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IMF Working Paper

A Requiem for the Fiscal Theory of the Price Level

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IMF Working Paper

Monetary Capital Markets Department

A Requiem for the Fiscal Theory of the Price Level*

Prepared by Roger E.A. Farmer[†] and Pawel Zabczyk[‡]

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Abstract

The Fiscal Theory of the Price Level (FTPL) is the claim that, in a popular class of theoretical models, the price level is sometimes determined by fiscal policy rather than monetary policy. The models where this claim has been established assume that all decisions are made by an infinitely-lived representative agent. We present an alternative, arguably more realistic model, populated by sixty-two generations of people. We calibrate our model to an income profile from U.S. data and we show that the FTPL breaks down. In our model, the price level and the real interest rate are indeterminate, even when monetary and fiscal policy are both active. Our findings challenge established views about what constitutes a good combination of fiscal and monetary policies.

JEL Classification Numbers: E31, E58, H62

Keywords: fiscal policy, monetary policy, fiscal theory of the price level, price level determination, overlapping generations model, dynamic efficiency

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1. INTRODUCTION

New Keynesian economists classify monetary and fiscal policies as *active* or *passive*. If the central bank raises the interest rate more than one-for-one in response to inflation, monetary policy is said to be *active*. If the central bank raises the interest rate less than one-for-one in response to inflation, monetary policy is said to be *passive*. If the fiscal authority borrows to finance an arbitrary path of expenditure and taxes, fiscal policy is said to be *active*. If the fiscal authority adjusts its expenditures and the tax rate to ensure fiscal solvency for all possible paths of the real interest rate, fiscal policy is said to be *passive*.¹

In the New Keynesian (NK) model, uniqueness of equilibrium requires either that monetary policy is active and fiscal policy is passive, or that monetary policy is passive and fiscal policy is active. The fact that a passive monetary policy in combination with an active fiscal policy leads to a unique price level is referred to as the fiscal theory of the price level (FTPL).

After the 2008 global financial crisis, the Federal Reserve System, the European Central Bank, and the Bank of England, maintained passive interest rate policies with a constant nominal interest rate peg for more than a decade. When the central bank pegs the interest rate, standard economic theory predicts that the price level is indeterminate (Sargent and Wallace, 1975; McCallum, 1981). Advocates of the FTPL claim that the price level is nevertheless uniquely determined even when the interest rate is pegged. Their argument rests on a reinterpretation of the government's debt accumulation equation which is seen, not as a budget constraint, but, as a debt valuation equation. The fact that interest rates have been characterized by a peg at or close to zero for a decade or more in many developed economies is one reason why the FTPL has received considerable attention in recent years.²

The FTPL argument for price-level uniqueness is typically made in the context of an infinitely-lived, representative agent (RA) model.³ In this paper, we study the implications of the FTPL in the Overlapping Generations (OLG) model. We provide an algorithm to construct the steady-state equilibria of a class of T-period OLG models and we study the local properties of dynamic equilibria around the steady-states of these models. The OLG model is known to possess an indeterminate steady-state equilibrium where money has value (Samuelson, 1958;

¹ The terminology of active and passive policies originated with Leeper (1991).

² The FTPL began with papers by Leeper (1991), Sims (1994), and Woodford (1994), and it has been promoted by Cochrane (2005, 2018, 2019) as a theory of price level determination. Prominent critics of the FTPL include Buiter (2002, 2017), McCallum (1999, 2001, 2003) and Niepelt (2004).

³ Bennett McCallum and Edward Nelson, (2005) survey this literature.

Gale, 1973). Although indeterminacy of a monetary steady-state equilibrium could theoretically be of arbitrary degree, previous known examples of this phenomenon have been widely considered to be unrealistic.

We provide an example of a sixty-two generation OLG model, where people are endowed with an income profile calibrated to fit U.S. data. We show that our model possesses a steady-state equilibrium where money has positive value. When monetary policy is passive and fiscal policy is active, this steady state equilibrium displays two degrees of indeterminacy. It is not just the initial price level that remains unexplained by economic fundamentals; our sixty-two generation calibrated model also fails to uniquely determine the real interest rate. If we assume that monetary and fiscal policy are *both* active, the degree of indeterminacy is reduced from two to one. In this case a linear combination of the real interest rate and the price level is determinate, but the model is still unable to uniquely determine the price level as a function of economic fundamentals.

Our model has implications for the constraints placed on the fiscal authority by the dictates of fiscal sustainability. It is often argued that the government should spend less or tax more in a recession. Proponents of this argument claim that failure to adjust the deficit in response to endogenous fluctuations in tax revenues could lead to an exploding and unsustainable level of government debt. In our calibrated example, a fiscal policy that does not respond to endogenous fluctuations in debt can be pursued without leading to an exploding debt level.⁴

In Section 2 we review related literature and in Section 3 we explain the idea behind the FTPL. In Sections 4 and 5 we establish the conditions for price-level and interest rate determinacy in the T -generation OLG model and in Section 6 we develop a calibrated sixty-two generation example. Our example contains a locally stable and dynamically efficient equilibrium in which government debt converges to a positive number for arbitrary values of the initial price level and the initial real interest rate.⁵

⁴ Within limits that depend on the parameters of the model, a fiscal authority can increase its expenditure without raising taxes. This is possible in the OLG model because a higher level of initial debt redistributes resources to current generations from future generations and is associated with a different equilibrium path of real interest rates. In Farmer and Zabczyk (2018) we showed how this mechanism operates in a tractable two-generation example. This paper extends our previous analysis to a more realistically calibrated sixty-two generation model.

⁵ An equilibrium in which the interest rate is greater than or equal to the growth rate is said to be dynamically efficient. A steady-state equilibrium in which the interest rate is less than the growth rate is said to be dynamically inefficient. In the representative agent model, dynamically inefficient equilibria cannot exist because they imply that the wealth of the representative agent is unbounded. But in the OLG model, dynamically inefficient equilibria are common, and, in many examples of OLG models, dynamic inefficiency is associated with indeterminacy. In our sixty-two generation model however, there exist indeterminate equilibria that are dynamically efficient.

Sections 4 and 5 study the case of an active fiscal policy and a passive monetary policy. This corresponds to the OLG model with fiat money. In Section 7 we check the robustness of our results to alternative calibrations for the discount rate and the coefficient of risk aversion and in Section 8 we explain how our results would change if monetary policy and fiscal policy were both passive or both active. Finally, in Section 9 we present a brief conclusion.

2. THE RELATIONSHIP OF OUR WORK TO PREVIOUS LITERATURE

Our argument rests on the fact that in OLG models, the local dynamics of prices close to any given steady-state may be indeterminate of arbitrary degree. This result was established by Kehoe and Levine (1985) in the context of a two-period-lived model with multiple goods and multiple agents.⁶ To map the Kehoe-Levine results into the literature on the FTPL, we study instead, a model with a single good in each period and a single type of person who lives for T -periods and discounts the future at a constant rate. We also depart from Kehoe-Levine's assumption that the outside asset is a fixed stock of money. We assume instead that the government issues an interest-bearing nominal liability and we allow the government to create more of this liability to pay interest on outstanding debt as well as to purchase goods.

Previous examples of indeterminate equilibria in overlapping generations models have been restricted to two-generation or three-generation models in which indeterminacy was associated with the absence of money (Samuelson, 1958), negative money (Gale, 1973; Farmer, 1986), or unrealistic calibrations generally considered to be empirically irrelevant (Azariadis, 1981; Farmer and Woodford, 1997; Kehoe and Levine, 1983). Our paper is the first to provide a long-lived example of an economy where the endowment profile is matched to U.S. micro data, and where there exists an indeterminate steady-state equilibrium when both monetary and fiscal policy are active. In contrast to Benhabib et al. (2001), our results are not driven by global or non-linear dynamics since we explicitly restrict attention to the properties of a linearized system of equilibrium conditions around a monetary steady state.

An extensive literature uses OLG models to answer questions of political economy (Auerbach, 2003; Auerbach and Kotlikoff, 1987; Ríos-Rull, 1996), but almost all of it either ignores the possibility of indeterminacy or calibrates models to explicitly rule it out. We suspect this oversight may be related, at least in part, to an influential paper by Aiyagari (1985) who showed that, under some assumptions, the OLG model becomes close to the RA model as the length of

⁶ It follows from the results of Balasko and Shell (1981) that the two-period assumption is unrestrictive as long as there are multiple agents and multiple goods.

life increases. Even though average life expectancy has increased significantly we consider the limiting case of immortality a theoretical curiosity.⁷

A number of authors have studied the wealth distribution in stochastic overlapping generations models with and without complete securities markets. One branch of this literature includes papers by Ríos-Rull and Quadrini (1997), Castañeda et al. (2003), and Kubler and Schmedders (2011). Our work is peripherally related to that literature, but we study a different question. A second branch of this literature investigates the existence of non-fundamental equilibria in overlapping generations models. A non-exhaustive list of papers, following Tirole’s seminal contribution (Tirole, 1985), would include Martin and Ventura (2011, 2012), Miao and Wang (2012), Miao et al. (2012), and Azariadis et al. (2015). The model we develop in this paper contains what Tirole would call ‘bubbly equilibria’, but unlike these papers, we develop our argument in the context of a complete markets overlapping generations model without the credit constraints introduced by these authors.

Bassetto and Cui (2018) revisit the implications of the FTPL in models in which assets differ in characteristics because of risk, or because debt provides liquidity services. In contrast to their work, our results do not rely on dynamic inefficiency, risk premia or liquidity effects. We show that the price level is indeterminate in a model where the steady state equilibrium is dynamically efficient and where there are no frictions or rigidities of any kind other than the natural assumption that people are born and die at different dates.

Eggertsson et al. (2019) study steady-state equilibria in a fifty-six generation overlapping generations model with sticky prices. They use their model to discuss the idea that a negative real interest rate may be inconsistent with full employment, a concept that they refer to as ‘secular stagnation’. In one section of their paper, Eggertsson et al. study the transition path from one steady-state equilibrium to another. Their solution method assumes that this transition path is unique. This is an assumption that is called into question by the analysis in our paper.

3. THE FISCAL THEORY OF THE PRICE LEVEL

In this section we outline the idea behind the Fiscal Theory of the Price Level and we explain why it fails to hold in the overlapping generations model.

⁷ Importantly, the Aiyagari (1985) result requires the endowments of agents to be bounded away from zero. Pietro Reichlin (1992) has shown that, when one drops that assumption, OLG models display very rich behaviours even when people live potentially forever, but face a probability of death each period as in Blanchard (1985).

3.1. The Debt Accumulation Equation.

The government purchases g_t units of a consumption good, which it finances with dollar-denominated pure discount bonds and lump-sum taxes, τ_t . Let B_t be the quantity of pure-discount bonds each of which promises to pay one dollar at date $t + 1$ and let Q_t be the date t dollar price of a discount bond. Further, let p_t be the date t dollar price of a consumption good. Using these definitions, government debt accumulation is represented by the following equation,

$$Q_t B_t + p_t \tau_t = B_{t-1} + p_t g_t.$$

Define i_t to be the net nominal interest rate from period t to period $t + 1$, and let Π_{t+1} , be the gross inflation rate. These variables are given by,

$$i_t \equiv \frac{1}{Q_t} - 1 \quad \text{and,} \quad \Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}.$$

Further, let

$$b_t \equiv \frac{B_{t-1}}{p_t},$$

be the real value of government debt maturing in period t and define the *real primary deficit* as

$$d_t \equiv g_t - \tau_t,$$

where the negative of d_t is the *real primary surplus*. Let R_{t+1} represent the gross real return from t to $t + 1$, which from the Fisher-parity condition equals

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}. \tag{1}$$

We can combine these definitions to rewrite the government budget equation in purely real terms

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{2}$$

Although Equation (2) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

Advocates of the FTPL argue that Equation (2) is not a budget equation in the usual sense; it is a debt valuation equation. To understand their argument, let Q_t^k ,

$$Q_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad Q_t^t = 1,$$

be the relative price at date t of a commodity for delivery at date k . Now, iterate Equation (2) forwards to write the current real value of debt outstanding as the present value of all future surpluses,

$$\frac{B_0}{p_1} = - \sum_{t=1}^{\infty} Q_1^t d_t + \lim_{T \rightarrow \infty} Q_1^T b_T. \quad (3)$$

If the government were to be treated in the same way as other agents, Equation (3) would act as a constraint on feasible paths for the sequence of surpluses, $-\{d_t\}_{t=1}^{\infty}$, that would be required to hold for all paths of $\{Q_1^t\}_{t=1}^{\infty}$ and all initial price levels, p_1 . In New-Keynesian models, in which the central bank follows a passive monetary policy, the initial price level would be indeterminate if the government were constrained to balance its budget in this way. Advocates of the FTPL argue that the government should be treated differently from other agents in a general equilibrium model. When monetary policy is passive, Equation (3) should, they claim, be treated as a debt valuation equation that determines the value of p_1 as a function of the *specific* path of primary surpluses $-\{d_t\}_{t=1}^{\infty}$ chosen by the treasury. All initial price levels, other than the specific value of p_1 that satisfies Equation (3), are infeasible since they lead to paths of government debt that eventually become unbounded.

In contrast, in the example we construct in Section 6, equilibrium real interest rates are not pinned down uniquely. It follows that Equation (3) may hold for more than one value of the price level p_1 . Our example implies that the logic behind the FTPL cannot be extended to the overlapping generations model and it suggests that indeterminacy may be more prevalent in realistically calibrated models than previously believed.

4. EQUILIBRIA IN THE T-GENERATION OVERLAPPING GENERATIONS MODEL

Sections 4 and 5 contain our theoretical results. The reader who is interested in the practical application of our work is invited to skip ahead to Section 6, where we provide a sixty-two generation model that illustrates our key findings.

We study a T-generation overlapping generations model in which the government issues a nominal liability, B_t , which it uses to finance a real budget deficit, d_t , and to pay interest rate on outstanding debt at a constant rate, \bar{i} . The inflation rate can be recovered from the

equilibria of our model through the Fisher-parity condition, which implies that,

$$\frac{p_{t+1}}{p_t} = \frac{1 + \bar{i}}{R_{t+1}}, \quad t = 1, \dots, \infty.$$

The fact that d_t is not responsive to variations in the value of outstanding debt implies that fiscal policy is active. The fact that the interest rate is constant implies that monetary policy is passive.

To characterize equilibria we derive two sets of equations. The first set, which we refer to as *generic equations*, characterizes market clearing in period $T - 1$ and in all later periods. The second set, which we refer to as *non-generic equations*, characterizes market clearing in the first $T - 2$ periods. In period 1, the model is populated by the members of a young *generic generation* who live for T periods and by the members of $T - 1$ *non-generic generations* with horizons that vary from 1 period to $T - 1$ periods.

The non-generic generations are people born before the first period of the model. For example, in a three-generation OLG model there is a generic young person who lives for three periods, a non-generic middle-aged person who lives for two periods and a non-generic old person who lives for one period. The T -generation model generalizes this concept and, in the T -generation overlapping generations model, there are $T - 1$ non-generic generations. The existence of non-generic generations gives rise to a set of non-generic equations that characterize equilibrium in the first $T - 2$ periods. The non-generic generations begin life with dollar-denominated financial assets or liabilities. The behaviour of these people is constrained by their initial wealth positions, which influences the market clearing equations in periods 1 through $T - 2$.

$T - 1$ is the first period in which all market participants are generic. For periods $T - 1$ and later we characterize equilibrium as the solution to a difference equation in a vector of economic variables that includes $2(T - 2)$ interest rates and the real value of government debt. The non-generic equations in periods 1 to $T - 2$ provide the initial conditions for this difference equation and the issue of determinacy comes down to the question: Do the non-generic equilibrium conditions in periods 1 through $T - 2$ provide enough initial conditions to determine a unique convergent path for the economic variables, close to the steady state?

4.1. The Generic Consumer's Problem.

We refer to a person born in period t as *generation t* and we use a superscript on a variable to denote generation and a subscript to denote calendar time. For example, c_t^t is consumption

of generation t in period τ . We assume that after-tax endowments, denoted by $\tilde{\omega}_{\tau-t+1}$, are independent of calendar time and we index them by age. If τ is generation and t is calendar time, then age is related to t and τ by the identity $age \equiv \tau - t + 1$.

Generation t has a utility function defined over consumption in periods t through $t + T - 1$ and the members of generation t solve the problem

$$\max_{\{c_t^t, \dots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t),$$

such that

$$\sum_{k=t}^{t+T-1} Q_t^k (c_k^t - \tilde{w}_{k-t+1}) \leq 0.$$

The solution to this problem is characterized by a set of T excess demand functions, one for each period of life

$$\tilde{x}_k^t(R_{t+1}, \dots, R_{t+T-2}, R_{t+T-1}) \equiv \hat{c}_k^t - \tilde{w}_{k-t+1}, \quad \text{for } k \in \{t, \dots, t + T - 1\}.$$

We provide an explicit solution to this problem for the case of Constant Elasticity of Substitution Preferences (CES), in Appendix A, Section A.1.

To smooth their consumption, the members of generation t trade in the asset markets by buying or selling one-period dollar-denominated government bonds. Corresponding to the T excess demand functions there are $T - 1$ savings functions $\{s_k^t\}, k \in \{t, \dots, t + T - 2\}$ which are computed recursively from the excess demand functions using the following expressions,

$$\begin{aligned} s_t^t &= -\tilde{x}_t^t, \\ s_{k+1}^t &= R_{k+1} s_k^t - \tilde{x}_{k+1}^t, \quad k = t, \dots, t + T - 3. \end{aligned} \tag{4}$$

There are $T - 1$, rather than T of these savings functions because we assume no bequest motive and hence the optimal amount to save in the T 'th period of life is zero. We express the dependence of the savings functions on the sequence of interest factors by the notation,

$$s_k^t(R_{t+1}, \dots, R_{t+T-2}, R_{t+T-1}), \quad k = t, \dots, t + T - 2. \tag{5}$$

4.2. The Non-generic Consumers' Problem.

We index the non-generic generations by their date of birth, which we measure by an index $1 - j$ where j runs from 1 to $T - 1$. The financial wealth of generation $1 - j$ is represented by a real number λ_{1-j} , which may be positive or negative and which represents the share, owned

by generation $1 - j$, of the period 1 dollar-valued government debt.

$$\nu_{1-j} \equiv \lambda_{1-j} \frac{B_0}{p_1}.$$

We denote initial government debt by a positive number, B_0 , and we refer to the vector of shares as $\mathbb{L} \equiv \{\lambda_0, \lambda_{-1}, \dots, \lambda_{2-T}\}$. These shares sum to 1 as a consequence of asset market clearing.⁸

The non-generic generations solve the problem,

$$\max_{\{c_1^{1-j}, \dots, c_{1-j+T-1}^{1-j}\}} U^{1-j} \left(c_1^{1-j}, c_2^{1-j}, \dots, c_{1-j+T-1}^{1-j} \right),$$

such that

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left(c_k^{1-j} - \tilde{w}_{k-(1-j)+1} \right) \leq \lambda_{1-j} b_1,$$

where $k - (1 - j) + 1$ is the age of a member of generation $1 - j$ in period k . The solution to this problem is characterized by a set of excess-demand functions, one for each generation and each period. In Appendix A, Section A.2, we derive explicit functional forms for these excess-demand functions for the case of CES preferences.

In Section 4.3 we use the generic asset demand functions to characterize equilibria as solutions to a vector valued difference equation in a vector of economic variables. These variables include $2(T - 2)$ real interest rates and the value of government debt. In Section 4.4, we use the non-generic asset demand functions to derive a set of initial conditions to this equation.

4.3. The Generic Equilibrium Equations.

There are two components to the difference equation that characterizes equilibrium sequences. The first component is an expression for asset market equilibrium. The second component is the government's debt-accumulation equation.

We characterize asset market clearing by defining a function $f(\cdot)$ that we equate to government borrowing in period t ,

$$f(R_{t-T+3}, R_{t-T+4}, \dots, R_{t+T-2}, R_{t+T-1}) = R_{t+1}^{-1} b_{t+1}. \quad (6)$$

$f(\cdot)$ is the sum of the savings functions, defined in Equation (5), of generations 1 to $T - 1$. This is equated to the term $R_{t+1}^{-1} b_{t+1}$ which represents the government borrowing requirement

⁸ We require $B_0 \neq 0$ for the shares λ to be well-defined. See Niepelt (2004) for a discussion of a conceptual issue associated with non-zero values of B_0 .

at date t . Government borrowing is financed by issuing discount bonds b_{t+1} that sell for price R_{t+1}^{-1} in units of current consumption.

The second component to the equilibrium equation is government asset accumulation, which we reproduce below,

$$b_{t+1} = R_{t+1}(b_t + d_t). \quad (2)$$

4.4. The Non-generic Equilibrium Equations.

Asset market equilibrium in periods 1 through $T-2$ is characterized by a family of functions, $g_t^T(\cdot)$, one family for each value of T . These functions are different at each date t , because the asset demand functions of the non-generic generations depend on the initial wealth distribution and the initial price level as well as on real interest rates. To cut down on notation, we have suppressed the dependence of $g_t^T(\cdot)$ on \mathbb{L} .

To understand the structure of non-generic asset demand equations it helps to build intuition by considering the case of $T = 4$. This is the simplest example where we must keep track of the debt-accumulation equations in the $T - 2$ initial periods. For the case of $T = 4$, there are two non-generic asset demand equations and one non-generic debt accumulation equation. These equations are described by the expression

$$G^4(Z_0^4) \equiv \begin{bmatrix} g_1^4(R_2, R_3, R_4, b_1) & - & (b_1 + d) \\ g_2^4(R_2, R_3, R_4, R_5, b_1) & - & (b_2 + d) \\ b_2 & - & R_2(b_1 + d) \end{bmatrix} = 0,$$

where the vector Z_0^4 , is a set of variables that are determined, in equilibrium, by asset market clearing and government asset accumulation in the initial $T - 2$ periods of the model. For the case of $T = 4$, the equation $G^4(\cdot) = 0$ places 3 restrictions on the 6 elements of Z_0^4 .

$$Z_0^4 \equiv [R_5, R_4, R_3, R_2, b_2, b_1]^\top \equiv [X_{T-2}, Y_{T-2}]^\top,$$

where Y_{T-2} is the vector

$$Y_{T-2} \equiv [b_1]^\top,$$

and X_{T-2} is a vector of initial conditions to the vector-valued difference equation characterized by equations (6) and (2). This example can be generalized. For the T -generation model, Z_0^T contains $3T - 6$ elements, equal to the union of the terms in the over-braces of the following

expression,

$$Z_0^T \equiv \left[\overbrace{R_{2T-3}, R_{2T-4}, \dots, R_2, b_{T-2}}^{2T-3}, \overbrace{b_{T-3}, \dots, b_2, b_1}^{T-3} \right]^\top \equiv [X_{T-2}, Y_{T-2}]^\top.$$

The function $G(\cdot)$ contains $2T - 5$ rows,

$$G^T(Z_0^T) \equiv \begin{bmatrix} g_1^T(R_2, \dots, R_{1+T-2}, R_{1+T-1}, b_1) & - & (b_1 + d) \\ \vdots & & \vdots \\ g_{T-2}^T(R_2, \dots, R_{2T-4}, R_{2T-3}, b_1) & - & (b_{T-2} + d) \\ & b_2 & - & R_2(b_1 + d) \\ & \vdots & & \vdots \\ & b_{T-2} & - & R_{T-1}(b_{T-3} + d) \end{bmatrix} = 0, \quad (7)$$

and the equation $G^T(\cdot) = 0$ places $2T - 5$ restrictions on the $3T - 6$ elements of the unknown vector Z_0^T . Subtracting the number of these restrictions from the number of initial variables leaves $T - 1$ non-predetermined elements of X_{T-2} . In Section 5 we describe how the non-generic equilibrium equations may be combined with the assumption that equilibrium sequences must remain bounded to characterize the determinacy properties of equilibria.

5. THE DETERMINACY PROPERTIES OF EQUILIBRIA

Blanchard and Kahn (1980) derived a set of conditions on linear rational expectations models that guarantee the uniqueness of a solution. In this section, we derive these conditions for our model in the neighbourhood of a steady-state equilibrium.

5.1. Steady State Equilibria.

A *steady-state equilibrium* is a non-negative real number \bar{R} and a (possibly negative) real number \bar{b} that solve the equations,

$$f(\bar{R}, \bar{R}, \dots, \bar{R}) = \bar{b} + d, \quad \bar{b}(1 - \bar{R}) = \bar{R}d.$$

This system of non-linear equations has at least two solutions and in the important special case when $d = 0$, one of these solutions is given by $\bar{R} = 1$ and the others are solutions to the equation $f(\bar{R}, \bar{R}, \dots, \bar{R}) = 0$. Following Gale (1973) we refer to the first of these solutions as the *golden-rule* and to the others as *autarkic*.

5.2. Local Dynamic Equilibria.

Let $\{\bar{R}, \bar{b}\}$ be a steady state equilibrium and let

$$\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b},$$

represent deviations of b_t and R_t from their steady state values. Define a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \dots, R_{t-T+4}, b_t]^\top,$$

and a function $F(\cdot)$,

$$F(X_t, X_{t-1}) \equiv \begin{bmatrix} f(R_{t-T+3}, R_{t-T+4}, \dots, R_{t+T-2}, R_{t+T-1}) - b_t + d_t \\ b_t - R_t(b_{t-1} + d_{t-1}) \end{bmatrix},$$

and let J_1 and J_2 represent the partial derivatives of this function with respect to X_t and X_{t-1} .

Using these definitions, consider the following matrix expressions,

$$J_1 X_t \equiv \begin{array}{c} \overbrace{\hspace{10em}}^{J_1} \\ \left[\begin{array}{cccccccc} f_{t+T-1} & f_{t+T-2} & \cdots & f_{t+1} & \cdots & f_{t-T+5} & f_{t-T+4} & -1 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array} \right] \overbrace{\hspace{1em}}^{\tilde{X}_t} \left[\begin{array}{c} \tilde{R}_{t+T-1} \\ \tilde{R}_{t+T-2} \\ \vdots \\ \tilde{R}_{t+1} \\ \vdots \\ \tilde{R}_{t-T+5} \\ \tilde{R}_{t-T+4} \\ \tilde{b}_t \end{array} \right], \end{array}$$

$$J_2 X_{t-1} \equiv \begin{array}{c} \overbrace{\hspace{10em}}^{J_2} \\ \left[\begin{array}{cccccc} 0 & \cdots & 0 & \cdots & 0 & -f_{t-T+3} & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \ddots & \vdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & \bar{b} + d & \cdots & 0 & 0 & \bar{R} \end{array} \right] \overbrace{\hspace{1em}}^{\tilde{X}_{t-1}} \left[\begin{array}{c} \tilde{R}_{t+T-2} \\ \tilde{R}_{t+T-3} \\ \vdots \\ \tilde{R}_t \\ \vdots \\ \tilde{R}_{t-T+4} \\ \tilde{R}_{t-T+3} \\ \tilde{b}_{t-1} \end{array} \right], \end{array}$$

where f_k is the partial derivative of the function f with respect to R_k evaluated at the steady state $\{\bar{R}, \bar{b}\}$. Using this notation, the local dynamics of equilibrium sequences close to the steady state can be approximated as solutions to the linear difference equation

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad t = T - 1, \dots \quad (8)$$

with initial condition

$$\tilde{X}_{T-2} = \bar{X}_{T-2}. \quad (9)$$

The local stability of these equations depends on the eigenvalues of the matrix

$$J \equiv J_1^{-1} J_2.$$

If one or more roots of the matrix J is outside the unit circle, then there is no guarantee that sequences of interest factors and government debt generated by Equation (8) will remain bounded. To ensure stability, we must choose initial conditions that place \tilde{X}_{T-2} in the linear subspace associated with the stable eigenvalues of J . In Appendix B, we derive the conditions that guarantee local boundedness of equilibrium sequences and we combine them with the non-generic market clearing equations to find a set of conditions under which an equilibrium is determinate.

Our results can be summarized in the following proposition.⁹

Proposition 1 (Blanchard-Kahn). *Let K denote the number of eigenvalues of J with modulus greater than 1.*

- *If $K > T - 1$ there are no bounded sequences that satisfy the equilibrium conditions in the neighbourhood of \bar{X} . In this case equilibrium does not exist.*
- *If $K = T - 1$ there is a unique bounded sequence that satisfies the equilibrium equations. Further, this sequence converges asymptotically to the steady state (\bar{R}, \bar{b}) . In this case the steady state equilibrium (\bar{R}, \bar{b}) is determinate.*
- *If $K \in \{0, \dots, T - 2\}$ there is a $T - 1 - K$ dimensional subspace of initial conditions that satisfy the equilibrium equations. All of these initial conditions are associated with sequences that converge asymptotically to the steady state (\bar{R}, \bar{b}) . In this case the steady state equilibrium (\bar{R}, \bar{b}) is indeterminate with degree of indeterminacy equal to $T - 1 - K$.*

⁹ An updated solution method for rational expectations models that allows either J_1 or J_2 or both to be singular can be found in Sims (2001). Solution methods for models with indeterminacy are provided in Farmer et al. (2015).

Proof. See Appendix B.

In our model fiscal policy is active but monetary policy is passive. According to the FTPL this policy mix should lead to a unique initial price level. In Section 6 we provide an example of an economy with a steady-state equilibrium where money has value and where the FTPL fails to hold. In this example, it is not only the initial price level that is indeterminate; it is also the initial real interest rate. In Section 7 we show that our example is robust to alternative parameterizations and in Section 8 we discuss the behaviour of our model under alternative monetary and fiscal policy regimes.

6. A SIXTY-TWO GENERATION EXAMPLE

In this section we construct a sixty-two generation model where each generation begins its economic life at age 18 and in which a period corresponds to one year. Our sixty-two generation example is inspired by Kehoe and Levine (1983) who provide a three-generation OLG model with CES preferences, an endowment profile of [3, 15, 2] and utility weights on the three periods of life of [2, 2, 1]. Their example displays four steady-state equilibria, two of which display one degree of indeterminacy, one of which is determinate and one of which displays two degrees of indeterminacy.¹⁰

To see if the Kehoe-Levine example might provide a plausible explanation of a real-world economy, we calibrated the income profile of a representative generation to U.S. data and we modified the preference weights to allow for a constant discount rate. We found that the key features of their example are the hump-shaped income profile and a coefficient of relative risk aversion of 6 which is well within the bounds of calibrated models in the macro-finance literature.

In our sixty-two generation example, people maximize the utility function,

$$u(c_t^t, \dots, c_{t+61}^t) = \sum_{i=1}^{62} \beta^{i-1} \left(\frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right).$$

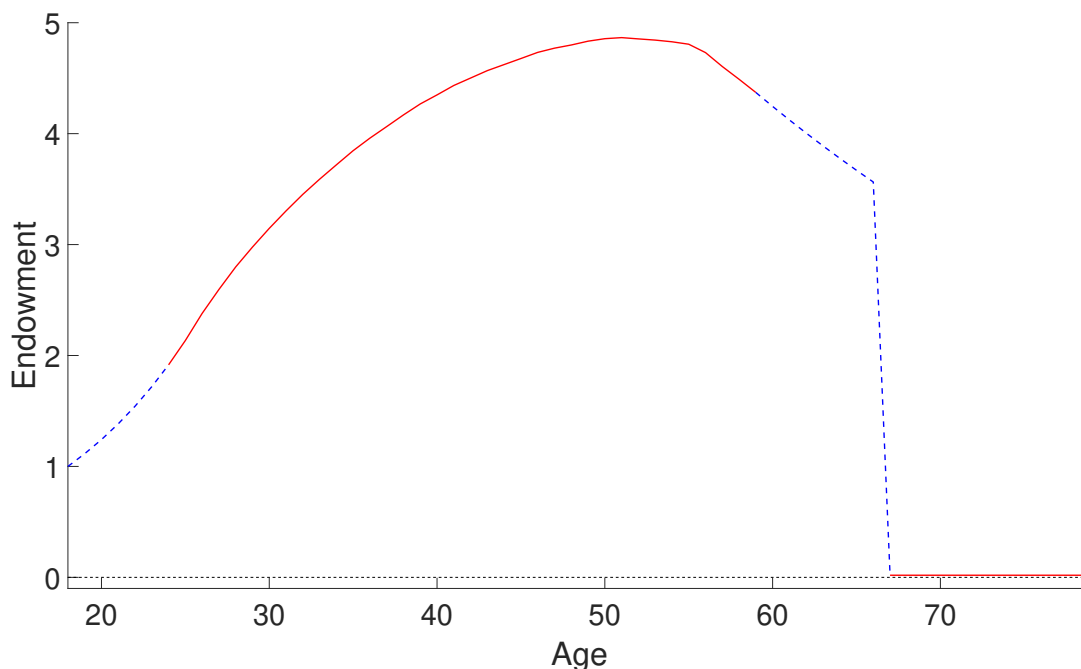
Explicit formulas for the excess demand functions for this functional form are provided in Appendix A.

We graph our calibrated income profile in Figure 1. Our representative generation enters the labour force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age at which a U.S. adult becomes eligible for social security benefits. For the working-age portion

¹⁰ The code used to generate all of our results is available online and is documented in an accompanying online document “Numerical Recipes”. Our code also replicates the findings reported in Kehoe and Levine (1983).

of this profile we use data from Guvenen et al. (2015) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

FIGURE 1. Normalized Endowment Profile.



Source: Guvenen et al. (2015). U.S. Data in Solid Red; Interpolated Data in Dashed Blue.

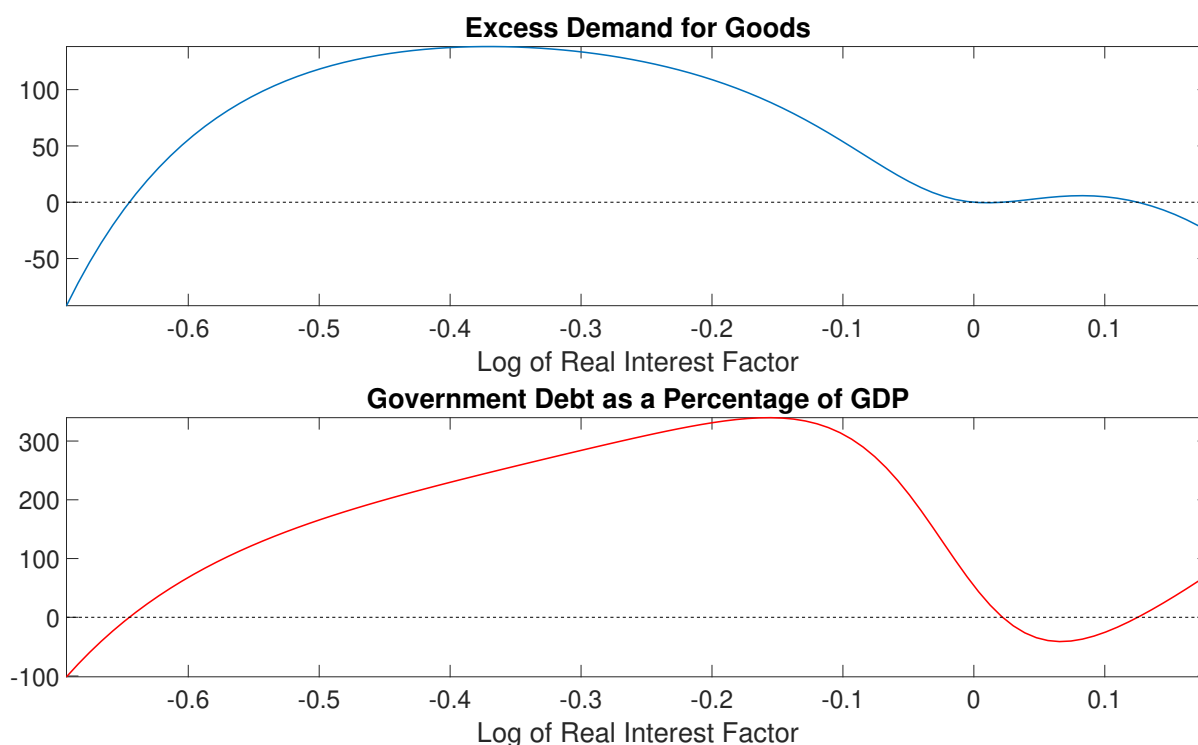
U.S. retirement income comes from three sources; private pensions, government social security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income that is independent of contributions, we used Supplemental Security Income which, for the U.S., we estimate at 0.137% of GDP.¹¹

For the remaining parameters of our model we chose a budget deficit of $d_t = 0$, an annual discount rate of 0.953 and an elasticity of substitution of 0.17. This corresponds to $\alpha = -5$ and a corresponding measure of Arrow-Pratt risk aversion of 6. For the calibrated income profile depicted in Figure 1 and for this choice of parameters, our model exhibits four steady-state

¹¹ From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled US \$447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled US \$1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of US \$8,618,700 million per annum, or US \$718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labour income.

equilibria. In Section 7 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the risk aversion parameter.

FIGURE 2. Steady States in the Sixty-Two Generation Model.



In Figure 2 we graph the steady-state equilibria of our model. The upper panel of this figure plots the logarithm of the gross real interest rate on the horizontal axis and the steady-state excess demand for goods on the vertical axis. The lower panel plots government debt as a percentage of GDP at the steady state. We see from the upper panel that the excess demand function crosses the horizontal axis four times. And we see from the lower panel that three of these crossings are associated with steady-state equilibria in which steady-state government debt is equal to zero.

The three steady-state equilibria in which debt equals zero are examples of what Gale (1973) refers to as autarkic steady-state equilibria. In these equilibria there is no trade with future unborn generations. The fourth steady-state equilibrium is what Gale refers to as the golden-rule. This steady-state equilibrium always exists in OLG models and in models with population growth it has the property that the real interest rate equals the rate of population growth. But although the golden-rule steady-state equilibrium always exists, *it is not true* that the golden-rule value of \bar{b} is always non-negative.

The golden-rule steady state occurs when the logarithm of the real interest factor equals zero. By inspecting the lower panel of Figure 2, it is apparent that government debt is positive at the golden rule steady-state and, since debt is denominated in dollars, the price level is also positive in the golden-rule steady-state equilibrium. This is important because it is the empirically relevant case in most western democracies. For example, in the United States, government debt in the first quarter of 2019 exceeded US \$22 trillion.

The values and properties of all four steady-state equilibria are reported in Table 1. We refer to the autarkic steady-state equilibria as Steady-State A, Steady-State C and Steady-State D and to the golden-rule steady-state equilibrium as Steady-State B. We see from this table that Steady-States B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady-State A is associated with a negative interest rate of -47.5% and is therefore dynamically inefficient.

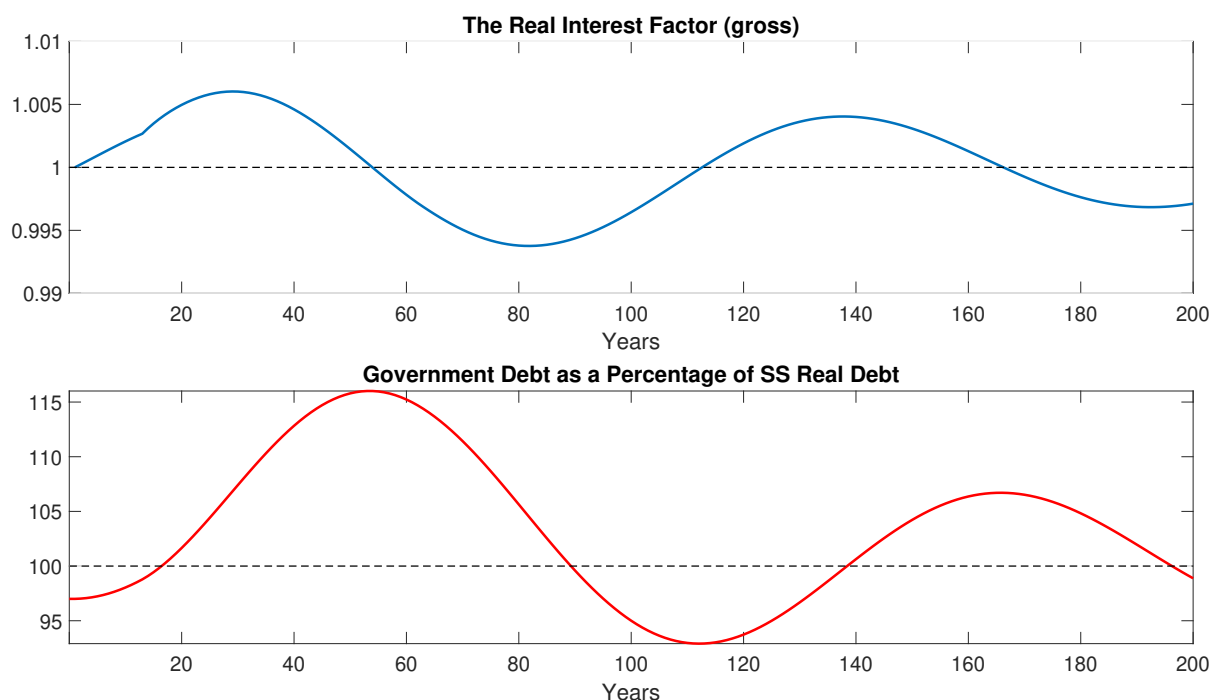
TABLE 1. Steady States of the Sixty-Two Generation Model.

Type	Equilibrium Real Interest Rates				
	Value of \bar{R}	Value of \bar{b}	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady-State A	0.525	0	60	61	1
Steady-State B	1	53.7% of GDP	59	61	2
Steady-State C	1.022	0	60	61	1
Steady-State D	1.13	0	61	61	0

The sixty-two generation model with a calibrated income profile is similar in many respects to Kehoe-Levine’s (1983) three generation model. In both examples, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both examples, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady-state is second-degree indeterminate, indeterminacy of the price level can hold even when both monetary and fiscal policy are active.

In Figure 3 we show the result of an experiment in which we perturb the initial value of b_1 by 3% and we perturb the real value of the initial wealth of all of the non-generic generations by the same amount. We refer to this perturbation as a 3% shock to the initial price level. We restrict R_2 to equal its steady state value, but all other elements of Z_0 are allowed to respond to the shock to keep the path of interest rates and debt on a convergent path back to the

FIGURE 3. The Impact of the Initial Price Level Exceeding Its Steady State Value by 3% .



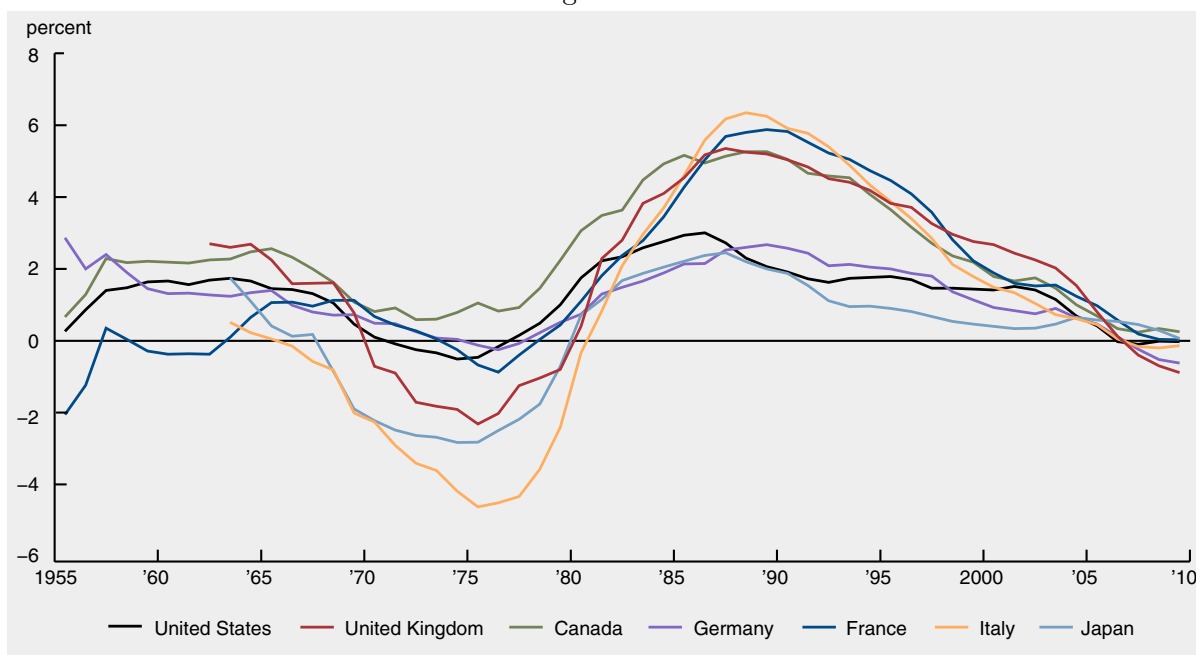
steady state. Figure 3 demonstrates that the return to the steady state from an arbitrary initial condition is extremely slow.¹²

We also see from Figure 3 that our model can endogenously generate prolonged periods of negative real interest rates. The upper panel of this figure plots the path by which the real interest rate returns to its steady-state value and the lower panel plots the return path of the real value of government debt expressed as a percentage of GDP. The figure demonstrates that small deviations of initial conditions from the steady state can have long-lasting effects, and that during the convergence process the real interest rate may be negative for periods well in excess of ten years.

One may question whether the high degree of real interest rate persistence implied by our model is excessive. Have such long swings in real interest rates actually ever been observed? To address this question, Figure 4, reproduced from Yi and Zhang (2017), compares long run real interest rates in the G7 and documents that low-frequency real rate cycles, similar to those

¹² Note that our choice of 3% is entirely arbitrary and that this is one of many admissible equilibrium paths. In particular, since we endow the non-generic cohorts with steady state asset shares, therefore the steady state equilibrium with $\forall t \geq 1 : b_t \equiv \bar{b}$ and $R_{t+1} \equiv \bar{R}$ would have also been feasible.

FIGURE 4. G7 Long-Run Real Interest Rates.



Source: Figure 1 in Yi and Zhang (2017). Long-Run Real Interest Rates are 11-Year Centered Moving Averages of Annual Real Interest Rates.

generated by our model, have characterized the evolution of real interest rates in all of these economies.¹³

7. ROBUSTNESS TO DIFFERENT CALIBRATIONS

We have presented an example of a sixty-two generation OLG model in which there is a golden-rule steady-state equilibrium with two degrees of indeterminacy. To explore the robustness of our findings to alternative calibrations, in Table 2 we record the properties of our model for different values of the annual discount rate and the coefficient of relative risk aversion. The example we featured in Section 6 had two degrees of indeterminacy and positive valued debt at the monetary steady state. Table 2 demonstrates that that property is not particularly special.

Table 2 provides 40 different parameterizations of our model with risk aversion parameters ranging from 5 to 9 and discount rates ranging from 0.91 to 0.98. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure 1. For each calibration Table 2 displays the number of steady-state equilibria in the interval $\bar{R} \in [0.5, 1.5]$, and the number of degrees of indeterminacy at the golden-rule steady-state equilibrium. There are 15

¹³ See Yi and Zhang (2017) for a discussion of why long-run moving averages are likely to characterize trends in fundamental forces underlying real interest rates.

TABLE 2. Robustness of Indeterminacy to Alternative Calibrations, Focusing Only on Steady States with $\bar{R} \in [0.5, 1.5]$.

		Annual Discount Factor								
		0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	
Risk Aversion										
RA = 5	# Steady States	4	4	4	4	4	2	2	2	
	Degree of Indeterminacy	1	1	1	1	1	2	0	0	
	Value of Debt	-2.8	-2.2	-1.5	-0.8	-0.2	0.5	1.1	1.8	
RA = 6	# Steady States	4	4	4	4	4	4	2	2	
	Degree of Indeterminacy	1	1	1	1	2	2	0	0	
	Value of Debt	-1.9	-1.3	-0.7	-0.2	0.4	0.9	1.5	2.0	
RA = 7	# Steady States	3	3	3	3	3	3	1	1	
	Degree of Indeterminacy	1	1	1	2	2	0	0	0	
	Value of Debt	-1.2	-0.7	-0.2	0.3	0.8	1.2	1.7	2.2	
RA = 8	# Steady States	3	3	3	3	3	3	3	1	
	Degree of Indeterminacy	1	1	2	2	2	0	0	0	
	Value of Debt	-0.6	-0.2	0.2	0.6	1.0	1.5	1.9	2.3	
RA = 9	# Steady States	3	3	3	3	3	3	3	3	
	Degree of Indeterminacy	1	2	2	2	2	0	0	0	
	Value of Debt	-0.2	0.1	0.5	0.9	1.3	1.6	2.0	2.4	

parameterizations in which the golden-rule steady state displays one degree of indeterminacy and 12 in which it displays two degrees of indeterminacy. In all 12 of these parameterizations, debt has positive value in the steady state.

In Section 6, we focused on the golden-rule steady where $\beta = 0.953$ and $\rho = 6$. An example with two-degrees of indeterminacy is interesting because it is not only the price level that is free to be determined by the beliefs of market participants; it is also the real rate of interest. We want to reiterate, however, that only one degree of indeterminacy is required for violations of the FTPL. And that occurs more frequently in our model than second degree violations.¹⁴

8. FISCAL AND MONETARY POLICY

In Sections 4 and 5, we characterized conditions under which the real interest rate and the real value of government debt remain stable when fiscal policy is active and the monetary authority operates an interest rate peg. While the assumption of active fiscal policy played a key role in our argument, the assumption of an interest rate peg was used only to show that

¹⁴ If we hold constant the degree of risk aversion and increase the discount rate, the number of unstable eigenvalues decreases initially from 60 to 59 and then changes abruptly to 61. We see this behaviour in Table 2 by moving along a typical row and observing that we pass from one degree of indeterminacy to two degrees of indeterminacy and then jump abruptly to 0 degrees of indeterminacy. At this last transition, a pair of complex roots crosses the unit circle, a phenomenon associated with a Hopf Bifurcation and the creation of a limit cycle. See Guckenheimer and Holmes (1983) for a discussion of the Hopf Bifurcation. We have not explored the phenomenon in this paper, but it is likely that for discount rates close to 1, this model displays endogenous limit cycles that are second-degree indeterminate.

inflation remains bounded. The boundedness of inflation follows directly from the Fisher parity condition,

$$\lim_{T \rightarrow \infty} \Pi_T = \lim_{T \rightarrow \infty} \frac{1 + \bar{i}}{R_T} = \frac{1 + \bar{i}}{\bar{R}}.$$

In this section, we discuss what would happen if we were to relax either the assumption that fiscal policy is active or the assumption that monetary policy is passive. We first show that passive fiscal policy makes indeterminacy more likely. We then demonstrate that ensuring bounded inflation under an active Taylor rule requires an additional restriction on the set of equilibrium paths. This additional restriction reduces the degree of indeterminacy by one.

8.1. Passive Fiscal Policy.

Consider first what happens when fiscal policy is passive. To model a passive fiscal policy we assume that the treasury raises taxes, τ_t , in proportion to the real value of outstanding debt to ensure that the primary deficit d_t satisfies the equation

$$d_t = -\delta b_t,$$

where $\delta \geq 0$ is a debt repayment parameter. Combining this assumption with the definition of the government debt accumulation equation leads to the following amended debt accumulation equation,

$$b_{t+1} = [R_{t+1} - \delta]b_t.$$

For values of $[\bar{R} - \delta] < 1$ the effect of making fiscal policy passive is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states whenever δ is large enough. Passive fiscal policy makes indeterminacy more likely.

8.2. The Case of a Taylor Rule.

We next assume that fiscal policy is active and the central bank follows a Taylor rule (Taylor, 1999),

$$1 + i_t = \left(\frac{\bar{R}}{\bar{\Pi}^\eta} \right) \Pi_t^{1+\eta}, \quad t = 1, \dots, \infty. \quad (10)$$

Because this equation begins at date 1, the nominal interest rate in period 1 depends on p_0 through the definition, $\Pi_1 = p_1/p_0$. We treat p_0 as an initial condition that has the same status as the initial value of nominal debt, B_0 . $\bar{\Pi}$ is the inflation target, and \bar{R} is the steady state real interest rate: the Taylor Rule is passive if $-1 \leq \eta \leq 0$ and active if $\eta > 0$.

When the central bank follows a Taylor Rule, the real interest rate and the real value of government debt continue to be determined by the bond market clearing equation and the

debt accumulation equation. It follows that the conditions we have characterized in previous sections continue to ensure that the real interest rate and the real value of government debt remain bounded.

8.2.1. *A Passive Taylor Rule.* When the central bank follows a passive Taylor Rule, (see Appendix C.1) the following equation characterizes the asymptotic behaviour of the future inflation rate,

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \eta)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \eta)^{T-s} \tilde{R}_{T+1}, \quad (11)$$

where $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. The limit of the first term on the right side of Equation (C2) is zero because $1 + \eta < 1$ and the second term is finite as a consequence of the boundedness of R_t . It follows that inflation is bounded whenever R_t is bounded. This is a generalization of the argument we made for the boundedness of the inflation rate when the central bank follows an interest rate peg and it does not impose any additional restrictions on the equations of the model for an equilibrium to be determinate.

8.2.2. *An Active Taylor Rule.* When the central bank follows an active Taylor Rule, (see Appendix C.2), the initial price level is determined by the forward-looking equation

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \eta} \right)^s (R_{1+s} - \bar{R}) \right). \quad (12)$$

Importantly, this restriction on the set of equilibrium paths is additional to the restriction

$$p_1 = \frac{B_0}{b_1},$$

that we used to generate the equilibrium sequence of interest rates. It follows that we are no longer free to pick R_2 and p_1 independently of each other. This establishes that an active monetary policy eliminates one degree of indeterminacy.

If a model has one degree of indeterminacy when the policy combination is passive-active, an active-active policy combination would now admit a unique solution. When the steady-state equilibrium displays second-degree indeterminacy, as in our sixty-two generation example, it is not just the initial price level that is indeterminate; it is also the initial real interest rate.

For any *given* choice of the initial interest rate, R_2 , active monetary policy removes nominal indeterminacy. Once we have specified the initial real interest rate, the initial price level cannot be freely chosen. If we seek an equilibrium path in which inflation is bounded then p_1 is uniquely determined by the formula in Equation 12. Crucially, however, active monetary policy does not

remove real indeterminacy and there continue to be many possible choices for the initial real interest rate, each of them associated with a different initial price level and a different equilibrium path for all future real interest rates and all future inflation rates.

9. CONCLUSIONS

We have demonstrated an important difference between the infinitely lived representative agent model and the overlapping generations model. In the RA model, government debt is both an asset and a liability of the representative agent. Because these two aspects exactly offset each other, the representative agent is indifferent about the quantity of debt she holds and in the simplest case the real interest rate in the corresponding model reflects time preferences and the evolution of the endowment.

In the OLG model, the situation is different. When the stock of government debt is not paid off during the lifetime of any single generation, the assets and liabilities of the treasury do not cancel each other out as they would in the representative agent model. As a consequence, changes in real interest rates are redistributive across generations, and they may lead to fluctuations in the demand for government bonds that are self-stabilizing. It is because of these underlying wealth effects that the golden-rule steady-state equilibrium of our calibrated model can be both dynamically efficient and second-degree indeterminate. Since wealth effects can be very persistent, the propagation mechanism of our model generates prolonged periods of negative real interest rates, similar to those we have observed in recent data.

Our findings challenge established views about what constitutes a good combination of fiscal and monetary policies. Our agents are rational and have rational expectations. Nevertheless, the price level and the real interest rate are not uniquely determined by what most economists would recognize as economic fundamentals, even when the central bank pursues an active monetary policy and the treasury pursues an active fiscal policy. These features of our model lead to very different conclusions from those of the representative agent model. If the FTPL holds, a benevolent monetary policy maker who pursues an interest rate peg might rely on fiscal policy to anchor the price level. In the OLG model we study here that is no longer possible.

Our model also leads to non-standard advice to fiscal policy makers. In an RA model, when monetary policy is active, the fiscal policy maker must raise taxes or lower expenditures in response to recessions, however they are caused. In Farmer and Zabczyk (2018) we showed, in a two-generation OLG model, that equilibrium debt dynamics can be self-stabilizing. In this paper we have extended our previous analysis to a calibrated sixty-two generation OLG model.

We have shown that an active fiscal policy can safely be pursued without the fear of causing explosive debt dynamics even when monetary policy is active.

A.1. The Generic Optimization Problem.

Consider a person with Constant Elasticity of Substitution (CES) preferences who lives for T periods and has perfect foresight of future prices. This person solves the problem,

Problem 1.

$$\max_{\{c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t\}} \frac{a_1(c_t^t)^\alpha + a_2(c_{t+1}^t)^\alpha + \dots + a_T(c_{t+T-1}^t)^\alpha}{\alpha}, \quad (\text{A1})$$

subject to the lifetime budget constraint

$$\sum_{i=1}^T Q_t^{t-1+i} c_{t-1+i} = \sum_{i=1}^T Q_t^{t-1+i} \tilde{w}_i. \quad (\text{A2})$$

Here, c_s^t is consumption in period s of a person born in period t , $i \in 1, \dots, T$ is age, and \tilde{w}_i is after-tax endowment. The parameters a_i are utility weights and $\alpha \leq 1$ is a curvature parameter which is related to intertemporal substitution, η , by the identity

$$\eta \equiv \frac{1}{1 - \alpha}.$$

The term Q_t^k , defined by the expression

$$Q_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad Q_t^t = 1,$$

is the relative price at date t of a commodity for delivery at date k .

This optimization problem includes the case of a constant discount factor β for which

$$[a_1, a_2, \dots, a_T] = [1, \beta, \dots, \beta^{T-1}]$$

and logarithmic preferences which is the limiting case when $\alpha \rightarrow 0$. We permit the discount factor to vary with age to nest the Kehoe and Levine (1983) example which we use to cross-check our results.

Proposition 2. *The solution to Problem 1 is given by*

$$\hat{c}_{t-1+k}^t = \frac{a_k^\eta \sum_{i=1}^T (Q_t^{t-1+i} \tilde{w}_i)}{\left(Q_t^{t-1+k}\right)^\eta \sum_{i=1}^T (Q_t^{t-1+i})^{1-\eta} a_i^\eta}, \quad k = 1, \dots, T. \quad (\text{A3})$$

where \hat{c}_{t-1+k}^t denotes the consumption, at time $t - 1 + k$, of an agent born at time t .

Proof. The result follows directly from substituting the first-order conditions into the budget constraint and rearranging terms. □

A.2. Non-generic Optimization Problems.

Let j be an index that runs from 1 to $T - 1$. Consider a non-generic person born in period $1 - j$ with real assets $\nu_{1-j} \equiv \lambda_{1-j}b_1$ who lives for $T - j$ periods. This person solves Problem 2

Problem 2.

$$\max_{\{c_1^{1-j}, \dots, c_{1-j+T-1}^{1-j}\}} \frac{a_{T-j+1}(c_1^{1-j})^\alpha + a_{T-j+2}(c_2^{1-j})^\alpha + \dots + a_T(c_{1-j+T-1}^{1-j})^\alpha}{\alpha}, \quad j = 1, \dots, T - 1 \quad (\text{A4})$$

subject to the lifetime budget constraint

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left(c_k^{1-j} - \tilde{w}_{k-(1-j)+1} \right) \leq \lambda_{1-j}b_1, \quad (\text{A5})$$

Proposition 3. Let $k \in \{1, \dots, T - j\}$. The solution to Problem 2 is given by

$$\hat{c}_k^{1-j} = \frac{a_{k+j}^\eta \left(\nu_{1-j} + \sum_{i=1}^{T-j} \mathcal{Q}_t^i \tilde{w}_{j+i} \right)}{\left(\mathcal{Q}_t^{t+k-1} \right)^\eta \sum_{i=1}^{T-j} \left(\mathcal{Q}_t^i \right)^{1-\eta} a_{j+i}^\eta}, \quad k = 1 \dots 1 - j + T - 1. \quad (\text{A6})$$

Proof. The problem above is identical to a generic one solved by an agent who has $T - j$ periods to live, whose endowments are $\{\tilde{w}_{j+1} + \nu_{1-j}, \tilde{w}_{j+2}, \dots, \tilde{w}_T\}$, and whose preference parameters in the utility function are $\{a_{j+1}, a_{j+2}, \dots, a_T\}$. \square

APPENDIX B. CONDITIONS FOR LOCAL UNIQUENESS OF EQUILIBRIUM

To find conditions under which equilibrium sequences remain bounded, we exploit the properties of the Jordan decomposition of J , which we write as

$$J = Q\Lambda Q^{-1}, \quad (\text{B1})$$

where Λ is an upper triangular matrix with the eigenvalues of J on the diagonal and Q is the matrix of left eigenvectors of J . The restrictions that prevent X_t from becoming unbounded are found by premultiplying \tilde{X}_{T-2} by the rows of Q^{-1} associated with the unstable eigenvalues of J and equating the product to zero. We refer to the matrix that contains these rows as Q_u^{-1} .¹⁵

Let K be the number of eigenvalues of J that lie outside the unit circle. The requirement that equilibrium sequences remain bounded places K linear restrictions on the initial vector

¹⁵ In practice, the Jordan decomposition is numerically unstable but there are several good computational methods to compute Q_u^{-1} that are implemented in all modern programming languages. The reader is referred to Golub and VanLoan (1996) for a description of the algorithms used to solve problems of this kind and to Farmer (1999) for an accessible introduction to solution methods for rational expectations models with indeterminate equilibria.

X_{T-2} which we express with the matrix equation

$$Q_u^{-1}X_{T-2} = 0, \quad (\text{B2})$$

where Q_u is a $K \times 2T - 3$ matrix.

The function $G^T(\cdot)$ depends not only on Z_0^T but also on the initial nominal wealth distribution \mathbb{L} . To ensure that our local analysis remains valid we first compute the steady state distribution of wealth from the equation¹⁶

$$\nu_{1-j} = \bar{R}s_1^{2-j}(\bar{R}, \bar{R}, \dots, \bar{R}), \quad j = 1, \dots, T-1, \quad (\text{B3})$$

and we use the identity,

$$\lambda_{1-j} \equiv \frac{\nu_{1-j}}{\bar{b}}, \quad (\text{B4})$$

to recover the steady-state values of λ_{1-j} . In what follows we set the initial asset shares to these steady state values.

Let G_X, G_Y be the Jacobians of the function $G^T(\cdot)$ with respect to X_{T-2} and Y_{T-2} evaluated at the steady-state equilibrium $\{\bar{R}, \bar{b}\}$ and let $\tilde{X}_{T-2}, \tilde{Y}_{T-2}$ be deviations of these vectors from their steady-state values. Using this notation, the predetermined equilibrium conditions place the following $2T - 5$ restrictions on the $3T - 6$ unknowns $Z_0^T \equiv [X_{T-2}, Y_{T-2}]$.

$$G_X \tilde{X}_{T-2} + G_Y \tilde{Y}_{T-2} = 0. \quad (\text{B5})$$

The uniqueness of equilibrium then comes down to the question of the number of solutions to the equations

$$\begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_{T-2} \\ \tilde{Y}_{T-2} \end{bmatrix} = \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}, \quad (\text{B6})$$

where $\bar{G} \neq 0$ represents the perturbations of the initial wealth distribution \mathbb{L} from its steady-state value. It follows immediately that a sufficient condition for the existence of a unique equilibrium is that the matrix

$$\mathcal{M} \equiv \begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix}, \quad (\text{B7})$$

is square and non singular.

¹⁶ Here, s_1^{1-j} is the savings function defined in Equation (5) evaluated at the steady state sequence of real interest rates.

If $K > T - 1$ then \mathcal{M} has more rows than columns and, except for the special case where the rows of \mathcal{M} are linearly dependent, Equation (B6) has no solution.

If $K = T - 1$ then \mathcal{M} is square. If \mathcal{M} also has full row rank, there exists a unique solution to this equation given by the expression

$$\begin{bmatrix} X_{T-2} \\ Y_{T-2} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}. \quad (\text{B8})$$

Finally, if $K \leq T - 1$ then \mathcal{M} has fewer rows than columns. In this case we are free to append a $(T - 1 - K) \times (3T - 6)$ matrix of linear restrictions to Equation (B6). For example, if $T - 1 - K = 1$ we can append a row that sets

$$b_1 = \bar{b}_1. \quad (\text{B9})$$

APPENDIX C. APPENDIX: INFLATION UNDER A TAYLOR RULE

In this Appendix we derive equations that characterize the behaviour of the inflation rate when the Taylor Rule is passive and when it is active.

C.1. The Case of a Passive Taylor Rule.

Using the Taylor rule to substitute for $1 + i_t$ in the Fisher parity condition yields the following difference equation for inflation

$$\Pi_{t+1} = \left(\frac{\bar{R}}{R_{t+1}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^\eta \Pi_t, \quad \text{for all } t = 1, \dots, \infty \quad (\text{C1})$$

which we linearize around a steady state to obtain

$$\tilde{\Pi}_{t+1} = (1 + \eta) \tilde{\Pi}_t - \kappa \tilde{R}_{t+1}, \quad \text{for all } t = 1, \dots, \infty. \quad (\text{C2})$$

Here, $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. Iterating Equation (C2) we obtain

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \eta)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \eta)^{T-s} \tilde{R}_{T+1}. \quad (\text{C3})$$

This is Equation (11) in Section 8.2.

C.2. The Case of an Active Taylor Rule.

To find conditions under which inflation is bounded when the Taylor Rule is active, we use Equation C2 to write the inflation rate at date t as a function of all future real interest rates

and all future inflation rates,

$$\tilde{\Pi}_t = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\eta} \right)^s \tilde{R}_{t+s} + \lim_{T \rightarrow \infty} \left(\frac{1}{1+\eta} \right)^T \tilde{\Pi}_{t+T}. \quad (\text{C4})$$

If inflation is bounded, and if the Taylor Rule is active, the second term on the right side of Equation (12) is zero. Evaluating Equation (C4) at $t = 1$, we arrive the following expression for the initial gross inflation rate.

$$\tilde{\Pi}_1 \equiv (\tilde{\Pi}_1 - \bar{\Pi}) = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\eta} \right)^s \tilde{R}_{1+s} \quad (\text{C5})$$

Using the definition of inflation in period 1, Equation (C5) places the following restriction on the initial price level,

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\eta} \right)^s (R_{1+s} - \bar{R}) \right). \quad (\text{C6})$$

This is Equation (12) in Section 8.2.

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