding to $\eta - \eta'$ mixing. Pions, connected with the fields $\widetilde{\phi}_i(x)$ $(i = 1, 2, 3)$, acquire non-zero masses due to the presence of the CP-violating condensates even for zero current masses $m_u = m_d = 0$. At the same time in the case $\widetilde{m}_1 = \widetilde{m}_2$, $\widetilde{m}_3 = 0$ there is less contribution of CP violating condensates to the masses of scalar mesons.

To obtain the diagonal matrix m_{AB}, one can make the transformation of the rotation group for fields $\phi_a(x)$, $\widetilde{\phi}_a(x)$ $(a = 0, 3, 8)$. For the simple mixing of fields $\widetilde{\phi}_0(x)$, $\widetilde{\phi}_8(x)$, one obtains

$$
\begin{aligned}
\widetilde{\phi}'_0(x) &= \widetilde{\phi}_0(x) \cos\theta_P - \widetilde{\phi}_8(x) \sin\theta_P, \\
\widetilde{\phi}'_8(x) &= \widetilde{\phi}_0(x) \sin\theta_P + \widetilde{\phi}_8(x) \cos\theta_P,
\end{aligned}
\tag{6.68}
$$

where $\tan 2\theta_P = 2m^2_{\tilde{0}\tilde{8}}/(m^2_{\tilde{8}\tilde{8}} - m^2_{\tilde{0}\tilde{0}})$. The masses of bosonic fields $\widetilde{\phi}'_0(x)$, $\widetilde{\phi}'_8(x)$ become:

$$
m'^2_{\tilde{0}\tilde{0}} = m^2_{\tilde{0}\tilde{0}} \cos^2\theta_P + m^2_{\tilde{8}\tilde{8}} \sin^2\theta_P - m^2_{\tilde{0}\tilde{8}} \sin 2\theta_P,
$$

$$m'^2_{\tilde{8}\tilde{8}} = m^2_{\tilde{0}\tilde{0}} \sin^2\theta_P + m^2_{\tilde{8}\tilde{8}} \cos^2\theta_P + m^2_{\tilde{0}\tilde{8}} \sin 2\theta_P. \tag{6.69}$$

We consider the case $m_1 = m_2$ when the isotopic symmetry is conserved. It follows then from Eqs. (6.40) that this requires the vacuum expectation value $\sigma_3 = 0$. The relation $m_1 - m_u = -g(\sqrt{2}\sigma_0 + \sigma_8)/\sqrt{3}$ (after the renormalization of the constant g_0) is treated as the quark level version of the Goldberger–Treiman identity [451] with the pion decay constant $(\sqrt{2}\sigma_0 + \sigma_8)/\sqrt{3} = f_\pi = 93$ MeV. We imply here a very small possible contribution of CP violating condensates to the real masses of mesons. We use here the value of the constant [330] $g = 3.628$, so that the parameter of expansion is $g^2/4\pi^2 = 1/N_C = 1/3$. Using the freedom in the choice of the bare quark mass, we put $m_u = m_d = 5.3$ MeV. From the Goldberger–Treiman relation one obtains the constituent masses of the light quarks $m_1 = m_2 = 342.7$ MeV ($\widetilde{m}_j = 0$). Following from equation (6.64): the covariant cut-off Λ is given by $\Lambda = em_1 = 931.5$ MeV. To find the constituent mass of the s-quark, we find from the gap equations (6.50) the self-consistent relation $m_1(m_s - m_3)I_1 = m_3(m_u - m_1)I_3$. Setting the free parameter of the s-quark current mass $m_s = 166$ MeV, for a given cut-off, one obtains the dynamical strange quark mass: $m_3 = 570$ MeV. With the help of these masses and the cut-off Λ, we calculate from equations (6.66) the masses of π, K mesons and quark condensates

$$\begin{gathered} m_\pi = 139 \text{ MeV}, \quad m_K = 494 \text{ MeV}, \\ \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = m_1 I_1 = (-252 \text{ MeV})^3 , \\ \langle \bar{s}s \rangle = m_3 I_3 = (-268 \text{ MeV})^3 . \end{gathered} \tag{6.70}$$

Masses of K mesons are degenerated here as well as masses of pions. The masses and condensates (6.70) are agreed with the phenomenology. The pseudoscalar $\eta' - \eta$ mixing angle, evaluated from equations (6.68), is $\theta_P = -35°$. Masses of η, η' and their mixing angle are not described correctly here because we did not take into consideration the anomaly and the axial symmetry $U_A(1)$ is not broken. It is easy to verify that the Gell-Mann−Oakes−Renner [452] relation $f_\pi^2 m_\pi^2 = -2m_u \langle \overline{u}u \rangle$ is approximately valid.

From Eqs.(6.65) we obtain the elements of the mass matrix corresponding to nonets of scalar mesons

$$\begin{gathered} m_{00} = 938 \text{ MeV}, \quad m_{11} = m_{22} = m_{33} = 699 \text{ MeV}, \\ m_{44} = m_{55} = m_{66} = m_{77} = 1013 \text{ MeV}, \\ m_{88} = 1128 \text{ MeV}. \end{gathered} \tag{6.71}$$

The mixing angle of the ϕ_0 and ϕ_8 fields is $\theta_S = -35°$. Scalar and pseudoscalar fields are not mixed in the case (see equations (6.67)) when the equality $\widetilde{m}_j = 0$ is valid.

We do not identify here the scalar mesons $\phi_a(x)$ with nonets of known scalar mesons: $\sigma(560)$, $f_0(980)$, $\kappa(900)$, $a_0(980)$ due to their complicated nature: there are contributions of four-quark states and gluons in these mesons [453] - [455].

The model under consideration can describe CP violation in strong interactions. There is a contribution of CP violating condensates, $\widetilde{m}_j$, to constituent masses of u, d and s quarks and to masses of scalar and pseudoscalar mesons. If the current masses of quarks equal zero, and CP-violating condensate $\widetilde{m}_j = 0$, all pseudoscalar mesons π, K, η, η' become massless Goldstone bosons. Masses of all K-mesons are degenerated in the case $m_{01} = m_{02}$. In this model, the appearance of CP-violating condensates leads to the chiral limit: $m_u = m_d = 0$. In the particular case $\widetilde{m}_j = 0$, when there is no CP violation, the model gives reasonable dynamical quark masses, masses of π, K mesons and quark condensates. At the same time η and η' mesons cannot be described correctly in the framework of the model considered because the $U_A(1)$ symmetry is not broken. To take into consideration the $U_A(1)$-anomaly, one may generalize the model by including the determinant of six-quark interactions (due to instantons) violating $U_A(1)$ symmetry [217].

6.1.7 Effective chiral Lagrangians of the $SU(3)$-group and low-energy physics of hadrons

It is known that the QCD Lagrangian at zero current masses of the quarks has a chiral $U(3)_L \otimes U(3)_R$ symmetry. Due to Adler−Bell−Jackiw anomalies at the quantum level the original symmetry is spontaneous breaking, which leads to a lack of parity doublets in the physical spectrum:

$$U(3)_L \otimes U(3)_R / U(1)_A \to SU(3)_L \otimes SU(3)_R \otimes U(1)_V \\ \to SU(3)_V \otimes U(1)_V . \tag{6.72}$$

The nonet of pseudo-scalar mesons: π^0, $\pi^\pm$, K^0, $K^\pm$, η, η' occurs as Goldstone particles. The chiral field, performing a non-linear realization of the group $SU(3)_L \otimes SU(3)_R$, is parameterized as follows:

$$U(x) = \exp\left(\frac{2i}{F_\pi}\Phi(x)\right), \\ \Phi(x) = \Phi_a(x)\lambda^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{6.73}$$

where $F_\pi = \sqrt{2} f_\pi = 135$ MeV.

The effective action is invariant under the symmetry group $SU(3)_L \otimes SU(3)_R$, and can be composed of two parts, a normal part of Γ_N and anomalous Γ_{WZ} (calibrated Wess−Zumino action) [456] - [461]:

$$\Gamma = \Gamma_N + \Gamma_{WS}. \tag{6.74}$$

The action (6.74) includes U, (6.73), as well as the octet of vector and axial mesons

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}(\rho^0 + \omega) & \rho^+ & K^{*+} \\ \rho^- & \frac{1}{\sqrt{2}}(-\rho^0 + \omega) & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \varphi \end{pmatrix}, \tag{6.75}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}}(A_1^0 + D) & A_1^+ & Q^+ \\ A_1^- & \frac{1}{\sqrt{2}}(-A_1^0 + D) & Q^0 \\ Q^- & \bar{Q}^0 & -E \end{pmatrix}. \tag{6.76}$$

It is assumed that the vector and axial-vector mesons interact with other fields with the same constant g.

The action of Γ_N is invariant under the substitution $U \to U^{-1}$, and meets the symmetry $\Phi \to -\Phi$ (see [155]). This part of the action describes a process in which the number of pseudoscalar mesons N_P is even. Besides the action Γ_N is invariant under the reflection operation of spatial coordinates P_0: $P_0\Phi(\mathbf{x}, t)P_0^{-1} = -\Phi(-\mathbf{x}, t)$. Witten has noted [155], that there are processes that retain only the combination of $P = P_0(-1)^{N_P}$, which is the actual symmetry of strong interactions. For example, the real, pure strong, processes

$$K^+K^-, \quad \eta\pi^0, \quad K^0\bar{K}^0 \to \pi^+\pi^-\pi^0 \tag{6.77}$$

break apart symmetry P_0 and $(-1)^{N_P}$, but keep P. Processes of this type are due to anomalies of QCD and are described by the actions Γ_{WZ}.

In this approach (see [457], [458]) vector and axial-vector mesons V_μ, A_μ appear as gauge fields and define strength

$$F_{\mu\nu}^{L,R} = \partial_\mu A_\nu^{L,R} - \partial_\nu A_\mu^{L,R} - ig[A_\mu^{L,R}, A_\nu^{L,R}], \tag{6.78}$$

where

$$A_\mu^L = \frac{1}{2}\left(V_\mu + A_\mu\right), \qquad A_\mu^R = \frac{1}{2}\left(V_\mu - A_\mu\right).$$

In this case, the divergence of the axial current is not equal to zero ([134]). This situation is true and this approach is called the theory of massive Yang−Mills fields (MYM) [459]. There is another approach based on the hidden symmetry [460], [462]. It should be noted that the effective action approach MYM was obtained in [95] based on the NJL Lagrangian (1.9). At the same time, as shown in Sec. 5.1, NJL Lagrangians are obtained directly from QCD as its low energy limit. Thus, the considered approach is justified in the framework of QCD. In this case, however, the parameters included in the Γ_N, are not independent, but are calculated accurately. In addition, in the normal part of the action, Γ_N, other terms with higher derivatives of the chiral field appear. However, to describe the actual processes of strong interactions in the energy range ~ 1 GeV it is sufficient consideration of the normal part of the action Γ_N.

In this approach, the spontaneous breaking of the global $SU(3)_L \otimes SU(3)_R$ symmetry is violated by the mass matrix

$$M = \frac{2}{3}\left(m_K^2 + \frac{1}{2}m_\pi^2\right) I_3 - \frac{2}{\sqrt{3}}\left(m_K^2 - m_\pi^2\right)\lambda^8 \tag{6.79}$$
$$= \mathrm{diag}\left(m_\pi^2,\ m_\pi^2,\ m_K^2\right),$$

where m_π is the mass of π-meson, and m_K is the mass of K-meson.

In contrast to the normal part of the Γ_N, the anomalous part of Γ_{WZ} contains no adjustable parameters and has great predictive power. It describes a large number of processes with unnatural parity P. Such action Γ_{WZ} describes the processes of decay $\pi^0 \to 2\gamma$, $\gamma^* \to 3\pi$ in accordance with the results of current algebra [155]. This approach naturally turns the KSFR-relation [463], [464] and improved Weinberg's ratio. Interaction Lagrangians of pseudoscalar, vector and axial-vector fields $\mathcal{L}_{V\Phi\Phi}$, $\mathcal{L}_{AV\Phi}$, $\mathcal{L}_{AAV}$, $\mathcal{L}_{VVV}$, $\mathcal{L}_{A\Phi\Phi\Phi}$, $\mathcal{L}_{AA\Phi\Phi}$, $\mathcal{L}_{VV\Phi\Phi}$, $\mathcal{L}_{AAA\Phi}$, $\mathcal{L}_{AAAA}$, $\mathcal{L}_{VVVV}$, $\mathcal{L}_{\Phi\Phi\Phi\Phi}$, $\mathcal{L}_{A\Phi VV}$, $\mathcal{L}_{AAVV}$ also follow from the action Γ_N. The expressions for the $\mathcal{L}_{V\Phi\Phi}$, $\mathcal{L}_{AV\Phi}$ were found in [457], [458] to calculate the decay widths of vector and axial-vector mesons. Using the $\mathcal{L}_{V\Phi\Phi}$, $\mathcal{L}_{AV\Phi}$ to describe the decays $\rho^0 \to \pi^+\pi^-$, $A^+ \to \pi^0\rho^+$ and comparing the respective widths of the experimental data [458] allow the fixing of the parameters of the model. After fixing the parameters, a large number of processes described by these Lagrangians can be calculated. Thus, the considered approach based on symmetries has great predictive power. At energies ≥ 1 GeV the vector dominance hypothesis can be used, according to which the neutral vector bosons can be transformed into an electromagnetic field [323], [465], [466]. This statement is well tested, and the corresponding interaction Lagrangian has the form

$$\mathcal{L}_{em} = \frac{\sqrt{2}e}{g} A_\mu \left(m_\rho^2 \rho_\mu^0 + \frac{1}{3} m_\omega^2 \omega_\mu - \frac{\sqrt{2}}{3} m_\Phi^2 \Phi_\mu \right) + \mathcal{O}(A_\mu^2), \tag{6.80}$$

where $m_\rho = 770$ MeV, $m_\omega = 783$ MeV, $m_\Phi = 1020$ MeV, $g = 9.6$, $e^2 = 4\pi\alpha$, $\alpha = 1/137$.

We now turn to the anomalous part of the effective action of Γ_{WZ}. On the basis of the Wess–Zumino action

Γ_{WZ} there were calculated the widths of the radiative decays of vector mesons [458], [459], which are in reasonable agreement with the experimental data. It should be taken into account the mixing η and η'-mesons [458]:

$$\eta = \eta_P \cos\theta + \eta'_P \sin\theta, \quad \eta = -\eta_P \sin\theta + \eta'_P \cos\theta, \tag{6.81}$$

at $\theta = -18^\circ$, and ω, φ-mesons

$$\omega_\mu = \omega_\mu^P - \xi\varphi_\mu^P, \quad \varphi_\mu = \xi\omega_\mu^P + \varphi_\mu^P, \tag{6.82}$$

where $|\xi| = 0.076$. To obtain correct decay widths of η, η', and φ-mesons we must also take into account the relations: $F_K/F_\pi = 1.28$, $F_\eta/F_\pi = 1.28$, $F_{\eta'}/F_\pi = 1$.

It should be noted that the anomalous vertexes compared to normal vertexes contain the additional factor of $1/(16\pi^2)$. So, for example, the contribution from the two anomalous vertexes will be suppressed compared to the contribution from the normal part of the Γ_N.

The effective Lagrangian should be applied at the tree level. This follows from the fact that it is obtained from the accounting loops. If we consider the additional loop diagrams, they will only lead to a renormalization of the corresponding quantities. The dependence on the kinematic variables is negligible near threshold processes that hold the effective Lagrangian.

Note also that the anomalous vertexes necessarily contain at least one pseudoscalar meson. This follows from the form of the Γ_{WZ} [458]. This implies the following predictions about the lack of decay channels:

$$D \to \rho\rho, \quad D \to \omega\omega, \quad A_1 \to \rho\omega,$$
$$E^{\pm} \to K^*\rho, \quad E^{\pm} \to K^*\omega, \quad E^{\pm} \to K^{*\pm}\varphi. \tag{6.83}$$

Most hadronic processes that can be observed in e^+e^--collisions have the energy thresholds of $1 - 3$ GeV. The widths of the purely hadronic processes $\omega \to 3\pi$ b $K^* \to K\pi\pi$, are described by the anomalous part of the Lagrangian $\mathcal{L}_{WZ}$, in good agreement with the experimental data [458].

We note here also that the effective action obtained by the calibration of the Wess$-$Zumino$-$Witten Lagrangian under the group $U(1)$ does not violate low energy theorems [95], [467].

It should be borne in mind that the use of the vector-dominance does not affect low-energy theorems for the processes $\gamma \to \pi\pi\pi$, $\pi \to \gamma\gamma$ [468].

Thus, the advantage of this approach is to use the symmetry of the strong interactions. The accuracy obtained ($\sim$ 15 per cent) is the same as the accuracy of $SU(3)$-symmetry. To improve the accuracy of the calculations one can include in a normal part of the Lagrangian additional terms with higher powers of derivatives of chiral fields. In this case, however, there will be additional unknown parameters.

As noted above, the quantum loop corrections to the ECL should not be taken into account. In the papers of Gasser and Leutwyller [100] - [102] the Lagrangian was constructed which does not lead to divergences of loop corrections, i.e. it is renormalizable. In this case, the loop corrections to the normal parts have linear combinations of terms of $\mathcal{L}_N$, and the amendment to the $\mathcal{L}_{WZ}$ gives a proportional change to all terms of $\mathcal{L}_{WZ}$, due to the symmetry requirement.

This approach does not consider the question of the excited states of the higher spin octet of mesons (Regge partners). These issues require further development.

The verification of the predictions of this approach to low-energy meson physics experiments is feasible in e^+e^--colliders.

Figure 6.1 shows processes that can be expected on the basis of $\mathcal{L}_{WZ}$.

Thus, predictions of ECL can be checked in the processes shown in Figure 6.1, where in the final state pseudoscalar, vector and axial-vector mesons can be created. Readers can find the detailed descriptions of $\mathcal{L}_N$, $\mathcal{L}_{WZ}$ Lagrangians and corresponding processes in [445].

6.1.8 The decay $\pi^+ \to \mu^+ \bar{\nu}_\mu$ in the field of the plane electromagnetic wave by taking into account pion polarizabilities

Here we consider the pion decay in the field of the plane electromagnetic wave with allowance for the polarizability of the pion. The interest in this problem is primarily due to the possibility of creating real high-power laser beams [469]. Accounting for the internal structure of the pion requires the introduction of coefficients of electromagnetic polarizabilities. It is convenient to use the exact solutions of wave equations for the pion in external electromagnetic fields found in Appendices A, B, C in assessing the impact of electromagnetic polarizabilities on the observed characteristics of the decay.

The amplitude of the process of π-meson: $\pi^+ \to \mu^+ + \bar{\nu}_\mu$, written in the contact approximation of the standard model, has the form [32]

$$A = \frac{G_F f_\pi \cos\theta_c}{\sqrt{2}} \int d^4x \; \psi_\pi P_\pi^\alpha \tilde{\psi}_\mu \gamma_\alpha (1+\gamma_5)\psi_\nu, \qquad (6.84)$$

where $P_\pi^\alpha = i\partial_\alpha - eA_\alpha$ is the kinetic momentum of the pion, G_F is the Fermi constant, θ_c is the Kabibbo angle, and ψ_π, ψ_ν are wave functions of the pion and neutrino, respectively.

Since the process is seen here in the electromagnetic wave, then we take the exact solution (7.31) of the corresponding equation, where the normalization constant $C = 1/\sqrt{2p_0}$. In this case, the quasi-momentum of the pion is given by

$$q_\mu = p_\mu - \frac{a_\nu^2}{4} k_\mu \left[\frac{e^2}{(kp)} - \frac{(\alpha+\beta)(kp)}{m_\pi} \right] (1+\delta_0^2). \quad (6.85)$$

(See the notation in Appendix C). Summing up the polarizations of the particles after the relevant calculations, we obtain the following expression for the total pion decay probability (W^l is for a linearly polarized wave and W^c is for a circularly polarized wave):

$$W^l(x,\chi) = \frac{G_F^2 f_\pi^2 m_\mu^2 m_\pi^2 \cos^2\theta_c}{16\pi^2 q_0}$$

$$\times \sum_{s>s_0} \int_0^{2\pi} d\varphi \int_0^{u_s} \frac{du}{(1+u)^2} \Big\{ \Delta A_0^2(s,\gamma,\tau)$$

$$+x^2 u \Big[A_1^2(s,\gamma,\tau) - A_0(s,\gamma,\tau) A_2(s,\gamma,\tau)$$

$$\times \left(1 - \frac{(\alpha+\beta)\chi^2 m_\pi^3 u}{e^2 x^2} \right) \Big] \Big\},$$

$$W^c(x,\chi) = \frac{G_F^2 f_\pi^2 m_\mu^2 m_\pi^2 \cos^2\theta_c}{8\pi^2 q_0}$$

$$\times \sum_{s>s_0} \int_0^{u_s} \frac{du}{(1+u)^2} \left\{ \Delta J_s^2(z) + \frac{x^2 u}{2} \left[J_{s-1}^2(z) \right.\right. \tag{6.86}$$

$$\left. + J_{s+1}^2(z) - 2J_s^2(z) \right] + \varepsilon \left(\frac{x^2 us}{z} - \frac{\chi z}{x} \right)$$

$$\left. \times J_s(z) \left[J_{s-1}(z) - J_{s+1}(z) \right] - \frac{(\alpha+\beta)\chi^2 m_\pi^3}{e^2} J_s^2(z) \right\},$$

where $u = (kl)/(kq')$, l is neutrino momentum, q' is quasi-momentum of μ-meson,

$$\varepsilon = \frac{i}{a^2(kp)} \varepsilon_{\mu\nu\lambda\sigma} a_{1\mu} a_{2\nu} k_\lambda p_\sigma = \pm 1$$

for right ($\varepsilon = +1$) and left ($\varepsilon = -1$) polarization of the wave ($a^2 \equiv a_\mu a_\mu$),

$$x = \frac{ea}{m_\pi}, \qquad \chi = \frac{e\sqrt{(F_{\mu\nu} q_\nu)^2}}{m_\pi^3},$$

$$A_n(s,\gamma,\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \cos^n \lambda$$

$$\times \exp\left[i(\gamma \sin\lambda - \tau \sin 2\lambda - s\lambda) \right],$$

$$\tau = \frac{x^3 u}{8\chi} - \frac{(\alpha+\beta)\chi m_\pi^3 x}{8e^2}, \quad \gamma = z\cos\varphi,$$

$$z = \frac{x^2 m_\mu}{\chi m_\pi}\sqrt{u(u_s - u)\left[1 + \frac{x^2}{2}(1+\delta_0^2)\right]}, \tag{6.87}$$

$$u_s = \frac{x\Delta + 2s\chi - (\alpha+\beta)\chi^2 m_\pi^3 x/e^2}{x(1-\Delta+x^2)}$$

$$s_0 = \frac{x[-\Delta + (\alpha+\beta)\chi^2 m^3/(xe^2)]}{2\chi}, \quad \Delta = 1 - \left(\frac{m_\mu}{m_\pi}\right)^2,$$

φ is the angle between planes $(\mathbf{k}, \mathbf{q}')$ and $(\mathbf{k}, \mathbf{a})$ in the center of mass system, m_μ is the mass of μ-meson, and $J_s(z)$ is the Bessel function.

The analytical expression (6.87) can be evaluated by numerical integration.

Note that the characteristic value, part of (6.87) and related to the polarizability of the pion is

$$\zeta = \frac{(\alpha+\beta)\chi^2 m_\pi^3}{e^2}. \tag{6.88}$$

For real-life high-power lasers the parameter is $\chi \simeq 1$ [469]. Then, using the experimental value for the polarizabilities of charged pions (6.19), we obtain a value of $\zeta \simeq 10^{-3}$. Thus, the effects due to the internal structure of the pion field to the currently existing lasers are small. However, increasing the beam intensity when the parameter $\chi > 1$, we can expect the appearance of the structure of the pions in corresponding experiments.

Note that for $x \gg 1$, following from (6.87) are the expressions for the probability of the pion decay in the constant crossed fields. This case is interesting in that it has substantially reduced the decay of ultra-relativistic particles in a constant electromagnetic field.

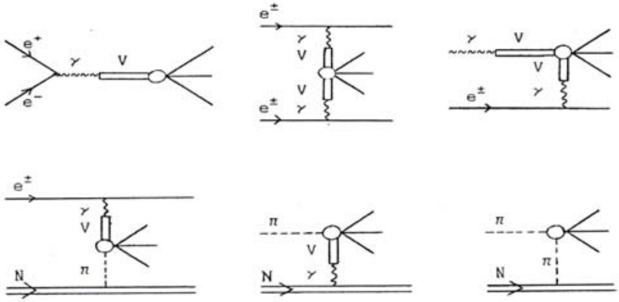

Figure 6.1: Processes that can be expected on the basis of $\mathcal{L}_{WZ}$.

Chapter 7

7.1 Conclusion

So, the four-fermion models are used in the theory of electro-weak and strong interactions in the theory of elementary particles.

Entered in the locally $SU(2)\otimes U(1)$-invariant field model, the four-fermion interaction for the initial leptons and quarks, and then taking into account the dynamical symmetry breaking provides a similar outcome to the Higgs procedure of the mass generation of leptons e, μ, τ, which leaves neutrinos ν_e, ν_μ, ν_τ massless and gives masses to all quarks and $W^\pm$, Z-bosons. The analogue of the scalar Higgs field arises naturally as a unified field of collective fermion-antifermion excitations, whose mass is approximately $2m_t$. Detection and measurement of the mass of the scalar particle may shed light on the validity of this approach as a possible alternative to the standard theory of electro-weak interactions. It should be noted that despite the discovery of the 125 GeV scalar boson at LHC, developed and discussed in this monograph, the direction associated with a composite Higgs boson continues to grow (see [476], [477]).

It turned out that the need for a four-quark interaction leads also to considering the low-energy limit of QCD. This breaks the chiral symmetry as Goldstone bosons are the π-mesons. This conclusion is consistent with the approach based on the concept of the instanton vacuum. The four-quark interaction does not provide a description of the confinement, but covers the intermediate region of the strong interaction, which lies between the regions of confinement and the asymptotic freedom of quarks. This is consistent with the studies of other authors. The resulting mass of the pion, σ-meson and quark condensates matches the values which follow from phenomenology. The calculated values of the pion electromagnetic polarizabilities derived from the chiral Lagrangian, are consistent with experimental data.

The effective chiral Lagrangian includes a normal part with two arbitrary parameters and the anomalous part of the Wess−Zumino allows the description of a large number of processes, and, in particular, the processes in e^+e^--collisions, and in this sense has a sufficiently great predictive force.

Exact solutions of the wave equation for the pions in external electromagnetic fields, taking into account their polarizabilities made it possible to find the probability of the decay $\pi^+ \to \mu^+\bar{\nu}_\mu$ in the field of a plane electromagnetic wave. Although the polarizability leads to a weak effect in supercritical fields it must be taken into account.

Chapter 8

8.1 Appendix

8.1.1 Pion in the field of the uniform external magnetic field

Consider the motion of the pion in a constant and homogeneous magnetic field $\mathbf{H} = (0, 0, H)$. The vector potential can be chosen in the following way:

$$A_1 = \frac{1}{2}Hx_2, \quad A_2 = -\frac{1}{2}Hx_1, \quad A_0 = A_3 = 0. \tag{8.1}$$

From the Lagrangian (6.14), we obtain the equation of motion for the pion field

$$D_\mu^2\varphi - D_\mu \left[(D_\nu\varphi) K_{\mu\nu} \right] - m_\pi^2 + \frac{m_\pi d}{2} F_{\mu\nu}^2 \varphi = 0, \tag{8.2}$$

where $\varphi \equiv \pi^-$,

$$K_{\mu\nu} = \frac{a+b}{m_\pi} F_{\mu\alpha} F_{\nu\alpha} - \frac{b}{2m_\pi} F_{\alpha\beta}^2 \delta_{\mu\nu}.$$

Using a variational procedure [328], we find the four-current density

$$j_\mu = \frac{e}{2im_\pi} \left[\varphi^* \partial_\nu \varphi - \varphi \partial_\nu \varphi^* - 2ieA_\nu \varphi\varphi^* \right] \left(\delta_{\mu\nu} - K_{\mu\nu} \right). \tag{8.3}$$

It is easy to see that the current continuity equation $\partial_\mu j_\mu(x) = 0$ holds. If $a = b = 0$ ($K_{\mu\nu} = 0$), expression (7.3) coincides with the corresponding expression for the point-like scalar particle [470].

Note that the kinetic term for the pion (see (6.10) and (6.14)) will have the standard form, if you do go to the renormalization of physical field $\bar{\varphi}$:

$$\bar{\varphi} = \left(1 + \frac{b}{2m_\pi} F_{\mu\nu}^2\right)^{1/2} \varphi. \tag{8.4}$$

Given the link $E_k = iF_{k4}$, $H_k = \frac{1}{2}\varepsilon_{kmn}F_{mn}$ ($\varepsilon_{123} = 1$), we find that the charge density for the physical fields is

$$\rho_{phys} = \frac{ie}{2m_\pi}\left[\bar{\varphi}^*\partial_0\bar{\varphi} - \bar{\varphi}\partial_0\bar{\varphi}^* + 2ieA_0\bar{\varphi}\bar{\varphi}^* \right. \\ \left. + \left(\bar{\varphi}^*\partial_n\bar{\varphi} - \bar{\varphi}\partial_n\bar{\varphi}^* - 2ieA_n\bar{\varphi}\bar{\varphi}^*\right)\frac{(\alpha+\beta)}{m_\pi}\varepsilon_{nkm}E_k H_m\right]. \tag{8.5}$$

Formula (7.5) can be used to normalize the wave equation solutions. The wave equation (7.2) with (7.1) can be written as

$$\left[A\left(\partial_1^2 + \partial_2^2\right) + B\left(\partial_3^2 - \partial_0^2\right) - eHAJ_3 \right. \tag{8.6}$$

$$-\frac{e^2H^2}{4}A(x_1^2+x_2^2)-m_\pi^2 D\Big]\varphi=0,$$

where

$$A=1-\frac{aH^2}{m_\pi},\quad B=1+\frac{bH^2}{m_\pi},\quad D=1-\frac{dH^2}{m_\pi},$$
$$J_3=[\mathbf{r},\mathbf{p}]_3=i\left(x_2\partial_1-x_1\partial_2\right),$$

and J_3 is the operator of projection of the angular momentum on the x_3 axis.

Introducing cylindrical coordinates, $x_1=r\cos\phi$, $x_2=r\sin\phi$, and assuming

$$\varphi(x)=\frac{\exp(il\phi)}{\sqrt{2\pi}}\frac{\exp i(k_3x_3-k_0t)}{\sqrt{\Lambda}}\psi(r), \tag{8.7}$$

where l is the orbital quantum number, k_0 is the particle energy, and k_3 is the projection of the momentum on the x_3 axis, from (7.6), one obtains the equation

$$\begin{gathered}A\left(\psi''-\frac{l^2}{r^2}+\frac{1}{r}\psi'\right)-eHlA\psi-\frac{e^2H^2}{4}Ar^2\psi\\ +B(k_0^2-k_3^2)\psi-m_\pi^2D\psi=0.\end{gathered}\tag{8.8}$$

The requirement that the final solutions of (7.8) are finite with $r\to\infty$ (see [470], [471]) gives the expression for the energy (squared) of the particle

$$k_0^2=k_3^2+B^{-1}\left[m_\pi^2D+eHA(2n+1)\right], \tag{8.9}$$

where $n = s + l$ is the principal quantum number.

In the case of a point-like scalar particle we must put $a = b = d = 0$ $(A = B = D = 1)$. Then from (7.9) we obtain the well-known [470] expression for the energy of a point-like scalar particle in a magnetic field $k_0^2 = k_3^2 + m_\pi^2 + eH(2n+1)$.

We now consider the smallness of coefficients a, b, d. Then, assuming $aH^2/m_\pi \ll 1$, $bH^2/m_\pi \ll 1$, $dH^2/m_\pi \ll 1$, we use the approximation of $B^{-1} \simeq 1 - bH^2/m_\pi$.

In this case, using (6.18), the expression for the energy (7.9) takes the form

$$k_0^2 = k_3^2 + m_\pi^2 + eH(2n+1) - \beta H^2 m_\pi - \frac{eH^3}{m_\pi}(2n+1)(\alpha+\beta). \tag{8.10}$$

The last two terms in (7.10) give the correction to the energy of the pion due to its internal structure. For strong external fields (e.g. in neutron stars) it is to be taken into account. The solution of equation (7.8) has the same form as for a point-like particle, and it is expressed by the Laguerre polynomials.

8.1.2 Pion in the field of the uniform external electric field

The vector potential in this case can be chosen in the form

$$A_\mu(x) = (0, 0, 0, -if(x_3)) \,, \tag{8.11}$$

which implies that $\mathbf{E} = (0,\ 0,\ 0,\ f'(x_3))$.

The solution of equation (7.2) is sought in the form

$$\varphi(x) = \eta(x_3) \exp\left[i\left(p_1 x_1 + p_2 x_2 + p_4 x_4\right)\right]. \tag{8.12}$$

Substituting (7.11) and (7.12) in (7.2), we obtain

$$\begin{aligned} &\left[\left(1 + \frac{a}{m_\pi} f'^2\right)\left(\partial_3^2 + 2ep_0 f + e^2 f^2\right)\right. \\ &\quad + \frac{2a}{m_\pi} f' f'' \partial_3 + \Delta + \frac{a}{m_\pi} f'^2 p_0^2 \\ &\left. + f'^2 \left(\frac{b}{m_\pi}\left(p_1^2 + p_2^2\right) - m_\pi d\right)\right] \eta(x_3) = 0, \end{aligned} \tag{8.13}$$

where $\Delta = p_0^2 - p_1^2 - p_2^2 - m_\pi^2$. Consider the case of a constant and uniform electric field when

$$f(x_3) = C x_3. \tag{8.14}$$

Here $C = E_3 = E$. Equation (7.13) takes the form

$$\left(\partial_3^2 + a_1 + b_1 x_3 + c_1 x_3^2\right) \eta(x_3) = 0, \tag{8.15}$$

$$a_1 = \frac{\frac{b}{m_\pi}E^2(p_1^2 + p_2^2) - m_\pi dE^2 + \Delta + \frac{a}{m_\pi}E^2 p_0^2}{1 + \frac{a}{m_\pi}E^2}, \tag{8.16}$$

$$b_1 = 2ep_0E, \qquad c_1 = e^2E^2.$$

If we introduce a new variable $z = \gamma_1 + \beta_1 x_3$, then (7.15) can be reduced to the form of Weber's equation [472]

$$\left(\partial_z^2 - \frac{1}{4}z^2 + \varepsilon\right)\eta(z) = 0, \tag{8.17}$$

where the constants are defined as follows:

$$\varepsilon = \frac{a_1}{\beta_1^2} - \frac{b_1\gamma_1}{\beta_1^3} + \frac{c_1\gamma_1^2}{\beta_1^4}, \quad \frac{\gamma_1}{\beta_1} = \frac{b_1}{2c_1}, \quad \frac{c_1}{\beta_1^4} = -\frac{1}{4}. \tag{8.18}$$

The solution of (7.17) is well known [472] and expressed in terms of parabolic cylinder functions $D_\nu(z)$:

$$\eta(z) = ND_\nu(z), \tag{8.19}$$

where $\nu = \varepsilon - 1/2$, N is the normalization constant. Using (7.16) and (7.18) we find the value

$$\varepsilon = \frac{i}{2eE}\left[\frac{(p_1^2 + p_2^2)\left(1 - \frac{b}{m_\pi}E^2\right) + m_\pi^2 + m_\pi dE^2}{1 + \frac{a}{m_\pi}E^2}\right]. \tag{8.20}$$

Given the smallness of a, b, d, using (6.18), we transform (7.20) to the form

$$\varepsilon = \frac{i}{2eE}\left[m_\pi^2 + p_1^2 + p_2^2 - (p_1^2 + p_2^2)\frac{\alpha+\beta}{m_\pi}E^2 - m_\pi \alpha E^2\right]. \tag{8.21}$$

8.1.3 Pion in the field of the electromagnetic plane wave

These results are consistent with those that have been received on the basis of an exact solution of the Duffin−Kemmer equation for a scalar particle in the field of the electromagnetic wave [473].

Equation (7.2) for this case can be written as

$$\Big[\partial_\mu^2 - 2ieA_\mu\partial_\mu - e^2A_\mu^2 - m_\pi^2 + m_\pi d(H^2 - E^2)$$

$$-K_{\mu\nu}\Big(\partial_\mu\partial_\nu - 2ieA_\mu\partial_\nu \tag{8.22}$$

$$-ie(\partial_\mu A_\nu) - e^2A_\mu A_\nu\Big) - (\partial_\mu K_{\mu\nu})(\partial_\nu - ieA_\nu)\Big]\varphi(x) = 0.$$

It is assumed that the Lorentz condition $\partial_\mu A_\mu = 0$ holds. The vector potential of the field will be given as

$$A_\mu(x) = a_\mu^i f_i(\vartheta) \qquad (i = 1, 2). \tag{8.23}$$

Here $\vartheta = k_\mu x_\mu = \mathbf{kx} - k_0x_0$, $k_\mu^2 = 0$, $k_\mu a_\mu^i = 0$, $a_\mu^i a_\mu^j = \delta_{ij}$, $f_i(\vartheta)$ are arbitrary scalar functions.

We find an expression for the symmetric tensor

$$K_{\mu\nu} = \frac{\alpha + \beta}{m_\pi}(f_i')^2 k_\mu k_\nu, \tag{8.24}$$

where $f_i' = df_i/d\vartheta$.

The solution of (7.22) can be obtained in the form [474], [475] (see also [325])

$$\varphi(x) = \chi(\vartheta)\exp\left[i(px - \epsilon)\right]. \tag{8.25}$$

Here, p_μ is the four-momentum of a free particle, i.e. $p^2 = -m_\pi^2$, $px = p_\mu x_\mu = \mathbf{px} - p_0 x_0$,

$$\epsilon = \int_0^\vartheta d\alpha g(\alpha).$$

Substituting (7.25) into (7.22), taking into account (7.23), (7.24) yields

$$\begin{gathered} 2i(pk)\chi' + 2(pk)g\chi + 2e(p_\mu a_\mu^i)f_i\chi \\ -e^2 f_i^2\chi + \frac{\alpha+\beta}{m_\pi}(f_i')^2(pk)^2\chi = 0. \end{gathered} \tag{8.26}$$

Since there is an arbitrary function $g(\vartheta)$, we can set

$$g(\vartheta) = \frac{e^2 f_i^2 - 2e(p_\mu a_\mu^i)f_i}{2(pk)} - \frac{\alpha+\beta}{2m_\pi}(f_i')^2(pk). \tag{8.27}$$

Then the solution of (7.26) in view of (7.27) is

$$\chi = C = const. \tag{8.28}$$

Finally the function (7.25) can be written as

$$\varphi(x) = C\exp\Big\{i\Big[px - \int_0^{kx} d\alpha\Big(\frac{e^2 f_i^2(\alpha) - 2e(p_\mu a_\mu^i)f_i(\alpha)}{2(pk)} - \frac{\alpha+\beta}{2m_\pi}(f_i'(\alpha))^2(pk)\Big)\Big]\Big\}. \tag{8.29}$$

The constant C is chosen from the normalization condition (see [470]).

We now consider various special cases of the potential (7.23).

a). If you put $f_i(\vartheta) = q\vartheta$ $(i = 1)$, the electric and magnetic fields are

$$\mathbf{E} = (k_0\mathbf{a} - a_0\mathbf{k})q, \qquad \mathbf{H} = [\mathbf{k}, \mathbf{a}]q,$$

and $\mathbf{E} \perp \mathbf{H}$. Thus, we come to the case of constant crossed electromagnetic fields. The solution (7.29) for crossed constant fields becomes

$$\varphi(x) = C\exp\Big\{i\Big[px - \frac{e^2(kx)^3q^2}{6(pk)} + \frac{e(p_\mu a_\mu^i)(kx)^2 q}{2(pk)} + \frac{\alpha+\beta}{2m_\pi}(pk)(kx)q^2\Big]\Big\}. \tag{8.30}$$

b). Let us consider the case

$$A_\mu(\vartheta) = a_{1\mu}\cos\vartheta + \delta_0 a_{2\mu}\sin\vartheta,$$

where $\vartheta = kx$, $(a_{1\mu})^2 = (a_{2\mu})^2 = (a_\mu)^2$, $-1 \leq \delta_0 \leq 1$, δ_0 is a degree of polarization of the wave. When $\delta_0 = 0$ we have the linear polarization of an electromagnetic wave, and $\delta_0 = \pm 1$ corresponds to a circular polarization of an electromagnetic wave. In this case, after the integration in (7.29), we obtain

$$\varphi(x) = C \exp\Bigg\{ i \Bigg[px - \frac{e^2 (a_\mu)^2}{2(pk)} \left[\frac{kx}{2}(1+\delta_0^2) + \frac{\sin(2kx)}{4} \right.$$

$$\left. \times (1-\delta_0^2) \right] + \frac{e}{(pk)} \left[p_\mu (a_{1\mu} \sin(kx) - a_{2\mu} \delta_0 \cos(kx)) \right] \quad (8.31)$$

$$+ \frac{\alpha + \beta}{2m_\pi}(pk)(a_\mu)^2 \left[\frac{kx}{2}(1+\delta_0^2) - \frac{\sin(2kx)}{4}(1-\delta_0^2) \right] \Bigg] \Bigg\}.$$

Readers can find solutions of equations for a pion in the external electromagnetic fields of other configurations in [478]-[481].

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