Kindergarten of Fractional Calculus

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Dedicated to my mother, the late Smt.Purabi Das (1934-2018), and my blind father, the late Sri.Soumendra Kumar Das (1926-2009), as well as all of my school and college teachers.

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যাদবপুর বিশ্ববিদ্যালয়
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Boundaries between different branches of science seem to be melting away, as knowledge progresses. It is as if we are moving towards a unified interdisciplinary ‘natural science’, held together by a framework of mathematics. Novel materials, as well as familiar, but less understood complex systems, lie on the overlapping areas between the compartments we use for classifying ‘subjects’, such as physics, biology, geoscience, chemical engineering and so on. We need new techniques for understanding and modelling these systems. As we know, Hooke’s law works well for many solid materials within the elastic limit, as does Newton’s law for viscous forces, in the case of many simple fluids. However, what about the other familiar materials we always use – things like sticky pastes, ductile metals, starch gels, and so on? The simple linear laws which work quite well under limiting conditions are not of much use here. An ingenious way of treating such systems, suggested by a group of scientists, was to implement *generalized calculus* in the governing equations, allowing the order of differentiation to take on non-integer values. This opens up a huge field to play with concerning such peculiar and non-conformist systems. However, of course, a vast amount of intimidating mathematical techniques have to be worked out in order to actually *solve* problems using this crazy idea.

In this kindergarten, or play-school, Shantanu Das develops the ideas behind this new field of mathematics, starting from a most elementary level up to actual applications in different areas of science. Shantanu Das has already published a very successful book on this subject, and has been working in this field for many years having applied generalized calculus in control systems, as well as in the study of visco-elastic materials, ion-conducting polymers and other fields. He has also delivered lectures and courses on the subject at different universities and institutes. I believe the reader will enjoy the experience at this kindergarten and graduate to higher levels in order to make fruitful use of the ideas introduced here.

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PREFACE

This book has been compiled from various ‘hand-written teaching notes’ for classroom lectures and ‘presentations’ that I delivered on the topic of fractional calculus at various universities and institutes, from 2005 onwards. The classroom lectures were termed as a ‘Kindergarten of Fractional Calculus’. I must say that book writing is very tough compared to delivering lectures interactively on a blackboard. The name ‘Kindergarten of Fractional Calculus’ (for the classroom talks) suggests that the treatment is very simple; unconventional, yet serious. It suggests that, without going into ‘formal’ theories ,this topic could be developed following a ‘just do it’ attitude, building on our knowledge of classical calculus and mathematics gained at high school and college. I employ this same, informal style in this book. Therefore, I request purists excuse me for this unconventional and non- formal approach, as it is particularly practical.

Why did I write this book in such an unconventional way? It is because my students liked the way I presented this complex subject in class, as it made it easy to grasp and interesting for them, as I first discussed the paradoxes and conditions of the subject, before going on, at a later stage, to discuss its formalism. In this book, I have not dealt with physical interpretations; rather, I took time to write each small step that goes towards every derivation, which is essential in order to develop the subject. This being said, the first chapter and appendix contain a number of relevant definitions and descriptions for terms such as analyticity, analytic continuation, residue calculus, Jordan’s lemma, Laplace transformation, gamma function and its properties, psi function, and Stirling’s number, among others. I also use generalized functions (such as the Mittag-Leffler function and its variants), in other chapters. The first chapter deals with the concept of mixed differentiation and integration, product rules and chain rules, etc., of classical calculus; that I used subsequently for concept generalization. In all other chapters, I carry out detailed derivations in their totality, with each step elaborated and explained. Thus, I am not economical with page space and the descriptive language used. Purists may term this as a verbose treatment, but students nevertheless appreciate this unconventional approach of opening out each step in the derivation process with descriptive explanations, especially those studying Engineering and Applied Sciences. This type of detailed treatment is missing in most of the existing literature as details of derivations are skipped, which students and users usually find difficult to absorb. I wrote this book in this unconventional way mainly because of demand from the student community in my part of the globe.

In 2008, the organizers of the ‘International Mathematics Olympiad’ (INMO) at Mumbai asked me to introduce this subject of fractional calculus to a select group of class XI and XII school students. That was a challenge. I started with the idea of tossing several coins. I then arrived at a formula for a Probability Generating Function (PGF) - for one coin, for two coins and then extended for n - coins. Then, I said to the students, “let us put n equal to ½ and see how ‘a half-coin’ should behave”. Mathematically I demonstrated to the students that we have constructed this half-coin, but I stated, “we have our limitations today, because we are presently unable to attach much physical sense (especially in terms of the notion of ‘negative’ probability) to this new construct”. Hence, we are in a paradoxical situation, and I made it clear to the students that “the paradox is because mathematics goes far beyond our physical understanding”. This is how I began the concept generalization.

With this example, I stated, “as we have one whole differentiation and two whole differentiations, and n-whole differentiation, we can have ½- differentiations or ½ - integrations”, and proceeded in a very simple way to develop the concept, which was once considered a paradox. Today however, we have a physical and engineering idea of it. This physical and engineering sense I have already dealt with in my previous books and other publications, listed in the Bibliography section. Thus, we can say fractional calculus is concept generalization for the existing classical calculus theory; it is also termed as generalized calculus and sometimes even called on-Newtonian calculus. It is not a paradox anymore.

This introduction of the subject to the students gave me confidence that, if I develop the paradoxical, complex-looking mathematics in a different, easy and interesting way, perhaps the students will be hooked on this subject. To some extent I was successful as I delivered detailed classes at Jadavpur University and Calcutta University, and some short classes at Pune University, Mumbai University, IIT- Kharagpur, VNIT- Nagpur, and at several other places. The result is that I see the growth of this subject in this part of globe. Students of Mathematics, Physics, and Engineering have taken up this subject for further research. Some students are presently working, or have previously worked with me in such fields on this subject. Here, in this book I aim to deliver detailed derivations, in a simplified though unconventional way, of the various aspects of the beautiful mathematics of fractional calculus.

Recalling the simple classical integration process that is viewed as the area under the curve (that is, the area under the original function being integrated), we take into account all of the values of the function from the present point of interest to the start of the function. We recall that we memorize all the past points (in a causal sense), and sum them up in order to carry out the classical integration process. The ‘fractional integration’ that we will learn is also the area under the curve, but it is an ‘area under a shape-changing curve’, which keeps on changing as we move ahead. Fractional integration, too, has to account for all the previous values of the function, but here different weights (with decreasing value) are multiplied by each previous and past value of the function, giving a real ‘fading memory’ effect. This is reality as the past memory always fades as we move ahead.

Differentiation, as we know classically, is a slope at a particular point in a curve. The ‘fractional differentiation’ that we will learn will be a slope at a particular point of a function, which is the ‘area under the shape changing curve’. Therefore, in order to evaluate fractional differentiation at a desired point, I should consider all the past values of the function. That is the fractional derivative, which has an in-built fractional integration process. We will be calling this the ‘differ-integration’ process. We will also see the embedded ‘memory’ in the concept of fractional differentiation This means that all the past points with decreasing weights are considered in order to evaluate fractional derivatives at the point of interest; unlike classical differentiation, which is a point property or local property.

Today, we have several engineered systems based on fractional calculus. These engineered systems, are based on the use of the ‘fractional Laplace variable’, which appears when we conduct Laplace transforms of fractional derivatives and fractional integration operators. We will also learn this here in this book. With the fractional Laplace variable, we have developed analog and digital electronic circuits allowing fractional differentiation and fractional integration. We will also study the use of the Laplace transform method to solve fractional differential equations here.

The picture in Figure-P1 below shows a magnetic levitation system, where the metal ball is floating in the air, and is being controlled by current in the coil of a magnet. This current is being governed proportionately by fractional derivatives and fractional integration of the error signal for the ball’s position (Courtesy: BRNS funded joint project of VNIT Nagpur and BARC, which developed a digital fractional order controller for industrial applications. This system also demonstrates that by using fractional calculus in control, we are getting efficient controls, as compared to the classical schemes using classical calculus. This is depicted in Figure-P2.

Comparing these two pictures, we see that, in order to do the same job (that is, to position the floating ball and slowly make it follow sinusoidal command), the voltage output of the controller in the case of the second case is fluctuating severely. Many critics will say it is ‘noise’, but why is the ‘so called noisy’ output absent in fractional calculus-based systems? We are using the same electronics and only changing the program of the processor (in this case, the micro-controller); once for classical control governed via classical differentiation and integration, and in the next case for control governed via fractional calculus. The justification for having better efficient control via fractional calculus is that the fractional differentiation and integration operations possess inherent memory. This memory in the system works in order to govern the ball’s position based on its previous or past experience. Therefore, this fractional differentiation and integration process gives us an ideal filtering action, whereas classical differentiation is a point (or local) property that does not therefore have memory, and which acts instantly with no previous experience. Also, classical integration is the summation of all the previous values of function when all are equally weighted; whereas fractional integration is also the summation of previous values, but with decreasing weights. Thus, fractional differentiation and fractional integration give us a memory action where memory fades as we move on. We will study how memory is inherently embedded in fractional calculus.

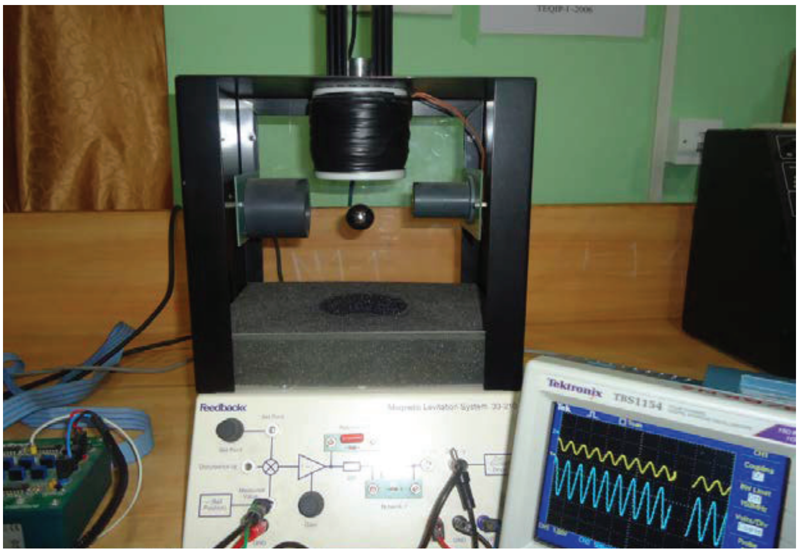
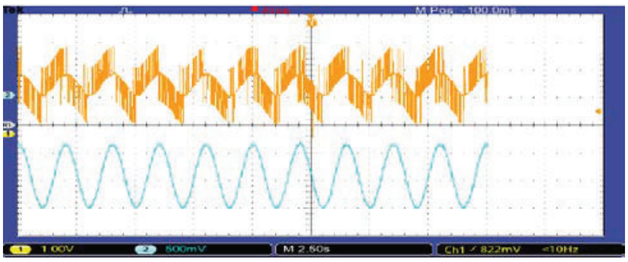


Figure-P1: Picture showing magnetic levitation where fractional calculus is used for controls

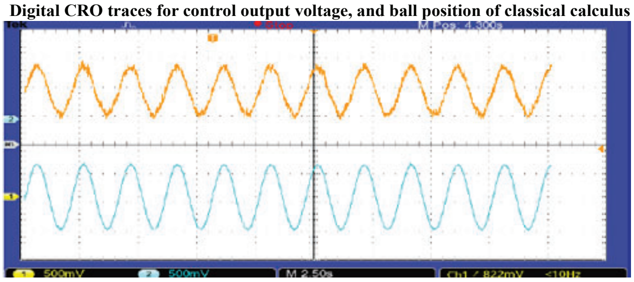
Thus, in the second case, the manoeuvring signal is instantaneously oscillating very dramatically, again giving us the notion that some ‘noise’ is being injected into the system. We also infer in the second case that the controller (with classical calculus) is making a lot of effort to control this ball and keep it afloat, while in the first case the controller’s action (based on fractional calculus) is smooth and effortless. So we can see that the fractional calculus-based system does the control action with less effort than the conventional classical calculus based controllers, and that it is, therefore, better and more efficient. Figure-P3 gives us the experimental records for CRO traces of control voltage (upper trace) and ball position (lower trace) in detail.

Figure-P2: Picture showing magnetic levitation where classical calculus is used for controls

Now we pose a question- what is the implication of ‘lesser effort’ by a control system doing the same job? We have carried out another development where we use classical and fractional calculus to regulate (control) the speed of a DC motor. Figure-P4 gives the picture of the full setup of the DC motor speed control system. Figure-P5 gives us the record of armature voltage and current, while the DC motor is controlled by classical calculus. The multiplication of armature voltage and armature current gives 231.07 Watts as input power to the armature of the DC motor, while regulating the speed at 1000 RPM. Figure-P6 gives the same motor running at the same speed of 1000 RPM but controlled by fractional calculus, which shows an intake power by an armature of 181.61 Watts. Thus, to run a DC motor at a speed of 1000 RPM, the classical calculus-based system takes 17.3% more power than the fractional calculus-based control system. The experiment is done at several speed settings from 500RPM to 1300RPM and at all speeds we observe lesser power intake, when control is based on fractional calculus. This gives us a clue that by using fractional calculus, we are achieving ‘energy/fuel efficiency’. Maybe in future, the industry will adapt this new mathematics to make a fuel/energy efficient control system; at least, I hope so!



Digital CRO traces for control output voltage, and ball position of classical calculus



Digital CRO traces for control output voltage, and ball position of fractional calculus

Figure-P3: Picture showing CRO traces for control voltage and ball position for both systems with classical calculus and with fractional calculus

Figure-P4: DC motor speed control setup

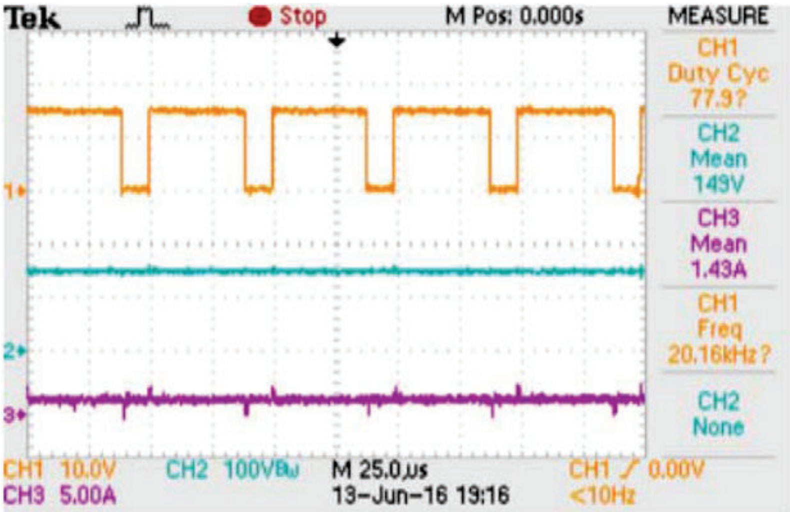


Figure-P5: Armature voltage and current at 1000 RPM DC motor with classical calculus

Figure-P6: Armature voltage and current at 1000 RPM DC motor with fractional calculus

Anyway, how is the efficiency achieved? Is it because by using fractional calculus in my control action, I am interacting with the actual plant (i.e. the DC motor or the magnetic levitation system) in a better and more efficient way? Therefore, can I say that fractional calculus is the mathematics with which natural dynamic systems operate better? This thought came to my mind while I was using logarithmic logics and ratio control and doing derivative operations on these logarithmic domains, for nuclear reactor control. The usage of logarithmic logic gave me better and more efficient nuclear reactor control than conventional use of linear logics, and these 'new' formulas were implemented in nuclear plants, in 2002. When asked why, I advised that the reason was that the way to govern 'a natural, exponential system' (i.e. the nuclear reactor) was using logarithmic power error and its derivative in logarithmic domains which closely match the language of the system. Therefore, maybe this hypothesis is increasing the efficiency of the control action. In the servo-system, the servant, i.e. the plant (the nuclear reactor), is efficiently able to understand the master that is the controller's command (language, in a logarithmic and exponential domain). Perhaps, in the logarithmic case, 'we are talking to the plant to be controlled (i.e. the nuclear reactor)', which is naturally exponential in the language of the process. This delivers an efficient way of achieving better control.

The point, which is emphasized here, is that if we communicate in the language of the dynamic system then we will be communicating better. Thus for efficient communication, 'communicate in French with persons in France'! The above experiments do point out that perhaps fractional calculus is the language that dynamic systems understand better! I had an opportunity to talk on the topic of 'fuel efficient controls', in Beijing, China, at the International Conference on Nuclear Engineering (ICONE- 13), in 2005; for the first time worldwide. At that point, it was my conjecture or hypothesis that we should apply non-Newtonian calculus to achieve fuel efficiency. Today, we have energy/fuel efficient controls practically realized, via non-Newtonian calculus. I feel blessed that I could see, within my life span, the original conjecture or hypothesis of mine regarding energy/fuel efficient controls realized practically and become a reality. Therefore, this is one motivation to provide a course on fractional calculus for Science and Engineering students.

Here, I may also mention that tuning the controller of a plant to either classical calculus or fractional calculus (say DC motor, magnetic levitation system, nuclear reactor, etc.) uses minimization of the chosen performance index (PI). The PI is a function of the controller's parameters that we set, and the plant to be controlled is described by plant's transfer function. By using the minimization technique, we obtain the values of the controller's parameters. This minimization technique, for minimizing the PI is like minimizing the least square error (LSE), call it 'E' for the curve fitting the polynomial. Here I give one example that we can get a better fit with a lower value of E if the curve is composed of fractional power monomials.

Let us say we have n data points $x_1, x_2, x_3, ..., x_n$ and that the corresponding values are $y_1, y_2, y_3, ..., y_n$. Assume by examination of the plotted points; we say linear fitting is good here for this set. Therefore, we need a linear function; $y = f(x) = ax + b$ is to be fitted so that LSE, that is, E is a function of a and b (i.e. $\mathrm{E}(a,b) = \sum_{i=1}^{n}\left(y_i - (ax_i + b)\right)^2$) and is minimized. Following simple mathematics:

$$
\begin{aligned}
&\frac{\partial}{\partial\, a}\left[\mathrm{E}(a,b)\right] = -2\sum_{i=1}^{n} x_i\left(y_i - (ax_i + b)\right) = 0\\
&a\left(x_1^2 + x_2^2 + ... + x_n^2\right) + b\left(x_1 + x_2 + ... + x_n\right) = x_1y_1 + x_2y_2 + ...x_ny_n\\
&\frac{\partial}{\partial\, b}\left[\mathrm{E}(a,b)\right] = -2\sum_{i=1}^{n}\left(y_i - (ax_i + b)\right) = 0\\
&a\left(x_1 + x_2 + ... + x_n\right) + \overbrace{b + b + ...b}^{n} = y_1 + y_2 + ...y_n
\end{aligned}
\tag{P1}
$$

Compactly we use a set of equations, i.e.

$$
\begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}
\tag{P2}
$$

to get the values of a and b, which minimise LSE, i.e. $\mathrm{E}(a,b)$. The above expressions ((P1) and (P2)) can be used for the linear fitting which has two degrees of freedom, namely a and b. Now, if $\mathrm{E}(a,b) \neq 0$, then we try another function ($y = ax^{\alpha} + b$), and search for the value of, which is close to one (i.e. $\alpha \sim 1.00$) to make the obtained E from the linear fit still lower.

This we demonstrate with an example. Take six data points with $n = 6$ as follows where we want to fit a linear function, $y = ax + b$:

$$\left\{x_i\right\}_{i=1}^{i=6} = \left\{0, 0.5, 1, 1.5, 2, 2.5\right\}$$
$$\left\{y_i\right\}_{i=1}^{i=6} = \left\{-0.43, -0.17, 3.12, 4.79, 4.85, 8.69\right\} \tag{P3}$$

Using the formula as indicated above (P2) with $n = 6$ using the above (P3) values of the data points, we get E minimized with $a = 3.56$ and $b = -0.97$. Therefore, we have a linear function $y = 3.56x - 0.97$, giving us minimum E as $\mathrm{E}(a,b) = \sum_{i=1}^{6}\left(y_i - (3.56x_i - 0.97)\right)^2 = 9.47$. The plot of data fitting by a linear curve $y = 3.56x - 0.97$ gives minimum E.

Now we search for $\alpha \sim 1.0$ with the new function $y = 3.56x^{\alpha} - 0.97$, and see the following results for $\mathrm{E}(a,b,\alpha) = \sum_{i=1}^{6}\left(y_i - (ax_i^{\alpha} + b)\right)^2$, which now also depend on α:

$$\begin{gathered}
\mathrm{E}(a,b,\alpha) = \sum_{i=1}^{6}\left(y_i - (ax_i^{\alpha} + b)\right)^2 \\
a = 3.56; \quad b = -0.97 \\
\alpha = 1.05 \qquad \mathrm{E} = 10.93 \\
\alpha = 1.00 \qquad \mathrm{E} = 9.47 \\
\alpha = 0.95 \qquad \mathrm{E} = 8.56 \\
\alpha = 0.90 \qquad \mathrm{E} = 8.14
\end{gathered} \tag{P4}$$

Thus, we have the possibility of lowering E from the obtained value 9.47, by choosing the non-integer α. What did the fractional order monomial do? The obvious answer is that it gave me an extra degree of freedom in the parameter $\alpha \sim 1.0$. Therefore, in this case, we have three degrees of freedom, namely a, b and α as compared to two degrees of freedom in a linear curve fitting. Now the question comes, how did this curve-fitting example suit our control system example?

If we say that the variable x is a Laplace variable called s, then we already know from classical calculus theory, that operator s corresponds to one-whole differentiation, and operator s^{-1} corresponds to a one-whole integration operation. We will learn in this book that s^{α} corresponds to fractional differentiation if $\alpha > 0$, while $\alpha < 0$ corresponds to a fractional integration operation. The polynomial representation of a classical calculus-based controller in the Laplace domain is $f_{\text{int}}(s) = k_p + k_i s^{-1} + k_d s$. This controller has three degrees of freedom, namely k_p, k_i and k_d in order to minimize a control system PI. The latter is a function of controller settings (i.e. k_p, k_i, k_d) and the plant to be controlled. After evaluating these three parameters in order to minimize PI, we set these values into the control system, while the fractional calculus-based controller structure in the Laplace domain is $f_{\text{fract}}(s) = k_p + k_i s^{-\lambda} + k_d s^{\mu}$. This is with five degrees of freedom, namely k_p, k_i, k_d, λ and μ; with $0 < \left(\lambda, \mu\right) < 2$; that is enhancing the degree of freedom with these two extra fractional indices. With the extra freedom of non-integers λ and μ, we are enabled to minimize the chosen PI still further as compared to classical case; as we demonstrated in the curve fitting example. This is the essence of fuel/energy efficient controls using fractional calculus.

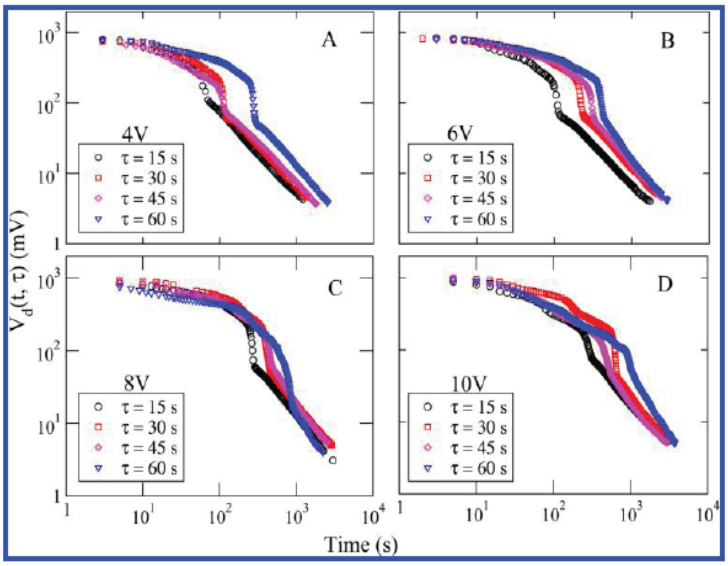


Figure-P7: Memorizing charging time of applied electric- field in experiments with Laponite for crack formation studies

Figure-P7 presents experimental evidence that a relaxing system (in this case, a Laponite stressed with a DC electric field) has a memory of being connected to a voltage supply during time τ and continues as if it is still connected to a non-zero voltage source, even after the voltage source is switched off. The figure shows the variation of discharge voltage $V_d(t,\tau)$ with time after the electric field is switched off at time $t=\tau$ for different applied voltages (Courtesy of the Condensed Matter Physics Research Center (CMPRC), Dept. of Physics, Jadavpur University Kolkata). The self- discharging curve is a function of time (τ); therefore in a way the relaxing system is memorizing its history of the charging profile that is described by τ .

This explanation is only possible if we have a fractional capacitor present in the relaxing system. The impedance of the fractional capacitor is given as $Z(s)=s^{-\alpha}C_\alpha^{-1}$, which is described in the Laplace variable s. This fractional capacitor has a voltage current relation via fractional calculus, i.e. $\mathrm{i}(t)=C_\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\mathrm{v}(t)$. This means that the current $\mathrm{i}(t)$ through a fractional capacitor is a fractional derivative (of the fractional order α with $0<\alpha<1$) of the voltage $\mathrm{v}(t)$ across it. For $\alpha=1$, we get the classical capacitor that we studied in textbooks which has a terminal relationship as $\mathrm{i}(t)=C\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(t)$. We note here that the unit of classical capacitor (C) is in Farads while the unit of fractional capacity C_α is in 'new units' of $\mathrm{Farads}/\mathrm{sec}^{1-\alpha}$. The study of these 'new types' of units is also a topic of fundamental research. The memory phenomena are described in the self-discharging curves of Figure- P7 which are different for 15, 30, 45 and 60seconds of charging time τ, and which would not be observed if the capacitor behaved as a classical 'geometric' capacitor, as described in textbooks. This memory phenomenon is also observed in supercapacitor self-discharging curves.

The presence of a fractional capacitor is identified by the charge-discharge curve of the super-capacitor. See Figure-P8(courtesy of the Department of Electrical Engineering, NIT- Raipur) for a joint project with BARC on the identification of fractional orders of super-capacitors. Here we charge a super-capacitor with a constant current source of $\mathrm{I}=10\mathrm{mA}$, 20mA and 30mA. These charging currents raise the voltage across the super-capacitor from zero to about 3V, and at this 3V point we discharge the super-capacitor by reversing the constant current to 30mA, until the voltage across it reaches zero.

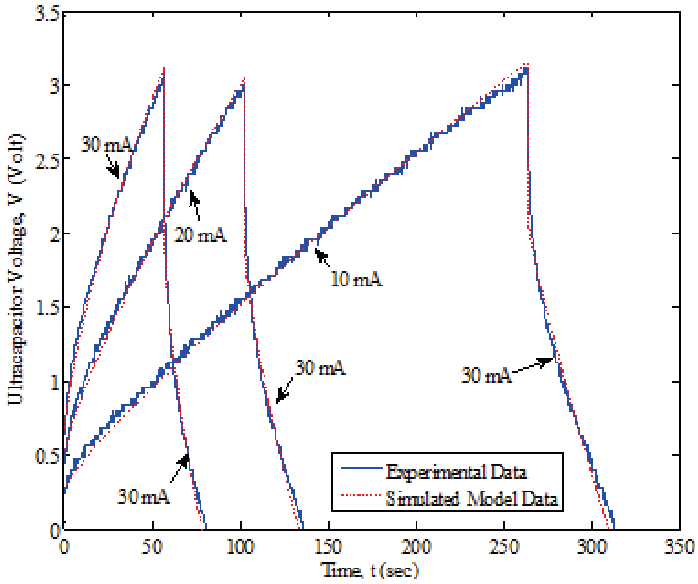


Figure-P8: Charge discharge of a super-capacitor showing the presence of a fractional capacitor

If the super-capacitor possessed a textbook type of ideal capacity, then the voltage rise across it would have been a linear function of time (ramp- up i.e. $\mathrm{v}(t) \sim t$ as $\mathrm{v}(t) = \int_0^t \mathrm{I}\,d\tau$) while charging (i.e. proportional to the integration of a constant current; that is, I). Instead, what we observe is a curved path of voltage rise and fall, which is proportional to $\mathrm{v}(t) \sim t^{\alpha}$ with $0 < \alpha < 1$, instead of the ideal which is $\mathrm{v}(t) \sim t$.

These charge-discharge curves are used to measure or find the fractional capacity C_{α} and the value of α present in a super-capacitor. This interesting experiment shows that when the value of α is close to one, the ideal value (but not one) as the charging rate (Coulombs/second) is smaller, and that α is much less than one when charging current is high. Similarly, it is observed that the fractional capacity C_{α} is close to 'geometric' capacity ($\alpha \sim 1.00$, i.e. the ideal textbook capacity) for lower charge rates. Not only do super-capacitors show this fractional capacity and memory effect, but large numbers of experiments on insulators also exhibit this. A paper dielectric capacitor which is experimented with is found to have $\alpha \sim 0.9$. Thus, $\alpha \neq 1$ is a reality; therefore, fractional calculus comes to describe circuits with fractional differential equations, and takes into account fractional capacity, which is a reality.

I have been a collaborator and guide of all these engineering and physics experiments that I have described above along with my colleagues (both teachers and students) from Jadavpur University, VNIT- Nagpur, NIT Raipur, and IIT Bombay.

Much experimental evidence gives memory-based dynamics like: in the characterization of visco-elastic materials; dielectric relaxation; super-capacitor charging/discharging, wherein the super-capacitor memorizes the 'time' that it has been kept in a charging process; and in impedance spectroscopy study of transport phenomena for electro-chemical systems, plus several other phenomena, where we have applied fractional calculus. Similarly, the distributed system behaves with fractional order dynamics where we represent it not as a point, but rather having a spread, and thus they are non-local in terms of behavior. As an example, a distributed system behaves like a semi- infinite system, with half-order dynamics. The transport phenomena in porous, rough and fractal media have fractional order dynamics; we have also observed fractional order dynamics in the cooling of liquids. These facts do motivate us to learn fractional calculus and apply the concepts. These above mentioned physics and engineering aspects will not be discussed in this book.

One interesting fact is that the points in the graph that are non-differentiable have finite value in their fractional derivative. We use this feature to characterize the points of an ECG graph for differential diagnostic purposes. Thus, fractional calculus is a tool to describe non-differentiable systems, and we will study this aspect in this book. This shows the reality of having a course in fractional calculus, at least at the post-graduate level, for the benefit of further research in mathematics, science and engineering.

Calculus is about evaluating rate, i.e. changing the dependent-quantity (say Δf) per unit of differential independent quantity (say Δx). In classical calculus, we take this differential independent quantity (variable) towards zero, i.e. in the limit $\Delta x \downarrow 0$, which gives rise to the notion of a classical derivative ($f^{(1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$) and (anti-derivative) classical integration ,i.e. $f^{(-1)}(x) = \int \big(f(x)\big)\mathrm{d}x$. Both classical derivative and classical integration are conducted with respect to the differential quantity, i.e. Δx, which is made zero. The question is if we have a finite spread of a differential element, by which we mean a classical differential element to the power of a fractional number (less than one); i.e. $(\Delta x)^{\alpha}$, with $0 < \alpha < 1$; then we will have derivative and integration with respect to this finite spread of the element i.e. $(\mathrm{d}x)^{\alpha}$, with $(\Delta x)^{\alpha} > \Delta x$, while we take the limit of $\Delta x \downarrow 0$. Thus, we will be dealing with a fractional derivative and fractional integration, taking rate (derivative) and integration with respect to $(\Delta x)^{\alpha}$ into fractional calculus.

This argument shows that, when we are observing a system with coarse-grained phenomena, we will be using fractional calculus to describe its dynamics. Why is this coarse graining required? The reason is that this method gives us a way to view the system, taking into account the homogeneities and the roughness, etc. present in real physical systems. On the other hand, the conventional fine graining approach gives us a view of macroscopic behavior. Thus, we move away from a point quantity (which is ideal) to a non- local quantity, which is reality, and thus we use fractional calculus. We will also discuss the development of fractional calculus in conjugation with classical calculus for non- differentiable functions.

We may use fractional derivatives to deal with differentiable functions, but we must ask questions about whether this is essential. As far as a function is differentiable and a system is Markovian (i.e. the system's present state is not affected by its past states), then its dynamic equation is then quite well-defined. If its dynamics are differentiable, strictly speaking, we should not need fractional derivatives in order to analyze its behavior. We may suggest that a given system, which is defined by a differential equation (such as the classical one, i.e. $x^{(1)}(t) = f(t)$) will show dynamics with a fractional derivative i.e. $x^{(\alpha)}(t) = g(t)$ and will therefore contribute extra information. Many system dynamics show memory-based behaviour where a non- Markovian approach is required; there the dynamics $x^{(\alpha)}(t) = g(t)$, with fractional derivative have been employed. Those ones that we described in Figures P7 and P8, are dynamics with memory. However, if the function in consideration is not differentiable, fractional derivatives should be of use. This concept of using a fractional derivative suitably defining non-differential functions in conjugation with classical calculus is a developing subject.

As I mentioned, this book is mainly comprised of all my previous hand-written classroom lecture notes that I have used for more than a decade for teaching in several classes on the subject of fractional calculus, and the treatment here is unconventionally simple, without rigorous formalism, and with elaboration of every step of the discussion. I can guarantee that a high school/college student who has studied differentiation and integration will be hooked on this book, and he/she may develop a liking for this subject, and perhaps even later work towards the advancement of fractional calculus and its applications. I cannot claim that each aspect of fractional calculus will be addressed in full in this book, but I can definitely claim that it will generate immense interest amongst new researchers in order to further investigate and use, develop and apply this subject. I have learnt this subject from various pioneering studies that I have listed in detail in the Bibliography section; and I provide a few notable works at the end of each chapter.

I humbly submit that my primary motivation was to make and then engineer electronics circuits and systems, and use fractional calculus in automatic controls for 'robust and efficient systems'. I started on this subject on my own, more than a decade and a half ago without any guide, and picked up the mathematics part, and simplified the same all by myself; primarily for my own sake. I enjoyed taking this course and enjoyed every bit of interaction with all my students and professors, learning a lot from them. At this time, the subject did not exist for students at most universities and institutes in India, although it is gradually growing in prominence in this part of the world.

I am still trying to develop this subject, and its applications in science and engineering, and I am still learning, too. I thank all my colleagues who have taken up this subject passionately with me. I am grateful to my colleagues (both students and professors) who have allowed me to propagate this "new subject" and deliberate with them, even though I am from an engineering background and not qualified with any higher academic degrees. Even today, this subject is not widely accepted, and still I feel no harm in pursuing it. I have several light-years to travel before I fully know what this subject is, and, if our nature truly follows the mathematics of fractional calculus. I wish you pleasant reading.

Shantanu Das

Shantanu Das

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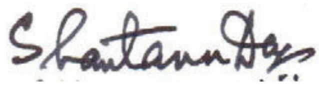

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CHAPTER ONE

CONCEPT GENERALISATION

1.1 Introduction

In this chapter, we will do a 'concept generalisation' of what we already know about $\frac{d}{dx} f(x)$ or $\int_0^x \left(f(y)\right) dy$ i.e. the usual classical derivative and classical integration operation in order to get expressions for n fold differentiation and integration. The further generalisation trick is to have n as non-integers; that we will derive in subsequent chapters. The concept generalisation is like that from natural numbers i.e. $n = 1, 2, 3, ...$ we generalise the number line to the negative side i.e. $n = -1, -2, -3, ...$ and call them integer numbers. Then we have, in between these numbers: $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{3}{4}, \pm\frac{3}{2}, ...$ or even irrational ones like π, $\sqrt{2}$, $-\sqrt{3}$ etc. and we call them the real line. Similarly, we extend one whole differentiation i.e. $\frac{d[f(x)]}{dx}$ to have twice the differentiation i.e. $\frac{d^2[f(x)]}{dx^2}$, then thrice i.e. $\frac{d^3[f(x)]}{dx^3}$ and generalise to get the expression for n folds, that is $\frac{d^n[f(x)]}{dx^n}$. Similarly we generalise one whole integration i.e. $\int_0^x \left(f(y)\right) dy$ to n folds integration, that is $\int_0^x \left(f(y)\right)(dy)^n$, or $\int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} ... \int_0^{x_2} dx_1 \int_0^{x_1} \left(f(y)\right) dy$. The extension of our classical calculus to fractional calculus is a generalisation of the theory of calculus. This extension in terms of mathematics is an 'analytic-continuation' of the operation i.e. n- fold differentiation and integration from the n integer to entire complex plane z. So we can have the derivatives (or integrals) in between like $\frac{d^{1/2}}{dx^{1/2}} f(x)$. That is, we have operation $\frac{d^\alpha}{dx^\alpha} f(x)$, where α is arbitrary, a 'real number', a 'complex number', even a 'function' as a continuous distribution. A positive α will signify a differentiation process and a negative α will give us integration (an anti-differentiation process). This is a generalised theory of calculus, a subject as old as Leibniz's or Newtonian calculus, and is called fractional calculus. Like further generalisation, can we have α as a complex number? The answer is why not. Similarly, can we have α as a continuous distribution function? The answer again is why not. We can also have a system where α is a variable. Therefore, there is fun in learning the subject of fractional calculus. In this chapter, we will be dealing with concept generalisation, with examples, and describing various important functions that are the outcome of the process of generalisation. In addition, we will apply this generalisation to obtain interesting relations in classical calculus theory, for $n-$ fold integration and $n-$ fold differentiation, like chain rule, Leibniz's rule etc. This chapter thus demonstrates the concept generalisation process.

1.2 The thought problem for constructing a 'half-coin': an example of concept generalisation

1.2.1 The idea of Negative Probability through the generalisation of existing theories

The idea of negative probability was considered by Richard P. Feynman and M. S. Bartlett as a generalisation of existing theories. Bartlett wrote in 'Mathematical Proceedings of Cambridge Philosophical Society' (1945), "It has been shown that orthodox probability theory may consistently be extended to include probability numbers outside the conventional range, and in particular negative probabilities. Random variables are correspondingly 'generalised' to include extraordinary random variables; these have been defined in general, however, only through their characteristic functions. This 'generalised theory' implies redundancy, and its use is a matter of convenience. Negative probabilities must always be combined with positive ones to give an ordinary probability before a physical interpretation is admissible." Therefore, the notion of negative probability is the process of concept generalisation.

In the following example, we elucidate the possibility of having negative probability, by constructing a hypothetical 'half-coin'. This notion of a half-coin construct was first introduced in 2005 by Gabor. J. Szekely, and further worked on by J. A. T. Machado, for possible applications in physical systems and in relation to fractional calculus. This concept of negative probability, though it sounds absurd, has recently been getting attention regarding its possible usages in physical systems.

1.2.2 The tossing of coins and Probability Generation Functions (PGFs)

We know about the full process of coin tossing, and we construct this process in order to realize a Probability Generating Function (PGF). A fair coin has two sides, Heads (H) and Tails (T), which can be represented as 'states' say $\mathrm{X}=0$ and $\mathrm{X}=1$ with equal probabilities that are $\frac{1}{2}$. We write PGF as $\mathrm{G}_{1\text{-}\mathrm{X}}(x)$, indicating the PGF of 'one-whole coin' as having two states $\mathrm{X}=0$ and $\mathrm{X}=1$ as in the following expression:

$$\begin{aligned} \mathrm{G}_{1\text{-}\mathrm{X}}(x) &= \tfrac{1}{2}(1+x) = \tfrac{1}{2}x^0 + \frac{1}{2}x^1 \\ &= \left(\mathrm{P}(\mathrm{X}:0)x^0 + \mathrm{P}(\mathrm{X}:1)x^1\right) \end{aligned} \tag{1.1}$$

In (1.1) we call $\mathrm{P}(\mathrm{X}:0)=\frac{1}{2}$ the probability of getting a 'Heads', i.e. for the first state corresponding to the variable x^0. Then we have $\mathrm{P}(\mathrm{X}:1)=\frac{1}{2}$ as the probability of getting a 'Tails', i.e. for the second state, corresponding to the variable x^1. In the PGF of a fair coin, we have a function that is $\frac{1}{2}(1+x)$. If we write the PGF as $\frac{1}{2}(1-x)$, then we will have $\mathrm{P}(\mathrm{X}:1)=-\frac{1}{2}$, assigning negative probability for state two (i.e. for $\mathrm{X}=1$). This looks absurd but mathematically we can have it. Up until now, we have been unable to attach any useful physical meaning to a probability which can be negative or which can have a value of probability greater than one. One may argue that probability as a positive number signifies the occurrence of an "event", so a negative probability may signify an "anti-event"!

With two such fair coins, we have the PGF as the following expression, represented as $\mathrm{G}_{2\text{-}\mathrm{X}}(x)$

$$\begin{aligned} \mathrm{G}_{2\text{-}\mathrm{X}}(x) &= \left(\mathrm{G}_{1\text{-}\mathrm{X}}(x)\right)\left(\mathrm{G}_{1\text{-}\mathrm{X}}(x)\right) \\ &= \left(\tfrac{1}{2}(1+x)\right)^2 \\ &= \tfrac{1}{4}x^0 + \tfrac{1}{2}x^1 + \tfrac{1}{4}x^2 \end{aligned} \tag{1.2}$$

In (1.2) we have the probability $\mathrm{P}(\mathrm{X}:0)=\frac{1}{4}$ corresponding to the state-variable x^0 that is state one, and is for the event HH. The probability $\mathrm{P}(\mathrm{X}:1)=\frac{1}{2}$ corresponds to the state variable x^1 that is state two and is for event HT (or TH). The probability $\mathrm{P}(\mathrm{X}:2)=\frac{1}{4}$ corresponds to the state variable x^2 that is state three and is for event TT. We extend this PGF to three coins and write the PGF i.e. $\mathrm{G}_{3\text{-}\mathrm{X}}(x)$ as follows

$$\begin{aligned} \mathrm{G}_{3\text{-}\mathrm{X}}(x) &= \left(\tfrac{1}{2}(1+x)\right)^3 \\ &= \tfrac{1}{8}x^0 + \tfrac{3}{8}x^1 + \tfrac{3}{8}x^2 + \tfrac{1}{8}x^3 \end{aligned} \tag{1.3}$$

We have four states with probabilities $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$ respectively, for the events HHH, TTH, HTT, HHT, HTH, and TTT.

We extend this (1.3) to the n-coin system; and the $(n+1)$ states with their corresponding probabilities are expressed in the following steps

$$\begin{aligned} \mathrm{G}_{n-\mathrm{X}}(x) &= \left(\tfrac{1}{2}(1+x)\right)^n \\ &= \frac{1}{2^n}\left(1+nx+\tfrac{n(n-1)}{2!}x^2+\tfrac{n(n-1)(n-2)}{3!}x^3+\dots.x^n\right) \\ &= \tfrac{1}{2^n}x^0+\tfrac{n}{2^n}x^1+\tfrac{n(n-1)}{2^n 2!}x^2+\tfrac{n(n-1)(n-2)}{2^n 3!}x^3+\dots.\tfrac{1}{2^n}x^n \end{aligned} \tag{1.4}$$

1.2.3 Generalizing the PGF to get a half-coin construct: a paradox?

We ask ourselves what happens if $n=\frac{1}{2}$, that is in effect, a 'half-coin'! We place $n=\frac{1}{2}$ in the above formula (1.4) and get the PGF of the half-coin denoted as $\mathrm{G}_{\frac{1}{2}\text{-}\mathrm{X}}(x)$, following a series of infinite 'events' and 'anti-events' (we use the term 'anti-events' for state-variables with negative probabilities).

$$
\begin{aligned}
G_{\frac{1}{2}\text{-X}}(x) &= \left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left(1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+.....\right) \\
&= \frac{1}{\sqrt{2}}x^0+\frac{1}{2\sqrt{2}}x^1-\frac{1}{8\sqrt{2}}x^2+\frac{1}{16\sqrt{2}}x^3+...
\end{aligned}
\tag{1.5}
$$

Now with two half coins we get a PGF of one-whole coin as demonstrated in the following steps:

$$
\begin{aligned}
G_{1\text{-X}}(x) &= \left(G_{\frac{1}{2}\text{-X}}(x)\right)\left(G_{\frac{1}{2}\text{-X}}(x)\right) \\
&= \left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}}\left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}} \\
&= \tfrac{1}{2}(1+x)
\end{aligned}
\tag{1.6}
$$

That is (1.6) the PGF we have for a single fair coin (1.1). But what about an infinite number of 'anti-events' appearing in the 'half-coin' construct above in (1.6)? Those are cancelled as we demonstrate in the following steps:

$$
\begin{aligned}
G_{1\text{-X}}(x) &= \left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}}\left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}} \\
&= \left(\left(\tfrac{1}{2}(1+x)\right)^{\frac{1}{2}}\right)^2 \\
&= \left(\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}x-\frac{1}{8\sqrt{2}}x^2+...\right)^2 \\
&= \frac{1}{2}+\frac{1}{2\times 2}x-\frac{1}{2\times 8}x^2+... \\
&\quad +\frac{1}{2\times 2}x+\frac{1}{2\times 4}x^2+.... \\
&\quad -\frac{1}{2\times 8}x^2+... \\
&= \frac{1}{2}+\frac{1}{2}x+0x^2+0x^3+...
\end{aligned}
\tag{1.7}
$$

It is interesting to consider what the case of an n-coins system will be if we determine that $n=\frac{1}{2}$ gives us a construct of a 'half-coin' with the existence of 'negative probabilities'. We are in a paradoxical situation, unable to attach physical meaning to this notion of negative-probability or anti-event. A similar query was posed by L'Hospital to Leibniz in respect to $\frac{d^n}{dx^n}f(x)$; questioning what it would become if n is determined to be half? This brief demonstration of a half coin construct indicates the method of generalisation of our existing formulations of processes. It would be interesting to use this half-coin to generate, say, a Brownian motion.

1.3 The question posed by L'Hospital to Leibniz

In a letter to L'Hospital on September 30th 1695, Leibniz raised the possibility of generalizing the operation of differentiation to non-integer orders, that is finding $\frac{d^n}{dx^n}f(x)$, and L'Hospital asked what would be the result of half-differentiating the function $f(x)=x$; that is: $\frac{d^{1/2}}{dx^{1/2}}\left[x\right]$. Leibniz replied: "It leads to a paradox, from which one day useful consequences will be drawn". The paradoxical aspects arise because there are several different ways of generalizing the differentiation operator into non-integer orders, leading to inequivalent results. This aspect we will discuss in subsequent chapters. We can say with accuracy that this query was dated the 30th September, 1695 and that it gave birth to "Fractional Calculus"; therefore this subject of fractional calculus with half derivatives and half integrals is as old as conventional Newtonian or Leibniz's calculus. Classical calculus is also termed Newtonian calculus. However, this subject of fractional-calculus (some even term this as non-Newtonian calculus) was dormant until the beginning of the century, and only now has it started finding applications in science and engineering.

In due course, we will be replying to the question posed by L'Hospital. However, in the case of negative probability as described in Section 1.2, the difficultly in attaching physical meaning was noted, but in the case of $\frac{d^\alpha}{dx^\alpha}f(x)$ with α as a non-integer, we do have physical meaning. Today we have systems based on these fractional differential equations. Here we will be doing detailed mathematical derivations without diverging much towards physics or engineering; rather, we will be generating fun in doing and discussing various aspects of $\frac{d^\alpha}{dx^\alpha}f(x)$ where α is arbitrary and not restricted only to integer numbers, as well as discussing paradoxes and difficulties. Our classical calculus knowledge is enough and we thus generalise in the same way in the next chapters and bring out the essence of fractional calculus, which is a generalisation of our classical integer order and Newtonian or Leibniz's calculus. In this book, there will be various aspects of classical calculus that we will be using in a fractional calculus context. Therefore, recapitulation of some of these classical concepts is essential.

1.4 A recall for analytic functions

If the function $f(z)$ is an analytic at point z then derivative $f^{(1)}(z)$ is continuous at z. This implies that if $f(z)$ has continuous derivatives of all orders at the point z, the function is analytic. A function is called an analytic function if it is locally given by a convergent power series. There are 'real analytic' and 'complex analytic' function categories that are similar in some ways, but different in other aspects. Functions of each analytic type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. We will revise this concept.

1.4.1 Real analytic functions

A function is analytic if, and only if, its Taylor series about a point, say x_0, converges to the function in some neighborhood for every x_0 in its domain. Formally, a function f is a real analytic function on an open set A on the real line $\mathbb{R}$, if for any x_0 in A one can write the following series expansion:

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+........ \tag{1.8}$$

In (1.8), the terms a_0, a_1, a_2, etc., are real numbers and the series is convergent to $f(x)$ for x in the neighborhood of x_0. Alternatively, an analytic function is infinitely differentiable such that the Taylor series at any point x_0 in its domain is the following

$$T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad f^{(n)}(x_0)=\left.\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right|_{x=x_0} \tag{1.9}$$

which converges to $f(x)$ for all x in the neighborhood of x_0 (locally and uniformly).

A function is complex analytic if and only if it is 'holomorphic' – that is, it is a 'complex differentiable'. Examples of analytic functions are polynomials, exponentials, trigonometric functions, logarithmic functions, power functions, special functions like Mittag-Leffler functions, Miller-Ross functions, hyper-geometric functions, Bessel's functions (refer to Appendix A), or Gamma functions. Special functions are analytic at least in some ranges of the complex plane.

Non-analytic functions like $f(x)=|x|+b$, $f(x)=x^{\alpha}+c;\quad 0<\alpha<1,\quad x\geq 0$ are not everywhere analytic, as they are not differentiable at zero, i.e. at $x=0$. The functions that are defined at different zones with different functions are not analytic where the zones meet. We have discussed the point that analytic functions are infinitely differentiable, indicating that they are smooth, but all smooth functions need not be analytic.

1.4.2 Complex analytic functions

Now we shall look at a complex differentiable concept and a complex analytic condition. In complex variables a function $f(z)$ where $z=x+iy$ is said to be analytic in region 'A' of a complex plane $\mathbb{C}$ if $f(z)$ has a derivative at each point of A and if $f(z)$ is said to be single valued. In other words, $f(z)$ is said to be analytic at a point z, if z is an interior point of some region where $f(z)$ is analytic. Hence, the concept of an analytic function at a point implies that the function is analytic in some circles with the centre at that point. The condition for a complex function $f(z)=u(x,y)+iv(x,y)$, with $z=x+iy$ and conjugate $\bar{z}=x-iy$ to be analytic is something we will discuss.

Since $x=\left(\frac{z+\bar{z}}{2}\right)$ and $y=\left(\frac{z-\bar{z}}{2i}\right)$, we write $f(z,\bar{z})=u(x,y)+iy(x,y)$. A necessary condition for $f(z,\bar{z})$ to be analytic is:

$$\frac{\partial}{\partial\ \bar{z}}f(z,\bar{z})=0 \tag{1.10}$$

Therefore, a necessary condition for $f=u+iv$ to be analytic is that f depends on z. In terms of real and imaginary parts u, v of f, the above (1.10) condition is equivalent to:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \tag{1.11}$$

This expression (1.11) is also called a Cauchy-Riemann equation. The necessary and sufficient condition therefore is that the four partial derivatives satisfy the Cauchy-Riemann equation, and that these four partial derivatives are continuous. From the above expression (1.11) we can also write the following expressions, called Laplace equations:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \tag{1.12}$$

The real and imaginary parts of an analytic function are 'harmonic conjugate functions'. That is to say that the complex analytic function $f=u+iv$ is a solution to the above (1.12) Laplace equations, and satisfies the Cauchy-Riemann equations (1.11).

1.5 Recalling the Cauchy integral formula

In the complex-variables review, we have studied contour integration. We revise those parts here, as in subsequent chapters we will be using this method; especially in order to find inverse Laplace transformations (Appendix G). We studied the Cauchy-Goursat theorem (or simply Cauchy's theorem); which says that if $f(z)$ is analytic in and on closed contour C, then the contour integral $\int_{\mathrm{C}}\big(f(z)\big)\mathrm{d}z=0$. We recall that the positive sense of travel on the closed contour is in the anti-clockwise direction. A consequence of this is that $\int_a^b\big(f(z)\big)\mathrm{d}z$ does not depend on a path. Cauchy's integral formula states that if $f(z)$ is analytic within and on closed contour C, then

$$f(z_0)=\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{f(z)}{(z-z_0)}\mathrm{d}z \tag{1.13}$$

Taking one-whole derivative of Cauchy's integral formula in (1.13), with respect to z_0 we write $f^{(1)}(z_0)=\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{f(z)\mathrm{d}z}{(z-z_0)^2}$; and by continuing this differentiation n times, we get the following important relationship

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{\mathrm{C}}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z \tag{1.14}$$

1.5.1 The derivative is an integration process

An interesting observation from (1.14) is that differentiation is an integration process, that is $\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)\big]=\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{(f(y))\mathrm{d}y}{(y-x)^2}$ and $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big[f(x)\big]=\frac{n!}{2\pi i}\int_{\mathrm{C}}\frac{(f(y))\mathrm{d}y}{(y-x)^{n+1}}$.

It is a very important fact that the derivative is actually an integration process that will be helpful in later chapters, where we generalise our earlier classical calculus theories. In the formula (1.14) i.e. $f^{(n)}(z_0)=\left(\frac{n!}{2\pi i}\right)\int_{\mathrm{C}}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$; the integration path on the closed contour C is anti-clockwise (say), and a small circle encircling the point z_0; for n as a zero or positive integer. If n is any real number, then the contour is differently drawn. This concept we will be using in subsequent chapters in order to evaluate the following $f^{(\alpha)}(z)$ i.e. $\alpha-$ order derivative is:

$$f^{(\alpha)}(z)=\frac{\alpha!}{2\pi i}\int_{\mathrm{C}}\frac{f(\omega)}{(\omega-z)^{\alpha+1}}\mathrm{d}\omega \tag{1.15}$$

In (1.15) α is a non-integer number.

1.5.2 The Taylor series from Cauchy's integral formula

With Cauchy's integral formula, we can now derive the Taylor series. As we stated, an analytic function has a convergent expansion about any point z within its domain of analyticity. We write Cauchy's integral formula (1.13) as $f(z)=\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{(f(z_1))\mathrm{d}z_1}{(z_1-z)}$, where C is circle centered at z_0; and z is within C so $\left|\frac{z-z_0}{z_1-z_0}\right|<1$. With this, we write the following steps

$$\frac{1}{z_1 - z} = \frac{1}{(z_1 - z_0)\left(1 - \frac{z - z_0}{z_1 - z_0}\right)}$$
$$= \frac{1}{(z_1 - z_0)}\left(1 - \left(\frac{z - z_0}{z_1 - z_0}\right)\right)^{-1} = \frac{1}{(z_1 - z_0)}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{z_1 - z_0}\right)^n \tag{1.16}$$
$$= \sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(z_1 - z_0)^{n+1}}$$

We have used the expansion $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$, that is simply a summation of a geometric series in (1.16). Therefore, Cauchy's integral formula becomes

$$f(z) = \frac{1}{2\pi i}\int_{\mathrm{C}}\frac{\big(f(z_1)\big)\mathrm{d}z_1}{(z_1 - z)}$$
$$= \frac{1}{2\pi i}\int_{\mathrm{C}}\left(\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(z_1 - z_0)^{n+1}}\right)\big(f(z_1)\big)\big(\mathrm{d}z_1\big)$$
$$= \frac{1}{2\pi i}\int_{\mathrm{C}}\left(\sum_{n=0}^{\infty}\frac{\big(f(z_1)\big)(z - z_0)^n}{(z_1 - z_0)^{n+1}}\right)\mathrm{d}z_1 \tag{1.17}$$
$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{\big(f(z_1)\big)\mathrm{d}z_1}{(z_1 - z_0)^{n+1}}\big((z - z_0)^n\big)$$

Using (1.14) i.e. $\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{(f(z_1))\mathrm{d}z_1}{(z_1 - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}$ (this is coming from Cauchy's integral formula), in the above derived expression (1.17) we get the following:

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{f^{(n)}(z_0)}{n!}\right)(z - z_0)^n \tag{1.18}$$

In (1.18) we have achieved the Taylor series for an analytic function $f(z)$.

1.6 Singularity pole branch point and branch cut

We shall now revise the concept of singularities of analytic functions, that are points at which $f(z)$ is not analytic and which are called singular points or singularities. That is, the function blows up at those points.

1.6.1 Isolated singularity

The 'isolated singularity' of $f(z)$ which is analytic everywhere throughout some neighborhoods of a point $z = a$, say inside a circle $|z - a| = R$ (with radius R and centered at $z = a$); except at the point $z = a$. Then the point $z = a$ is called an isolated singularity of $f(z)$. Thus, the function $f(z)$ cannot be bounded near an isolated singularity.

1.6.2 Pole singularity (removable singularity)

The next type of singularity is called a 'pole'. If $f(z)$ has an isolated singular point at $z = a$, and if in addition there exists an integer i.e. n such that $(z - a)^n f(z)$ is analytic at $z = a$, then $f(z)$ has a pole of order n at $z = a$. We note here that because $(z - a)^n f(z)$ is analytic at $z = a$, such a singularity is termed as a 'removable singularity'. For example, $f(z) = z^{-2}$ have poles of order 2 at $z = 0$

1.6.3 Essential singularity

'Essential singularity' is an isolated singular point, which is not a pole (that is not a removable singularity); for example, $f(z) = \sin\left(\frac{1}{z}\right)$ has essential singularity at $z = 0$.

We ought to clarify the above discussions regarding singularity. Suppose $f(z)$ has isolated singularity at z_0 but is analytic in the neighborhood of z_0, one can write a Laurent series as the following

$$f(z)=\left(\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+.........\right)+\left(\sum_{n=0}^{\infty}a_n(z-z_0)^n\right) \tag{1.19}$$

We classify the isolated singularity as following on from (1.19):

Simple pole: $b_n=0, \quad n>1$, then $f(z)=\frac{b_1}{(z-z_0)}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$

Pole of order N : $b_n=0, \quad n>N$, then $f(z)=\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+....+\frac{b_N}{(z-z_0)^N}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$

Essential singularity: Infinite terms of $(z-z_0)^{-1}$ then $f(z)=\sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$

1.6.4 Multivalued functions, branch points and branch-cuts

Branch points and branch-cuts are the methods used to tackle multi-valued functions. There are three types of multi-valued functions: a) the logarithmic type, b) the algebraic type and c) the trigonometric type, and they have respective branch cuts and branch points.

Knowing this, we can discuss multi-valued functions, for example $f(z)=\ln z$, which blow out at $z=0$. If we write $z=re^{i\theta}$ i.e. in polar form, then we have $\ln z=\ln r+i\theta$. Thus for apparently the same point i.e. $z=re^{i(\theta+2\pi n)}$ with n as integers (also termed winding numbers) and for say $\theta\sim 0$; the function $\ln r+2\pi in$ can have different values as we go around the circle around point $z=0$ (depending on the winding number, i.e. n). Thus this function blows out at $z=re^{i\theta}$ when r tends towards 0, at infinite θ values say $\theta=0,2\pi,3\pi,4\pi...$ spread on a primary and infinite number of secondary complex planes, one beneath the other; called Riemann sheets (Appendix E).

Therefore the point $z=0$, is not a simple essential singularity or pole, rather it is a 'branch-point'. In this case, an infinite number of values are possible and $\ln z$ has an infinite number of 'branches'. This requires us to introduce 'branch-cuts', which are barriers through which z cannot go (an example being the line integral), and thus $f(z)=\ln z$ remains a single value.

The logarithm has a jump discontinuity of $2\pi i$ when crossing the branch cut. The logarithm can be made continuous by gluing together many copies, called sheets, of the complex plane along the branch cut. On each sheet, the value of the log differs from its principal value by a multiple of $2\pi i$. These surfaces are glued to each other along the branch cut in a unique way to make the logarithm continuous. Each time the variable goes around the origin, the logarithm moves to a different branch (Appendix E).

Alternatively one can think of Riemann sheets whereby crossing a branch-cut one moves onto a different Riemann sheet of the function. The number of branches is equal to the number of Riemann sheets. This allows closed contours to be formed by going around branch cuts as many times as required to get back to the original (or primary) Riemann sheet. In the case of $\ln z=\ln r+2\pi in$, the primary Riemann-sheet is for $n=0$. This example is of a 'logarithmic branch point'. In the function $f(z)=\ln z$ we can argue the same at the point $z=\infty$ by making a substitution i.e. $z\equiv\frac{1}{z}$ and then say that the point $z=\infty$ is also a branch point. The branch cut therefore joins these two points $z=0$ and $z=\infty$, and makes the barrier that one should not cross, and the function stays single valued (with its 'principal value'). Similarly the function $f(z)=z^{\alpha}$ for $\alpha\in\mathbb{R}$, is multi-valued and has a logarithmic branch point at $z=0$. We will not be discussing the trigonometric multi-valued function.

Another type of branch point is the 'algebraic branch point'. For example point zero ($z=0$) is a branch point of the square root function i.e. $f(z)=\sqrt{z}$ this we explain here ., , Take the function $f(z)=z^{1/2}$ and take the point z that starts at $z=4$ and moves along a circle with a radius equalling 4 in the complex plane centered at $z=0$. When z has made one full circle (0 to 2π), going from $z=4$ back to $z=4$ again, $f(z)$ will have made one half-circle (0 to π), going from the positive square root of $z=4$ i.e., from $f(z)=2$ to the negative square root of $z=4$ i.e. $f(z)=-2$.

Roughly speaking, the branch points are the points where the various sheets of a multiple valued function come together. The branches of the function are the various sheets of the function. For example, the squre root function

$f(z) = z^{1/2}$ has two branches: one where the square root comes in with a plus sign, and the other with a minus sign and these two branches meet at $z = 0$ that is the branch point. A 'branch cut' is a curve in the complex plane from which it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve. Branch cuts are usually, but not always, taken between pairs of branch points. Branch cuts allow one to work with a collection of single-valued functions, 'glued' together along the branch cut instead of a multi-valued function. For example, to make the function single-valued, one makes a branch cut along the interval $[0,1]$ on the real axis, connecting the two branch points of the function. The same idea can be applied to the function $f(z) = z^{1/2}$; but in that case one has to perceive that the 'point at infinity' is the appropriate 'other' branch point to connect to from $z = 0$, for example along the whole negative real axis.

We will use these concepts of branch cuts, branch points and Riemann-sheets in obtaining an inverse Laplace transform of some special functions. In this chapter too we will deal with branch cuts and branch points depicted in Figure 1.4, while we do contour integrations in the section of Beta functions (Section 1.15). The standard branch cut that we will use in complex planes is a line in the negative real axis from a point of interest (i.e. $z = 0$ or some other branch point, say $z = \operatorname{Re} z_0$) to point the $z = \operatorname{Re} z = -\infty$. Refer to Appendix E for more details and discussion on branch cuts and Appendix G for an elucidation on obtaining an inverse Laplace transform.

1.7 Recalling residue calculus

We will be using the concept of residue calculus in subsequent chapters. We recall that the residue calculus is used for complex differentiation, complex integration, and power series expansion (PSE), and provides three approaches to holomorphic (complex analytic) functions. The Cauchy integral formula provides the relationship between complex differentiation and complex integration. The residue serves to formulate the relationship between complex integration and PSE.

Let us recall that the residue of a function f can be holomorphic everywhere within and on closed curve C except possibly at point z_0 in the interior of C where f may have isolated singularity. Then the residue of f at z_0 is

$$\text{Residue}_{z_0} f \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{\mathbf{C}} \big(f(z)\big)\,\mathrm{d}z \tag{1.20}$$

If z_0 is a non-singular point then the residue at z_0 is zero, else it is non-zero. The use of the definition of residue is to calculate the contour integral $\int_{\mathbf{C}} \big(f(z)\big)\,\mathrm{d}z$. If the closed contour encloses say n poles at e.g. $z = z_1, z_2, ...z_n$ then we write the following rule:

$$\begin{aligned}\int_{\mathbf{C}} \big(f(z)\big)\,\mathrm{d}z &= 2\pi i\Big(\text{Residue}_{z_1} f + \text{Residue}_{z_2} f + ...\text{Residue}_{z_n} f\Big)\\ &= 2\pi i\sum_{i=1}^{n} \text{Residue}_{z_i} f\end{aligned} \tag{1.21}$$

Therefore, calculations of residue are very important for the evaluation of $\int_{\mathbf{C}} \big(f(z)\big)\,\mathrm{d}z$.

1.7.1 Finding the residue by power series expansion

Consider the Laurent series expansion of f at z_0 i.e. $f(z) = \sum_{n=-\infty}^{n=\infty} a_n \left(z - z_0\right)^n$, where $a_n = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{f(\xi)}{(\xi-\xi_0)^{n+1}}\mathrm{d}\xi$. Here residue is nothing but a coefficient of $(z - z_0)^{-1}$, in the expansion. For example say $f(z) = \frac{1-z}{(1-2z)^2}$, has a pole at $z = \frac{1}{2}$ of order $n = 2$. We write:

$$f(z) = \frac{1-z}{(1-2z)^2} = \left(\frac{1}{8}\right)\left(\frac{1}{\left(z-\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4}\right)\left(\frac{1}{\left(z-\frac{1}{2}\right)}\right) \tag{1.22}$$

The coefficient of term $(z - z_0)^{-1}$ here in (1.22) is $-\frac{1}{4}$ and that is residue of f at $z = \frac{1}{2}$. Similarly, the function $f(z) = \exp\left(\frac{1}{z^2}\right)$ has essential singularity at $z = 0$, and if we write the series as the following:

$$f(z) = \exp\left(\tfrac{1}{z^2}\right) = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + ... \tag{1.23}$$

In (1.23) we find that no term with $(z - z_0)^{-1}$ for $z_0 = 0$ exists; hence the residue of f at $z = 0$ in this case is zero. We used the series expansion of $\exp(z) = 1 + z + \frac{1}{2!}z^2 + \ldots$, with this one potentially verifying that for $f(z) = \exp\left(-\frac{1}{z}\right)$, the residue at essential singularity point $z = 0$ is -1. We see that for $f(z) = \frac{\sin z}{z^6}$ which has the pole of order $n = 5$ at $z = 0$. For $f(z) = \frac{\sin z}{z^6}$ by using an expansion formula i.e. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$, we get the following power series representation:

$$f(z) = \frac{\sin z}{z^6} = \frac{1}{z^6}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right) = \frac{1}{z^5} - \frac{1}{6z^3} + \frac{1}{120z} + \varphi(z) \tag{1.24}$$

In (1.24) the function $\varphi(z)$ is an analytic part, i.e. $\varphi(z) = \sum_{m=0}^{\infty} a_m z^m$. From the above series (1.24) we observe the residue is $\frac{1}{120}$ at point $z = 0$.

1.7.2 The formula for calculating residue

Now we shall derive a formula for the calculations of residues in the case of poles of a function $f(z) = \frac{h(z)}{(z-z_0)^n}$, with function $h(z)$ as holomorphic in the entire complex plane $\mathbb{C}$. Thus, we have $h(z) = (z - z_0)^n f(z)$. From the definition of the residue and then by utilizing the Cauchy formula, we write the following steps:

$$\begin{aligned} \text{Residue}_{z_0} f &= \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{h(\xi)}{(\xi - z)^n}\mathrm{d}\xi \\ &= \frac{1}{(n-1)!}\left(\frac{\mathrm{d}^{n-1}h(z)}{\mathrm{d}z^{n-1}}\right) \\ &= \lim_{z\to z_0}\left(\frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(z - z_0)^n f(z)\right) \end{aligned} \tag{1.25}$$

We used $\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{h(z_1)}{(z_1 - z)^m}\mathrm{d}z_1 = \frac{h^{(m)}(z)}{m!}$ i.e. the Cauchy formula (1.14) and then used $h(z) = (z - z_0)^n f(z)$, in the above (1.25) steps of derivation. Let us apply this to our earlier example (1.22) i.e. $f(z) = \frac{1-z}{(1-2z)^2}$ where we have a pole at $z = \frac{1}{2}$ of the order $n = 2$. We obtain the residue by formula (1.25) as demonstrated in the following steps:

$$\begin{aligned} \text{Residue}_{z=\frac{1}{2}} f &= \lim_{z\to\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}z}\left(\left(z - \tfrac{1}{2}\right)^2 f(z)\right) \\ &= \lim_{z\to\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1-z}{4}\right) = -\frac{1}{4} \end{aligned} \tag{1.26}$$

That is what we got from the Laurent series expansion (1.22) also. We have calculated in an earlier example of $f(z) = \frac{\sin z}{z^6}$ (1.24) the residue at $z = 0$ (five-order pole) as $z = \frac{1}{120}$. Now we can use this above described formulation (1.25) to calculate the contour integral on a closed curve C i.e. a circular or radial one, call it Ω centered at the point $z = 0$, and write that as the following:

$$\int_{\Omega} \frac{\sin z}{z^6}\mathrm{d}z = 2\pi i\left(\text{Residue}_{z=0} f\right) = 2\pi i\left(\frac{1}{120}\right) = \frac{\pi i}{60} \tag{1.27}$$

The above integral of (1.27) i.e. $\int_{\Omega} f(z)\mathrm{d}z$ is valid for any circle centered at origin with any radius, even an infinite radius.

We shall conclude this part by writing a few integrals obtained via residue calculus. Assume the contour C is a circle centered at origin, with a radius large enough so that it is enclosing all the singularities of functions $f(z)$ taken as $\frac{5z-2}{z(z-1)}$, $\exp\left(-\frac{1}{z}\right)$ and $\frac{\exp(\pi z/4)}{z^2+1}$, which are presented in the following expressions:

$$\int_C \frac{5z-2}{z(z-1)}\mathrm{d}z = 2\pi i\left(\text{Residue}_{z=0} f + \text{Residue}_{z=1} f\right) = 2\pi i\left(2+3\right) = 10\pi i$$

$$\int_C \exp\left(-\tfrac{1}{z}\right)\mathrm{d}z = 2\pi i\left(\text{Residue}_{z=0} f\right) = 2\pi i(-1) = -2\pi i$$

$$\begin{aligned}\int_C \frac{\mathrm{d}z}{z^2+1}\exp\left(z\tfrac{\pi}{4}\right) &= 2\pi i\left(\text{Residue}_{z=i} f + \text{Residue}_{z=-i} f\right) \\ &= 2\pi i\left(\frac{\exp\left(i\left(\frac{\pi}{4}\right)\right)}{2i} + \frac{\exp\left(-i\left(\frac{\pi}{4}\right)\right)}{(-2i)}\right) \\ &= 2\pi i \sin\left(\tfrac{\pi}{4}\right) = i\pi\sqrt{2}\end{aligned} \tag{1.28}$$

Appendix G gives us the use of residue calculus in order to obtain inverse Laplace transforms; for complex functions with poles and for multi-valued functions, where we employ branch-cuts in the primary Riemann sheet.

1.8 The basics of analytic continuation

We will be using the concept of analytic continuation for several concept generalisation processes in this chapter and in subsequent chapters. We start with a simple example of the function e^x, the exponential function originally defined on the real axis $\mathbb{R}$, the analytical continuation to this is $f(z) = 1 + z + \frac{z^2}{2!} + ...$ a unique function $f(z)$, where z is in the complex domain $\mathbb{C}$ that is equal to e^x on the real line. This is what is meant by an analytical continuation in this simple example. From the real domain we continued to a complex domain in this example of an exponential function.

The intersection of two domains (in the region of the complex plane in our case); say $\mathbb{C}_1$ and $\mathbb{C}_2$ denoted as $\mathbb{C}_1 \cap \mathbb{C}_2$ is a set of all points common to both $\mathbb{C}_1$ and $\mathbb{C}_2$. The union of two domains $\mathbb{C}_1$ and $\mathbb{C}_2$ denoted by $\mathbb{C}_1 \cup \mathbb{C}_2$ is the set of all points in either $\mathbb{C}_1$ or $\mathbb{C}_2$. Now suppose we have two domains $\mathbb{C}_1$ and $\mathbb{C}_2$, and a function f_1 that is an analytic in domain $\mathbb{C}_1$, if there exists a function f_2 that is analytic in domain $\mathbb{C}_2$ and such that $f_1 = f_2$ on the intersection $\mathbb{C}_1 \cap \mathbb{C}_2$, then we say that f_2 is an analytic continuation of f_1 into domain $\mathbb{C}_2$.

Now, whenever the analytic continuation exists it is unique. This comes from a theory of complex variables: 'A function is analytic in domain $\mathbb{C}$ and is uniquely determined over $\mathbb{C}$ by its values over the domain or along an arc interior to $\mathbb{C}$'. The function $F(z)$ analytic over union $\mathbb{C}_1 \cup \mathbb{C}_2$ can be defined as:

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } \mathbb{C}_1 \\ f_2(z) & \text{when } z \text{ is in } \mathbb{C}_2 \end{cases} \tag{1.29}$$

In other words F is given by f_1 over $\mathbb{C}_1$ and f_2 over $\mathbb{C}_2$, and since $f_1 = f_2$ over the intersection of $\mathbb{C}_1$ and $\mathbb{C}_2$ this is a well-defined holomorphic function. By the definition at the beginning of this section, since F is analytic in $\mathbb{C}_1 \cup \mathbb{C}_2$ it is uniquely determined by f_1 on $\mathbb{C}_1$ (for that matter it is also uniquely determined by f_2 on $\mathbb{C}_2$). In other words there is one possible holomorphic function on $\mathbb{C}_1 \cup \mathbb{C}_2$ that matches f_1 on $\mathbb{C}_1$. In this case $F(z)$ is said to be an analytic continuation over $\mathbb{C}_1 \cup \mathbb{C}_2$ of either f_1 or f_2. We therefore can immediately use the result to extend the originally defined function on the real axis to the complex plane example (as we stated at the beginning) with $e^z = 1 + z + \frac{z^2}{2!} + ...$ being a unique function $f(z)$ that is equal to e^x on the real line.

Let us consider $f_1(z) = \sum_{n=0}^{\infty} z^n$; this power series converges when $|z| < 1$ to $\frac{1}{1-z}$ and is not defined for $|z| \geq 1$. One understands this series as a geometric series and we can calculate this geometric series so long as we are in the region of convergence. On the other hand, function $f_2(z) = \frac{1}{1-z}$ is analytic everywhere except at point $z = 1$. Since $f_1 = f_2$ on the disk $|z| < 1$; we can view f_2 as an analytic continuation of f_1 to the rest of the complex plane (minus the point $z = 1$).

Consider the function g_1 represented as an integral in the equation $g_1(z) = \int_0^\infty e^{-zx}\mathrm{d}x$; and this integral exists only when $\mathrm{Re}\left[z\right] > 0$; and for such a condition this has a value of $\frac{1}{z}$. We can check by performing this integration that gives the result $\left(-\frac{e^{-zx}}{z}\right)_{x=0}^{x=\infty}$; which equals $\frac{1}{z}$ only for $z > 0$. Since the function $\frac{1}{z}$ matches g_1 on $\mathrm{Re}\left[z\right] > 0$ we say the function $\left(\frac{1}{z}\right)$ is an analytic continuation of g_1 to non-zero complex numbers. While we are at it, we can define $g_2(z) = i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n$. This series converges for $\left|z+i\right| < 1$; and so g_2 is defined only within a unit-disk centered on point $z = -i$. Using the substitution $\left(\frac{z+i}{i}\right) = p$ one can show that $g_2(z) = \frac{1}{z}$, by using the fact that $\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$ which is a summation of the geometric series when $\left|p\right| < 1$. That gives $\left|\frac{z+i}{i}\right| < 1$; this implies $\left|z+i\right| < 1$, (note that $\left|i\right| = 1$). Then $i\sum_{n=0}^{\infty} p^n = \frac{i}{1-p}$ and substituting $p = \left(\frac{z+i}{i}\right)$ gives $i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n = \frac{1}{z}$. Since g_2 matches $\frac{1}{z}$ on a disk, we can see that $\frac{1}{z}$ is an analytic continuation of $g_2(z) = i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n$ to non-zero complex numbers. In addition, we can view $g_2(z) = i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n$ as an analytic continuation of $g_1(z) = \int_0^\infty e^{-zx}\mathrm{d}x$ to the disk $\left|z+i\right| < 1$.

The above are examples for describing the concept of analytic continuity. However, there are exceptions that may be noted, in particular the fact that not all functions can be continued indefinitely. The functions may have natural barriers of singularities through which one cannot pass. It may happen that the function obtained through this continuation is a multi-valued one. For example, we may carry out analytical continuation along some path that returns to the region of convergence. Then if the path has crossed a branch-cut the values of analytically continued functions will not agree with the original value. The principle of analytic continuation may be involved in the context of integrals, generalising the recurrence relations which we will be seeing in due course.

1.9 The factorial and its several representations

The generalisation of factorial $n!$ to a non-integer number was first done by Bernoulli and Goldback (1720), then later on by Euler (1729) who gave the formulation using integral representation, series representation and then by Gamma function. The concept of a factorial was introduced by French mathematician Louis Franosis Antoine Arbogast. We know the factorial as $n! = n\times(n-1)\times(n-2)\times....2\times1$, this is called a falling factorial, or decreasing factorial.

1.9.1 Representation of a falling factorial

In a compact way we write $n! = \prod_{k=1}^{n} k$, or explicitly we state $n!$ as the following:

$$n! = \begin{cases} 1 & n = 0 \\ (n-1)!(n) & n > 0 \end{cases} \tag{1.30}$$

The number n in the above definition is a positive integer.

1.9.2 Double factorial

From here the double factorial is defined as the following:

$$n!! = \begin{cases} n\times(n-2)\times...\times5\times3\times1 & n > 0 \quad n:\text{odd} \\ n\times(n-2)\times...\times6\times4\times2 & n > 0 \quad n:\text{even} \\ 1 & n = -1,\quad 0 \end{cases} \tag{1.31}$$

A few values of the double factorial are $0!! = 1$, $1!! = 1$, $2!! = 2$, $3!! = 3$, $4!! = 8$, $5!! = 15$, $6!! = 48$, $7!! = 105$, $8!! = 384$, and $9!! = 945$. We have a relationship of the double factorial expressed as:

$$n! = n!!(n-1)!! \tag{1.32}$$

We are familiar with the above definition of $n!$ for n as a positive integer.

1.9.3 Truncated factorial

In addition, a truncated factorial is represented as $n^{\underline{k}}$ and is defined as:

$$n^{\overset{k}{--}} = n \times (n-1) \times (n-2) \times ...(n-k+1) \tag{1.33}$$

We have the following relationship from (1.33) and (1.30):

$$\frac{n^{\overset{k}{--}}}{k!} = \frac{n(n-1)(n-2)....(n-k+1)}{k(k-1)(k-2).....(1)} \tag{1.34}$$

The above expression (1.34) we reckon as binomial coefficients i.e. ${}^{n}C_k$, which appears in the binomial expansion of $(1+x)^n$. We write the following expression for the binomial coefficients in different forms:

$${}^{n}C_k \equiv \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} = \frac{n^{\overset{k}{--}}}{k!} = \frac{n^{\overset{n-k}{---}}}{(n-k)!} = \begin{pmatrix} n \\ n-k \end{pmatrix} \equiv {}^{n}C_{n-k} \tag{1.35}$$

1.9.4 Some other historical representations of factorials

Initially Bernoulli and Goldback wrote in an attempt to generalise the factorial as the following expression:

$$n! = \left(\frac{2^n}{1+n}\right) \times \left(\frac{\left(\frac{3}{2}\right)^n}{1+\frac{n}{2}}\right) \times \left(\frac{\left(\frac{4}{3}\right)^n}{1+\frac{n}{3}}\right) \times ..... = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^n}{\left(1+\frac{n}{k}\right)} \tag{1.36}$$

Euler gave the formula originally as the following expression:

$$(z-1)! = \lim_{n \uparrow \infty} \left(\frac{n^z (n!)}{\prod_{k=0}^{\infty} (z+k)} \right) \tag{1.37}$$

In expression (1.37) we have used $\uparrow$ as a symbol indicating 'tending to high' i.e. ∞, in the rest of the chapters we also use this symbol; $\downarrow$ we will use to indicate 'tending to low i.e. 0. For any other value other than 0 or ∞ we will use the usual symbol b i.e. $\to b$, saying 'tending to b'.

From the basic definition of a factorial that is $n! = 1 \times 2 \times .... \times (n-1) \times (n)$ we also say that $\log(n!) = \sum_{x=1}^{n} \log x$. On this Ramanujan worked and came out with the following representation i.e. approximately representing the factorial in two ways:

$$\begin{aligned} \log(n!) &\approx n \log n - n + \frac{\log\left(n\left(1+4n(1+2n)\right)\right)}{6} + \frac{\log \pi}{2} \\ &= n \log n - n + \frac{\log\left(\frac{1}{2n} + \frac{1}{8n^2}\right)}{6} + \frac{\log(2n)}{2} + \frac{\log \pi}{2} \end{aligned} \tag{1.38}$$

Stirling gave the approximation of a factorial as the following:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \tag{1.39}$$

where in (1.39) e is $y = e^x$ with $x = 1$.

These above expressions are approximation formulas to find the factorial even if n is a non-negative number not essentially an integer.

1.9.5 Increasing truncated factorials (Pochhammer numbers)

Here we'll talk about increasing the truncated factorial which is defined as:

$$(n)_k = \begin{cases} n(n+1)(n+2)...(n+k-1) & k > 0 \\ 1 & k = 0 \end{cases} \tag{1.40}$$

The above expression (1.40) of $(n)_k$ is also called the Pochhammer number (or polynomial). In terms of factorial notation, we can write an increasing factorial for $k > 0$, as:

$$(n)_k = \frac{(n+k-1)!}{(n-1)!} \tag{1.41}$$

The truncated factorial $n^{\underline{k}}$ defined above in (1.33) i.e. $n^{\underline{k}} = n(n-1)(n-2)...(n-k+1)$ is also a Pochhammer number; we may represent this decreasing truncated factorial as $(n-)_k$ depicted in the following representation:

$$(n-)_k = n(n-1)(n-2)...(n-(k-1)) = n(n-1)(n-2).....(n-k+1) \tag{1.42}$$

This is in order to distinguish it from the increasing truncated factorial i.e. $(n)_k = n(n+1)(n+2)...(n+k-1)$.

1.9.6 Euler's integral representation for factorials

In terms of integral representation Euler proposed $n!$ as the following formula:

$$n! = \int_0^1 \left(\ln \tfrac{1}{x}\right)^n \mathrm{d}x \tag{1.43}$$

The above expression (1.43) i.e. $n! = \int_0^1 (-\ln x)^n \mathrm{d}x$ can be put in other forms. Put $(-\ln x) = t$, so we get $x = e^{-t}$ and $-\left(\frac{1}{x}\right)\mathrm{d}x = \mathrm{d}t$. The limits of integration in the variable x, i.e. from 0 to 1, become the variable t from ∞ to 0. So, with these changes of variables we write $n! = \int_\infty^0 t^n \left(-x\mathrm{d}t\right) = \int_\infty^0 t^n e^{-t}(-\mathrm{d}t)$. This gives another integral representation of the factorial as:

$$n! = \int_0^\infty e^{-t} t^n \mathrm{d}t \tag{1.44}$$

This integral is also called 'Euler's integral of a second kind', and we will see its usage, shortly as a generalisation of factorial to non-integer numbers. The most popular generalisation of factorials is by use of the Gamma-function, which we will describe in the next section (Section 1.10).

1.10 The gamma function

One of the basic functions of fractional calculus is Euler's gamma function. This function generalises the factorial $n!$, and allows n to take non-integer values. The definition of the gamma function in integral form is the following:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t \tag{1.45}$$

This above (1.45) improper integral defined in complex plane $z \in \mathbb{C}$ converges in the right half of the complex plane i.e. $\mathrm{Re}\left[z\right] > 0$. This convergent improper integral for $\mathrm{Re}\left[z\right] > 0$ was derived by Daniel Bernaulli. We note here the expression (1.45) is Mellin transform of the function $f(t) = e^{-t}$ we represent as $\Gamma(z) = \mathcal{M}\left\{e^{-t}\right\}$. Also in the above (1.45) expression we have $t^{z-1} = e^{(z-1)\ln t}$ and $\ln t \in \mathbb{R}$. Considering - z to be a real number this statement implies that the gamma function is defined continuously for positive real values of z. Compare the above integral (1.45) i.e. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$ with Euler's integral of a second kind obtained for factorial representation in Section 1.9, (1.44), i.e. $n! = \int_0^\infty e^{-t} t^n \mathrm{d}t$ and observe the similarity.

The other representations of gamma functions are the following:

$$\Gamma(z) = 2\int_0^\infty y^{2z-1} e^{-y^2} \mathrm{d}y \qquad \Gamma(z) = \int_0^1 \left(\ln \tfrac{1}{y}\right)^{z-1} \mathrm{d}y \tag{1.46}$$

We get the above expression (1.46) by using $t = y^2$ and $e^{-t} = y$ respectively in the original integral (1.45) i.e. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$. For small iterations of z, between 0 and 1, we have following approximation:

$$\frac{1}{\Gamma(z)} \approx z \tag{1.47}$$

1.10.1 The recurring relationship of the gamma function

The basic properties of the gamma function is the recurrence relation which is as follows:

$$\Gamma(z+1) = z\Gamma(z) \tag{1.48}$$

The proof is attained through integration by parts formula that is $\int (I)(II)\mathrm{d}t = I\int (II)\mathrm{d}t - \int\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}(I)\right)\int (II)\mathrm{d}t\right)\mathrm{d}t$ with $I = t^z$ and $II = e^{-t}$ demonstrated in the following steps

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t}t^{(z+1)-1}\mathrm{d}t = \int_0^\infty e^{-t}t^z\mathrm{d}t \\ &= t^z\int_0^\infty e^{-t}\mathrm{d}t - \int_0^\infty\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[t^z\right]\int e^{-t}\mathrm{d}t\right)\mathrm{d}t \\ &= \left[-e^{-t}t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t \\ &= z\left(\Gamma(z)\right)\end{aligned} \tag{1.49}$$

This above equality (1.48) i.e. $\Gamma(z+1) = z\left(\Gamma(z)\right)$ is true only at the right half plane (RHP), i.e. $\mathrm{Re}\left[z\right] > 0$, otherwise the integral in (1.49) steps i.e. $\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t = \Gamma(z)$ is ∞, i.e. diverging. In the above steps (1.49) $\left[-e^{-t}t^z\right]_{t=0}^{t=\infty} = 0$. With this initial recurrence relationship $\Gamma(z+1) = z\left(\Gamma(z)\right)$, we will be able to 'analytically continue' the gamma function for $\mathrm{Re}\left[z\right] > -1$ (excluding the origin), by writing $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. This process we will describe in a short while in this section.

Obviously we have $\Gamma(1) = 1$, as $\Gamma(1) = \int_0^\infty e^{-t}\mathrm{d}t = -e^{-t}\Big|_0^\infty = 1$ and using the above property (1.48) of the recurrence relation of the gamma function, we obtain values for $z = 1, 2, 3, \ldots$

$$\begin{aligned}&\Gamma(2) = 1\times\Gamma(1) = 1! \\ &\Gamma(3) = 2\times\Gamma(2) = 2! \\ &\Gamma(4) = 3\times\Gamma(3) = 3! \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\Gamma(n+1) = n\times\Gamma(n) = n(n-1)! = n!\end{aligned} \tag{1.50}$$

The above property (1.50) is valid for positive values of z.

1.10.2 Factorials related to the gamma function

With this observation, we write the decreasing factorial as:

$$(z)! = \Gamma(z+1) \tag{1.51}$$

Increasing the truncated factorial for $k > 0$ is done as follows:

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} \tag{1.52}$$

1.10.3 Calculation of the factorial of non-integer numbers by gamma function

We have a recurrence of the relationship for the Gamma-function that is $\Gamma(z+1) = (z)\left(\Gamma(z)\right)$. We calculate $\Gamma\left(\frac{1}{2}\right)$ which is equivalent to stating $\left(-\frac{1}{2}\right)!$, as follows:

$$\Gamma\left(\tfrac{1}{2}\right)=\int_0^\infty e^{-t}t^{-(1/2)}\mathrm{d}t=\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\mathrm{d}t \qquad t=u^2 \qquad \mathrm{d}t=2u\mathrm{d}u$$
$$=\int_0^\infty \frac{e^{-u^2}}{u}2u\mathrm{d}u \tag{1.53}$$
$$=2\int_0^\infty e^{-u^2}\mathrm{d}u=\int_{-\infty}^{\infty} e^{-u^2}\mathrm{d}u=\sqrt{\pi}$$

We have used a known integral that is $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x=\sqrt{\pi}$ in the above (1.53) derivation. This integral i.e. $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x=\sqrt{\pi}$ was obtained by Pierre-Simon Laplace by considering the square of the integral i.e. $\mathrm{I}^2=\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right)^2$. Accordingly, we do the following manipulations:

$$\mathrm{I}^2=\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right)^2$$
$$=\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right) \tag{1.54}$$
$$=\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

We recognise $(\mathrm{d}x)(\mathrm{d}y)$ as an elemental area in Cartesian (x,y) coordinates with the domain of integration as on the entire X-Y plane. We have polar representation in (r,θ) coordinates with transformations as $x=r\cos\theta$ and $y=r\sin\theta$. In the polar coordinates (r,θ) the elemental area is transformed as $(\mathrm{d}x)(\mathrm{d}y)=r(\mathrm{d}r)(\mathrm{d}\theta)$ with $r^2=x^2+y^2$. The entire X-Y plane is now represented by r varying from zero to infinity, and θ varying from zero to 2π. With this change in the coordinate system from (x,y) to (r,θ) we write the following steps, with substitution $r^2=u$ and $2r\mathrm{d}r=\mathrm{d}u$:

$$\mathrm{I}^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$
$$=\int_0^\infty\int_0^{2\pi} e^{-r^2}r\mathrm{d}r\mathrm{d}\theta$$
$$=\int_0^\infty e^{-r^2}r\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta=\left(\int_0^\infty e^{-r^2}r\mathrm{d}r\right)(2\pi) \tag{1.55}$$
$$=\left(\int_0^\infty e^{-u}\left(\tfrac{1}{2}\mathrm{d}u\right)\right)(2\pi)=(\pi)\int_0^\infty e^{-u}\mathrm{d}u=\pi$$

The above derivation (1.55) is done by Laplace which gives us $\mathrm{I}=\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x=\sqrt{\pi}$; a very useful result. Thus, here we demonstrate the factorial of a non-integer getting defined, i.e. $\left(-\tfrac{1}{2}\right)!=\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$.

1.10.4 Poles of the gamma function

Another important property of the gamma function is that it has simple poles at $z=0,-1,-2,-3,\dots$. The proof is explained as splitting the function into two intervals as indicated below

$$\Gamma(z)=\int_0^1 e^{-t}t^{z-1}\mathrm{d}t+\int_1^\infty e^{-t}t^{z-1}\mathrm{d}t \tag{1.56}$$

The first integral of (1.56) can be evaluated by using a series expansion for the exponential function i.e. $e^{-t}=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}$. With $\mathrm{Re}[z]=x>0$, then we have $\mathrm{Re}[(z+k)]=x+k>0$ and thus $t^{(z+k)}\Big|_{t=0}=0$. Therefore, we write the first term, that is $\int_0^1 e^{-t}t^{z-1}\mathrm{d}t$, in series form as follows:

$$\begin{aligned}\int_0^1 e^{-t}t^{z-1}\mathrm{d}t &= \int_0^1 \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}t^{z-1}\mathrm{d}t\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^1 t^{k+z-1}\mathrm{d}t = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\right)\left(\frac{t^{k+z}}{k+z}\Bigg|_{t=0}^{t=1}\right)\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+z)}\end{aligned} \tag{1.57}$$

The second term of (1.56) i.e. $\int_1^{\infty} e^{-t}t^{z-1}\mathrm{d}t$ may be represented as an "entire-function", call it $\varphi(z)$, which permits us to write the split integral for $\Gamma(z)$ as follows:

$$\begin{aligned}\Gamma(z) &= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\frac{1}{(k+z)}\right)+\varphi(z)\\ &= \left(\frac{(-1)^0}{0!}\frac{1}{0+z}+\frac{(-1)^1}{1!}\frac{1}{1+z}+\frac{(-1)^2}{2!}\frac{1}{2+z}+.....\right)+\varphi(z)\end{aligned} \tag{1.58}$$

Clearly this derivation (1.58) is indicating simple poles at $z = 0,-1,-2,-3,...$. This means that at negative integer points the gamma function asymptotically approaches infinity and is discontinuous at those negative integer values. So by the above (1.45) integral representation of the gamma function $\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}\mathrm{d}t$ defined originally for positive z, we extended our argument to point zero, and all negative integer points, where the gamma function blows up. What does that not say about the values of the gamma function at other negative non-integer points?

1.10.5 Extending the gamma function for negative arguments by analytic continuation

What happens to the Gamma function at say $z=-\frac{1}{2}$, $z=-\frac{3}{2}$ etc.? The recurrence relation (1.48) $\Gamma(z+1)=(z)\big(\Gamma(z)\big)$ which we re-write as $\Gamma(x-1)=\frac{\Gamma(x)}{(x-1)}$, also serves as an analytical continuation, extending the definition of the gamma function to the negative arguments to which the definition (1.45) i.e. $\Gamma(x)=\int_0^{\infty} y^{x-1}e^{-y}\mathrm{d}y$, $x>0$ is inapplicable. This extension shows $\Gamma(0)$ to be infinite as with $\Gamma(-1)$ and values of the gamma function at all negative integers (Section-1.10.4).

The integral $\int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t$ does not make sense in the left half plane (LHP). However if we define $\Gamma(z)=\frac{\Gamma(z+1)}{z}$ on the LHP (and this is not equal to the integral on the LHP); this is consistent with the fact that $\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t$ on the RHP and provides us with a possible extension on the LHP, except the points of negative integers. Further, this has to be analytic continuation since $\Gamma(z)=\frac{\Gamma(z+1)}{z}$ is analytic everywhere except at zero and negative integers and matches with the integral on the RHP. Hence it is the uniqueness of analytic continuation and this is the only possible extension.

We can verify this by taking a strip on the LHP i.e. $(-1,0)$ such that $\mathrm{Re}\big[z\big]\in(-1,0)$ and define $\Gamma(z)=\frac{\Gamma(z+1)}{z}$ on this strip. First, note that $\Gamma(z+1)$ is analytic on the strip $\mathrm{Re}\big[z\big]\in(-1,0)$, since $(z+1)\in\mathrm{RHP}$, and $\frac{1}{z}$ is analytic on the strip $\mathrm{Re}\big[z\big]\in(-1,0)$. Hence $\Gamma(z)=\frac{\Gamma(z+1)}{z}$ is analytic on the strip $\mathrm{Re}\big[z\big]\in(-1,0)$. In this way, we have extended $\Gamma(z)$ to the region $\mathrm{Re}\big[z\big]>-1$ (except at $z=0$). Now repeat the same reasoning for the strip $\mathrm{Re}\big[z\big]\in(-2,-1)$, since now $\Gamma(z)$ is analytic on region $\mathrm{Re}\big[z\big]>-1$, and so on. This way the gamma function is extended to the entire complex plane except at points at zero, and all negative integers.

1.10.6 Representations of the gamma function via the contour integral (Hankel's formula)

The gamma function is also represented by the contour integral, is also called Hankel's formula, and is valid for all $z \neq 0, \pm 1, \pm 2, ...$ as in the following:

$$\Gamma(z) = \frac{-1}{2i\sin\left(\pi z\right)} \int_{\mathrm{C}} (-t)^{z-1} e^{-t} \mathrm{d}t \tag{1.59}$$

We will derive this (1.59) subsequently. The contour C starts slightly above the real axis at $+\infty$, runs down to $t = 0^{+}$, where it goes counter-clockwise in a small circle, say of radius $\in$, and returns to $+\infty$ just below the real axis. This integral on C we can represent by $\int_{\infty}^{(0+)} (-t)^{z-1} e^{-t} \mathrm{d}t$. The complex plane t-plane is cut from 0 to ∞; the branch cut is the positive real axis (Figure 1.1). To prove the above result (1.59), consider the following integral with $\mathrm{Re}\left[z\right] > 0$:

$$\mathrm{I} = \int_{\mathrm{C}} (-t)^{z-1} e^{-t} \mathrm{d}t \tag{1.60}$$

where $(-t)^{z-1} = e^{(z-1)\ln(-t)}$ and $\ln(-t) \in \mathbb{R}$. That is, we are defining the logarithm such that $\ln(-t)$ is real on the negative t-axis. Thus, on the negative real t-axis we have $\angle\left[(-t)\right] = 0$ or $\arg(-t) = 0$, for defining $\ln(-t) \in \mathbb{R}$. Thus from our convention of positive angles, if negative real axis corresponds to an angle of 0^{0}, then the positive real axis should have an angle of $\pm 180^{0}$, as depicted in Figure 1.1a. On the contour, i.e. C; as described above (i.e. the contour enclosing the positive real t-axis), we have $-\pi < \arg(-t) < \pi$; such that $\arg(-t) = 0$ on the negative real t-axis. Therefore just above the positive real t-axis we have $\arg(-t) = -\pi$, whereas just below we have $\arg(-t) = +\pi$; the angle being measured counter-clockwise (Figure 1.1a). Thus, we write for the part of a contour, which is just above the positive real axis as $(-t)^{z-1} = t^{z-1} e^{-i\pi(z-1)}$ in polar form with its magnitude and its argument. For the part of the contour just below the positive real axis, we write $(-t)^{z-1} = t^{z-1} e^{+i\pi(z-1)}$. For the part of the contour i.e. the small circle enclosing $t = 0$ we have $(-t) = \left(\in\right) e^{i\theta}$ where in limit $\in \downarrow 0$ and θ change from $-\pi$ to $+\pi$; this makes $-\mathrm{d}t = \left(\in\right) i e^{i\theta} \mathrm{d}\theta$. Thus we write the contour integral as the sum of (1) the integral on the line just above the positive real axis, (2) the integral on the small circle encircling the origin, and (3) the integral on the line just below the positive real axis, as follows:

$$\begin{aligned} \mathrm{I} &= \int_{\mathrm{C}} (-t)^{z-1} e^{-t} \mathrm{d}t \\ &= \int_{\infty}^{\in} t^{z-1} e^{-i\pi(z-1)} e^{-t} \mathrm{d}t \\ &\quad + \left(\int_{-\pi}^{\pi} \left(\left(\left(\in\right) e^{i\theta} \right)^{z-1} e^{\in(\cos\theta + i\sin\theta)} \left(\left(-\in\right) e^{i\theta} i \mathrm{d}\theta \right) \right) \right) \\ &\quad + \int_{\in}^{\infty} t^{z-1} e^{+i\pi(z-1)} e^{-t} \mathrm{d}t \end{aligned} \tag{1.61}$$

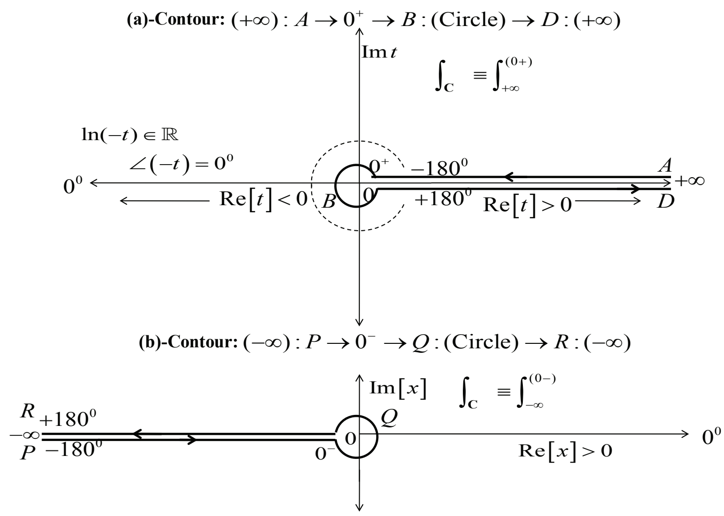

Figure 1.1: Hankel's path for integration

In the second integral (1.61) we have written $e^{\in(\cos\theta + i\sin\theta)}$ for e^{-t} with $(-t) = \left(\in\right)e^{i\theta} = \in\left(\cos\theta + i\sin\theta\right)$. The second integral of (1.61) goes to zero as in the limit $\in\downarrow 0$; i.e. the integral over the small circle vanishes, as does $\in^{z}$, which is demonstrated in the following steps:

$$\begin{aligned}
&\lim_{\in\downarrow 0}\left(\int_{-\pi}^{\pi}\left(\left(\in e^{i\theta}\right)^{z-1} e^{\in(\cos\theta+i\sin\theta)}\left(\in e^{i\theta} i\mathrm{d}\theta\right)\right)\right); \qquad \lim_{\in\downarrow 0} e^{\in(\cos\theta+i\sin\theta)} = 1 \\
&= \lim_{\in\downarrow 0}\left(\int_{-\pi}^{\pi}\left(\left(\in e^{i\theta}\right)^{z-1}\left(\in e^{i\theta} i\mathrm{d}\theta\right)\right)\right) \qquad (1.62) \\
&= \lim_{\in\downarrow 0}\int_{-\pi}^{\pi}\mathrm{d}\theta\left(\in^{z} i e^{i\theta z} e^{\in(\cos\theta+i\sin\theta)}\right) = 0
\end{aligned}$$

So we have the following steps for evaluation I as defined in (1.61)

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbf{C}}(-t)^{z-1}e^{-t}\mathrm{d}t \\
&= \int_{\infty}^{\in} t^{z-1}e^{-i\pi(z-1)}e^{-t}\mathrm{d}t + \int_{\in}^{\infty} t^{z-1}e^{+i\pi(z-1)}e^{-t}\mathrm{d}t \\
&= \int_{\in}^{\infty} t^{z-1}e^{+i\pi(z-1)}e^{-t}\mathrm{d}t - \int_{\in}^{\infty} t^{z-1}e^{-i\pi(z-1)}e^{-t}\mathrm{d}t \qquad (1.63) \\
&= \left(e^{+i\pi(z-1)} - e^{-i\pi(z-1)}\right)\int_{\in}^{\infty} t^{z-1}e^{-t}\mathrm{d}t \\
&= 2i\sin\left(\pi(z-1)\right)\int_{\in}^{\infty} t^{z-1}e^{-t}\mathrm{d}t
\end{aligned}$$

We used the known identity $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ in the above steps of (1.63). Using the trigonometric identity $\sin(-\pi + x) = -\sin x$ and recognising $\lim_{\in\downarrow 0}\left(\int_{\in}^{\infty} t^{z-1}e^{-t}\mathrm{d}t\right) = \Gamma(z)$ from the definition of the gamma function in (1.45), we write the following:

$$\lim_{\in\downarrow 0}\left(\mathrm{I}\right) = \int_{\mathbf{C}}(-t)^{z-1}e^{-t}\mathrm{d}t = -2i\left(\sin(\pi z)\right)\left(\Gamma(z)\right) \qquad (1.64)$$

Thus, we get Hankel's formula valid for all $z \neq 0, \pm 1, \pm 2, ...$ for representation of the gamma function as $\Gamma(z) = -\frac{1}{2i\sin(\pi z)}\int_{\mathbf{C}}(-t)^{z-1}e^{-t}\mathrm{d}t$ that is described in (1.59).

1.10.7 Ratio of gamma functions at negative integer points

Even though the values of the gamma function at negative integer points blow up the ratios of gamma functions of negative integers, they are however, finite; thus if N and n are positive integers then

$$\frac{\Gamma(-n)}{\Gamma(-N)} = (-N)(-N+1)....(-n-2)(-n-1) = (-1)^{N-n}\frac{N!}{n!} \qquad (1.65)$$

The proof of (1.65) is left out. The other properties of the gamma function are listed in the following lines (though we are not going to be proving all of them).

1.10.8 The duplication formula for the gamma function

The duplication formula is described as follows:

$$\Gamma(2x) = \frac{4^{x}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right)}{2\sqrt{\pi}} \qquad (1.66)$$

The above (1.66) one is from the Gauss-multiplication formula, which is set out as follows:

$$\Gamma(nx) = \sqrt{\frac{2\pi}{n}}\left(\frac{n^{x}}{\sqrt{2\pi}}\right)^{n}\prod_{k=0}^{n-1}\left(x + \frac{k}{n}\right) \qquad (1.67)$$

1.10.9 Reflection formula of the gamma function

The reflection formula is as follows (detailed proof is given in Section 1.15, pertaining to the beta function):

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad \Gamma(-x) = \frac{-\pi \csc(\pi x)}{\Gamma(x+1)} \tag{1.68}$$

1.10.10 Asymptotic representation of the gamma function

For $x > 0$, the asymptotic representation of the gamma function is as follows:

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi x} \tag{1.69}$$

Note earlier that we wrote Stirling's approximation for factorials (1.39) which is the same as above.

1.10.11 Reciprocal gamma function

The reciprocal gamma function is defined as follows:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{\left(-z/k\right)} \tag{1.70}$$

In (1.70) $\gamma = 0.577215666....$; it is Euler's constant. This reciprocal gamma function is continuous everywhere for the real values of z. The value of the reciprocal gamma function is zero at all the negative integer points and at point zero (this is an important point), whereas the gamma function has discontinuities at all the negative integer points and at point zero. The reciprocal of the gamma function $\frac{1}{\Gamma(x)}$ is single valued and finite for all x. The asymptotic representation, for a large approach of x is the following (with x tending to ∞):

$$\frac{1}{\Gamma(x)} \sim \frac{x^{\left(\frac{1}{2}\right)-x}}{\sqrt{2\pi}} \tag{1.71}$$

Figure 1.2 gives the plot of a gamma function and Figure 1.3 the plot of a reciprocal gamma function:

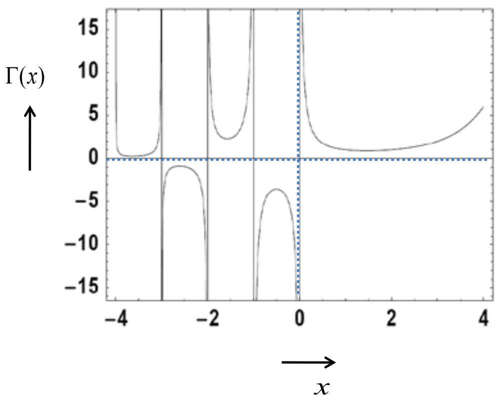

Figure 1.2: Plot of a gamma function

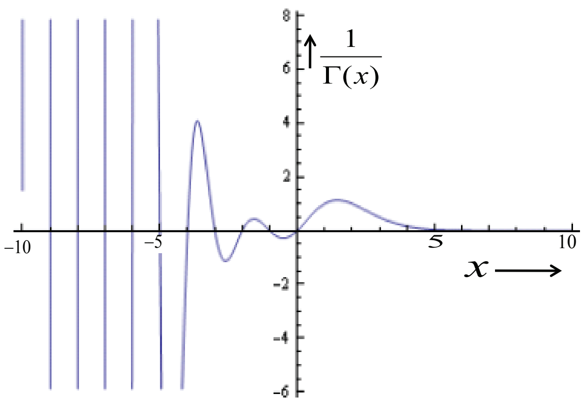

Figure 1.3: Plot of a reciprocal of the gamma function. Note that values are zero for $x = 0$ and at x equal to negative integers -1, -2, -3 …

We have derived Hankel's formula (1.59) for the gamma function, i.e. $\Gamma(z)=\frac{-1}{2i\sin(\pi z)}\int_{\mathrm{C}}(-t)^{z-1}e^{-t}\mathrm{d}t$. We use the reflection formula $\Gamma(z)\Gamma(1-z)=\pi\left(\csc(\pi z)\right)$ and write the following steps:

$$\begin{aligned}\Gamma(z)&=\frac{-1}{2i\sin(\pi z)}\int_{\mathrm{C}}(-t)^{z-1}e^{-t}\mathrm{d}t\\&=\frac{-\left(\Gamma(z)\Gamma(1-z)\right)}{2i\pi}\int_{\mathrm{C}}(-t)^{z-1}e^{-t}\mathrm{d}t\end{aligned}\tag{1.72}$$

From the above (1.72) steps we write the following useful expression:

$$\frac{1}{\Gamma(1-z)}=\frac{-1}{2\pi i}\int_{\mathrm{C}}(-t)^{z-1}e^{-t}\mathrm{d}t\tag{1.73}$$

Writing $(1-z)=\omega$ we obtain Hankel's integral representation for the reciprocal gamma function with the following expression:

$$\frac{1}{\Gamma(\omega)}=\frac{i}{2\pi}\int_{\infty}^{(0+)}(-t)^{-\omega}e^{-t}\mathrm{d}t\tag{1.74}$$

Here we used the symbol $\int_{\infty}^{(0+)}$ representing the contour integral on C, i.e. $\int_{\mathrm{C}}$, meaning thereby a path starts at infinity on the positive real axis, encircles 0 in a positive sense (counter-clockwise), and returns to infinity along the positive real axis, respecting the branch-cut along the positive real axis, as we have described previously too (Figure 1.1a). We will be using this expression (1.74) in the subsequent chapter. In the above expression (1.74) we make the substitution, $-t=x$ then the above integral (1.74) is re-written as the following:

$$\frac{1}{\Gamma(\omega)}=\frac{-i}{2\pi}\int_{-\infty}^{(0-)}(x)^{-\omega}e^{x}\mathrm{d}x\tag{1.75}$$

Here we have used the symbol $\int_{-\infty}^{(0-)}$ representing the contour integral on C, i.e. $\int_{\mathrm{C}}$, meaning thereby a path starts at minus infinity ($-\infty$) on the negative real axis, encircling 0 in a positive sense (counter-clockwise), and returns back to minus infinity, again on the negative real axis, respecting the branch cut along the negative real axis (Figure 1.1b). We write Hankel's integral for the reciprocal gamma function as the following:

$$\frac{1}{\Gamma(\alpha)}=\frac{1}{2\pi i}\int_{Ha}\frac{e^{t}}{t^{\alpha}}\mathrm{d}t\tag{1.76}$$

Where, for the above expression (1.76), Hankel's path i.e. Ha is defined as in Figure 1.1b.

1.10.12 Representing the gamma function as a limit form

The gamma function can also be represented as a limit, which is from the Euler formula for factorial $(z-1)!$ that is expressed in the following expression:

$$\Gamma(z)=\lim_{n\uparrow\infty}\left(\frac{(n!)n^{z}}{z(z+1)...(z+n)}\right)\tag{1.77}$$

Here we initially assume the right half plane $\mathrm{Re}\left[z\right]>0$, in case of real number positive values. Let us introduce an auxiliary function to prove this part, $f_n(z)=\int_0^n\left(1-\frac{t}{n}\right)^n t^{z\text{-}1}\mathrm{d}t$. Substitute $\tau=\frac{t}{n}$, and then performing integration by parts repeatedly, we get the following steps:

$$
\begin{aligned}
f_n(z) &= \int_0^n \left(1-\tfrac{t}{n}\right)^n t^{z-1}\mathrm{d}t \\
&= n^z \int_0^1 (1-\tau)^n \tau^{z-1}\mathrm{d}\tau \\
&= n^z \left[(1-\tau)^n \left(\tfrac{\tau^z}{z}\right)\right]_0^1 - n^z \int_0^1 \left(-n(1-\tau)^{n-1}\left(\tfrac{\tau^z}{z}\right)\right)\mathrm{d}\tau \\
&= \frac{n^z}{z} n \int_0^1 (1-\tau)^{n-1}\tau^z \mathrm{d}\tau \\
&= \left(\tfrac{n^z}{z} n\right)\left[(1-\tau)^{n-1}\left(\tfrac{\tau^{z+1}}{z+1}\right)\right]_0^1 - \left(\tfrac{n^z}{z} n\right)\int_0^1 \left(-(n-1)(1-\tau)^{n-2}\left(\tfrac{\tau^{z+1}}{z+1}\right)\right)\mathrm{d}\tau \\
&= \left(\frac{n^z}{z(z+1)} n(n-1)\right)\int_0^1 (1-\tau)^{n-2}\tau^{z+1}\mathrm{d}\tau \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \frac{n^z n!}{z(z+1)\ldots(z+n-1)}\int_0^1 \tau^{z+n-1}\mathrm{d}\tau \\
&= \frac{n^z n!}{z(z+1)\ldots(z+n-1)(z+n)}
\end{aligned}
\tag{1.78}
$$

Taking into account the well known identity i.e. $\lim_{n\uparrow\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}$, we expect the following:

$$
\begin{aligned}
\lim_{n\uparrow\infty}\left(f_n(z)\right) &= \lim_{n\uparrow\infty}\left(\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\mathrm{d}t\right) \\
&= \int_0^\infty e^{-t} t^{z-1}\mathrm{d}t \\
&= \Gamma(z)
\end{aligned}
\tag{1.79}
$$

Therefore, we have $\Gamma(z) = \lim_{n\uparrow\infty}\left(\frac{n! n^z}{z(z+1)\ldots(z+n)}\right)$.

1.10.13 Some interesting values of gamma functions that we frequently encountered

As we have seen, the gamma function of a positive integer n is itself a positive integer, while the gamma function $\Gamma(-n)$ of a negative integer is infinite. The gamma functions $\Gamma\left(\frac{1}{2}+n\right)$ and $\Gamma\left(\frac{1}{2}-n\right)$ turn out to be multiples of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, thus we can write another formula as:

$$
\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\tfrac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}, \qquad \Gamma\left(\tfrac{1}{2}-n\right) = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}
\tag{1.80}
$$

We list the few frequently encountered values of gamma functions as the following:

$$
\begin{aligned}
&\Gamma\left(-\tfrac{3}{2}\right) = \tfrac{4}{3}\sqrt{\pi}, \quad \Gamma(-1) = \pm\infty, \quad \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}, \quad \Gamma(0) = \pm\infty, \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \\
&\Gamma(1) = 1, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(2) = 1, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}, \quad \Gamma(3) = 2
\end{aligned}
\tag{1.81}
$$

1.11 Ratio of gamma functions

We will be using in subsequent chapters the expression $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$ frequently, where j is a non-negative integer and q may take any value. For small numerical values of j this expression is readily simplified to a polynomial in q by application of $\Gamma(x+1)=x\left(\Gamma(x)\right)$ and $\Gamma(n+1)=n!$. This procedure generalises to give Stirling's approximation. Stirling's approximation for the ratio of Gamma functions is given by Stirling's number, shown by the following expression:

$$\begin{aligned}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}&=\frac{(-1)^j}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m\\&=\frac{(-1)^j}{j!}\left(S_j^{(0)}+S_j^{(1)}q+S_j^{(2)}q^2+\dots S_j^{(j)}q^j\right)\end{aligned}\tag{1.82}$$

Where $S_j^{(m)}$ is Stirling's number of the first kind with a recurring relation given as $S_{j+1}^{(m)}\equiv S_j^{(m-1)}-jS_j^{(m)}$ with $S_0^{(m)}=S_j^{(0)}=1$, except $S_0^{(0)}=1$. A few Stirling numbers of the first kind are listed in Table 1.1. We use this table to calculate $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$ for a few terms for $j=0,1,2,3...$ and for $q=\frac{1}{2}$ as follows:

$$\begin{aligned}
&j=0\quad q=\tfrac{1}{2}\quad \frac{(-1)^j}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m=\frac{(-1)^0}{0!}S_0^{(0)}\left(\tfrac{1}{2}\right)^0=1\\
&j=0\quad q=\tfrac{1}{2}\quad \frac{(-1)^j}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m=\frac{(-1)^1}{1!}\left(S_1^{(0)}\left(\tfrac{1}{2}\right)^0+S_1^{(1)}\left(\tfrac{1}{2}\right)^1\right)=-\frac{1}{2}\\
&j=0\quad q=\tfrac{1}{2}\quad \frac{(-1)^j}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m\\
&\qquad=\frac{(-1)^2}{2!}\left(S_2^{(0)}\left(\tfrac{1}{2}\right)^0+S_2^{(1)}\left(\tfrac{1}{2}\right)^1+S_2^{(2)}\left(\tfrac{1}{2}\right)^2\right)=-\frac{1}{8}\\
&j=3\quad q=\tfrac{1}{2}\quad \frac{(-1)^j}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m\\
&\qquad=\frac{(-1)^3}{3!}\left(S_3^{(0)}\left(\tfrac{1}{2}\right)^0+S_3^{(1)}\left(\tfrac{1}{2}\right)^1+S_3^{(2)}\left(\tfrac{1}{2}\right)^2+S_3^{(3)}\left(\tfrac{1}{2}\right)^3\right)\\
&\qquad=-\frac{1}{16}
\end{aligned}\tag{1.83}$$

If we observe the following power series expansion of $f(x)=(1+x)^{\frac{1}{2}}$ as:

$$\begin{aligned}(1+x)^{\frac{1}{2}}&=\left(1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+.....\right)\\&=x^0+\frac{1}{2}x^1-\frac{1}{8}x^2+\frac{1}{16}x^3+...\end{aligned}\tag{1.84}$$

the terms for various $j=0,1,2,3...$ in $\frac{\Gamma\left(j-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma(j+1)}=(-1)^j\frac{1}{j!}\sum_{m=0}^{j}S_j^{(m)}\left(\frac{1}{2}\right)^m$ are pointing towards binomial coefficients of the power series expansion which are $(-1)^j\left({}^{1/2}C_j\right)$. This observation we will be used to generalise the binomial coefficients.

Representation (1.82), i.e. $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}=(-1)^j\frac{1}{j!}\sum_{m=0}^{j}S_j^{(m)}q^m$ as the polynomial in q, establishes that the expression $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$ is finite and single valued for all finite q and j. This relation with the Stirling number provides an expression for the gamma function quotient $\frac{\Gamma(j-q)}{\Gamma(-q)}$. The other quotient $\frac{\Gamma(j-q)}{\Gamma(j+1)}$ appearing in expression (1.82) is also of interest. This quotient has asymptotic expansion for large j, i.e. in limit $j\uparrow\infty$, represented as the following expression:

$$\frac{\Gamma(j-q)}{\Gamma(j+1)}\sim j^{-1-q}\left(1+\frac{q(q+1)}{2j}+O(j^{-2})\right)\tag{1.85}$$

j	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	-1	1	0	0	0
3	0	2	-3	1	0	0
4	0	-6	11	-6	1	0
5	0	24	-50	35	-10	1

Table 1.1: Some examples of Stirling numbers of the first kind

The approximation, of the ratio of the gamma function as a polynomial in q establishes the ratio of the gamma function as finite and a single value function for all q and j. Two examples of the Stirling approximations for $x \uparrow \infty$ are:

$$\begin{aligned}\frac{\Gamma(x+a)}{\Gamma(x+b)} &= x^{a-b}\left(1+O(x^{-1})\right)\\ \frac{\Gamma(x-q)}{\Gamma(x+1)} &= x^{-(1+q)}\left(1+\frac{q(q+1)}{2x}+O(x^{-2})\right)\end{aligned} \tag{1.86}$$

The result of (1.86) i.e. $\frac{\Gamma(j-q)}{\Gamma(j+1)} \sim j^{-1-q}\left(1+\frac{q(q+1)}{2j}+O(j^{-2})\right)$ is a representation that establishes itself as in the limit $j \uparrow \infty$ then the limit of $j^{q+1}\left(\frac{\Gamma(j-q)}{\Gamma(j+1)}\right)$ is unity. Thus, one may write the following expression:

$$\lim_{j\uparrow\infty}\left(j^{c+q+1}\frac{\Gamma(j-q)}{\Gamma(j+1)}\right) = \lim_{j\uparrow\infty}\left(j^{c+q}\frac{\Gamma(j-q)}{\Gamma(j)}\right) = \begin{cases}+\infty, & c>0\\ 1, & c=0\\ 0, & c<0\end{cases} \tag{1.87}$$

From the above expression (1.87) we get the following, by setting $c=0$ and writing $j=N$:

$$\lim_{N\uparrow\infty}\left(N^{\alpha}\frac{\Gamma(N-\alpha)}{\Gamma(N)}\right)=1 \tag{1.88}$$

The result that may be generalised to the following relationship:

$$\lim_{j\uparrow\infty}\left(j^{c+q+1}\frac{\Gamma(j+k-q)}{\Gamma(j+k+1)}\right) = \begin{cases}+\infty, & c>0\\ 1, & c=0\\ 0, & c<0\end{cases} \tag{1.89}$$

1.12 Generalising binomial coefficients by gamma function

The expression that we discussed in (1.82), i.e. $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$, may be regarded as a generalised binomial coefficient; given by the following relationship:

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \binom{j-q-1}{j} = (-1)^j\binom{q}{j} \tag{1.90}$$

Equivalently we write $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \left({}^{(j-q-1)}C_j\right) = (-1)^j\left({}^{q}C_j\right)$. The above (1.90) equivalents are readily established from the definition of a binomial coefficient i.e. ${}^nC_r = \frac{n!}{r!(n-r)!}$, using what we learnt about factorials and the gamma function and use of a reflection formula (1.68) i.e. $\Gamma(-x)\Gamma(x+1) = -\pi\csc(\pi x)$. This is described in the following steps:

$$\begin{aligned}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} &= \frac{\Gamma\left(-(q-j)\right)}{\Gamma(-q)\Gamma(j+1)}\\ &= \left(\frac{-\pi}{\left(\sin\pi(q-j)\right)\left(\Gamma(q-j+1)\right)}\right)\left(\frac{\left(\sin\pi q\right)\left(\Gamma(q+1)\right)}{(-\pi)}\right)\left(\frac{1}{\Gamma(j+1)}\right)\\ &= \left(\frac{\sin\pi q}{\sin\pi(q-j)}\right)\left(\frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)}\right)\end{aligned} \tag{1.91}$$

The ratio $\frac{\sin \pi q}{\sin \pi (q-j)}$ in (1.91) for $j = 0,1,2,3,4...$ takes the values as $\frac{\sin \pi q}{\sin \pi q} = 1$, $\frac{\sin \pi q}{\sin \pi (q-1)} = \frac{\sin \pi q}{\sin (\pi q - \pi)} = -1$, $\frac{\sin \pi q}{\sin \pi (q-2)} = \frac{\sin \pi q}{\sin (\pi q - 2\pi)} = +1$, $\frac{\sin \pi q}{\sin \pi (q-3)} = \frac{\sin \pi q}{\sin (\pi q - 3\pi)} = -1$ …. and so on. Therefore, we can write $\frac{\sin \pi q}{\sin \pi (q-j)} = (-1)^j$. From this we can write the following result:

$$\begin{aligned} \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} &= \left(\frac{\sin \pi q}{\sin \pi (q-j)}\right)\left(\frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)}\right) \\ &= (-1)^j \frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)} = (-1)^j \frac{q!}{(q-j)!\,j!} \\ &= (-1)^j \binom{q}{j} \end{aligned} \tag{1.92}$$

We can write the following expression from the above derived relationship in (1.92), by putting $j = n$ and $q = -z$:

$$\frac{\Gamma(j-q)}{\Gamma(-q)} = (-1)^j \frac{\Gamma(q+1)}{\Gamma(q-j+1)} \quad , \quad \frac{\Gamma(n+z)}{\Gamma(z)} = (-1)^n \frac{\Gamma(-z+1)}{\Gamma(-z-n+1)} \tag{1.93}$$

Besides the obvious relationship $\Gamma(z+1) = z\big(\Gamma(z)\big)$, we observe from the above expression (1.93), that if n is an integer, then $\big(\Gamma(z+n)\big)\big(\Gamma(-z-n+1)\big) = (-1)^n \big(\Gamma(z)\big)\big(\Gamma(1-z)\big)$, for all n. Using $\alpha! = \Gamma(\alpha+1)$, when α is not a positive integer, then we may generalise the binomial coefficients ${}^{(-z)}C_u$ as the following:

$$\binom{-z}{u} = \frac{(-z)!}{u!(-z-u)!} = \frac{\Gamma(1-z)}{\Gamma(u+1)\Gamma(1-z-u)} \tag{1.94}$$

In particular if u is a non-negative integer, say n, then using the above (1.94) identity, we have following the relationship:

$$\binom{-z}{n} = \frac{\Gamma(1-z)}{n!\Gamma(1-z-n)} = (-1)^n \frac{\Gamma(z+n)}{n!\Gamma(z)} = (-1)^n \binom{z+n-1}{n} \tag{1.95}$$

Of particular use is the expression we wrote earlier (1.92), i.e. $\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = {}^{j-\alpha-1}C_j = (-1)^j \left({}^{\alpha}C_j\right)$, resulting in the following (here we have changed our q to α):

$$\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = \binom{j-\alpha-1}{j} = (-1)^j \binom{\alpha}{j} \tag{1.96}$$

We simplify (1.96) by following with ${}^{n}C_k = \frac{n!}{k!(n-k)!}$:

$$\begin{aligned} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} &= (-1)^j \binom{\alpha}{j} = (-1)^j \frac{\alpha!}{j!(\alpha-j)!} \\ &= (-1)^j \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \end{aligned} \tag{1.97}$$

$$\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = (-1)^j \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}$$

We can then write a useful relationship from a derivation of (1.97) that is the following:

$$\frac{\Gamma(p+1)}{\Gamma(p-j+1)} = (-1)^j \frac{\Gamma(j-p)}{\Gamma(-p)} \tag{1.98}$$

A useful identity from a standard *Handbook of Mathematical Functions* is listed below:

$$\sum_{j=0}^{n} \binom{j-\alpha-1}{j} = \binom{n-\alpha}{n} \tag{1.99}$$

(which symbolically also is $\sum_{j=0}^{n} {}^{j-\alpha-1}C_j = {}^{n-\alpha}C_n$), deriving the following relationship:

$$\sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)} \tag{1.100}$$

This is followed by setting $N = n+1$ and expressing the binomial coefficients as their equivalent gamma function combination. Similarly after multiplication by $-\alpha$, it is readily shown as the following expression:

$$\sum_{j=1}^{n}\binom{j-\alpha-1}{j-1}=\binom{n-\alpha}{n-1} \tag{1.101}$$

This is then straight forward from the above expression that is $\sum_{j=0}^{n} {}^{j-\alpha-1}C_j = {}^{n-\alpha}C_n$. On redefining α , j and n we are led to the following identity:

$$\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j)}=\frac{-\alpha\left(\Gamma(N-\alpha)\right)}{\Gamma(2-\alpha)\Gamma(N-1)} \tag{1.102}$$

Likewise, the well-known formula, from the *Handbook of Mathematical Functions* that is:

$$\sum_{k=0}^{j}\binom{q}{k}\binom{Q+k}{j-k}=\binom{q+Q}{j} \tag{1.103}$$

becomes the following identity:

$$\sum_{k=0}^{j}\frac{\Gamma(q+1)\Gamma(p+1)\Gamma(j+1)}{\Gamma(q-k+1)\Gamma(k+1)\Gamma(p-q+k+1)\Gamma(j-k+1)}=\frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)} \tag{1.104}$$

on setting in (1.103) $Q \equiv (p-q+j)$ and multiplication by $\Gamma(j+1)$. Yet again, the binomial relationship is:

$$\sum_{j=0}^{n}\binom{j-q-1}{j-m}=\binom{n-q}{n-m} \tag{1.105}$$

which is obtainable from our earlier noted relationship (1.99) $\sum_{j=0}^{n} {}^{j-\alpha-1}C_j = {}^{n-\alpha}C_n$, which leads easily to the following identity:

$$\sum_{j=0}^{n}\binom{j-q-1}{j}\binom{j}{m}=\binom{m-q-1}{m}\binom{n-q}{n-m} \tag{1.106}$$

Table 1.2 gives some values for ${}^{1/2}C_n$, $\sum_{j=0}^{n}\left({}^{1/2}C_n\right)$, ${}^{-1/2}C_n$ and $\sum_{j=0}^{n}\left({}^{-1/2}C_n\right)$; and also values for a very large n.

We have used values from column one of Table 1.2 in the expansion of $(1+x)^{1/2}$, in the half-coin problem of Section-1.2. We then have the following expansion:

$$\begin{aligned}(x+1)^n &= {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + \dots\dots\dots + {}^nC_nx^n \\ &= x^0 + nx^1 + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots.x^n \\ (x+1)^{1/2} &= {}^{1/2}C_0x^0 + {}^{1/2}C_1x^1 + {}^{1/2}C_2x^2 + \dots\dots + {}^{1/2}C_nx^n \\ &= x^0 + \frac{1}{2}x^1 - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 \dots\dots\end{aligned} \tag{1.107}$$

that was used earlier in Section 1.2:

n	${}^{1/2}C_n$	$\sum_{j=0}^{n}\left({}^{1/2}C_j\right)$	${}^{-1/2}C_n$	$\sum_{j=0}^{n}\left({}^{-1/2}C_n\right)$
0	1	1	1	1
1	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
2	$-\frac{1}{8}$	$\frac{11}{8}$	$\frac{3}{8}$	$\frac{7}{8}$
3	$\frac{1}{16}$	$\frac{23}{16}$	$-\frac{5}{16}$	$\frac{9}{16}$
4	$-\frac{5}{128}$	$\frac{179}{128}$	$\frac{35}{128}$	$\frac{107}{128}$
5	$\frac{7}{256}$	$\frac{365}{256}$	$-\frac{63}{256}$	$\frac{151}{256}$
6	$-\frac{21}{1024}$	$\frac{1439}{1024}$	$\frac{231}{1024}$	$\frac{835}{1024}$
∞	$\frac{-(-1)^n}{2\sqrt{\pi}(n)^{3/2}}$	$\sqrt{2}$	$\frac{(-1)^n}{\sqrt{\pi}(n)}$	$\frac{1}{\sqrt{2}}$

Table 1.2: Some values of binomial coefficients, ${}^{1/2}C_j$, ${}^{-1/2}C_j$ and their cumulative sums

1.13 Incomplete gamma functions

1.13.1 Tricomi's incomplete gamma function and the complementary incomplete gamma function

Tricomi's incomplete gamma function is defined in the integral form:

$$\gamma^*(a,x) = \frac{x^{-a}}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}\mathrm{d}t \tag{1.108}$$

where we have defined Euler's gamma function, that is $\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}\mathrm{d}t$, whereas a complementary incomplete gamma function is defined as the following integral:

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\mathrm{d}t \qquad x \geq 0, \qquad -\infty < a < \infty \tag{1.109}$$

The asymptotic expansion for large x of the complementary incomplete gamma function is the following in limit $x \uparrow \infty$:

$$\Gamma(a,x) \cong x^{a-1}e^{-x}\left(1 + \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + \ldots.\right) \tag{1.110}$$

1.13.2- Incomplete gamma function

The incomplete gamma function is defined as follows:

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}\mathrm{d}t \qquad \mathrm{Re}\left[a\right] > 0 \tag{1.111}$$

We thus have:

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a) \tag{1.112}$$

where we have defined Euler's gamma function, that is $\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}\mathrm{d}t$.

1.13.3 The elementary relationships of Tricomi's incomplete gamma function

Tricomi's incomplete gamma function is defined in series form as depicted below:

$$\gamma^*(v,z) = e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(v+k+1)} \qquad \text{and} \qquad \gamma^*(v,0) = \frac{1}{\Gamma(v+1)} \tag{1.113}$$

It is also represented by Kummer's function, that is $M(a,b,z) \overset{\text{def}}{=} 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots$ as follows:

$$\gamma^*(a,x) = \frac{e^{-x}}{\Gamma(a+1)}M(1,a+1;x) = \frac{M(a,a+1;-x)}{\Gamma(a+1)} \tag{1.114}$$

We have the following identities:

$$\begin{gathered}\Gamma(a,x) = \left(1 - \frac{x^a}{\gamma^*(a,x)}\right)\Gamma(a) \\ \gamma^*(a,x) = x^{-a}\left(1 - \frac{\Gamma(a,x)}{\Gamma(a)}\right) \qquad \gamma^*(a,x) = \frac{x^{-a}}{\Gamma(a)}\gamma(a,x)\end{gathered} \tag{1.115}$$

If p is a non-negative integer, the following properties are deduced from a series definition as:

$$\gamma^*(p,ax) = e^{-ax}\sum_{k=p}^{\infty}\frac{(ax)^{k-p}}{k!} \qquad \text{and} \qquad \gamma^*(-p,ax) = (ax)^p \tag{1.116}$$

The second expression above in (1.116) is proven as follows:

$$\begin{aligned}
\gamma^*(-p,ax) &= e^{-ax}\sum_{k=-p}^{\infty}\frac{(ax)^{k+p}}{k!} \\
&= e^{-ax}(ax)^p\sum_{k=-p}^{\infty}\frac{(ax)^k}{k!} \\
&= e^{-ax}(ax)^p\left[\frac{(ax)^{-p}}{(-p)!}+\frac{(ax)^{-p+1}}{(-p+1)!}+.....+\frac{(ax)^0}{0!}+\frac{(ax)^1}{1!}+\frac{(ax)^2}{2!}+......\right] \\
&= e^{-ax}(ax)^p\sum_{k=0}^{\infty}\frac{(ax)^k}{k!} \\
&= e^{-ax}(ax)^p e^{ax} \\
&= (ax)^p
\end{aligned} \tag{1.117}$$

In the above steps of (1.117), the $\frac{1}{p!}=\frac{1}{\Gamma(p+1)}=0$ for all $p+1<0$. The special values of this version of Tricomi's incomplete gamma functions are $\gamma^*(1,ax)=\frac{1-e^{-ax}}{ax}$, $\gamma^*(0,ax)=1$ and $\gamma^*(-1,ax)=ax$. Tricomi's incomplete gamma function is an entire function of both z and v ; and its integral representation is for $\mathrm{Re}\left[v\right]>0$ is from (1.108) is: $\gamma^*(v,t)\overset{\text{def}}{=}\frac{1}{t^v\Gamma(v)}\int_0^t u^{v-1}e^{-u}\mathrm{d}u$ and $\lim_{t\uparrow\infty} t^v\gamma^*=1$.

Some elementary relationships that exist amongst Tricomi's incomplete gamma functions are as follows:

$$\gamma^*(v-1,t)-t\gamma^*(v,t)=e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(v+k)}-e^{-t}\sum_{k=0}^{\infty}\frac{t^{k+1}}{\Gamma(v+k+1)}=\frac{e^{-t}}{\Gamma(v)} \tag{1.118}$$

Replacing in (1.118) the variable t, with at, we get $\gamma^*(v-1,at)-(at)\gamma^*(v,at)=\frac{e^{-at}}{\Gamma(v)}$. Iterating this $(p-1)$ times we arrive at the following identity:

$$\gamma^*(v,at)=(at)^p\gamma^*(v+p,at)+e^{-at}\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(v+k+1)} \tag{1.119}$$

Using the series definition (1.113) of Tricomi's incomplete gamma function on the above expression (1.119) we get the following identity:

$$(at)^v e^{at}\gamma^*(v,at)=\sum_{k=0}^{\infty}\frac{(at)^{k+v}}{\Gamma(v+k+1)} \tag{1.120}$$

and taking the p-th derivative of (1.120), leads to the following identity:

$$\begin{aligned}
\frac{\mathrm{d}^p}{\mathrm{d}t^p}\left[t^v e^{at}\gamma^*(v,at)\right] &= t^{v-p}\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(v+k+1-p)} \\
&= t^{v-p}e^{at}\gamma^*(v-p,at)
\end{aligned} \tag{1.121}$$

From this (1.121) and the earlier identity in (1.119) we get the following expression:

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p}\left[t^v e^{at}\gamma^*(v,at)\right]=a^p\left(t^v e^{at}\gamma^*(v,at)\right)+t^{v-p}\sum_{k=0}^{p-1}\frac{(at)^k}{\Gamma(v+k+1-p)} \tag{1.122}$$

In this form (1.122) we see that Tricomi's incomplete gamma function on both sides of the equation have the same arguments. From this with $p=1$, we find the integral form to be the following:

$$\int_0^t u^v e^{au}\gamma^*(v,au)\mathrm{d}u=t^{v+1}e^{at}\gamma^*(v+1,at);\qquad \mathrm{Re}\left[v\right]>-1 \tag{1.123}$$

Tricomi's incomplete Gamma function is used in obtaining fractional differentiation and fractional integration of periodic functions, used as the sinusoidal response studies of fractional operators; we will use this in subsequent chapters.

1.14 Power numbers

Power numbers may be expressed in terms of complete gamma function as follows:

$$q^j = \sum_{m=0}^{j} (-1)^m S_j^{[m]} \frac{\Gamma(m-q)}{\Gamma(-q)} \tag{1.124}$$

The coefficients $S_j^{[m]}$ appearing in this expression are Stirling numbers of a second kind. Examples of these numbers, which obey recurrence $S_j^{[m]} = S_j^{[m-1]} + mS_j^{[m]}$, $S_0^{[m]} = S_j^{[0]} = 0$ except $S_0^{[0]} = 1$, are listed in Table 1.3.

j	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

Table 1.3: Some values of the Stirling Number of the second kind

Note that $S_j^{[j]} = 1$ for all j and that $S_j^{[m]} = 0$ for $j = 0,1,2,...,(m-1)$. The formula

$$\sum_{l=0}^{m} (-1)^{l+m} \binom{m}{l} l^k = m!S_k^{[m]} \tag{1.125}$$

may be regarded as defining Stirling numbers of the second kind.

This is used in an expansion of the analytic function φ of the argument of $(x + jy)$ in a rather special way. We give the gist of this here, firstly by relating $\varphi(x + jy)$ to the values $\varphi(x)$, $\varphi(x + y)$, $\varphi(x + 2y)$,..., $\varphi(x + jy)$, by the formula:

$$\varphi(x + jy) = \sum_{m=0}^{j} \binom{j}{m} \sum_{l=0}^{m} (-1)^{l+m} \binom{m}{l} \varphi(x + ly) \tag{1.126}$$

The inner summation in (1.126) is symbolized as $G_m(\varphi, x, y)$ and is expanded in the Taylor series as follows:

$$\begin{aligned} G_m(\varphi, x, y) &= \sum_{l=0}^{m} (-1)^{l+m} \binom{m}{l} \sum_{k=0}^{\infty} \frac{(ly)^k}{k!} \varphi^{(k)}(x) \\ &= \sum_{k=0}^{\infty} \frac{y^k}{k!} \varphi^{(k)}(x) \sum_{l=0}^{m} (-1)^{l+m} \binom{m}{l} l^k \end{aligned} \tag{1.127}$$

It requires now an application of summation (1.125) i.e. $\sum_{l=0}^{m} (-1)^{l+m} \left({}^m C_l\right) l^k = m!S_k^{[m]}$ to give the final result:

$$\varphi(x + jy) = \sum_{m=0}^{j} G_m(\varphi, x, y) \binom{j}{m} \tag{1,128}$$

In (1.128) $G_m(\varphi, x, y) = \sum_{k=0}^{\infty} \left(\frac{m!}{k!}\right) S_k^{[m]} y^k \varphi^{(k)}(x)$. Since ${}^j C_m$ vanishes when m exceeds integer j, the upper summation limit in the above expression (1.128) may be replaced by ∞. Similarly, (as this is a consequence of $S_j^{[m]} = 0$, for $j = 0,1,...,(m-1)$), the $k = 0$ that is the lower summation limit of the expression that is $G_m(\varphi, x, y) = \sum_{k=0}^{\infty} \left(\frac{m!}{k!}\right) S_k^{[m]} y^k \varphi^{(k)}(x)$, may be replaced by $k = m$.

1.15 The beta function and the incomplete beta function

The derived functions from the complete gamma function are called beta functions, defined as:

$$\mathrm{B}(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1} \mathrm{d}u = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1.129}$$

For the positive values of $p>0$ and $q>0$ the definite integral $\int_0^1 u^{p-1}(1-u)^{q-1}\mathrm{d}u$ is called the beta integral, also known as Euler's integral of a second kind. If either p or q is non-positive the integral diverges and then the beta function is defined by the relationship $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, valid for all p and q. We also note that $\mathrm{B}(p,q)=\mathrm{B}(q,p)$. The proofs we are omitting.

1.15.1 Integral representations of the beta function

With the above definition (1.129) we can form interesting and useful integral representations. Say, we use $u=\frac{z}{z+1}$ and thus we get $\mathrm{d}u=\frac{1}{(z+1)^2}\mathrm{d}z$; this gives limits as for $u=0$ we get $z=0$ and for $u=1$ we get $z+1=z$ which means $z\uparrow\infty$. With these substitutions we get the following steps:

$$
\begin{aligned}
\mathrm{B}(p,q) &= \int_0^1 u^{p-1}(1-u)^{q-1}\mathrm{d}u \\
&= \int_0^\infty \left(\frac{z}{z+1}\right)^{p-1}\left(1-\frac{z}{z+1}\right)^{q-1}\frac{\mathrm{d}z}{(z+1)^2} \\
&= \int_0^\infty \frac{z^{p-1}}{(z+1)^{p-1}}\left(\frac{1}{(z+1)^{q-1}}\right)\left(\frac{1}{(z+1)^2}\right)\mathrm{d}z \\
&= \int_0^\infty \frac{z^{p-1}}{(z+1)^{p+q}}\mathrm{d}z
\end{aligned}
\tag{1.130}
$$

This gives the formula as:

$$\int_0^\infty u^{p-1}(u+1)^{-p-q}\mathrm{d}u=\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1.131}$$

If say we have $p+q=r$, then we get another useful formula from (1.131):

$$\int_0^\infty u^{p-1}(u+1)^{-r}\mathrm{d}u=\mathrm{B}(p,r-p)=\frac{\Gamma(p)\Gamma(r-p)}{\Gamma(r)} \tag{1.132}$$

From these derived expressions, we can write another expression:

$$\int_0^\infty u^{\upsilon-1}(u+k)^{-p}\mathrm{d}u=k^{-(p-\upsilon)}\mathrm{B}(p-\upsilon,\upsilon) \tag{1.133}$$

From the formula of (1.132) i.e. $\int_0^\infty u^{p-1}(u+1)^{-r}\mathrm{d}u=\mathrm{B}(p,r-p)=\frac{\Gamma(p)\Gamma(r-p)}{\Gamma(r)}$ we can write the useful integral as

$$\mathrm{B}(x,1-x)=\Gamma(x)\Gamma(1-x)=\int_0^\infty u^{x-1}(u+1)^{-1}\mathrm{d}u \tag{1.134}$$

1.15.2 The derivation for the reflection formula of the gamma function

We will use expression (1.134) to prove the important formula for gamma functions i.e. $\Gamma(z)\Gamma(1-z)=\pi\csc\pi z$ for $z\neq 0,\pm1,\pm2,\ldots$ as well using contour integration, first by restricting the real values of $x=z$ with the restriction $0<x<1$.

For this we consider the contour integration i.e. $\int_{\mathrm{C}}\omega^{x-1}(1-\omega)^{-1}\mathrm{d}\omega$; with $0<x<1$, in a complex plane ω. Take the function $f(\omega)=\omega^{x-1}(1-\omega)^{-1}$. Clearly $\omega=1$ is a singularity (pole of order one). For $x=0$, we call $\omega=0$ as a singularity (pole of order one); but when $0<x<1$ the point $\omega=0$ is a logarithmic branch point. Writing $\omega=\in e^{i\theta}$, and going about point zero in a circle gives values for ω^{x-1} as $\in^{x-1}e^{i(x-1)\theta}$ i.e. from $\in^{\alpha}$ to $\in^{\alpha}e^{2\pi i(x-1)}$; showing discontinuity at every rotation, as was the case for the complex logarithm function ($\ln\omega$), described in Section 1.6.4. Thus at $\omega=0$ we draw a branch cut on the negative real axis shown in Figure 1.4. We take contour C in complex plane ω with the branch point at $\omega=0$ as depicted in Figure 1.4 (with a standard branch cut). We notice that the contour encloses the pole at $\omega=1$, and bypasses the branch point at $\omega=0$. Therefore, the only residue required is for $\omega = 1$. The contour is broken as $\mathrm{C} = C_1 \cup C_2 \cup C_3 \cup C_4$, which is depicted in Figure 1.4. We use residue calculus as follows:

$$\int_{\mathrm{C}} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} = 2\pi i\left(\mathrm{Residue}_{\omega=1} f\right) = 2\pi i\left(\lim_{\omega\to 1}\left(\omega-1\right)\frac{\omega^{x-1}}{1-\omega}\right) = -2\pi i \tag{1.135}$$

On the C_1 large circle with radius R ; $\omega = Re^{i\theta}$ we write for $0 < x < 1$ the following:

$$\begin{aligned}\int_{C_1} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} &= \lim_{R\uparrow\infty}\int_{-\pi}^{\pi} \frac{R^{x-1}e^{i(x-1)\theta}\mathrm{d}\left(Re^{i\theta}\right)}{1-Re^{i\theta}} = \lim_{R\uparrow\infty}\int_{-\pi}^{\pi} \frac{i\theta R^{x}e^{ix\theta}\mathrm{d}\theta}{1-Re^{i\theta}} \\ &\cong \lim_{R\uparrow\infty}\int_{-\pi}^{\pi} i\theta R^{x-1}e^{i\theta(x-1)}\mathrm{d}\theta = 0\end{aligned} \tag{1.136}$$

On the small circle with radius $\in$; $\omega = \in e^{i\theta}$ we write for $0 < x < 1$ the following:

$$\begin{aligned}\int_{C_3} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} &= \lim_{\in\downarrow 0}\int_{\pi}^{-\pi} \frac{\in^{x-1}\ e^{i(x-1)\theta}}{1-\in e^{i\theta}}\mathrm{d}\left(\in e^{i\theta}\right) \\ &= \lim_{\in\downarrow 0}\int_{\pi}^{-\pi} \frac{i\theta \in^{x}\ e^{ix\theta}\mathrm{d}\theta}{1-\in e^{i\theta}} \cong \lim_{\in\downarrow 0}\int_{\pi}^{-\pi} i\theta \in^{x} e^{ix\theta}\mathrm{d}\theta = 0\end{aligned} \tag{1.137}$$

On the line C_2 from $-R$ to $-\in$, we write $\omega = -\rho = \rho e^{i\pi}$ and we get:

$$\int_{C_2} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} = \int_{R}^{\in} \frac{\rho^{x-1}e^{i(x-1)\pi}\mathrm{d}\left(\rho e^{i\pi}\right)}{1+\rho} = \int_{R}^{\in} \frac{\rho^{x-1}e^{ix\pi}\mathrm{d}\rho}{1+\rho} \tag{1.138}$$

On the line C_4 from $-\in$ to $-R$ we write $\omega = -\rho = \rho e^{-i\pi}$ and we get the following:

$$\int_{C_4} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} = \int_{\in}^{R} \frac{\rho^{x-1}e^{-i(x-1)\pi}\mathrm{d}\left(\rho e^{-i\pi}\right)}{1+\rho} = \int_{\in}^{R} \frac{\rho^{x-1}e^{-ix\pi}\mathrm{d}\rho}{1+\rho} \tag{1.139}$$

The total contour integration on C is the sum of all four segments and we know that is equal to $-2\pi i$ (calculated from residues). Thus, in the limit $R \uparrow \infty$ and $\in\downarrow 0$ we have:

$$\int_{\mathrm{C}} \frac{\omega^{x-1}\mathrm{d}\omega}{1-\omega} = \int_{\infty}^{0} \frac{\rho^{x-1}e^{i\pi x}\mathrm{d}\rho}{1+\rho} + \int_{0}^{\infty} \frac{\rho^{x-1}e^{-i\pi x}\mathrm{d}\rho}{1+\rho} = -2\pi i \tag{1.140}$$

From above (1.140) we can write the following steps:

$$\begin{aligned}&-2\pi i = \left(e^{-i\pi x} - e^{i\pi x}\right)\int_0^\infty \frac{\rho^{x-1}\mathrm{d}\rho}{1+\rho}; \qquad \sin(ay) = \frac{e^{iay}-e^{-iay}}{2\pi i} \\ &\int_0^\infty \frac{\rho^{x-1}\mathrm{d}\rho}{1+\rho} = \frac{2\pi i}{e^{i\pi x}-e^{-i\pi x}} = \frac{\pi}{\sin \pi x}; \qquad \int_0^\infty u^{x-1}(1+u)^{-1}\mathrm{d}u = \mathrm{B}(x,1-x) \\ &\mathrm{B}(x,1-x) = \pi\left(\csc \pi x\right)\end{aligned} \tag{1.141}$$

Recognizing the LHS in (1.141) as $\mathrm{B}(x,1-x) = \Gamma(x)\Gamma(1-x) = \int_0^\infty \omega^{x-1}(\omega+1)^{-1}\mathrm{d}\omega$, we have proved the following reflection formula as in (1.68):

$$\Gamma(x)\Gamma(1-x) = \pi \csc \pi x \tag{1.142}$$

here for $0 < x < 1$ real $z = x$.

1.15.3 Application of analytic continuation to the reflection formula

The above obtained result of (1.142) is for $0 < x < 1$ real $z = x$. The full result follows from the concept of analytic continuation. Alternatively, the result is obtained by the following argument. If the identity $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ holds for real values of z for $0 < z < 1$, then it holds for all complex z with $0 < \mathrm{Re}\, z < 1$ by analyticity. Then it also holds for $\mathrm{Re}\, z = 0$ with $z \neq 0$ by continuity. Finally, the full result follows for z shifted by integers using $\Gamma(z+1) = z\left(\Gamma(z)\right)$ and $\sin(z+\pi) = -\sin z$. Note that $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ holds for all complex values of z, with $z \neq 0,-1,-2,\ldots$. Instead of $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ we may write:

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi} \tag{1.143}$$

which holds for all z in complex plane $\mathbb{C}$.

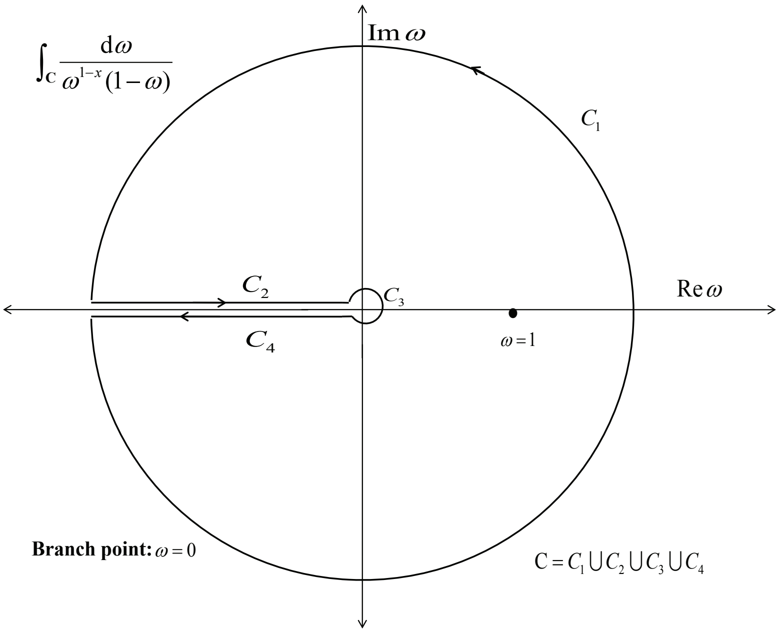

Figure 1.4: Contour for integration with pole and branch point with branch cut

1.15.4 Incomplete beta functions

In addition, there is an incomplete beta function (analogous to the gamma function and its incomplete counterpart) which is defined as:

$$\mathrm{B}_x(u,v) \overset{\text{def}}{=} \int_0^x t^{u-1}(1-t)^{v-1}\,\mathrm{d}t; \qquad 0 < x < 1 \tag{1.144}$$

If $\mathrm{Re}\left[v\right] > 0$ and k is a non-negative integer then we have the following definition:

$$\int_0^x u^{v-1}(x-u)^k\,\mathrm{d}u = x^{k+v}\mathrm{B}(k+1,v) = \frac{(k!)x^{k+v}\Gamma(v)}{\Gamma(v+k+1)} \tag{1.145}$$

Hence, we may write the same as the following:

$$\frac{1}{k!x^v\Gamma(v)}\int_0^x u^{v-1}(x-u)^k\,\mathrm{d}u = \frac{x^k}{\Gamma(v+k+1)} \tag{1.146}$$

Using the series definition of an incomplete gamma function (Section 1.13) and applying the integration by parts formula i.e. $\int uv\mathrm{d}x = u\int v\mathrm{d}x - \int u^{(1)}\left(\int v\mathrm{d}x\right)\mathrm{d}x$, we obtain:

$$\begin{aligned} e^x\gamma^*(v,x) &= \frac{1}{x^v\Gamma(v)}\left[\int_0^x u^{v-1}\mathrm{d}u + \frac{1}{v}\int_0^x u^v\left(\sum_{j=0}^{\infty}\frac{(x-u)^j}{j!}\right)\mathrm{d}u\right] \\ &= \frac{1}{x^v\Gamma(v)}\int_0^x u^{v-1}e^{(x-u)}\mathrm{d}u \end{aligned} \tag{1.147}$$

Thus, we see that the integral representation of incomplete gamma functions is described earlier (Section 1.13). Another useful representation of the incomplete Beta function comes from the hypergeometric function (refer to Appendix A), as follows:

$$\mathrm{B}_x(c,1+b-c) = x^c(1-x)^{1+b-c}\frac{\Gamma(c)}{\Gamma(1+b)}\left(\sum_{j=0}^{\infty}x^j\frac{\Gamma(j+1+b)}{\Gamma(j+1+c)}\right) \tag{1.148}$$

1.16 Psi function

The psi function is also called the 'digamma function'. The derivative of the complete gamma function is a 'psi' function $\psi(x)$ and is defined as following, for $x \neq -1, -2, -3, \dots$:

$$\psi(x) = \frac{1}{\Gamma(x)} \frac{\mathrm{d}\left[\Gamma(x)\right]}{\mathrm{d}x} \quad \text{or} \quad \psi(x) = \frac{\mathrm{d}\left[\ln \Gamma(x)\right]}{\mathrm{d}x} = \frac{\Gamma^{(1)}(x)}{\Gamma(x)} \tag{1.149}$$

One property 'psi' function, that is:

$$\psi(c) - \psi(c-b) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} \left(\frac{\Gamma(b+k)}{k\Gamma(c+k)} \right) \tag{1.150}$$

for $\mathrm{Re}\left[c\right] > \mathrm{Re}\left[b\right]$ (we are not proving this property), gets derived via use of the 'hypergeometric function'.

1.16.1 The recurring series formula for psi function

However, by the following steps and manipulating the gamma function we get a recurring series representation for $\psi(x+1) - \psi(x+1+m)$, where $x > -(m+1)$ with $m \in \mathbb{Z}^{+}$.

$$\begin{aligned} \psi(c) - \psi(c-b) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} \frac{\Gamma(b+k)}{k\Gamma(c+k)} \\ &= \frac{\Gamma(c)}{(b-1)(b-2)\dots\dots\dots(2)(1)} \sum_{k=1}^{\infty} \frac{(b+k-1)(b+k-2)\dots\dots(b)(b-1)(b-2)\dots.(2)(1)}{k\Gamma(c+k)} \\ &= \Gamma(c) \sum_{k=1}^{\infty} \frac{(b+k-1)(b+k-2)\dots.b}{k\Gamma(c+k)} \end{aligned} \tag{1.151}$$

and $(b+k-1)(b+k-2)\dots.(b+1)(b) = (-1)^{k}(-b)(-b-1)\dots.(-b-k+1)$. Here we substitute in the above expression (1.151) $c = (x+1)$, $b = -m$, so $x > -(m+1)$ where m is a positive integer, so that the condition $\mathrm{Re}\left[c\right] > \mathrm{Re}\left[b\right]$ is also maintained, to get the following expression:

$$\begin{aligned} \psi(x+1) - \psi(x+1+m) &= \Gamma(x+1) \sum_{k=1}^{\infty} \frac{(-1)^{k} m(m-1)\dots(m-k+1)}{k\Gamma(x+1+k)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k} m!\,\Gamma(x+1)}{k(m-k)!\,\Gamma(x+1+k)} \qquad x > -(m+1) \end{aligned} \tag{1.152}$$

The psi function obeys recursion relation $\psi(x+1) - \psi(x) = \frac{1}{x}$ and it has a special value of $-\psi(1) = -\Gamma^{(1)}(1) = \gamma = 0.5772156666\dots$ which is called Euler's constant. Euler's constant is also noted as $\gamma = \frac{1}{2}\left(\sqrt[3]{10} - 1\right)$, also $\gamma = \lim_{n \uparrow \infty} \sum_{k=1}^{n} \left(\frac{1}{k}\right) - \ln n$. For $\alpha \in \mathbb{R}$ a real number which tends to a natural number $n \in \mathbb{N}$ the expression $\lim_{\alpha \to n} \left[\frac{\psi(1-\alpha)}{\Gamma(1-\alpha)}\right] = (-1)^{-n}\Gamma(n)$ is true. In addition, we observe that $\psi(n+1) - \psi(1) = \sum_{j=1}^{n} \left(\frac{1}{j}\right)$ and $\psi(x+1) - \psi(x) = \frac{1}{x}$ are the properties of the 'psi' function. The other interesting properties are $\psi\left(\frac{1}{2}\right) = \psi(1) - \ln 4 = -\gamma - \ln 4$. If z is not a negative integer then we write the relationship as $\psi(z+1) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{z}{k(z+k)}\right)$. Similarly there is the relationship $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ and $\psi\left(z\right) + \psi\left(z + \frac{1}{2}\right) + 2\ln 2 = 2\psi(2z)$.

1.16.2 Analytical continuation of a finite harmonic series by using psi function

We can make an interesting application of 'analytical continuation' of the finite harmonic series $h(n)$:

$$h(n)=\sum_{k=1}^{n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}$$
$$=\psi(1+n)+\gamma \tag{1.153}$$
$$=\ln n+\gamma+O\left(\tfrac{1}{n}\right)$$

The above derivation (1.153) is from number theory. Here the use of the psi function with Euler's constant is demonstrated. Now we observe that the finite harmonic series $h(n)$ appears as the coefficients of x^n terms when we repeatedly integrate $\ln x$ (using integration by parts formula i.e. $\int uv\mathrm{d}x=u\int v\mathrm{d}x-\int u^{(1)}\left(\int v\mathrm{d}x\right)\mathrm{d}x$), as follows:

$$\begin{aligned}\int_0^x\left(\ln y\right)\left(\mathrm{d}y\right)&=\ln y\int(1)\mathrm{d}y\Big|_{y=0}^{y=x}-\int\left(\tfrac{\mathrm{d}}{\mathrm{d}y}\ln y\int(1)\mathrm{d}y\right)\mathrm{d}y\Big|_{y=0}^{y=x}\\&=\left(\ln x\right)(x)-\int\left(\tfrac{1}{y}\right)\left(y\mathrm{d}y\right)\Big|_{y=0}^{y=x}=x\left(\ln x-1\right)\end{aligned} \tag{1.154}$$

That we write as $\int_0^x\left(\ln y\right)\mathrm{d}y=x(\ln x-1)$. Integrating it once more, we get the following:

$$\begin{aligned}\int_0^x\ln y(\mathrm{d}y)^2&=\int(y\ln y-y)\mathrm{d}y\Big|_{y=0}^{y=x}\\&=\ln y\int y\mathrm{d}y\Big|_{y=0}^{y=x}-\int\tfrac{\mathrm{d}}{\mathrm{d}y}\ln y\left(\int y\mathrm{d}y\right)\mathrm{d}y\Big|_{y=0}^{y=x}-\left(\tfrac{x^2}{2}\right)\\&=\left(\ln x\right)\left(\tfrac{x^2}{2}\right)-\int\left(\tfrac{y}{2}\right)\mathrm{d}y\Big|_{y=0}^{y=x}-\left(\tfrac{x^2}{2}\right)=\left(\tfrac{1}{2}x^2\right)\ln x-\tfrac{3}{4}x^2\end{aligned} \tag{1.155}$$

$$\int_0^x\ln y(\mathrm{d}y)^2=\frac{x^2}{2}\left(\ln x-\tfrac{3}{2}\right)=\frac{x^2}{2}\left(\ln x-\left(1+\tfrac{1}{2}\right)\right)$$

Continuing n times, we arrive at the following expression:

$$\int_0^x\ln y(\mathrm{d}y)^n=\frac{x^n}{n!}\left(\ln x-\left(1+\tfrac{1}{2}+\tfrac{1}{3}+.....+\tfrac{1}{n}\right)\right)=\frac{x^n}{n!}\left(\ln x-h(n)\right) \tag{1.156}$$

Therefore, by analogy to the concept of generating functions, we can take $\int_0^x\ln y(\mathrm{d}y)^n$ as the generating integral of the harmonic series $h(n)$. Thus by analytically continuing for a non-integer n, (here α), we get the following:

$$\int_0^x\ln y\left(\mathrm{d}y\right)^{\alpha}=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\left(\ln x-h(\alpha)\right) \tag{1.157}$$

We will take up this integral (1.157) in a subsequent chapter.

1.17 A unified notation to represent multiple differentiation and integration

We know the notation $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ represents the $n-$th derivative of a function $f(x)$ with respect to x, when n is a non-negative integer. Because integration and differentiation are inverse operations (with exceptions that we will revise later), it is logical to associate the symbol $\frac{\mathrm{d}^{-1}f(x)}{\mathrm{d}x^{-1}}$ with the indefinite integration of $f(x)$ with respect to x. However, it is mandatory to define a lower limit of integration so that the indefinite integration is completely specific, that is $\frac{\mathrm{d}^{-1}f(x)}{\mathrm{d}x^{-1}}\equiv\int_0^x f(y)\mathrm{d}y$. Multiple integrations with a zero-lower terminal limit are symbolized as follows:

$$\begin{aligned}\int_0^x\left(f(y)\right)(\mathrm{d}y)^2&=\frac{\mathrm{d}^{-2}f(x)}{\left[\mathrm{d}x\right]^{-2}}\equiv\int_0^x\mathrm{d}x_1\int_0^{x_1}\left(f(x_0)\right)\mathrm{d}x_0\\\int_0^x\left(f(y)\right)(\mathrm{d}y)^n&=\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}x\right]^{-n}}\equiv\int_0^x\mathrm{d}x_{n-1}\int_0^{x_{n-1}}\mathrm{d}x_{n-2}\ldots\int_0^{x_2}\mathrm{d}x_1\int_0^{x_1}\left(f(x_0)\right)\mathrm{d}x_0\end{aligned} \tag{1.158}$$

Using the identity $\int_a^x \left(f(y)\right)\mathrm{d}y = \int_0^{x-a}\left(f(y+a)\right)\mathrm{d}y$, we extend the above symbolism (1.158) to a non-zero lower limit as follows:

$$\begin{aligned}\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}} &\equiv \int_a^x \left(f(y)\right)\mathrm{d}y \\ \frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} &\equiv \int_a^x \mathrm{d}x_{n-1}\int_a^{x_{n-1}}\mathrm{d}x_{n-2}\ldots\int_a^{x_2}\mathrm{d}x_1\int_a^{x_1}\left(f(x_0)\right)\mathrm{d}x_0\end{aligned} \tag{1.159}$$

The notation $f^{(n)}(x)$ will find frequent usage as the short-form of $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$. We will also use $f^{(-n)}(x)$ to represent $n-$ fold integration with respect to x, where the lower limits are not specified. That is

$$f^{(-n)}(x) \equiv \int_{a_n}^x \mathrm{d}x_{n-1}\int_{a_{n-1}}^{x_{n-1}}\mathrm{d}x_{n-2}\ldots\int_{a_2}^{x_2}\mathrm{d}x_1\int_{a_1}^{x_1}\left(f(x_0)\right)\mathrm{d}x_0 \tag{1.160}$$

where $a_1, a_2, \ldots. a_n$ are completely arbitrary. However, for $f^{(-n)}(x) - f^{(-n)}(a)$ we attach the same lower limits $a_1, a_2, \ldots. a_n$ to each integral. We have introduced simple symbols in the subsequent chapters; we will be using these and many other symbols that are used historically and for convenience.

1.18 Differentiation and integration of series

This idea of classical calculus is very important as the technique we will be using in finding fractional differentiation and fractional integration for functions expanded in a power series. A great many functions are traditionally described by infinite series expansions. It is of very much importance to understand the conditions that permit term-by-term differentiation or integration of such functions. Suppose that $f_0(x)$, $f_1(x)$ are functions defined and are continuous on the interval $a \le x \le b$, then:

$$\frac{\mathrm{d}^{-1}}{\left[\mathrm{d}(x-a)\right]^{-1}}\left[\sum_{j=0}^{\infty} f_j(x)\right] = \sum_{j=0}^{\infty}\frac{\mathrm{d}^{-1}\left[f_j(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}, \qquad a \le x \le b \tag{1.161}$$

provided the series $\sum f_j$ converges uniformly in the interval $a \le x \le b$.

The condition required for a derivative to be distributed through the terms of an infinite series are somewhat different. For this, we need each f_j to have a continuous derivative on interval $a \le x \le b$ then, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{j=0}^{\infty} f_j\right] = \sum_{j=0}^{\infty}\frac{\mathrm{d}\left[f_j\right]}{\mathrm{d}x} \qquad a \le x \le b \tag{1.162}$$

provided that $\sum f_j$ converges point-wise and $\sum \frac{\mathrm{d}\left[f_j\right]}{\mathrm{d}x}$ converges uniformly on the interval $a \le x \le b$.

Therefore we note that a uniform convergent series of continuous function (which itself defines a continuous function), may be integrated term by term, and that a continuously differentiable function may be differentiated term by term provided that the derived series is uniformly convergent.

1.19 Revising the concepts of mixed differentiation and integration

In classical calculus the identity:

$$\frac{\mathrm{d}^n}{\left[\mathrm{d}x\right]^n}\left(\frac{\mathrm{d}^N f(x)}{\left[\mathrm{d}x\right]^N}\right) = \frac{\mathrm{d}^{n+N}f(x)}{\left[\mathrm{d}x\right]^{n+N}} = \frac{\mathrm{d}^N}{\left[\mathrm{d}x\right]^N}\left(\frac{\mathrm{d}^n f(x)}{\left[\mathrm{d}x\right]^n}\right) \tag{1.163}$$

and the identity:

$$\frac{\mathrm{d}^{-n}}{\left[\mathrm{d}(x-a)\right]^{n}}\left(\frac{\mathrm{d}^{-N}f(x)}{\left[\mathrm{d}(x-a)\right]^{-N}}\right)=\frac{\mathrm{d}^{-n-N}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n-N}}$$
$$=\frac{\mathrm{d}^{-N}}{\left[\mathrm{d}(x-a)\right]^{-N}}\left(\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}\right) \tag{1.164}$$

hold for n and N non-negative integers. These (1.163) and (1.164) identities are basics to multiple differentiations and integration.

Let us try mixed differentiation and integration. For example, take $f(x)=e^{2x}+1$ and $a=0$. First, we do integrations three times. That is, first, we have $\int_0^x (e^{2y}+1)\mathrm{d}y=\frac{e^{2x}}{2}+x-\frac{1}{2}$; doing it once more, we have $\int_0^x \left(\frac{e^{2y}}{2}+y-\frac{1}{2}\right)\mathrm{d}y=\frac{e^{2x}}{4}+\frac{x^2}{2}-\frac{x}{2}-\frac{1}{4}$. Using this, we do one more integration and write $\int_0^x \left(\frac{e^{2y}}{4}+\frac{y^2}{2}-\frac{y}{2}-\frac{1}{4}\right)\mathrm{d}y=\frac{e^{2x}}{8}+\frac{x^3}{6}-\frac{x^2}{4}-\frac{x}{4}-\frac{1}{8}$. Now differentiating once the last expression, we get $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{e^{2x}}{8}+\frac{x^3}{6}-\frac{x^2}{4}-\frac{x}{4}-\frac{1}{8}\right]=\frac{e^{2x}}{4}+\frac{x^2}{2}-\frac{x}{2}-\frac{1}{4}$. Therefore, following our notation (of Section 1.17) we write the result as below:

$$\frac{\mathrm{d}}{\left[\mathrm{d}x\right]}\left(\frac{\mathrm{d}^{-3}\left[e^{2x}+1\right]}{\left[\mathrm{d}x\right]^{-3}}\right)=\frac{\mathrm{d}^{-2}\left[e^{2x}+1\right]}{\left[\mathrm{d}x\right]^{-2}}=\frac{1}{4}e^{2x}+\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{4} \tag{1.165}$$

Following a similar procedure, we first differentiate the function $f(x)=e^{2x}+1$ once, and then we integrate three folds (from $a=0$). Then we write in the following form the result in our described notation (of Section 1.17) to get the following expression:

$$\frac{\mathrm{d}^{-3}}{\left[\mathrm{d}x\right]^{-3}}\left(\frac{\mathrm{d}\left[e^{2x}+1\right]}{\mathrm{d}x}\right)=\frac{1}{4}e^{2x}-\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{4}\neq\frac{\mathrm{d}^{-2}\left[e^{2x}+1\right]}{\left[\mathrm{d}x\right]^{-2}} \tag{1.166}$$

We note that the above two methods (1.164) and (1.165) give different results.

Here we will deal with mixing differentiation and integration, that is $\frac{\mathrm{d}^{-n}}{[\mathrm{d}(x-a)]^{-n}}\left(\frac{\mathrm{d}^{N}f(x)}{[\mathrm{d}(x-a)]^{N}}\right)$, $\frac{\mathrm{d}^{N-n}f(x)}{[\mathrm{d}(x-a)]^{N-n}}$ and $\frac{\mathrm{d}^{N}}{[\mathrm{d}(x-a)]^{N}}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)$; and then derive similar identities when any two are unequal. If $f(x)$ is a function that is $N-$fold differentiable with $N\geq 1$ then by, the identity (1.163) i.e. $\frac{\mathrm{d}^{n}}{[\mathrm{d}x]^{n}}\left(\frac{\mathrm{d}^{N}f(x)}{[\mathrm{d}x]^{N}}\right)=\frac{\mathrm{d}^{n+N}f(x)}{[\mathrm{d}x]^{n+N}}=\frac{\mathrm{d}^{N}}{[\mathrm{d}x]^{N}}\left(\frac{\mathrm{d}^{n}f(x)}{[\mathrm{d}x]^{n}}\right)$ and by using a fundamental theorem of calculus we write the following expression (with understanding $f^{(0)}(x)\equiv f(x)$):

$$\frac{\mathrm{d}^{-1}f^{(N)}(x)}{\left[\mathrm{d}(x-a)\right]^{-1}}=f^{(N-1)}(x)-f^{(N-1)}(a) \tag{1.167}$$

Let us see how (1.167) gets formed. For that take $N=1$, the LHS of (1.167) reads $\int_a^x \left(f^{(1)}(y)\right)\mathrm{d}y$ equalling the RHS of (1.67) i.e. $f(x)-f(a)$. It is an integration of a once-differentiated function and returns function minus the value of function at the start-point. That we write it again as the following:

$$\frac{\mathrm{d}^{-1}f^{(1)}(x)}{\left[\mathrm{d}(x-a)\right]^{-1}}=\int_a^x \left(f^{(1)}(y)\right)\mathrm{d}y=f(x)-f(a) \tag{1.168}$$

This also states, that operation d and d^{-1} are not inverse in operation, unless $f(a)=0$. For $N=0$ we write the following:

$$\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}}=\int_a^x \left(f(y)\right)\mathrm{d}y=f^{(-1)}(x)-f^{(-1)}(a) \tag{1.169}$$

where in (1.169) $f^{(-1)}(a) \equiv \int \left(f(x) \right) \mathrm{d}x \Big|_{x=a}$. Integrating once more and using a fundamental theorem again we get the following expression:

$$\frac{\mathrm{d}^{-2} f(x)}{\left[\mathrm{d}(x-a)\right]^{-2}} = f^{(-2)}(x) - f^{(-2)}(a) - (x-a) f^{(-1)}(a) \tag{1.170}$$

where in (1.170) $f^{(-1)}(a) \equiv \int \left(f(x) \right) \mathrm{d}x \Big|_{x=a}$ and $f^{(-2)}(a) \equiv \int\int \left(f(x) \right) \mathrm{d}x \Big|_{x=a}$.

The above identity (1.170) in a way is generalised for function differentiated $N-$ folds.
Next we integrate a second time making use of identity (1.164) i.e. $\frac{\mathrm{d}^{-n}}{[\mathrm{d}(x-a)]^{n}} \left(\frac{\mathrm{d}^{-N}[f(x)]}{[\mathrm{d}(x-a)]^{-N}} \right) = \frac{\mathrm{d}^{-n-N}[f(x)]}{[\mathrm{d}(x-a)]^{-n-N}} = \frac{\mathrm{d}^{-N}}{[\mathrm{d}(x-a)]^{-N}} \left(\frac{\mathrm{d}^{-n}[f(x)]}{[\mathrm{d}(x-a)]^{-n}} \right)$. The LHS of (1.164) is $\frac{\mathrm{d}^{-2} f^{(N)}(x)}{[\mathrm{d}(x-a)]^{-2}}$, for $n=2$.

When $N \geq 2$, using (1.163) i.e. $\frac{\mathrm{d}^{n}}{[\mathrm{d}x]^{n}} \left(\frac{\mathrm{d}^{N} f(x)}{[\mathrm{d}x]^{N}} \right) = \frac{\mathrm{d}^{n+N} f(x)}{[\mathrm{d}x]^{n+N}} = \frac{\mathrm{d}^{N}}{[\mathrm{d}x]^{N}} \left(\frac{\mathrm{d}^{n} f(x)}{[\mathrm{d}x]^{n}} \right)$, and using the fundamental theorem we have the following for $n=2$:

$$\begin{aligned}\frac{\mathrm{d}^{-2} f^{(N)}(x)}{\left[\mathrm{d}(x-a)\right]^{-2}} &= f^{(N-2)}(x) - f^{(N-2)}(a) - \left(x-a\right) f^{(N-1)}(a) \\ &= f^{(N-2)}(x) - \sum_{k=0}^{(2-1)} \frac{(x-a)^k}{k!} f^{(N-2+k)}(a)\end{aligned} \tag{1.171}$$

From the above expression (1.171) the following generalisation follows:

$$\frac{\mathrm{d}^{-n} f^{(N)}(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} = f^{(N-n)}(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(N-n+k)}(a) \tag{1.172}$$

In the above expression (1.172) by setting $N=n$, we find the following identity:

$$\frac{\mathrm{d}^{-n} f^{(n)}(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} = f(x) - \sum_{k=0}^{n-1} \left(\frac{(x-a)^k}{k!} \right) f^{(k)}(a) \tag{1.173}$$

That is, d^{n} and d^{-n} are not inverse of each other unless all derivatives $f^{(1)}(x)$, $f^{(2)}(x) \ldots . f^{(n-1)}(x)$ are zero at $x=a$, along with $f(a)=0$. After setting $N=n$ and rearrangement of (1.173), the formula becomes:

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n \tag{1.174}$$

This (1.174) is a Taylor expansion where the remainder is $R_n = \frac{\mathrm{d}^{-n} f^{(n)}(x)}{[\mathrm{d}(x-a)]^{-n}}$, which gets expressed as $n-$ fold integral of $n-$ fold derivative of $f(x)$.

Now we use (1.172) i.e. $\frac{\mathrm{d}^{-n} f^{(N)}(x)}{[\mathrm{d}(x-a)]^{-n}} = f^{(N-n)}(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(N-n+k)}(a)$ and put $N=0$ and then write the following:

$$\begin{aligned}\frac{\mathrm{d}^{-n} f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} &= f^{(-n)}(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k-n)}(a) \\ &= f^{(-n)}(x) - \left(\begin{aligned} &f^{(-n)}(a) + (x-a) f^{(1-n)}(a) + \frac{(x-a)^2}{2!} f^{(2-n)}(a) + \\ &\ldots\ldots\ldots\ldots\ldots \frac{(x-a)^{n-1}}{(n-1)!} f^{(-1)}(a) \end{aligned} \right)\end{aligned} \tag{1.175}$$

Differentiating once the above expression (1.175), we write the following:

$$\frac{\mathrm{d}}{[\mathrm{d}(x-a)]}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)=f^{(1-n)}(x)-\left(\begin{array}{l}0+f^{(1-n)}(a)+(x-a)f^{(2-n)}(a)+\\ \ldots\ldots\ldots\ldots(n-1)\dfrac{(x-a)^{n-2}}{(n-1)!}f^{(-1)}(a)\end{array}\right)$$

$$=f^{(1-n)}(x)-\left(\begin{array}{l}f^{(1-n)}(a)+(x-a)f^{(2-n)}(a)+\\ \ldots\ldots\ldots\ldots\dfrac{(x-a)^{n-2}}{(n-2)!}f^{(-1)}(a)\end{array}\right) \tag{1.176}$$

$$=f^{(1-n)}(x)-\sum_{k=1}^{n-1}\frac{(x-a)^{k-1}}{(k-1)!}f^{(k-n)}(a)$$

We used $\frac{m!}{m}=(m-1)!$ and then redefined the series in (1.176) in compact form. Again, if we repeat the process on the above-obtained relationship (1.176), i.e. once again differentiating (1.176), we will get the following:

$$\frac{\mathrm{d}^2}{[\mathrm{d}(x-a)]^2}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)=f^{(2-n)}(x)-\sum_{k=2}^{n-1}\frac{(x-a)^{k-2}}{(k-2)!}f^{(k-n)}(a) \tag{1.177}$$

Therefore repeating N times, we obtain the following:

$$\frac{\mathrm{d}^N}{[\mathrm{d}(x-a)]^N}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)=f^{(N-n)}(x)-\sum_{k=N}^{n-1}\frac{(x-a)^{k-N}}{(k-N)!}f^{(k-n)}(a) \tag{1.178}$$

Let us see for the case $N=2$, and $n=1$, then write from (1.178) the following:

$$\frac{\mathrm{d}^2}{[\mathrm{d}(x-a)]^2}\left(\frac{\mathrm{d}^{-1}f(x)}{[\mathrm{d}(x-a)]^{-1}}\right)=f^{(1)}(x)-\sum_{k=2}^{0}\frac{(x-a)^{k-2}}{(k-2)!}f^{(k-1)}(a) \tag{1.179}$$

$$=f^{(1)}(x)$$

For $N=2$ and $n=2$:

$$\frac{\mathrm{d}^2}{[\mathrm{d}(x-a)]^2}\left(\frac{\mathrm{d}^{-2}f(x)}{[\mathrm{d}(x-a)]^{-2}}\right)=f(x)-\sum_{k=2}^{1}\frac{(x-a)^{k-2}}{(k-2)!}f^{(k-2)}(a) \tag{1.180}$$

$$=f(x)$$

Note that the summation is zero for $N\geq n$. Therefore for $N\geq n$ we write the following identity:

$$\frac{\mathrm{d}^N}{[\mathrm{d}(x-a)]^N}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)=f^{(N-n)}(x) \tag{1.181}$$

As choice of the lower terminal does not affect (integer order) differentiation, we have $\frac{\mathrm{d}^N f(x)}{[\mathrm{d}(x-a)]^N}=\frac{\mathrm{d}^N f(x)}{[\mathrm{d}(x)]^N}$, with N as a positive integer.

We have earlier noted (1.173) that $\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}=f^{(-n)}(x)-\sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}f^{(k-n)}(a)$; thus for $N<n$ with this formula we write the following:

$$\frac{\mathrm{d}^{N-n}f(x)}{[\mathrm{d}(x-a)]^{N-n}}=f^{(N-n)}(x)-\sum_{k=0}^{n-N-1}\frac{(x-a)^k}{k!}f^{(k+N-n)}(a) \tag{1.182}$$

Earlier we obtained in (1.178) $\frac{\mathrm{d}^N}{[\mathrm{d}(x-a)]^N}\left(\frac{\mathrm{d}^{-n}f(x)}{[\mathrm{d}(x-a)]^{-n}}\right)=f^{(N-n)}(x)-\sum_{k=N}^{n-1}\frac{(x-a)^{k-N}}{(k-N)!}f^{(k-n)}(a)$, and just above in (1.182) we obtained $\frac{\mathrm{d}^{N-n}f(x)}{[\mathrm{d}(x-a)]^{N-n}}=f^{(N-n)}(x)-\sum_{k=0}^{n-N-1}\frac{(x-a)^k}{k!}f^{(k+N-n)}(a)$. The summation expression for both (1.178) and (1.182) is the same so we have to re-adjust the summation index. That makes the RHS of both of these expressions the same as those which imply their LHSs are also the same. This discussion demonstrated the following rule:

$$\frac{\mathrm{d}^N}{\left[\mathrm{d}(x-a)\right]^N}\left(\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}\right)=\frac{\mathrm{d}^{N-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{N-n}}$$
$$=\frac{\mathrm{d}^{-n}}{\left[\mathrm{d}(x-a)\right]^{-n}}\left(\frac{\mathrm{d}^N f(x)}{\mathrm{d}(x-a)^N}\right)+\sum_{k=n-N}^{n-1}\frac{(x-a)^k}{k!}f^{(k+N-n)}(a) \tag{1.183}$$

Therefore, we summarise that identity as follows:

$$\frac{\mathrm{d}^m}{\left[\mathrm{d}(x-a)\right]^m}\left(\frac{\mathrm{d}^M f(x)}{\left[\mathrm{d}(x-a)\right]^M}\right)=\frac{\mathrm{d}^{m+M}f(x)}{\left[\mathrm{d}(x-a)\right]^{m+M}} \tag{1.184}$$

When in (1.184) m and M are integers necessarily holds 'unless' M is positive and m is negative, or in other words 'unless' the function $f(x)$ is first differentiated and then integrated. The expression:

$$\frac{\mathrm{d}^{-n}}{\left[\mathrm{d}(x-a)\right]^{-n}}\left(\frac{\mathrm{d}^N f(x)}{\left[\mathrm{d}(x-a)\right]^N}\right)=\frac{\mathrm{d}^{N-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{N-n}}$$
$$=\frac{\mathrm{d}^N}{\left[\mathrm{d}(x-a)\right]^N}\left(\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}\right) \tag{1.185}$$

holds only when $f(a)=0$, and all derivatives of $f(x)$ through $(N-1)$-th, are also zero at $x=a$. We revert to our example, that is $f(x)=e^{2x}+1$ with $a=0$ and write the following:

$$\frac{\mathrm{d}}{\left[\mathrm{d}x\right]}\left(\frac{\mathrm{d}^{-3}\left[e^{2x}+1\right]}{\left[\mathrm{d}x\right]^{-3}}\right)=\frac{\mathrm{d}^{-2}\left[e^{2x}+1\right]}{\left[\mathrm{d}x\right]^{-2}}=\frac{1}{4}e^{2x}+\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{4}$$
$$\frac{\mathrm{d}^{-3}}{\left[\mathrm{d}x\right]^{-3}}\left(\frac{\mathrm{d}\left[e^{2x}+1\right]}{\mathrm{d}x}\right)=\frac{1}{4}e^{2x}-\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{4} \tag{1.186}$$

The difference between them is x^2. From our formula (1.183) we have for $N=1$, $n=3$ and $a=0$ the following, with $f(x)=e^{2x}+1$:

$$\frac{\mathrm{d}}{\left[\mathrm{d}x\right]}\left(\frac{\mathrm{d}^{-3}f(x)}{\left[\mathrm{d}x\right]^{-3}}\right)=\frac{\mathrm{d}^{-2}f(x)}{\left[\mathrm{d}x\right]^{-2}}=\frac{\mathrm{d}^{-3}}{\left[\mathrm{d}x\right]^{-3}}\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)+\sum_{k=2}^{2}\frac{(x)^k}{k!}f^{(k-2)}(0) \tag{1.187}$$

Note that $\sum_{k=2}^{2}\frac{(x)^k}{k!}f^{(k-2)}(0)=\frac{x^2}{2}f(0)=\frac{x^2}{2}(2)=x^2$ which is the difference between the two operations.

In subsequent chapters, we will be carrying on this generalisation to any arbitrary number n, N (not integers) say α, β i.e. as non-integer numbers.

1.20 The lower limit as an important factor in multiple integrals

We will see how multiple integrals behave in a complicated way when their lower limit is changed. In an earlier section (1.175), we derived the following:

$$\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}=f^{(-n)}(x)-\sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}f^{(k-n)}(a) \tag{1.188}$$

Say the lower terminal of the n- fold integration gets changed to b, with $a<b<x$ then using the above formula (1.188) we write the difference:

$$\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}-\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-b)\right]^{-n}}$$
$$=\sum_{k=0}^{n-1}\frac{1}{k!}\left((x-b)^k f^{(k-n)}(b)-(x-a)^k f^{(k-n)}(a)\right) \tag{1.189}$$

For $n=1$:

$$\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}}-\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-b)\right]^{-1}} \\ =\sum_{k=0}^{0}\frac{1}{k!}\left((x-b)^{k}f^{(k-1)}(b)-(x-a)^{k}f^{(k-1)}(a)\right) \\ =f^{(-1)}(b)-f^{(-1)}(a) \tag{1.190}$$

This is an obvious case that is $\int_a^x\left(f(y)\right)\mathrm{d}y-\int_b^x\left(f(y)\right)\mathrm{d}y=\int_a^b\left(f(x)\right)\mathrm{d}x$, obtained in the same way as $f^{(-1)}(x)\Big|_{x=b}-f^{(-1)}(x)\Big|_{x=a}$. The obtained difference expression (1.189), for n other than one; depends on both the lower terminals of multiple integration, that is a and b; and on the variable x. We do the following manipulation from the obtained expression above (1.189):

$$\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}-\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-b)\right]^{-n}}-\sum_{k=0}^{n-1}\frac{(x-b)^{k}}{k!}f^{(k-n)}(b) \\ =-\sum_{k=0}^{n-1}\frac{(x-a)^{k}}{k!}f^{(k-n)}(a) \\ =-\sum_{k=0}^{n-1}\frac{1}{k!}\left((x-b)+(b-a)\right)^{k}f^{(k-n)}(a) \\ =-\sum_{k=0}^{n-1}\frac{1}{k!}\left(\sum_{j=0}^{k}\tfrac{k!}{j!(k-j)!}(x-b)^{j}(b-a)^{k-j}\right)f^{(k-n)}(a) \\ =-\sum_{k=0}^{n-1}f^{(k-n)}(a)\left(\sum_{j=0}^{k}\frac{(x-b)^{j}(b-a)^{k-j}}{j!(k-j)!}\right) \tag{1.191}$$

In the above derivation (1.191) we have used a binomial expansion that is $(x+y)^n=\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}x^r y^{n-r}$. The sum that $\sum_{k=0}^{n-1}f^{(k-n)}(a)\sum_{j=0}^{k}\frac{(x-b)^j(b-a)^{k-j}}{j!(k-j)!}$ appears above (1.191) may be expressed in various ways, like $\sum_{K=0}^{n-1}\frac{(x-b)^K}{K!}\sum_{J=K}^{n-1}\frac{(b-a)^{J-K}}{(J-K)!}f^{(J-n)}(a)$ or $\sum_{K=0}^{n-1}\frac{(x-b)^K}{K!}\sum_{j=0}^{n-K-1}\frac{(b-a)^j}{j!}f^{(j+K-n)}(a)$. We identify the index K with the k of (1.191) and write the following expression:

$$\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}-\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-b)\right]^{-n}} \\ =\sum_{K=0}^{n-1}\frac{(x-b)^{K}}{K!}\left(f^{(K-n)}(b)-\sum_{j=0}^{n-K-1}\frac{(b-a)^{j}}{j!}f^{(K+j-n)}(a)\right) \\ =\sum_{k=1}^{n}\frac{(x-b)^{n-k}}{(n-k)!}\left(f^{(-k)}(b)-\sum_{j=0}^{k-1}\frac{(b-a)^{j}}{j!}f^{(j-k)}(a)\right) \tag{1.192}$$

using $\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}=f^{(-n)}(x)-\sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}f^{(k-n)}(a)$ i.e. (1.173) for the bracketed expression in (1.192) so that it becomes $f^{(-k)}(b)-\sum_{j=0}^{k-1}\frac{(b-a)^j}{j!}f^{(j-k)}(a)=\frac{\mathrm{d}^{-k}f(b)}{\left[\mathrm{d}(b-a)\right]^{-k}}$. Substituting this in the above derivation (1.192), we get the following expression:

$$\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}}-\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-b)\right]^{-n}}=\sum_{k=1}^{n}\frac{(x-b)^{n-k}}{(n-k)!}\frac{\mathrm{d}^{-k}f(b)}{\left[\mathrm{d}(b-a)\right]^{-k}} \tag{1.193}$$

Therefore, we have the expression above (1.193) showing explicit dependence on $b-a$, shifting the lower terminal of multiple integration from a to b. We will extend this and generalise when n is a non-integer number in later chapters.

1.21 Generalising product rule for multiple integration and differentiations

Leibniz's rule for $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[f(x)g(x)\right]$ is obtained as follows. Writing $f(x)=f$ and $g(x)=g$, we know the formula:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right]=f^{(1)}g+g^{(1)}f \tag{1.194}$$

Applying this again, we get:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[fg\right]=f^{(2)}g+2f^{(1)}g^{(1)}+fg^{(2)} \tag{1.195}$$

Repeating this process we get the n-th order generalisation represented below:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[fg\right]=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}g^{(k)}=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}f}{\mathrm{d}x^{n-k}}\frac{\mathrm{d}^k g}{\mathrm{d}x^k} \tag{1.196}$$

Now we will see Leibniz's formula for $-n$ which is the product rule for multiple integrations. Starting with a well-known formula for integration by parts i.e. $\int_a^x\left(g(y)\right)\left(\mathrm{d}v(y)\right)=\left(g(x)\right)\left(v(x)\right)-\left(g(a)\right)\left(v(a)\right)-\int_a^x\left(v(y)\right)\left(\mathrm{d}g(y)\right)$ we will derive some interesting results. Let $v(y)=\int_a^y\left(f(z)\right)\mathrm{d}z$, then we write the integration by parts of the formula as follows:

$$\int_a^x\left(g(y)\right)\left(f(y)\right)\mathrm{d}y=g(x)\int_a^x\left(f(z)\right)\mathrm{d}z-\int_a^x\left(\int_a^y\left(f(z)\right)\mathrm{d}z\right)\left(\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right)\mathrm{d}y \tag{1.197}$$

With our new symbolism (Section 1.17) and writing $f(x)=f$ and $g(x)=g$, we shall re-write the above expression (1.197) as:

$$\frac{\mathrm{d}^{-1}\left[fg\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}=g\left(\frac{\mathrm{d}^{-1}f}{\left[\mathrm{d}(x-a)\right]^{-1}}\right)-\frac{\mathrm{d}^{-1}}{\left[\mathrm{d}(x-a)\right]^{-1}}\left[g^{(1)}\frac{\mathrm{d}^{-1}f}{\left[\mathrm{d}(x-a)\right]^{-1}}\right] \tag{1.198}$$

We apply this again to the product in the brackets of the second term in the above expression (1.198) and write the following:

$$\begin{aligned}\frac{\mathrm{d}^{-1}\left[fg\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}&=g\left(\frac{\mathrm{d}^{-1}f}{\left[\mathrm{d}(x-a)\right]^{-1}}\right)-g^{(1)}\frac{\mathrm{d}^{-2}f}{\left[\mathrm{d}(x-a)\right]^{-2}}\\&+\frac{\mathrm{d}^{-1}}{\left[\mathrm{d}(x-a)\right]^{-1}}\left[g^{(2)}\frac{\mathrm{d}^{-2}f}{\left[\mathrm{d}(x-a)\right]^{-2}}\right]\end{aligned} \tag{1.199}$$

Doing this process an infinite number of times, we get the following expression:

$$\frac{\mathrm{d}^{-1}\left[fg\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}=\sum_{j=0}^{\infty}(-1)^j g^{(j)}\frac{\mathrm{d}^{-1-j}f}{\left[\mathrm{d}(x-a)\right]^{-1-j}} \tag{1.200}$$

The alternate sign change by $(-1)^j$ for $j=0,1,2,3...$ can be cast by a binomial expression, ${}^{-1}C_j=\frac{(-1)!}{j!(-1-j)!};\quad j=0,1,2,3....$ This is like an infinite series expansion i.e. $(1+x)^{-1}=\sum_{j=0}^{\infty}\frac{(-1)!}{j!(-1-j)!}(1)^j(x)^{-1-j}=1-x^{-1}+x^{-2}+....$

For verification, taking $j=0$, we obtain $\frac{(-1)!}{j!(-1-j)!}=\frac{(-1)!}{0!(-1)!}=1$, then taking $j=1$, we obtain $\frac{(-1)!}{j!(-1-j)!}=\frac{(-1)!}{(1)!(-2)!}=\frac{(-1)(-2)!}{(1)(-2)!}=-1$; after that we take $j=2$ and we get $\frac{(-1)!}{j!(-1-j)!}=\frac{(-1)!}{2!(-3)!}=\frac{(-1)(-2)(-3)!}{2(-3)!}=1$ and so on. With this observation, we can write the following formula:

$$\frac{\mathrm{d}^{-1}\left[fg\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}=\sum_{j=0}^{\infty}\binom{-1}{j}g^{(j)}\frac{\mathrm{d}^{-1-j}f}{\left[\mathrm{d}(x-a)\right]^{-1-j}} \tag{1.201}$$

We perform integration again to the above formula (1.201) with the identity we noted earlier, i.e. those from (1.163) and (1.164) which are written again below:

$$\frac{\mathrm{d}^n}{[\mathrm{d}x]^n}\left(\frac{\mathrm{d}^N f}{[\mathrm{d}x]^N}\right)=\frac{\mathrm{d}^{n+N} f}{[\mathrm{d}x]^{n+N}}=\frac{\mathrm{d}^N}{[\mathrm{d}x]^N}\left(\frac{\mathrm{d}^n f}{[\mathrm{d}x]^n}\right)$$

$$\frac{\mathrm{d}^{-n}}{[\mathrm{d}(x-a)]^{-n}}\left(\frac{\mathrm{d}^{-N} f}{[\mathrm{d}(x-a)]^{-N}}\right)=\frac{\mathrm{d}^{-n-N} f}{[\mathrm{d}(x-a)]^{-n-N}} \qquad (1.202)$$

$$=\frac{\mathrm{d}^{-N}}{[\mathrm{d}(x-a)]^{-N}}\left(\frac{\mathrm{d}^{-n} f}{[\mathrm{d}(x-a)]^{-n}}\right)$$

Equation (1.202) holds true for non-negative integers N and n. That is demonstrated below:

$$\frac{\mathrm{d}^{-2}[fg]}{[\mathrm{d}(x-a)]^{-2}}=\frac{\mathrm{d}^{-1}}{[\mathrm{d}(x-a)]^{-1}}\left[\sum_{j=0}^{\infty}\binom{-1}{j}g^{(j)}\frac{\mathrm{d}^{-1-j} f}{[\mathrm{d}(x-a)]^{-1-j}}\right]$$

$$=\sum_{j=0}^{\infty}\binom{-1}{j}\left(\frac{\mathrm{d}^{-1}}{[\mathrm{d}(x-a)]^{-1}}\left[g^{(j)}\frac{\mathrm{d}^{-1-j} f}{[\mathrm{d}(x-a)]^{-1-j}}\right]\right) \qquad (1.203)$$

$$\frac{\mathrm{d}^{-1}}{[\mathrm{d}(x-a)]^{-1}}\left[g^{(j)}\frac{\mathrm{d}^{-1-j} f}{[\mathrm{d}(x-a)]^{-1-j}}\right]=\sum_{k=0}^{\infty}\binom{-1}{k}g^{(j+k)}\frac{\mathrm{d}^{-j-k-2} f}{[\mathrm{d}(x-a)]^{-j-k-2}}$$

We now proceed with the following steps:

$$\frac{\mathrm{d}^{-2}[fg]}{[\mathrm{d}(x-a)]^{-2}}=\sum_{j=0}^{\infty}\binom{-1}{j}\sum_{k=0}^{\infty}\binom{-1}{k}g^{(j+k)}\frac{\mathrm{d}^{-j-k-2} f}{[\mathrm{d}(x-a)]^{-j-k-2}}$$

$$=\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\binom{-1}{j}\binom{-1}{m-j}g^{(m)}\frac{\mathrm{d}^{-2-m} f}{[\mathrm{d}(x-a)]^{-2-m}}$$

$$=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{-1}{j}\binom{-1}{m-j}g^{(m)}\frac{\mathrm{d}^{-2-m} f}{[\mathrm{d}(x-a)]^{-2-m}} \qquad (1.204)$$

$$=\sum_{m=0}^{\infty}\binom{-2}{m}g^{(m)}\frac{\mathrm{d}^{-2-m} f}{[\mathrm{d}(x-a)]^{-2-m}}$$

In the above derivation (1.204) we have used a permutation that is $\sum_{k=0}^{\infty}\sum_{j=0}^{k} \;=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}$. This permutation is analogous to the permutation of variables in double integration that is $\int_0^\infty \mathrm{d}y\int_0^y \mathrm{d}x=\int_0^\infty \mathrm{d}x\int_x^\infty \mathrm{d}y$. In (1.204) we used the following formula of binomial coefficients:

$$\sum_{k=0}^{j}\binom{n}{k}\binom{N}{j-k}=\binom{n+N}{j} \qquad (1.205)$$

The iteration of the above procedure (1.204) further leads to the following formula:

$$\frac{\mathrm{d}^{-n}[fg]}{[\mathrm{d}(x-a)]^{-n}}=\sum_{j=0}^{\infty}\binom{-n}{j}g^{(j)}\frac{\mathrm{d}^{-n-j} f}{[\mathrm{d}(x-a)]^{-n-j}} \qquad n=1,2,3,... \qquad (1.206)$$

A similarity between product rule for multiple differentiations and the expression for product rule for multiple integrations can be seen. Writing both of them below (for $n>0$) gives us:

$$\frac{\mathrm{d}^n[fg]}{\mathrm{d}x^n}=\sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^{n-j} f}{\mathrm{d}x^{n-j}}\frac{\mathrm{d}^j g}{\mathrm{d}x^j}$$

$$\frac{\mathrm{d}^{-n}[fg]}{[\mathrm{d}(x-a)]^{-n}}=\sum_{j=0}^{\infty}\binom{-n}{j}\frac{\mathrm{d}^{-n-j} f}{\mathrm{d}x^{-n-j}}\frac{\mathrm{d}^j g}{\mathrm{d}x^j} \qquad (1.207)$$

We will use this developed formula for further generalisation when n is a non-integer in later chapters.

1.22 Chain rule for multiple derivatives

We write what we call chain rule as:

$$\frac{\mathrm{d}}{\mathrm{d}x} g\left(f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}u} g(u) \frac{\mathrm{d}}{\mathrm{d}x} f(x) = g^{(1)} f^{(1)} \tag{1.208}$$

This enables us to differentiate $g(u)$ with respect of x, where $u = f(x)$, if the derivative of $g(u)$ with respect to u and the derivative of $u = f(x)$ are known. By application of this chain rule, and Leibniz's rule for differentiating a product of two functions we use the following steps:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2} g\left(f(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x} g\left(f(x)\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}u} g(u)\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x} f(x)\right)\right] \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\left[g(u)\right]}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}\left[f(x)\right]}{\mathrm{d}x}\right) + \left(\frac{\mathrm{d}\left[g(u)\right]}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2\left[f(x)\right]}{\mathrm{d}x^2}\right) \\
&= \left(\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}^2\left[g(u)\right]}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}\left[f(x)\right]}{\mathrm{d}x}\right) + \left(\frac{\mathrm{d}\left[g(u)\right]}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2\left[f(x)\right]}{\mathrm{d}x^2}\right) \\
&= \left(\frac{\mathrm{d}\left[f(x)\right]}{\mathrm{d}x}\frac{\mathrm{d}^2\left[g(u)\right]}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}\left[f(x)\right]}{\mathrm{d}x}\right) + \left(\frac{\mathrm{d}\left[g(u)\right]}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2\left[f(x)\right]}{\mathrm{d}x^2}\right) \\
&= \left(\frac{\mathrm{d}\left[f(x)\right]}{\mathrm{d}x}\right)^2\left(\frac{\mathrm{d}^2\left[g(u)\right]}{\mathrm{d}x^2}\right) + \left(\frac{\mathrm{d}\left[g(u)\right]}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2\left[f(x)\right]}{\mathrm{d}x^2}\right) \\
&= g^{(1)} f^{(2)} + g^{(2)}\left(f^{(1)}\right)^2
\end{aligned} \tag{1.209}$$

We repeat it and then write the following expressions:

$$\begin{aligned}
\frac{\mathrm{d}^3\left[g\left(f(x)\right)\right]}{\mathrm{d}x^3} &= g^{(1)} f^{(3)} + 3g^{(2)} f^{(1)} f^{(2)} + g^{(3)}\left(f^{(1)}\right)^3 \\
\frac{\mathrm{d}^4\left[g\left(f(x)\right)\right]}{\mathrm{d}x^4} &= g^{(1)} f^{(4)} + 4g^{(2)} f^{(1)} f^{(3)} + 6g^{(2)}\left(f^{(2)}\right)^2 \\
&\quad + 6g^{(3)}\left(f^{(1)}\right)^2 f^{(2)} + g^{(4)}\left(f^{(1)}\right)^4 \\
\frac{\mathrm{d}^5\left[g\left(f(x)\right)\right]}{\mathrm{d}x^5} &= g^{(1)} f^{(5)} + 5g^{(2)} f^{(1)} f^{(4)} + 10g^{(3)}\left(f^{(1)}\right)^2 f^{(3)} \\
&\quad + 30g^{(3)} f^{(1)}\left(f^{(2)}\right)^2 + 10g^{(4)}\left(f^{(1)}\right)^3 f^{(2)} + g^{(5)}\left(f^{(1)}\right)^5
\end{aligned} \tag{1.210}$$

The continuation of the above steps (1.210) to n would give the generalised formula $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[g\left(f(x)\right)\right]$. However, it is tough and requires a special formula, such as Faa' di Bruno's formula (we are not proving this). Faa' di Bruno's formula is too complicated to be of any utility for large n. We write that, however, as the following:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} g\left(f(x)\right) = n!\sum_{m=1}^{n} g^{(m)} \sum \prod_{k=1}^{n} \frac{1}{P_k!}\left(\frac{f^{(k)}}{k!}\right)^{P_k} \tag{1.211}$$

Where in (1.211) the summation $\sum$ extends over all combinations of non-negative integer values of $P_1, P_2, ..., P_n$ such that $\sum_{k=1}^{n} kP_k = n$ and $\sum_{k=1}^{n} P_k = m$. However, this complicated formula reveals its utility in some specific cases. When $g = u^2$, then we have:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} g\left(f(x)\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(f(x)\right)^2 = 2f^{(n)} f + \sum_{k=1}^{n-1}\binom{n}{k} f^{(k)} f^{(n-k)} \tag{1.212}$$

When $f(x) = e^x$, then we have:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} g\left(f(x)\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} g\left(e^x\right) = e^{nx}\sum_{m=1}^{n} S_n^{[m]} g^{(m)} \tag{1.213}$$

where $S_n^{[m]}$ is a Stirling number of a second kind. Again if $g = u^2$, then we find:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[f(x)\right]^2 = 2\left(f^{(n)}(x)\right)\left(f(x)\right) + \sum_{k=1}^{n-1}\binom{n}{k}\left(f^{(k)}(x)\right)\left(f^{(n-k)}(x)\right) \tag{1.214}$$

a result that may also be obtained from Leibniz's theorem for multiple integrals.

1.23 Differentiation and integration of power functions: unification & generalisation

In Section 1.18, we have dealt with differentiation and the integration of infinite series of functions. We will be encountering in the rest of the chapters, the series whose general term is a simple power function i.e. of type $(x-a)^p$, where p is considered as an integer. We collect here elementary formulae to express $\frac{\mathrm{d}^n}{[\mathrm{d}(x-a)]^n}\left[(x-a)^p\right]$ for positive and negative values of n. For differentiation we have the following:

$$\begin{aligned} n=0 &\qquad \frac{\mathrm{d}^0\left[(x-a)^p\right]}{\mathrm{d}x^0} = (x-a)^p \\ n=1 &\qquad \frac{\mathrm{d}^1\left[(x-a)^p\right]}{\mathrm{d}x^1} = p(x-a)^{p-1} \\ n=2 &\qquad \frac{\mathrm{d}^2\left[(x-a)^p\right]}{\mathrm{d}x^2} = p(p-1)(x-a)^{p-2} \end{aligned} \tag{1.215}$$

From the above steps of (1.215) we write the following:

$$\begin{aligned} \frac{\mathrm{d}^n\left[(x-a)^p\right]}{\mathrm{d}x^n} &= p(p-1).......(p-n+1)(x-a)^{p-n} \\ &= \frac{p!}{(p-n)!}(x-a)^{p-n} \qquad n=0,1,2,... \end{aligned} \tag{1.216}$$

We note in (1.216) the term $\frac{p!}{(p-n)!} = \frac{p(p-1)(p-2)....(p-n+1)(p-n)(p-n-1)....(2)(1)}{(p-n)(p-n-1)....(2)(1)} = p(p-1)....(p-n+1)$. For the integration we have the following steps:

$$\begin{aligned} n=-1 &\qquad \int_a^x (x_0-a)^p \mathrm{d}x_0 = \frac{1}{(p+1)}(x-a)^{p+1} \\ n=-2 &\qquad \int_a^x \mathrm{d}x_1 \int_a^{x_1} (x_0-a)^p\,\mathrm{d}x_0 = \frac{1}{(p+1)(p+2)}(x-a)^{p+2} \end{aligned} \tag{1.217}$$

From the above steps of (1.217), we write the following:

$$\begin{aligned} \frac{\mathrm{d}^{-n}\left[(x-a)^p\right]}{\left[\mathrm{d}(x-a)\right]^{-n}} &\equiv \int_a^x \mathrm{d}x_{n-1}\int_a^{x_{n-1}} \mathrm{d}x_{n-2}........\int_a^{x_1}(x_0-a)^p\mathrm{d}x_0 \\ &= \begin{cases} \dfrac{(x-a)^{p+n}}{(p+1)(p+2)...(p+n)} = \dfrac{p!}{(p+n)!}(x-a)^{p+n} & p>-1 \\ \infty & p\le -1 \end{cases} \end{aligned} \tag{1.218}$$

$$n=1,2,3,...$$

We note in the above (1.218) that we have $\frac{p!}{(p+n)!} = \frac{p(p-1)(p-2)......(2)(1)}{(p+n)(p+n-1)......(p+2)(p+1)(p)(p-1).....(2)(1)} = \frac{1}{(p+n)(p+n-1)...(p+2)(p+1)}$.

From the above formulas (1.216) and (1.218) for differentiation and integration we write the following table, where positive n is differentiation and negative n is integration, and instead of the values we indicate positive and negative signs for the result. Thus, Table 1.4 unifies the above two formulas, for integer p and integer n.

$p\downarrow$ $n\rightarrow$	-3	-2	-1	0	1	2	3
2	+	+	+	+	+	+	0
1	+	+	+	+	+	0	0
0	+	+	+	+	0	0	0
-1	∞	∞	∞	+	$-$	+	$-$
-2	∞	∞	∞	+	$-$	+	$-$

Table 1.4: The sign of $\mathrm{d}^n\left[(x-a)^p\right]/\left[\mathrm{d}(x-a)\right]^n$ for the positive and negative integers n and p

By invoking the properties of the gamma function for the factorials both formulas (1.217) and (1.218) can be generalised for any p by writing $\frac{p!}{(p-n)!}$ as $\frac{\Gamma(p+1)}{\Gamma(p-n+1)}$ for positive and negative n.

$$\frac{\mathrm{d}^n\left[(x-a)^p\right]}{\left[\mathrm{d}(x-a)\right]^n}=\begin{cases}\dfrac{\Gamma(p+1)}{\Gamma(p-n+1)}(x-a)^{p-n} & \begin{cases}n=0,1,2,.....; & \text{For all } p\\ n=-1,-2,....; & p>-1\end{cases}\\ \infty & \quad n=-1,-2,....; \quad p\le -1\end{cases} \tag{1.219}$$

The coefficient $\frac{\Gamma(p+1)}{\Gamma(p-n+1)}$ may be positive, negative or zero according to the values of p and n. Figure 1.5 displays these values graphically. We will study the above generalisation in subsequent chapters, for n as a non-integer.

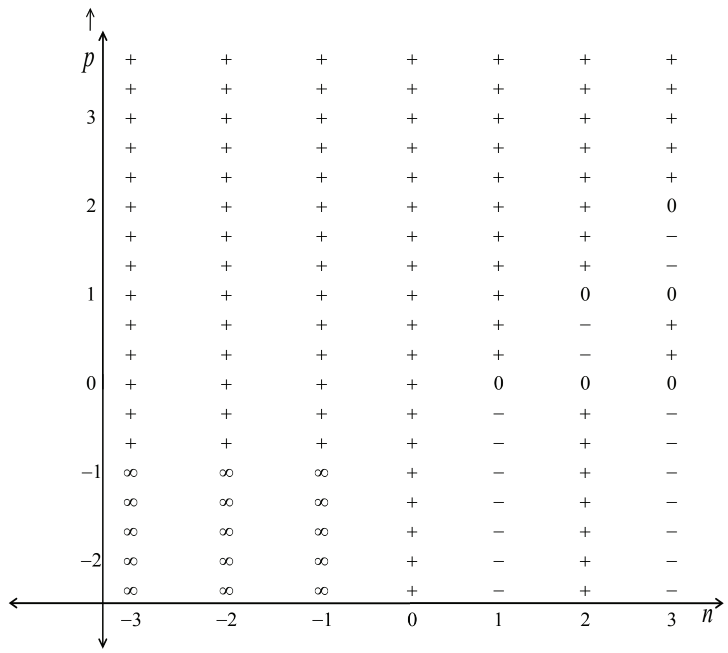

Figure 1.5: The sign of $\mathrm{d}^n\left[(x-a)^p\right]/\left[\mathrm{d}(x-a)\right]^n$ for positive and negative integer values of n

1.24 Non-differentiable functions: the motivation for fractional calculus

1.24.1 Hurst exponent

The main aim or at least the motive in investigating fractional differential manifolds is the fact that fractional calculus seems quite relevant to investigate some of the problems, which occur in fractal space-time. The first elementary point of view is the mathematics framework of fractal physics that deals with differentials satisfying the condition:

$$\mathrm{d}\big(f(x)\big)\propto\big(\mathrm{d}x\big)^H \tag{1.220}$$

for the function $f(x)$, where $H>0$ (need not be an integer). This H is referred to as the 'Hurst exponent'. The above differential condition means that the classical equality $\mathrm{d}\big(f(x)\big)=\big(f^{(1)}(x)\big)\mathrm{d}x$ no longer holds; in other words, the function $f(x)$ is not differentiable, and that instead we should be describing it in the following form:

$$\mathrm{d}\big(f(x)\big)=\big(g(x)\big)\big(\mathrm{d}x\big)^H \tag{1.221}$$

This expression (1.221) leads to fractional calculus. Observe that we have here differential quantity as fractional differential quantity which leads to a derivative and integration with respect to fractional differential elements, and that we call it $(\mathrm{d}x)^{\alpha}$, where α is a non-integer, and generally $0<\alpha<1$.

1.24.2 Self-similarity

Qualitatively speaking a self-similar function is a function which exhibits similar patterns when one changes the scale of observation. For example, the pattern generated by $f(x)$ and $f(\lambda x)$ with $\lambda>0$ looks the same. Formally, $f(x)$ with $x\in\mathbb{R}$ is of a similar order to H if one has equality:

$$f(\lambda x)=\lambda^{H}f(x) \qquad \lambda>0 \qquad H>0 \tag{1.222}$$

which means that the landscape in the vicinity of x, and that of λx look alike. Such a function satisfies the condition, i.e. $f(0)=0$. It also provides the following relation via substitution of $x\leftarrow 1$ making $f(\lambda)=\lambda^{H}f(1)$ and then by substituting $\lambda\leftarrow x$ giving the following expression:

$$f(x)=x^{H}f(1) \tag{1.223}$$

1.24.3 Nowhere differentiable functions

As a special case when $0<H<1$ the function $f(x)$ is non-differentiable but is fractionally differentiable for the fractional order of H. Like $f(x)=x^{1/2}$; $x\geq 0$ is non-differentiable at $x=0$, but is fractionally differentiable by order $1/2$, as we will learn in subsequent chapters. Then how can we form a Taylor series for a non-differentiable function? This is only possible if we have non-integer (fractional) derivatives; this we will learn subsequently too.

We shall soon be dealing with nowhere differentiable functions. The function $f(x)=x^{1/2}$ is not differentiable at $x=0$, but surely has derivatives at points other than zero. However, what is a function that is nowhere differentiable?. Let us construct one such case. Let us take interval $[0,1]$ on which we will define the nowhere differentiable function. In this interval we have a set of discrete points $x_1,x_2,....x_N$ such that $x_1=0$ and $x_N=1$ with $x_1<x_2<x_3<...<x_N$. We construct a function $f_{\text{fract}}(x)=\sum_{k=1}^{N}(x-x_k)^{1/2}$ for $x\geq x_k$ and $f_{\text{fract}}(x)=0$ for $x<x_k$, with $k=1,2,3...,N$. We observe that at points x_k the function value is 0 but is not differentiable, for any $k=1,2,3...,N$. Thus, the function $f_{\text{fract}}(x)$ is continuous at all points x_k but not differentiable at any point x_k. Therefore, we will write for infinite points N and the function in the interval $[0,1]$ as:

$$f_{\text{fract}}(x)=\begin{cases}\lim_{N\uparrow\infty}\sum_{k=1}^{N}(x-x_k)^{1/2} & ;\quad x\geq x_k\\ 0 & ;\quad x<x_k\end{cases} \tag{1.224}$$

It is continuous everywhere but nowhere differentiable. These types of functions are called 'fractal functions'.

The arrangement of points in the interval is via some 'fractal' arrangement, say as a Cantor set. A line segment $[0,1]$ is cut into three parts. Those are $\left[0,\frac{1}{3}\right]$, $\left[\frac{1}{3},\frac{2}{3}\right]$ and $\left[\frac{2}{3},1\right]$. Delete the middle part i.e. the segment $\left[\frac{1}{3},\frac{2}{3}\right]$ from the interval $[0,1]$ leaving the two parts, i.e. $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$. In the next step the open middle third of each segment is deleted leaving the segments: $\left[0,\frac{1}{9}\right]$, $\left[\frac{2}{9},\frac{1}{3}\right]$, $\left[\frac{2}{3},\frac{7}{9}\right]$, $\left[\frac{8}{9},1\right]$. This process is continued ad infinitum. Thus, the Cantor set contains all points in the interval $[0,1]$ that are not deleted at any steps in this infinite process. This idea gives a fractal function continuous but nowhere differentiable.

In classical calculus, everything happens when an elemental point (the differential element, say Δx) is made infinitesimally small i.e. made zero. Thereby we take the rate with respect to this element i.e. Δx, and classically we term this rate as a usual derivative i.e. $\mathrm{d}(f(x))/\mathrm{d}x$ in the limit as $\Delta x\downarrow 0$; that is normal differentiation. In a coarse-grained system (or phenomena), everything happens as if this elemental point or the differential element Δx has some spread (or thickness), and is non-zero. This can be viewed with fractional differential quantity $(\Delta x)^{\alpha}$ with $0<\alpha<1$;

so $\left(\Delta x\right)^{\alpha} > \Delta x$ as we take the limit $\Delta x \downarrow 0$. This gives rate as per unit of this fractional differential i.e. $\mathrm{d}\left(f(x)\right)/(\mathrm{d}x)^{\alpha}$; suggesting the need of a fractional derivative. These are some motivations to learn from fractional calculus that we will discuss and develop in subsequent chapters.

1.25 Short summary

Here in this chapter we have learnt about several generalisation examples. We have come across various functions that generalise our classical concepts like factorial, binomial formulas etc. We have also seen the generalisation of some concepts of classical calculus in terms of multiple whole numbers of differentiation and integration and derived some important formulas and then inferences. We will use all these developed concepts to further generalise classical calculus when the whole number of integration and differentiation that is n or N is an arbitrary non-integer value. The methods of residue calculus, contour integration, analytical continuation etc that we revised in this chapter will be of help in understanding further concepts in subsequent chapters. In subsequent chapters, we shall be trying to answer the question posed by L'Hospital to Leibniz, and discuss whether or not it is a paradox**.**

1.26 References

This chapter is based on pioneering literature. Those which provided motivation are detailed in the bibliography section below.

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CHAPTER TWO

FRACTIONAL INTEGRATION

2.1 Introduction

In this chapter, we will discuss the process and formulation of fractional integration, which is a concept generalisation of what we know in classical calculus as repeated integration or n-fold integration with n as a positive integer 1, 2, 3 … and so on. We will learn how to formulate the fractional integration when n is a non-integer say $\frac{1}{2}$, $\frac{3}{2}$ etc. We shall start with various notations that are used to represent fractional integration, followed by a review of classical iterative integration, or n-fold repeated integration. Thereafter we will use generalisations of factorials by use of the gamma function to get a fractional integration formula; and apply the obtained formula to get expressions of fractionally integrating various useful and simple functions. We will introduce a few types of higher transcendental functions. In this chapter, we will discuss the notion that if we change the sign of fractional integration processes we get some indication of the formulation of fractional differentiation. In doing so we will discover that there is an indication that the fractional derivative process is similar to the integration process. Here we will give the Riemann-Liouville formulation of fractional integration, and discuss Weyle's definition of fractional integration, with an application to some simple problems. We will also try to give a geometric interpretation to fractional integration as an area under a curve whose shape changes. In this chapter while discussing the various methods, we will also learn many more aspects about fractional calculus.

Various notations have been used for the last three hundred years for representing fractional integration operators. Leibniz, Lagrange, and Liouville used $\int^{\alpha}$, and Grunwald wrote $\int^{\alpha}\left[...\mathrm{d}x^{\alpha}\right]_{x=a}^{x=x}$, where function is integrated by the fractional order $\alpha > 0$ and $\alpha \in \mathbb{R}$, from lower terminal $a > -\infty$. Riemann used the symbol $\partial_x^{-\alpha}$ to represent fractional integration. Several authors use the symbols $\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}x^{-\alpha}}$, f_{α}, I^{α}, I_x^{α}, I_{a+}^{α}, ${}_aI_x^{\alpha}$, $\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}(x-a)^{-\alpha}}$, ${}_aD_x^{-\alpha}$ and ${}_xD_b^{-\alpha}$. We will be using some of these notations.

2.2 Iterative integration: a review

What we mean by iterative integration is $n-$ fold (or many times repeated) integration of a function. Let us symbolise the $n-$ fold integration as ${}_0I_x^n$, with n as a positive integer or as an anti-derivative ${}_0D_x^{-n}$, meaning that this operator will do $n-$ fold integration from the lower limit zero to the upper limit x. That we symbolise as the following:

$$ {}_0I_x^n\left[f(x)\right] = \underbrace{\int_0^x\int_0^{x_1}.....\int_0^{x_{n-1}}}_{n} f(x_0)\mathrm{d}x_0\mathrm{d}x_1...\mathrm{d}x_{n-2}\mathrm{d}x_{n-1} \tag{2.1} $$

Here in (2.1) we have $n \geq 0$; therefore, for $n = 2$, for a function $f(x)$ we have the following operation:

$$ {}_0I_x^2\left[f(x)\right] = {}_0D_x^{-2}\left[f(x)\right] = \int_0^x\left(\int_0^{x_1}\left(f(x_0)\right)\mathrm{d}x_0\right)\mathrm{d}x_1 \tag{2.2} $$

2.2.1 Double integration

Take $f(x) = \sqrt{x}$, and apply a double integration operation of (2.2) as follows:

$$
\begin{aligned}
{}_0I_x^2\left[\sqrt{x}\right] = {}_0D_x^{-2}\left[\sqrt{x}\right] &= \int_0^x\left(\int_0^{x_1}\left(\sqrt{x_0}\right)\mathrm{d}x_0\right)\mathrm{d}x_1 \\
&= \int_0^x\left[\tfrac{2}{3}x_0^{3/2}\right]_0^{x_1}\mathrm{d}x_1 = \int_0^x \tfrac{2}{3}x_1^{3/2}\mathrm{d}x_1 = \left(\tfrac{2}{3}\right)\left[\tfrac{2}{5}x_1^{5/2}\right]_0^x \\
&= \tfrac{4}{15}x^{5/2}
\end{aligned}
\tag{2.3}
$$

For $f(x)=1$, we explain the above procedure with the aid of Figure 2.1. The two variables x_0 and x_1 are shown on the X-axis and Y-axis respectively. The first integration $\int \mathrm{d}x_0$ gives a straight line passing through the origin $x_1 = x_0$. The second integration is with respect to x_1 for this straight line; where $\mathrm{d}x_1$ is segment DG, placed at position D with AD having the length x_1. The area of the rectangle $DEFG$ is $\mathrm{d}A$, which is almost $x_1\mathrm{d}x_1$, that is also $\left(\int_0^{x_1}\mathrm{d}x_0\right)\mathrm{d}x_1 = \mathrm{d}A$. The area of AED is $\int_0^{x_1}\mathrm{d}A = \frac{x_1^2}{2}$. The total area of the triangle ABC, is $\int_0^x\left(\int_0^{x_1}\mathrm{d}x_0\right)\mathrm{d}x_1$, that is $\int_0^x \mathrm{d}A$ which is $\frac{x^2}{2}$.

2.2.2 Double integration with a changed order of integration giving the idea of convolution

Now we interchange the order of integration and derive the following:

$$
\begin{aligned}
{}_0I_x^2\left[\sqrt{x}\right] = {}_0D_x^{-2}\left[\sqrt{x}\right] &= \int_0^x\left(\int_{x_0}^{x}\sqrt{x_0}\,\mathrm{d}x_1\right)\mathrm{d}x_0 \\
&= \int_0^x\left(\sqrt{x_0}\int_{x_0}^{x}\mathrm{d}x_1\right)\mathrm{d}x_0 \\
&= \int_0^x\sqrt{x_0}\left(x-x_0\right)\mathrm{d}x_0 \\
&= \int_0^x x\sqrt{x_0}\,\mathrm{d}x_0 - \int_0^x x_0\sqrt{x_0}\,\mathrm{d}x_0 \\
&= x\left[\tfrac{2}{3}x_0^{3/2}\right]_0^x - \left[\tfrac{2}{5}x_0^{5/2}\right]_0^x \\
&= \tfrac{4}{15}x^{5/2}
\end{aligned}
\tag{2.4}
$$

We note that with this change of order of integration we are getting the same result as obtained earlier (2.3) i.e. $\frac{4}{15}x^{5/2}$.

Look at the third step of the above derivation (2.4) i.e. ${}_0I_x^2\left[\sqrt{x}\right] = {}_0D_x^{-2}\left[\sqrt{x}\right] = \int_0^x\sqrt{x_0}\left(x-x_0\right)\mathrm{d}x_0$, which allows us to write the following general step, with $f(x_0)=\sqrt{x_0}$

$$
\begin{aligned}
{}_0D_x^{-2}\left[f(x)\right] &= \int_0^x\left(f(x_0)\right)\left((x-x_0)\right)\mathrm{d}x_0\,; \quad x_0 = y \\
&= \int_0^x\left(f(y)\right)\left((x-y)\right)\mathrm{d}y
\end{aligned}
\tag{2.5}
$$

Here we mention that the integral $\int_0^x\left(f(y)\right)\left(g(x-y)\right)\mathrm{d}y = \left(f(x)\right)*\left(g(x)\right)$ is a convolution integral with $g(x)=x$. Therefore, we say that the double integration operation is a convolution operation i.e. given as ${}_0I_x^2\left[f(x)\right] = \left(f(x)\right)*\left(x\right)$. We note that this convolution operation became visible while doing the double integration by interchanging the order of integration, in steps (2.4).

2.2.3 Generalising the double integration to an $n-$fold integration

We may guess here that an $n-$fold integration in convolution form will look like the following:

$$ {}_0I_x^n\left[f(x)\right]=\frac{1}{(n-1)!}\left(\left(f(x)\right)*\left(x^{n-1}\right)\right) \tag{2.6}$$

For $n=1$ we have ${}_0I_x^1\left[f(x)\right]=\frac{\left(\left(f(x)\right)*\left(x^0\right)\right)}{0!}=f(x)$

For $n=2$ we have ${}_0I_x^2\left[f(x)\right]=\frac{\left(f(x)\right)*\left(x\right)}{(1)!}=\left(f(x)\right)*\left(x\right)$, that we just obtained earlier in (2.5). This discussion suggests that we are likely to have the following formula:

$$\begin{aligned} {}_0I_x^n\left[f(x)\right]={}_0D_x^{-n}\left[f(x)\right]&=\frac{1}{(n-1)!}\left(\left(f(x)\right)*\left(x^{n-1}\right)\right)\\ &=\frac{1}{(n-1)!}\int_0^x\left(f(y)\right)\left((x-y)^{n-1}\right)\mathrm{d}y \end{aligned} \tag{2.7}$$

for generalised $n-$fold integration. We will further discuss this convolution aspect:

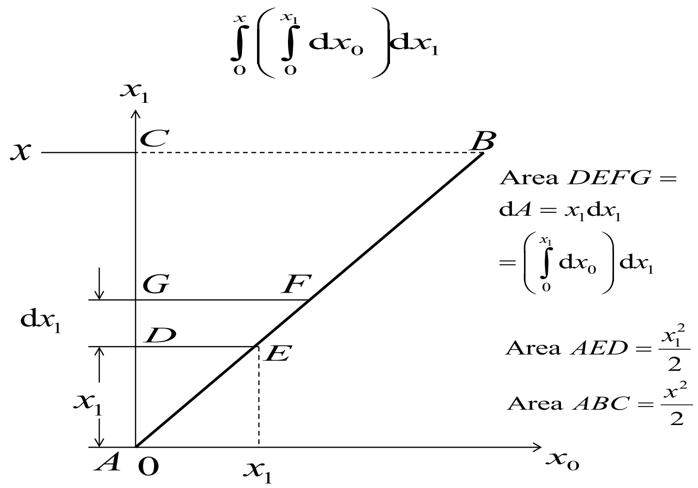

Figure 2.1: Geometrically explaining the double integration $\int_0^x\left(\int_0^{x_1}\mathrm{d}x_0\right)\mathrm{d}x_1$

For $f(x)=1$, we explain the above procedure with the aid of Figure 2.2. The two variables x_0 and x_1 are shown on the X-axis and Y-axis respectively. The first integration $\int\mathrm{d}x_1$ gives a straight line passing through the origin $x_0=x_1$. The second integration is with respect to x_0 for this straight line; where $\mathrm{d}x_0$ is segment UV , placed at position V with RV having the length x_0. The area of a rectangle $STUV$ is $\mathrm{d}A$, which is almost $SV\times UV$, that equals $\mathrm{d}A=(x-x_0)\mathrm{d}x_0$, which is also $\left(\int_{x_0}^x\mathrm{d}x_1\right)\mathrm{d}x_0=\mathrm{d}A$. The area of $PSVR$ is half of the product of RV with $(PR+SV)$, giving $xx_0-\left(\frac{x_0^2}{2}\right)$. The total area of the triangle PQR, is $\int_0^x\mathrm{d}A=\int_0^x\left(\int_{x_0}^x\mathrm{d}x_1\right)\mathrm{d}x_0$, that is $\int_0^x\mathrm{d}A$ which is $\frac{x^2}{2}$.

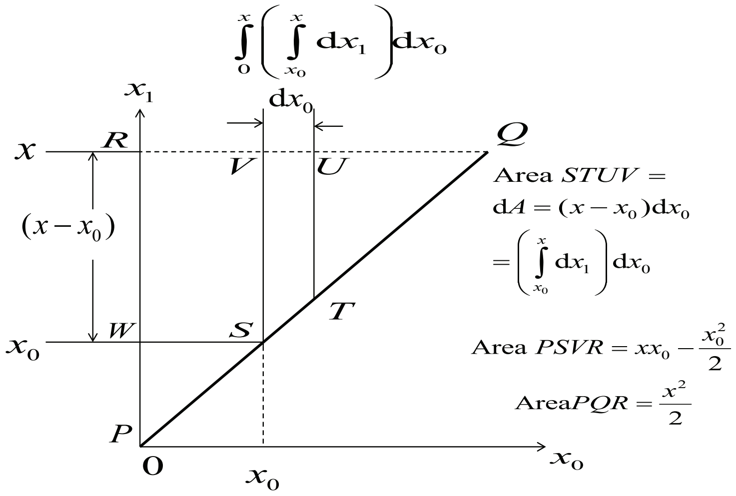

Figure 2.2: Geometrically explaining the double integration $\int_0^x \left(\int_{x_0}^{x_1} \mathrm{d}x_1 \right) \mathrm{d}x_0$ by interchanging the order of integration and giving the idea of convolution

2.3 Repeated integration formula by induction

Let us write:

$$\int_0^x f(x_0)\mathrm{d}x_0 = F(x) - F(0) \tag{2.8}$$

For $F(0) = 0$ we then have $F(x) = \int_0^x f(x_0)\mathrm{d}x_0 = {}_0D_x^{-1}\left[f(x) \right]$. Further if $F(0) = F(F(0))... = 0$, then for $n \geq 0$ we have the following:

$${}_0I_x^n \left[f(x) \right] = {}_0D_x^{-n} \left[f(x) \right] = \int_0^x \frac{f(x_0)(x - x_0)^{n-1}}{(n-1)!} \mathrm{d}x_0 \tag{2.9}$$

If the above expression (2.9) is true for n, then replacing n with $(n+1)$ gives us the following steps:

$$\begin{aligned}
{}_0I_x^{(n+1)} \left[f(x) \right] = {}_0D_x^{-(n+1)} \left[f(x) \right] &= {}_0D_x^{-1} \left[\int_0^x \frac{f(x_0)(x - x_0)^{n-1}}{(n-1)!} \mathrm{d}x_0 \right] \\
&= \int_0^x \left(\int_0^x \frac{f(x_0)(x_1 - x_0)^{n-1}}{(n-1)!} \mathrm{d}x_0 \right) \mathrm{d}x_1 \\
&= \int_0^x \left(\int_{x_0}^x \frac{f(x_0)(x_1 - x_0)^{n-1}}{(n-1)!} \mathrm{d}x_1 \right) \mathrm{d}x_0 \\
&= \int_0^x \left(\frac{f(x_0)}{(n-1)!} \int_{x_0}^x (x_1 - x_0)^{n-1} \mathrm{d}x_1 \right) \mathrm{d}x_0 \\
&= \int_0^x \frac{f(x_0)}{(n-1)!} \left[\frac{(x_1 - x_0)^n}{n} \right]_{x_0}^x \mathrm{d}x_0 \\
&= \int_0^x \frac{f(x_0)(x - x_0)^n}{n!} \mathrm{d}x_0
\end{aligned} \tag{2.10}$$

which is also true for $n+1$. In the above derivation (2.10), i.e. demonstration with induction method, we used the concept of obtaining double integration by changing the order of integration, as we described in Figure 2.2, that is:

$$
\begin{aligned}
{}_0I_x^n\left[f(x)\right] &= \frac{1}{(n-1)!}\left(f(x)\right)*\left(x^{n-1}\right) \\
&= \frac{1}{(n-1)!}\int_0^x \left(f(y)\right)\left((x-y)^{n-1}\right)\mathrm{d}y
\end{aligned}
\tag{2.11}
$$

Now, we formally write for the iterated integrals, for a locally integrable real valued function $f:\mathbb{G}\to\mathbb{R}$, when domain definition is $\mathbb{G}\equiv\left[a,b\right]\subseteq\mathbb{R}$, that is an interval with $-\infty\le a<b\le\infty$:

$$
\begin{aligned}
{}_aI_x^n\left[f(x)\right] &= \int_a^x\int_a^{x_1}\cdots\cdots\int_a^{x_n-1} f(x_n)\mathrm{d}x_n\mathrm{d}x_{n-1}...\mathrm{d}x_2\mathrm{d}x_1 \\
&= \frac{1}{(n-1)!}\int_a^x \left((x-y)^{n-1}\right)\left(f(y)\right)\mathrm{d}y
\end{aligned}
\tag{2.12}
$$

where $a<x<b$, and n is a natural number, (2.12) is also called Cauchy's formula for repeated iterated integration. Thus, we have n fold integration reducing to a single 'convolution' integral. Note that if we take $g(x)=x^{n-1}$, then the integral above in (2.12) i.e. $\int(x-y)^{n-1}\left(f(y)\right)\mathrm{d}y$, is a convolution integral as $g*f=\int\left(g(x-y)\right)\left(f(y)\right)\mathrm{d}y$. Generally, the limit of integration of a convolution integral is from $y=-\infty$ to $y=x$. As an example, we wish to perform the convolution of two functions, say $f(x)=e^{-\lambda x}$ with λ as a constant and $g(x)=\lambda K$ as a constant function for $x\ge 0$ and $g(x)=0$ for $x<0$. The following steps demonstrate this process:

$$
\begin{aligned}
g*f &= \int_{-\infty}^x \left(g(x-y)\right)\left(f(y)\right)\mathrm{d}y \\
&= \int_0^x \left(\lambda K\right)\left(e^{-\lambda y}\right)\mathrm{d}y = \lambda K\left[\frac{e^{-\lambda y}}{-\lambda}\right]_{y=0}^{y=x} \\
&= -K\left(e^{-\lambda x}-1\right) = K\left(1-e^{-\lambda x}\right)
\end{aligned}
\tag{2.13}
$$

After this comes:

$$
\begin{aligned}
f*g &= \int_{-\infty}^x \left(f(x-y)\right)\left(g(y)\right)\mathrm{d}y \\
&= \int_0^x e^{-\lambda(x-y)}\left(\lambda K\right)\mathrm{d}y \\
&= \lambda Ke^{-\lambda x}\int_0^x e^{\lambda y}\mathrm{d}y = Ke^{-\lambda x}\left(e^{\lambda x}-1\right) \\
&= K\left(1-e^{-\lambda x}\right)
\end{aligned}
\tag{2.14}
$$

The above operations (2.13) and (2.14) show that $g*f=f*g$. The convolution is defined as:

$$
f*g = \int_{-\infty}^x \left(f(x-y)\right)\left(g(y)\right)\mathrm{d}y \tag{2.15}
$$

If the integration is from say point x_0 instead of $-\infty$, we write the convolution as:

$$
f *^{x_0} g = \int_{x_0}^x \left(f(x-y)\right)\left(g(y)\right)\mathrm{d}y \tag{2.16}
$$

Here we also state that this convolution is used in finding a particular solution to non-homogeneous differential equations, which we will use in later chapters.

Analogous to the above (2.12) explanation for left integration, or forward integration ${}_aI_x^n\left[f(x)\right]$, we define the backward integration or right integration as ${}_xI_b^n\left[f(x)\right]$, and write it as follows:

$$
{}_xI_b^n\left[f(x)\right] = \frac{1}{(n-1)!}\int_x^b \left((y-x)^{n-1}\right)\left(f(y)\right)\mathrm{d}y \tag{2.17}
$$

This above expression (2.16) of ${}_xI_b^n\left[f(x)\right]$ we will discuss subsequently.

2.4 Explaining the convolution process in repeated integration numerically

Say we have a function $f(x)$, the values of which are f_0 at $x = x_0 = 0$, f_1 at $x = h$, f_2 at $x = 2h$, f_3 at $x = 3h$, f_4 at $x = 4h$ and f_5 at $x = 5h$. With these six values of function, the integration $\int_{x_0}^{x} f(y)\mathrm{d}y$ can be approximated by the sum of these values, as:

$$S = h\left(f_0 + f_1 + f_2 + f_3 + f_4 + f_5\right) \tag{2.18}$$

The above expression (2.17) is nothing but a sum of an area of rectangles. With h as a common term, we can say first that the sum is:

$$S_1 = f_0 + f_1 + f_2 + f_3 + f_4 + f_5 \tag{2.19}$$

The growing sum as x grows is $S_1 = f_0$ (at $x = 0$), for the next point (at $x = h$) it is $S_1 = f_0 + f_1$, for the next point (at $x = 2h$) the sum is $S_1 = f_0 + f_1 + f_2$ and so on, as depicted in the following steps:

$$\begin{array}{ccc} x & f(x) & S_1 \\ x_0 & f_0 & f_0 \\ x_0 + h & f_1 & f_0 + f_1 \\ x_0 + 2h & f_2 & f_0 + f_1 + f_2 \\ x_0 + 3h & f_3 & f_0 + f_1 + f_2 + f_3 \\ x_0 + 4h & f_4 & f_0 + f_1 + f_2 + f_3 + f_4 \\ x_0 + 5h & f_5 & f_0 + f_1 + f_2 + f_3 + f_4 + f_5 \end{array} \tag{2.20}$$

Now the next step is to derive a second sum (equivalent to double integration), that is taking S_1 as a function, and the point wise summing to get S_2. The growing sum as x grows is $S_2 = f_0$ for the first point and then for the next point (at $x = h$) is $S_2 = f_0 + (f_0 + f_1) = 2f_0 + f_1$. We note here that the value of S is $S_1 = f_0 + f_1$ at $x = h$. Now we go further at $x = 2h$ and write $S_2 = f_0 + (f_0 + f_1) + (f_0 + f_1 + f_2) = 3f_0 + 2f_1 + f_2$, and so on, as depicted in the following steps:

$$\begin{array}{ccc} x & f(x) & S_2 \\ x_0 & f_0 & f_0 \\ x_0 + h & f_1 & 2f_0 + f_1 \\ x_0 + 2h & f_2 & 3f_0 + 2f_1 + f_2 \\ x_0 + 3h & f_3 & 4f_0 + 3f_1 + 2f_2 + f_3 \\ x_0 + 4h & f_4 & 5f_0 + 4f_1 + 3f_2 + 2f_3 + f_4 \\ x_0 + 5h & f_5 & 6f_0 + 5f_1 + 4f_2 + 3f_3 + 2f_4 + f_5 \end{array} \tag{2.21}$$

We write the second sum for the six points as follows:

$$S_2 = 6f_0 + 5f_1 + 4f_2 + 3f_3 + 2f_4 + f_5 \tag{2.22}$$

Now we generate the third sum S_3 (equivalent to a triple integration) from the obtained S_2 as x grows. The first point is $S_3 = f_0$. At the second point at $x = h$, the triple sum is $S_3 = f_0 + (2f_0 + f_1) = 3f_0 + f_1$, furthermore, as we go at $x = 2h$, the sum is $S_3 = f_0 + (2f_0 + f_1) + (3f_0 + 2f_1 + f_2) = 6f_0 + 3f_1 + f_2$; and so on; as depicted in the following steps:

$$\begin{array}{ccc} x & f(x) & S_3 \\ x_0 & f_0 & f_0 \\ x_0 + h & f_1 & 3f_0 + f_1 \\ x_0 + 2h & f_2 & 6f_0 + 3f_1 + f_2 \\ x_0 + 3h & f_3 & 10f_0 + 6f_1 + 3f_2 + f_3 \\ x_0 + 4h & f_4 & 15f_0 + 10f_1 + 6f_2 + 3f_3 + f_4 \\ x_0 + 5h & f_5 & 21f_0 + 15f_1 + 10f_2 + 6f_3 + 3f_4 + f_5 \end{array} \tag{2.23}$$

The six point third sum at $x = 5h$ is the following:

$$S_3 = 21f_0 + 15f_1 + 10f_2 + 6f_3 + 3f_4 + f_5 \tag{2.24}$$

For the six points, we write the first second and third sums as:

$$
\begin{aligned}
S_1 &= f_0 + f_1 + f_2 + f_3 + f_4 + f_5 \\
S_2 &= (6-0)f_0 + (6-1)f_1 + (6-2)f_2 + (6-3)f_3 + (6-4)f_4 + (6-5)f_5 \\
S_3 &= \frac{(6+1)(6-0)}{2}f_0 + \frac{(6-0)(6-1)}{2}f_1 + \frac{(6-1)(6-2)}{2}f_2 \\
&\quad + \frac{(6-2)(6-3)}{2}f_3 + \frac{(6-3)(6-4)}{2}f_4 + \frac{(6-4)(6-5)}{2}f_5
\end{aligned}
\tag{2.25}
$$

Look at S_1; multiplying by h and then as h is made very close to zero so we say S_1h is indicative of $\int_{x_0}^{x} f(u)\mathrm{d}u$. Now we look at S_2 and for small h we may say that S_2h is indicative of $\int_{x_0}^{x}\int_{x_0}^{x} f(u)\mathrm{d}u$. Let us see, then, how we get $\int_{x_0}^{x}\int_{x_0}^{x} f(u)\mathrm{d}u$.

The first term of S_2 in (2.25) is $(6-0)f_0$; by taking 6 as x and 0 as u we can write f_0 as $f(u)$ at $u = 0 = x_0$. Therefore $(6-0)f_0$ is $(x-u)f(u)$, is the first term at $u = 0 = x_0$. We use the same argument for $(6-1)f_1$ and all other terms of S_2. Therefore, with $h = \mathrm{d}u$, we write S_2h as $\int_{x_0}^{x}(x-u)\big(f(u)\big)\mathrm{d}u$. This is the same as what we derived in an earlier section, that is $\int_{x_0}^{x}\int_{x_0}^{x_1}\big(f(y)\mathrm{d}y\big)\mathrm{d}y = \int_{x_0}^{x}(x-u)f(u)\mathrm{d}u$, which is a double integration, as a convolution i.e. $\big(x\big)*\big(f(x)\big)$ in Sections 2.2 and 2.3.

Now we take the sum S_3 and with h being very small we analyse this triple sum. The first term is $\frac{(6+1)(6-0)}{2}f_0$. Take $x = 6$, with $u = 0$ and f_0 as $f(u)$ at $u = 0$; the term approximately represented as $\frac{(x-u)^2}{2}f(u)$. This approximation is valid when h is close to zero. This argument can be extended to all the other terms of S_3 and we can see that S_3h is $\int_{x_0}^{x}\frac{(x-u)^2}{2}f(u)\mathrm{d}u$; this is a formula for triple integration $\int_{x_0}^{x}\int_{x_0}^{x_2}\int_{x_0}^{x_1}\Big(\big(f(y)\mathrm{d}y\big)\mathrm{d}y\Big)\mathrm{d}y$, that is $\frac{1}{2}\Big(\big(x^2\big)*\big(f(x)\big)\Big)$ or $\frac{1}{2}\int_{x_0}^{x}\big((x-u)^2\big)\big(f(u)\big)\mathrm{d}u$. Therefore, we have the following generalisation for $n-$fold integration with the $n-$th sum S_n i.e.

$$
\begin{aligned}
S_nh &= \int_a^{x}\int_a^{x_1}.....\int_a^{x_n-1} f(x_n)\mathrm{d}x_n\mathrm{d}x_{n-1}...\mathrm{d}x_2\mathrm{d}x_1 \\
&= \frac{1}{(n-1)!}\int_a^{x}(x-y)^{n-1}\big(f(y)\big)\mathrm{d}y
\end{aligned}
\tag{2.26}
$$

This is another way we could explain the process of repeated integration – and a convolution process. Now, with this background, we can subsequently derive a formula for arbitrary n.

We have discussed a very basic way the concept of iterative integration and the concept of convolution. This is necessary as this process that we discussed in this section and previous sections is very important in the generalisation of the concept of $n-$fold integration to fractional integration (and then fractional differentiation) when $n-$is not a natural number, i.e. for $n-$as a real number. We also stress that the convolution process is the basis of fractional integration as well as fractional differentiation.

2.5 Generalising the expression for a repeated integration to get the Riemann-Liouville fractional integration formula

We write the following from expression (2.12):

$$
\begin{aligned}
{}_0D_x^{-1}\left[f(x)\right] &= \int_0^x \left(f(y)\right)\mathrm{d}y \\
{}_0D_x^{-2}\left[f(x)\right] &= \int_0^x \left(f(y)\right)(x-y)\mathrm{d}y \\
{}_0D_x^{-3}\left[f(x)\right] &= \frac{1}{2}\int_0^x \left(f(y)\right)(x-y)^2\mathrm{d}y \\
{}_0D_x^{-4}\left[f(x)\right] &= \frac{1}{(2)(3)}\int_0^x \left(f(y)\right)(x-y)^3\mathrm{d}y \\
{}_0D_x^{-n}\left[f(x)\right] &= \frac{1}{(n-1)!}\int_0^x \left(f(y)\right)(x-y)^{n-1}\mathrm{d}y
\end{aligned}
\tag{2.27}
$$

2.5.1 A generalisation of a repeated integration formula by the gamma function

Now we have to generalise $n!$ for non-integer n. This Euler gamma function has the integral representation $\Gamma(n) \overset{\text{def}}{=} \int_0^\infty x^{n-1}e^{-x}\mathrm{d}x$; $\mathrm{Re}[n]>0$ (see Section 1.10). Comparing the integral obtained with the definition of the gamma function, we have the following observation:

$$n! = \int_0^\infty t^n e^{-t}\mathrm{d}t = \Gamma(n+1) \qquad (n-1)! = \int_0^\infty t^{n-1}e^{-t}\mathrm{d}t = \Gamma(n) \tag{2.28}$$

The above integral representation (2.28) is a generalised factorial function known as a gamma function, for any positive n, real or complex, which need not be a natural number. The following is the proof of this gamma function, as a generalisation of factorial function. Let $\mathrm{I}(n)$ be the integral representation of the gamma function, and to that apply the formula for integration by parts, that is:

$$
\begin{aligned}
\mathrm{I}(n) &= \int_0^\infty e^{-t}t^n\mathrm{d}t \\
&= -\left[e^{-t}t^n\right]_0^\infty + n\int_0^\infty e^{-t}t^{n-1}\mathrm{d}t \\
&= n\left(\mathrm{I}(n-1)\right)
\end{aligned}
\tag{2.29}
$$

In addition, we have $\mathrm{I}(0) = \int_0^\infty e^{-t}\mathrm{d}t = 1$. Therefore, we write $\mathrm{I}(n) = n! = \Gamma(n+1)$. Now, consider continuing the $\mathrm{I}(n)$ of (2.29) to a complex plane, i.e. where n is replaced by a complex number, $z\in\mathbb{C}$. We require $\mathrm{Re}[z]>-1$, so that $\mathrm{I}(z) = \int_0^\infty e^{-t}t^z\mathrm{d}t$ converges (Refer Section-1.10.1) Thus, we have the definition $\Gamma(z+1) \overset{\text{def}}{=} \int_0^\infty e^{-t}t^z\mathrm{d}t$, with $\mathrm{Re}[z]>-1$, which satisfies $\Gamma(z+1) = z\left(\Gamma(z)\right)$. This $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ we have discussed previously in Section 1.10 as an analytic continuation of the gamma function to the left half plane (LHP), though the integral representation of the gamma function, $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t$, is defined for the RHP, $\mathrm{Re}[z]>0$.

We have for the gamma function, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(n+1) = n\left(\Gamma(n)\right)$. Therefore, for n a positive non-integer number from the above generalisation (2.28), we get an iterated integral formula as follows:

$${}_0I_x^n\left[f(x)\right] = {}_0D_x^{-n}\left[f(x)\right] = \frac{1}{\Gamma(n)}\int_0^x \left(f(y)\right)(x-y)^{n-1}\mathrm{d}y \tag{2.30}$$

This expression (2.30) was developed by Liouville (1832) and Riemann (1876), who developed logical definitions of this fractional integration operation when in the above formula, n is a non-integer, in general i.e. arbitrary positive or negative real number. The above definite integral (2.30) i.e. $\int_0^x \left(f(y)\right)(x-y)^{n-1}\,\mathrm{d}y$ is a Riemann-Liouville integral.

2.5.2 Applying the Riemann-Liouville formula to get double-integration and semi-integration for the function $f(x)=\sqrt{x}$

Let us apply the formula ${}_0I_x^n\left[f(x)\right]=\frac{1}{\Gamma(n)}\int_0^x \left(f(y)\right)(x-y)^{n-1}\,\mathrm{d}y$ for $f(x)=\sqrt{x}$, for $n=2$.

$$\begin{aligned}
{}_0D_x^{-2}\left[f(x)\right] &= \frac{1}{\Gamma(2)}\int_0^x \sqrt{y}(x-y)\mathrm{d}y \\
&= \int_0^x \left(xy^{1/2}-y^{3/2}\right)\mathrm{d}y \\
&= \left[\tfrac{2}{3}xy^{3/2}-\tfrac{2}{5}y^{5/2}\right]_0^x \\
&= \tfrac{4}{15}x^{5/2}
\end{aligned} \tag{2.31}$$

The result of (2.31) we got in (2.3) and (2.4) earlier.

Now let us obtain semi-integration of $f(x)=\sqrt{x}$ i.e. with $n=\frac{1}{2}$ by the above generalised expression i.e. ${}_0I_x^n\left[f(x)\right]=\frac{1}{\Gamma(n)}\int_0^x \left(f(y)\right)(x-y)^{n-1}\,\mathrm{d}y$ obtained for repeated integration and written in the following manner:

$$\begin{aligned}
{}_0I_x^{1/2}\left[\sqrt{x}\right] = {}_0D_x^{-1/2}\left[\sqrt{x}\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \sqrt{y}(x-y)^{\left(\frac{1}{2}\right)-1}\mathrm{d}y \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{xy-y^2}}\mathrm{d}y \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{\frac{x^2}{4}-y^2-\frac{x^2}{4}+2x\left(\frac{y}{2}\right)}}\mathrm{d}y \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{\frac{x^2}{4}-\left(y-\frac{x}{2}\right)^2}}\mathrm{d}y
\end{aligned} \tag{2.32}$$

Now in (2.23) put $y=\frac{1}{2}\left(x+x\sin\theta\right)$, then we have $\mathrm{d}y=\left(\frac{x}{2}\right)(\cos\theta)\mathrm{d}\theta$ and at $y=0$, we have $\theta=-\frac{\pi}{2}$, and at $y=x$ we have $\theta=\frac{\pi}{2}$. With these values, we get the following steps:

$$\begin{aligned}
{}_0D_x^{-1/2}\left[\sqrt{x}\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2} \frac{\left(\frac{1}{2}\right)}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}}\left(\frac{x}{2}\right)\left(\cos\theta\right)\mathrm{d}\theta \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\right)\left(x+x\sin\theta\right)\mathrm{d}\theta \\
&= \frac{1}{2\left(\Gamma\left(\frac{1}{2}\right)\right)}\left[x\theta-x\cos\theta\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \\
&= \frac{\pi x}{2\left(\Gamma\left(\frac{1}{2}\right)\right)} \\
&= \frac{\sqrt{\pi}}{2}x
\end{aligned} \tag{2.33}$$

We used the value $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, and with great effort (and with skill in doing the integration) we applied the generalised expression (2.30) of ${}_0D_x^{-n}\left[f(x)\right]$ to get the value of semi-integration of $\sqrt{x}$ as ${}_0I_x^{\frac{1}{2}}\left[\sqrt{x}\right]=\frac{\sqrt{\pi}}{2}x$.

2.5.3 Formally defining forward (left) and backward (right) Riemann-Liouville fractional integration

Let $-\infty \le a < x \le b \le \infty$, then the Riemann-Liouville fractional integration of order $\alpha > 0$ is defined for the locally integrable function $f(x)$, as $f:\left[a,b\right]\to\mathbb{R}$, which are followed by:

$$
\begin{aligned}
{}_aI_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)\mathrm{d}y \\
{}_bI_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1}f(y)\mathrm{d}y
\end{aligned}
\tag{2.34}
$$

The first one is integration in the left sense, and the second one is integration in the right sense. We may call the first expression a causal one and the second expression an anti-causal one. From above, for $x < b$ and for $\alpha = 0$, we write the following identity:

$$
{}_aI_x^{0}\left[f(x)\right] = {}_xI_b^{0}\left[f(x)\right] = f(x) \tag{2.35}
$$

2.6 Fractional integration of a power function

Next, we deal with the expression of a generalisation of the repeated integration:

$$
{}_cD_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y\,; \qquad \alpha > 0 \tag{2.36}
$$

Having taken note that in (2.36) the start point of the fractional integration above is $x = c$, which is the lower limit of integration, earlier in our derivation (2.30) we maintained the lower limit of integration was zero i.e. $x = 0$. We now try to achieve the fractional integration of the power function:

$$
f(x) = x^{\mu} \qquad \mu > -1 \tag{2.37}
$$

The condition $\mu > -1$ in (2.37) will be clear as we derive this fractional integration expression. In the expression of the fractional integration (2.36), we put $y = x(1-u)$ and re-wrote the expression with this change of variable as follows with $-x\left(\mathrm{d}u\right)=\mathrm{d}y$ and the limits of integration from the lower limit $(x-c)/x$ to zero.

$$
{}_cD_x^{-\alpha}\left[f(x)\right] = \frac{x^{\alpha}}{\Gamma(\alpha)}\int_0^{\chi} u^{\alpha-1}\left(f(x-xu)\right)\mathrm{d}u\,; \qquad \chi = \frac{x-c}{x} \tag{2.38}
$$

Here $x > c \ge 0$. For $f(x) = x^{\mu}$, we have to put $f(x-xu) = x^{\mu}(1-u)^{\mu}$, and write the following:

$$
\begin{aligned}
{}_cD_x^{-\alpha}\left[x^{\mu}\right] &= \frac{x^{\mu+\alpha}}{\Gamma(\alpha)}\int_0^{\chi} u^{\alpha-1}(1-u)^{\mu}\mathrm{d}u \\
&= \frac{x^{\mu+\alpha}}{\Gamma(\alpha)}\mathrm{B}_{\chi}(\alpha,\mu+1)
\end{aligned}
\tag{2.39}
$$

We have defined the incomplete beta function (1.144) as $\mathrm{B}_{\chi}(x,y) \overset{\text{def}}{=} \int_0^{\chi} t^{x-1}(1-t)^{y-1}\mathrm{d}t$ and using this we write the above $\mathrm{B}_{\chi}(\alpha,\mu+1)$ for the definite integral. For this beta integral both the arguments α and $\mu+1$ need to be positive; this also gives us the condition that $\mu > -1$. When $c = 0$, we have $\chi = 1$, therefore the integral is $\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$; which we recognise as a complete beta function (1.129) i.e. $\mathrm{B}(x,y)$ which is defined as $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
With this we write $\mathrm{B}(\alpha,\mu+1) = \frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}$.

2.6.1 Fractional integration of power functions with the lower limit of integration as zero

Therefore in (2.39) by use of the beta function as a ratio of the gamma function (as defined in Section 1.15), we write the fractional integration of $f(x)=x^{\mu}$ with a lower limit of integration $c=0$ as follows:

$$
\begin{aligned}
{}_0I_x^{\alpha}\left[x^{\mu}\right] = {}_0D_x^{-\alpha}\left[x^{\mu}\right] &= \frac{x^{\mu+\alpha}}{\Gamma(\alpha)}\mathrm{B}\left(\alpha,\mu+1\right) \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha} \qquad \alpha>0,\quad \mu>-1
\end{aligned}
\tag{2.40}
$$

The beta function $\mathrm{B}(\alpha,\mu+1)=\frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}$ diverges for $\mu\le -1$ thus we safely write:

$$
{}_0I_x^{\alpha}\left[x^{\mu}\right] = {}_0D_x^{-\alpha}\left[x^{\mu}\right] = \infty \qquad \mu>-1 \qquad \alpha>0 \tag{2.41}
$$

2.6.2 Fractional integration of power functions with the lower limit of integration as non-zero

Let us evaluate the fractional integration of $f(x)=(x-a)^{\mu}$ from the lower terminal $x=c$ where $x>c>a$ and $c\ge 0$. Thus, we have from the formula the following expression:

$$
\begin{aligned}
{}_cD_x^{-\alpha}\left(x-a\right)^{\mu} &= \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y, \qquad f(y)=(y-a)^{\mu} \\
&= \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\left((y-a)^{\mu}\right)\mathrm{d}y
\end{aligned}
\tag{2.42}
$$

We substitute in (2.42) $y-a=(x-a)(1-u)$ so the lower limit $y=c$ becomes $u=\frac{(x-c)}{(x-a)}$ and the upper limit $y=x$ is changed to $u=0$. We then have $x-y=u(x-a)$ and $\mathrm{d}y=(x-a)(-\mathrm{d}u)$. With these substitutions, we can write the following expression, on similar lines as obtained for the above derivations (2.39) and (2.40) with the use of an incomplete beta function:

$$
\begin{aligned}
{}_cD_x^{-\alpha}\left[\left(x-a\right)^{\mu}\right] &= \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\left((y-a)^{\mu}\right)\mathrm{d}y \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{(x-c)}{(x-a)}}^{0}\left(u(x-a)\right)^{\alpha-1}\left((x-a)(1-u)\right)^{\mu}(x-a)\left(-\mathrm{d}u\right) \\
&= \frac{(x-a)^{\alpha+\mu}}{\Gamma(\alpha)}\int_0^{\frac{x-c}{x-a}} u^{\alpha-1}(1-u)^{\mu}\,\mathrm{d}u; \qquad \chi=\frac{x-c}{x-a} \\
&= \frac{(x-a)^{\alpha+\mu}}{\Gamma(\alpha)}\mathrm{B}_{\chi}(\alpha,\mu+1)
\end{aligned}
\tag{2.43}
$$

Therefore, we have the formula:

$$
{}_cD_x^{-\alpha}\left[\left(x-a\right)^{\mu}\right] = \frac{(x-a)^{\alpha+\mu}}{\Gamma(\alpha)}\mathrm{B}_{\chi}(\alpha,\mu+1), \qquad \chi=\frac{x-c}{x-a} \qquad \alpha>0 \tag{2.44}
$$

For the start point or the lower limit of integration as non-zero i.e. $c\ne 0$, can we have the following result?

$$
{}_cI_x^{\alpha}\left[x^{\mu}\right] = {}_cD_x^{-\alpha}\left[x^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(x-c)^{\mu+\alpha}; \qquad \mu>-1 \qquad \alpha>0 \tag{2.45}
$$

This expression (2.45) we get by manipulating the derivation of ${}_cD_x^{-\alpha}(x-a)^{\mu}$ as done in (2.43) above by setting $a=0$, and making $\chi=\frac{x-c}{x}=1$, giving $x=(x-c)$ and using the definition of a complete beta function i.e. $\mathrm{B}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, (1.129) that is described in the following steps:

$$
\begin{aligned}
{}_cD_x^{-\alpha}\left[x^\mu\right] &= \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y; \qquad f(y)=x^\mu, \qquad y = x(1-u)\\
&= \frac{x^{\alpha+\mu}}{\Gamma(\alpha)}\int_0^{\frac{x-c}{x}} u^{\alpha-1}(1-u)^\mu \,\mathrm{d}u \qquad \chi=\frac{x-c}{x}=1, \quad x=(x-c)\\
&= \frac{(x-c)^{\alpha+\mu}}{\Gamma(\alpha)}\int_0^1 u^{\alpha-1}(1-u)^\mu\,\mathrm{d}u \qquad (2.46)\\
&= \frac{(x-c)^{\alpha+\mu}}{\Gamma(\alpha)}\mathrm{B}(\alpha,\mu+1)\\
&= \frac{(x-c)^{\alpha+\mu}}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}\right) = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(x-c)^{\mu+\alpha}
\end{aligned}
$$

The above (2.46) derivation shows that for it to be valid we have taken $\chi=\frac{x-c}{x-a}=1$, thus here we have $c=a$. For the above case (2.46) to be valid we thus have $c=0$, i.e. the starting point of the fractional integration process is zero $x=0$. Thus, we recover ${}_0D_x^{-\alpha}\left[x^\mu\right]=\left(\frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}\right)(x)^{\mu+\alpha}$. For $\mu=0$ and $c=0$, we have fractional integration of a constant function $f(x)=1$ as follows:

$$
{}_0I_x^\alpha\left[1\right] = {}_0D_x^{-\alpha}\left[1\right] = \frac{\Gamma(1)}{\Gamma(\alpha+1)}x^\alpha = \frac{x^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\alpha!}x^\alpha \qquad (2.47)
$$

2.6.3 Region of validity of the integer order integration of the power function

Let us take the formula for fractional integration i. e. ${}_0I_x^\alpha\left[x^\mu\right]=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha}$, and observe the following simple calculations for cases where α and μ are integers. When both are integers, the formula is:

$$
\begin{aligned}
{}_0I_x^\alpha\left[x^\mu\right] &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha} = \frac{\mu!}{(\mu+\alpha)!}x^{\mu+\alpha}, \qquad \alpha=1,2,3,..\\
&= \begin{cases} \dfrac{1}{(\mu+1)(\mu+2)...(\mu+\alpha)}x^{\mu+\alpha} & \mu>-1\\ \infty & \mu\le -1\end{cases}. \qquad (2.48)
\end{aligned}
$$

This we have discussed in Section 1.23. With this, we can compose Table 2.1, showing the region of validity. This is a safe bet, to exclude the region for $f(x)=x^\mu$, when $\mu\le -1$ (refer to Table 2.1) for doing fractional integration. Nevertheless, can we extend our derived formula for this region? We discuss this issue subsequently.

$\mu\downarrow \quad \alpha\rightarrow$	3	2	1	0
2	$\frac{2!}{5!}x^5$	$\frac{2!}{4!}x^4$	$\frac{2!}{3!}x^3$	x^2
1	$\frac{1}{4!}x^4$	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x
0	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x	1
-1	∞	∞	∞	x^{-1}
-2	∞	∞	∞	x^{-2}
-3	∞	∞	∞	x^{-3}

Table 2.1: Integer order integration of the power function with integer exponents with a fractional integration formula

2.7 The extended region of fractional integration of a power function: analytic continuation

This is called 'extended fractional calculus'. In this concept the ratio of the gamma function that appears as a coefficient in fractional integration, the formula of the power-function (2.45) i.e. $\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\alpha+\in)}$ is taken as a fundamental defining expression. Unlike the Riemann-Liouville integral which we saw is defined by the region $\mu > -1$ for function $f(x) = x^{\mu}$ (2.45), the limit $\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\alpha+\in)}$ is well defined in the plane $\alpha, \mu \in \mathbb{R}$; except along the line interval $\mu \in \mathbb{Z}^{-}, \alpha \in \mathbb{R}\setminus\mathbb{Z}$. Here, '\'is a "mod" operation between set $\mathbb{R}$ and set $\mathbb{Z}$. The "mod" operation example is: $A = \{1,2,3,4,5,6\}$ $B = \{1,3,4\}$ then $A \setminus B = \{2,5,6\}$.

Here we will discuss the analytic continuation to these intervals and show it to be:

$$ {}_0I_x^{\alpha}\left[x^{\mu}\right] = \lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)x^{\alpha+\mu}}{\Gamma(\in)\Gamma(\alpha+\mu+1)}\left(\ln x - \psi(\alpha+\mu+1) - \gamma\right) \quad (2.49)$$

where in (2.49) ψ is the 'psi' function and γ is Euler's constant (we have discussed this function in the previous chapter, in Section 1.16).

The demonstration in an earlier discussion (Section 2.6.3) shows that when $\mu < 0$, the formula (2.45) i.e. ${}_0I_x^{\alpha}\left[x^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha}$ becomes uncomfortable when μ is a negative integer $-1,-2,-3,....$, because the value of the gamma function at point zero or at points of negative integer is infinity. Hence, safely we specify the region of validity $\mu > -1$, for this formula to be valid, also due to the divergence of the beta-integral in the derivation of this formula (2.45).

However, we see that for $\mu \le -1$, at the integer points $\mu = -1,-2,-3,...$ the following are some relations after doing a normal classical integration as demonstrated below:

$$\begin{aligned}
&\alpha = 1 \quad \mu = -1 \quad && \int\left(\tfrac{1}{x}\right)\mathrm{d}x = \ln x \\
&\alpha = 1 \quad \mu = -2 \quad && \int\left(\tfrac{1}{x^2}\right)\mathrm{d}x = -x^{-1} \\
&\alpha = 1 \quad \mu = -3 \quad && \int\left(\tfrac{1}{x^3}\right)\mathrm{d}x = -\tfrac{1}{2}x^{-2} \\
&\alpha = 2 \quad \mu = -1 \quad && \int\left(\tfrac{1}{x}\right)(\mathrm{d}x)^2 = \int \ln x(\mathrm{d}x) \\
&\alpha = 3 \quad \mu = -3 \quad && \int\left(\tfrac{1}{x^3}\right)(\mathrm{d}x)^3 = \int\int -\left(\tfrac{x^2}{2}\right)\mathrm{d}x\mathrm{d}x = \int \tfrac{1}{2!x}\mathrm{d}x = \tfrac{1}{2!}\ln x
\end{aligned} \quad (2.50)$$

We construct the following table (Table 2.2), showing the integration of x^{μ}, with μ as an integer in this case of ${}_0I_x^{\alpha}\left[x^{\mu}\right]$, also with α as a positive integer. This table has finite entries whereas the entries are ∞ in the corresponding table (Table 2.1).

$\mu\downarrow \quad \alpha\rightarrow$	3	2	1	0
2	$\frac{2!}{5!}x^5$	$\frac{2!}{4!}x^4$	$\frac{2!}{3!}x^3$	x^2
1	$\frac{1}{4!}x^4$	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x
0	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x	1
-1	$\int \ln x(\mathrm{d}x)^2$	$\int \ln x(\mathrm{d}x)$	$\ln x$	x^{-1}
-2	$-\int \ln x(\mathrm{d}x)$	$-\ln x$	$-x^{-1}$	x^{-2}
-3	$\frac{1}{2!}\ln x$	$\frac{1}{2!}x^{-1}$	$-\frac{1}{2!}x^{-2}$	x^{-3}

Table 2.2: Integer order integration of the power function with integer exponents

With this observation of the breakdown of our obtained formula i. e. ${}_0I_x^{\alpha}\left[x^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha}$, for μ as an integer and with $\mu < 0$, we note that ,in Table 2.2, a log-region is formed with a pattern. Let us take

${}_0I_x^1\left[x^{-2}\right]=\frac{\Gamma(-2+1)}{\Gamma(-2+1+1)}x^{-2+1}=\frac{\Gamma(-1)}{\Gamma(0)}x^{-1}$; this is getting troublesome with formula (2.45). Nevertheless, doing normal integration once, we get $\int\frac{1}{x^2}\mathrm{d}x=-x^{-1}$. We use the following trick, by writing the formula in a limit for $\mu<0$:

$$ {}_0I_x^\alpha\left[x^\mu\right]=\lim_{\in\downarrow 0}\frac{\Gamma(\mu+1+\in)}{\Gamma(\mu+\alpha+1+\in)}x^{\mu+\alpha} \tag{2.51}$$

Which gives us ${}_0I_x^\alpha\left[x^\mu\right]=\lim_{\in\downarrow 0}\frac{\Gamma(\mu+1+\in)}{\Gamma(\mu+\alpha+1+\in)}x^{\mu+\alpha}=\lim_{\in\downarrow 0}\frac{\Gamma(\in-1)}{\Gamma(\in)}x^{-1}$. Now use $\Gamma(x)=\frac{\Gamma(x+1)}{x}$ to write $\Gamma(\in-1)=\frac{\Gamma(\in)}{(\in-1)}$ and using this we get $\lim_{\in\downarrow 0}\frac{\Gamma(\in-1)}{\Gamma(\in)}x^{-1}=\lim_{\in\downarrow 0}\frac{x^{-1}}{(\in-1)}=-x^{-1}$. Therefore, we get the same answer as when doing normal integration; thus we may say the following:

$$ {}_0I_x^1\left[x^{-2}\right]=\lim_{\in\downarrow 0}\left[\frac{\Gamma(\mu+1+\in)}{\Gamma(\mu+\alpha+1+\in)}x^{\mu+\alpha}\right]_{\mu=-2,\alpha=1}=-x^{-1} \tag{2.52}$$

One more example we take is ${}_0I_x^1\left[x^{-3}\right]=\frac{\Gamma(-2)}{\Gamma(-1)}x^{-2}$ which has ratios of $\frac{\Gamma(-2)}{\Gamma(-1)}=(-1)^{2+1}\frac{1!}{2!}=-\frac{1}{2!}$. Well if we write again $\lim_{\in\downarrow 0}\frac{\Gamma(\in-2)}{\Gamma(\in-1)}x^{-2}$, and use $\Gamma(\in-2)=\frac{\Gamma(\in-1)}{(\in-2)}$, we will get $\lim_{\in\downarrow 0}\frac{\Gamma(\in-2)}{\Gamma(\in-1)}x^{-2}=-\frac{1}{2!}x^{-2}$. Therefore, we write the following:

$$ {}_0I_x^1\left[x^{-3}\right]=\lim_{\in\downarrow 0}\left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha}\right)_{\mu=-3,\alpha=1}=-\frac{1}{2!}x^{-2} \tag{2.53}$$

Perhaps a better way to write the formula (except for the log-region) in Table 2.1 is as follows:

$$ {}_0I_x^\alpha\left[x^\mu\right]=\lim_{\in\downarrow 0}\left(\frac{\Gamma(\mu+1+\in)}{\Gamma(\mu+\alpha+1+\in)}x^{\mu+\alpha}\right) \tag{2.54}$$

Take an example from Table 2.1, and write the few entries of the log-region as follows:

$$\begin{aligned}
{}_0I_x^3\left[x^{-1}\right]&=\lim_{\in\downarrow 0}\frac{\Gamma(1-1+\in)}{\Gamma(\in)}\int\ln x(\mathrm{d}x)^{3-1}=\int\ln x(\mathrm{d}x)^2 &\qquad \lim_{\in\downarrow 0}\frac{\Gamma(\in)}{\Gamma(\in)}=1\\
{}_0I_x^3\left[x^{-2}\right]&=\lim_{\in\downarrow 0}\frac{\Gamma(1-2+\in)}{\Gamma(\in)}\int\ln x(\mathrm{d}x)^{3-2}=-\int\ln x(\mathrm{d}x) &\qquad \lim_{\in\downarrow 0}\frac{\Gamma(\in-1)}{\Gamma(\in)}=-1\\
{}_0I_x^3\left[x^{-3}\right]&=\lim_{\in\downarrow 0}\frac{\Gamma(1-3+\in)}{\Gamma(\in)}\int\ln x(\mathrm{d}x)^{3-3}=\frac{1}{2!}\ln x &\qquad \lim_{\in\downarrow 0}\frac{\Gamma(\in-2)}{\Gamma(\in)}=\frac{1}{2!}
\end{aligned} \tag{2.55}$$

Looking at the log-region in Table 2.2 and analyzing it, we get a formula as in the following expression:

$$\begin{aligned}
{}_0I_x^\alpha\left[x^\mu\right]&=\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}\int\ln x(\mathrm{d}x)^{\alpha+\mu}\\
&=\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}\,{}_0I_x^{\alpha+\mu}\left[\ln x\right]
\end{aligned} \tag{2.56}$$

We will use these interesting results (2.56) when we formally do analytical continuation of this integral operator. What is the fractional integration of $\ln x$, that is $\int\ln x(\mathrm{d}x)^{\alpha+\mu}$ or ${}_0I_x^{\alpha+\mu}\left[\ln x\right]$? If we write the fractional integration formula in the log-region as follows:

$$ {}_0I_x^\alpha\left[x^\mu\right]=\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)x^{\alpha+\mu}}{\Gamma(\in)\Gamma(\alpha+\mu+1)}\left(\ln x-\psi(\alpha+\mu+1)-\gamma\right) \tag{2.57}$$

with the concept of the analytic continuation of the harmonic series represented above in terms of $\psi(z)$ a psi-function and γ as Euler's constant, we will not be inaccurate. The discussion of this is in the next section, though we have introduced the concept in an earlier chapter (Section 1.16).

2.8 An analytic continuation of a finite harmonic series-derived via a fractional integration concept

We have seen in an earlier chapter (Section 1.16) the multiple $n-$fold integration of $\ln x$ and we write that as follows:

$$
\begin{aligned}
{}_0I_x^n\left[\ln x\right] = \int_0^x \ln y(\mathrm{d}y)^n &= \frac{x^n}{n!}\left(\ln x - \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{n}\right)\right) \\
&= \frac{x^n}{n!}\left(\ln x - h(n)\right)
\end{aligned}
\tag{2.58}
$$

In (2.57), $h(n)=\sum_{n=1}^{n}\left(\frac{1}{k}\right)$ is the harmonic series. We say that $\int_0^x \ln y(\mathrm{d}y)^n$ is the 'generating-integral' of the harmonic series. When we change n to any real number, say υ, and use the concept of analytic continuity for writing $\upsilon! = \Gamma(\upsilon+1)$, we write the following as an analytic continuation of the 'generating integral' for the harmonic series:

$$
{}_0I_x^\upsilon\left[\ln x\right] = \int_0^x \ln y(\mathrm{d}y)^\upsilon = \frac{x^\upsilon}{\Gamma(\upsilon+1)}\left(\ln x - h(\upsilon)\right) \tag{2.59}
$$

We use the formula i.e. ${}_0I_x^\upsilon\left[x^\mu\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)}x^{\mu+\upsilon}$ in the derivation of the analytic continuation of $h(n)$ and by writing $\ln x = \lim_{\in\downarrow 0}\frac{1}{\in}\left(x^{\in}-1\right)$, in the integral $\int_0^x \ln y(\mathrm{d}y)^\upsilon$ as shown below:

$$
\begin{aligned}
\ln x = \ln x + \ln 1 &= \int y^{-1}\mathrm{d}y\Big|_1^x \\
&= \lim_{\in\downarrow 0}\int_1^x y^{-1+\in}\mathrm{d}y = \lim_{\in\downarrow 0}\frac{1}{\in}y^{\in}\Big|_{y=1}^{y=x} \\
&= \lim_{\in\downarrow 0}\left(\frac{x^{\in}}{\in}-\frac{1}{\in}\right)
\end{aligned}
\tag{2.60}
$$

Therefore, we have $\ln x = \lim_{\in\downarrow 0}\frac{1}{\in}\left(x^{\in}-1\right)$. The analytic continuation of the integral $\int_0^x \ln y(\mathrm{d}y)^n$ thus can be evaluated as follows, with this substitution:

$$
\begin{aligned}
\int_0^x \ln y(\mathrm{d}y)^\upsilon &= \lim_{\in\downarrow 0}\frac{1}{\in}\int_0^x\left(y^{\in}-1\right)(\mathrm{d}y)^\upsilon \\
&= \lim_{\in\downarrow 0}\frac{1}{\in}\left(\int_0^x y^{\in}(\mathrm{d}y)^\upsilon - \int_0^x 1(\mathrm{d}y)^\upsilon\right) \\
&= \lim_{\in\downarrow 0}\frac{1}{\in}\left({}_0I_x^\upsilon\left[x^{\in}\right] - {}_0I_x^\upsilon\left[1\right]\right) \\
&= \lim_{\in\downarrow 0}\frac{1}{\in}\left(\frac{\Gamma(1+\in)}{\Gamma(1+\in+\upsilon)}x^{\in+\upsilon} - \frac{\Gamma(1)}{\Gamma(1+\upsilon)}x^\upsilon\right) \\
&= \lim_{\in\downarrow 0}\left(\frac{x^\upsilon}{\in}\right)\left(\frac{\Gamma(1+\in)}{\Gamma(1+\in+\upsilon)}x^{\in} - \frac{1}{\Gamma(1+\upsilon)}\right)
\end{aligned}
\tag{2.61}
$$

We equate the above (2.61), that is $\int_0^x \ln y(\mathrm{d}y)^\upsilon = \lim_{\in\downarrow 0}\left(\frac{x^\upsilon}{\in}\right)\left(\frac{\Gamma(1+\in)}{\Gamma(1+\in+\upsilon)}x^{\in} - \frac{1}{\Gamma(1+\upsilon)}\right)$, with our earlier obtained (2.59) expression, $\int_0^x \ln y(\mathrm{d}y)^\upsilon = \frac{x^\upsilon}{\Gamma(\upsilon+1)}\left(\ln x - h(\upsilon)\right)$, and arrive at the following after simple manipulations i.e. a relation for $h(\upsilon)$ as the analytic continuation of the harmonic series $h(n)=\sum_{n=1}^{n}\left(\frac{1}{k}\right)$:

$$
h(\upsilon) = \ln x - \lim_{\in\downarrow 0}\left(\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\in\Gamma(1+\in+\upsilon)}x^{\in} - \frac{1}{\in}\right) \tag{2.62}
$$

The following steps are important to further simplify the above expression (2.62):

$$h(\upsilon)=\ln x-\lim_{\in\downarrow 0}\frac{1}{\in}\left(\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}x^{\in}-1\right)$$

$$=\ln x-\lim_{\in\downarrow 0}\frac{1}{\in}\left(\begin{array}{l}\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}x^{\in}-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}+\\ \frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}-1\end{array}\right)$$

$$=\ln x-\lim_{\in\downarrow 0}\frac{1}{\in}\left(x^{\in}-1\right)\left(\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right)$$

$$+\lim_{\in\downarrow 0}\frac{1}{\in}\left(1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right)$$

$$=\ln x-\lim_{\in\downarrow 0}\frac{1}{\in}\left(x^{\in}-1\right)+\lim_{\in\downarrow 0}\frac{1}{\in}\left(1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right)$$

Taking $\lim_{\in\downarrow 0}\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}=1$

$$h(\upsilon)=\lim_{\in\downarrow 0}\left(\frac{1}{\in}\left(1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right)\right)\qquad \text{use}\qquad \ln x=\lim_{\in\downarrow 0}\tfrac{1}{\in}\left(x^{\in}-1\right)$$

$$=\lim_{\in\downarrow 0}\frac{\frac{\mathrm{d}}{\mathrm{d}\in}\left[1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right]}{\frac{\mathrm{d}}{\mathrm{d}\in}[\in]}$$

$$=\lim_{\in\downarrow 0}\left(-\Gamma(1+\upsilon)\frac{\Gamma(1+\in+\upsilon)\Gamma^{(1)}(1+\in)-\Gamma(1+\in)\Gamma^{(1)}(1+\in+\upsilon)}{\left(\Gamma(1+\in+\upsilon)\right)^{2}}\right)\tag{2.63}$$

$$=-\Gamma^{(1)}(1)+\frac{\Gamma^{(1)}(1+\upsilon)}{\Gamma(1+\upsilon)}=\gamma+\psi(1+\upsilon)$$

We have applied L'Hospital's rule in (2.63) to calculate the limit in the above derivation (2.63), since we are getting $\frac{0}{0}$ for the following expression:

$$\lim_{\in\downarrow 0}\left(\frac{1}{\in}\left(1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right)\right)=\lim_{\in\downarrow 0}\frac{\frac{\mathrm{d}}{\mathrm{d}\in}\left[1-\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}\right]}{\frac{\mathrm{d}}{\mathrm{d}\in}[\in]}\tag{2.64}$$

as $\lim_{\in\downarrow 0}\frac{\Gamma(1+\in)\Gamma(1+\upsilon)}{\Gamma(1+\in+\upsilon)}=1$. In addition, we have used a derivative of the quotient formula that is:

$$f(x)=\frac{g(x)}{h(x)}\qquad f^{(1)}(x)=\frac{h(x)g^{(1)}(x)-g(x)h^{(1)}(x)}{\left(h(x)\right)^{2}}\tag{2.65}$$

in the above steps (2.63) during derivation.

Using the definition of the 'psi' function (or the di-gamma function), $\frac{\Gamma^{(1)}(z)}{\Gamma(z)}=\psi(z)$, and the Euler constant $\gamma=-\Gamma^{(1)}(1)$, in the above derivation (2.63) we get analytical continuation of the harmonic series as:

$$h(\upsilon)=\psi(1+\upsilon)+\gamma\tag{2.66}$$

The above (2.66) result is standard formulation proved in 'number theory'; but here it is also derived from analytic continuation the of integration operation, that is, operator ${}_0I_x^{\upsilon}$ where υ is a non-integer. With this, we can also write fractional integration of $\ln x$ as:

$${}_0I_x^{\upsilon}\left[\ln x\right]=\int_0^x \ln y(\mathrm{d}y)^{\upsilon}=\frac{x^{\upsilon}}{\Gamma(\upsilon+1)}\left(\ln x-\psi(\upsilon)-\gamma\right)\tag{2.67}$$

This above expression (2.67) we have used to extend the formula of fractional integration to the log-region (2.57) for the formula ${}_0I_x^{\alpha}\left[x^{\mu}\right]=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\alpha+\mu}$, where we had difficulty for cases $\mu<0$ and $\alpha\le\mu$.

2.9 Fractional integral of exponential function by series

The earlier derived formula (2.45) for fractional integration ${}_0I_x^{\upsilon}\left[x^{\mu}\right]$ that is ${}_0D_x^{-\upsilon}\left[x^{\mu}\right]=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)}x^{\mu+\alpha}$ for $\mu>-1$, $\upsilon>0$ and $\upsilon,\mu\in\mathbb{R}$, can be used as a term by term formula for the series expanded function e^{ax}, as indicated below:

$$e^{ax}=1+(ax)+\frac{(ax)^2}{2!}+\frac{(ax)^3}{3!}+\ldots=\sum_{k=0}^{\infty}\frac{(ax)^k}{k!}=\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(k+1)} \tag{2.68}$$

2.9.1 Fractional integration of the exponential function with a start point of integration as zero

We use (2.45) and then write the following steps:

$$\begin{aligned}{}_0I_x^{\upsilon}\left[e^{ax}\right]&={}_0I_x^{\upsilon}\left[1\right]+{}_0I_x^{\upsilon}\left[(ax)\right]+{}_0I_x^{\upsilon}\left[\frac{(ax)^2}{2!}\right]+\ldots\\&=\frac{x^{\upsilon}}{\Gamma(\upsilon+1)}+\frac{a\Gamma(2)x^{\upsilon+1}}{1!\Gamma(\upsilon+2)}+\frac{a^2\Gamma(3)x^{\upsilon+2}}{2!\Gamma(\upsilon+3)}+\frac{a^3\Gamma(4)x^{\upsilon+3}}{3!\Gamma(\upsilon+4)}+\ldots\\&=x^{\upsilon}\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(k+\upsilon+1)}\\{}_0I_x^{\upsilon}\left[e^{ax}\right]&={}_0D_x^{-\upsilon}\left[e^{ax}\right]=x^{\upsilon}\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(\upsilon+k+1)}\end{aligned} \tag{2.69}$$

2.9.2 Introducing a higher transcendental function in the result of the fractional integration of the exponential function

We have a two-parameter Mittag-Leffler function i.e. $E_{\alpha,\beta}(x)$ defined as follows in the series representation which is also detailed in Appendix A:

$$E_{\alpha,\beta}(x)\stackrel{\text{def}}{=}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}\qquad \alpha,\beta>0\qquad \alpha,\beta\in\mathbb{R} \tag{2.70}$$

In Appendix F, we extend the definition of $E_{\alpha,\beta}(x)$ for negative order α. However, we will be considering $E_{\alpha,\beta}(x)$ positive order α, in our discussions. Thus in terms of the above (2.69) ${}_0I_x^{\upsilon}\left[e^{ax}\right]=x^{\upsilon}\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(\upsilon+k+1)}$ and by defining the Mittag-Leffler function (2.70) i.e. $E_{\alpha,\beta}(x)$, we obtain the fractional integration of the exponential function as follows:

$${}_0I_x^{\upsilon}\left[e^{ax}\right]=x^{\upsilon}E_{1,(\upsilon+1)}(ax) \tag{2.71}$$

We note here that function $E_{\alpha,\beta}(ax)$ is one of several higher transcendental functions. The series that appeared in the above derivation (2.69), that is, $x^{\upsilon}\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(\upsilon+k+1)}$, is called the Miller-Ross function, represented as $E_x(\upsilon,a)$, which is another higher exponential function. Its relation to the two-parameter Mittag-Leffler function is the following (noted in Appendix A):

$$E_x(\upsilon,a)\stackrel{\text{def}}{=}x^{\upsilon}\sum_{k=0}^{\infty}\frac{(ax)^k}{\Gamma(\upsilon+k+1)}=x^{\upsilon}E_{1,\upsilon+1}(ax) \tag{2.72}$$

Thus in (2.72) we have $\upsilon=0$ $E_x(0,a)=e^{ax}$ and $E_{\theta}(0,ia)=e^{i(a\theta)}=\cos(a\theta)+i\sin(a\theta)$.

What we note here is that while fractionally integrating the exponential function (a transcendental function) we are getting a higher exponential (higher transcendental) function that is:

$${}_0I_x^{\upsilon}\left[e^{ax}\right]={}_0D_x^{-\upsilon}\left[e^{ax}\right]=E_x(\upsilon,a)=x^{\upsilon}E_{1,\upsilon+1}(ax) \tag{2.73}$$

The complex argument Miller-Ross function is derived from the above definition (2.72) as follows:

$$E_x(\upsilon, ia) = (x)^{\upsilon}\left(\sum_{k=\text{even}}^{\infty}\frac{(-1)^{\frac{k}{2}}(ax)^k}{\Gamma(\upsilon+k+1)} + i\sum_{k=\text{odd}}^{\infty}\frac{(-1)^{\frac{k-1}{2}}(ax)^k}{\Gamma(\upsilon+k+1)}\right) \tag{2.74}$$
$$= C_x(\upsilon, a) + iS_x(\upsilon, a)$$

From this we obtain (in Miller-Ross notation) the higher cosine and sine functions as:

$$C_x(\upsilon, a) \overset{\text{def}}{=} x^{\upsilon}\sum_{k=\text{even}}^{\infty}\frac{(-1)^{k/2}(ax)^k}{\Gamma(\upsilon+k+1)}$$
$$S_x(\upsilon, a) \overset{\text{def}}{=} x^{\upsilon}\sum_{k=\text{odd}}^{\infty}\frac{(-1)^{\frac{k-1}{2}}(ax)^k}{\Gamma(\upsilon+k+1)} \tag{2.75}$$

When $\upsilon = 0$, we have:

$$C_x(0, a) = \sum_{k=\text{even}}^{\infty}\frac{(-1)^{k/2}(ax)^k}{\Gamma(k+1)} = 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} + ... = \cos(ax)$$
$$S_x(0, a) = \sum_{k=\text{odd}}^{\infty}\frac{(-1)^{\frac{k-1}{2}}(ax)^k}{\Gamma(k+1)} = (ax) - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + ... = \sin(ax) \tag{2.76}$$

Like we have demonstrated by taking the series expansion of the exponential function and then using the obtained fractional integration formula for a power function, a higher exponential function is obtained; similarly we can derive the series expansion of the cosine and sine function and obtain the following with the method described in (2.69):

$${}_0D_{\theta}^{-\upsilon}\left[\cos(a\theta)\right] = C_{\theta}(\upsilon, a) \quad , \quad {}_0D_{\theta}^{-\upsilon}\left[\sin(a\theta)\right] = S_{\theta}(\upsilon, a) \tag{2.77}$$

2.9.3 Fractional integration of an exponential function with the start point of integration as non-zero

Now we do the fractional integration of the exponential function with a non-zero start point of integration, i.e. we evaluate ${}_aI_x^{\alpha}\left[e^{\lambda(x-a)}\right]$ by using ${}_aI_x^{\alpha}\left[(x-a)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}$, which we derived earlier in (2.46). We write the series for $e^{\lambda(x-a)}$ as:

$$e^{\lambda(x-a)} = 1 + \frac{\lambda(x-a)}{1!} + \frac{\lambda^2(x-a)^2}{2!} + \frac{\lambda^3(x-a)^3}{3!} + ... \tag{2.78}$$

Now we apply a term-by-term fractional integration formula to (2.78) to get the following:

$${}_aI_x^{\alpha}\left[e^{\lambda(x-a)}\right] = \frac{\Gamma(1)(x-\alpha)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2)\lambda(x-a)^{\alpha+1}}{1!\Gamma(\alpha+2)} + \frac{\Gamma(3)\lambda^2(x-a)^{\alpha+2}}{2!\Gamma(\alpha+3)} + ... \tag{2.79}$$
$$= (x-a)^{\alpha}E_{1,\alpha+1}\left(\lambda(x-a)\right)$$

We have used the series definition of the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(x)$ (refer to Appendix A). Having derived the above expression, we now get the expression for the fractional integration of ${}_aI_x^{\alpha}\left[e^{\lambda x}\right]$. We write $e^{\lambda x} = e^{\lambda x - \lambda a + \lambda a} = e^{\lambda a}e^{\lambda(x-a)}$. Using the above derived expression (2.79) for ${}_aI_x^{\alpha}\left[e^{\lambda(x-a)}\right]$, we get the following:

$${}_aI_x^{\alpha}\left[e^{\lambda x}\right] = e^{\lambda a}(x-a)^{\alpha}\left(E_{1,\alpha+1}\left(\lambda(x-a)\right)\right) \tag{2.80}$$

The one parameter Mittag-Leffler function (refer to Appendix A) is defined in series form as follows:

$$E_{\alpha}(ax^{\alpha}) \overset{\text{def}}{=} 1 + \frac{ax^{\alpha}}{\Gamma(1+\alpha)} + \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)} + ...; \tag{2.81}$$
$$\alpha > 0; \quad \alpha \in \mathbb{R}; \quad a \in \mathbb{C}$$

We note that $E_{\alpha,1}(x) = E_{\alpha}(x)$, i.e. $E_{\alpha,\beta}(x) = E_{\alpha}(x)$ for $\beta = 1$ (refer to Appendix A).

We integrate the above Mittag-Leffler functions series (2.81) from 0 to x, by fractional order $\alpha > 0$ and write the following steps:

$$
\begin{aligned}
{}_0I_x^\alpha\left[E_\alpha(ax^\alpha)\right] &= {}_0I_x^\alpha[1] + {}_0I_x^\alpha\left[\frac{ax^\alpha}{\Gamma(1+\alpha)}\right] + {}_0I_x^\alpha\left[\frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)}\right] + \\
&\quad {}_0I_x^\alpha\left[\frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)}\right] + \dots \\
&= \frac{1}{\Gamma(1+\alpha)}x^\alpha + \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}ax^{2\alpha} + \\
&\quad \frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)}a^2x^{3\alpha} + \dots \\
&= \frac{1}{a}\left(\frac{1}{\Gamma(1+\alpha)}ax^\alpha + \frac{1}{\Gamma(1+2\alpha)}a^2x^{2\alpha} + \frac{1}{\Gamma(1+3\alpha)}a^3x^{3\alpha} + \dots\right) \\
&= \frac{1}{a}\left(\left(1 + \frac{ax^\alpha}{\Gamma(1+\alpha)} + \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \dots\right) - 1\right) \\
&= \frac{1}{a}\left(\left(E_\alpha(ax^\alpha)\right) - 1\right)
\end{aligned}
\tag{2.82}
$$

2.9.4 Fractional integration of a one-parameter Mittag-Leffler function as conjugation to classical integration of the exponential function

Therefore, from (2.82) we have ${}_0I_x^\alpha\left[E_\alpha(ax^\alpha)\right] = \frac{1}{a}E_\alpha(ax^\alpha) - \frac{1}{a}$. Note that the value of $E_\alpha(ax^\alpha)$ at $x = 0$ is 1. The integration obtained here for the one-parameter Mittag-Leffler function is a conjugation to the exponential function that is $\int_0^x e^{ay}\mathrm{d}y = \frac{1}{a}e^{ax} - \frac{1}{a}$. This is a very important observation. Like in classical calculus, we get the integration of exponential as exponential; we can make a similar observation for the one-parameter Mittag-Leffler function too. Therefore, this Mittag-Leffler function may play an important role like the exponential function plays, for solving a fractional differential equation. We note this observation for future usage.

2.10 Tricks required for solving fractional integrals

The integral of the type that has appeared in the fractional integral formula requires proficiency to be solved. The integrals are of the type $\int_c^x (x-y)^{\alpha-1}f(y)\mathrm{d}y$, that is, convolution integral, require some tricks, as indicated in finding the fractional integral of $f(x) = x^\mu$. However, there is no standard universal method. Because the kernel of this integral is of the type $(x-y)^{\alpha-1}$, there is some possibility to have some tricks developed. The tricks that we mention give us useful identities.

2.10.1 Obtaining a useful identity for fractional integration with a lower limit of integration as zero

Let us write the formulation of fractional integral of the function $x\left(f(x)\right)$ as follows

$$
{}_0D_x^{-\alpha}\left[x\left(f(x)\right)\right] = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\left(y\left(f(y)\right)\right)\mathrm{d}y \tag{2.83}
$$

Make the substitution $y\left(f(y)\right) = \left(x-(x-y)\right)\left(f(y)\right)$ in the above (2.83) expression (we have added and subtracted x), and with several simple steps of expansion, we get the following:

$$
\begin{aligned}
{}_0D_x^{-\alpha}\left[x\big(f(x)\big)\right] &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\big(x-(x-y)\big)\big(f(y)\big)\,\mathrm{d}y \\
&= x\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\big(f(y)\big)\mathrm{d}y\right) - \\
&\qquad \left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha}\big(f(y)\big)\,\mathrm{d}y\right)
\end{aligned} \tag{2.84}
$$

The first integral term in (2.84) is recognised as ${}_0D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\big(f(y)\big)\,\mathrm{d}y$. We multiply and divide the second term in (2.84) by α and use $n\big(\Gamma(n)\big) = \Gamma(n+1)$, to write the following:

$$
\begin{aligned}
{}_0D_x^{-\alpha}\left[x\big(f(x)\big)\right] &= x\Big({}_0D_x^{-\alpha}\left[f(x)\right]\Big) - \frac{\alpha}{\alpha\big(\Gamma(\alpha)\big)}\int_0^x (x-y)^{\alpha}\big(f(y)\big)\,\mathrm{d}y \\
&= x\Big({}_0D_x^{-\alpha}\left[f(x)\right]\Big) - \frac{\alpha}{\Gamma(\alpha+1)}\int_0^x (x-y)^{\alpha}\big(f(y)\big)\,\mathrm{d}y
\end{aligned} \tag{2.85}
$$

Now the second integral in (2.85) is recognised as ${}_0D_x^{-(\alpha+1)}\left[f(x)\right]$, and we get one useful identity that is represented below:

$$
\begin{aligned}
{}_0D_x^{-\alpha}\left[x\big(f(x)\big)\right] &= x\Big({}_0D_x^{-\alpha}\left[f(x)\right]\Big) - \alpha\Big({}_0D_x^{-(\alpha+1)}\left[f(x)\right]\Big) \\
{}_0I_x^{\alpha}\left[x\big(f(x)\big)\right] &= x\Big({}_0I_x^{\alpha}\left[f(x)\right]\Big) - \alpha\Big({}_0I_x^{(\alpha+1)}\left[f(x)\right]\Big)
\end{aligned} \tag{2.86}
$$

From the above obtained expression we can write the fractional integration for some functions with the use of the Miller-Ross function (a higher transcendental function [see Appendix A]) as follows:

$$
\begin{aligned}
{}_0D_x^{-\upsilon}\left[xe^{ax}\right] &= x\Big({}_0D_x^{-\upsilon}\left[e^{ax}\right]\Big) - \upsilon\Big({}_0D_x^{-(\upsilon+1)}\left[e^{ax}\right]\Big) \\
&= xE_x(\upsilon,a) - \upsilon E_x(\upsilon+1,a) \\
{}_0D_x^{-\upsilon}\left[x\cos(ax)\right] &= x\Big({}_0D_x^{-\upsilon}\left[\cos(ax)\right]\Big) - \upsilon\Big({}_0D_x^{-(\upsilon+1)}\left[\cos(ax)\right]\Big) \\
&= xC_x(\upsilon,a) - \upsilon C_x(\upsilon+1,a) \\
{}_0D_x^{-\upsilon}\left[x\sin(ax)\right] &= x\Big({}_0D_x^{-\upsilon}\left[\sin(ax)\right]\Big) - \upsilon\Big({}_0D_x^{-(\upsilon+1)}\left[\sin(ax)\right]\Big) \\
&= xS_x(\upsilon,a) - \upsilon S_x(\upsilon+1,a)
\end{aligned} \tag{2.87}
$$

We can generalise the expression ${}_0D_x^{-\upsilon}\left[x\big(f(x)\big)\right] = x\Big({}_0D_x^{-\upsilon}\left[f(x)\right]\Big) - \upsilon\Big({}_0D_x^{-(\upsilon+1)}\left[f(x)\right]\Big)$ for function $x^p f(x)$ by the following steps:

$$
{}_0D_x^{-\upsilon}\left[x^p f(x)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^x (x-y)^{\upsilon-1}\big(y^p f(y)\big)\mathrm{d}y \tag{2.88}
$$

We make $y^p = \left[x-(x-y)\right]^p$ and expand as follows:

$$
y^p = \big(x-(x-y)\big)^p = \sum_{k=0}^{p}(-1)^k\binom{p}{k}x^{p-k}(x-y)^k \tag{2.89}
$$

Substituting this expansion (2.89) in the fractional integration expression of (2.88) we write the following:

$$
{}_0D_x^{-\upsilon}\left[x^p f(x)\right] = \frac{1}{\Gamma(\upsilon)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}x^{p-k}\int_0^x (x-y)^{\upsilon+k-1}\big(f(y)\big)\,\mathrm{d}y \tag{2.90}
$$

Write $\int_0^x (x-y)^{\upsilon+k-1}\big(f(y)\big)\,\mathrm{d}y$ appearing in the above expression (2.90), using the formula of the Riemann-Liouville fractional integration to give:

$$\int_0^x (x-y)^{\upsilon+k-1}\left(f(y)\right)\mathrm{d}y = \Gamma(\upsilon+k)\left(\frac{1}{\Gamma(\upsilon+k)}\int_0^x (x-y)^{(\upsilon+k)-1}\left(f(y)\right)\mathrm{d}y\right)$$
$$= \Gamma(\upsilon+k)\left({}_0D_x^{-(\upsilon+k)}\left[f(x)\right]\right) \tag{2.91}$$

Using the above obtained expressions (2.90) and (2.91) we get the useful expression:

$$ {}_0D_x^{-\upsilon}\left[x^p f(x)\right] = \frac{1}{\Gamma(\upsilon)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\upsilon+k)\left({}_0D_x^{-(\upsilon+k)}\left[f(x)\right]\right)$$
$$ {}_0I_x^{\upsilon}\left[x^p f(x)\right] = \frac{1}{\Gamma(\upsilon)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\upsilon+k)\left({}_0I_x^{(\upsilon+k)}\left[f(x)\right]\right) \tag{2.92}$$

By use of the identity of the generalisation of binomial coefficients, which is the following:

$$\binom{-z}{n} = \frac{\Gamma(1-z)}{n!\Gamma(1-z-n)} = (-1)^n\frac{\Gamma(z+n)}{n!\Gamma(z)} = (-1)^n\binom{z+n-1}{n} \tag{2.93}$$

we express the obtained formula (2.92) for ${}_0D_x^{-\upsilon}\left[x^p f(x)\right]$ in another form as follows:

$$ {}_0D_x^{-\upsilon}\left[x^p f(x)\right] = \frac{1}{\Gamma(\upsilon)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\upsilon+k)\left({}_0D_x^{-(\upsilon+k)}\left[f(x)\right]\right)$$
$$= \sum_{k=0}^{p}(-1)^k\frac{p!}{k!(p-k)!}\frac{\Gamma(\upsilon+k)}{\Gamma(v)}x^{p-k}\left({}_0D_x^{-(\upsilon+k)}\left[f(x)\right]\right)$$
$$= \sum_{k=0}^{p}\binom{-\upsilon}{k}\left[D_x^k x^p\right]\left[{}_0D_x^{-(\upsilon+k)}f(x)\right] \tag{2.94}$$

We have used the following identities in the above derivation (2.94):

$$(-1)^k\frac{\Gamma(\upsilon+k)}{k!\Gamma(\upsilon)} = \binom{-\upsilon}{k} \qquad D^k x^p = \frac{\mathrm{d}^k}{\mathrm{d}x^k}x^p = \frac{p!}{(p-k)!}x^{p-k} \tag{2.95}$$

That is the $k-$th derivative of x^p, and an expression of generalised binomial coefficients, in (2.95).

2.10.2 Obtaining a useful identity for a fractional integration with a lower limit of integration as non-zero

Now we try to find a general expression for fractional integration for $f(x)$ with a non-zero lower limit:

$$ {}_cD_x^{-\upsilon}\left[f(x)\right] = \frac{1}{\Gamma(\upsilon)}\int_c^x (x-y)^{\upsilon-1}\left(f(y)\right)\mathrm{d}y \qquad \upsilon>0 \qquad 0\le c\le x \tag{2.96}$$

Put $y = x(1-z)$ giving $\mathrm{d}y = -x\mathrm{d}z$. Then $(x-y) = xz$ and for $y = c$ the new lower limit is $z = \frac{(x-c)}{x}$, and the new upper limit for $y = x$ becomes $z = 0$. The above integral (2.96) with changed variables substituted is:

$$ {}_cD_x^{-\upsilon}\left[f(x)\right] = \frac{1}{\Gamma(\upsilon)}\int_{(x-c)/x}^{0}(xz)^{\upsilon-1}\left(f(x-xz)\right)\left(-x\mathrm{d}z\right)$$
$$= \frac{x^{\upsilon}}{\Gamma(\upsilon)}\int_0^{(x-c)/x} z^{\upsilon-1}\left(f(x-xz)\right)\mathrm{d}z \tag{2.97}$$

Therefore, we get an identity that is as follows:

$$ {}_cD_x^{-\upsilon}\left[f(x)\right] = \frac{x^{\upsilon}}{\Gamma(\upsilon)}\int_0^{(x-c)/x} z^{\upsilon-1}\left(f(x-xz)\right)\mathrm{d}z$$
$$ {}_cI_x^{\upsilon}\left[f(x)\right] = \frac{x^{\upsilon}}{\Gamma(\upsilon)}\int_0^{(x-c)/x} z^{\upsilon-1}\left(f(x-xz)\right)\mathrm{d}z \tag{2.98}$$

2.10.3 Application of the obtained identity for acquiring a fractional integration of a power function

We apply (2.98) for $c = 0$ and for function $f(x) = x^{\mu}$ with $\mu > -1$, from the above to get:

$$
\begin{aligned}
{}_0D_x^{-\upsilon}\left[x^{\mu}\right] &= \frac{x^{\upsilon}}{\Gamma(\upsilon)}\int_0^1 z^{\upsilon-1}\left(x^{\mu}(1-z)^{\mu}\right)\mathrm{d}z \\
&= \frac{x^{\mu+\upsilon}}{\Gamma(\upsilon)}\int_0^1 z^{\upsilon-1}(1-z)^{\mu}\,\mathrm{d}z = \frac{x^{\mu+\upsilon}}{\Gamma(\upsilon)}\mathrm{B}(\upsilon,\mu+1) \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)}x^{\mu+\upsilon}
\end{aligned}
\tag{2.99}
$$

The same expression we obtained earlier too with the beta function, in Section 2.6.

2.10.4 Application of the obtained identity for acquiring fractional integration of the exponential function in terms of Tricomi's incomplete gamma function

Let us try to find a fractional integral of order $v > 0$ and of exponential function $f(t) = e^{at}$. We write the fractional integration operation as:

$$
{}_0D_t^{-v}\left[e^{at}\right] = \frac{1}{\Gamma(v)}\int_0^t (t-u)^{v-1}e^{au}\,\mathrm{d}u \tag{2.100}
$$

Make the substitution $x = t - u$, then the integral expression is changed as follows:

$$
{}_0D_t^{-v}\left[e^{at}\right] = \frac{e^{at}}{\Gamma(v)}\int_0^t x^{v-1}e^{-ax}\,\mathrm{d}x \tag{2.101}
$$

Clearly, this integral is not an elementary function, but is related to Tricomi's incomplete gamma function and thus the fractional integral of the exponential function is expressed as follows:

$$
{}_0D_t^{-v}\left[e^{at}\right] = t^{v}e^{at}\gamma^{*}(v,at) \tag{2.102}
$$

where Tricomi's incomplete gamma function is $\gamma^{*}(a,x) = \frac{x^{-a}}{\Gamma(a)}\int_0^x e^{-y}y^{a-1}\mathrm{d}y$, which we discussed earlier in Section1.13. We will use this result in detail in the next chapters.

2.11 Fractional integration of the analytical function

The Riemann-Liouville fractional integral is expressed in the following convolution integral:

$$
\begin{aligned}
{}_0D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}\left(f(u)\right)\mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\left(f(x-y)\right)\mathrm{d}y
\end{aligned}
\tag{2.103}
$$

This expression (2.103) is achieved by the simple substitution $(x-u) = y$. An analytic function is infinitely differentiable such that the Taylor series at any point of x_0 in its domain is the following (Section 1.4):

$$
T(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \qquad f^{(n)}(x_0) = \left.\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right|_{x=x_0} = \left.D^n f(x)\right|_{x=x_0} \tag{2.104}
$$

The series $T(x)$ represents the function $f(x)$ in the neighborhood of x_0 such that $x = x_0 + \Delta x$.

2.11.1 Fractional integration related to a series of ordinary whole derivatives of an analytical function

We write the following steps (with $n = 0,1,2,3...$ i.e. positive integers), representing the function $f(x)$ in the neighborhood of x_0 such that $x = x_0 + \Delta x$, by using (2.104):

$$
\begin{aligned}
f(x_0+\Delta x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\
&= f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \\
&\qquad \dots \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots \\
&= f(x_0) + f^{(1)}(x_0)\left(\Delta x\right) + \frac{f^{(2)}(x_0)}{2!}\left(\Delta x\right)^2 + \\
&\qquad \dots \frac{f^{(n)}(x_0)}{n!}\left(\Delta x\right)^n + \dots
\end{aligned}
\tag{2.105}
$$

With a change in variables $x_0 = x$ and $\Delta x = -y$ we can re-write the above (2.105) series in the following way:

$$
\begin{aligned}
f(x-y) &= f(x) - \left(f^{(1)}(x)\right)(y) + \frac{f^{(2)}(x)}{2!}(y)^2 + \dots(-1)^n \frac{f^{(n)}(x)}{n!} y^n + \dots \\
&= f(x) - \left(D^1 f(x)\right) y + \frac{D^2 f(x)}{2!} y^2 + \dots(-1)^n \frac{D^n f(x)}{n!} y^n + \dots
\end{aligned}
\tag{2.106}
$$

In the above derivation (2.106), assuming $f(x)$ is analytic we thus write a power series expansion for $f(x-y)$ as follows:

$$
f(x-y) = f(x) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k f(x)}{k!} y^k \tag{2.107}
$$

We substitute the above series expansion of $f(x-y)$ (2.107) into the fractional integration formula (2.103), i.e. ${}_0D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\left(f(x-y)\right)\mathrm{d}y$ which yields the following steps:

$$
\begin{aligned}
{}_0D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\left(f(x) + \sum_{k=1}^{\infty} \frac{D^k f(x)}{k!} y^k\right)\mathrm{d}y \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1} f(x)\mathrm{d}y + \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1+k}\left(\sum_{k=1}^{\infty} \frac{D^k f(x)}{k!}\right)\mathrm{d}y \\
&= \frac{f(x)}{\Gamma(\alpha)}\left[\frac{y^\alpha}{\alpha}\right]_0^x + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{\infty} \frac{D^k f(x)}{k!}\left[\frac{y^{\alpha+k}}{\alpha+k}\right]_0^x \\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{f(x)}{\alpha}x^\alpha + \sum_{k=1}^{\infty} \frac{D^k f(x)}{k!}\frac{x^{\alpha+k}}{\alpha+k}\right) \\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{f(x)}{\alpha}x^\alpha - \frac{f^{(1)}(x)}{\alpha+1}x^{\alpha+1} + \frac{f^{(2)}(x)}{2!(\alpha+2)}x^{\alpha+2} - \dots\right)
\end{aligned}
\tag{2.108}
$$

In the above steps (2.108) we have interchanged the order of summation and integration, assuming uniform convergence of the infinite series in the interval of interest. Thus, we have an interesting expression relating to the fractional integration of the function to a series of ordinary whole derivatives of an analytical function, i.e.

$$
{}_0D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\left(\frac{f(x)}{\alpha}x^\alpha - \frac{f^{(1)}(x)}{\alpha+1}x^{\alpha+1} + \frac{f^{(2)}(x)}{2!(\alpha+2)}x^{\alpha+2} - \dots\right) \tag{2.109}
$$

2.11.2 Application of the Fractional integration of an analytical function to an exponential function for integer order integration

Let us put $\alpha = 1$ and see what happens for $f(x) = e^x$. This exponential function (e^x) is an analytical function for all x, with $f^{(1)}(x) = f^{(2)}(x) = \dots = e^x$. Therefore, using the above expansion (2.109) for ${}_0D_x^{-\alpha}\left[f(x)\right]$ we get the following steps:

$$
\begin{aligned}
{}_0D_x^{-1}\left[f(x)\right] &= \frac{1}{\Gamma(1)}\left[\left(f(x)\right)x - \frac{f^{(1)}(x)}{2}x^2 + \frac{f^{(2)}(x)}{2!(3)}x^3 - ....\right] \\
{}_0D_x^{-1}\left[e^x\right] &= \left[xe^x - \frac{x^2}{2}e^x + \frac{x^3}{3!}e^x - ...\right] \\
&= e^x\left[x - \frac{x^2}{2} + \frac{x^3}{3!} - ...\right] \\
&= e^x\left[1 - 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - ...\right] \\
&= e^x\left[1 - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + ...\right)\right] \\
&= e^x\left[1 - (e^{-x})\right] \\
&= e^x - 1
\end{aligned}
\tag{2.110}
$$

Indeed the above is true: ${}_0D_x^{-1}\left[e^x\right] = \int_0^x e^y \mathrm{d}y = e^x - 1$ is the classical integration of an exponential function.

2.11.3 An alternate representation and generalisation of the Riemann-Liouville fractional integration formula of analytical function

From the discussion and observations of Sections 2.11.1 and 2.11.2 we can now generalise and proceed to discuss the alternate representation of ${}_aD_x^{\alpha}\left[\varphi(x)\right]$, where $\varphi(x)$ is a real analytic function, i.e. this function has a convergent power series expansion in the interval $a \le y \le x$ of interest. While $\alpha < 0$ this permits us to use the Riemann-Liouville (RL) definition, which is as follows:

$$
{}_aD_x^{\alpha}\left[\varphi(x)\right] = \frac{\mathrm{d}^{\alpha}\left[\varphi(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{1}{\Gamma(-\alpha)}\int_a^x \frac{\varphi(y)\mathrm{d}y}{(x-y)^{\alpha+1}} = \frac{1}{\Gamma(-\alpha)}\int_0^{x-a} \frac{\varphi(x-\upsilon)\mathrm{d}\upsilon}{\upsilon^{\alpha+1}} \tag{2.111}
$$

We have used $\upsilon \equiv (x-y)$. Upon the Taylor expansion of $\varphi(x-\upsilon)$ about the point x, as we did earlier in (2.106), we write the following:

$$
\varphi(x-\upsilon) = \varphi(x) - \upsilon\varphi^{(1)}(x) + \frac{\upsilon^2}{2!}\varphi^{(2)}(x) + ... = \sum_{k=0}^{\infty}\frac{(-1)^k \upsilon^k \varphi^{(k)}(x)}{k!} \tag{2.112}
$$

The above representation (2.112) involves no remainder since we have assumed $\varphi(x-\upsilon)$ to have convergent power series. We insert this expansion (2.112) into the RL formula (2.111) and write the following:

$$
\begin{aligned}
{}_aD_x^{\alpha}\left[\varphi(x)\right] &= \frac{\mathrm{d}^{\alpha}\left[\varphi(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{1}{\Gamma(-\alpha)}\int_0^{x-a}\frac{\varphi(x-\upsilon)\mathrm{d}\upsilon}{\upsilon^{\alpha+1}} \\
&= \frac{1}{\Gamma(-\alpha)}\int_0^{x-a}\sum_{k=0}^{\infty}\frac{(-1)^k\upsilon^k\varphi^{(k)}(x)}{\upsilon^{\alpha+1}k!}\mathrm{d}\upsilon \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(x-a)^{k-\alpha}\varphi^{(k)}(x)}{\Gamma(-\alpha)(k-\alpha)k!}
\end{aligned}
\tag{2.113}
$$

We note here that the analyticity argument in α may be used here to establish the above obtained formula (2.113) for all α positive or negative (differentiation or integration) even though this was derived for $\alpha < 0$; (integration) which we will formally elaborate upon subsequently, in a later chapter.

2.12 A simple hydrostatic problem of water flowing through the weir of a dam with the use of a fractional integration formula

Here we take a simple problem, where we determine the shape of a notch or a weir of a dam as a function depending on the water flow rate through it. Figure 2.3 gives a side view of the dam and water flow (on the X-Z plane) and a front view showing the shape of the notch (on the Y-Z plane). Figure 2.4 gives the details of the function i.e.

$y = f(z)$ that describes the shape of the notch. In Figure 2.3 the water element I is at position (x_0, y_0, z_0) at height z_0, whereas the water level height is $z = h$, which keeps element-I under hydrostatic pressure from the water height above it $(h - z_0)$, apart from the atmospheric pressure (P_{atm}) that is always acting. After some time element-I is at position II, that is $(0, y_0, z_0)$, which means element-II of the water mass has no pressure from the water head above it and is free from the atmosphere; it is just on the verge of falling off.

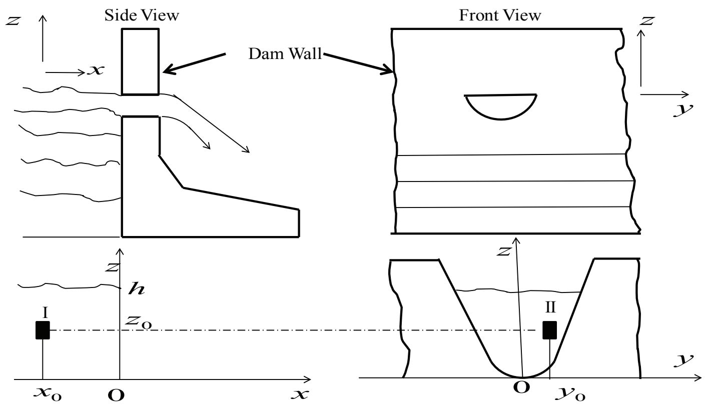

Figure 2.3: Water flowing through a dam weir

Consider element I and element II as unit volumes with density ρ flowing with respective velocities as shown in Figure 2.3. Here we write pressure heads at I and II from Bernoulli's law of hydrostatics, given in the following equation:

$$P_{\text{I}} + \rho g z_0 + \frac{1}{2}\rho V_{\text{I}}^2 = P_{\text{II}} + \rho g z_0 + \frac{1}{2}\rho V_{\text{II}}^2 \tag{2.114}$$

In the above equation (2.114) the expression g is the gravitational constant (acceleration $\approx 10\,\text{m}/\text{s}^2$); V_{I} is the velocity of element-I, which is far from the opening; and V_{II} is the velocity of element-II gushing out of the notch. Obviously, this gushing out of the element's velocity will be much greater than the velocity of element-I, which is far from the opening. Thus, we can assume $V_{\text{I}} << V_{\text{II}}$, and make an approximation as $V_{\text{I}} \approx 0$. The pressure is $P_{\text{I}} \propto (h - z_0) + P_{\text{atm}}$ and $P_{\text{II}} \propto P_{\text{atm}}$. Therefore, we have $P_{\text{I}} - P_{\text{II}} = (\rho g)(h - z_0)$. Using all these arguments and substituting these in the obtained hydrostatic equation i.e. $P_{\text{I}} + \rho g z_0 + \frac{1}{2}\rho V_{\text{I}}^2 = P_{\text{II}} + \rho g z_0 + \frac{1}{2}\rho V_{\text{II}}^2$; we naturally get the following:

$$\begin{aligned} V_{\text{II}}(z_0) &= \sqrt{2g(h - z_0)} \\ V_{\text{II}}(z) &= \sqrt{2g}\,(h - z)^{1/2} \end{aligned} \tag{2.115}$$

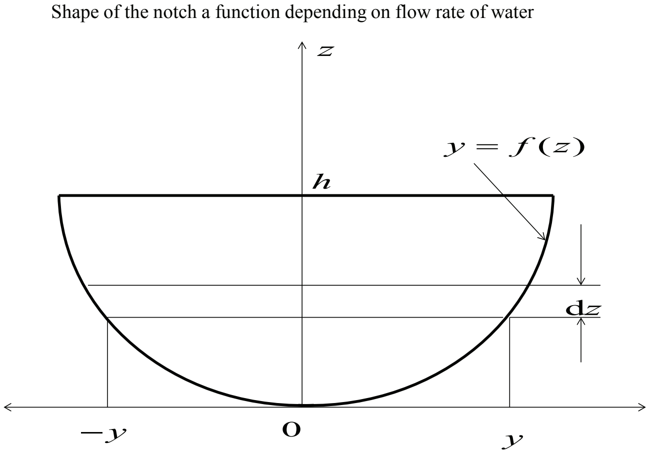

Figure 2.4: A function describing the shape of the notch of a dam that depends on flow rate

Let the profile of the notch on the Y-Z plane as depicted in Figure 2.4 be represented by the function $y = f(z)$. Looking at Figure 2.4, we get the elemental area of a section of the notch as follows:

$$\mathrm{d}A = 2y\mathrm{d}z = 2\big(f(z)\big)\mathrm{d}z \qquad (2.116)$$

From this area element, the elemental flow rate is velocity at II times are $\mathrm{d}A$, that is:

$$\begin{aligned}\mathrm{d}Q &= V_{\mathrm{II}}\mathrm{d}A = V_{\mathrm{II}}\big(2f(z)\big)\mathrm{d}z \\ &= 2\sqrt{2g}(h-z)^{1/2}\big(f(z)\big)\mathrm{d}z\end{aligned} \qquad (2.117)$$

The flow Q as a function of h is:

$$Q = \int_0^h \mathrm{d}Q = 2\sqrt{2g}\int_0^h (h-z)^{1/2}\big(f(z)\big)\mathrm{d}z \qquad (2.118)$$

Let us write the fractional integral of the order $\frac{3}{2}$ by the Riemann-Liouville formula, that is:

$$\begin{aligned}{}_0D_x^{-3/2}\big[f(x)\big] &= \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^x (x-y)^{\left(\frac{3}{2}\right)-1}\big(f(y)\big)\mathrm{d}y \\ &= \frac{1}{\Gamma\left(1+\frac{1}{2}\right)}\int_0^x (x-y)^{1/2}\big(f(y)\big)\mathrm{d}y \\ &= \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-y)^{1/2}\big(f(y)\big)\mathrm{d}y \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \\ &= \frac{2}{\sqrt{\pi}}\int_0^x (x-y)^{1/2}\big(f(y)\big)\mathrm{d}y\end{aligned} \qquad (2.119)$$

Comparing the obtained expression for Q in (2.118) i.e. $Q(h) = 2\sqrt{2g}\int_0^h (h-z)^{1/2}\big(f(z)\big)\mathrm{d}z$ with the expression of ${}_0D_x^{-3/2}\big[f(x)\big] = \frac{2}{\pi}\int_0^x (x-y)^{1/2}\big(f(y)\big)\mathrm{d}y$, in (2.119) we conclude the following integral (rather than the fractional integral) equation relating to the flow rate $Q(z)$ and the notch-profile $f(z)$:

$$Q(h) = \sqrt{2g\pi}\,{}_0D_h^{-3/2}\big[f(h)\big] \qquad Q(z) = \sqrt{2g\pi}\,{}_0D_z^{-3/2}\big[f(z)\big] \qquad (2.120)$$

For example if the notch profile is a linear function say $y = f(z) = K_1 z$, then flow rate as a function of z is $Q(z) = k_1 z^{5/2}$. If the notch profile is curved as in $y = k_2\sqrt{z}$, the flow rate is $Q(z) = k_2 z^2$. Thus, we have used the Riemann-Liouville fractional integration formula to relate the flow rate to the shape of the notch.

2.13 Introducing an initialising process of fractional integration

Consider a function $f(x)$ such as it starts from $x = a$ that is $f(x) = 0$ for points $x \le a$. Consider two processes of fractional integration of $f(x)$, the first is in the interval $\big[a, x\big]$ and the second is in the interval $\big[c, x\big]$ such that $c > a$. These two integrals are represented as follows:

$$\begin{aligned}{}_aD_x^{-\alpha}\big[f(x)\big] &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\big(f(y)\big)\mathrm{d}y \\ {}_cD_x^{-\alpha}\big[f(x)\big] &= \frac{1}{\Gamma(\alpha)}\int_c^x (x-y)^{\alpha-1}\big(f(y)\big)\mathrm{d}y\end{aligned} \qquad (2.121)$$

In order to have the continuation of the integration process from $c > a$ to $x > c > a$, we have to add e.g. ψ to the integration process ${}_cD_x^{-\alpha}$ such that it equals the process ${}_aD_x^{-\alpha}$. Therefore, for $x > c > a$, we write the following:

$${}_cD_x^{-\alpha}\big[f(x)\big] + \psi = {}_aD_x^{-\alpha}\big[f(x)\big] \qquad (2.122)$$

Thus, we have ψ as follows:

$$\begin{aligned}\psi &= {}_aD_x^{-\alpha}\left[f(x)\right] - {}_cD_x^{-\alpha}\left[f(x)\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_a^c (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y\end{aligned} \tag{2.123}$$

What we obtained, ψ as $\psi(x)$, is a function of the independent variable x, and not a constant of integration. Here ψ is not a 'psi' function, as we have defined and used earlier (Section 1.16). This symbol ψ was used by Riemann, and thus we are retaining this symbol for the initialisation function.

Therefore, unlike the addition of the constant of integration in the ordinary process of integer order integration, here in the case of the fractional order of integration one has to use the initialisation function. For $\alpha = 1$, we have ψ as:

$$\begin{aligned}\psi &= \frac{1}{\Gamma(\alpha)}\int_a^c (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y = \frac{1}{\Gamma(1)}\int_a^c (x-y)^{1-1}\left(f(y)\right)\mathrm{d}y \\ &= \int_a^c \left(f(y)\right)\mathrm{d}y = K\end{aligned} \tag{2.124}$$

a constant of integration in the case of an integer order for which the whole integration is recovered.

Let us take an example with the function $f(x) = x$ with $f(x) = 0$ for $x < 0$. The semi-integration of this function from $x = 0$ to $x = 1$ is, with the substitution of $(x - y) = z$ following on in order to get the initialisation function:

$$\begin{aligned}\psi(x) = {}_0D_1^{-1/2}\left[x\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (x-y)^{\left(\frac{1}{2}\right)-1} y\mathrm{d}y \\ &= \frac{1}{\sqrt{\pi}}\int_x^{x-1} \frac{(x-z)(-\mathrm{d}z)}{z^{1/2}} \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{2}{3}z^{3/2} - 2xz^{1/2}\right]_x^{x-1} \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{2}{3}(x-1)^{3/2} - 2x(x-1)^{1/2} - \frac{2}{3}x^{3/2} + 2xx^{1/2}\right] \\ &= \frac{2}{3\sqrt{\pi}}\left[\frac{2}{3}x^{3/2} - (2x+1)(x-1)^{1/2}\right]\end{aligned} \tag{2.125}$$

The semi-integration of this function from $x = 0$ to x is, with the substitution of $(x - y) = z$ is the following:

$$\begin{aligned}{}_0D_x^{-1/2}\left[x\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-y)^{\left(\frac{1}{2}\right)-1} y\mathrm{d}y \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_x^0 \frac{(x-z)(-\mathrm{d}z)}{x^{1/2}} \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{2}{3}z^{3/2} - 2xz^{1/2}\right]_x^0 = \frac{4}{3\sqrt{\pi}}x^{3/2}\end{aligned} \tag{2.126}$$

Now we do the same semi-integration of $f(x) = x$ from $x = 1$, as below:

$$\begin{aligned}{}_1D_x^{-1/2}\left[x\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^x (x-y)^{\left(\frac{1}{2}\right)-1} y\mathrm{d}y \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{x-1}^0 \frac{(x-z)(-\mathrm{d}x)}{x^{1/2}} \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{2}{3}z^{3/2} - 2xz^{1/2}\right]_{(x-1)}^0 \\ &= \frac{2}{3\sqrt{\pi}}\left[(x-1)^{1/2}(2x+1)\right]\end{aligned} \tag{2.127}$$

Thus, we get the initialisation function as follows:

$$\begin{aligned}\psi(x) &= {}_0D_x^{-1/2}\left[x\right] - {}_1D_x^{-1/2}\left[x\right]\\ &= \frac{2}{3\sqrt{\pi}}\left[2x^{3/2} - (2x+1)\sqrt{(x-1)}\right]\end{aligned} \tag{2.128}$$

We have obtained the same initialisation function in the beginning of this section. We stop here and we will take up this discussion further in Chapter 5, where we will use this to state that fractional derivatives too require an initialisation function.

2.14 Can we change the sign of fractional order in a fractional integration formula to write a formula for the fractional derivative?

In the Riemann-Liouville formula for fractional integration that we reproduce as ${}_0D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y$, we try to replace $-\alpha$ with $+\alpha$, and write the following:

$${}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y \tag{2.129}$$

The, $\Gamma(-\alpha)$ is a finite for a non-integer with negative value. With the $x-y=u$ substitution, the integral in (2.129) becomes:

$${}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(-\alpha)}\int_0^x u^{-\alpha-1}\left(f(x-u)\right)\mathrm{d}u \tag{2.130}$$

2.14.1 Divergence of the Riemann-Liouville integral for a change of sign of order of integration

The integral $\int_0^x u^{-\alpha-1}\left(f(x-u)\right)\mathrm{d}u$ will diverge near $u=0$ no matter how smooth the function f is. For generality's sake take the upper limit of integration and make it ∞. Take a small number $\in$ so the integral is approximated near to the point $u=0$ by writing $f(x-u)\cong f(x)$ in the following way:

$$\begin{aligned}\int_0^\infty u^{-\alpha-1}\left(f(x-u)\right)\mathrm{d}u &\cong \int_\in^\infty u^{-\alpha-1}\left(f(x)\right)\mathrm{d}u\\ &= \left[-\frac{f(x)}{\alpha}u^{-\alpha}\right]_\in^\infty = \frac{f(x)}{\alpha\in^\alpha}\end{aligned} \tag{2.131}$$

As $\in\downarrow 0$ the above expression (2.131) diverges, thus making it uncomfortable with a change of sign of the order i.e. α.

In equation (2.129), ${}_aD_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(-\alpha)}\int_a^x (x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y$ for $\alpha<0$, in the Riemann-Liouville fractional integration formula worked on by Reisz in 1949, α was regarded as a complex variable; this integral converges for $\mathrm{Re}\left[\alpha\right]<0$ and defines for a fixed $f(x)$ an analytic function of α on the left half α-plane.

2.14.2 Using Reisz's logic of analytic continuation to geta meaningful representation of the Riemann-Liouville fractional integration formula with a changed sign of the order of integration

Although the integral i.e. RL formula diverges for $\mathrm{Re}\left[\alpha\right]\ge 0$ (2.131) a meaning may be attached to the RL operator ${}_aD_x^{\alpha}$ defined above in (1.129), by proper analytic continuation across the line $\mathrm{Re}\left[\alpha\right]=0$ provided the function $f(x)$ is sufficiently differentiable in $a\le y\le x$. In fact if the function is n times differentiable the formula

$${}_aD_x^{\alpha}\left[f(x)\right] = \sum_{k=0}^{n-1}\frac{(x-a)^{-\alpha+k}f^{(k)}(a)}{\Gamma(-\alpha+k+1)} + {}_aD_x^{\alpha-n}\left[f^{(n)}(x)\right] \tag{2.132}$$

defines an analytic function of α for $\mathrm{Re}\left[\alpha\right]<n$ and thus provides a valid analytic continuation. This above formula (2.132) will be clear shortly by following explanations and steps.

For say $\alpha < 1$, considering α as a real number, then we write the above (2.132) expression as:

$$ {}_aD_x^{\alpha}\left[f(x)\right] = \frac{(x-a)^{-\alpha} f(a)}{\Gamma(-\alpha+1)} + {}_aD_x^{\alpha-1}\left[f^{(1)}(x)\right] \tag{2.133}$$

Well, we know that if $\alpha < 0$, then ${}_aD_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(-\alpha)}\int_a^x (x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y$ is valid. For say $\alpha > 0$ but $\alpha < 1$, we have $(\alpha-1) < 0$, then $\frac{1}{\Gamma(-(\alpha-1))}\int_a^x (x-y)^{-(\alpha-1)-1}\left(f(y)\right)\mathrm{d}y = {}_aD_x^{\alpha-1} f(x)$ is also valid; we recognise this $(1-\alpha)$ as an order of fractional integration. We write the following steps carefully by applying integration by parts i.e. $\int (u)(v)\mathrm{d}x = (u)\int (v)\mathrm{d}x - \int\left(\int (u)\mathrm{d}x\right)v^{(1)}\mathrm{d}x$:

$$\begin{aligned}
{}_aD_x^{\alpha-1} f(x) &= \frac{1}{\Gamma\left(-(\alpha-1)\right)}\int_a^x (x-y)^{-(\alpha-1)-1}\left(f(y)\right)\mathrm{d}y \\
&= \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-y)^{-\alpha}\left(f(y)\right)\mathrm{d}y \\
&= \frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} f(y)\int_a^x (x-y)^{-\alpha}\,\mathrm{d}y \\ -\int_a^x\left(\int (x-y)^{-\alpha}\,\mathrm{d}y\right)\left(f^{(1)}(y)\right)\end{pmatrix}\mathrm{d}y \\
&= \frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} -f(y)\left.\dfrac{(x-y)^{-\alpha+1}}{(-\alpha+1)}\right|_{y=a}^{y=x} \\ +\int_a^x\left(\dfrac{(x-y)^{-\alpha+1}}{(-\alpha+1)}\right)\left(f^{(1)}(y)\right)\mathrm{d}y\end{pmatrix} \\
&= \frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} \dfrac{f(a)}{(-\alpha+1)}(x-a)^{-\alpha+1} \\ +\int_a^x\left(\dfrac{(x-y)^{-\alpha+1}}{(-\alpha+1)}\right)\left(f^{(1)}(y)\right)\mathrm{d}y\end{pmatrix}
\end{aligned} \tag{2.134}$$

Now we do one whole differentiation of (2.134) with respect to x, i.e. we operate $D_x^1 \equiv \frac{\mathrm{d}}{\mathrm{d}x}$ and write the following steps:

$$\begin{aligned}
D_x^1\left({}_aD_x^{\alpha-1} f(x)\right) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \dfrac{f(a)}{(-\alpha+1)}(x-a)^{-\alpha+1} \\ +\int_a^x\left(\dfrac{(x-y)^{-\alpha+1}}{(-\alpha+1)}\right)\left(f^{(1)}(y)\right)\mathrm{d}y\end{pmatrix} \\
&= \frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} \left(\dfrac{f(a)}{(-\alpha+1)}\right)\left((-\alpha+1)(x-a)^{-\alpha}\right) \\ +\dfrac{\mathrm{d}}{\mathrm{d}x}\int_a^x\left(\dfrac{(x-y)^{-\alpha+1}}{(-\alpha+1)}\right)\left(f^{(1)}(y)\right)\mathrm{d}y\end{pmatrix} \\
&= \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} \\
&\quad + \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{1}{(-\alpha+1)}\frac{\mathrm{d}(x-y)^{-\alpha+1}}{\mathrm{d}x}\left(f^{(1)}(y)\right)\mathrm{d}y \\
&= \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-y)^{-\alpha}\left(f^{(1)}(y)\right)\mathrm{d}y
\end{aligned} \tag{2.135}$$

We recognise the second term of the last line in the above steps (2.135) as the fractional integration of order $(1-\alpha)$ for function $f^{(1)}(x)$ that is:

$$ {}_aD_x^{-(1-\alpha)}\left[f^{(1)}(x)\right]=\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-y)^{-\alpha}\left(f^{(1)}(y)\right)\mathrm{d}y,\quad 0<\alpha<1 \tag{2.136}$$

This RL formula (2.136) is valid as $(\alpha-1)<0$. In this way we get a clue, that if $\alpha>0$ but is less than one, i.e. $\alpha<1$, we can write the following:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= D_x^1\left({}_aD_x^{\alpha-1}f(x)\right)\\ &=\frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}+\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-y)^{-\alpha}\left(f^{(1)}(y)\right)\mathrm{d}y \\ &=\frac{(x-a)^{-\alpha}f(a)}{\Gamma(1-\alpha)}+{}_aD_x^{\alpha-1}\left[f^{(1)}(x)\right]\qquad \alpha>0\qquad \alpha<1\end{aligned} \tag{2.137}$$

For $\alpha>0$ and $\alpha<2$, by repeating the above procedures i.e. (2.135) and (2.136), we will get the following expression:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= D_x^2\left({}_aD_x^{\alpha-2}f(x)\right)\\ &=\frac{(x-a)^{-\alpha}f(a)}{\Gamma(-\alpha+1)}+\frac{(x-a)^{-\alpha+1}f^{(1)}(a)}{\Gamma(-\alpha+2)}+{}_aD_x^{\alpha-2}\left[f^{(2)}(x)\right]\end{aligned} \tag{2.138}$$

Continuing this for $\alpha<n$ we get the following expression:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right)\\ &=\frac{(x-a)^{-\alpha}f(a)}{\Gamma(-\alpha+1)}+\frac{(x-a)^{-\alpha+1}f^{(1)}(a)}{\Gamma(-\alpha+2)}+\\ &\quad ....+\frac{(x-a)^{-\alpha+(n-1)}f^{(n-1)}(a)}{\Gamma(-\alpha+n)}+{}_aD_x^{\alpha-n}\left[f^{(n)}(x)\right]\\ &=\sum_{k=0}^{n-1}\frac{(x-a)^{-\alpha+k}f^{(k)}(a)}{\Gamma(-\alpha+k+1)}+{}_aD_x^{\alpha-n}\left[f^{(n)}(x)\right]\end{aligned} \tag{2.139}$$

This analytic continuation is very analogous to the way one extends the definition of the gamma function on the complex plane (Section 1.10). This is not surprising since the gamma function plays such a fundamental role in defining the fractional integrals (and the derivatives too). We shall make frequent use of this analyticity in α to simplify the discussions on the properties of ${}_aD_x^{\alpha}$, the defined RL operator for general order α, a real number ($\mathbb{R}$), positive as well as negative.

2.14.3 Attaching meaning to the Riemann-Liouville fractional integration formula for a positive order elementary approach

We prefer to take a different and more elementary approach attaching a meaning to the RL operator for $\alpha\geq 0$. The formula ${}_aD_x^{\alpha}\left[f(x)\right]=\frac{1}{\Gamma(-\alpha)}\int_a^x (x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y$ for $\alpha<0$ will be retained only for $\alpha<0$, indicating fractional integration.

It is extended to $\alpha\geq 0$, by insisting that ${}_aD_x^{n}\left({}_aD_x^{\alpha}\left[f(x)\right]\right)={}_aD_x^{n+\alpha}\left[f(x)\right]$ is satisfied for all positive integers n and that all α also be satisfied by the RL integral. That is, we shall require that:

$$ {}_aD_x^{\alpha}\left[f(x)\right]\equiv D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right) \tag{2.140}$$

in (2.140) where $D_x^n\equiv\frac{\mathrm{d}^n}{\mathrm{d}x^n}$ is n-th derivative (ordinary whole derivative operator). Choosing the positive integer i.e. n so large that we have $\alpha-n<0$; the relationship ${}_aD_x^{\alpha}\left[f(x)\right]\equiv D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right)$ is for all α. We see that if we choose α equal to a negative integer say $-n$, and then we have ${}_aD_x^{-n}\left[f(x)\right]=\frac{1}{\Gamma(n)}\int_a^x (x-y)^{n-1}\left(f(y)\right)\mathrm{d}y$, this generates n-fold integration. If we chose $n=1$ and $\alpha=0$, we obtain the following:

$$ {}_aD_x^0\left[f(x)\right] = D_x^1\left({}_aD_x^{-1}\left[f(x)\right]\right) = f(x) \tag{2.141}$$

Courant and Hilbert in 1962 define semi-derivatives by use of the following expression:

$$\frac{\mathrm{d}^{\frac{1}{2}}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(y)\mathrm{d}y}{\sqrt{x-y}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{-\frac{1}{2}}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-\frac{1}{2}}}\right) \tag{2.142}$$

This (2.142) is the same as we described for ${}_aD_x^{\alpha}\left[f(x)\right] \equiv D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right)$ i.e. the RL definition (2.140). A careful integration by part for (2.142) shows the following:

$$\begin{aligned}\frac{\mathrm{d}^{\frac{1}{2}}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\frac{1}{2}}} &= \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(y)\mathrm{d}y}{\sqrt{x-y}} \\ &= \frac{1}{\sqrt{\pi}}\frac{f(a)}{\sqrt{x-a}} + \frac{1}{\sqrt{\pi}}\int_a^x \frac{f^{(1)}(y)\mathrm{d}y}{\sqrt{x-y}} \\ &= \frac{f(a)}{\Gamma\left(\frac{1}{2}\right)\sqrt{x-a}} + \left(\frac{\mathrm{d}^{-\frac{1}{2}}\left[f^{(1)}(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-\frac{1}{2}}}\right)\end{aligned} \tag{2.143}$$

Equation (2.143) agrees with the earlier expression (2.139) to give

$$ {}_aD_x^{\alpha}\left[f(x)\right] = \sum_{k=0}^{n-1}\frac{(x-a)^{-\alpha+k}f^{(k)}(a)}{\Gamma(-\alpha+k+1)} + {}_aD_x^{\alpha-n}\left[f^{(n)}(x)\right] \tag{2.144}$$

as it must, for $\alpha = \frac{1}{2}$ and $n = 1$. The semi-derivative defined in this way requires the knowledge of $f^{(1)}(x)$. This we will discuss in Chapter 3.

2.15 Fractional integration of the Weyl type

We have developed the concept of the Riemann-Liouville fractional integration for a function $f:\left[a,b\right]\to\mathbb{R}$, $a < x < b$ where we are doing integration from a to x, from the left side and proceeding towards the right. That is also called a left integration, if the integration axis is time we say that we are evolving the integration from past to present, also called a forward integration. This is a causal way of thinking here in that the past states contribute to the present states. When $a = -\infty$, we have the following representation:

$$ {}_{-\infty}I_x^{\alpha}\left[f(x)\right] = {}_{-\infty}D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \tag{2.145}$$

2.15.1 Weyl transform of the fractional integration of a function on future points to infinity

We have backward integration that is from the present point to all future points, that is from x to ∞ of a function. This fractional integration is defined as the Weyl (1917) fractional integration as follows for $\mathrm{Re}\left[\alpha\right] > 0$:

$$ {}_xI_{\infty}^{\alpha}\left[f(x)\right] = {}_xW_{\infty}^{-\alpha}\left[f(x)\right] \overset{\text{def}}{=} \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(y-x)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \tag{2.146}$$

The above description is also called the Weyl transform.

The RL fractional integration of order $\alpha > 0$ is ${}_{-\infty}D_x^{-\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-u)^{\alpha-1}\left(f(u)\right)\mathrm{d}u$.

Here we put $u = -v$ to get the following:

$$\begin{aligned}{}_{-\infty}D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_{+\infty}^{-x}(x+v)^{\alpha-1}\left(f(-v)\right)(-\mathrm{d}v) \\ &= (-1)\frac{1}{\Gamma(\alpha)}\int_{\infty}^{-x}(x+v)^{\alpha-1}\left(f(-v)\right)\mathrm{d}v\end{aligned} \tag{2.147}$$

Now we put in the above $x = -y$ and write the following:

$$
\begin{aligned}
{}_{-\infty}D_{-y}^{-\alpha}\left[f(-y)\right] &= \frac{(-1)}{\Gamma(\alpha)}\int_{\infty}^{y}(v-y)^{\alpha-1}\left(f(-v)\right)\mathrm{d}v \\
&= \frac{1}{\Gamma(\alpha)}\int_{y}^{\infty}(v-y)^{\alpha-1}f(-v)\mathrm{d}v
\end{aligned}
\tag{2.148}
$$

Call $f(-y)=g(y)$, then the above formula (2.148) with an obvious change in notation becomes Weyl fractional integration. The RHS as described in the abovementioned definition (2.148) is a Weyl transform.

$$
\begin{aligned}
{}_{-\infty}D_{-y}^{-\alpha}\left[g(y)\right] &= \frac{1}{\Gamma(\alpha)}\int_{y}^{\infty}(v-y)^{\alpha-1}\left(g(v)\right)\mathrm{d}v \\
{}_{y}W_{\infty}^{-\alpha}\left[g(y)\right] &= \frac{1}{\Gamma(\alpha)}\int_{y}^{\infty}(v-y)^{\alpha-1}\left(g(v)\right)\mathrm{d}v
\end{aligned}
\tag{2.149}
$$

The above expression (2.149) of ${}_{y}W_{\infty}^{-\alpha}\left[g(y)\right]$ is a Weyl fractional integration of the right sense. This is a non-causal type of integration, as it requires the point ahead (or future points), whereas the classical RL integration is of a causal type, which is evaluated on the past points.

Put in the Weyl fractional integration formula (2.149) $v-y=z$, so $\mathrm{d}v=\mathrm{d}z$; the limit of integration $v=y$ to $v=\infty$ will now be $z=0$ to $z=\infty$, and we thus get:

$$
\begin{aligned}
{}_{y}W_{\infty}^{-\alpha}\left[g(y)\right] &= \frac{1}{\Gamma(\alpha)}\int_{y}^{\infty}(v-y)^{\alpha-1}\left(g(v)\right)\mathrm{d}v \\
&= \frac{1}{\Gamma(\alpha)}\int_{0}^{\infty}z^{\alpha-1}\left(g(z+y)\right)\mathrm{d}z
\end{aligned}
\tag{2.150}
$$

The general Weyl integration can be seen in the following steps with substitution $y=x(z+1)$ and $\mathrm{d}y=x\mathrm{d}z$

$$
\begin{aligned}
{}_{x}W_{c}^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_{x}^{c}(y-x)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \\
{}_{x}W_{c}^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(\alpha)}\int_{0}^{(c-x)/x}(xz)^{\alpha-1}\left(f(xz+x)\right)\left(x\mathrm{d}z\right) \\
&= \frac{x^{\alpha}}{\Gamma(\alpha)}\int_{0}^{(c-x)/x}z^{\alpha-1}\left(f(x+xz)\right)\mathrm{d}z
\end{aligned}
\tag{2.151}
$$

With $c=\infty$, we get:

$$
{}_{x}W_{\infty}^{-\alpha}\left[f(x)\right] = \frac{x^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}z^{\alpha-1}\left(f(x+xz)\right)\mathrm{d}z \tag{2.152}
$$

2.15.2 Weyl fractional integration of the power function

We take $f(x)=x^{-p}$, with $p>0$, and find the Weyl fractional integral as follows in these steps:

$$
\begin{aligned}
{}_{x}W_{\infty}^{-\alpha}\left[x^{-p}\right] &= \frac{x^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}z^{\alpha-1}(x+xz)^{-p}\mathrm{d}z \\
&= \frac{x^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}z^{\alpha-1}x^{-p}(1+z)^{-p}\mathrm{d}z \\
&= \frac{x^{-p+\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}z^{\alpha-1}(1+z)^{-p}\mathrm{d}z
\end{aligned}
\tag{2.153}
$$

We recognise that $\int_{0}^{\infty}z^{\alpha-1}(1+z)^{-p}\mathrm{d}z=\mathrm{B}(\alpha,p-\alpha)=\frac{\Gamma(\alpha)\Gamma(p-\alpha)}{\Gamma(p)}$, we derived this in an earlier chapter describing the beta integral (Section 1.15). By using this formula for the beta function, we get the following:

$$
\begin{aligned}
{}_xW_\infty^{-\alpha}\left[x^{-p}\right] &= \frac{x^{-p+\alpha}}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\Gamma(p-\alpha)}{\Gamma(p)}\right) \\
&= \frac{\Gamma(p-\alpha)}{\Gamma(p)}x^{-p+\alpha}
\end{aligned}
\tag{2.154}
$$

Now we put $\alpha = 1$, so we get:

$$
\begin{aligned}
{}_xW_\infty^{-1}\left[x^{-p}\right] &= \frac{\Gamma(p-1)}{\Gamma(p)}x^{-p+1} = \frac{\Gamma(p-1)}{(p-1)\Gamma(p-1)}x^{-p+1} \\
&= \frac{1}{p-1}x^{-p+1}
\end{aligned}
\tag{2.155}
$$

The above expression (2.155) is a classical one-whole integration, i.e.:

$$
\begin{aligned}
{}_xW_\infty^{-1}\left[x^{-p}\right] &= \int_x^\infty y^{-p}\mathrm{d}y = \left.\frac{y^{-p+1}}{(-p+1)}\right|_{y=x}^{y=\infty} = -\frac{x^{-p+1}}{(-p+1)}; \qquad p>0 \\
&= \frac{1}{p-1}x^{-p+1}
\end{aligned}
\tag{2.156}
$$

2.15.3 Weyl fractional integration of the exponential function

We apply this Weyl integration to find the fractional integration of e^{-px}

$$
{}_xW_\infty^{-\alpha}\left[e^{-px}\right] = \frac{1}{\Gamma(\alpha)}\int_x^\infty (u-x)^{\alpha-1}e^{-pu}\mathrm{d}u \tag{2.157}
$$

Substituting $(u-x) = \frac{y}{p}$, $\mathrm{d}u = \frac{\mathrm{d}y}{p}$, we write:

$$
\begin{aligned}
{}_xW_\infty^{-\alpha}\left[e^{-px}\right] &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{y^{\alpha-1}}{p^{\alpha-1}}e^{-(y+px)}\frac{\mathrm{d}y}{p} \\
&= \frac{e^{-px}}{p^\alpha\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}\mathrm{d}y \\
&= \frac{1}{p^\alpha}e^{-px}
\end{aligned}
\tag{2.158}
$$

We have used the definition of the gamma function, $\Gamma(\alpha) = \int_0^\infty e^{-y}y^{\alpha-1}\mathrm{d}y$, in the above derivation (2.158). Now we do one-whole integration of e^{-px} in Weyl's way, giving the following:

$$
{}_xW_\infty^{-1}e^{-px} = \int_x^\infty e^{-py}\mathrm{d}y = \left.\frac{e^{-py}}{(-p)}\right|_x^\infty = \frac{e^{-px}}{p} \tag{2.159}
$$

We note that the Weyl fractional integration of exponential function (2.158) returns an exponential function like that in the case of classical integration. Whereas we observed that the Riemann-Liouville fractional integration returns higher transcendental functions on fractionally integrating exponential functions as derived in (2.71), (2.73), (2.80), and (2.102).

2.16 Applications using the Weyl fractional integration

With reference to Figure 2.5, we need to find the potential function $\varphi(r)$ for the two perpendicular long wires with negative charges, such that the force on a test charge Q^+ kept at $x = L$ from the wire CD is twice the force from the wire AB; the condition is thus $F_{\mathrm{CD}} = 2F_{\mathrm{AB}}$.

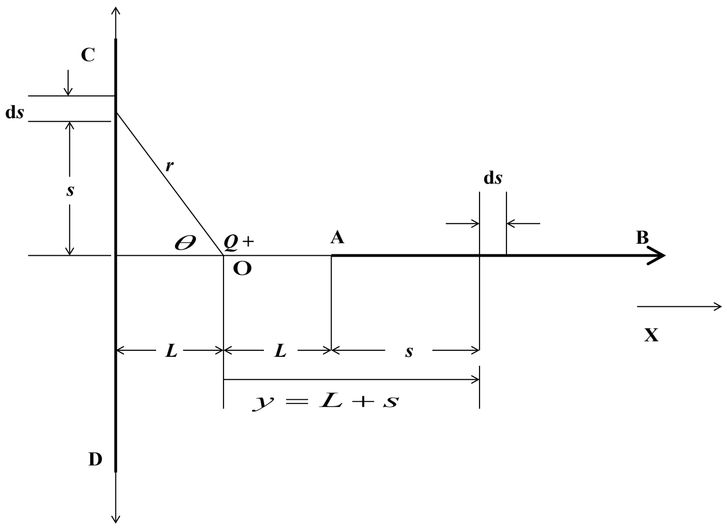

Figure 2.5: Two wires with negative charges attracting a test charge with force

Now take an infinitesimal small segment on AB i.e. $\mathrm{d}s$. The infinitesimal force due to this element on the test charge Q is $\mathrm{d}F_{\mathrm{AB}} = \big(\varphi(L+s)\big)\mathrm{d}s$, thus F_{AB}, which is directed in X-direction is:

$$F_{\mathrm{AB}} = \int_0^\infty \big(\varphi(L+s)\big)\mathrm{d}s \tag{2.160}$$

From Figure 2.5, we have $y = L+s$, thus $\mathrm{d}y = \mathrm{d}s$, and $s=0$ implies $y = L$. With these changes in variables we have the following expression:

$$F_{\mathrm{AB}} = \int_L^\infty \big(\varphi(y)\big)\mathrm{d}y \tag{2.161}$$

Similarly an element in segment CD has a force directed in r direction meaning $\mathrm{d}F'_{\mathrm{CD}} = \big(\varphi(r)\big)\mathrm{d}s$, the X-component of this force is $\mathrm{d}F_{\mathrm{CD}} = \varphi(r)\big(\cos\theta\big)(\mathrm{d}s)$. The total force due to wire CD on the test charge in X-direction is therefore as follows:

$$F_{\mathrm{CD}} = \int_{-\infty}^{\infty} \varphi(r)\big(\cos\theta\big)\mathrm{d}s \tag{2.162}$$

From the geometry of Figure 2.5, we write $\cos\theta = \frac{L}{r}$; therefore (2.162) should be changed to the following:

$$F_{\mathrm{CD}} = L\int_{-\infty}^{\infty} \frac{\varphi(r)}{r}\mathrm{d}s = 2L\int_0^\infty \frac{\varphi(r)}{r}\mathrm{d}s \tag{2.163}$$

Again from the geometry of Figure 2.5 we have $r^2 = s^2 + L^2$ therefore $\mathrm{d}s = \left(\frac{r}{s}\right)\mathrm{d}r$, this substitution gives the limit of integration of F_{CD} in the variable r, from L to ∞ for the integral describing F_{CD}, i.e. $2L\int_0^\infty \left(\frac{\varphi(r)}{r}\right)\mathrm{d}s$ as in the above steps (2.163), when the limit of integration of s is from 0 to ∞. Therefore, we get the following expression:

$$F_{\mathrm{CD}} = 2L\int_L^\infty \frac{\varphi(r)}{\sqrt{r^2 - L^2}}\mathrm{d}r \tag{2.164}$$

In the above expression (2.164) we've now put $r^2 = k$, so $2r\mathrm{d}r = \mathrm{d}k$, or $\mathrm{d}r = \frac{1}{2\sqrt{k}}\mathrm{d}k$; and $L^2 = x$, and written the expression of integration describing F_{CD} with these new substitutions to get the following steps:

$$
\begin{aligned}
F_{\mathrm{CD}} &= 2\sqrt{x}\int_{\sqrt{x}}^{\infty}\frac{\left(\varphi(\sqrt{k})\right)}{\sqrt{k-x}}\left(\frac{1}{2\sqrt{k}}\right)\mathrm{d}k \\
&= \sqrt{x}\int_{\sqrt{x}}^{\infty}\frac{\left(\varphi(\sqrt{k})\right)}{\sqrt{k}}(k-x)^{-1/2}\,\mathrm{d}k \qquad (2.165) \\
&= \sqrt{x}\int_{\sqrt{x}}^{\infty}\left(g\sqrt{k}\right)\left(k-x\right)^{-1/2}\,\mathrm{d}k
\end{aligned}
$$

In (2.165) we call $\frac{1}{\sqrt{k}}\left(\varphi(\sqrt{k})\right)=\left(g(\sqrt{k})\right)=f(k)$, this is a function of k and we do the integration from $k=x$ to $k=\infty$ instead of from $\sqrt{x}$ to ∞ and obtain the following steps:

$$
\begin{aligned}
F_{\mathrm{CD}} &= \sqrt{x}\int_{x}^{\infty}\left(f(k)\right)(k-x)^{-1/2}\mathrm{d}k \\
&= \sqrt{x}\left(\Gamma\left(\tfrac{1}{2}\right)\right)\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{x}^{\infty}\left(f(k)\right)(k-x)^{-1/2}\,\mathrm{d}k\right] \qquad (2.166) \\
&= \sqrt{x}\sqrt{\pi}\left({}_{x}W_{\infty}^{-1/2}\left[f(x)\right]\right)
\end{aligned}
$$

In the above derivation (2.166), we used the Weyl fractional integration definition (given in 2.149), with $\alpha=\frac{1}{2}$, and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ to give:

$$
{}_{x}W_{\infty}^{-1/2}\left[f(x)\right]=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{x}^{\infty}(k-x)^{\left(\frac{1}{2}\right)-1}\left(f(k)\right)\mathrm{d}k \qquad (2.167)
$$

With the substitution $f(k)=\frac{1}{\sqrt{k}}\left(\varphi(\sqrt{k})\right)$, or $\varphi(y)=k\left(f(k^2)\right)$ we write the obtained force from AB, which is $F_{\mathrm{AB}}=\int_{L}^{\infty}\left(\varphi(y)\right)\mathrm{d}y$, as:

$$
F_{\mathrm{AB}}=\int_{\sqrt{x}}^{\infty}k\left(f(k^2)\right)\mathrm{d}k \qquad (2.168)
$$

Put in the above expression (2.168), $\xi=k^2$ that gives the following expression:

$$
F_{\mathrm{AB}}=\frac{1}{2}\int_{x}^{\infty}f(\xi)\mathrm{d}\xi=\frac{1}{2}\left({}_{x}W_{\infty}^{-1}\left[f(x)\right]\right) \qquad (2.169)
$$

Our condition is $F_{\mathrm{CD}}=2F_{\mathrm{AB}}$, and that gives the following integral equation:

$$
\sqrt{\pi x}\left({}_{x}W_{\infty}^{-1/2}\left[f(x)\right]\right)={}_{x}W_{\infty}^{-1}\left[f(x)\right] \qquad (2.170)
$$

Now physically we know that an attractive force between charge Q and the AB and CD wires decreases as r increases. Therefore, we can write $f(x)$ in the following form:

$$
\begin{aligned}
f(x) &= \frac{a_1}{x^{1+q}}+\frac{a_2}{x^{2+q}}+\frac{a_3}{x^{3+q}}+\ldots \qquad q>0 \\
&= \sum_{n=1}^{\infty}a_n x^{-(n+q)}
\end{aligned} \qquad (2.171)
$$

Use ${}_{x}W_{\infty}^{-\alpha}\left[x^{-p}\right]=\left(\frac{\Gamma(p-\alpha)}{\Gamma(p)}\right)x^{-p+\alpha}$ from (2.154) which is the Weyl fractional integral of x^{-p}, and substitute the $f(x)$ written in the series form as above in the integral equation, and write the following steps:

$$\sqrt{\pi}(x)^{1/2}\left({}_xW_\infty^{-1/2}\left[f(x)\right]\right) = {}_xW_\infty^{-1}\left[f(x)\right]$$

$$f(x)=\sum_{n=1}^{\infty}a_n x^{-(n+q)} \tag{2.172}$$

$$\sqrt{\pi}(x)^{1/2}\sum_{n=1}^{\infty}a_n\frac{\Gamma\left(n+q-\frac{1}{2}\right)}{\Gamma(n+q)}\frac{1}{x^{n+q-\left(\frac{1}{2}\right)}}=\sum_{n=1}^{\infty}\frac{1}{n+q-1}\frac{1}{x^{n+q-1}}$$

Equating the like powers on both sides of (2.172) and along with the simplification by using $x\left(\Gamma(x)\right)=\Gamma(x+1)$ leads to the following steps:

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+q-\frac{1}{2}\right)}{\Gamma(n+q)}=\frac{1}{n+q-1}$$
$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+q-\frac{1}{2}\right)}{(n+q-1)\Gamma(n+q-1)}=\frac{1}{n+q-1} \tag{2.173}$$
$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(n+q-\tfrac{1}{2}\right)=\Gamma(n+q-1)$$

We get the above equality if $n+q=\frac{3}{2}$, that is $\Gamma\left(\frac{1}{2}\right)\Gamma(1)=\Gamma\left(\frac{1}{2}\right)$. Therefore, we write:

$$f(x)=\frac{a}{x^{3/2}}=\frac{\left(\varphi(\sqrt{x})\right)}{\sqrt{x}} \tag{2.174}$$

We get the potential function as follows the expression, i.e. an inverse square law:

$$\left(\varphi(\sqrt{x})\right)=\frac{a}{x} \qquad \varphi(r)=\frac{a}{r^2} \tag{2.175}$$

This shows an application of the Weyl fractional integration formula.

2.17 Use of the kernel function in a convolution integral in the fractional integration formula

2.17.1 Defining kernel functions for forward and backward fractional integration

From the Weyl representation, we write the forward and backward fractional order integration formula as follows:

$$\begin{aligned}{}_{-\infty}I_x^{\alpha}\left[f(x)\right]&=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y\\ {}_xI_\infty^{\alpha}\left[f(x)\right]&=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(y-x)^{\alpha-1}\left(f(y)\right)\mathrm{d}y\end{aligned} \tag{2.176}$$

The kernel in the above integration (2.176) is a power function, namely ${}_\pm k_\alpha(x)$ i.e. $\frac{(x)^{\alpha-1}}{\Gamma(\alpha)}$, and $\frac{(-x)^{\alpha-1}}{\Gamma(\alpha)}$ respectively for ${}_+k_\alpha(x)$ and ${}_-k_\alpha(x)$. With this, we can write the left and right integrals as

$$I_\pm^{\alpha}\left[f(x)\right]={}_\pm k_\alpha(x)*f(x)\overset{\text{def}}{=}\int_{-\infty}^{\infty}\left({}_\pm k_\alpha(x-y)\right)\left(f(y)\right)\mathrm{d}y \tag{2.177}$$

The $+$ is for a left fractional integration operator i.e. ${}_{-\infty}I_x^{\alpha}$, and $-$ is for the right fractional integration operator i.e. ${}_xI_\infty^{\alpha}$. The kernel we define as:

$${}_\pm k_\alpha(x)=\frac{(\pm x)^{\alpha-1}}{\Gamma(\alpha)}\left(u(\pm x)\right) \tag{2.178}$$

where $u(x)$ is the Heaviside unit step function defined as $u(x)=1$ for $x>0$ and $u(x)=0$ for $x\le 0$. We have $x^{\alpha}=e^{\alpha\ln x}$ and the conventions that function $\ln x$ are real for $x>0$. For $\alpha=0$, the kernel is the delta function at $x=0$, that is ${}_+k_0(x)={}_-k_0(x)=\delta(x)$. This is the usual kernel called the 'power-law' kernel as described in (2.178).

2.17.2 Additive rule for fractional integrals

We write the following convolution of kernels as:

$$\begin{aligned}\left({}_{+}k_{\alpha}(x)\right)*\left({}_{-}k_{\beta}(x)\right) &= \int_{0}^{x}\left({}_{+}k_{\alpha}(x-y)\right)\left({}_{-}k_{\beta}(y)\right)\mathrm{d}y \\ &= \int_{0}^{x}\frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}\frac{y^{\beta-1}}{\Gamma(\beta)}\mathrm{d}y\end{aligned} \tag{2.179}$$

Putting $\frac{y}{x}=z$ gives us $\mathrm{d}y = x\mathrm{d}z$ and new limits of integration for variable z from 0 to 1. Then, substituting these changes in variables for the above steps (2.179), we get the following:

$$\begin{aligned}\left({}_{+}k_{\alpha}(x)\right)*\left({}_{-}k_{\beta}(x)\right) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{1}x^{\alpha-1}(1-z)^{\alpha-1}(xz)^{\beta-1}(x\mathrm{d}z) \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)}\frac{x^{\beta-1}}{\Gamma(\beta)}\int_{0}^{1}(1-z)^{\alpha-1}z^{\beta-1}x\mathrm{d}z \\ &= \frac{x^{\alpha}x^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{1}(1-z)^{\alpha-1}z^{\beta-1}\mathrm{d}z \\ &= \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{B}(\alpha,\beta)\end{aligned} \tag{2.180}$$

The integral $\int_0^1 (1-x)^{p-1}x^{q-1}\mathrm{d}x = \mathrm{B}(p,q)$ is called the beta function, and $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$; using this property (as in Section 1.15), we get the following:

$$\begin{aligned}\left({}_{+}k_{\alpha}(x)\right)*\left({}_{-}k_{\beta}(x)\right) &= \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &= {}_{+}k_{\alpha+\beta}(x)\end{aligned} \tag{2.181}$$

With this derivation (2.180) and using (2.181) we imply that the semi-group additive law of exponents in a fractional integration operation is

$$\begin{aligned}\left({}_{a}I_{x}^{\alpha}\right)\left({}_{a}I_{x}^{\beta}\right) &= {}_{a}I_{x}^{\alpha+\beta}, \qquad \alpha,\beta>0 \\ \left({}_{a}D_{x}^{-\alpha}\right)\left({}_{a}D_{x}^{-\beta}\right) &= {}_{a}D_{x}^{-(\alpha+\beta)}\end{aligned} \tag{2.182}$$

The above is an index law which holds true for fractional integration. This we will use further and formalise in Chapter-9, for fractional differentiation and integration composition laws.

2.17.3 The choice of a different convolution kernel defines different types of fractional integration operators

We saw that fractional integration is defined via a convolution integral and the choice of kernel defines the type of fractional integration. We write in a generalised form the following convolution integrals:

$$\begin{aligned}{}_{a}\mathcal{I}_{t}^{\alpha}\left[f(t)\right] &= \int_{a}^{t}\left(k_{\alpha}(t-\tau)\right)\left(f(\tau)\right)\mathrm{d}\tau \\ {}_{t}\mathcal{I}_{b}^{\alpha}\left[f(t)\right] &= \int_{t}^{b}\left(k_{\alpha}(\tau-t)\right)\left(f(\tau)\right)\mathrm{d}\tau, \quad \alpha>0\end{aligned} \tag{2.183}$$

Obviously, the kernels are $k_{\alpha}(t-\tau)=\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ for the first one and $k_{\alpha}(\tau-t)=\frac{1}{\Gamma(\alpha)}(\tau-t)^{\alpha-1}$ for the second one, giving us our derived fractional integration formulas. We are deleting the plus and minus signs for the symbol of a kernel as used in previous discussion in (2.178). With the choice of a different kernel, we can have different types of fractional integration formulas.

2.17.3.(a) Riesz fractional integration formula

One example of Riesz fractional integration is defined via the following kernel:

$$k_\alpha(t-\tau) = \frac{1}{\gamma_n(\alpha)} \begin{cases} \left|t-\tau\right|^{\alpha-n} & \text{for} \quad (\alpha-n) \neq 0,2,4,\ldots \\ \left|t-\tau\right|^{\alpha-n} \log\left(\dfrac{1}{\left|t-\tau\right|}\right) & \text{for} \quad (\alpha-n) = 0,2,4,\ldots \end{cases} \tag{2.184}$$

With constants as:

$$\gamma_n(\alpha) = \begin{cases} 2^\alpha \pi^{n/2} \left(\dfrac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{(n-\alpha)}{2}\right)}\right) & \text{for} \quad (\alpha-n) \neq 0,2,4,\ldots \\ (-1)^{(n-\alpha)/2} 2^{\alpha-1} \pi^{n/2} \left(\Gamma\left(1+\frac{(\alpha-n)}{2}\right)\right)\Gamma\left(\frac{\alpha}{2}\right) & \text{for} \quad (\alpha-n) = 0,2,4,\ldots \end{cases} \tag{2.185}$$

which gives a Riesz fractional integral operator defined as ${}^{\mathrm{R}}I_{\mathbb{R}^n}^{\alpha}\left[f(t)\right] = \int_{\mathbb{R}^n}\left(k_\alpha(t-\tau)\right)f(\tau)\mathrm{d}\tau$, for $n=1$, and $(\alpha-n) \neq 0,2,4,\ldots$ we get a Riesz fractional integral that is:

$${}^{\mathrm{R}}I_{\mathbb{R}}^{\alpha}\left[f(t)\right] = \int_{-\infty}^{+\infty} \frac{f(\tau)}{\left|t-\tau\right|^{1-\alpha}}\mathrm{d}\tau \tag{2.186}$$

2.17.3.(b) Further generalisation of the fractional integration formula

Now we further generalise the concept of fractional integration. This comes from the work of Erdelyi (1964) and Osler (1971).

Let function $f(x)$ be defined in the interval $a \le x \le b$ and integrable in the interval $\left[a,b\right]$. With respect to $z(x)$ and with weight $w(x)$, the left/forward causal integration is thus defined as follows:

$$\begin{aligned} &{}_a\mathcal{I}_{x,[z(x),w(x)]}^{\alpha} f(x) \\ &\overset{\text{def}}{=} \frac{1}{\left(w(x)\right)\left(\Gamma(\alpha)\right)} \int_a^x \left(z(x)-z(t)\right)^{\alpha-1} w(t)\left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t \qquad \alpha > 0 \end{aligned} \tag{2.187}$$

With respect to $z(x)$ and with weight $w(x)$, the right/backward non-causal integration is thus defined as:

$${}_x\mathcal{I}_{b,[z(x),w(x)]}^{\alpha} f(x) \overset{\text{def}}{=} \frac{w(x)}{\Gamma(\alpha)} \int_x^b \frac{\left(z(t)-z(x)\right)^{\alpha-1}}{w(t)} \left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t \qquad \alpha > 0 \tag{2.188}$$

Here we assume that the function $z(x)$ is positive and that the monotonically increasing function of x, in the interval $a < x < b$ also has the continuous derivative $z^{(1)}(x)$ in the interval, with $z^{(1)}(x) \neq 0$, and with the weight function $w(x)$ assumed to be sufficiently good.

Choosing $z(t) = t$, $w(t) = 1$ and $z^{(1)}(t) = 1$, the above reduces to a usual Riemann integration (a causal one):

$$\begin{aligned} {}_a\mathcal{I}_{x,[z,w]}^{\alpha} f(x) &= \frac{1}{\left(w(x)\right)\left(\Gamma(\alpha)\right)} \int_a^x \left(z(x)-z(t)\right)^{\alpha-1} w(t)\left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t \\ {}_a\mathcal{I}_{x,[t,1]}^{\alpha} f(x) &= {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\left(f(t)\right)\mathrm{d}t, \qquad \alpha > 0 \end{aligned} \tag{2.189}$$

2.17.3.b(i) Hadamard type fractional integration

By choosing $z(t) = \ln t$, $w(t) = 1$ and $z^{(1)}(t) = \frac{1}{t}$ we get Hadamard type fractional integration operators. Following this, we write Hadamard type left/forward integration operators:

$$
\begin{aligned}
&{}_a\mathcal{I}^{\alpha}_{x,[z,w]}f(x)=\frac{1}{\big(w(x)\big)\big(\Gamma(\alpha)\big)}\int_a^x\big(z(x)-z(t)\big)^{\alpha-1}w(t)\big(z^{(1)}(t)\big)\big(f(t)\big)\,\mathrm{d}t\\
&{}_a\mathcal{I}^{\alpha}_{x,[\ln t,1]}f(x)={}^{H}_{a}I^{\alpha}_{x}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\frac{\ln x}{\ln t}\right)^{\alpha-1}\left(\frac{f(t)}{t}\right)\mathrm{d}t,\qquad \alpha>0
\end{aligned}
\tag{2.190}
$$

2.17.3.b(ii) The modified Erdelyi-Kober (MEK) fractional integration formula

By choosing $z(t)=t^{\sigma}$, $w(t)=t^{\sigma\eta}$ and $z^{(1)}(t)=\sigma t^{\sigma-1}$ we obtain a modified Erdelyi-Kober (MEK) integration operator:

$$
\begin{aligned}
&{}_a\mathcal{I}^{\alpha}_{x,[z,w]}f(x)=\frac{1}{\big(w(x)\big)\big(\Gamma(\alpha)\big)}\int_a^x\big(z(x)-z(t)\big)^{\alpha-1}w(t)\big(z^{(1)}(t)\big)\big(f(t)\big)\,\mathrm{d}t\\
&\alpha>0\\
&{}_a\mathcal{I}^{\alpha}_{x,\left[t^{\sigma},t^{\sigma\eta}\right]}f(x)={}^{MEK}_{\ \ \ a}I^{\alpha}_{x}f(x)=\frac{\sigma t^{-\sigma\eta}}{\Gamma(\alpha)}\int_a^x\left(x^{\sigma}-t^{\sigma}\right)^{\alpha-1}t^{\sigma(1-\eta)-1}\big(f(t)\big)\,\mathrm{d}t
\end{aligned}
\tag{2.191}
$$

With these different types of fractional integral operators, we will define several types of fractional derivative operators in the next chapter.

2.17.4 Generalisation of the fractional integration formula with respect to function

Let us take $f(x)=\big(z(x)-z(a)\big)^{\beta-1}$ with $\beta>0$ and $w(x)=1$.

On the application of the generalised formula (2.187), i.e. for $\alpha>0$ we write ${}_a\mathcal{I}^{\alpha}_{x,[z,w]}f(x)=\frac{1}{(w(x))(\Gamma(\alpha))}\int_a^x\big(z(x)-z(t)\big)^{\alpha-1}w(t)\big(z^{(1)}(t)\big)\big(f(t)\big)\,\mathrm{d}t$ which gives us the following steps:

$$
\begin{aligned}
{}_a\mathcal{I}^{\alpha}_{x,[z,1]}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x\big(z(x)-z(t)\big)^{\alpha-1}\big(z^{(1)}(t)\big)\big(z(t)-z(a)\big)^{\beta-1}\,\mathrm{d}t\\
&=\frac{1}{\Gamma(\alpha)}\int_a^x\big(z(x)-z(t)\big)^{\alpha-1}\left(\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\big(z(t)-z(a)\big)^{\beta-1}\big(\mathrm{d}z(t)\big)\\
&=\frac{1}{\Gamma(\alpha)}\int_{z(a)}^{z(x)}\big(z(x)-z(t)\big)^{\alpha-1}\big(z(t)-z(a)\big)^{\beta-1}\big(\mathrm{d}z(t)\big)
\end{aligned}
\tag{2.192}
$$

In the last step of the above derivation (2.192), we have integration with respect to $z(t)$ and hence we re-wrote the lower limit as $z(t)=z(a)$ and upper limit as $z(t)=z(x)$. For the last integral in (2.192) we see that it is similar to an RL fractional integration formula i.e. ${}_aI^{\alpha}_{x}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\big(f(t)\big)\,\mathrm{d}t$, with $f(x)=\big(z(x)-z(a)\big)^{\beta-1}$ and integration with respect to $z(x)$ in this case. Therefore, we write the following formula:

$$
\begin{aligned}
&{}_a\mathcal{I}^{\alpha}_{x,[z,1]}\left[\big(z(x)-z(a)\big)^{\beta-1}\right]\\
&\qquad=\frac{1}{\Gamma(\alpha)}\int_{z(a)}^{z(x)}\big(z(x)-z(t)\big)^{\alpha-1}\big(z(t)-z(a)\big)^{\beta-1}\big(\mathrm{d}z(t)\big)\\
&\qquad={}_aI^{\alpha}_{z(x)}\left[\big(z(x)-z(a)\big)^{\beta-1}\right]
\end{aligned}
\tag{2.193}
$$

Using the result obtained in Section 2.6, i.e. ${}_cI^{\alpha}_{x}\left[x^{\mu}\right]=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(x-c)^{\mu+\alpha}$, $\mu>-1$, $\alpha>0$, we write the result as follows in this expression:

$$
{}_aI^{\alpha}_{z(x)}\left[\big(z(x)-z(a)\big)^{\beta-1}\right]=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\big(z(x)-z(a)\big)^{\beta-1+\alpha}
\tag{2.194}
$$

2.18 Scaling law in a fractional integral

We write the following law called the scaling law:

$$ {}_aI_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_{\lambda a}I_x^{\alpha}\left[f(x)\right]\right) \qquad \lambda>0 \tag{2.195}$$

We apply the RL definition of I^{α} to $f(\lambda x)$ then do changes in the variable i.e. $\lambda y=z$ and later do the substitution $u=\lambda x$. This is demonstrated in the following steps, for the derivation:

$$\begin{aligned}
{}_aI_x^{\alpha}\left[f(\lambda x)\right]&=\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left(f(\lambda y)\right)\mathrm{d}y\\
&=\frac{1}{\Gamma(\alpha)}\int_{\lambda a}^{\lambda x}\left(x-\tfrac{z}{\lambda}\right)^{\alpha-1}\left(f(z)\right)\left(\frac{\mathrm{d}z}{\lambda}\right)\\
&=\frac{1}{\Gamma(\alpha)}\int_{\lambda a}^{\lambda x}\frac{(\lambda x-z)^{\alpha-1}}{\lambda^{\alpha-1}}\left(f(z)\right)\left(\frac{1}{\lambda}\right)\mathrm{d}z\\
&=\frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_{\lambda a}^{\lambda x}(\lambda x-z)^{\alpha-1}\left(f(z)\right)\mathrm{d}z\\
&=\lambda^{-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_{\lambda a}^{u}(u-z)^{\alpha-1}\left(f(z)\right)\mathrm{d}z\right)\\
&=\lambda^{-\alpha}\left({}_{\lambda a}I_x^{\alpha}\left[f(x)\right]\right)
\end{aligned}\tag{2.196}$$

In the above derivation (2.196), for $a=0$ we have ${}_0I_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_0I_x^{\alpha}\left[f(x)\right]\right)$; this is also the same as ${}_0I_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_0I_x^{\alpha}\left[f(x)\right]\right)=\lambda^{-\alpha}\left({}_0I_{\lambda x}^{\alpha}\left[f(\lambda x)\right]\right)$. The following are the steps to prove this result:

$$\begin{aligned}
{}_0I_x^{\alpha}\left[f(\lambda x)\right]&=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(\lambda y)}{(x-y)^{-\alpha+1}}\mathrm{d}y; \qquad z=\lambda y\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{\lambda x}\frac{f(z)}{\left(\frac{\lambda x-z}{\lambda}\right)^{-\alpha+1}}\left(\frac{\mathrm{d}z}{\lambda}\right)\\
&=\frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\int_0^{\lambda x}\frac{f(z)}{(\lambda x-z)^{-\alpha+1}}\mathrm{d}z; \qquad z=\lambda y\\
&=\frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\int_0^{\lambda x}\frac{f(\lambda y)}{(\lambda x-\lambda y)^{-\alpha+1}}\left(\mathrm{d}(\lambda y)\right)\\
&=\lambda^{-\alpha}\left({}_0I_{\lambda x}^{\alpha}\left[f(\lambda x)\right]\right)
\end{aligned}\tag{2.197}$$

Thus, we obtained an important identity, namely ${}_aI_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_{\lambda a}I_x^{\alpha}\left[f(x)\right]\right)$ and ${}_0I_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_0I_{\lambda x}^{\alpha}\left[f(\lambda x)\right]\right)$ which is known as the scaling law of fractional integration.

2.19 The formula for fractional integration is derived from the generalisation of Cauchy's integral theorem of complex variables

Interestingly the derivative is in someway related to the integration process. For that we revised Cauchy's integral theorem (in Section 1.5), and here we further generalise the same. Let $f(z)$ be an analytic single valued function on a complex plane i.e. $z,\omega\in\mathbb{C}$. Cauchy's integral theorem (1825) statement is the following:

$$f(z)=\frac{1}{2\pi i}\int_{\mathbf{C}}\frac{f(\omega)}{(\omega-z)}\mathrm{d}\omega \tag{2.198}$$

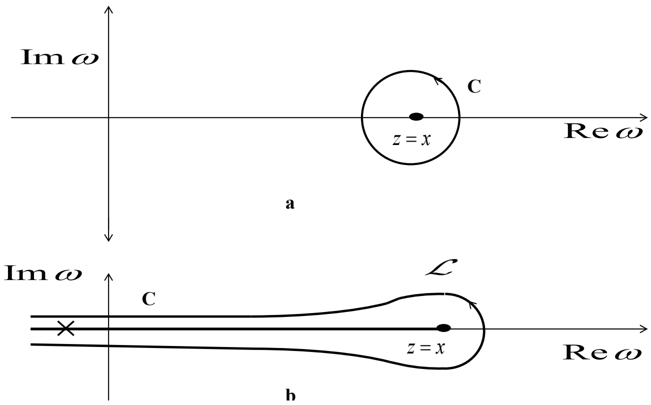

Figure 2.6: Contours of integration on the complex plane

The $n-$th differentiation ($n \in \mathbb{Z}^+$ being a positive integer) of $f(z)$ from Cauchy's integral formula is:

$$D_z^n\left[f(z)\right] = \frac{n!}{2\pi i}\int_{\mathrm{C}} \frac{f(\omega)}{(\omega - z)^{n+1}}\mathrm{d}\omega \tag{2.199}$$

We call $\omega = z$ a pole of order $(n+1)$.

In a way Cauchy was first to state that differentiation of a function is related to the integration process in complex variable theory. For n as a positive integer, $x = z$ is a pole with C as a closed contour (as in Figure 2.6a). For n as a positive real number (say $\alpha \in \mathbb{R}^+$) we call $x = z$ a branch point (not a pole), having a branch cut (depicted in Figure 2.6b), and $\mathcal{L}$ a loop (refer to Appendix E). Then Cauchy's integral formula is generalised as the following expression, by using $\Gamma(n+1) = n!$:

$$D_z^\alpha\left[f(z)\right] = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{\mathrm{C}} \frac{f(\omega)}{(\omega - z)^{\alpha+1}}\mathrm{d}\omega \tag{2.200}$$

We re-draw Figure 2.6b as a simplified loop, shown in Figure 2.7. We write the following:

$$\begin{aligned} {}_cD_x^\alpha\left[f(x)\right] &\overset{\text{def}}{=} \frac{\Gamma(\alpha+1)}{2\pi i}\int_c^{(x+)} (\omega - x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &= \frac{\Gamma(\alpha+1)}{2\pi i}\int_{\mathcal{L}} (\omega - x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \end{aligned} \tag{2.201}$$

The above expression (2.201) is in a way defining fractional derivatives of the order α in terms of the contour integral in Cauchy's formulation. We will do the contour integration now:

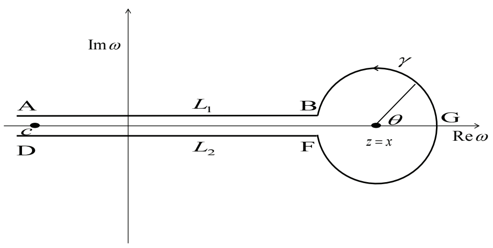

Figure 2.7: The loop redrawn

The loop ($\mathcal{L}$) is composed of the circle FGB (γ), segments DF (L_2) and segment AB (L_1), making $\mathcal{L} = L_1 \cup L_2 \cup \gamma$. In addition, the complex quantity $(\omega - x)$ we write in polar form as follows:

$$(\omega - x) = \left|(\omega - x)\right| e^{i\theta} \tag{2.202}$$

We do the following arithmetic steps:

$$\begin{aligned}(\omega-x)^{-\alpha-1} &= \left|(\omega-x)\right|^{(-\alpha-1)} e^{i\theta(-\alpha-1)} \\ &= e^{\ln\left|(\omega-x)\right|^{(-\alpha-1)}} e^{i\theta(-\alpha-1)} \\ &= e^{(-\alpha-1)\ln\left|(\omega-x)\right|} e^{i\theta(-\alpha-1)} \\ &= e^{\left[(-\alpha-1)\ln\left|(\omega-x)\right|+i\theta(-\alpha-1)\right]} \\ &= e^{(-\alpha-1)\left(\ln\left|(\omega-x)\right|+i\theta\right)}\end{aligned} \tag{2.203}$$

We note that $\left|\omega-x\right|=(x-\omega)$ refers to Figure 2.7, the point $z=x$ is always at the right side of any other variable ω, near or on the real axis, while also on segments AB or DF, thus $\omega \le x$, on L_1 & L_2.

For L_1 we have $\theta=\pi$, so, we do the following arithmetic steps:

$$(\omega-x)^{-\alpha-1} = e^{(-\alpha-1)\ln(x-\omega)} e^{-(\alpha+1)i\pi} = e^{-i\pi(\alpha+1)}\left((x-\omega)^{(-\alpha-1)}\right) \tag{2.204}$$

For L_2 we have $\theta=-\pi$, therefore, we get the following relationship:

$$(\omega-x)^{-\alpha-1} = e^{(-\alpha-1)\ln(x-\omega)} e^{i\pi(\alpha+1)} = e^{i\pi(\alpha+1)}\left((x-\omega)^{(-\alpha-1)}\right) \tag{2.205}$$

We write the loop integral as in the following steps:

$$\begin{aligned}\int_{c}^{(x+)} (\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega &= \int_{L_2}(\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &\quad+\int_{\gamma}(\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &\quad+\int_{L_1}(\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &= e^{i(\alpha+1)\pi}\int_{c}^{(x-r)}(x-\omega)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &\quad+\int_{\gamma}(\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega \\ &\quad+e^{-i\pi(\alpha+1)\pi}\int_{x-r}^{c}(x-\omega)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega\end{aligned} \tag{2.206}$$

Figure 2.7 shows the integration path as counter clockwise. The radius of the circle is r, so the path L_2 is from point c to point $(x-r)$; whereas the path L_1 is from $(x-r)$ to point c. The circle FGB of Figure 2.7 has a very small radius, i.e. $r\downarrow 0$. The paths considered in the segments AB and DF therefore almost overlap with each other and merge on the real axis (X-axis) and the ω-plane of Figure 2.7; but are on different Riemann sheets of the Riemann surface for $\left(\omega-x\right)^{-\alpha-1}$ (refer to Appendix E). Figure 2.7 shows only the primary Riemann sheet.

Returning to expression $\frac{\Gamma(\alpha+1)}{2\pi i}\int_{\mathcal{L}}(\omega-x)^{-\alpha-1}\left(f(\omega)\right)\mathrm{d}\omega$, we analyse it for $\alpha<0$. For integration on the circle FGB, set $(\omega-x)=re^{i\theta}$, for θ change from $-\pi$ to π, in addition $\omega=x+re^{i\theta}$, and $\mathrm{d}\omega=(ri\theta)e^{i\theta}\mathrm{d}\theta$. Now as $r\downarrow 0$, we write the following integral:

$$\int_{\gamma}(\omega-x)^{-\alpha-1}f(\omega)\mathrm{d}\omega = \int_{-\pi}^{\pi} r^{-\alpha-1}e^{-i(\alpha+1)\theta}f(x+re^{i\theta})(ire^{i\theta})\mathrm{d}\theta \tag{2.207}$$

and:

$$\left|\int_{\gamma}(\omega-x)^{-\alpha-1}f(\omega)\mathrm{d}\omega\right| \le r^{-\alpha}\int_{-\pi}^{\pi}\left|f(x+re^{i\theta})\right|\mathrm{d}\theta, \qquad \alpha<0 \tag{2.208}$$

Notice that for a small circle $r\downarrow 0$, and for $\alpha<0$ the above integral (2.208) on the circle FGB tends to 0, i.e.

$$\lim_{r\downarrow 0}\int_{\gamma}(\omega-x)^{-\alpha-1}f(\omega)\mathrm{d}\omega = 0 \tag{2.209}$$

Therefore, as $r \downarrow 0$ we are left with integration on L_1 and L_2 only, and thus we write the following steps:

$$\int_c^{(x+)} (\omega - x)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$$

$$= \lim_{r\downarrow 0} \begin{pmatrix} e^{i(\alpha+1)\pi} \int_c^{(x-r)} (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega + \\ e^{-i\pi(\alpha+1)\pi} \int_{x-r}^{c} (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega \end{pmatrix}$$

$$= e^{i(\alpha+1)\pi} \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$$

$$+ e^{-i\pi(\alpha+1)\pi} \int_x^c (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega \tag{2.210}$$

$$= \left(e^{i(\alpha+1)\pi} - e^{-i(\alpha+1)\pi}\right) \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$$

Use $\sin(p\pi) = \frac{1}{2i}\left(e^{ip\pi} - e^{-ip\pi}\right)$

$$\int_c^{(x+)} (\omega - x)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega = 2i \sin\left(\pi(\alpha+1)\right) \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$$

Therefore from definition (2.201) i.e. ${}_cD_x^\alpha\left[f(x)\right] = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\mathcal{L}} (\omega - x)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$, which is the generalised Cauchy's integral formula, and by using the above derived expression (2.210) i.e. $\int_c^{(x+)} (\omega - x)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega = 2i \sin\left(\pi(\alpha+1)\right) \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega$ we get the following expression for $\alpha < 0$:

$$\begin{aligned} {}_cD_x^\alpha\left[f(x)\right] &\overset{\text{def}}{=} \frac{\Gamma(\alpha+1)}{2\pi i} \int_c^{(x+)} (\omega - x)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega \\ &= \frac{\left(\Gamma(\alpha+1)\right)\left(\sin \pi(\alpha+1)\right)}{\pi} \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega \end{aligned} \tag{2.211}$$

Using one of the reflection properties of the gamma function from Chapter-1 (1.68), $\Gamma(-x)\Gamma(x+1) = -\pi \csc(\pi x)$, we get the following simplification:

$$\begin{aligned} \frac{\left(\Gamma(\alpha+1)\right)\left(\sin \pi(\alpha+1)\right)}{\pi} &= \frac{\left(\Gamma(\alpha+1)\right)\left(\sin(\pi + \alpha\pi)\right)}{\pi} \\ &= \frac{\left(\Gamma(\alpha+1)\right)\left(-\sin(\alpha\pi)\right)}{\pi} \\ &= -\frac{\Gamma(\alpha+1)}{\pi \csc(\alpha\pi)} = \frac{1}{\Gamma(-\alpha)} \end{aligned} \tag{2.212}$$

Now we write the final formula as follows:

$${}_cD_x^\alpha\left[f(x)\right] = \frac{1}{\Gamma(-\alpha)} \int_c^x (x-\omega)^{-\alpha-1} \left(f(\omega)\right) \mathrm{d}\omega, \qquad \alpha < 0 \tag{2.213}$$

This is what is obtained by changing the sign of a fractional order in the RL fractional integration formula, as discussed previously, in Section 2.14.

2.20 Repeated integrals as a limit of a sum and its generalisation

2.20.1 Limit of a sum formula to have classical repeated $n-$fold integration

The following notation:

$$\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}} \equiv \int_a^x f(y)\mathrm{d}y \tag{2.214}$$

represents integration as an anti-derivative of a function from the start point of integration $x=a$. This notation we discussed in Section 1.17. Let the interval of integration from a to x be divided into N slices such that $h_N = \frac{x-a}{N}$ are the slice sizes on an infinitesimal width.

Then we write:

$$\begin{aligned}\frac{\mathrm{d}^{-1}f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}} &\equiv \int_a^x f(y)\mathrm{d}y = \lim_{h_N\downarrow 0}\begin{pmatrix} h_N f(x) + h_N f(x-h_N) + \\ h_N f(x-2h_N) + \ldots . f(a+h_N)\end{pmatrix} \\ &= \lim_{h_N\downarrow 0}\left[h_N\left(f(x)+f(x-h_N)+f(x-2h_N)+\ldots . f(a+h_N)\right)\right]\end{aligned} \tag{2.215}$$

The application of the above definitions (2.214) and (2.215); to double integration gives us the following:

$$\begin{aligned}\frac{\mathrm{d}^{-2}f(x)}{\left[\mathrm{d}(x-a)\right]^{-2}} &\equiv \int_a^x \mathrm{d}x_1 \int_a^{x_1} f(x_0)\mathrm{d}x_0 \\ &= \lim_{h_N\downarrow 0}\left[h_N^2\begin{pmatrix} f(x)+2f(x-h_N)+3f(x-2h_N)+ \\ \ldots . Nf(a+h_N)\end{pmatrix}\right] \\ &= \lim_{h_N\downarrow 0}\left(h_N^2\sum_{j=0}^{N-1}(j+1)f(x-jh_N)\right)\end{aligned} \tag{2.216}$$

We do one more step as follows, writing the triple integration:

$$\begin{aligned}\frac{\mathrm{d}^{-3}f(x)}{\left[\mathrm{d}(x-a)\right]^{-3}} &\equiv \int_a^x \mathrm{d}x_2 \int_a^{x_2} \mathrm{d}x_1 \int_a^{x_1} f(x_0)\mathrm{d}x_0 \\ &= \lim_{h_N\downarrow 0}\left(h_N^3\sum_{j=0}^{N-1}\frac{(j+1)(j+2)}{2}f(x-jh_N)\right)\end{aligned} \tag{2.217}$$

We notice that coefficients of summation are building up with the order of integration n as described below, and all the signs of the coefficients are positive:

$$\begin{pmatrix} j+n-1 \\ j \end{pmatrix} = \frac{(j+n-1)!}{j!n!} \tag{2.218}$$

With this, we may write the n-fold repeated integration as a limit of a sum as follows in this formula:

$$\begin{aligned}\frac{\mathrm{d}^{-n}f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} &= \lim_{h_N\downarrow 0}\left(h_N^n\sum_{j=0}^{N-1}\begin{pmatrix} j+n-1 \\ j \end{pmatrix}f(x-jh_N)\right) \\ &= \lim_{N\uparrow\infty}\left(\left(\tfrac{x-a}{N}\right)^n\sum_{j=0}^{N-1}\begin{pmatrix} j+n-1 \\ j \end{pmatrix}f\left(x-j\left(\tfrac{x-a}{N}\right)\right)\right)\end{aligned} \tag{2.219}$$

The expression (2.219) is the formula for repeated integration in classical calculus, taking the limit of a sum.

2.20.2 Limit of a sum formula to have fractional integration: the generalisation of classical formula

We take formula (2.219) as derived above using a generalisation formula for binomial coefficients by the gamma function (Chapter-1), that is

$$\begin{pmatrix} j+q-1 \\ j \end{pmatrix} = \frac{\Gamma(j+q)}{\Gamma(q)\Gamma(j+1)} \tag{2.220}$$

We write the fractional integration of order q as follows:

$$\frac{\mathrm{d}^{-q} f(x)}{\left[\mathrm{d}(x-a)\right]^{-q}} = \lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{q}}{\Gamma(q)}\sum_{j=0}^{N-1}\frac{\Gamma(j+q)}{\Gamma(j+1)} f\left(x - j\left(\tfrac{x-a}{N}\right)\right)\right) \tag{2.221}$$

The above obtained formula (2.221) is well valid for any q, say we replace $-q$ by α, then we write the following expression:

$$\frac{\mathrm{d}^{\alpha} f(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - j\left(\tfrac{x-a}{N}\right)\right)\right) \tag{2.222}$$

We will discuss (2.222) in Chapters 3 and 4.

Notice also that the above expression (2.222) involves only an evaluation of the function itself; no explicit use is made of the derivatives or the integrals of $f(x)$.

2.20.3 Using a generalised limit of the sum formula derivation of an additive composition rule

We would like to establish $D_x^n\left({}_aD_x^{\alpha}\left[f(x)\right]\right) = {}_aD_x^{n+\alpha}\left[f(x)\right]$ (called the additive composition rule) for all positive integers n and for all α. To establish this composition, let $h_N = \frac{x-a}{N}$, and we will then write from (2.222) the following:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - jh_N\right)\right) \tag{2.223}$$

We subdivide the interval that is $a \le y \le x - h_N$ into only $N-1$ equally spaced subintervals resulting in the following steps:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[f\left(x-h_N\right)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} &= \lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-2}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - h_N - jh_N\right)\right)\\ &= \lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-2}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - (j+1)h_N\right)\right)\\ &= \lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha-1)}{\Gamma(j)} f\left(x - jh_N\right)\right)\end{aligned} \tag{2.224}$$

On the differentiation of (2.223) and by defining $\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)\right] = \lim_{h_N\downarrow 0} h_N^{-1}\left(g(x) - g(x-h_N)\right)$ we write the following by making use of the above derived expressions, (2.223) and (2.224):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\right) &= \lim_{N\uparrow\infty}\left(h_N^{-1}\left(\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} - \frac{\mathrm{d}^{\alpha}\left[f(x-h_N)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\right)\right)\\ &\lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha-1}}{\Gamma(-\alpha)}\left(\Gamma(-\alpha)f(x) + \sum_{j=0}^{N-1}\left(\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} - \frac{\Gamma(j-\alpha-1)}{\Gamma(j)}\right)\right)\right)\end{aligned} \tag{2.225}$$

Then by making use of recurrence properties of the gamma function $\Gamma(x+1) = x\Gamma(x)$ (as seen in Chapter-1), we obtain the following:

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} - \frac{\Gamma(j-\alpha-1)}{\Gamma(j)} = \frac{\Gamma(-\alpha)\Gamma(j-\alpha-1)}{\Gamma(-\alpha-1)\Gamma(j+1)} \tag{2.226}$$

Using (2.226) in (2.225) gives us:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\right)=\lim_{N\uparrow\infty}\left(\frac{h_N^{-\alpha-1}}{\Gamma(-\alpha-1)}\left(\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha-1)}{\Gamma(j+1)}f(x-jh_N)\right)\right)$$
$$=\frac{\mathrm{d}^{\alpha+1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha+1}} \tag{2.227}$$

So we have $D_x^1\left({}_aD_x^{\alpha}\left[f(x)\right]\right)={}_aD_x^{1+\alpha}\left[f(x)\right]$, and thus for all α and the positive integer n we have from induction the following expression:

$$D_x^n\left({}_aD_x^{\alpha}\left[f(x)\right]\right)={}_aD_x^{n+\alpha}\left[f(x)\right] \tag{2.228}$$

2.21 Fractional integration is the area under the shape changing curve

2.21.1 Analytical explanation of the area under the shape changing curve for the Riemann-Liouville fractional integration

The Riemann-Liouville formula is ${}_0I_t^{\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left(f(\tau)\right)\mathrm{d}\tau$. In this formula we change the variable to $\tau=t-x^{1/\alpha}$ so we have $\mathrm{d}\tau=-\left(\frac{1}{\alpha}\right)x^{\frac{1-\alpha}{\alpha}}\mathrm{d}x$ and substitute in this expression to get the following steps:

$${}_0D_t^{-\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_{t^{\alpha}}^{0}x^{\frac{\alpha-1}{\alpha}}\left(-\frac{1}{\alpha}\right)x^{\frac{1-\alpha}{\alpha}}\mathrm{d}x$$
$$=\frac{-1}{\alpha\left(\Gamma(\alpha)\right)}\int_{t^{\alpha}}^{0}\left(f(t-x^{1/\alpha})\right)\mathrm{d}x \tag{2.229}$$
$$=\frac{1}{\Gamma(\alpha+1)}\int_0^{t^{\alpha}}\left(f(t-x^{1/\alpha})\right)\mathrm{d}x$$

Let us take $\alpha=\frac{1}{2}$, and see what is the area; i.e. $\int_0^{\sqrt{t}}\left(f(t-x^2)\right)\mathrm{d}x$ indicates for the function f say $f(x)=(x+2)^2=x^2+4+4x$, when the upper limit of integration keeps changing. The changed function is, $f(t-x^2)$ that is $f(t-x^2)=\left((t-x^2)+2\right)^2$.

First we take a small interval of the integration variable t that is from 0 to 0.001. Here we have $t=0.001$ and the integration is thus $\int_0^{\sqrt{0.001}}f(0.001-x^2)\mathrm{d}x=\int_0^{\sqrt{0.001}}(0.001-x^2+2)^2\,\mathrm{d}x$. This is the area under the curve $(0.001-x^2+2)^2\cong(4-4x^2+x^4)$ from 0 to $\sqrt{0.001}$. Next is the calculation we do at $t=1$, here the function is $f(1-x^2)$ that is $\left((1-x^2)+2\right)^2$ or $9-6x^2+x^4$. Therefore, at point $t=1$, we calculate the area under a new curve, that is $f(1-x^2)=9-6x^2+x^4$ from 0 to $\sqrt{1}$. Thus the fractional integration of $f(t)$ from 0 to t indicates from this example that the area under the shape changing curve known as t, grows. An interesting observation is contrary to the classical integration where the shape of the curve is fixed.

2.21.2 Numerical demonstration of fractional integration as the area under a shape changing curve

We noted in a previous section that the fractional integration is the area under the curve whose shape keeps on changing as we advance. For a demonstration, let us take a constant function i.e. $f(x)=C$. From the discussion of the previous section (Section 2.20), we write the integration process of a classical type of one whole order as follows, with the lower limit as zero, i.e. $a=0$ as:

$${}_0I_x^1\left[f(x)\right]=\lim_{h_N\downarrow 0}\left[h_N\left(f(x)+f(x-h_N)+f(x-2h_N)+....f(h_N)\right)\right] \tag{2.230}$$

We are not going to show that for $f(x)=C$ the curve $f(x)$ remains the same while we do the process of integration at $x_0=h_N$, $x_1=2h_N$, $x_3=3h_N$ and so on i.e. by increasing the variable x, which is obvious from the above formula, and from our classical calculus theory.

Now let us write the formula for fractional integration that we discussed in sections (2.221) and (2.223), and do the following manipulation:

$$\frac{\mathrm{d}^{-\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-\alpha}}=\lim_{N\uparrow\infty}\left(\frac{h_N^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j+\alpha)}{\Gamma(j+1)}f\left(x-jh_N\right)\right),\quad a=0$$

$$\begin{aligned}{}_0I_x^{\alpha}\left[f(x)\right]&=\lim_{N\uparrow\infty}\left(h_N^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\left(j!\right)}f\left(x-jh_N\right)\right)\\&=h_N^{\alpha-1}\lim_{N\uparrow\infty}\left(h_N\sum_{j=0}^{N-1}\frac{\Gamma(j+\alpha)}{\left(\Gamma(\alpha)\right)\left(j!\right)}f\left(x-jh_N\right)\right)\\&=h_N^{\alpha-1}\lim_{N\uparrow\infty}\left(h_N\sum_{j=0}^{N-1}c_jf\left(x-jh_N\right)\right)\\&=K\begin{pmatrix}h_N\left(c_0f(x)+c_1f(x-h_N)+c_2f(x-2h_N\right)+\\ ....c_{N-1}f(x-(N-1)h_N\end{pmatrix}\end{aligned}\tag{2.231}$$

Where in (2.231) we have taken the proportionality constant as $K=h_N^{\alpha-1}$; and the above manipulation (2.231) shows that the fractional integration is:

$${}_0I_x^{\alpha}\left[f(x)\right]\propto\begin{pmatrix}h_N\left(c_0f(x)+c_1f(x-h_N)+c_2f(x-2h_N)\right)\\ +.....c_{N-1}f(x-(N-1)h_N\end{pmatrix}\tag{2.232}$$

Comparing the above expression (2.232) with the classical one of order integration (2.230), we find that the function values are modified by coefficients $c_j=\frac{\Gamma(j+\alpha)}{(\Gamma(\alpha))(j!)}$.

We take $\alpha=\frac{1}{2}$, and $f(x)=C$. For this we have coefficients such as $c_0=1$, $c_1=\frac{1}{2}$, $c_2=\frac{3}{8}$, $c_4=\frac{35}{128}$ and $c_5=\frac{63}{256}$, for the five points of the function, derived for $c_j=\frac{\Gamma\left(j+\frac{1}{2}\right)}{\left(\Gamma\left(\frac{1}{2}\right)\right)(j!)}$; $j=0,1,2,3,4,5$ by using $\Gamma(1+\alpha)=\alpha\left(\Gamma(\alpha)\right)$. Thus at point $x=x_5$, we write the following expression:

$$\begin{aligned}{}_0I_{x_5}^{1/2}\left[f(x)\right]&\propto h_N\begin{pmatrix}c_0f(x_5)+c_1f(x_4)+c_2f(x_3)+c_3f(x_2)\\ +c_4f(x_1)+c_5f(x_0)\end{pmatrix}\\&=h_N\begin{pmatrix}f(x_5)+\frac{1}{2}f(x_4)+\frac{3}{8}f(x_3)+\frac{5}{16}f(x_2)\\ +\frac{35}{128}f(x_1)+\frac{63}{256}f(x_0)\end{pmatrix}\\&=h_NC\left(1+\frac{1}{2}+\frac{3}{8}+\frac{5}{16}+\frac{35}{128}+\frac{63}{256}\right)=2.7069h_NC\end{aligned}\tag{2.233}$$

The above expression (2.233) is the area under Curve-5 as depicted in Figure-2.8; clearly this is not the area under the original curve i.e. $f(x)=C$. In Figure-2.8 the value of the slice on the X-axis is $h_N=x_1-x_0=x_2-x_1=....$ i.e. the slices are equally spaced; we have drawn the piece-wise linear curves for demonstration.

When examining Figure-2.8 and writing the fractional integration at point $x=x_0$, which is:

$${}_0I_{x_0}^{1/2}\left[f(x)\right]\propto h_Nf(x_0)=h_NC\tag{2.234}$$

here the fractional integration, i.e. ${}_0I_{x_0}^{1/2}\left[f(x)\right]$, is proportional to the area under the curve $f(x)=C$, the original curve shown by Curve-0 in Figure-2.8. We proceed and at point $x=x_1$ we write the following:

$${}_0I_{x_1}^{1/2}\left[f(x)\right]\propto h_N\left(f(x_1)+\frac{1}{2}f(x_0)\right)=h_NC\left(1+\frac{1}{2}\right)=1.5h_NC\tag{2.235}$$

Here ${}_0I_{x_1}^{1/2}\left[f(x)\right]$ is proportional to the area under Curve-1, as shown in Figure-2.8:

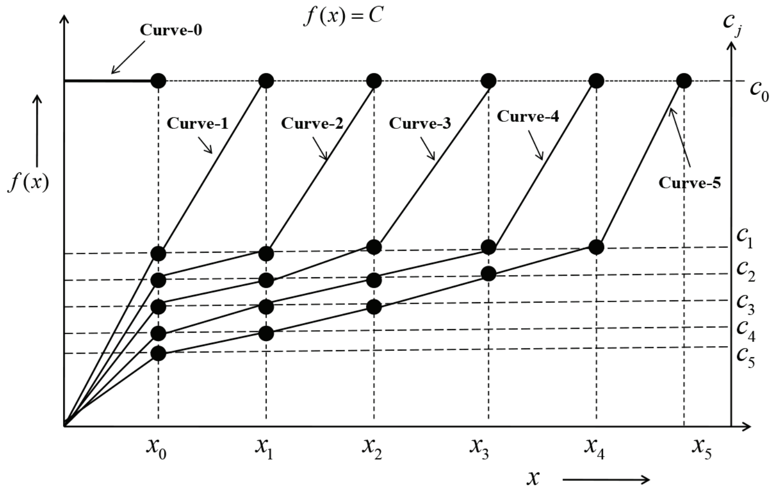

Figure 2.8: Fractional integration is the area under the shape-changing curve

At $x = x_2$ we write:

$$ {}_0I_{x_2}^{1/2}\left[f(x)\right] \propto h_N\left(f(x_2)+\tfrac{1}{2}f(x_1)+\tfrac{3}{8}f(x_0)\right) = 1.875h_NC \tag{2.236}$$

Here ${}_0I_{x_2}^{1/2}\left[f(x)\right]$ is proportional to the area under Curve-2 of Figure 2.8. Similarly:

$$ {}_0I_{x_3}^{1/2}\left[f(x)\right] \propto h_N\left(f(x_3)+\tfrac{1}{2}f(x_2)+\tfrac{3}{8}f(x_1)+\tfrac{5}{16}f(x_0)\right) = 2.1875h_NC \tag{2.237}$$

is the area under Curve-3 of Figure-2.8, at $x = x_3$. As we move to the next point $x = x_4$, we will obtain the area under Curve-4 of Figure 2.8, i.e.

$$ \begin{aligned} {}_0I_{x_4}^{1/2}\left[f(x)\right] &\propto h_N\left(f(x_4)+\tfrac{1}{2}f(x_3)+\tfrac{3}{8}f(x_2)+\tfrac{5}{16}f(x_1)+\tfrac{35}{128}f(x_0)\right) \\ &= 2.4609h_NC \end{aligned} \tag{2.238}$$

The evaluation at point $x = x_5$ is already stated earlier.

We note that as we proceed to new points from x_0 to x_5 for the semi-integration of the constant function $f(x)=C$ we get the value of an increasing area (from h_NC, $1.5h_NC$, $1.875h_NC$, $2.1875h_NC$, $2.4609h_NC$, $2.7069h_NC$) but the increments or as we say rate of change of the area under Curves-0 to 5, keep on decreasing (the differences are $0.5h_NC$, $0.375h_NC$, $0.31252h_NC$, $0.2734h_NC$, $0.2460h_NC$). This indicates that the semi-integration of the constant function $f(x)=C$ is a monotonically increasing function; say $g(x)$ with a rate of change tapering as we increase x, like $g(x) = Ax^{\alpha}$, where $0<\alpha<1$. It is true since we have already found out that ${}_0I_x^{\alpha}\left[C\right] = \frac{C}{\Gamma(1+\alpha)}x^{\alpha}$, and for semi-integration $\alpha = \frac{1}{2}$.

2.21.3 A fractional integration is the case of fading memory while a classical one-whole integration is a case with a constant past memory

Now we see that the formula of fractional integration $0<\alpha<1$

$$ {}_0I_x^{\alpha}\left[f(x)\right] = h_N^{\alpha-1}\begin{pmatrix} h_N\left(c_0f(x)+c_1f(x-h_N)+c_2f(x-2h_N)\right)+ \\ \ldots\ldots c_{N-1}f(x-(N-1)h_N \end{pmatrix} \tag{2.239}$$

is the weighted sum at various points (specifically previous points in view of a causal case) of the function right from its start point, and we see that the weights keeps on decreasing as we take past and still past points. Conversely, in the classical case of one-whole integration the weights are equal (constant) for all the past points of the function from the start. If we say in a different way the classical one-whole integration is a process with constant memory of all the past happenings, while fractional integration is a process with decreasing/fading memory of all of the past happenings, we would not be inaccurate. We all have fading memory in reality, like fractional integration!

2.22 The functions which can be fractionally integrated and differentiated

We have discussed the arbitrary order α of the integral transform equation in the Riemann-Liouville way. There is a class of functions that we can fractionally integrate (or differentiate). In this section, we shall confine ourselves to classically defined functions rather than distributions. These are sometimes referred to as symbolic or generalised functions like delta distribution etc. We take from our classical calculus the definition of such a function that is defined on a closed interval i.e. $\left[a,x\right]$ and is better behaved than $(x-a)^{-1}$ at the lower limit $x=a$.

We define the class of a differ-integrable series as

$$\begin{aligned} f(x) &= a_0(x-a)^p + a_1(x-a)^{p+\left(\frac{1}{n}\right)} + a_2(x-a)^{p+\left(\frac{2}{n}\right)} + a_3(x-a)^{p+\left(\frac{3}{n}\right)} + \ldots\ldots \\ &= (x-a)^p \sum_{j=0}^{\infty} a_j (x-a)^{\left(j/n\right)} \qquad a_0 \neq 0 \qquad p > -1 \end{aligned} \tag{2.240}$$

The above series (2.240) is a product of the power of $(x-a)$ and an analytic function of $(x-a)^{\left(1/n\right)}$, where n is a positive integer. However, it is likely that the arbitrary powers of $(x-a)$ could be treated in the infinite series factor of $f(x)$. This inclusion will complicate the discussion; thus we restrict our attention to the simple series as presented above.

Notice that this p has been chosen to ensure that the leading coefficient is non-zero in the above series. The above series satisfies the following condition

$$\begin{aligned} \lim_{x\to a}\big((x-a)f(x)\big) &= \lim_{x\to a}\begin{pmatrix} a_0(x-a)^{p+1} + a_1(x-a)^{p+1+\left(\frac{1}{n}\right)} + \\ a_2(x-a)^{p+1+\left(\frac{2}{n}\right)} + a_3(x-a)^{p+1+\left(\frac{3}{n}\right)} + \ldots \end{pmatrix} \\ &= 0 \qquad \text{for} \qquad p > -1 \end{aligned} \tag{2.241}$$

The expression $\lim_{x\to a}\big((x-a)f(x)\big)=0$, in (2.241) means that the area under the curve, described via the function $f(x)=(x-a)^p\sum_{j=0}^{\infty}a_j(x-a)^{\left(j/n\right)}$, with $a_0 \neq 0$, and $p>-1$ just near the start point $x=a$, is zero. This is what we mean when we say that $f(x)$ is better behaved at the lower terminal i.e. $x=a$ than $(x-a)^{-1}$.

The functions $f(x)=(x-a)^{-1/2}$, $f(x)=(x-a)^{-5/4}\sin\left(\sqrt{x-a}\right)$, $f(x)=\frac{1}{2}\sin(\pi(x-a))-1$, $f(x)=\ln(x-a)$ are such a class of functions. The important consequence of this representation of $f(x)$ is that the function $f(x)$ may be further decomposed as a finite sum of n differ-integrable "units" $f_\upsilon(x)$ each of which is a product of a power (greater than -1) of $(x-a)$ and a function analytic in $(x-a)$, that is represented below.

$$\begin{aligned} f(x) &= \left((x-a)^p\sum_{j_1=0}^{\infty}a_{j_1}(x-a)^{j_1}\right) + \left((x-a)^{p+\frac{1}{n}}\sum_{j_2=0}^{\infty}a_{j_2}(x-a)^{j_2}\right) + \ldots \\ &\qquad \ldots\ldots + \left((x-a)^{p+1-\frac{1}{n}}\sum_{j_n=0}^{\infty}a_{j_n}(x-a)^{j_n}\right) \\ &= \left((x-a)^p\sum_{j_1=0}^{\infty}a_{j_1}(x-a)^{j_1}\right) + \left((x-a)^{\frac{np+1}{n}}\sum_{j_2=0}^{\infty}a_{j_2}(x-a)^{j_2}\right) + \\ &\qquad ..+\left((x-a)^{\frac{np+n-1}{n}}\sum_{j_n=0}^{\infty}a_{j_n}(x-a)^{j_n}\right) \end{aligned} \tag{2. 242}$$

The functions that cannot be differ-integrated are $f(x)=(x-a)^{-2}$ and $f(x)=\sqrt{\frac{9}{4}-(x-a)^2}$. Many functions are not expansible as the differ-integrable series are nevertheless differ-integrable; for example, logarithmic and Heaviside's unit step function and even the trivial function $f(x) \equiv 0$. We used the nomenclature differ-integrable function to include all the above and in fact any function whose differ-integrals can be obtained.

2.23 Discrete numerical evaluation of the Riemann-Liouville fractional integration

In this section, we will approximate $\frac{d^\alpha}{dx^\alpha} f(x)$ for arbitrary $\alpha < 0$, with known values of $f(x)$ at $N+1$ and evenly spaced points in the range 0 to x. We designate symbols $f_N \equiv f(0)$, $f_{N-1} \equiv f\left(\frac{x}{N}\right)$, … $f_j \equiv f\left(x - j\frac{x}{N}\right)$, and … $f_0 \equiv f(x)$. The Riemann-Liouville (RL) fractional integration formula for $\alpha < 0$ we write as alternative representations that are obtained from the original formula by a change of variables see (2.129) and (2.130), we re-write as follows:

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(y)}{(x-y)^{\alpha+1}} dy = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(x-y)}{y^{\alpha+1}} dy \qquad \alpha < 0 \tag{2.243}$$

Re-writing the above integration (2.243) from 0 to x by splitting the integration into small intervals and then by summing all small integrations in all the discrete intervals we get the following:

$$\begin{aligned} \frac{d^\alpha f(x)}{dx^\alpha} &= \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(x-y)}{y^{\alpha+1}} dy \\ &= \frac{1}{\Gamma(-\alpha)} \left(\int_0^{x/N} \frac{(f(x-y))dy}{y^{\alpha+1}} + \int_{x/N}^{2x/N} \frac{(f(x-y))dy}{y^{\alpha+1}} + .. \int_{(N-1)x/N}^{x} \frac{(f(x-y))dy}{y^{\alpha+1}} \right) \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \int_{jx/N}^{(j+1)x/N} \frac{f(x-y)}{y^{\alpha+1}} dy \end{aligned} \tag{2.244}$$

2.23.1 Using the average of function values at the end of intervals to have a formula for numerical evaluation of the Riemann-Liouville fractional integration

Now we use approximation for $f(x-y)$ in the interval of integration i.e. $\left[j\Delta x, (j+1)\Delta x\right]$ as $f(x-y) \approx \frac{f(x-j\Delta x)+f(x-(j+1)\Delta x)}{2}$, averaging the values at $y = j\Delta x$ and $y = (j+1)\Delta x$ i.e. at the end-points of intervals. We then use them in the following steps:

$$\begin{aligned} \int_{jx/N}^{(j+1)x/N} \frac{\left(f(x-y)\right)dy}{y^{\alpha+1}} &\approx \frac{f\left(x - \frac{jx}{N}\right) + f\left(x - \frac{(j+1)x}{N}\right)}{2} \int_{jx/N}^{(j+1)x/N} \frac{dy}{y^{\alpha+1}} \\ &= \frac{f\left(x - \frac{jx}{N}\right) + f\left(x - \frac{x}{N} - \frac{jx}{N}\right)}{2} \int_{jx/N}^{(j+1)x/N} \frac{dy}{y^{\alpha+1}} \\ &= \frac{f_j + f_{j+1}}{-2\alpha} \left(\left(\frac{(j+1)x}{N}\right)^{-\alpha} - \left(\frac{jx}{N}\right)^{-\alpha} \right) \\ &= \frac{f_j + f_{j+1}}{-2\alpha} \left(\frac{x}{N}\right)^{-\alpha} \left((j+1)^{-\alpha} - j^{-\alpha}\right) \end{aligned} \tag{2.245}$$

In the above steps (2.245) for the evaluation of $\int_{jx/N}^{(j+1)x/N} \frac{f(x-y)}{y^{\alpha+1}} dy$ the approximated $f(x-y)$ i.e. $\frac{f(x-j\Delta x)+f(x-(j+1)\Delta x)}{2}$ comes out of the integration process as this is now not a variable in y. The point should also be noted that while we are writing an approximation as $f(x-y) \approx \frac{f(x-j\Delta x)+f(x-(j+1)\Delta x)}{2}$, we have used the end points of $y = j\Delta x$ and $y = (j+1)\Delta x$; these are *a-priori* defined as discrete points.

Using the above obtained expression (2.245) i.e. $\int_{jx/N}^{(j+1)x/N} \frac{f(x-y)}{y^{\alpha+1}} dy = \frac{f_j + f_{j+1}}{-2\alpha} \left(\frac{x}{N}\right)^{-\alpha} \left((j+1)^{-\alpha} - j^{-\alpha}\right)$ and by using the expression $\left(-\alpha\left(\Gamma(-\alpha)\right)\right) = \Gamma(-\alpha+1)$ we get the following formula:

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \left(\frac{x}{N}\right)^{-\alpha} \sum_{j=0}^{N-1} \frac{f_j + f_{j+1}}{2} \left((j+1)^{-\alpha} - j^{-\alpha}\right); \qquad \alpha < 0$$
$$= \frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{N-1} \frac{f_j + f_{j+1}}{2} \left((j+1)^{-\alpha} - j^{-\alpha}\right) \tag{2.246}$$

Here $\Delta x = \left(\frac{x}{N}\right)$ is a discrete step size. From the above derivation, we write a final formula for the discrete numerical evaluation of an RL fractional integration of order $\alpha > 0$ as follows:

$${}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(1+\alpha)} \sum_{j=0}^{N-1} \frac{f_j + f_{j+1}}{2} \left((j+1)^{\alpha} - j^{\alpha}\right); \qquad \alpha > 0 \tag{2.247}$$

Use of the other approximation will lead to a different formula.

2.23.2 Using the weighted average of the function values at the end of intervals to have a formula for the numerical evaluation of the Riemann-Liouville fractional integration

We use the approximation for integration i.e. the one based on a linear interpolation between f_{j+1} and f_j. We will use the weighted sum of f_{j+1} and f_j, to have an interpolation in this case. The interval of integration (and our interpolation) is from $y = j\Delta x$ to $y = (j+1)\Delta x$. Say we take any point y between $j\Delta x$ and $(j+1)\Delta x$. The distance between $j\Delta x$ to this point y is $(y - j\Delta x)$ and the distance between y and $(j+1)\Delta x$ is $\left((j+1)\Delta x - y\right)$. The relative distances from y to these two end-points, in the discrete step size Δx are $\frac{(y-j\Delta x)}{\Delta x}$ and $\frac{((j+1)\Delta x - y)}{\Delta x}$ respectively. With this, we write interpolation (or approximation) as

$$f(y) \approx \left(\frac{\left((j+1)\Delta x - y\right)}{\Delta x}\right) f_j + \left(\frac{(y - j\Delta x)}{\Delta x}\right) f_{j+1}; \qquad \Delta x = \frac{N}{x}$$
$$= \left((j+1) - y\left(\tfrac{N}{x}\right)\right) f_j + \left(y\left(\tfrac{N}{x}\right) - j\right) f_{j+1} \tag{2.248}$$

The weighted average technique used above for interpolation, gives a larger weight to f_j when the point y is close to $j\Delta x$ and gives more weight to f_{j+1} when the point is close to $(j+1)\Delta x$. When the point y is at the mid-point of the interval i.e. at $y = \left(j + \frac{1}{2}\right)\Delta x$, the two weights are equal to half. This central point interpolation is what we followed in the previous derivation (2.247), where we approximated it as $\frac{f_j + f_{j+1}}{2}$. This weighted sum (2.248) is the interpolation formula we now use in the following derivation:

$$\begin{aligned}
\int_{jx/N}^{(j+1)x/N} \frac{\left(f(x-y)\right)\mathrm{d}y}{y^{\alpha+1}} &\approx \int_{jx/N}^{(j+1)x/N} \frac{\begin{pmatrix}\left(1+j-\frac{Ny}{x}\right) f\left(x-\frac{jx}{N}\right)+ \\ \left(\frac{Ny}{x}-j\right) f\left(x-\frac{(j+1)x}{N}\right)\end{pmatrix}}{y^{\alpha+1}}\mathrm{d}y \\
&= \int_{jx/N}^{(j+1)x/n} \left(\frac{(1+j)f_j - \frac{Ny}{x} f_j + \frac{Ny}{x} f_{j+1} - j\ f_{j+1}}{y^{\alpha+1}}\right)\mathrm{d}y \\
&= \int_{jx/N}^{(j+1)x/N} \left(\frac{(1+j)f_j - j\ f_{j+1}}{y^{\alpha+1}}\right)\mathrm{d}y \\
&\quad + \left(\frac{N}{x}\right)\int_{jx/N}^{(j+1)x/N} \left(\frac{f_{j+1}-f_j}{y^{\alpha}}\right)\mathrm{d}y \\
&= \left((1+j)f_j - j\ f_{j+1}\right)\left[\frac{y^{-\alpha}}{-\alpha}\right]_{jx/N}^{(j+1)x/N} + \\
&\quad \left(\frac{x}{N}\right)^{-1}\left(f_{j+1}-f_j\right)\left[\frac{y^{-\alpha+1}}{1-\alpha}\right]_{jx/N}^{(j+1)x/N} \\
&= \left((1+j)f_j - j\ f_{j+1}\right)\left(\frac{x}{N}\right)^{-\alpha}\left(-\frac{(j+1)^{-\alpha}}{\alpha}+\frac{j^{-\alpha}}{\alpha}\right) \\
&\quad + \left(f_{j+1}-f_j\right)\left(\frac{x}{N}\right)^{-1}\left(\frac{x}{N}\right)^{-\alpha+1}\left(\frac{(j+1)^{-\alpha+1}}{1-\alpha}-\frac{j^{-\alpha+1}}{1-\alpha}\right) \\
&= \left(\frac{x}{N}\right)^{-\alpha}\begin{pmatrix}\dfrac{\left((1+j)f_j - j\ f_{j+1}\right)\left((j+1)^{-\alpha}-j^{-\alpha}\right)}{-\alpha} \\ +\dfrac{\left(f_{j+1}-f_j\right)\left((j+1)^{-\alpha+1}-j^{-\alpha+1}\right)}{1-\alpha}\end{pmatrix}
\end{aligned} \tag{2. 249}$$

The above steps in derivation (2.249) lead to a more complicated formula described as follows for $\alpha < 0$:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} &= \frac{1}{\Gamma(-\alpha)}\left(\frac{x}{N}\right)^{-\alpha}\sum_{j=0}^{N-1}\begin{pmatrix}\dfrac{\left((j+1)f_j - j\ f_j\right)\left((j+1)^{-\alpha}-j^{-\alpha}\right)}{-\alpha} \\ +\dfrac{\left(f_{j+1}-f_j\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)}{1-\alpha}\end{pmatrix} \\
&\frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\begin{pmatrix}\dfrac{\left((j+1)f_j - j\ f_j\right)\left((j+1)^{-\alpha}-j^{-\alpha}\right)}{-\alpha} \\ +\dfrac{\left(f_{j+1}-f_j\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)}{1-\alpha}\end{pmatrix}
\end{aligned} \tag{2.250}$$

From the above obtained expression (2.250) we write the fractional integration formula for $\alpha > 0$ as a discrete numerical evaluation for RL fractional integration as follows:

$${}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\begin{pmatrix}\dfrac{\left((j+1)f_j - j\ f_j\right)\left((j+1)^{\alpha}-j^{\alpha}\right)}{\alpha} \\ +\dfrac{\left(f_{j+1}-f_j\right)\left((j+1)^{1+\alpha}-j^{1+\alpha}\right)}{1+\alpha}\end{pmatrix} \tag{2.251}$$

We demonstrated the use of the numerical integration method to approximately evaluate the Riemann-Liouville fractional integration formula.

2.24 Short summary

In this chapter, we have discussed the various concepts of the classical integration process and concluded that there is an extension of what we know about classical integration. The discussion on fractional integration has given us a thought provoking 'question' as to how to unify this concept and then generalise to a differentiation process. This aspect we will take up subsequently. We have also hinted that fractional differentiation is also an integration process; on this issue, we shall build the concept of fractional differentiation further. We took the topic of fractional integration as a start point as we will note subsequently that this formulation is the core process of fractional differentiation. We noted here that the fractional calculus calls for various other types of functions called higher-transcendental functions, which we will be using subsequently too and several of them are described in the Appendices, which are used in the context of fractional calculus. Now we move now to a fractional differentiation process.

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CHAPTER THREE

FRACTIONAL DERIVATIVES

3.1 Introduction

We have stated while discussing concept generalisation in Chapter-1 about Leibniz stating to L'Hospital that putting $\frac{1}{2}$ in the formula of the n-th derivative of a function will lead to a paradox. Why? The paradoxes arise because there are several ways of generalising the differentiation operations to non-integer orders leading to equivalents but not always with the same results. Here we will deal with the most fundamental approaches of using a fractional derivative operator. We will also see that fractional differentiation needs be termed as a differ-integration process, and reason out why the fractional derivative is a non-local property, requiring memory and history. In short we will try to resolve the 'paradoxes' that seemed to exist three hundred years ago. We will try to find an answer to L'Hospital's question and will carry on the generalising of classical integer order calculus.

There are several symbols used to represent a fractional derivative operator of order $\alpha > 0$ and with $\alpha \in \mathbb{R}$. Leibniz and Euler used the symbol d^{α}, while Riemann used ∂_x^{α}. Liouville's notation is $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}$, while Grunwald gave a lower and upper limit in the representation of a fractional derivative operator, that is $\left[\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}}\right]_{x=a}^{x=x}$, for the function $f(x)$ from $a > -\infty$ to x, while $x > a$. Marchaud used the notation D_a^{α}. Hardy-Littlewood used $f^{(\alpha)}$. Many authors in the modern day use $I^{-\alpha}$, $I_x^{-\alpha}$, ${}_aD_x^{\alpha}$, $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}$, $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}(x-a)^{\alpha}}$ and D_{a+}^{α}, for $\alpha > 0$. We will be using some of these symbols.

3.2 Iterative differentiation and its generalisation

Like in Chapter-2, where we talked about iterative integration, here we start with a repeated n – fold differentiation or iterative differentiation.

3.2.1 Euler's scheme of generalisation for repeated differentiation

This is Euler's scheme of a repeated n – fold differentiation (with n as a positive integer), of a power function $f(x) = x^p$, $p > 0$ which is reproduced below (with the understanding that $D^k \equiv \frac{\mathrm{d}^k}{\mathrm{d}x^k}$, $k = 0,1,2,....n$):

$$D^0\left[x^p\right] = x^p, \qquad D^1\left[x^p\right] = px^{p-1}, \qquad D^2\left[x^p\right] = p(p-1)x^{p-2} \tag{3.1}$$

For n – times differentiation, we obtain the following:

$$\begin{aligned} D^n\left[x^p\right] &= p(p-1)(p-2).....(p-n+1)x^{p-n} \\ &= \frac{p(p-1)...(p-n+1)\big((p-n)(p-n-1)...1\big)}{\big((p-n)(p-n-1)...1\big)}x^{p-n} \\ &= \frac{p!}{(p-n)!}x^{p-n} \end{aligned} \tag{3.2}$$

Equation (3.2) is called Euler's formula. Recall that we did this in Chapter-2. Now we generalize (3.2) for a positive real number α and $p > 0$, by using the gamma function as a generalisation for a factorial; as we did in the case of integration (Chapter-2) to get the following expression:

$$D^{\alpha}\left[x^p\right] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha} \tag{3.3}$$

The above expression (3.3) we may term as a fractional differentiation of $f(x) = x^p$. The expression is valid for $p > -1$, the simple reason being that otherwise $\Gamma(p+1)$ will blow up at $p = -1$. In addition to what we have discussed in the last chapter (Section 2.22) that notes; the function (say x^p) needs to be better behaved than x^{-1} at the start point, (which in this case is $x = 0$), for it to be fractionally differ-integrated. When $\alpha < 0$ say $\alpha = -\upsilon$ with $\upsilon > 0$, we have a formula for fractional integration as follows, which we have derived in an earlier chapter (Section 2.6), that we get to by placing $\alpha = -\upsilon$, $\upsilon > 0$ as $D^{-\upsilon}\left[x^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}x^{p+\upsilon} = I^{\upsilon}\left[x^p\right]$.

This is an indication that there is a concept unifying fractional integration and fractional differentiation into a single expression. In the previous chapter (Sections 2.14 and 2.19) we hinted at that. We note this point here too, and write the unified formula for the fractional-differ-integration of a power function (discussed in Section 1.23) as for the integer order n case, that is the following:

$$\frac{\mathrm{d}^n\left[(x-a)^p\right]}{\left[\mathrm{d}(x-a)\right]^n} = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-n+1)}(x-a)^{p-n} & \begin{cases} n = 0,1,2,.....; & \text{For all } p \\ n = -1,-2,....; & p > -1 \end{cases} \quad (3.4) \\ \infty & n = -1,-2,....; \quad p \le -1 \end{cases}$$

We may now generalize (3.4) for a non-integer order α and write the following:

$$\frac{\mathrm{d}^\alpha\left[(x-a)^p\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha} & p > -1 \quad \text{For all } \alpha \\ \infty & p \le -1 \quad \text{For all } \alpha \end{cases} \tag{3.5}$$

3.2.2 A seeming paradox in Euler's scheme of generalisation for repeated differentiation

We should now give thought to Euler's formula (3.5) putting $p = -\frac{1}{2}$, and $\alpha = \frac{1}{2}$, with $a = 0$ and applying the above formula (3.5):

$$\begin{aligned} D^{1/2}\left[x^{-1/2}\right] &= \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}-\frac{1}{2}+1\right)}x^{-\frac{1}{2}-\frac{1}{2}} \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(0)}x^{-1} = 0 \end{aligned} \tag{3.6}$$

As $\Gamma(0) = \infty$ we get the above result (3.6).

This seems absurd presently, as we know that a constant function should give zero while doing derivative operations. We will deliberate this observation of getting a zero for a half derivative for non-constant functions, later.

3.2.3 Euler's formula applied to find the fractional derivative and fractional integration of power functions

Applying this formula to find a half derivative and a half integral of $f(x) = x$, using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, and $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$ we get the following:

$$\begin{aligned} D^{1/2}\left[x\right] &= \frac{\Gamma(2)}{\Gamma\left(1-\frac{1}{2}+1\right)}x^{1-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{x} = 2\sqrt{\frac{x}{\pi}} \\ D^{-1/2}\left[x\right] &= \frac{\Gamma(2)}{\Gamma\left(1+\frac{1}{2}+1\right)}x^{1+\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}} = \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}} \end{aligned} \tag{3.7}$$

The above expression (3.7) shows the applicability of Euler's formula to get a fractional derivative and a fractional integral of functions like x^p, with $p > -1$. Here we mention that the start point of the process of fractional differentiation and the fractional integration itself is at $x = 0$, so the fractional derivative operator is written as ${}_0D_x^\alpha$.

Figure-3.1 shows various fractional orders of ${}_0D_x^\alpha\left[x\right]$, for $\alpha = 0, 1/4, 1/3, 1/2$. We observe that in the case of Figure-3.1, all the fractional derivatives pass through the origin and its value is zero at the start point of the fractional derivative process i.e. ${}_0D_x^\alpha\left[f(x)\right]\Big|_{x=0} = 0$ for $f(x) = x$. In addition, we observe that as α increases from zero towards one the fractional derivatives tend to be curved towards $y = 1$ which is one whole derivative of $f(x) = x$; i.e. $f^{(1)}(x) = 1$.

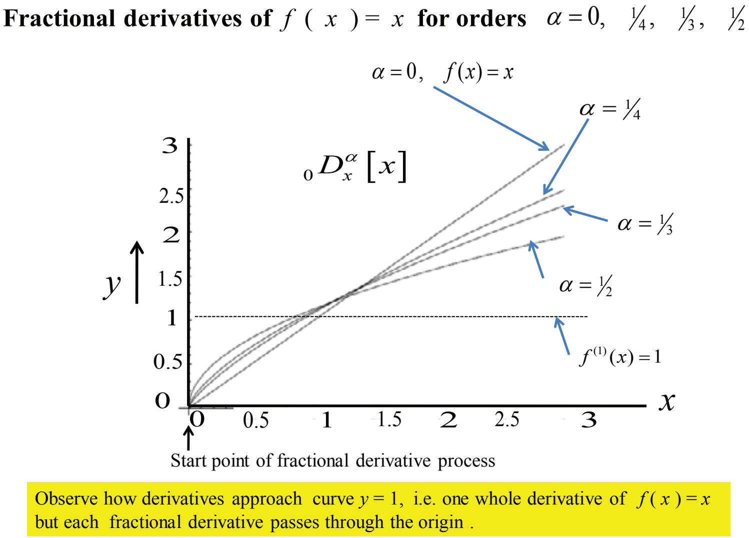


Figure-3.1: Fractional derivative of $f(x) = x$ for $\alpha = 0$, $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$

3.2.4 That Euler's formula gives us non-zero as a fractional derivative for a constant function seems to be a paradox

With this Euler's formula, we write the fractional derivative and fractional integral of the constant $f(x) = 1$ as follows:

$$D^{\frac{1}{2}}\left[1\right] = \frac{\Gamma(1)}{\Gamma\left(0 - \frac{1}{2} + 1\right)} x^{0-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}$$
$$D^{-\frac{1}{2}}\left[1\right] = \frac{\Gamma(1)}{\Gamma\left(0 + \frac{1}{2} + 1\right)} x^{0+\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}} \tag{3.8}$$

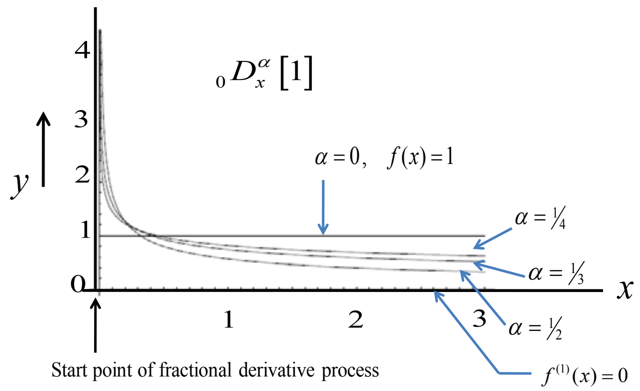
Here we again come across a paradox that is a constant function, which is not returning a zero after the fractional derivative operation.

Figure-3.2 gives the curves of fractional derivatives ${}_0D_x^{\alpha}\left[f(x)\right]$ for $\alpha = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ of a constant function i.e. $f(x) = 1$, with a start point of the fractional derivative operation which is at $x = 0$. We observe in Figure-3.2 that at $x = 0$ the values of fractional derivatives blow up, i.e. ${}_0D_x^{\alpha}\left[f(x)\right]\Big|_{x=0} = \infty$, for $f(x) = 1$.

This implies that for a constant function we get a singularity at the start point of the fractional derivative process. In addition, we observe that all the fractional derivatives tend towards a value of 'zero' at $x \uparrow \infty$, i.e. $\lim_{x\uparrow\infty}\left({}_0D_x^{\alpha}\left[f(x)\right]\right) = 0$, for $f(x) = 1$ which incidentally is one-whole derivative i.e. ${}_0D_x^{(1)}\left[1\right] = f^{(1)}(x) = 0$.

We observe the singularity at the start point of a fractional differentiation process in Figure-3.2, while there is no singularity at the start point $x = 0$ in Figure-3.1. This is a very important observation, and we say that in Figure-3.1 the start value of the function at the start of a fractional differentiation process ($x = 0$) is zero i.e. $f(x)\big|_{x=0} = 0$ whereas in Figure-3.2 the start value of the function is non-zero i.e. $f(x)\big|_{x=0} \neq 0$. This observation we will make use of in subsequent chapters.

Fractional derivatives of $f(x) = 1$ for orders $\alpha = 0, \ ^1\!/_4, \ ^1\!/_3, \ ^1\!/_2$



Observe that the fractional derivatives converge to $y = 0$ for large x; except for an asymptote at $x = 0$ where there is singularity i.e. ${}_0D_x^\alpha[1]\Big|_{x=0} = \infty$

Figure-3.2: Fractional derivatives of a constant

Note that all the fractional derivatives of a constant go to zero at a far point; while at the start point of fractional differentiation (that is in this case $x = 0$), the value of a fractional derivative is infinity (i.e. we get singularity at the start point).

3.2.5 The observation of Euler's formula: the fractional derivative of a constant becomes zero when the fractional order of the derivative tends to one-whole number in the classical case

Well for a function that is constant such as $f(x) = C$, we get a fractional derivative like $D^\alpha\left[C\right] = \frac{1}{\Gamma(1-\alpha)} C x^{-\alpha}$, by applying Euler's formula. This is counterintuitive as we are used to having zero as a derivative of the constant. However, here in the fractional derivative case, we are getting a non-zero. Rather we are getting a decaying power function which tends to zero at large values of x (see Figure-3.2) In the expression $D^\alpha\left[C\right] = \frac{1}{\Gamma(1-\alpha)} C x^{-\alpha}$ for the fractional derivative of a constant, when α tends to 1, the denominator, i.e. $\Gamma(1-\alpha)$, tends to $\Gamma(0)$ which tends towards ∞; thus, one-whole derivative of a constant tends to zero. The fractional derivative of the constant obtained is $D^\alpha\left[C\right] = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)}$, and tends to zero only when α is a natural number. This comes from the property of the gamma function that is $\Gamma(-z) = \frac{-\pi}{z\Gamma(z)\sin\pi z}$. For z a Natural number $\mathbb{N}$, $\sin\pi z = 0$ giving $\Gamma(-z) = \infty$.

This non-zero value for a fractional derivative of a constant function, gives irritation to many scientists, thus they tend not to use fractional calculus. Well, let us carry on for the time being, with this observation seemingly as a paradox.

3.2.6 Some interesting formulas for the semi-derivative and semi-integration with Euler's formula

We obtain an interesting semi-derivative and semi-integration with the Euler generalisation

$$D^{1/2}\left[x^{n+\frac{1}{2}}\right] = \frac{\Gamma\left(n + \frac{1}{2} + 1\right)}{\Gamma(n+1)} x^n \qquad n = 0, 1, 2, 3, \dots \tag{3.9}$$

We use the formula $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$, $\Gamma(n+1) = n!$, $(n+1)! = (n+1)(n!)$ and write the following steps and thereby achieve interesting results for the semi-derivative:

$$
\begin{aligned}
D^{1/2}\left[x^{n+\frac{1}{2}}\right] &= \frac{\Gamma\left(\frac{1}{2}+(n+1)\right)}{n!}x^n \\
&= \frac{\left(2(n+1)\right)!\sqrt{\pi}}{4^{(n+1)}(n+1)!n!}x^n = \frac{(2n+2)!\sqrt{\pi}}{4(n+1)!n!}\left(\frac{x}{4}\right)^n \\
&= \frac{(2n+2)(2n+1)!\sqrt{\pi}}{4(n+1)(n!)^2}\left(\frac{x}{4}\right)^n \\
&= \frac{(2n+1)!\sqrt{\pi}}{2(n!)^2}\left(\frac{x}{4}\right)^n
\end{aligned}
\tag{3.10}
$$

The result of semi-integration is as follows:

$$
\begin{aligned}
D^{-1/2}\left[x^{n+\frac{1}{2}}\right] &= \frac{\Gamma\left(n+1+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+1+\frac{1}{2}\right)}x^{n+1} \qquad n=0,1,2,3,\ldots \\
&= \frac{\Gamma\left(\frac{1}{2}+(n+1)\right)}{\Gamma(n+2)}x^{n+1} = \frac{\Gamma\left(\frac{1}{2}+(n+1)\right)}{(n+1)!}x^{n+1} \\
&= \frac{\left(2(n+1)\right)!\sqrt{\pi}}{4^{(n+1)}(n+1)!(n+1)!}x^{n+1} \\
&= \frac{(2n+2)!\sqrt{\pi}}{\left((n+1)!\right)^2}\left(\frac{x}{4}\right)^{n+1}
\end{aligned}
\tag{3.11}
$$

A further result on the semi-derivative is as follows:

$$
\begin{aligned}
D^{1/2}\left[x^n\right] &= \frac{\Gamma(n+1)}{\Gamma\left(n+1-\frac{1}{2}\right)}x^{n-\frac{1}{2}} \qquad n=0,1,2,3,\ldots \\
&= \frac{(n!)4^n(n!)}{(2n)!\sqrt{\pi}}x^n x^{-\frac{1}{2}} = \frac{(n!)^2(4x)^n}{(2n)!\sqrt{\pi x}}
\end{aligned}
\tag{3.12}
$$

The following is the result of semi-integration:

$$
\begin{aligned}
D^{-1/2}\left[x^n\right] &= \frac{\Gamma(n+1)}{\Gamma\left((n+1)+\frac{1}{2}\right)}x^{n+\frac{1}{2}} \qquad n=0,1,2,3,\ldots \\
&= \frac{(n!)(4)^{(n+1)}(n+1)!}{\left(2(n+1)\right)!\sqrt{\pi}}x^n x^{\frac{1}{2}} \\
&= \frac{(n!)(n+1)(n!)(4)^n(4)x^n x^{\frac{1}{2}}}{(2n+2)(2n+1)!\sqrt{\pi}} = \frac{(n!)^2(4x)^n(4)^{\frac{1}{2}}x^{\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} \\
&= \frac{(n!)^2(4x)^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}}
\end{aligned}
\tag{3.13}
$$

3.2.7 Applying Euler's formula for a function represented as power series expansion

Let a function be represented as the series expansion $f(x)=\sum_{n=0}^{\infty}a_n x^n$. Then we can take the help of Euler's formula and apply it term by term to get the following:

$$
\begin{aligned}
D^{\alpha}\left[f(x)\right] &= \sum_{n=0}^{\infty}a_n D^{\alpha}\left[x^n\right] \\
&= \sum_{n=0}^{\infty}a_n\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}
\end{aligned}
\tag{3.14}
$$

Let us evaluate a $\frac{1}{2}$ derivative of $f(x)=e^x$ by applying Euler's formula term by term for expansion of $f(x)=\sum_{n=0}^{\infty} a_n x^n$, which is, $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$. To this series, if we apply Euler's formula we get a semi-derivative as follows

$$D^{\frac{1}{2}}\left[e^x\right]=\frac{1}{\sqrt{\pi x}}\left(1+2x+\frac{4}{3}x^2+\frac{8}{15}x^3+\frac{16}{105}x^4+.....\right) \tag{3.15}$$

Figure-3.3 plots this semi-derivative of e^x and the function e^x.

Fractional Derivative of Exponential Function is not Exponential

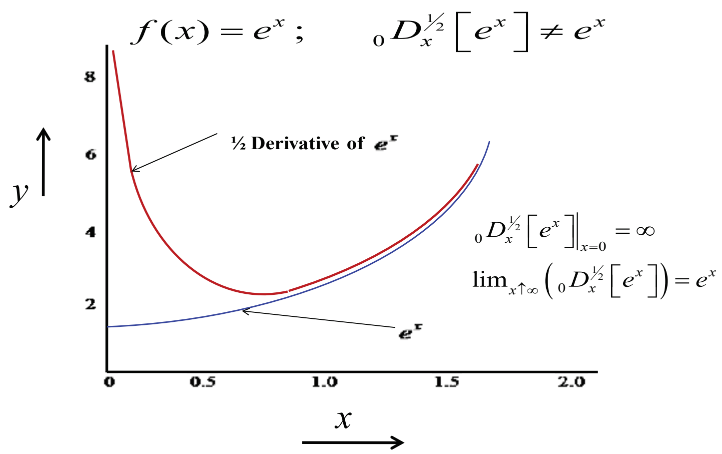


Figure-3.3: The semi-derivative of an exponential function

Figure-3.3 elicits that the fractional derivative of an exponential function is not an exponential function, from Euler's formula. Due to this perhaps, Leibniz thought it a paradox. This is an important observation, and we will discuss it further, subsequently. Also, note that in Figure-3.3, there is a singularity at the start point of the fractional differentiation process i.e. at $x=0$ and the curve converges towards e^x for large x, with the knowledge that e^x is a classical derivative of $f(x)=e^x$. This we write as $\lim_{x\uparrow\infty}\left(D^{\frac{1}{2}}\left[e^x\right]\right)=e^x$.

3.2.8 Generalising the iterated differentiation for x^{-p} to get a fractional derivative formula, a different result is obtained by Euler's scheme

Now we do the iterated differentiation of the function $f(x)=x^{-p}$, with $p>0$, as follows:

$$\begin{aligned} D^1\left[x^{-p}\right] &= -px^{-p-1} \\ D^2\left[x^{-p}\right] &= -p(-p-1)x^{-p-2} \\ &= (-1)^2 p(p+1)x^{-(p+2)} \\ D^3\left[x^{-p}\right] &= (-p)(-p-1)(-p-2)x^{-p-3} \\ &= (-1)^3 p(p+1)(p+2)x^{-(p+3)} \end{aligned} \tag{3.16}$$

Continuing this for n – times, we have the following expression:

$$D^n\left[x^{-p}\right]=(-1)^n p(p+1)(p+2)....(p+n-1)x^{-(p+n)} \tag{3.17}$$

We will manipulate the above obtained expression (3.17), as depicted in the following steps:

$$D^n\left[x^{-p}\right]=(-1)^n\begin{pmatrix}(p+n-1)(p+n-2)(p+n-3)...\\(p+3)(p+2)(p+1)p\end{pmatrix}x^{-(p+n)}$$
$$=(-1)^n\frac{\big((p+n-1)(p+n-2)...(p+1)p\big)\big((p-1)(p-2)...1\big)}{\big((p-1)(p-2)...1\big)}x^{-(p+n)} \tag{3.18}$$
$$=(-1)^n\frac{(p+n-1)!}{(p-1)!}x^{-(p+n)}=(-1)^n\frac{\Gamma(p+n)}{\Gamma(p)}x^{-(p+n)}$$

For the positive real numbers α and p, we generalise the above obtained expression (3.18) by using the gamma function instead of the factorial and get the following:

$$D^\alpha\left[x^{-p}\right]=(-1)^\alpha\frac{\Gamma(p+\alpha)}{\Gamma(p)}x^{-(p+\alpha)} \tag{3.19}$$

We mention here that if we take $\alpha<0$ say $\alpha=-\upsilon$, we write the above expression (3.19) as $D^{-\upsilon}\left[x^{-p}\right]=(-1)^{-\upsilon}\left(\frac{\Gamma(p-\upsilon)}{\Gamma(p)}\right)x^{-p+\upsilon}$.

Note that we have obtained the Weyl fractional integration of x^{-p} as ${}_xW_\infty^{-\upsilon}\left[x^{-p}\right]=\frac{\Gamma(p-\upsilon)}{\Gamma(p)}x^{-p+\upsilon}$ (in Section 2.15.2), which is similar to the above derivation except (3.19) with regards to the term $(-1)^{-\upsilon}$. In addition, we observe the above expression (3.19) returns a complex quantity for the fractional derivative, since $(-1)^\alpha$ is a complex quantity, that is $(-1)^\alpha=e^{i\pi\alpha}=\cos(\pi\alpha)+i\sin(\pi\alpha)$. Therefore, we write the following expression:

$$D^\alpha\left[x^{-p}\right]=e^{i\alpha\pi}\frac{\Gamma(p+\alpha)}{\Gamma(p)}x^{-(p+\alpha)} \tag{3.20}$$

Here if we place $\alpha=\frac{1}{2}$ and $p=\frac{1}{2}$, we obtain the following result:

$$D^{1/2}\left[x^{-1/2}\right]=e^{i\frac{\pi}{2}}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}x^{-\left(\frac{1}{2}+\frac{1}{2}\right)}$$
$$=i\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}x^{-1} \tag{3.21}$$
$$=\frac{i}{\sqrt{\pi}}x^{-1}$$

We had used earlier in (3.5) and (3.6) the formula $D^\alpha\left[x^p\right]=\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}$, to find that $D^{1/2}\left[x^{-1/2}\right]=0$, revealing some contradictions. Whereas in this derivation (3.21) we observe $D^{1/2}\left[x^{-1/2}\right]=\frac{1}{\sqrt{\pi}}ix^{-1}$, a complex (imaginary) quantity. Therefore, Euler partially resolved Leibniz's paradox to state that:

$$\frac{\mathrm{d}^\alpha x^\beta}{\mathrm{d}x^\alpha}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha} \tag{3.22}$$

which is valid for the integer and non-integer α and β, and we can possibly generalisethe above Euler formula (3.21) to all functions that are expanded into a power series, which might seem a natural step.

3.3 A natural extension of normal integer order classical derivative to obtain a fractional derivative: the exponential approach of Leibniz and Liouville

Leibniz implicitly assumed that for any real number α, the fractional order derivative of $f(x)=e^{\lambda x}$, was:

$$\frac{\mathrm{d}^\alpha e^{\lambda x}}{\mathrm{d}x^\alpha}=\lambda^\alpha e^{\lambda x} \tag{3.23}$$

This was a natural way to extend the concept of classical calculus. For an integer n we are used to formulae i.e. $\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{\lambda x}=\lambda^n e^{\lambda x}$; in a classical sense. This approach (3.23) is also termed as an exponential approach. Therefore, the above postulate (3.23) could equally take a natural start point and define the fractional derivative, for functions that are expanded in series with exponentials, such as $f(x)\approx\sum_k c_k e^{\lambda_k x}$, which we can write as:

$$\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}=\sum_{k}c_k\lambda_k^{\alpha}e^{\lambda_k x} \tag{3.24}$$

3.3.1 Obtaining the fractional derivative by exponential approach for $x^{-\beta}$ by using the Laplace integral

We shall now use the above postulate (3.23) to find the fractional derivative of function $f(x)=x^{-\beta}$ by using the representation of $x^{-\beta}$ by the Laplace integral as in the following expression:

$$x^{-\beta}=\frac{1}{\Gamma(\beta)}\int_0^{\infty}e^{-yx}y^{\beta-1}\mathrm{d}y \tag{3.25}$$

The above expression (3.25) has an integral representation of $x^{-\beta}$ in terms of the exponential function (the derivation of this integral representation of $x^{-\beta}$ we will describe shortly). Now, applying the above postulate of Leibniz, we can write the following:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}x^{-\beta}}{\mathrm{d}x^{\alpha}}&=\frac{1}{\Gamma(\beta)}\int_0^{\infty}y^{\beta-1}\left(\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}(e^{-yx})\right)\mathrm{d}y\\&=\frac{1}{\Gamma(\beta)}\int_0^{\infty}y^{\beta-1}\left((-y)^{\alpha}e^{-yx}\right)\mathrm{d}y\\&=\frac{(-1)^{\alpha}}{\Gamma(\beta)}\int_0^{\infty}y^{\alpha+\beta-1}e^{-yx}\mathrm{d}y\end{aligned} \tag{3.26}$$

Take the integral in the above expression (3.26) i.e. $\mathrm{I}=\int_0^{\infty}e^{-yx}y^{\alpha+\beta-1}\mathrm{d}y$, put $z=yx$, that gives $\mathrm{d}y=\frac{\mathrm{d}z}{x}$. We write the integral i.e. I with this variable change as $\mathrm{I}=\int_0^{\infty}e^{-z}\left(\frac{z}{x}\right)^{\alpha+\beta-1}\left(\frac{1}{x}\right)\mathrm{d}z$, which after simplification is $\mathrm{I}=\frac{1}{x^{\alpha+\beta}}\int_0^{\infty}e^{-z}z^{(\alpha+\beta)-1}\mathrm{d}z$.

Here we recognise the integral $\int_0^{\infty}e^{-z}z^{(\alpha+\beta)-1}\mathrm{d}z=\Gamma(\alpha+\beta)$, which comes from the integral definition of the gamma function. Therefore using all these steps, we obtain the following:

$$\frac{\mathrm{d}^{\alpha}x^{-\beta}}{\mathrm{d}x^{\alpha}}=\frac{(-1)^{\alpha}\Gamma(\alpha+\beta)}{x^{\alpha+\beta}\Gamma(\beta)} \tag{3.27}$$

If we generalise the above to $-\beta$ (presently disregarding the existence of the integral), we get:

$$\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{(-1)^{\alpha}\Gamma(-\beta+\alpha)}{\Gamma(-\beta)}x^{\beta-\alpha} \tag{3.28}$$

This is not similar to $\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, as obtained by generalising the repeated differentiation of x^{p} earlier in (3.5).

The difference becomes obvious for $\alpha=\frac{1}{2}$, and $\beta=-\frac{1}{2}$, as we get from the formula $\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{(-1)^{\alpha}\Gamma(-\beta+\alpha)}{\Gamma(-\beta)}x^{\beta-\alpha}$, $D^{1/2}\left[x^{-1/2}\right]=\frac{i}{\sqrt{\pi}}x^{-1}$; and from the formula $\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, we get $D^{1/2}\left[x^{-1/2}\right]=0$. This is revealing an inconsistency again.

3.3.2 Representing $x^{-\beta}$ by using the Laplace integral in terms of the exponential function by using the Laplace integral derivation

Now we discuss the Laplace integral representing $x^{-\beta}$ that we used in the above discussion regarding (3.25). From the table of the Laplace transforms we have a property called the 'frequency integration', while the parameter t is represented as the Laplace complex-frequency (say s), for a function $f(t)$ whose Laplace Transform is $F(s)$. The property is $\mathcal{L}\left\{\frac{f(t)}{t}\right\} \equiv \int_s^\infty \left(F(\sigma)\right)\mathrm{d}\sigma$, and is 'frequency integration'. From here we apply the definition of the Laplace transform, which is $\mathcal{L}\left\{f(t)\right\} = F(s) \overset{\text{def}}{=} \int_0^\infty e^{-st}\left(f(t)\right)\mathrm{d}t$, to the LHS of the property of frequency integration as follows;

$$\int_0^\infty e^{-st}\frac{f(t)}{t}\mathrm{d}t = \int_s^\infty \left(F(\sigma)\right)\mathrm{d}\sigma \tag{3.29}$$

Here $f(t) = t^\beta$, so we have from the LHS of (3.29) $\mathrm{I} = \int_0^\infty e^{-st}\frac{(t^\beta)}{t}\mathrm{d}t = \int_0^\infty e^{-st}t^{\beta-1}\mathrm{d}t$. From the Tables of Laplace Transforms we have $\mathcal{L}\left\{t^\beta\right\} = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$. Use this in the above property of the frequency integration with $F(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$, that is at the RHS of (3.29) to get the following:

$$\begin{aligned} \int_s^\infty \left(F(\sigma)\right)\mathrm{d}\sigma &= \int_s^\infty \frac{\Gamma(\beta+1)}{\sigma^{\beta+1}}\mathrm{d}\sigma \\ &= \Gamma(\beta+1)\left[\frac{\sigma^{-(\beta+1)+1}}{-(\beta+1)+1}\right]_s^\infty = \Gamma(\beta+1)\left[\frac{\sigma^{-\beta}}{-\beta}\right]_s^\infty \\ &= \frac{s^{-\beta}}{\beta}\Gamma(\beta+1) = s^{-\beta}\left(\Gamma(\beta)\right) \end{aligned} \tag{3.30}$$

In the above steps (3.30) we used $\Gamma(z+1) = z\left(\Gamma(z)\right)$ as the property of the gamma function. Therefore, from this derivation we have the following expression:

$$s^{-\beta}\left(\Gamma(\beta)\right) = \int_0^\infty e^{-st}t^{\beta-1}\mathrm{d}t \tag{3.31}$$

Change the variables s to x and t to y in the above expression to get the following expressions:

$$x^{-\beta}\left(\Gamma(\beta)\right) = \int_0^\infty e^{-yx}y^{\beta-1}\mathrm{d}y \quad , \quad x^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-yx}y^{\beta-1}\mathrm{d}y \tag{3.32}$$

The above expression (3.32) is the Laplace integral representation of $x^{-\beta}$, which we had used to derive $\frac{\mathrm{d}^\alpha x^{-\beta}}{\mathrm{d}x^\alpha} = \frac{(-1)^\alpha\Gamma(\beta+\alpha)}{\Gamma(\beta)}x^{-\beta-\alpha}$, via an 'exponential approach'.

Thus (3.32) is an integral representation of the power law function $f(x) = x^{-\alpha}$, $x^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-1}e^{-\lambda x}\mathrm{d}\lambda$ and is a useful expression.

3.3.3 A fractional derivative of the trigonometric and exponential functions using an exponential approach

From our classical calculus, we write the following for repeated differentiation of $f(x) = \sin x$:

$$D^0\left[\sin x\right] = \sin x, \quad D^1\left[\sin x\right] = \cos x, \quad D^2\left[\sin x\right] = -\sin x \tag{3.33}$$

Every differentiation operation (D) on $\sin x$ gives a 90^0 or $\frac{\pi}{2}$ phase lead. So for $n-$ fold differentiation of $\sin x$, we write:

$$D^n\left[\sin x\right] = \sin\left(x + \tfrac{n\pi}{2}\right) \tag{3.34}$$

For n as a real number, say α, we may write the following:

$$D^\alpha\left[\sin x\right] = \sin\left(x + \tfrac{\alpha\pi}{2}\right), \quad D^\alpha\left[\cos x\right] = \cos\left(x + \tfrac{\alpha\pi}{2}\right) \tag{3.35}$$

Therefore, like for classical calculus, for the exponential function we have the n – th derivative as $D^n\left[e^{ax}\right]=a^n e^{ax}$ and we can extend this as for any order α, real complex, positive, negative, etc. as in the following:

$$D^\alpha\left[e^{ax}\right]=a^\alpha e^{ax} \tag{3.36}$$

3.3.4 An exponential approach postulate of Leibniz and extended by Liouville

This exponential approach was a postulate of Leibniz, and then Liouville used it further. There is a Liouville class of functions, $f(x)$ which can be extended in an exponential Fourier Series as $f(x)=\sum_{-\infty}^{+\infty} c_n e^{inx}$. Then by Liouville's postulate, one can write the fractional derivative as:

$$D^\alpha\left[f(x)\right]=\sum_{-\infty}^{+\infty} i^\alpha n^\alpha c_n e^{inx} \tag{3.37}$$

With this Liouville wrote the following as also described in (3.35):

$$D^\alpha\left[\sin ax\right]=a^\alpha \sin\left(ax+\tfrac{\alpha\pi}{2}\right) \tag{3.38}$$

This Liouville (1832) postulate is the direct application of a classical integer order derivative formula to get fractional derivatives. If it were so, then there would be no need for any elaborate development of fractional calculus. However, at this stage, we state that this formulation is true in certain cases, which we will discuss later.

3.4 Liouville's way of looking at a fractional derivative contradicts Euler's generalisation

3.4.1 Liouville and Euler's fractional derivatives approach is applied to an exponential function

Like an ordinary integer order derivative, Liouville too postulated that for exponential function the fractional derivative is taken as follows:

$$D^\alpha\left[e^{x}\right]=e^{x} \tag{3.39}$$

Let us apply Euler's generalisation on $f(x)=e^x$, by a series expansion the exponential function $e^x=\sum_{n=0}^{\infty}\left(x^n/n!\right)$. We apply (3.5) i. e. $D^\alpha\left[x^n\right]=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$ term by term to the series representing e^x, while using $n!=\Gamma(n+1)$ and write the following:

$$D^\alpha\left[e^{x}\right]=\sum_{n=0}^{\infty}\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)} \tag{3.40}$$

We note that:

$$D^\alpha\left[e^{x}\right]=\sum_{n=0}^{\infty}\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}\neq e^x \tag{3.41}$$

Therefore, with Euler's generalisation, we say that $D^\alpha\left[e^x\right]\neq e^x$, unless α is a whole number. That is, if α is a positive real number, then e^x and $\sum_{n=0}^{\infty}\left(\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}\right)$ are different. Thus, we have hit at a contradiction in Liouville's postulate.

We also had earlier contradictions as we demonstrated earlier. The contradictions were, for $\frac{d^\alpha x^\beta}{dx^\alpha}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$ (3.22) achieved by the use of Euler's formula, and $\frac{d^\alpha x^\beta}{dx^\alpha}=\frac{(-1)^\alpha\Gamma(-\beta+\alpha)}{\Gamma(-\beta)}x^{\beta-\alpha}$ (3.28) obtained by Leibniz's postulate.

The inconsistency becomes obvious for $\alpha=\frac{1}{2}$, and $\beta=-\frac{1}{2}$, as we get from the formula (3.28) i.e. $\frac{d^\alpha x^\beta}{dx^\alpha}=\frac{(-1)^\alpha\Gamma(-\beta+\alpha)}{\Gamma(-\beta)}x^{\beta-\alpha}$, $D^{1/2}\left[x^{-1/2}\right]=\frac{i}{\sqrt{\pi}}x^{-1}$ and from the formula (3.22) $\frac{d^\alpha x^\beta}{dx^\alpha}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, we get $D^{1/2}\left[x^{-1/2}\right]=0$.

No other way to see this inconsistency expands the exponential function by a power series; as done in (3.40) and as demonstrated in Figure-3.3, and applies Euler's generalisation scheme. We summarise:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[e^{x}\right]=\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\sum_{k=0}^{\infty}\frac{x^{k}}{k!}\right]$$
$$=\sum_{k=0}^{\infty}\frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}\neq e^{x} \tag{3.42}$$

These two inconsistencies show that Euler's rule is not consistent with Leibniz's postulate. Liouville also found this inconsistency while calculating the fractional derivative with the rule of Euler, i.e. $\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}}=\sum_{k} c_{k}\lambda_{k}^{\alpha}e^{\lambda_{k}x}$ for functions represented by $f(x)\approx\sum_{k} c_{k}e^{\lambda_{k}x}$.

3.4.2 Discussing the paradoxes of Euler's and Leibniz's by considering formulas to get classical one-whole integration–and a resolution

A resolution of Leibniz's paradox emerges when the formulas of Euler, $\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, and of Leibniz, $\frac{\mathrm{d}^{\alpha}e^{\lambda x}}{\mathrm{d}x^{\alpha}}=\lambda^{\alpha}e^{\lambda x}$, are compared for $\alpha=-1$. That is, when they are interpreted as classical one-whole integrations.

Actually, Leibniz suggested this interpretation as the following:

$$\frac{\mathrm{d}^{-1}e^{x}}{\mathrm{d}x^{-1}}=e^{x}=\int_{-\infty}^{x}e^{y}\mathrm{d}y$$
$$\neq\int_{0}^{x}e^{y}\mathrm{d}y=e^{x}-1=\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\left[\sum_{k=0}^{\infty}\frac{x^{k}}{k!}\right] \tag{3.43}$$

This suggests that Euler's fractional derivative on the RHS of (3.43) differs from Leibniz's/Liouville's fractional derivative at the LHS of (3.43).

Similarly, $\frac{\mathrm{d}^{\alpha}x^{\beta}}{\mathrm{d}x^{\alpha}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$ with $\alpha=-1$, Euler's generalisation, corresponds to the following:

$$\frac{\mathrm{d}^{-1}x^{\beta}}{\mathrm{d}x^{-1}}=\frac{x^{\beta+1}}{\beta+1}=\int_{0}^{x}y^{\beta}\mathrm{d}y \tag{3.44}$$

On the other hand Leibniz's postulate i.e. $\frac{\mathrm{d}^{\alpha}x^{-\beta}}{\mathrm{d}x^{\alpha}}=\frac{(-1)^{\alpha}\Gamma(\beta+\alpha)}{\Gamma(\beta)}x^{-(\beta+\alpha)}$ with $\alpha=-1$ corresponds to the following:

$$\frac{\mathrm{d}^{-1}x^{-\beta}}{\mathrm{d}x^{-1}}=\frac{x^{1-\beta}}{1-\beta}=-\int_{x}^{\infty}y^{-\beta}\mathrm{d}y=\int_{\infty}^{x}y^{-\beta}\mathrm{d}y \tag{3.45}$$

Note how (3.44) and (3.45) differ in the limit of integration.

3.4.3 The difference between Euler's and Leibniz's postulate is due to the limits of integration

The argument discussed suggests that Euler's and Liouville's definitions differ with the limit of integration in (3.44) and (3.45). Let us write the following integration:

$${}_{b}D_{x}^{-1}\left[e^{ax}\right]=\int_{b}^{x}e^{ax}\mathrm{d}x=\frac{1}{a}e^{ax}=a^{-1}e^{ax} \tag{3.46}$$

For what value of b this is (3.46) true? Such a question we address as follows:

$$\int_{b}^{x}e^{ax}\mathrm{d}x=\frac{1}{a}e^{ax}-\frac{1}{a}e^{ab} \tag{3.47}$$

The above integration (3.47) is equal to $a^{-1}e^{ax}$ (3.46), for $ab=-\infty$. So for $a>0$ and $b=-\infty$, we have ${}_{-\infty}D_{x}^{-1}\left[e^{ax}\right]=a^{-1}e^{ax}$ with $a>0$. Therefore, the Liouville postulate should be valid if we write ${}_{-\infty}D_{x}^{\alpha}\left[e^{ax}\right]=a^{\alpha}e^{ax}$.

We have discussed regarding Figure-3.3, for $x \uparrow \infty$ we get $\lim_{x\uparrow\infty}\left({}_0D_x^{\alpha}\left[e^x\right]\right)=e^x$, and observe the singularity at $x=0$, the start point of fractional differentiation. If the singularity point i.e. the start point of a fractional differentiation is pushed to $x=-\infty$, we will get $\lim_{c\downarrow-\infty}\left({}_cD_x^{\alpha}\left[e^x\right]\right)=e^x$; that is what we discussed for the formula i.e. ${}_{-\infty}D_x^{\alpha}\left[e^{ax}\right]=a^{\alpha}e^{ax}$.

3.4.4 The fractional derivative requires the limits of a start-point and end-point for integration

Now consider the integration for x^p as depicted below:

$$\begin{aligned}{}_bD_x^{-1}\left[x^p\right]&=\int_b^x x^p\mathrm{d}x=\frac{x^{p+1}}{p+1}-\frac{b^{p+1}}{p+1}\\&=\frac{x^{p+1}}{p+1}\qquad\text{for}\qquad b=0\end{aligned}\tag{3.48}$$

The limit of integration plays an important role in the fractional differentiation or fractional integration. This we would like to state here. Therefore, we cannot rule out Liouville's postulate as a contradiction, but it is dependent on a lower limit which in this case is $-\infty$. These arguments also point to a question: 'does fractional differentiation require a lower and higher limit as in integration?' Indeed, it is true that unlike a normal classical integer order derivative, the fractional derivative does have lower and upper limits like with integration.

3.5 Applying Liouville's logic to get a fractional derivative of x^{-a} and a cosine function

This approach is an exponential one which concerns what we state as Liouville's postulate. Thus if we write, in terms of Liouville's logic:

$${}_{-\infty}D_x^{\pm\upsilon}\left[e^{ax}\right]=a^{\pm\upsilon}e^{ax}\tag{3.49}$$

Then this (3.49) becomes a natural way to extend the derivative of any arbitrary order that is υ (a natural number, integer, real number or complex number). We will see how Liouville's logic is applied for finding a fractional derivative of x^{-a}, with $a>0$. For that, consider the following integral:

$$\mathrm{I}=\int_0^{\infty}u^{a-1}e^{-xu}\mathrm{d}u\qquad a>0\qquad x>0\tag{3.50}$$

Put $xu=t$, in the above integral (3.50) to get:

$$\mathrm{I}=x^{-a}\int_0^{\infty}t^{a-1}e^{-t}\mathrm{d}t=x^{-a}\Gamma(a)\tag{3.51}$$

Note that the definition $\Gamma(a)=\int_0^{\infty}t^{a-1}e^{-t}\mathrm{d}t$ is used in (3.51). We thus get the following:

$$x^{-a}=\frac{1}{\Gamma(a)}(\mathrm{I})\qquad;\qquad \mathrm{I}=x^{-a}\int_0^{\infty}t^{a-1}e^{-t}\mathrm{d}t\tag{3.52}$$

We will use Liouville's logic, ${}_{-\infty}D_x^{\upsilon}\left[e^{ax}\right]=a^{\upsilon}e^{ax}$, in the following steps to get ${}_{-\infty}D_x^{\upsilon}\left[x^{-a}\right]$, by using an expression obtained for x^{-a} in terms of the integral defined as I in (3.52).

It is depicted in the following steps:

$$
\begin{aligned}
{}_{-\infty}D_x^{\upsilon}\left[x^{-a}\right] = {}_{-\infty}D_x^{\upsilon}\left[\frac{1}{\Gamma(a)}(\mathrm{I})\right] &= \frac{1}{\Gamma(a)}\,{}_{-\infty}D_x^{\upsilon}\left[\int_0^{\infty} u^{a-1}e^{-xu}\mathrm{d}u\right] \\
&= \frac{1}{\Gamma(a)}\left(\int_0^{\infty} u^{a-1}\left({}_{-\infty}D_x^{\upsilon}\left[e^{-xu}\right]\right)\mathrm{d}u\right) \\
&= \frac{1}{\Gamma(a)}\left(\int_0^{\infty} u^{a-1}(-u)^{\upsilon}e^{-xu}\mathrm{d}u\right) \\
&= \frac{1}{\Gamma(a)}\int_0^{\infty} u^{a-1}(-1)^{\upsilon}u^{\upsilon}e^{-xu}\mathrm{d}u \\
&= \frac{(-1)^{\upsilon}}{\Gamma(a)}\int_0^{\infty} u^{a+\upsilon-1}e^{-xu}\mathrm{d}u
\end{aligned}
\tag{3.53}
$$

We will do some tricks here. In our defined integral I, i.e. $\mathrm{I}=\int_0^{\infty} u^{a-1}e^{-xu}\mathrm{d}u = x^{-a}\Gamma(a)$ we will replace a with $(a+\upsilon)$, and get the following expression:

$$\int_0^{\infty} u^{a+\upsilon-1}e^{-xu}\mathrm{d}u = x^{-(a+\upsilon)}\left(\Gamma(a+\upsilon)\right) \tag{3.54}$$

Using this expression in the last line of our derivation in the above steps of (3.53), i.e. ${}_{-\infty}D_x^{\upsilon}\left[x^{-a}\right] = \frac{(-1)^{\upsilon}}{\Gamma(a)}\int_0^{\infty} u^{a+\upsilon-1}e^{-xu}\mathrm{d}u$, we can write the following expression:

$${}_{-\infty}D_x^{\upsilon}\left[x^{-a}\right] = \frac{(-1)^{\upsilon}}{\Gamma(a)}\left(\Gamma(a+\upsilon)\right)x^{-a-\upsilon} \tag{3.55}$$

Therefore, we have used Liouville's logic to obtain ${}_{-\infty}D_x^{\upsilon}\left[x^{-a}\right] = \frac{(-1)^{\upsilon}\Gamma(a+\upsilon)}{\Gamma(a)}x^{-a-\upsilon}$ – a difficult way to derive it. However, we have derived this via Laplace integration which earlier (3.28) we wrote as $\frac{\mathrm{d}^{\upsilon}x^{-a}}{\mathrm{d}x^{\upsilon}} = \frac{(-1)^{\upsilon}\Gamma(\upsilon+\alpha)}{\Gamma(\upsilon)}x^{-(\upsilon+\alpha)}$.

Now we apply this Liouville logic to get a fractional derivative of the cosine function, with an exponential approach ${}_{-\infty}D_x^{\pm\upsilon}\left[e^{ax}\right] = a^{\pm\upsilon}e^{ax}$ as depicted in the following steps:

$$
\begin{aligned}
{}_{-\infty}D_x^{\upsilon}\left[\cos(x)\right] &= {}_{-\infty}D_x^{\upsilon}\left(\frac{e^{ix}+e^{-ix}}{2}\right) \\
&= \frac{(i)^{\upsilon}e^{ix}+(-i)^{\upsilon}e^{-ix}}{2}; \qquad (\pm i)^{\upsilon} = \left(e^{\pm i\pi/2}\right)^{\upsilon} = \left(e^{\pm i\pi\upsilon/2}\right) \\
&= \frac{(e^{i\upsilon\pi/2})e^{ix}+(e^{-i\upsilon\pi/2})e^{-ix}}{2} = \frac{e^{i\left(x+\frac{\upsilon\pi}{2}\right)}+e^{-i\left(x+\frac{\upsilon\pi}{2}\right)}}{2} \\
&= \cos\left(x+\tfrac{\upsilon\pi}{2}\right)
\end{aligned}
\tag{3.56}
$$

We have a nice result, i.e.

$${}_{-\infty}D_x^{\upsilon}\left[\cos(x)\right] = \cos\left(x+\tfrac{\upsilon\pi}{2}\right) \tag{3.57}$$

Thus, the generalised differential operator simply shifts the phase of the cosine function (and likewise the sine function) by $\upsilon\times 90^0$ that is in proportion to the order of the differentiation.

For differentiation the process advances the phase, the integration makes the phase lagged. After all for $\upsilon=1$ and, $\upsilon=-1$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = \cos\left(x+\tfrac{\pi}{2}\right) \quad , \quad \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\cos(x) = \cos\left(x-\tfrac{\pi}{2}\right) \tag{3.58}$$

Needless to say that, this approach can be applied to the exponential, i.e. the $\frac{1}{2}$ derivative of e^{ax} should be:

$$ {}_{-\infty}D_x^{\frac{1}{2}}\left[e^{ax}\right] = (a^{\frac{1}{2}})e^{ax} \tag{3.59}$$

With this, we do indeed find that the $\upsilon-$th derivative of e^{ax} is simply $a^{\upsilon}e^{ax}$ consistent with the purely exponential approach.

Now we write the fractional derivative as ${}_aD_x^{\alpha}$ following on with a lower terminal as $a=-\infty$ and $a=0$ in the following expressions:

$$\begin{aligned}
&{}_{-\infty}D_x^{\alpha}\left[e^{x}\right] = e^{x}\\
&{}_{0}D_x^{\alpha}\left[x^{p}\right] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}\\
&{}_{0}D_x^{\alpha}\left[e^{x}\right] = \sum_{n=0}^{\infty}\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}\\
&{}_{-\infty}D_x^{\alpha}\left[e^{ax}\right] = a^{\alpha}e^{ax}\\
&{}_{a}D_x^{\alpha}\left[(x-c)^{p}\right] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-c)^{p-\alpha} \qquad \text{only if} \quad a=c
\end{aligned} \tag{3.60}$$

3.6 Liouville's approach of a fractional derivative to arrive at formulas as a limit of difference quotients

3.6.1 Liouville's formula for fractional integration

It has been mentioned that Liouville defined a fractional derivative as $\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}} = \sum_k c_k\lambda_k^{\alpha}e^{\lambda_k x}$ for functions represented as the sum of the exponentials $f(x)\approx\sum_k c_k e^{\lambda_k x}$. Liouville seems not to have recognised the necessity of a limit of integration. Thus from this postulate Liouville derives several integrals and series representations, particularly, he finds the fractional integral of order $\alpha>0$, as follows:

$$\begin{aligned}
\int^{\alpha}\left(f(x)\right)(\mathrm{d}x)^{\alpha} &= \int_0^{-\infty}\left(f(x+y)\right)y^{\alpha-1}\mathrm{d}y\\
\int^{\alpha}\left(f(x)\right)(\mathrm{d}x)^{\alpha} &= \int_0^{\infty}\left(f(x-y)\right)y^{\alpha-1}\mathrm{d}y
\end{aligned} \tag{3.61}$$

The first one in (3.61) has the condition $f(-\infty)=0$, and the second one, also in (3.61) has the condition $f(\infty)=0$.

3.6.2 Liouville's formula for a fractional derivative in integral representation

Then Liouville gave the formula of a fractional derivative of the order $\alpha>0$, as:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}} &= \frac{1}{(-1)^{n-\alpha}\Gamma(n-\alpha)}\int_0^{-\infty}\frac{\mathrm{d}^{n}f(x+y)}{\mathrm{d}x^{n}}y^{n-\alpha-1}\mathrm{d}y\\
\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}} &= \frac{1}{(-1)^{n-\alpha}\Gamma(n-\alpha)}\int_0^{\infty}\frac{\mathrm{d}^{n}f(x-y)}{\mathrm{d}x^{n}}y^{n-\alpha-1}\mathrm{d}y
\end{aligned} \tag{3.62}$$

where $(n-1)<\alpha<n$, with n as a positive integer.

Liouville restricts the discussion to functions represented as $f(x)\approx\sum_k c_k e^{\lambda_k x}$, where $\lambda_k>0$ so that $f(-\infty)=0$. Let us stop for a moment and study these Liouville formulas. We will derive Liouville's formulas of (3.62).

3.6.3 Derivation of Liouville's fractional derivative formulas

We will apply the Liouville postulates i.e. $\frac{d^\alpha f(x)}{dx^\alpha}=\sum_k c_k\lambda_k^\alpha e^{\lambda_k x}$ for functions $f(x)$ that are represented as the sum of exponentials i.e. $f(x)=\sum_k c_k e^{\lambda_k x}$. Put $f(x)=\sum_k c_k e^{\lambda_k x}$ and using the fractional integration of the order $\alpha>0$, that is, $\frac{d^{-\alpha}e^{\lambda_k x}}{dx^{-\alpha}}=\lambda_k^{-\alpha}e^{\lambda_k x}$, we write Liouville's fractional integral formula for $f(x)=\sum_k c_k e^{\lambda_k x}$. This integral is finite (bounded) for limits 0 and $-\infty$, for $\lambda_k>0$. We write the following:

$$\int^\alpha \left(f(x)\right)\left(dx^\alpha\right)=\left[\sum_k c_k(\lambda_k^{-\alpha})e^{\lambda_k x}\right]_0^{-\infty} \tag{3.63}$$

Now we substitute $f(x)=\sum_k c_k e^{\lambda_k x}$, in the RHS of (3.61), i.e. Liouville's fractional integration formula, and get the following expression:

$$\begin{aligned}\int_0^{-\infty}\left(f(x+y)\right)y^{\alpha-1}dy&=\int_0^{-\infty}\sum_k c_k e^{\lambda_k(x+y)}y^{\alpha-1}dy\\&=\int_0^{-\infty}\sum_k c_k e^{\lambda_k x}e^{\lambda_k y}y^{\alpha-1}dy\end{aligned} \tag{3.64}$$

We substitute in the above expression (3.64), $\lambda_k y=-z$, $dy=-\frac{dz}{\lambda_k}$. In it the y limits are from 0 to $-\infty$; so, the changed variable z will have limits from 0 to ∞ and we express the above (3.64) integral $\int_0^{-\infty}\left(f(x+y)\right)y^{\alpha-1}dy$ as in the following steps:

$$\begin{aligned}\int_0^{-\infty}\left(f(x+y)\right)y^{\alpha-1}dy&=\int_{y=0}^{y=-\infty}\sum_k c_k e^{\lambda_k x}e^{\lambda_k y}y^{\alpha-1}dy\\&=\int_{z=0}^{z=\infty}\sum_k c_k e^{\lambda_k x}e^{-z}\left(-\tfrac{z}{\lambda_k}\right)^{\alpha-1}\left(-\tfrac{dz}{\lambda_k}\right)\\&=\sum_k c_k e^{\lambda_k x}\left(\tfrac{(-1)^\alpha}{\lambda_k^\alpha}\right)\int_0^\infty e^{-z}z^{\alpha-1}dz\\&=(-1)^\alpha\Gamma(\alpha)\sum_k\frac{c_k e^{\lambda_k x}}{\lambda_k^\alpha}\\&=(-1)^\alpha\Gamma(\alpha)\int^\alpha\left(f(x)\right)\left(dx^\alpha\right)\end{aligned} \tag{3.65}$$

We have used $\Gamma(\alpha)=\int_0^\infty e^{-z}z^{\alpha-1}dz$, i.e. the definition of the gamma function in the above derivation (3.65) and we get:

$$\int^\alpha\left(f(x)\right)\left(dx^\alpha\right)=\frac{1}{(-1)^\alpha\Gamma(\alpha)}\int_0^{-\infty}\left(f(x+y)\right)y^{\alpha-1}dy \tag{3.66}$$

The above expression (3.66) is for $\lambda_k>0$ and $f(-\infty)=0$. Say we write the above fractional integration of the order $(1-\alpha)>0$, the formula will look as follows:

$$\int^{(1-\alpha)}\left(f(x)\right)(dx)^{(1-\alpha)}=\frac{1}{(-1)^{1-\alpha}\Gamma(1-\alpha)}\int_0^{-\infty}\left(f(x+y)\right)y^{-\alpha}dy \tag{3.67}$$

Differentiating one whole time (3.67), we get by using $D^1I^{1-\alpha}=D^1D^{-(1-\alpha)}=D^\alpha$ the following:

$$\frac{d^\alpha f(x)}{dx^\alpha}=\frac{1}{(-1)^{1-\alpha}\Gamma(1-\alpha)}\int_0^{-\infty}\frac{d\ \left(f(x+y)\right)}{dx}y^{-\alpha}dy \tag{3.68}$$

Therefore, for $(n-\alpha)>0$ order integration and then followed by n whole differentiation we get the fractional order derivative formula of Liouville, for $(n-1)<\alpha<n$ as:

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{1}{(-1)^{n-\alpha}\Gamma(n-\alpha)}\int_0^{-\infty}\frac{\mathrm{d}^n\left(f(x+y)\right)}{\mathrm{d}x^n}y^{n-\alpha-1}\mathrm{d}y \tag{3.69}$$

The use of $D^nI^{n-\alpha} = D^nD^{-(n-\alpha)} = D^\alpha$ is valid as an initial condition, $f(-\infty)=0$ and the differentiation integration process is from the zero state.

Put $f(x)=\sum_k c_k e^{\lambda_k x}$, using the fractional integration of the order $\alpha > 0$, which is $\frac{\mathrm{d}^{-\alpha}e^{\lambda_k x}}{\mathrm{d}x^{-\alpha}} = \lambda_k^{-\alpha}e^{\lambda_k x}$, and write Liouville's fractional integral formula. This integral is finite for limits 0 and $+\infty$, for $\lambda_k < 0$. Thus we write:

$$\int^\alpha \left(f(x)\right)\left(\mathrm{d}x\right)^\alpha = \left[\sum_k c_k(\lambda_k^{-\alpha})e^{\lambda_k x}\right]_0^\infty \tag{3.70}$$

Now we substitute $f(x)=\sum_k c_k e^{\lambda_k x}$, in the RHS of (3.61) i.e. Liouville's fractional integration formula and get the following:

$$\begin{aligned}\int_0^\infty \left(f(x-y)\right)y^{\alpha-1}\mathrm{d}y &= \int_0^\infty\sum_k c_k e^{\lambda_k(x-y)}y^{\alpha-1}\mathrm{d}y\\ &= \int_0^\infty\sum_k c_k e^{\lambda_k x}e^{-\lambda_k y}y^{\alpha-1}\mathrm{d}y\end{aligned} \tag{3.71}$$

We put in the above expression (3.71) $\lambda_k y = z$, $\mathrm{d}y = \frac{\mathrm{d}z}{\lambda_k}$; we get the following steps:

$$\begin{aligned}\int_0^\infty \left(f(x-y)\right)y^{\alpha-1}\mathrm{d}y &= \int_0^\infty\sum_k c_k e^{\lambda_k x}e^{-z}\left(\frac{z}{\lambda_k}\right)^{\alpha-1}\left(\frac{\mathrm{d}z}{\lambda_k}\right)\\ &= \sum_k c_k e^{\lambda_k x}\left(\frac{1}{\lambda_k^\alpha}\right)\int_0^\infty e^{-z}z^{\alpha-1}\mathrm{d}z\\ &= \Gamma(\alpha)\sum_k(\lambda_k)^{-\alpha}c_k e^{\lambda_k x}\\ &= \Gamma(\alpha)\int^\alpha\left(f(x)\right)(\mathrm{d}x)^\alpha\end{aligned} \tag{3.72}$$

From the above steps (3.72) we get the following expression:

$$\int^\alpha\left(f(x)\right)(\mathrm{d}x)^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^\infty\left(f(x-y)\right)y^{\alpha-1}\mathrm{d}y \tag{3.73}$$

The above expression (3.73) is for $\lambda_k < 0$ and $f(\infty)=0$. Therefore for the $(n-\alpha)>0$ integration and then followed by the n whole differentiation we get the fractional order derivative formula of Liouville, for $(n-1)<\alpha<n$:

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^\infty\frac{\mathrm{d}^n\left(f(x-y)\right)}{\mathrm{d}x^n}y^{n-\alpha-1}\mathrm{d}y \tag{3.74}$$

The use of $D^nI^{n-\alpha} = D^nD^{-(n-\alpha)} = D^\alpha$ is valid as an initial condition, $f(\infty)=0$, and the differentiation integration process is from the zero state.

3.6.4 Extension of Liouville's approach to arrive at formulas of the fractional derivative as a limit of difference quotients

In Liouville's method, we have $\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \sum_k c_k(\lambda_k^\alpha)e^{\lambda_k x}$, for the function $f(x)=\sum_k c_k e^{\lambda_k x}$ expanded as the sum of exponentials. What are the values of λ_k, in the exponential expansion of $f(x)=\sum_k c_k e^{\lambda_k x}$? The expression of $(\lambda_k)^\alpha$ for $\lambda_k > 0$ and $\lambda_k < 0$ are the following:

$$\lambda_k^\alpha = \lim_{h\downarrow 0}\frac{1}{h^\alpha}\left(1-e^{-h\lambda_k}\right)^\alpha \qquad \lambda_k > 0$$
$$\lambda_k^\alpha = (-1)^\alpha \lim_{h\downarrow 0}\frac{1}{h^\alpha}\left(1-e^{h\lambda_k}\right)^\alpha \qquad \lambda_k < 0 \tag{3.75}$$

We will find out how in (3.75) the expressions are obtained.

Take the function $g_1(x)=e^{\lambda_k x}$ with $\lambda_k>0$. Differentiating this function i.e. $g_1(x)=e^{\lambda_k x}$ at $x=0$, we get $g_1^{(1)}(x)\Big|_{x=0}=\left[\lambda_k e^{\lambda_k x}\right]_{x=0}=\lambda_k$. The derivative of $g_1(x)$ at $x=0$, with h as an infinitesimal increment in x, we write the following:

$$g_1^{(1)}(0)=\lim_{h\downarrow 0}\frac{g_1(0)-g_1(0-h)}{h} \tag{3.76}$$

Placing $g_1(x)=e^{\lambda_k x}$, and $g_1^{(1)}(0)=\lambda_k$ in (3.76), we have:

$$\lambda_k = \lim_{h\downarrow 0}\frac{g_1(0)-g_1(-h)}{h} = \lim_{h\downarrow 0}\frac{e^{\lambda_k(0)}-e^{\lambda_k(-h)}}{h}$$
$$\lambda_k = \lim_{h\downarrow 0}\frac{1-e^{-\lambda_k h}}{h} \tag{3.77}$$
$$\lambda_k^\alpha = \lim_{h\downarrow 0}\left(\frac{1-e^{-\lambda_k h}}{h}\right)^\alpha = \lim_{h\downarrow 0}\frac{1}{h^\alpha}\left(1-e^{-\lambda_k h}\right)^\alpha \qquad \lambda_k>0$$

Now we write $g_2(x)=e^{-\lambda_k x}, \lambda_k>0$, and $g_2^{(1)}(x)=-\lambda_k e^{-\lambda_k x}$. At $x=0$ therefore we have $\lambda_k=-g_2^{(1)}(0)$. As done in the above case for $g_1^{(1)}(0)$, we write the following steps:

$$g_2^{(1)}(0)=\lim_{h\downarrow 0}\frac{g_2(0)-g_2(0-h)}{h}=\lim_{h\downarrow 0}\frac{g_2(0)-g_2(-h)}{h}$$
$$=\lim_{h\downarrow 0}\frac{1-e^{-\lambda_k(-h)}}{h} \tag{3.78}$$
$$=\lim_{h\downarrow 0}\frac{1-e^{\lambda_k h}}{h}=-\lambda_k$$

Therefore, we have:

$$\lambda_k = -\lim_{h\downarrow 0}\frac{1-e^{\lambda_k h}}{h} \tag{3.79}$$

From the above expression (3.79) we write:

$$\lambda_k^\alpha = (-1)^\alpha \lim_{h\downarrow 0}\frac{1}{h^\alpha}\left(1-e^{\lambda_k h}\right)^\alpha \qquad \lambda_k<0 \tag{3.80}$$

In Liouville's method, we have $\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}=\sum_k c_k(\lambda_k^\alpha)e^{\lambda_k x}$ for the function $f(x)=\sum_k c_k e^{\lambda_k x}$ expanded as a sum of exponentials. To use the value of λ_k^α, we need the expansion for $\left(1-e^{-h\lambda_k}\right)^\alpha$ or $\left(1-e^{h\lambda_k}\right)^\alpha$. For that, we use the power series expansion expressed in the following formula:

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+... \tag{3.81}$$

We use the above (3.81) to expand $\left(1-e^{-h\lambda_k}\right)^\alpha$, for $\lambda_k>0$ and get the following:

$$\left(1-e^{-h\lambda_k}\right)^\alpha = 1-\alpha e^{-h\lambda_k}+\frac{\alpha(\alpha-1)}{2!}e^{-2h\lambda_k}-... \tag{3.82}$$

Then come the following steps:

$$
\begin{aligned}
\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} &= \sum_{k} c_k \lambda_k^{\alpha} e^{\lambda_k x} \qquad \lambda_k > 0 \\
&= \lim_{h\downarrow 0} \sum_{k} c_k \left[\tfrac{1}{h^{\alpha}} \left(1 - e^{-h\lambda_k}\right)^{\alpha} \right] e^{\lambda_x x} \\
&= \lim_{h\downarrow 0} \sum_{k} \tfrac{c_k e^{\lambda_k x}}{h^{\alpha}} \left(1 - e^{-h\lambda_k}\right)^{\alpha} \\
&= \lim_{h\downarrow 0} \sum_{k} \tfrac{c_k e^{\lambda_k x}}{h^{\alpha}} \left[1 - \alpha e^{-\lambda_k h} + \frac{\alpha(\alpha-1)}{2!} e^{-2\lambda_k h} - \dots \right] \\
&= \lim_{h\downarrow 0} \sum_{k} \left[\tfrac{1}{h^{\alpha}} c_k e^{\lambda_k x} - \tfrac{1}{h^{\alpha}} \alpha c_k e^{\lambda_k (x-h)} + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!} \right) c_k e^{\lambda_k (x-2h)} - \dots \right] \\
&= \lim_{h\downarrow 0} \left[\begin{array}{l} \tfrac{1}{h^{\alpha}} \sum_{k} c_k e^{\lambda_k x} - \tfrac{1}{h^{\alpha}} \alpha \sum_{k} c_k e^{\lambda_k (x-h)} \\ + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!} \right) \sum_{k} c_k e^{\lambda_k (x-2h)} - \dots \end{array} \right] \\
&= \lim_{h\downarrow 0} \left[\tfrac{1}{h^{\alpha}} f(x) - \tfrac{1}{h^{\alpha}} \alpha f(x-h) + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!} \right) f(x-2h) - \dots \right] \\
&= \lim_{h\downarrow 0} \frac{1}{h^{\alpha}} \left[\sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x-mh) \right]
\end{aligned} \tag{3.83}
$$

We have used the binomial coefficients as appearing in the expansion above (3.83) which are 1, α, $\frac{\alpha(\alpha-1)}{2!}$ …, generalised for α as a non-integer, as follows:

$$
\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha-m)!} = \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)!} \tag{3.84}
$$

For $\lambda_k < 0$, we have $\lambda_k^{\alpha} = \lim_{h\downarrow 0} (-1)^{\alpha} \frac{1}{h^{\alpha}} \left(1 - e^{\lambda_k h}\right)^{\alpha}$, we use the power series expansion for $(1+x)^{\alpha}$ as mentioned above (3.81) to write the following expression

$$
\left(1 - e^{h\lambda_k}\right)^{\alpha} = 1 - \alpha e^{h\lambda_k} + \frac{\alpha(\alpha-1)}{2!} e^{2h\lambda_k} - \dots \tag{3.85}
$$

From this, we derive the following steps:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} &= \sum_k c_k \lambda_k^{\alpha} e^{\lambda_k x} \qquad \lambda_k > 0 \\
&= \lim_{h\downarrow 0} \sum_k c_k \left[\tfrac{(-1)^{\alpha}}{h^{\alpha}} \left(1 - e^{h\lambda_k}\right)^{\alpha} \right] e^{\lambda_x x} \\
&= \lim_{h\downarrow 0} \sum_k (-1)^{\alpha} \tfrac{c_k e^{\lambda_k x}}{h^{\alpha}} \left(1 - e^{h\lambda_k}\right)^{\alpha} \\
&= \lim_{h\downarrow 0} \sum_k (-1)^{\alpha} \tfrac{c_k e^{\lambda_k x}}{h^{\alpha}} \left[1 - \alpha e^{\lambda_k h} + \tfrac{\alpha(\alpha-1)}{2!} e^{2\lambda_k h} - \ldots\right] \\
&= \lim_{h\downarrow 0} (-1)^{\alpha} \sum_k \begin{bmatrix} \tfrac{1}{h^{\alpha}} c_k e^{\lambda_k x} - \tfrac{1}{h^{\alpha}} \alpha c_k e^{\lambda_k (x+h)} \\ + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!}\right) c_k e^{\lambda_k (x+2h)} - \ldots \end{bmatrix} \\
&= \lim_{h\downarrow 0} (-1)^{\alpha} \begin{bmatrix} \tfrac{1}{h^{\alpha}} \sum_k c_k e^{\lambda_k x} - \tfrac{1}{h^{\alpha}} \alpha \sum_k c_k e^{\lambda_k (x+h)} \\ + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!}\right) \sum_k c_k e^{\lambda_k (x+2h)} - \ldots \end{bmatrix} \\
&= \lim_{h\downarrow 0} (-1)^{\alpha} \left[\tfrac{1}{h^{\alpha}} f(x) - \tfrac{1}{h^{\alpha}} \alpha f(x+h) + \tfrac{1}{h^{\alpha}} \left(\tfrac{\alpha(\alpha-1)}{2!}\right) f(x+2h) - \ldots \right] \\
&= \lim_{h\downarrow 0} \frac{(-1)^{\alpha}}{h^{\alpha}} \left[\sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x+mh) \right]
\end{aligned} \tag{3.86}$$

Therefore from Liouville's postulate for functions that are represented as a series with a sum of exponentials, that is $f(x) = \sum_k c_k e^{\lambda_k x}$, the fractional derivative is represented as $\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \sum_k c_k \lambda_k^{\alpha} e^{\lambda_k x}$. We arrive at a formula that contains the representation of the integer order derivatives as limits of difference quotients, those are:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} &= \lim_{h\downarrow 0} \left[\frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x-mh) \right] \qquad \lambda_k > 0 \\
\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} &= \lim_{h\downarrow 0} (-1)^{\alpha} \left[\frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x+mh) \right] \qquad \lambda_k < 0
\end{aligned} \tag{3.87}$$

where the generalised binomial coefficient (Section 1.12) is the following:

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha-m+1)} \tag{3.88}$$

This idea (3.87) was taken up by Grunwald who defined fractional derivatives as limits of the generalised difference quotients; we call this Grunwald's formula. Grunwald wanted to free the definition of fractional derivatives from their special form of function. Liouville's postulate was on functions, which are only expanded as an exponential series.

Grunwald emphasised that fractional derivatives are 'interoderivatives'. From calculus based on the limits of the difference quotients, the Grunwald formula is thus the following:

$$F\left[a, x, \alpha, h\right]_f = \sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} \frac{f(x-kh)}{h^{\alpha}}; \qquad N = \frac{x-a}{h} \tag{3.89}$$

Grunwald then calls (3.89) the following expression:

$$D^{\alpha} \left[f(x)\right]_{x=a}^{x=x} = \lim_{h\downarrow 0} F\left[a, x, \alpha, h\right]_f \tag{3.90}$$

that is the α -th, differential quotient taken over a straight line from a to x .

3.7 A repeated integration approach to get the fractional derivative in Riemann-Liouville and Caputo formulations

We have seen in Chapter-2 the iterated integrals, and they take the following form:

$$\frac{\mathrm{d}^{-2}}{\mathrm{d}x^{-2}}f(x)=\int_0^x\int_0^{u_1}\left(f(u_1)\right)\mathrm{d}u_2\mathrm{d}u_1=\int_0^x(x-u)\left(f(u)\right)\mathrm{d}u$$
$$\frac{\mathrm{d}^{-3}}{\mathrm{d}x^{-3}}f(x)=\int_0^x\int_0^{u_1}\int_0^{u_2}\left(f(u_1)\right)\mathrm{d}u_3\mathrm{d}u_2\mathrm{d}u_1=\frac{1}{2!}\int_0^x(x-u)^2\left(f(u)\right)\mathrm{d}u \tag{3.91}$$

and so on, like in the following:

$$\frac{\mathrm{d}^{-n}}{\mathrm{d}x^{-n}}f(x)=\int_0^x\int_0^{u_1}\int_0^{u_2}\underbrace{............\int_0^{u_{n-1}}}_{n}\left(f(u_1)\right)\mathrm{d}u_n\mathrm{d}u_{n-1}.....\mathrm{d}u_1$$
$$=\frac{1}{(n-1)!}\int_0^x(x-u)^{n-1}\left(f(u)\right)\mathrm{d}u \tag{3.92}$$

3.7.1 Fractional integration is a must to get a fractional derivative

The common point between the ways of obtaining a fractional derivative is fractional integration. This (3.92) we can generalize using the gamma function instead of factorials, i.e. $(n-1)!=\Gamma(n)$, for fractional order, as a positive real number n call it υ. This we have discussed in Chapter-2, and we express Riemann fractional integration as below:

$$\frac{\mathrm{d}^{-\upsilon}}{\mathrm{d}x^{-\upsilon}}f(x)={}_0D_x^{-\upsilon}\left[f(x)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-u)^{\upsilon-1}\left(f(u)\right)\mathrm{d}u \tag{3.93}$$

The convergence properties of this formula are best when υ has a value between 0 and 1. Let us evaluate the double integral, in the interval $\left[0,x\right]$ of $\sqrt{x}$, taking $\upsilon=2$, and apply formula (3.93):

$${}_0D_x^{-2}\left[\sqrt{x}\right]=\frac{\mathrm{d}^{-2}\left[\sqrt{x}\right]}{\mathrm{d}x^{-2}}=\frac{1}{\Gamma(2)}\int_0^x(x-u)\sqrt{u}\mathrm{d}u$$
$$=\int_0^x\left(x\sqrt{u}-\sqrt{u^3}\right)\mathrm{d}u=\left[\frac{2}{3}x\sqrt{u^3}-\frac{2}{5}\sqrt{u^5}\right]_{u=0}^{u=x} \tag{3.94}$$
$$=\frac{4}{15}\sqrt{x^5}$$

Now let us perform a semi-integration in the interval $\left[0,x\right]$ of $\sqrt{x}$ taking $\upsilon=\frac{1}{2}$ and applying it to the Riemann formula (3.93):

$${}_0D_x^{-1/2}\left[\sqrt{x}\right]=\frac{\mathrm{d}^{-1/2}\left[\sqrt{x}\right]}{\mathrm{d}x^{-1/2}}=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{\sqrt{u}}{(x-u)^{-\left(\frac{1}{2}\right)+1}}\mathrm{d}u$$
$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{u\mathrm{d}u}{\sqrt{u}(\sqrt{x-u})}$$
$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{u}{\sqrt{xu-u^2}}\mathrm{d}u=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{u\mathrm{d}u}{\sqrt{\frac{x^2}{4}-u^2-\frac{x^2}{4}+2x\left(\frac{u}{2}\right)}} \tag{3.95}$$
$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{u\mathrm{d}u}{\sqrt{\frac{x^2}{4}-\left(u-\frac{x}{2}\right)^2}}$$

We put $u=\left(\frac{1}{2}\right)\left(x+x\sin\theta\right)$ so $\mathrm{d}u=\left(\frac{x}{2}\right)\left(\cos\theta\right)\mathrm{d}\theta$ and as limits for $u=0$ we have $\theta=-\frac{\pi}{2}$ and for $u=x$ we have $\theta=\frac{\pi}{2}$. With this change we write the following and proceed:

$$
\begin{aligned}
{}_0D_x^{-1/2}[\sqrt{x}] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2}\frac{\left(\frac{1}{2}\right)\left(x+x\sin\theta\right)}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}}\left(\frac{x}{2}\right)(\cos\theta)\mathrm{d}\theta \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2}\frac{\left(\frac{1}{2}\right)\left(x+x\sin\theta\right)}{\left(\frac{x}{2}\right)\sqrt{1-\sin^2\theta}}\left(\frac{x}{2}\right)(\cos\theta)\mathrm{d}\theta \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2}\left(\left(\frac{1}{2}\right)\left(x+x\sin\theta\right)\right)\mathrm{d}\theta = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x\theta}{2}-\frac{x\cos\theta}{2}\right)\Big|_{\theta=-\pi/2}^{\theta=+\pi/2} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\pi x}{2}\right) \\
&= \left(\frac{\sqrt{\pi}}{2}\right)x
\end{aligned}
\tag{3.96}
$$

The same we will get via the Euler formula as shown below by putting $m=\frac{1}{2}$, and $\upsilon=-\frac{1}{2}$:

$$
\begin{aligned}
{}_0D_x^{\upsilon}[x^m] &= \frac{\Gamma(m+1)}{\Gamma(m-\upsilon+1)}x^{m-\upsilon} \\
{}_0D_x^{-1/2}\left[x^{1/2}\right] &= \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}x^{\frac{1}{2}+\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}x = \Gamma\left(\frac{3}{2}\right)(x) = \left(\frac{\sqrt{\pi}}{2}\right)x
\end{aligned}
\tag{3.97}
$$

The above formulas (3.96) and (3.97) demonstrate that the Riemann-fractional integration is the same as we get from Euler's generalisation.

3.7.2 Different ways to apply a fractional integration formula to obtain a fractional derivative

There are two different ways, in which this formula (3.93) of fractional integration might be applied. For example, if we wish to find the $\frac{7}{3}$-rd (say order $\mu=\frac{7}{3}$) derivative of a function that is $D_x^{\mu}f(x)$ or $D_x^{7/3}f(x)$ then we could begin by applying the above formula with $\upsilon=\frac{2}{3}$ and thereafter differentiate the resulting function three whole times ($m=3$). Here we are doing fractional integration first and following that process by a whole differentiation. This is called the Riemann-Liouville or RL (1872) method of fractional differentiation, or left hand definition (LHD).

The equation as written above (3.93) giving the Riemann fractional integration formula i.e. ${}_0D_x^{-\upsilon}\left[f(x)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-u)^{\upsilon-1}\left(f(u)\right)\mathrm{d}u$, demonstrates the non-local character of fractional operations, because it explicitly involves an integral, which we have stipulated to range from 0 to x. For any whole number of differentiations we don't need to invoke this integral, but for a non-integer number of differentiations we must include the effect of this integral, which implies that the result depends not just on the values of functions at x but over the stipulated range from 0 to x.

To illustrate the use of the equation for the Riemann fractional integration expression to obtain a fractional derivative we will (again) determine the half-derivative of $f(x)=x$. Using the left hand definition, we first apply a half integration to this function using the equation of the Riemann integration with $\upsilon=\frac{1}{2}$, giving the following:

$$
\begin{aligned}
{}_0D_x^{-\frac{1}{2}}\left[x\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-u)^{-\frac{1}{2}}(u)\mathrm{d}u \qquad \text{put} \qquad x-u=z\,,\quad \mathrm{d}u=-\mathrm{d}z \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_x^0 (-\mathrm{d}z)\frac{(x-z)}{\sqrt{z}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\int_x^0 z^{1/2}\mathrm{d}z - \int_x^0 xz^{-1/2}\mathrm{d}z\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[-\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{3}{2}}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{4}{3}\right)x^{\frac{3}{2}} \\
&= \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}
\end{aligned}
\tag{3.98}
$$

Then we apply one whole differentiation to give the net result of a half-derivative as:

$$
{}_0D_x^{\frac{1}{2}}\left[x\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}\right) = 2\sqrt{\frac{x}{\pi}} \tag{3.99}
$$

Formula (3.99) is in agreement with the derived semi-derivative from Euler's generalisation.
In the operator sense we have for Riemann-Liouville (RL) the following fractional derivative:

$$
\begin{aligned}
&{}_0D_x^{\frac{1}{2}}[f(x)] = D^1\left[{}_0D_x^{-\left(1-\frac{1}{2}\right)}f(x)\right] \qquad \text{here} \qquad m=1 \qquad D^1 \equiv \frac{\mathrm{d}}{\mathrm{d}x} \\
&{}_0D_x^{\upsilon}\left[f(x)\right] = D^m\left[{}_0D_x^{-(m-\upsilon)}f(x)\right] \qquad \upsilon > 0 \\
&(m-1) < \upsilon < m\,, \qquad m \in \mathbb{Z}^+ \qquad D^m \equiv \frac{\mathrm{d}^m}{\mathrm{d}x^m}
\end{aligned}
\tag{3.100}
$$

Alternatively, we could begin by differentiating the function three whole times (taking the nearest integer, say m is just greater than μ; that is, $m=3$), and then apply the above formula (3.93) of the Riemann fractional integration with $\upsilon = \frac{2}{3}$ to "deduct" two thirds i.e. $\upsilon = (m-\mu) = \left(3-\frac{7}{3}\right) = \frac{2}{3}$ of a thrice-differentiation. This process of first taking a whole derivative (in this case three) and then applying (the remaining) fractional integration is Caputo's (1967) method of obtaining fractional differentiation. This process is opposite to that we described for an RL process (3.100). This is also termed the right hand definition (RHD).

In this case, the right hand definition (Caputo) gives the same result as (3.99), for $f(x)=x$, to get a semi-derivative. Choose $m=1$ then differentiate the function $f(x)=x$ once to have $f^{(1)}(x)=1$, then apply the semi-integration on this function $f^{(1)}(x)=1$ that is described in the following expression by using (3.93):

$$
\begin{aligned}
{}_0D_x^{-\frac{1}{2}}\left[1\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-u)^{-\frac{1}{2}}(1)\mathrm{d}u \\
&= \frac{1}{\sqrt{\pi}}\left[-2(x-u)^{\frac{1}{2}}\right]_{u=0}^{u=x} = 2\sqrt{\frac{x}{\pi}}
\end{aligned}
\tag{3.101}
$$

We will be getting the same result as (3.101) by using Euler's formula that is ${}_0D_x^m[x^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-m)}x^{p-m}$, placing $p=0$ and $m=-\frac{1}{2}$. We get the result i.e. ${}_0D_x^{-\frac{1}{2}}[1] = \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$, same as obtained in (3.101).

In this process, we first differentiate the function m times, then follow up by a remainder fraction and integrate it by that fraction. In the operator sense the Caputo derivative is as follows (we write C here to distinguish from the Riemann-Liouville fractional derivative):

$$
\begin{aligned}
&{}_0^CD_x^{\frac{1}{2}}[f(x)] = {}_0D_x^{-\frac{1}{2}}\left[D^1 f(x)\right] \qquad \text{here} \qquad m=1 \qquad D^1 \equiv \frac{\mathrm{d}}{\mathrm{d}x} \\
&{}_0^CD_x^{\upsilon}\left[f(x)\right] = {}_0D_x^{-(m-\upsilon)}\left[D^m f(x)\right] \qquad \upsilon > 0 \\
&(m-1) < \upsilon < m\,, \qquad m \in \mathbb{Z}^+ \qquad D^m \equiv \frac{\mathrm{d}^m}{\mathrm{d}x^m}
\end{aligned}
\tag{3.102}
$$

The two processes for a semi-derivative of $f(x)=x$ is shown in Figure-3.4. These two alternatives for fractional derivatives are called the right hand definition (Caputo) and the left hand definition (Riemann-Liouville). Although, these two definitions give the same result in many circumstances especially when the start point of the process is at $-\infty$ or the value of the function at the start point is zero, however, they are not equivalent. This is because (for example) the half-derivative of a constant is zero by the right hand definition. (In Caputo's process we are first differentiating a constant a whole number of times resulting zero). Whereas the left hand definition gives for the half-derivative of a constant the result given previously as obtained by Euler's generalisation that is a decaying function. These two are depicted as $D^{\frac{1}{2}}\left[K\right]=K\frac{x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}=K\left(\frac{1}{\sqrt{\pi x}}\right)$ and ${}^{C}D^{\frac{1}{2}}\left[K\right]=0$ with pre-fix C denoting Caputo.

In general, the left hand definition (RL) is more uniformly consistent with the previous methods, but the right hand definition (Caputo) has also found some applications.

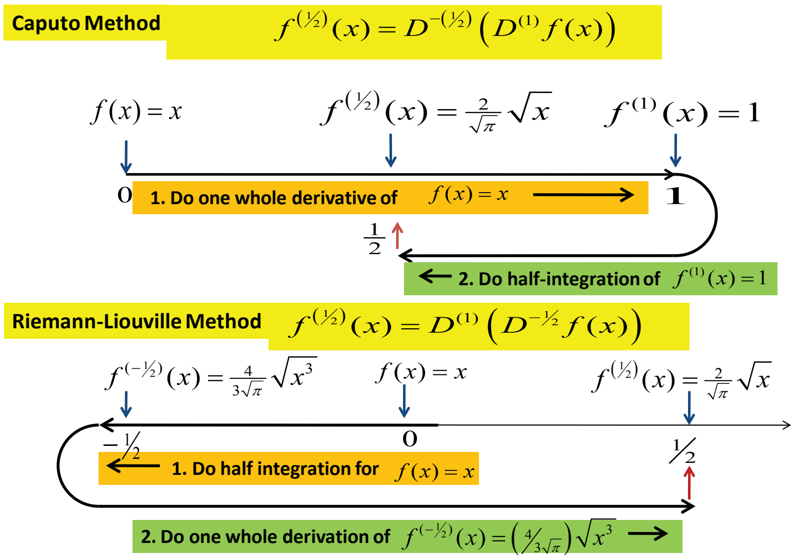


Figure-3.4: Taking the half-derivative of a function by Caputo and the Riemann-Liouville methods

We note that by both methods we are getting a half derivative of $f(x)=x$ as $\frac{2}{\sqrt{\pi}}\sqrt{x}$ i.e. the same result. In this case it is true because $f(0)=0$ i.e. the start point value of the function and the start point of the half derivative is also from $x=0$ i.e. we are taking ${}_{0}D_{x}^{\frac{1}{2}}$ for $f(x)=x$. We will elaborate upon this point later. Here we just point out that in all cases the Caputo and Riemann-Liouville derivative need not give the same results.

In the integral form, we write the $\alpha>0$ order fractional derivative in RL form, for a function $f(x)$ in the interval $\left[a,x\right]$ as follows:

$$ {}_{a}D_{x}^{\alpha}\left[f(x)\right]=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-u)^{m-\alpha-1}\left(f(u)\right)\mathrm{d}u\right] \tag{3.103} $$

In the integral form, we write the $\alpha>0$ order fractional derivative in Caputo form, for a function $f(x)$ in the interval $\left[a,x\right]$ as:

$$ \begin{aligned} {}_{a}^{C}D_{x}^{\alpha}\left[f(x)\right] &=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-u)^{m-\alpha-1}\left(f^{(m)}(u)\right)\mathrm{d}u \\ &=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-u)^{m-\alpha-1}\left(\frac{\mathrm{d}^m f(u)}{\mathrm{d}u^m}\right)\mathrm{d}u \end{aligned} \tag{3.104} $$

3.8 Fractional derivatives for the Riemann-Liouville (RL): 'left hand definition' (LHD)

The formulation of this definition is done according to the following:
Select an integer m greater than the fractional number $\alpha > 0$

(i) Integrate the function $(m-\alpha)$ folds by the RL integration method.

(ii) Differentiate the above result by m-folds

The expression is given as:

$$ {}_aD_t^{\alpha}\left[f(t)\right] = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau\right] \tag{3.105} $$

Figure-3.5 gives the process block diagram & Figure-3.6 gives the process of differentiation 2.3 times for a function.

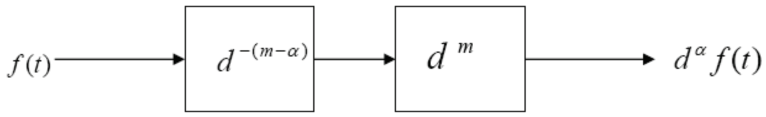


Figure-3.5: Fractional differentiation of the left hand definition (LHD) block diagram

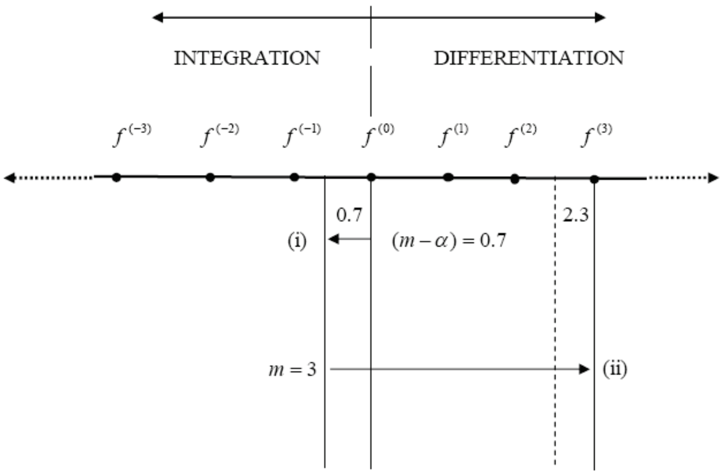


Figure-3.6: Fractional differentiation of 2.3 times in the LHD

In this LHD the limit of integration is from 0 to t, we denote the derivative by notation ${}_0D_t^{\alpha} f(t)$. In fractional calculus we find the limit of the derivative i.e., derivatives are taken in an interval. We call this a 'forward derivative'. Now if the limits of integration are changed to (t to 0) the derivative is denoted as ${}_tD_0^{\alpha}\left[f(t)\right]$ the 'backward derivative'. The backward derivative is related to the forward derivative by:

$$ {}_tD_0^{\alpha}\left[f(t)\right] = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m}\, {}_tI_0^{m-\alpha}\left[f(t)\right] \tag{3.106} $$

Therefore, in order to obtain a fractional derivative of a function at a point (say 0) we should have the values of these two derivatives be the same: the forward derivative should equal the backward derivative. This implies not only that one should know the function from past to the point of interest (say 0) but also the function should be known into the future – in order to have a fractional derivative at a point!

A fractional derivative of a purely imaginary order i.e. $\alpha = i\theta$, ($\theta \neq 0$) with a real part as 'zero' is expressed in the Riemann-Liouvelli notation (with $m = 1$):

$$ {}_aD_x^{i\theta}\left[f(x)\right] = \frac{1}{\Gamma(1-i\theta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(u)}{(x-u)^{i\theta}}\mathrm{d}u \tag{3.107} $$

and the associated integral of a purely imaginary order in Riemann-Liouvelli definition is the following:

$$
\begin{aligned}
{}_aD_x^{-i\theta}\left[f(x)\right] &= {}_aI_x^{i\theta}\left[f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\,{}_aI_x^{1+i\theta}f(x) \\
&= \frac{1}{\Gamma(1+i\theta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x (x-u)^{i\theta}\left(f(u)\right)\mathrm{d}u
\end{aligned}
\tag{3.108}
$$

3.9 Fractional derivatives of the Caputo: 'right hand definition' (RHD)

The formulation is exactly opposite to the LHD.
Select an integer m greater than the fractional number $\alpha > 0$

(i) Differentiate the function m times.
(ii) Integrate the above result $(m-\alpha)$ folded by the RL integration method.

In the LHD and RHD the integer selection is made such that $(m-1) < \alpha < m$. For example for differentiation of the function by the order $\alpha = 3.14$ we will select $m = 4$. The formulation of the Caputo RHD is as follows:

$$
{}_a^CD_t^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau \tag{3.109}
$$

Figure-3.7 gives the block diagram representation of the RHD process and Figure-3.8 represents graphically where the RHD is used for a fractionally differentiating function 2.3 times.

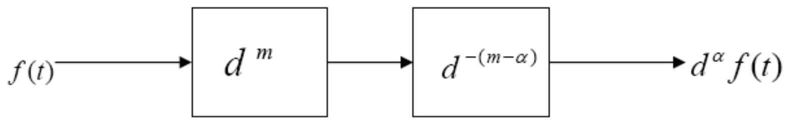


Figure-3.7: Block diagram representation of the Caputo RHD

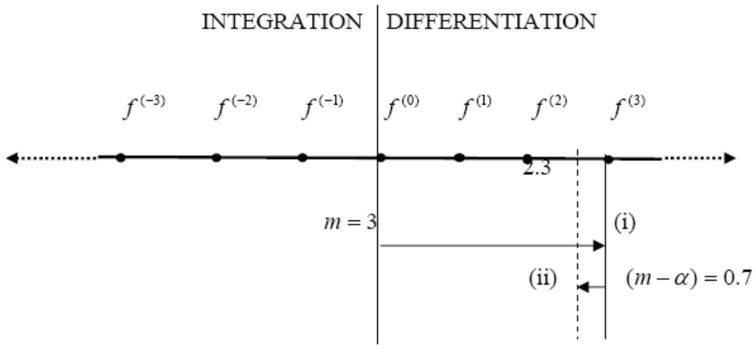


Figure-3.8: Differentiation of 2.3 times by the RHD

The definitions of Riemann-Liouville fractional differentiation played an important role in the development of fractional calculus. However, the demands of modern science and engineering require a certain revision of the well-established pure mathematical approaches. Applied problems require definitions of fractional derivatives allowing the utilisation of physically interpretable "initial conditions" which contain $f(a), f^{(1)}(a), f^{(2)}(a)...$; where a is the initial point; and not fractional quantities (presently unthinkable!).

The RL definitions require values of fractional initial states like ${}_aD_t^{\alpha-1}\left[f(t)\right]\Big|_{t=a}$, ${}_aD_t^{\alpha-2}\left[f(t)\right]\Big|_{t=a}$,.... In spite of the fact that initial value problems with such conditions can be successfully solved mathematically, their solutions are practically useless, because there is no known physical interpretation for such initial conditions, presently. It is tough to interpret.

Caputo requirements are usual initial states i. e. $f(a), f^{(1)}(a), f^{(2)}(a)....$ For the LHD $D^{\alpha}C \neq 0 = \frac{C}{\Gamma(1-\alpha)}t^{-\alpha}$, the derivative of constant C is not zero. This fact led to using the RL or LHD approach with the lower limit of differentiation as $-\infty$. In the physical world this poses a problem.

The physical meaning of this lower limit extending towards minus infinity is starting the physical process at time immemorial! In such cases, transient effects cannot then be studied. However making start point a as $-\infty$ is a necessary abstraction for consideration of the steady state process, for example for study of sinusoidal analysis for a steady state fractional order system. While today we are familiar with the interpretation of the physical world with integer order differential equations, we do not (currently) have a practical understanding of the world with fractional order differential equations. Our mathematical tools go beyond the practical limitation of our understanding. Therefore, the process is still on to 'generalise' the concepts for use in the practical world.

3.10 Relation between Riemann-Liouville and Caputo derivatives

3.10.1 Riemann-Liouvelli and Caputo derivatives related by initial value of the function at start point of fractional derivative process

The RL derivative of a function $y(t)$ from 0 to t of fractional order n such that $0<n<1$ is expressed as the derivative of the fractional integral as follows:

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n}=\frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau \tag{3.110}$$

We will evaluate $\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t (t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau\right]$ by first doing the evaluation of $\int_0^t (t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau$, by using the integration by parts formula

$$\int_0^t \left(f(\tau)\right)\left(g(\tau)\right)\mathrm{d}\tau=\left[f(\tau)\int g(\tau)\mathrm{d}\tau\right]_{\tau=0}^{\tau=t}-\int_0^t\left(\left(f^{(1)}(\tau)\right)\int_0^t\left(g(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}\tau \tag{3.111}$$

This is given by assuming in the interval 0 to t that the first derivative of $y(t)$ exists, call it $y^{(1)}(t)$, and that it is represented by the following expression:

$$\begin{aligned}\int_0^t (t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau&=\left[y(\tau)\int(t-\tau)^{-n}\mathrm{d}\tau\right]_{\tau=0}^{\tau=t}-\int_0^t\left(\left(y^{(1)}(\tau)\right)\int(t-\tau)^{-n}\mathrm{d}\tau\right)\mathrm{d}\tau\\&=\left[y(\tau)\left(-\tfrac{(t-\tau)^{(-n+1)}}{(-n+1)}\right)\right]_{\tau=0}^{\tau=t}-\int_0^t\left(\left(y^{(1)}(\tau)\right)\left(\tfrac{(-1)(t-\tau)^{-n+1}}{(-n+1)}\right)\right)\mathrm{d}\tau\\&=\frac{y(0)}{(-n+1)}t^{-n+1}-\int_0^t\left(\left(y^{(1)}(\tau)\right)\left(\tfrac{(-1)(t-\tau)^{-n+1}}{(-n+1)}\right)\right)\mathrm{d}\tau\end{aligned} \tag{3.112}$$

Now we take one-whole derivative of (3.112) with respect to t and write the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\left(y(\tau)\right)(t-\tau)^{-n}\mathrm{d}\tau\right]&=y(0)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(t)^{-n+1}}{-n+1}\right]\\&\quad-\int_0^t\left(y^{(1)}(\tau)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(-1)(t-\tau)^{-n+1}}{(-n+1)}\right]\right)\mathrm{d}\tau\\&=\frac{y(0)}{(-n+1)}\left[(-n+1)t^{-n+1-1}\right]+\int_0^t\frac{y^{(1)}(\tau)}{(-n+1)}\left((-n+1)(t-\tau)^{-n+1-1}\right)\mathrm{d}\tau\\&=\frac{y(0)}{t^n}+\int_0^t\left(y^{(1)}(\tau)\right)(t-\tau)^{-n}\mathrm{d}\tau\end{aligned} \tag{3.113}$$

After the above result (3.113), we multiply both sides by $\frac{1}{\Gamma(1-n)}$ and write the following expression:

$$\begin{aligned}&\frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau\\&=\frac{1}{\Gamma(1-n)}\int_0^t(t-\tau)^{-n}\left(y^{(1)}(\tau)\right)\mathrm{d}\tau+\left(\frac{y(0)}{\Gamma(1-n)}\right)t^{-n},\qquad 0<n<1\end{aligned} \tag{3 .114}$$

The LHS of the above expression (3.114) is the RL derivative of the fractional order $0<n<1$ and the first term of the RHS of the expression is the Caputo derivative. Therefore, we write the RL fractional derivative of order $0<n<1$ as the Caputo derivative plus a decaying term based on the initial value of function $y(0)$ at the start of the fractional differentiation:

$$\begin{aligned}\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} &= \frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-n}\left(y(\tau)\right)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-n)}\int_0^t (t-\tau)^{-n}\left(y^{(1)}(\tau)\right)\mathrm{d}\tau + \left(\frac{y(0)}{\Gamma(1-n)}\right)t^{-n}; \qquad 0<n<1\end{aligned} \tag{3.115}$$

Therefore, for $0<\alpha<1$, we write the relationship between the RL and Caputo derivative of a function $f(x)$ as follows:

$${}_aD_x^\alpha\left[f(x)\right] = {}_a^CD_x^\alpha\left[f(x)\right] + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{3.116}$$

For $\alpha=\frac{1}{2}$, we get ${}_aD_x^{1/2}\left[f(x)\right] = {}_a^CD_x^{1/2}\left[f(x)\right] + \frac{f(a)(x-a)^{-1/2}}{\Gamma\left(1-\frac{1}{2}\right)}$. With $a=0$ and using $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, we write ${}_0D_x^{1/2}\left[f(x)\right] = {}_0^CD_x^{1/2}\left[f(x)\right] + \frac{f(0)}{\sqrt{\pi x}}$. We note that $f(a)$ is constant, and $f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$ is the RL fractional derivative of the constant $f(a)$, with start point a, that is ${}_aD_x^\alpha\left[f(a)\right] = f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$. Therefore, we can write the following steps:

$$\begin{aligned}{}_aD_x^\alpha\left[f(x)\right] &= {}_a^CD_x^\alpha\left[f(x)\right] + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= {}_a^CD_x^\alpha\left[f(x)\right] + {}_aD_x^\alpha\left[f(a)\right]\end{aligned} \tag{3.117}$$

Form (3.117) we have the following relation:

$${}_a^CD_x^\alpha\left[f(x)\right] = {}_aD_x^\alpha\left[f(x)-f(a)\right] \tag{3.118}$$

That is for the Caputo derivative for a fractional order $0<\alpha<1$ the function $f(x)$ one needs to offset the function by subtracting the constant value at the start point $f(a)$, and take the Riemann-Liouville (RL) derivative of the offset function i.e. $f(x)-f(a)$.

3.10.2 A generalisation of a fundamental theorem of integral calculus is the relation between Riemann-Liouville and Caputo fractional derivatives

Look at the expression we wrote that is:

$$\begin{aligned}{}_aD_x^\alpha\left[f(x)\right] &= {}_a^CD_x^\alpha\left[f(x)\right] + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\ {}_aD_x^1\left[{}_aD_x^{-(1-\alpha)}f(x)\right] &= {}_aD_x^{-(1-\alpha)}\left[D_x^1 f(x)\right] + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\end{aligned} \tag{3.119}$$

In this (3.119) we take the limit $\alpha\downarrow 0^+$; and write the following:

$$\begin{aligned}D^1\left[{}_aD_x^{-1}f(x)\right] &= {}_aD_x^{-1}\left[D^1 f(x)\right] + f(a) \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_a^x f(u)\mathrm{d}u &= \left[\int_a^x\left(\frac{\mathrm{d}}{\mathrm{d}u}f(u)\right)\mathrm{d}u\right] + f(a)\end{aligned} \tag{3.120}$$

The above expression (3.120) is a fundamental theorem of integral calculus, where the differentiation of integration is a constant (i.e. the value of the function at the start point of integration) plus the integration of a differentiated function.

Well if the start point of the function is zero the process of differentiation of integration commutes with the process of integration of differentiation; that is,that they are equal. Therefore, we can state that the relationship between the RL and Caputo derivatives is the generalisation of this fundamental theorem of integral calculus.

3.10.3 Further generalisation of the relationship between the Caputo and Riemann-Liouville fractional derivatives

We generalize further, for any α need not be between zero and one, and we write it as:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= {}_a^CD_x^{\alpha}\left[f(x)\right]+\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \\ &= {}_a^CD_x^{\alpha}\left[f(x)\right]+f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}+f^{(1)}(a)\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad +\dots f^{(m-1)}(a)\frac{(x-a)^{m-1-\alpha}}{\Gamma(m-\alpha)} \end{aligned} \tag{3 .121}$$

Here in (3.121), we have $(m-1)<\alpha<m$, where m is a positive integer number just greater than the real number for the fractional order α. The above (3.121) formula is applicable, only if the $(m-1)$ integer order derivatives exist at the start point of the function. The above (3.121) relationship between the RL and Caputo derivatives is the correction by an offset factor at the start point of the function. If the start point of the function is zero that is $f(a)=0$, then, the RL and Caputo derivatives are the same, or if the start point a is at $-\infty$, and $f(-\infty)=0$ at which all the function values are zero, then also both are the same.

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= {}_a^CD_x^{\alpha}\left[f(x)\right] \qquad f(a)=0 \\ {}_{-\infty}D_x^{\alpha}\left[f(x)\right] &= {}_{-\infty}^{\;\;C}D_x^{\alpha}\left[f(x)\right] \qquad f(-\infty)=0 \end{aligned} \tag{3.122}$$

We take the following:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= {}_a^CD_x^{\alpha}\left[f(x)\right]+f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}+f^{(1)}(a)\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad +\dots f^{(m-1)}(a)\frac{(x-a)^{m-1-\alpha}}{\Gamma(m-\alpha)} \\ &= {}_a^CD_x^{\alpha}\left[f(x)\right]+f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}+f^{(1)}(a)\frac{(x-a)^{-(\alpha-1)}}{\Gamma\left(1-(\alpha-1)\right)} \\ &\quad +f^{(2)}(a)\frac{(x-a)^{-(\alpha-2)}}{\Gamma\left(1-(\alpha-2)\right)}+ \\ &\quad \dots+f^{(m-1)}(a)\frac{(x-a)^{-(\alpha+1-m)}}{\Gamma\left(1-(\alpha+1-m)\right)} \end{aligned} \tag{3 .123}$$

The values of integer order whole derivatives of $f(x)$, at $x=a$ i.e. at the start point, of fractional derivative, namely $f^{(1)}(a)$, $f^{(2)}(a)$, $f^{(3)}(a)$..........$f^{(m-1)}(a)$ are constants. We can write the above (3.123) in terms of fractional derivatives that is, by using ${}_aD_x^{k\alpha}\left[f(a)\right]=f(a)\frac{(x-a)^{-k\alpha}}{\Gamma(1-k\alpha)}$, as shown below:

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= {}_a^CD_x^{\alpha}\left[f(x)\right]+f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}+f^{(1)}(a)\frac{(x-a)^{-(\alpha-1)}}{\Gamma\left(1-(\alpha-1)\right)} \\ &\quad +f^{(2)}(a)\frac{(x-a)^{-(\alpha-2)}}{\Gamma\left(1-(\alpha-2)\right)}+..+f^{(m-1)}(a)\frac{(x-a)^{-(\alpha+1-m)}}{\Gamma\left(1-(\alpha+1-m)\right)} \\ {}_aD_x^{\alpha}\left[f(x)\right] &= {}_a^CD_x^{\alpha}\left[f(x)\right]+{}_aD_x^{\alpha}\left[f(a)\right]+{}_aD_x^{\alpha-1}\left[f^{(1)}(a)\right] \\ &\quad +{}_aD_x^{\alpha-2}\left[f^{(2)}(a)\right]+....+{}_aD_x^{\alpha-m+1}\left[f^{(m-1)}(a)\right] \end{aligned} \tag{3 .124}$$

For $(m-1)<\alpha<m$ we have a useful relationship between the Caputo & RL derivative i.e.

$${}_a^CD_x^{\alpha}\left[f(x)\right]={}_aD_x^{\alpha}\left[f(x)\right]-\begin{pmatrix} {}_aD_x^{\alpha}\left[f(a)\right]+{}_aD_x^{\alpha-1}\left[f^{(1)}(a)\right] \\ +\dots{}_aD_x^{\alpha-m+1}\left[f^{(m-1)}(a)\right] \end{pmatrix} \tag{3.125}$$

3.10.4 The Caputo and Riemann-Liouville fractional derivatives need an area under the shape changing curve

Concluding this section, we mention that the RL or Caputo derivatives both have a fractional integration process, therefore we are dealing with the area under the shape changing curve, as we discussed in the previous chapter (Section-2.21). The RL derivative is a process first of fractional integration, then followed by an integer-order whole differentiation. Thus we are required to find the area under a 'shape-changing' curve ($f(x_0)$, $f_1(x_1)$, $f_2(x_2)$ …) as we move along the points x_0, x_1, x_2,… and after that at each point of interest we have to find the integer-order whole derivative. Thus, the process is of finding a slope of area under the shape changing curve!

The Caputo process is the reverse of this; we have to find an integer order whole derivative first at the point of interest, and then find the area under the shape changing curve ($f^{(n)}(x_0)$, $f_1^{(n)}(x_1)$, $f_2^{(n)}(x_2)$ …) as we move along the points x_0, x_1, x_2,…. and so on. An interesting point is that both these processes require values of the function or values of the integer whole-derivative of the function from the start point to the point of interest; indicating that the RL or Caputo fractional derivative processes have memory.

3.11 Can we change the sign of order of the Riemann-Liouville fractional derivative to write the RL fractional integration formula?

In the previous chapter (Section-2.14), we noted that it was difficult to change the sign of a fractional order for a fractional integration formula; to directly get the formula of a fractional derivative. Then we explained via the concept of an analytic continuation that we sought to obtain the validity of this process of sign change. Here we see the reverse operation. The expression given for the RL fractional derivative of the order $\alpha > 0$ such that $(m-1) \le \alpha < m$ where $m = 1, 2, 3, ...$ is the following:

$$ {}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^x \frac{f(y)}{(x-y)^{\alpha+1-m}}\mathrm{d}y \tag{3.126} $$

The formula for the RL fractional derivative for order α so that $0 \le \alpha < 1$ is the following:

$$ {}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{f(y)\mathrm{d}y}{(x-y)^{\alpha}}; \qquad 0 \le \alpha < 1 \tag{3.127} $$

In the above formula (3.127) we reverse the sign and put $-\alpha$ instead of α and write the following steps to see that this change of sign for the RL fractional derivative formula leads to a fractional integration formula:

$$ \begin{aligned} {}_0D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(1+\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{-\alpha}} \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-y)^{\alpha}\right]\left(f(y)\right)\mathrm{d}y \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^x \alpha(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y, \quad \Gamma(\alpha+1)=\alpha(\Gamma(\alpha)) \\ &= \frac{1}{\alpha\left(\Gamma(\alpha)\right)}\int_0^x \alpha(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \\ &= {}_0I_x^{\alpha}\left[f(x)\right] \end{aligned} \tag{3.128} $$

The above (3.128) steps indicate that if we change the sign of the order in the RL-fractional derivative we get the formula of the RL fractional integration as represented below for $0 \le \alpha < 1$:

$$ \begin{aligned} {}_0D_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha}} \\ {}_0I_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(1+\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{-\alpha}} \end{aligned} \tag{3.129} $$

However, the converse was not quite true; that we have seen in the last chapter. Let us take the case of the RL fractional derivative formula for $1 \le \alpha < 2$, i.e.

$$ {}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha-1}}; \qquad 1 \le \alpha < 2 \qquad (3.130)$$

Now we do the sign change as we did in the earlier case (3.128), demonstrated in the following steps:

$$\begin{aligned} {}_0D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(2+\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{-\alpha-1}} \\ &= \frac{1}{\Gamma(2+\alpha)}\int_0^x \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[(x-y)^{\alpha+1}\right]\left(f(y)\right)\mathrm{d}y \\ &= \frac{1}{\Gamma(2+\alpha)}\int_0^x (\alpha+1)(\alpha)(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \end{aligned} \qquad (3\ .131)$$

Using $\Gamma(2+\alpha) = \alpha(\alpha+1)\Gamma(\alpha)$

$$\begin{aligned} {}_0D_x^{-\alpha}\left[f(x)\right] &= \frac{1}{\alpha(\alpha+1)\Gamma(\alpha)}\int_0^x (\alpha+1)(\alpha)(x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\left(f(y)\right)\mathrm{d}y = {}_0I_x^{\alpha}\left[f(x)\right] \end{aligned}$$

The above steps (3.131) indicate that if we change the sign of the order in the RL-fractional derivative we get the formula for the RL fractional integration represented below for $1 \le \alpha < 2$:

$$\begin{aligned} {}_0D_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha-1}} \\ {}_0I_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(2+\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{-\alpha-1}} \end{aligned} \qquad (3.132)$$

We can carry on for other intervals $(m-1) \le \alpha < m$ and write a different form of a fractional integration formula by changing of sign of fractional order α from the RL fractional derivative formula; as indicated below:

$$\begin{aligned} {}_0D_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^x \frac{f(y)}{(x-y)^{\alpha+1-m}}\mathrm{d}y \\ {}_0I_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(m+\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^x \frac{f(y)}{(x-y)^{-\alpha+1-m}}\mathrm{d}y \end{aligned} \qquad (3.133)$$

The above (3.133) is a useful observation that we will apply in a unifying fractional differentiation and in a fractional integration formula for discrete numerical evaluation. Thus, fractional integration of the order $(m-1) \le \alpha < m$ is the process of first fractionally integrating $(m+\alpha)$ folds by the RL fractional integration formula, and is then followed by m – whole differentiation. Thus, we have:

$$I^{\alpha}\left[f(x)\right] = D^m\left(I^{(m+\alpha)}\left[f(x)\right]\right); \quad (m-1) \le \alpha < m \qquad (3.134)$$

If we require the finding of a fractional integration of order $\alpha = 2.3$, we choose $m = 3$ then we do the fractional integration $m+\alpha = 5.3$ folds, followed by thrice differentiating the result, to get 2.3 fold fractional integration of the function.

To get $f^{(-2.3)}$ i.e. fractional integration of order 2.3, we refer Figure-3.6. In this figure we first take the function $f^{(0)}$ by process of fractional integration to the point in number line towards left side at $f^{(-5.3)}$ and then we perform three whole differentiations, i.e. moving on the number line of Figure-3.6 towards right to the point $f^{(-2.3)}$.

3.12 The Weyl fractional derivative

We have seen in the previous chapter (Section-2.15) that the fractional integration of the Weyl type is given as the following expression:

$$ {}_xW_\infty^{-\upsilon}\left[f(x)\right]=\frac{1}{\Gamma(\upsilon)}\int_x^\infty (y-x)^{\upsilon-1}\left(f(y)\right)\mathrm{d}y \qquad \upsilon>0 \tag{3.135}$$

Like Riemann-Liouville, the definition of Weyl's fractional derivative of the function is defined for an integer just greater than the fractional index of the derivative. For $m-1\le\alpha<m$, where $m\in\mathbb{N}$, the Weyl's fractional derivative definition is:

$$ {}_xW_\infty^{\alpha}\left[f(x)\right]=(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left({}_xW_\infty^{m-\alpha}\left[f(x)\right]\right) \tag{3.136}$$

In (3.136), ${}_xW_\infty^{m-\alpha}\left[f(x)\right]$ is the Weyl fractional integration of the order $(m-\alpha)$. In the integral transform representation, we have the following:

$$ {}_xW_\infty^{\alpha}\left[f(x)\right]=(-1)^m\left(\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\int_x^\infty (y-x)^{m-\alpha-1}\left(f(y)\right)\mathrm{d}y\right)\right) \tag{3.137}$$

$$m-1\le\alpha<m$$

Let $f(x)=e^{-px}$, with $p>0$, and under the substitution $u-x=(y/p)$, and with the integral representation of the gamma function defined as $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$, we get the following expression for the Weyl fractional integration:

$$\begin{aligned} {}_xW_\infty^{-\alpha}\left[e^{-px}\right]&=\frac{1}{\Gamma(\alpha)}\int_x^\infty (u-x)^{\alpha-1}e^{-pu}\mathrm{d}u \\ &=\frac{e^{-px}}{p^\alpha\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}\mathrm{d}y \\ &=p^{-\alpha}e^{-px} \end{aligned} \tag{3.138}$$

Therefore in (3.138) for $\alpha=(1-\upsilon)$, where $0<\upsilon<1$, we have the Weyl fractional integration of e^{-px} as ${}_xW_\infty^{-(1-\upsilon)}\left[e^{-px}\right]=p^{-(1-\upsilon)}e^{-px}$. Now we calculate the υ-th order Weyl fractional derivative, from (3.137), where $0<\upsilon<1$. The nearest integer for $0<\upsilon<1$ is $m=1$, this gives the Weyl fractional derivative as:

$$\begin{aligned} {}_xW_\infty^{\upsilon}\left[e^{-px}\right]&=(-1)\frac{\mathrm{d}}{\mathrm{d}x}\left({}_xW_\infty^{-(1-\upsilon)}\left[e^{-px}\right]\right) \\ &=(-1)\frac{\mathrm{d}}{\mathrm{d}x}\left[p^{-(1-\upsilon)}e^{-px}\right]=(-1)\left(p^{-(1-\upsilon)}(-p)e^{-px}\right) \\ &=p^{\upsilon}e^{-px} \end{aligned} \tag{3.139}$$

We observe that the Weyl fractional derivative of e^{-px} could be obtained by a sign change in the Weyl fractional integral.

3.13 The most fundamental approach for repeated differentiation

This approach is also called the limit of finite difference. In classical calculus, we define the derivative as a limit of finite difference as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\lim_{h\downarrow 0}\frac{f(x)-f(x-h)}{h} \tag{3.140}$$

Similarly:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)&=\lim_{h\downarrow 0}\frac{f(x)-2f(x-h)+f(x-2h)}{h^2} \\ \frac{\mathrm{d}^3}{\mathrm{d}x^3}f(x)&=\lim_{h\downarrow 0}\frac{f(x)-3f(x-h)+3f(x-2h)-f(x-3h)}{h^3} \end{aligned} \tag{3.140}$$

Repeating this n times for any positive integer we get the following expression:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)=\lim_{h\downarrow 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(f(x-jh)\right) \tag{3.142}$$

If the n-th derivative of $f(x)$ exists, this equation (3.142) does indeed define $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ as an unrestricted limit, i.e. as the limit h tends to zero through values that are totally unrestricted. In order to unify this formula (3.142) with one that defines an integral as the limit of a sum, it is desirable to define derivatives in terms of a restricted limit, namely, as a limit in that h tends to zero through discrete values only. To do this, choose $h_N \equiv \frac{x-a}{N}$, where a is the start point of the function, and $N = 1, 2, 3, \ldots$. The number $a < x$ plays the role of a lower limit. If the unrestricted limit exists so does the restricted limit and they are equal; the n-th derivative may be defined as follows:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = \lim_{h_N \downarrow 0} \frac{1}{h_N^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(f\left(x - jh_N\right)\right) \tag{3.143}$$

Now since:

$$\binom{n}{j} = 0 \qquad \text{for} \qquad j > n \tag{3.144}$$

when n is an integer, the above expression (3.143) may be re-written as:

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) &= \lim_{h_N \downarrow 0} \frac{1}{h_N^n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} \left(f\left(x - jh_N\right)\right) \\ &= \lim_{N \uparrow \infty} \frac{1}{\left(\frac{x-a}{N}\right)^n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} \left(f\left(x - j\left(\tfrac{x-a}{N}\right)\right)\right)\end{aligned} \tag{3.145}$$

The above expression (3.145) is defining $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)$ with the understanding that the limit exists in the usual unrestricted sense. We verify the above expression for $n = 2$ and $f(x) = x^4$ in the following steps:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^4\right] &= \lim_{h_N \downarrow 0} \frac{1}{h_N^2} \sum_{j=0}^{2} (-1)^j \binom{n}{j} \left(x - jh_N\right)^4 \\ &= \lim_{h_N \downarrow 0} \frac{1}{h_N^2} \left[x^4 - 2\left(x - h_N\right)^4 + \left(x - 2h_N\right)^4\right] \\ &= \lim_{h_N \downarrow 0} \left[12x^2 - 24xh_N + 14h_N^2\right] \\ &= 12x^2\end{aligned} \tag{3.146}$$

We can generalise the summation in (3.145), i.e. $\sum^{N-1}[\ldots]$ by generalising the binomial coefficients upper summation limit that is n, with the following formula using with the gamma function (Chapter-1):

$$(-1)^j \binom{n}{j} = \binom{j-n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} \tag{3.147}$$

We write the n-th derivative to any arbitrary order q by the following formula:

$$\frac{\mathrm{d}^q f(x)}{\left[\mathrm{d}(x-a)\right]^q} = \lim_{N \uparrow \infty} \left(\frac{\left(\frac{x-a}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} \left(f\left(x - j\left(\tfrac{x-a}{N}\right)\right)\right)\right) \tag{3.148}$$

Note that here q is an arbitrary positive or negative. A negative q implies fractional integration, while a positive q implies fractional differentiation. Even for q a non-negative integer say n, so that $\Gamma(-n)$ is infinite, but the ratio $\frac{\Gamma(j-n)}{\Gamma(-n)}$ is finite. Note that the above expression (3.148) involves only the evaluation of the function itself while no explicit use is made of the derivatives or integrals of the function $f(x)$.

3.14 The unified context differentiation/integration Grunwald-Letnikov formula

3.14.1 Generalisation by use of the backward shift operator defining the repeated differentiation and integration

To clarify the situation, as we stated in the previous section, let us start from scratch for the operation of differentiation and integration in a unified context. Let us consider the arbitrary smooth function $f(x)$ and divide it with slices of h, as depicted in Figure-3.9.

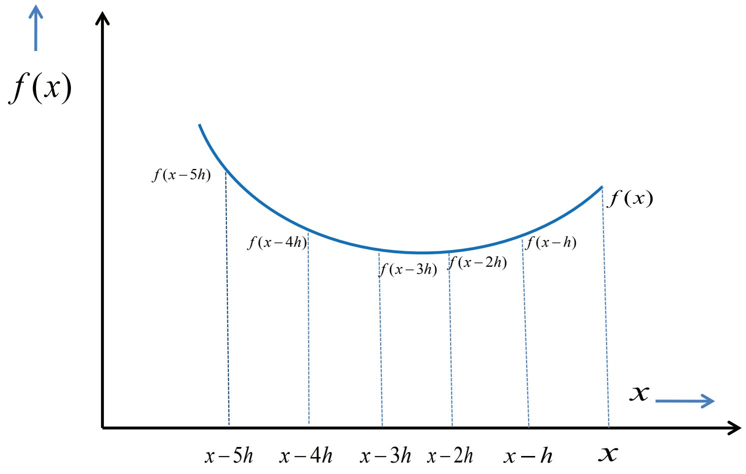


Figure-3.9: The function is divided into a slice

Here we should take note that h we regard as h_N, as discussed previously in Section-3.13, without restriction on the slice. We define a backward shift operator E_h operating on $f(x)$ at x giving the value of the function at $x-h$ that is $f(x-h)$.

$$\mathrm{E}_h\left[f(x)\right] \overset{\text{def}}{=} f(x-h) \tag{3.149}$$

The shift operator E_h^N denotes N back shifts as:

$$\mathrm{E}_h^N\left[f(x)\right] = f(x-Nh) \tag{3.150}$$

We apply this defined back shift operator as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= \lim_{h\downarrow 0}\frac{f(x)-f(x-h)}{h}\\ &= \lim_{h\downarrow 0}\frac{f(x)-\mathrm{E}_h\left[f(x)\right]}{h}\\ &= \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)\left[f(x)\right]\end{aligned} \tag{3.151}$$

Therefore, with this shift operator E_h we can write the one whole differentiation and one-whole integration. The one whole differentiation is:

$$D^1\left[f(x)\right] = \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)^1 f(x) = \lim_{h\downarrow 0}\frac{f(x)-f(x-h)}{h} \tag{3.152}$$

The one whole integration is:

$$\begin{aligned}D^{-1}\left[f(x)\right] &= \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)^{-1} f(x)\\ &= \lim_{h\downarrow 0}\left[h\left(1+\mathrm{E}_h+\mathrm{E}_h^2+....\right)f(x)\right]\\ &= \lim_{h\downarrow 0}\left[h\left(f(x)+f(x-h)+f(x-2h)+....+f(0)\right)\right]\end{aligned} \tag{3.153}$$

We have used the infinite series expansion $(1-\mathrm{x})^{-1} = 1+\mathrm{x}+\mathrm{x}^2+\mathrm{x}^3+...$ in the above derivation (3.153). However, instead of taking the infinite series expansion of $\left(1-\mathrm{E}_h\right)^{-1}$, we truncate at $f(0)$. As the limit, $h\downarrow 0$ the operation D^1 is simply differentiation, and operation D^{-1} is simply integration. We do n times and write:

$$D^n\left[f(x)\right] = \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)^n f(x) \tag{3.154}$$

This reproduces ordinary whole derivatives. For example:

$$\begin{aligned} D^2\left[f(x)\right] &= \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)^2 f(x) = \lim_{h\downarrow 0}\frac{\left(1-2\mathrm{E}_h+\mathrm{E}_h^2\right)\left[f(x)\right]}{h^2} \\ &= \lim_{h\downarrow 0}\frac{f(x)-2\left(\mathrm{E}_h\left[f(x)\right]\right)+\mathrm{E}_h^2\left[f(x)\right]}{h^2} \\ &= \lim_{h\downarrow 0}\frac{f(x)-2\left(f(x-h)\right)+f(x-2h)}{h^2} \end{aligned} \tag{3.155}$$

In the limit $h\downarrow 0$, which is illustrated above (3.155), how we recover the binomial coefficients for any whole number of differentiation $D^n\left[f(x)\right]$:

$$D^n\left[f(x)\right] = \lim_{h\downarrow 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(f(x-jh)\right) \tag{3.156}$$

However, strictly speaking for (3.155) i.e. the expansion of $D^2\left[f(x)\right]$, this context makes it clear that we should actually write the second derivative as:

$$\begin{aligned} D^2\left[f(x)\right] &= \lim_{h\downarrow 0}\left(\frac{1-\mathrm{E}_h}{h}\right)^2 f(x) \\ &= \lim_{h\downarrow 0}\frac{(1)\left(f(x)\right)-2\left(f(x-h)\right)+(1)\left(f(x-2h)\right)-(0)\left(f(x-3h)\right)+....+(0)\left(f(0)\right)}{h^2} \end{aligned} \tag{3 .157}$$

As in this case n as for the integer in $D^n\left[f(x)\right]$, all the binomial coefficients after $(n+1)$ are identical to zero. That is ${}^nC_j = \frac{n!}{j!(n-j)!}$ for $j \le n$, and ${}^nC_j = 0$ for $j > n$. Thus, we truncate the series at $j = n$ for n as a positive integer. Note here that if n is a negative or non-integer then the binomial coefficients do not truncate.

Consequently, the upper summation limit in the formula (3.156) i.e. n is $n = \frac{x-a}{h}$, where the lower bound on the range of the evaluation of the function is a. We often choose $a = 0$, by convention, but it is actually arbitrary, and we write:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) = \lim_{h\downarrow 0}\frac{1}{h^n}\sum_{j=0}^{\left\lfloor\frac{x-a}{h}\right\rfloor}(-1)^j\binom{n}{j}\left(f(x-jh)\right) \tag{3.158}$$

where $\left\lfloor\frac{x-a}{h}\right\rfloor$ is the floor operator returning the integer number just lower than computed $n = \frac{x-a}{h}$. For any arbitrary n say $\upsilon \in \mathbb{R}$, we write a generalised formula as:

$$\begin{aligned} {}_aD_x^{\upsilon}\left[f(x)\right] &= \frac{\mathrm{d}^{\upsilon}f(x)}{\left[\mathrm{d}(x-a)\right]^{\upsilon}}\lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{x-a}{h}\right\rfloor}(-1)^j\binom{\upsilon}{j}\left(f(x-jh)\right) \\ &= \lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{x-a}{h}\right\rfloor}(-1)^j\frac{\Gamma(\upsilon+1)}{j!\Gamma(\upsilon+1-j)}\left(f(x-jh)\right) \end{aligned} \tag{3.159}$$

For $a = -\infty$ we thus have the following formula:

$${}_{-\infty}D_x^{\upsilon}\left[f(x)\right] = \frac{\mathrm{d}^{\upsilon}f(x)}{\left[\mathrm{d}(x+\infty)\right]^{\upsilon}} = \lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(\upsilon+1)}{j!\Gamma(\upsilon+1-j)}\left(f(x-jh)\right) \tag{3.160}$$

3.14.2 Using the generalised formula obtained by a backward shift operator of fractional integration/differentiation on an exponential function

For $f(x) = e^{ax}$, we do fractional differentiation from $-\infty$ to x by the above formula (3.160) as in the following steps:

$$
\begin{aligned}
{}_{-\infty}D_x^{\upsilon}\left[e^{ax}\right] &= \lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\infty}(-1)^j\binom{\upsilon}{j}e^{a(x-jh)} \\
&= \lim_{h\downarrow 0}\frac{e^{ax}}{h^{\upsilon}}\left(\sum_{j=0}^{\infty}(-1)^j\binom{\upsilon}{j}e^{-ajh}\right) \qquad (3.161) \\
&= \lim_{h\downarrow 0}\frac{e^{ax}}{h^{\upsilon}}\left(1-\upsilon e^{-ah}+\frac{\upsilon(\upsilon-1)}{2!}e^{-2ah}-....\right)
\end{aligned}
$$

Comparing the power series expansion formula that is $(1+\mathrm{x})^{\upsilon}=1+\upsilon\mathrm{x}+\frac{\upsilon(\upsilon-1)}{2!}\mathrm{x}^2+...$ with the above obtained expression (3.161) in the brackets i.e. $1-\upsilon e^{-ah}+\frac{\upsilon(\upsilon-1)}{2!}e^{-2ah}-...$, we can write the following:

$$
\begin{aligned}
{}_{-\infty}D_x^{\upsilon}\left[e^{ax}\right] &= \lim_{h\downarrow 0}\frac{e^{ax}}{h^{\upsilon}}\left(1-e^{-ah}\right)^{\upsilon} \\
&= \lim_{h\downarrow 0}e^{ax}\left(\frac{1-e^{-ah}}{h}\right)^{\upsilon} \qquad (3.162) \\
&= e^{ax}a^{\upsilon}
\end{aligned}
$$

In the above steps we have used $\lim_{h\downarrow 0}\left(\frac{1-e^{-ah}}{h}\right)=a$; which we have derived earlier in this chapter (Section-3.6), that gives the final result ${}_{-\infty}D_x^{\upsilon}\left[e^{ax}\right]=a^{\upsilon}e^{ax}$. This $\lim_{h\downarrow 0}\left(\frac{1-e^{-ah}}{h}\right)=a$ can be simply obtained too by using a series expansion for e^{-ah} and then simplifying it.

3.14.3 Generalisation by use of the forward shift operator defining repeated differentiation and integration

We have used a backward shift operator in the above derivation. We can define a forward shift operator, and from the following definition of one whole derivative, we can build the generalisation. We proceed with the following steps:

$$
\begin{aligned}
D^1\left[f(x)\right] &= \lim_{h\downarrow 0}\frac{f(x+h)-f(x)}{h} \\
D^2\left[f(x)\right] &= \lim_{h\downarrow 0}\frac{f(x+2h)-f(x+h)+f(x)}{h^2}
\end{aligned} \qquad (3.163)
$$

Continuing n times, we obtain the following useful expression:

$$D^n\left[f(x)\right]=\lim_{h\downarrow 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Big(f\big(x+(n-j)h\big)\Big) \qquad (3.164)$$

Say we have a function with the end-point $x=b$, then $n=\left\lfloor\frac{b-x}{h}\right\rfloor$, and we generalise this derivative with a forward shift operator for any non-integer υ as:

$$ {}_xD_b^{\upsilon}\left[f(x)\right]=\lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{b-x}{h}\right\rfloor}(-1)^j\binom{\upsilon}{j}\Big(f\big(x+(\upsilon-j)h\big)\Big) \qquad (3.165)$$

For $b=\infty$ we get:

$$ {}_xD_{\infty}^{\upsilon}\left[f(x)\right]=\lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\infty}(-1)^j\binom{\upsilon}{j}\Big(f\big(x+(\upsilon-j)h\big)\Big) \qquad (3.166)$$

3.14.4 Using the generalised formula obtained by a forward shift operator of fractional integration/differentiation on an exponential function

Now, if we want to obtain the α -derivative of the function $f(x)=e^{ax}$ from point x to ∞; we write the following steps:

$$
\begin{aligned}
{}_xD_\infty^\alpha\left[e^{ax}\right] &= \lim_{h\downarrow 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{a\left(x+(\alpha-k)h\right)}\\
&= \lim_{h\downarrow 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{ax}\left(e^{ah}\right)^{\alpha-k}\\
&= \lim_{h\downarrow 0}\frac{e^{ax}}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\left(e^{ah}\right)^{\alpha-k} = \lim_{h\downarrow 0}\frac{e^{ax}}{h^\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(e^{ah}\right)^{\alpha-k}(-1)^k\\
&= \lim_{h\downarrow 0}\frac{e^{ax}}{h^\alpha}\left(\left(e^{ah}\right)^{\alpha}+\alpha\left(e^{ah}\right)^{\alpha-1}(-1)+\frac{\alpha(\alpha-1)}{2!}\left(e^{ah}\right)^{\alpha-2}(-1)^2+\ldots\right) \qquad (3.167)\\
&= \lim_{h\downarrow 0}\frac{e^{ax}}{h^\alpha}\left(e^{ah}-1\right)^{\alpha}\\
&= \lim_{h\downarrow 0}e^{ax}\left(\frac{e^{ah}-1}{h}\right)^{\alpha}\\
&= e^{ax}a^{\alpha}
\end{aligned}
$$

In the above derivation (3.167) we have used the power series expansion formula $(\mathrm{x}+\mathrm{y})^n=\mathrm{x}^n+n\mathrm{x}^{n-1}\mathrm{y}+\frac{n(n-1)}{2!}\mathrm{x}^{n-2}\mathrm{y}^2+\ldots$, with $n=\alpha$, $\mathrm{x}=e^{ah}$ and $\mathrm{y}=-1$. The $\lim_{h\downarrow 0}\left(\frac{e^{ah}-1}{h}\right)=a$, gives us the result ${}_xD_\infty^\alpha\left[e^{ax}\right]=a^\alpha e^{ax}$.

This is proved via similar lines as done for $\lim_{h\downarrow 0}\left(\frac{1-e^{-ah}}{h}\right)=a$. That is take $g(x)=e^{ax}$ then $g^{(1)}(x)=ae^{ax}$, $g(0)=1$, and $g^{(1)}(0)=a$. The $g^{(1)}(x)=\lim_{h\downarrow 0}\left(\frac{g(x+h)-g(x)}{h}\right)$, and at $x=0$, we have $g^{(1)}(x)\Big|_{x=0}=\lim_{h\downarrow 0}\left(\frac{g(h)-g(0)}{f}\right)$, which is equal to $g^{(1)}(x)\Big|_{x=0}=\lim_{h\downarrow 0}\left(\frac{e^{ah}-1}{h}\right)=a$. These results are otherwise obtained via the series expansion of e^{ax}.

3.14.5 The final unified formula for fractional derivatives and integration: the Grunwald-Letnikov formula

In this section, what we observed is the fractional differentiation which is the like weighted sum of all the values of the function from a to x or from x to b; just like integration. In an earlier chapter (Section-2.20) we have generalised the repeated integration and obtained the formula for fractional integration of the order $\alpha>0$ as a limit of the sum, and we write that as:

$$\frac{\mathrm{d}^{-\alpha}f(x)}{\left[\mathrm{d}(x-a)\right]^{-\alpha}}=\lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j+\alpha)}{\Gamma(j+1)}\left(f\left(x-j\left(\frac{x-a}{N}\right)\right)\right)\right) \qquad (3.168)$$

We write the β-th derivative to any arbitrary $\beta>0$ by the following formula, which we derived previously (Section-3.13), that is:

$$\frac{\mathrm{d}^{\beta}f(x)}{\left[\mathrm{d}(x-a)\right]^{\beta}}=\lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\beta}}{\Gamma(-\beta)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}\left(f\left(x-j\left(\frac{x-a}{N}\right)\right)\right)\right) \qquad (3.169)$$

Therefore, in light of these deductions we write a unified formula for fractional differ-integration (differentiation and integration) of any arbitrary order α as:

$$
\begin{aligned}
\frac{\mathrm{d}^{\alpha}f(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}} &= \lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}\left(f\left(x-j\left(\frac{x-a}{N}\right)\right)\right)\right)\\
&= \lim_{N\uparrow\infty}\left(\left(\frac{N}{x-a}\right)^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\left(f\left(\frac{Nx-jx+ja}{N}\right)\right)\right)
\end{aligned}
\qquad (3.170)
$$

With $\alpha=\frac{1}{2}$, and $a=0$ we write:

$$ {}_0D_x^{1/2}\left[f(x)\right]=\frac{\mathrm{d}^{1/2}\left[f(x)\right]}{\mathrm{d}x^{1/2}}=\lim_{N\uparrow\infty}\left(\left(\sqrt{\frac{N}{x}}\right)\sum_{j=0}^{N-1}\left(\frac{-(2j)!}{(2j-1)(2^j j!)^2}\right)\left(f\left(x-\tfrac{jx}{N}\right)\right)\right) \tag{3.171}$$

With $\alpha=-\frac{1}{2}$ and $a=0$ we write:

$$ {}_0D_x^{-1/2}\left[f(x)\right]=\frac{\mathrm{d}^{-1/2}\left[f(x)\right]}{\mathrm{d}x^{-1/2}}=\lim_{N\uparrow\infty}\left(\left(\sqrt{\frac{x}{N}}\right)\sum_{j=0}^{N-1}\left(\frac{(2j)!}{(2^j j!)^2}\right)f\left(x-\tfrac{jx}{N}\right)\right) \tag{3.172}$$

We have used the formula $\Gamma\left(\frac{1}{2}+j\right)=\frac{(2j)!\sqrt{\pi}}{4^j j!}$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, to simplify $\frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(j+1)}$ as described, for the case when $\alpha=-\frac{1}{2}$ in the following expression:

$$\frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(j+1)}=\left(\frac{1}{\sqrt{\pi}}\right)\left(\frac{(2j)!\sqrt{\pi}}{(2^{2j})j!}\right)\left(\frac{1}{j!}\right)=\frac{(2j)!}{(2^j j!)^2} \tag{3.173}$$

For $\alpha=\frac{1}{2}$, we do the following steps, with $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$ and using $(2j-2)!=\frac{(2j)!}{(2j)(2j-1)}$ and also $(j-1)!=\frac{j!}{j}$ with $\Gamma\left(\frac{1}{2}+k\right)=\frac{(2k)!\sqrt{\pi}}{4^k k!}$:

$$\begin{aligned}\frac{\Gamma\left(j-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma(j+1)}&=\frac{\Gamma\left((j-1)+\frac{1}{2}\right)}{\left(-2\sqrt{\pi}\right)(j!)}=\left(-\frac{1}{2\sqrt{\pi}}\right)\left(\frac{\left(2(j-1)\right)!\sqrt{\pi}}{4^{j-1}(j-1)!}\right)\left(\frac{1}{j!}\right)\\&=\left(-\frac{1}{2}\right)\left(\frac{\frac{(2j)!}{(2j)(2j-1)}}{4^j4^{-1}\left(\frac{j!}{j}\right)}\right)\left(\frac{1}{j!}\right)=-\frac{(2j)!}{(2j-1)(2^j j!)^2}\end{aligned} \tag{3.174}$$

When α is negative, the formula gives us a fractional integration and for positive α the formula is a fractional differentiation. These above formulae are useful to compute the fractional differ-integration of a function $f(x)$ from a to x.

3.14.6 Reduction of a unified fractional derivative/integration unified formula to a classical derivative and integration

The above Grunwald-Letnikov (GL) formula also correctly reproduces the standard classical definition when α is the integer of either sign. Thus when $\alpha=-1$ or $\alpha=1$, the GL formula reduces to the Riemann sum limit as follows:

$$\frac{\mathrm{d}^{-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}=\lim_{N\uparrow\infty}\left(\left(\frac{x-a}{N}\right)\sum_{j=0}^{N-1}\left(f\left(x-j\left(\tfrac{x-a}{N}\right)\right)\right)\right) \tag{3.175}$$

or the backward difference quotient limit as follows:

$$\frac{\mathrm{d}^{1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{1}}=\lim_{N\uparrow\infty}\left(\left(\frac{x-a}{N}\right)^{-1}\sum_{j=0}^{1}(-1)^j f\left(x-j\left(\tfrac{x-a}{N}\right)\right)\right) \tag{3.176}$$

However, notice that the Riemann sum would converge more rapidly to the integral $\int_a^x f(x)\mathrm{d}x\equiv\frac{\mathrm{d}^{-1}}{\left[\mathrm{d}(x-a)\right]^{-1}}\left[f(x)\right]$ as $N\uparrow\infty$ if we write the following expression:

$$\frac{\mathrm{d}^{-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{-1}}=\lim_{N\uparrow\infty}\left(\left(\frac{x-a}{N}\right)\sum_{j=0}^{N-1}f\left(x-\left(j+\tfrac{1}{2}\right)\left(\tfrac{x-a}{N}\right)\right)\right) \tag{3.177}$$

Similarly, the difference quotient would converge more rapidly to the true derivative if we write the following expression:

$$\frac{\mathrm{d}^{1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{1}}=\lim_{N\uparrow\infty}\left(\left(\frac{x-a}{N}\right)^{-1}\sum_{j=0}^{1}(-1)^j f\left(x-\left(j-\tfrac{1}{2}\right)\left(\tfrac{x-a}{N}\right)\right)\right) \tag{3.178}$$

This type of modification gives us the modified Grunwald-Letnikov formula for any arbitrary order α, which is expressed below:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}f\left(x-\left(j-\tfrac{1}{2}\alpha\right)\left(\tfrac{x-a}{N}\right)\right)\right) \tag{3.179}$$

This modified GL formula we will use in the subsequent section of formulating numerical algorithms to evaluate fractional differ-integrals.

3.15 The whole derivative is local property whereas the fractional derivative is non-local property

For n as a non-integer say υ; we write the generalised expression as:

$$\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}x^{\upsilon}}f(x)=\lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{x-a}{h}\right\rfloor}(-1)^{j}\,\frac{\Gamma(\upsilon+1)}{j!\,\Gamma(\upsilon+1-j)}f(x-jh) \tag{3.180}$$

If υ is a positive integer the vanishing of binomial coefficients for all $j>n$ implies we do not really need to carry the summation in the above formula beyond $j=n$, and in the limit $h\downarrow 0$, the n values of $f(x-jh)$ with non-zero coefficients all converge on point x. Therefore, the n whole derivative is the local property of the function. This process of classical Newtonian calculus, where whole derivatives are local, or the point property is referred to as a Markovian process. However, in general, the binomial expansion behaves infinitely and may have non-zero coefficients so the result depends on the value of the function at x plus all the way down to a; that is, the start point of the function. In that case, we are evaluating the derivative (note this is not the same as slope at point) for the entire interval a to x. The processes where the value of the function at all the previous points is considered to arrive at the derivative are called non-Markovian processes. Therefore the fractional derivative (generalisation of the whole order derivative), is not a local property but a non-local operation and suited for dynamics with non-Markovian systems. Thus, we note that the generalised derivative depends on the value of the function $f(x)$ over the entire range a to x. In the formula the term $f(x-jh)$, has j which ranges from $j=0,1,2,....\left\lfloor\frac{x-a}{h}\right\rfloor$. Meaning that, the argument of the function f ranges from point x, down to the start point of the function.

It just so happens that this non-locality disappears for positive whole derivatives, when υ is a positive integer, say n. However, for $n=\upsilon$ as a real number, the property is non-local, thus requiring all the points of the function to construct the generalised derivative of the function. This non-locality of the fractional derivative says that the fractional derivative of the function at any point depends on the history in the sense if we consider E_h a backward shift operator. While constructing the derivative by forward shift operator, we consider forward history. That is, the future points from x to b are required to compute the derivative.

3.16 Applying the limit of a finite difference formula result to get the answer to L'Hospital's question

We chose the start point $a=0$ and function $f(x)=x$ and in the following formula, we chose $\upsilon=\frac{1}{2}$.

$$\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}x^{\upsilon}}f(x)=\lim_{h\downarrow 0}\frac{1}{h^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{x}{h}\right\rfloor}(-1)^{j}\,\frac{\Gamma(\upsilon+1)}{j!\,\Gamma(\upsilon+1-j)}f(x-jh) \tag{3.181}$$

Using the binomial expansion in the above formula (3.181) we write:

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[x\right]=\lim_{h\downarrow 0}\frac{1}{\sqrt{h}}\left(\begin{array}{l}1(x)-\frac{1}{2}(x-h)-\frac{1}{8}(x-2h)-\frac{1}{16}(x-3h)\\ -\frac{5}{128}(x-4h)-...-\left(\begin{array}{c}\frac{1}{2}\\ \left\lfloor\frac{x}{h}\right\rfloor-2\end{array}\right)(2h)-\left(\begin{array}{c}\frac{1}{2}\\ \left\lfloor\frac{x}{h}\right\rfloor-1\end{array}\right)(h)-0\end{array}\right) \tag{3.182}$$

In this expression (3.182), the binomial coefficient symbol is understood to denote the generalised function, with the factorials expressed in terms of the gamma function. As explained earlier (Section-3.14), the coefficients in the above expressions are just the coefficients in the binomial expansion of $\left(1-\mathrm{E}_h\right)^{1/2}$. The RHS of the above obtained series (3.182) can be evaluated with various values of x, and taking a certain slice h; and we then see that approximately the expression converges to $2\sqrt{\frac{x}{\pi}}$. Obviously, this numerical evaluation of the series representation is approximate. The point is that the convergence of the above to the earlier obtained semi-derivative of $f(x)=x$, via the Euler formula is:

$$
\begin{aligned}
{}_0D_x^{\upsilon}\left[x^m\right] &= \frac{\Gamma(m+1)}{\Gamma(m-\upsilon+1)}x^{m-\upsilon} \\
{}_0D_x^{1/2}\left[x\right] &= \frac{1!}{\Gamma\left(1-\frac{1}{2}+1\right)}x^{1-\frac{1}{2}} = \frac{\sqrt{x}}{\Gamma\left(\frac{3}{2}\right)} = 2\sqrt{\frac{x}{\pi}}
\end{aligned} \tag{3.183}
$$

We used $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. From the Euler formula, we do the anti-derivative of $2\sqrt{\frac{x}{\pi}}$, that is:

$$
\begin{aligned}
{}_0D_x^{-1/2}\left[2\sqrt{\tfrac{x}{\pi}}\right] &= \frac{2}{\sqrt{\pi}}\,{}_0D_x^{-1/2}\left[x^{1/2}\right] = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}x^{\frac{1}{2}-\left(-\frac{1}{2}\right)} \\
&= \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}\right)}{1!}x = x
\end{aligned} \tag{3.184}
$$

3.17 Fractional finite difference (from local finite difference to non-local finite difference)

In this section, we will find what we have developed in an earlier part of the chapter, the formula for fractional differentiation and fractional integration via the most fundamental definition, applied to the concept of finite differences. Thereby we analytically continue the concept of finite differences to the non-integer order. We here too see the non-local effect and the memory effect described earlier. In a finite difference if we choose the matrix $\left[D^1\right]$ as:

$$
\left[D_x^1\right] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left(\Delta x\right)^{-1} \tag{3.185}
$$

Then $\left[D_x^2\right] = \left[D_x^1\right]\left[D_x^1\right] = \left[D_x^1\right]^2$ is the following:

$$
\left[D_x^2\right] = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left(\Delta x\right)^{-2} \tag{3.186}
$$

For integration $\left[D^{-1}\right]$ is an inverse of $\left[D^1\right]$, that is $\left[D_x^{-1}\right] = \left[D_x^1\right]^{-1}$; we then write the following:

$$
\left[D_x^{-1}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & .. & 1 \\ 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 1 & 1 & .. & 1 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left(\Delta x\right) \tag{3.187}
$$

For $\left[D_x^m\right]$, where m is a positive integer we write the following:

$$
\left[D_x^m\right] = \left[D_x^1\right]^m = \overbrace{\left[D_x^1\right]\left[D_x^1\right]........................\left[D_x^1\right]}^{m-\text{times}} \tag{3.188}
$$

For $\left[D_x^{-m}\right]$, where m is a positive integer we write the following:

$$\left[D_x^{-m}\right]=\left[D_x^{-1}\right]^m=\underbrace{\left[D_x^{-1}\right]\left[D_x^{-1}\right]\ldots\ldots\ldots\ldots\ldots\ldots\left[D_x^{-1}\right]}_{m-\text{times}} \tag{3.189}$$

With the function $f(x)$ having values at x_1, x_2, $x_3\ldots$ x_{n-1} and x_n, that is with $x_k=x_1+(k-1)\Delta x$, for $k=1,2,\ldots n$ the matrix representation of the finite difference as per the above definitions are the following:

$$\left[D_x^1\right]f(x)=\begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\..&..&..&..&..&..\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}f(x_n)\\f(x_{n-1})\\f(x_{n-2})\\..\\f(x_2)\\f(x_1)\end{bmatrix}\left(\Delta x\right)^{-1}$$

$$=\begin{bmatrix}D_x^1f(x_n)\\D_x^1f(x_{n-1})\\D_x^1f(x_{n-2})\\..\\D_x^1f(x_2)\\\left(f(x_1)\right)\left(\Delta x\right)^{-1}\end{bmatrix} \tag{3.190}$$

For integration we have the following:

$$\left[D_x^{-1}\right]f(x)=\begin{bmatrix}1&1&1&1&1&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\..&..&..&..&..&..\\0&0&0&0&1&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}f(x_n)\\f(x_{n-1})\\f(x_{n-2})\\..\\f(x_2)\\f(x_1)\end{bmatrix}\left(\Delta x\right)$$

$$=\begin{bmatrix}D_x^{-1}f(x)\Big|_{x_1}^{x_n}\\D_x^{-1}f(x)\Big|_{x_1}^{x_{n-1}}\\D_x^{-1}f(x)\Big|_{x_1}^{x_{n-2}}\\..\\D_x^{-1}f(x)\Big|_{x_1}^{x_2}\\\left(f(x_1)\right)\left(\Delta x\right)\end{bmatrix} \tag{3.191}$$

Now with a fractional generalisation of the m integer to $\alpha>0$, and $\alpha<0$, a real number, we have the analytic continuation to write the following:

$$\left[D_x^{\alpha}\right]=\begin{bmatrix}w_{\alpha,1}&w_{\alpha,2}&w_{\alpha,3}&..&..&w_{\alpha,n}\\0&w_{\alpha,1}&w_{\alpha,2}&w_{\alpha,3}&..&w_{\alpha,(n-1)}\\0&0&w_{\alpha,1}&w_{\alpha,2}&..&w_{\alpha,(n-2)}\\0&0&0&..&..&..\\0&0&0&0&w_{\alpha,1}&w_{\alpha,2}\\0&0&0&0&0&w_{\alpha,1}\end{bmatrix}\left(\Delta x\right)^{-\alpha} \tag{3.192}$$

$\alpha>0$

$$\left[D_x^{\alpha}\right]=\begin{bmatrix} w_{-\alpha,1} & w_{-\alpha,2} & w_{-\alpha,3} & .. & .. & w_{-\alpha,n} \\ 0 & w_{-\alpha,1} & w_{-\alpha,2} & w_{-\alpha,3} & .. & w_{-\alpha,(n-1)} \\ 0 & 0 & w_{-\alpha,1} & w_{-\alpha,2} & .. & w_{-\alpha,(n-2)} \\ 0 & 0 & 0 & .. & .. & .. \\ 0 & 0 & 0 & 0 & w_{-\alpha,1} & w_{-\alpha,2} \\ 0 & 0 & 0 & 0 & 0 & w_{-\alpha,1} \end{bmatrix}^{-1} (\Delta x)^{-\alpha} \tag{3.193}$$

$\alpha < 0$

which satisfies the condition $\left[D_x^{\alpha}\right]=\left[D_x^{\alpha_1}\right]\left[D_x^{\alpha_2}\right]......\left[D_x^{\alpha_p}\right]$; for $\alpha=\alpha_1+\alpha_2+\alpha_3+....+\alpha_p$. The weights are:

$$w_{\alpha,k}=\frac{(-1)^{(k-1)}}{\Gamma(k)}\frac{\Gamma(1+\alpha)}{\Gamma(2+\alpha-k)} \tag{3.194}$$

which is incidentally the $k-$th term coefficient of binomial expansion of $\left(1+(-1)\right)^{\alpha}$. Note that we have derived the weight, i.e. $w_{\alpha,k}$, for the fractional differentiation and integration formula. These weights ($w_{\alpha,k}$) are the same as those obtained in our earlier section where we wrote $\frac{(-1)^j}{j!}\frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-j)}$ for υ-th order differ-integration. If we replace j by $k-1$ and υ with α, we obtain our coefficient $w_{\alpha,k}$, that is $\frac{(-1)^{(k-1)}}{(k-1)!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k+1)}=\frac{(-1)^{(k-1)}}{\Gamma(k)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+2)}=w_{\alpha,k}$. We will learn in the next chapter (Chapter-4) to write in limit form to have a more general representation of the coefficients $w_{\alpha,k}$ as in the following form:

$$w_{\alpha,k}=\frac{(-1)^{(k-1)}}{\Gamma(k)}\lim_{\in\downarrow 0}\frac{\Gamma(1+\alpha+\in)}{\Gamma(2+\alpha-k+\in)} \tag{3.195}$$

The matrix representation D^m is sparse, while D^{α} is generally dense; all the elements in the upper tri-diagonal blocks are non-zero. In finite difference, a sparse matrix entails taking the difference among the neighboring set of points, while a dense matrix entails taking the difference among almost every point in the domain; this is exactly the non-local effect; also called the memory effect. From the above matrix representations, we say:

$$\begin{aligned} D^{\alpha}f(x_m) &= \lim_{n\uparrow\infty}\left[D^{\alpha}\right]\left[f(x)\right]\Big|_{x=x_m} \\ &= \lim_{n\uparrow\infty}\sum_{k=1}^{m} w_{\alpha,k}\left(f(x_{m-k+1})\right)\left(\Delta x\right)^{-\alpha}, \quad 1<<m\le n \end{aligned} \tag{3.196}$$

We have derived the above (3.196) in an earlier section too. If $f(x)$ is an integer power series, then $D^{\alpha}f(x_m)$ on the LHS in general is a non-integer power series; for example as we have demonstrated (in an earlier section) the half-derivative of e^x represented as an integer order power-series, when operated by $D^{1/2}$ gives a non-integer order power-series. The corresponding matrix representation on the RHS above is a sum of ordinary power series. The non-local effect can be then seen to arise from approximating the non-integer power series by sum of an ordinary power series.

3.18 Dependence on the lower limit

What is the effect of fractional differentiation and integration if the lower limit changes from say $x=a$ to $x=b$, obviously there will be a different result. In other words, we want to evaluate $\Delta\equiv {}_aD_x^{\alpha}\left[f(x)\right]-{}_bD_x^{\alpha}\left[f(x)\right]$. Here we derive a formula that precisely demonstrates these phenomena. We consider for study an analytic function $\varphi(x)$, in the interval a to x and assume that $a<b<x$. Writing $\Delta=\frac{\mathrm{d}^{\alpha}\varphi(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}}-\frac{\mathrm{d}^{\alpha}\varphi(x)}{\left[\mathrm{d}(x-b)\right]^{\alpha}}$, one might expect that the difference Δ is expressible in terms of integrals of $\varphi(x)$ in the interval $a\le x\le b$; the formula is $\Delta=\sum_{k=1}^{\infty}\frac{\mathrm{d}^{\alpha+k}[1]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k}}\frac{\mathrm{d}^{-k}\varphi(b)}{\left[\mathrm{d}(b-a)\right]^{-k}}$; analogous to the case that we obtained in the earlier chapter (Section-1.20) for a repeated n-fold integration. Assume $\alpha<0$, and we have the Riemann-Liouville formula and state the following:

$$\begin{aligned}
\Delta &\equiv {}_aD_x^{\alpha}\left[\varphi(x)\right] - {}_bD_x^{\alpha}\left[\varphi(x)\right] \\
&= \frac{\mathrm{d}^{\alpha}\varphi(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}} - \frac{\mathrm{d}^{\alpha}\varphi(x)}{\left[\mathrm{d}(x-b)\right]^{\alpha}} \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^x \frac{\varphi(y)\mathrm{d}y}{(x-y)^{\alpha+1}} - \frac{1}{\Gamma(-\alpha)}\int_b^x \frac{\varphi(y)\mathrm{d}y}{(x-y)^{\alpha+1}} \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^b \frac{\varphi(y)\mathrm{d}y}{(x-y)^{\alpha+1}} \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^b \frac{\varphi(y)\mathrm{d}y}{(x-b+b-y)^{\alpha+1}} \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^b \left(\sum_{j=0}^{\infty}\binom{-1-\alpha}{j}(x-b)^{-1-\alpha-j}(b-y)^{j}\right)\left(\varphi(y)\right)\mathrm{d}y
\end{aligned} \tag{3.197}$$

In the above steps (3.197) we have used a binomial expansion for $\left((x-b)+(b-y)\right)^{-1-\alpha}$. We have the following property of the binomial coefficients and its generalisation by the gamma function i.e.

$$\binom{j-q-1}{j} = (-1)^j\binom{q}{j} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \tag{3.198}$$

Replacing in the above formula (3.198) j with r and q with $\alpha+r$ we get:

$$\binom{-1-\alpha}{r} = (-1)^r\binom{\alpha+r}{r} = \frac{\Gamma(-\alpha)}{\Gamma(-\alpha-r)\Gamma(r+1)} \tag{3.199}$$

and then using the above relationship from (3.199) in (3.197) we write the following steps:

$$\begin{aligned}
\Delta &\equiv {}_aD_x^{\alpha}\left[\varphi(x)\right] - {}_bD_x^{\alpha}\left[\varphi(x)\right] \\
&= \int_a^b \sum_{k=0}^{\infty}\frac{(x-b)^{-1-\alpha-k}}{\Gamma(-\alpha-k)}\frac{(b-y)^k}{\Gamma(k+1)}\left(\varphi(y)\right)\mathrm{d}y \\
&= \sum_{k=0}^{\infty}\frac{\mathrm{d}^{\alpha+k+1}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k+1}}\int_a^b \frac{\left(\varphi(y)\right)\mathrm{d}y}{\left(\Gamma(k+1)\right)(b-y)^{-k}} \\
&= \sum_{k=0}^{\infty}\frac{\mathrm{d}^{\alpha+k+1}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k+1}}\frac{\mathrm{d}^{-k-1}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k-1}} \\
&= \sum_{k=0}^{\infty}\frac{\mathrm{d}^{\alpha+k}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k}}\frac{\mathrm{d}^{-k}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k}} = \sum_{k=0}^{\infty}\left({}_bD_x^{\alpha+k}\left[1\right]\right)\left({}_aD_b^{-k}\left[\varphi(b)\right]\right)
\end{aligned} \tag{3.200}$$

In the above derivation (3.200) we used ${}_bD_x^{\alpha+r+1}\left[1\right] = \frac{(x-b)^{-(\alpha+r+1)}}{\Gamma(-\alpha-r)}$ which comes from ${}_aD_x^{\alpha}\left[C\right] = \frac{C}{\Gamma(1-\alpha)}(x-a)^{-\alpha}$, where $C=1$.

In addition, we used ${}_aD_x^{-(r+1)}\left[\varphi(x)\right] = \int_a^x \frac{(\varphi(y))\mathrm{d}y}{(\Gamma(r+1))(x-y)^{-r}} = \int_a^b \frac{(\varphi(y))\mathrm{d}y}{(\Gamma(r+1))(b-y)^{-r}}$ in (3.199); from this, we write, by using $x=b$, the expression ${}_aD_b^{-(r+1)}\left[\varphi(b)\right] = \int_a^b \frac{(\varphi(y))\mathrm{d}y}{(\Gamma(r+1))(b-y)^{-r}} = \frac{\mathrm{d}^{-(r+1)}[\varphi(b)]}{[\mathrm{d}(b-a)]^{-(r+1)}}$. We state that

$$\Delta = \sum_{k=1}^{\infty}\frac{\mathrm{d}^{\alpha+k}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k}}\frac{\mathrm{d}^{-k}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k}} = \sum_{k=1}^{\infty}\left({}_bD_x^{\alpha+k}\left[1\right]\right)\left({}_aD_b^{-k}\left[\varphi(b)\right]\right) \tag{3.201}$$

The expression (3.201) is valid for $\alpha<0$. Therefore, we have the above formula (3.201), and argue that, since both sides of the equation are analytic in α by the identity theorem for analytic functions, this above formula gets valid for all α.

For $\alpha = n = 0,1,2,3,\dots$, we have the following:

$$\Delta = \sum_{k=1}^{\infty} \frac{\mathrm{d}^{\alpha+k}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k}} \frac{\mathrm{d}^{-k}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k}} \qquad \alpha = n = 0,1,2,3...$$

$$= \sum_{k=1}^{\infty} \frac{\mathrm{d}^{n+k}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{n+k}} \frac{\mathrm{d}^{-k}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k}}$$

$$= \frac{\mathrm{d}^{n+1}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{n+1}} \frac{\mathrm{d}^{-1}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-1}} + \frac{\mathrm{d}^{n+2}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{n+2}} \frac{\mathrm{d}^{-2}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-2}} \tag{3.202}$$

$$+ \frac{\mathrm{d}^{n+3}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{n+3}} \frac{\mathrm{d}^{-3}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-3}} + \dots$$

$$= 0 \qquad n = 1,2,3...$$

Therefore, when α are zero or a positive integer, then all integer order derivatives of unity are zero so we obtain $\Delta = 0$ for $\alpha = n = 0,1,2,3....$. For $\alpha = -1$, we obtain the classical result, as we demonstrate below:

$$\Delta = \sum_{k=1}^{\infty} \frac{\mathrm{d}^{\alpha+k}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{\alpha+k}} \frac{\mathrm{d}^{-k}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-k}} \qquad \alpha = -1$$

$$= \frac{\mathrm{d}^{-1+1}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{-1+1}} \frac{\mathrm{d}^{-1}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-1}} + \frac{\mathrm{d}^{-1+2}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{-1+2}} \frac{\mathrm{d}^{-2}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-2}}$$

$$+ \frac{\mathrm{d}^{-1+3}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{-1+3}} \frac{\mathrm{d}^{-3}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-3}} + \dots \tag{3.203}$$

$$= \frac{\mathrm{d}^{0}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{0}} \frac{\mathrm{d}^{-1}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-1}} + 0 = \frac{\mathrm{d}^{-1}\left[\varphi(b)\right]}{\left[\mathrm{d}(b-a)\right]^{-1}}$$

$$= \int_a^b \varphi(y)\mathrm{d}y = \varphi^{-1}(x)\Big|_{x=b} - \varphi^{-1}(x)\Big|_{x=a} = \varphi^{-1}(b) - \varphi^{-1}(a)$$

This result we have obtained in an earlier chapter (Section-1.20).

For all other values of α, the difference Δ is non-zero and its value depends not only on a, b but also on x. For example, for a negative integer $\alpha = -n$, we find:

$$\Delta = \sum_{k=1}^{n} \frac{\mathrm{d}^{k-n}\left[1\right]}{\left[\mathrm{d}(x-b)\right]^{k-n}} \frac{\mathrm{d}^{-k}\varphi(b)}{\left[\mathrm{d}(b-a)\right]^{-k}}, \quad \alpha = -n = -1,-2,-3,....$$

$$= \sum_{k=1}^{n} \frac{(x-b)^{n-k}}{\Gamma(1-k+n)} \frac{\mathrm{d}^{-k}\varphi(b)}{\left[\mathrm{d}(b-a)\right]^{-k}} = \sum_{k=1}^{n} \frac{(x-b)^{n-k}}{(k-n)!} \frac{\mathrm{d}^{-k}\varphi(b)}{\left[\mathrm{d}(b-a)\right]^{-k}} \tag{3.204}$$

Note, for $\alpha = -n$ the sum terminates at $k = n$, as at $k = n+1$ and beyond $\frac{\mathrm{d}^{k-n}[1]}{\left[\mathrm{d}(x-b)\right]^{k-n}} = 0$; nevertheless for a non-integer α, the sum extends until infinity.

3.19 Translation property

By translation property, we mean $f(x)$ is replaced by $f(C+x)$, where C is a constant, which we take as positive. We assume that $\frac{\mathrm{d}^{\alpha}[f(x)]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}$ is defined and of course $f(x)$ is defined between $\min(a, a+C)$ and $\max(x, x+C)$. We apply the RL formula:

$$\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{1}{\Gamma(-\alpha)}\int_{a}^{x}\frac{\left(f(y+C)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}\qquad \text{put}\qquad \xi=y+C$$
$$=\frac{1}{\Gamma(-\alpha)}\int_{a+C}^{x+C}\frac{\left(f(\xi)\right)\mathrm{d}\xi}{(x+C-\xi)^{\alpha+1}} \tag{3.205}$$

It is evident that translation by a distance C is equivalent to a shift in the upper limit from x to $x+C$, and shift in lower limit from a to $a+C$. Representing the effect of this shift in the independent variable, by the symbol Δ as we derived in the earlier section, we write the following:

$$\Delta=\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x+C-a)\right]^{\alpha}}-\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} \tag{3.206}$$

From here, we write:

$$\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x+C-a)\right]^{\alpha}}-\Delta$$
$$=\frac{\mathrm{d}^{\alpha}\left[f(x+C)\right]}{\left[\mathrm{d}(x+C-a)\right]^{\alpha}}-\sum_{k=1}^{\infty}\frac{\mathrm{d}^{\alpha+k}\left[1\right]}{\left[\mathrm{d}(x+C-a)\right]^{\alpha+k}}\frac{\mathrm{d}^{-k}\left[f(a+C)\right]}{\left[\mathrm{d}(a+C-a)\right]^{-k}} \tag{3.207}$$

Though the use of the RL definition used in calculating Δ requires $\alpha<0$; (3.205) the usual analytical argument based on the identity theorem serves to remove this restriction. The above-obtained formula involves an infinite sum that is not, in general, represented by a closed form expression. Thus, this translation represents a process, which is difficult to handle in fractional differentiation and integration. For $\alpha=\frac{1}{2}$, and $a=0$ then with Euler's formula we write $\frac{\mathrm{d}^{(1/2)+k}[1]}{\left[\mathrm{d}(x+C)\right]^{(1/2)+k}}=\frac{1}{\Gamma\left(1-\frac{1}{2}-k\right)}(x+C)^{-(1/2)-k}$; with this we have:

$$\frac{\mathrm{d}^{\frac{1}{2}}\left[f(x+C)\right]}{\left[\mathrm{d}(x)\right]^{\frac{1}{2}}}=\frac{\mathrm{d}^{\frac{1}{2}}\left[f(x+C)\right]}{\left[\mathrm{d}(x+C)\right]^{\frac{1}{2}}}-\sum_{k=1}^{\infty}\frac{(x+C)^{-\left(\frac{1}{2}\right)-k}}{\Gamma\left(\frac{1}{2}-k\right)}\frac{\mathrm{d}^{-k}\left[f(C)\right]}{\mathrm{d}C^{-k}} \tag{3.208}$$

For $\alpha=-\frac{1}{2}$ and $a=0$ we write $\frac{\mathrm{d}^{(-1/2)+k}[1]}{\left[\mathrm{d}(x+C)\right]^{(-1/2)+k}}=\frac{1}{\Gamma\left(1+\frac{1}{2}-k\right)}(x+C)^{(1/2)-k}$ after which we write:

$$\frac{\mathrm{d}^{-\frac{1}{2}}\left[f(x+C)\right]}{\left[\mathrm{d}(x)\right]^{-\frac{1}{2}}}=\frac{\mathrm{d}^{-\frac{1}{2}}\left[f(x+C)\right]}{\left[\mathrm{d}(x+C)\right]^{-\frac{1}{2}}}-\sum_{k=1}^{\infty}\frac{(x+C)^{\left(\frac{1}{2}\right)-k}}{\Gamma\left(\frac{3}{2}-k\right)}\frac{\mathrm{d}^{-k}\left[f(C)\right]}{\mathrm{d}C^{-k}} \tag{3.209}$$

3.20 Scaling property

By the scale change of a function $f(x)$ with respect of the start point of differentiation (or integration) that is the lower limit $x=a$, it means that the function is replaced by $f(\lambda x-\lambda a+a)$, where λ is a constant, called a scale-factor. To simplify this take $a=0$, then the scale change converts $f(x)$ to $f(\lambda x)$. Here we will obtain $\frac{\mathrm{d}^{\alpha}}{\left[\mathrm{d}(x-a)\right]^{\alpha}}$ an RL operation on $f(\lambda x-\lambda a+a)$. We assume $\frac{\mathrm{d}^{\alpha}f(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}}$ is known. Call the variable $\chi=x+\left(\frac{a-a\lambda}{\lambda}\right)$, and call $\xi=\lambda y-\lambda a+a$, then the following steps are carried out:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[f(\lambda\chi)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}&=\frac{\mathrm{d}^{\alpha}\left[f(\lambda x-\lambda a+a)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\\&=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{f(\lambda y-\lambda a+a)}{(x-y)^{\alpha+1}}\mathrm{d}y\\&=\frac{1}{\Gamma(-\alpha)}\int_a^{\lambda\chi}\frac{f(\xi)}{\left(\frac{\lambda\chi-\xi}{\lambda}\right)^{\alpha+1}}\left(\frac{\mathrm{d}\xi}{\lambda}\right)\\&=\frac{\lambda^{\alpha}}{\Gamma(-\alpha)}\int_a^{\lambda\chi}\frac{\left(f(\xi)\right)\mathrm{d}\xi}{(\lambda\chi-\xi)^{\alpha+1}}\\&=\lambda^{\alpha}\frac{\mathrm{d}^{\alpha}\left[f(\lambda\chi)\right]}{\left[\mathrm{d}(\lambda\chi-a)\right]^{\alpha}}\end{aligned}\tag{3.210}$$

When $a=0$ then we have $\chi=x$ and the scale change of the property is simply a multiplication of the independent variable by the constant λ; the above formula is then:

$$\frac{\mathrm{d}^{\alpha}\left[f(\lambda x)\right]}{\left[\mathrm{d}x\right]^{\alpha}}=\lambda^{\alpha}\frac{\mathrm{d}^{\alpha}\left[f(\lambda x)\right]}{\left[\mathrm{d}(\lambda x)\right]^{\alpha}},\qquad {}_0D_x^{\alpha}\left[f(\lambda x)\right]=\lambda^{\alpha}\left({}_0D_{\lambda x}^{\alpha}\left[f(\lambda x)\right]\right)\tag{3.211}$$

In the previous chapter (Section-2.18), we did a similar derivation for the fractional integration process, that is:

$${}_0D_x^{-\alpha}\left[f(\lambda x)\right]=\lambda^{-\alpha}\left({}_0D_x^{-\alpha}\left[f(x)\right]\right)=\lambda^{-\alpha}\left({}_0D_{\lambda x}^{-\alpha}\left[f(\lambda x)\right]\right)\tag{3.212}$$

Here we mention that the above obtained derivations (3.210) and (3.211) are for a fractional derivative and fractional integration respectively. Say we have $a=0$, for $\alpha=\frac{1}{2}$, and $\alpha=-\frac{1}{2}$ we write the following:

$$\frac{\mathrm{d}^{1/2}\left[f(\lambda x)\right]}{\left[\mathrm{d}x\right]^{1/2}}=\sqrt{\lambda}\frac{\mathrm{d}^{1/2}\left[f(\lambda x)\right]}{\left[\mathrm{d}(\lambda x)\right]^{1/2}}\quad,\quad\frac{\mathrm{d}^{-1/2}\left[f(\lambda x)\right]}{\left[\mathrm{d}x\right]^{-1/2}}=\frac{1}{\sqrt{\lambda}}\frac{\mathrm{d}^{-1/2}\left[f(\lambda x)\right]}{\left[\mathrm{d}(\lambda x)\right]^{-1/2}}\tag{3.213}$$

and with $\lambda=-1$ we write the following:

$$\frac{\mathrm{d}^{1/2}\left[f(-x)\right]}{\left[\mathrm{d}x\right]^{1/2}}=i\frac{\mathrm{d}^{1/2}\left[f(-x)\right]}{\left[\mathrm{d}(-x)\right]^{1/2}}\quad,\quad\frac{\mathrm{d}^{-1/2}\left[f(-x)\right]}{\left[\mathrm{d}x\right]^{-1/2}}=-i\frac{\mathrm{d}^{-1/2}\left[f(-x)\right]}{\left[\mathrm{d}(-x)\right]^{-1/2}}\tag{3.214}$$

3.21 Fractional differentiation-integration behavior near the lower limit

When α is a positive integer or zero, the operation $\frac{\mathrm{d}^{\alpha}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha}}$ or ${}_aD_x^{\alpha}\left[f(x)\right]$ for $\alpha=0,1,2,3...$ is 'local' and is a point property, this we have discussed earlier; and the behavior of ${}_aD_x^{\alpha}\left[f(x)\right]$ near $x=a$ is exceptional. For all other values, i.e. non-integer values of α we will demonstrate that ${}_aD_x^{\alpha}\left[f(x)\right]$ usually approaches either zero or infinity as x approaches a. We write $f(x)=f_{\upsilon}(x)=(x-a)^p\sum_{j=0}^{\infty}a_j(x-a)^j$; with $p>-1$ and $a_0\neq 0$. We will show that $\frac{\mathrm{d}^{\alpha}[f_{\upsilon}(x)]}{[\mathrm{d}(x-a)]^{\alpha}}$ normally approaches either zero or infinity as x approaches a:

$$\frac{\mathrm{d}^{\alpha}\left[f_{\upsilon}(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\sum_{j=0}^{\infty}\frac{a_j\left(\Gamma(p+j+1)\right)}{\Gamma(p+j-\alpha+1)}(x-a)^{p+j-\alpha}\tag{3.215}$$

We know that $\frac{1}{\Gamma(p+j-\alpha+1)}$ is always finite while $\Gamma(p+j+1)$ is finite since $p>-1$ (from the properties of the gamma function). Therefore, the RHS of the above (3.215) is dominated by its first term ($j=0$) for small $x-a$, that is:

$$\lim_{x\to a}\left(\frac{\mathrm{d}^\alpha f_\upsilon(x)}{\left[\mathrm{d}(x-a)\right]^\alpha}\right)=\lim_{x\to a}\left(\frac{(x-a)^{p-\alpha}a_0\left(\Gamma(p+1)\right)}{\Gamma(p-\alpha+1)}\right)$$
$$=\begin{cases}0 & \text{for} \quad p-\alpha>0\\ a_0\left(\Gamma(p+1)\right) & \text{for} \quad p-\alpha=0\\ \infty & \text{for} \quad p-\alpha<0\end{cases} \tag{3.216}$$

since $a_0\neq 0$.

3.22 The fractional differentiation-integration behavior far from the lower limit

We here demonstrate restricting the discussion to an analytical function $\varphi(x)$ and for $x>>a$ we write a few terms of the binomial expansion of $(x-a)^{k-\alpha}$ as the following steps:

$$\begin{aligned}(x-a)^{k-\alpha}&=x^{k-\alpha}\left(1-\frac{a}{x}\right)^{k-\alpha}\\ &=x^{k-\alpha}\left(1-(k-\alpha)\tfrac{a}{x}+O\left(\tfrac{a^2}{x^2}\right)\right)\\ &\sim x^{k-\alpha}+\frac{(\alpha-k)ax^k}{x^{\alpha+1}}\end{aligned} \tag{3.217}$$

The above approximation (3.217) we substitute in the formula which we obtained previously in Chapter-2 (Section-2.11.3) which was:

$$\begin{aligned}{}_aD_x^\alpha\left[\varphi(x)\right]=\frac{\mathrm{d}^\alpha\left[\varphi(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha}&=\sum_{k=0}^{\infty}\frac{(-1)^k(x-a)^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{\left(\Gamma(-\alpha)\right)(k-\alpha)k!}\\ &\sim\sum_{k=0}^{\infty}\frac{(-1)^k\left(x^{k-\alpha}+\frac{(\alpha-k)ax^k}{x^{\alpha+1}}\right)\left(\varphi^{(k)}(x)\right)}{\left(\Gamma(-\alpha)\right)(k-\alpha)k!}\end{aligned} \tag{3.218}$$

In (3.218) we have assumed $\varphi(x)$ as an analytic function. Use the reflection formula of the gamma function, $\Gamma(-\alpha)=-\frac{\pi\csc(\alpha\pi)}{\Gamma(\alpha+1)}$ (Section-1.10), to get the following expression for $x>>a$

$$\begin{aligned}\frac{\mathrm{d}^\alpha\left[\varphi(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha}={}_aD_x^\alpha\left[\varphi(x)\right]&\sim\sum_{k=0}^{\infty}\frac{(-1)^k\left(x^{k-\alpha}+\frac{(\alpha-k)ax^k}{x^{\alpha+1}}\right)\left(\varphi^{(k)}(x)\right)}{\left(\Gamma(-\alpha)\right)(k-\alpha)\left(k!\right)}\\ &=\frac{\sin(\alpha\pi)\Gamma(\alpha+1)}{\pi}\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{(\alpha-k)\left(k!\right)}+\frac{a}{x^{\alpha+1}}\sum_{k=0}^{\infty}\frac{(-1)^k x^k\left(\varphi^{(k)}(x)\right)}{\left(k!\right)}\right)\\ &=\frac{\mathrm{d}^\alpha\varphi(x)}{\mathrm{d}x^\alpha}+\frac{a\left(\varphi(0)\right)\left(\sin(\alpha\pi)\right)\Gamma(\alpha+1)}{\pi x^{\alpha+1}}\end{aligned} \tag{3.219}$$

The following expression for the second term is used in the above derivation (3.219) i.e.

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k x^k\left(\varphi^{(k)}(x)\right)}{k!}&=1-x\varphi^{(1)}(x)+\frac{x^2}{2!}\varphi^{(2)}(x)\\ &\quad-\frac{x^3}{3!}\varphi^{(3)}(x)+...=\varphi(x)\big|_{x=0}\end{aligned} \tag{3.220}$$

which is the Taylor expansion of $\varphi(x)$ at the origin $x=0$; that is $\varphi(x)\big|_{x=0}=\varphi(0)$.

We also used for the first term in (3.219) the following expression:

$$\frac{\sin(\alpha\pi)\Gamma(\alpha+1)}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k x^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{(\alpha-k)\left(k!\right)}=\frac{-1}{\Gamma(-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k x^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{(\alpha-k)\left(k!\right)}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k x^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{\left(\Gamma(-\alpha)\right)(k-\alpha)\left(k!\right)} \tag{3.221}$$
$$={}_0D_x^{\alpha}\left[\varphi(x)\right]=\frac{\mathrm{d}^{\alpha}\varphi(x)}{\mathrm{d}x^{\alpha}}$$

Here we used the expression i.e. ${}_aD_x^{\alpha}\left[\varphi(x)\right]=\sum_{k=0}^{\infty}\frac{(-1)^k(x-a)^{k-\alpha}\left(\varphi^{(k)}(x)\right)}{(\Gamma(-\alpha))(k-\alpha)(k!)}$ with $a=0$ (Section-2.11.3), in derivation (3.221). We observe that if α is a positive integer say $\alpha=1,2,3,\ldots$, the second term of the RHS of the above expression vanishes, indicating again that the integer order derivatives are local (or point) property.

3.23 A numerical evaluation of fractional differ-integrals using the Grunwald-Letnikov formula

In this section, we will approximate $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}f(x)$ for arbitrary α, with known values of $f(x)$ at $N+1$ and evenly spaced points in the range 0 to x. We designate symbols $f_N\equiv f(0)$, $f_{N-1}\equiv f\left(\frac{x}{N}\right)$, $\ldots f_j\equiv f\left(x-j\frac{x}{N}\right)$, $\ldots$ $f_0\equiv f(x)$; and discrete steps as $\Delta x=\left(\frac{x}{N}\right)$. We write the simplest definition as we saw earlier from the Grunwald-Letnikov (GL) formula; that is:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}f\left(x-j\left(\frac{x-a}{N}\right)\right)\right) \tag{3.222}$$

We omit the limit $N\uparrow\infty$ and make the start point $a=0$ and write the following:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\mathrm{d}x^{\alpha}}\approx\frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+`1)}f\left(x-j\frac{x}{N}\right)$$
$$=\frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}f_j=\frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}f_j \tag{3.223}$$

This is an approximate algorithm to evaluate the $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[f(x)\right]$. If we call ${}^{GL}w_j=\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$, ${}^{GL}w_{j-1}=\frac{\Gamma(j-1-\alpha)}{\Gamma(-\alpha)\Gamma(j)}$ as Grunwald's coefficients then we write:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\mathrm{d}x^{\alpha}}=\left(\Delta x\right)^{-\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}f_j=\left(\Delta x\right)^{-\alpha}\sum_{j=0}^{N-1}\left({}^{GL}w_j\right)f_j \tag{3.224}$$

We have a recursion relationship that is $\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}=\frac{j-1-\alpha}{j}\left(\frac{\Gamma(j-1-\alpha)}{\Gamma(j)}\right)$. With this, we write a recursive expression for the Grunwald Coefficient as:

$${}^{GL}w_j=\left(\frac{j-1-\alpha}{j}\right)\left({}^{GL}w_{(j-1)}\right) \tag{3.225}$$

This algorithm may be implemented by the following multiplication-addition, multiplication-addition, multiplication-addition scheme

$$\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}=\left(\frac{x}{N}\right)^{-\alpha}\left[\left[\left[\left[\cdot\cdot\left[f_{N-1}\left(\frac{N-\alpha-2}{N-1}\right)+f_{N-2}\right]\left(\frac{N-\alpha-3}{N-2}\right)+f_{N-3}\right]\cdot\cdot\right]\left(\frac{1-\alpha}{2}\right)+f_1\right]\left(\frac{-\alpha}{1}\right)+f_0\right] \tag{3.226}$$

The above formula (3.226) avoids explicit usage of the gamma function.

We have a modified GL definition which has rapid convergence. We write with $a=0$ the following:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\mathrm{d}x^{\alpha}} = \lim_{N\uparrow\infty}\left(\frac{x^{-\alpha}N^{\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x-\left(j-\alpha\tfrac{1}{2}\right)\left(\tfrac{x}{N}\right)\right)\right)$$
$$= \lim_{N\uparrow\infty}\left(\frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x-j\Delta x+\alpha\tfrac{\Delta x}{2}\right)\right) \tag{3.227}$$

We notice that unless $\alpha = 0, \pm 2, \pm 4, \ldots$ this modified formula requires an evaluation of $f(x)$ at other points than as f_j values. We therefore approximate $f\left(x+\alpha\tfrac{x}{2N}-j\tfrac{x}{N}\right)$ based upon Lagrange's three point interpolation formula, that is, upon:

$$f\left(x+\alpha\tfrac{x}{2N}-j\tfrac{x}{N}\right) \approx \left(\frac{\alpha}{4}+\frac{\alpha^2}{8}\right) f\left(x+\tfrac{x}{N}-j\tfrac{x}{N}\right)+\left(1-\frac{\alpha^2}{4}\right) f\left(x-j\tfrac{x}{N}\right)$$
$$+\left(\frac{\alpha^2}{8}-\frac{\alpha}{4}\right) f\left(x-\tfrac{x}{N}-j\tfrac{x}{N}\right) \tag{3.228}$$

Thus, our modified GL formula becomes with the above substitution (3.228) the following:

$$\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\mathrm{d}x^{\alpha}} \approx \frac{\left(\tfrac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}\begin{pmatrix} f_j+\frac{1}{4}\alpha\left(f_{j-1}-f_{j+1}\right) \\ +\frac{1}{8}\alpha^2\left(f_{j-1}-2f_j+f_{j+1}\right)\end{pmatrix} \tag{3.229}$$

3.24 The discrete numerical evaluation of the Riemann-Liouville fractional derivative formula

We continue from the previous section and try to get a numerical scheme to evaluate the Riemann-Liouville (RL) fractional derivative formula; with known values of $f(x)$ at $N+1$ evenly spaced points in the range 0 to x. We designate the symbols $f_N \equiv f(0)$, $f_{N-1} \equiv f\left(\tfrac{x}{N}\right)$, $\ldots f_j \equiv f\left(x-j\tfrac{x}{N}\right)$, $\ldots f_0 \equiv f(x)$. We continue the method of numerical integration that we applied to the RL fractional integration formula in the previous chapter (Section-2.23). The RL fractional derivative formula for $0 \le \alpha < 1$; for a function with a lower terminal set as $a = 0$ is:

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \frac{x^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-y)^{-\alpha}\left(f^{(1)}(y)\right)\mathrm{d}y \tag{3.230}$$

This above expression (3.230) is a relationship between the RL and Caputo derivative for $0 \le \alpha < 1$ with $f^{(1)}$ the one-whole derivative of function $f(x)$ assumed to exist and with an initial value $f(0)$ as finite. We re-write the above (3.230) in the following form, with a change of variables of integration and writing N discrete integration per interval Δx and then summing them up, as was demonstrated in the previous chapter (Section-2.23):

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \frac{x^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(1-\alpha)}\int_0^x \left(\frac{\mathrm{d}\ f(y)}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}y}{(x-y)^{\alpha}}\right)$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(0)}{x^{\alpha}}+\sum_{j=0}^{N-1}\int_{jx/N}^{(j+1)x/N}\left(\frac{\mathrm{d}\ f(x-y)}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}y}{y^{\alpha}}\right)\right) \tag{3.231}$$

We use an approximation $\frac{\mathrm{d}\ f(x-y)}{\mathrm{d}y} \approx \frac{f(x-j\Delta x)-f(x-(j+1)\Delta x)}{\Delta x}$ and use this in the following derivation:

$$\int_{jx/N}^{(j+1)x/N}\left(\frac{\mathrm{d}\ f(x-y)}{\mathrm{d}y}\right)\frac{\mathrm{d}y}{y^{\alpha}} \approx \left(\frac{f\left(x-\frac{jx}{N}\right)-f\left(x-\frac{(j+1)}{N}x\right)}{\left(\frac{x}{N}\right)}\right)\int_{jx/N}^{(j+1)x/N}\frac{\mathrm{d}y}{y^{\alpha}}$$
$$= \frac{1}{1-\alpha}\left(\frac{x}{N}\right)^{-\alpha}\left(f_j-f_{j+1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right) \tag{3.232}$$

Using the above expression (3.232) we get the following:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}&=\frac{1}{\Gamma(1-\alpha)}\left(\frac{f(0)}{x^{\alpha}}+\sum_{j=0}^{N-1}\int_{jx/N}^{(j+1)x/N}\left(\frac{\mathrm{d}\ f(x-y)}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}y}{y^{\alpha}}\right)\right);\qquad f(0)=f_N\\&=\frac{1}{\Gamma(1-\alpha)}\left(f_N x^{-\alpha}+\frac{1}{(1-\alpha)}\left(\frac{x}{N}\right)^{-\alpha}\sum_{j=0}^{N-1}\Big(\big(f_j-f_{j+1}\big)\big((j+1)^{1-\alpha}-j^{1-\alpha}\big)\Big)\right)\\&=\frac{\left(\frac{x}{N}\right)^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\left((1-\alpha)N^{-\alpha}f_N+\sum_{j=0}^{N-1}\Big(\big(f_j-f_{j+1}\big)\big((j+1)^{1-\alpha}-j^{1-\alpha}\big)\Big)\right)\\&=\frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{(1-\alpha)f_N}{N^{\alpha}}+\sum_{j=0}^{N-1}\Big(\big(f_j-f_{j+1}\big)\big((j+1)^{1-\alpha}-j^{1-\alpha}\big)\Big)\right)\end{aligned}\tag{3.233}$$

We have the following final formula:

$$\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}=\frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{(1-\alpha)}{N^{\alpha}}f_N+\sum_{j=0}^{N-1}\Big(\big((j+1)^{1-\alpha}-j^{1-\alpha}\big)\big(f_j-f_{j+1}\big)\Big)\right)\tag{3.234}$$

$$0\le\alpha<1$$

We take a case for $1\le\alpha<2$ and write the RL derivative formula as:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}&=\frac{x^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{x^{1-\alpha}f^{(1)}(0)}{\Gamma(2-\alpha)}+\frac{1}{\Gamma(2-\alpha)}\int_0^x\frac{\left(f^{(2)}(y)\right)\mathrm{d}y}{(x-y)^{\alpha-1}}\\&=\frac{1}{\Gamma(2-\alpha)}\begin{pmatrix}\dfrac{(1-\alpha)f(0)}{x^{\alpha}}+\dfrac{f^{(1)}(0)}{x^{\alpha-1}}\\+\displaystyle\sum_{j=0}^{N-1}\int_{jx/N}^{(j+1)x/N}\frac{\left(f^{(2)}(x-y)\right)\mathrm{d}y}{y^{\alpha-1}}\end{pmatrix}\end{aligned}\tag{3.235}$$

The above expression (3.235) is an RL–Caputo derivative relation for $1\le\alpha<2$ with the assumption that a second derivative of the function i.e. $f^{(2)}(x)$ exists and the initial values $f(0)$ and $f^{(1)}(0)$ are finite. We make approximations $f^{(1)}(0)\approx\frac{f(\Delta x)-f(0)}{\Delta x}$ i.e. the first derivative at the initial point:

$$f^{(1)}(0)\approx\frac{f\left(\frac{x}{N}\right)-f(0)}{\left(\frac{x}{N}\right)}=\left(\frac{N}{x}\right)\left(f_{N-1}-f_N\right)\tag{3.236}$$

and $f^{(2)}(x-y)\approx\frac{f\left(x-(j-1)\Delta x\right)-2f\left(x-j\Delta x\right)+f\left(x-(j+1)\Delta x\right)}{(\Delta x)^2}$ i.e. the second derivative and its use as follows:

$$\begin{aligned}&\int_{jx/N}^{(j+1)x/N}\left(f^{(2)}(x-y)\right)\left(\frac{\mathrm{d}y}{y^{\alpha-1}}\right)\\&\approx\left(\frac{f\left(x-\frac{jx}{N}+\frac{x}{N}\right)-2f\left(x-\frac{jx}{N}\right)+f\left(x-\frac{jx}{N}-\frac{x}{N}\right)}{\left(\frac{x}{N}\right)^2}\right)\int_{jx/N}^{(j+1)x/N}\frac{\mathrm{d}y}{y^{\alpha-1}}\\&=\frac{\left(\frac{x}{N}\right)^{-\alpha}}{2-\alpha}\left(f_{j-1}-2f_j+f_{j+1}\right)\left((j+1)^{2-\alpha}-j^{2-\alpha}\right)\end{aligned}\tag{3 .237}$$

Using the above approximations of (3.236) and (3.237) we arrive at the following final result (by a similar procedure as done in the previous (3.234) derivation thus not showing all the steps):

$$\frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}x^{\alpha}}=\frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(3-\alpha)}\begin{pmatrix}\dfrac{(1-\alpha)(2-\alpha)}{N^{\alpha}}f_N\\+\dfrac{(2-\alpha)}{N^{\alpha-1}}\displaystyle\sum_{j=0}^{N-1}\left(f_{j-1}-2f_j+f_{j+1}\right)\left((j+1)^{2-\alpha}-j^{2-\alpha}\right)\end{pmatrix}\tag{3.238}$$

$$1\le\alpha<2$$

Similar steps may be followed for the RL derivative formulas for $2\le\alpha<3$ and more.

3.25 Generalisation of the Riemann-Liouville and Caputo derivatives with the choice of a different kernel, weights and base-function

In the last chapter (Section-2.17) we have seen that the formula of the Riemann-Liouville fractional integration yields several other kinds of formulas by choice of the kernel of integration, the weight function and the base-function; here we apply the same concept for fractional derivatives. We have discussed that the Riemann-Liuoville (RL) fractional derivative is the first fractional integration followed by a whole number of classical differentiation, and the Caputo is just the opposite. Therefore choosing the different kind of integral operators based on a different kernel, weight function and base function and performing either the RL or Caputo process will yield various different kinds of fractional derivative operators.

Here we will define the left/forward causal, and right/backward non-causal fractional integrals and fractional derivatives of a function $f(x)$ with respect to another function $z(x)$, call it the base function and weight scale $w(x)$; a sufficiently good function. We assume that the base function $z(x)$ is positive and a monotonically increasing function of x in the interval $a < x \le b$; having a continuous derivative $z^{(1)}(x)$ on the interval $a < x < b$.

3.25.1 Generalisation of the left and right fractional derivatives with respect to the base function $z(x)$ and weights $w(x)$

A generalised left/forward fractional derivative of the RL sense, with respect to $z(x)$ and with the weight $w(x)$, is defined with $m \in \mathbb{N}$ in the following relationships, just greater than $\alpha \in \mathbb{R}$, i.e. $m-1 < \alpha < m$; as follows:

$$ {}_a\mathscr{D}^{\alpha}_{x,[z(x),w(x)]} f(x) = \big(w(x)\big)^{-1} \left(\frac{1}{\big(z^{(1)}(x)\big)} D_x \right)^m \big(w(x)\big)\Big({}_a\mathcal{I}^{m-\alpha}_{x,[z(x),w(x)]} f(x)\Big) \tag{3.239}$$

$$\alpha > 0$$

Where we had defined in an earlier chapter (Section 2.17) ${}_a\mathcal{I}^{\alpha}_{x;[z(x),w(x)]}$ as a left/forward causal generalised integral, as in the following expression:

$$ {}_a\mathcal{I}^{\alpha}_{x,[z(x),w(x)]} f(x) = \frac{1}{\big(w(x)\big)\Gamma(\alpha)} \int_a^x \big(z(x)-z(t)\big)^{\alpha-1} w(t)\big(z^{(1)}(t)\big)\big(f(t)\big)\,\mathrm{d}t \tag{3.240}$$

$$\alpha > 0$$

a generalised right/backward non-causal fractional derivative of the RL sense with respect to $z(x)$ and with weight $w(x)$, is defined as follows:

$$ {}_x\mathscr{D}^{\alpha}_{b,[z(x),w(x)]} f(x) = \big(w(x)\big) \left(\frac{-1}{\big(z^{(1)}(x)\big)} D_x \right)^m \big(w(x)\big)^{-1}\Big({}_x\mathcal{I}^{m-\alpha}_{b,[z(x),w(x)]} f(x)\Big) \tag{3.241}$$

$$\alpha > 0$$

We had defined in an earlier chapter (Section-2.17) ${}_x\mathcal{I}^{\alpha}_{b;[z(x),w(x)]}$ as a right/backward non-causal generalised integral as follows:

$$ {}_x\mathcal{I}^{\alpha}_{b,[z(x),w(x)]} f(x) = \frac{w(x)}{\Gamma(\alpha)} \int_x^b \big(z(t)-z(x)\big)^{\alpha-1} \big(w(t)\big)^{-1}\big(z^{(1)}(t)\big)\big(f(t)\big)\,\mathrm{d}t \tag{3.242}$$

$$\alpha > 0$$

Where $D_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}$, is a normal integer order derivative operator. We further define the left and right integer order derivative operator as follows:

$$D_{x+;[z,w];L} f(x) = \frac{1}{w(x)}\left(\frac{1}{\left(z^{(1)}(x)\right)} D_x\right)\left(w(x) f(x)\right)$$
$$= \frac{1}{w(x)}\left[\left(\frac{\mathrm{d}}{\mathrm{d}[z(x)]}\right)\left(w(x) f(x)\right)\right]$$
$$D_{x-;[z,w];R} f(x) = w(x)\left(\frac{-1}{\left(z^{(1)}(x)\right)} D_x\right)\left(\frac{1}{w(x)} f(x)\right)$$
$$= w(x)\left(-\frac{\mathrm{d}}{\mathrm{d}[z(x)]}\right)\left(\frac{1}{w(x)} f(x)\right) \tag{3.243}$$

Note that the above expressions (3.243) appear in the formula of ${}_a\mathfrak{D}^{\alpha}_{x,[z(x),w(x)]} f(x)$ (3.239) and ${}_x\mathfrak{D}^{\alpha}_{b,[z(x),w(x)]} f(x)$ (3.241) which is the original statement written at the start of this section. These formulas (3.243) are like the left/forward causal and the right/backward non-causal operators, with D and $-D$, except they contain the functions $w(x)$ and $z(x)$. The above formulas (3.243) are new integer order operators. Therefore, in terms of these new integer order operators, we re-write for the RL type ${}_a\mathfrak{D}^{\alpha}_{x,[z(x),w(x)]} f(x)$ and ${}_x\mathfrak{D}^{\alpha}_{b,[z(x),w(x)]} f(x)$ operators in the following construct:

$${}_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x) = D^m_{x;[z,w];L}\left({}_a\mathcal{I}^{m-\alpha}_{x;[z,w]} f(x)\right)$$
$${}_x\mathfrak{D}^{\alpha}_{b,[z,w]} f(x) = D^m_{x;[z,w];R}\left({}_x\mathcal{I}^{m-\alpha}_{b;[z,w]} f(x)\right) \tag{3.244}$$

Similarly, one can construct the Caputo type ${}^C_a\mathfrak{D}^{\alpha}_{x,[z(x),w(x)]}$ and ${}^C_x\mathfrak{D}^{\alpha}_{b,[z(x),w(x)]}$ left or right generalised fractional derivative as follows:

$${}^C_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x) = {}_a\mathcal{I}^{m-\alpha}_{x,[z,w]}\left(D^m_{x,[z,w]} f(x)\right)$$
$${}^C_x\mathfrak{D}^{\alpha}_{b,[z,w]} f(x) = {}_x\mathcal{I}^{m-\alpha}_{b,[z,w]}\left(D^m_{x,[z,w]} f(x)\right) \tag{3.245}$$

Explicitly for $w(x)=1$, the generalised Caputo derivative is expressed as follows:

$${}^C_a\mathfrak{D}^{\alpha}_{x,[z,1]} f(x)$$
$$= \frac{1}{\Gamma(m-\alpha)}\int_a^x \left(z^{(1)}(t)\right)\left(z(x)-z(t)\right)^{m-\alpha-1}\left(\frac{1}{\left(z^{(1)}(t)\right)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left(f(t)\right)\mathrm{d}t \tag{3.246}$$

3.25.2 The relationship between the generalised Riemann-Liouville and Caputo fractional derivatives with weight and base function

The generalised RL and Caputo fractional derivatives are related by the following formula, which is for left/forward derivatives:

$${}_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x)$$
$$= \frac{w(a)}{w(x)}\sum_{k=0}^{m-1}\left(\frac{\left(D^{(k)}_{x,[z,w]} f(x)\right)\Big|_{x=a}}{\Gamma(k+1-\alpha)}\right)\left(z(x)-z(a)\right)^{k-\alpha} + {}^C_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x) \tag{3.247}$$

One could similarly write the expression for a right/backward derivative's relation. The above relationship (3.247) shows a similarity to the expression for the relationship between a normal Caputo and a normal RL derivative. From the above expression, we write the following:

$${}_a\mathfrak{D}^{\alpha}_{x;[z,w]} f(x) = {}^C_a\mathfrak{D}^{\alpha}_{x;[z,w]} f(x) \quad \text{iff} \quad f^{(k)}(a)=0; \quad k=0,1,\ldots,m-1$$
$${}_x\mathfrak{D}^{\alpha}_{b;[z,w]} f(x) = {}^C_x\mathfrak{D}^{\alpha}_{b;[z,w]} f(x) \quad \text{iff} \quad f^{(k)}(b)=0; \quad k=0,1,\ldots,m-1 \tag{3.248}$$

3.25.3 The composition properties of generalised fractional derivatives

We now list the properties of these generalised fractional derivatives, and generalised fractional integral operators. The operators ${}_{a}\mathcal{I}^{\alpha}_{x;[z,w]}$ and ${}_{x}\mathcal{I}^{\alpha}_{b;[z,w]}$ satisfy the semi-group property:

$$\begin{aligned}\left({}_{a}\mathcal{I}^{\alpha}_{x;[z,w]}\right)\left({}_{a}\mathcal{I}^{\beta}_{x;[z,w]}\right) &= {}_{a}\mathcal{I}^{\alpha+\beta}_{x;[z,w]} = \left({}_{a}\mathcal{I}^{\beta}_{x;[z,w]}\right)\left({}_{a}\mathcal{I}^{\alpha}_{x;[z,w]}\right)\\ \left({}_{x}\mathcal{I}^{\alpha}_{b;[z,w]}\right)\left({}_{x}\mathcal{I}^{\beta}_{b;[z,w]}\right) &= {}_{x}\mathcal{I}^{\alpha+\beta}_{b;[z,w]} = \left({}_{x}\mathcal{I}^{\beta}_{b;[z,w]}\right)\left({}_{x}\mathcal{I}^{\alpha}_{b;[z,w]}\right)\end{aligned} \tag{3.249}$$

The operator ${}_{a}\mathfrak{D}^{\alpha}_{x;[z,w]}$ and the operator ${}_{x}\mathfrak{D}^{\alpha}_{b;[z,w]}$ are left inverse of the respective ${}_{*}\mathcal{I}^{\alpha}_{**;[z,w]}$ operators resulting in the following rules:

$${}_{a}\mathfrak{D}^{\alpha}_{x;[z,w]}\left({}_{a}\mathcal{I}^{\alpha}_{x;[z,w]}f(x)\right) = f(x) \qquad {}_{x}\mathfrak{D}^{\alpha}_{b;[z,w]}\left({}_{x}\mathcal{I}^{\alpha}_{b;[z,w]}f(x)\right) = f(x) \tag{3.250}$$

We will prove the composition of these rules as described above in Chapter-9.

3.25.4 The various forms of generalised fractional derivatives

The choice of the base function $z(x)$ and weight function $w(x)$ reduces the described generalised definitions to the various types of fractional derivative operators.

3.25.4.(a) A normal Riemann-Liouville/Caputo derivative from a generalised definition

The generalized definition of fractional derivative (3.244) and (3.245) is reduced to a normal RL or Caputo type (as we discussed) and by choosing $z(t) = t$, $w(t) = 1$ and $z^{(1)}(t) = 1$, we have the generalised fractional integration as the Riemann fractional integration which is recovered as follows:

$$\begin{aligned}&{}_{a}\mathcal{I}^{\alpha}_{x,[z,w]}f(x) = \frac{1}{w(x)\Gamma(\alpha)}\int_a^x \left(z(x)-z(t)\right)^{\alpha-1} w(t)\left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t\\ &\alpha > 0\\ &{}_{a}\mathcal{I}^{\alpha}_{x,[t,1]}f(x) = {}_{a}I^{\alpha}_{x}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left(f(t)\right)\mathrm{d}t\end{aligned} \tag{3.251}$$

The Riemann-Liouville fractional derivative is recovered as follows:

$$\begin{aligned}&{}_{a}\mathfrak{D}^{\alpha}_{x,[z,w]}f(x) = \left(w(x)\right)^{-1}\left(\frac{1}{\left(z^{(1)}(x)\right)}D_x\right)^{m}\left(w(x)\right)\left({}_{a}\mathcal{I}^{m-\alpha}_{x,[z(x),w(x)]}f(x)\right)\\ &\alpha > 0, \quad m-1 < \alpha < m\\ &{}_{a}\mathfrak{D}^{\alpha}_{x,[t,1]}f(x) = {}_{a}D^{\alpha}_{x}f(x) = D^{m}_{x}\left({}_{a}\mathcal{I}^{m-\alpha}_{x,[t,1]}f(x)\right)\\ &\qquad = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}\left(f(t)\right)\mathrm{d}t\right)\end{aligned} \tag{3.252}$$

These are left/forward causal RL fractional integrals and derivatives. In a similar way, we have the other reductions in the RL or Caputo type as follows:

$$\begin{aligned}&{}_{x}\mathcal{I}^{\alpha}_{b,[t,1]}f(x) = {}_{x}I^{\alpha}_{b}f(x); \quad {}_{x}\mathfrak{D}^{\alpha}_{b,[t,1]}f(x) = {}_{x}D^{\alpha}_{b}f(x)\\ &{}^{C}_{a}\mathfrak{D}^{\alpha}_{x,[t,1]}f(x) = {}^{C}_{a}D^{\alpha}_{x}f(x); \quad {}^{C}_{x}\mathfrak{D}^{\alpha}_{b,[t,1]}f(x) = {}^{C}_{x}D^{\alpha}_{b}f(x)\end{aligned} \tag{3.253}$$

3.25.4.(b) Hadamard type fractional derivative from generalised definitions

By choosing $z(t) = \ln t$, $w(t) = 1$ and $z^{(1)}(t) = \left(\frac{1}{t}\right)$ we get Hadamard type fractional operators. Following this we write a Hadamard type left/forward causal RL type integral and derivative operators:

$$ {}_a\mathcal{I}^{\alpha}_{x,[z,w]} f(x) = \frac{1}{w(x)\Gamma(\alpha)} \int_a^x \left(z(x)-z(t)\right)^{\alpha-1} w(t)\left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t $$

$$ \alpha>0 \tag{3.254} $$

$$ {}_a\mathcal{I}^{\alpha}_{x,[\ln t,1]} f(x) = {}^{H}_{a}I^{\alpha}_{x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\tfrac{\ln x}{\ln t}\right)^{\alpha-1}\left(\tfrac{1}{t}\right)\left(f(t)\right)\mathrm{d}t $$

$$ {}_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x) = \left(w(x)\right)^{-1}\left(\frac{1}{\left(z^{(1)}(x)\right)} D_x\right)^m \left(w(x)\right)\left({}_a\mathcal{I}^{m-\alpha}_{x,[z(x),w(x)]} f(x)\right) $$

$$ \alpha>0 \qquad m-1<\alpha<m \tag{3.255} $$

$$ {}_a\mathfrak{D}^{\alpha}_{x,[\ln t,1]} f(x) = {}^{H}_{a}D^{\alpha}_{x} f(x) = D^m_x\left({}_a\mathcal{I}^{m-\alpha}_{x,[\ln t,1]} f(x)\right) $$

$$ = \frac{1}{\Gamma(m-\alpha)}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left(\int_a^x \left(\tfrac{\ln t}{\ln x}\right)^{m-\alpha-1}\left(\tfrac{f(t)}{t}\right)\mathrm{d}t\right) $$

The other Hadamard right/backward operators, Caputo left and right, are thus:

$$ {}_x\mathcal{I}^{\alpha}_{b,[\ln t,1]} f(x) = {}^{H}_{x}I^{\alpha}_{b} f(x); \quad {}_x\mathfrak{D}^{\alpha}_{b,[\ln t,1]} f(x) = {}^{H}_{x}D^{\alpha}_{b} f(x) $$

$$ {}^{C}_{a}\mathfrak{D}^{\alpha}_{x,[\ln t,1]} f(x) = {}^{H-C}_{\quad a}D^{\alpha}_{x} f(x); \quad {}^{C}_{x}\mathfrak{D}^{\alpha}_{b,[\ln t,1]} f(x) = {}^{H-C}_{\quad x}D^{\alpha}_{b} f(x) \tag{3.256} $$

3.25.4.(c) Modified Erdelyi-Kober (MEK) type fractional derivatives from the generalised definition

By choosing $z(t)=t^{\sigma}$ $w(t)=t^{\sigma\eta}$ and $z^{(1)}(t)=\sigma t^{\sigma-1}$, we obtain the modified Erdelyi-Kober (MEK) operator. The following is a left causal integral and derivative:

$$ {}_a\mathcal{I}^{\alpha}_{x,[z,w]} f(x) = \frac{1}{w(x)\Gamma(\alpha)} \int_a^x \left(z(x)-z(t)\right)^{\alpha-1} w(t)\left(z^{(1)}(t)\right) f(t)\mathrm{d}t \qquad \alpha>0 $$

$$ {}_a\mathcal{I}^{\alpha}_{x,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK}_{\quad a}I^{\alpha}_{x} f(x) = \frac{\sigma t^{-\sigma\eta}}{\Gamma(\alpha)} \int_a^x \left(x^{\sigma}-t^{\sigma}\right)^{\alpha-1} t^{\sigma(1-\eta)-1}\left(f(t)\right)\mathrm{d}t \tag{3.257} $$

$$ {}_a\mathfrak{D}^{\alpha}_{x,[z,w]} f(x) = \left(w(x)\right)^{-1}\left(\frac{1}{\left(z^{(1)}(x)\right)} D_x\right)^m \left(w(x)\right)\left({}_a\mathcal{I}^{m-\alpha}_{x,[z(x),w(x)]} f(x)\right) $$

$$ \alpha>0 \qquad m-1<\alpha<m \tag{3 .258} $$

$$ {}_a\mathfrak{D}^{\alpha}_{x,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK}_{\quad a}D^{\alpha}_{x} f(x) = D^m_x\left({}_a\mathcal{I}^{m-\alpha}_{x,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x)\right) $$

$$ = x^{-\sigma\eta}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m x^{\sigma\eta}\left(\frac{\sigma t^{-\sigma\eta}}{\Gamma(\alpha)}\int_a^x \frac{t^{\sigma(1-\eta)-1} f(t)}{\left(x^{\sigma}-t^{\sigma}\right)^{1-\alpha}}\mathrm{d}t\right) $$

The other MEK operators are right/backward, Caputo left and right, are thus the following:

$$ {}_x\mathcal{I}^{\alpha}_{b,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK}_{\quad x}I^{\alpha}_{b} f(x); \quad {}_x\mathfrak{D}^{\alpha}_{b,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK}_{\quad x}D^{\alpha}_{b} f(x) $$

$$ {}^{C}_{a}\mathfrak{D}^{\alpha}_{x,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK-C}_{\quad a}D^{\alpha}_{x} f(x); \quad {}^{C}_{x}\mathfrak{D}^{\alpha}_{b,\left[t^{\sigma},t^{\sigma\eta}\right]} f(x) = {}^{MEK-C}_{\quad x}D^{\alpha}_{b} f(x) \tag{3.259} $$

However, the types of operators described in this section we are not going to use this in the rest of the chapters. In subsequent chapters we will use the regular Caputo and Riemann-Liouville fractional derivatives.

3.26 The Caputo fractional derivative with a non-singular exponential type kernel

From (3.109) we write the formulation of the Caputo derivative, as ${}^{C}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f^{(1)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha}}$, for a fractional order $0<\alpha<1$. This is applicable if the function $f(t)$ has one-whole derivative $f^{(1)}(t)$ in the interval under consideration. As per our discussions, the Caputo derivative is a fractional integration of order $1-\alpha$ for the function

$f^{(1)}(t)$ i.e. ${}_a^C D_t^\alpha f(t) = {}_a I_t^{1-\alpha} f^{(1)}(t)$. Here the kernel of integration is a 'power function' and is a singular function at $t=0$ i.e. ${}^C k_\alpha(t) \sim t^{-\alpha}$. In terms of the convolution the integral ${}_a^C D_t^\alpha f(t)$ is ${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\left({}^C k_\alpha(t)\right) * \left(f^{(1)}(t)\right)\right)$. The kernel if considered as ${}^{CF} k_\alpha(t) \sim \exp\left(-\frac{\alpha}{1-\alpha} t\right)$, instead of the usual 'power function' then we have the Caputo-Fabrizo (CF) definition given as:

$$ {}_a^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\tfrac{\alpha}{1-\alpha}(t-\tau)\right) f^{(1)}(\tau)\, \mathrm{d}\tau ; \quad 0<\alpha<1 \tag{3.260} $$

Where $M(\alpha)$ is a normalization constant $M(0) = M(1) = 1$.

Now if we have a kernel as a Mittag-Leffler 'higher transcendental function', i.e., ${}^{ABC} k_\alpha(t) \sim E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right)$, $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ (Appendix-A) then we have the Atangana-Baleanu-Caputo (ABC) definition given as:

$$ {}_a^{ABC} D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \left(E_\alpha\left(-\tfrac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\right)\left(f^{(1)}(\tau)\right) \mathrm{d}\tau, \quad 0<\alpha<1 \tag{3.261} $$

where $B(\alpha)$ is a normalisation constant $B(0) = B(1) = 1$.

We note that our classical definition of a fractional derivative (RL or Caputo) is connected to the Riemann-Liouville fractional integration, with the kernel in convolution integration as a 'power function'. We observe that in those classical definitions, the kernel of fractional integration is $\sim t^{-\alpha}$ which is singular at $t=0$; whereas definitions (3.260) and (3.261) have a kernel of the convolution integration as a non-singular function– described by the exponential and Mittag-Leffler functions respectively.

One may verify in (3.260) by expanding $\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)$ in the series and then using Cauchy's formula for repeated integration, that this expression is a weighted series and the sum of various orders of integer integrals. Similarly by expanding $E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)$ in (3.261) one may verify that this actually is a weighted series sum of various orders of fractional integrals of the RL type.

We have mentioned these new developments, and it is still a developing subject, i.e. using non-singular kernels, in these new definitions of fractional derivatives of the Caputo types. The applicability of these two new definitions is yet a matter of discussions and developments in the fractional order dynamics of the physical systems. We will use only the classical definitions (Riemann-Liouville, Caputo, Grunwald-Letnikov) that are described via a singular kernel, in the rest of the discussions in further chapters.

3.27 Short summary

In this chapter, we have unified the context of the integration and differentiation and arrived at several useful definitions and properties of the fractional derivative. Importantly we have seen that unlike the classical integer order derivative, the fractional derivative is a non-local operator requiring the values of the function from the start – that is, this operation requires a lower-limit and upper limit like the integration process. We have also discussed that there is a memory effect due to this non-local effect for fractional differentiation. We have discussed several paradoxes that appeared in the due course of the development of this fractional calculus over the last three centuries. We now proceed with the knowledge developed here and apply it in the next chapters. In the next chapters we will be using the Riemann-Liouville (RL), Caputo, and Grunwald-Letnikov fractional derivatives for an application to the fractional differential equations and discussions on dynamic systems. In the last chapter, we will modify the RL fractional derivative to give a conjugation to classical integer order derivatives, as well as for non-differentiable functions, and that will give a formulation of a 'local' fractional derivative operator.

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The details of the above works, many of which are pioneering, are listed in the bibliography section in alphabetical order.

Chapter Four

Fractional Differ-integrations' further Extension

4.1 Introduction

The previous chapters highlighted the various ways in which mathematicians described and, by various methods, developed the notion of fractional calculus – namely, fractional integrations and fractional derivatives. We have come across a number of the approaches that can be used to accomplish this. A natural and widely used approach starts from repeated integrations and then extends to fractional integration. Fractional derivatives are then defined either by continuation of the fractional integrals into negative orders (following Leibniz's idea) or by taking the integer order derivatives of the fractional integrals, as we saw in the Riemann-Liouville integral. We have also discussed why the fractional derivative operation is called 'differ-integration'. In this chapter, we will be using the formulations developed earlier in the book, and will apply them in order to derive various fractional differ-integrations of some important functions. We will show that fractional derivatives of a non-differentiable point (of a function) exist, and derive their value, in order to provide some examples. We will also extend our general approach to formulate Leibniz' rule for fractional differ-integrations of the product of two functions and the 'chain-rule' for fractional differ-integration, and discuss the limitations thereof.

4.2 Applying the Grunwald-Letnikov (GL) formula of fractional derivatives to simple functions

4.2.1 Finding the fractional derivative of a constant using the Grunwald-Letnikov (GL) formula

We find the differ-integration of the unit step function, which is $f(x)=1$ for $x \ge a$ to order α, and $f(x)=0$ for $-\infty \le x < a$. This is generally denoted by $f(x)=u(x-a)$, and is termed as the Heaviside step function. We have a GL formula as follows (Section-3.23):

$$\begin{aligned} {}_aD_x^{\alpha}\left[f(x)\right] &= \frac{\mathrm{d}^{\alpha} f(x)}{\left[\mathrm{d}(x-a)\right]^{\alpha}} \\ &= \lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x-j\left(\tfrac{x-a}{N}\right)\right)\right) \end{aligned} \tag{4.1}$$

Straight forward application of the above formula is:

$${}_aD_x^{\alpha}\left[1\right]=\frac{\mathrm{d}^{\alpha}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\lim_{N\uparrow\infty}\left(\left(\frac{N}{x-a}\right)^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right) \tag{4.2}$$

We first apply the property of the binomial coefficient, that is $\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}=\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)}$ (Section-1.12), and write the following expression:

$$\begin{aligned} {}_aD_x^{\alpha}\left[1\right] &= \lim_{N\uparrow\infty}\left(\left(\frac{N}{x-a}\right)^{\alpha}\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)}\right) \\ &= \frac{1}{(x-a)^{\alpha}\Gamma(1-\alpha)}\lim_{N\uparrow\infty}\left(N^{\alpha}\frac{\Gamma(N-\alpha)}{\Gamma(N)}\right) \end{aligned} \tag{4.3}$$

From the property of the Gamma function (Section-1.11) described below:

$$\lim_{j\uparrow\infty}\left(j^{c+\alpha+1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}\right)=\lim_{j\uparrow\infty}\left(j^{c+\alpha}\frac{\Gamma(j-\alpha)}{\Gamma(j)}\right)=\begin{cases}+\infty & c>0\\ 1 & c=0\\ 0 & c<0\end{cases} \tag{4.4}$$

We recognize that $\lim_{N\uparrow\infty}\left(N^{\alpha}\frac{\Gamma(N-\alpha)}{\Gamma(N)}\right)=1$; i.e. it is obtained by setting $j=N$ and $c=0$ in the above (4.4) formula. Therefore, we obtain:

$$ {}_aD_x^{\alpha}\left[1\right]=\frac{\mathrm{d}^{\alpha}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.5}$$

This (4.5) we have noted earlier in Section-3.2, but this demonstration is intended to bring out the same result, while noting that here special properties of the Gamma function are used. From the above (4.5), we get for $f(x)=C$ any constant including zero that is the following result:

$$ {}_aD_x^{\alpha}\left[C\right]=\frac{\mathrm{d}^{\alpha}\left[C\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=C\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.6}$$

Since ${}_aD_x^{\alpha}\left[1\right]=\frac{\mathrm{d}^{\alpha}[1]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$ is never infinite for $x>a$, we conclude that by setting $C=0$ ${}_aD_x^{\alpha}\left[0\right]=\frac{\mathrm{d}^{\alpha}[0]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=0$ for all α. This may sound trivial, and appears obvious. As an example of its importance, however, it provides a powerful counter example to the belief that if $\frac{\mathrm{d}^{\alpha}[f(x)]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=g(x)$ then $\frac{\mathrm{d}^{-\alpha}[g(x)]}{\left[\mathrm{d}(x-a)\right]^{-\alpha}}=f(x)$, and if $f(x)$ gives zero on α-th order differentiation, then $f(x)$ cannot be restored by $\alpha-$th order integration.

4.2.2 Finding a fractional derivative of a linear function using the Grunwald-Letnikov (GL) formula

Now we take $f(x)=x-a$ and apply the GL formula in the following steps:

$$\begin{aligned}
{}_aD_x^{\alpha}\left[x-a\right] &= \lim_{N\uparrow\infty}\left(\left(\tfrac{N}{x-a}\right)^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\left((x-a)-j\left(\tfrac{N-a}{N}\right)\right)\right)\\
&= \lim_{N\uparrow\infty}\left(\left(\tfrac{N}{x-a}\right)^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\left(\tfrac{Nx-jx+ja}{N}-a\right)\right)\\
&= \lim_{N\uparrow\infty}\left(\left(\tfrac{N}{x-a}\right)^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\left(\tfrac{Nx-jx+ja-Na}{N}\right)\right)\\
&= \lim_{N\uparrow\infty}\left(\left(\tfrac{N^{\alpha-1}}{(x-a)^{\alpha}}\right)\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\left(N(x-a)-j(x-a)\right)\right)\\
&= \lim_{N\uparrow\infty}\left(\left(\tfrac{N^{\alpha}}{(x-a)^{\alpha-1}}\right)\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right)\\
&\qquad -\lim_{N\uparrow\infty}\left(\left(\tfrac{N^{\alpha-1}}{(x-a)^{\alpha}}\right)\sum_{j=0}^{N-1}j\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right)\\
&= (x-a)^{1-\alpha}\left(\begin{aligned}&\lim_{N\uparrow\infty}\left(N^{\alpha}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right)\\ &-\lim_{N\uparrow\infty}\left(N^{\alpha-1}\sum_{j=0}^{N-1}j\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right)\end{aligned}\right)
\end{aligned} \tag{4.7}$$

We first apply the property of the Gamma function, that is $\Gamma(j+1)=j\left(\Gamma(j)\right)$, and then use the property of the binomial coefficient, that is $\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}=\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)}$ and $\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j)}=\frac{-\alpha\Gamma(N-\alpha)}{\Gamma(2-\alpha)\Gamma(N-1)}$, before using $\Gamma(N)=(N-1)\left(\Gamma(N-1)\right)$ to write the following steps:

$$
{}_aD_x^{\alpha}\left[x-a\right]=(x-a)^{1-\alpha}\begin{pmatrix}\lim\limits_{N\uparrow\infty}\left(N^{\alpha}\sum\limits_{j=0}^{N-1}\dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}\right)\\ -\lim\limits_{N\uparrow\infty}\left(N^{\alpha-1}\sum\limits_{j=0}^{N-1}\dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j)}\right)\end{pmatrix}
$$

$$
=(x-a)^{1-\alpha}\begin{pmatrix}\lim\limits_{N\uparrow\infty}\left(N^{\alpha}\dfrac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)}\right)\\ +\lim\limits_{N\uparrow\infty}\left(N^{\alpha-1}\dfrac{\alpha\Gamma(N-\alpha)}{\Gamma(2-\alpha)\Gamma(N-1)}\right)\end{pmatrix}
$$

$$
=(x-a)^{1-\alpha}\begin{pmatrix}\dfrac{1}{\Gamma(1-\alpha)}\lim\limits_{N\uparrow\infty}\left(N^{\alpha}\dfrac{\Gamma(N-\alpha)}{\Gamma(N)}\right)\\ +\dfrac{\alpha}{\Gamma(2-\alpha)}\lim\limits_{N\uparrow\infty}\left(N^{\alpha-1}\dfrac{\Gamma(N-\alpha)}{\Gamma(N-1)}\right)\end{pmatrix}
$$

$$
=(x-a)^{1-\alpha}\begin{pmatrix}\dfrac{1}{\Gamma(1-\alpha)}\lim\limits_{N\uparrow\infty}\left(N^{\alpha}\dfrac{\Gamma(N-\alpha)}{\Gamma(N)}\right)\\ +\dfrac{\alpha}{\Gamma(2-\alpha)}\lim\limits_{N\uparrow\infty}\left(N^{\alpha}(N-1)\dfrac{\Gamma(N-\alpha)}{\Gamma(N)}\right)\end{pmatrix} \tag{4 .8}
$$

We recognize that $\lim_{N\uparrow\infty}\left(N^{\alpha}\frac{\Gamma(N-\alpha)}{\Gamma(N)}\right)=1$; i.e. it is obtained by setting $j=N$ and $c=0$ in the above (4.4) formula as used previously (Section-1.11). We consequently get the following:

$$
{}_aD_x^{\alpha}\left[x-a\right]=(x-a)^{1-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(2-\alpha)}\right) \tag{4.9}
$$

We use $\Gamma(x+1)=x\left(\Gamma(x)\right)$ to write $\Gamma(2-\alpha)=(1-\alpha)\left(\Gamma(1-\alpha)\right)$ in order to simplify $\left(\frac{1}{\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(2-\alpha)}\right)$ as $\frac{1}{\Gamma(2-\alpha)}$, and arrive at the final result, which is:

$$
{}_aD_x^{\alpha}\left[x-a\right]=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \tag{4.10}
$$

4.2.3 Verifying the GL fractional derivative of a linear function using the Riemann-Liouville (RL) formula of the fractional integration

We verify this from the Riemann-Liouville (RL) formula for fractional integration for $\alpha<0$ is ${}_aD_x^{\alpha}\left[f(x)\right]=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{f(y)}{(x-y)^{\alpha+1}}\mathrm{d}y$. Applying this to $f(x)=x-a$, we get the following:

$$
\begin{aligned}
{}_aD_x^{\alpha}\left[x-a\right]&=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{(y-a)}{(x-y)^{\alpha+1}}\mathrm{d}y \qquad \text{put} \qquad w=x-y\\
&=\frac{1}{\Gamma(-\alpha)}\int_0^{x-a}\frac{(x-a-w)}{w^{\alpha+1}}\mathrm{d}w\\
&=\frac{1}{\Gamma(-\alpha)}\left(\int_0^{x-a}\frac{(x-a)}{w^{\alpha+1}}\mathrm{d}w-\int_0^{x-a}\frac{1}{w^{\alpha}}\mathrm{d}w\right)\\
&=\frac{1}{\Gamma(-\alpha)}\left(\frac{(x-a)^{1-\alpha}}{-\alpha}-\frac{(x-a)^{1-\alpha}}{1-\alpha}\right)\\
&=\frac{(x-a)^{1-\alpha}}{(-\alpha)(1-\alpha)\Gamma(-\alpha)}=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \qquad \alpha<0
\end{aligned} \tag{4.11}
$$

We obtain the denominator in (4.11), the last step of which is $(-\alpha)(1-\alpha)\left(\Gamma(-\alpha)\right)=\Gamma(2-\alpha)$, by using the recurrence formula, i.e. $x\left(\Gamma(x)\right)=\Gamma(x+1)$. As such, we achieve the earlier obtained result (4.10).

4.2.4 Verifying the GL fractional derivative of a linear function by usingthe Riemann-Liouville (RL) formula of the fractional derivative

We can now use the RL-derivative formula ${}_aD_x^{\alpha}\left[x-a\right]=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\mathrm{d}^{\alpha-n}[x-a]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}\right)$. Here, we remove the restrictive condition on α. Thus, for any arbitrary α, one may select an integer n so large that $\alpha-n<0$, and the above obtained result (4.11) will give:

$$\frac{\mathrm{d}^{\alpha-n}\left[x-a\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}=\frac{(x-a)^{1-\alpha+n}}{\Gamma(2-\alpha+n)} \tag{4.12}$$

We use the above expression (4.12) in the following form:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[x-a\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}&=\frac{\mathrm{d}^{n}}{\left[\mathrm{d}(x-a)\right]^{n}}\left(\frac{\mathrm{d}^{\alpha-n}\left[x-a\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}\right)\\&=\frac{\mathrm{d}^{n}}{\left[\mathrm{d}(x-a)\right]^{n}}\left[\frac{(x-a)^{1-\alpha+n}}{\Gamma(2-\alpha+n)}\right]\\&=\left(\frac{1}{\Gamma(2-\alpha+n)}\right)\left(\frac{\Gamma(2-\alpha+n)}{\Gamma(2-\alpha)}\right)(x-a)^{1-\alpha}\\&=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}\end{aligned} \tag{4.13}$$

The above (4.13) follows from $\frac{\mathrm{d}^n}{\left[\mathrm{d}(x-a)\right]^n}\left[(x-a)^p\right]=p(p-1)...(p-n+1)(x-a)^{p-n}$, which is equal to $\frac{\mathrm{d}^n}{\left[\mathrm{d}(x-a)\right]^n}\left[(x-a)^p\right]=\frac{\Gamma(p+1)}{\Gamma(p-n+1)}(x-a)^{p-n}$.

We note, as expected, the formula ${}_aD_x^{\alpha}\left[x-a\right]=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}$ which reduces to zero when $\alpha=2,3,4...$; reduces to unity when $\alpha=1$; reduces to $(x-a)$ when $\alpha=0$; and reduces to $\frac{(x-a)^{n+1}}{(n+1)!}$ when $\alpha=-n=-1,-2,-3,\dots$. We also notice that ${}_aD_x^{\alpha}\left[x-a\right]=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}$ when compared to ${}_aD_x^{\alpha}\left[1\right]=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$, and we observe that the $\alpha-$th differ-integral of the $(x-a)$ equals the $(\alpha-1)$ -th differ-integral of the unity function i.e.

$${}_aD_x^{\alpha}\left[x-a\right]={}_aD_x^{\alpha-1}\left[1\right]=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \tag{4.14}$$

4.3 Applying the Riemann-Liouville formula to get the fractional derivative for some simple functions

4.3.1 Left fractional derivative of the power function through application of the integral formula of the Riemann-Liouville fractional derivative

From the RL fractional derivative formula applied to a function $f(x)=x^b$, we have the following:

$${}_0D_x^{\alpha}\left[x^b\right]=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-u)^{n-\alpha-1}u^b\mathrm{d}u\right] \tag{4.15}$$

Put $u=xy$; $\mathrm{d}u=x\mathrm{d}y$, $(x-u)=x(1-y)$ in the above expression and obtain the following steps:

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[x^b\right] &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\int_0^1 x^{n-\alpha-1}(1-y)^{n-\alpha-1}x^b y^b (x\mathrm{d}y)\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n-\alpha+b}\int_0^1 (1-y)^{n-\alpha-1}y^b \mathrm{d}y\right] \\
&= \frac{\int_0^1 (1-y)^{n-\alpha-1}y^b\mathrm{d}y}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}^n\left[x^{n-\alpha+b}\right]}{\mathrm{d}x^n}\right) \\
&= \frac{\mathrm{B}(n-\alpha,b+1)}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}^n\left[x^{n-\alpha+b}\right]}{\mathrm{d}x^n}\right)
\end{aligned}
\tag{4.16}
$$

$$n-\alpha>0 \qquad b+1>0$$

Here we used the Beta-integral, that is $\mathrm{B}(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}\mathrm{d}u=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, with $p>0$ and $q>0$, giving us the condition $n>\alpha$, and $b>-1$; so that the Beta-integral does not diverge. Using the Gamma-function representation of the Beta-function, that is $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, while also using $\frac{\mathrm{d}^n x^p}{\mathrm{d}x^n}=\frac{\Gamma(p+1)}{\Gamma(p+1-n)}x^{n-p}$, where n is integer (as discussed in Section-1.15), we write the following:

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[x^b\right] &= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(b+1)\Gamma(n-\alpha)}{\Gamma(b+1+n-\alpha)}\left(\frac{\mathrm{d}^n x^{n-\alpha+b}}{\mathrm{d}x^n}\right) \\
&= \frac{\Gamma(b+1)}{\Gamma(b+1+n-\alpha)}\frac{\Gamma(n-\alpha+b+1)}{\Gamma(n-\alpha+b-n+1)}x^{b-\alpha} \\
&= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)}x^{b-\alpha} \qquad b>-1
\end{aligned}
\tag{4.17}
$$

The above expression (4.17) we also obtained from Euler's formula. For $b=0$, we get ${}_0D_x^{\alpha}\left[1\right]=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$, that is the fractional derivative of the constant function as non-zero in the RL sense.

Using the formula for RL fractional integration for $f(t)=(t-a)^{\gamma}$, we see the fractional integral for order $\alpha>0$ as:

$$
{}_aD_t^{-\alpha}\left[(t-a)^{\gamma}\right]=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\gamma}\,\mathrm{d}\tau \tag{4.18}
$$

Using the substitution in (4.18) $\tau=a+\xi(t-a)$, as we have for $\tau=a$, $\xi=0$ and for $\tau=t$, $\xi=1$; $\mathrm{d}\tau=(t-a)\mathrm{d}\xi$, $(t-\tau)=t-a-\xi(t-a)=(t-a)(1-\xi)$; $(\tau-a)=\xi(t-a)$, we get the following:

$$
\begin{aligned}
{}_aD_t^{-\alpha}\left[(t-a)^{\gamma}\right] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\gamma}\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-a)^{\alpha-1}(1-\xi)^{\alpha-1}\xi^{\gamma}(t-a)^{\gamma}(t-a)\mathrm{d}\xi \\
&= \frac{(t-a)^{\gamma+\alpha}}{\Gamma(\alpha)}\int_0^1 \xi^{\gamma}(1-\xi)^{\alpha-1}\mathrm{d}\xi \\
&= \frac{(t-a)^{\gamma+\alpha}}{\Gamma(\alpha)}\mathrm{B}(\alpha,\gamma+1) \qquad \alpha>0 \qquad \gamma+1>0 \\
&= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(t-a)^{\gamma+\alpha} \qquad \gamma>-1
\end{aligned}
\tag{4.19}
$$

We used the Beta-integral $\mathrm{B}(\alpha,\gamma+1)=\int_0^1 \xi^{\gamma}(1-\xi)^{\alpha-1}\mathrm{d}\xi$, and $\mathrm{B}(\alpha,\gamma+1)=\frac{\Gamma(\alpha)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}$. Applying the above obtained result (4.19) to the fractional integral of the order $(1-\upsilon)$, with $0\le\upsilon<1$, we get:

$$ {}_aD_t^{-(1-\upsilon)}\left[(t-a)^\gamma\right]=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}(t-a)^{\gamma+1-\upsilon} \tag{4.20}$$

Taking one whole derivative of the above (4.20), we get a fractional derivative of the order $0\le\upsilon<1$ for the function $f(t)=(t-a)^\gamma$.

$$\begin{aligned} D^1\left[{}_aD_t^{-(1-\upsilon)}(t-a)^\gamma\right] &= {}_aD_t^\upsilon\left[(t-a)^\gamma\right] \\ &=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}(t-a)^{\gamma+1-\upsilon}\right] \\ &=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\upsilon+2)}(\gamma+1-\upsilon)(t-a)^{\gamma-\upsilon} \\ &=\frac{\Gamma(\gamma+1)}{(\gamma-\upsilon+1)\Gamma(\gamma+1-\upsilon)}(\gamma+1-\upsilon)(t-a)^{\gamma-\upsilon} \\ &=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\upsilon)}(t-a)^{\gamma-\upsilon} \qquad \gamma>-1 \end{aligned} \tag{4.21}$$

We have therefore calculated the fractional derivative using the RL formula for fractional order υ such that $0\le\upsilon<1$; thus our nearest integer is one that is $n=1$ and we write this below:

$$\begin{aligned} {}_aD_t^\upsilon\left[f(t)\right] &=\frac{1}{\Gamma(1-\upsilon)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_a^t(t-\tau)^{-\upsilon}\left(f(\tau)\right)\mathrm{d}\tau \\ &= D^1\left[D_t^{-(1-\upsilon)}f(t)\right] \end{aligned} \tag{4.22}$$

Thus for $a=0$, the fractional RL derivative of the $f(t)=t^\gamma$ as

$$ {}_0D_t^\upsilon\left[t^\gamma\right]=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\upsilon)}t^{\gamma-\upsilon} \tag{4.23}$$

4.3.2 Right fractional derivative of the power function through application of the integral formula of the Riemann-Liouville fractional derivative

Let us conduct the integration in a reversed way, i.e. from a lower limit t to a higher limit A, for a function $f(t)=(A-t)^\gamma$ with the formula ${}_tD_A^{-\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_t^A(\tau-t)^{\alpha-1}f(\tau)\mathrm{d}\tau$. This is in line with the Weyl theorem as discussed (Section-2.15) for ${}_xW_\infty^{-\upsilon}$ or fractional integration, with the upper limit as A instead of ∞. Using the substitution $\tau=A-\xi(A-t)$ we have for $\tau=A$, $\xi=0$ and for $\tau=t$, $\xi=1$; $\mathrm{d}\tau=-(A-t)\mathrm{d}\xi$, $(\tau-t)=A-t-\xi(A-t)=(A-t)(1-\xi)$; $(A-\tau)=\xi(A-t)$. We get the following results:

$$\begin{aligned} {}_tD_A^{-\alpha}\left[(A-t)^\gamma\right] &=\frac{1}{\Gamma(\alpha)}\int_t^A(\tau-t)^{\alpha-1}(A-\tau)^\gamma\,\mathrm{d}\tau \\ &=\frac{1}{\Gamma(\alpha)}\int_1^0(A-t)^{\alpha-1}(1-\xi)^{\alpha-1}\xi^\gamma(A-t)^\gamma\left(-(A-t)\right)\mathrm{d}\xi \\ &=\frac{(A-t)^{\gamma+\alpha}}{\Gamma(\alpha)}\int_0^1(1-\xi)^{\alpha-1}\xi^\gamma\,\mathrm{d}\xi \\ &=\frac{(A-t)^{\gamma+\alpha}}{\Gamma(\alpha)}\mathrm{B}(\alpha,\gamma+1)=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(A-t)^{\gamma+\alpha} \end{aligned} \tag{4.24}$$

Therefore, we have ${}_tD_A^{-\alpha}\left[(A-t)^\gamma\right]=\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\right)(A-t)^{\gamma+\alpha}$.

Now take $\alpha=1-\upsilon$, to write ${}_tD_A^{-(1-\upsilon)}\left[(A-t)^\gamma\right]=\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}\right)(A-t)^{\gamma+1-\upsilon}$. We now conduct one whole derivation on (4.24) with a minus sign as depicted below:

$$(-1)D^1\left({}_tD_A^{-(1-\upsilon)}\left[(A-t)^\gamma\right]\right)=(-1)\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}\right)(A-t)^{\gamma+1-\upsilon}\right]$$
$$=(-1)\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}(\gamma+1-\upsilon)(A-t)^{\gamma-\upsilon}(-1) \tag{4.25}$$
$$=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}(A-t)^{\gamma-\upsilon}={}_tD_A^{\upsilon}\left[(A-t)^\gamma\right]$$

We get the following formula for a fractional derivative in the right sense as:

$${}_tD_A^{\upsilon}\left[(A-t)^\gamma\right]=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2-\upsilon)}(A-t)^{\gamma-\upsilon}\qquad \gamma>-1 \tag{4.26}$$

We prefer to take a different and more elementary approach attaching a meaning to the RL operator for $\alpha\ge 0$.

The formula ${}_aD_x^\alpha\left[f(x)\right]=\frac{1}{\Gamma(-\alpha)}\int_a^x(x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y$ for $\alpha<0$ will be retained for only $\alpha<0$, indicating fractional integration; it is extended to $\alpha\ge 0$, by insisting that ${}_aD_x^n\left({}_aD_x^\alpha\left[f(x)\right]\right)={}_aD_x^{n+\alpha}\left[f(x)\right]$ is satisfied for all positive integers n and for all α; that it will also be satisfied by the RL integral. That is, we shall require ${}_aD_x^\alpha\left[f(x)\right]\equiv D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right)$ where $D_x^n\equiv\frac{\mathrm{d}^n}{\mathrm{d}x^n}$ is n-th derivative (the ordinary derivative operator), choosing the positive integer so large that $\alpha-n<0$. The relationship ${}_aD_x^\alpha\left[f(x)\right]\equiv D_x^n\left({}_aD_x^{\alpha-n}\left[f(x)\right]\right)$ holds for all α. We described this earlier in the book (Section-2.14), while performing an extension of the order α, where $\alpha<0$, and ${}_aD_x^\alpha\left[f(x)\right]=\frac{1}{\Gamma(-\alpha)}\int_a^x(x-y)^{-\alpha-1}\left(f(y)\right)\mathrm{d}y$; to all α. We will use this concept subsequently.

4.4 Applying the Caputo formula to get a fractional derivative for simple functions

4.4.1 Left fractional derivative of the power function x^b with respect to x through application of the integral formula of the Caputo fractional derivative

Let us take the function $f(x)=x^b$, with $b>-1$, and we will analyse the Caputo derivative from the starting point $x=0$, as described below:

$${}_0^CD_x^\alpha\left[x^b\right]=\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-u)^{n-\alpha-1}\left(\frac{\mathrm{d}^nu^b}{\mathrm{d}u^n}\right)\mathrm{d}u$$
$$=\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-u)^{n-\alpha-1}\left(\frac{\Gamma(b+1)}{\Gamma(b+1-n)}u^{b-n}\right)\mathrm{d}u$$
$$=\frac{\Gamma(b+1)}{\Gamma(b+1-n)}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-u)^{n-\alpha-1}u^{b-n}\mathrm{d}u\right) \tag{4.27}$$
$$=\frac{\Gamma(b+1)}{\Gamma(b+1-n)}\left({}_0I_x^{n-\alpha}\left[x^{b-n}\right]\right)$$
$$=\frac{\Gamma(b+1)}{\Gamma(b+1-n)}\left(\frac{\Gamma(b-n+1)}{\Gamma(b-n+1+n-\alpha}x^{b-n+n-\alpha}\right)$$
$$=\frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}x^{b-\alpha}$$

We note that $\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-u)^{n-\alpha-1}u^{b-n}\mathrm{d}u={}_0I_x^{n-\alpha}\left[x^{b-n}\right]$, in the above steps, (4.27) is an expression for the RL fractional integration of the order $n-\alpha$, for function $f(x)=x^{b-n}$, and we used the derived formula ${}_0I_x^p\left[x^q\right]=\frac{\Gamma(q+1)}{\Gamma(q+1+p)}x^{q+p}$ (Section-2.6). Note that this (4.27) is the same result as that obtained for the fractional derivative using the RL formula (4.17). The reason for this is that the value of the function at the starting point of the differentiation is zero. In a similar way, one can find the Caputo derivative for all other functions described in the previous section.

4.4.2 Left fractional derivative of the power function $\left(\psi(x)\right)^{\upsilon}$ with respect to x through application of the integral formula of the generalized Caputo fractional derivative

We have demonstrated the use of the generalized RL fractional integration formula, that is ${}_{a}\mathcal{I}^{\alpha}_{x,[z,w]}f(x)=\frac{1}{(w(x))(\Gamma(\alpha))}\int_a^x\left(z(x)-z(t)\right)^{\alpha-1}w(t)\left(z^{(1)}(t)\right)\left(f(t)\right)\mathrm{d}t$, in an earlier chapter (Sections 2.17 and 3.25). Now, we demonstrate the use of the generalized fractional derivative formula that we noted earlier with weight function $w(x)=1$, and $n-1<\alpha<n$, i.e.

$$ {}_{a}^{C}\mathfrak{D}^{\alpha}_{x,[z,1]}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x z^{(1)}(t)\left(z(x)-z(t)\right)^{n-\alpha-1}\left(\frac{1}{z^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(f(t)\right)\mathrm{d}t \tag{4.28}$$

Let us take $f(x)=\left(\psi(x)-\psi(a)\right)^{\upsilon}$; $\upsilon=\beta-1$, $z(x)=\psi(x)$ with $\beta>0$ and write the following steps:

$$\begin{aligned}
&{}_{a}^{C}\mathfrak{D}^{\alpha}_{x,[\psi,1]}\left[\left(\psi(x)-\psi(a)\right)^{\beta-1}\right]\\
&\frac{1}{\Gamma(n-\alpha)}\int_a^x \psi^{(1)}(t)\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}\left(\frac{1}{\psi^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(\psi(t)-\psi(a)\right)^{\beta-1}\mathrm{d}t\\
&=\frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}\left(\frac{\mathrm{d}^n\left(\psi(t)-\psi(a)\right)^{\beta-1}}{\mathrm{d}\left[\psi(t)\right]^n}\right)\left(\psi^{(1)}(t)\right)\mathrm{d}t\\
&=\frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-n)}\left(\psi(t)-\psi(a)\right)^{\beta-1-n}\right)\left(\psi^{(1)}(t)\right)\mathrm{d}t\\
&=\frac{\Gamma(\beta)}{\Gamma(\beta-n)}\left(\frac{1}{\Gamma(n-\alpha)}\int_{\psi(a)}^{\psi(x)}\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}\left(\psi(t)-\psi(a)\right)^{\beta-1-n}\mathrm{d}\left(\psi(t)\right)\right)\\
&=\frac{\Gamma(\beta)}{\Gamma(\beta-n)}\left({}_{\psi(a)}I^{(n-\alpha)}_{\psi(x)}\left[\left(\psi(x)-\psi(a)\right)^{\beta-1-n}\right]\right)\\
&=\frac{\Gamma(\beta)}{\Gamma(\beta-n)}\left(\frac{\Gamma(\beta-1-n+1)}{\Gamma(\beta-n+n-\alpha)}\left(\psi(x)-\psi(a)\right)^{\beta-1-n+n-\alpha}\right)\\
&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\psi(x)-\psi(a)\right)^{\beta-1-\alpha}
\end{aligned}\tag{4.29}$$

We note that we have used $\left(\psi^{(1)}(t)\right)\mathrm{d}t=\frac{\mathrm{d}(\psi(t))}{\mathrm{d}t}\mathrm{d}t=\mathrm{d}\left(\psi(t)\right)$ in the above steps (4.29). We also note that we recognize $\frac{1}{\Gamma(n-\alpha)}\int_{\psi(a)}^{\psi(x)}\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}\left(\psi(t)-\psi(a)\right)^{\beta-1-n}\mathrm{d}\left(\psi(t)\right)$ as a formula for fractional integration of the order $(n-\alpha)$ of the function $f(x)=\left(\psi(x)-\psi(a)\right)^{\beta-1-n}$, with respect to $\psi(x)$ from the lower terminal limit $\psi(a)$ to the higher terminal limit $\psi(x)$ i.e. compactly expressed as ${}_{\psi(a)}I^{(n-\alpha)}_{\psi(x)}\left[\left(\psi(x)-\psi(a)\right)^{\beta-1-n}\right]$.

We use ${}_{a}I^{\upsilon}_{x}\left[(x-a)^{b}\right]=\frac{\Gamma(b+1)}{\Gamma(b+1+\upsilon)}(x-a)^{b+\upsilon}$ and obtained the result reported in the above step of the derivation (4.29).

4.5 The fractional differ-integration of the binomial function

We developed the fractional differ-integration of the simple function. The most important application of these techniques is using the fractional differ-integration of functions like $f(x)=\sum_{j=0}^{\infty}a_j(x-a)^{p+\left(\frac{j}{n}\right)}$. A series representation is presented in the following way:

$$f(x) = a_0(x-a)^p + a_1(x-a)^{p+\left(\frac{1}{n}\right)} + a_2(x-a)^{p+\left(\frac{2}{n}\right)} +$$
$$\ldots a_{n-1}(x-a)^{p+\left(\frac{n-1}{n}\right)} + a_n(x-a)^{p+1}\ldots \tag{4.30}$$
$$= (x-a)^p \sum_{j=0}^{\infty} a_j (x-a)^{\left(\frac{j}{n}\right)}$$

We refer to the above represented $f(x)$ (4.30) as a differ-integrable function with the condition $p > -1$, n as its positive integer, and $a_0 \neq 0$ as its non-zero constant. This type of function with $p > -1$ is called a Riemann-Class function. We conduct term-by-term differ-integration using the arbitrary number α and write the following

$${}_aD_x^{\alpha}\left[\sum\nolimits_{j=0}^{\infty} a_j (x-a)^{p+\left(\frac{j}{n}\right)}\right] = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{pn+j+n}{n}\right)}{\Gamma\left(\frac{pn-\alpha n+j+n}{n}\right)}(x-a)^{p-\alpha+\left(\frac{j}{n}\right)} \tag{4.31}$$

The above expression (4.31) is obtained using the earlier derived formula detailed in Section-4.3, the differ-integration formula for function $(x-a)^p$.

For the binomial function i.e. $f(x) = (C-cx)^p$, we can write $C - cx = C - ac - c(x-a)$, and then via the binomial theorem, i.e. $(x+y)^m = \sum\nolimits_{j=0}^{\infty} \frac{m!}{j!(m-j)!} x^{m-j} y^j$, we write the power series expansion as expressed below:

$$(C-cx)^p = \left((C-ac) - c(x-a)\right)^p$$
$$= \sum_{j=0}^{\infty} \frac{p!}{j!(p-j)!}\left((C-ac)^{p-j}\right)\left(\left(-c(x-a)\right)^j\right) \tag{4.32}$$
$$= \sum_{j=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(j+1)\Gamma(p-j+1)}(-c)^j (C-ac)^{p-j}(x-a)^j$$

provided $-1 < \frac{c(x-a)}{C-ca} < 1$. Note that the expansion (4.32) is valid even if p is a positive integer, but that in this event the sum is automatically finite, while all terms for which j exceeds p have finite denominators. Using the identity $\frac{\Gamma(p+1)}{\Gamma(p-j+1)} = (-1)^j \frac{\Gamma(j-p)}{\Gamma(-p)}$, we write the α -th differ-integral as follows:

$${}_aD_x^{\alpha}\left[(C-cx)^p\right]$$
$$= {}_aD_x^{\alpha}\left[\sum_{j=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(j+1)\Gamma(p-j+1)}(-c)^j (C-ac)^{p-j}(x-a)^j\right]$$
$$= \sum_{j=0}^{\infty}\left(\frac{(C-ac)^p(-1)^j c^j \Gamma(p+1)}{(C-ac)^j \Gamma(j+1)\Gamma(p-j+1)}\, {}_aD_x^{\alpha}\left[(x-a)^j\right]\right) \tag{4.33}$$
$$= \sum_{j=0}^{\infty}\left(\frac{(C-ac)^p(-1)^j c^j \Gamma(p+1)}{(C-ac)^j \Gamma(j+1)\Gamma(p-j+1)}\left(\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\right)(x-a)^{j-\alpha}\right)$$
$$= \frac{(C-ac)^p (x-a)^{-\alpha}}{\Gamma(-p)} \sum_{j=0}^{\infty} \frac{\Gamma(j-p)}{\Gamma(j-\alpha+1)}\left(\frac{c(x-a)}{C-ac}\right)^j$$

We try to obtain the closed form of the above derivation using the identity of the hyper-geometric function and the Beta function (Appendix-A), i.e. $\mathrm{B}_y(c, 1+b-c) = \frac{y^c(1-y)^{1+b-c}\Gamma(c)}{\Gamma(1+b)}\left(\sum\nolimits_{j=0}^{\infty} y^j \frac{\Gamma(j+1+b)}{\Gamma(j+1+c)}\right)$, in order to simplify the summation in the expression (4.33). Taking $y = \frac{c(x-a)}{(C-ca)}$ and taking parameters $b = -(p+1)$ and $c = -\alpha$, we obtain the following steps as a final result:

$$\frac{\mathrm{d}^{\alpha}\left[(C-cx)^{p}\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{(C-ac)^{p}}{(x-a)^{\alpha}\Gamma(-p)}\sum_{j=0}^{\infty}\left(\frac{cx-ac}{C-ac}\right)^{j}\frac{\Gamma\left(j+(-p)\right)}{\Gamma\left(j+1+(-\alpha)\right)}$$
$$=\frac{(C-ac)^{p}}{(x-a)^{\alpha}\Gamma(-p)}\sum_{j=0}^{\infty}y^{j}\frac{\Gamma\left(j+1+(-(p+1))\right)}{\Gamma\left(j+1+(-\alpha)\right)}$$
$$=\frac{(C-ac)^{p}}{(x-a)^{\alpha}\Gamma(-p)}\left(\frac{(C-ac)^{\alpha-p}}{(C-cx)^{\alpha-p}}\right)\left(\frac{(cx-ac)^{\alpha}}{(C-ac)^{\alpha}}\right)\frac{\Gamma(-p)}{\Gamma(-\alpha)}\mathrm{B}_{y}\left(-\alpha,\alpha-p\right) \tag{4.34}$$
$$=\frac{c^{\alpha}(C-cx)^{p-\alpha}}{\Gamma(-\alpha)}\mathrm{B}_{y}\left(-\alpha,\alpha-p\right)$$

For $a=0$ and $C=c=1$, we write the following simple result:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[(1-x)^{p}\right]=\frac{(1-x)^{p-\alpha}}{\Gamma(-\alpha)}\mathrm{B}_{x}(-\alpha,\alpha-p) \tag{4.35}$$

4.6 Fractional differ-integration exponential function using the Riemann-Liouville fractional derivative formula

Let us try to find the fractional integral of the order $v>0$ of the exponential function $f(t)=e^{at}$. We write the fractional integration operation as follows:

$$ {}_{0}D_{t}^{-v}\left[e^{at}\right]=\frac{1}{\Gamma(v)}\int_{0}^{t}(t-u)^{v-1}e^{au}\mathrm{d}u \tag{4.36}$$

Once the substitution $x=t-u$ is made, the above integral expression (4.36) is changed as follows:

$$ {}_{0}D_{t}^{-v}\left[e^{at}\right]=\frac{e^{at}}{\Gamma(v)}\int_{0}^{t}x^{v-1}e^{-ax}\mathrm{d}x \tag{4.37}$$

Clearly, this integral is not an elementary function, but is closely related to Tricomi's incomplete gamma function (discussed in Section-1.13), and thus the fractional integral of the exponential function is expressed as follows:

$$ {}_{0}D_{t}^{-v}\left[e^{at}\right]=\frac{e^{at}}{\Gamma(v)}\int_{0}^{t}x^{v-1}e^{-ax}\mathrm{d}x$$
$$=t^{v}e^{at}\frac{t^{-v}}{\Gamma(v)}\int_{0}^{t}x^{v-1}e^{-ax}\mathrm{d}x \qquad \text{put} \qquad ax=y$$
$$=t^{v}e^{at}\frac{t^{-v}}{\Gamma(v)}\int_{0}^{at}\frac{y^{v-1}}{a^{v-1}}e^{-y}\left(\frac{\mathrm{d}y}{a}\right) \tag{4.38}$$
$$=t^{v}e^{at}\frac{(at)^{-v}}{\Gamma(v)}\int_{0}^{at}y^{v-1}e^{-y}\mathrm{d}y$$

In the above, Tricomi's incomplete gamma function is $\gamma^{*}(v,x)=\frac{x^{-v}}{\Gamma(v)}\int_{0}^{x}e^{-t}t^{v-1}\mathrm{d}t$. Therefore, in this derivation we write $\frac{(at)^{-v}}{\Gamma(v)}\int_{0}^{at}y^{v-1}e^{-y}\mathrm{d}y=\gamma^{*}(v,at)$, and with that, we use the following expression

$$ {}_{0}D_{t}^{-v}\left[e^{at}\right]=t^{v}e^{at}\gamma^{*}(v,at) \tag{4.39}$$

We will use this formula (4.39) in the following section. When we take $-v=\alpha$, $a=\pm1$, we get the following expression as an important formula with a change in variable:

$$ {}_{0}D_{x}^{\alpha}\left[e^{\pm x}\right]=\frac{\mathrm{d}^{\alpha}\left[e^{\pm x}\right]}{\left[\mathrm{d}x\right]^{\alpha}}=\frac{e^{\pm x}}{x^{\alpha}}\gamma^{*}(-\alpha,\pm x) \tag{4.40}$$

With C and c as arbitrary constants, using the power series expansion of the $f(x)=e^{(C-cx)}$, we write the Taylor expansion at $x=a$ in the following steps:

$$\begin{aligned}f(x) &= f(a)+(x-a)f^{(1)}(a)+\frac{(x-a)^2}{2!}f^{(2)}(a)+.....\\ &= e^{C-ca}+(x-a)(-c)e^{C-ca}+\frac{(x-a)^2}{2!}c^2e^{C-ca}+\frac{(x-a)^3}{3!}(-c)^3e^{C-ca}+....\\ &= \left(e^{C-ca}\right)\left(1-c(x-a)+\tfrac{1}{2!}c^2(x-a)^2-\tfrac{1}{3!}c^3(x-a)^3+...\right)\\ &= e^{C-ca}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j+1)}\end{aligned} \tag{4.41}$$

Thus, we write $e^{C-cx}=e^{C-ca}\sum_{j=0}^{\infty}\frac{(-c(x-a))^j}{\Gamma(j+1)}$, a power series expansion of e^{C-cx} at $x=a$, which is valid for all $x-a$. We differ-integrate term by term by order α with respect to cx and with the starting point ca, with the formula ${}_{ca}D_{cx}^{\alpha}\left[(cx-ca)^j\right]=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(cx-ca)^{j-\alpha}$ in the above series, and write the following steps

$$\begin{aligned}{}_{ca}D_{cx}^{\alpha}\left[e^{C-cx}\right] &= \frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(cx-ca)\right]^{\alpha}}=\frac{\mathrm{d}^{\alpha}}{\left[\mathrm{d}(cx-ca)\right]^{\alpha}}\left[e^{C-ca}\sum_{j=0}^{\infty}\frac{\left(-(cx-ca)\right)^j}{\Gamma(j+1)}\right]\\ &= e^{C-ca}\sum_{j=0}^{\infty}\frac{\mathrm{d}^{\alpha}}{\left[\mathrm{d}(cx-ca)\right]^{\alpha}}\left[\frac{\left((-1)^j(cx-ca)\right)^j}{\Gamma(j+1)}\right]\\ &= e^{C-ca}\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+1)}\left(\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(cx-ca)^{j-\alpha}\right)\\ &= e^{C-ca}\sum_{j=0}^{\infty}\frac{\left((-1)^j(cx-ca)^j(cx-ca)^{-\alpha}\right)}{\Gamma(j-\alpha+1)}\\ &= \left(c(x-a)\right)^{-\alpha}e^{C-ca}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j-\alpha+1)}\end{aligned} \tag{4.42}$$

The sum in (4.42) may be expressed as Tricomi's incomplete gamma function, which is the following:

$$\gamma^*(\alpha,x)=\frac{\alpha^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\mathrm{d}y=e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+\alpha+1)} \tag{4.43}$$

We take $\gamma^*\left(-\alpha,-c(x-a)\right)=e^{c(x-a)}\sum_{j=0}^{\infty}\frac{(-c(x-a))^j}{\Gamma(j-\alpha+1)}$, and use this in the following derivation

$$\begin{aligned}{}_{ca}D_{cx}^{\alpha}\left[e^{C-cx}\right] &= \frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(cx-ca)\right]^{\alpha}}=\frac{e^{C-ca}}{c^{\alpha}(x-a)^{\alpha}}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j-\alpha+1)}\\ &= \frac{e^{C-ca-cx+cx}}{c^{\alpha}(x-a)^{\alpha}}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j-\alpha+1)}\\ &= \frac{e^{C-cx+c(x-a)}}{c^{\alpha}(x-a)^{\alpha}}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j-\alpha+1)}\\ &= \left(e^{c(x-a)}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^j}{\Gamma(j-\alpha+1)}\right)\\ &= \frac{e^{C-cx}\gamma^*\left(-\alpha,-c(x-a)\right)}{c^{\alpha}(x-a)^{\alpha}}\end{aligned} \tag{4.44}$$

The scaling property that we evaluated earlier (in Section-3.20) gives the following formula:

$$\lambda^{\alpha}\frac{\mathrm{d}^{\alpha}\left[f(\lambda x)\right]}{\left[\mathrm{d}(\lambda x)\right]^{\alpha}}=\frac{\mathrm{d}^{\alpha}\left[f(\lambda x)\right]}{\left[\mathrm{d}x\right]^{\alpha}} \tag{4.45}$$

Using these above identities (4.45), we write the following steps:

$$
\begin{aligned}
\frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} &= c^{\alpha}\,\frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(cx-ca)\right]^{\alpha}} \\
&= c^{\alpha}\left(\left(c(x-a)\right)^{-\alpha}\, e^{C-ca}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^{j}}{\Gamma(j-\alpha+1)}\right) \\
&= \frac{e^{C-ca+cx-cx}}{(x-a)^{\alpha}}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^{j}}{\Gamma(j-\alpha+1)} \\
&= \frac{e^{C-cx}}{(x-a)^{\alpha}}\left(e^{cx-ca}\sum_{j=0}^{\infty}\frac{\left(-c(x-a)\right)^{j}}{\Gamma(j-\alpha+1)}\right) \\
&= \frac{e^{C-cx}}{(x-a)^{\alpha}}\,\gamma^{*}\left(-\alpha,-c(x-a)\right)
\end{aligned}
\tag{4.46}
$$

Finally, we have derived the following formula:

$$
{}_{a}D_{x}^{\alpha}\left[e^{C-cx}\right] = \frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{e^{C-cx}}{(x-a)^{\alpha}}\,\gamma^{*}\left(-\alpha,-c(x-a)\right) \tag{4.47}
$$

Take $\alpha = n$, $C = a = 0$, and $c = \mp 1$:

$$
{}_{0}D_{x}^{\alpha}\left[e^{\pm x}\right] = \frac{\mathrm{d}^{\alpha}\left[e^{\pm x}\right]}{\left[\mathrm{d}(x)\right]^{\alpha}} = \frac{e^{\pm x}}{x^{\alpha}}\,\gamma^{*}\left(-\alpha,\pm x\right) \tag{4.48}
$$

If n is a non-negative integer, then $\gamma^{*}(-n,x) = x^{n}$, and the above obtained result (4.48) reduces to a formula of the multiple differentiation of an exponential function for the integer order case below:

$$
\begin{aligned}
{}_{0}D_{x}^{n}\left[e^{\pm x}\right] &= \frac{\mathrm{d}^{n}\left[e^{\pm x}\right]}{\left[\mathrm{d}(x)\right]^{x}} = \frac{e^{\pm x}}{x^{n}}\,\gamma^{*}\left(-n,\pm x\right) \\
&= \frac{e^{\pm x}}{x^{n}}(\pm x)^{n} \\
&= (\pm 1)^{n} e^{\pm x} \qquad n > 0
\end{aligned}
\tag{4.49}
$$

The incomplete gamma function has an asymptotic expansion for large x and is given by the following expression:

$$
\frac{\gamma^{*}(-\alpha,x)}{x^{\alpha}} \sim 1 - \frac{e^{-x}}{x^{\alpha+1}\Gamma(-\alpha)}\left(1-\frac{\alpha+1}{x}+O(x^{-2})\right) \tag{4.50}
$$

The above formula (4.50) does not come directly, but results from the asymptotic expansion formula of the complementary incomplete gamma function $\Gamma(a,x)$, which is defined for $x \ge 0$ and $-\infty < a < \infty$, as an integral, that is $\Gamma(a,x) = \int_{x}^{\infty} e^{-t} t^{a-1}\mathrm{d}t$. Note that the lower limit is other than zero, as it was in the case of the Euler gamma function $\Gamma(a)$. The asymptotic expansion for large x is given by the following relationship:

$$
\Gamma(a,x) \cong x^{a-1}e^{-x}\left(1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^{2}}+\ldots\right) \tag{4.51}
$$

We use the identity $\gamma^{*}(a,x) = x^{-a}\left(1-\frac{\Gamma(a,x)}{\Gamma(a)}\right)$, and use the asymptotic expansion of the $\Gamma(a,x)$ (4.51) to write the following:

$$\gamma^*(a,x) = x^{-a}\left(1-\frac{e^{-x}x^{a-1}}{\Gamma(a)}\left(1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^2}+...\right)\right)$$

$$= x^{-a} - \frac{e^{-x}x^{-1}}{\Gamma(a)}\left(1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^2}+...\right) \tag{4.52}$$

$$= x^{-a} - \frac{e^{-x}}{\Gamma(a)}\left(\frac{1}{x}+\frac{a-1}{x^2}+\frac{(a-1)(a-2)}{x^3}+...\right)$$

Putting $a=-\alpha$, we get $\gamma^*(-\alpha,x)=x^{\alpha}-\frac{e^{-x}}{\Gamma(-\alpha)}\left(\frac{1}{x}+\frac{(-\alpha-1)}{x^2}+\frac{(-\alpha-1)(-\alpha-2)}{x^3}+...\right)$. From here, we write the expression $\frac{\gamma^*(-\alpha,x)}{x^{\alpha}} \sim 1-\frac{e^{-x}}{x^{\alpha+1}\Gamma(-\alpha)}\left(1-\frac{\alpha+1}{x}+O(x^{-2})\right)$, for large x i.e. $x\uparrow\infty$. We will use this asymptotic expansion formula in the following derivation. We have the formula below:

$$_aD_x^{\alpha}\left[e^{C-cx}\right]=\frac{\mathrm{d}^{\alpha}\left[e^{C-cx}\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\frac{e^{C-cx}}{(x-a)^{\alpha}}\gamma^*\left(-\alpha,-c(x-a)\right) \tag{4.53}$$

We write the following steps, by putting $C=0$, $a=-\infty$ and $c=-c$ in the above expression (4.53):

$$_{-\infty}D_x^{\alpha}\left[e^{cx}\right]=\frac{\mathrm{d}^{\alpha}\left[e^{cx}\right]}{\left[\mathrm{d}(x+\infty)\right]^{\alpha}}=\frac{e^{cx}}{(x+\infty)^{\alpha}}\gamma^*\left(-\alpha,c(x+\infty)\right)$$

$$=e^{cx}\lim_{y\uparrow\infty}\left(\frac{\gamma^*(-\alpha,cy)}{y^{\alpha}}\right) \qquad x+\infty=y$$

$$=e^{cx}\lim_{y\uparrow\infty}\left(c^{\alpha}\frac{\gamma^*(-\alpha,cy)}{(cy)^{\alpha}}\right)=c^{\alpha}e^{cx}\lim_{z\uparrow\infty}\left(\frac{\gamma^*(-\alpha,z)}{z^{\alpha}}\right) \tag{4.54}$$

Substituting $cy=z$

$$_{-\infty}D_x^{\alpha}\left[e^{cx}\right]=c^{\alpha}e^{cx}\left(1-\frac{e^{-z}}{z^{\alpha+1}\Gamma(-\alpha)}\left(1-\frac{\alpha+1}{z}+O(z^{-2})\right)\right)=c^{\alpha}e^{cx}$$

This simple result for differ-integration with a lower terminal limit is $a=-\infty$, as discussed earlier in Sections 3.4, 3.5 and 3.6, which is provided by Liouville, with postulates about fractional differ-integration of the exponential function. Functions other than exponentials, however, seldom yield finite results for the differ-integrations when the lower limit is pushed towards $-\infty$. Here, the function e^{C-cx} is analytic in $(x-a)$, and was subject to a term-by-term differ-integration process.

Let us evaluate the α-th differ-integral of the $f(x)=x^{\alpha-1}e^{-1/x}$. For $\alpha<0$ using the RL definition we write $_0D_x^{\alpha}\left[x^{\alpha-1}e^{-1/x}\right]=\frac{1}{\Gamma(-\alpha)}\int_0^x (x-y)^{-\alpha-1}y^{\alpha-1}e^{-1/y}\mathrm{d}y$ and, followed by the substitution $y=\frac{x}{xz+1}$, we get the following steps:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\frac{e^{-1/x}}{x^{1-\alpha}}\right]=\frac{1}{\Gamma(-\alpha)}\int_0^x\frac{y^{\alpha-1}e^{-1/y}}{(x-y)^{\alpha+1}}\mathrm{d}y$$

$$=\frac{1}{\Gamma(-\alpha)}\int_{\infty}^0\left(\frac{x}{xz+1}\right)^{\alpha-1}\frac{e^{-\left(z+\frac{1}{x}\right)}}{x^{\alpha+1}\left(\frac{xz}{xz+1}\right)^{\alpha+1}}\left(-\left(\frac{x}{xz+1}\right)^2\right)\mathrm{d}z \tag{4.55}$$

$$=\frac{e^{-1/x}}{x^{\alpha+1}\Gamma(-\alpha)}\int_0^{\infty}z^{-\alpha-1}e^{-z}\mathrm{d}z$$

Using the definition of the gamma function $\Gamma(n)=\int_0^{\infty}z^{n-1}e^{-z}\mathrm{d}z$, we write $\Gamma(-\alpha)=\int_0^{\infty}z^{-\alpha-1}e^{-z}\mathrm{d}z$, then the following formula:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\frac{e^{-1/x}}{x^{1-\alpha}}\right]=\frac{e^{-1/x}}{x^{\alpha+1}} \tag{4.56}$$

4.7 Fractional differ-integration for logarithmic functions

We conducted fractional integration as an analytic continuation of the repeated n-integer order integration of the function i.e. $\ln x$ earlier in the book (in Section-2.8). Here, we apply the Riemann-Liouville formula to achieve the same result that we obtained in that section. When $f(x)=\ln x$, we first find the Riemann-Liouville fractional integration of the order $0<\alpha<1$ with a change of the variable .i.e. $u=x(1-v)$. As such, we write the following steps:

$$\begin{aligned}
{}_0D_x^{-\alpha}\left[\ln x\right] = {}_0I_x^{\alpha}\left[\ln x\right] &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}\left(\ln(u)\right)\mathrm{d}u, \quad u=x(1-v) \\
&= \frac{1}{\Gamma(\alpha)}\int_1^0 \left(xv\right)^{\alpha-1}\left(\ln\left(x(1-v)\right)\right)\left(-x\mathrm{d}v\right) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{\alpha}v^{\alpha-1}\left(\ln x+\ln(1-v)\right)\mathrm{d}v \\
&= \frac{(\ln x)x^{\alpha}}{\Gamma(\alpha)}\int_0^1 v^{\alpha-1}\mathrm{d}v + \frac{x^{\alpha}}{\Gamma(\alpha)}\int_0^1 v^{\alpha-1}\left(\ln(1-v)\right)\mathrm{d}v
\end{aligned} \tag{4.57}$$

We write here $\mathrm{d}\left[(1-v^{\alpha})\right]=\left(-\alpha v^{\alpha-1}\right)\mathrm{d}v$ and write the second integral in the above obtained expression (4.57), i.e. $\int_0^1 v^{\alpha-1}\left(\ln(1-v)\right)\mathrm{d}v$ as $\int_0^1 v^{\alpha-1}\ln(1-v)\mathrm{d}v = -\frac{1}{\alpha}\int_0^1 \ln(1-v)\left(\mathrm{d}\left[(1-v^{\alpha})\right]\right)$, and substitute it to write the following process:

$$\begin{aligned}
{}_0I_x^{\alpha}\left[\ln x\right] &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\ln x - \frac{x^{\alpha}}{\alpha\left(\Gamma(\alpha)\right)}\int_0^1 \ln(1-v)\left(\mathrm{d}\left[(1-v^{\alpha})\right]\right) \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\ln x - \left((1-v^{\alpha})\ln(1-v)\Big|_{v=0}^{v=1} - \int_0^1 (1-v^{\alpha})\left(\frac{\mathrm{d}\left[\ln(1-v)\right]}{\mathrm{d}v}\right)\mathrm{d}v\right)\right) \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\ln x - \left((1-v^{\alpha})\ln(1-v)\Big|_{v=0}^{v=1} - \int_0^1 (1-v^{\alpha})\left(\frac{(-1)}{(1-v)}\right)\mathrm{d}v\right)\right) \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\ln x - \left((1-v^{\alpha})\ln(1-v)\right)\Big|_0^1 - \int_0^1 \frac{1-v^{\alpha}}{1-v}\mathrm{d}v\right)
\end{aligned} \tag{4.58}$$

By doing integration in parts, we obtained the last steps of (4.58).

Noting $\int_0^1 \frac{v^x-v^y}{1-v}\mathrm{d}v=\psi(y+1)-\psi(x+1)$, with $\mathrm{Re}[x],\mathrm{Re}[y]>-1$, with a 'psi' function defined as the derivative of the complete gamma function that is $\psi(x)=\frac{1}{\Gamma(x)}\frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x}$, or $\psi(x)=\frac{\mathrm{d}\left[\ln\Gamma(x)\right]}{\mathrm{d}x}=\frac{\Gamma^{(1)}(x)}{\Gamma(x)}$, we find that it obeys recursion $\psi(x+1)-\psi(x)=(x)^{-1}$ and $-\psi(1)=\gamma=0.5772156666....$ is Euler's constant (Section-1.16). As such, we write:

$${}_0D_x^{-\alpha}\left[\ln x\right] = {}_0I_x^{\alpha}\left[\ln x\right] = \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\ln x-\psi(\alpha+1)+\psi(1)\right) \tag{4.59}$$

We previously derived the same result differently in Section-2.8. Thus for $m-1\le \mathrm{Re}[\alpha]\le m$, and using the above obtained expression (4.59) for ${}_0D_x^{-\alpha}\left[\ln x\right]$, we write the following:

$$\begin{aligned}
{}_0D_x^{\alpha}\left[\ln x\right] &= D_x^m\left({}_0D_x^{-(m-\alpha)}\left[\ln x\right]\right) = \frac{\mathrm{d}^m\left[{}_0D_x^{-(m-\alpha)}\left[\ln x\right]\right]}{\mathrm{d}x^m} \\
&= D_x^m\left({}_0D_x^{-\upsilon}\left[\ln x\right]\right)
\end{aligned} \tag{4.60}$$

Letting $m-\alpha=\upsilon$ in the above (4.60) and using ${}_0D_x^{-\upsilon}\left[\ln x\right]=\frac{x^{\upsilon}}{\Gamma(\upsilon+1)}\left(\ln x-\psi(\upsilon+1)+\psi(1)\right)$, from (4.59) we have the following expression:

$$ {}_0D_x^{\alpha}\left[\ln x\right] = D^m\left[\frac{x^{\upsilon}}{\Gamma(\upsilon+1)}\left(\ln x - \psi(\upsilon+1)+\psi(1)\right)\right] \tag{4.61}$$

We introduce $\Psi \equiv \psi(1)-\psi(\upsilon+1)$ which is a constant and independent of the value of x. With this, we write ${}_0D_x^{\alpha}\left[\ln x\right] = \frac{1}{\Gamma(\upsilon+1)}D^m\left[x^{\upsilon}\left(\ln x+\Psi\right)\right]$. Now by using Leibniz's rule regarding the m-th derivative of the product of the two functions, i.e. $D^m\left[f(x)g(x)\right] = \sum_{k=0}^{m}\left({}^mC_k\right)\left(D_x^{m-k}\left[f(x)\right]\right)\left(D_x^k\left[g(x)\right]\right)$, we write the following expression:

$$ {}_0D_x^{\alpha}\left[\ln x\right] = \frac{1}{\Gamma(\upsilon+1)}\sum_{k=0}^{m}\binom{m}{k}\left(D_x^{m-k}\left[x^{\upsilon}\right]\right)\left(D_x^k\left[\ln x+\Psi\right]\right) \tag{4.62}$$

Now using $D^{m-k}x^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-m+k+1)}x^{\upsilon-m+k}$; for $k=0,1,....,m$ and the integer derivatives of the $\ln x+\Psi$ as follows:

$$ D^0\left[\ln x+\Psi\right] = \ln x+\Psi\,,\quad D^k\left[\ln x+\Psi\right] = \frac{(-1)^{k-1}(k-1)!}{x^k} \tag{4.63}$$

$$k = 1,2,3...,m$$

With the above expressions ((4.62) and (4.63)), we write the following steps:

$$ D^{\alpha}\left[\ln x\right] = \frac{1}{\Gamma(\upsilon+1)}\left(\begin{aligned} &\frac{\Gamma(\upsilon+1)x^{\upsilon-m}}{\Gamma(\upsilon-m+1)}\left(\ln x+\Psi\right)+\frac{m\Gamma(\upsilon+1)x^{\upsilon-m+1}}{\Gamma(\upsilon-m+2)}\left(\frac{1}{x}\right)+...\\ &...+\frac{m!\Gamma(\upsilon+1)x^{\upsilon-m+k}}{k!(m-k)!\Gamma(\upsilon-m+k+1)}\left(\frac{(-1)^{k-1}(k-1)!}{x^k}\right)+\\ &...+x^{\upsilon}\left(\frac{(-1)^{m-1}(m-1)!}{x^m}\right)\end{aligned}\right)$$

$$ = \left(\begin{aligned} &\frac{x^{\upsilon-m}}{\Gamma(\upsilon-m+1)}\left(\ln x+\Psi\right)+\frac{mx^{\upsilon-m}}{\Gamma(\upsilon-m+2)}+......\\ &+\frac{m!x^{\upsilon-m}\left((-1)^{k-1}(k-1)!\right)}{k!(m-k)!\Gamma(\upsilon-m+k+1)}+...+\frac{x^{\upsilon-m}\left((-1)^{m-1}(m-1)!\right)}{\Gamma(\upsilon+1)}\end{aligned}\right)$$

$$ = \left(\begin{aligned} &\frac{x^{\upsilon-m}\left(\ln x+\Psi\right)}{\Gamma(\upsilon-m+1)}-\frac{x^{\upsilon-m}(-1)m!\Gamma(\upsilon-m+1)}{\Gamma(\upsilon-m+1)\left((m-1)!\Gamma(\upsilon-m+2)\right)}-......\\ &-\frac{x^{\upsilon-m}m!\left((-1)^k\right)\Gamma(\upsilon-m+1)}{\Gamma(\upsilon-m+1)\left(k(m-k)!\Gamma(\upsilon-m+k+1)\right)}\\ &....-\frac{x^{\upsilon-m}\left((-1)^m(m-1)!\right)\Gamma(\upsilon-m+1)}{\Gamma(\upsilon-m+1)\Gamma(\upsilon+1)}\end{aligned}\right)$$

$$ = \frac{x^{\upsilon-m}}{\Gamma(\upsilon-m+1)}\left(\ln x+\Psi-\sum_{k=1}^{m}\frac{(-1)^k m!\Gamma(\upsilon-m+1)}{k(m-k)!\Gamma(\upsilon-m+k+1)}\right)$$

$$ = \frac{x^{\upsilon-m}}{\Gamma(\upsilon-m+1)}\left(\ln x+\psi(1)-\psi(\upsilon+1)-\sum_{k=1}^{m}\frac{(-1)^k m!\Gamma(\upsilon-m+1)}{k(m-k)!\Gamma(\upsilon-m+k+1)}\right) \tag{4.64}$$

Since $\upsilon = m-\alpha$, we have, as found above in (4.64), the following expression:

$$ D^{\alpha}\left[\ln x\right] = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\left(\ln x+\psi(1)-\psi(m-\alpha+1)-\sum_{k=1}^{m}\frac{(-1)^k m!\Gamma(1-\alpha)}{k(m-k)!\Gamma(k+1-\alpha)}\right) \tag{4.65}$$

We use the expression $\psi(x+1)-\psi(x+1+m) = \sum_{k=1}^{\infty}\frac{(-1)^k m!\Gamma(x+1)}{k(m-k)!\Gamma(x+1+k)}$ with $x > -(m+1)$, as discussed earlier in the book (Section-1.16), to write the following:

$$D^{\alpha}\left[\ln x\right]=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\Big(\ln x+\psi(1)-\psi(m+1-\alpha)-\big(\psi(-\alpha+1)-\psi(-\alpha+1+m)\big)\Big)$$

$$=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\big(\ln x+\psi(1)-\psi(1-\alpha)\big) \tag{4.66}$$

$$=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\big(\ln x-\gamma-\psi(1-\alpha)\big);\qquad -\psi(1)=\gamma$$

When α is a natural number (i.e. $\alpha=n\in\mathbb{N}$), the fractional derivative for the RHS of the above expression (4.66) should be interpreted as the limit for $\alpha\to n$. In fact, in $\lim_{\alpha\to n}\frac{\psi(1-\alpha)}{\Gamma(1-\alpha)}=(-1)^{-n}\Gamma(n)$, the rule $\frac{\mathrm{d}^n[\ln x]}{\mathrm{d}x^n}=-\big((n-1)!\big)\big(-x\big)^{-n}$ is fulfilled. That is

$$D^{n}\left[\ln x\right]=\lim_{\alpha\to n}x^{-\alpha}\left(\frac{\ln x}{\Gamma(1-\alpha)}+\frac{\psi(1)}{\Gamma(1-\alpha)}-\frac{\psi(1-\alpha)}{\Gamma(1-\alpha)}\right)$$

$$=\lim_{\alpha\to n}\left(-x^{-\alpha}\frac{\psi(1-\alpha)}{\Gamma(1-\alpha)}\right)=-x^{-n}(-1)^{n}\Gamma(n) \tag{4.67}$$

$$=-\big((n-1)!\big)(-x)^{-n}$$

The first two terms in the bracket of the first line of the steps above (4.67) reach zero as $\lim_{\alpha\to n}\frac{1}{\Gamma(1-\alpha)}=0$, as the reciprocal of the gamma function at points zero and negative integers is zero. However, for $\alpha=-n; n\in\mathbb{N}$ there holds the classical result, as in the following:

$$\begin{aligned}D^{-n}\left[\ln x\right]&=\frac{x^{n}}{\Gamma(1+n)}\big(\ln x+\psi(1)-\psi(1+n)\big)\\&=\frac{x^{n}}{n!}\left(\ln x-\sum_{k=1}^{n}\left(\tfrac{1}{k}\right)\right)\end{aligned} \tag{4.68}$$

This (4.68) is obtained by using $\psi(n+1)-\psi(1)=\sum_{k=1}^{n}\frac{1}{k}$ i.e. the properties of the 'psi' function (Section-1.16).

Now we do the same thing for $f(x)=\ln x$ with RL-formulation with α replaced by $-\alpha$. We first state that $\alpha<0$, before providing the reasoning behind the analytic continuation.

$$\frac{\mathrm{d}^{\alpha}\ln(x)}{\mathrm{d}x^{\alpha}}=\frac{1}{\Gamma(-\alpha)}\int_{0}^{x}\frac{\ln(y)\mathrm{d}y}{(x-y)^{\alpha+1}}\qquad \alpha<0\qquad \text{put}\qquad \upsilon=\frac{x-y}{x}$$

$$=\frac{x^{-\alpha}\ln(x)}{\Gamma(-\alpha)}\int_{0}^{1}\frac{\mathrm{d}\upsilon}{\upsilon^{\alpha+1}}+\frac{x^{-\alpha}}{\Gamma(-\alpha)}\int_{0}^{1}\frac{\ln(1-\upsilon)\mathrm{d}\upsilon}{\upsilon^{\alpha+1}} \tag{4.69}$$

$$=\frac{x^{-\alpha}\ln(x)}{\Gamma(-\alpha)}\left(-\frac{1}{\alpha}\right)+\frac{x^{-\alpha}}{\Gamma(-\alpha)}\int_{0}^{1}\frac{\ln(1-\upsilon)\mathrm{d}\upsilon}{\upsilon^{\alpha+1}}$$

Taking the second definite integral in (4.69) and doing integration by parts, we write the following using $\gamma=-\psi(1)=0.5772157...$, and $\int_0^1\frac{\left(\upsilon^{x}-\upsilon^{y}\right)}{1-\upsilon}\mathrm{d}\upsilon=\psi(y+1)-\psi(x+1)$, as shown earlier in Section-1.16. We get

$$\int_{0}^{1}\frac{\ln(1-\upsilon)\mathrm{d}\upsilon}{\upsilon^{\alpha+1}}=\frac{1}{\alpha}\int_{0}^{1}\ln(1-\upsilon)\left[\mathrm{d}(1-\upsilon^{-\alpha})\right]$$

$$=\left.\frac{\big(1-\upsilon^{-\alpha}\big)\ln(1-\upsilon)}{\alpha}\right|_{0}^{1}-\frac{1}{\alpha}\int_{0}^{1}\frac{1-\upsilon^{-\alpha}}{1-\upsilon}\mathrm{d}\upsilon \tag{4.70}$$

$$=0-\frac{\gamma+\psi(1-\alpha)}{\alpha}=-\frac{\gamma}{\alpha}-\frac{\psi(1-\alpha)}{\alpha}$$

We have used the 'psi' function of the argument $1-\alpha$ in the above (4.70). Combining the above result (4.70) together with a recurrence of the relationship of the gamma function i.e. $(-x)\big(\Gamma(-x)\big)=\Gamma(1-x)$, we write

$$\frac{d^\alpha \ln(x)}{dx^\alpha} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\big(\ln(x) - \gamma - \psi(1-\alpha)\big) \tag{4.71}$$

Though the result (4.71) is derived for $\alpha < 0$ (as in Section-2.8), the usual argument of the analytic-continuation says it is valid for all α.

4.8 Fractional differ-integration for some complicated functions described by the power series expansion

Let us take the function $f(x) = \frac{x^\alpha}{1-x}$ and differ-integrate by order α from the starting point or lower limit $a = 0$, in order to get ${}_0D_x^\alpha\left[\frac{x^\alpha}{1-x}\right]$. We use the binomial expansion formula (i.e. $(1-x)^{-1} = 1 + x + x^2 + ... = \sum_{j=0}^{\infty} x^j$) here and write the following

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{x^\alpha}{1-x}\right] = \sum_{j=0}^{\infty}\frac{d^\alpha\left[x^{j+\alpha}\right]}{dx^\alpha} \tag{4.72}$$

We subject the above expansion (4.72) to the condition that the magnitude of x is less than unity, that is $0 < x < 1$. If the condition $\alpha > -1$ is maintained, then we have the following. Using Euler's formula for fractional differentiation (as explored in Section-3.2.3), we write:

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{x^\alpha}{1-x}\right] = \sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}x^j \tag{4.73}$$

We have obtained the following from the property of the gamma function (in Sections 1.11 and 1.12) in terms of the binomial coefficients in q and then put $q = -\alpha - 1$, i.e.

$$\begin{aligned}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} &= (-1)^j\binom{q}{j} \qquad \text{with } q = -\alpha - 1 \\ \frac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)\Gamma(j+1)} &= (-1)^j\binom{-\alpha-1}{j}\end{aligned} \tag{4.74}$$

We manipulate $\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}$ using $\Gamma(\alpha+1)\left(\frac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)\Gamma(j+1)}\right)$, to write in terms of the binomial coefficients in (4.73) as follows:

$$\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)} = (-1)^j\binom{-\alpha-1}{j}\Gamma(\alpha+1) \tag{4.75}$$

By using the above obtained expression (4.75), we thus write the following expression:

$$\begin{aligned}\frac{d^\alpha}{dx^\alpha}\left[\frac{x^\alpha}{1-x}\right] &= \sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}x^j \\ &= \Gamma(\alpha+1)\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}(-x)^j\end{aligned} \tag{4.76}$$

Using the sum in (4.76) as in the following representation

$$\begin{aligned}\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}(-x)^j &= 1 - \frac{(-\alpha-1)!}{1!(-\alpha-2)!}x + \frac{(-\alpha-1)!}{2!(-\alpha-3)!}x^2 - ... \\ &= (1-x)^{-\alpha-1}\end{aligned} \tag{4.77}$$

we write the simple result as in the following form

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{x^\alpha}{1-x}\right] = \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}} \tag{4.78}$$

In the next function, we take $f(x) = \frac{x^\beta}{1-x}$, with a similar procedure to that used for derivation ${}_0D_x^\alpha\left[\frac{x^\alpha}{1-x}\right]$ (4.78). We write the following expression

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[\tfrac{x^{\beta}}{1-x}\right] &= \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\frac{x^{\beta}}{1-x}\right] \\
&= x^{\beta-\alpha}\sum_{j=0}^{\infty}\frac{\Gamma(j+\beta+1)}{\Gamma(j+\beta-\alpha+1)}x^{j}
\end{aligned}
\tag{4.79}
$$

with the restrictions $0<x<1$ and $\beta>-1$. To further manipulate the above (4.79) and get a closed form is difficult in this case as it requires a hyper-geometric function (refer to Appendix-A) and its relationship to the incomplete beta-function, that is $\left[x\,\frac{b}{c}\right]=\frac{\Gamma(1+b)}{\Gamma(c)}\frac{\mathrm{B}_x(c,1+b-c)}{x^{c}(1-x)^{1+b-c}}$. From this we write

$$
\frac{\mathrm{B}_x(c,1+b-c)}{(1-x)^{1+b-c}}=\frac{x^{c}\Gamma(c)}{\Gamma(1+b)}\left(\left[x\,\frac{b}{c}\right]\right)=\frac{x^{c}\Gamma(c)}{\Gamma(1+b)}\left(\sum_{j=0}^{\infty}x^{j}\,\frac{\Gamma(j+1+b)}{\Gamma(j+1+c)}\right)
\tag{4.80}
$$

Putting $c=p-q$ and $b=p$, we obtain

$$
\begin{aligned}
\frac{\mathrm{B}_x(p-q,1+q)}{(1-x)^{1+q}} &= \frac{x^{p-q}\Gamma(p-q)}{\Gamma(1+p)}\left(\left[x\,\frac{p}{p-q}\right]\right) \\
&= \frac{x^{p-q}\Gamma(p-q)}{\Gamma(1+p)}\left(\sum_{j=0}^{\infty}x^{j}\,\frac{\Gamma(j+1+p)}{\Gamma(j+1+p-q)}\right)
\end{aligned}
\tag{4.81}
$$

We use the above identity (4.81) and apply it in $\frac{\mathrm{d}^{\alpha}\left[x^{\beta}(1-x)^{-1}\right]}{\mathrm{d}x^{\beta}}=x^{\beta-\alpha}\sum_{j=0}^{\infty}x^{j}\left(\frac{\Gamma(j+\beta+1)}{\Gamma(j+\beta-\alpha+1)}\right)$ (4.79) to write the following steps:

$$
\begin{aligned}
\frac{\mathrm{d}^{\alpha}\left[x^{\beta}(1-x)^{-1}\right]}{\mathrm{d}x^{\alpha}} &= x^{\beta-\alpha}\sum_{j=0}^{\infty}x^{j}\,\frac{\Gamma\left(j+\beta+1\right)}{\Gamma\left(j+\beta-\alpha+1\right)} \\
&= x^{\beta-\alpha}\left(\frac{\mathrm{B}_x\left(\beta-\alpha,1+\alpha\right)}{(1-x)^{1+\alpha}}\,\frac{\Gamma\left(1+\beta\right)}{x^{\beta-\alpha}\Gamma\left(\beta-\alpha\right)}\right) \\
&= \frac{\left(\Gamma(1+\beta)\right)\left(\mathrm{B}_x\left(\beta-\alpha,1+\alpha\right)\right)}{\left(\Gamma(\beta-\alpha)\right)(1-x)^{1+\alpha}}
\end{aligned}
\tag{4.82}
$$

If we apply the operator ${}_0D_x^{-\alpha}$ on both sides of the above obtained expression (4.78) i.e. ${}_0D_x^{\alpha}\left[x^{\alpha}/(1-x)\right]=\Gamma(\alpha+1)\left((1-x)^{-(\alpha+1)}\right)$ and then letting the composition rule i.e. ${}_0D_x^{-\alpha}\left({}_0D_x^{\alpha}f(x)\right)=f(x)$, we write the following:

$$
\frac{x^{\alpha}}{1-x}=\Gamma(\alpha+1)\,\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}x^{-\alpha}}\left[(1-x)^{-\alpha-1}\right]
\tag{4.83}
$$

The above obtained relationship (4.83) for $\alpha<1$ follows from the fact that the function $x^{\alpha}/(1-x)$ is bound at $x=0$ and the condition $\alpha>-1$, was imposed while deriving the fractional differ-integration of the function $x^{\alpha}/(1-x)$. Rearrangement and reversal of the sign of the α in (4.83) gives us the following

$$
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[(1-x)^{\alpha-1}\right]=\frac{x^{-\alpha}}{(1-x)\Gamma(1-\alpha)}
\tag{4.84}
$$

with the condition $|\alpha|<1$. This above result (4.84) may be considered as a special case for the following

$$
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[(1-x)^{\beta}\right]=\frac{(1-x)^{\beta-\alpha}}{\Gamma(-\alpha)}\mathrm{B}_x\left(-\alpha,\alpha-\beta\right)
\tag{4.85}
$$

which we have derived in an earlier section (Section-4.5) for the binomial function.

We will use the identity of the incomplete beta-function i.e. $\mathrm{B}_x(c,1+b-c)=(1-x)^{1+b-c}x^{c}\,\frac{\Gamma(c)}{\Gamma(1+b)}\left(\sum_{j=0}^{\infty}x^{j}\,\frac{\Gamma(j+1+b)}{\Gamma(j+1+c)}\right)$, which comes from the hyper-geometric function

(Appendix-A), in the above expression (4.85), by putting $\beta=\alpha-1$, $c=-\alpha$, $1+b-c=1$, $b=c=-\alpha$, and using $\sum_{j=0}^{\infty} x^j=(1-x)^{-1}$ to write the following:

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[(1-x)^{\alpha-1}\right]&=\frac{(1-x)^{-1}}{\Gamma(-\alpha)}\mathrm{B}_x\left(-\alpha,1\right)\\&=\frac{(1-x)^{-1}}{\Gamma(-\alpha)}\left(\frac{x^{-\alpha}(1-x)\Gamma(-\alpha)}{\Gamma(1-\alpha)}\right)\sum_{j=0}^{\infty}x^j\\&=\frac{(1-x)^{-1}}{\Gamma(-\alpha)}\left(\frac{x^{-\alpha}(1-x)\Gamma(-\alpha)}{\Gamma(1-\alpha)}\right)\left(1-x\right)^{-1}\\&=\frac{x^{-\alpha}}{(1-x)\Gamma(1-\alpha)}\end{aligned}\tag{4.86}$$

That is what we obtained earlier (4.84) in this section too.

4.9 Fractional differ-integration of the hyperbolic and trigonometric function using the series expansion method

Here, we follow the series expansion method. First, take $f(x)=\sinh\left(\sqrt{x}\right)$ and write its series expansion as

$$f(x)=\sinh\left(\sqrt{x}\right)=\frac{x^{\frac{1}{2}}}{\Gamma(2)}+\frac{x^{\frac{3}{2}}}{\Gamma(4)}+\frac{x^{\frac{5}{2}}}{\Gamma(6)}+....=\sum_{j=0}^{\infty}\frac{x^{j+\frac{1}{2}}}{\Gamma(2j+2)}\tag{4.87}$$

Using this series, we conduct term-by-term differ-integration and write the following:

$$\begin{aligned}{}_0D_x^\alpha\left[\sinh\left(\sqrt{x}\right)\right]&=\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\sinh\left(\sqrt{x}\right)\\&=\sum_{j=0}^{\infty}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[\frac{x^{j+\frac{1}{2}}}{\Gamma(2j+2)}\right]\\&=\sum_{j=0}^{\infty}\frac{\Gamma\left(j+\frac{3}{2}\right)}{\Gamma(2j+2)\Gamma\left(j+\frac{3}{2}-\alpha\right)}x^{j+\frac{1}{2}-\alpha}\end{aligned}\tag{4.88}$$

We use the simplification $\frac{\Gamma\left(j+\frac{3}{2}\right)}{\Gamma(2j+2)}=\frac{\sqrt{\pi}}{2^{2j+1}\Gamma(j+1)}$, which is a consequence of the duplication property of the Gamma function (as discussed in Section-1.10.8). Using this, we write

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\sinh\left(\sqrt{x}\right)&=\sum_{j=0}^{\infty}\frac{\Gamma\left(j+\frac{3}{2}\right)}{\Gamma(2j+2)\Gamma\left(j+\frac{3}{2}-\alpha\right)}x^{j+\frac{1}{2}-\alpha}\\&=\frac{\sqrt{\pi}}{2}x^{\left(\frac{1}{2}\right)-\alpha}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}x\right)^j}{\Gamma\left(j-\alpha+\frac{3}{2}\right)\Gamma(j+1)}\\&=\frac{1}{2}\sqrt{\pi}\left(2\sqrt{x}\right)^{\left(\frac{1}{2}\right)-\alpha}\left(I_{\left(\frac{1}{2}\right)-\alpha}\left(\sqrt{x}\right)\right)\end{aligned}\tag{4.89}$$

where $I_{\frac{1}{2}-\alpha}\left(\sqrt{x}\right)$ denotes the $\left(\frac{1}{2}-\alpha\right)$-th order hyperbolic Bessel function of the argument $\sqrt{x}$. This comes from a generalization of Rayleigh's formula. The differ-integration of the $f(x)=\sin\left(\sqrt{x}\right)$ is expressed in a similar way, as in

$${}_0D_x^\alpha\left[\sin\left(\sqrt{x}\right)\right]=\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[\sin\left(\sqrt{x}\right)\right]=\frac{1}{2}\sqrt{\pi}\left(2\sqrt{x}\right)^{\frac{1}{2}-\alpha}\left(J_{\frac{1}{2}-\alpha}\left(\sqrt{x}\right)\right)\tag{4.90}$$

where $J_{\frac{1}{2}-\alpha}\left(\sqrt{x}\right)$ denotes the $\left(\frac{1}{2}-\alpha\right)$-th order Bessel function of the first kind of the argument $\sqrt{x}$. This also comes from a generalization of Rayleigh's formula.

4.10 Fractional differ-integration of the Bessel function using the series expansion method

We will first conduct differ-integration of the $f(x)=x^{(\upsilon/2)}J_\upsilon\left(2\sqrt{x}\right)$. The υ-th order Bessel function of the argument $2\sqrt{x}$ is represented via the series as follows

$$J_\upsilon\left(2\sqrt{x}\right)=x^{(\upsilon/2)}\sum_{j=0}^{\infty}\frac{(-1)^j(x)^j}{\Gamma(j+1)\Gamma(j+\upsilon+1)} \tag{4.91}$$

Therefore, we have from (4.91) the following expression:

$$f(x)=x^{(\upsilon/2)}J_\upsilon\left(2\sqrt{x}\right)=\sum_{j=0}^{\infty}\frac{(-1)^j x^{j+\upsilon}}{\Gamma(j+1)\Gamma(j+\upsilon+1)} \tag{4.92}$$

(4.92) represents a differ-integrable function with condition $\upsilon>-1$. Below, we conduct term-by-term differ-integration and write the following expression

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[x^{\upsilon/2}J_\upsilon\left(2\sqrt{x}\right)\right]&=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(j+\upsilon+1)x^{j+\upsilon-\alpha}}{\Gamma(j+1)\Gamma(j+\upsilon+1)\Gamma(j+\upsilon+1-\alpha)}\\&=\sum_{j=0}^{\infty}\frac{(-1)^j x^{j+\upsilon-\alpha}}{\Gamma(j+1)\Gamma(j+\upsilon-\alpha+1)}\\&=x^{\left(\frac{\upsilon-\alpha}{2}\right)}J_{\upsilon-\alpha}\left(2\sqrt{x}\right)\end{aligned} \tag{4.93}$$

for $f(x)=x^{\upsilon/2}I_\upsilon\left(2\sqrt{x}\right)$, where $I_\upsilon\left(2\sqrt{x}\right)$ is modified by the Bessel function, and is represented as the series

$$I_\upsilon\left(2\sqrt{x}\right)=x^{(\upsilon/2)}\sum_{j=0}^{\infty}\frac{(x)^j}{\Gamma(j+1)\Gamma(j+\upsilon+1)} \tag{4.94}$$

The difference between I_υ and J_υ is that, in the modified Bessel function, the alternate sign change in infinite summation does not exist. Carrying out the steps of the term-by-term differ-integration, we obtain the following

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[x^{\upsilon/2}I_\upsilon\left(2\sqrt{x}\right)\right]&=\sum_{j=0}^{\infty}\frac{\Gamma(j+\upsilon+1)x^{j+\upsilon-\alpha}}{\Gamma(j+1)\Gamma(j+\upsilon+1)\Gamma(j+\upsilon+1-\alpha)}\\&=\sum_{j=0}^{\infty}\frac{x^{j+\upsilon-\alpha}}{\Gamma(j+1)\Gamma(j+\upsilon-\alpha+1)}\\&=x^{\left(\frac{\upsilon-\alpha}{2}\right)}I_{\upsilon-\alpha}\left(2\sqrt{x}\right)\end{aligned} \tag{4.95}$$

Setting $y=2\sqrt{x}$, or $x=\frac{y^2}{4}$; giving $\mathrm{d}x=\frac{1}{2}\left(y\right)\mathrm{d}y$, we get the following result

$$\frac{\mathrm{d}^\alpha}{\left[y\mathrm{d}y\right]^\alpha}\left[y^\upsilon J_\upsilon(y)\right]=y^{\upsilon-\alpha}J_{\upsilon-\alpha}(y) \tag{4.96}$$

4.11 Fractional differ-integration of the distribution functions using the Riemann-Liouville formula

4.11.1 Uniform distribution function and its fractional differ-integration

An example of the uniform distribution function is the Heaviside unit step function. The Heaviside unit step function is described as

$$u(x-x_0)=\begin{cases}0 & x<x_0\\ 1 & x>x_0\end{cases} \tag{4.97}$$

The differ-integration of the above is piecewise defined as the function for $\alpha<0$, and with $a<x_0<x$ via the Riemann-Liouville formula it is

$$
\begin{aligned}
{}_aD_x^{\alpha}\left[u(x-x_0)\right] &= \frac{\mathrm{d}^{\alpha}\left[u(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{1}{\Gamma(-\alpha)}\int_a^x \frac{\left(u(y-x_0)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}, \qquad \alpha<0 \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^{x_0}\frac{(0)\mathrm{d}y}{(x-y)^{\alpha+1}} + \frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}\frac{(1)\mathrm{d}y}{(x-y)^{\alpha+1}} \\
&= 0 + \frac{\mathrm{d}^{\alpha}\left[1\right]}{\left[\mathrm{d}(x-x_0)\right]^{\alpha}} \qquad x>x_0 \\
&= \frac{(x-x_0)^{-\alpha}}{\Gamma(1-\alpha)} \qquad x>x_0
\end{aligned}
\tag{4.98}
$$

Here the results of the earlier discussed differ-integration of the constant function of the unity, i.e. ${}_aD_x^{\alpha}\left[1\right] = \frac{\mathrm{d}^{\alpha}[1]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$ (as in Section-4.2), are used. The above expression (4.98) applies for all α. One use of the Heaviside unit step function is to delimit the range of the definition of the function so that we have the product:

$$
f(x)u(x-x_0) = \begin{cases} 0 & x<x_0 \\ f(x) & x>x_0 \end{cases} \tag{4.99}
$$

By analogy of the above derivation, we can write:

$$
\begin{aligned}
{}_aD_x^{\alpha}\left[f(x)u(x-x_0)\right] &= \frac{\mathrm{d}^{\alpha}\left[f(x)u(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} \\
&= u(x-x_0)\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-x_0)\right]^{\alpha}}, \quad a<x_0<x
\end{aligned}
\tag{4.100}
$$

4.11.2 The delta distribution function and its fractional differ-integration

The delta function (distribution) is also termed as the Dirac delta function, and is often used in mathematical physics to state or describe a point source (a monopole). The integer order derivative of a delta function returns two delta functions in opposite directions placed at the same point. This is a dipole placed at that point, so the delta function is a point source or a monopole that moves to a dipole placed at that point, after doing one whole differentiation.
We have a property of the delta function as follows

$$
\int_b^d \left(\delta(x-c)\right)\mathrm{d}x = 1 \qquad \int_b^d \left(\delta(x-c)\right)\left(f(x)\right)\mathrm{d}x = f(c) \qquad b\le c\le d \tag{4.101}
$$

We will use this property for a function $f(x)$ defined in the interval $\left[b,d\right]$ while integrating that function in the interval after multiplying it by a delta function residing at $x=c$, returning the value of the function at that point, $f(c)$.

We write the Riemann-Liouville formulation, in order to get a fractional derivative of the order α of the delta function, as follows:

$$
{}_{-\infty}D_x^{\alpha}\left[\delta(x-c)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{-\infty}^{x}\left(\delta(y-c)\right)(x-y)^{n-\alpha-1}\mathrm{d}y \tag{4.102}
$$

We have chosen a positive integer n just greater than the real number α, before conducting fractional integration of the order $(n-\alpha)$ followed up by whole differentiation of the integer order n. This is what is contained in the above formulation. For $0<\alpha<1$, we have $n=1$; the above formulation becomes

$$
{}_{-\infty}D_x^{\alpha}\left[\delta(x-c)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x}\left(\delta(y-c)\right)(x-y)^{-\alpha}\mathrm{d}y \tag{4.103}
$$

Now we use the property of the delta function (4.101) i.e. $\int \delta(y-c)\left((x-y)^{-\alpha}\right)\mathrm{d}y = (x-c)^{-\alpha}$, and write the following steps

$$
\begin{aligned}
{}_{-\infty}D_x^{\alpha}\left[\delta(x-c)\right] &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}(x-c)^{-\alpha} \\
&= \frac{(-\alpha)(x-c)^{-\alpha-1}}{\Gamma(1-\alpha)} \\
&= \frac{(x-c)^{-\alpha-1}}{\Gamma(-\alpha)}
\end{aligned}
\tag{4.104}
$$

We have used the property of the gamma function, which is $\Gamma(-\alpha+1) = -\alpha\left(\Gamma(-\alpha)\right)$. For $\alpha = \frac{1}{2}$ and $c = 0$, we write the semi-derivative of the delta function as follows with the value $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

$$
{}_{-\infty}D_x^{1/2}\left[\delta(x)\right] = \frac{x^{-3/2}}{\Gamma\left(-\frac{1}{2}\right)} = -\frac{1}{2\sqrt{\pi}}x^{-3/2} \tag{4.105}
$$

We may state that by conducting a semi-derivative of the delta function, we are getting a 'fractional pole' which is in between a monopole and a dipole.

Now we try an approach similar to that which we used for the Heaviside unit step function in order to obtain the fractional differ-integral of the delta function, $f(x) = \delta(x-x_0)$, by means of the property we reproduce again, that is $\int_a^x\left(\delta(y-x_0)\right)\left(f(y)\right)\mathrm{d}y = f(x_0)$ where $a < x_0 < x$, for any function $f(x)$. After selecting $f(x) = (x-y)^{-\alpha-1}$, we write the following relationship:

$$
\int_a^x \frac{\left(\delta(y-x_0)\right)\mathrm{d}y}{(x-y)^{\alpha+1}} = (x-x_0)^{-\alpha-1} \qquad \alpha < 0 \tag{4.106}
$$

Dividing the above by $\frac{1}{\Gamma(-\alpha)}$, we recognize that the LHS of the above (4.106) is an α-th differ-integral of the Dirac delta function, that is

$$
\frac{\mathrm{d}^{\alpha}\left[\delta(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{1}{\Gamma(-\alpha)}\int_a^x \frac{\left(\delta(y-x_0)\right)\mathrm{d}y}{(x-y)^{\alpha+1}} = \frac{(x-x_0)^{-\alpha-1}}{\Gamma(-\alpha)} \tag{4.107}
$$

The formula (4.107) can be extended for all α.

4.11.3 The relationship between uniform distribution and the delta distribution function

Now comparing the earlier derived result, i.e. $\frac{\mathrm{d}^{\alpha}\left[u(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{(x-x_0)^{-\alpha}}{\Gamma(1-\alpha)}$ (4.98), with $\frac{\mathrm{d}^{\alpha}\left[\delta(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} = \frac{(x-x_0)^{-\alpha-1}}{\Gamma(-\alpha)}$ (4.107), we notice the following useful relationships

$$
\begin{aligned}
&\frac{\mathrm{d}^{\alpha+1}\left[u(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha+1}} = \frac{\mathrm{d}^{\alpha}\left[\delta(x-x_0)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}} \\
a = 0 \qquad &\frac{\mathrm{d}^{\alpha+1}}{\mathrm{d}x^{\alpha+1}}\left[u(x-x_0)\right] = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\delta(x-x_0)\right]
\end{aligned}
\tag{4.108}
$$

Taking $\alpha = 0$, we get the well-known relationship for $a = 0$, that is

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[u(x-x_0)\right] = \delta(x-x_0) \tag{4.109}
$$

The one whole derivative of the Heaviside unit step function is the Dirac delta function.

4.12 Fractional differ-integration of the saw-tooth function

First, we take the general piece-wise function and find its α-th differ-integration. Consider the following function

$$
f(x) = \begin{cases} f_1(x) & a \le x < x_1 \\ f_2(x) & x_1 < x \end{cases} \tag{4.110}
$$

With the use of the Heaviside unit step function, we write

$$\begin{aligned} f(x) &= f_1(x)u(x_1 - x) + f_2(x)u(x - x_1) \\ &= \left(f_1(x) - f_1(x)u(x - x_1)\right) + f_2(x)u(x - x_1) \\ &= f_1(x) + \left(f_2(x) - f_1(x)\right)u(x - x_1) \end{aligned} \tag{4.111}$$

since $u(x - x_1) + u(x_1 - x) = 1$ for $x \neq x_1$. After applying the linearity and the result of the Heaviside unit step function (4.100) i.e. $\frac{\mathrm{d}^\alpha[f(x)u(x-x_0)]}{[\mathrm{d}(x-a)]^\alpha} = u(x - x_0)\left(\frac{\mathrm{d}^\alpha[f(x)]}{[\mathrm{d}(x-x_0)]^\alpha}\right)$, we establish the following:

$$\frac{\mathrm{d}^\alpha\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \frac{\mathrm{d}^\alpha\left[f_1(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} + u(x - x_1)\frac{\mathrm{d}^\alpha\left[f_2(x) - f_1(x)\right]}{\left[\mathrm{d}(x - x_1)\right]^\alpha} \qquad x \neq x_1 \tag{4.112}$$

Using the above logic (4.112), we find several piece-wise sections and define a function

$$\begin{aligned} f(x) &= f_k(x) \qquad x_{k-1} < x < x_k \qquad k = 1,2,3,...,N; \quad k = 1, \quad x_0 \equiv a \\ &= f_1(x) + u(x - x_1)\left(f_2(x) - f_1(x)\right) + ... \\ &\quad + u(x - x_{N-1})\left(f_N(x) - f_{N-1}(x)\right) \qquad x \neq x_k \end{aligned} \tag{4 .113}$$

where as in (4.113), $x_0 \equiv a$.

The general differ-integral result is the following for the above function:

$$\frac{\mathrm{d}^\alpha\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \frac{\mathrm{d}^\alpha\left[f_1(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} + \sum_{k=1}^{N-1} u(x - x_k)\left(\frac{\mathrm{d}^\alpha\left[f_{k+1}(x) - f_k(x)\right]}{\left[\mathrm{d}(x - x_k)\right]^\alpha}\right) \tag{4.114}$$

We express the saw-tooth function as:

$$\begin{aligned} f_{\mathrm{SAW}}(x) &= (-1)^k(2k - x - 2) \\ &= x + \sum_{k=1}^{N-1}(-1)^k(2x - 4k + 2)u(x - 2k + 1) \end{aligned} \tag{4.115}$$

$$k = 1,2,3,...,N, \quad (2k–3)<x<(2k–1)$$

The application of the formula (4.114), i.e. $\frac{\mathrm{d}^\alpha[f(x)]}{[\mathrm{d}(x-a)]^\alpha} = \frac{\mathrm{d}^\alpha[f_1(x)]}{[\mathrm{d}(x-a)]^\alpha} + \sum_{k=1}^{N-1} u(x - x_k)\frac{\mathrm{d}^\alpha[f_{k+1}(x) - f_k(x)]}{[\mathrm{d}(x-x_k)]^\alpha}$, with a lower limit as $a = 0$, yields the following expression:

$$\begin{aligned} \frac{\mathrm{d}^\alpha\left[f_{\mathrm{SAW}}(x)\right]}{\mathrm{d}x^\alpha} &= \frac{\mathrm{d}^\alpha\left[x\right]}{\mathrm{d}x^\alpha} + \sum_{k=1}^{N-1} u(x - 2k + 1)\left(\frac{\mathrm{d}^\alpha\left[(-1)^k(2x - 4k + 2)\right]}{\left[\mathrm{d}(x - 2k + 1)\right]^\alpha}\right) \\ &= \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + 2\sum_{k=1}^{N-1} u(x - 2k + 1)\left((-1)^k\frac{(x - 2k + 1)^{1-\alpha}}{\Gamma(2-\alpha)}\right) \end{aligned} \tag{4.116}$$

4.13 Fractional differ-integration of the generalized periodic function

The function $f_{\mathrm{PER}}(t)$, which can be expressed as the sum of the complex conjugated exponential functions is described as $f_{\mathrm{PER}}(t) = \sum_{k=1}^\infty A_k \exp\left(i\frac{2\pi kt}{T}\right) + \sum_{k=1}^\infty \bar{A}_k \exp\left(-i\frac{2\pi kt}{T}\right)$, where A_k and $\bar{A}_k$ are complex-conjugate constants and T is the period of the function. Using the scaling property, that is $\frac{\mathrm{d}^\alpha[f(cx)]}{\mathrm{d}x^\alpha} = c^\alpha\frac{\mathrm{d}^\alpha[f(cx)]}{[\mathrm{d}(cx)]^\alpha}$, and the differ-integration of the exponential function, i.e. $\frac{\mathrm{d}^\alpha[\exp(cx)]}{[\mathrm{d}(cx)]^\alpha} = \left(cx\right)^{-\alpha}\exp(cx)\left(\gamma^*(-\alpha, cx)\right)$, both discussed earlier (in Section-3.20), we write the differ-integration expression for the exponential as follows with $c = \pm\frac{2\pi ik}{T}$ and $x \equiv t$

$$\frac{\mathrm{d}^{\alpha}\left[\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\right]}{\mathrm{d}t^{\alpha}}=\left(\pm i\,\frac{2\pi k}{T}\right)^{\alpha}\frac{\mathrm{d}^{\alpha}\left[\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\right]}{\left[\mathrm{d}\left(\pm\,\frac{2\pi ikt}{T}\right)\right]^{\alpha}}$$
$$=\left(\pm i\,\frac{2\pi ik}{T}\right)^{\alpha}\left(\pm i\,\frac{2\pi kt}{T}\right)^{-\alpha}\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\left(\gamma^{*}\left(-\alpha,\pm i\,\frac{2\pi kt}{T}\right)\right) \tag{4.117}$$
$$=t^{-\alpha}\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\left(\gamma^{*}\left(-\alpha,\pm i\,\frac{2\pi kt}{T}\right)\right)$$

Putting this (4.117) into the expression of the $f_{\mathrm{PER}}(t)$, that is

$$f_{\mathrm{PER}}(t)=\sum_{k=1}^{\infty}\left(A_k\exp\left(i\,\frac{2\pi kt}{T}\right)+\bar{A}_k\exp\left(-i\,\frac{2\pi kt}{T}\right)\right) \tag{4.118}$$

gives us a complete description of the fractional differ-integration of a periodic function which is

$$\frac{\mathrm{d}^{\alpha}\left[f_{\mathrm{PER}}(t)\right]}{\mathrm{d}t^{\alpha}}=\sum_{k=1}^{\infty}A_k t^{-\alpha}\exp\left(i\,\frac{2\pi kt}{T}\right)\gamma^{*}\left(-\alpha,i\,\frac{2\pi kt}{T}\right)$$
$$+\sum_{k=1}^{\infty}\bar{A}_k t^{-\alpha}\exp\left(-i\,\frac{2\pi kt}{T}\right)\gamma^{*}\left(-\alpha,-i\,\frac{2\pi kt}{T}\right) \tag{4.119}$$

For small values of x, and by using the series expansion of Tricomi's incomplete gamma function, that is $\lim_{x\downarrow 0}\gamma^{*}(-\alpha,x)=\exp(-x)\left(\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j-\alpha+1)}\right)$, we write the following steps:

$$\frac{\mathrm{d}^{\alpha}\left[\exp(\pm x)\right]}{\mathrm{d}x^{\alpha}}=x^{-\alpha}e^{\pm x}\gamma^{*}(-\alpha,\pm x)$$
$$=x^{-\alpha}e^{\pm x}\left(e^{\mp x}\sum_{j=0}^{\infty}\frac{\pm x^j}{\Gamma(j-\alpha+1)}\right) \tag{4.119}$$
$$=x^{-\alpha}\sum_{j=0}^{\infty}\frac{\pm x^j}{\Gamma(j-\alpha+1)}$$

Therefore, for small t, we write $\lim_{t\downarrow 0}\frac{\mathrm{d}^{\alpha}\left[\exp\left(\pm i\frac{2\pi kt}{T}\right)\right]}{\mathrm{d}t^{\alpha}}=t^{-\alpha}\sum_{j=0}^{\infty}\frac{\left(\pm i\frac{2\pi kt}{T}\right)^j}{\Gamma(j-\alpha+1)}$, and elaborating upon this we have the following form:

$$\lim_{t\downarrow 0}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\right]=t^{-\alpha}\left(\sum_{j=0}^{\infty}\frac{\left(\pm i\,\frac{2\pi kt}{T}\right)^j}{\Gamma(j-\alpha+1)}\right)$$
$$=t^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}+\frac{\left(\pm i\,\frac{2\pi kt}{T}\right)}{\Gamma(2-\alpha)}+\frac{\left(\pm i\,\frac{2\pi kt}{T}\right)^2}{\Gamma(3-\alpha)}+\ldots\right) \tag{4.120}$$
$$=\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}+\frac{\left(\pm i\,\frac{2\pi k}{T}\right)}{\Gamma(2-\alpha)}t^{1-\alpha}+\frac{\left(\pm i\,\frac{2\pi k}{T}\right)^2}{\Gamma(3-\alpha)}t^{2-\alpha}+\ldots$$

Therefore, in the limit of the small t, i.e. $t\downarrow 0$, the fractional differ-integration of the periodic function is given in the following steps

$$\lim_{t\downarrow 0}\frac{\mathrm{d}^{\alpha}\left[f_{\mathrm{PER}}(t)\right]}{\mathrm{d}t^{\alpha}}=\sum_{k=1}^{\infty}A_k t^{-\alpha}\sum_{j=0}^{\infty}\frac{\left(i\,\frac{2\pi kt}{T}\right)^{j}}{\Gamma(j-\alpha+1)}+\sum_{k=1}^{\infty}\bar{A}_k t^{-\alpha}\sum_{j=0}^{\infty}\frac{\left(-i\,\frac{2\pi kt}{T}\right)^{j}}{\Gamma(j-\alpha+1)}$$

$$=\sum_{k=1}^{\infty}A_k t^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}+\frac{\left(i\,\frac{2\pi kt}{T}\right)}{\Gamma(2-\alpha)}+\frac{\left(i\,\frac{2\pi kt}{T}\right)^{2}}{\Gamma(3-\alpha)}+\ldots\right)$$

$$+\sum_{k=1}^{\infty}\bar{A}_k t^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}-\frac{\left(i\,\frac{2\pi kt}{T}\right)}{\Gamma(2-\alpha)}+\frac{\left(i\,\frac{2\pi kt}{T}\right)^{2}}{\Gamma(3-\alpha)}+\ldots\right) \tag{4.121}$$

$$=\sum_{k=1}^{\infty}\tfrac{\left(A_k+\bar{A}_k\right)}{\Gamma(1-\alpha)}t^{-\alpha}+\sum_{k=1}^{\infty}\left(i\,\tfrac{2\pi k}{T}\right)\tfrac{\left(A_k-\bar{A}_k\right)}{\Gamma(2-\alpha)}t^{1-\alpha}+\sum_{k=1}^{\infty}\left(i\,\tfrac{2\pi k}{T}\right)^{2}\tfrac{\left(A_k+\bar{A}_k\right)}{\Gamma(3-\alpha)}t^{2-\alpha}+\ldots$$

$$=\sum_{k=1}^{\infty}\tfrac{\left(A_k+\bar{A}_k\right)}{\Gamma(1-\alpha)}t^{-\alpha}+\sum_{k=1}^{\infty}\left(\tfrac{2\pi k}{T}\right)\tfrac{i\left(A_k-\bar{A}_k\right)}{\Gamma(2-\alpha)}t^{1-\alpha}-\sum_{k=1}^{\infty}\left(\tfrac{2\pi k}{T}\right)^{2}\tfrac{\left(A_k+\bar{A}_k\right)}{\Gamma(3-\alpha)}t^{2-\alpha}+\ldots$$

Notice that the coefficients $\left(A_k+\bar{A}_k\right)$ and $i\left(A_k-\bar{A}_k\right)$ in (4.121) are real numbers. The leading term of the above expression (4.121) is the differ-integral of the initial value of the periodic function $f_{\mathrm{PER}}(t)\big|_{t=0}=\sum_{k=1}^{\infty}\left(A_k+\bar{A}_k\right)$, that is, we say $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[f_{\mathrm{PER}}(t)\big|_{t=0}\right]=\sum_{k=1}^{\infty}t^{-\alpha}\left(\frac{A_k+\bar{A}_k}{\Gamma(1-\alpha)}\right)$.

The asymptotic formulation for large x for Tricomi's incomplete gamma function gives the expression $\lim_{x\uparrow\infty}\gamma^{*}(-\alpha,x)=x^{\alpha}-\frac{e^{-x}}{\Gamma(-\alpha)}\left(\frac{1}{x}+\frac{(-\alpha-1)}{x^{2}}+\frac{(-\alpha-1)(-\alpha-2)}{x^{3}}+\ldots\right)$, which we use in the following derivation:

$$\lim_{t\uparrow\infty}\frac{\mathrm{d}^{\alpha}\left[\exp\left(\pm i\,\frac{2\pi kt}{T}\right)\right]}{\mathrm{d}t^{\alpha}}=\lim_{t\uparrow\infty}\left(t^{-\alpha}e^{\pm i\frac{2\pi kt}{T}}\gamma^{*}\left(-\alpha,\pm i\,\frac{2\pi kt}{T}\right)\right)$$

$$=t^{-\alpha}e^{\pm i\frac{2\pi kt}{T}}\left(\begin{array}{l}\left(\pm i\,\frac{2\pi kt}{T}\right)^{\alpha}\\ -\dfrac{e^{\mp i\frac{2\pi kt}{T}}}{\Gamma(-\alpha)}\left(\begin{array}{l}\left(\pm i\,\frac{2\pi kt}{T}\right)^{-1}+(-\alpha-1)\left(\pm i\,\frac{2\pi kt}{T}\right)^{-2}+\ldots\\ +(-\alpha-1)(-\alpha-2)\left(\pm i\,\frac{2\pi kt}{T}\right)^{-3}+\ldots\end{array}\right)\end{array}\right)$$

$$=t^{-\alpha}e^{\pm i\frac{2\pi kt}{T}}\left(\begin{array}{l}\left(\pm i\,\frac{2\pi kt}{T}\right)^{\alpha}\\ -e^{\mp i\frac{2\pi kt}{T}}\left(\begin{array}{l}\dfrac{1}{\Gamma(-\alpha)}\left(\pm i\,\frac{2\pi kt}{T}\right)^{-1}+\dfrac{(-\alpha-1)}{\Gamma(-\alpha)}\left(\pm i\,\frac{2\pi kt}{T}\right)^{-2}+\ldots\\ +\dfrac{(-\alpha-1)(-\alpha-2)}{\Gamma(-\alpha)}\left(\pm i\,\frac{2\pi kt}{T}\right)^{-3}+\ldots\end{array}\right)\end{array}\right)$$

$$=\left(\pm i\,\tfrac{2\pi k}{T}\right)^{\alpha}e^{\pm i\frac{2\pi kt}{T}}-\left(\left(\begin{array}{l}\dfrac{\left(\pm i\,\frac{2\pi kt}{T}\right)^{-1}t^{-\alpha}}{\Gamma(-\alpha)}+\dfrac{\left(\pm i\,\frac{2\pi kt}{T}\right)^{-2}t^{-\alpha}}{\Gamma(-\alpha-1)}\\ +\dfrac{\left(\pm i\,\frac{2\pi kt}{T}\right)^{-3}t^{-\alpha}}{\Gamma(-\alpha-2)}\end{array}\right)+\ldots\right)$$

$$=\left(\pm i\,\tfrac{2\pi k}{T}\right)^{\alpha}\exp\left(\pm i\,\tfrac{2\pi kt}{T}\right)-\sum_{j=0}^{\infty}t^{-\alpha}\left(\frac{\left(\pm i\frac{2\pi kt}{T}\right)^{-1-j}}{\Gamma(-\alpha-j)}\right) \tag{4.122}$$

In the above (4.122), we have used $(n-1)\left(\Gamma(n-1)\right)=\Gamma(n)$ with $n=-\alpha$. Now, we use the fact that $(\pm i)^{\alpha}$ is a complex number $\exp\left(\pm i\,\frac{\pi\alpha}{2}\right)$ to yield the following:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[e^{\left(\frac{2\pi ikt}{T}\right)}\right] &= \left(\tfrac{2\pi ik}{T}\right)^{\alpha} e^{\left(\frac{2\pi ikt}{T}\right)} - \left(\frac{Tt^{-\alpha}}{2\pi ikt\Gamma(-\alpha)} + \frac{T^2t^{-\alpha}}{4\pi^2 i^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots\right) \\
&= \left(\tfrac{2\pi k}{T}\right)^{\alpha} e^{\frac{i\pi\alpha}{2}} e^{\left(\frac{2\pi ikt}{T}\right)} - \left(\frac{Tt^{-\alpha}}{2\pi ikt\Gamma(-\alpha)} - \frac{T^2t^{-\alpha}}{4\pi^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots\right) \\
&= \left(\tfrac{2\pi k}{T}\right)^{\alpha} e^{\left(2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)} + i\frac{Tt^{-\alpha}}{2\pi kt\Gamma(-\alpha)} + \frac{T^2t^{-\alpha}}{4\pi^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots \\
&= \left(\tfrac{2\pi k}{T}\right)^{\alpha} e^{\left(2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)} + i\frac{Tt^{-1-\alpha}}{2\pi k\Gamma(-\alpha)} + \frac{T^2t^{-2-\alpha}}{4\pi^2 k^2 \Gamma(-\alpha-1)} \\
&\quad - i\frac{T^3t^{-3-\alpha}}{8\pi^3k^3\Gamma(-\alpha-2)} - \frac{T^4t^{-4-\alpha}}{16\pi^4k^4\Gamma(-\alpha-3)}\ldots
\end{aligned} \tag{4 .123}$$

Similarly, we write the following steps:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[e^{\left(-\frac{2\pi ikt}{T}\right)}\right] &= \left(-\tfrac{2\pi ik}{T}\right)^{\alpha} e^{\left(-\frac{2\pi ikt}{T}\right)} - \left(\frac{Tt^{-\alpha}}{(-2\pi ikt)\Gamma(-\alpha)} + \frac{T^2t^{-\alpha}}{4\pi^2 i^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots\right) \\
&= \left(\tfrac{2\pi k}{T}\right)^{\alpha} e^{-\frac{i\pi\alpha}{2}} e^{\left(-\frac{2\pi ikt}{T}\right)} - \left(\frac{Tt^{-\alpha}}{(-2\pi ikt)\Gamma(-\alpha)} - \frac{T^2t^{-\alpha}}{4\pi^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots\right) \\
&= \left(\tfrac{2\pi k}{T}\right)^{\alpha} e^{\left(-2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)} - i\frac{Tt^{-\alpha}}{2\pi kt\Gamma(-\alpha)} + \frac{T^2t^{-\alpha}}{4\pi^2 k^2 t^2 \Gamma(-\alpha-1)} + \ldots
\end{aligned} \tag{4.124}$$

Using the above two expressions ((4.123) and (4.124)), and putting $f_{\text{PER}}(t) = \sum_{k=1}^{\infty} A_k \exp\left(i\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} \overline{A}_k \exp\left(-i\frac{2\pi kt}{T}\right)$ into the definition of the periodic function for large t, we write the following:

$$\begin{aligned}
\lim_{t\uparrow\infty} \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[f_{\text{PER}}(t)\right] &= \sum_{k=1}^{\infty} \left(\tfrac{2\pi k}{T}\right)^{\alpha} \left(A_k e^{\left(2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)} + \overline{A}_k e^{\left(-2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)}\right) \\
&\quad + \sum_{k=1}^{\infty}\left(t^{-1-\alpha}\left(\tfrac{i\left(A_k-\overline{A}_k\right)T}{2\pi k\Gamma(-\alpha)}\right) + t^{-2-\alpha}\left(\tfrac{\left(A_k+\overline{A}_k\right)T^2}{4\pi^2k^2\Gamma(-\alpha-1)}\right) + \ldots\right)
\end{aligned} \tag{4.125}$$

In the above expression (4.125), the terms grouped inside the first summation are periodic. They show that the effect of the differ-integration of the order α is to change the amplitude of each component by $\left(\frac{2\pi k}{T}\right)^{\alpha}$ and to change the phase by an angle $\alpha\left(\frac{\pi}{2}\right)$. Within the second summation, the terms are aperiodic. Provided $\alpha > -1$, as the condition for standard fractional differentiability, the second grouped summation terms of (4.125) represent transients, which die down, and become insignificant at certain (large) times.

Let $f_{\text{PER}}(t) = \cos(t) = \sum_{k=1}^{\infty} A_k e^{\left(i\frac{2\pi kt}{T}\right)} + \sum_{k=1}^{\infty} \overline{A}_k e^{\left(-i\frac{2\pi kt}{T}\right)} = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$. Note that the generalized periodic function here when $f_{\text{PER}}(t) = \cos(t)$ has only a term for $k = 1$, with all the others being zero. With $\frac{2\pi}{T} = 1$ and $A_1 = \overline{A}_1 = \frac{1}{2}$, and then applying the fractional derivative formula as obtained above (4.125), we write the following:

$$\begin{aligned}\lim_{t\uparrow\infty}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[\cos(t)\right]&=\sum_{k=1}^{\infty}\left(\tfrac{2\pi k}{T}\right)^{\alpha}\left(A_k e^{\left(2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)}+\bar{A}_k e^{\left(-2\pi i\left(\frac{kt}{T}+\frac{\alpha}{4}\right)\right)}\right)\\&+\sum_{k=1}^{\infty}\begin{pmatrix}t^{-1-\alpha}\left(\frac{i\left(A_k-\bar{A}_k\right)T}{2\pi k\Gamma(-\alpha)}\right)\\+t^{-2-\alpha}\left(\frac{\left(A_k+\bar{A}_k\right)T^2}{4\pi^2k^2\Gamma(-\alpha-1)}\right)\\-t^{-3-\alpha}\left(\frac{i\left(A_k-\bar{A}_k\right)T^3}{8\pi^3k^3\Gamma(-\alpha-2)}\right)\\-t^{-4-\alpha}\left(\frac{\left(A_k+\bar{A}_k\right)T^4}{16\pi^4k^4\Gamma(-\alpha-3)}\right)\ldots\end{pmatrix}\\&=\left(\tfrac{1}{2}e^{i\left(t+\frac{\alpha\pi}{2}\right)}+\tfrac{1}{2}e^{-i\left(t+\frac{\alpha\pi}{2}\right)}\right)+\frac{t^{-2-\alpha}}{\Gamma(-\alpha-1)}-\frac{t^{-4-\alpha}}{\Gamma(-\alpha-3)}+\ldots\\&=\cos\left(t+\tfrac{\alpha\pi}{2}\right)+\frac{t^{-2-\alpha}}{\Gamma(-\alpha-1)}-\frac{t^{-4-\alpha}}{\Gamma(-\alpha-3)}+\ldots\end{aligned}\tag{4.126}$$

One can use this argument and can write the following for large x:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\cos(x)\right]&=\cos\left(x+\tfrac{\alpha\pi}{2}\right)+\frac{x^{-2-\alpha}}{\Gamma(-\alpha-1)}-\frac{x^{-4-\alpha}}{\Gamma(-\alpha-3)}+\ldots\\\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\sin(x)\right]&=\sin\left(x+\tfrac{\alpha\pi}{2}\right)+\frac{x^{-1-\alpha}}{\Gamma(-\alpha)}-\frac{x^{-3-\alpha}}{\Gamma(-\alpha-2)}+\ldots\end{aligned}\tag{4.127}$$

This above described method (4.126) is one of the procedures used to obtain the fractional differ-integrals of the periodic function. When the fractional order α is the positive or negative integer $\alpha=\pm1,\pm2,\ldots$, the reciprocal of the gamma function at a negative integer point is zero, the transient term vanishes, and we get well-known classical formulas of the derivatives and integrals of the sine and cosine functions. However, other methods are available, as fractional derivatives of the trigonometric functions may be represented by higher transcendental functions or transcendental trigonometric functions in terms of the Mittag-Leffler or Miller-Ross functions, among others (some of which were discussed in Chapters 2 and 3). For fractional integration of the cosine and sine functions, we reverse the sign of α in the formula obtained for fractional differentiation, and write the following set of expressions, with $\frac{2\pi}{T}=\omega$ and for large t.

$$\begin{aligned}D^{\alpha}\left(\sin(\omega t)\right)&=\omega^{\alpha}\sin\left(\omega t+\tfrac{\pi\alpha}{2}\right)+\frac{\left(\omega t\right)^{-1-\alpha}}{\omega\left(\Gamma(-\alpha)\right)}-\frac{\left(\omega t\right)^{-3-\alpha}}{\omega^{3}\left(\Gamma(-\alpha-2)\right)}+\ldots.\\I^{\alpha}\left(\sin(\omega t)\right)&=\omega^{-\alpha}\sin\left(\omega t-\tfrac{\pi\alpha}{2}\right)+\frac{\left(\omega t\right)^{-1+\alpha}}{\omega\left(\Gamma(\alpha)\right)}-\frac{\left(\omega t\right)^{-3+\alpha}}{\omega^{3}\left(\Gamma(\alpha-2)\right)}+\ldots.\\D^{\alpha}\left(\cos(\omega t)\right)&=\omega^{\alpha}\cos\left(\omega t+\tfrac{\pi\alpha}{2}\right)+\frac{\left(\omega t\right)^{-2-\alpha}}{\omega^{2}\left(\Gamma(-\alpha-1)\right)}-\frac{\left(\omega t\right)^{-4-\alpha}}{\omega^{4}\left(\Gamma(-\alpha-3)\right)}+\ldots\\I^{\alpha}\left(\cos(\omega t)\right)&=\omega^{-\alpha}\cos\left(\omega t-\tfrac{\pi\alpha}{2}\right)+\frac{\left(\omega t\right)^{-2+\alpha}}{\omega^{2}\left(\Gamma(\alpha-1)\right)}-\frac{\left(\omega t\right)^{-4+\alpha}}{\omega^{4}\left(\Gamma(\alpha-3)\right)}+\ldots\end{aligned}\tag{4.128}$$

4.14 The Eigen-functions for the Riemann-Liouville and Caputo fractional derivative operators

This is a very important section. We use $f(x)=e^{\lambda x}$ as the eigen-function for the classical derivative operator as $D_x^1 f(x)=\lambda f(x)$ for $f(x)=e^{\lambda x}$. Similarly, we will also deal with the Mittag-Leffler function (described in Appendix-A), which is a higher transcendental function (similar to the exponential function), and derive the eigen-functions for the Caputo and Riemann-Liouville fractional derivative operators.

4.14.1 Caputo derivative of the order α for the 'one-parameter Mittag-Leffler function': $f(x)=E_\alpha(\lambda x^\alpha)$

Let us find the Caputo derivative of $f(x)=E_\alpha(\lambda x^\alpha)$ (the one parameter Mittag-Leffler function) as follows:

$$\begin{aligned} E_\alpha(\lambda x^\alpha) = E_{\alpha,1}(\lambda x^\alpha) &= \sum_{k=0}^{\infty}\lambda^k \frac{x^{k\varepsilon}}{\Gamma(k\alpha+1)} \\ &= 1+\frac{\lambda x^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(2\alpha+1)}+.... \end{aligned} \tag{4.129}$$

Noting that ${}_0^C D_x^\alpha\left[1\right]=0$ and ${}_0^C D_x^\alpha\left[x^{n\alpha}\right]=\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)}x^{n\alpha-\alpha}$, we write the following

$$\begin{aligned} {}_0^C D_x^\alpha\left[E_\alpha(\lambda x^\alpha)\right] &= {}_0^C D_t^\alpha\left[\left(1+\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\lambda^3 x^{3\alpha}}{\Gamma(1+3\alpha)}+...\right)\right] \\ &= {}_0^C D_t^\alpha\left[\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\lambda^3 x^{3\alpha}}{\Gamma(1+3\alpha)}+...\right] \\ &= \lambda\frac{\Gamma(1+\alpha)}{\Gamma(1)\Gamma(1+\alpha)}+\lambda^2\frac{\Gamma(1+2\alpha)x^\alpha}{\Gamma(1+2\alpha)\Gamma(1+\alpha)}+\lambda^3\frac{\Gamma(1+3\alpha)x^{2\alpha}}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)}+... \\ &= \lambda\left(1+\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\lambda^3 x^{3\alpha}}{\Gamma(1+3\alpha)}+...\right) \\ &= \lambda E_\alpha(\lambda x^\alpha) \end{aligned} \tag{4.130}$$

Therefore, for the Caputo derivative operator we have $f(x)=E_\alpha(\lambda x^\alpha)$ as an eigen-function.

4.14.2 Riemann-Liouville derivative of the order α for the 'one-parameter Mittag-Leffler function': $f(x)=E_\alpha(\lambda x^\alpha)$

We conduct the Riemann-Liouville fractional derivative on $f(x)=E_\alpha(\lambda x^\alpha)$ as depicted in the following steps, noting that the RL derivative of the constant is non-zero, i.e. ${}_0D_x^\alpha\left[1\right]\neq 0$, but, as discussed earlier, is rather ${}_0D_x^\alpha\left[1\right]=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$

$$\begin{aligned} {}_0D_x^\alpha\left[E_\alpha(\lambda x^\alpha)\right] &= {}_0D_x^\alpha\left[\left(1+\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\lambda^3 x^{3\alpha}}{\Gamma(1+3\alpha)}+................\right)\right] \\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}+\lambda\frac{\Gamma(1+\alpha)}{\Gamma(1)\Gamma(1+\alpha)}+\lambda^2\frac{\Gamma(1+2\alpha)x^\alpha}{\Gamma(1+2\alpha)\Gamma(1+\alpha)}+\lambda^3\frac{\Gamma(1+3\alpha)x^{2\alpha}}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)}+.. \\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}+\lambda\left(1+\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}+\frac{\lambda^2 x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\lambda^3 x^{3\alpha}}{\Gamma(1+3\alpha)}+.............\right) \\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}+\lambda E_\alpha(\lambda x^\alpha) \\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}+{}_0^C D_x^\alpha\left[E_\alpha(\lambda x^\alpha)\right] \end{aligned} \tag{4.131}$$

Thus, we note that in the RL case ${}_0D_x^\alpha\left[E_\alpha(\lambda x^\alpha)\right]\neq E_\alpha(\lambda x^\alpha)$ so $f(x)=E_\alpha(\lambda x^\alpha)$ is not an eigen-function for the RL operator, i.e. for ${}_0D_x^\alpha f(x)\neq\lambda f(x)$. In the above steps of the derivation (4.131), we also observe that ${}_0^C D_x^\alpha\left[f(x)\right]={}_0D_x^\alpha\left[f(x)-f(0)\right]$; for $0<\alpha<1$; and here $f(0)=1$ and $D_x^\alpha f(0)=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$.

4.14.3 Riemann-Liouville derivative of the order α for a function of the 'two-parameter Mittag-Leffler function': $f(x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$

Let us take now $f(x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$, where the function $E_{\alpha,\beta}(\lambda x^{\alpha})$ is the two-parameter Mittag-Leffler function, i.e. $E_{\alpha,\beta}(\lambda x^{\alpha}) = \sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha})^k}{\Gamma(\beta+k\alpha)}$. With $\beta = \alpha > 0$, we write the following series:

$$\begin{aligned} f(x) &= x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha}) \\ &= x^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + \frac{\lambda x^{\alpha}}{\Gamma(2\alpha)} + \frac{\lambda^2 x^{2\alpha}}{\Gamma(3\alpha)} + ...\right) \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(3\alpha)} + ... \end{aligned} \tag{4.132}$$

We note that $f(0) = \infty$ for this case. Now we explore the term-by-term fractional derivative of the RL type:

$$\begin{aligned} {}_0D_x^{\alpha}\left[f(x)\right] &= {}_0D_x^{\alpha}\left[x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})\right] \\ &= {}_0D_x^{\alpha}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(3\alpha)} + ...\right) \\ &= \frac{\Gamma(\alpha)x^{-1}}{\Gamma(\alpha-\alpha)\Gamma(\alpha)} + \lambda\frac{\Gamma(2\alpha)x^{\alpha-1}}{\Gamma(2\alpha-\alpha)\Gamma(2\alpha)} + \lambda^2\frac{\Gamma(3\alpha)x^{2\alpha-1}}{\Gamma(3\alpha-\alpha)\Gamma(3\alpha)} + ... \\ &= \frac{x^{-1}}{\Gamma(0)} + \lambda\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda^2\frac{x^{2\alpha-1}}{\Gamma(2\alpha)} + ... \\ &= \lambda\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(3\alpha)} + ...\right) = \lambda\left(x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})\right) \end{aligned} \tag{4.133}$$

In the above derivation (4.133) $\frac{1}{\Gamma(0)} = 0$ is used. Thus, we obtained the eigen-function as $f(x) = x^{\alpha-1}\left(E_{\alpha,\alpha}(\lambda x^{\alpha})\right)$ for the Riemann-Liouville fractional derivative operator.

4.15 The fractional derivative of the zero-corrected function and relationship to Caputo derivative

Consider the function $f(x) = -\left(x-\frac{1}{2}\right)$ in the interval 0 to x. At the starting point ,i.e. $x = 0$, the function has a non-zero value, which is $f(0) = \frac{1}{2}$. If we construct a function that is offset by this starting point value (calling that new function $f_0(x) = f(x) - f(0)$), then at the starting point $f_0(0) = 0$. The starting point may be any general value (say $x = a$), where the function has a value $f(a)$; and in that case the zero-corrected function will be $f_0(x) = f(x) - f(a)$. This is what we mean by the zero-corrected function. In addition, the RL fractional derivative of $f_0(x)$ when function $f(x)$ is a constant will be zero too.

Regarding some functions where the starting point value is undefined (say $\pm\infty$), then we take for our calculations $f(a^+)$, making $f_0(x) = f(x) - f(a^+)$ finite. These are practical considerations for calculations to take the finite part of the zero corrected function.

4.15.1 The Riemann-Liouville fractional derivative of the zero-corrected function $f(x) = a - x$ with a derivative start point at $x = 0$

In the interval 0 to x, where the function is $f(x) = -\left(x-\frac{1}{2}\right)$, we apply an RL fractional derivative of the order α with $0 < \alpha < 1$ to the offset function, that is $f_0(x) = f(x) - f(0) = f(x) - \frac{1}{2}$. To obtain a fractional derivative ${}_0D_x^{\alpha}\left[f_0(x)\right]$ for the zero corrected function $f_0(x)$, we must follow the below steps:

$$
\begin{aligned}
&{}_0D_x^{\alpha}\left[f_0(x)\right]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x(x-y)^{-\alpha}\left(f_0(y)\right)\mathrm{d}y\\
&f_0(x)=-\left(x-\tfrac{1}{2}\right)-\tfrac{1}{2}\qquad 0<\alpha<1\\
&{}_0D_x^{\alpha}\left[f_0(x)\right]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x(x-y)^{-\alpha}\left(-\left(y-\tfrac{1}{2}\right)-\tfrac{1}{2}\right)\mathrm{d}y\qquad 0<x\le\tfrac{1}{2}\\
&\qquad=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x(x-y)^{-\alpha}\left(-y\right)\mathrm{d}y
\end{aligned}
\tag{4 .134}
$$

Now we use $\int f_1 f_2\mathrm{d}x=f_1\int f_2\mathrm{d}x-\int\left(f_1^{(1)}\right)\left(\int f_2\mathrm{d}x\right)\mathrm{d}x$, for $\mathrm{I}=\int_0^x(x-y)^{-\alpha}\left(-y\right)\mathrm{d}y$ in (4.134) and write the following steps:

$$
\begin{aligned}
\mathrm{I}&=\int_0^x(x-y)^{-\alpha}\left(-y\right)\mathrm{d}y=\left[(-y)\frac{(x-y)^{-\alpha+1}}{-\alpha+1}(-1)-\int\left(\int(x-y)^{-\alpha}\mathrm{d}y\right)(-1)\mathrm{d}y\right]_0^x\\
&=\left[\frac{y(x-y)^{1-\alpha}}{1-\alpha}+\int\frac{(x-y)^{1-\alpha}}{1-\alpha}(-1)\mathrm{d}y\right]_0^x=\left[\frac{y(x-y)^{1-\alpha}}{1-\alpha}+\frac{(x-y)^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right]_0^x\\
&=-\frac{x^{2-\alpha}}{(1-\alpha)(2-\alpha)}
\end{aligned}
\tag{4.135}
$$

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[f_0(x)\right]&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{x^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right]=-\frac{1}{\Gamma(1-\alpha)}\times(2-\alpha)\frac{x^{1-\alpha}}{(1-\alpha)(2-\alpha)}\\
&=-\frac{x^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}=-\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

Therefore, we write from (4.135)

$$
{}_0D_x^{\alpha}\left[-\left(x-\tfrac{1}{2}\right)-\tfrac{1}{2}\right]=-\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}
\tag{4.136}
$$

With the fractional derivative of the constant as ${}_0D_x^{\alpha}\left[\tfrac{1}{2}\right]=\frac{1}{2(\Gamma(1-\alpha))}x^{-\alpha}$, we write

$$
{}_0D_x^{\alpha}\left[-\left(x-\tfrac{1}{2}\right)\right]=-\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{x^{-\alpha}}{2\left(\Gamma(1-\alpha)\right)}
\tag{4.137}
$$

4.15.2 The Caputo fractional derivative function $f(x)=a-x$ with a derivative start point at $x=0$

Now we derive the Caputo derivative for the original function $f(x)=-\left(x-\tfrac{1}{2}\right)$, i.e. without taking zero correction, as described in the following steps.

$$
\begin{aligned}
{}_0^C D_x^{\alpha} f(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} \left(\frac{\mathrm{d}\, f(y)}{\mathrm{d}y} \right) \mathrm{d}y ; \quad 0<\alpha<1 \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}y} \left(-(y-\tfrac{1}{2}) \right) \right) \mathrm{d}y \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} \left(-1 \right) \mathrm{d}y \\
&= -\left(\frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} (1) \mathrm{d}y \right) \\
&= -\left({}_0 I_x^{(1-\alpha)} \left[1 \right] \right) = -\frac{\Gamma(1)}{\Gamma(1+1-\alpha)} x^{1-\alpha} \\
&= -\frac{1}{\Gamma(2-\alpha)} x^{1-\alpha}
\end{aligned}
\tag{4.138}
$$

The result is the same as it was for the RL fractional derivative of the zero-corrected function (4.136), i.e. ${}_{x_0}^C D_x^{\alpha} \left[f(x) \right] = {}_{x_0} D_x^{\alpha} \left[f(x) - f(x_0) \right]$. This is the relationship between the RL and Caputo derivatives for $0<\alpha<1$.

In the above derivation (4.138), we have used $\frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} (1) \mathrm{d}y$ as the RL fractional integration formula of the order $1-\alpha$, for a constant function $f(x)=1$.

4.15.3 The Riemann-Liouville fractional derivative of the zero-corrected function $f(t)=t^{v}$ from the start point as $t \neq 0$

Let us now take $f(t)=t^{\gamma}$, with the starting point $t=a$, $f(a)=a^{\gamma}$, thus $f_0(t)=f(t)-f(a)=t^{\gamma}-a^{\gamma}$. Now we apply the definition of the RL fractional derivative as indicated in the following steps.

$$
\begin{aligned}
{}_a D_t^{\alpha} \left[f_0(t) \right] &= f^{(\alpha)} \left[t^{\gamma} - a^{\gamma} \right]_a^t \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\xi)^{-\alpha} \left(\xi^{\gamma} - a^{\gamma} \right) \mathrm{d}\xi ; \quad 0<\alpha<1 \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\xi)^{-\alpha} \left(\xi^{\gamma} \right) \mathrm{d}\xi - \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\xi)^{-\alpha} \left(a^{\gamma} \right) \mathrm{d}\xi \\
&= D^1 \left[{}_a D_t^{-(1-\alpha)} \left[t^{\gamma} \right] - {}_a D_t^{-(1-\alpha)} \left[a^{\gamma} \right] \right] \\
&= {}_a D_t^{\alpha} \left[t^{\gamma} \right] - {}_a D_t^{\alpha} \left[a^{\gamma} \right]
\end{aligned}
\tag{4.139}
$$

The above expression (4.139) shows that this derivative of the zero-corrected function is composed of the two RL derivatives, which are ${}_a D_t^{\alpha} \left[t^{\gamma} \right]$ minus the RL derivative of a constant ${}_a D_t^{\alpha} \left[a^{\gamma} \right]$. Going by a similar method to that used for ${}_0 D_t^{\alpha} \left[t^{\gamma} \right]$, we get first the fractional integral in terms of the Beta function as

$$
\begin{aligned}
{}_a D_t^{-(1-\alpha)} \left[t^{\gamma} \right] &= \frac{t^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} \int_0^{(t-a)/t} z^{-\alpha} (1-z)^{\gamma} \mathrm{d}z \\
&= \frac{t^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} \mathrm{B}_{\eta}(1-\alpha, \gamma+1) \qquad \eta = \frac{t-a}{t}
\end{aligned}
\tag{4.140}
$$

The fractional derivative of the order α with $0<\alpha<1$ of the function t^{γ} is found by taking one whole derivative of the above expression. As such, we get the following:

$$
{}_a D_t^{\alpha} \left[t^{\gamma} \right] = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{t^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} \mathrm{B}_{\eta}(1-\alpha, \gamma+1) \right] \qquad \eta = \frac{t-a}{t} \tag{4.141}
$$

The RL-fractional derivative of the a^{γ} (constant) is:

$$ {}_aD_t^{\alpha}\left[a^{\gamma}\right]=\frac{a^{\gamma}}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{4.142}$$

We write the following further steps:

$$\begin{aligned}
{}_aD_t^{\alpha}\left[f_0(t)\right]&=f^{(\alpha)}\left[t^{\gamma}-a^{\gamma}\right]_a^t=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(t-\xi)^{-\alpha}\left(\xi^{\gamma}-a^{\gamma}\right)\mathrm{d}\xi\\
&=D^1\left[{}_aD_t^{-(1-\alpha)}\left[t^{\gamma}\right]-{}_aD_t^{-(1-\alpha)}\left[a^{\gamma}\right]\right];\quad 0<\alpha<1\\
&={}_aD_t^{\alpha}\left[t^{\gamma}\right]-{}_aD_t^{\alpha}\left[a^{\gamma}\right]\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^{\gamma+1-\alpha}}{\Gamma(1-\alpha)}\mathrm{B}_{\eta}(1-\alpha,\gamma+1)\right]-\frac{a^{\gamma}}{\Gamma(1-\alpha)}(t-a)^{-\alpha}
\end{aligned} \tag{4.143}$$

Therefore, we have:

$$ {}_aD_t^{\alpha}\left[t^{\gamma}\right]=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^{\gamma+1-\alpha}}{\Gamma(1-\alpha)}\mathrm{B}_{\eta}(1-\alpha,\gamma+1)\right]+\frac{a^{\gamma}(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.144}$$

For $a=0$, we have:

$$\begin{aligned}
{}_0D_t^{\alpha}\left[f_0(t)\right]&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\xi)^{-\alpha}\left[\xi^{\gamma}-0\right]\mathrm{d}\xi\\
&=D^1\left[{}_0D_t^{-(1-\alpha)}t^{\gamma}\right]={}_0D_t^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}
\end{aligned} \tag{4.145}$$

4.15.4 The Riemann-Liouville fractional derivative of the order α for the zero-corrected 'one-parameter Mittag-Leffler function': $f(t)=E_{\alpha}(at^{\alpha})$ with a derivative start point at $t=0$

The one parameter Mittag-Leffler $E_{\alpha}(at^{\alpha})$ is defined by an infinite series as:

$$E_{\alpha}(at^{\alpha})\overset{\mathrm{def}}{=}1+\frac{at^{\alpha}}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\ldots\ldots\ldots\ldots \tag{4.146}$$

By taking $f(t)=E_{\alpha}(at^{\alpha})$, we have $f(0)=1$. We note that for $E_{\alpha}(at^{\alpha})$, the first derivative does not exist at $t=0$. The zero-corrected function is $f_0(t)=E_{\alpha}(at^{\alpha})-1$. The fractional derivative of the order α, with $0\le\alpha<1$ and with its starting point as $a=0$ for $f(t)=t^{n\alpha}$, is ${}_0D_t^{\alpha}\left[t^{n\alpha}\right]=\frac{\Gamma(n\alpha+1)}{\Gamma[\alpha(n-1)+1]}t^{\alpha(n-1)}$, for $n=1,2,3,\ldots$; we will use this expression in the following derivation for ${}_0D_t^{\alpha}\left[E_{\alpha}(at^{\alpha})-1\right]$

$$\begin{aligned}
&{}_0D_t^{\alpha}\left[E_{\alpha}(at^{\alpha})-1\right]\\
&={}_0D_t^{\alpha}\left[\left(1+\frac{at^{\alpha}}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\ldots\ldots\ldots\ldots\right)-1\right]\\
&={}_0D_t^{\alpha}\left[\frac{at^{\alpha}}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\ldots\ldots\ldots\ldots\right]\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1)\Gamma(1+\alpha)}a+\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\alpha)}a^2t^{\alpha}+\frac{\Gamma(1+3\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)}a^3t^{2\alpha}+..\\
&=a\left(1+\frac{at^{\alpha}}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\ldots\ldots\ldots\right)\\
&=aE_{\alpha}(at^{\alpha})
\end{aligned} \tag{4.147}$$

Using ${}_0D_t^{\alpha}\left[1\right]=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$; we write the following:

$$ {}_0D_t^{\alpha}\left[E_\alpha(at^\alpha)\right]=aE_\alpha(at^\alpha)+\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} \qquad (4.148)$$

4.15.5 The Caputo fractional derivative of the order α for the 'one-parameter Mittag-Leffler function': $E_\alpha(at^\alpha)$ with a derivative start point at $t=0$

Now we explore the Caputo derivative of the function $f(t)=E_\alpha(at^\alpha)$, depicted in the following steps

$$\begin{aligned}
&{}_0^CD_t^{\alpha}\left[E_\alpha(at^\alpha)\right]\\
&={}_0^CD_t^{\alpha}\left[\left(1+\frac{at^\alpha}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+...\right)\right];\qquad {}_0^CD_t^{\alpha}[1]=0\\
&={}_0^CD_t^{\alpha}\left[\frac{at^\alpha}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+................\right]\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1)\Gamma(1+\alpha)}a+\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\alpha)}a^2t^\alpha \qquad (4.\ 149)\\
&\quad+\frac{\Gamma(1+3\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)}a^3t^{2\alpha}+..........\\
&=a\left(1+\frac{at^\alpha}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)}+.............\right)\\
&=aE_\alpha(at^\alpha)
\end{aligned}$$

The Caputo fractional derivative of the constant is zero, and, along with ${}_0^CD_t^{\alpha}\left[t^{n\alpha}\right]=\frac{\Gamma(n\alpha+1)}{\Gamma[\alpha(n-1)+1]}t^{\alpha(n-1)}$ is used in the above steps. Therefore, we get the following expression from (4.147) and (4.149):

$$ {}_0^CD_t^{\alpha}\left[E_\alpha(at^\alpha)\right]={}_0D_t^{\alpha}\left[E_\alpha(at^\alpha)-1\right]=aE_\alpha(at^\alpha) \qquad (4.150)$$

4.15.6 The Riemann-Liouville fractional derivative of the order α of the zero-corrected function $f(t)=\cos_\alpha(t^\alpha)$ and $f(t)=\sin_\alpha(t^\alpha)$ with a derivative start point at $t=0$

We defined the complex Mittag-Leffler function in the following form (see Appendix-A for more detail):

$$\begin{aligned}
&E_\alpha(it^\alpha)\overset{\text{def}}{=}\cos_\alpha(t^\alpha)+i\sin_\alpha(t^\alpha)\\
&\cos_\alpha(t^\alpha)=\frac{E_\alpha(it^\alpha)+E_\alpha(-it^\alpha)}{2}=\sum_{k=1}^{\infty}(-1)^k\frac{t^{2k\alpha}}{(2k\alpha)!}=\sum_{k=1}^{\infty}(-1)^k\frac{t^{2k\alpha}}{\Gamma(2k\alpha+1)} \qquad (4.151)\\
&\sin_\alpha(t^\alpha)=\frac{E_\alpha(it^\alpha)-E_\alpha(-it^\alpha)}{2}=\sum_{k=1}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{(2k\alpha+\alpha)!}=\sum_{k=1}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(2k\alpha+\alpha+1)}
\end{aligned}$$

The series presentation of $\cos_\alpha(t^\alpha)$ is:

$$\cos_\alpha(t^\alpha)=1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}+..... \qquad (4.152)$$

Take $f(t)=\cos_\alpha(t^\alpha)$, where $f(0)=1$, and we have the zero-corrected function $f_0(t)=\cos_\alpha(t^\alpha)-1$, and we will find ${}_0D_t^{\alpha}\left[f_0(t)\right]={}_0D_t^{\alpha}\left[\cos_\alpha(t^\alpha)-1\right]$, for $0<\alpha<1$, in the following steps (as we did for the Mittag-Leffler function):

$$
\begin{aligned}
&{}_0D_t^{\alpha}\left[\cos_{\alpha}(t^{\alpha})-1\right] \\
&= {}_0D_t^{\alpha}\left[\left(1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}+.....\right)-1\right] \\
&= {}_0D_t^{\alpha}\left[-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}+....\right] \\
&= -\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\alpha)}t^{2\alpha-\alpha}+\frac{\Gamma(1+4\alpha)}{\Gamma(1+4\alpha)\Gamma(1+3\alpha)}t^{4\alpha-\alpha} \\
&\quad -\frac{\Gamma(1+6\alpha)}{\Gamma(1+6\alpha)\Gamma(1+5\alpha)}t^{6\alpha-\alpha}+..... \\
&= -\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+.......\right)=-\sin_{\alpha}(t^{\alpha})
\end{aligned}
\tag{4.153}
$$

Thus, we have:

$$
{}_0D_t^{\alpha}\left[\cos_{\alpha}(t^{\alpha})-1\right]=-\sin_{\alpha}(t^{\alpha}) \tag{4.154}
$$

With the Caputo derivative, we can show ${}_0^CD_t^{\alpha}\left[\cos_{\alpha}(t^{\alpha})\right]=-\sin_{\alpha}(t^{\alpha})$. From here we write:

$$
{}_0D_t^{\alpha}\left[\cos_{\alpha}(t^{\alpha})\right]=-\sin_{\alpha}(t^{\alpha})+\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} \tag{4.155}
$$

The series representation of $\sin_{\alpha}(t^{\alpha})$ is the following:

$$
\sin_{\alpha}(t^{\alpha})=\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}+..... \tag{4.156}
$$

Taking a term-by-term fractional derivative, we get:

$$
\begin{aligned}
{}_0D_t^{\alpha}\left[\sin_{\alpha}(t^{\alpha})\right] &= {}_0D_t^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}+.....\right] \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\alpha)}t^{\alpha-\alpha}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+3\alpha)\Gamma(1+3\alpha-\alpha)}t^{3\alpha-\alpha} \\
&\quad +\frac{\Gamma(1+5\alpha)}{\Gamma(1+5\alpha)\Gamma(1+5\alpha-\alpha)}t^{5\alpha-\alpha}-\frac{\Gamma(1+7\alpha)}{\Gamma(1+7\alpha)\Gamma(1+7\alpha-\alpha)}t^{7\alpha-\alpha}+..... \\
&= 1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}+..... \\
&= \cos_{\alpha}(t^{\alpha})
\end{aligned}
\tag{4 .157}
$$

Therefore, we have, using $f(t)=\sin_{\alpha}(t^{\alpha})$ with $f(0)=0$, the following useful identity:

$$
{}_0D_t^{\alpha}\left[\sin_{\alpha}(t^{\alpha})\right]=\cos_{\alpha}(t^{\alpha}) \tag{4.158}
$$

In addition, we have in a similar way, the following:

$$
{}_0^CD_t^{\alpha}\left[\sin_{\alpha}(t^{\alpha})\right]=\cos_{\alpha}(t^{\alpha}) \tag{4.159}
$$

4.15.7 The Riemann-Liouville fractional derivative of the order β of the zero-corrected 'one-parameter Mittag-Leffler function' $f(x)=E_{\alpha}(x^{\alpha})$ with a derivative start point at $x=0$ and $\alpha\neq\beta$

Let $f(x)=E_{\alpha}(x^{\alpha})$, which has $f(0)=1$. Now we evaluate D^{β} for the function $f_0(x)=E_{\alpha}(x^{\alpha})-1$, and using the fractional derivative of the order α, with $0\le\alpha<1$ and with the start point as $a=0$ for $f(x)=x^{k\alpha}$, ${}_0D_x^{\alpha}\left[x^{k\alpha}\right]=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k-1)+1)}x^{\alpha(k-1)}$, for $k=1,2,3,...$, we write the following:

$$
\begin{aligned}
{}_0D_x^{\beta}\left[E_{\alpha}(x^{\alpha})-1\right] &= {}_0D_x^{\beta}\left[\left(1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\right)-1\right] \\
&= {}_0D_x^{\beta}\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\right] \\
&= \frac{x^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}+\frac{x^{2\alpha-\beta}}{\Gamma(1+2\alpha-\beta)}+\frac{x^{3\alpha-\beta}}{\Gamma(1+3\alpha-\beta)}+\ldots \\
&= x^{\alpha}x^{-\beta}\left(\frac{1}{\Gamma(1+\alpha-\beta)}+\frac{x^{\alpha}}{\Gamma(1+2\alpha-\beta)}+\frac{x^{2\alpha}}{\Gamma(1+3\alpha-\beta)}+\ldots\right) \\
&= x^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(x^{\alpha})^k}{\Gamma(1+\alpha-\beta+k\alpha)} \\
&= x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(x^{\alpha})
\end{aligned}
\tag{4.160}
$$

Where the 'two-parameter Mittag-Leffler function 'is placed in the above derivation (4.160), we get the following (see Appendix-A for more details)

$$
E_{\alpha,\mu}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{(x^{\alpha})^k}{\Gamma(\mu+k\alpha)}=\frac{1}{\Gamma(\mu)}+\frac{(x)^{\alpha}}{\Gamma(\alpha+\mu)}+\frac{(x^{\alpha})^2}{\Gamma(2\alpha+\mu)}+\frac{(x^{\alpha})^3}{\Gamma(3\alpha+\mu)}+\ldots \tag{4.161}
$$

where $\mu=\alpha-\beta+1$ in our derivation as above (4.160). Noting that ${}_0D_x^{\beta}[1]=\frac{x^{-\beta}}{\Gamma(1-\beta)}$, we write the following useful expression:

$$
{}_0D_x^{\beta}\left[E_{\alpha}(x^{\alpha})\right]=x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(x^{\alpha})+\frac{x^{-\beta}}{\Gamma(1-\beta)} \tag{4.162}
$$

In addition, we may write the Caputo derivative as:

$$
{}_0^CD_x^{\beta}\left[E_{\alpha}(x^{\alpha})\right]=x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(x^{\alpha}) \tag{4.163}
$$

4.15.8 The Riemann-Liouville and Caputo fractional derivatives of the order α for the function $f(x)=x^{\beta-1}\cos_{\alpha,\beta}(x^{\alpha})$ and $f(x)=x^{\beta-1}\sin_{\alpha,\beta}(x^{\alpha})$ with a derivative start point at $x=0$

Here we have, as above (4.161), a two-parameter Mittag-Leffler function. Similar to the earlier cosine and sine functions ($\cos_{\alpha}(x^{\alpha})$ and $\sin_{\alpha}(x^{\alpha})$), we define the two parameters as $\cos_{\alpha,\beta}(x^{\alpha})$ and $\sin_{\alpha,\beta}(x^{\alpha})$, as depicted below:

$$
\begin{aligned}
\cos_{\alpha,\alpha}(x^{\alpha}) &\overset{\text{def}}{=} \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{2k\alpha}}{\Gamma(\beta+2\alpha k)}=\frac{1}{\Gamma(\beta)}-\frac{x^{2\alpha}}{\Gamma(2\alpha+\beta)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+\beta)}-\ldots \\
\sin_{\alpha,\beta}(x^{\alpha}) &\overset{\text{def}}{=} \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{(2k+1)\alpha}}{\Gamma\left(\beta+(1+2k)\alpha\right)}=\frac{x^{\alpha}}{\Gamma(\alpha+\beta)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+\beta)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+\beta)}+\ldots
\end{aligned}
\tag{4.164}
$$

Now placing this (4.164) and the definition of the two-parameter Mittag-Leffler function (4.161) in an imaginary argument, we get the following useful identity:

$$
\begin{aligned}
E_{\alpha,\beta}(ix^{\alpha}) &= \sum_{k=0}^{\infty}\frac{(ix^{\alpha})^k}{\Gamma(\beta+k\alpha)} \\
&= \frac{1}{\Gamma(\beta)}+\frac{(ix)^{\alpha}}{\Gamma(\alpha+\beta)}+\frac{(ix^{\alpha})^2}{\Gamma(2\alpha+\beta)}+\frac{(ix^{\alpha})^3}{\Gamma(3\alpha+\beta)}+\ldots \\
&= \left(\frac{1}{\Gamma(\beta)}-\frac{x^{2\alpha}}{\Gamma(2\alpha+\beta)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+\beta)}-\ldots\right) \\
&\quad +i\left(\frac{x^{\alpha}}{\Gamma(\alpha+\beta)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+\beta)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+\beta)}+\ldots\right) \\
&= \cos_{\alpha,\beta}(x^{\alpha})+i\sin_{\alpha,\beta}(x^{\alpha})
\end{aligned}
\tag{4 .165}
$$

Now for $\beta > \alpha$, we take a fractional derivative of the order α on the function $f(x) = x^{\beta-1}\cos_{\alpha,\beta}(x^{\alpha})$ (here $f(0)=0$) as depicted in the following steps, using the formula ${}_0D_x^{\alpha}\left[x^{\upsilon}\right] = \frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-\alpha)}x^{\upsilon-\alpha}$.

$$\begin{aligned}
&{}_0D_x^{\alpha}\left[x^{\beta-1}\cos_{\alpha,\beta}(x^{\alpha})\right] \\
&= {}_0D_x^{\alpha}\left[x^{\beta-1}\left(\frac{1}{\Gamma(\beta)} - \frac{x^{2\alpha}}{\Gamma(2\alpha+\beta)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+\beta)} - \dots\right)\right] \\
&= {}_0D_x^{\alpha}\left[\frac{x^{\beta-1}}{\Gamma(\beta)} - \frac{x^{\beta+2\alpha-1}}{\Gamma(2\alpha+\beta)} + \frac{x^{\beta+4\alpha-1}}{\Gamma(4\alpha+\beta)} - \dots\right] \\
&= \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} - \frac{x^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} + \frac{x^{\beta+3\alpha-1}}{\Gamma(\beta+3\alpha)} - \dots \\
&= x^{\beta-\alpha-1}\left(\frac{1}{\Gamma(\beta-\alpha)} - \frac{x^{2\alpha}}{\Gamma(\beta-\alpha+2\alpha)} + \frac{x^{\beta+3\alpha-1}}{\Gamma(\beta-\alpha+4\alpha)} - \dots.\right) \\
&= x^{\beta-\alpha-1}\cos_{\alpha,\beta-\alpha}(x^{\alpha})
\end{aligned} \tag{4.166}$$

Thus, we get the very useful relationship:

$${}_0D_x^{\alpha}\left[x^{\beta-1}\cos_{\alpha,\beta}(x^{\alpha})\right] = x^{\beta-\alpha-1}\cos_{\alpha,\beta-\alpha}(x^{\alpha}) \tag{4.167}$$

Here, the Caputo derivative will also be the same as above because $f(0)=0$, that is:

$${}_0^{C}D_x^{\alpha}\left[x^{\beta-1}\cos_{\alpha,\beta}(x^{\alpha})\right] = x^{\beta-\alpha-1}\cos_{\alpha,\beta-\alpha}(x^{\alpha}) \tag{4.168}$$

Similarly, it can be shown that:

$${}_0D_x^{\alpha}\left[x^{\beta-1}\sin_{\alpha,\beta}(x^{\alpha})\right] = x^{\beta-\alpha-1}\sin_{\alpha,\beta-\alpha}(x^{\alpha}) \tag{4.169}$$

Here, the Caputo derivative will also be the same as above because $f(0)=0$, that is:

$${}_0^{C}D_x^{\alpha}\left[x^{\beta-1}\sin_{\alpha,\beta}(x^{\alpha})\right] = x^{\beta-\alpha-1}\sin_{\alpha,\beta-\alpha}(x^{\alpha}) \tag{4.170}$$

4.15.9 The Riemann-Liouville fractional derivative of the zero-corrected exponential function: $f(x)=e^{ax}$ with a derivative start point at $x=0$ and its relationship to the Caputo derivative

Now we calculate the fractional order derivative of $f(x)=e^{ax}$ as we did for $E_{\alpha}(x^{\alpha})$ using the formula ${}_0D_x^{\alpha}\left[x^{\upsilon}\right] = \frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-\alpha)}x^{\upsilon-\alpha}$ and $D^{\alpha}\left[1\right] = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$. The function is $f(x)=e^{ax}$, and $f(0)=1$. In series form, we write it as $e^{ax} = 1 + \frac{ax}{\Gamma(2)} + \frac{a^2x^2}{\Gamma(3)} + \frac{a^3x^3}{\Gamma(4)} + \dots$. Now we evaluate the fractional derivative of $f_0(x) = e^{ax} - 1$:

$$\begin{aligned}
{}_0D_x^{\alpha}\left[e^{ax}-1\right] &= {}_0D_x^{\alpha}\left[\left(1 + \frac{ax}{\Gamma(2)} + \frac{a^2x^2}{\Gamma(3)} + \frac{a^3x^3}{\Gamma(4)} + \dots\right) - 1\right] \\
&= {}_0D_x^{\alpha}\left[\frac{ax}{\Gamma(2)} + \frac{a^2x^2}{\Gamma(3)} + \frac{a^3x^3}{\Gamma(4)} + \dots\right] \\
&= \frac{ax^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{a^2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{a^3x^{3-\alpha}}{\Gamma(4-\alpha)} + \dots \\
&= ax^{1-\alpha}\left(\frac{1}{\Gamma(2-\alpha)} + \frac{ax}{\Gamma(3-\alpha)} + \frac{a^2x^2}{\Gamma(4-\alpha)} + \dots\right) \\
&= ax^{1-\alpha}\left(E_{1,2-\alpha}(ax)\right)
\end{aligned} \tag{4.171}$$

$$E_{\alpha,\beta}(\lambda z) = \frac{1}{\Gamma(\beta)} + \frac{\lambda z}{\Gamma(\alpha+\beta)} + \frac{\lambda^2 z^2}{\Gamma(2\alpha+\beta)} + \dots$$

We write the following:

$$ {}_0D_x^{\alpha}\left[e^{ax}\right]=ax^{1-\alpha}\left(E_{1,2-\alpha}(ax)\right)+\frac{x^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.172}$$

In a similar way, we write:

$$ {}_0^CD_x^{\alpha}\left[e^{ax}\right]=ax^{1-\alpha}\left(E_{1,2-\alpha}(ax)\right) \tag{4.173}$$

4.15.10 The Riemann-Liouville fractional derivative of the zero-corrected $f(x)=\cos(ax)$ and $f(x)=\sin(ax)$ functions with a derivative start point at $x=0$ and its relationship to the Caputo derivative

We find the fractional order derivative of $f(x)=\cos(ax)$ in the same way as we did for $\cos_{\alpha}(x^{\alpha})$, by using the formula $D^{\alpha}\left[x^{\upsilon}\right]=\frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-\alpha)}x^{\upsilon-\alpha}$ and $D^{\alpha}\left[1\right]=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$. We calculate the fractional derivative of the $f_0(x)=\cos(x)-1$, with $\cos(ax)=1-\frac{a^2x^2}{\Gamma(3)}+\frac{a^4x^4}{\Gamma(5)}+\frac{a^6x^6}{\Gamma(7)}+..$; as in the following steps:

$$\begin{aligned}
{}_0D_x^{\alpha}\left[\cos(ax)-1\right] &= {}_0D_x^{\alpha}\left[\left(1-\frac{a^2x^2}{\Gamma(3)}+\frac{a^4x^4}{\Gamma(5)}+\frac{a^6x^6}{\Gamma(7)}+..\right)-1\right]\\
&=-\frac{a^2x^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{a^4x^{4-\alpha}}{\Gamma(5-\alpha)}-\frac{a^6x^{6-\alpha}}{\Gamma(7-\alpha)}+\ldots\\
&=-ax^{1-\alpha}\sin_{1,2-\alpha}(ax)
\end{aligned} \tag{4.174}$$

We obtain:

$$ {}_0D_x^{\alpha}\left[\cos(ax)\right]=-ax^{1-\alpha}\sin_{1,2-\alpha}(ax)+\frac{x^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.175}$$

Similarly, we can derive the Caputo derivative as:

$$ {}_0^CD_x^{\alpha}\left[\cos(ax)\right]=-ax^{1-\alpha}\sin_{1,2-\alpha}(ax) \tag{4.176}$$

Similarly, the fractional order derivative of $f(x)=\sin(x)$ is the following:

$$ {}_0D_x^{\alpha}\left[\sin(ax)\right]=ax^{1-\alpha}\cos_{1,2-\alpha}(ax) \tag{4.177}$$

Here the singular term will not be added as $\sin(ax)=0\big|_{x=0}$ so in this case $f(0)=0$ and thus $f_0(x)=f(x)$. Here too one may verify that:

$$ {}_0^CD_x^{\alpha}\left[\sin(ax)\right]=ax^{1-\alpha}\cos_{1,2-\alpha}(ax) \tag{4.178}$$

4.16 The fractional derivative at a non-differentiable point of the continuous function

Consider the function $f(x)=\left|x-\frac{1}{2}\right|$ in the interval $0\le x\le 1$. Clearly $f^{(1)}$ does not exist at $x=\frac{1}{2}$, though at this point the function is continuous. Now we obtain the fractional derivative RL type on the zero-corrected function for the interval $0\le x\le\frac{1}{2}$ and $\frac{1}{2}\le x<1$. The value $f(0)=\frac{1}{2}$ is still used to offset both regions. Thus, we conduct the integration of the $f_0(x)=f(x)-f(0)$ in two regions that are first 0 to $\frac{1}{2}$, and then in the interval $\frac{1}{2}$ to x (as we show in the following steps), and then take one whole derivative to find the fractional derivative at $x=\frac{1}{2}$. The integral I_1 is $\mathrm{I}_1=\int_0^{1/2}(x-y)^{-\alpha}\left(f_0(y)\right)\mathrm{d}y$. In the interval 0 to $x\le\frac{1}{2}$, $f(x)=-\left(x-\frac{1}{2}\right)$ so the zero-corrected function is $f_0(x)=f(x)-\frac{1}{2}=-x$. With this, we calculate I_1, which is described as follows:

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/2}(x-y)^{-\alpha}\left(-\left(y-\tfrac{1}{2}\right)-\tfrac{1}{2}\right)\mathrm{d}y \qquad f_0(x)=-\left(x-\tfrac{1}{2}\right)-\tfrac{1}{2};\quad 0<x\le\tfrac{1}{2}\\
\mathrm{I}_1 &= \int_0^{1/2}(x-y)^{-\alpha}\left(-y\right)\mathrm{d}y
\end{aligned} \tag{4.179}$$

Writing $-y = x - y - x$, we get the following steps:

$$
\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/2} (x-y)^{-\alpha}\left((x-y)-x\right)\mathrm{d}y \\
&= \int_0^{1/2} \left((x-y)^{1-\alpha} - x(x-y)^{-\alpha}\right)\mathrm{d}y \\
&= -\left(\frac{(x-y)^{2-\alpha}}{2-\alpha} - \frac{x(x-y)^{1-\alpha}}{1-\alpha}\right)_{y=0}^{y=\frac{1}{2}} \\
&= -\frac{\left(x-\frac{1}{2}\right)^{2-\alpha} - x^{2-\alpha}}{2-\alpha} + \frac{x\left(x-\frac{1}{2}\right)^{1-\alpha} - x^{1-\alpha}}{1-\alpha}
\end{aligned}
\tag{4.180}
$$

I_2 is $\mathrm{I}_2 = \int_{1/2}^{x} (x-y)^{-\alpha}\left(f_0(y)\right)\mathrm{d}y$ in the region $x \ge \frac{1}{2}$ $f_0(x) = f(x) - \frac{1}{2}$, and $f(x) = x - \frac{1}{2}$. Therefore, in the interval $\frac{1}{2}$ to x, the zero-corrected function is $f_0(x) = x - 1$. With this, I_2 is calculated as follows:

$$
\mathrm{I}_2 = \int_{1/2}^{x} (x-y)^{-\alpha}\left(y-1\right)\mathrm{d}y \qquad f_0(x) = \left(x-1\right); \qquad x \ge \frac{1}{2}
\tag{4.181}
$$

Writing $y - 1 = (x-1) - (x-y)$, we obtain the following steps:

$$
\begin{aligned}
\mathrm{I}_2 &= \int_{1/2}^{x} (x-y)^{-\alpha}\left((x-1)-(x-y)\right)\mathrm{d}y \\
&= \int_{1/2}^{x} \left((x-1)(x-y)^{-\alpha} - (x-y)^{1-\alpha}\right)\mathrm{d}y \\
\mathrm{I}_2 &= -\left((x-1)\frac{(x-y)^{1-\alpha}}{1-\alpha} - \frac{(x-y)^{2-\alpha}}{2-\alpha}\right)_{y=\frac{1}{2}}^{y=x} \\
&= \frac{(x-1)\left(x-\frac{1}{2}\right)^{1-\alpha}}{1-\alpha} - \frac{\left(x-\frac{1}{2}\right)^{2-\alpha}}{2-\alpha}
\end{aligned}
\tag{4.182}
$$

After having performed the integrations in the two regions as above, we take one whole derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ of $\mathrm{I}_1 + \mathrm{I}_2$ and then multiply it by $\frac{1}{\Gamma(1-\alpha)}$ to get the fractional derivative ${}_0D_x^{\alpha}\left[f_0(x)\right]$, as follows for $0 < \alpha < 1$.

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[f_0(x)\right] &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{I}_1 + \mathrm{I}_2\right] \\
&= \frac{1}{\Gamma(2-\alpha)}\left(2\left(x-\frac{1}{2}\right)^{1-\alpha} - x^{1-\alpha}\right)
\end{aligned}
\tag{4.183}
$$

Therefore, we have the following derivative as above (4.183), and find the fractional derivative at the non-differentiable point, i.e. $x = \frac{1}{2}$

$$
\begin{aligned}
0 < \alpha < 1, \qquad {}_0D_x^{\alpha}\left[f_0(x)\right] &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x}\left(f(y) - f(0)\right)(x-y)^{-\alpha}\,\mathrm{d}y \\
&= \frac{1}{\Gamma(2-\alpha)}\left(2\left(x-\frac{1}{2}\right)^{1-\alpha} - x^{1-\alpha}\right)
\end{aligned}
\tag{4.184}
$$

$$
f^{(\alpha)}(x)\Big|_{x=\frac{1}{2}} = -\frac{\left(\frac{1}{2}\right)^{1-\alpha}}{\Gamma(2-\alpha)} \qquad ; \qquad f^{(1)}(x)\Big|_{x=\frac{1}{2}} = \infty
$$

This example suggests that there exists a finite value of the fractional derivative for points where the integer order's whole derivative is undefined. From the above expression, it is clear that although $f^{(1)}\left(\frac{1}{2}\right)$ does not exist, the fractional derivative ${}_0D_x^{\alpha}\left[f(x)-f(0)\right]_{x=\frac{1}{2}}$ exists at $x=\frac{1}{2}$ and is equal to ${}_0D_x^{\alpha}\left[f(x)-f(0)\right]_{x=\frac{1}{2}}=-\frac{1}{\Gamma(2-\alpha)}\left(\frac{1}{2}\right)^{1-\alpha}$. For $\alpha=\frac{1}{2}$, we get the value $-\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{1}{2}\right)^{1/2}$. By putting $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$, we get at $x=\frac{1}{2}$, ${}_0D_x^{\alpha}\left[f(x)-f(0)\right]_{x=\frac{1}{2}}=-\sqrt{\frac{2}{\pi}}$, whereas $f^{(1)}\left(\frac{1}{2}\right)=\infty$. We solve this example by following the consolidated steps again

$$
{}_0D_x^{\alpha}\left[f(x)-f(0)\right]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\mathrm{d}\xi \qquad 0<\alpha<1 \tag{4.185}
$$

The above (4.185) provides the definition of the RL fractional derivative of an order less than one. In the above expression, the function for $0\le x\le 1/2$ is $f(x)=-x+\frac{1}{2}$.

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[f(x)-f(0)\right]&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\mathrm{d}\xi\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}\left(-\xi+\tfrac{1}{2}-\tfrac{1}{2}\right)\mathrm{d}\xi\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}\left((x-\xi)-x\right)\mathrm{d}\xi\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \left((x-\xi)^{1-\alpha}-x(x-\xi)^{-\alpha}\right)\mathrm{d}\xi\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{(x-\xi)^{2-\alpha}}{2-\alpha}+x\frac{(x-\xi)^{1-\alpha}}{1-\alpha}\right]_0^x\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{x^{2-\alpha}}{(2-\alpha)(1-\alpha)}\right]=-\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
\tag{4.186}
$$

Again when $1/2\le x\le 1$, then $f(x)=x-\frac{1}{2}$. The fractional derivatives from zero to point half and beyond half in two separate segments are depicted below

$$
{}_0D_x^{\alpha}\left[f(x)-f(0)\right]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\begin{array}{l}\int\limits_0^{1/2}\left((x-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\right)\mathrm{d}\xi\\ +\int\limits_{1/2}^{x}\left((x-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)\right)\mathrm{d}\xi\end{array}\right)\right]
$$

$$
\begin{aligned}
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{l}\int\limits_0^{1/2}(x-\xi)^{-\alpha}\left(-\xi+\frac{1}{2}-\frac{1}{2}\right)\mathrm{d}\xi\\ +\int\limits_{1/2}^{x}(x-\xi)^{-\alpha}\left(\xi-\frac{1}{2}-\frac{1}{2}\right)\mathrm{d}\xi\end{array}\right]\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{l}\left(-\frac{(x-\xi)^{2-\alpha}}{2-\alpha}+x\frac{(x-\xi)^{1-\alpha}}{1-\alpha}\right)_0^{1/2}\\ +\int\limits_{1/2}^{x}(x-\xi)^{-\alpha}\left(\xi-1\right)\mathrm{d}\xi\end{array}\right]\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{l}\left(-\frac{(x-\xi)^{2-\alpha}}{2-\alpha}+x\frac{(x-\xi)^{1-\alpha}}{1-\alpha}\right)_0^{1/2}\\ +\int\limits_{1/2}^{x}(x-\xi)^{-\alpha}\left((x-1)-(x-\xi)\right)\mathrm{d}\xi\end{array}\right]\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{l}\left(-\frac{(x-\xi)^{2-\alpha}}{2-\alpha}+x\frac{(x-\xi)^{1-\alpha}}{1-\alpha}\right)_0^{1/2}\\ -\left(-\frac{(x-\xi)^{2-\alpha}}{2-\alpha}+(x-1)\frac{(x-\xi)^{1-\alpha}}{1-\alpha}\right)_{1/2}^{x}\end{array}\right]\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{l}-\frac{\left(x-\frac{1}{2}\right)^{2-\alpha}-x^{2-\alpha}}{2-\alpha}+x\frac{\left(x-\frac{1}{2}\right)^{1-\alpha}-x^{1-\alpha}}{1-\alpha}\\ -\frac{\left(x-\frac{1}{2}\right)^{2-\alpha}}{2-\alpha}+(x-1)\frac{\left(x-\frac{1}{2}\right)^{1-\alpha}}{1-\alpha}\end{array}\right]\\
&=\frac{2\left(x-\frac{1}{2}\right)^{1-\alpha}-x^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
\tag{4.187}
$$

Therefore, we obtain the following result:

$$
{}_0D_x^{\alpha}\left[f(x)-f(0)\right]=\begin{cases}-\dfrac{x^{1-\alpha}}{\Gamma(2-\alpha)}, & 0\le x\le\dfrac{1}{2}\\[2ex] \dfrac{2\left(x-\frac{1}{2}\right)^{1-\alpha}-x^{1-\alpha}}{\Gamma(2-\alpha)}, & \dfrac{1}{2}\le x\le 1\end{cases}
\tag{4.188}
$$

By putting $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$, we get at $x=\frac{1}{2}$, ${}_0D_x^{\alpha}\left[f(x)-f(0)\right]_{x=\frac{1}{2}}=-\sqrt{\frac{2}{\pi}}$ whereas $f^{(1)}\left(\frac{1}{2}\right)=\infty$.

4.17 Computation of the fractional derivative and integration-a review and comparison of the various schemes

From the most fundamental approach, we write the derivative of the function as:

$$
\frac{\mathrm{d}\ f(x)}{\mathrm{d}x}=\lim_{\Delta x\downarrow 0}\frac{f(x)-f(x-\Delta x)}{\Delta x}
\tag{4.189}
$$

The double derivative is:

$$
\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}=\lim_{\Delta x\downarrow 0}\frac{f(x)-2\left(f(x-\Delta x)\right)+f(x-2\Delta x)}{(\Delta x)^2}
\tag{4.190}
$$

The three-whole derivative is:

$$\frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} = \lim_{\Delta x \downarrow 0} \frac{f(x) - 3\big(f(x-\Delta x)\big) + 3\big(f(x-2\Delta x)\big) - f(x-3\Delta x)}{(\Delta x)^3} \tag{4.191}$$

For the $n-$ th whole derivative, we obtain the following:

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} = \lim_{\substack{\Delta x \downarrow 0 \\ N\uparrow\infty}} \frac{\sum_{j=0}^{N-1} (-1)^j \binom{n}{j} f(x - j\Delta x)}{(\Delta x)^n} \qquad \Delta x = \frac{x-a}{N} \tag{4.192}$$

Now for any real order when n is changed by order α, we can generalize the above (4.192) with j changed to index k, as in:

$$D_x^\alpha f(x) = \lim_{\Delta x \downarrow 0} \frac{1}{(\Delta x)^\alpha} \sum_{k=0}^{\infty} w_k \big(f(x-k\Delta x)\big) \qquad w_k = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \tag{4.193}$$

We write the anti-derivative, one-whole integration as follows, with $\Delta x = \frac{(x-a)}{N}$:

$$\begin{aligned} \frac{\mathrm{d}^{-1} f(x)}{\left[\mathrm{d}(x-a)\right]^{-1}} &\equiv \int_a^x f(y)\mathrm{d}y \\ &= \lim_{\Delta x \downarrow 0} (\Delta x)\big(f(x) + f(x-\Delta x) + f(x-2\Delta x) + \ldots + f(a+\Delta x)\big) \\ &= \lim_{\Delta x \downarrow 0} \left((\Delta x) \sum_{j=0}^{N-1} f(x - j\Delta x) \right) \end{aligned} \tag{4.194}$$

Two-whole integration is expressed as in the following expression:

$$\begin{aligned} \frac{\mathrm{d}^{-2} f(x)}{\left[\mathrm{d}(x-a)\right]^{-2}} &\equiv \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_0 f(x_0) \\ &= \lim_{\Delta x \downarrow 0} (\Delta x)^2 \begin{pmatrix} f(x) + 2\big(f(x-\Delta x)\big) + 3\big(f(x-2\Delta x)\big) + \\ \ldots. + N\big(f(a+\Delta x)\big) \end{pmatrix} \end{aligned} \tag{4.195}$$

Therefore, for $n-$ th fold integration we write:

$$\frac{\mathrm{d}^{-n} f(x)}{\left[\mathrm{d}(x-a)\right]^{-n}} \equiv \lim_{\Delta x \downarrow 0} \left((\Delta x)^n \sum_{j=0}^{N-1} \binom{j+n-1}{j} f(x - j\Delta x) \right) \tag{4.196}$$

We use the following property of the binomial expansions (as detailed in Section-1.12) and generalize as follows:

$$(-1)^j \binom{n}{j} = \binom{j-n-1}{j} = \binom{-n}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} \tag{4.197}$$

Using this generalization for the fractional differ-integration of the α order, we get;

$$\begin{aligned} \frac{\mathrm{d}^\alpha f(x)}{\left[\mathrm{d}(x-a)\right]^\alpha} &= \lim_{\Delta x \downarrow 0} \left((\Delta x)^{-\alpha} \sum_{k=0}^{N-1} w_k \big(f(x-k\Delta x)\big) \right) \\ w_k &= \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}, \qquad \Delta x = \frac{x-a}{N} \end{aligned} \tag{4.198}$$

This (4.198) for $\alpha > 0$ is a fractional differentiation; and for $\alpha < 0$ it is a fractional integration.

In Chapter-3, we used the Grunwald-Letnikov (GL) formula to determine the numerical evaluation technique that is reproduced again. This is the approximate algorithm to evaluate $\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[f(x)\right]$, with weights called Grunwald's coefficients i.e. ${}^{GL}w_j = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$, ${}^{GL}w_{j-1} = \frac{\Gamma(j-1-\alpha)}{\Gamma(-\alpha)\Gamma(j)}$. Note that w_k, which we used above, is the same as ${}^{GL}w_j$; that we described in the previous chapter. Furthermore, in Section-3.23, we wrote the following (with $a = 0$)

$${}_0D_x^\alpha\left[f(x)\right] = \left(\Delta x\right)^{-\alpha} \sum_{j=0}^{N-1} \left({}^{GL}w_j\right) f_j; \qquad {}^{GL}w_j = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}, \qquad \Delta x = \frac{x}{N} \tag{4.199}$$

In all of these numerical schemes discussed in previous chapters (specifically Chapters 2 and 3), we have, using known values of $f(x)$ at $N+1$, evenly spaced points in the range 0 to x. We designate symbols $f_N \equiv f(0)$,

$f_{N-1} \equiv f\left(\frac{x}{N}\right)$, $\ldots f_j \equiv f\left(x - j\frac{x}{N}\right)$, $\ldots f_0 \equiv f(x)$; and the discrete step size as $\Delta x = \left(\frac{x}{N}\right)$. Although this GL scheme was derived for $\alpha > 0$, we proved in the beginning of this section that this formula is also valid for $\alpha < 0$.

The above mentioned GL formulation (i.e. ${}_0D_x^{\alpha} f(x) = \left(\Delta x\right)^{-\alpha} \sum_{j=0}^{N-1} \left({}^{GL}w_j\right) f_j$) is unique as this scheme does not utilize f_N (the function value) at the start point ($f_N = f(0)$). This permits this formula to be used for functions like $f(x) = \ln x$, $f(x) = \frac{1}{\sqrt{x}}$, where the function is singular at the start point, meaning $\lim_{x \downarrow 0} f(x) = \infty$.

In the GL formula, we need not restrict the function where the value of the function at the start point needs to be merely finite. Thus, the GL formulations for numerical evaluations are important as they span all of the values of α, including positives and negatives and various functions which have singularity at the start point; and can be implemented by convenient multiplication-addition schemes. That is we reproduce them again as:

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \left(\Delta x\right)^{-\alpha} \left[\left[\left[\left[\cdot\cdot\left[f_{N-1}\left(\frac{N-\alpha-2}{N-1}\right) + f_{N-2}\right]\left(\frac{N-\alpha-3}{N-2}\right) + f_{N-3}\right]\cdot\cdot\right]\left(\frac{1-\alpha}{2}\right) + f_1\right]\left(\frac{-\alpha}{1}\right) + f_0\right] \tag{4.200}$$

The modified GL method is as follows, and here we require the value of the function at the start point, i.e. $f_N = f(0)$, which needs to be finite. We reproduce this as follows:

$$\begin{aligned} \frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\mathrm{d}x^{\alpha}} &\approx \frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} \left(f_j + \tfrac{1}{4}\alpha\left(f_{j-1} - f_{j+1}\right) + \tfrac{1}{8}\alpha^2\left(f_{j-1} - 2f_j + f_{j+1}\right)\right) \\ &= \left(\Delta x\right)^{-\alpha} \sum_{j=0}^{N-1} \left({}^{GL}w_j\right)\left(f_j + \tfrac{1}{4}\alpha\left(f_{j-1} - f_{j+1}\right) + \tfrac{1}{8}\alpha^2\left(f_{j-1} - 2f_j + f_{j+1}\right)\right) \end{aligned} \tag{4.201}$$

We discussed the first formula in Chapter-2 as an RL-integration formula; i.e. the formula for discrete numerical evaluation is obtained from the RL fractional integration formula, as below

$${}_0I_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(y)}{(x-y)^{-\alpha+1}} \mathrm{d}y; \qquad \alpha > 0 \tag{4.202}$$

The formula for discrete numerical evaluation is the following:

$${}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(1+\alpha)} \sum_{j=0}^{N-1} \frac{f_j + f_{j+1}}{2}\left((j+1)^{\alpha} - j^{\alpha}\right); \qquad \alpha > 0 \tag{4.203}$$

The above formula (4.203) requires the values at the start point of the function $f_N = f(0)$ which needs to be finite. It is valid only for fractional integration and not for fractional differentiation.

The second formula in Chapter-2 that we obtained for RL fractional integration is:

$${}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \left(\begin{array}{l} \dfrac{\left((j+1)f_j - j\ f_j\right)\left((j+1)^{\alpha} - j^{\alpha}\right)}{\alpha} \\ + \dfrac{\left(f_{j+1} - f_j\right)\left((j+1)^{1+\alpha} - j^{1+\alpha}\right)}{1+\alpha} \end{array} \right) \tag{4.204}$$

The above formula (4.204) also requires the values at the starting point of the function $f_N = f(0)$, which needs to be finite.

In Chapter-3, for $0 \le \alpha < 1$, we obtained, a fractional differentiation formula for discrete numerical evaluation, from the RL fractional derivative as below:

$${}_0D_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(2-\alpha)} \left(\frac{(1-\alpha)}{N^{\alpha}} f_N + \sum_{j=0}^{N-1} \left(\left((j+1)^{1-\alpha} - j^{1-\alpha}\right)\left(f_j - f_{j+1}\right)\right) \right) \tag{4.205}$$

$$0 \le \alpha < 1$$

We note here that the above formula also requires the values at the start point of the function $f_N = f(0)$, which needs to be finite. The above formula is derived from the basic formula of the RL fractional derivative i.e.

$${}_0D_x^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_0^x \frac{f(y)}{(x-y)^{\alpha+1-m}} \mathrm{d}y \qquad 0 \le \alpha < 1; \quad m = 1 \tag{4.206}$$

We noted in the previous chapter that we get the fractional integration formula, for the fractional order $(m-1) \le \alpha < m$, by changing the sign of the fractional order α from an RL fractional derivative formula, as indicated below:

$$
\begin{aligned}
{}_0D_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^x \frac{f(y)}{(x-y)^{\alpha+1-m}}\mathrm{d}y \\
{}_0I_x^{\alpha}\left[f(x)\right] &= \frac{1}{\Gamma(m+\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^x \frac{f(y)}{(x-y)^{-\alpha+1-m}}\mathrm{d}y
\end{aligned}
\tag{4.207}
$$

Thus, with a change of the sign we get the numerical integration formula as follows:

$$
{}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(2+\alpha)}\left(\frac{(1+\alpha)}{N^{-\alpha}}f_N + \sum_{j=0}^{N-1}\left(\left((j+1)^{1+\alpha} - j^{1+\alpha}\right)\left(f_j - f_{j+1}\right)\right)\right)
\tag{4.208}
$$

$0 \le \alpha < 1$

Similarly, we saw the discrete numerical evaluation formula for the RL fractional derivative for $1 \le \alpha < 2$ that is:

$$
{}_0D_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{-\alpha}}{\Gamma(3-\alpha)}\left(\begin{aligned}&\frac{(1-\alpha)(2-\alpha)}{N^{\alpha}}f_N + \\ &\frac{(2-\alpha)}{N^{\alpha-1}}\sum_{j=0}^{N-1}\left(f_{j-1} - 2f_j + f_{j+1}\right)\left((j+1)^{2-\alpha} - j^{2-\alpha}\right)\end{aligned}\right)
\tag{4.209}
$$

The fractional integration formula for discrete numerical evaluation for $1 \le \alpha < 2$ is:

$$
{}_0I_x^{\alpha}\left[f(x)\right] = \frac{\left(\Delta x\right)^{\alpha}}{\Gamma(3+\alpha)}\left(\begin{aligned}&\frac{(1+\alpha)(2+\alpha)}{N^{-\alpha}}f_N + \\ &\frac{(2+\alpha)}{N^{-\alpha-1}}\sum_{j=0}^{N-1}\left(f_{j-1} - 2f_j + f_{j+1}\right)\left((j+1)^{2+\alpha} - j^{2+\alpha}\right)\end{aligned}\right)
\tag{4.210}
$$

In this section, we discussed various formulae in order to provide discrete numerical evaluation for fractional derivatives and fractional integration. The GL method has been shown to be more universal and unified, though several RL methods have also been developed and are useful.

4.18 Fractional derivatives of the same-order but of different RL-Caputo types: A unified formula

We have seen two types of fractional derivatives and discussed them. The following formula (4.211) provides fractional derivative definition using a combination of the RL and Caputo types, with β as the type defining parameter as $0 \le \beta \le 1$. When $\beta = 1$, the definition is the Caputo derivative with the symbol ${}^{C}D_t^{\alpha}f(t)$; while $\beta = 0$ gives the RL fractional derivative with the symbol $D_t^{\alpha}f(t)$. The generalized definition is as follows:

$$
{}_0^{\beta}D_t^{\alpha}\left[f(t)\right] = {}_0I_t^{\beta(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-\beta)(1-\alpha)}\left[f(t)\right]\right]\right]
\tag{4.211}
$$

The fractional order α is the order of the derivative; here in this definition it is $0 < \alpha < 1$ and β is the type of derivative with $0 \le \beta \le 1$. Therefore, the nearest integer is $m = 1$ in the above generalization. Let us see how it works- by using $\beta = 1$, we get:

$$
\begin{aligned}
&{}_0^{\beta}D_t^{\alpha}f(t) = {}_0I_t^{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-\beta)(1-\alpha)}f(t)\right] \qquad \beta = 1 \\
&{}_0^{1}D_t^{\alpha}f(t) = {}_0I_t^{(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-1)(1-\alpha)}f(t)\right] = {}_0I_t^{(1-\alpha)}\frac{\mathrm{d}\left[f(t)\right]}{\mathrm{d}t} = {}_0^{C}D_t^{\alpha}f(t)
\end{aligned}
\tag{4.212}
$$

$0 < \alpha < 1$

When $\beta = 0$, we get the following:

$$
\begin{aligned}
{}_0^0D_t^{\alpha} f(t) &= {}_0I_t^{0(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-0)(1-\alpha)} f(t) \right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-\alpha)} f(t) \right] = {}_0^{RL}D_t^{\alpha} f(t), \qquad 0<\alpha<1
\end{aligned} \tag{4.213}
$$

In a way, we could unify the two formulae for a fractional derivative. This is a different kind of generalization; say with $\beta = \frac{1}{2}$, we get:

$$
{}_0^{1/2}D_t^{\alpha} f(t) = {}_0I_t^{\frac{1}{2}(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{\frac{1}{2}(1-\alpha)} f(t) \right] \tag{4.214}
$$

This (4.214) shows that the fractional derivative operator with the order $0<\alpha<1$ is a half Caputo and half RL type. The physical meaning of such a type has yet to be developed.

4.19 Generalizing Leibniz's rule

In this section, let us write $f(x) = f$ and $g(x) = g$. We have a rule for the differentiation of the product of the two functions as discussed earlier in the book (Section-1.21), which is:

$$
\frac{\mathrm{d}^n [fg]}{\mathrm{d}x^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{\mathrm{d}^{n-j} f}{\mathrm{d}x^{n-j}} \frac{\mathrm{d}^j g}{\mathrm{d}x^j} \tag{4.215}
$$

In Section-1.21, we derived, via the use of integration by parts, a product rule for multiple-integration, which is:

$$
\frac{\mathrm{d}^{-n} [fg]}{[\mathrm{d}(x-a)]^{-n}} = \sum_{j=0}^{\infty} \binom{-n}{j} \frac{\mathrm{d}^{-n-j} f}{[\mathrm{d}(x-a)]^{-n-j}} \frac{\mathrm{d}^j g}{[\mathrm{d}(x-a)]^j} \tag{4.216}
$$

In the differentiation formula, the sum terminates at $j = n$, because the binomial coefficients $\frac{n!}{j!(n-j)!}$ are zero when $j > n$ for n positive integer; this sum could very well be extended to infinity (making the coefficients zero for the terms $j > n$). We noted this argument while discussing the local properties of the integer order for differentiation. With this, we expect the generalized product rule for any $n = \alpha$ real number to be:

$$
\frac{\mathrm{d}^{\alpha} [fg]}{[\mathrm{d}(x-a)]^{\alpha}} = \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{\mathrm{d}^{\alpha-j} f}{[\mathrm{d}(x-a)]^{\alpha-j}} \frac{\mathrm{d}^j g}{[\mathrm{d}(x-a)]^j} \tag{4.217}
$$

We have discussed in Section-2.11 that, if $\varphi(x)$ is an analytic function, then we have fractional differ-integration of the analytic function in the following expression:

$$
\begin{aligned}
\frac{\mathrm{d}^{\alpha} [\varphi(x)]}{[\mathrm{d}(x-a)]^{\alpha}} &= \sum_{k=0}^{\infty} \frac{(-1)^k (x-a)^{k-\alpha} \varphi^{(k)}(x)}{\big(\Gamma(-\alpha)\big)\big((k-\alpha)k!\big)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \big(\Gamma(1+k-\alpha)\big)(x-a)^{k-\alpha} \varphi^{(k)}(x)}{\big(\Gamma(1+k-\alpha)\Gamma(-\alpha)\big)\big((k-\alpha)k!\big)}
\end{aligned} \tag{4.218}
$$

Noting that the fractional derivative of the Heaviside unit step is the function $\frac{\mathrm{d}^{\alpha-k}[1]}{[\mathrm{d}(x-a)]^{\alpha-k}} = \frac{(x-a)^{-\alpha}}{\Gamma(1+k-\alpha)}$, we can re-write, with the above formula (4.218), the fractional differ-integration of the analytical function as follows

$$
\frac{\mathrm{d}^{\alpha} [\varphi(x)]}{[\mathrm{d}(x-a)]^{\alpha}} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(1+k-\alpha)}{\big(\Gamma(-\alpha)\big)(k-\alpha)k!} \left(\frac{\mathrm{d}^{\alpha-k}[1]}{[\mathrm{d}(x-a)]^{\alpha-k}} \right) \varphi^{(k)}(x) \tag{4.219}
$$

Using the properties of the gamma functions (namely those discussed in Sections 1.10 and 1.12)

$$
\begin{aligned}
&\Gamma(x+1) = x\big(\Gamma(x)\big), \qquad \Gamma(n+1) = n\big(\Gamma(n)\big) = n(n-1)\Gamma(n-1) = \ldots = n! \\
&\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \binom{j-q-1}{j} = (-1)^j \binom{q}{j}
\end{aligned} \tag{4.220}
$$

we do the following manipulation

$$(-1)^k \frac{\Gamma(1+k-\alpha)}{\left(\Gamma(-\alpha)\right)\left((k-\alpha)k!\right)} = (-1)^k \frac{(k-\alpha)\Gamma(k-\alpha)}{(k-\alpha)\Gamma(-\alpha)\Gamma(k+1)}$$
$$= (-1)^k \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k\left((-1)^k\binom{\alpha}{k}\right) = \binom{\alpha}{k} \tag{4.221}$$

Thus, we can re-write compactly the fractional differ-integration of the analytical function as

$$\frac{\mathrm{d}^\alpha\left[\varphi\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\mathrm{d}^{\alpha-k}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-k}}\frac{\mathrm{d}^k\varphi}{\left[\mathrm{d}(x-a)\right]^k} \tag{4.222}$$

The above (4.222) is similar to Leibniz's rule with one function $f=1$ and $g=\varphi$. Now we start with the above formula and, substituting φ with the product $\phi\psi$, we obtain the following

$$\frac{\mathrm{d}^\alpha\left[\phi\psi\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\mathrm{d}^{\alpha-k}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-k}}\left(\phi\psi\right)^{(k)}$$
$$= \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\mathrm{d}^{\alpha-k}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-k}}\sum_{j=0}^{k}\binom{k}{j}\phi^{(k-j)}\psi^{(j)} \tag{4.223}$$

We have made use of the formula (4.215) of $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[fg\right]$ as noted at the beginning of this section for an n positive integer. Note that since j is an integer, then the repeated derivative $\psi^{(j)}$ with respect to x equals that with respect to $x-a$. We use the permutation $\sum_{k=0}^{\infty}\sum_{j=0}^{k} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}$. This identity is easily established, as it is analogous to the permutation of the variables in a double integral that is $\int_0^\infty \mathrm{d}y\int_0^y \mathrm{d}x = \int_0^\infty \mathrm{d}x\int_x^\infty \mathrm{d}y$; with these we write the following:

$$\frac{\mathrm{d}^\alpha\left[\phi\psi\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \sum_{j=0}^{\infty}\psi^{(j)}\sum_{k=j}^{\infty}\binom{\alpha}{k}\binom{k}{j}\frac{\mathrm{d}^{\alpha-k}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-k}}\phi^{(k-j)} \tag{4.224}$$

We set $k-j=m$ the second summation of (4.224).Thus, it will be from $m=0$ to ∞, and we use the following identity:

$$\binom{\alpha}{m+j}\binom{m+j}{j} = \binom{\alpha}{j}\binom{\alpha-j}{m} \tag{4.225}$$

This (4.225) can be easily proven using a factorial notation (or gamma function) for the binomial coefficients with simple manipulation, as demonstrated below:

$$\binom{\alpha}{m+j}\binom{m+j}{j} = \left(\frac{\alpha!}{(m+j)!(\alpha-m-j)!}\right)\left(\frac{(m+j)!}{j!m!}\right) = \frac{\alpha!}{m!j!(\alpha-m-j)!}$$
$$\binom{\alpha}{j}\binom{\alpha-j}{m} = \left(\frac{\alpha!}{j!(\alpha-j)!}\right)\left(\frac{(\alpha-j)!}{m!(\alpha-j-m)!}\right) = \frac{\alpha!}{m!j!(\alpha-j-m)!} \tag{4.226}$$

Using (4.224) and (4.226), we re-write the above expression as follows:

$$\frac{\mathrm{d}^\alpha\left[\phi\psi\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \sum_{j=0}^{\infty}\psi^{(j)}\sum_{m=0}^{\infty}\binom{\alpha}{m+j}\binom{m+j}{j}\frac{\mathrm{d}^{\alpha-j-m}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-j-m}}\phi^{(m)}$$
$$= \sum_{j=0}^{\infty}\binom{\alpha}{j}\psi^{(j)}\sum_{m=0}^{\infty}\binom{\alpha-j}{m}\frac{\mathrm{d}^{\alpha-j-m}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-j-m}}\phi^{(m)} \tag{4.227}$$
$$= \sum_{j=0}^{\infty}\binom{\alpha}{j}\psi^{(j)}\frac{\mathrm{d}^{\alpha-j}\phi}{\left[\mathrm{d}(x-a)\right]^{\alpha-j}}$$

In the above steps (4.227), we have used the following relationship

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\mathrm{d}^{\alpha-k}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-k}}\frac{\mathrm{d}^k\varphi}{\left[\mathrm{d}(x-a)\right]^k} = \frac{\mathrm{d}^\alpha\left[\varphi\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} \tag{4.228}$$

with the symbols changed from α to $\alpha-j$ and from k to m. Therefore, we obtained our expression as:

$$\frac{\mathrm{d}^{\alpha}\left[\phi\psi\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{\mathrm{d}^{\alpha-j}\phi}{\left[\mathrm{d}(x-a)\right]^{\alpha-j}}\frac{\mathrm{d}^{j}\psi}{\left[\mathrm{d}(x-a)\right]^{j}} \tag{4.229}$$

In all of the above discussions, we assumed that ϕ and ψ are analytic functions. A somewhat different argument may be required when one of the functions is a polynomial. Consider the product $x\left(f(x)\right)$. For $\alpha<0$, we write the Riemann-Liouville formula as:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}&=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{y\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{\left(x-(x-y)\right)\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}\\&=\frac{x}{\Gamma(-\alpha)}\int_a^x\frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}-\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{(x-y)\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}\\&=\frac{x}{\Gamma(-\alpha)}\int_a^x\frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha+1}}-\frac{(-\alpha)}{(-\alpha)\Gamma(-\alpha)}\int_a^x\frac{\left(f(y)\right)\mathrm{d}y}{(x-y)^{\alpha}}\\&=x\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}+\alpha\frac{\mathrm{d}^{\alpha-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-1}}\end{aligned} \tag{4.230}$$

We have used $(-\alpha)\left(\Gamma(-\alpha)\right)=\Gamma(-\alpha+1)$ and $\frac{1}{\Gamma(-(\alpha-1))}\int_a^x\frac{(f(y))\mathrm{d}y}{(x-y)^{\alpha}}=\frac{\mathrm{d}^{\alpha-1}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-1}}$. The above technique (4.230) has also been discussed in Section-2.11. The extension of this result for $\alpha>0$ is easy, as, if $(n-1)\le\alpha<n$, with $n=1,2,3...$, then:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}&=\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\frac{\mathrm{d}^{\alpha-n}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}\right]\\&=\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[x\frac{\mathrm{d}^{\alpha-n}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}+(\alpha-n)\frac{\mathrm{d}^{\alpha-n-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n-1}}\right]\end{aligned} \tag{4.231}$$

We have to use Leibniz's rule for the first term $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)\right]$ with the following formula

$$\begin{aligned}\frac{\mathrm{d}^{n}\left[fg\right]}{\mathrm{d}x^{n}}&=\left(\frac{\mathrm{d}^{n}f}{\mathrm{d}x^{n}}\right)\left(g\right)+n\left(\frac{\mathrm{d}^{n-1}f}{\mathrm{d}x^{n-1}}\right)\left(\frac{\mathrm{d}\ g}{\mathrm{d}x}\right)+n(n-1)\left(\frac{\mathrm{d}^{n-2}f}{\mathrm{d}x^{n-2}}\right)\left(\frac{\mathrm{d}^{2}g}{\mathrm{d}x^{2}}\right)+......\\&......+n\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^{n-1}g}{\mathrm{d}x^{n-1}}\right)+\left(f\right)\left(\frac{\mathrm{d}^{n}g}{\mathrm{d}x^{n}}\right)\end{aligned} \tag{4.232}$$

With $f=x$, we see that all of the derivatives greater than one are zero only terms with $\frac{\mathrm{d}[f]}{\mathrm{d}x}$, and that f exists. So within the above formula (4.232), only the last two terms exist with all other terms vanishing. Therefore, we write $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)\right]$ with $f=x$ and $g=\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)$ using the above formula as follows:

$$\begin{aligned}\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[x\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)\right]&=n\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[x\right]\right)\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)\right]\right)\\&\quad+\left(x\right)\left(\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\left(\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right)\right]\right)\\&=n\left(\frac{\mathrm{d}^{\alpha-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-1}}\right)+x\left(\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\right)\end{aligned} \tag{4.233}$$

Also noting that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(\frac{\mathrm{d}^{\alpha-n-1}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n-1}}\right)\right]=\frac{\mathrm{d}^{\alpha-1}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-1}}$, we complete the formula as in the following steps:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}&=\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[x\frac{\mathrm{d}^{\alpha-n}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n}}\right]+(\alpha-n)\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\frac{\mathrm{d}^{\alpha-n-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-n-1}}\right]\\&=\left(n\frac{\mathrm{d}^{\alpha-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-1}}+x\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\right)+(\alpha-n)\frac{\mathrm{d}^{\alpha-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-1}}\\&=x\frac{\mathrm{d}^{\alpha}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}+\alpha\frac{\mathrm{d}^{\alpha-1}\left[f(x)\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-1}}\end{aligned}\tag{4.234}$$

For $\alpha=\frac{1}{2}$ and $\alpha=-\frac{1}{2}$ with $a=0$, we write the following results:

$$\begin{aligned}\frac{\mathrm{d}^{1/2}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x)\right]^{1/2}}&=x\frac{\mathrm{d}^{1/2}\left[f(x)\right]}{\left[\mathrm{d}(x)\right]^{1/2}}+\frac{1}{2}\frac{\mathrm{d}^{-1/2}\left[f(x)\right]}{\left[\mathrm{d}(x)\right]^{-1/2}}\\\frac{\mathrm{d}^{-1/2}\left[x\left(f(x)\right)\right]}{\left[\mathrm{d}(x)\right]^{-1/2}}&=x\frac{\mathrm{d}^{-1/2}\left[f(x)\right]}{\left[\mathrm{d}(x)\right]^{-1/2}}-\frac{1}{2}\frac{\mathrm{d}^{-3/2}\left[f(x)\right]}{\left[\mathrm{d}(x)\right]^{-3/2}}\end{aligned}\tag{4.235}$$

Notice that we have used the relationship $\frac{\mathrm{d}^{\alpha}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha}}\equiv\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\frac{\mathrm{d}^{\alpha-n}[f(x)]}{[\mathrm{d}(x-a)]^{\alpha-n}}\right]$, i.e. the definition of the Riemann-Liouville fractional derivative. The inductive argument establishes the following formula:

$$\frac{\mathrm{d}^{\alpha}\left[fg\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha}}=\sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{\mathrm{d}^{\alpha-j}f}{\left[\mathrm{d}(x-a)\right]^{\alpha-j}}\frac{\mathrm{d}^{j}g}{\left[\mathrm{d}(x-a)\right]^{j}}\tag{4.236}$$

with $g(x)=x^{k}$ for any non-negative integer k and for any $f(x)$, and thus for $g(x)$ any polynomial and any arbitrary $f(x)$. When $g(x)$ is a polynomial in $(x-a)$, the RHS of the above formula (4.236) is finite. We may write for analytic functions $\phi(x)$ and $\psi(x)$, with $a=0$ for the following expression:

$$\frac{\mathrm{d}^{\alpha}\left[\phi\psi\right]}{\left[\mathrm{d}(x)\right]^{\alpha}}=\sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{\mathrm{d}^{\alpha-j}\phi}{\left[\mathrm{d}(x)\right]^{\alpha-j}}\frac{\mathrm{d}^{j}\psi}{\left[\mathrm{d}(x)\right]^{j}}\tag{4.237}$$

For $\alpha=\frac{1}{2}$ and $\alpha=-\frac{1}{2}$, we write the following results:

$$\begin{aligned}\frac{\mathrm{d}^{1/2}\left[\phi\psi\right]}{\left[\mathrm{d}(x)\right]^{1/2}}&=\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\frac{\mathrm{d}^{\left(\frac{1}{2}\right)-j}\phi}{\left[\mathrm{d}(x)\right]^{\left(\frac{1}{2}\right)-j}}\psi^{(j)}\\\frac{\mathrm{d}^{-1/2}\left[\phi\psi\right]}{\left[\mathrm{d}(x)\right]^{-1/2}}&=\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\frac{\mathrm{d}^{\left(-\frac{1}{2}\right)-j}\phi}{\left[\mathrm{d}(x)\right]^{\left(-\frac{1}{2}\right)-j}}\psi^{(j)}\end{aligned}\tag{4.238}$$

However, sometimes the formula has convergence issues. For example, take $f(x)=1$ and $g(x)=\frac{1}{1-x}=1+x+x^{2}+.....=\sum_{k=0}^{\infty}x^{k}$; for $-1<x<1$. This is certainly a differ-integrable series; in fact, it is analytic. As such, we write with term-by-term differ-integration the following

$$\frac{\mathrm{d}^{\alpha}\left[fg\right]}{\mathrm{d}x^{\alpha}}=\frac{\mathrm{d}^{\alpha}\left[\frac{1}{1-x}\right]}{\mathrm{d}x^{\alpha}}=\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[\sum_{k=0}^{\infty}x^{k}\right]=\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}\tag{4.239}$$

The above (4.239) converges for $0<|x|<1$. By using the previously discussed Leibniz's rule, we obtain the following steps

$$\frac{\mathrm{d}^{\alpha}\left[fg\right]}{\mathrm{d}x^{\alpha}}=\frac{\mathrm{d}^{\alpha}\left[(1)\left(\frac{1}{1-x}\right)\right]}{\mathrm{d}x^{\alpha}}=\sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{\mathrm{d}^{\alpha-j}\left[1\right]}{\mathrm{d}x^{\alpha-j}}\frac{\mathrm{d}^{j}\left[(1-x)^{-1}\right]}{\mathrm{d}x^{j}}$$
$$=\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}\frac{x^{j-\alpha}}{\Gamma(j-\alpha+1)}\left(\Gamma(j+1)\right)\left(1-x\right)^{-1-j}$$
$$=\frac{\Gamma(\alpha+1)}{1-x}x^{-\alpha}\sum_{j=0}^{\infty}\frac{1}{\Gamma(\alpha-j+1)\Gamma(j-\alpha+1)}\frac{x^{j}}{(1-x)^{j}}$$
$$=\frac{\Gamma(\alpha+1)}{1-x}x^{-\alpha}\sum_{j=0}^{\infty}\frac{x^{j}}{(1-x)^{j}}\frac{1}{(j-\alpha)\Gamma(j-\alpha)\Gamma(\alpha-j+1)}\tag{4.240}$$
$$=\frac{\Gamma(\alpha+1)}{1-x}x^{-\alpha}\sum_{j=0}^{\infty}\frac{1}{(j-\alpha)\Gamma\left(-(\alpha-j)\right)\Gamma\left((\alpha-j)+1\right)}\frac{x^{j}}{(1-x)^{j}}$$
$$=\frac{\Gamma(\alpha+1)}{1-x}x^{-\alpha}\sum_{j=0}^{\infty}\frac{(-1)}{(j-\alpha)\pi\csc\left((\alpha-j)\pi\right)}\frac{x^{j}}{(1-x)^{j}}$$
$$=\frac{\Gamma(\alpha+1)}{1-x}x^{-\alpha}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(j-\alpha)\pi\csc(\pi\alpha)}\frac{x^{j}}{(1-x)^{j}}$$

We have used the identity $\Gamma(x+1)=x\left(\Gamma(x)\right)$ and $\left(\Gamma(-x)\right)\left(\Gamma(x+1)\right)=-\pi\csc(\pi x)$ in the above derivation (4.240). The last line of the above formula is elaborated upon in the following steps:

$$\sum_{j=0}^{\infty}\frac{(1)}{(\alpha-j)\pi\csc\left((\alpha-j)\pi\right)}$$
$$=\frac{1}{(\alpha)\pi\csc(\alpha\pi)}+\frac{1}{(\alpha-1)\pi\csc(\alpha\pi-\pi)}+\frac{1}{(\alpha-2)\pi\csc(\alpha\pi-2\pi)}+\ldots$$
$$=\frac{1}{(\alpha)\pi\csc(\alpha\pi)}-\frac{1}{(\alpha-1)\pi\csc(\pi-\alpha\pi)}-\frac{1}{(\alpha-2)\pi\csc(2\pi-\alpha\pi)}+\ldots$$
$$=\frac{1}{(\alpha)\pi\csc(\alpha\pi)}-\frac{1}{(\alpha-1)\pi\csc(\alpha\pi)}+\frac{1}{(\alpha-2)\pi\csc(\alpha\pi)}+\ldots\tag{4.241}$$
$$=\frac{-1}{(-\alpha)\pi\csc(\alpha\pi)}+\frac{1}{(1-\alpha)\pi\csc(\alpha\pi)}-\frac{1}{(2-\alpha)\pi\csc(\alpha\pi)}+\ldots$$
$$=\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(j-\alpha)\pi\csc(\alpha\pi)}$$

The above step (4.241) uses the trigonometric identities $\sin(-\theta)=-\sin\theta$, $\sin(\pi-\theta)=\sin\theta$ and $\sin(2\pi-\theta)=-\sin\theta$. The result obtained above is very different to what we obtained when conducting term-by-term differ-integration. Therefore, if we extend Leibniz's rule to the case with any arbitrary $f(x)$ and analytic $g(x)$, the convergence may be a problem.

4.20 Chain rule for fractional differ-integration

As we know, the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}g\left(f(x)\right)=\frac{\mathrm{d}}{\mathrm{d}[f(x)]}\left[g\left(f(x)\right)\right]\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right]$ is a rule for differentiation. Unfortunately, there is no analogous rule for integration in integral calculus. If there were an analogous rule for integration, then the process of integration would pose no greater difficulty than differentiation does. We discussed the generalization process for multiple differentiations for the chain rule in Section-1.22; and we emphasise its difficulty in the following expressions:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} g\left(f(x)\right) = g^{(1)} f^{(2)} + g^{(2)}\left(f^{(1)}\right)^2$$
$$\frac{\mathrm{d}^3\left[g\left(f(x)\right)\right]}{\mathrm{d}x^3} = g^{(1)} f^{(3)} + 3g^{(2)} f^{(1)} f^{(2)} + g^{(3)}\left(f^{(1)}\right)^3$$
$$\begin{aligned}\frac{\mathrm{d}^4\left[g\left(f(x)\right)\right]}{\mathrm{d}x^4} &= g^{(1)} f^{(4)} + 4g^{(2)} f^{(1)} f^{(3)} + 6g^{(2)}\left(f^{(2)}\right)^2 \\ &\quad + 6g^{(3)}\left(f^{(1)}\right)^2 f^{(2)} + g^{(4)}\left(f^{(1)}\right)^4\end{aligned} \tag{4.242}$$
$$\begin{aligned}\frac{\mathrm{d}^5\left[g\left(f(x)\right)\right]}{\mathrm{d}x^5} &= g^{(1)} f^{(5)} + 5g^{(2)} f^{(1)} f^{(4)} + 10g^{(3)}\left(f^{(1)}\right)^2 f^{(3)} \\ &\quad + 30g^{(3)} f^{(1)}\left(f^{(2)}\right)^2 + 10g^{(4)}\left(f^{(1)}\right)^3 f^{(2)} + g^{(5)}\left(f^{(1)}\right)^5\end{aligned}$$

The continuation of the above would give $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[f\left(g(x)\right)\right]$, $n > 0$ and an integer but we previously pointed out the difficulties in Section-1.22. However, there is no generalization for $n < 0$ that can be use for integration in integral calculus. Since the general formula for $\frac{\mathrm{d}^\alpha}{[\mathrm{d}(x-a)]^\alpha}\left[g\left(f(x)\right)\right]$ must encompass integration as a special case, we expect difficulty in the further generalization for the chain rule for arbitrary α. However, we will develop a chain rule. We start with a formula for analytic function $\phi(x)$ that we used in (4.222), that is:

$$\frac{\mathrm{d}^\alpha\left[\phi(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{\mathrm{d}^{\alpha-j}\left[1\right]}{\left[\mathrm{d}(x-a)\right]^{\alpha-j}}\frac{\mathrm{d}^j\left[\phi(x)\right]}{\mathrm{d}x^j} \tag{4.243}$$

In Section-4.2, we derived the formula $\frac{\mathrm{d}^\alpha[1]}{[\mathrm{d}(x-a)]^\alpha} = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$; and using this in the above (4.243), we write the following by separating the term for $j = 0$ from the rest of the equation;

$$\frac{\mathrm{d}^\alpha\left[\phi(x)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\phi(x) + \sum_{j=1}^{\infty}\binom{\alpha}{j}\frac{(x-a)^{j-\alpha}}{\Gamma(1+j-\alpha)}\frac{\mathrm{d}^j\left[\phi(x)\right]}{\mathrm{d}x^j} \tag{4.244}$$

Now consider ϕ as function of $f(x)$, that is $\phi \equiv \phi\left(f(x)\right)$; with this we discuss the difficulties for obtaining $\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[\phi\left(f(x)\right)\right]$ and apply Faa de Bruno's formula as we did in Section-1.22 of the book.

$$\begin{aligned}\frac{\mathrm{d}^\alpha\left[\phi\left(f(x)\right)\right]}{\left[\mathrm{d}(x-a)\right]^\alpha} &= \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\phi\left(f(x)\right) \\ &\quad + \sum_{j=1}^{\infty}\binom{\alpha}{j}\frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)} j!\sum_{m=1}^{j}\phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{P_k!}\left(\frac{f^{(k)}}{k!}\right)^{P_k}\end{aligned} \tag{4.245}$$

We provided a description of the third summation in (4.245) and P_k in the same section. We describe the complicacy of the above general result (4.245) by using $\alpha = -1$ as a 'chain-rule' for the case of simple integration:

$$\begin{aligned}&\int_a^x \phi\left(f(y)\right)\mathrm{d}y \\ &= (x-a)\phi\left(f(x)\right) + \sum_{j=1}^{\infty}(-1)^j\frac{(x-a)^{j+1}}{j+1}\sum_{m=1}^{j}\phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{P_k}\left(\frac{f^{(k)}}{k!}\right)^{P_k}\end{aligned} \tag{4.246}$$

This gives an infinite series, and there exists a difficulty in expressing the closed form formula.
As in Section-1.22, we write the formula when $f(x) = e^x$ as follows:

$$\frac{\mathrm{d}^{\alpha}}{\left[\mathrm{d}(x-a)\right]^{\alpha}}\left[\phi\left(e^{x}\right)\right] = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\phi\left(e^{x}\right)+\sum_{j=1}^{\infty}\binom{\alpha}{j}\frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)}e^{(jx)}\sum_{m=1}^{j}S_{j}^{[m]}\phi^{(m)} \tag{4.247}$$

Where $S_j^{[m]}$ is Stirling number of the second kind.

4.21 Analytical continuation of the differ-integral operator from integer to real order: Extended fractional calculus

The operator D_x^n for integer n is well defined as it corresponds to $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$ for $n \ge 1$ and $\int (\mathrm{d}x)^{-n}$ for $n \le -1$. We have also on many occasions referred to analytic-continuation and applied this concept. We have studied the Euler formula and we write the following formula for a power function $f(x) = x^m$, that is $D_x^n x^m$ where $m, n \in \mathbb{Z}$ (i. e. integers):

$$\frac{\mathrm{d}^n x^m}{\mathrm{d}x^n} = \begin{cases} 0 & \text{for } 0 < m \le n \\ \dfrac{m!}{(m-n)!}x^{m-n} & \text{for } m > n > 0 \end{cases} \tag{4.248}$$

$$\int x^m (\mathrm{d}x)^{-n} = \frac{m!}{(m-n)!}x^{m-n} \qquad \text{for } m > 0,\quad n < 0$$

We reproduce this point here and write the unified formula for fractional-differ-integration of a power function (as discussed in Sections 1.23, 2.6, 2.7, and 4.3) as the following expression:

$$\frac{\mathrm{d}^n\left[x^m\right]}{\mathrm{d}x^n} = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n} & \begin{cases} n = 0,1,2,.....; & \text{For all } m \\ n = -1,-2,....; & m > -1 \end{cases} \\ \infty & \quad n = -1,-2,....; \quad m \le -1 \end{cases} \tag{4.249}$$

We may now generalize the above (4.249) for a non-integer order α and write the following for the differ-integration of the power function $f(x) = x^{\mu}$, $\mu \in \mathbb{R}$, and $\alpha \in \mathbb{R}$

$$\frac{\mathrm{d}^{\alpha}\left[x^{\mu}\right]}{\mathrm{d}x^{\alpha}} = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha} & \mu > -1 \quad \text{For all } \alpha \\ \infty & \mu \le -1 \quad \text{For all } \alpha \end{cases} \tag{4.250}$$

The breakdown of the above formula for $\mu \le -1$ is associated with the pole of the order unity which occurs at $x = a$ for the function $f(x) = (x-a)^{\mu}$; $\mu \le -1$ (in the above formula (4.250), it is $a = 0$). The functions for which such a pole occurs anywhere in the open interval a to x lead to similar difficulties; and for such reasons we have until now purposely excluded these functions from the differ-integrable series.

Figure-4.1 here shows the sign of the ratio of the gamma-function in the coefficient of the unified differ-integral formula. Note that the figure stops short of $\mu = -1$.

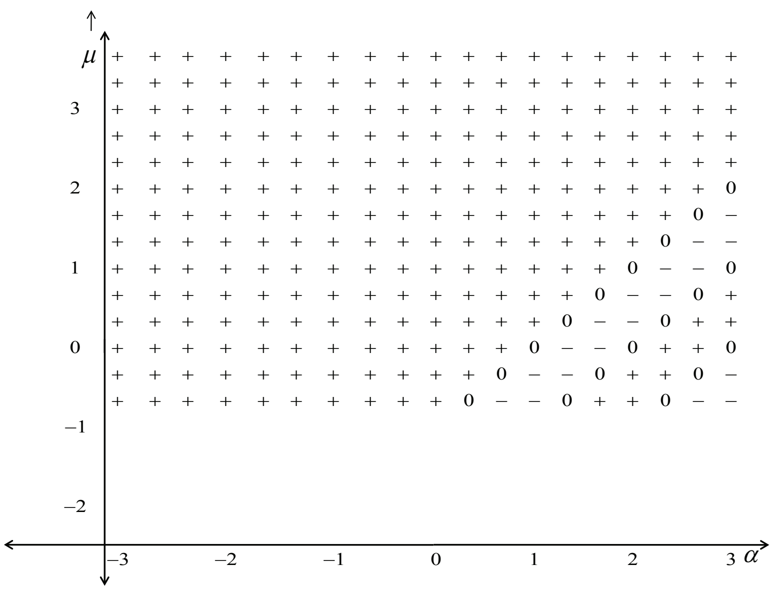

Figure-4.1: The sign of the coefficients $\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}$ in the differ-integration $\frac{\mathrm{d}^{\alpha}x^{\mu}}{\mathrm{d}x^{\alpha}}$ for various μ and α values

Now can we extend the generalized formula to the region $\mu \le -1$ for $f(x) = x^{\mu}$. However, if we tabulate $D_x^n x^m$, (for integers n and m), we observe emerging pattern (see Table-4.1). Recall that we previously discussed the integration side of this table in Chapter-2.

$m \downarrow n \rightarrow$	-3	-2	-1	0	1	2	3
2	$\frac{2!}{5!}x^5$	$\frac{2!}{4!}x^4$	$\frac{2!}{3!}x^3$	x^2	$2!x$	$2!$	0
1	$\frac{1}{4!}x^4$	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x	1	0	0
0	$\frac{1}{3!}x^3$	$\frac{1}{2!}x^2$	x	1	0	0	0
−1	$\int \ln x(\mathrm{d}x)^2$	$\int \ln x\mathrm{d}x$	$\ln x$	x^{-1}	$-x^{-2}$	$2!x^{-3}$	$-3!x^{-4}$
−2	$-\int \ln x\mathrm{d}x$	$-\ln x$	$-x^{-1}$	x^{-2}	$-2!x^{-3}$	$3!x^{-4}$	$-4!x^{-5}$
−3	$\frac{1}{2!}\ln x$	$\frac{1}{2!}x^{-1}$	$-\frac{1}{2!}x^{-2}$	x^{-3}	$\frac{3!}{4!}x^{-4}$	$\frac{4!}{2!}x^{-5}$	$-\frac{5!}{2!}x^{-6}$

Table-4.1: Tabulating the values of the $D_x^n x^m$ for integer m and integer n

We have started with an n-fold integral $\int_a^x \big(f(y)\big)(\mathrm{d}y)^n = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}\big(f(y)\big)\mathrm{d}y$ as a fundamental defining expression, for an n integer. Then we obtained the Riemann-Liouville fractional integral which is an analytic continuation of this, giving us ${}_aD_x^{\alpha} f(x) = \int_a^x \big(f(y)\big)(\mathrm{d}y)^{-\alpha} = \frac{1}{\Gamma(-\alpha)}\int_a^x (x-y)^{-\alpha-1}\big(f(y)\big)\mathrm{d}y$, for $\alpha < 0$ and $\alpha, a \in \mathbb{R}$. After this, we stated that the fractional derivative of the Riemann-Liouville type is an ordinary m differentiation with $m > \alpha$, that is $D_x^{\alpha} f(x) = D_x^m\left(D_x^{-(m-\alpha)} f(x)\right)$ for $\alpha > 0$, and $m \in \mathbb{Z}^+$.

However, $D_x^{\alpha}x^{\mu}$ the RL integral is well defined for $\alpha \in \mathbb{R}$ and $\mu > -1$, as we have seen in the previous chapter (Section-3.2). Can we extend this for $\mu \le -1$, and conduct the analytical continuation? Let us discuss the Euler formula that we obtained for fractional differ-integration as a generalization, that is ${}_0D_x^{\alpha}\left[x^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}x^{\mu-\alpha}$, for any arbitrary α.

Extended fractional calculus takes the ratio of the gamma-function $\lim_{\epsilon\downarrow 0}\frac{\Gamma(1+\mu+\epsilon)}{\Gamma(1+\mu+\epsilon-\alpha)}$ as a fundamental defining expression. Unlike RL integration, giving ${}_0D_x^{\alpha}\left[x^{\mu}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}x^{\mu-\alpha}$ where $f(x) = x^{\mu}$ with $\mu > -1$, this limit i.e.

$\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\in-\alpha)}$, is well-defined in the plane $\alpha,\mu\in\mathbb{R}$, apart from along the line intervals $\mu\in\mathbb{Z}^{-}, \alpha\in\mathbb{R}\setminus\mathbb{Z}$. The operation denoted by symbol '\' is a "mod" operation in sets, and it is akin to a remainder operation. For example, we have $\mathrm{A}=\{1,2,3,4,5,6\}$, $\mathrm{B}=\{1,3,4\}$ then, A "mod" B (symbolically $\mathrm{A}\setminus\mathrm{B}$) is $\mathrm{A}\setminus\mathrm{B}=\{2,5,6\}$.

The analytical continuation along the line intervals $\mu\in\mathbb{Z}^{-}$, $\alpha\in\mathbb{R}\setminus\mathbb{Z}$ will be derived subsequently. The log-region belongs to the integration operation, and, as we discussed in Section-2.7, the analytic continuation of the fractional differ-integration to this log zone (in the line intervals $\mu\in\mathbb{Z}^{-}$, $\alpha\in\mathbb{R}\setminus\mathbb{Z}$) is found using the following formula

$$\begin{aligned} D_x^{\alpha}x^{\mu} &= \lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}D_x^{\alpha-\mu}\ln x \\ &= \lim_{\in\downarrow 0}\left(\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}\right)\left(\frac{x^{\mu-\alpha}}{\Gamma(1+\mu-\alpha)}\right)\left(\ln x-\psi(1+\mu-\alpha)-\gamma\right) \end{aligned} \tag{4.251}$$

We documented this in Sections 2.6 and 2.7.

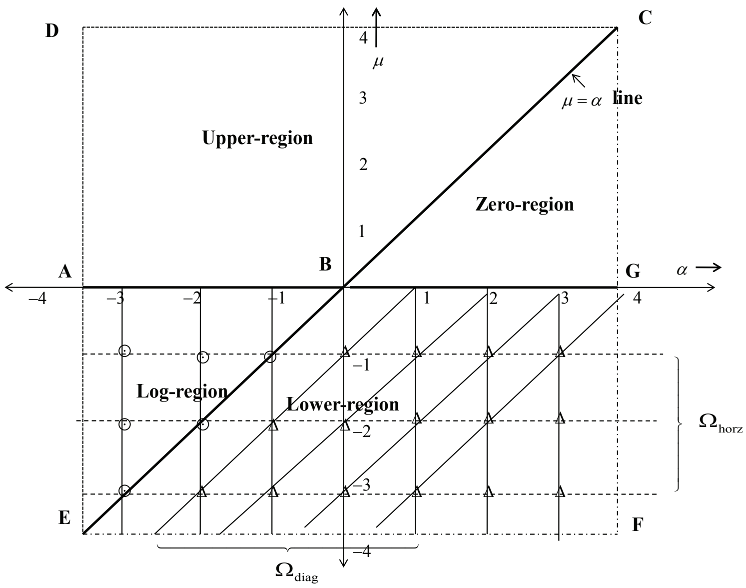

Figure-4.2: Different regions of the $D_x^{\alpha}x^{\mu}$ for analytic continuation in $\alpha-\mu$ plane

We draw a $\alpha-\mu$ diagram, (Figure-4.2) based on our discussion about $D_x^{\alpha}x^{\mu}$, and its Euler's generalization, plus the difficulties encountered in integration, with an observation from Table-4.1, showing four different regions; namely the Upper-region (ABCD), Lower-region (EFGB), Log-region (AEB), and Zero-region (BGC). We define them as follows in zones in the $\alpha-\mu$ plane

$$\begin{array}{lll} \text{Upper - region} & : \ \mu\ge\alpha, \ \ \mu\ge 0 & \\ \text{Lower - region} & : \ \mu<\alpha, \ \ \mu<0 & \text{Differentiation Right - Half - Plane} \ \ \alpha>0 \\ \text{Log - region} & : \ \mu\ge\alpha \ \ \ \mu<0 & \text{Integration Left - Half - Plane} \ \ \alpha<0 \\ \text{Zero - region} & : \ \mu<\alpha \ \ \ \mu\ge 0 & \end{array}$$

The 'Δ points' are 'grid-points' in the Lower-region, and $\odot$ denotes a 'grid-point' in the Log-region, in Figure-4.2. In the Upper-region (region ABCD in Figure-4.2), the ratio $\frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)}$ is finite for $\mu\ge\alpha$ and $\mu\ge 0$. Therefore, in this Upper-region the formula is as follows:

$${}_0D_x^{\alpha}x^{\mu}=\frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)}x^{\mu-\alpha} \tag{4.252}$$

In the Lower-region (region EFGB in Figure-4.2) and Log-region (region AEB-in Figure-4.1), a region of $\Omega_{\mu<0}$ is defined as a union of the Lower and Log-regions, such that $\mu<0$ and $\alpha\in\mathbb{R}$ (in the lower half plane region AEFG-in Figure-4.2). In this region, the AEFG defines Ω_{horz} as a set of horizontal lines at negative integer points as with

$\mu = -n_+$, with $n_+ = 1,2,3,...$, i.e. $n_+ \in \mathbb{Z}^+$ (integers). In this region, we define Ω_{grid} as a set of integer points in the $\Omega_{\mu<0}$ region AEFG, such that $\alpha \in \mathbb{Z}$, $\mu \in \mathbb{Z}^-$. Also in Figure-4.2, Ω_{diag} are a set of right-sloping diagonal lines in the lower region (EFGB), such that $\mu = (\alpha - n_+) < 0$, $n_+ \in \mathbb{Z}^+$, $\alpha, \mu \in \mathbb{R}$.

The ratio $\frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)}$ is finite everywhere in the region $\Omega_{\mu<0}$ except along the Ω_{horz} horizontal lines "mod" and the Ω_{grid} grid points, (that is in $\Omega_{\text{horz}} \setminus \Omega_{\text{grid}}$). The ratio is zero along the diagonals in the lower region Ω_{diag} "mod" grid points Ω_{grid} (that is $\Omega_{\text{diag}} \setminus \Omega_{\text{grid}}$). However, the $\lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\in-\alpha)}$ evaluated at the points in Ω_{grid} is convergent.

Thus we write in the region $\Omega_{\mu<0}$ "mod" (Ω_{horz} "mod" Ω_{grid}) or $\Omega_{\mu<0} \setminus (\Omega_{\text{horz}} \setminus \Omega_{\text{grid}})$.

$${}_0D_x^{\alpha} x^{\mu} = \lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\in-\alpha)} x^{\mu-\alpha} \tag{4.253}$$

From Table-4.1, the natural analytic continuation of ${}_0D_x^{\alpha} x^{\mu}$ in part of the $\Omega_{\text{horz}} \setminus \Omega_{\text{grid}}$ lying within the Log-region is the following:

$$\lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(\in)} \int \ln x (\mathrm{d}x)^{\mu-\alpha} \equiv \lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(\in)} \left({}_0D_x^{\alpha-\mu} \ln x \right) \tag{4.254}$$

This is derived from the chapter on fractional integration (Sections 2.6 and 2.7), where the sign of α is shown to be the opposite of the above (4.254). We have also derived the relationship highlighted in Section-2.8, i.e. $\int \ln x (\mathrm{d}x)^{\upsilon} = D_x^{-\upsilon} \ln x = \frac{x^{\upsilon}}{\Gamma(1+\upsilon)} \left(\ln x - \left(\psi(1+\upsilon) + \gamma \right) \right)$. With this relationship and the above (4.254) correlation, we can analytically continue from $\Omega_{\text{horz}} \setminus \Omega_{\text{grid}}$ lying in the Log-region into that lying in the lower region. Thus in the $\Omega_{\text{horz}} \setminus \Omega_{\text{grid}}$, we have $\mu \in \mathbb{Z}^-$, $\alpha - \mu \notin \mathbb{Z}$ and analytic continuation as follows:

$$\begin{aligned} D_x^{\alpha} x^{\mu} &= \lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(\in)} D_x^{(\alpha-\mu)} \ln x \\ &= \lim_{\in\downarrow 0} \frac{\Gamma(1+\mu+\in)}{\Gamma(\in)} \frac{x^{(\mu-\alpha)}}{\Gamma(1+\mu-\alpha)} \left(\ln x - \left(\psi(1+\mu-\alpha) + \gamma \right) \right) \end{aligned} \tag{4.255}$$

Now we deal with the Zero-region (in Figure-4.1, the region BGC). Here we have two possible choices, from two views. The first view is that, with a postulate, the fractional derivative of a constant can be non-zero. As such, the Riemann-Liouville integral should be taken as a fundamental defining relationship and then used to derive a fractional derivative in that Zero-region. Then we write

$$D_x^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha} \tag{4.256}$$

in the Zero-region.

In the second postulate, we say as the ordinary whole integer order derivative of a constant is zero, the fractional derivative shall inherit this property, i.e. the fractional derivative of a constant will be zero. In addition, here we also postulate that the fractional derivatives commute. For a constant function $f(x) = c$, we write in this case

$$D_x^1 c = 0 \qquad D_x^{\alpha} c = D_x^{(\alpha-1)} \left(D_x^1 c \right) = D_x^{(\alpha-1)} \left[0 \right] = 0, \quad \text{for } \alpha > 1 \tag{4.257}$$

As $D_x^{\alpha} x^{\alpha} = \Gamma(1+\alpha)$, for $\alpha \geq 0$ by continuity $D_x^{\alpha} x^{\mu} = 0$ for $\alpha > \mu$, $\mu \geq 0$ implies that

$$D_x^{\alpha} x^{\mu} = 0 \tag{4.258}$$

in the Zero-region, and commutability is preserved in this region.

Therefore, we have a mutual trade-off between these two postulates, with the options:

1. We choose analyticity and lose commutability in the Zero-region.
2. We choose to preserve and carry commutability and lose analyticity at the edges of the Zero-region.

The first type is recognized as the Riemann-Liouville type fractional derivative and the second type is recognized as being similar to the Caputo type fractional derivative. With the above points, we write the analytic continuation for the fractional derivative formula of the Euler formula for the entire $\alpha-\mu$ plane. As such, we write the following for the two types of derivatives.

a. Riemann-Liouville type: That is

$$
{}_0D_x^{\alpha}x^{\mu}=\begin{cases}\lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}\frac{x^{\mu-\alpha}}{\Gamma(1+\mu-\alpha)}\left(\ln x-\left(\psi(1+\mu-\alpha)+\gamma\right)\right) & \text{in }\Omega_{\text{horz}}\setminus\Omega_{\text{grid}}\\ \lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\in-\alpha)}x^{\mu-\alpha} & \text{elsewhere}\end{cases} \tag{4.259}
$$

b. Caputo Type: That is

$$
{}_0D_x^{\alpha}x^{\mu}=\begin{cases}0 & \text{in Zero - region}\\ \lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(\in)}\frac{x^{\mu-\alpha}}{\Gamma(1+\mu-\alpha)}\left(\ln x-\left(\psi(1+\mu-\alpha)+\gamma\right)\right) & \text{in }\Omega_{\text{horz}}\setminus\Omega_{\text{grid}}\\ \lim_{\in\downarrow 0}\frac{\Gamma(1+\mu+\in)}{\Gamma(1+\mu+\in-\alpha)}x^{\mu-\alpha} & \text{elsewhere}\end{cases} \tag{4.260}
$$

4.22 Analytic continuation of the differ-integral operator from a real to complex order

To analytically continue D^{α} to D^{z} where $z=\sigma+it$ is a complex number, we just generalize $D_x^z x^w$, with complex z and w. For the Caputo type fractional derivative, we assume that commutability is preserved. That means the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ commutes with itself and its inverse, which is the integral operator $\int \mathrm{d}x$; and so does D^z. This assumption makes the complex differ-integration easy, and we discuss this type.

With this assumption, the commutative operator D^z splits into two. Each of these products acts on the function independently, as follows:

$$
D^z=D^{\sigma+it}=D^{\sigma}D^{it}=D^{it}D^{\sigma} \tag{4.261}
$$

Consider $D_x^z x^w$ with $z=\sigma+it$ and $w=u+iv$. In the Zero-region of the $(\sigma-u)$-plane we have:

$$
D_x^{\sigma}x^{u}=0 \quad \text{implies} \quad D_x^{it}\left(D_x^{\sigma}x^{u}\right)=D_x^{z}x^{u}=0 \tag{4.262}
$$

Similarly in the Zero-region of the $(it-iv)$-plane:

$$
D_x^{it}x^{iv}=0 \quad \text{implies} \quad D_x^{\sigma}\left(D_x^{it}x^{iv}\right)=D_x^{z}x^{iv}=0 \tag{4.263}
$$

We can think of the above (4.263) as D^{it} expanding into the triangular Zero-region of the $(\sigma-u)$-plane into a wedge shaped volume of infinite length along t-direction, and similarly expanding that $(it-iv)$ plane into another along the σ-direction, in order to define $D^z x^w=0$. This is depicted in Figure-4.3.

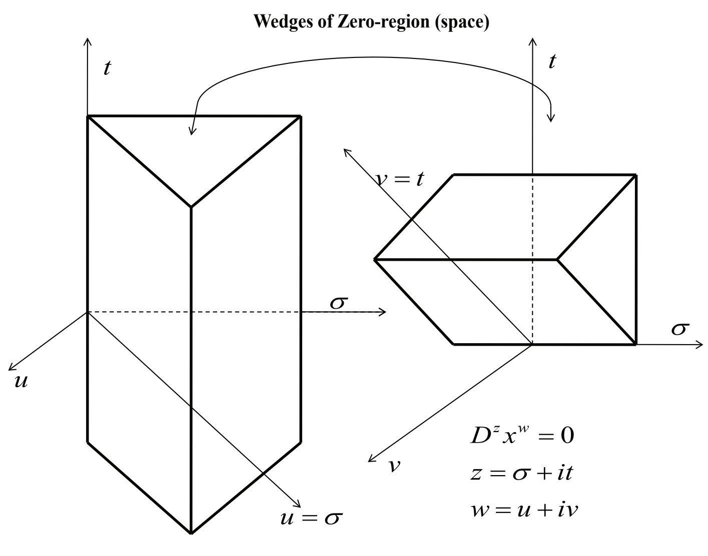

Figure-4.3: The Zero-region (space) wedges in the $(z-w)$ diagram where $D^z x^w=0$

For the line interval in Ω_{horz} "mod" Ω_{grid}, the operator D^{it} extends these intervals in the $(\sigma-u)$-plane, along the $t-$direction to planer sections, as is the case for D^{σ}. By now, we have completed the task of defining the action and have obtained a complete picture of the operator D^z . In essence, what we have done is an analytic continuation of an operator to act on a function complex $z-$times. Let us stop here, and we will find out more about the operator D^z via Laplace transforms in Chapter-5.

4.23 Short summary

In this chapter, we have further extended the concept of fractional differ-integration. We have seen a number of interesting results of various fractionally differ-integrating functions. In addition, we have noted the very interesting fact of the existence of the finite value of fractional differ-integration for a non-differentiable point. Here, we have further generalized Leibniz's product rule and saw the difficulties in generalizing the chain rule, and dealt with the analytic-continuation of fractional differ-integration to complex orders.

4.24 References

This chapter is motivated by a number of pioneering works. They are:

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Further details of these works are listed in the bibliography section,in alphabetical order

CHAPTER FIVE

GENERALIZED INITIALIZATION OF FRACTIONAL INTEGRALS/ DERIVATIVES AND GENERALIZED LAPLACE TRANSFORMS

5.1 Introduction

In this chapter, we will see how the initialization process of fractional calculus is different from that of classical calculus theory, and the impact this difference has on the Riemann-Liouville fractional derivative and the Caputo fractional derivative. We will discuss why fractional derivatives should require initialization, unlike in the case of classical calculus. We will see that, for fractional integration, the initialization value is not a constant, as it is for classical integration, but is, in fact, a function. We will elaborate upon all these concepts using examples, and adapting Lorenzo and Hartley's approach. Then we will explore the formulation of Laplace and Fourier transforms in the case of fractional differ-integration operators. Here, we will also discuss the concept of generalized initial conditions and introduce the complex order differ-integrals.

Mathematicians have used several different types of notations since the birth of fractional calculus, as mentioned earlier in the book. Here, we attempt to standardize these notations as differ-integral (differentiation integral) operators, with respect to uninitialized and initialized parts. The same operator is used as an integrator when the index is negative and as a differentiator when the index is positive, as we have discussed in previous chapters. Separate notation will be used to indicate the initialized differ-integral operator and the un-initialized operator.

The symbol ${}_cD_t^q\left[f(t)\right]$ represents the 'initialized' $q-$th order differ-integration of $f(t)$ from the start point c to t. The notation ${}_c\mathrm{d}_t^q\left[f(t)\right]$ represents an 'un-initialized' generalized (or fractional) $q-$th order differ-integral. This is also the same as $\frac{\mathrm{d}^q f(t)}{\left[\mathrm{d}(t-c)\right]^q} \equiv {}_c\mathrm{d}_t^q\left[f(t)\right]$ i.e. shifting the origin of the function at the point from where the differ-integration process starts. This un-initialized operator can also be presented in short form as $\mathrm{d}^q f(t)$.

The initialization function, which is not a constant in the case of fractional calculus, is represented as $\psi(f,q,a,c,t)$ meaning that this is function ψ of independent variable t, and is the differ-integral operator of order q, for function f born at $t=a$; before that the function is zero; i.e. $f(t)=0$ for $t<a$, and the differ-integral process starts at $t=c$, where $c>a$. This initialization function can be presented in short form as $\psi(t)$ or $\psi(f,q,t)$. Therefore, the expression between an initialized differ-integral and an un-initialized one is ${}_cD_t^q\left[f(t)\right]={}_c\mathrm{d}_t^q\left[f(t)\right]+\psi(f,q,a,c,t)$ for the function i.e. $f(t)$ where the function $f(t)$ is defined for $t\geq a$ and $f(t)=0$ for $t<a$. The fractional differentiation or fractional integration starts at point $t=c$, where $c>a$. This is a causal or left fractional differ-integration process. In practical applications, it is usually the case that the problem to be solved is in some way isolated from the past. That is, it should not be necessary to retreat to $-\infty$ in time to start the analysis.

Usually, the analyst desires to start the analysis at some time $t=a=t_0$, with the knowledge (or assumption) of all values of the function and its derivatives, specifically $f(t_0)$, $f^{(1)}(t_0)$, and $f^{(2)}(t_0)$, and so on, in the case of integer order classical calculus. In modern parlance, this collection is called the 'state' and contains the effect of all history. Note here the symbol ψ stands for the initialization function. This symbol ψ was used by Riemann; and this should not be confused with the 'psi' function that we used in earlier chapters.

5.2 Initialization fractional integration (the Riemann-Liouville approach)

5.2.1 The origin of the initialization function (or history function) in the fractional integration process

This non-constant initialization function $\psi(t)$, which shall be elucidated clearly and brings out the past history, also allows to define in the fractional integral the effect of the past; namely, the effect of fractionally integrating the function from its birth or start point, before which the function was zero. This added effect will also influence the process after time $t = c$, the start of the integration process.

Consider the fractional order q, with $q > 0$ integration of $f(t)$, the first starting at $t = a$, and the second starting at $t = c > a$ as depicted in the following expressions:

$$
\begin{aligned}
{}_a\mathrm{d}_t^{-q}\left[f(t)\right] &= \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau \\
{}_c\mathrm{d}_t^{-q}\left[f(t)\right] &= \frac{1}{\Gamma(q)}\int_c^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau
\end{aligned}
\tag{5.1}
$$

Assume that the function was born at $t = a$, that is $f(t) = 0$ i.e. $f(t) = 0$ for $t < a$. As such, the region between a and c may be considered as 'history'. The assumption is that the integral ${}_c\mathrm{d}_t^{-q}\left[f(t)\right]$ is properly initialized so that it should work as a continuation of the integral starting at $t = a$.

Therefore an initialization must be added to ${}_c\mathrm{d}_t^{-q}\left[f(t)\right]$, so that the fractional integral starting at $t = c$ should be identical to the result starting at $t = a$ for $t > c$. We call this what Riemann proposed a complementary function to the initialization function ψ. We have for the above argument the following statement

$$
{}_c\mathrm{d}_t^{-q}\left[f(t)\right] + \psi = {}_a\mathrm{d}_t^{-q}\left[f(t)\right] \tag{5.2}
$$

when $t > c$. Then

$$
\psi = {}_a\mathrm{d}_t^{-q}\left[f(t)\right] - {}_c\mathrm{d}_t^{-q}\left[f(t)\right] \tag{5.3}
$$

for $t > c$. Therefore,

$$
\psi = \frac{1}{\Gamma(q)}\int_a^c (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau \equiv {}_a\mathrm{d}_c^{-q} f(t)\,.\ t > c \tag{5.4}
$$

5.2.2 The initialization function is a constant function in the classical integration process

Here in (5.4), ψ is a function of the independent variable t, and is thus not a constant. For integer order integration, we put $q = 1$ into (5.4) and see that

$$
\psi = \int_a^c \left(f(\tau)\right)\mathrm{d}\tau = K \tag{5.5}
$$

i.e. we get K as a constant. Thus for the integer order integration ψ, the initialization function is a constant. Because of the increased complexity of the initialization relative to the integer order calculus, it is important to formalize the initialization process in the case of fractional calculus. This formalization will include the initialization term within the definition of these fundamental fractional order calculus operators.

5.2.3 Types of initialization for fractional integration process

Two types of initialization are considered here. The first is the terminal initialization, where it is assumed that the differ-integral operator can be initialized by effectively differ-integrating prior to the start point, i.e. say time $t = c$. The second is the side-initialization where a fully arbitrary initialization may be applied to the differ-integral operator at time $t = c$. In contemporary terms, these may be stated as 'terminal charging' and 'side charging', as used by Lorenzo and Hartley.

First, we restrict it to the RL type of differ-integrals for formalizing these definitions. This initialization function ψ has the effect of allowing the function $f(t)$ and its derivatives to start at a value other than zero, namely ${}_aD_c^{-q}\left[f(t)\right]$, and continues to contribute to the differ-integral response after $t = c$. That is, a function of time is added to the uninitialized integral, (not just a constant at $t = c$ as in the integer order calculus case).

5.3 Terminal and side initialization for fractional integration

5.3.1 Terminal charging for fractional integration

The process of terminal initialization is also termed as terminal charging. The standard contemporary definition of a fractional integral (RL) is accepted if and only if the differ-integrand function $f(t)=0$ for all $t \le a$. The initialization region is $a \le t \le c$. The fractional integration takes place for $t > c \ge a$. Furthermore, the fractional integration starts at $t=c$ (i.e. the point of initialization).

$$ {}_aD_t^{-q}\left[f(t)\right] \equiv \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau\,,\; q \ge 0 \;\; t > a \tag{5.6}$$

subject to $f(t)=0$ for all $t \le a$. The following definition of fractional integration will apply generally (at any $t > c \ge a$)

$$ {}_cD_t^{-q}\left[f(t)\right] \equiv \left(\frac{1}{\Gamma(q)}\int_c^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau\right) + \psi(f,-q,a,c,t) \tag{5.7}$$

with $q \ge 0$, $t > a$, $c \ge a$, and $f(t)=0$ for $t \le a$. The function $\psi(f,-q,a,c,t)$ is called the initialization function and will be chosen such that

$$ {}_aD_t^{-q}\left[f(t)\right] = {}_cD_t^{-q}\left[f(t)\right] \qquad t > c \tag{5.8}$$

This above (5.8) condition gives us the following expression

$$ \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau = \left(\frac{1}{\Gamma(q)}\int_c^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau\right) + \left(\psi(f,-q,a,c,t)\right) \tag{5.9}$$

Since, $\int_a^t \left(g(\tau)\right)\mathrm{d}\tau = \int_a^c \left(g(\tau)\right)\mathrm{d}\tau + \int_c^t \left(g(\tau)\right)\mathrm{d}\tau$ we write the following

$$ \psi(f,-q,a,c,t) = {}_aD_c^{-q}\left[f(t)\right] = \frac{1}{\Gamma(q)}\int_a^c (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau \tag{5.10}$$

where we have $t > c$ and $q > 0$. This expression (5.10) for $\psi(t)$ gives us 'terminal initialization', and brings out in the definition of the fractional integral the effect of 'history'. This history is the effect of fractionally integrating the function $f(t)$ from a to c. This effect is also called 'terminal initialization'.

5.3.2 Evaluation of initialization function for $f(t)=t$ for semi-integration process starting at $t=1$

As an example, let us take function $f(t)=t$ for $t \ge 0$ and $f(t)=0$ for $t<0$; thus we say the start point of a function is $a=0$. We need to find the initialized fractional integration of this function for order $q = \tfrac{1}{2}$, from the start point $c=1 \ne a$. The semi-integral process of initialization is demonstrated below ((5.11) to (5.13)) from the start point of the integration, that is at $c=t=1$. By applying RL formulations to fractional integration, we obtain the following fractional integration from 0 to t.

$$\begin{aligned} {}_0D_t^{-\frac{1}{2}}\left[t\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\frac{1}{2}-1}(\tau)\mathrm{d}\tau \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_t^0 \frac{(t-x)}{x^{\frac{1}{2}}}(-\mathrm{d}x) = \frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} \end{aligned} \tag{5.11}$$

Now we conduct the fractional integration process in the interval 1 to t, i.e. for $t \ge 1$, as in the following steps, with the initialization function, i.e.:

$$\begin{aligned}
{}_1D_t^{-1/2}\left[t\right] &= {}_1\mathrm{d}_t^{-1/2}\left[t\right] + \psi\left(t,-\tfrac{1}{2},0,1,t\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t (t-\tau)^{(1/2)-1}\tau\,\mathrm{d}\tau + \psi\left(t,-\tfrac{1}{2},0,1,t\right) \\
&= \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t \left(t-\tau\right)^{-1/2}\left(\tau\right)(\mathrm{d}\tau)\right) + +\psi\left(t,-\tfrac{1}{2},0,1,t\right), \quad \left(t-\tau\right)=x \\
&= \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{t-1}^{0} \frac{(t-x)}{x^{1/2}}(-\mathrm{d}x)\right) + \psi\left(t,-\tfrac{1}{2},0,1,t\right), \qquad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi} \\
&= \frac{2}{3\sqrt{\pi}}\left[(t-1)^{1/2}(2t+1)\right] + \psi\left(t,-\tfrac{1}{2},0,1,t\right)
\end{aligned} \tag{5.12}$$

Therefore, the initialization function for semi-integration is derived by subtracting the values obtained in the above process as follows:

$$\begin{aligned}
{}_1D_t^{-1/2}\left[t\right] &= {}_1\mathrm{d}_t^{-1/2}\left[t\right] + \psi\left(t,-\tfrac{1}{2},0,1,t\right) \\
&= {}_0D_t^{-1/2}\left[t\right] = \frac{4}{3\sqrt{\pi}}t^{3/2} \\
\psi\left(t,-\tfrac{1}{2},0,1,t\right) &= {}_0D_t^{-1/2}\left[t\right] - {}_1\mathrm{d}_t^{-1/2}\left[t\right] \\
\psi\left(t,-\tfrac{1}{2},0,1,t\right) &= \frac{2}{3\sqrt{\pi}}\left[2t^{3/2} - (2t+1)(t-1)^{1/2}\right]
\end{aligned} \tag{5.13}$$

The nature of the initialization function should be noted, as it is a semi-integration of the function from 0 to 1, that is $\psi\left(t,-\tfrac{1}{2},0,1,t\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (t-\tau)^{-1/2}\tau\,\mathrm{d}\tau$; $\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$, and is a function of variable t. This we calculate in the following steps:

$$\begin{aligned}
\psi\left(t,-\tfrac{1}{2},0,1,t\right) &= {}_0D_1^{-1/2}\left[t\right] \\
&= \frac{1}{\sqrt{\pi}}\int_0^1 (t-\tau)^{-1/2}\tau\,\mathrm{d}\tau; \qquad (t-\tau)=x, \quad \mathrm{d}\tau=-\mathrm{d}x \\
&= \frac{1}{\sqrt{\pi}}\int_{t-1}^t x^{-1/2}(t-x)\left(\mathrm{d}x\right) = \frac{1}{\sqrt{\pi}}\left(t\int_{t-1}^t x^{-1/2}\mathrm{d}x - \int_{t-1}^t x^{1/2}\mathrm{d}x\right) \\
&= \frac{1}{\sqrt{\pi}}\left(t\left(2t^{1/2} - 2(t-1)^{1/2}\right) - \left(\tfrac{2}{3}t^{3/2} - \tfrac{2}{3}(t-1)^{3/2}\right)\right) \\
&= \frac{1}{\sqrt{\pi}}\left(2t^{3/2} - 2t(t-1)^{1/2} - \tfrac{2}{3}t^{3/2} + \tfrac{2}{3}(t-1)^{3/2}\right) \\
&= \frac{2}{3\sqrt{\pi}}\left(2t^{3/2} - (t-1)^{1/2}(2t+1)\right)
\end{aligned} \tag{5.14}$$

In the above steps for fractionally integrating the function (5.14), there is no requirement for having an initialization function. We observe in the present example that the initialization function $\psi(t)$ has the component $-\frac{2(t-1)^{1/2}(2t+1)}{3\sqrt{\pi}}$ and a second part $\frac{4}{3\sqrt{\pi}}t^{3/2}$ in (5.14). The un-initialized fractional integral, i.e. ${}_1\mathrm{d}_t^{-1/2}\left[t\right] = \frac{1}{\sqrt{\pi}}\int_1^t (t-\tau)^{-1/2}\tau\,\mathrm{d}\tau$ is $\frac{2(t-1)^{1/2}(2t+1)}{3\sqrt{\pi}}$. With the addition of these two, i.e. ${}_1\mathrm{d}_t^{-1/2}\left[t\right]+\psi(t)$, we get the function $\frac{4}{3\sqrt{\pi}}t^{3/2}$, which is an initialized fractional integration ${}_1D_t^{-1/2}\left[t\right]$ (5.13). This is a continuation of the fractional integration process from the start of the function, from $t=0$, i.e. ${}_0D_t^{-1/2}\left[t\right]$ (5.11). Figure-5.1 shows the above described process of initialization of fractional integration.

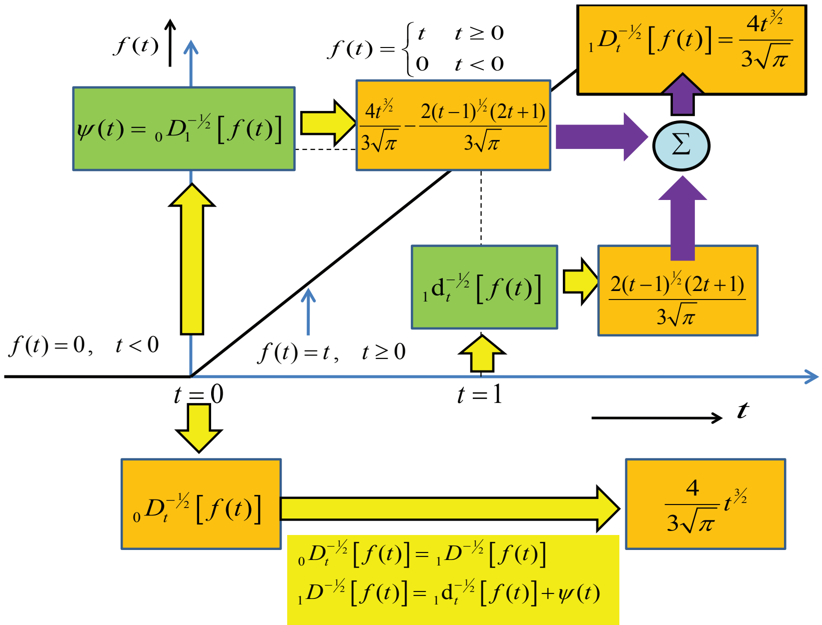

Figure-5.1: Initialized fractional integration example

5.3.3 The initialization concept of classical integration in the context of a developed concept of the initialization function for fractional integration

For the integer $q=1$, we write the following for a function $f(t)=t$ for $t\ge 0$ and $f(t)=0$ for $t<0$

$$\begin{aligned}\lim_{q\to 1}\left(\psi\left(t,-q,0,1,t\right)\right) &= \lim_{q\to 1}\left({}_0D_1^{-q}\left[t\right]\right)\\ &= \lim_{q\to 1}\left(\frac{1}{\Gamma(q)}\int_0^1 (t-\tau)^{q-1}\tau\,\mathrm{d}\tau\right) = \frac{1}{2}\end{aligned} \tag{5.15}$$

We achieve initialization as constant, i.e. $\int_0^1 \tau\,\mathrm{d}\tau = \frac{1}{2}$.

The following steps are an initialized integration in classical calculus although proven in a fractional calculus context.

$$\lim_{q\to 1}\left({}_1D_t^{-q}\left[t\right]\right) = \lim_{q\to 1}\left({}_1\mathrm{d}_t^{-q}\left[t\right]\right) + \lim_{q\to 1}\left(\psi\left(t,-q,0,1,t\right)\right)$$

With $\lim_{q\to 1}\left(\psi\left(t,-q,0,1,t\right)\right) = \frac{1}{2}$

$$\begin{aligned}\lim_{q\to 1}\left({}_1D_t^{-q}\left[t\right]\right) &= \lim_{q\to 1}\left(\frac{1}{\Gamma\left(q\right)}\int_1^t (t-\tau)^{(q)-1}\tau\,\mathrm{d}\tau\right) + \frac{1}{2} = \left(\frac{t^2}{2}-\frac{1}{2}\right) + \frac{1}{2} = \frac{t^2}{2}\\ &= \lim_{q\to 1}\left(\frac{1}{\Gamma\left(q\right)}\int_0^t (t-\tau)^{(q)-1}\tau\,\mathrm{d}\tau\right)\\ &= \lim_{q\to 1}\left({}_0D_t^{-q}\left[t\right]\right)\end{aligned} \tag{5.16}$$

5.3.4 Side charging for fractional integration

When the initial function ψ is taken as an arbitrary function, where the terminal initialization equation (5.10) is not valid, then the effect is called ‘side-initialization’. For side initialization, the initialization function $\psi(t)$ could be anything, such as Dirac’s delta function or Heaviside’s step function or the operation of obtaining the initialization function focusing on any other function than the function that is fractionally integrated i.e. $f(t)$, call it $f_i(t)\neq f(t)$. This process is also called side-charging.

5.4 Initializing the fractional derivative Riemann-Liouville approach

5.4.1 Fractional derivative for a non-local operation requires initialization

This is contrary to the integer order derivative, which is a point-property and a local quantity (or property) where initialization is not called for. However, the definitions of fractional derivatives do contain fractional integration, and thus the fractional derivative of a function is not a point quantity. Therefore, the process of initialization for a fractional derivative is also necessary. The fractional derivative is a non-local operator, and therefore has history and memory. In a solution of differential equations, the initialization constants, which set the initial values of the derivatives, have the effect of accounting for the integration of the derivative from $-\infty$ to the starting time of the integration of the dynamic system given by the differential equation.

5.4.2 The generalization of the initialization process

Extending the generalization concept, the integer order derivative also calls for initialization in a 'fractional context'. Thus, a generalized integer order differentiation is defined (as with initialization) as the following

$$ {}_cD_t^m\left[f(t)\right] \equiv \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m} + \psi(f,m,a,c,t), \quad t > c \tag{5.17}$$

Here in (5.17), m is a positive integer and $\psi(f,m,a,c,t)$ is an initialization function. Now a 'bare' or 'un-initialized' fractional derivative is defined as the following with $q = m - p$

$$ {}_aD_t^q\left[f(t)\right] \equiv {}_aD_t^m\left({}_aD_t^{-p}\left[f(t)\right]\right) \tag{5.18}$$

where $q \geq 0$, $t > a$ and $f(t) = 0$ for $t \leq a$. This means that m is the integer just greater than the fractional order q, by amount p.

The function starts or is born at $t = a$ and before that the value is zero. The differentiation starts at $t > c$. Now as in the fractional integration case the function ψ i.e. $\psi(f,-p,a,a,t) = 0$. Furthermore, consider the function $h(t) = {}_aD_t^{-p}\left[f(t)\right]$ i.e. the fractional integral of a function starting at a with an initialized term $\psi(h,m,a,a,t) = 0$; the initialized fractional derivative looks like, and is defined as, $q \geq 0$ and $t > c \geq a$ in the following expression:

$$ {}_cD_t^q\left[f(t)\right] \equiv {}_cD_t^m\left({}_cD_t^{-p}\left[f(t)\right]\right) \tag{5.19}$$

5.5 Terminal initialization for the fractional derivative

5.5.1 Derivation of terminal charging for the fractional derivative

The definition and concept is similar to that obtained as a terminal initialization for fractional integrals. The requirement is also the same as in the case for fractional integrals, that is

$$ {}_cD_t^q\left[f(t)\right] = {}_aD_t^q\left[f(t)\right] \tag{5.20}$$

for all $t > c \geq a$. Specifically this requires the compatibility of the derivatives starting at $t = a$ and at $t = c$ for all $t > c$. Therefore, it follows that

$$ {}_cD_t^m\left({}_cD_t^{-p}\left[f(t)\right]\right) = {}_aD_t^m\left({}_aD_t^{-p}\left[f(t)\right]\right) \tag{5.21}$$

Expanding the fractional integrals in the above expression (5.21) with initialization, we obtain the following

$$ \begin{aligned} &{}_cD_t^m\left[\int_c^t (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau + \left(\psi(f,-p,a,c,t)\right)\right] \\ &= {}_aD_t^m\left[\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau + \left(\psi(f,-p,a,a,t)\right)\right] \end{aligned} \tag{5.22}$$

for $t > c$, while we also have $\psi(f,-p,a,a,t) = 0$, in the above expansion (5.22). Using, the definition of a generalized integer order derivative (5.7), and as defined above in Section-5.4.2, we get the following

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_c^t (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau + \left(\psi(f,-p,a,c,t)\right)\right] + \psi(h_1,m,a,c,t)$$
$$= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau\right] + \psi(h_2,m,a,a,t) \tag{5.23}$$

where we now call the new functions $h_1 = {}_cD_t^{-p}\left[f(t)\right]$ and $h_2 = {}_aD_t^{-p}\left[f(t)\right]$. In the RHS of the above expression (5.23), the integer order derivative is initialized at $t = a$, thus $\psi(h_2,m,a,a,t) = 0$. This means that there is no initialization for the integer order derivative. After rearranging the integrals from the above expression, we get:

$$\psi(h_1,m,a,c,t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau\right] - \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\psi(f,-p,a,c,t)\right]$$
$$= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau - \psi(f,-p,a,c,t)\right] \tag{5.24}$$

This formula (5.24) is the requirement for the initialization of the derivative in general. The initialization function of fractional integration under the condition of the terminal charging of the fractional integral $\psi(f,-p,a,c,t) = \frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau$ is as defined and derived earlier (5.10) i.e.

$$\psi(f,-p,a,c,t) = {}_aD_c^{-p}\left[f(t)\right] = \frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau \tag{5.25}$$

Therefore, in the derived expression for $\psi(h_1,m,a,c,t)$ (in (5.24)), we place the expression for $\psi(f,-p,a,c,t) = {}_aD_c^{-p}\left[f(t)\right] = \frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau$ and get:

$$\psi(h_1,m,a,c,t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau - \psi(f,-p,a,c,t)\right]$$
$$= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau - \frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau\right] \tag{5.26}$$
$$= 0$$

Thus, we have $\psi(h_1,m,a,c,t) = 0$, and a very important result is seen, that is 'integer order differentiation' which cannot be initialized through the terminal initialization. This is what classical calculus says, that there is no initialization for classical derivatives.

5.5.2 Evaluation of the initialized semi-derivative for $f(t) = (t+2)^2$ starting at $t = 0$

Let us take the function $f(t) = (t+2)^2$ for $t \ge -2$ and $f(t) = 0$ for $t < 0$; thus, this function starts at point $a = -2$, with $f(-2) = 0$.

We need to evaluate the initialized fractional derivative of order $q = \frac{1}{2}$, from the start point $c = 0$ for $t > c$, and $c > a$. Thus, we have the start point of the function as $t = -2$. For $q = \frac{1}{2}$, we have $m = 1$, which means $p = \frac{1}{2}$. The initialized semi-derivative from $c = t = 0$ is the sum of the uninitialized semi-derivative plus the initialization function, as expressed below

$${}_0D_t^{1/2}\left[(t+2)^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right]$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2}^0 (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right] \tag{5 .27}$$
$$= {}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] + \psi\left((t+2)^2, \tfrac{1}{2}, -2, 0, t\right)$$

The uninitialized fractional integration of order p, i.e. semi-integration from the start point $c = t = 0$ for $f(t) = (t+2)^2$, is as follows:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{16}{15}t^{5/2} + \frac{16}{3}t^{3/2} + 8t^{1/2}\right]
\end{aligned}
\tag{5.28}
$$

We will discuss this expression (5.28), along with others from Figure-5.2, in detail in the subsequent section in this chapter.

Differentiating by one whole integer order the above expression, we obtain the RL un-initialized semi-derivative of the function as expressed in the following:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{16}{15}t^{5/2} + \frac{16}{3}t^{3/2} + 8t^{1/2}\right)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\tfrac{8}{3}t^{3/2} + 8t^{1/2} + 4t^{-1/2}\right]
\end{aligned}
\tag{5.29}
$$

We evaluate as total initialization the terminal initialization case for the semi-derivative operation as in the following steps:

$$
\begin{aligned}
\psi\left((t+2)^2, 1/2, -2, 0, t\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2}^{0} (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}(t+2)^{\left(\frac{3}{2}\right)} - \frac{8}{3}t^{\left(\frac{3}{2}\right)} - 8t^{\left(\frac{1}{2}\right)} - 4t^{\left(-\frac{1}{2}\right)}\right]
\end{aligned}
\tag{5.30}
$$

Thus, we get an initialized semi-derivative as follows:

$$
\begin{aligned}
{}_0D_t^{1/2}\left[(t+2)^2\right] &= {}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] + \psi\left((t+2)^2, 1/2, -2, 0, t\right) \\
&= \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{3/2}
\end{aligned}
\tag{5.31}
$$

This (5.31) is the same as the semi-derivative of the function $f(t) = (t+2)^2$; $t \geq 0$ from the start point of the function, i.e. $a = t = -2$, given as:

$$
{}_{-2}D_t^{1/2}\left[(t+2)^2\right] = \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{3/2}
\tag{5.32}
$$

Figure-5.2 gives us a diagrammatic representation of the initialization of a fractional derivative for this example.

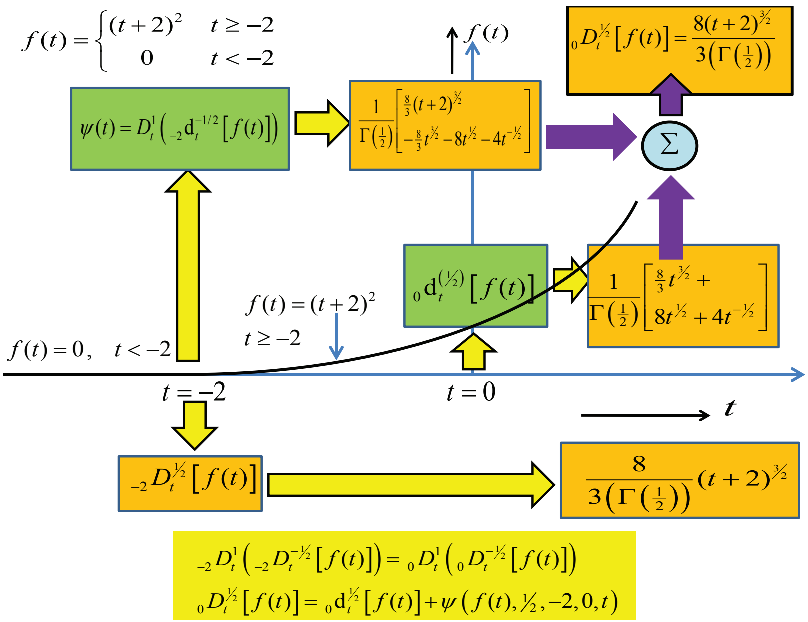

Figure-5.2: Initialized fractional derivative example

5.6 Side-initialization of the fractional derivative

Refer to expression (5.24) that we have obtained i.e.:

$$\psi(h_1,m,a,c,t)=\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)d\tau\right]-\frac{d^m}{dt^m}\left[\psi(f,-p,a,c,t)\right] \tag{5.33}$$

This (5.33) is the requirement for the initialization for a general derivative as obtained in the terminal initialization case (as discussed in Section-5.5). If the side initialization is employed, the function $\psi(f,-p,a,c,t)$ is arbitrary. Thus, we can infer from the requirement equation (5.33), the following:

$$\psi(h_1,m,a,c,t)=D_t^m\left[\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)d\tau\right]-D_t^m\left[\psi(f,-p,a,c,t)\right] \tag{5.34}$$

which is that either $\psi(f,-p,a,c,t)$ or $\psi(h_1,m,a,c,t)$ can be arbitrary, but not both together. In addition, should this then satisfy the requirement expression derived above (5.34), the generalized expression for the side charging case for $t>c$ can be stated as follows:

$${}_cD_t^q\left[f(t)\right]={}_cD_t^m\left(\frac{1}{\Gamma(p)}\int_c^t (t-\tau)^{p-1}f(\tau)d\tau+\left(\psi(f,-p,a,c,t)\right)\right) \tag{5.35}$$

m is a positive integer and $m>q$ with $q=m-p$. From the above expression (5.35), we write the following

$${}_cD_t^q\left[f(t)\right]=\frac{d^m}{dt^m}\frac{1}{\Gamma(p)}\int_c^t (t-\tau)^{p-1}\left(f(\tau)\right)d\tau+\left(\begin{array}{l}\frac{d^m}{dt^m}\left[\psi(f,-p,a,c,t)\right]\\ +\psi(h,m,a,c,t)\end{array}\right) \tag{5.36}$$

where $h(t)={}_aD_t^{-p}\left[f(t)\right]$. Here in (5.36), both the initialization terms are arbitrary and thus may be considered as a single (arbitrary) term, namely:

$$\psi(f,q,a,c,t)\equiv\frac{d^m}{dt^m}\psi(f,-p,a,c,t)+\psi(h,m,a,c,t) \tag{5.37}$$

In the case of the terminal initialization of the fractional integral, the initialization part is as follows:

$$\psi(f,-p,a,c,t)=\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)d\tau\;;\; t>c \tag{5.38}$$

5.7 Initializing fractional differ-integrals using the Grunwald-Letnikov approach

Here in this approach too, we must take the function's starting point as a, and the differ-integration process starting point as $t = c$. An initialization (the notation for which is the same as for the RL-approach) is introduced to account for past-history and goes back to $t = a$, with $f(t) = 0$ at all points before $t = a$. Then the differ-integration with an arbitrary order q is

$$ {}_aD_t^q\left[f(t)\right] = \frac{\mathrm{d}^q\left[f(t)\right]}{\left[\mathrm{d}(t-a)\right]^q} \equiv \lim_{N\uparrow\infty}\left(\frac{\left(\frac{t-a}{N}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(t-j\left(\frac{t-a}{N}\right)\right)\right) \tag{5.39}$$

with $t > a$ and $f(t) = 0$ for $t \le a$. The Grunwald-Letnikov (GL) definition for differ-integrals will generally apply as:

$$ {}_cD_t^q\left[f(t)\right] \equiv \frac{\mathrm{d}^q\left[f(t)\right]}{\left[\mathrm{d}(t-c)\right]^q} + \psi(f,q,a,c,t) \tag{5.40}$$

Here again $\psi(f,q,a,c,t)$ is selected such that ${}_cD_t^q\left[f(t)\right]$ will produce the same result as ${}_aD_t^q\left[f(t)\right]$ for $t > c$. Expressed as the following:

$$\begin{aligned} {}_cD_t^q\left[f(t)\right] &= {}_c\mathrm{d}_t^q\left[f(t)\right] + \psi(f,q,a,c,t) = {}_aD_t^q\left[f(t)\right] \\ \psi(f,q,a,c,t) &= {}_aD_t^q\left[f(t)\right] - {}_c\mathrm{d}_t^q\left[f(t)\right] \end{aligned} \tag{5.41}$$

it is self-explanatory, for all $t > c$, and $f(t) = 0$ for all $t \le a$. Therefore, with $\psi(f,q,a,c,t) = {}_aD_t^q\left[f(t)\right] - {}_c\mathrm{d}_t^q\left[f(t)\right]$ or identifying ${}_aD_t^q\left[f(t)\right]$ as ${}_a\mathrm{d}_t^q\left[f(t)\right]$ i.e. an un-initialized differ-integral, as per the standard notation, we write the same as $\psi(f,q,a,c,t) = {}_a\mathrm{d}_t^q\left[f(t)\right] - {}_c\mathrm{d}_t^q\left[f(t)\right]$. In this, with (5.41) substituting the GL series we obtain the following:

$$\begin{aligned} \psi(f,q,a,c,t) = &\lim_{N_1\uparrow\infty}\left(\frac{\left(\frac{t-a}{N_1}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N_1-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(t-j\left(\frac{t-a}{N_1}\right)\right)\right) \\ &- \lim_{N_2\uparrow\infty}\left(\frac{\left(\frac{t-c}{N_2}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N_2-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(t-j\left(\frac{t-c}{N_2}\right)\right)\right) \end{aligned} \tag{5.42}$$

For all $t > c$ and $f(t) = 0$ for $t < a$. After considerable manipulation by adjusting the delay element as equal, that is $N_2 = \left(\frac{t-c}{t-a}\right)N_1$, and adjusting with $\Delta T = \left(\frac{t-a}{N_1}\right)$ and $N_3 = \left(\frac{c-a}{t-a}\right)N_1$ we write the following expression for the initialization function:

$$\psi(f,q,a,c,t) = \lim_{N_1\uparrow\infty}\left(\frac{\Delta T^{-q}}{\Gamma(-q)}\sum_{j=0}^{N_3-1}\frac{\Gamma(N_1-1-q-j)}{\Gamma(N_1-j)}f\left(t-(N_1-1-j)\Delta T\right)\right) \tag{5.43}$$

5.8 Criteria for the generalized differ-integration composition

5.8.1 Composition rules

One of the fundamental problems of fractional calculus is the requirement that the function and its derivatives be identically equal to zero at the start of initialization (at the start of the differ-integration process) at time $t = c$. This was needed to ensure that the composition or index law holds, implying that:

$$ {}_cD_t^v\left({}_cD_t^u\left[f(t)\right]\right) = {}_cD_t^u\left({}_cD_t^v\left[f(t)\right]\right) = {}_cD_t^{u+v}\left[f(t)\right] \tag{5.44}$$

It is difficult in engineering and physical sciences, to always require that the functions and their derivatives be at zero (or at rest) at initialization instants. This fundamentally implies that 'there can be no initialization or that composition is lost'. Thus, it is not in general true that:

$$f(t) - \frac{\mathrm{d}^{-q}}{\mathrm{d}t^{-q}}\frac{\mathrm{d}^q f}{\mathrm{d}t^q} = 0 \tag{5.45}$$

Therefore, while solving a fractional differential equation of the form in the following:

$$\frac{\mathrm{d}^q f}{\mathrm{d}t^q} = F \tag{5.46}$$

additional terms must be added, such as:

$$f(t) - \frac{\mathrm{d}^{-q}}{\mathrm{d}t^{-q}}\frac{\mathrm{d}^q f}{\mathrm{d}t^q} = C_1 t^{q-1} + C_2 t^{q-2} + ...C_m t^{q-m} \tag{5.47}$$

to achieve the most general solution, which is:

$$f(t) = \frac{\mathrm{d}^{-q} F}{\mathrm{d}t^{-q}} + C_1 t^{q-1} + C_2 t^{q-2} + ... + C_m t^{q-m} \tag{5.48}$$

These issues describe the reality that the index law or composition law is inadequate. A minimal set of criteria have been thought fit to be applied for fractional (or generalized) calculus. They are listed as follows and are called the Ross criteria (1974):

(i) If $f(z)$ is an analytic function of the complex variable z; the differ-integral ${}_cD_z^{v}\left[f(z)\right]$ is an analytic function of z and v;

(ii) The operator ${}_cD_x^{v}\left[f(x)\right]$ must produce the same result for the differentiation, when v is a positive integer;

(iii) If v is a negative integer (say $v = -n$) then ${}_cD_x^{-n}\left[f(x)\right]$ must produce the same result of the $n-$th fold integration of the function $f(x)$ and ${}_cD_x^{-n}\left[f(x)\right]$ must vanish along with $f^{(1)}, f^{(2)}, ...f^{(n-1)}$, and all the $(n-1)$ derivatives at the start point that is $x = c$;

(iv) 'Zero' operation leaves the function unchanged, that is ${}_cD_x^{0}\left[f(x)\right] = f(x)$;

(v) The linearity of the fractional (generalized) differ-integral operator is:

$${}_cD_x^{-q}\left[af(x) + bg(x)\right] = a\left({}_cD_x^{-q}\left[f(x)\right]\right) + b\left({}_cD_x^{-q}\left[g(x)\right]\right) \tag{5.49}$$

(vi) The law of exponents for the arbitrary order holds, meaning:

$${}_cD_x^{-u}\left({}_cD_x^{-v}\left[f(x)\right]\right) = {}_cD_x^{-u-v}\left[f(x)\right] = {}_cD_x^{-v-u}\left[f(x)\right] \tag{5.50}$$

The above notations are used by Ross.

It should be noted that there is a minor conflict contained in these criteria. Also, a clearly noted explanation can be given as ${}_cD_x^{q}\left[f(x)\right]$ in the above criteria, which is un-initialized by differ-integrals. It is correct as the function itself starts at $t = c$ and before that is zero. So at $t = c$, ${}_cD_x^{q}\left[f(x)\right] = {}_c\mathrm{d}_x^{q}\left[f(x)\right]$.

The criteria (ii) & (iii) call for backward compatibility and criterion (vi) calls for the index law to hold vis-à-vis integer order calculus.

The fundamental theorem of an integer order calculus violates this "zero law" as:

$$\mathrm{d}^{-m}\mathrm{d}^{m} f(x) \neq \mathrm{d}^0 f(x) = f(x) \tag{5.51}$$

for all $f(x)$ the integer is m. The fundamental theorem states that (take $m = 1$):

$$\int_c^x \left(f^{(1)}(y)\right)\mathrm{d}y = f(x) - f(c), \qquad {}_c\mathrm{d}_x^{-1}\,{}_c\mathrm{d}_x^{1} f(x) = f(x) - f(c) \tag{5.52}$$

and it can thus be observed that the reversal of differentiation and integration differs from $f(x)$ by $f(c)$, that is by the initialization which is a constant in the classical integer order calculus.

This failure in backward compatibility and index law is handled in the integer order calculus by a constant of integration and by a complementary function for the solution of the differential equations (in an ad-hoc manner).

We summarize the composition law of the index with a simplified example for the following:

1. $D^{-\mu}\left(D^{-v}\left[f(t)\right]\right) = D^{-(\mu+v)}\left[f(t)\right] = D^{-v}\left(D^{-\mu}\left[f(t)\right]\right)$ for $.\mu, v > 0$: this implies that commutation and a computation (additive) are valid for fractional integration.
2. The integer derivative of the fractional integration of a function is not equal to the fractional integration of the integer derivative of the same function. They are related by $D\left(D^{-v}\left[f(t)\right]\right) = D^{-v}\left(D\left[f(t)\right]\right) + \frac{t^{v-1}}{\Gamma(v)} f(0)$; that is the initial condition. Here, the order is $0 < v < 1$. This follows from the fundamental theorem of calculus-a

generalization of the same. Thus, the Riemann-Liouville and Caputo derivatives are not the same unless the initial conditions are zero (or static).

3. Fractional integration of order $(v+1)$ of the integer order derivative of a function is not the same as the fractional integration of order $0<v<1$, but again is related by an initial condition as shown in $D^{-(v+1)}\left(D\left[f(t)\right]\right)=D^{-v}\left[f(t)\right]-\frac{t^{v}}{\Gamma(v+1)}f(0)$.
4. The arbitrary composition is invalid, that is $D^{u}\left(D^{v}\left[f(t)\right]\right)\neq D^{u+v}\left[f(t)\right]$, for any $u,v\in\mathbb{R}$. This composition and commutation is only valid for fractional integrations.

5.8.2 Demonstration of the composition of a fractional derivative, and the fractional integration and mixed operation that is applied to functions $f(t)=t^{1/2}$ & $f(t)=t^{-1/2}$, and discussion

Let us take $f(t)=\sqrt{t}$ with $u=\frac{1}{2}$ and $v=\frac{3}{2}$. First, we calculate $D^{u}\left[f(t)\right]$ as $D^{u}\left[\sqrt{t}\right]=D^{1/2}\left[t^{1/2}\right]=\frac{\sqrt{\pi}}{2}$. Similarly, we get $D^{v}\left[f(t)\right]$, that is $D^{v}\left[\sqrt{t}\right]=D^{3/2}\left[t^{1/2}\right]=0$. Therefore, in this example we have $D^{u}\left(D^{v}\left[f(t)\right]\right)=0$ that is $D^{1/2}\left(D^{3/2}\left[\sqrt{t}\right]\right)=0$.

Now calculate $D^{v}\left(D^{u}\left[f(t)\right]\right)$ as $D^{v}\left(D^{u}\left[\sqrt{t}\right]\right)=D^{3/2}\left(D^{1/2}\left[t^{1/2}\right]\right)=D^{3/2}\left[\frac{\sqrt{\pi}}{2}\right]=-\frac{1}{4\sqrt{t^{3}}}$. Then we calculate $D^{u+v}\left[\sqrt{t}\right]=D^{2}\left[t^{1/2}\right]=-\frac{1}{4\sqrt{t^{3}}}$. This calculation demonstrates that $D^{u}\left(D^{v}\left[f(t)\right]\right)\neq D^{v}\left(D^{u}\left[f(t)\right]\right)$ and $D^{u}\left(D^{v}\left[f(t)\right]\right)\neq D^{u+v}\left[f(t)\right]$. This demonstrates that $D^{u}\left(D^{v}\left[f(t)\right]\right)$ and $D^{v}\left(D^{u}\left[f(t)\right]\right)$ both exist but are not equal.

Things could be worse, specifically in that one of them may not exist. Say we take $f(t)=t^{-1/2}$ and let $u=-\frac{1}{2}$ and $v=1$, then $D^{v}\left(D^{u}\left[f(t)\right]\right)=0$ but $D^{v}\left[f(t)\right]=-\frac{1}{2}t^{-3/2}$ and $D^{u}\left(D^{v}\left[f(t)\right]\right)$ do not exist since the integral $\int_{0}^{t}(t-\tau)^{-1/2}\tau^{-3/2}\mathrm{d}\tau$ diverges.

The discussion point of all these differ-integrations is limited to the real domain. Under the condition of the terminal initialization of the $u-$th and $v-$th differ-integrations

$$ {}_{c}D_{t}^{u}\left({}_{c}D_{t}^{v}\left[f(t)\right]\right)={}_{c}D_{t}^{v}\left({}_{c}D_{t}^{u}\left[f(t)\right]\right)={}_{c}D_{t}^{u+v}\left[f(t)\right] \tag{5.53}$$

for $t>0$ holds under the following condition:

a) $u<0, v<0$ for continuous $f(t)$

b) $u>0, v>0$ for $f(t)$ are m times the whole-differentiable i.e. ${}_{a}D_{t}^{m}\left[f(t)\right]$ exists and is a non-zero continuous function of t for $t>a$, where m is an integer larger than an integer that is part of $\left[u\right]$ or $\left[v\right]$.

c) $u<0, v>0$ is the same as (b).

5.8.3 Composition rules with initialized differ-integrations

Under the conditions of the terminal initialization case, the above-discussed properties and criteria hold, and this in turn provides credibility to the initialized fractional (generalized) calculus. Some conditions are, however, imposed on the linearity of fractional integrals (for $t>c$):

$$ {}_{c}D_{t}^{-v}\left[bf(t)+kg(t)\right]=b\left({}_{c}D_{t}^{-v}\left[f(t)\right]\right)+k\left({}_{c}D_{t}^{-v}\left[g(t)\right]\right) \tag{5.54}$$

holds only if the initialization functions satisfy the following condition

$$\psi(bf+kg,-v,a,c,t)=b\psi(f,-v,a,c,t)+k\psi(g,-v,a,c,t) \tag{5.55}$$

Relative to the criteria of backward compatibility with the integer order calculus, the addition of the initialized function is clearly a generalization relative to integer order calculus. In a strict sense, $\psi(t)\neq 0$ violates criterion 2. However, we are looking for the generalization of integer order calculus and it is clear that this generalization (the addition of the initialization function) will be very useful in many applications.

i. Relative to the criteria of zero order property as holds for terminal initialization;
ii. Relative to the linearity as holds for the terminal initialization subject to the above rule;
iii. Relative to the composition rule the above should follow.

It is noted that the condition, $f^{(k)}(c)=0$ for all k no longer exists. This constraint has effectively been contained (shifted) to the requirement $f(t)=0$ for all $t\le a$. This allows for the initialization of fractional differential equations.

In summary, the terminal initialization case is backwardly compatible with the integer order calculus, which satisfies the applicable criteria established by Ross.

The case for side charging is less definitive. The criteria for backward compatibility are the same as those for the case for terminal charging. Relative to zero property, the condition:

$$\psi(f,-p,a,c,t)=\frac{1}{\Gamma(p)}\int_a^c (t-\tau)^{p-1}\left(f(\tau)\right)\mathrm{d}\tau=0=\psi(h,m,a,c,t) \quad (5.56)$$

is required for side initialization since ψ is arbitrary. When these conditions are not met, the zero order operation on $f(t)$ will return to $f(t)+g(t)$, i.e. the original function with an extra time function $g(t)$, showing the effect of initialization. Relative to the linearity of the side initialization, it demands additional requirements for initialization.

These are not so much of an issue as they initially appear for practical applications. In the solution of fractional differential equations, $\psi(t)$ will be chosen in much the same manner as initializations are currently chosen for ordinary differential equations in classical calculus. This will imply the nature of $f(t)$ from a to c. The new aspect is that to achieve a particular initialization for a given composition, attention must now be paid to the initialization of the composing elements.

5.9 The Relationship between uninitialized Caputo and Riemann-Liouville (RL) fractional derivatives

5.9.1 Expression for the RL fractional integration of $f(t)$ from $0-t$ as integration of function $f(t-\sqrt[q]{x})$ from $0-t^q$

We derived the relationship between the RL and Caputo formulations in Section-3.10; here we will derive the same in a different way. The generalized integration (RL) of any real number q-folds is as follows:

$$ {}_0\mathrm{d}_t^{-q}\left[f(t)\right]=\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau, \qquad q>0 \quad (5.57)$$

In the above (5.57) expression, let us put $\tau=t-x^{\frac{1}{q}}$, giving $\mathrm{d}\tau=-\frac{1}{q}x^{\frac{1-q}{q}}\mathrm{d}x$, substituting this change of variable in above, to get (5.58). The generalized (fractional) order integration relationship is:

$$\begin{aligned} {}_0\mathrm{d}_t^{-q}\left[f(t)\right]&=\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau)\mathrm{d}\tau \\ &=\frac{1}{\Gamma(q)}\int_{t^q}^0\left(t-t+x^{1/q}\right)^{q-1}\left(-\frac{1}{q}x^{\frac{1-q}{q}}\right)\left(f(t-x^{1/q})\right)\mathrm{d}x \\ &=\frac{1}{q\left(\Gamma(q)\right)}\int_{t^q}^0-\left(x^{1/q}\right)^{q-1}x^{\frac{1-q}{q}}\left(f(t-x^{1/q})\right)\mathrm{d}x \\ &=\frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(f(t-x^{1/q})\right)\mathrm{d}x \end{aligned} \quad (5.58)$$

This above expression (5.58) is also discussed in the chapter on fractional integration, and its geometric interpretation is highlighted in Section-2.21. We will stop here and will use the above expression (5.58) later on in this chapter.

5.9.2 Finding an un-initialized Caputo derivative and demonstrating that no singularity term appears at the start point of fractional differentiation

To understand the basic issues of the differences between Caputo and RL derivatives, let us consider a simple function $f(t)=(t+2)^2$ for $t \ge -2$ and $f(t)=0$ for $t<-2$. This means that the system $f(t)$ is at rest before time $t=-2$. Our aim is to differentiate from $t \ge 0$, by order α, we take $\alpha=\frac{1}{2}$ as a demonstration, where the function value is $f(0)=4$, using the forward (left) differentiation method. Therefore, the highest integer order in this case is $m=1>\frac{1}{2}=\alpha$, as required by the Caputo and RL definitions. The rule of a fractional differentiable is maintained as per the existence criteria for this function, that is, the candidate function is defined and bounded in the interval, and at the lower terminal $a=-2$, is 'better' behaved than $(t-a)^{-1}$.

Using the Caputo definition, choosing the integer order derivative as unity, and then obtaining a semi-integration of that derivative of the function from $c=0$, we get the following steps:

$$
\begin{aligned}
{}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= {}_0\mathrm{d}_t^{-1/2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(t+2)^2\right)\\
&= {}_0\mathrm{d}_t^{(-1/2)}\left[(2t)\right]+{}_0\mathrm{d}_t^{(-1/2)}\left[(4)\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(2\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}\tau\,\mathrm{d}\tau+4\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}\mathrm{d}\tau\right)
\end{aligned}
\tag{5.59}
$$

With the change of variable $(t-\tau)=x$, we get $\mathrm{d}\tau=-\mathrm{d}x$ and $\tau=t-x$. By substituting this and integrating it with changed limits, we get the following:

$$
\begin{aligned}
{}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(2\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}\tau\,\mathrm{d}\tau+4\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}\mathrm{d}\tau\right)\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(2\int_t^0 x^{-\frac{1}{2}}(t-x)(-\mathrm{d}x)+4\int_t^0 x^{-\frac{1}{2}}(-\mathrm{d}x)\right)\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(2t\left[\frac{x^{1/2}}{1/2}\right]_0^t-2\left[\frac{x^{3/2}}{3/2}\right]_0^t+4\left[\frac{x^{1/2}}{1/2}\right]_0^t\right)
\end{aligned}
\tag{5.60}
$$

From the above (5.60), after simplification, we write the following:

$$
{}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right]=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}t^{3/2}+8t^{1/2}\right] \tag{5.61}
$$

Note that, in the above expression (5.61), there is no singularity term at the start point, i.e. at $t=0$. Note also that the Caputo definition of a fractional derivative is the 'fractional integration of a derivative of a function'.

5.9.3 Finding an un-initialized Riemann-Liouville (RL) derivative and demonstrating the appearance of a singularity term at the start point of a fractional derivative

The un-initialized RL derivative is as follows:

$$
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right]=\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right]\right) \tag{5.62}
$$

The un-initialized semi-integration is:

$$
{}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right]=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau \tag{5.63}
$$

Changing the variable as $(t-\tau)=x$ gives us the following:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_t^0 x^{\left(-\frac{1}{2}\right)}(t-x+2)^2(-\mathrm{d}x) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_t^0 x^{\left(-\frac{1}{2}\right)}\left[\left((t+2)-x\right)^2\right](-\mathrm{d}x) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_t^0 \left[-(t+2)^2 x^{-\frac{1}{2}} + 2(t+2)x^{\frac{1}{2}} - x^{\frac{3}{2}}\right]\mathrm{d}x \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[2(t+2)^2 t^{1/2} - \frac{4}{3}(t+2)t^{3/2} + \frac{2}{5}t^{5/2}\right]
\end{aligned}
\tag{5.64}
$$

Simplifying the above (5.64) yields the following result:

$$
{}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{16}{15}t^{5/2} + \frac{16}{3}t^{3/2} + 8t^{1/2}\right] \tag{5.65}
$$

Differentiating by a one-whole integer order, we obtain the RL un-initialized semi-derivative of the function as expressed below:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{16}{15}t^{5/2} + \frac{16}{3}t^{3/2} + 8t^{1/2}\right)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}t^{3/2} + 8t^{1/2} + 4t^{-1/2}\right] \\
&= {}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right] + \frac{4t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}
\end{aligned}
\tag{5.66}
$$

Note that the above expression (5.66) has a singularity at the starting point of differentiation, i.e. $t = 0$.

5.9.4 Adding singularity function at the start point of a fractional derivative to equate uninitialized RL and Caputo derivatives

In (5.66), note that $f(0) = 4$, and ${}_0D_t^{1/2}\left[f(0)\right] = \frac{4}{\Gamma\left(\frac{1}{2}\right)}$, i.e. a semi-derivative of a constant such as $f(0) = 4$. From the above example, we have the two following semi-derivatives:

$$
\begin{aligned}
{}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}t^{3/2} + 8t^{1/2}\right] \\
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}t^{3/2} + 8t^{1/2} + 4t^{\left(-1/2\right)}\right]
\end{aligned}
\tag{5.67}
$$

Thus we find that the relationship between the un-initialized RL derivative and the Caputo derivative is ${}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] = {}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right] + \frac{4}{\Gamma\left(\frac{1}{2}\right)}t^{\left(-1/2\right)}$, which we write as the following:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{1/2}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{1/2}\left[f(t)\right] + \frac{f(0)}{\Gamma\left(1-\frac{1}{2}\right)}t^{-\left(1/2\right)} \\
{}_0\mathrm{d}_t^{1/2}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{1/2}\left[f(t)\right] + {}_0D_t^{1/2}\left[f(0)\right]
\end{aligned}
\tag{5.68}
$$

Noting $f(0) = 4$, the above expressions also equate the two definitions (RL and Caputo) through a singularity at the starting point of the differentiation. We highlighted this relationship between RL and Caputo in Section-3.10, and further explore this process in Figure-5.3.

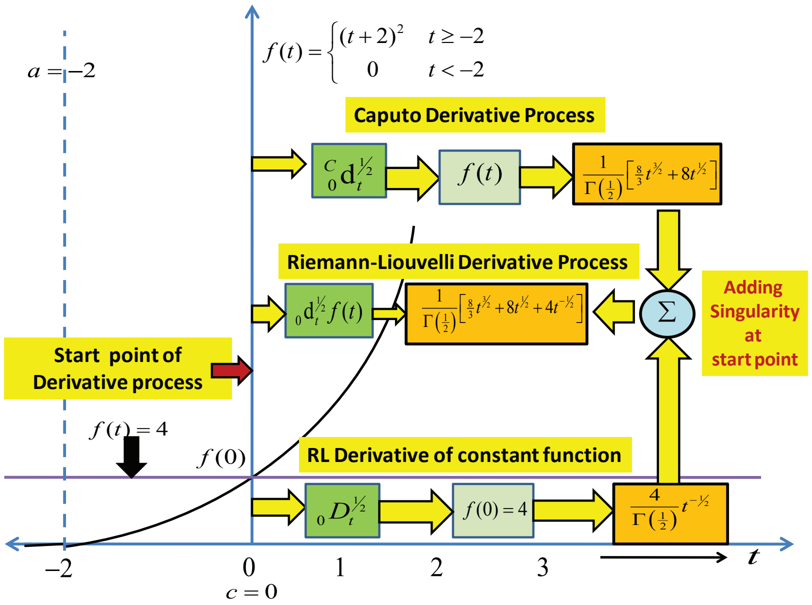

Figure-5.3: Adding a singularity at the start point to the Caputo derivative to provide an example of Riemann-Liouville (RL) to Caputo conversion

We can therefore write a general expression for an $\alpha<1$ order RL and the Caputo derivative as follows for the function $f(t)$:

$$ {}_a\mathrm{d}_t^{\alpha}\left[f(t)\right]={}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right]+\frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{5.69}$$

These two derivatives are equal only if at the starting point of fractional differentiation the function is at rest i.e. $f(a)=0$, or if the start point of differentiation coincides with the start of the function or the process (as demonstrated below).

We say the starting point of the function is $t=t_0$. For $t\le t_0$, we have $f(t)=0$. Generally, if the starting point of differentiation differs from the start of the function (or process), these two derivatives are not equal. The Caputo derivative is added by a function, which is singular at the start point, of the differentiation process (see Figure-5.3), to get the Riemann-Liouville fractional derivative. However, at a steady state (i.e. for t tending to ∞ we have $\lim_{t\uparrow\infty}(t-a)^{-\alpha}=0$; $\alpha>0$), these two fractional derivatives converge, as is expressed below:

$$\lim_{t\uparrow\infty}\left({}_a\mathrm{d}_t^{\alpha}\left[f(t)\right]\right)=\lim_{t\uparrow\infty}\left({}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right]\right) \tag{5.70}$$

Similarly, if we push the start point of the fractional differentiation process, i.e. $t=a$, to $-\infty$, the two fractional derivatives converge; that is:

$$\lim_{a\downarrow-\infty}\left({}_a\mathrm{d}_t^{\alpha}\left[f(t)\right]\right)=\lim_{a\downarrow-\infty}\left({}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right]\right) \tag{5.71}$$

5.9.5 Finding out the relationship of the RL-Caputo fractional derivative via Leibniz's formula

For the generalization proof of ${}_a\mathrm{d}_t^{\alpha}\left[f(t)\right]={}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right]+\frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$, let us consider $a=0$, and observe the effect of the differentiation of the integral. From the definition of the fractional integral as discussed above in (5.58), we express the equivalence obtained as follows:

$$\begin{aligned}{}_0\mathrm{d}_t^{-q}\left[f(t)\right]&=\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau, \qquad q>0\\&=\frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(f(t-x^{1/q})\right)\mathrm{d}x\end{aligned} \tag{5.72}$$

To differentiate the above, we use Leibniz's rule, of differentiation of an integration (of several variables), namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\left(G(x,t)\right)\mathrm{d}x=G(t,t)+\int_0^t\left(\frac{\partial G}{\partial t}\right)\mathrm{d}x \tag{5.73}$$

The rule (5.73) has a function in (x,t). When $x^{1/q}=y$ we obtain $x=y^q$ and $\mathrm{d}x=\left(qy^{q-1}\right)\mathrm{d}y$ in order to obtain $G(t,x)=f(t-x^{1/q})=f(t-y)$. Then the fractional integral expression becomes the following:

$$\begin{aligned} {}_0\mathrm{d}_t^{-q}\left[f(t)\right] &= \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(f(t-x^{1/q})\right)\mathrm{d}x \\ &= \frac{1}{\Gamma(q+1)}\int_0^{t}\left(f(t-y)\right)\left(qy^{q-1}\right)\mathrm{d}y \\ &= \frac{1}{\Gamma(q+1)}\int_0^{t} G(t,y)\mathrm{d}y \end{aligned} \tag{5.74}$$

With the differentiation of the above expression (5.74) and using Leibniz's rule and $G(t,x)=qx^{q-1}f(t-x)$, one obtains the following:

$$\begin{aligned} \mathrm{d}_t^1\left({}_0\mathrm{d}_t^{-q}\left[f(t)\right]\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(q+1)}\int_0^t G(t,x)\mathrm{d}x\right) \\ \mathrm{d}_t^1\left({}_0\mathrm{d}_t^{-q}\left[f(t)\right]\right) &= \frac{1}{\Gamma(q+1)}G(t,t)+\frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(\frac{\partial}{\partial t}\left[f(t-x^{1/q})\right]\right)\mathrm{d}x \end{aligned} \tag{5.75}$$

The first term in RHS of the above (5.75) is obtained by placing $x=t$ in the expression $G(t,x)=q\left(x^{q-1}\right)\left(f(t-x)\right)$, thus we obtain $G(t,t)=q\left(t^{q-1}\right)\left(f(0)\right)$. The second in RHS of the above (5.75) expression term, is obtained by putting $t-x^{1/q}=\xi$; we get $x=(t-\xi)^q$, $\mathrm{d}\xi=-\frac{1}{q}x^{(1/q)-1}\mathrm{d}x$ as depicted in the following steps:

$$\begin{aligned} \frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(\frac{\partial}{\partial t}\left[f(t-x^{1/q})\right]\right)\mathrm{d}x &= \frac{1}{\Gamma(q+1)}\int_t^0 -(\mathrm{d}\xi)\left(\frac{q}{x^{(1/q)-1}}\right)\frac{\mathrm{d}}{\mathrm{d}t}f(\xi) \\ &= \frac{1}{q\left(\Gamma(q)\right)}\int_0^t \mathrm{d}\xi\left(\frac{q}{\left[(t-\xi)^q\right]^{\left(\frac{1-q}{q}\right)}}\right)\frac{\mathrm{d}}{\mathrm{d}t}f(\xi) \\ &= \frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}\frac{\mathrm{d}}{\mathrm{d}t}f(\xi)\mathrm{d}\xi \\ &= {}_0\mathrm{d}_t^{-q}\left(\mathrm{d}_t^1\left[f(t)\right]\right) \end{aligned} \tag{5.76}$$

Substituting these two terms, and simplifying what we obtain, the derivative of the (fractional) integral operation is as follows:

$$\begin{aligned} \mathrm{d}_t^1\left({}_0\mathrm{d}_t^{-q}\left[f(t)\right]\right) &= \frac{1}{\Gamma(q+1)}G(t,t)+\frac{1}{\Gamma(q+1)}\int_0^{t^q}\left(\frac{\partial}{\partial t}\left[f(t-x^{1/q})\right]\right)\mathrm{d}x \\ &= \frac{1}{q\left(\Gamma(q)\right)}f(0)qt^{q-1}+\frac{1}{\Gamma(q)}\int_0^t (t-\xi)^{q-1}\frac{\mathrm{d}}{\mathrm{d}t}f(\xi)\mathrm{d}\xi \\ &= \frac{t^{q-1}}{\Gamma(q)}f(0)+{}_0\mathrm{d}_t^{-q}\left(\mathrm{d}_t^1\left[f(t)\right]\right) \end{aligned} \tag{5.77}$$

Rearranging the above, we obtain the useful expression:

$$\mathrm{d}_t^1\left({}_0\mathrm{d}_t^{-q}\left[f(t)\right]\right)={}_0\mathrm{d}_t^{-q}\left(\mathrm{d}_t^1\left[f(t)\right]\right)+\frac{t^{q-1}}{\Gamma(q)}f(0) \tag{5.78}$$

Putting $q=1-\alpha$, and applying the definition of the RL and Caputo derivatives we find the relation between RL and Caputo, which is:

$$ {}_0\mathrm{d}_t^{\alpha}\left[f(t)\right] = {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0) \tag{5.79} $$

5.9.6 Evaluation of the RL and Caputo derivatives from a non-zero start point where the value of the function is zero

Let us now evaluate the fractional Caputo and RL derivatives from the start point of the function itself, that is say at $a = -2$, for the function $f(t) = (t+2)^2$ where the value of the function is zero, at $f(-2) = 0$. The Caputo derivative is derived as follows:

$$
\begin{aligned}
{}_{-2}^{C}\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= {}_{-2}\mathrm{d}_t^{-1/2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(t+2)^2\right] = {}_{-2}\mathrm{d}_t^{-1/2}\left[(2t)\right] + {}_{-2}\mathrm{d}_t^{-1/2}\left[(4)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[2\int_{-2}^{t}(t-\tau)^{\left(\frac{1}{2}\right)-1}\tau\,\mathrm{d}\tau + 4\int_{-2}^{t}(t-\tau)^{\left(\frac{1}{2}\right)-1}\mathrm{d}\tau\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[2\int_{t+2}^{0}x^{-\left(\frac{1}{2}\right)}(t-x)(-\mathrm{d}x) + 4\int_{t+2}^{0}x^{-\left(\frac{1}{2}\right)}(-\mathrm{d}x)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[4t(t+2)^{\left(\frac{1}{2}\right)} - \tfrac{4}{3}(t+2)^{\left(\frac{3}{2}\right)} + 8(t+2)^{\left(\frac{1}{2}\right)}\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}(t+2)^{\left(\frac{1}{2}\right)}\left[4t - \tfrac{4}{3}(t+2) + 8\right] \\
&= \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{\left(\frac{3}{2}\right)}
\end{aligned}
\tag{5.80}
$$

The RL derivative is derived as per the below steps:

$$
\begin{aligned}
{}_{-2}\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{-2}\mathrm{d}_t^{-1/2}\left[(t+2)^2\right]\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-2}^{t}(t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t+2}^{0}x^{\left(-\frac{1}{2}\right)}(t-x+2)^2(-\mathrm{d}x) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t+2}^{0}x^{\left(-\frac{1}{2}\right)}\left[(t+2)^2 - 2(t+2)x + x^2\right](-\mathrm{d}x) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t+2}^{0}-(t+2)^2x^{\left(-\frac{1}{2}\right)}\mathrm{d}x + \int_{t+2}^{0}2(t+2)x^{\left(\frac{1}{2}\right)}\mathrm{d}x - \int_{t+2}^{0}x^{\left(\frac{3}{2}\right)}\mathrm{d}x\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[2(t+2)^2(t+2)^{\left(\frac{1}{2}\right)} - \tfrac{4}{3}(t+2)(t+2)^{\left(\frac{3}{2}\right)} + \frac{2}{5}(t+2)^{\left(\frac{5}{2}\right)}\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{16}{15}(t+2)^{\left(\frac{5}{2}\right)}\right] = \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{\left(\frac{3}{2}\right)}
\end{aligned}
\tag{5.81}
$$

The above examples ((5.80) and (5.81)) state that the Caputo derivative and the RL derivative are equal when the differentiation process starts at the point where the value of the function is zero, and also before the start point of the function.

5.10 Initialization of Caputo derivatives

5.10.1 Initialization of Caputo derivatives and the associated difficulties

In real applications, it is usually the case that the problem to be solved is in some way isolated from the past. That is, it should not be necessary to retreat to $-\infty$ in time; that is, time immemorial, to start the analysis. Usually, the analyst desires to start the analysis at some time with $t = t_0$, with the knowledge (or assumption) of all values of the function and its derivatives. Specifically, $f(t_0), f^{(1)}(t_0), f^{(2)}(t_0).....f^{(n)}(t_0)$, in the case of integer order calculus. In modern parlance, this collection is called a 'state', and contains the effect of all that came before it. However, in classical calculus these 'initial-conditions' are constants.

A non-constant initialization or function appears as history in the initializing differentiation (anti-differentiation), as is required for fractional calculus whilst also being well-documented for Riemann-Liouville (RL) and Grunwald-Letnikov (GL) formulations, as described in the earlier sections of this chapter. However, for fractional initialization, conditions of type, $f(t_0)$ $f^{(1/2)}(t_0)$ $f^{(-1/2)}(t_0)$ or $f^{(n-\alpha)}(t_0)$ where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ are a natural number, are required. This seems to become physically unrealizable for those formulations, however, great an effort may be required to relate to them physically.

The Caputo Derivative definition (1967) was made in order to realise integer order initializations and the making of the fractional derivative of a constant as zero. However, although these may be mathematically pure, as the requirement in the Caputo formulation is for integer order initial conditions, the mathematical purity makes them difficult to realise physically. A popular belief is that the Caputo derivative formulation for a fractional derivative properly accounts for the initialization effect, which is not generally true when applied to fractional differential equations.

Constant initialization of the past is insufficiently general, and the widely used contemporary equations for the Laplace transform for differ-integrals based on that assumption lack generality. Therefore, the generalized form requires an initialization function. In a solution for the fractional differential equations with an assumed history, the set of initializing constants representing the values of fractional differ-integrals at $t = 0$ or at start points are ineffective, thus what is required is a 'non-constant initialization' for the generalized concept of integration and differentiation, as explained in the earlier sections of this chapter. The requirement of this initialization function for initializing generalized derivatives in RL formulations and the difficulty encountered in Caputo formulation to find an equivalent history function ate demonstrated in this chapter through the use of a few examples. An attempt is made to clarify that initialization function, or assumed history function, which is required for the Caputo formulation, and can give a drastic condition tracing to $-\infty$, which is much further away from the actual functional behavior that is under consideration.

In RL formulation with terminal initialization, the continuity and the nature of the function are maintained throughout the interval. In the next section, an attempt is made through simple examples to point out the difficulties in initializing the Caputo formulation, against the physical reality.

5.10.2 The initialized fractional derivative theory

Let $\psi(t)$ be the time dependent initialization function, taking into account the history of the system, function $f(t)$, which starts at point $t = a$, and the differentiation process, which starts at time $t = c$, where $a < c < t$, and $f(t) = 0$, for $t \le a$. For an un-initialized fractional integration for $t > a$ of order q with $q > 0$, the RL formulation is ${}_a\mathrm{d}_t^{-q}\left[f(t)\right] = \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau$ and for $t > c$ is ${}_c\mathrm{d}_t^{-q}\left[f(t)\right] = \frac{1}{\Gamma(q)}\int_c^t (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau$.

The region between a and c is considered to be the history of the fractional integral ${}_c\mathrm{d}_t^{-q}\left[f(t)\right]$, an-initialization function.

Thus the expression, for the entire integration should be, for $t \ge c$, ${}_c\mathrm{d}_t^{-q}\left[f(t)\right] + \psi(t) = {}_a\mathrm{d}_t^{-q}\left[f(t)\right]$ with $\psi(t)$ as $\psi(t) = {}_a\mathrm{d}_t^{-q}\left[f(t)\right] - {}_c\mathrm{d}_t^{-q}\left[f(t)\right]$.

The initialization function in the sense of terminal initialization is about fractionally integrating the function $f(t)$ from time a to c, i.e. $\psi(t) = \frac{1}{\Gamma(q)}\int_a^c (t-\tau)^{q-1}\left(f(\tau)\right)\mathrm{d}\tau$. For this terminal initialization, the integral can be initialized prior to the start time, $t = c$, of the differ-integration process. Using the standard symbols, one may thus express the initialization as ${}_cD_t^{-q}\left[f(t)\right] \equiv {}_c\mathrm{d}_t^{-q}\left[f(t)\right] + \psi(f, -q, a, c, t)$. The meaning of $\psi(f, -q, a, c, t)$ is fractionally integrating (by order of $q > 0$) the function $f(t)$ from point a to c, to provide a time varying function ψ, for the purpose of initializing the fractional integration process for $t > c$.

We can, therefore, generalize the fractional derivative (as has been done for the fractional integral) for the RL definition as ${}_cD_t^q\left[f(t)\right] \equiv {}_cD_t^m\left({}_cD_t^{-p}\left[f(t)\right]\right)$, where m is an integer such that $(m-1) < q < m$ with $p = m - q$, $q \geq 0$ and $t > c \geq a$. This expression allows for an initialized derivative and for the converting of the same by expansion with the un-initialized derivative and the initialization function (as done for the fractional integration case). We get this by taking $h(t) = {}_a\mathrm{d}_t^{-p}\left[f(t)\right]$, as ${}_cD_t^m\left[h(t)\right] = {}_c\mathrm{d}_t^m\left[h(t)\right] + \psi(h,m,a,c,t)$, where ${}_c\mathrm{d}_t^m \equiv \frac{\mathrm{d}^m}{\mathrm{d}t^m}$, is expanded as ${}_cD_t^q\left[f(t)\right] = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left({}_c\mathrm{d}_t^{-p}\left[f(t)\right] + \psi(f,-p,a,c,t)\right) + \psi(h,m,a,c,t)$.

For the terminal initialization for an integer (m) order derivative, the initialization is zero, that is $\psi(h,m,a,c,t) = 0$. For terminal initialization, the $\psi(t)$ is the operation on the function itself from time a to c.

We calculate the initialized α order derivative $0 < \alpha < 1$, for a function $f(t)$ where for $t \leq a < 0$, $f(t) = 0$. That is, a function is born at $t = a < 0$. We calculate the fractional initialized derivative from $t = 0 > a$.

$$
\begin{aligned}
{}_0D_t^\alpha\left[f(t)\right] &= {}_0D_t^1\left({}_0D_t^{-(1-\alpha)}\left[f(t)\right]\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left({}_0D_t^{-(1-\alpha)}\left[f(t)\right]\right) + \psi(h,1,a,0,t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left({}_0\mathrm{d}_t^{-(1-\alpha)}\left[f(t)\right] + \psi\left(f,-(1-\alpha),a,0,t\right)\right) + 0 \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left(f(\tau)\right)\mathrm{d}\tau\right) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^0 (t-\tau)^{-\alpha}\left(f(\tau)\right)\mathrm{d}\tau\right)
\end{aligned}
\tag{5 .82}
$$

For the side initialization, the initialization function $\psi(t)$ could be anything, such as Dirac's delta function or Heaviside's step function, or the operation of obtaining the initialization of any function other than $f(t)$, call it $f_i(t) \neq f(t)$.

5.10.3 Demonstration of the initialized function for a fractional derivative (RL and Caputo) as zero when the function value at the start point is zero and the function is zero before the start point

For the considered example $f(t) = (t+2)^2$, take $m = 1, \alpha = \frac{1}{2}$, $c = 0$ and $a = -2$. We will use the above formulation (5.82) for an initialized fractional semi-derivative for the RL formulation, for the terminal initialization case for our function $f(t) = (t+2)^2$, and we write the following:

$$
\begin{aligned}
{}_0D_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right] \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2}^0 (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right]
\end{aligned}
\tag{5 .83}
$$

For the first term in RHS of the above (5.83) we have already calculated it as an un-initialized semi integration (5.28) i.e. ${}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau$, which is ${}_0\mathrm{d}_t^{-1/2}\left[(t+2)^2\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{16}{15}t^{(5/2)} + \frac{16}{3}t^{(3/2)} + 8t^{(1/2)}\right)$. We use this derived result in the following expression for the first term of RHS of (5.83), i.e.

$$
\begin{aligned}
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\,\mathrm{d}\tau\right] \\
{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{16}{15}t^{\left(\frac{5}{2}\right)}+\frac{16}{3}t^{\left(\frac{3}{2}\right)}+8t^{\left(\frac{1}{2}\right)}\right)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{8}{3}t^{\left(\frac{3}{2}\right)}+8t^{\left(\frac{1}{2}\right)}+4t^{\left(-\frac{1}{2}\right)}\right)
\end{aligned}
\tag{5.84}
$$

For the second term of RHS (5.83), we evaluate it as a total initialization, for a terminal initialization case as follows:

$$
\begin{aligned}
\psi\left((t+2)^2,\tfrac{1}{2},-2,0,t\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2}^{0}(t-\tau)^{\left(\frac{1}{2}\right)-1}(\tau+2)^2\mathrm{d}\tau\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{t+2}^{t}\frac{1}{\Gamma\left(\frac{1}{2}\right)}x^{\left(-\frac{1}{2}\right)}(t-x+2)^2(-\mathrm{d}x) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t+2}^{t}x^{\left(-\frac{1}{2}\right)}\left((t+2)-x\right)^2(-\mathrm{d}x)\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t+2}^{t}x^{\left(-\frac{1}{2}\right)}\left((t+2)^2-2(t+2)x+x^2\right)(-\mathrm{d}x)\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\int_{t+2}^{t}(t+2)^2x^{\left(-\frac{1}{2}\right)}\mathrm{d}x+2\int_{t+2}^{t}(t+2)x^{\frac{1}{2}}\mathrm{d}x-\int_{t+2}^{t}x^{\frac{3}{2}}\mathrm{d}x\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(-2(t+2)^2x^{\left(\frac{1}{2}\right)}\Big|_{t+2}^{t}+\frac{4}{3}(t+2)x^{\left(\frac{3}{2}\right)}\Big|_{t+2}^{t}-\frac{2}{5}x^{\left(\frac{5}{2}\right)}\Big|_{t+2}^{t}\right)\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{8}{3}(t+2)^{\left(\frac{3}{2}\right)}-\frac{8}{3}t^{\left(\frac{3}{2}\right)}-8t^{\left(\frac{1}{2}\right)}-4t^{\left(-\frac{1}{2}\right)}\right]
\end{aligned}
\tag{5.85}
$$

From the above derivation (5.85), what we get is an initialized RL semi-derivative shown as:

$$
\begin{aligned}
{}_0D_t^{1/2}\left[(t+2)^2\right] &= {}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right]+\psi\left((t+2)^2,\tfrac{1}{2},-2,0,t\right) \\
{}_0D_t^{1/2}\left[(t+2)^2\right] &= \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{\left(\frac{3}{2}\right)}
\end{aligned}
\tag{5.86}
$$

This above expression (i.e. ${}_0D_t^{1/2}\left[(t+2)^2\right]=\frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{\left(\frac{3}{2}\right)}$) is the same as the RL and Caputo derivatives in being a derivative un-initialized starting point at $a=-2$ where the value of the function $f(t)=(t+2)^2$ is zero, that is $f(-2)=0$ as derived earlier in the chapter (see equations (5.80) and (5.81)). This means that it is ${}_{-2}\mathrm{d}_t^{1/2}\left[(t+2)^2\right]$ or ${}_{-2}^{C}\mathrm{d}_t^{1/2}\left[(t+2)^2\right]$, that are equal to $\frac{8}{3\Gamma\left(\frac{1}{2}\right)}(t+2)^{3/2}$, that is:

$$
\begin{aligned}
{}_{-2}D_t^{1/2}\left[(t+2)^2\right] = {}_{-2}\mathrm{d}_t^{1/2}\left[(t+2)^2\right] &= {}_{-2}^{C}\mathrm{d}_t^{1/2}\left[(t+2)^2\right] \\
&= \frac{8}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}(t+2)^{3/2}
\end{aligned}
\tag{5.87}
$$

As it is obvious that in this case $\psi(t)$ that the required initialization will be zero, then:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left((t+2)^2,-\tfrac{1}{2},-2,-2,t\right)\right]=\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,a,t\right)\right]=0 \tag{5.88}
$$

This is at the starting point of the function where the function value is zero, and prior to that too, where the initialization is also zero and the two definitions (the RL and the Caputo derivatives) are equal.

5.10.4 Making the initialized Caputo derivative equal to the initialized RL derivative in order to provide an initialization function for the Caputo derivative

We have seen the relationship between un-initialized Caputo and RL derivatives as follows (5.89), where, at the start point of the derivative, the function value is non-zero. In our example, this is shown by $c = 0$ with $f(0) = 4$:

$$
\begin{aligned}
{}_c\mathrm{d}_t^{\alpha}\left[f(t)\right] &= {}_c^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + f(c)\frac{(t-c)^{-\alpha}}{\Gamma(1-\alpha)} \\
{}_0\mathrm{d}_t^{\alpha}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}
\tag{5.89}
$$

For case $0 < \alpha < 1$, to this we add on both sides the expression of the initialization function of the fractional derivative and obtain the following:

$$
\begin{aligned}
&{}_0\mathrm{d}_t^{\alpha}\left[f(t)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right] \\
&\qquad = {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right] \\
&{}_0D_t^{\alpha}\left[f(t)\right] = {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right]
\end{aligned}
\tag{5.90}
$$

where the expression in LHS of (5.90), i.e. ${}_0\mathrm{d}_t^{\alpha}\left[f(t)\right] + \frac{\mathrm{d}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right]}{\mathrm{d}t} \equiv {}_0D_t^{\alpha}\left[f(t)\right]$, is the initialized RL derivative. The Caputo derivative will be equal to the initialized RL derivative if we make for the above (5.90) obtained expression the following:

$$
\begin{aligned}
&f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right] = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\psi\left(f,-(1-\alpha),a,0,t\right) = -f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \\
&\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^0 (t-\tau)^{-\alpha}\left(f(\tau)\right)\mathrm{d}\tau = -f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}
\tag{5.91}
$$

where we have $\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^0 (t-\tau)^{-\alpha} f(\tau)\mathrm{d}\tau = {}_aD_0^{\alpha} f(t)$ which is equated to $\left(-\frac{1}{\Gamma(1-\alpha)}\right)t^{-\alpha} f(0)$ via above (5.91) steps. The term $\frac{-f(0)}{\Gamma(1-\alpha)}t^{-\alpha}$ is recognized as ${}_0D_t^{\alpha}\left[-f(0)\right]$, that is a RL-fractional derivative of a constant; in this case the constant is $-f(0)$.

In our example, considered here as $f(0) = 4$, we can thus write this in the form as expressed below:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-\tfrac{1}{2},-2,0,t\right)\right] &= -4\frac{t^{-\frac{1}{2}}}{\Gamma\left(1-\frac{1}{2}\right)} \\
&= -4\frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = {}_0D_t^{\frac{1}{2}}\left[-4\right]
\end{aligned}
\tag{5.92}
$$

In this case, the initialization is equal to the fractionally differentiating constant function by order of half i.e. $f_i(t) = -4$, i.e. ${}_0D_t^{\frac{1}{2}}\left[-4\right]$, so that:

$$
\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}t^{\frac{1}{2}}}\left[-4\right] = {}_0\mathrm{d}^{\frac{1}{2}}\left[-4\right] = -\frac{4t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}
\tag{5.93}
$$

In this chosen example, one can choose the history function or the initialization function, $f_i(t) = -4$ as a constant to allow the Caputo derivative to be the same as the RL initialized derivative. That function i.e. $f_i(t) = -4$ is semi-differentiated i.e. ${}_0D_t^{\frac{1}{2}}\left[f_i(t)\right]$ and then subtracted from the Caputo to make an RL initialized derivative the same as the Caputo.

However, choosing this constant as -4 will make the function $f(t)$ discontinuous at $t=0$, as $f_i(0^-)=-4\neq f(0^+)=4$. However, the argument suggests, that in order to make the RL initialized derivative and the Caputo process the same, one has to use a constant function (at least in this chosen example) and the concept of terminal initialization will not work, since the choice here is arbitrary. These calls for side-initialization have an initialization function $\psi(t)$ derived from any function other than the one being fractionally differentiated as $f(t)$, referred to here as $f_i(t)\neq f(t)$. In the above case, $f_i(t)=-4\neq f(t)$, where $f(t)=(t+2)^2$.

5.10.5 The Caputo derivative initialization function: A case of a side initialization process

The initialization function is formed for the Caputo derivative by a history function, which is analytical at $t=c=0$, and is represented by the Taylor series, for $t\leq 0$ as in the following:

$$f_i(t)=\sum_{n=0}^{\infty}\frac{f_i^{(n)}(0^-)}{n!}t^n,\qquad t<0 \tag{5.94}$$

For the Caputo initialization, we have, $f_i(0^-)=-f(0^+)$ as described in the previous section. For $0<\alpha<1$, we have $m=1$, and we replace $f(t)$ by $f_i(t)$ and write the following steps:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\psi\left(f_i,-(1-\alpha),a,0,t\right)=f_i(0^-)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^0(t-\tau)^{-\alpha}f_i(\tau)\mathrm{d}\tau\right)=f_i(0^-)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}=-f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=0}^{\infty}\frac{f_i^{(n)}(0^-)}{n!}\frac{1}{\Gamma(1-\alpha)}\int_a^0(t-\tau)^{-\alpha}\tau^n\mathrm{d}\tau\right]=-f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}\tag{5.95}$$

The LHS of (5.95) is ${}_aD_0^{\alpha}\left[f_i(t)\right]$ i.e. α is the order derivative of $f_i(t)$ from a to $c=0$. For $n=0$ we get:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f_i(0^-)}{\Gamma(1-\alpha)}\int_a^0(t-\tau)^{-\alpha}\mathrm{d}\tau\right]=-f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\\
&f_i(0^-)\frac{\mathrm{d}}{\mathrm{d}t}\int_a^0(t-\tau)^{-\alpha}\mathrm{d}\tau=-t^{-\alpha}f(0^+)\\
&f_i(0^-)\left((t-a)^{-\alpha}-t^{-\alpha}\right)=-t^{-\alpha}f(0^+)
\end{aligned}\tag{5.96}$$

Now, as a tends to $-\infty$, we get $\lim_{a\downarrow-\infty}\left(f_i(0^-)\left((t-a)^{-\alpha}-t^{-\alpha}\right)\right)=-t^{-\alpha}f_i(0^-)$. With this, we write the following:

$$\lim_{a\downarrow-\infty}\left(f_i(0^-)\frac{\mathrm{d}}{\mathrm{d}t}\int_a^0(t-\tau)^{-\alpha}\mathrm{d}\tau\right)=-t^{-\alpha}f_i(0^-)=-t^{-\alpha}f(0^+) \tag{5.97}$$

For this above identity (5.97) to be valid, as the initial point a tends to $-\infty$, it must give us $f_i(0^-)=f(0^+)$ for $-\infty<t<0$.

In our example, therefore, to have the Caputo and RL derivatives initialized and to get them equated, we have to take the function $f(t)=(t+2)^2$ and abruptly take a constant value that is the value of the function at $t=0$ and a constant at value $f_i(t)=f(0)=4$, from the start point $t=-\infty$, until $t=0$.

Therefore, the history function for the Caputo derivative in this example is a constant in Heaviside's step function, multiplied by the constant of value 4 (that is the value of function $f(t)$ and the point $t=0$, i.e. the start point of fractional differentiation), starting at $t=-\infty$. This is, however, in no way a backward continuation of the function $f(t)$ being considered for the fractional derivative process. It is different from the original form of the function $f(t)$ from $t=-\infty$ to the start point of the differentiation process i.e. $t=0$.

This condition seems to be physically demanding. Say the function represents a velocity that started at some point in the past, for instance at point $t=-2$, and we are obtaining fractional acceleration through this process of the use of a half derivative, in order to obtain forces that are acting. The situation is such that if we want to initialize this with the Caputo formulation then we have to consider that the system has a constant velocity from time immemorial ($t=-\infty$) up to point $t=0$, and then suddenly this takes on a defined trajectory of the function. This seems to be unrealistic physically, at least in this case.

In terms of RL, the initialization lets us take $f_i(t)=f(0^+)=4$, as a historical function from $-2\le t\le 0$. In the case of the terminal initialization as demonstrated above, the history function for the RL process was taken as $f_i(t)=f(t)=(t+2)^2$ for $-2\le t\le 0$. We evaluate the initialization term with a revised history function $f_i(t)=4$, and in so doing we obtain the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(4,-\tfrac{1}{2},-2,0,t\right)\right]&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-2}^{0}\frac{1}{\Gamma\left(\frac{1}{2}\right)}(t-\tau)^{\left(\frac{1}{2}\right)-1}(4)\mathrm{d}\tau\right]\\&=\frac{4}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t+2}^{t}x^{\left(-\frac{1}{2}\right)}(-\mathrm{d}x)\right]\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{8}{\Gamma\left(\frac{1}{2}\right)}t^{\left(\frac{1}{2}\right)}+\frac{8}{\Gamma\left(\frac{1}{2}\right)}(t+2)^{\left(\frac{1}{2}\right)}\right]\\&=-\frac{4}{\Gamma\left(\frac{1}{2}\right)}t^{\left(-\frac{1}{2}\right)}+\frac{4}{\Gamma\left(\frac{1}{2}\right)}(t+2)^{\left(-\frac{1}{2}\right)}\end{aligned}\tag{5.98}$$

The un-initialized RL derivative as obtained earlier is reproduced below:

$$\begin{aligned}{}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right]&=\frac{8t^{\left(\frac{3}{2}\right)}}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}+\frac{8t^{\left(\frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2}\right)}+\frac{4t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\\&={}_0^C\mathrm{d}_t^{(1/2)}\left[(t+2)^2\right]+\frac{4t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\end{aligned}\tag{5.99}$$

Therefore the initialized RL derivative with a history function taken as $f_i(t)=4$ in the region $t=-2$ to $t=0$ which is different from $f(t)=(t+2)^2$ is:

$$\begin{aligned}{}_0D_t^{1/2}\left[(t+2)^2\right]&={}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right]+\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(4,\tfrac{1}{2},-2,0,t\right)\right]\\&=\frac{8t^{3/2}}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}+\frac{8t^{1/2}}{\Gamma\left(\frac{1}{2}\right)}+\frac{4(t+2)^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}\end{aligned}\tag{5.100}$$

The observation from the above expression (5.100) is that the singularity is shifted at $t=-2$ from $t=0$, by taking this initialization function as a constant of ($f_i(t)=4$), from the start point of the function ($t=-2$) to that of the fractional derivative ($t=0$) process. However, pushing the start point towards minus infinity, i.e. taking $a=-\infty$, makes the initialized RL derivative equal to the Caputo Derivative, exactly as follows:

$$\begin{aligned}{}_0D_t^{1/2}\left[(t+2)^2\right]&={}_0\mathrm{d}_t^{1/2}\left[(t+2)^2\right]+\frac{\mathrm{d}}{\mathrm{d}t}\psi\left(4,\tfrac{1}{2},-\infty,0,t\right)\\&=\frac{8t^{3/2}}{3\left(\Gamma\left(\frac{1}{2}\right)\right)}+\frac{8t^{1/2}}{\Gamma\left(\frac{1}{2}\right)}\\&={}_0^C\mathrm{d}_t^{1/2}\left[(t+2)^2\right]\end{aligned}\tag{5.101}$$

The exactness in the RL and Caputo derivatives is thus achieved by pushing the singularity to $-\infty$, at least for this example.

We have set the initialization function as $\psi_i(t)$, which is derived from a history function in our case, which is $f_i(t)=4$ for $t=-\infty$ to $t=0$, given in the following expression:

$$\psi_i(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f_i(t)\right)\right]\qquad t\in\left(-\infty,0\right]\qquad f_i(t)=4\tag{5.102}$$

We calculate the value of the initialization function in this case in the following steps:

$$\begin{aligned}
\psi_i(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(4,-\tfrac{1}{2},-\infty,0,t\right)\right] = \lim_{a\downarrow-\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(4,-\tfrac{1}{2},a,0,t\right)\right] \\
&= \lim_{a\downarrow-\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^0 \frac{1}{\Gamma\left(\frac{1}{2}\right)}(t-\tau)^{\left(\frac{1}{2}\right)-1}(4)\mathrm{d}\tau\right] \\
&= \lim_{a\downarrow-\infty}\frac{4}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t-a}^{t} x^{\left(-\frac{1}{2}\right)}(-\mathrm{d}x)\right] \\
&= \lim_{a\downarrow-\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{8}{\Gamma\left(\frac{1}{2}\right)}t^{\left(\frac{1}{2}\right)}+\frac{8}{\Gamma\left(\frac{1}{2}\right)}(t-a)^{\left(\frac{1}{2}\right)}\right] \\
&= \lim_{a\downarrow-\infty}\left(-\frac{4}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}}+\frac{4}{\Gamma\left(\frac{1}{2}\right)}(t-a)^{-\frac{1}{2}}\right) \\
&= -\frac{4}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}} = -\frac{f(0^+)}{\Gamma(1-\alpha)}t^{-\alpha}
\end{aligned} \tag{5.103}$$

This is much the same as using a constant function from time immemorial ($-\infty$) and putting the formulation of the terminal initialization of the RL derivative for obtaining an initialization function to equate to the Caputo formulation, and to cancel the singularity function which appears as a difference to the un-initialized RL and Caputo derivatives. However, it is observed that this history function makes the first derivative at $t=0$ a discontinuous one -that is, with this type of history function, the integer order differentiability is lost. This restriction may not be physically viable for physical processes, even though mathematically it sounds correct!

5.11 Generalization of the RL and Caputo formulations and the initialization function

The expressions obtained for $0<\alpha<1$ in the previous sections are:

$$\begin{aligned}
{}_a\mathrm{d}_t^{\alpha}\left[f(t)\right] &= {}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right]+\frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \\
{}_0D_t^{\alpha}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right]+f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right]
\end{aligned} \tag{5.104}$$

This (5.104) may be extended by the above example for cases $1<\alpha<2$; the greatest integer for the RL and Caputo formulation is $m=2$. As such, the initialization expression is as follows:

$$\begin{aligned}
{}_0D_t^{\alpha}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right]+\frac{(t-0)^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{(t-0)^{1-\alpha}f^{(1)}(0)}{\Gamma(2-\alpha)} \\
&\quad +\frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi\left(f,-(2-\alpha),a,0,t\right) \qquad t>0
\end{aligned} \tag{5.105}$$

For the RL derivative initialized case to be equal to the Caputo derivative initialization, it needs to follow the below expression:

$$\frac{(t-0)^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{(t-0)^{1-\alpha}f^{(1)}(0)}{\Gamma(2-\alpha)}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi\left(f,-(2-\alpha),a,0,t\right)=0 \tag{5.106}$$

Note that $\frac{\mathrm{d}^2\left[\psi\left(f,-(2-\alpha),a,0,t\right)\right]}{\mathrm{d}t^2} = {}_aD_0^{\alpha}\left[f(t)\right]$, with $1<\alpha<2$. This implies the following:

$$\begin{aligned}
\frac{\mathrm{d}^2\left[\psi\left(f,-(2-\alpha),a,0,t\right)\right]}{\mathrm{d}t^2} &= {}_aD_0^{\alpha}\left[f(t)\right] \\
&= \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_a^0 (t-\tau)^{2-\alpha-1}\left(f(\tau)\right)\mathrm{d}\tau
\end{aligned} \tag{5.107}$$

Again considering the history function $f_i(t)$ as an analytical for $t\le 0$ (as in the Taylor series).it is as follows:

$$f_i(t)=\sum_{n=0}^{\infty}\frac{f_i^{(n)}(0^-)}{n!}t^n \tag{5.108}$$

We obtain the condition for the initialization as described below with $f_i(t) \neq f(t)$

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_a^0 (t-\tau)^{2-\alpha-1}\sum_{n=0}^{\infty}\frac{f_i^{(n)}(0^-)}{n!}\tau^n \mathrm{d}\tau \\ &= -\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0^+) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}f^{(1)}(0^+)\end{aligned} \tag{5.109}$$

Equating the $n=0$ and $n=1$ in this case provides the history function that will make the RL and Caputo derivatives equal, and which will have the condition $f_i(0^-) = f(0^+)$, and $f_i^{(1)}(0^-) = f^{(1)}(0^+)$. This causes the history function to be $f_i(t) = f(0^+) + t\left(f^{(1)}(0^+)\right)$ for $-\infty \le t \le 0$.

For the case of $f(t) = (t+2)^2$ and considering the fractional derivative starting point as $t = 0$ and the fractional order derivative as $\alpha = 1.5$, the history function will be $f_i(t) = 4t + 4$, for $t \le 0$. Clearly, this history function will have a second discontinuous derivative at $t = 0$. Again, this poses a physical restriction; as we look at the function from time immemorial as being very different from the original function (process) itself.

Therefore, we can generalize the expressions obtained earlier as follows:

$$\begin{aligned}&{}_a\mathrm{d}_t^{\alpha}\left[f(t)\right] = {}_a^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad 0<\alpha<1 \\ &{}_0D_t^{\alpha}\left[f(t)\right] = {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right] \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left[\psi\left(f,-(1-\alpha),a,0,t\right)\right] = -f(0^+)\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)}; \quad 0<\alpha<1 \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_a^0 (t-\tau)^{-\alpha}\left(f_i(\tau)\right)\mathrm{d}\tau = -f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}; \quad 0<\alpha<1\end{aligned} \tag{5.110}$$

Thus, by induction, for any integer m, those fractional RL and Caputo derivatives are initialized, for $(m-1) < \alpha < m$, as follows:

$$\begin{aligned}{}_0D_t^{\alpha}\left[f(t)\right] &= {}_0^C\mathrm{d}_t^{\alpha}\left[f(t)\right] + \sum_{n=0}^{m-1}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}f^{(n)}(0^+) \\ &+ \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\psi\left(f,-(m-\alpha),a,0,t\right)\right]\end{aligned} \tag{5.111}$$

To make sure the RL and Caputo derivatives are the same, for $t > 0$ we have the following rule:

$$\begin{aligned}&\frac{\mathrm{d}^m}{\mathrm{d}t^m}\psi\left(f,-(m-\alpha),a,0,t\right) = -\sum_{n=0}^{m-1}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}f^{(n)}(0^+) \\ &\frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{1}{\Gamma(m-\alpha)}\int_a^0 (t-\tau)^{m-\alpha-1}\left(f_i(\tau)\right)\mathrm{d}\tau = -\sum_{n=0}^{m-1}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}f^{(n)}(0^+)\end{aligned} \tag{5.112}$$

Using the Taylor expression for $f_i(t)$ and the history function for $-\infty \le t \le 0$, the expression is as follows:

$$\begin{aligned}&\frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^0 (t-\tau)^{m-\alpha-1}\left(\sum_{n=0}^{\infty}\frac{f_i^{(n)}(0^-)}{n!}\tau^n\right)\mathrm{d}\tau \\ &= -\sum_{n=0}^{m-1}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}f^{(n)}(0^+)\end{aligned} \tag{5.113}$$

The above formula (5.113) will thus be valid for $f_i^{(n)}(0^-) = f^{(n)}(0^+)$, and for $n = 0,1,2,...(m-1)$. Therefore, for the generalized history function to have initialized the RL and Caputo derivatives as being the same as the fractional derivative, which is a polynomial of order $(m-1)$ for $-\infty \le t \le 0$, it must be represented as the following:

$$f_i(t) = f^{(m-1)}(0^+)t^{m-1} + f^{(m-2)}(0^+)t^{m-2} + ... \tag{5.114}$$

This initialization law for the Caputo derivative that we developed provides a discontinuity for derivatives higher than $(m-1)$ for the function with a history at $t = 0$.

5.12 Observations regarding difficulties in Caputo initialization and demanding physical conditions vis-à-vis RL initialization conditions

The discussion in the earlier sections suggests that a Caputo inferred history has discontinuous integer order derivatives at the starting point of the fractional differentiation process. Meanwhile, the RL initialization is a smooth continuation of the function that is being fractionally differentiated, for the terminal initialization. There is a difference in the two RL initialized derivatives compared with the Caputo initialized ones, and it is profound at certain times (that is large). However, the functional forms of the two derivatives are identical in the case of a power function with time for times significantly larger than the start point of fractional differentiation.

The RL un-initialized fractional derivative of the function, which begins at the start point of the function itself, is a smooth backward continuation of the initialized fractional RL derivative beginning somewhere other than the start point of the function, for terminal initialization. It is also noted that the RL fractional derivative, using the Caputo inferred history for initialization for $-\infty \le t \le c = 0$, is the same as the Caputo fractional derivative. The two definitions of derivatives are the same if and only if the initialization part is zero or the derivative process for both the RL and Caputo derivatives starts at the birth of the function in our example, as we have $f(t) = f^{(1)}(t) = 0$ at $t \le a = -2$. Otherwise, the two derivatives will be equated by a polynomial that is singular at the start point of differentiation ($t = c = 0$).

Say we have a differential equation

$$y(t) = {}_0^C\mathrm{d}_t^{\alpha}\left[x(t)\right] \tag{5.115}$$

Using the Caputo inferred history may be acceptable if:

(a) it is found that the history is acceptable to the physics of the problem;

(b) it is acceptable to have a discontinuity of the integer order derivative for $x(t)$ of the order $m-1$, where $(m-1) < \alpha < m$.

Suppose we consider a slightly complicated fractional order differential equation such as

$$y(t) = {}_0^C\mathrm{d}_t^{3/2}\left[x(t)\right] + {}_0^C\mathrm{d}_t^{1/2}\left[x(t)\right] + x(t) \tag{5.116}$$

and suppose we assume (for simplicity's sake) that both the fractional derivative terms have the same history, i.e. $x_i(t)$, for $t \le 0$.

For this example, the orders of the terms lay between different integer orders. For the first term of the fractional derivative, the order is $1 < \alpha = \frac{3}{2} < 2$, and for the second fractional derivative, the order is between $0 < \alpha = \frac{1}{2} < 1$. From our discussion on the generalization of the initialization, the history for the first term is $x_i(t) = x(0) + t\left(x^{(1)}(0)\right)$ and for the second term the history will be $x_i(t) = x(0)$ for $-\infty \le t \le 0$. Clearly the only history that will satisfy both fractional terms is $x_i(t) = x(0)$, for $-\infty \le t \le 0$, with $x^{(1)}(0) = 0$ (which is not the case, as with this history the first derivative will be discontinuous at $t = 0$).

If on the other hand each has a separate history, each of the fractional differential terms in the equation (5.116) will be disconnected from one other, and will be acted upon by their own individual history, in negative time. Then at time zero, the entire individual position $x(t)$ and velocity $x^{(1)}(t)$ will have the same value, and under this condition and scenario the initialization of each term reverts to the case, as discussed for the case $y(t) = {}_0^C\mathrm{d}_t^{\alpha}\left[x(t)\right]$, (5.115). This elaborate discussion is as per theory proposed by Lorenzo and Hartley.

5.13 The Fractional Derivative of a sinusoidal function with a lower terminal not at minus infinity, and an initialization function

As described in Section 4-13, the fractional derivative of $\sin \lambda t$ from $t = -\infty$ to t is:

$${}_{-\infty}D_t^q\left[\sin(\lambda t)\right] = \lambda^q \sin\left(\lambda t + \frac{q\pi}{2}\right) \tag{5.117}$$

meaning that, if the derivative (or anti-derivative) process begins from time immemorial, then we will get a steady state condition., and the periodic function will be phase shifted, either leading, or lagging by $q\left(\frac{\pi}{2}\right)$. However, the situation is different if the lower limit of differentiation (integration) is shifted from time immemorial ($t = -\infty$) to some other time (say time $t = c$). In that case, we write the initialized differ-integral of a function as an uninitialized part, plus the initialization function as the following:

$$ {}_cD_t^q\left[f(t)\right] = {}_a\mathrm{d}_t^q\left[f(t)\right] + \psi(f,q,a,c,t) \qquad (5.118)$$

where the function $f(t)=0$ for $t<a$, and the initialization function is the result of differ-integrating the function from time a to c by order q, for the terminal-initialization case. The initialization function can be chosen as arbitrary as per the requirement for the side-initialization case, whereas the uninitialized differ-integral is ${}_a\mathrm{d}_t^q\left[f(t)\right] \equiv \frac{\mathrm{d}^q f(t)}{\mathrm{d}[t-a]^q}$.

The differ-integration of the sinusoidal function can be represented as follows (as discussed in Section-4.13 of this book):

$$\frac{\mathrm{d}^q\left[\sin(t)\right]}{\mathrm{d}[t-a]^q} = \sin\left(t+q\tfrac{\pi}{2}\right) + \frac{\left(t-a\right)^{-1-q}}{\Gamma(-q)} - \frac{\left(t-a\right)^{-3-q}}{\Gamma(-q-2)} + ... \qquad (5.119)$$

This expression suggests that fractional differ-integration of the sinusoidal periodic function provides a phase change in the periodic part; and there are transients associated with the non-periodic part which die down at large times, becoming insignificant. Setting $a=0$, we get:

$$ {}_0\mathrm{d}_t^q\left[\sin(t)\right] = \sin\left(t+q\tfrac{\pi}{2}\right) + \frac{t^{-1-q}}{\Gamma(-q)} - \frac{t^{-3-q}}{\Gamma(-q-2)} + ... \qquad (5.120)$$

If we put $q=\frac{1}{2}$, and $a\neq 0$, the semi-differentiation is:

$$ {}_a\mathrm{d}_t^{\left(\frac{1}{2}\right)}\left[\sin(t)\right] = \sin\left(t+\tfrac{\pi}{4}\right) + \frac{\left(t-a\right)^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)} - \frac{\left(t-a\right)^{-\frac{7}{2}}}{\Gamma\left(-\frac{5}{2}\right)} + .. \qquad (5.121)$$

Taking $a=-\infty$, the polynomial part in the above formula (5.121) becomes zero. Thus, we get:

$$ {}_{-\infty}\mathrm{d}_t^{\left(\frac{1}{2}\right)}\left[\sin(t)\right] = \sin\left(t+\tfrac{\pi}{4}\right) \qquad (5.122)$$

Putting the value $q=-\frac{1}{2}$ and $a\neq 0$, we have semi-integration set as:

$$ {}_a\mathrm{d}_t^{-\frac{1}{2}}\left[\sin(t)\right] = \sin\left(t-\tfrac{\pi}{4}\right) + \frac{\left(t-a\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} - \frac{\left(t-a\right)^{-\frac{5}{2}}}{\Gamma\left(-\frac{3}{2}\right)} + ... \qquad (5.123)$$

In this case we are also putting $a=-\infty$, which gives higher order terms in the remainder polynomial as zero, yielding:

$$ {}_{-\infty}\mathrm{d}_t^{-\frac{1}{2}}\left[\sin(t)\right] = \sin\left(t-\tfrac{\pi}{4}\right) \qquad (5.124)$$

Using scaling law (as described in Section-3.20), that is $\frac{\mathrm{d}^q[f(\lambda x)]}{[\mathrm{d}x]^q} = \lambda^q \frac{\mathrm{d}^q[f(\lambda x)]}{[\mathrm{d}(\lambda x)]^q}$, we get:

$$ {}_0\mathrm{d}_t^q\left[\sin(\lambda t)\right] = \lambda^q \sin\left(t+q\tfrac{\pi}{2}\right) + \lambda^q \frac{t^{-1-q}}{\Gamma(-q)} - \lambda^q \frac{t^{-3-q}}{\Gamma(-q-2)} + ... \qquad (5.125)$$

Putting the value $q=1$, and $\lambda=1$ in the above expression (5.125), we get:

$$\begin{aligned} {}_0\mathrm{d}_t^1\left[\sin(t)\right] &= \sin\left(t+\tfrac{\pi}{2}\right) + \frac{t^{-2}}{\Gamma(-1)} - \frac{t^{-4}}{\Gamma(-3)} + ... \\ &= \sin\left(t+\tfrac{\pi}{2}\right) + 0 \qquad (5.126) \\ &= \cos(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sin(t)\right] \end{aligned}$$

Here (5.126), we have used the properties of the reciprocal Gamma function, which are zero at all negative integer points (Section-1.10.11).

Putting $q=-1$ in the expression ${}_0\mathrm{d}_t^q\left[\sin(t)\right] = \sin\left(t+q\tfrac{\pi}{2}\right) + \frac{t^{-1-q}}{\Gamma(-q)} - \frac{t^{-3-q}}{\Gamma(-q-2)} + \dots$, we get:

$$
\begin{aligned}
{}_0\mathrm{d}_t^{-1}\left[\sin(t)\right] &= \sin\left(t-\tfrac{\pi}{2}\right)+\frac{t^0}{\Gamma(1)}-\frac{t^{-2}}{\Gamma(-1)}+\ldots. \\
&= \sin\left(t-\tfrac{\pi}{2}\right)+1 \qquad (5.127)\\
&= -\cos(t)+1=\int_0^t \sin(t)\mathrm{d}t
\end{aligned}
$$

The above formula shows that the expression for fractional differ-integration of $\sin(t)$ is correct for one complete integer order of differentiation and one-complete integration of integer order. In the above derivation (5.125), we have assumed that the arbitrary initialization function is zero, i.e. $\psi\left(\sin(t),q,a,c,t\right)=0$, and the $f(t)=\sin t=0$, for $t<a=0$.

5.14 The Laplace transform of fractional differ-integrals

5.14.1 The generalization of classical Laplace transform formulas for differentiation and integration

Here we form Laplace transforms of all $\mathrm{d}^\alpha\left[f(x)\right]/\mathrm{d}x^\alpha$ for all α and differ-integrable functions $f(x)$, which are:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right\}=\int_0^\infty e^{-sx}\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\mathrm{d}x \qquad (5.128)$$

We wish to relate the above (5.128) with $\mathcal{L}\left\{f(x)\right\}$, which is a Laplace transform of $f(x)$ (see Appendix-G for more details) defined as:

$$\mathcal{L}\left\{f(x)\right\}\overset{\text{def}}{=}\int_0^\infty e^{-sx}\left(f(x)\right)\mathrm{d}x \qquad (5.129)$$

We note that $\mathcal{L}\left\{f(x)\right\}$ is a function in complex frequency s; it is also expressed as $\mathcal{L}\left\{f(x)\right\}=F(s)$. In terms of engineering science, the variable x is the time variable t. For an inverse Laplace transform, we have a contour integration (see Appendix-G for further information) noted as:

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{f(x)\right\}\right\}=\mathcal{L}^{-1}\left\{F(s)\right\}\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(F(s)\right)e^{sx}\mathrm{d}s \qquad (5.130)$$

The expression (5.130) is derived from the use of the Fourier integral (see Appendix-G).
From a classical Laplace transform of integer order calculus, we write the following for multiple derivative operations as:

$$
\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\} &= s^n\mathcal{L}\left\{f(x)\right\}-s^{n-1}f(0)-s^{n-2}\left.\frac{\mathrm{d}\ f(x)}{\mathrm{d}x}\right|_{x=0} \\
&\quad -\ldots s^0\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}f(x)\right|_{x=0};\quad n=1,2,3,.. \qquad (5.131)\\
&= s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^{n-1-k}\left.\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right|_{x=0}.
\end{aligned}
$$

and for multiple iterated integrals as:

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}\qquad n=0,-1,-2,\ldots \qquad (5.132)$$

We write the terms in summation of an expression in reverse of the above formula (5.131) and re-index it to give the following:

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}-s^0\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}f(x)\right|_{x=0}-s\left.\frac{\mathrm{d}^{n-2}f(x)}{\mathrm{d}x^{n-2}}\right|_{x=0}$$
$$\ldots.-s^{n-2}\left.\frac{\mathrm{d}\ f(x)}{\mathrm{d}x}\right|_{x=0}-s^{n-1}f(0) \tag{5.133}$$
$$=s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\left.\frac{\mathrm{d}^{n-1-k}f(x)}{\mathrm{d}x^{n-1-k}}\right|_{x=0}$$

We note that both of the above formulas ((5.131) and (5.133)) are similar and can be transformed into a common formula, (for $n=0,\pm1,\pm2,..$) which is as follows:

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\left.\frac{\mathrm{d}^{n-1-k}f(x)}{\mathrm{d}x^{n-1-k}}\right|_{x=0} \tag{5.134}$$

Here in the above formulation (5.134), the upper summation limit is written as $n-1$, and may be replaced by any integer larger than $n-1$ and even by ∞. In (5.134), the effect is to add terms containing $\left.\frac{\mathrm{d}^{-1}f(x)}{\mathrm{d}x^{-1}}\right|_{x=0}$, $\left.\frac{\mathrm{d}^{-2}f(x)}{\mathrm{d}x^{-2}}\right|_{x=0}$ etc. for $n=0,-1,-2,\ldots$; such terms are necessarily zero for any function $f(x)$ whose Laplace transform exists.

If n is a non-integer, say α, we generalize the above formula for all α and write the following

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right\}=s^\alpha\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\left.\frac{\mathrm{d}^{\alpha-1-k}f(x)}{\mathrm{d}x^{\alpha-1-k}}\right|_{x=0} \tag{5.135}$$

where, n is the largest integer such that $(n-1)<\alpha\le n$. Notice that the sum in (5.135) is zero when $\alpha\le 0$. In satisfying the above (5.135) generalization, first consider $\alpha<0$, i.e. it is a fractional integration, for which the RL formula is:

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}=\frac{1}{\Gamma(-\alpha)}\int_0^x(x-y)^{\alpha-1}\big(f(y)\big)\mathrm{d}y \qquad \alpha<0 \tag{5.136}$$

We apply a convolution theorem of Laplace transforms that results in:

$$\begin{gathered}\mathcal{L}\left\{\int_0^x\big(f_1(x-y)\big)\big(f_2(y)\big)\mathrm{d}y\right\}=\big(\mathcal{L}\left\{f_1(x)\right\}\big)\big(\mathcal{L}\left\{f_2(x)\right\}\big)\\ \mathcal{L}\left\{\big(f_1(x)\big)*f_2(x)\right\}=\big(\mathcal{L}\left\{f_1(x)\right\}\big)\big(\mathcal{L}\left\{f_2(x)\right\}\big)\end{gathered} \tag{5.137}$$

to the formula given in (5.136), and write for $\alpha<0$ as follows :

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right\}=\frac{1}{\Gamma(-\alpha)}\mathcal{L}\left\{x^{-1-\alpha}\right\}\mathcal{L}\left\{f(x)\right\}=s^\alpha\mathcal{L}\left\{f(x)\right\} \tag{5.138}$$

We have used known Laplace transforms, i.e. $\mathcal{L}\left\{x^m\right\}=\frac{\Gamma(m+1)}{s^{m+1}}$. We see that for $\alpha<0$, negative integers from the generalized formula described above in (5.138) remain unchanged. For the positive α, i.e. fractional differentiation, we have the RL composition as:

$$\frac{\mathrm{d}^\alpha\left[f(x)\right]}{\mathrm{d}x^\alpha}=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{\mathrm{d}^{\alpha-n}\left[f(x)\right]}{\mathrm{d}x^{\alpha-n}}\right] \tag{5.139}$$

Here, n is a positive integer such that $(n-1)<\alpha<n$. Now on the application of the formula as obtained using $\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\sum_{k=0}^{n-1}s^k\left.\frac{\mathrm{d}^{n-1-k}f(x)}{\mathrm{d}x^{n-1-k}}\right|_{x=0}$ for $n=0,\pm1,\pm2,\ldots$, (5.134), we find that:

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right\}&=\mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{\mathrm{d}^{\alpha-n}f(x)}{\mathrm{d}x^{\alpha-n}}\right]\right\}\\&=s^n\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha-n}f(x)}{\mathrm{d}x^{\alpha-n}}\right\}-\sum_{k=0}^{n-1}s^k\frac{\mathrm{d}^{n-1-k}}{\mathrm{d}x^{n-1-k}}\left[\frac{\mathrm{d}^{\alpha-n}f(x)}{\mathrm{d}x^{\alpha-n}}\right]_{x=0}\end{aligned} \tag{5.140}$$

The difference $\alpha-n<0$ is a negative. The first term of the RHS of (5.140) is evaluated using $\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha-n}f(x)}{\mathrm{d}x^{\alpha-n}}\right\}=s^{\alpha-n}\mathcal{L}\left\{f(x)\right\}$ as obtained above in (5.138) for $\alpha-n<0$. Since $\alpha-n<0$, the composition rule may be applied to the second term of (5.140), within the summation sign, and we therefore write:

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}}\right\} = s^{\alpha}\mathcal{L}\left\{f(x)\right\} - \sum_{k=0}^{n-1} s^{k}\left(\frac{\mathrm{d}^{\alpha-1-k} f(x)}{\mathrm{d}x^{\alpha-1-k}}\right)_{x=0} \tag{5.141}$$

$$0 < \alpha \neq 1,2,3,...$$

Therefore, we have proved our generalization process. The Laplace transform formula in (5.141) is a very simple generalization of the Laplace transform formula of classical integral calculus; that is, of the Laplace transform of the derivative or the integral of a function $f(x)$.

5.14.2 Generalization is not possible for a few classical Laplace transform identities

A point may be noted, that no generalization exists for the following classical formulas of integration and differentiation of the Laplace transformed function $\mathcal{L}\left\{f(x)\right\} = F(s)$:

$$\mathcal{L}\left\{-\frac{f(x)}{x}\right\} = \int_{\infty}^{s}\left(\mathcal{L}\left\{f(x)\right\}\right)\mathrm{d}s = \frac{\mathrm{d}^{-1}\mathcal{L}\left\{f(x)\right\}}{\mathrm{d}s^{-1}} - \left(\frac{\mathrm{d}^{-1}\mathcal{L}\left\{f(x)\right\}}{\mathrm{d}s^{-1}}\right)_{s=\infty}$$

$$\mathcal{L}\left\{-x\left(f(x)\right)\right\} = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{f(x)\right\} \tag{5.142}$$

$$\mathcal{L}\left\{(-x)^{n} f(x)\right\} = \frac{\mathrm{d}^{n}}{\mathrm{d}s^{n}}\mathcal{L}\left\{f(x)\right\} \qquad n = 1,2,3,...$$

To prove this statement, that the generalization in (5.142) is not possible for fractional differ-integration of the Laplace transform function; let us take for example $F(s) = \mathcal{L}\left\{f(x)\right\} = s^{\beta}$ with $-1 < \beta < 0$. Using the expression of the Laplace transform identity i.e. $\mathcal{L}\left\{x^{m}\right\} = \frac{\Gamma(m+1)}{s^{m+1}}$, we recognize that for $F(s) = s^{\beta}$ we have $f(x) = \frac{x^{-\beta-1}}{\Gamma(-\beta)}$.

Now we carry out fractional differ-integration for both sides of $\mathcal{L}\left\{f(x)\right\} = s^{\beta}$ as in the following steps:

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}s^{\alpha}}\mathcal{L}\left\{f(x)\right\} &= \frac{\mathrm{d}^{\alpha}\left[s^{\beta}\right]}{\mathrm{d}s^{\alpha}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} s^{\beta-\alpha}, \quad \mathcal{L}\left\{x^{m}\right\} = \Gamma(m+1)s^{-(m+1)} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\mathcal{L}\left\{\frac{x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)\Gamma(\alpha-\beta)}\mathcal{L}\left\{x^{\alpha}x^{-\beta-1}\right\} \\
&= \frac{\Gamma(\beta+1)\Gamma(-\beta)}{\Gamma(\beta-\alpha+1)\Gamma(\alpha-\beta)}\mathcal{L}\left\{x^{\alpha}\frac{x^{-\beta-1}}{\Gamma(-\beta)}\right\}; \qquad f(x) = \frac{x^{-\beta-1}}{\Gamma(-\beta)} \\
&= \frac{\Gamma(\beta+1)\Gamma(-\beta)}{\Gamma(\beta-\alpha+1)\Gamma(\alpha-\beta)}\mathcal{L}\left\{x^{\alpha} f(x)\right\} \\
&= \frac{\csc(-\beta\pi)}{\csc\left((\alpha-\beta)\pi\right)}\mathcal{L}\left\{x^{\alpha} f(x)\right\}
\end{aligned} \tag{5 .143}$$

We used the formula $\left(\Gamma(-\upsilon)\right)\left(\Gamma(\upsilon+1)\right) = -\pi\csc(\pi\upsilon)$ and the Laplace transform identity $\mathcal{L}\left\{x^{m}\right\} = \frac{\Gamma(m+1)}{s^{m+1}}$ with $f(x) = \frac{x^{-\beta-1}}{\Gamma(-\beta)}$ as depicted in the above steps (5.143).

This derivation shows that the fractional differ-integration of $\mathcal{L}\left\{x^{\alpha} f(x)\right\} \neq \frac{\mathrm{d}^{\alpha}\left[\mathcal{L}\{f(x)\}\right]}{\mathrm{d}s^{\alpha}}$, similar to cases with α as an integer. This we proved via the example of $f(x) = \frac{x^{-\beta-1}}{\Gamma(-\beta)}$. Therefore $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}s^{\alpha}}\left[\mathcal{L}\left\{f(x)\right\}\right] = \left(\frac{\csc(-\beta\pi)}{\csc\left((\alpha-\beta)\pi\right)}\right)\mathcal{L}\left\{x^{\alpha} f(x)\right\}$ does not hold for any function $f(x)$.

5.14.3 Evaluating the Laplace transform for function $f(x)=e^{-kx}\left({}_0D_x^{\alpha}\left[e^{kx}f(x)\right]\right)$, $\alpha \le 0$ by using the Riemann-Liouville integral formula

Let us evaluate the Laplace transform below:

$$\mathcal{L}\left\{e^{-kx}\left(\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[e^{kx}f(x)\right]\right)\right\}=(s+k)^{\alpha}\mathcal{L}\left\{f(x)\right\}; \qquad \alpha \le 0 \tag{5.144}$$

For $\alpha = 0$, the result is a trivial identity, i.e. at $k=0$, we have $\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha}[f(x)]}{\mathrm{d}x^{\alpha}}\right\}=s^{\alpha}\mathcal{L}\left\{f(x)\right\}$. For $\alpha<0$, we use the RL definition of a fractional integral and write the following:

$$\begin{aligned}\mathcal{L}\left\{e^{-kx}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left[e^{kx}f(x)\right]\right\}&=\mathcal{L}\left\{\frac{e^{-kx}}{\Gamma(-\alpha)}\int_0^x\frac{e^{ky}f(y)}{(x-y)^{\alpha+1}}\mathrm{d}y\right\}\\&=\frac{1}{\Gamma(-\alpha)}\mathcal{L}\left\{\int_0^x\left(\frac{e^{-k(x-y)}}{(x-y)^{\alpha+1}}\right)\left(f(y)\right)\mathrm{d}y\right\}\\&=\frac{1}{\Gamma(-\alpha)}\left(\mathcal{L}\left\{e^{-kx}/x^{\alpha+1}\right\}\right)\left(\mathcal{L}\left\{f(x)\right\}\right)\\&=(s+k)^{\alpha}\mathcal{L}\left\{f(x)\right\}\end{aligned} \tag{5.145}$$

In (5.145), we have used the definition of the Laplace transform of a convolution integral of two functions, that is $\mathcal{L}\left\{\int_0^x\left(f_1(x-y)\right)\left(f_2(y)\right)\mathrm{d}y\right\}=\left(\mathcal{L}\left\{f_1(x)\right\}\right)\left(\mathcal{L}\left\{f_2(x)\right\}\right)$. Also, having drawn upon known Laplace transform tables, we have used $\mathcal{L}\left\{\frac{e^{-kx}}{x^{\alpha+1}}\right\}=\Gamma(-\alpha)(s+k)^{\alpha}$.

5.15 Approximate representation of a fractional Laplace variable by rational polynomials in the integer power of the Laplace variable

In the fractional order Laplace transform, we will encounter the fractional power of Laplace variable s, like s^{α} or $s^{-\alpha}$. For example, $\sqrt{s}$ represents a semi-differentiation, whereas $\frac{1}{\sqrt{s}}$ is a semi-integration operation. These fractional Laplace variables, s^{α} for $\alpha\in\mathbb{R}$, are irrational functions of s; which we shall express first approximately as $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in an s-variable; and second by simple polynomial expansion. This is called a rational approximation for irrational functions.

For the first representation, we have used a Continued Fraction Expansion (CFE) method as one way to represent the irrational function. The CFE formula is as follows:

$$(1+x)^{\alpha}\overset{\text{def}}{=}\cfrac{1}{1-\alpha\cfrac{x}{1+\left(\frac{1}{2}\right)(\alpha+1)\cfrac{x}{1-\left(\frac{1}{6}\right)(\alpha-1)\cfrac{x}{1+\left(\frac{1}{6}\right)(\alpha+2)\cfrac{x}{1-\left(\frac{1}{10}\right)(\alpha-2)\cfrac{x}{1+\ldots}}}}}} \tag{5.146}$$

The other way to write (5.146) is:

$$(1+x)^{\alpha}\overset{\text{CFE}}{=}\frac{1}{1-}\,\frac{\alpha x}{1+}\,\frac{(1+\alpha)x}{2+}\,\frac{(1-\alpha)x}{3+}\,\frac{(2+\alpha)x}{2+}\,\frac{(2-\alpha)x}{5+\ldots\ldots\ldots\ldots} \tag{5.147}$$

CFE converges in a finite complex plane along a negative real axis for $x=-\infty$ to $x=1$. The formula given in (5.146) is an infinite expansion, although we may terminate it into a few terms, in order to approximate the LHS. For approximating say $\sqrt{s}$, set $x=s-1$ and set $\alpha=\frac{1}{2}$; and by use of these two terms we get:

$$\sqrt{s} \approx \frac{3s+1}{s+3} \tag{5.148}$$

By the use of four terms of the CFE formula, we get:

$$\sqrt{s} \approx \frac{5s^2+10s+1}{s^2+10s+5} \tag{5.149}$$

For six terms, we have $\sqrt{s} \approx \frac{7s^3+35s^2+21s+1}{s^3+21s^2+35s+7}$; for eight terms, $\sqrt{s} \approx \frac{11s^5+165s^4+462s^3+330s^2+55s+1}{s^5+55s^4+330s^3+462s^2+165s+11}$. In this way, we approximate the fractional Laplace operator as a ratio of polynomials. For semi-integration $\frac{1}{\sqrt{s}}$, the ratio will be 'reciprocal' for that which we obtained for $\sqrt{s}$ (5.148), (5.149).

The second means of approximation is via the Power Series Expansion (PSE) method, as depicted by the following formula:

$$\begin{aligned}(1+x)^{\alpha} &= 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots \\ &= \sum_{n=0}^{\infty}\frac{(\alpha-)_n}{n!}x^n\end{aligned} \tag{5.150}$$

Where $\alpha^{\underline{n}} = \alpha(\alpha-1)\ldots(\alpha-n+1) = (\alpha-)_n$ is a (truncated) falling factorial, such as a Pochhammer number (which we discussed in Section-1.9). Again, when approximating $\sqrt{s}$, set $x = s-1$ and $\alpha = \frac{1}{2}$; and with PSE we get two terms, set as the following:

$$\begin{aligned}\sqrt{s} &\approx 1+\frac{1}{2}(s-1)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(s-1)^2 \\ &= -\tfrac{1}{8}(s^2-6s+9)\end{aligned} \tag{5.151}$$

5.16 Generalized Laplace transform

We have seen two types of fractional derivatives, and previously discussed them in Section-4.18. The following formula gives us a fractional derivative definition with an in-between RL-Caputo type, with β as a type-defining parameter $0 \le \beta \le 1$. When $\beta = 1$, the definition is a Caputo derivative with representation ${}^{C}D_t^{\alpha}\left[f(t)\right]$, while $\beta = 0$ gives an RL fractional derivative with representation $D_t^{\alpha}\left[f(t)\right]$. The generalized definition (as shown in Section-4.18) is as follows:

$${}_0^{\beta}D_t^{\alpha}\left[f(t)\right] = {}_0I_t^{\beta(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0I_t^{(1-\beta)(1-\alpha)}\left[f(t)\right]\right)\right] \tag{5.152}$$

The fractional order α is the order of a derivative; here in this definition it is $0<\alpha<1$ and β is the type of derivative with $0 \le \beta \le 1$. Therefore, the nearest integer is $n=1$ in the above generalization (5.152).

The two types of fractional derivatives are the Caputo type and the Riemann-Liouville type. We previously generalized these two types in Section-4.18 using the type parameter β.

The generalized Laplace transform in integer order theory (5.131), for $n-$ fold differentiation, is:

$$\mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)\right\} = s^nF(s)-\sum_{k=0}^{n-1}s^kf^{(n-1-k)}(0) \tag{5.153}$$

Here, we use the symbol $f^{(n-1-k)}(0) = \frac{\mathrm{d}^{n-1-k}}{\mathrm{d}x^{n-1-k}}f(x)\Big|_{x=0}$, where $n \in \mathbb{N}$ is a natural number.

5.16.1 Laplace transform for the Riemann-Liouville fractional derivative

Following this, we have generalized the above (5.153) for fractional order α, with $(n-1)<\alpha<n$ for the Riemann-Liouville derivative as follows:

$$
\begin{aligned}
\mathcal{L}\left\{{}_0D_x^{\alpha} f(x)\right\} &= \mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_0I_x^{n-\alpha} f(x)\right\} \\
&= s^n \mathcal{L}\left\{{}_0I_x^{n-\alpha} f(x)\right\} - \sum_{k=0}^{n-1} s^k \frac{\mathrm{d}^{n-1-k}}{\mathrm{d}x^{n-1-k}}\left[{}_0I_x^{n-\alpha} f(x)\right]_{x=0} \\
&= s^n s^{-(n-\alpha)} \mathcal{L}\left\{f(x)\right\} - \sum_{k=0}^{n-1} s^k \left(D_x^{n-1-k}\right)\left[{}_0D_x^{-(n-\alpha)} f(x)\right]_{x=0} \\
&= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k\,{}_0D_x^{\alpha-1-k} f(0) \\
&= s^{\alpha} F(s) - f^{(\alpha-1)}(0) - s\ f^{(\alpha-2)}(0) - \ldots\ldots. - s^{n-1} f^{(\alpha-n)}(0) \\
\mathcal{L}\left\{{}_0D_x^{\alpha} f(x)\right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k f^{(\alpha-1-k)}(0)
\end{aligned}
\tag{5.154}
$$

The symbol ${}_0D_x^{\alpha-1-k} f(0)$ is represented as ${}_0D_x^{\alpha-1-k} f(0) \equiv \frac{\mathrm{d}^{\alpha-1-k}}{\mathrm{d}x^{\alpha-1-k}} f(x)\Big|_{x=0} \equiv f^{(\alpha-1-k)}(0)$.

5.16.2 Laplace transform for the Caputo fractional derivative

For the Caputo derivative, the Laplace transform can be derived using the above method (5.154), shown as:

$$
\begin{aligned}
\mathcal{L}\left\{{}_0^C D_x^{\alpha} f(x)\right\} &= \mathcal{L}\left\{{}_0I_x^{n-\alpha}\,\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\} \\
&= s^{-(n-\alpha)} \mathcal{L}\left\{{}_0D_x^{n} f(x)\right\} \\
&= s^{-(n-\alpha)}\left[s^n \mathcal{L}\left\{f(x)\right\} - \sum_{k=0}^{n-1} s^{n-1-k} \frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\bigg|_{x=0}\right] \\
&= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} \frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\bigg|_{x=0}
\end{aligned}
\tag{5.155}
$$

Thus, for the Caputo derivative, we have the Laplace transform expression as follows:

$$
\begin{aligned}
\mathcal{L}\left\{{}_0^C D_x^{\alpha} f(x)\right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0) \\
&= s^{\alpha} F(s) - s^{\alpha-1} f(0) - s^{\alpha-2} f^{(1)}(0) - \ldots\ldots\ldots - s^{\alpha-n} f^{(n-1)}(0)
\end{aligned}
\tag{5.156}
$$

5.16.3 Requirement of the fractional order of the initial states of the RL derivative and the integer order (classical) initial states for Caputo derivative initializations

The essential difference between the two is that the RL derivative requires ${}_0D_x^{\alpha-1-k} f(0)$, $k = 0,1,2...(n-1)$ known as fractional initial states, whereas the Caputo requires $f^{(k)}(0)$, $k = 0,1,2,...(n-1)$, known as (usual) integer order initial states. For example, in the case where $0 < \alpha < 1$, the Laplace transforms for RL and Caputo are expressed as follows:

$$
\begin{aligned}
\mathcal{L}\left\{{}_0D_x^{\alpha} f(x)\right\} &= s^{\alpha} F(s) - f^{(\alpha-1)}(0) \\
\mathcal{L}\left\{{}_0^C D_x^{\alpha} f(x)\right\} &= s^{\alpha} F(s) - s^{\alpha-1} f(0)
\end{aligned}
\tag{5.157}
$$

One may generalize the above two fractional derivatives ((5.154) and (5.157)) as a solution $g(x)$ of the integral equation:

$$
{}_0I_x^{\alpha}\left[g(x)\right] = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} \left(g(u)\right) \mathrm{d}u = f(x) \qquad \alpha > 0 \tag{5.158}
$$

in cases where it exists. Let us denote the solution by adding ${}_0^{*}D_x^{\alpha} f(x)$, meaning that the following assertions are equivalent for $n-1 \le \alpha \le n$:

a) $s^{\alpha}F(s)=G(s)$, where $\mathcal{L}\{g(x)\}=G(s)$ and $\mathcal{L}\{f(x)\}=F(s)$

b) $g(x)={}_0^{*}D_x^{\alpha}f(x)$

c) $f^{(\alpha-1-k)}(x)$ are equal to ZERO at $x=0$ for $k=0,1,2,...(n-1)$, together with ${}_0D_x^{\alpha}f(x)=g(x)$, the Riemann-Liouville derivative.

d) $f^{(k)}(x)$ are equal to ZERO at $x=0$ for $k=0,1,2,3,...(n-1)$, together with ${}_0^{C}D_x^{\alpha}f(x)=g(x)$, the Caputo derivative.

5.16.4 Generalized Laplace transforms formula for a fractional derivative with a type parameter

With type parameter $0\le\beta\le1$, we have a generalized definition in the Laplace transform, for a fractional order derivative of order $0<\alpha<1$:

$$\mathcal{L}\left\{{}_a^{\beta}D_t^{\alpha}\left[f(x)\right]\right\}=s^{\alpha}F(s)-s^{\beta(\alpha-1)}\left({}_0D_t^{(1-\beta)(\alpha-1)}\left[f(x)\right]\right)_{x=0} \tag{5.159}$$

Where ${}_0D_t^{(1-\beta)(\alpha-1)}f(x)$ is the Riemann-Liouville fractional derivative for the fractional order $0<\alpha<1$, and for any type described by β we define the fractional derivative as:

$${}_a^{\beta}D_x^{\alpha}\left[f(x)\right]={}_aI_x^{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\left({}_aI_x^{(1-\beta)(1-\alpha)}\left[f(x)\right]\right) \tag{5.160}$$

5.16.5 Generalized Laplace transform demonstrated for use in a fractional differential equation

As an example, consider the differential equation of order $0<\alpha<1$, with type variable $0\le\beta\le1$, and with K as a constant. The given initial condition is:

$${}_0D_t^{(1-\beta)(\alpha-1)}f(t)\Big|_{t=0}={}_0I_t^{(1-\beta)(1-\alpha)}f(0)\Big|_{t=0}=f_0 \tag{5.161}$$

For RL case we have $\beta=0$, the initial condition is ${}_0D_t^{(\alpha-1)}f(t)\Big|_{t=0}=f_0$ that is fractional order initial state (5.161). This implies in RL case we write $1-\alpha$ order fractional integration of function $f(t)$ at the initial point is a constant f_0. Therefore, the initialization function at $t=0$ for total solution, is obtained by fractional integration by order $\alpha-1$ of the constant function f_0; that is $f(t)\Big|_{t=0}=f_0\frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

For Caputo case we get initial condition as ${}_0D_t^{0}f(t)\Big|_{t=0}=f(t)\Big|_{t=0}=f_0$ by putting $\beta=1$ in (5.161) we write usual integer order initial state. We will use these concepts and demonstrate in the use in fractional differential equation.

We have a fractional differential equation expressed as the following:

$${}_0^{\beta}D_t^{\alpha}\left[f(t)\right]=K \tag{5.162}$$

Using a generalized Laplace transform identity as obtained above (5.159), we write the Laplace transformed equation, recognizing that ${}_0I_t^{(1-\beta)(1-\alpha)}\equiv{}_0D_t^{(1-\beta)(\alpha-1)}$, as the following:

$$\begin{aligned}&\mathcal{L}\left\{{}_0^{\beta}D_t^{\alpha}\left[f(t)\right]\right\}=\mathcal{L}\{K\}\qquad t>0\\&s^{\alpha}F(s)-s^{\beta(\alpha-1)}\left({}_0D_t^{(1-\beta)(\alpha-1)}\left[f(0)\right]\right)=\frac{K}{s}\\&s^{\alpha}F(s)-s^{\beta(\alpha-1)}f_0=\frac{K}{s}\end{aligned} \tag{5.163}$$

giving:

$$F(s)=\frac{K}{s^{\alpha+1}}+\frac{f_0}{s^{\alpha+\beta(1-\alpha)}}=\frac{K}{s^{\alpha+1}}+\frac{f_0}{s^{((\alpha+\beta-\alpha\beta-1)+1)}} \tag{5.164}$$

Doing an inverse Laplace transform of the above obtained $F(s)$ (5.164), using a known identity, i.e. $\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right\}=t^{\alpha}$, we get the following solution for $t\ge0$:

$$f(t)=\frac{K}{\Gamma(\alpha+1)}t^{\alpha}+\frac{f_0}{\Gamma\left((1-\beta)(\alpha-1)+1\right)}t^{(1-\beta)(\alpha-1)} \tag{5.165}$$

For an RL type, i.e. with $\beta=0$, we have the solution for $t\geq 0$:

$$f(t)=\frac{K}{\Gamma(\alpha+1)}t^{\alpha}+\frac{f_0}{\Gamma(\alpha)}t^{(\alpha-1)} \tag{5.166}$$

We observe that in (5.166) the first term is uninitialized solution, got from ${}_0D_t^{\alpha}\left[f(t)\right]=K$ by fractional integration, i.e. ${}_0I_t^{\alpha}\left[K\right]$ giving $f_{\text{un-init}}(t)=K\frac{t^{\alpha}}{\Gamma(1+\alpha)}$. The second term of (5.166) is initialization function; we get from fractional initial state given by ${}_0D_t^{(\alpha-1)}f(t)\Big|_{t=0}=f_0$; that is by finding ${}_0I_t^{(\alpha-1)}f_0$, that we write as $f_{\text{init}}(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}f_0$. We note that (5.166) is $f(t)=f_{\text{un-init}}(t)+f_{\text{init}}(t)$. We also observe that for $0<\alpha<1$ the RL case has $f(t)\big|_{t=0}=\infty$. We verify by obtaining RL fractional derivative of order α for (5.166), that is $K+f_0t^{-1}\frac{\Gamma(\alpha)}{\Gamma(0)}=K$. We obtain the original fractional differential equation with RL derivative i.e. ${}_0D_t^{\alpha}\left[f(t)\right]=K$.

For a Caputo type, i.e. with $\beta=1$, in (5.165) we have the solution:

$$f(t)=\frac{K}{\Gamma(\alpha+1)}t^{\alpha}+f_0 \tag{5.167}$$

In this case the uninitialized function is $f_{\text{un-init}}(t)=K\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, that we get by ${}_0I_t^{\alpha}\left[K\right]$; as we discussed in the RL case, and the $f_{\text{init}}(t)=f_0$. The (5.167) is $f(t)=f_{\text{un-init}}(t)+f_{\text{init}}(t)$. Taking Caputo fractional derivative of order α for (5.167); recognizing Caputo fractional derivative of constant is zero, we get the original fractional differential equation, that is ${}_0^CD_t^{\alpha}\left[f(t)\right]=K$. In this case we observe $f(t)\big|_{t=0}=f_0$.

Consider another example as a generalized fractional differential equation:

$${}_0^{\beta}D_t^{\alpha}f(t)=-K\left(f(t)\right) \tag{5.168}$$

with the given initial condition as $\left[{}_0I_t^{(1-\beta)(1-\alpha)}f(t)\right]_{t=0}=f_0$ with K as a constant, with $0<\alpha<1$ and $0\leq\beta\leq 1$. Using the generalized Laplace transform identity as in the above example we get the following steps:

$$\begin{aligned}
&{}_0^{\beta}D_t^{\alpha}f(t)=-K\left(f(t)\right)\\
&\mathcal{L}\left\{{}_0^{\beta}D_t^{\alpha}\left[f(t)\right]\right\}=\mathcal{L}\left\{-K\left(f(t)\right)\right\} \qquad t>0\\
&s^{\alpha}F(s)-s^{\beta(\alpha-1)}\left({}_0D_t^{(1-\beta)(\alpha-1)}\left[f(0)\right]\right)=-K\\
&s^{\alpha}F(s)+K\left(F(s)\right)=s^{\beta(\alpha-1)}f_0\\
&\left(s^{\alpha}+K\right)F(s)=s^{\beta(\alpha-1)}f_0
\end{aligned} \tag{5.169}$$

Therefore, we have the following expression in the Laplace domain:

$$\begin{aligned}
F(s)&=\frac{s^{\beta(\alpha-1)}f_0}{s^{\alpha}+K}=\frac{s^{\alpha-\left(\alpha+\beta(1-\alpha)\right)}}{s^{\alpha}+K}f_0\\
&=\frac{s^{\alpha-\gamma}}{s^{\alpha}+K}f_0
\end{aligned} \tag{5.170}$$

where $\gamma=\alpha+\beta(1-\alpha)$. To get an inverse Laplace transform, we re-write the above expression (5.170) as depicted in the following steps:

$$\begin{aligned}F(s)=\frac{s^{\alpha-\gamma}}{K+s^{\alpha}}f_0&=s^{-\gamma}\left(\frac{1}{Ks^{-\alpha}+1}\right)f_0\\&=f_0\left(s^{-\gamma}\left(1+Ks^{-\alpha}\right)^{-1}\right)\\&=f_0\left(s^{-\gamma}\left(1-\left(Ks^{-\alpha}\right)+\left(Ks^{-\alpha}\right)^2+...\right)\right)\\&=f_0\left(s^{-\gamma}-(K)s^{-\alpha-\gamma}+(K)^2s^{-2\alpha-\gamma}+...\right)\\&=f_0\sum_{k=0}^{\infty}(-K)^k s^{-\alpha k-\gamma}\end{aligned}\tag{5.171}$$

We used $(1+\mathrm{x})^{-1}=1-\mathrm{x}+\mathrm{x}^2-\mathrm{x}^3+...$ with $\mathrm{x}=Ks^{-\alpha}$. Then by using $\mathcal{L}^{-1}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}=s^{-\alpha}$, and applying this term by term for the above expression, we obtain the response as:

$$\begin{aligned}f(t)&=\mathcal{L}^{-1}\left\{f_0\sum_{k=0}^{\infty}(-K)^k s^{-\alpha k-\gamma}\right\}\\&=f_0\sum_{k=0}^{\infty}\frac{(-K)^k t^{\alpha k+\gamma-1}}{\Gamma(\alpha k+\gamma)}\\f(t)&=t^{\gamma-1}\sum_{k=0}^{\infty}\frac{(-Kt^{\alpha})^k}{\Gamma(\alpha k+\gamma)}\\&=f_0\left(t^{(1-\beta)(\alpha-1)}\right)E_{\alpha,\alpha+\beta(1-\alpha)}(-Kt^{\alpha})\end{aligned}\tag{5.172}$$

From this, we write:

$$\begin{aligned}f(t)&=f_0\left(t^{\gamma-1}\sum_{k=0}^{\infty}\frac{(-Kt^{\alpha})^k}{\Gamma(\alpha k+\gamma)}\right)\\&=f_0\left(t^{\alpha+\beta(1-\alpha)-1}\sum_{k=0}^{\infty}\frac{(-Kt^{\alpha})^k}{\Gamma\left(\alpha k+\alpha+\beta(1-\alpha)\right)}\right)\\&=f_0\left(t^{(\alpha-1)(1-\beta)}\sum_{k=0}^{\infty}\frac{(-Kt^{\alpha})^k}{\Gamma\left(\alpha k+\alpha+\beta(1-\alpha)\right)}\right)\\&=f_0\left(t^{(1-\beta)(\alpha-1)}\right)E_{\alpha,\alpha+\beta(1-\alpha)}(-Kt^{\alpha})\end{aligned}\tag{5.173}$$

Here, we have used the definition of a two parameter Mittag-Leffler function (refer to Appendix-A), that is:

$$\begin{aligned}E_{a,b}(x)&=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(ak+b)}\qquad a>0\qquad b>0\\&=\frac{1}{\Gamma(b)}+\frac{x}{\Gamma(a+b)}+\frac{x^2}{\Gamma(2a+b)}+...\end{aligned}\tag{5.174}$$

for all $a>0$ and $b\in\mathbb{C}$. This is an entire function and completely monotone for $0<a\le 1$ and $b\ge a$. Also, note that $E_{a,b}(0)=\frac{1}{\Gamma(b)}$ (as described in Appendix-A).

1) For $K=0$, the result is $f(t)=\frac{f_0}{\Gamma((1-\beta)(\alpha-1)+1)}t^{(1-\beta)(\alpha-1)}$, because $E_{a,b}(0)=\frac{1}{\Gamma(b)}$. In this case, if $\beta=0$, that is the RL case, then the solution is $f(t)=\frac{f_0}{\Gamma(\alpha)}t^{\alpha-1}$.

   Take ${}_0D_t^{\alpha}\left[t^{\alpha-1}\right]$, which is $\frac{\Gamma(\alpha)}{\Gamma(0)}t^{-1}$ and is zero, since $\left(\Gamma(0)\right)^{-1}=0$. For $\beta=1$ in the Caputo case, the solution is $f(t)=f_0$, revealing that the Caputo fractional derivative of a constant is zero.

2) For $\beta=1$ in the Caputo derivative type, $f(t)=f_0\left(E_{\alpha}(-Kt^{\alpha})\right)$, where $E_{\alpha}(x)=E_{\alpha,1}(x)$, and it denotes the one parameter Mittag-Leffler function. For the RL case with $\beta=0$, the solution is $f(t)=f_0t^{\alpha-1}E_{\alpha,\alpha}(-Kt^{\alpha})$.

These examples make us think about the concept of 'stationarity' in a fresh way, within the domain of fractional calculus, leading us to initial conditions, such as ${}_0D_t^{(1-\beta)(\alpha-1)}f(t)\Big|_{t=0}$. In addition, we saw that, depending on the type of fractional derivative which composes a fractional differential equation, the solution varies.

5.17 Generalized stationary conditions

Consider a fractional 'stationary' differential equation with order $0<\alpha<1$ and type $0\le\beta\le 1$ as ${}_0^{\beta}D_t^{\alpha}f(t)=0$, with the initial condition shown as ${}_0D^{(1-\beta)(\alpha-1)}f(0)=f_0$, which is a fractional Riemann-Liouville derivative at the initial condition. We can re-write this initial condition in order to state it as a fractional integral initial value given as a constant, that is ${}_0I_t^{(1-\beta)(1-\alpha)}f(0)=f_0$. For $\alpha=1$, the conventional definition of the stationary condition is recovered as ${}_0^{\beta}D_t^{1}f(t)=0$, i.e. $f(0)=f_0$ is a constant, for any type β. The solution of this fractional differential equation is as follows:

$$f(t)=\frac{f_0}{\Gamma\big((1-\beta)(\alpha-1)+1\big)}t^{(1-\beta)(\alpha-1)} \tag{5.175}$$

This solution was stated in the previous section. This may also be seen by inserting the above expression of $f(t)$ into the generalized definition of a fractional derivative as follows:

$${}_a^{\beta}D_t^{\alpha}\left[f(t)\right]={}_aI_t^{\beta(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left({}_aI_t^{(1-\beta)(1-\alpha)}\left[f(t)\right]\right)\right] \tag{5.176}$$

for $0<\alpha<1$ and using the fractional integral of the power function expression ${}_aI_t^{\alpha}\left[(t-a)^{\lambda}\right]=\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}(t-a)^{\alpha+\lambda}$.

For the RL case with $\beta=0$, the function $f(t)=\frac{f_0}{\Gamma(\alpha)}t^{\alpha-1}$, and for the Caputo case with $\beta=1$, the function is $f(t)=f_0$, a constant.

Note that the integral ${}_0I_t^{(1-\beta)(1-\alpha)}f(t)=f_0$ remains conserved and constant for all t, while the function $f(t)$ itself varies. In particular, $\lim_{t\downarrow 0}f(t)=\infty$ and $\lim_{t\uparrow\infty}f(t)=0$. For $\beta=1$ and $\alpha=1$, one recovers $f(t)=f_0$ as usual. The fractional derivative of the constant for the Caputo type is zero, and the integer order derivative of the constant of any type is zero.

The new type of stationary states (in the initial condition) that a fractional integral (rather than the function itself) is a constant that is arrived at. This seems to rather lack knowledge about the fractional 'stationary-conditions' and is particularly responsible for the difficulty of deciding which type of fractional derivative should be used when generalizing a traditional equation of the motion of thermodynamics and electrodynamics control systems, among others. Nevertheless, let us carry on with this dichotomy.

5.18 Demonstration of the generalized Laplace transform for solving the initialized fractional differential equation

5.18.1 Composition of a fractional differential equation for a Riemann-Liouville fractional derivative

Say we need to find the solution of ${}_0D_t^{1/2}\left[f(t)\right]+bf(t)=0$ for $t>0$. We must apply the Laplace transform rule for an RL fractional derivative, which is:

$$\mathcal{L}\left\{{}_0D_t^{\alpha}f(t)\right\}=s^{\alpha}F(s)-\sum_{k=0}^{n-1}s^k\left({}_0D_t^{\alpha-1-k}f(0)\right)$$
$$(n-1)<\alpha<n \tag{5.177}$$

With $\alpha=\frac{1}{2}$, we have $n=1$, so we get $\mathcal{L}\left\{{}_0D_t^{1/2}\left[f(t)\right]\right\}=s^{1/2}F(s)-\sum_{k=0}^{0}s^k\,{}_0D_t^{\alpha-1-k}f(0)$, that is $\mathcal{L}\left\{{}_0D_t^{1/2}\left[f(t)\right]\right\}=s^{1/2}F(s)-{}_0D_t^{-1/2}f(0)$. Using this expression in the given fractional differential equation, we get the Laplace transformed equation as follows:

$$\left(s^{\frac{1}{2}}F(s) - {}_0D_t^{-\frac{1}{2}}f(0)\right) + bF(s) = 0 \tag{5.178}$$

Here we require a fractional initialization state, that is, the value of ${}_0D_t^{-\frac{1}{2}}\left[f(t)\right]$ at $t=0$; meaning the value of the semi-integral of a function at the initial point. Say that the value is given as a constant, namely ${}_0D_t^{-\frac{1}{2}}f(0) = C$, with this we write:

$$s^{\frac{1}{2}}F(s) - C + bF(s) = 0 \tag{5.179}$$

From the above (5.178), we get $F(s) = \frac{C}{\sqrt{s}+b}$; the inverse Laplace transform of this (tables of Laplace transforms of the Mittag-Leffler function, refer to Appendix-A) gives the following result:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{C}{\sqrt{s}+b}\right\} = C\left(t^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\left(-b\sqrt{t}\right)\right) \tag{5.180}$$

The solution is formed with a two-parameter Mittag-Leffler function. The two-parameter function is defined as $E_{\alpha,\beta}(at) = \sum_{k=0}^{\infty}\frac{a^k t^k}{\Gamma(\alpha k+\beta)}$; $\alpha,\beta > 0$ (refer to Appendix-A). The Laplace transform is given by the following relationship (again, see Appendix-A):

$$\mathcal{L}\left\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(at^{\alpha})\right\} = \frac{s^{\alpha-\beta}k!}{(s^{\alpha}-a)^{k+1}}; \quad \mathrm{Re}(s) > |a|^{\frac{1}{\alpha}}$$

$$\mathcal{L}\left\{t^{\alpha-1}E_{\alpha,\alpha}(at^{\alpha})\right\} = \frac{1}{s^{\alpha}-a}; \qquad \alpha=\beta, \quad k=0 \tag{5.181}$$

$$\mathcal{L}\left\{t^{\left(\frac{1}{2}\right)k-\left(\frac{1}{2}\right)}E_{\frac{1}{2},\frac{1}{2}}^{(k)}\left(a\sqrt{t}\right)\right\} = \frac{k!}{(\sqrt{s}-a)^{k+1}} \qquad \alpha=\beta$$

In the above expressions $k = 0,1,2...$, indicating an integer order whole derivatives of the Mittag-Leffler function. From the Laplace transform relationships, we set $k=0$ and $a=-b$ for our case to find the solution. In addition, we note from the Laplace transform tables (Appendix-A) that $\frac{1}{s^{\alpha}+b} = \mathcal{L}\left\{F_{\alpha}(-b,t)\right\}$, where $F_{\alpha}(k,t)$ is the Robotnov-Hartley function. Thus, the other form of solution of the above is as follows:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{C}{\sqrt{s}+b}\right\} = C\left(F_{\frac{1}{2}}\left(-b,t\right)\right) \tag{5.182}$$

We observe the $F_{\alpha}\left(-b,t\right) = t^{\alpha-1}E_{\alpha,\alpha}(-bt^{\alpha})$ relationship of the Robotnov-Hartley function and the two parameter Mittag-Leffler function; we write the series form which is given as $F_{\alpha}(-b,t) = t^{\alpha-1}E_{\alpha,\alpha}(-bt^{\alpha}) = t^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-b)^n t^{\alpha n}}{\Gamma(\alpha n+\alpha)}$. This can also be cast with a generalized R function i.e. $R_{\frac{1}{2},0}\left(-b,0,t\right)$, where $R_{\alpha,v}[a,c,t] = \sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)\alpha-1-v}}{\Gamma\left((n+1)\alpha-v\right)}$ (refer to Appendix-A).

5.18.2 Composition of the fractional differential equation for the Caputo fractional derivative

Now we compose the same differential equation with the Caputo derivative as ${}_0^CD_t^{\frac{1}{2}}\left[f(t)\right] + bf(t) = 0$. By using the Laplace transform rule of the Caputo derivative (as in 5.156); i.e. $\mathcal{L}\left\{{}_0^CD_x^{\alpha}f(x)\right\} = s^{\alpha}F(s) - s^{\alpha-1}f(0) - s^{\alpha-2}f^{(1)}(0)... - s^{\alpha-n}f^{(n-1)}(0)$, we get for $\alpha = \frac{1}{2}$, the Laplace transformed equation as $\left(s^{\frac{1}{2}}F(s) - s^{-\frac{1}{2}}f(0)\right) + bF(s) = 0$. With the initial condition $f(0) = K = 1$ a constant at $t=0$, we write:

$$F(s) = \frac{s^{-\frac{1}{2}}K}{s^{\frac{1}{2}}+b} \tag{5.183}$$

Using the Laplace transform identity of one parameter of the Mittag-Leffler function (i.e. $\mathcal{L}\left\{E_{\alpha}\left(\lambda t^{\alpha}\right)\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda}$), we write the solution as:

$$f(t) = E_{\frac{1}{2}}\left(-b\sqrt{t}\right) \tag{5.184}$$

5.18.3 Applying the concept of the initialization function of fractional derivatives in a fractional differential equation

Now we apply the concept of an initialized fractional derivative with an initialization function that we have evolved as $\psi(t)$. We write:

$$ {}_0D_t^{1/2}\left[f(t)\right] = {}_0\mathrm{d}_t^{1/2}\left[f(t)\right] + \psi(t) \tag{5.185}$$

The given fractional differential equation (i.e. ${}_0D_t^{1/2}\left[f(t)\right] + bf(t) = 0$) is, therefore, as follows in terms of the initialization function:

$$\left({}_0\mathrm{d}_t^{1/2}\left[f(t)\right] + \psi(t)\right) + b\left(f(t)\right) = 0 \tag{5.186}$$

The term ${}_0\mathrm{d}_t^{1/2}\left[f(t)\right]$ is an uninitialized semi-derivative with an initial condition of zero, and its Laplace transform is thus $\mathcal{L}\left\{{}_0\mathrm{d}_t^{1/2}\left[f(t)\right]\right\} = s^{1/2}F(s)$. The Laplace transform of the fractional differential equation is, with $\mathcal{L}\left\{\psi(t)\right\} = \psi(s)$, as follows:

$$s^{1/2}F(s) + \psi(s) + bF(s) = 0 \tag{5.187}$$

From here, we write $F(s) = -\frac{\psi(s)}{\sqrt{s}+b}$; the inverse Laplace transform of this will give $f(t)$. Specifically, for a side-initialization we give $\psi(t) = -C\delta(t)$, that is, the initialization function is a delta function at $t = 0$, which gives us $\psi(s) = \mathcal{L}\left\{\psi(t)\right\} = \mathcal{L}\left\{-C\delta(t)\right\} = -C$. Using this initialization, we get $F(s) = \frac{C}{\sqrt{s}+b}$, and the inverse Laplace transform will give the same answer as we got above (namely, $f(t) = Ct^{-1/2}E_{1/2,1/2}\left(-b\sqrt{t}\right) = CF_{1/2}(-b,t)$) in (5.182).

For any other $\psi(t)$, we will have a different result. We note that $F(s) = -\left(\frac{1}{\sqrt{s}+b}\right)\psi(s)$. In addition, we note that $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}+b}\right\} = R_{1/2,0}\left(-b,0,t\right)$, in terms of the 'Generalized - R 'function defined below with its Laplace transform:

$$\begin{aligned}
R_{\alpha,v}[a,c,t] &= \sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)\alpha-1-v}}{\Gamma\left((n+1)\alpha-v\right)} \equiv R_{\alpha,v}[a,t-c] \\
\mathcal{L}\left\{R_{\alpha,v}(a,0,t)\right\} &= \frac{s^v}{s^\alpha - a};\quad \mathrm{Re}\left[(\alpha-v)\right]>0;\quad \mathrm{Re}\left[s\right]>0 \\
\mathcal{L}\left\{R_{\alpha,v}(a,c,t)\right\} &= \frac{e^{-cs}s^v}{s^\alpha - a};\quad c\ge 0\quad \mathrm{Re}\left[(\alpha-v)\right]>0; \mathrm{Re}\left[s\right]>0
\end{aligned} \tag{5.188}$$

Note that $F(s)$ is a product of two functions in the Laplace domain (complex frequency s-domain) and that is a convolution integral in the time domain. Therefore, for a general solution we write the convolution as follows:

$$\begin{aligned}
f(t) &= -\left(R_{1/2,0}\left(-b,0,t\right)\right)*\left(\psi(t)\right) \\
&= -\int_0^t R_{1/2,0}\left(-b,0,(t-\tau)\right)\left(\psi(\tau)\right)\mathrm{d}\tau
\end{aligned} \tag{5.189}$$

This demonstrates the use of generalized Laplace transforms and initialized fractional derivatives in solving fractional differential equations. We will devote the next few chapters solely to various types of fractional differential equations. As in the earlier case, the $f(t)$ can be casted via the use of a two parameter Mittag-Leffler function, and also via the use of a Robotnov-Hartley function, meaning there are several functions (refer to Appendix-A), which are used for fractional differential equations.

5.19 Fourier transform of a fractional derivative operator

5.19.1 Fourier transform from a Laplace transform

The Laplace transform of a fractional derivative-integral of order α operation is:

$$\mathcal{L}\left\{{}_0D_x^\alpha\left[f(x)\right]\right\} = s^\alpha\mathcal{L}\left\{f(x)\right\} - \sum_{k=0}^{n-1}s^k\left({}_0D_x^{\alpha-1-k}f(x)\right)\Big|_{x=0} \tag{5.190}$$

Where the Laplace transform is defined as:

$$\mathcal{L}\left\{f(x)\right\} = \int_0^\infty e^{-sx}\left(f(x)\right)\mathrm{d}x \tag{5.191}$$

In the Laplace transform definition above, the order of differ-integration $\alpha \in \mathbb{R}$; and the positive integer $n \in \mathbb{Z}^+$ is such that $(n-1) < \alpha \le n$. In this expression, when $\alpha < 0$ (that is for fractional integration), the term involving summation in the above expression becomes zero for any function, $f(x)$ with the available Laplace transform, i.e. $\mathcal{L}\left\{f(x)\right\}$. In addition, one can have something similar to the Laplace transform of fractional differ-integrals of $f(x)$; a Fourier transform of a fractional differ-integral operation. For a function $f(x)$, which is "well-behaved" at $x = -\infty$, we can have:

$$\mathcal{F}\left\{{}_{-\infty}D_x^{\alpha}\left[f(x)\right]\right\} = (i\omega)^{\alpha}\mathcal{F}\left\{f(x)\right\} \tag{5.192}$$

and therefore we have a fractional derivative/integral operation as an inverse Fourier transformed one, which is:

$${}_{-\infty}D_x^{\alpha}\left[f(x)\right] = \mathcal{F}^{-1}\left\{(i\omega)^{\alpha}\mathcal{F}\left\{f(x)\right\}\right\} \tag{5.193}$$

Where the Fourier and an inverse Fourier transform are depicted as the following formulae:

$$\begin{aligned}\mathcal{F}\left\{f(x)\right\} = \mathcal{F}(\omega) &\overset{\text{def}}{=} \int_{-\infty}^{+\infty}\mathrm{d}x\left(e^{-i\omega x}f(x)\right)\\ f(x) = \mathcal{F}^{-1}\left\{F(\omega)\right\} &\overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega\left(e^{i\omega x}F(\omega)\right)\end{aligned} \tag{5.194}$$

In some cases (especially for steady state systems with a lower terminal of differ-integration $a = -\infty$), the Fourier transform method is another way to find a fractional derivative/fractional integration of the function $f(x)$. That is:

(i) Obtain the Fourier transform of $f(x)$ as $F(\omega)$;

(ii) This is then transformed into $F(\omega)$ in the frequency ω domain, which we multiply by $(i\omega)^{\alpha}$;

(iii) With the resulting function $(i\omega)^{\alpha}F(\omega)$, we conduct an inverse Fourier transform, to get ${}_{-\infty}D_x^{\alpha}\left[f(x)\right]$.

We check the result of the Fourier transform of the derivative of a function, that is, $\mathcal{F}\left\{f^{(1)}(x)\right\} = (i\omega)\mathcal{F}\left\{f(x)\right\}$. With our definition, we write:

$$\mathcal{F}\left\{f^{(1)}(x)\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}f^{(1)}(x)\mathrm{d}x \tag{5.195}$$

Now we write $u = e^{-i\omega x}$, then $\mathrm{d}u = -i\omega e^{-i\omega x}\mathrm{d}x$ and $\mathrm{d}v = f^{(1)}\mathrm{d}x$, $v = f(x)$. Applying the formula for integration by parts (i.e. $\int_{-\infty}^{+\infty}u\mathrm{d}v = uv\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}v\mathrm{d}u$), we get the following steps:

$$\begin{aligned}\mathcal{F}\left\{f^{(1)}(x)\right\} &= \int_{-\infty}^{+\infty} e^{-i\omega x}f^{(1)}(x)\mathrm{d}x\\ &= e^{-i\omega x}f(x)\Big|_{x=-\infty}^{x=+\infty} - \int_{-\infty}^{+\infty}(-i\omega e^{-i\omega x})f(x)\mathrm{d}x\end{aligned} \tag{5.196}$$

Since we take the function $f(x)$ as absolutely integrable, thus we have, $e^{-i\omega x}\, f(x)\Big|_{x=-\infty}^{x=+\infty} = 0$, so we write:

$$\begin{aligned}\mathcal{F}\left\{f^{(1)}(x)\right\} &= \int_{-\infty}^{+\infty} i\omega(e^{-i\omega x})\left(f(x)\right)\mathrm{d}x\\ &= (i\omega)\int_{-\infty}^{+\infty} e^{-i\omega x}\left(f(x)\right)\mathrm{d}x = (i\omega)\mathcal{F}\left\{f(x)\right\}\end{aligned} \tag{5.197}$$

Repeating the above n-times, we can write $\mathcal{F}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\} = (i\omega)^n\mathcal{F}\left\{f(x)\right\}$. We can also verify the same for an inverse Fourier transform formula and then differentiate the same as described below:

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\left(F(\omega)\right)\mathrm{d}\omega \\
F(\omega) &= \mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}\left(f(x)\right)\mathrm{d}x \\
f^{(1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\left(F(\omega)\right)\mathrm{d}\omega\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\left((i\omega)F(\omega)\right)\mathrm{d}\omega = \mathcal{F}^{-1}\left\{(i\omega)F(\omega)\right\}
\end{aligned}
\tag{5.198}
$$

From the steps in (5.198), we write $\mathcal{F}\left\{f^{(1)}(x)\right\} = (i\omega)F(\omega)$, repeating n -times so that we get $\mathcal{F}\left\{f^{(n)}(x)\right\} = (i\omega)^n F(\omega)$. For n non-integer, say α, we have:

$$\mathcal{F}\left\{f^{(\alpha)}(x)\right\} = (i\omega)^{\alpha} F(\omega) \tag{5.199}$$

5.19.2 Similarity and dissimilarity in Laplace and Fourier transforms

The two integral expressions are rewritten as the following:

$$\mathcal{L}\left\{f(x)\right\} = \int_0^{\infty} e^{-sx}\left(f(x)\right)\mathrm{d}x \qquad \mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}\left(f(x)\right)\mathrm{d}x \tag{5.200}$$

Refer to Appendix-G where the derivation of an inverse Laplace transform is shown using Fourier transforms. They are very similar where we assume the representations of $f(x)$ in both cases of (5.200) are integrable, except within the limits of integration. Similarly, we see $\mathcal{L}\left\{{}_0D_x^{\alpha}\left[f(x)\right]\right\} = s^{\alpha}\mathcal{L}\left\{f(x)\right\}$ with all the initial values as zero with $\mathcal{F}\left\{{}_{-\infty}D_x^{\alpha}\left[f(x)\right]\right\} = (i\omega)^{\alpha}\mathcal{F}\left\{f(x)\right\}$; where s is replaced by $i\omega$. We term the Laplace variable $s = \mathrm{Re}[s] + i\,\mathrm{Im}[s]$ as a complex number, expressed as $s = \mathrm{Re}[s] + i\omega$; $\omega = \mathrm{Im}[s]$ (s is called a complex frequency), whereas the Fourier variable ω is a real frequency, and the quantity $i\omega$ is an imaginary frequency. We can term the Fourier frequency as the Laplace frequency with $\mathrm{Re}[s] = 0$.

The limits of integration in the Fourier transform expression are from $-\infty$ to $+\infty$, meaning that we consider all the initial terms as zero; whereas the Laplace transform integration has a limit from 0 to ∞, and has the effect of initial terms. Physically we mean that the Fourier transform is a steady state representation, while the Laplace transform pictures transient and steady state behavior. However, we can have the definition of a double-sided Laplace transform by extending the lower limit of integration from zero to minus infinity. Therefore, for a given Laplace transform of a function, can we change the Laplace variable s to $i\omega$ and get the Fourier transform?

For example, if $\mathcal{L}\left\{u(x)\right\} = \frac{1}{s}$ then should we have $\mathcal{F}\left\{u(x)\right\} = \frac{1}{i\omega}$? Where $u(x)$ is the Heaviside unit step function; i.e. $u(x) = 0$ for $x < 0$ and $u(x) = 1$ for $x > 0$. We also know from $\frac{\mathrm{d}}{\mathrm{d}x}u(x) = \delta(x)$, that it is a Dirac-delta function.

Let us construct a function $y(x) = -\frac{1}{2}$ for $x < 0$ and $y(x) = +\frac{1}{2}$ for $x > 0$. Clearly, we can write $y(x) = u(x) - \frac{1}{2}$ and $\frac{\mathrm{d}}{\mathrm{d}x}y(x) = \delta(x)$. In addition, we write here from the Fourier transform tables that $\mathcal{F}\left\{\delta(x)\right\} = 1$. In the previous section, we have derived the expression $\mathcal{F}(\omega) = \mathcal{F}\left\{f(x)\right\} = \frac{1}{i\omega}\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right\}$, and using this we write the following:

$$
\begin{aligned}
\mathcal{F}\left\{u(x)\right\} &= \frac{1}{i\omega}\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right\} = \frac{1}{i\omega}\mathcal{F}\left\{\delta(x)\right\} = \frac{1}{i\omega} \\
\mathcal{F}\left\{y(x)\right\} &= \frac{1}{i\omega}\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right\} = \frac{1}{i\omega}\mathcal{F}\left\{\delta(x)\right\} = \frac{1}{i\omega}
\end{aligned}
\tag{5.201}
$$

We are therefore having the same Fourier transform for two different functions. The functions are different in terms of a constant that is $\frac{1}{2}$. This is obviously caused by the fact that the derivatives of the constant are zero. Now we discuss where the difference should exist.

Take the average value as $\langle u(x)\rangle_{-X}^{+X} = \frac{1}{2X}\int_{-X}^{+X} u(x)\mathrm{d}x = \frac{1}{2X}\int_{0}^{X}(1)\mathrm{d}x = \frac{1}{2}$ and similarly $\langle y(x)\rangle_{-X}^{+X} = \frac{1}{2X}\int_{-X}^{+X} y(x)\,\mathrm{d}\,x = \frac{1}{2X}\int_{-X}^{0}\left(-\frac{1}{2}\right)\mathrm{d}x + \frac{1}{2X}\int_{0}^{+X}\left(\frac{1}{2}\right)\mathrm{d}x = 0$. In a physical sense, this average value signifies that the function $u(x)$ has an average value (dc component) as $\frac{1}{2}$ and the function $y(x)$ has no such dc-component. The dc-component corresponds to the frequency $\omega = 0$ in the Fourier transformed function. Therefore, we need to add this dc-component in our $\mathcal{F}\{u(x)\}$; as follows:

$$\begin{aligned}\mathcal{F}\{u(x)\} &= \mathcal{F}\{y(x)\} + \mathcal{F}\{\tfrac{1}{2}\} \\ &= \frac{1}{i\omega} + \pi\delta(\omega)\end{aligned} \tag{5.202}$$

We used $\mathcal{F}\{e^{i\omega_0 x}\} = 2\pi\left(\delta(\omega-\omega_0)\right)$ by setting $\omega_0 = 0$ to get $\mathcal{F}\{1\} = 2\pi\left(\delta(\omega)\right)$, and used in the above derivation (5.202). Therefore, we see similarity through relations like the Laplace and Fourier transforms of derivatives of a function, yet the two are different.

5.20 Complex order differ-integrations described via the Laplace transform

We have discussed the differ-integral operation $g(t) = {}_0D_t^{\alpha}\left[f(t)\right]$ with α as an arbitrary real number. Say if $\alpha = u + iv$, with u and v as real numbers. Then we have a complex order differ-integration as $g(t) = \frac{\mathrm{d}^{u+iv}}{\mathrm{d}t^{u+iv}}\left[f(t)\right]$, which is an analytical continuation of the differ-integral operator, which we discussed in an earlier chapter (Section 4.22).

Assuming that our initial values are zero i.e. $f(t) = 0$ at $t = 0$; then we can use our generalized Laplace transform formula to write the following steps:

$$\begin{aligned}G(s) = \mathcal{L}\{g(t)\} &= s^{u+iv}F(s) \\ &= s^u s^{iv}F(s) = s^u e^{\ln s^{iv}}F(s) \\ &= s^u e^{iv\ln s}F(s) \\ &= s^u\left(\cos(v\ln s) + i\sin(v\ln s)\right)F(s)\end{aligned} \tag{5.203}$$

We write the invert of $g(t) = {}_0D_t^{(u+iv)}f(t)$ as $f(t) = {}_0D_t^{-(u+iv)}g(t)$ and its Laplace transform as $F(s) = s^{-(u+iv)}G(s)$.

Using a known Laplace transform (i.e. $\mathcal{L}^{-1}\{s^{-\alpha}\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and taking $G(s) = 1$ or $g(t) = \delta(t)$), we get the impulse response $f(t)$, which is as follows, by taking an inverse Laplace transform as:

$$\begin{aligned}f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{s^{-(u+iv)}G(s)\} \\ &= \mathcal{L}^{-1}\{s^{-(u+iv)}\} \\ &= \frac{t^{u+iv-1}}{\Gamma(u+iv)} = \frac{t^{u-1}}{\Gamma(u+iv)}e^{iv\ln t} \\ &= \frac{t^{u-1}\left(\cos(v\ln t) + i\sin(v\ln t)\right)}{\Gamma(u+iv)}\end{aligned} \tag{5.204}$$

The derivation of $f(t)$ in (5.204) says we get an imaginary response component along with the real response component, in a time domain.

Now we use conjugated differ-integrals where α and $\bar{\alpha}$ are complex conjugates, as follows

$$\begin{aligned}g(t) &= {}_0D_t^{\alpha}\left[f(t)\right] + {}_0D_t^{\bar{\alpha}}\left[f(t)\right] \\ &= {}_0D_t^{u+iv}\left[f(t)\right] + {}_0D_t^{u-iv}\left[f(t)\right]\end{aligned} \tag{5.205}$$

The Laplace transform of the above conjugated differ-integrals is as follows

$$\begin{aligned}\mathcal{L}\left\{g(t)\right\} &= G(s) = \left(s^{u+iv} + s^{u-iv}\right)F(s) \\ &= \left(s^u s^{iv} + s^u s^{-iv}\right)F(s) = s^u\left(e^{iv\ln s} + e^{-iv\ln s}\right)F(s) \\ &= 2s^u\left(\cos(v\ln s)\right)F(s)\end{aligned} \tag{5.206}$$

We write $f(t) = {}_0D_t^{-\alpha}\left[g(t)\right] + {}_0D_t^{-\bar{\alpha}}\left[g(t)\right] = {}_0D_t^{-(u+iv)}\left[g(t)\right] + {}_0D_t^{-(u-iv)}\left[g(t)\right]$, and take the Laplace transform, which is $\mathcal{L}\left\{f(t)\right\} = F(s) = s^{-u}\left(s^{-iv} + s^{iv}\right)G(s)$. With $G(s) = 1$, that is $g(t) = \delta(t)$, and using $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, the impulse response of a conjugated differ-integral is as follows:

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\left\{s^{-(u+iv)} + s^{-(u-iv)}\right\} \\ &= \frac{t^{u+iv-1}}{\Gamma(u+iv)} + \frac{t^{u-iv-1}}{\Gamma(u-iv)}\end{aligned} \tag{5.207}$$

From the property of the gamma function we have $\frac{1}{\Gamma(u+iv)} = \mathrm{Re}\left[\frac{1}{\Gamma(u+iv)}\right] + i\,\mathrm{Im}\left[\frac{1}{\Gamma(u+iv)}\right]$ and $\frac{1}{\Gamma(u-iv)} = \mathrm{Re}\left[\frac{1}{\Gamma(u+iv)}\right] - i\,\mathrm{Im}\left[\frac{1}{\Gamma(u+iv)}\right]$, using these we write the following (impulse response for complex conjugated differ-integration) as:

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\left\{s^{-(u+iv)} + s^{-(u-iv)}\right\} = \mathcal{L}^{-1}\left\{2s^{-u}\cos(v\ln s)\right\} \\ &= \frac{t^{u+iv-1}}{\Gamma(u+iv)} + \frac{t^{u-iv-1}}{\Gamma(u-iv)} \\ &= t^{u-1}\left((t^{-iv} + t^{iv})\,\mathrm{Re}\left[\frac{1}{\Gamma(u+iv)}\right] + i(t^{-iv} - t^{iv})\,\mathrm{Im}\left[\frac{1}{\Gamma(u+iv)}\right]\right) \\ &= 2t^{u-1}\left(\left(\cos(v\ln t)\right)\mathrm{Re}\left[\frac{1}{\Gamma(u+iv)}\right] + \left(\sin(v\ln t)\right)\mathrm{Im}\left[\frac{1}{\Gamma(u+iv)}\right]\right)\end{aligned} \tag{5.208}$$

We are getting a pure and real response in time, for these conjugated differ-integrals.

Note that we have $\left((s^{u+iv})(s^{u-iv})\right)F(s) = s^{2u}F(s)$ and $\left(\frac{s^{u+iv}}{s^{u-iv}}\right)F(s) = s^{2iv}F(s)$. This means that any real order differ-integral can be composed of the product of two complex conjugated differ-integral Laplace operators. A pure, imaginary order differ-integral can be composed of the ratio of two complex conjugated differ-integral Laplace operators. This gives a 'complex' order calculus; which is still a developing subject.

5.21 Short summary

The primary focus of this chapter was the concept of initialization for fractional derivatives and fractional integration. We showed here that the fractional integrations and fractional derivatives require an initialization function, unlike classical integer order calculus. We discussed the initializing processes that require demanding issues for the Riemann-Liouville (RL) and Caputo fractional derivatives. We also stated that to equate the two initializations for RL and Caputo, is a difficult task. We then discussed Laplace transforms for fractional derivatives and fractional integral operations and how initial states matter in both Caputo and RL formulations. We touched briefly upon the Fourier transform of fractional differ-integrations and pointed out the differences between Laplace and Fourier transforms. We also described the approximation method to represent a fractional Laplace operator via continued fraction expansion or the power series expansion method; this is useful for circuit realization in electronic engineering. Then we state the interesting concept of generalized stationary conditions; and we have seen the generalization of the fractional derivative of Caputo and RL types. We also introduced the usage of generalized initialized fractional derivatives and the use of the generalized Laplace transform to solve a fractional differential equation. We also have briefly introduced the composition of complex order differ-integrals via complex Laplace variables, and obtained interesting responses. We will now devote the next few chapters to various aspects of solving fractional differential and fractional integral equations, especially using generalized Laplace transform techniques.

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The above pioneering works are furnished with further detail and listed in alphabetical order in the Bibliography section.

Chapter Six

Fractional Differential Equations and their Analysis

6.1 Introduction

Fractional differential equations appear in several physical systems. In systems involving coarse-grained phenomena everything happens as if the elemental point is not infinitesimally zero; rather, it has some thickness (or spread), that could be pictured using, say, $(\mathrm{d}x)^{\alpha}$ with $0<\alpha<1$ instead of $\mathrm{d}x$ with $(\mathrm{d}x)^{\alpha}>\mathrm{d}x$ as $\mathrm{d}x \downarrow 0$. In other words, we would be led to consider the rate of variation as $\mathrm{d}y/\mathrm{d}x^{\alpha}$, (or $\mathrm{d}^{\alpha}y/\mathrm{d}x^{\alpha}$), and therefore, in a way, we are going towards a fractional derivative. This finite thickness (or spread) of the fractional differential quantity allows us to view scales where the in-homogeneity and roughness of the systems are captured on micro and meso scales. This is in contrast to fine-grained phenomena where the normal differential element tends towards zero, which make the system uniform. This is one of the reasons for having fractional differential equations, where the differential quantity is $(\mathrm{d}x)^{\alpha}$, different from $\mathrm{d}x$. This $\mathrm{d}x$ is a standard differential quantity for 'finely-grained' phenomena, where normal derivatives and integrals appear. The solution to these fractional differential equations is no more rigorous than their integer order counterparts. The Laplace transform technique is very popular, although several other analytical approaches do exist. However, in this chapter, it is mainly the Laplace transform technique that is considered, as this technique is the most easily understood and the most popular amongst engineers and scientists. We will study methods to obtain an inverse Laplace transform via contour integration and via a power series expansion, and we will discuss the new method of obtaining an inverse Laplace transform via the Berberan-Santos method where the contour integration is not required. We will study the response of a fractional differential equation, obtain response functions, and evolve their properties. We will study how we can determine decay rates for complex decay functions. In this chapter, the Mittag-Leffler function will be used and studied. In addition, various simple types of fractional differential equations will be taken, and their respective solutions will be obtained.

6.2 Tricks in solving some fractional differential equations

6.2.1 A method using the Laplace transform for a fractional differential equation

Let us take the fractional differential equation (FDE), i.e. $t\left(D^{1/2}\left[y(t)\right]\right)-y(t)=0$. This equation is of the type $\frac{\mathrm{d}^{\alpha}y}{\mathrm{d}t^{\alpha}}=\frac{y}{t}$, with $\alpha=1/2$. With $D^{1/2}$ as a Riemann-Liouville (RL) semi-derivative operator fractional, the initial condition is required to solve this FDE; which is given as $D_t^{-1/2}y(t)\Big|_{t=0}=D_t^{-1/2}y(0)=0$. We could also solve this FDE with the tricks of the Laplace transform and via an alternative method.

The known Laplace transform is $\mathcal{L}\left\{t\left(f(t)\right)\right\}=-\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{L}\left\{f(t)\right\}\right]=-D_s^1\left[\mathcal{L}\left\{f(t)\right\}\right]$. Using this, we write $\mathcal{L}\left\{t\left(D^{1/2}\left[y(t)\right]\right)\right\}=-D_s^1\left[\mathcal{L}\left\{D^{1/2}\left[y(t)\right]\right\}\right]=-D_s^1\left[s^{1/2}y(s)-D_t^{-1/2}y(0)\right]$. Here we have used $\mathcal{L}\left\{D^{1/2}\left[y(t)\right]\right\}=s^{1/2}y(s)-\left(D^{-1/2}\left[y(t)\right]\right)_{t=0}$, with $y(s)=\mathcal{L}\left\{y(t)\right\}$. Thus, we write the Laplace transform of the given fractional differential equation as follows:

$$t\left(D^{1/2}\left[y(t)\right]\right)-y(t)=0,\qquad -D_s^1\left[s^{1/2}y(s)-D_t^{-1/2}y(0)\right]-y(s)=0 \qquad (6.1)$$

Using the given initial condition, i.e. $D_t^{-1/2}y(0)=0$, and expanding (6.1), noting that $D_s^1\equiv\frac{\mathrm{d}}{\mathrm{d}s}$ operation, we get $-\left(\frac{1}{2}s^{-1/2}y(s)+s^{1/2}D_s^1\left[y(s)\right]\right)-y(s)=0$. Rearranging this, we get the first order ordinary differential equation in the s-variable that is expressed as follows:

$$\frac{\mathrm{d}\ y(s)}{\mathrm{d}s}+\left(\frac{1}{2}s^{-1}+s^{-\frac{1}{2}}\right)y(s)=0,\qquad \frac{\mathrm{d}\ y(s)}{y(s)}=\left(-\frac{1}{2s}-s^{-\frac{1}{2}}\right)\mathrm{d}s \tag{6.2}$$

Integrating both sides of the second equation of (6.2), we get the following:

$$\begin{aligned}\ln\left(y(s)\right) &= -\tfrac{1}{2}\ln(s)-2s^{\frac{1}{2}}+k\\ &= -\tfrac{1}{2}\ln(s)+\ln e^{-2\sqrt{s}}+\ln K\end{aligned} \tag{6.3}$$

With k as an integration constant in (6.3) we can write $k=\ln K$ and simplify (6.3) to write $\ln y=\ln\left(Ks^{-\frac{1}{2}}e^{-2\sqrt{s}}\right)$ or $y(s)=Ks^{-\frac{1}{2}}e^{-2\sqrt{s}}$. By using the inverse Laplace transform of this expression from standard tables, $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}\right\}=\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$, we get $y(t)=K_1t^{-\frac{1}{2}}e^{-\frac{1}{t}}$, for $t>0$, with constant K_1 including $\sqrt{\pi}$.

6.2.2 Alternative method to solve a fractional differential equation

We now provide an alternative solution for our example (i.e. $t\left(D_t^{\frac{1}{2}}\left[y(t)\right]\right)=y(t)$). Taking a half derivative of this, we get $D_t^{\frac{1}{2}}\left[t\left(D_t^{\frac{1}{2}}\left[y(t)\right]\right)\right]=D_t^{\frac{1}{2}}\left[y(t)\right]$. Then applying Leibniz's rule (that is, $D_x^{\alpha}\left[x\left(f(x)\right)\right]=xD_x^{\alpha}\left[f(x)\right]+\alpha D_x^{\alpha-1}\left[f(x)\right]$), as described in Section-4.19 of this book, to the LHS of this equation, we get a new equation $t\left(D_t^{\frac{1}{2}}\left[D_t^{\frac{1}{2}}y(t)\right]\right)+\frac{1}{2}D_t^{-\frac{1}{2}}\left[D_t^{\frac{1}{2}}y(t)\right]=D_t^{\frac{1}{2}}y(t)$. Simplifying this, we get $t\left(D_t^{1}y(t)\right)+\frac{1}{2}y(t)=D_t^{\frac{1}{2}}y(t)$. Using our original equation, that is, $D_t^{\frac{1}{2}}y(t)=\frac{y(t)}{t}$, and substituting this in the obtained equation (i.e. $t\left(D_t^{1}y(t)\right)+\frac{1}{2}y(t)=D_t^{\frac{1}{2}}y(t)$), we get a first order differential equation as follows:

$$t\left(\frac{\mathrm{d}\ y(t)}{\mathrm{d}t}\right)+\frac{1}{2}y(t)=\frac{y(t)}{t}\quad,\quad \frac{\mathrm{d}\ y(t)}{y(t)}=\left(\frac{1}{t^2}-\frac{1}{2t}\right)\mathrm{d}t \tag{6.4}$$

Integrating both sides of the second equation of (6.4) we get $\ln y=-t^{-1}-\frac{1}{2}\ln t+k$, where k is a different integration constant. We write $k=\ln K$ and that gives us a further simplification, as $\ln y=\ln\left(Kt^{-\frac{1}{2}}e^{-\frac{1}{t}}\right)$. From this, we write a solution, that is, $y(t)=Kt^{-\frac{1}{2}}e^{-\frac{1}{t}}$, which we also obtained via the Laplace transform tricks in Section-6.2.1.

6.2.3 A method using the Laplace transform for a fractional integral equation

Now we take a fractional integral equation (i.e. $\left(ty(t)-\sqrt{\pi}D_t^{-\frac{1}{2}}\left[y(t)\right]\right)=0$). Using the Laplace transform identity (i.e. $\mathcal{L}\left\{t\left(f(t)\right)\right\}=-D_s^{1}\left[\mathcal{L}\left\{f(t)\right\}\right]$ and $\mathcal{L}\left\{D_t^{-\frac{1}{2}}y(t)\right\}=s^{-\frac{1}{2}}y(s)$), we write the Laplace transform of our fractional integral equation, that is, $-D_s^{1}y(s)-\sqrt{\pi}s^{-\frac{1}{2}}y(s)=0$, which is a first order differential equation in Laplace variable s. We can rewrite this as follows:

$$\frac{\mathrm{d}\ y(s)}{\mathrm{d}s}+\sqrt{\pi}s^{-\frac{1}{2}}y(s)=0 \tag{6.5}$$

Following the steps of the previous example in Section-6.2.2, we can write $\ln y(s)=\ln K-2\sqrt{\pi}s^{\frac{1}{2}}=\ln K+\ln\left(e^{-2\sqrt{\pi s}}\right)$, or $\ln y(s)=Ke^{-2\sqrt{\pi s}}$. Using the standard Laplace transform tables (i.e. $\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}=\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$), we can write $t>0$, $y(t)=Kt^{-\frac{3}{2}}e^{-\frac{\pi}{t}}$.

6.2.4 An alternative method to solve fractional integral equations

Here, we carry out alternative methods for obtaining the solution of $\left(ty(t)-\sqrt{\pi}D_t^{-1/2}y(t)\right)=0$. Now, take the half-derivative of the equation and write $D_t^{1/2}\left[ty(t)\right]=\sqrt{\pi}D_t^{1/2}\left[D_t^{-1/2}y(t)\right]$, and apply Leibniz's rule to the LHS (that is $D_x^{\alpha}\left[x\left(f(x)\right)\right]=xD_x^{\alpha}\left[f(x)\right]+\alpha D_x^{\alpha-1}\left[f(x)\right]$) described in Section-4.19 and write; $tD_t^{1/2}y(t)+\frac{1}{2}\left(D_t^{-1/2}y(t)\right)=\sqrt{\pi}y(t)$. Here, substitute $D_t^{-1/2}y(t)=\frac{t}{\sqrt{\pi}}y(t)$ and write the following expression:

$$t\left(D_t^{1/2}y(t)\right)+\frac{t}{2\sqrt{\pi}}y(t)=\sqrt{\pi}y(t) \tag{6.6}$$

Taking an ordinary whole derivative of $ty(t)-\sqrt{\pi}D_t^{-1/2}y(t)=0$, we get another equation, expressed in the following step:

$$t\left(D_t^1y(t)\right)+y(t)=\sqrt{\pi}D_t^{1/2}y(t) \tag{6.7}$$

With these $t\left(D_t^{1/2}y(t)\right)+\frac{t}{2\sqrt{\pi}}y(t)=\sqrt{\pi}y(t)$ and $t\left(D_t^1y(t)\right)+y(t)=\sqrt{\pi}D_t^{1/2}y(t)$, we eliminate the term (i.e. $D_t^{1/2}y(t)$) and get $t^2D_t^1y(t)+\left(\frac{3}{2}t-\pi\right)y(t)=0$. Therefore, we have $\frac{\mathrm{d}\,y(t)}{y(t)}=\left(\frac{\pi}{t^2}-\frac{3}{2t}\right)\mathrm{d}t$, and upon integrating and rearranging it, we get $y(t)=Kt^{-3/2}e^{-\pi/t}$ (i.e. by following the steps as we did for earlier examples). Here we have demonstrated the various tricks used to obtain a solution for a fractional differential equation.

6.3 Abel's fractional integral equation of 'tautochrone'-a classical problem

Abel's problem is to find a curve where the time of descent is the same irrespective of the position of release of the ball in a frictionless system. In Figure-6.1, S is the arc length measured along curve C from point O (the origin) to an arbitrary point Q on C (Figure-6.1). The gain in kinetic energy while the ball is descending represents the loss in potential energy as given by the following law:

$$\frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=mg(y-\eta) \qquad \mathrm{d}s=-\mathrm{d}t\sqrt{2g(y-\eta)} \tag{6.8}$$

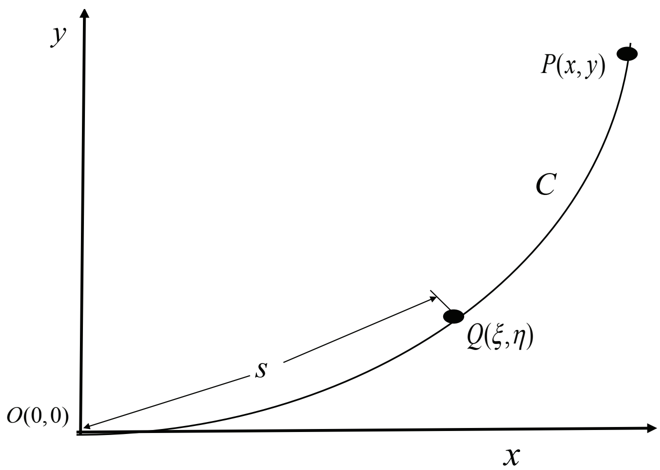


Figure-6.1: Abel's tautochrone curve

The negative square root indicates that the distance (i.e. arc-length) decreases as the time increases. The equation to be solved is, therefore, the following:

$$\mathrm{d}t=-\frac{1}{\sqrt{2g(y-\eta)}}\mathrm{d}s \tag{6.9}$$

The time of descent from point 'P' to 'O' is T, which is constant and is represented as follows:

$$T=-\frac{1}{\sqrt{2g}}\int_P^O\frac{1}{\sqrt{y-\eta}}\mathrm{d}s \tag{6.10}$$

Now take the arc length as a function $s = h(\eta)$ or $\frac{\mathrm{d}s}{\mathrm{d}\eta} = \left(h^{(1)}(\eta)\right)$, where h depends on the shape of curve 'C'. We can write it in differential form to show the curve as $\mathrm{d}s = \left(h^{(1)}(\eta)\right)\mathrm{d}\eta$. Therefore, this substitution in (6.10) gives us the following expression:

$$T = -\frac{1}{\sqrt{2g}}\int_y^0 (y-\eta)^{-1/2}\left(\left(h^{(1)}(\eta)\right)\mathrm{d}\eta\right) \tag{6.11}$$

Re-writing the above expression (6.11), we obtain the following:

$$\left(\sqrt{2g}\right)T = \int_0^y (y-\eta)^{-1/2}\left((h^{(1)}\eta)\right)\mathrm{d}\eta \tag{6.12}$$

The expression (6.12) means that the integral, that is, the RHS of the previous equation, when 'a constant', will be the solution by which we get a constant time of descent (T). In the above expression, with k as a constant, the integral of the RHS is as follows:

$$k \equiv \int_0^x (x-t)^{-1/2}\left(f(t)\right)\mathrm{d}t \tag{6.13}$$

This (6.13) is the standard Abel integral equation.

Thus, when the half derivative of a constant k is computed, the function $f(x)$ is obtained. Therefore, $\frac{\mathrm{d}^{1/2}[k]}{\mathrm{d}x^{1/2}} = \sqrt{\pi}\left(f(x)\right)$. We relate the above Abel's integral equation (6.13) to the RL-fractional integration formula by taking $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The solution to $\left(\sqrt{2g}\right)T = \int_0^y (y-\eta)^{-1/2}\left(h^{(1)}(\eta)\right)\mathrm{d}\eta$ (6.12) can be obtained where $h^{(1)}(\eta) = \frac{\mathrm{d}s}{\mathrm{d}\eta}$, and through letting $f(y) \equiv \left(h^{(1)}(y)\right)$ Subsequently, we will divide both sides of the expression by $\Gamma\left(\frac{1}{2}\right)$ and apply the Riemann-Liouville definition of a fractional integration, through which we get the following:

$$\begin{aligned} D^{-1/2}\left[f(x)\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{-1/2}\left(f(t)\right)\mathrm{d}t \\ \sqrt{\pi}D^{-1/2}\left[f(x)\right] &= \int_0^x (x-t)^{-1/2}\left(f(t)\right)\mathrm{d}t \end{aligned} \tag{6.14}$$

Thereby inverting the operator $D^{-1/2}$ in (6.14), we get the following:

$$\frac{\sqrt{2g}}{\Gamma\left(\frac{1}{2}\right)}T = D^{-1/2}\left[f(y)\right] \qquad D^{1/2}\left[\sqrt{\frac{2g}{\pi}}T\right] = f(y) \tag{6.15}$$

putting the half derivative with regard to y for a constant (i.e. T as $D_y^{1/2}\left[T\right] = \frac{T}{\sqrt{\pi y}}$) to get $f(y) = \frac{\sqrt{2g}}{\pi}\frac{T}{\sqrt{y}}$. By carrying out algebraic manipulation, the following expressions are obtained:

$$\begin{aligned} f(y) &\equiv \left(h^{(1)}(y)\right) = \frac{\mathrm{d}s}{\mathrm{d}y} = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{\mathrm{d}y} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \\ \frac{\mathrm{d}x}{\mathrm{d}y} &= \sqrt{\left(f(y)\right)^2 - 1} \\ x &= \left(\int_0^y \left(\sqrt{\frac{2gT^2}{\pi^2\eta} - 1}\right)\mathrm{d}\eta\right) + c \end{aligned} \tag{6.16}$$

We continue with the following assertions:

$$c = 0, \qquad x = 0 = y \qquad\qquad x = \int_0^y \left(\sqrt{\frac{2gT^2}{\pi^2\eta} - 1}\right)\mathrm{d}\eta \tag{6.17}$$

Let $a = \frac{gT^2}{\pi^2}$, and with the change of variables $\eta = 2a\sin^2\xi$, which gives $x = 4a\int_0^{\beta}\left(\cos^2\xi\right)\mathrm{d}\xi$, with $\beta = \sin^{-1}\left(\sqrt{\frac{y}{2a}}\right)$. The parametric equation of the cycloid is thus obtained as follows:

$$x = 2a\left(\beta + \tfrac{1}{2}\sin 2\beta\right) \qquad y = 2a\sin^2\beta \qquad \theta = 2\beta \qquad a = \frac{gT^2}{\pi^2}$$
$$x = a\left(\theta + \sin\theta\right) \qquad y = a\left(1 - \cos\theta\right) \tag{6.18}$$

Cycloid is the tautochrone. It should be mentioned that the cycloid curve shape C - is too curved for the brachistochrone problem as solved by Bernoulli (that is, the determination of the shape of the curve giving us a 'minimum' time of descent). Therefore, the constant time T is also the minimum time of descent in a cycloid.

6.4 Using the power series expansion method to obtain inverse Laplace transforms

The easiest and most conventional and popular method to solve a differential equation is to form a polynomial equation in the Laplace variable s from the given differential equation and then take the inverse Laplace transform to get the solution. In fractional differential equations, one may follow a similar method. However, to get an inverse Laplace transform, one requires certain skills.

In a fractional differential equation, say we need to (for example) obtain an inverse Laplace transform of $y(s) = \frac{k}{s(s^{\alpha}+k)}$ with $k > 0$ to arrive at $y(t)$. The Laplace transformed expression corresponds to the fractional differential equation:

$$\frac{1}{k}\frac{\mathrm{d}^{\alpha}y(t)}{\mathrm{d}t^{\alpha}} + y(t) = x(t) \tag{6.19}$$

with $y(t) = 0$ for $t < 0$; and $x(t)$ as Heaviside's unit step at $t = 0$, that is $x(t) = 1$ for $t \geq 0$ and $x(t) = 0$ for $t < 0$.

The Laplace transform of the equation $\frac{1}{k}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}y(t) + y(t) = x(t)$ is $\frac{1}{k}s^{\alpha}\left(y(s)\right) + y(s) = x(s)$, which gives $y(s) = \frac{k}{s^{\alpha}+k}x(s)$. In addition, with $x(s) = \frac{1}{s}$ (that is, the Laplace transform of the unit step function), we have $y(s) = \frac{k}{s(s^{\alpha}+k)}$.

There could be a direct method used here by employing the integral definition of an inverse Laplace transform formula, namely $y(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{k}{s(s^{\alpha}+k)}e^{st}\mathrm{d}s$ (see Appendix-G for further details). Here, performing the numerical integration is tough, and we will elucidate upon that later. We will first resort to a power series expansion by using $(1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$ with $x = \frac{k}{s^{\alpha}}$, $|x| < 1$ and for $\alpha > 0$ we have the following derivation:

$$\begin{aligned} y(s) &= \frac{k}{s(s^{\alpha}+k)} = \frac{k}{ss^{\alpha}\left(1+\frac{k}{s^{\alpha}}\right)} \\ &= \frac{k}{s^{\alpha+1}}\left(1 - \frac{k}{s^{\alpha}} + \frac{k^2}{s^{2\alpha}} - \frac{k^3}{s^{3\alpha}} + \ldots\right) \\ &= \frac{k}{s^{\alpha+1}} - \frac{k^2}{s^{2\alpha+1}} + \frac{k^3}{s^{3\alpha+1}} - \frac{k^4}{s^{4\alpha+1}} + \ldots\ldots \end{aligned} \tag{6.20}$$

We use a known Laplace transform pair (that is, $\mathcal{L}^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$) by applying it term by term to the above expression in order to write the following:

$$
\begin{aligned}
y(t) &= \frac{kt^{\alpha}}{\Gamma(1+\alpha)} - \frac{k^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{k^3 t^{3\alpha}}{\Gamma(1+3\alpha)} - ...... \\
&= 1 - 1 + \frac{kt^{\alpha}}{\Gamma(1+\alpha)} - \frac{k^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{k^3 t^{3\alpha}}{\Gamma(1+3\alpha)} - ...... \\
&= 1 - \left(1 - \frac{kt^{\alpha}}{\Gamma(1+\alpha)} + \frac{k^2 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{k^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + ...\right) \\
&= 1 - \sum_{m=0}^{\infty} \frac{\left(-kt^{\alpha}\right)^m}{\Gamma\left(m\alpha+1\right)} = 1 - E_{\alpha}\left(-kt^{\alpha}\right)
\end{aligned}
\tag{6.21}
$$

In the above (6.21) we have used $E_{\alpha}(x)$ as a one-parameter Mittag-Leffler function (i.e. $E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{(x)^n}{\Gamma(1+n\alpha)}$). The series expansion formula we use is $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, and this is valid for $|x| \ll 1$; meaning in our case $\left|\frac{k}{s^{\alpha}}\right| \ll 1$. This means our series expansion (for the case of $\alpha > 0$) is fine for a higher s (frequency) or is true for early times t.

However, as shown by the tables of the Laplace transforms, for the Mittag-Leffler function we have the Laplace transform pair $\mathcal{L}\left\{t^{\alpha p+\beta-1} E_{\alpha,\beta}^{(p)}(at^{\alpha})\right\} = p!\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$. The function $E_{\alpha,\beta}^{(p)}(at^{\alpha})$, $p = 0,1,2,3,...$ is the p-th integer order derivative of the two-parameter Mittag-Leffler function, $E_{\alpha,\beta}(x)$. The two-parameter Mittag-Leffler function is defined as $E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha m+\beta)}$ (see Appendix-A). In this, the Laplace transforms the identity by setting $p = 0$ and $\beta = \alpha + 1$, meaning that we have $\mathcal{L}\left\{t^{\alpha} E_{\alpha,\alpha+1}(at^{\alpha})\right\} = \frac{s^{-1}}{s^{\alpha}-a}$. Using this, we write for our function: $\mathcal{L}^{-1}\left\{\frac{k}{s(s^{\alpha}+k)}\right\} = k\left(t^{\alpha}\left(E_{\alpha,\alpha+1}(-kt^{\alpha})\right)\right)$. Therefore, we have the two equivalent results as depicted below:

$$
y(t) = 1 - E_{\alpha}(-kt^{\alpha}) = kt^{\alpha}\left(E_{\alpha,\alpha+1}(-kt^{\alpha})\right) \tag{6.22}
$$

For the case $\alpha < 0$, let us represent that by considering $\alpha = -\beta$ with $\beta > 0$, and for the same function we conduct the following derivation along the same lines as the power series expansion and by using a term-by-term inverse Laplace transform:

$$
\begin{aligned}
y(s) &= \frac{k}{s(s^{\alpha}+k)} = \frac{k}{s(s^{-\beta}+k)} \\
&= \frac{1}{s(1+k^{-1}s^{-\beta})} \\
&= \frac{1}{s}\left(1 - k^{-1}s^{-\beta} + k^{-2}s^{-2\beta} - k^{-3}s^{-3\beta} + ...\right) \\
&= \frac{1}{s} - \frac{1}{ks^{\beta+1}} + \frac{1}{k^2 s^{2\beta+1}} - \frac{1}{k^3 s^{3\beta+1}} + ...
\end{aligned}
\tag{6.23}
$$

Now conducting a term by term inverse Laplace transform with $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$, we achieve the following:

$$
\begin{aligned}
y(t) &= 1 - \frac{t^{\beta}}{k\left(\Gamma(\beta+1)\right)} + \frac{t^{2\beta}}{k^2\left(\Gamma(2\beta+1)\right)} - \frac{t^{3\beta}}{k^3\left(\Gamma(3\beta+1)\right)} + ... \\
&= \sum_{n=0}^{\infty} \frac{\left(-\frac{t^{\beta}}{k}\right)^n}{\Gamma(1+n\beta)} = E_{\beta}\left(-\frac{t^{\beta}}{k}\right)
\end{aligned}
\tag{6.24}
$$

For the case $\alpha < 0$, our series expansion requires $\left|k^{-1}s^{-\beta}\right| \ll 1$, with $\beta > 0$. This is valid for a large s (frequency) and thus an early time for t. In the above cases, we have obtained the solution in the form of a Mittag-Leffler function, for the chosen Laplace domain (a complex frequency domain) expression. This type of Laplace domain expression does appear in several physical systems like the low pass filter, the high pass filter, the charging/discharging of fractional order capacitors, etc.

6.5 Using Laplace transform techniques and power series expansion to solve simple fractional differential equations

Let us apply the Laplace transform to the FDE, that is:

$$ {}_0^C D^{1/3}\left[x(t)\right]+x(t)=y(t) \qquad y(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases} \tag{6.25}$$

With $x(0)=0$ for $t<0$, and the fractional derivative operator ${}_0^C D^{1/3}$ is of Caputo's type. The source term (i.e. $y(t)$) is Heaviside's unit step function, whose Laplace transform is $Y(s)=\frac{1}{s}$. With a generalized Laplace transform, as we developed in Section-5.16.4, assuming that the fractional derivative operator in the above dynamic equation is of Caputo's formulation, then $\mathcal{L}\left\{\mathrm{d}^{\pm\alpha} f(t)\right\}\equiv s^{\pm\alpha}F(s)$, when $f(0)=0$, that is, with zero initial conditions. Taking the Laplace transform of the above equation (6.25), we get the following:

$$ s^{1/3}X(s)+X(s)=Y(s) \tag{6.26}$$

Therefore, rearranging the above and using $(1+x)^{-1}=\sum_{n=0}^{\infty}(-1)^n x^n$, with $x=s^{-1/3}$, and $|x|<1$, we write the following steps:

$$\begin{aligned}X(s)&=\frac{1}{s\left(s^{1/3}+1\right)}=\frac{1}{s^{4/3}\left(1+s^{-1/3}\right)}\\ &=\frac{\left(1-(-s)^{-1/3}\right)^{-1}}{s^{4/3}}=\frac{\left(1+s^{-1/3}\right)^{-1}}{s^{4/3}}\\ &=\frac{1}{s^{4/3}}\sum_{k=0}^{\infty}(-1)^k s^{-k/3}=\sum_{k=0}^{\infty}(-1)^k s^{-\frac{(k+4)}{3}}\\ &=\sum_{n=4}^{\infty}(-1)^n s^{-n/3}=\sum_{n=4}^{\infty}\frac{(-1)^n}{s^{n/3}}\end{aligned} \tag{6.27}$$

In (6.27) we have used a series expansion approximation with the assumption $\left|s^{-1/3}\right|<<1$. Now using the Laplace transform pair (i.e. $\mathcal{L}\left\{t^n\right\}=\frac{(n)!}{s^{n+1}}=\frac{\Gamma(n+1)}{s^{n+1}}$), we write the solution as:

$$ x(t)=\sum_{n=4}^{\infty}\frac{(-1)^n t^{\frac{n}{3}-1}}{\Gamma\left(\frac{n}{3}\right)} \tag{6.28}$$

Here we are unable to place any closed form result like the one we obtained in Section-6.4.

Now consider a ball falling freely under gravity (another case of fractionally damped motion) in a viscous fluid, with a constituent equation as follows:

$$\frac{\mathrm{d}\ v(t)}{\mathrm{d}t}+\frac{\mathrm{d}^{\alpha}v(t)}{\mathrm{d}t^{\alpha}}+v(t)=1 \qquad 0<\alpha<1 \tag{6.29}$$

with the initial condition (velocity) as $v(0)=0$. With a generalized Laplace transform (as we developed in Section-5.16), assuming that the fractional derivative operator in the above dynamic equation is of Caputo's formulation, then $\mathcal{L}\left\{\mathrm{d}^{\pm\alpha} f(t)\right\}\equiv s^{\pm\alpha}F(s)$, with zero initial condition (i.e. $f(0)=0$). We apply the Laplace transform to both sides of the above equation (6.29) to get $sV(s)+s^{\alpha}V(s)+V(s)=\frac{1}{s}$; where we used the Laplace transform identity, i.e. $\mathcal{L}\left\{1\right\}=\frac{1}{s}$. After some algebraic manipulation, the following expression is obtained:

$$ V(s)=\frac{1}{s(1+s+s^{\alpha})}=\frac{\left(1-(-s^{-1}-s^{\alpha-1})\right)^{-1}}{s^2}=\frac{\left(1+(s^{-1}+s^{\alpha-1})\right)^{-1}}{s^2} \tag{6.30}$$

Expanding the numerator as a binomial series with $(1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$, we write $x = (s^{-1} + s^{\alpha-1})$ and with $|x| < 1$ the following:

$$V(s) = \frac{\left(1+(s^{-1}+s^{\alpha-1})\right)^{-1}}{s^2} = \frac{1}{s^2}\sum_{n=0}^{\infty}(-1)^n\left(s^{-1}+s^{\alpha-1}\right)^n \tag{6.31}$$

Now we use the binomial expansion formula $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^n b^{n-r}$ with ${}^nC_r = \frac{n!}{r!(n-r)!}$ as binomial coefficients, and expand $\left(s^{-1}+s^{\alpha-1}\right)^n$ in the above expression to write the following:

$$\begin{aligned} V(s) &= \frac{1}{s^2}\sum_{n=0}^{\infty}(-1)^n\sum_{r=0}^{r}\binom{n}{r}\left(s^{-1}\right)^n\left(s^{\alpha-1}\right)^{n-r} \\ &= \sum_{n=0}^{\infty}(-1)^n\sum_{r=0}^{n}\binom{n}{r}\frac{1}{s^{2+(2-\alpha)n-(1-\alpha)r}} \end{aligned} \tag{6.32}$$

We use the Laplace transform identity (i.e. $\mathcal{L}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$) to write the solution as follows:

$$v(t) = \sum_{n=0}^{\infty}(-1)^n\sum_{r=0}^{n}\binom{n}{r}\frac{t^{1+(2-\alpha)n-(1-\alpha)r}}{\Gamma\left(2+(2-\alpha)n-(1-\alpha)r\right)} \tag{6.33}$$

Here too, we are unable to obtain a closed form solution, as we had achieved earlier in the book (Section-6.4). In the above examples, we wrote that the fractional derivative was Caputo's formulation. In both cases, our initial conditions are zero no matter whether the fractional derivative operator is of Riemann-Liouville or Caputo type.

6.6 The contour integration method for obtaining inverse Laplace transforms

We aim to obtain $y(t) = \mathcal{L}^{-1}\left\{\frac{k}{s(s^\alpha+k)}\right\}$, by use of the formula $y(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{k}{s(s^\alpha+k)}e^{st}\mathrm{d}s$, for our example of the Laplace expression $y(s) = \frac{k}{s(s^\alpha+k)}$, with $k > 0$. Here $s = \mathrm{Re}[s] + i\,\mathrm{Im}[s]$ is a complex quantity or a complex frequency. It may also be expressed as $s = \sigma + i\omega$. The inverse Laplace transform is an integration that is on an imaginary axis (refer to Figure-6.2), which is on the line FA. This is a Bromwich path (written in short form as Br) on which the Laplace inversion integration is done in a line on the complex plane, from $s = \gamma - i\infty$ to $s = \gamma + i\infty$ (see Appendix-G). Take the function $F(s) = \frac{k}{s(s^\alpha+k)}e^{st}$; this has a pole at $s = 0$ and is infinite as many numbers of pole positions are at $s^\alpha = -k$. Pole implies that the function $F(s)$ goes to infinity at those pole points. Writing $-k$ as $ke^{i(\pi+2\pi m)}$, with m as integers $0, \pm1, \pm2, ...$ we get $s^\alpha = ke^{i(\pi+2\pi m)}$. Therefore, there are multiple poles, in several Riemann sheets (see Appendix-E and Appendix-G for further details). The real solution $y(t)$ is obtained from performing integration in a primary Riemann sheet using a complex plane, as shown in Figure-6.2. The primary Riemann sheet has the region $-\pi \le \arg[s] \le \pi$, with a branch cut at location $s = 0$ and $s = -\infty$, which is a standard branch-cut for a complex plane, for a multi-valued function (refer to Appendix-E).

Looking at $F(s)$, for $m = 0$ we write $s_1 = \sqrt[\alpha]{k}e^{i\left(\frac{\pi}{\alpha}\right)}$. For $m = 1$, we write $s_{2'} = \sqrt[\alpha]{k}e^{i\left(\frac{\pi+2\pi}{\alpha}\right)} = \sqrt[\alpha]{k}e^{i\left(\frac{3\pi}{\alpha}\right)}$; this does not fall into the primary Riemann sheet, i.e. $-\pi \le \arg[s] \le \pi$. For $m = -1$, we write $s_2 = \sqrt[\alpha]{k}e^{i\left(\frac{\pi-2\pi}{\alpha}\right)} = \sqrt[\alpha]{k}e^{-i\left(\frac{\pi}{\alpha}\right)}$, which is in the primary Riemann sheet, i.e. $-\pi \le \arg[s] \le \pi$. So for the case $m = \pm1$, which says the corresponding poles should be in the secondary Riemann sheet. However, for $m = -1$, we see that the pole in the neighboring Riemann sheet influences the response in the primary Riemann sheet of integration:

Inverse Laplace transform of a function by integration in complex plane

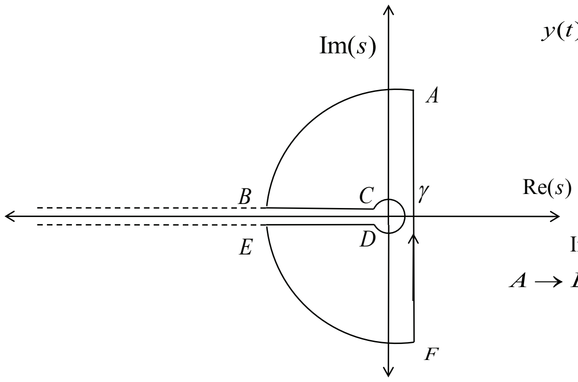
$$y(t) = \mathcal{L}^{-1}\left\{\frac{k}{s(s^\alpha + k)}\right\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ke^{st}}{s(s^\alpha + k)}\mathrm{d}s \qquad F(s) = \frac{ke^{st}}{s(s^\alpha + k)} \qquad k > 0$$

Pole at $s = 0$ and $s = \sqrt[\alpha]{k}e^{i\left(\frac{\pi + 2\pi m}{\alpha}\right)}$ $m = 0, \pm 1, \pm 2, \ldots$

The power function of non-integer order that is $s^\alpha = -k$; a multi-valued and we have standard branch cut

The standard branch cut in complex plane is $(-\infty, 0]$

$$y(t) = \frac{1}{2\pi i}\lim_{T \uparrow \infty}\int_{\gamma - iT}^{\gamma + iT} e^{st}y(s)\mathrm{d}s$$



Integration contour is

$A \to B \to C \to D \to E \to F \to A$

Figure-6.2: Branch cut and contour in a complex plane for integration

Let us take first the case with $0 < \alpha < 1$; say we can choose $\alpha = 0.7$. For $m = 0$, we have $s_1 = \sqrt[0.7]{k}e^{i\left(\frac{\pi}{0.7}\right)}$; here $\arg\left[s_1\right] = \frac{10\pi}{7} > \pi$. For $m = 1$, we have $s_{2'} = \sqrt[0.7]{k}e^{i\left(\frac{\pi + 2\pi}{0.7}\right)}$; in this case, $\arg\left[s_{2'}\right] = \frac{30\pi}{7} > \pi$. For the case $m = -1$, $s_2 = \sqrt[0.7]{k}e^{i\left(\frac{\pi}{0.7} - \frac{2\pi}{0.7}\right)}$; in this case, $\arg\left[s_2\right] = -\frac{10\pi}{7} < -\pi$. Therefore, in this case, $\alpha = 0.7$ and $m = 0, 1, -1$, the poles s_1 and s_2 are not in the sheet of integration, that is, the primary Riemann sheet (i.e. $-\pi \le \arg\left[s\right] \le \pi$).

Therefore, only pole $s = 0$ gives us the result of integration. Only pole $s = 0$ exists and this will not provide any influence in the integration since the contour of Figure-6.2 encircles that and leaves this pole out of the region. Therefore, for the case $0 < \alpha < 1$, there is no influence of any other poles as they do not exist within this region of integration.

Let us now consider $1 < \alpha < 2$, say with $\alpha = 1.4$. For $m = 0$, we have $s_1 = \sqrt[1.4]{k}e^{i\left(\frac{10}{14}\pi\right)}$; here $\arg\left[s_1\right] = \frac{10}{14}\pi < \pi$. For the case $m = 1$, we have $s_{2'} = \sqrt[1.4]{k}e^{i\left(\frac{30}{14}\pi\right)}$; in this case, $\arg\left[s_{2'}\right] = \frac{30}{14}\pi > \pi$. For the case $m = -1$, we have $s_2 = \sqrt[1.4]{k}e^{i\left(-\frac{10}{14}\pi\right)}$; here $\arg\left[s_2\right] = -\frac{10}{14}\pi > -\pi$. Therefore, for the case of $1 < \alpha < 2$, the poles in primary sheets (s_1 and s_2 corresponding to $m = 0$ and $m = -1$) are in the sheet of integration. Since these two poles are within the region of contour integration, their influence (by their residues) is placed within the contour integration and thus the solution. The pole $s = 0$ is still outside the region. We shall solve these two cases separately.

The closed contour integration from the 'Residue Calculus' of a complex variable states the following, with s_j poles in the region of integration, i.e. the primary Riemann sheet for a branch-cut system (see Appendix-G for more information)

$$\begin{aligned}\int_{\mathbf{C}} F(s)\mathrm{d}s \\ &= \int_A^B F(s)\mathrm{d}s + \int_B^C F(s)\mathrm{d}s + \int_C^D F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s + \int_F^A F(s)\mathrm{d}s \\ &= 2\pi i \sum \mathrm{Residues}_{s_j} F\end{aligned} \tag{6.34}$$

The contour is a closed path as shown in Figure-6.2; that is, $C \equiv A, B, C, D, E, F, A$. Therefore, we get from (6.34), the following:

$$\frac{1}{2\pi i}\int_F^A F(s)\mathrm{d}s = -\frac{1}{2\pi i}\begin{pmatrix} \int_A^B F(s)\mathrm{d}s + \int_B^C F(s)\mathrm{d}s + \int_C^D F(s)\mathrm{d}s \\ + \int_D^E F(s)\mathrm{d}s + \int_E^F F(s)\mathrm{d}s \end{pmatrix} + \sum \mathrm{Residues}_{s_j} F \tag{6.35}$$

We note that $\frac{1}{2\pi i}\int_F^A F(s)\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)\mathrm{d}s$, which is $\mathcal{L}^{-1}\{y(s)\} = y(t)$ so we have to evaluate the following in order to get an inverse Laplace transform of $y(s)$ to find $y(t)$; that is:

$$y(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ke^{st}}{s(s^\alpha + k)}\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left(F(s)\right)\mathrm{d}s$$

$$= -\frac{1}{2\pi i}\begin{pmatrix} \int_A^B F(s)\mathrm{d}s + \int_B^C F(s)\mathrm{d}s + \\ \int_C^D F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s + \int_E^F F(s)\mathrm{d}s \end{pmatrix} + \sum \mathrm{Residues}_{s_j} F \tag{6.36}$$

The case of $\alpha < 1$ has no effect on poles as only pole $s = 0$ remains outside the contour of integration, and there are no other poles present in that region of Figure-6.2. Therefore, in that case the quantity $\sum \mathrm{Residues}_{s_j} F = 0$ for $0 < \alpha < 1$. We notice that for the case $1 < \alpha < 2$ we have two poles, namely s_1 and s_2, which are in the region of integration. Thus, in this case we have a non-zero value of $\sum \mathrm{Residues}_{s_j} F$, that is the sum given as $\mathrm{Residue}_{s_1} F + \mathrm{Residue}_{s_2} F$ These residue values are required for calculations.

Now, first, we must calculate the integral $\int_A^B F(s)\mathrm{d}s + \int_B^C F(s)\mathrm{d}s + \int_C^D F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s + \int_E^F F(s)\mathrm{d}s$, required for obtaining a total solution. With $F(s) = \frac{k}{s(s^\alpha + k)}e^{st} = y(s)e^{st}$, we see that if we put $s = re^{i\theta}$, the function $y(s) = \frac{k}{s(s^\alpha + k)}$ tends to zero as $r \uparrow \infty$. Therefore, we write the following as $r \uparrow \infty$ and the integrations on the large arcs AB and EF are zero; i.e. $\int_A^B F(s)\mathrm{d}s + \int_E^F F(s)\mathrm{d}s = 0$; this is as per Jordan's Lemma (see Appendix-G).

Now, we try to find $\int_C^D F(s)\mathrm{d}s$. The region CD is a circle encircling the point $s = 0$; Figure-6.2 shows it to be enlarged; in fact, it is a very small circle with its overall radius tending to zero. For $s = re^{i\theta}$, we write $\mathrm{d}s = ire^{i\theta}\mathrm{d}\theta$, the angle θ varies from π to $-\pi$ as we move from point C to D. Therefore, with this substitution we write the following for $\alpha > 0$:

$$\int_C^D F(s)\mathrm{d}s = \int_\pi^{-\pi} \frac{ke^{st}(ir)e^{i\theta}\mathrm{d}\theta}{(re^{i\theta})\left((re^{i\theta})^\alpha + k\right)} = \int_\pi^{-\pi} \frac{ke^{st}i\mathrm{d}\theta}{\left(r^\alpha e^{i\theta\alpha} + k\right)}$$

$$\int_C^D F(s)\mathrm{d}s = \lim_{r\downarrow 0, s\downarrow 0}\left(\int_\pi^{-\pi} \frac{k\left(e^{st}\right)i\mathrm{d}\theta}{\left(r^\alpha e^{i\theta\alpha} + k\right)}\right) \tag{6.37}$$

$$= \lim_{r\downarrow 0}\left(\int_\pi^{-\pi} \frac{ki\mathrm{d}\theta}{\left(r^\alpha e^{i\theta\alpha} + k\right)}\right) = \int_\pi^{-\pi} i\mathrm{d}\theta = -2\pi i$$

When the circle CD is tending towards $s = 0$ from all sides, we may argue that e^{st} tends towards unity. Now with this argument and making the circle very small, by which we mean making $r \downarrow 0$, we obtain $\int_C^D F(s)\mathrm{d}s = \int_\pi^{-\pi} i\mathrm{d}\theta = -2\pi i$. That is derived in the above steps (6.37).

Now we take the segments BC and DE in Figure-6.2. For BC, we write $s = xe^{i\pi}$ so $\mathrm{d}s = (\mathrm{d}x)e^{i\pi}$ and for DE we write $s = xe^{-i\pi}$ and $\mathrm{d}s = (\mathrm{d}x)e^{-i\pi}$. Note that in Figure-6.2, the segments BC and DE are very close to the negative real axis, so for both the cases we may write $s = -x = xe^{\pm i\pi}$. Therefore, from B to C, the variable s varies from $-\infty$ to zero, or $x = -s$ varies from ∞ to zero. On D to E, the variable s varies from zero to $-\infty$, or $x = -s$ varies from zero to ∞. With these substitutions and observations, we write the following:

$$\begin{aligned}
&\int_B^C F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s \\
&= \int_\infty^0 \frac{ke^{-xt}(\mathrm{d}x)e^{i\pi}}{(xe^{i\pi})(x^\alpha e^{i\pi\alpha}+k)} + \int_0^\infty \frac{ke^{-xt}(\mathrm{d}x)e^{-i\pi}}{(xe^{-i\pi})(x^\alpha e^{-i\pi\alpha}+k)} \\
&= k\int_0^\infty \left(\frac{e^{-xt}\mathrm{d}x}{x(x^\alpha e^{-i\pi\alpha}+k)} - \frac{e^{-xt}\mathrm{d}x}{x(x^\alpha e^{i\pi\alpha}+k)} \right)
\end{aligned} \tag{6.38}$$

Now we simplify the expression (6.38) within the bracket as in the following steps, using $e^{\pm i\pi\alpha} = \cos(\pi\alpha) \pm i\sin(\pi\alpha)$, and we get:

$$\begin{aligned}
&\frac{e^{-xt}}{x(x^\alpha e^{-i\pi\alpha}+k)} - \frac{e^{-xt}}{x(x^\alpha e^{i\pi\alpha}+k)} \\
&= \frac{e^{-xt}}{x}\left(\frac{1}{x^\alpha e^{-i\pi\alpha}+k} - \frac{1}{x^\alpha e^{i\pi\alpha}+k} \right) \\
&= \frac{e^{-xt}}{x}\left(\begin{array}{c} \dfrac{1}{\left(x^\alpha\cos(\pi\alpha)+k\right) - ix^\alpha\sin(\pi\alpha)} \\ -\dfrac{1}{\left(x^\alpha\cos(\pi\alpha)+k\right) + ix^\alpha\sin(\pi\alpha)} \end{array} \right) \\
&= \frac{e^{-xt}}{x}\left(\frac{2ix^\alpha\sin(\pi\alpha)}{\left(x^\alpha\cos(\pi\alpha)+k\right)^2 + \left(x^\alpha\sin(\pi\alpha)\right)^2} \right) \\
&= \frac{e^{-xt}}{x}\left(\frac{2ix^\alpha\sin(\pi\alpha)}{x^{2\alpha}\cos^2(\pi\alpha) + 2x^\alpha k\cos(\pi\alpha) + k^2} \right)
\end{aligned} \tag{6.39}$$

Therefore, we write the following:

$$\begin{aligned}
\int_B^C F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s &= k\int_0^\infty \frac{e^{-xt}}{x}\left(\frac{2ix^\alpha\sin(\pi\alpha)\mathrm{d}x}{x^{2\alpha} + 2x^\alpha k\cos(\pi\alpha) + k^2} \right) \\
&= 2ik\sin(\pi\alpha)\int_0^\infty \frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(x^{2\alpha} + 2x^\alpha k\cos(\pi\alpha) + k^2\right)}
\end{aligned} \tag{6.40}$$

Now we write the expression for $y(t)$ as:

$$\begin{aligned}
y(t) &= -\frac{1}{2\pi i}\left(\int_C^D F(s)\mathrm{d}s + \int_B^C F(s)\mathrm{d}s + \int_D^E F(s)\mathrm{d}s \right) + \sum \mathrm{Residues}_{s_j} F \\
&= -\frac{1}{2\pi i}\left(-2\pi i + 2ik\sin(\pi\alpha)\int_0^\infty \frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(x^{2\alpha} + 2kx^\alpha\cos(\pi\alpha) + k^2\right)} \right) \\
&\quad + \sum \mathrm{Residues}_{s_j} F \\
&= 1 - \frac{k\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(x^{2\alpha} + 2kx^\alpha\cos(\pi\alpha) + k^2\right)} + \sum \mathrm{Residues}_{s_j} F
\end{aligned} \tag{6.41}$$

The above expression (6.41) is a solution for $y(t)=\mathcal{L}^{-1}\left\{\frac{k}{s(s^{\alpha}+k)}\right\}=\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{ke^{st}\mathrm{d}s}{s(s^{\alpha}+k)}$, for the case of positive α; especially with $0<\alpha<2$, the residues in the above solution matter. As we have seen that for $0<\alpha<1$ there are no residues, so the value of residue in that case is zero; but for the case $1<\alpha<2$, there are two poles. As such, the residues need be calculated. We will calculate them shortly.

First, we must say what happens if $\alpha<0$, that is, is negative? We represent this by considering $-\alpha$, where $\alpha>0$ and perform the calculations for $y(s)=\frac{k}{s(s^{-\alpha}+k)}$ with $k>0$. We see now that $F(s)=\frac{ke^{st}}{s(s^{-\alpha}+k)}=\frac{ks^{\alpha-1}e^{st}}{(1+ks^{\alpha})}$, and that it has no pole at $s=0$. Therefore, the integral is $\int_C^D F(s)\mathrm{d}s=0$. The integral on the arcs AB and FE will be zero for a large radius tending to infinity as the value of function $y(s)=\frac{ks^{\alpha-1}}{(1+ks^{\alpha})}$ is zero (one can apply L'Hospital's rule to $y(s)$, and see the limit as a radius which tends to infinity). Therefore we have to evaluate now only the integrals on lines BC and DE, that is, the integrals given as $\int_B^C F(s)\mathrm{d}s$ and $\int_D^E F(s)\mathrm{d}s$. Therefore, we write $y(t)=-\frac{1}{2\pi i}\left(\int_B^C F(s)\mathrm{d}s+\int_D^E F(s)\mathrm{d}s\right)+\sum \mathrm{Residues}_{s_j} F$.

Let us evaluate now the integrals first as depicted below, with the same substitution as done earlier, (i.e. with $s=-x$ and $\mathrm{d}s=-\mathrm{d}x$, and with the same description for the limit for the integrations for $s=-x$ i.e. B to C and D to E, that is, ∞ to zero, and zero to ∞ for x, respectively).

$$\begin{aligned}
&\int_B^C F(s)\mathrm{d}s+\int_D^E F(s)\mathrm{d}s\\
&=\int_\infty^0\frac{e^{-xt}kx^{\alpha-1}e^{i\pi(\alpha-1)}(-\mathrm{d}x)}{kx^{\alpha}e^{i\pi\alpha}+1}+\int_0^\infty\frac{e^{-xt}kx^{\alpha-1}e^{-i\pi(\alpha-1)}(-\mathrm{d}x)}{kx^{\alpha}e^{-i\pi\alpha}+1}\\
&=\int_0^\infty\frac{ke^{-xt}x^{\alpha-1}e^{i\pi(\alpha-1)}\mathrm{d}x}{kx^{\alpha}e^{i\pi\alpha}+1}-\int_0^\infty\frac{ke^{-xt}x^{\alpha-1}e^{-i\pi(\alpha-1)}\mathrm{d}x}{kx^{\alpha}e^{-i\pi\alpha}+1}\\
&=\int_0^\infty\frac{ke^{-xt}}{x^{1-\alpha}}\left(\frac{e^{i\pi(\alpha-1)}}{kx^{\alpha}e^{i\pi\alpha}+1}-\frac{e^{-i\pi(\alpha-1)}}{kx^{\alpha}e^{-i\pi\alpha}+1}\right)\mathrm{d}x\\
&=\int_0^\infty\left(\frac{ke^{-xt}}{x^{1-\alpha}}\right)\left(\frac{2i\sin\left(\pi(\alpha-1)\right)}{k^2x^{2\alpha}+2kx^{\alpha}\cos(\pi\alpha)+1}\right)\mathrm{d}x\\
&=2ki\sin\left(\pi(\alpha-1)\right)\int_0^\infty\frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(k^2x^{2\alpha}+2kx^{\alpha}\cos(\pi\alpha)+1\right)}
\end{aligned}\tag{6.42}$$

We write the solution $y(t)$ in this case as follows:

$$\begin{aligned}
y(t)&=-\frac{1}{2\pi i}\left(\int_B^C F(s)\mathrm{d}s+\int_D^E F(s)\mathrm{d}s\right)+\sum \mathrm{Residues}_{s_j} F\\
&=-\frac{k\sin\left(\pi(\alpha-1)\right)}{\pi}\int_0^\infty\frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(k^2x^{2\alpha}+2kx^{\alpha}\cos(\pi\alpha)+1\right)}\\
&\qquad+\sum \mathrm{Residues}_{s_j} F\\
&=\frac{k\sin\pi\alpha}{\pi}\int_0^\infty\frac{e^{-xt}\mathrm{d}x}{x^{1-\alpha}\left(k^2x^{2\alpha}+2kx^{\alpha}\cos(\pi\alpha)+1\right)}+\sum \mathrm{Residues}_{s_j} F
\end{aligned}\tag{6.43}$$

Now we calculate the residues of poles at s_j, for $F(s)$, which exist for the case $1<\alpha<2$ as s_1 and s_2. Let us discuss this for $1<\alpha<2$, so that we note that $m=0$, and that we have $s_1=\sqrt[\alpha]{k}e^{i\left(\frac{\pi}{\alpha}\right)}$. This we also represent as

$s_1 = \sqrt[\alpha]{k}\cos\left(\frac{\pi}{\alpha}\right) + i\sqrt[\alpha]{k}\sin\left(\frac{\pi}{\alpha}\right)$. For $m=-1$, the pole in the primary sheet is $s_2 = \sqrt[\alpha]{k}e^{i\left(\frac{\pi-2\pi}{\alpha}\right)}$, which turns out to be a complex conjugate of s_1 i.e. $s_2 = \bar{s}_1$, meaning that $s_2 = \bar{s}_1 = \sqrt[\alpha]{k}\cos\left(\frac{\pi}{\alpha}\right) - i\sqrt[\alpha]{k}\sin\left(\frac{\pi}{\alpha}\right)$. For simplicity, we represent $s_1 = -\sigma + i\omega_0$ and $s_2 = -\sigma - i\omega_0$, noting that $\sigma = \mathrm{Re}[s] = -\sqrt[\alpha]{k}\cos\left(\frac{\pi}{\alpha}\right)$ and $\omega_0 = \mathrm{Im}[s] = \sqrt[\alpha]{k}\sin\left(\frac{\pi}{\alpha}\right)$, with $\sigma^2 + \omega_0^2 = k^{2/\alpha}$. It may also be noted that s_1 and s_2 are roots of the equation $s^\alpha + k = 0$; which defines the poles of function $F(s)$. As such, we can write $s^\alpha + k = (s-s_1)(s-s_2)$. With this, let us compute the residues of poles s_1 and s_2 in the following steps (from the theory of residue calculus, as explained in Section-1.7)

$$\begin{aligned}
\sum_{j=1}^{2}\mathrm{Residues}_{s_j} F &= \lim_{s\to s_1}\frac{(s-s_1)ke^{st}}{s(s^\alpha+k)} + \lim_{s\to s_2}\frac{(s-s_2)ke^{st}}{s(s^\alpha+k)} \\
&= \lim_{s\to s_1}\frac{(s-s_1)ke^{st}}{s(s-s_1)(s-s_2)} + \lim_{s\to s_2}\frac{(s-s_2)ke^{st}}{s(s-s_1)(s-s_2)} \\
&= \lim_{s\to s_1}\frac{ke^{st}}{s(s-s_2)} + \lim_{s\to s_2}\frac{ke^{st}}{s(s-s_1)} \\
&= \frac{ke^{s_1t}}{s_1(s_1-s_2)} + \frac{ke^{s_2t}}{s_2(s_2-s_1)} \\
&= \frac{k}{s_1-s_2}\left(\frac{e^{s_1t}}{s_1} - \frac{e^{s_2t}}{s_2}\right) \\
&= \frac{k}{2i\omega_0}\left(\frac{e^{-\sigma t+i\omega_0t}}{-\sigma+i\omega_0} - \frac{e^{-\sigma t-i\omega_0t}}{-\sigma-i\omega_0}\right) \\
&= \frac{k}{2i\omega_0}\left(\frac{(-\sigma-i\omega_0)e^{-\sigma t+i\omega_0t} - (-\sigma+i\omega_0)e^{-\sigma t-i\omega_0t}}{\sigma^2+\omega_0^2}\right)
\end{aligned} \tag{6.44}$$

Using the above $\sigma^2+\omega_0^2 = k^{2/\alpha}$, $\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$, $\sigma = -k^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)$ and $\omega_0 = k^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)$, and then simplifying further, we get the following:

$$\begin{aligned}
\sum_{j=1}^{2}\mathrm{Residues}_{s_j} F &= \frac{ke^{-\sigma t}}{2i\omega_0k^{2/\alpha}}\left(\sigma(-e^{i\omega_0t}+e^{-i\omega_0t}) + i\omega_0(-e^{i\omega_0t}-e^{-i\omega_0t})\right) \\
&= \frac{ke^{-\sigma t}}{2i\omega_0k^{2/\alpha}}\left(-2i\sigma\sin\omega_0t - 2i\omega_0\cos\omega_0t\right) \\
&= \frac{-ke^{-\sigma t}}{\omega_0k^{2/\alpha}}\left(\sigma\sin\omega_0t + \omega_0\cos\omega_0t\right) \qquad (6.45) \\
&= \frac{-ke^{-\sigma t}}{\omega_0k^{2/\alpha}}\left(-k^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)\sin\omega_0t + k^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)\cos\omega_0t\right) \\
&= \frac{k^{\left(\frac{\alpha-1}{\alpha}\right)}e^{-\sigma t}}{\omega_0}\sin\left(\omega_0t - \frac{\pi}{\alpha}\right)
\end{aligned}$$

One may also use a similar method to calculate the residues of poles for $\alpha<0$, in the case of $-2<\alpha<-1$. The above elaborate method says that it is difficult (but possible) to have an inverse Laplace transform using contour integration for functions with a 'fractional' Laplace variable.

6.7 Operational calculus of applying Heaviside units to a partial differential equation

Oliver Heaviside (1871) developed the early theory for 'semi-infinite' systems described by partial differential equations. The one dimensional diffusion equation is:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \tag{6.46}$$

with the parameters $a^2 = \frac{c_p \rho}{k}$ and $a^2 = RC$ for a semi-infinite heat conductor and semi-infinite lossy transmission lines respectively. The equation is a standard Fick's diffusion equation in one dimension with a diffusing quantity shown as $u(x,t)$, a variable of position and time. The initial condition is $u(x,0) = 0$ for $x > 0$, and the boundary condition is $u(0,t) = u_0$ and $u(\infty,t) = 0$. Let the operator $s \equiv \frac{\partial}{\partial t} \equiv \mathrm{d}_t$ (as specified by Oliver Heaviside to solve differential equations). By putting this operator in the PDE (6.46), we obtain a differential equation in the position variable as follows, an ordinary second order differential equation in x:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - a^2 s u = 0 \tag{6.47}$$

The roots of the above linear differential equations (6.47) are $m = \pm a\sqrt{s}$, giving the standard solution as:

$$u(x,s) = A\exp\left(-a\sqrt{s}x\right) + B\exp\left(+a\sqrt{s}x\right) \tag{6.48}$$

Setting the condition for $x \uparrow \infty$, that is, $u(x,t) = 0$, we get $B = 0$. Setting the boundary condition, we get $x \downarrow 0$; $u(0,t) = u_0 = A$; giving the solution as $u(x,s) = u_0 \exp\left(-a\sqrt{s}x\right)$. We expand this exponential as a power series $e^{by} = \sum_{m=0}^{\infty} \frac{(by)^m}{m!}$ to get the following:

$$\begin{aligned} u(x,s) &= u_0 \exp\left(-a\sqrt{s}x\right) \\ &= u_0 + u_0 \sum_{n=1}^{\infty} \frac{\left(-ax\sqrt{s}\right)^n}{n!} \\ &= u_0 + \sum_{n=1}^{\infty} \frac{(-ax)^n (s)^{n/2}}{n!} u_0 \end{aligned} \tag{6.49}$$

Segregating odd and even terms in (6.49) and with re-arrangement, we get the following:

$$u(x,s) = u_0 - \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} s^m \left(s^{1/2} u_0\right) + \sum_{n=0}^{\infty} \frac{(ax)^{2n}}{(2n)!} s^n u_0 \tag{6.50}$$

The operators $s^{1/2}u_0$ and $s^n u_0$ in (6.50) are to be analysed further using a half (time) derivative of u_0, i.e. a constant as $\mathrm{d}_t^{1/2}\left[u_0\right] = u_0\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)} t^{-1/2}\right) = \frac{u_0}{\sqrt{\pi t}}$. Recognizing the fact that the n-th integer derivative of constant u_0 is zero and thereby putting the values of identities with $s^n \equiv \frac{\mathrm{d}^n}{\mathrm{d}t^n}$ where n is a positive-integer, and $s^{1/2} \equiv \frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}$, we have $\mathrm{d}_t^n\left[u_0\right] \to s^n u_0 \equiv 0$ and $\mathrm{d}_t^{1/2}\left[u_0\right] \to s^{1/2} u_0 \equiv u_0 / \sqrt{\pi t}$.

Taking the above formula (6.50), we will change the variable from s to t. We obtain the solution by making the above substitutions for $s^n u_0 \equiv 0$ and $s^{1/2} u_0 \equiv u_0 / \sqrt{\pi t}$ in the obtained solution $u(x,s)$, and thereby get $u(x,t)$ as follows:

$$\begin{aligned} u(x,t) &= u_0 - \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \left(\frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[\frac{u_0}{\sqrt{\pi}} t^{-1/2} \right] \right) \\ &= u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \left(\frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[t^{-1/2} \right] \right) \\ &= u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}-m+1\right)} t^{-\left(\frac{1}{2}\right)-m} \\ &= u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-m\right)} \frac{1}{t^{m+\left(\frac{1}{2}\right)}} \end{aligned} \tag{6.51}$$

Using the property of the gamma function as $\Gamma\left(-m+\frac{1}{2}\right)=\frac{(-4)^m m!\sqrt{\pi}}{(2m)!}$, we simplify the solution as follows:

$$u(x,t)=u_0-\frac{u_0}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(ax)^{2m+1}}{(2m+1)2^{2m}t^{m+\left(\frac{1}{2}\right)}} \tag{6.52}$$

writing:

$$\frac{(ax)^{2m+1}}{(2m+1)2^{2m}t^{m+\left(\frac{1}{2}\right)}}\equiv 2\int_0^{y=\frac{ax}{2\sqrt{t}}} y^{2m}\mathrm{d}y \tag{6.53}$$

and using $\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(x^2)^m\equiv\exp(-x^2)$, we get the exact solution as follows:

$$u(x,t)=u_0-\frac{2u_0}{\sqrt{\pi}}\int_0^{\frac{ax}{2\sqrt{t}}}\exp(-y^2)\mathrm{d}y=u_0\left(1-\operatorname{erf}\left(\frac{ax}{2\sqrt{t}}\right)\right) \tag{6.54}$$

This is the work of Heaviside and his operational calculus on a partial differential equation. Oliver Heaviside used the operator symbol as p for $\frac{\mathrm{d}}{\mathrm{d}t}$; since we knew that the Laplace operator replaces $\frac{\mathrm{d}}{\mathrm{d}t}$ with s, we have used that symbol. In a way, we have used the Laplace transform technique here and the concept of a half derivative too. We shall deal with a fractional partial differential equation with this Laplace and Fourier technique in Chapter-8.

6.8 The response of a fractional differential equation with detailed analysis of power law functions

6.8.1 Demonstration of power law functions as a solution to a simple fractional differential equation excited by a constant step function

Here, we will see the solutions of a fractional differential equation to the standard inputs of step type power-law (or power series) functions, offering either a decaying response or a rising response. Consider the following fractional differential equation of fractional order $0<\alpha<1$ as:

$$\sigma(t)=\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon(t)}{\mathrm{d}t^{\alpha}} \qquad \text{or} \qquad \varepsilon(t)=\frac{1}{\mathrm{E}\tau^{\alpha}}\frac{\mathrm{d}^{-\alpha}\sigma(t)}{\mathrm{d}t^{-\alpha}} \tag{6.55}$$

For the functions $\sigma(t)$ and $\varepsilon(t)$, we consider them to be 0 for $t<0$, and consider E and τ as constants. The symbol σ is a stress function, and the symbol ε is for a strain function; in (6.55) they represent the dynamic equation of visco-elastic systems.

We have generalized the differ-integral of order q for a monomial function $f(x)=x^p$ formula as $\frac{\mathrm{d}^q}{\mathrm{d}x^q}\left[x^p\right]={}_0D_x^q\left[x^p\right]=\frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q}$ for $p>-1$ and $q\in\mathbb{R}$, from the Euler formula. The positive order is a fractional derivative and the negative order is a fractional integration.

The above fractional differ-integrals of order α for $\sigma(t)$ and $\varepsilon(t)$ are not equivalent for a general function; however, they are equivalent for functions which are expandable in terms of differ-integrable units, particularly for functions which can be expressed as a series of powers of t^{α}, and which are bounded for $t=0$. The variable $\varepsilon(t)$ is found by the fractional integration of the constant $\sigma(t)=\sigma_0$; ($\sigma_0 t^p$; $p=0$) to order α. So, applying it in the above differ-integration formula (6.55) with $q=-\alpha$ and $p=0$, we get $\varepsilon(t)=\frac{\sigma_0}{\mathrm{E}}\frac{1}{\Gamma(1+\alpha)}\left(\frac{t}{\tau}\right)^{\alpha}$. This is a time response of $\varepsilon(t)$ for $t\geq 0$ governed by $\sigma(t)=\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon(t)}{\mathrm{d}t^{\alpha}}$ when $\sigma(t)=\sigma_0$ is a constant. The time response of $\varepsilon(t)$ as obtained from the equation $\sigma=\mathrm{E}\tau^{\alpha}\left(\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}\right)$, is like a power law $\varepsilon\left(t/\tau\right)\sim\left(t/\tau\right)^{\alpha}$.

6.8.2 Demonstration of a power series function as a solution to the fractional differential equation excited by a constant step function

We modify the above equation (6.55) as follows with E, E_M and τ as constants:

$$\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}=\sigma+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{6.56}$$

The time response of $\varepsilon(t)$ to a constant (that is, $\sigma(t)=\sigma_0$ for $t\geq 0$) is obtained by operating ${}_0D_t^{-\alpha}$ on both sides of the above equation (i.e. $\varepsilon(t)=\left(\frac{\sigma_0}{\tau^{\alpha}\mathrm{E}}\right){}_0D_t^{-\alpha}\left[1\right]+\frac{\sigma_0}{\mathrm{E}_M}$). Now applying the Euler formula, and after rearrangement, we get the following:

$$\varepsilon=\frac{\sigma_0}{\mathrm{E}_M}+\frac{\sigma_0}{\mathrm{E}}\frac{1}{\Gamma(1+\alpha)}\left(\frac{t}{\tau}\right)^{\alpha} \tag{6.57}$$

The above expression (6.57) is a time response function $\varepsilon(t)$ for $t\geq 0$; for constant input $\sigma(t)=\sigma_0$ applied at $t=0$ for $\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}=\sigma+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}$. The solution is slightly difficult, as a time response function $\sigma(t)$ to a step input $\varepsilon(t)=\varepsilon_0$ applied at $t=0$, for $\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}=\sigma+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}$.

In the equation (6.56), we consider a different fractional order derivative in time (β) for the variable $\varepsilon(t)$, and rewrite the equation without a loss of generality so that we can have $\mathrm{E}=\mathrm{E}_M$ and write following:

$$\mathrm{E}\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}}=\sigma+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}\quad,\quad \mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}}=\sigma+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{6.58}$$

We set $\varepsilon(t)=\varepsilon_0$ and assume that the time response will be a power series, shown as $\sigma(t)=\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha}$; $\delta>-1$, for $t\geq 0$, and put these in $\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}}=\sigma+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}$ to get the following:

$$\begin{aligned}&\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon(t)}{\mathrm{d}t^{\beta}}=\sigma(t)+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma(t)}{\mathrm{d}t^{\alpha}}\\&\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\left[\varepsilon_0\right]=\left(\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}\right)+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}\right]\end{aligned} \tag{6.59}$$

Using the formula $\left({}_0D_x^{\,q}\left[x^p\right]\right)=\frac{\Gamma(p+1)}{\Gamma(p+1-q)}x^{p-q}$; $p>-1$ in the above equation, we obtain the following:

$$\begin{aligned}\mathrm{E}_M\varepsilon_0\tau^{\beta}\frac{t^{-\beta}}{\Gamma(1-\beta)}&=\left(\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}\right)\\&+\tau^{\alpha}\sum a_k\left(\frac{\Gamma(\alpha k+\delta+1)}{\Gamma(\alpha k+\delta+1-\alpha)}\right)\left(\frac{1}{\tau^{\alpha k+\delta}}\right)\left(t^{\alpha k+\delta-\alpha}\right)\end{aligned} \tag{6.60}$$

which we re-write as follows:

$$\begin{aligned}&\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}\left(\frac{t}{\tau}\right)^{-\beta}=\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k+\delta}+\sum a_k b_k\left(\frac{t}{\tau}\right)^{\alpha k+\delta-\alpha}\\&b_k=\frac{\Gamma(\alpha k+\delta+1)}{\Gamma(\alpha k+\delta+1-\alpha)}\end{aligned} \tag{6.61}$$

We enter $x=\left({}^{t}\!/\!_{\tau}\right)$ in the above expression to get the following:

$$\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta)} x^{-\beta} = \sum a_k x^{\alpha k+\delta} + \sum a_k b_k x^{\alpha k+\delta-\alpha}$$
$$= (a_0 x^{\delta} + a_1 x^{\alpha+\delta} + a_2 x^{2\alpha+\delta} + ...) \tag{6.62}$$
$$+ (a_0 b_0 x^{\delta-\alpha} + a_1 b_1 x^{\delta} + a_2 b_2 x^{\alpha+\delta} + a_3 b_3 x^{2\alpha+\delta} + ...)$$
$$= a_0 b_0 x^{\delta-\alpha} + (a_0 + a_1 b_1) x^{\delta} + (a_1 + a_2 b_2) x^{\alpha+\delta} + (a_2 + a_3 b_3) x^{2\alpha+\delta}$$

Comparing the coefficient of the RHS and LHS for (6.62), we can write the following:

$$\delta - \alpha = -\beta; \quad \delta = \alpha - \beta; \quad a_0 b_0 = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta)}; \quad a_k + a_{k+1} b_{k+1} = 0$$

$$b_k = \frac{\Gamma\left(\alpha(k+1) - \beta + 1\right)}{\Gamma\left(\alpha k + 1 - \beta\right)}$$

$$b_0 = \frac{\Gamma(\alpha - \beta + 1)}{\Gamma(1-\beta)}; \quad a_0 = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta) b_0} = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(\alpha - \beta + 1)} \tag{6.63}$$

$$b_1 = \frac{\Gamma(2\alpha - \beta)}{\Gamma(\alpha - \beta + 1)}$$

$$a_1 = -\frac{a_0}{b_1} = -\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(\alpha - \beta + 1)} \frac{\Gamma(\alpha - \beta + 1)}{\Gamma(2\alpha - \beta)} = -\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(2\alpha - \beta)}$$

In general, therefore, we have:

$$a_k = \frac{(-1)^k \mathrm{E}_M \varepsilon_0}{\Gamma(\alpha k + \alpha - \beta + 1)} \tag{6.64}$$

Inserting these values in (6.63) and (6.64), we obtain the function $\sigma(t)$, for constant $\varepsilon(t) = \varepsilon_0$ as follows, with a change in variable of $x = \left(\frac{t}{\tau}\right)$:

$$\sigma(t) = \mathrm{E}_M \varepsilon_0 (x)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha - \beta + 1)} (x)^{\alpha k} \qquad x = \frac{t}{\tau} \tag{6.65}$$

This above treatment gives us the idea that the function $\sigma(t)$, for constant $\varepsilon(t) = \varepsilon_0$ for $t \geq 0$, is provided in terms of the power-series of time of the order $(k+1)\alpha - \beta$.

6.8.3 A demonstration of the power series function as a solution to a fractional differential equation excited by the constant step function through use of the Laplace transform

This (6.65) can also be obtained from the Laplace transform technique. The Laplace transform of the equation is as described as follows:

$$\mathcal{L}\left\{\mathrm{E}_M \tau^{\beta} \frac{\mathrm{d}^{\beta} \varepsilon(t)}{\mathrm{d}t^{\beta}}\right\} = \mathcal{L}\left\{\sigma(t)\right\} + \mathcal{L}\left\{\tau^{\alpha} \frac{\mathrm{d}^{\alpha} \sigma(t)}{\mathrm{d}t^{\alpha}}\right\} \tag{6.66}$$
$$\mathrm{E}_M \tau^{\beta} s^{\beta} \hat{\varepsilon}(s) = \hat{\sigma}(s) + \tau^{\alpha} s^{\alpha} \hat{\sigma}(s)$$

where $\hat{\sigma}(s) = \mathcal{L}\left\{\sigma(t)\right\}$ and $\hat{\varepsilon}(s) = \mathcal{L}\left\{\varepsilon(t)\right\}$ are Laplace transforms, with the initial conditions $\sigma(t) = 0$ and $\varepsilon(t) = 0$ for $t = 0$ and for $t < 0$. For a step function for $\varepsilon(t)$, say ε_0 at $t = 0$, we have $\hat{\varepsilon}(s) = \frac{\varepsilon_0}{s}$; by entering this value in the above Laplace transformed equation, and re-arranging terms, we get the following:

$$\hat{\sigma}(s) = \frac{\mathrm{E}_M \tau^{\beta} s^{\beta}}{1 + (\tau s)^{\alpha}} \frac{\varepsilon_0}{s} = \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{1 + (\tau s)^{\alpha}} \tag{6.67}$$

For this function, no tabulated Laplace transform exists, and therefore we must determine another way to arrive at the solution. We can use the term-wise inverse Laplace transform if we break the above as a power series of s that is, by power series expansion methods, as we discussed earlier in Section-6.4. The function $\left(1 + (\tau s)^{-\alpha}\right)^{-1}$ can also be expanded in a series with decreasing powers, and it will have a radius of convergence different from zero.

$$
\begin{aligned}
\hat{\sigma}(s) &= \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{1+(\tau s)^{\alpha}} \\
&= \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{(\tau s)^{\alpha}\left(1+(\tau s)^{-\alpha}\right)} \\
&= \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{(\tau s)^{\alpha}}\left(1+(\tau s)^{-\alpha}\right)^{-1} \\
&= \mathrm{E}_M \varepsilon_0 \tau (\tau s)^{\beta-1-\alpha}\left(1+(\tau s)^{-\alpha}\right)^{-1} \\
&= \mathrm{E}_M \varepsilon_0 \tau (\tau s)^{\beta-1-\alpha}\left(1-(\tau s)^{-\alpha}+(\tau s)^{-2\alpha}-(\tau s)^{-3\alpha}+\ldots\right)
\end{aligned}
\tag{6.68}
$$

We can rewrite (6.68) in a compact form, as follows:

$$
\hat{\sigma}(s) = \mathrm{E}_M \varepsilon_0 \left(\tau \sum_{k=0}^{\infty} (-1)^k (\tau s)^{\beta-1-\alpha-\alpha k}\right); \quad \beta-\alpha-1<0 \tag{6.69}
$$

We have used the expansion formula $(1+\mathrm{x})^{-1} = 1-\mathrm{x}+\mathrm{x}^2-\mathrm{x}^3+\ldots$ in the above derivation (6.69).

For the series $\hat{\sigma}(s) = \mathrm{E}_M \varepsilon_0 \left(\tau \sum_{k=0}^{\infty} (-1)^k (\tau s)^{\beta-1-\alpha-\alpha k}\right)$, we use a term-wise inverse Laplace transform for $\left(t/\tau\right) > 0$, using a known Laplace transform pair (that is, $\mathcal{L}^{-1}\left\{1/s^{1-q}\right\} = \frac{t^{-q}}{(-q)!} = \frac{t^{-q}}{\Gamma(1-q)}$), so that we invert the above (6.69) to get the result as described below:

$$
\begin{aligned}
\sigma(t) &= \mathrm{E}_M \varepsilon_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)} \left(\frac{t}{\tau}\right)^{\alpha k+\alpha-\beta}, \quad x>0: \quad x=\frac{t}{\tau}; \quad \alpha-\beta>-1 \\
&= \mathrm{E}_M \varepsilon_0 (x)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)} (x)^{\alpha k}
\end{aligned}
\tag{6.70}
$$

This (6.70) was obtained by a different method in (6.65):

Similarly for a step input $\sigma(t)=\sigma_0$ applied at point $t=0$ to the following fractional differential equation (given with its Laplace transformed equation), the response $\varepsilon(t)$ can be obtained as follows:

$$
\begin{aligned}
&\mathcal{L}\left\{\mathrm{E}_M \tau^{\beta} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} \varepsilon(t)\right\} = \mathcal{L}\left\{\sigma(t)\right\} + \mathcal{L}\left\{\tau^{\alpha} \frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}\right\} \\
&\mathrm{E}_M \tau^{\beta} s^{\beta} \hat{\varepsilon}(s) = \hat{\sigma}(s) + \tau^{\alpha} s^{\alpha} \hat{\sigma}(s)
\end{aligned}
\tag{6.71}
$$

For a step input function $\sigma(t)=\sigma_0$ applied at $t=0$, the Laplace transform is $\hat{\sigma}(s)=\frac{\sigma_0}{s}$. Putting this in the above form (6.71), we get the following:

$$
\begin{aligned}
\hat{\varepsilon}(s) &= \frac{\left(1+(\tau s)^{\alpha}\right)}{\mathrm{E}_M \tau^{\beta} s^{\beta}} \frac{\sigma_0}{s} \\
&= \frac{\sigma_0}{\mathrm{E}_M} \tau \frac{\left(1+(\tau s)^{\alpha}\right)}{(\tau s)^{\beta+1}} \\
&= \frac{\sigma_0}{\mathrm{E}_M}\left(\tau(\tau s)^{-(1+\beta)} + \tau(\tau s)^{\alpha-\beta-1}\right)
\end{aligned}
\tag{6.72}
$$

Conducting a term-wise inverse Laplace transform for $\left(t/\tau\right) > 0$, using a known Laplace transform pair (that is, $\mathcal{L}^{-1}\left\{1/s^{1-q}\right\} = \frac{t^{-q}}{(-q)!} = \frac{t^{-q}}{\Gamma(1-q)}$) we carry out an inverse Laplace transform of the above expression (6.72) to get the result below:

$$
\varepsilon(t) = \frac{\sigma_0}{\mathrm{E}_M}\left(\frac{\left(t/\tau\right)^{\beta}}{\Gamma(1+\beta)} + \frac{\left(t/\tau\right)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\right) \tag{6.73}
$$

For a case when $\alpha = \beta$, we have:

$$\varepsilon(t) = \frac{\sigma_0}{\mathrm{E}_M}\left(\frac{\left(t/\tau\right)^{\beta}}{\Gamma(1+\beta)} + 1\right) \tag{6.74}$$

This (6.74), we derived earlier too in (6.57).

6.8.4 Obtaining approximate short-time and long-time responses from a power series solution for a fractional differential equation excited by a constant step function

In (6.74), we obtained a value for $\varepsilon(t)$ that was easy to interpret, and show that the function is solely a power law type. It is easily seen that for the whole α and β parameter range, the function $\varepsilon(t)$ is strongly increasing if and only if $\beta - \alpha \geq 0$ is fulfilled. Now, we take both fractional orders to be the same in order to have a fractional differential equation as follows:

$$\begin{aligned} &\mathrm{E}\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}} = \sigma + \frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}}, \qquad \mathrm{E} = \mathrm{E}_M, \qquad \alpha = \beta, \\ &\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}} = \sigma + \tau^{\beta}\frac{\mathrm{d}^{\beta}\sigma}{\mathrm{d}t^{\beta}} \end{aligned} \tag{6.75}$$

With the function $\sigma(t)$ for a constant step input function, $\varepsilon(t) = \varepsilon_0$, we have the power law function as $\sigma(t) = \sum a_k x^{\beta k}$; $x = \frac{t}{\tau}$, $\beta k > -1$. We then repeat the earlier steps to get the following:

$$\begin{aligned} \frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}x^{-\beta} &= \sum a_k x^{\beta k} + \sum a_k b_k x^{\beta k - \beta} \\ &= (a_0 + a_1 x^{\beta} + a_2 x^{2\beta} + ...) \\ &\quad + (a_0 b_0 x^{-\beta} + a_1 b_1 + a_2 b_2 x^{\beta} + a_3 b_3 x^{2\beta} + ...) \\ &= (a_{-1} + a_0 b_0)x^{-\beta} + (a_0 + a_1 b_1) \\ &\quad + (a_1 + a_2 b_2)x^{\beta} + (a_2 + a_3 b_3)x^{2\beta} + ... \\ &= \sum \left(a_j + a_{j+1}b_{j+1}\right)(x)^{j\beta} \\ b_k = \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - \beta)}, \quad b_0 &= \frac{1}{\Gamma(1-\beta)}, \quad b_{j+1} = \frac{\Gamma\left((j+1)\beta + 1\right)}{\Gamma\left(j\beta + 1\right)} \end{aligned} \tag{6.76}$$

We have:

$$\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}x^{-\beta} = \sum\left(a_j + a_{j+1}b_{j+1}\right)(x)^{j\beta}, \qquad b_{j+1} = \frac{\Gamma\left((j+1)\beta + 1\right)}{\Gamma\left(j\beta + 1\right)} \tag{6.77}$$

From the above (6.77), comparing the RHS and LHS, we write the following for all $j \neq -1$:

$$a_j + a_{j+1}b_{j+1} = 0, \qquad a_{j+1} = -a_j / b_{j+1}, \qquad a_{j+1} = -a_j\frac{\Gamma\left(1 + j\beta\right)}{\Gamma\left(1 + (j+1)\beta\right)} \tag{6.78}$$

By comparing the $j = -1$ terms, we get the following condition:

$$\begin{aligned} a_{-1} + a_0 b_0 &= \frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)} \\ a_{-1} + a_0\frac{1}{\Gamma(1-\beta)} &= \frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)} \\ a_0 + a_{-1}\Gamma(1-\beta) &= \mathrm{E}_M\varepsilon_0 \end{aligned} \tag{6.79}$$

For a short time response, we can choose $a_{-1} = 0$ (actually it is so) and have $a_0 = \mathrm{E}_M\varepsilon_0$ (i.e. at $x = \left(t/\tau\right) \approx 0$), with:

$$a_1 = -a_0 \frac{\Gamma(1)}{\Gamma(1+\beta)} = -\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1+\beta)}$$
$$a_2 = -a_1 \frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)} = -\left(-\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1+\beta)}\right)\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)} = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1+2\beta)} \tag{6.80}$$

We write from the above (6.80), the following:

$$a_j = (-1)^j \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1+j\beta)} \tag{6.81}$$

Therefore, the function $\sigma(t)$ is obtained for the constant $\varepsilon(t) = \varepsilon_0$, which is:

$$\sigma(t) = \mathrm{E}_M \varepsilon_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\beta)} \left(\frac{t}{\tau}\right)^{\beta k} \tag{6.82}$$

The earlier obtained result (6.70) says that $\sigma(t) = \mathrm{E}_M \varepsilon_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)} \left(\frac{t}{\tau}\right)^{\alpha k+\alpha-\beta}$; here we set $\alpha = \beta$, to get the above obtained result (6.82). The instantaneous response $\sigma(t)$ at $t = 0$, for a constant $\varepsilon(t) = \varepsilon_0$ is nevertheless equal to $\mathrm{E}_M \varepsilon_0$, and, at short times, the response function $\sigma(t)$ is 'some type of exponential' decay function.

For viewing for longer periods of time the function $\sigma(t)$, we choose to make $a_0 = 0$ (it is expected that a function at longer periods of time will have a very small value of the initial value of a function). From the $j = -1$ term of the comparison from (6.81), we get:

$$a_{-1} = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta)}, \qquad a_{-j} = (-1)^{j+1} \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1+j\beta)} \tag{6.83}$$

The expansion for longer periods of time thus contains negative powers essentially limited to $j\beta < 1$: we have set the condition above that $j\beta > -1$, since the fractional derivative is not defined for the larger power of x^{-1}. Thus, for longer periods of time (i.e. $t \gg \tau$ or $x \gg 1$), the response is the power law of exponent β, as follows:

$$\sigma(t) \approx a_{-1} \left(\tfrac{t}{\tau}\right)^{-\beta} = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta)} \left(\frac{t}{\tau}\right)^{-\beta} \tag{6.84}$$

6.9 An analytical method to obtain inverse Laplace transforms 'without contour integration' - the Berberan-Santos method

6.9.1 Development of the Berberan-Santos technique to obtain a distribution function for the decay rates for the relaxation function of time

We offer an alternative method to obtain the inverse Laplace transform, rather than using residue calculus and the described contour integration method. This method was developed by Berberan-Santos, primarily for obtaining a relaxation rate distribution of a time decay function. We know that a simple decay function is of the type $e^{-k_0 t}$, with $k_0 > 0$ as a relaxation-rate of decay. This simple decaying (or say relaxing) function has only one rate, i.e. $k = k_0$. The function can have a complex way of decaying, meaning that it will have a distribution of several relaxation-rates. In this section, we carry out an inverse Laplace transform of the time domain response of decay of the function to call it $f(t)$, in order to obtain the 'distribution of relaxation rates', for a complex decay process that is governed by several rates (k's) of an exponential decay.

A generally complex decay in time is depicted in Figure-6.3 and Figure-6.4 denoted by the Mittag-Leffler functions $E_{\alpha,\beta}(-t)$. The physical decay function, however, cannot be like that shown in Figure-6.5, which is an unbounded function. Figures 6.3, 6.4 and 6.5 depict the Mittag-Leffler function $E_{\alpha,\beta}(-t)$ with $\beta = 1$ and its first derivative (i.e. $E^{(1)}_{\alpha,\beta}(-t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[E_{\alpha,\beta}(-t)\right]$) for $0 < \alpha < 1$, for $1 < \alpha < 2$ and $2 < \alpha < 3$ respectively. We see that for cases of $0 < \alpha < 1$, the function $E_{\alpha,1}(-t)$ is monotonically decreasing, and is, therefore, a decaying function. For the cases of

$1 < \alpha < 2$, the function $E_{\alpha,1}(-t)$ has a decaying nature with oscillations. On the other hand, for cases of $2 < \alpha < 3$, the function $E_{\alpha,1}(-t)$ unboundedly grows with oscillation.

The technique discussed here is developed to use an inverse Laplace transform to depict distribution rates or histogram functions (call them $H(k)$) or time functions (call them $f(t)$), i.e. $H(k) = \mathcal{L}^{-1}\{f(t)\}$. However, this technique can be used to provide an inverse Laplace transform of a frequency domain function called $G(s)$ to get $g(t)$ as $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Here the rigor of contour integration is eased, and what we get most of the time is an integral representation of an inverse Laplace transform function.

First, we will build the essential formulations for Laplace inversion using the Berberan-Santos method, by extracting a histogram function $H(k)$ from $f(t)$, and give some applications of an inverse Laplace transform.

The complex decay may be expressed as follows with several rate constants $k_1, k_2, k_3, ...$ with weights $a_1, a_2, a_3, ...$

$$\begin{aligned} f(t) &= a_1 e^{-k_1 t} + a_2 e^{-k_2 t} + a_3 e^{-k_3 t} + ... = \sum a_n e^{-k_n t} \\ f(0) &= a_1 + a_2 + a_3 + ... = \sum a_n \end{aligned} \tag{6.85}$$

For a normalized case, we can have $f(0) = \sum a_n = 1$. In the continuum limit, we may write the above (6.85) as follows:

$$f(t) = \int_0^\infty \left(H(k)\right) e^{-kt} \mathrm{d}k \qquad f(0) = 1 \tag{6.86}$$

where $H(k)$ is a distribution function (the histogram function) of the rate of the relaxation of the decay process, $f(t)$ depicted in Figure-6.6. While the discrete rates are shown here as a series of Dirac delta functions, the rate of the distribution function is the following:

$$\begin{aligned} H(k) &= a_1\delta(k-k_1) + a_2\delta(k-k_2) + a_3\delta(k-k_3) + ... \\ &= \sum a_n \left(\delta(k-k_n)\right) \end{aligned} \tag{6.87}$$

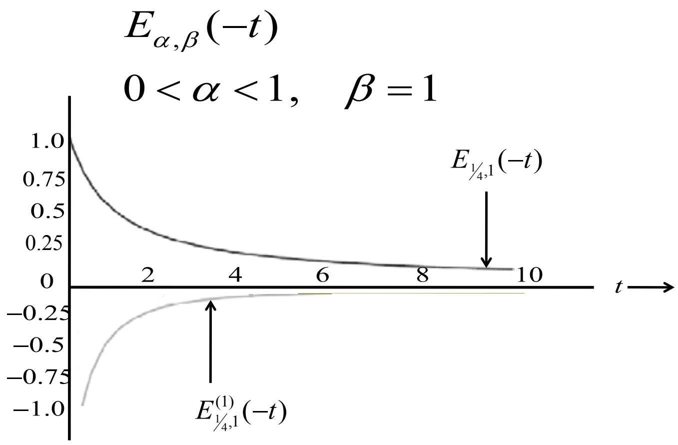


Figure-6.3: Plot of function $E_{\alpha,\beta}(-t)$, $\alpha = 0.25$, $\beta = 1$ and its derivative

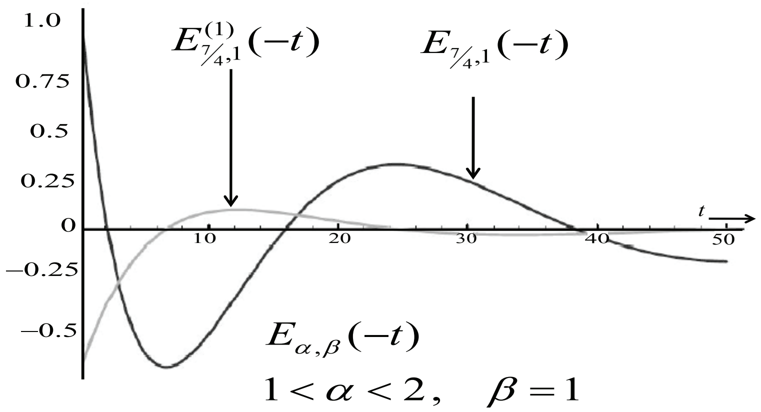


Figure-6.4: Plot of function $E_{\alpha,\beta}(-t)$, $\alpha = 1.75$, $\beta = 1$ and its derivative

The Laplace transform $F(s)$ of a function in a time domain $f(t)$ is defined as the Laplace integral (i.e. $F(s) = \int_0^\infty \left(f(t)\right)e^{-st}\mathrm{d}t$). This is the Laplace integral and is a standard integral transform from a time domain to a complex frequency domain $s = \mathrm{Re}\left[s\right] + i\,\mathrm{Im}\left[s\right]$; where the real part is significant in a transient response and the imaginary part of the frequency corresponds to a 'steady-state' response. Here, $f(t)$ is the inverse Laplace transform of $F(s)$ (i.e. $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$).

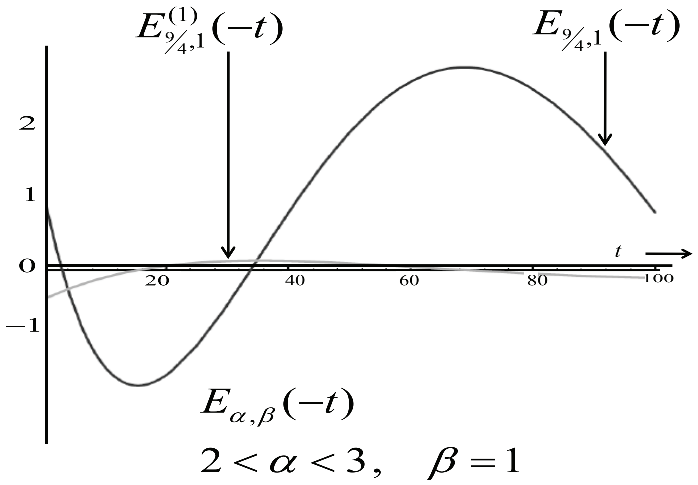


Figure-6.5: Plot of function $E_{\alpha,\beta}(-t)$, $\alpha = 2.25$, $\beta = 1$ and its derivative

Compare the Laplace integral with expression (6.86) (i.e. $f(t) = \int_0^\infty \left(H(k)\right)e^{-kt}\mathrm{d}k$) as follows:

$$f(t) = \int_0^\infty \left(H(k)\right)e^{-tk}\mathrm{d}k \qquad F(s) = \int_0^\infty \left(f(t)\right)e^{-st}\mathrm{d}t \tag{6.88}$$

Both of the above expressions are Laplace transform expressions, with the first one transforming the function $H(k)$ from k domain to t domain; while the second one is transforming $f(t)$ from t domain to s domain. However, both are Laplace transforms with only a change of variable and symbol. Therefore, we can say that the function $H(k)$ is an inverse Laplace transform of $f(t)$ in the first expression (6.88) and the function $f(t)$ is an inverse Laplace transform of $F(s)$ in the second expression (6.88). Therefore, in order to obtain the rate of the distribution-function $H(k)$ from the decay curve, say $f(t)$, we need to have an inverse Laplace transform of the time function $f(t)$ like the conventional definition of the inverse Laplace transform.

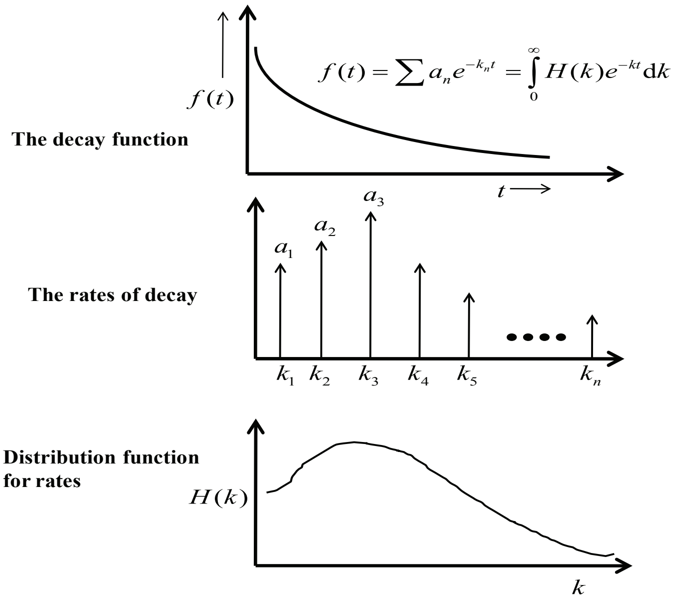
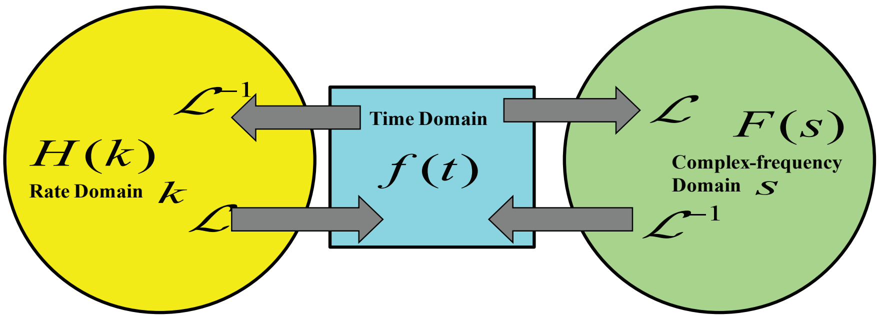


Figure-6.6: Rate of decay function



$$H(k)=\mathcal{L}^{-1}\{f(t)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(t)e^{tk}\mathrm{d}t \qquad f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}\mathrm{d}s$$

$$f(t)=\mathcal{L}\{H(k)\}=\int_0^\infty H(k)e^{-tk}\mathrm{d}k \qquad F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}\mathrm{d}t$$

Figure-6.7: Laplace inversion and Laplace transform

We get the following expressions for inverse Laplace transforms:

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(F(s)\right)e^{st}\mathrm{d}s \qquad H(k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(f(t)\right)e^{tk}\mathrm{d}t \tag{6.89}$$

In Section-6.6, we explored out how to carry out inverse Laplace transforms with contour integration. Figure-6.7 describes this process of inversion of the Laplace transform. In the above expression, c is a real number larger than c_0, where c_0 is such that $f(t)$ has some form of singularity on the real line $\mathrm{Re}\left[t\right]=c_0$ but is analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\left[t\right]>c_0$. The inverse Laplace transform is usually carried out by contour integration (see Appendix-G for more detail). We will develop the formula, where we carry out an inverse Laplace transform from complex $s=\mathrm{Re}\left[s\right]+i\,\mathrm{Im}\left[s\right]=\sigma+i\omega$ for a function in s domain called $G(s)$ to get the t domain function $g(t)$ such that the Laplace transform expression is $G(s)=\int_0^\infty\left(g(t)\right)e^{-st}\mathrm{d}t$ and the inverse Laplace transform expression is $g(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(G(s)\right)e^{st}\mathrm{d}s$.

6.9.2 Derivation of the Berberan-Santos method

We will not find $H(k)=\mathcal{L}^{-1}\left\{f(t)\right\}$, but we need to find $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$. However, the methods remain the same for finding $H(k)$, i.e. the relaxation rate distribution function from a relaxation function $f(t)$, where the time variable is considered a complex quantity. This was originally formulated by Berberan-Santos, and we shall be adapting this technique.

We develop in this section an analytical Laplace inversion technique in order to obtain the inverse Laplace transform given as $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}=\frac{1}{2\pi i}\int_{\sigma\text{-}i\infty}^{\sigma+i\infty}\left(G(s)\right)e^{st}\mathrm{d}s$. Usually the solution of this integral (i.e. the inverse Laplace integral $\frac{1}{2\pi i}\int_{\sigma\text{-}i\infty}^{\sigma+i\infty}\left(G(s)\right)e^{st}\mathrm{d}s$) is obtained via the contour integration that we discussed in Section-6.6 (see Appendix-G for more information). Here, we develop an analytical technique to find the inverse Laplace transform without using the usual contour integration method.

Here we describe the Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. Here is the real part, i.e. σ is constant as a vertical line (a Bromwich-path), with $\sigma=\sigma_0$ as a constant. The variable σ_0 is a real number being such that $G(s)$ has some form of singularity on the line $\mathrm{Re}\left[s\right]=\sigma_0$, but an analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\left[s\right]>\sigma_0$. Performing the variable change on a Laplace integral to $s=\sigma_0+i\omega$, we get the following steps:

$$\begin{aligned}
g(t)&=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(G(s)\right)e^{st}\mathrm{d}s;\qquad s\equiv\sigma_0+i\omega\\
&=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(G(\sigma_0+i\omega)\right)\left(e^{(\sigma_0+i\omega)t}\right)\left(\mathrm{d}(\sigma_0+i\omega)\right);\qquad \mathrm{d}\sigma_0=0\\
&=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(G(\sigma_0+i\omega)\right)\left(e^{t\sigma_0}e^{it\omega}\right)\left(i\mathrm{d}\omega\right)\\
&=\frac{e^{\sigma_0 t}}{2\pi}\int_{-\infty}^{+\infty}\left(G(\sigma_0+i\omega)\right)e^{i\omega t}\mathrm{d}\omega
\end{aligned}\tag{6.90}$$

Writing $e^{it\omega}=\cos\omega t+i\sin\omega t$, we get the following form:

$$g(t)=\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(G(\sigma_0+i\omega)\right)\cos(\omega t)\mathrm{d}\omega+i\int_{-\infty}^{+\infty}\left(G(\sigma_0+i\omega)\right)\sin(\omega t)\mathrm{d}\omega\right)\tag{6.91}$$

writing:

$$G(\sigma_0+i\omega)=\mathrm{Re}\left[G(\sigma_0+i\omega)\right]+i\,\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\tag{6.92}$$

and place (6.92) in the preceding expression (6.91) to get the following expression:

$$\begin{aligned}
g(t)&=\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\left(\cos(\omega t)\right)\right)-\left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right)\\
&+i\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\cos(\omega t)\right)\right)+\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right)
\end{aligned}\tag{6.93}$$

Given that $g(t)$ is a real function, we get the following (i.e. by equating the imaginary part to zero):

$$\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\cos(\omega t)\right)+\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right)=0\tag{6.94}$$

Thus, the above expression (6.93) for $g(t)$ reduces to the following (i.e. considering only the real part) and we get:

$$g(t)=\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega\right)\tag{6.95}$$

However, we have from a Laplace integral $G(s)=\int_0^\infty \left(g(t)\right)e^{-st}\mathrm{d}t$; and by putting here $s=\sigma_0+i\omega$, we get the following:

$$
\begin{aligned}
G(\sigma_0+i\omega) &= \int_0^\infty \left(g(t)\right)e^{-t(\sigma_0+i\omega)}\mathrm{d}t \\
&= \int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t - i\int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t
\end{aligned}
\tag{6.96}
$$

This gives the below:

$$
\begin{aligned}
\mathrm{Re}\left[G\right] &= \int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t \\
\mathrm{Im}\left[G\right] &= -\int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t
\end{aligned}
\tag{6.97}
$$

We find in (6.97) that function $\mathrm{Re}\left[G\right]$ is an even-function in the variable ω (call it $\mathrm{F}_{even}(\omega)$) and the function $\mathrm{Im}\left[G\right]$ is an odd-function in variable ω (call it $\mathrm{F}_{odd}(\omega)$). We get:

$$
\begin{aligned}
&\left(\mathrm{Re}\left[G\right]\cos\omega t - \mathrm{Im}\left[G\right]\sin\omega t\right) \\
&\quad = \left(\mathrm{F}_{even}(\omega)\right)\cos\omega t - \left(\mathrm{F}_{odd}(\omega)\right)\sin\omega t
\end{aligned}
\tag{6.98}
$$

as an 'even-function'. That is the even function $\mathrm{F}_{even}(\omega)$ multiplied by another even function (i.e. $\cos\omega t$), which gives a further even function; and an odd-function $\mathrm{F}_{odd}(\omega)$ multiplied by an odd function (i.e. $\sin\omega t$) gives an even function. Therefore for an overall even-function integrand, we have the following integral:

$$
\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[G\right]\cos\omega t - \mathrm{Im}\left[G\right]\sin\omega t\right)\mathrm{d}\omega \\
&= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G\right]\cos\omega t - \mathrm{Im}\left[G\right]\sin\omega t\right)\mathrm{d}\omega
\end{aligned}
\tag{6.99}
$$

Further simplification of (6.99) can be extended by considering 'causality' in $g(t)$ i.e. $g(t)=0$ for $t<0$ and $g(t)\neq 0$ for $t\geq 0$. For a case $\sigma_0=0$ we get $g(t)=\frac{2}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G\right]\cos\omega t\right)\mathrm{d}\omega = -\frac{2}{\pi}\int_0^\infty\left(\mathrm{Im}\left[G\right]\sin\omega t\right)\mathrm{d}\omega$ for $t\geq 0$. We are not proving this simplification. One can do this from (6.99) by putting positive t and writing $g(t)=0$ for negative t, then adding the two cases (or subtracting the two cases)

Write (6.92) in polar form as described below

$$
\begin{aligned}
G(\sigma_0+i\omega) &= \rho(\omega)e^{i\theta(\omega)} = \rho(\omega)\left(\cos\left(\theta(\omega)\right)+i\sin\left(\theta(\omega)\right)\right) \\
\rho(\omega) &= \left|G(\sigma_0+i\omega)\right| \qquad \theta(\omega)=\angle G(\sigma_0+i\omega)
\end{aligned}
\tag{6.100}
$$

to get alternate formulas from (6.99) as follows:

$$
\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\rho(\omega)\left(\cos\theta(\omega)\right)\cos(\omega t) - \rho(\omega)\left(\sin\theta(\omega)\right)\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\rho(\omega)\right)\left(\cos\left(\omega t+\theta(\omega)\right)\right)\mathrm{d}\omega
\end{aligned}
\tag{6.101}
$$

We will demonstrate the use of the formulae of (6.99) and (6.101) to obtain the inverse Laplace transforms of various functions and write the equivalent integral representations. The integrals obtained are, however, difficult to solve in closed form, but can be used to get presentation that is useful for numerical integration.

6.10 A few examples of the inverse Laplace transform of functions available in standard Laplace transform tables obtained using the Berberan-Santos method

6.10.1 The inverse Laplace transform of function: $G(s) = (s-a)^{-1}$

Putting $s = \sigma_0 + i\omega$ with $\sigma_0 > a$ in $G(s) = (s-a)^{-1}$, we write the following:

$$G(\sigma_0 + i\omega) = \frac{1}{(\sigma_0 - a) + i\omega} = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} - i\frac{\omega}{(\sigma_0 - a)^2 + \omega^2}$$

$$\operatorname{Re}\left[G(\sigma_0 + i\omega)\right] = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} \tag{6.102}$$

$$\operatorname{Im}\left[G(\sigma_0 + i\omega)\right] = -\frac{\omega}{(\sigma_0 - a)^2 + \omega^2}$$

Applying (6.99), we get the following:

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\operatorname{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) - \operatorname{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}(\sigma_0 - a)}{\pi}\int_0^\infty \frac{\cos(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t) + \omega\sin(\omega t)}{(\sigma_0 - a)^2 + \omega^2}\mathrm{d}\omega \end{aligned} \tag{6.103}$$

From the Laplace tables we know that $(s-a)^{-1} = \mathcal{L}\left\{e^{at}\right\}$. Therefore, we have an integral representation of e^{at} as follows:

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t) + \omega\sin(\omega t)}{(\sigma_0 - a)^2 + \omega^2}\mathrm{d}\omega = e^{at} \tag{6.104}$$

For $a = -b$, $b > 0$, we have $\mathcal{L}^{-1}\left\{(s+b)^{-1}\right\} = e^{-bt}$. We choose $\sigma_0 = 0 > -b$, in the formulae of (6.99) and (6.104) to write the following:

$$\frac{1}{\pi}\int_0^\infty \frac{b\cos(\omega t) + \omega\sin(\omega t)}{b^2 + \omega^2}\mathrm{d}\omega = e^{-bt} \tag{6.105}$$

In polar form, we obtain the following using the formula (6.101):

$$\begin{aligned} G(\sigma_0 + i\omega) &= \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} - i\frac{\omega}{(\sigma_0 - a)^2 + \omega^2} \\ \theta(\omega) &= \angle G(\sigma_0 + i\omega) = \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right), \\ \rho(\omega) &= \left|G(\sigma_0 + i\omega)\right| = \tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}} \end{aligned} \tag{6.106}$$

We apply (6.101) to get the below:

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\left(\cos\left(\omega t + \theta(\omega)\right)\right)\mathrm{d}\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}}\right)\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right)\right)\mathrm{d}\omega \end{aligned} \tag{6.107}$$

Thus, we have another formulation for e^{at} as follows:

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}}\right)\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right)\right)\mathrm{d}\omega = e^{at} \tag{6.108}$$

For the case of $G(s)=(s+1)^{-1}$, it has singularity at $s=-1$. We choose $\sigma_0=0>-1$, and we write the following:

$$\frac{1}{\pi}\int_0^\infty \frac{\cos\omega t+\omega\sin\omega t}{1+\omega^2}\mathrm{d}\omega=e^{-t}$$
$$\frac{1}{\pi}\int_0^\infty\left(\frac{1}{\sqrt{(1+\omega^2}}\right)\cos\left(\omega t+\tan^{-1}\left(-\omega\right)\right)\mathrm{d}\omega=e^{at} \tag{6.109}$$

6.10.2 The inverse Laplace transform of the function: $G(s)=s(s^2+1)^{-1}$

For a change of variable $s=\sigma_0+i\omega$ with $\sigma_0=1>0$, we have the following with $s=1+i\omega$:

$$G(s)=\frac{s}{s^2+1},\qquad G(1+i\omega)=\frac{1+i\omega}{(1+i\omega)^2+1} \tag{6.110}$$

The above expression has singularity at $s=\pm i$ and at line $\mathrm{Re}\left[s\right]=0$. We choose $\sigma_0=1$, and, in this case, we get:

$$\mathrm{Re}\left[G(1+i\omega)\right]=\frac{2+\omega^2}{4+\omega^4}\qquad \mathrm{Im}\left[G(1+i\omega)\right]=-\frac{\omega^2}{4+\omega^4} \tag{6.111}$$

Using (6.99), we write the following:

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega\\
&=\frac{e^{t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G(1+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(1+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega\\
&=\frac{e^{t}}{\pi}\int_0^\infty\left(\frac{2+\omega^2}{4+\omega^4}\cos(\omega t)+\frac{\omega^2}{4+\omega^4}\sin(\omega t)\right)\mathrm{d}\omega\\
&=\frac{e^{t}}{\pi}\int_0^\infty\frac{(2+\omega^2)\cos(\omega t)\mathrm{d}\omega}{4+\omega^4}+\frac{e^{t}}{\pi}\int_0^\infty\frac{\omega^2\sin(\omega t)\mathrm{d}\omega}{4+\omega^4}\\
&=\frac{e^{t}}{\pi}\int_0^\infty\frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega\end{aligned} \tag{6.112}$$

The known Laplace transform is $\mathcal{L}^{-1}\left\{s/(s^2+1)\right\}=\cos t$, so we have an integration representation of $\cos t$ as follows:

$$\frac{e^{t}}{\pi}\int_0^\infty\frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega=\cos t \tag{6.113}$$

In polar form, we have $G(s)=s(s^2+1)^{-1}$, $\rho(\omega)=\frac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}$; $\theta(\omega)=\tan^{-1}\left(-\frac{\omega^2}{\omega^2+2}\right)$. Using (6.101), we write the following:

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\rho(\omega)\right)\left(\cos\left(\omega t+\theta(\omega)\right)\right)\mathrm{d}\omega\\
&=\frac{e^{t}}{\pi}\int_0^\infty\left(\frac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}\right)\left(\cos\left(\omega t+\tan^{-1}\left(-\frac{\omega^2}{\omega^2+2}\right)\right)\right)\mathrm{d}\omega\end{aligned} \tag{6.114}$$

We have another representation of $\cos t$ as follows:

$$\frac{e^{t}}{\pi}\int_0^\infty\left(\frac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}\right)\left(\cos\left(\omega t+\tan^{-1}\left(-\frac{\omega^2}{\omega^2+2}\right)\right)\right)\mathrm{d}\omega=\cos t \tag{6.115}$$

6.10.3 The inverse Laplace transform of function $G(s)=e^{-\lambda_0 s}$

The function $G(s)=e^{-\lambda_0 s}$ has no singularity at $s>0$, so $\sigma_0=0$ must be chosen. Thus, we write in complex variable the function as $G(\sigma_0+i\omega)=G(i\omega)=e^{-i\omega\lambda_0}=\cos(\omega\lambda_0)-i\sin(\omega\lambda_0)$. Thus, we have real and imaginary parts as $\mathrm{Re}\left[G(\sigma_0+i\omega)\right]=\cos(\omega\lambda_0)$; $\mathrm{Im}\left[G(\sigma_0+i\omega)\right]=-\sin(\omega\lambda_0)$. Applying the formula (6.99), we get $g(t)=\mathcal{L}^{-1}\left\{e^{-\lambda_0 s}\right\}$ as follows:

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\Big(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\Big)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\Big(\cos(\omega\lambda_0)\cos(\omega t)+\sin(\omega\lambda_0)\sin(\omega t)\Big)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\cos\Big(\omega(t-\lambda_0)\Big)\mathrm{d}\omega\end{aligned}\tag{6.116}$$

From the standard Laplace tables, we have $\mathcal{L}^{-1}\left\{e^{-s\lambda_0}\right\}=\delta(t-\lambda_0)$; thus we get the integral representation as follows:

$$\frac{1}{\pi}\int_0^\infty\cos\Big(\omega(t-\lambda_0)\Big)\mathrm{d}\omega=\delta(t-\lambda_0)\tag{6.117}$$

In polar form, we have $\rho(\omega)=1,\quad \theta(\omega)=-\omega\lambda_0$. Using (6.119) we write:

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\Big(\rho(\omega)\Big)\Big(\cos\Big(\omega t+\theta(\omega)\Big)\Big)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\Big(\cos\Big(\omega t-\omega\lambda_0\Big)\Big)\mathrm{d}\omega\end{aligned}\tag{6.118}$$

This is the same that we obtained in (6.117) using the formula given in (6.99).

6.10.4 The inverse Laplace transform of function $G(s)=s^{-\alpha}$

The function $G(s)=s^{-\alpha}$ does not have a singularity at $s>0$. Choosing $\sigma_0=0$, we obtain:

$$G(\sigma_0+i\omega)=(i\omega)^{-\alpha}=\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)-i\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\tag{6.119}$$

Applying the formula (6.99), we get $g(t)=\mathcal{L}^{-1}\left\{s^{-\alpha}\right\}$ as the following:

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\Big(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\Big)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\Big(\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)\cos(\omega t)+\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\sin(\omega t)\Big)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\omega^{-\alpha}\cos\left(\omega t-\tfrac{\alpha\pi}{2}\right)\mathrm{d}\omega\end{aligned}\tag{6.120}$$

With the known Laplace transform (i.e. $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\}=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$), we write following:

$$\frac{1}{\pi}\int_0^\infty\omega^{-\alpha}\cos\left(\omega t-\tfrac{\alpha\pi}{2}\right)\mathrm{d}\omega=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}\tag{6.121}$$

Putting $(\alpha-1)=-n$, we write the integral representation of t^{-n} as follows:

$$\frac{\Gamma(1-n)}{\pi}\int_0^\infty\omega^{n-1}\cos\left(\omega t-\tfrac{(1-n)\pi}{2}\right)\mathrm{d}\omega=t^{-n}\tag{6.122}$$

With use of the polar form formula (6.101), we will get the same result as in (6.120):

6.10.5 The inverse Laplace transform of function: $G(s)=1$

Taking $G(s)=1$ and choosing $\sigma_0=0$, we have for $s=\sigma_0+i\omega$ and $G(\sigma_0+i\omega)=1+i(0)$. Using (6.99), we write $g(t)=\mathcal{L}^{-1}\{1\}$ as follows:

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\big[G(\sigma_0+i\omega)\big]\cos(\omega t)-\mathrm{Im}\big[G(\sigma_0+i\omega)\big]\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\Big((1)\cos(\omega t)-(0)\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\cos(\omega t)\mathrm{d}\omega \end{aligned} \tag{6.123}$$

Knowing $\mathcal{L}^{-1}\{1\}=\delta(t)$, we write the integral representation of $\delta(t)$ as:

$$\frac{1}{\pi}\int_0^{\infty}\cos(\omega t)\mathrm{d}\omega=\delta(t) \tag{6.124}$$

6.10.6 The inverse Laplace transform of the function $G(s)=s^{-1}$

We take $G(s)=s^{-1}$ and, with $\sigma_0=0$, we write $G(\sigma_0+i\omega)=(i\omega)^{-1}=\omega^{-1}\big(0-i(1)\big)$. We use the formula in (6.99) to write $g(t)=\mathcal{L}^{-1}\left\{s^{-1}\right\}$ as follows:

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\big[G(\sigma_0+i\omega)\big]\cos(\omega t)-\mathrm{Im}\big[G(\sigma_0+i\omega)\big]\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\Big(\omega^{-1}(0)\cos(\omega t)+(1)\omega^{-1}\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega t}{\omega}\mathrm{d}\omega \end{aligned} \tag{6.125}$$

We know that $\mathcal{L}^{-1}\left\{s^{-1}\right\}=u(t)$ (i.e. Heaviside unit step function at $t=0$), so we will write the integral representation of $u(t)$ as:

$$\begin{aligned} &\frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega t}{\omega}\mathrm{d}\omega=u(t)=1, \quad t\ge 0 \\ &\frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega t}{\omega}\mathrm{d}\omega=u(t)=0, \quad t<0 \end{aligned} \tag{6.126}$$

6.11 Examples of using the Berberan-Santos method to obtain the Laplace inversion of a few functions that are not provided in standard Laplace Transform tables

The Berberan-Santos method allows the provision of integral representation of the inverse Laplace transforms of some transfer functions that are useful in a fractional calculus context. Many such functions are however, not listed in the standard Laplace transform tables; we will utilize either the contour integration technique or the Berberan-Santos method to arrive at an expression for inverse Laplace transforms.

6.11.1 The inverse Laplace transform of function $G(s)=e^{-(s/s_0)^{\beta}}$

We obtain $G(s)=e^{-(s/s_0)^{\beta}}$ in a complex variable by putting $s=\sigma_0+i\omega$ with $\sigma_0=0$, as in the following representation in polar form:

$$
\begin{aligned}
G(i\omega) &= e^{-(i\omega/s_0)^{\beta}} = e^{-(\omega/s_0)^{\beta}(i)^{\beta}} \\
&= e^{-(\omega/s_0)^{\beta}\left[\cos\left(\frac{\beta\pi}{2}\right)+i\sin\left(\frac{\beta\pi}{2}\right)\right]} \\
&= e^{\left[-\left(\frac{\omega}{s_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]} e^{\left[-i\left(\frac{\omega}{s_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]} \\
&= \left(\rho(\omega)\right)\left(e^{i(\theta(\omega))}\right)
\end{aligned}
\tag{6.127}
$$

whereas in this case we have in polar form a representation for the use of (6.101):

$$
\begin{aligned}
\left|G(\sigma_0 + i\omega)\right| &= \rho(\omega) = e^{\left[-\left(\frac{\omega}{s_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]} \\
\angle G(\sigma_0 + i\omega) &= \theta(\omega) = -\left(\frac{\omega}{s_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)
\end{aligned}
\tag{6.128}
$$

Therefore, we write an inverse Laplace transform $\mathcal{L}^{-1}\left\{e^{-(s/s_0)^{\beta}}\right\}$ as $g(t)$ in the following expression:

$$
\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\cos\left(\omega t + \left(\theta(\omega)\right)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty} e^{\left[-\left(\frac{\omega}{s_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(\omega t - \left(\frac{\omega}{s_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}\omega
\end{aligned}
\tag{6.129}
$$

By changing the variable $u = \omega / s_0$, we obtain:

$$
g(t) = \frac{s_0}{\pi}\int_0^{\infty} e^{\left[-u^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(ts_0 u - u^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u
\tag{6.130}
$$

6.11.2 The inverse Laplace transform of function $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$

We take $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$. We use the following steps to find an inverse Laplace transform of this transfer function by putting $s = \sigma_0 + i\omega$ with $\sigma_0 = 0$:

$$
\begin{aligned}
& G(i\omega) = \frac{1}{\left(1+(1-\beta)\left(\frac{i\omega}{s_0}\right)\right)^{1/1-\beta}} \\
& \left|G(i\omega)\right| = \rho(\omega) = \left(1+\left(\frac{(1-\beta)\omega}{s_0}\right)^2\right)^{-1/2(1-\beta)}; \quad \angle G(i\omega) = \theta(\omega) = -\frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta}
\end{aligned}
\tag{6.131}
$$

Therefore, we write the inverse Laplace transform (i.e. $\mathcal{L}^{-1}\left\{\left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}\right\} = g(t)$) as follows:

$$
\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\cos\left(\omega t + \left(\theta(\omega)\right)\right)\mathrm{d}\omega; \qquad \sigma_0 = 0 \\
&= \frac{1}{\pi}\int_0^{\infty}\left[1+\left(\frac{(1-\beta)\omega}{s_0}\right)^2\right]^{-\frac{1}{2(1-\beta)}}\cos\left(\omega t - \frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta}\right)\mathrm{d}\omega
\end{aligned}
\tag{6.132}
$$

With a change of variable $u = \frac{(1-\beta)\omega}{s_0}$, we get the following expression:

$$
g(t) = \frac{s_0}{\pi(1-\beta)}\int_0^{\infty}\left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{t\tau_0 u - \tan^{-1}u}{1-\beta}\right)\mathrm{d}u; \qquad u = \frac{(1-\beta)\omega}{s_0}
\tag{6.133}
$$

6.11.3 The inverse Laplace transform of the function $G(s)=\left(1+\left(\frac{s}{a}\right)^{\alpha}\right)^{-1}$

We conduct an inverse Laplace transform of a simple power law in s domain (i.e. $G(s)=\frac{1}{1+(s/a)^{\alpha}}$; $\alpha<1$) to get the function $g(t)=\mathcal{L}^{-1}\left\{\left(1+\left(\frac{s}{a}\right)^{\alpha}\right)^{-1}\right\}$. Putting $s=0+i\omega$, we write the following steps:

$$\begin{aligned}G(i\omega)&=\frac{1}{1+\left(\frac{i\omega}{a}\right)^{\alpha}}=\frac{1}{1+\left(\frac{\omega}{a}\right)^{\alpha}(i)^{\alpha}}\\&=\frac{1}{1+\left(\left(\frac{\omega}{a}\right)^{\alpha}e^{i\frac{\alpha\pi}{2}}\right)}=\frac{1}{1+\left(\left(\frac{\omega}{a}\right)^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)}\\&=\frac{1}{\left(1+\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)}\end{aligned}\tag{6.134}$$

From the above (6.134), we get the following:

$$\begin{aligned}\mathrm{Re}\left[G(i\omega)\right]&=\frac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\\\mathrm{Im}\left[G(i\omega)\right]&=-\frac{\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\end{aligned}\tag{6.135}$$

We carry out an inverse Laplace transform (i.e. $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$) with $G(s)=\frac{1}{1+(s/a)^{\alpha}}$, as follows, using the formula (6.99):

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^{\infty}\left(\begin{array}{l}\left(\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\cos(\omega t)\\+\left(\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\sin(\omega t)\end{array}\right)\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^{\infty}\left(\frac{\left(\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\omega t+\left(\frac{\omega}{a}\right)^{\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\omega t}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}\omega;\quad u=\frac{\omega}{a}\\&=\frac{a}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos aut+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u\end{aligned}\tag{6.136}$$

6.11.4 An integral representation of the Mittag-Leffler function using the Berberan-Santos method

The listed inverse Laplace transform of a two-parameter Mittag-Leffler function (see Appendix-A) is defined as $E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{(z)^m}{\Gamma(\alpha m+\beta)}$ we have $g(t)=\mathcal{L}^{-1}\left\{\frac{1}{1+(s/a)^{\alpha}}\right\}=a\left(at\right)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)\right)$. From the obtained expression that is $\mathcal{L}^{-1}\left\{\frac{1}{1+(s/a)^{\alpha}}\right\}=\frac{a}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos aut+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u$ (6.136), we write following expressions:

$$a(at)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)\right)$$
$$=\frac{a}{\pi}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(atu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u\,;\quad u=\frac{\omega}{a}$$

$$E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)$$
$$=\frac{1}{\pi}(at)^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(atu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u \tag{6.137}$$
$$at=z$$
$$E_{\alpha,\alpha}\left(-z^{\alpha}\right)=\frac{1}{\pi}z^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(zu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin zu}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u$$

We obtained an integral representation of the Mittag-Leffler function via the Berberan-Santos method, i.e.

$$E_{\alpha,\alpha}\left(-x^{\alpha}\right)=\frac{1}{\pi}x^{1-\alpha}\int_0^{\infty}\frac{\left(y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(xy)\right)+\left(y^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin xy}{y^{2\alpha}+2y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}y \tag{6.138}$$

With $\alpha=1$ and $a=1$, we get $g(t)=\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=E_{1,1}\left(-t\right)=e^{-t}$. The integral representation is as follows:

$$E_{\alpha,\alpha}\left(-t^{\alpha}\right)=\frac{1}{\pi}t^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(tu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin tu}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u$$
$$\alpha=1 \tag{6.139}$$
$$E_{1,1}\left(-t\right)=\frac{1}{\pi}\int_0^{\infty}\frac{\cos(tu)+u\sin tu}{u^2+1}\mathrm{d}u=e^{-t}$$

Earlier in the expression (6.110), we obtained $\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=\frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega t+\omega\sin\omega t}{1+\omega^2}\mathrm{d}\omega=e^{-t}$, which is the same as we received from the Mittag-Leffler function in (6.139).

Also, using the expression (6.136), we can write the following useful identity of the Laplace transform pair:

$$\mathcal{L}^{-1}\left\{\frac{k}{k+s^{\alpha}}\right\}$$
$$=\frac{\sqrt[\alpha]{k}}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u \tag{6.140}$$
$$u=\frac{\omega}{\sqrt[\alpha]{k}}$$

6.11.5 The inverse Laplace transform of the function $G(s)=s^{\alpha-1}(s^{\alpha}+1)^{-1}$

We will apply the Berberan-Santos method to a known Laplace pair of the Mittag-Leffler function $\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}=s^{\alpha-1}(s^{\alpha}+1)^{-1}$ (see Appendix-A). For the function $G(s)=s^{\alpha-1}(s^{\alpha}+1)^{-1}$, in order to get an integral representation of the inverse Laplace transformed result, we write $s=\sigma_0+i\omega$ with $\sigma_0=0$. This is because we do not expect singularity in the right half plane of complex frequency s (i.e. in region $\mathrm{Re}\left[s\right]>0$) for function $G(s)$ for its well-meaning behavior. With this substitution, we get the following steps:

$$
\begin{aligned}
G(s) &= \frac{s^{\alpha-1}}{1+s^{\alpha}} \qquad\qquad s = 0 + i\omega \\
G(i\omega) &= \frac{(i\omega)^{\alpha-1}}{1+(i\omega)^{\alpha}} = \frac{\omega^{\alpha-1}(i)^{\alpha-1}}{1+(i\omega)^{\alpha}} \\
&= \frac{\omega^{\alpha-1}(i)^{-1}\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)} \\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)-i\cos\left(\frac{\alpha\pi}{2}\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)} \\
&= \omega^{\alpha-1}\frac{\left(\sin\left(\frac{\alpha\pi}{2}\right)-i\cos\left(\frac{\alpha\pi}{2}\right)\right)\left(\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)-i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} - i\omega^{\alpha-1}\frac{\omega^{\alpha}+\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}
\end{aligned}
\tag{6.141}
$$

From the above (6.141), we write the following:

$$
\begin{aligned}
\mathrm{Re}\left[G(i\omega)\right] &= \frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \\
\mathrm{Im}\left[G(i\omega)\right] &= -\frac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}
\end{aligned}
\tag{6.142}
$$

We write an inverse Laplace transform using the formula (6.99), as follows:

$$
\begin{aligned}
E_{\alpha}(-t^{\alpha}) &= g(t) = \mathcal{L}^{-1}\left\{G(i\omega)\right\} \\
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\left(\begin{array}{l}\left(\dfrac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\cos(\omega t) \\ +\left(\dfrac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\sin(\omega t)\end{array}\right)\mathrm{d}\omega \\
&\frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)\cos(\omega t)+\omega^{2\alpha-1}\sin(\omega t)+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega
\end{aligned}
\tag{6.143}
$$

From here, we write an integral representation of $E_{\alpha}(-z)$ as follows (i.e. placing $t^{\alpha} = z$):

$$
E_{\alpha}(-z) = \frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z}+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega \tag{6.144}
$$

Therefore, we can write from the above derivation the following expression:

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha}}{s(1+s^{\alpha})}\right\} \\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z}+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega, \qquad t^{\alpha} = z
\end{aligned}
\tag{6.145}
$$

6.11.6 The inverse Laplace transform of function $G(s) = ks^{-1}(s^{\alpha}+k)^{-1}$

Here, we note that we get functions like $\frac{s^{\alpha}}{s(1+s^{\alpha})}$ in the Laplace transform of a fractional differential equation (FDE). We need to carry out an inverse Laplace transform of the same to obtain a solution to the main differential equation. Using a contour integration as described earlier (in Section-6.6) or using this method of Laplace inversion through the Berberan-Santos formulas gives us an integral representation of the solution, which can be numerically solved via numerical integration. It should be noted that the FDE does not always result in a known functional form in solution. We take our earlier example, where we conducted contour integration to get $g(t) = \mathcal{L}^{-1}\left\{\frac{k}{s(s^{\alpha}+k)}\right\}$, which came from the FDE as follows (given the FDE with its Laplace transformed equation); with a zero initial condition and $x(t)$ as the Heaviside unit step excitation at $t=0$, $\mathcal{L}\left\{x(t)\right\} = X(s) = \frac{1}{s}$. The fractional differential equation is $\frac{1}{k}\frac{\mathrm{d}^{\alpha}g(t)}{\mathrm{d}t^{\alpha}} + g(t) = x(t)$. Carrying out the Laplace transform, we get $\frac{1}{k}\left(s^{\alpha}G(s)\right) + G(s) = \frac{1}{s}$. From this, we get $G(s) = \frac{k}{s(s^{\alpha}+k)}$.

We take $G(s) = ks^{-1}(s^{\alpha}+k)^{-1}$. Putting $s = i\omega$, we have the following steps:

$$
\begin{aligned}
G(i\omega) &= \frac{k}{(i\omega)\left((i\omega)^{\alpha}+k\right)} = \frac{k}{\omega\left(\omega^{\alpha} i^{\alpha+1} + ik\right)} \\
&= \frac{k}{\omega\left(\omega^{\alpha}\left(\cos\left(\frac{(\alpha+1)\pi}{2}\right) + i\omega^{\alpha}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + ik\right)\right)} \\
&= \frac{k}{\omega\left(\omega^{\alpha}\cos\left(\frac{(\alpha+1)\pi}{2}\right) + i\left(\omega^{\alpha}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)} \\
&= \frac{k\left(\omega^{\alpha}\cos\left(\frac{(\alpha+1)\pi}{2}\right) - i\left(\omega^{\alpha}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)}{\omega\left(\omega^{2\alpha}\cos^{2}\left(\frac{(\alpha+1)\pi}{2}\right) + \omega^{2\alpha}\sin^{2}\left(\frac{(\alpha+1)\pi}{2}\right) + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^{2}\right)} \\
&= \frac{k\left(\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right) - i\left(\omega^{\alpha-1}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^{2}}
\end{aligned} \tag{6.146}
$$

We get from (6.146) the following:

$$
\begin{aligned}
\mathrm{Re}\left[G(i\omega)\right] &= \frac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^{2}} \\
\mathrm{Im}\left[G(i\omega)\right] &= -\frac{k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^{2}}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^{2}}
\end{aligned} \tag{6.147}
$$

Now using the formula (6.99) and the Berberan-Santos method, we get the integral representation of $g(t) = \mathcal{L}^{-1}\left\{\frac{k}{s(s^{\alpha}+k)}\right\}$ as follows:

$$g(t) = \mathcal{L}^{-1}\left\{G(i\omega)\right\}$$

$$g(t) = \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \Big(\mathrm{Re}\big[G(\sigma_0 + i\omega)\big]\cos(\omega t) - \mathrm{Im}\big[G(\sigma_0 + i\omega)\big]\sin(\omega t)\Big)\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left(\begin{array}{l}\left(\dfrac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}\right)\cos(\omega t) \\ + \left(\dfrac{\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}\right)\sin(\omega t)\end{array}\right)\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\left(k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)\right)\cos(\omega t) + \left(k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2\right)\sin(\omega t)}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}\mathrm{d}\omega$$

$$= \frac{k}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\cos\left(\omega t - \left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\sin(\omega t)}{\omega^{2\alpha} + 2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}\mathrm{d}\omega \tag{6.148}$$

S. No.	**The transfer function as a function of complex frequency**	**In integral representation of function in time domain by an inverse Laplace transform using theBerberan-Santos method**
1	$\frac{1}{s+a}$	$\frac{1}{\pi}\int_0^\infty \frac{a\cos(\omega t)+\omega\sin(\omega t)}{\omega^2+a^2}\mathrm{d}\omega$
2	$\frac{s}{s^2+1}$	$\frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega$
3	e^{-sT_d}	$\frac{1}{\pi}\int_0^\infty \cos\big(\omega(t-T_d)\big)\mathrm{d}\omega$
4	1	$\frac{1}{\pi}\int_0^\infty \cos(\omega t)\mathrm{d}\omega$
5	$\frac{1}{s}$	$\frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega$
6	$s^{-\alpha}$	$\frac{1}{\pi}\int_0^\infty \frac{\cos\left(\omega t-\frac{\alpha\pi}{2}\right)}{\omega^\alpha}\mathrm{d}\omega$
7	$\left(1+(1-\beta)\left(\frac{s}{a}\right)\right)^{-\frac{1}{(1-\beta)}}$	$\frac{a}{\pi(1-\beta)}\int_0^\infty \left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{aut-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u;\quad u=\frac{(1-\beta)\omega}{a}$
8	$e^{-(s/a)^\beta}$	$\frac{a}{\pi}\int_0^\infty \left(e^{-u^\beta\cos(\beta\pi/2)}\right)\cos\left(atu-u^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u;\quad u=\frac{\omega}{a}$
9	$\frac{k}{k+s^\alpha}$	$\frac{\sqrt[\alpha]{k}}{\pi}\int_0^\infty \left(\frac{\left(u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u;\quad u=\frac{\omega}{\sqrt[\alpha]{k}}$
10	$\frac{s^\alpha}{s(1+s^\alpha)}$	$\frac{1}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega$
11	$\frac{k}{s(s^\alpha+k)}$	$\frac{k}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega$

Table-6.1: A few examples of Laplace inverted functions using the Berberan-Santos method

6.11.7 The inverse Laplace transform of function $G(s)=(\ln s)^{-1}(s^{b-1}-s^{a-1})$

This type of function appears in fractional differential equations where the fractional order is of a 'continuous distribution function'. We will explain this type of differential equation in Chapter-9. We take $G(s)=\left(\ln s\right)^{-1}\left(s^{b-1}-s^{a-1}\right)$, $b>a>0$. We may think that there is singularity at $s=1$. While performing the L'Hospital rule on $G(s)$, we find $\lim_{s\to 1} G(s)=b-a$; therefore $s=1$ is not a singular point. We also see $\lim_{s\uparrow\infty} G(s)=0$ and $G(s)=\frac{s^b-s^a}{s\ln s}$, as having singularity at $s=0$. Therefore, we put $s=\sigma_0+i\omega$ with $\sigma_0=0$. This provides the following steps:

$$G(i\omega)=\frac{(i\omega)^{b-1}-(i\omega)^{a-1}}{\ln(i\omega)};\quad \ln(i\omega)=\ln\omega+\ln i;\quad i=e^{i\pi/2};\quad \ln(i\omega)=\ln\omega+i\left(\tfrac{\pi}{2}\right)$$
$$=\frac{\omega^{b-1}\left(\cos\left(\frac{(b-1)\pi}{2}\right)+i\sin\left(\frac{(b-1)\pi}{2}\right)\right)-\omega^{a-1}\left(\cos\left(\frac{(a-1)\pi}{2}\right)+i\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\ln\omega+i\left(\frac{\pi}{2}\right)}$$
$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)+i\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\ln\omega+i\left(\frac{\pi}{2}\right)} \tag{6.149}$$
$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)+i\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\left(\ln\omega-i\left(\tfrac{\pi}{2}\right)\right)$$
$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\ln\omega\right)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$
$$-i\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$

This (6.149) gives real and imaginary parts as:

$$\mathrm{Re}\left[G\left(i\omega\right)\right]=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\ln\omega\right)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$
$$\mathrm{Im}\left[G\left(i\omega\right)\right]=-\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)} \tag{6.150}$$

We apply (6.99) to get the following:

$$g(t)=\mathcal{L}^{-1}\left\{\frac{s^b-s^a}{s\ln s}\right\}$$
$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega$$
$$=\frac{1}{\pi}\int_0^\infty\left(\begin{array}{l}\dfrac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\ln\omega\right)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\cos(\omega t)\\ +\dfrac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\sin(\omega t)\end{array}\right)\mathrm{d}\omega \tag{6.151}$$

For this type of function to be obtained, the inverse Laplace transform, the classical contour integration method is demonstrated in Appendix-G. Table-6.1 lists the various integral representations of the inverse Laplace transformed functions obtained by Berberran-Santos method.

6.12 The relaxation-response with the Mittag-Leffler function vis-à-vis the power law function as obtained for fractional differential equation analysis

We have obtained a relaxation function $\sigma(t)$ for a fractional differential equation of the following form (Section-6.8)

$$\begin{aligned} \mathrm{E}\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon(t)}{\mathrm{d}t^{\beta}} &= \sigma(t)+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma(t)}{\mathrm{d}t^{\alpha}} \\ \mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon(t)}{\mathrm{d}t^{\beta}} &= \sigma(t)+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma(t)}{\mathrm{d}t^{\alpha}} \end{aligned} \tag{6.152}$$

when the input function is a constant step $\varepsilon(t)=\varepsilon_0$ for $t\geq 0$. This response is the power series $\sigma(t)=\mathrm{E}_M\varepsilon_0\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k+\alpha-\beta}$, which we found in Section-6.8 of this book. We rewrite this as follows:

$$\sigma(t)=\mathrm{E}_M\varepsilon_0\left(\frac{t}{\tau}\right)^{\alpha-\beta}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k}\right) \tag{6.153}$$

Putting $\alpha=\beta=\frac{1}{2}$ gives:

$$\sigma(x)=\mathrm{E}_M\varepsilon_0\left(\exp(x)\right)\left(\operatorname{erfc}\left(x^{1/2}\right)\right) \qquad x=\tfrac{t}{\tau} \tag{6.154}$$

Where erfc is the 'complementary error function' (refer to Appendix-A for further information). Putting $\alpha=\beta=1$, we obtain a relaxation response of the ordinary integer order differential equation, as follows:

$$\mathrm{E}\tau\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\sigma+\frac{\mathrm{E}\tau}{\mathrm{E}_M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \qquad \mathrm{E}_M\tau\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\sigma+\tau\frac{\mathrm{d}\sigma}{\mathrm{d}t} \tag{6.155}$$

as $\sigma(x)=E_M\varepsilon_0\left(\exp(-x)\right)$, with $x=\frac{t}{\tau}$.

The sum contained in the relaxation function (6.153) is the generalized Mittag-Leffler function (GML), or the two-parameter Mittag-Leffler function (see Appendix-A), which reads as $E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{(x)^k}{\Gamma(\alpha k+\beta)}$ with $\alpha,\beta>0$ and $\alpha,\beta\in\mathbb{R}$, further $E_{\alpha,1}(x)=E_{\alpha}(x)$. Note that the notation of the Mittag-Leffler function uses $E_{(*),(**)}$, whereas E_M and E which denotes constants (not to be confused with the GML notation): while one is in italics, the other is not. For negative x, we have the expressions for the GML function which are $E_{\alpha,\beta}(-x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{\Gamma(\alpha k+\beta)}$ and $E_{\alpha,\beta}(-x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{\alpha k}}{\Gamma(\alpha k+\beta)}$.

Interestingly, for $\alpha=0$, we have $E_{0,\beta}(-x)=\left(\frac{1}{\Gamma(\beta)}\right)\sum_{k=0}^{\infty}(-1)^k x^k=\frac{1}{(\Gamma(\beta))(1+x)}$. With $\beta=1$, we have $E_{0,1}(-x)=E_0(-x)=\frac{1}{1+x}$ (see Appendix-A). Figures 6.3, 6.4, and 6.5 show the plots of $E_{\alpha,\beta}(-x)$ for $\beta=1$ with $\alpha=0.25$ $\alpha=1.75$, $\alpha=2.25$. Thus, for a decaying relaxation function, we restrict $\alpha<2$; as we observe from these figures the function is growing and not decaying for case $\alpha>2$.

$$\begin{aligned} \sigma(t) &= \mathrm{E}_M\varepsilon_0\left(\frac{t}{\tau}\right)^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k} \qquad \left(\frac{t}{\tau}\right)=x \\ &= \mathrm{E}_M\varepsilon_0(x)^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\gamma)}(x)^{\alpha k} \qquad \gamma=\alpha-\beta+1 \\ &= \mathrm{E}_M\varepsilon_0(x)^{\alpha-\beta}E_{\alpha,\gamma}(-x^{\alpha}) \end{aligned} \tag{6.156}$$

The asymptotic expansion for the Mittag-Leffler function for a negative argument at $x\uparrow\infty$ is the following (see Appendix-A):

$$\begin{aligned} E_{\alpha,\alpha}(-x) &\sim \frac{\alpha}{\Gamma(1-\alpha)}x^{-2}, \quad \alpha\neq 1 \qquad E_{\alpha,\gamma}(-x)\sim\frac{1}{\Gamma(\gamma-\alpha)}x^{-1}, \quad \gamma\neq\alpha \\ E_{\alpha,\alpha}(-x^{\alpha}) &\sim \frac{\alpha}{\Gamma(1-\alpha)}x^{-2\alpha}, \quad \alpha\neq 1 \qquad E_{\alpha,\gamma}(-x^{\alpha})\sim\frac{1}{\Gamma(\gamma-\alpha)}x^{-\alpha}, \quad \gamma\neq\alpha \end{aligned} \tag{6.157}$$

With these approximations, we express the asymptotic behavior of the relaxation function for short and long periods of time. The relaxation function is $\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-\beta} E_{\alpha,\gamma}(-x^{\alpha})$ with $\gamma = \alpha - \beta + 1$. For a case $\beta = 1$, (i.e. an order of differentiation of the function $\varepsilon(t)$ is unity and $0 < \alpha \le 1$), we have:

$$\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-1} E_{\alpha,\alpha} = \begin{cases} \mathrm{E}_M \varepsilon_0 \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{as} \quad x \downarrow 0 \\ \mathrm{E}_M \varepsilon_0 \dfrac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} & \text{as} \quad x \uparrow \infty \end{cases} \tag{6.158}$$

For the other case, we have the following:

$$\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-\beta} E_{\alpha,\gamma} = \begin{cases} \mathrm{E}_M \varepsilon_0 \dfrac{x^{\alpha-\beta}}{\Gamma(\gamma)} & \text{as} \quad x \downarrow 0 \\ \mathrm{E}_M \varepsilon_0 \dfrac{1}{\Gamma(1-\beta)} x^{-\beta} & \text{as} \quad x \uparrow \infty \end{cases} \qquad \gamma = \alpha - \beta + 1 \tag{6.159}$$

Figures 6.8 and 6.9 show the plot of $E_{\alpha}(-t^{\alpha})$, $0 \le \alpha \le 1$, $t > 0$ in a linear scale and log scale respectively. For $\alpha = 1$, the curve is a pure exponential curve. For other cases, for large t we get a power law behavior visible from Figure-6.9; after about $t = 1$, we get a constant slope.

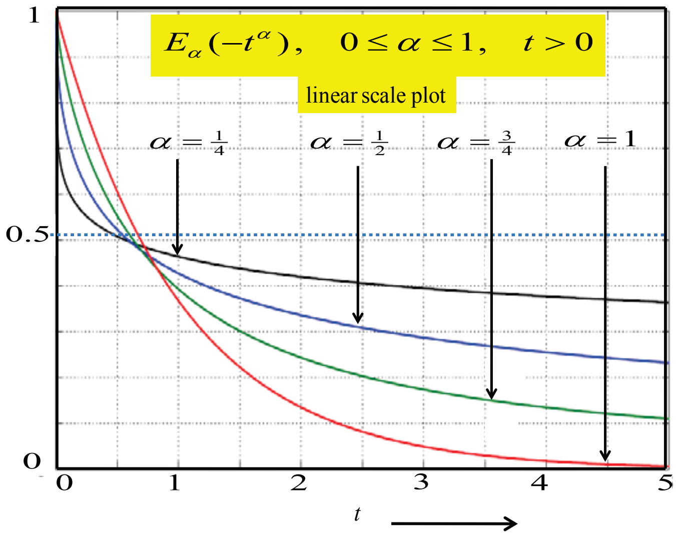


Figure-6.8: Plot of $E_{\alpha}(-t^{\alpha})$, $0 \le \alpha \le 1$, $t > 0$ in linear scale

A very practical observation from Figure-6.8 is inferred that is useful in studies of the relaxation dynamic. The classical relaxation function is proportional to the exponential decay function (i.e. $e^{-t} = E_1(-t)$). When the order of the Mittag-Leffler function is less than one (i.e. $E_{\alpha}(-t^{\alpha})$; $0 < \alpha < 1$), we find that for about the $0 < t < 1$ range (i.e. we call it early times) the decay is steeper (with a faster rate) than the classical exponential decay function. For late times for about $t > 1$, the $E_{\alpha}(-t^{\alpha})$, $0 < \alpha < 1$ falls (decays) at a slower rate compared to the classical exponential function. This means that we observe a 'long-tailed' decaying response in cases with $E_{\alpha}(-t^{\alpha})$, $0 < \alpha < 1$; as compared to $E_1(-t) = e^{-t}$. We point out that the function $E_{\alpha}(-t^{\alpha})$; $0 < \alpha < 1$ is not differentiable at $t = 0$, and at $t < 0$ the variable t^{α} becomes imaginary at some values of α. We will discuss the non-differentiability of this function and extend it to negative values in Chapter-9.

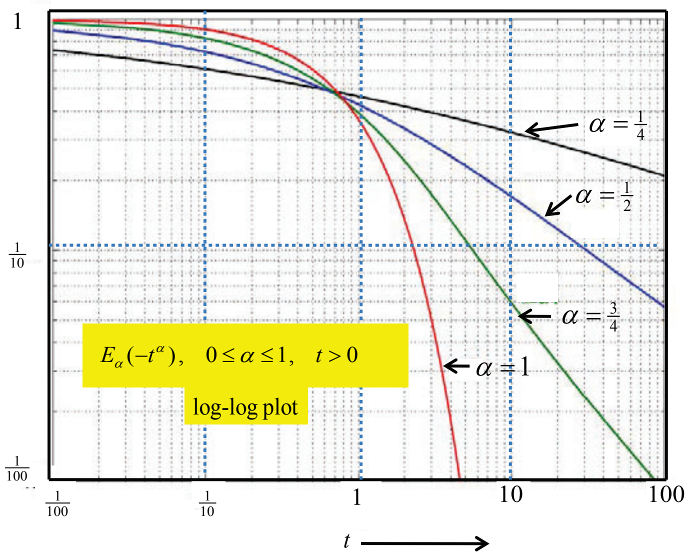


Figure-6.9: Plot of $E_\alpha(-t^\alpha)$, $0 \le \alpha \le 1$, $t > 0$ in log scale

Figure-6.9 says that for late times for about $t > 1$, the log-log plot shows an almost linear function (call it $\log y = -m \log t$), which implies that $y \sim t^{-m}$ which is called the power law decay.

6.13 Use of several fractional order derivatives in order to obtain a generalized fractional differential equation

Another way to generalize the integer order linear differential equation as stated above is via the use of several fractional derivatives operating on an input function (say $\varepsilon(t)$) and output functions (say $\sigma(t)$) as described below:

$$\sigma^{(\alpha_n)}(t) + \sum_{k=1}^{p} a_k D^{\alpha_k} \sigma(t) = \sum_{k=0}^{p} b_k D^{\beta_k} \varepsilon(t) \qquad (6.160)$$
$$a_k, b_k, \alpha_k, \beta_k \in \mathbb{R} \qquad k = 0,1,2,\ldots,p$$

The operators in the expression (6.160) (i.e. $D^\alpha ; D^\beta$) represent an appropriate fractional derivative; such that the frequency of the transformed equation or Laplace transformed equation for the above fractional differential equation model (6.160) is the following:

$$\left(s^{\alpha_n} + \sum_{k=1}^{p} a_k s^{\alpha_k} \right) \hat{\sigma}(s) = \left(\sum_{k=1}^{p} b_k s^{\beta_k} \right) \hat{\varepsilon}(s) \qquad (6.161)$$

The steady state frequency domain equation is achieved by using $s = i\omega$, through which we get:

$$\left((i\omega)^{\alpha_n} + \sum_{k=1}^{p} a_k (i\omega)^{\alpha_k} \right) \hat{\sigma}(i\omega) = \left(\sum_{k=0}^{p} b_k (i\omega)^{\beta_k} \right) \hat{\varepsilon}(i\omega) \qquad (6.162)$$

From the above expressions (6.161) and (6.162) we have reduced the parameter of the fractional differential equation, now comprising four parameters as depicted below:

$$\left(1 + bD^\alpha\right) \sigma(t) = \left(\mathrm{E}_0 + \mathrm{E}_1 D^\alpha\right) \varepsilon(t)$$
$$0 < \alpha < 1; \quad \mathrm{E}_0 \ge 0; \quad \mathrm{E}_1 > 0; \quad b \ge 0; \quad \mathrm{E}_1 \ge b\mathrm{E}_0 \qquad (6.163)$$

It should be noted here that the fractional derivative operator, although it is considered as an RL fractional derivative with a sole consideration, is as follows:

$$\mathcal{F}\left\{D^\alpha f(t)\right\} = (i\omega)^\alpha \hat{f}(i\omega) \qquad \hat{f}(i\omega) = \mathcal{F}\left\{f(t)\right\} \qquad (6.164)$$

With this sole consideration there are several definitions of fractional derivatives which can be of usage, like the Grunwald-Letnikov ${}^{G}_{-\infty}D^{\alpha}_{t,+}$ or the Liouville type forward derivative, the Caputo-Liouville type forward derivatives over an infinite interval $(-\infty, t]$), or the RL type ${}^{RL}_{0}D^{\alpha}_{t,+} f$ if $f(t) = 0$ for $t < 0$. We will continue further in the next chapters to give formal treatment to these fractional differential equations.

6.14 Short summary

In this chapter, we have discussed various tricks, especially the Laplace transform techniques that can be used to solve some fractional differential and integral equations. We observe that these methods are as rigorous as their counterparts in classical calculus. We also showed various ways of obtaining an inverse Laplace transform that we must use to provide a solution to fractional differential equation, and used a method of power series expansion, a contour-integration and a new technique, that is the Berberan-Santos method, to obtain an inverse Laplace transform. We have discussed the power-law function which appears in complex decay for some fractional differential equations and which has applied the Berberan-Santos method to get decay rate distributions of complex decay. In addition, we noted that this method gives us interesting definite integrals representing several functions. We have studied the Mittag-Leffler function as arising out of a complex decay function, and obtained asymptotic behavior and its integral representation. From the integral representation of the Mittag-Leffler function, it is easy to ascertain the values of the function, as compared to its series representation with gamma-functions, which are difficult to get for large values. The numerical integration is a rather simple way of computing the Mittag-Leffler function. In this chapter, we have informally discussed the various aspects of fractional differential equations and their solutions. We will now carry forward our study to further fractional differential equations.

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CHAPTER SEVEN

A FORMAL METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL AND FRACTIONAL INTEGRAL EQUATIONS

7.1 Introduction

In this chapter, we try to formulate a formal development regarding a fractional differential equation (FDE) and a fractional integral equation. We will be using the Laplace transformation technique to find these solutions. We will show the conjugation of the formal method with classical calculus. Here, several higher transcendental functions (like those of Mittag-Leffler, Miller-Ross, and Robotnov-Hartley, etc.) will be used to study various fractional differential and integral equations. We will start our discussion with a first order classical differential equation, and view its counterpart as a fractional differential equation composed by the Caputo derivative. We will then attempt to discern if we can find any similarity and conjugation between the two. We also will discuss the classical methods applied to solve a fractional differential equation composed via the Riemann-Liouville derivative in the Miller-Ross formulation. We will use the concept of a sequential fractional derivative, and will show how the solutions change when the fractional derivatives are either Riemann-Liouville or Caputo type. Here, we will also study a system of fractional differential equations, by introducing a state transition matrix, and explore its usage in the system of fractional differential equations.

7.2 A first order linear differential equation and its fractional generalization using the Caputo derivative

7.2.1 Demonstrating the similarity of a classical differential operator with the Caputo fractional differential operator in a differential equation solution pattern

We take simple examples of fractional differential equations in this section using the Caputo derivative, before looking at the Riemann-Liouville fractional derivative in the next section. These two parts form the basis for further sections of this chapter where we will formalize the observations of these two sections.

The simple dynamic equations with the Caputo derivative ${}_0^C D_t^\alpha \left[f(t)\right] = k$ with k as a constant, and $0 < \alpha < 1$ will offer the solution $f(t) = f_0 + {}_0 I_t^\alpha \left[k\right]$ for $t \geq 0$, where $f(t)\big|_{t=0} = f_0$ is a constant of initialization, and ${}_0 I_t^\alpha$ is a fractional integration operation. We can check this using the property of the Laplace transform (i.e. $\mathcal{L}\left\{{}_0^C D_t^\alpha \left[f(t)\right]\right\} = s^\alpha F(s) - s^{\alpha-1} f(0)$, $0 < \alpha < 1$ (as shown in Section-5.16)) in the following steps:

$$\begin{aligned} & s^\alpha F(s) - s^{\alpha-1} f_0 = \frac{k}{s}; \qquad \mathcal{L}\left\{f(t)\right\} = F(s) \\ & F(s) = \frac{f_0}{s} + \left(\frac{1}{s^\alpha}\right)\left(\frac{k}{s}\right) = \mathcal{L}\left\{f_0\left(u(t)\right)\right\} + \mathcal{L}\left\{I_t^\alpha\left[k\left(u(t)\right)\right]\right\} \end{aligned} \tag{7.1}$$

Where $u(t)$ is a Heaviside unit step function, $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. The Laplace transform of $u(t)$ is $\mathcal{L}\left\{u(t)\right\} = s^{-1}$. By conducting an inverse Laplace transform of (7.1), we get the following result:

$$\begin{aligned} f(t) &= f_0\left(u(t)\right) + {}_0 I_t^\alpha\left[k\right] \\ &= f_0 + \frac{k}{\Gamma(1+\alpha)} t^\alpha; \qquad t \geq 0 \end{aligned} \tag{7.2}$$

For $\alpha = 1$, we get a classical solution as $f(t) = f_0 + kt$.

The simple dynamic equations using the Caputo derivative ${}_0^C D_t^\alpha \left[f(t)\right] = x(t)$ with $x(t)$ as a function, and $0 < \alpha < 1$ will have the solution $f(t) = f_0 + {}_0 I_t^\alpha \left[x(t)\right]$ for $t \geq 0$, where $f(t)\big|_{t=0} = f_0$ is a constant of initialization. We can check this with the Laplace transformation, as we will get the following:

$$s^{\alpha}F(s) - s^{\alpha-1}f_0 = \mathcal{L}\{x(t)\}$$
$$F(s) = \frac{f_0}{s} + \left(\frac{1}{s^{\alpha}}\right)\left(\mathcal{L}\{x(t)\}\right) = \mathcal{L}\{f_0(u(t))\} + \mathcal{L}\{{}_0I_t^{\alpha}[x(t)]\} \tag{7.3}$$

An inverse Laplace transformation of the above (7.3) gives the following expression:

$$f(t) = f_0(u(t)) + {}_0I_t^{\alpha}[x(t)]$$
$$= f_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(x(\tau))\,\mathrm{d}\tau; \qquad t \geq 0 \tag{7.4}$$

For $\alpha = 1$, we get a classical solution as $f(t) = f_0 + \int_0^t (x(\tau))\,\mathrm{d}\tau$. These two examples show a definite similarity and conjugation of a differential equation of classical calculus with a fractional differential equation composed of the Caputo derivative. We will explore this further now.

Why do we specifically use the Caputo derivative operator? It is because, as shown in Section-4.14, fractional integration and the fractional derivative (Caputo type) of order α for the Mittag-Leffler function $E_{\alpha}(\lambda x^{\alpha})$ are the following:

$${}_0I_x^{\alpha}\left[E_{\alpha}(\lambda x^{\alpha})\right] = \tfrac{1}{\lambda}\left(E_{\alpha}(\lambda x^{\alpha}) - 1\right) \qquad {}_0^CD_x^{\alpha}\left[E_{\alpha}(\lambda x^{\alpha})\right] = \lambda E_{\alpha}(\lambda x^{\alpha}) \tag{7.5}$$

There is a similarity with the classical exponential function as ${}_0I_x^1\left[e^{\lambda x}\right] = \tfrac{1}{\lambda}\left(e^{\lambda x} - 1\right)$ and $D_x^1\left[e^{\lambda x}\right] = \lambda e^{\lambda x}$. We note that $E_{\alpha}(\lambda x^{\alpha}) = e^{\lambda x}$ for $\alpha = 1$. Moreover, like the classical case, the Caputo derivative of a constant function is zero (i.e. ${}_0^CD_x^{\alpha}[C] = 0$). With these observations we may state there may exist a conjugation and similarity in the fractional generalization with a Caputo derivative in order to construct a fractional differential equation system with classical differential equations. To use the Caputo derivative in the fractional differential equation, we will assume that the function is differentiable.

7.2.2 The first order differential equation of a relaxation process with an initial condition

Let us say that a process defined by a function $T(t)$ is a differentiable function of the time variable t, which has a rate relationship given by $T^{(1)}(t) = -\lambda(T(t) - T_a)$, where $\lambda > 0$ is a constant. Noting that $T(t)$ is an absolute instantaneous temperature, say for hot water, this law corresponds to the classical Newton's law of cooling, where T_a is room temperature or ambient temperature. Therefore, the rate of change of temperature of hot water is proportional to the difference in temperature. As such, we have the following first order differential equation:

$$\frac{\mathrm{d}\,T(t)}{\mathrm{d}t} = -\lambda(T(t) - T_a) \qquad 0 \leq t < \infty \tag{7.6}$$

We also set an initial condition as $T(0) = T_0$. Recalling our classical calculus, we put the homogeneous part of the above differential equation as $T^{(1)}(t) + \lambda T(t) = 0$. This is given by the solution $T_1(t)$ and a complementary part defined by constant λT_a, which gives us a particular integral component $T_2(t)$. Thus the total solution $T(t)$ is the sum of these two solutions $T_1(t)$ and $T_2(t)$. Adding a function, which is also called the forcing function to the RHS of the homogeneous differential equation, gives a non-homogeneous differential equation. In the above case (7.6), the forcing function, say $x(t)$, is a constant (i.e. $x(t) = \lambda T_a$).

7.2.3 A first order differential equation's homogeneous and complementary parts

We know that through the use of the property of the exponential function, we get a solution for the homogeneous part (i.e. $T^{(1)}(t) = -\lambda T(t)$), which represents the fundamental solution (also called Green's function). That is, $G(t) = e^{-\lambda t}$; and we use this to set the homogeneous solution as $T_1(t) = A(G(t)) = Ae^{-\lambda t}$, with the constant A, which is determined via a given initial condition. The initial condition for our case is, at $t = 0$, $T(t) = T_0$ gives $A = T_0$; therefore $T_1(t) = T_0 e^{-\lambda t}$. Now to get a particular integral, we use the fundamental solution:

$$G(t) = e^{-\lambda t} \tag{7.7}$$

In order to have particular solution, for the forcing function $x(t) = \lambda T_a$, we use the convolution integration (i.e. $T_2(t) = x(t) * G(t) = \int_{-\infty}^{t} \left(x(t-\tau)\right)\left(G(\tau)\right) \mathrm{d}\tau$). We note that the evolution of the given process (i.e. the cooling process) starts at $t = 0$. Thus, we conduct the following convolution integration to get the solution below:

$$\begin{aligned} T_2(t) &= \left(x(t)u(t)\right) * \left(G(t)\right); \qquad x(t) = \lambda T_a \\ &= \int_0^t \left(x(t-\tau)\right)\left(G(\tau)\right) \mathrm{d}\tau = \int_0^t \left(\lambda T_a\right)\left(e^{-\lambda\tau}\right) \mathrm{d}\tau \\ &= T_a\left(1 - e^{-\lambda t}\right) \end{aligned} \tag{7.8}$$

This above convolution is indicated as $\left(x(t)u(t)\right) * \left(G(t)\right)$, where $u(t)$ is a Heaviside step function at $t = 0$ (i.e. $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ otherwise). From these steps in (7.8), we get the total solution as $T(t) = T_1(t) + T_2(t) = T_0 e^{-\lambda t} + T_a\left(1 - e^{-\lambda t}\right)$, that is:

$$T(t) = T_a + \left(T_0 - T_a\right) e^{-\lambda t} \tag{7.9}$$

7.2.4 A fractional order differential equation composed using the Caputo derivative's homogeneous and complementary parts

Now we write the fractional differential equation of the fractional rate of change (using the Caputo fractional derivative), as proportional to the difference in temperature, as:

$${}_0^C D_t^\alpha \left[T(t)\right] = -\lambda\left(T(t) - T_a\right); \qquad 0 < \alpha < 1 \tag{7.10}$$

with the same initial conditions as in the integer order case, i.e. $T(0) = T_0$. With the property of the Mittag-Leffler function and the Caputo derivative, we have seen earlier in (7.5) that ${}_0^C D_t^\alpha \left[E_\alpha(-\lambda t^\alpha)\right] = -\lambda E_\alpha(-\lambda t^\alpha)$. As such, we say that the fundamental solution pertaining to the homogeneous part (i.e. ${}_0^C D_t^\alpha \left[T(t)\right] + \lambda T(t) = 0$) is given by:

$$G_\alpha(t) = E_\alpha(-\lambda t^\alpha) \tag{7.11}$$

and the homogeneous solution is $T_1(t) = T_0\left(E_\alpha(-\lambda t^\alpha)\right)$.

This comes from an initial condition, as $T_1(t) = A\left(E_\alpha(-\lambda t^\alpha)\right) = A\left(1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - ....\right)$. Thus we have at $t = 0$ $T_1(t)\big|_{t=0} = A = T_0$. The particular integral, if we write it in the same way via a convolution integral, then, is:

$$T_2(t) = \left(x(t)u(t)\right) * \left(G_\alpha(t)\right) = \int_0^t \lambda T_a\left(E_\alpha(-\lambda\tau^\alpha)\right) \mathrm{d}\tau \tag{7.12}$$

Therefore, we ask ourselves, can we write the total solution as $T = T_0\left(E_\alpha(-\lambda t^\alpha)\right) + \lambda T_a \int_0^t \left(E_\alpha(-\lambda\tau^\alpha)\right) \mathrm{d}\tau$? Can we just replace the $\exp(-\lambda t)$ obtained in the classical case with that obtained by $E_\alpha(-\lambda t^\alpha)$ to get the total solution?

7.2.5 Can we just replace the $\exp(-\lambda t)$ in the classical case with $E_\alpha(-\lambda t^\alpha)$ in order to get the total solution for the fractional differential equation composed using the Caputo derivative?

We will now verify the above logic of convolution (7.12), by using the Laplace transform. For the classical case, the Laplace transform of a given equation (i.e. $T^{(1)}(t) + \lambda T(t) = \lambda T_a$ for $0 \leq t < \infty$) is:

$$\left(sT(s) - T(0)\right) + \lambda T(s) = \frac{\lambda T_a}{s} \qquad \mathcal{L}\left\{T(t)\right\} = T(s) \tag{7.13}$$

The RHS of $T^{(1)}(t) + \lambda T(t) = \lambda T_a$ is transformed to $\lambda T_a\left(\frac{1}{s}\right)$ in (7.13) due to the use of the Heaviside step function, as we are considering the process from $t = 0$. Before that time, the process is zero. With $T(0) = T_0$, we obtain the following:

$$T(s) = \frac{T_0}{s + \lambda} + \frac{\lambda T_a}{s(s + \lambda)} \tag{7.14}$$

We note from the above expression (7.14) that the first term $T_0\left(\frac{1}{s+\lambda}\right)$ corresponds to a homogeneous solution, that is, an inverse Laplace transform of $\mathcal{L}^{-1}\left\{T_0\left(\frac{1}{s+\lambda}\right)\right\}$, which gives $T_1 = T_0 e^{-\lambda t}$, while the second term of (7.14) (i.e. $\left(\frac{\lambda T_a}{s}\right)\left(\frac{1}{s+\lambda}\right)$) corresponds to a convolution integral; giving a particular integral as $\mathcal{L}^{-1}\left\{\left(\lambda T_a\right)\left(\frac{1}{s}\right)\left(\frac{1}{s+\lambda}\right)\right\} = \left(\lambda T_a u(t)\right) * \left(e^{-\lambda t}\right)$. The $u(t)$ is a Heaviside step function and is $u(t) = 1$ for $t \geq 0$; and $u(t) = 0$, for $t < 0$. Carrying out an inverse Laplace transform of this second term of (7.14), we get $\mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s(s+\lambda)}\right)\right\} = T_a\left(1 - e^{-\lambda t}\right)$.

Now for the fractional differential equation ${}_0^C D_t^\alpha\left[T(t)\right] + \lambda T(t) = \lambda T_a$ the Laplace transformed equation using the Laplace transform for the Caputo derivative (as in Section-5.16) i.e. $\mathcal{L}\left\{{}_0^C D_t^\alpha\left[f(t)\right]\right\} = s^\alpha F(s) - s^{\alpha-1} f(0), \quad 0 < \alpha < 1$)) is the following:

$$\left(s^\alpha T(s) - s^{\alpha-1} T(0)\right) + \lambda T(s) = \frac{\lambda T_a}{s} \tag{7.15}$$

By using the initial condition $T(0) = T_0$, we obtain the following:

$$T(s) = \frac{s^{\alpha-1} T_0}{s^\alpha + \lambda} + \frac{\lambda T_a}{s(s^\alpha + \lambda)}; \qquad 0 < \alpha < 1 \tag{7.16}$$

For the first term in (7.16), we have $\mathcal{L}\left\{E_\alpha(at^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - a}$, i.e. from the Laplace transform of the Mittag-Leffler function (see Appendix-A).Thus the homogeneous solution is $T_1(t) = T_0\left(E_\alpha(-\lambda t^\alpha\right)$, that is, exactly $T_1(t) = T_0 G_\alpha(t)$, where we have $G_\alpha(t) = E_\alpha(-\lambda t^\alpha)$. The second term of (7.16) pertains to a particular integral, and we have derived from Section-6.4 of this book (via the concept of series expansion), the inverse Laplace transform, which is $\mathcal{L}^{-1}\left(\frac{k}{s(s^\alpha + k)}\right) = 1 - E_\alpha(-kt^\alpha)$; it gives $T_2(t) = T_\alpha\left(1 - E_\alpha(-\lambda t^\alpha)\right)$. This gives us the total solution as follows:

$$T(t) = T_a + \left(T_0 - T_a\right) E_\alpha(-\lambda t^\alpha) \tag{7.17}$$

The particular integral in this case is $T_2(t) = \mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s}\right)\left(\frac{1}{s^\alpha + \lambda}\right)\right\} = \int_{-\infty}^{\infty}\left(\lambda T_a u(t)\right)\left(F_\alpha(-\lambda, \tau\right) d\tau$. This is not as we have mentioned earlier (7.12)! Here, the Robotnov-Hartley function $F_\alpha(a,t) = t^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n t^{\alpha n}}{\Gamma(\alpha n + \alpha)}$ with the Laplace transform as $\mathcal{L}\left\{F_\alpha(a,t)\right\} = \frac{1}{s^\alpha - a}$ appears instead of $G_\alpha(t) = E_\alpha(-\lambda t^\alpha)$.

Thus, the particular integral is not the convolution integral that we wrote earlier (7.12), i.e. $T_2(t) \neq \int_{-\infty}^{t}\left(x(t-\tau)\right)\left(G_\alpha(\tau)\right) d\tau$. Let us now manipulate $T_2(s) = \lambda T_a\left(\frac{1}{s(s^\alpha + \lambda)}\right)$ in the following way

$$\begin{aligned} T_2(t) &= \mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s}\right)\left(\frac{1}{s^\alpha + \lambda}\right)\right\} \\ &= \mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s}\right)\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right)\left(\frac{1}{s^{\alpha-1}}\right)\right\} \\ &= \mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s^\alpha}\right)\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right)\right\} = {}_0I_t^\alpha\left[\left(\lambda T_a\right)\left(E_\alpha(-\lambda t^\alpha)\right)\right] \end{aligned} \tag{7.18}$$

We have manipulated $T_2(s) = s^{-\alpha}\left(\lambda T_a\right)\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right) = s^{-\alpha}\left(\left(\mathcal{L}\left\{\lambda T_a \delta(t)\right\}\right)\left(\mathcal{L}\left\{E_\alpha(-\lambda t^\alpha)\right\}\right)\right)$. We note that we used the function $\delta(t)$, as $\mathcal{L}\left\{\delta(t)\right\} = 1$. In the Laplace (frequency) domain multiplication is the time domain convolution, and, with $s^{-\alpha}$ representing a fractional integration operation, we can write the following inverse Laplace transform for:

$$T_2(s) = s^{-\alpha}\left(\lambda T_a\right)\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+\lambda}\right) = s^{-\alpha}\left(\left(\mathcal{L}\left\{\lambda T_a\delta(t)\right\}\right)\left(\mathcal{L}\left\{E_\alpha(-\lambda t^{\alpha})\right\}\right)\right) \tag{7.19}$$

as follows:

$$\begin{aligned} T_2(t) &= {}_0I_t^{\alpha}\left[\left(\lambda T_a\delta(t)\right)*\left(E_\alpha(-\lambda t^{\alpha}\right)\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\left(\lambda T_a\delta(\tau)\right)*\left(E_\alpha(-\lambda\tau^{\alpha}\right)\right)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_{-\infty}^{\tau}\lambda T_a\left(\delta(\tau-\tau_1)\right)\left(E_\alpha(-\lambda\tau_1^{\alpha})\right)\mathrm{d}\tau_1\right)\mathrm{d}\tau \end{aligned} \tag{7.20}$$

In the above expression (7.20), we have used RL fractional integration (i.e. ${}_0I_t^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau$). Then, with the property of the delta function (i.e. $\int_{-\infty}^{\infty}\left(\delta(x_0-x)\right)f(x)\mathrm{d}x = f(x_0)$), we write the integral in the above expression as $\int_{-\infty}^{\tau}\lambda T_a\left(\delta(\tau-\tau_1)\right)\left(E_\alpha(-\lambda\tau_1^{\alpha})\right)\mathrm{d}\tau_1 = \lambda T_a\left(E_\alpha(-\lambda\tau^{\alpha})\right)$. Therefore, the particular integral in the above case is as follows:

$$\begin{aligned} T_2(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_{-\infty}^{\tau}\lambda T_a\left(\delta(\tau-\tau_1)\right)\left(E_\alpha(-\lambda\tau_1^{\alpha})\right)\mathrm{d}\tau_1\right)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda T_a E_\alpha(-\lambda\tau^{\alpha})\right)\mathrm{d}\tau \\ &= {}_0I_t^{\alpha}\left[\left(\lambda T_a\right)E_\alpha(-\lambda t^{\alpha})\right] \end{aligned} \tag{7.21}$$

We note that the fractional integration operator appearing above is due to the operator $s^{-\alpha}$ and through recognizing ${}_0I_t^{\alpha}\left[E_\alpha(at^{\alpha})\right] = a^{-1}\left(E_\alpha(at^{\alpha})-1\right)$, we get the particular integral as:

$$\begin{aligned} T_2(t) &= {}_0I_t^{\alpha}\left[\left(\lambda T_a\right)\left(E_\alpha(-\lambda t^{\alpha})\right)\right] \\ &= \left(\lambda T_a\right)\left(\frac{E_\alpha(-\lambda t^{\alpha})-1}{(-\lambda)}\right) = T_a\left(1-E_\alpha(-\lambda t^{\alpha})\right) \end{aligned} \tag{7.22}$$

This is exactly what we have obtained earlier via the Laplace transform method (7.16). Thus, for a differential equation using the Caputo derivative:

$${}_0^CD_t^{\alpha}\left[T(t)\right] + \lambda T(t) = \lambda T_a; \qquad 0<\alpha<1; \qquad T(0)=T_0$$

We have a fundamental solution as $G_\alpha(t) = E_\alpha(-\lambda t^{\alpha})$, i.e. the homogeneous solution known as $T_1(t) = T_0G_\alpha(t)$, and the particular integral known as $T_2 = {}_0I_t^{\alpha}\left[\left(\lambda T_a\right)\left(E_\alpha(-\lambda\tau^{\alpha})\right)\right]$. The total solution is as follows:

$$\begin{aligned} T(t) &= T_0\left(E_\alpha(-\lambda t^{\alpha})\right) + {}_0I_t^{\alpha}\left[\left(\lambda T_a\right)\left(E_\alpha(-\lambda t^{\alpha})\right)\right] \\ &= T_a + \left(T_0 - T_a\right)\left(E_\alpha(-\lambda t^{\alpha})\right) \end{aligned} \tag{7.23}$$

7.2.6 Getting a particular solution to a fractional differential equation composed by the Caputo derivative through a fractional integration operation using a classical convolution process

Thus, in the case of a fractional differential equation composed via the Caputo fractional derivative, our classical convolution integral for obtaining a particular solution is changed to a fractional integral, and we write:

$$\begin{aligned} \left(f(t)*g(t)\right)_{(\alpha)} &= {}_{-\infty}I_t^{\alpha}\left[\left(f(t)*g(t)\right)\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-\tau)^{\alpha-1}\left(\int_{-\infty}^{\tau}\left(f(\tau-\tau_1)\right)\left(g(\tau_1)\right)\mathrm{d}\tau_1\right)\mathrm{d}\tau \end{aligned} \tag{7.24}$$

This is a fractional order convolution, the concept of which is still developing; it is based on the above discussion, and the fractional integration of a classical convolution operation is considered to be a fractional convolution. We will not use this concept in the subsequent sections of this chapter.

7.2.7 A homogeneous fractional differential equation composed using a Caputo derivative is driven by a fractional RL derivative of a Heaviside unit step function

The fundamental solution $G(t)=e^{-\lambda t}$ is obtained by an indicial polynomial corresponding to the differential equation $D^{(1)}\left[T(t)\right]+\lambda T(t)=0$, which is $P(x)=x+\lambda$. We call $\left(P(x)\right)T=0$ a homogeneous system, where x corresponds to operator D_t^1. The inverse Laplace transform of this indicial polynomial is $\mathcal{L}^{-1}\left\{\left(P(s)\right)^{-1}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s+\lambda}\right\}=e^{-\lambda t}$. We can say that this is a solution for a fundamental equation, which is as follows:

$$\begin{aligned}&\frac{\mathrm{d}\,T(t)}{\mathrm{d}t}+\lambda T(t)=\delta(t); \qquad T(s)=\frac{1}{s+\lambda}\\ &T(t)=\mathcal{L}^{-1}\left\{(s+\lambda)^{-1}\right\}=e^{-\lambda t}\end{aligned} \tag{7.25}$$

That is the homogeneous differential equation, but it is driven by forcing the function $\delta(t)$, noting that $\mathcal{L}\left\{\delta(t)\right\}=1$.

Now for ${}_0^CD_t^\alpha\left[T(t)\right]+\lambda T(t)=0$, we have an indicial polynomial such as $P(x^\alpha)=x^\alpha+\lambda$. Performing an inverse Laplace transform of this $\left(P(s^\alpha)\right)^{-1}$, we get $\mathcal{L}^{-1}\left\{\frac{1}{P(s^\alpha)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha+\lambda}\right\}=F_\alpha(-\lambda,t)$. Here we have the Robotnov-Hartley function (i.e. $F_\alpha(-\lambda,t)$) and not the Mittag-Leffler function (i.e. $G_\alpha(t)=E_\alpha(-\lambda t^\alpha)$), which we used in our discussion above. Therefore, we cannot use the equation ${}_0^CD_t^\alpha\left[T(t)\right]+\lambda T(t)=\delta(t)$ (i.e. with the Delta function as a forcing function to obtain a fundamental solution), which will give $T(s)=\frac{1}{s^\alpha+\lambda}$ and will result in the Robotnov-Hartley function.

Say we write the following homogeneous equation driven by a power function called $\delta_\alpha(t)=\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$:

$$\begin{aligned}&{}_0^CD_t^\alpha\left[T(t)\right]+\lambda T(t)=\delta_\alpha(t)\\ &{}_0^CD_t^\alpha\left[T(t)\right]+\lambda T(t)=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}; \qquad 0<\alpha<1\end{aligned} \tag{7.26}$$

The Laplace transform of the above equation (7.26) is $s^\alpha T(s)+\lambda T(s)=s^{\alpha-1}$, obtained by using $\mathcal{L}\left\{t^n\right\}=\left(n!\right)s^{-n-1}$ with $n=-\alpha$ and using $(-\alpha)!=\Gamma\left(-\alpha+1\right)$. We get here $T(s)=s^{\alpha-1}\left(s^\alpha+\lambda\right)^{-1}$, an inverse Laplace transform of which $G_\alpha(t)=E_\alpha(-\lambda t^\alpha)$. As such, we obtained a fundamental solution through the modification of using, for example, $\delta_\alpha(t)=\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$ - a new fractional order delta- function!

In Section-4.11 of this book, we showed that a fractional derivative (the Riemann-Liouville – RL type) of the Heaviside step function is ${}_0D_t^\alpha\left[u(t)\right]=\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$. In addition, we also found that one whole derivative of the Heaviside unit step function is a Dirac Delta function $D^1\left[u(t)\right]=\delta(t)$. Therefore, we have conceptualized the generalized rule to find the fundamental solution of the following homogeneous equation composed via the Caputo derivative:

$$\begin{aligned}&{}_0^CD_t^\alpha\left[T(t)\right]+\lambda T(t)={}_0D_t^\alpha\left[u(t)\right]=\delta_\alpha(t) \qquad 0<\alpha<1 \qquad T(0)=T_0\\ &\alpha=1\\ &D_t^1\left[T(t)\right]+\lambda T(t)=D_t^1\left[u(t)\right]=\delta(t)\end{aligned} \tag{7.27}$$

The solution to the homogeneous differential equation driven by the delta function is also called Green's function. Therefore, after obtaining Green's function, with the help of the initial condition, one gets a solution to the homogeneous equation, and, with Green's function, through carrying out a convolution one gets a particular integral. The sum of these two solutions is then the total solution. What we have observed is that a convolution integral is replaced by fractional integration, and the delta function, which is a derivative of the Heaviside step function, is replaced by a fractional derivative of the Heaviside step function, which we called a fractional delta function. This is also the case for a fractional differential equation using the Caputo derivative. The Riemann-Liouville case is different, however, and we will discuss this in subsequent sections.

7.3 The first order linear differential equation and its fractional generalization using the Riemann-Liouville (RL) derivative

7.3.1 Demonstrating the similarity of a classical differential operator with the Riemann-Liouville fractional differential operator in a differential equation solution

Continuing on from the previous section, we now change the fractional differential equation with the Riemann-Liouville fractional derivative. The simple dynamic equations with the Riemann-Liouville derivative ${}_0D_t^{\alpha}\left[f(t)\right]=k$ with k as a constant, and $0<\alpha<1$ will have the solution $f(t)=C\left({}_0I_t^{\alpha}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[k\right]$ for $t\ge 0$; where $f^{(\alpha-1)}(t)\Big|_{t=0}=C$ is a constant of initialization, that is, the value of fractional integration of $f(t)$, of order $1-\alpha$, at the initial point.

We can check this with the Laplace transform, as we will see via the use of Laplace transform property for the RL fractional derivative (i.e. $\mathcal{L}\left\{{}_0D_t^{\alpha}\left[f(t)\right]\right\}=s^{\alpha}F(s)-f^{(\alpha-1)}(t)\Big|_{t=0}$, $0<\alpha<1$ (as shown in Section-5.16)), which yields the following steps:

$$
\begin{aligned}
s^{\alpha}F(s)-C&=\frac{k}{s}\\
F(s)&=\frac{C}{s^{\alpha}}+\left(\frac{1}{s^{\alpha}}\right)\left(\frac{k}{s}\right)\\
&=\mathcal{L}\left\{{}_0I_t^{\alpha}\left[C\left(\delta(t)\right)\right]\right\}+\mathcal{L}\left\{{}_0I_t^{\alpha}\left[k\left(u(t)\right)\right]\right\}
\end{aligned}
\tag{7.28}
$$

Where the function $u(t)$ is the Heaviside unit step function (i.e. $u(t)=1$ for $t\ge 0$ and $u(t)=0$ for $t<0$), $\delta(t)$ is the Dirac-delta function. Performing an inverse Laplace transform on the above (7.28), we get:

$$
\begin{aligned}
f(t)&=C\left({}_0I_t^{\alpha}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[k\right];\qquad C=f^{(\alpha-1)}(t)\Big|_{t=0}\\
&=C\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{k}{\Gamma(1+\alpha)}t^{\alpha};\qquad t\ge 0
\end{aligned}
\tag{7.29}
$$

We have used the formula ${}_{-\infty}D_x^{\upsilon}\left[\delta(x-c)\right]=\frac{1}{\Gamma(-\upsilon)}(x-c)^{-\upsilon-1}$ by setting $\upsilon=-\alpha$ and $c=0$, which were described in Section-4.11. In order to verify the initial condition, we take a fractional integration of order $1-\alpha$ of the obtained function, i.e.

$$
\begin{aligned}
f^{(\alpha-1)}(t)&={}_0I_t^{1-\alpha}\left(C\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{k}{\Gamma(1+\alpha)}t^{\alpha}\right)\\
&=\left(\frac{C}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(\alpha-1+1)}{\Gamma(\alpha+1-\alpha)}\right)t^{\alpha-1+1-\alpha}+\left(\frac{k}{\Gamma(1+\alpha)}\right)\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+1-\alpha)}\right)t^{\alpha+1-\alpha}\\
&=C+kt
\end{aligned}
\tag{7.30}
$$

From the above (7.30), we find that $f^{(\alpha-1)}(t)\Big|_{t=0}=C$; that is, we have regained our initial condition. For the classical case, we have the solution as described in the previous section (i.e. $f(t)=f_0\left(u(t)\right)+{}_0I_t^{1}\left[k\right]$ for $t\ge 0$ and $f_0=f(0)$).

We may also write that through recognizing $u(t)=\int_{-\infty}^{t}\left(\delta(\tau)\right)\mathrm{d}\tau$ (i.e. the Heaviside step function) as an integration of the Delta function, we obtain the following modified expression:

$$
\begin{aligned}
f(t)&=f_0\left(I_t^{1}\left[\delta(t)\right]\right)+{}_0I_t^{1}\left[k\right]\qquad t\ge 0\qquad f_0=f(0)\\
&=f_0+{}_0I_t^{1}\left[k\right]=f_0+kt\qquad t\ge 0
\end{aligned}
\tag{7.31}
$$

For the Caputo case, we have the solution as:

$$
\begin{aligned}
f(t)&=f_0\left(u(t)\right)+{}_0I_t^{\alpha}\left[k\right];\qquad f_0=f(0)\\
&=f_0\left(I_t^{1}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[k\right]=f_0+\frac{k}{\Gamma(\alpha+1)}t^{\alpha}\qquad t\ge 0
\end{aligned}
\tag{7.32}
$$

For the RL case, we have the solution as:

$$f(t)=C\left({}_0I_t^{\alpha}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[k\right];\qquad C=f^{(\alpha-1)}(t)\Big|_{t=0}$$
$$=C\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{k}{\Gamma(1+\alpha)}t^{\alpha};\qquad t\geq 0 \tag{7.33}$$

We have seen similarities in the solutions of fractional differential equations composed via the Caputo derivative, the RL derivative and the classical one-order whole derivative, for simple cases. Note that the initial condition in the case of RL is $C=f^{(\alpha-1)}(t)\Big|_{t=0}$, and for the Caputo and classical cases, it is $f_0=f(t)\Big|_{t=0}$.

The simple dynamic equations with the RL derivative ${}_0D_t^{\alpha}\left[f(t)\right]=x(t)$ with $x(t)$ as a function, and $0<\alpha<1$ will have the solution $f(t)=C\left({}_0I_t^{\alpha}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[x(t)\right]$ for $t\geq 0$; where $f^{(\alpha-1)}(t)\Big|_{t=0}=C$ is a constant of initialization. We can check this with the Laplace transform, as we will get:

$$s^{\alpha}F(s)-f^{(\alpha-1)}(t)\Big|_{t=0}=\mathcal{L}\left\{x(t)\right\}$$
$$F(s)=\frac{C}{s^{\alpha}}+\left(\frac{1}{s^{\alpha}}\right)\left(\mathcal{L}\left\{x(t)\right\}\right) \tag{7.34}$$
$$=\mathcal{L}\left\{{}_0I_t^{\alpha}\left[C\delta(t)\right]\right\}+\mathcal{L}\left\{{}_0I_t^{\alpha}\left[x(t)\right]\right\}$$

An inverse Laplace transform of the above expression (7.34) gives the following:

$$f(t)=C\left({}_0I_t^{\alpha}\left[\delta(t)\right]\right)+{}_0I_t^{\alpha}\left[x(t)\right]$$
$$=C\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left(x(\tau)\right)\mathrm{d}\tau;\qquad t\geq 0 \tag{7.35}$$

We find a similarity between the classical case and the Caputo cases here too.

7.3.2 A fractional order differential equation composed with the Riemann-Liouville derivative is fundamental and a particular solution is offered by a given fractional order initial state

We have taken the first order classical system in the previous section as follows:

$$D_t^1\left[T(t)\right]=-\lambda\left(T(t)-T_a\right)\qquad 0\leq t<\infty$$
$$\left(\frac{\mathrm{d}\ T(t)}{\mathrm{d}t}-T_0\delta(t)\right)+\lambda T(t)=\lambda T_a \tag{7.36}$$
$$s\left(T(s)-T_0\right)+\lambda T(s)=\frac{\lambda T_a}{s}$$

With $T(t)\Big|_{t=0}=T_0\delta(t)$, we have solved the system. The case $T(t)\Big|_{t=0}=\delta(t)$ gives us a fundamental solution or Green's function (i.e. $G(t)=e^{-\lambda t}$). Now we rewrite the equation above this time using the RL fractional derivative, i.e.:

$${}_0D_t^{\alpha}\left[T(t)\right]=-\lambda\left(T(t)-T_a\right),\qquad 0\leq t<\infty,\quad 0<\alpha<1 \tag{7.37}$$

The Laplace transform of the RL derivative operator is $\mathcal{L}\left\{{}_0D_t^{\alpha}f(x)\right\}=s^{\alpha}F(s)-f^{(\alpha-1)}(t)\Big|_{t=0}$ (as shown in Section-5.16). This gives the following:

$$\left(s^{\alpha}T(s)-T^{(\alpha-1)}(t)\Big|_{t=0}\right)+\lambda T(s)=\frac{\lambda T_a}{s} \tag{7.38}$$

We assume that the fractional initial state is $T^{(\alpha-1)}(t)\Big|_{t=0}=\left(\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^t(t-\tau)^{\alpha-1}T(\tau)\mathrm{d}\tau\right)\Bigg|_{t=0}=C_0$; that is the fractional integration of function $T(t)$ at $t=0$ is a constant, namely C_0. With $C_0=1$, we attain the homogeneous part as:

$$\left(s^{\alpha}T(s)-1\right)+\lambda T(s)=0 \tag{7.39}$$

This means that we have the following homogeneous differential equation:

$$\left(\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}-\delta(t)\right)+\lambda T(t)=0 \qquad \left(\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}\right)+\lambda T(t)=\delta(t) \tag{7.40}$$

From the above (7.40), we have $\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}+\lambda}\right\}=G_{\alpha}(t)$ as the fundamental solution in this case or using Green's function, which is similar to the Robotnov-Hartley function $F_{\alpha}(-\lambda,t)$. We also know that $F_{\alpha}(-\lambda,t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, that is, in terms of a 'two-parameter' Mittag-Leffler function (see Appendix-A). The verification is that the function $f(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$ is the eigen-function for an RL fractional derivative operator for ${}_0D_t^{\alpha}\left[f(t)\right]=-\lambda f(t)$ that we described in an earlier chapter (Section-4.14).

We note that for $\alpha=1$ we have $f(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})=e^{-\lambda t}$. Therefore, in this case, the Green's function is a solution of a homogeneous system that is driven by the delta function. The full solution comes from the following steps:

$$\begin{aligned}&\left(s^{\alpha}T(s)-T^{(\alpha-1)}(t)\Big|_{t=0}\right)+\lambda T(s)=\frac{\lambda T_a}{s}\\&\left(s^{\alpha}T(s)-C_0\right)+\lambda T(s)=\frac{\lambda T_a}{s}\\&T(s)=C_0\left(\frac{1}{s^{\alpha}+\lambda}\right)+\lambda T_a\left(\frac{1}{s}\right)\left(\frac{1}{s^{\alpha}+\lambda}\right)\end{aligned} \tag{7.41}$$

While writing an inverse Laplace transform of the above expression, we get:

$$\begin{aligned}T(t)&=C_0\left(F_{\alpha}(-\lambda,t)\right)+\left(\lambda T_a u(t)\right)*\left(F_{\alpha}(-\lambda,t)\right)\\&=C_0\left(F_{\alpha}(-\lambda,t)\right)+\int_{-\infty}^{\infty}\left(\lambda T_a u(t)\right)\left(F_{\alpha}(-\lambda,\tau\right)\mathrm{d}\tau\end{aligned} \tag{7.42}$$

We have noted in the previous section that $\mathcal{L}^{-1}\left\{\lambda T_a\left(\frac{1}{s}\right)\left(\frac{1}{s^{\alpha}+\lambda}\right)\right\}=T_a\left(1-E_{\alpha}(-\lambda t^{\alpha})\right)$ and using $F_{\alpha}(-\lambda,t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, we write the following expression:

$$T(t)=C_0\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\right)+T_a\left(1-E_{\alpha}(-\lambda t^{\alpha})\right) \tag{7.43}$$

7.3.3 Verification of a fractional initial state from the obtained solution for a fractional order differential equation composed using the Riemann-Liouville derivative

For the system ${}_0D_t^{\alpha}\left[T(t)\right]+\lambda T(t)=\lambda T_a$, we have a homogeneous solution as $T_1(t)=C_0\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\right)$, which, in terms of Green's function, is $T_1(t)=AG_{\alpha}(t)$ where A is a constant depending on the fractional initial state, given as $T^{(\alpha-1)}(t)\Big|_{t=0}=C_0$. Let us verify this:

$$\begin{aligned}T_1(t)&=AF_{\alpha}(-\lambda,t)=At^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\\&=A\left(t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k t^{k\alpha}}{\Gamma\left((k+1)\alpha\right)}\right)=A\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}-\frac{\lambda t^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{\lambda^2t^{3\alpha-1}}{\Gamma(3\alpha)}-...\right)\end{aligned} \tag{7.44}$$

From above expression, we take $T^{(\alpha-1)}(t)$ by operating $D_t^{\alpha-1}$ term by term as in the following steps:

$$
\begin{aligned}
T_1^{(\alpha-1)}(t) &= {}_0D_t^{\alpha-1}\left(At^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k t^{k\alpha}}{\Gamma\left((k+1)\alpha\right)}\right) \\
&= {}_0D_t^{\alpha-1}\left(\frac{At^{\alpha-1}}{\Gamma(\alpha)}-\frac{\lambda At^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{\lambda^2 At^{3\alpha-1}}{\Gamma(3\alpha)}-...\right) \\
&= \frac{A\Gamma(\alpha)t^{\alpha-1-\alpha+1}}{\Gamma(\alpha)\Gamma(\alpha-\alpha+1)}-\frac{A\lambda\Gamma(2\alpha)t^{2\alpha-1-\alpha+1}}{\Gamma(2\alpha)\Gamma(2\alpha-\alpha+1)}+\frac{A\lambda^2\Gamma(3\alpha)t^{3\alpha-1-\alpha+1}}{\Gamma(3\alpha)\Gamma(3\alpha-\alpha+1)}+... \\
&= A\left(1-\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)}+....\right)=A\left(\sum_{k=0}^{\infty}\frac{(-\lambda)^k t^{k\alpha}}{\Gamma(k\alpha+1)}\right) \\
&= AE_{\alpha}(-\lambda t^{\alpha})
\end{aligned}
\tag{7.45}
$$

Here we see that $T^{(\alpha-1)}(t)\Big|_{t=0}=A$; thus, we get $A=C_0$.

7.3.4 Modification in a solution for a fractional differential equation composed using the Riemann-Liouville derivative solution with classical integer order initial states instead of fractional order initial states

What is the value of A when, instead of a fractional order initial state (i.e. $T^{(\alpha-1)}(t)\Big|_{t=0}=C_0$), the normal condition as in the case of a Caputo derivative, or a classical case, is given (i.e. $T(t)\Big|_{t=0}=T_0$).

In this case, with the RL fractional derivative, we should have the condition $T_1(t)\Big|_{t=0}=AG_{\alpha}(t)\Big|_{t=0}=T_0$. We note that $G_{\alpha}(t)\Big|_{t=0}=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\Big|_{t=0}=\infty$. Thus, we have difficulty in writing $A=T_0\left(G_{\alpha}^{-1}(t)\right)\Big|_{t=0}$.

Let us say that $G_{\alpha}^{-1}(t)=t^{1-\alpha}F(t)$ represents the inverse function of $G_{\alpha}(t)$. Therefore, we have an identity condition $\left(G_{\alpha}(t)\right)\left(G_{\alpha}^{-1}(t)\right)=1$ (i.e. $G_{\alpha}(t)\left(t^{1-\alpha}F(t)\right)=1$). This gives us the following steps, knowing that $G_{\alpha}(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$:

$$
\begin{aligned}
\left(G_{\alpha}(t)\right)\left(G_{\alpha}^{-1}(t)\right) &= \left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\right)\left(t^{1-\alpha}F(t)\right) \\
&= \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}-\lambda\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}+\lambda^2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}-......\right)\left(t^{1-\alpha}F(t)\right) \\
&= \frac{F(t)}{\Gamma(\alpha)}-\lambda F(t)\frac{t^{\alpha}}{\Gamma(2\alpha)}+\lambda^2 F(t)\frac{t^{2\alpha}}{\Gamma(3\alpha)}-....
\end{aligned}
\tag{7.46}
$$

From the above (7.46), we write that, for $\lim_{t\downarrow 0}\left(\left(G_{\alpha}(t)\right)\left(G_{\alpha}^{-1}(t)\right)\right)=1$, we should have $\lim_{t\downarrow 0}F(t)=\Gamma(\alpha)$. Thus; we get the inverse of $G_{\alpha}(t)$ in limit $t\downarrow 0$ as $G_{\alpha}^{-1}(t)\Big|_{t=0}=\left(t^{1-\alpha}\Gamma(\alpha)\right)$. Therefore, we get $A=T_0\left(t^{1-\alpha}\Gamma(\alpha)\right)$.

The homogeneous solution is:

$$
\begin{aligned}
T_1(t) = AG_{\alpha}(t) &= \left(T_0 t^{1-\alpha}\Gamma(\alpha)\right)\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\right) \\
&= T_0\left(\Gamma(\alpha)\right)\left(E_{\alpha,\alpha}(-\lambda t^{\alpha})\right)
\end{aligned}
\tag{7.47}
$$

In series form, we obtain the homogeneous solution as:

$$
\begin{aligned}
T_1(t) &= T_0\left(\Gamma(\alpha)\right)\left(E_{\alpha,\alpha}(-\lambda t^{\alpha})\right) \\
&= T_0\left(1-\lambda\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{\alpha}+\lambda^2\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}t^{2\alpha}-.....\right)
\end{aligned}
\tag{7.48}
$$

The particular integral in this case is a convolution for forcing the function with Green's function, i.e. $T_2 = \left(\lambda T_a u(t)\right) * \left(F_\alpha(-\lambda, t)\right) = \left(\lambda T_a u(t)\right) * \left(G_\alpha(t)\right)$. Thus, we write a fractional differential equation with an RL derivative and with a fractional order initial state (i.e. $T^{(\alpha-1)}(0) = C_0$) as follows:

$$ {}_0D_t^\alpha \left[T(t)\right] + \lambda T(t) = x(t) \qquad 0 \le t < \infty, \quad 0 < \alpha < 1 \qquad T^{(\alpha-1)}(t)\Big|_{t=0} = C_0 \tag{7.49}$$

and its solution as depicted (7.49) is $G_\alpha(t) = F_\alpha(-\lambda, t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$

$$T(t) = C_0\left(G_\alpha(t)\right) + \left(x(t)u(t)\right) * \left(G_\alpha(t)\right) \tag{7.50}$$

We write a fractional differential equation with an RL fractional derivative with an initial condition as an integer order initial state (i.e. $T(0) = T_0$) using the following system:

$$ {}_0D_t^\alpha \left[T(t)\right] + \lambda T(t) = x(t), \qquad 0 \le t < \infty, \quad 0 < \alpha < 1 \qquad T(t)\Big|_{t=0} = T_0 \tag{7.51}$$

The solution of (7.51) is:

$$T(t) = T_0\left(t^{1-\alpha}\Gamma(\alpha)\right)\left(G_\alpha(t)\right) + \left(x(t)u(t)\right) * \left(G_\alpha(t)\right) \tag{7.52}$$

7.3.5 The requirement of one Green's function for a fractional order differential equation composed using the Riemann-Liouville derivative and two Green's functions for the Caputo derivative

We see that for the Caputo derivative, it is based on a non-homogeneous fractional differential equation, say ${}_0^C D_t^\alpha \left[y(t)\right] + \lambda y(t) = x(t)$, with the initial condition $y(0) = y_0$, and, therefore, we obtain the fundamental solution (or Green's function) for its homogeneous part driven by ${}_0^C D_t^\alpha \left[u(t)\right] = \delta_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$. Here the Green's function is ${}^C G_\alpha(t) = E_\alpha(-\lambda t^\alpha)$.

The homogeneous solution is $y_1(t) = A\left({}^C G_\alpha(t)\right) = y_0 E_\alpha(-\lambda t^\alpha)$, and the particular integral is $y_2(t) = x(t) * \left(G_\alpha(t)\right)$, where $G_\alpha(t)$ is the Green's function in the case of the RL-fractional derivative (i.e. $G_\alpha(t) = F_\alpha(-\lambda, t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$, the Robotnov-Hartley function). This can be verified as a Laplace transform of ${}_0^C D_t^\alpha \left[y(t)\right] + \lambda y(t) = x(t)$, which is $\left(s^\alpha y(s) - s^{\alpha-1} y(0)\right) + \lambda y(s) = x(s)$, and gives:

$$y(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda} y(0) + x(s)\left(\frac{1}{s^\alpha + \lambda}\right) \tag{7.53}$$

The inverse Laplace transform of this above expression (7.53) is following:

$$\begin{aligned} y(t) &= y_0\left({}^C G_\alpha(t)\right) + \left(x(t)\right) * \left(G_\alpha(t)\right) \\ {}^C G_\alpha(t) &= E_\alpha(-\lambda t^\alpha) \\ G_\alpha(t) &= t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) = F_\alpha(-\lambda, t) \end{aligned} \tag{7.54}$$

The solution requires two Green's functions, when the fractional differential equation is formed via the Caputo fractional derivative. On the other hand, the fractional differential equation formed via the Riemann-Liouville derivative requires only one Green's function. This we will discuss in detail in subsequent sections, and we refer to these two separate fundamental solutions (Green's functions) as 'Alpha Exponential Function' type-1 and type-2. They are represented symbolically as $e_\alpha^{-\lambda t}$ for the RL-type operator, and $\tilde{e}_\alpha^{-\lambda t}$ for the Caputo type operator. Obviously for $\alpha = 1$, we get a classical exponential function (i.e. $e^{-\lambda t}$).

7.4 Formal description of a fractional differential and integral equation

7.4.1 Concepts of an ordinary differential equation as extended to fractional differential and integral equations

Consider the ordinary linear differential equation (ODE) as homogeneous with constant coefficients (i.e. $D^2 y(t) + aDy(t) + by(t) = 0$). Then if α and β are distinct zeros (roots) of an indicial polynomial (i.e. $P(x) = x^2 + ax + b$), we know that $\exp(\alpha t)$ and $\exp(\beta t)$ are linearly independent solutions of the ODE. If $\alpha = \beta$, then $\exp(\alpha t)$ and $t\exp(\alpha t)$ are the linearly independent solutions of the ODE. As an attempt to define a fractional differential equation (FDE), let $r_m, r_{m-1}, \ldots r_1, r_0$ be a strictly decreasing sequence of non-negative numbers. Then for the $a_1, a_2, \ldots a_m$ constants, we write the following:

$$\left(D^{r_m} + a_1 D^{r_{m-1}} + \ldots + a_m D^{r_0}\right)\left[y(t)\right] = x(t) \tag{7.55}$$

as a candidate to represent the general fractional differential equations. This equation is also complex looking, so we try to simplify it, through the addition of an imposed condition that r_j be rational numbers. Thus, with q as an LCM of the denominators of the nonzero r_j 's, we then formally define the FDE as

$$\left(D^{nv} + a_1 D^{(n-1)v} + \ldots + a_n D^0\right)\left[y(t)\right] = x(t) \tag{7.56}$$

$t \ge 0$, where $v = \frac{1}{q}$, and n is a positive integer.

Let m and q be positive integers and let $v = \left(\frac{1}{q}\right)$. Then the fractional integral equation is defined as follows:

$$\left(D^0 + a_1 D^{-v} + \ldots + a_m D^{mv}\right)\left[y(t)\right] = x(t) \tag{7.57}$$

In equation (7.56) (i.e. $\left(D^{nv} + a_1 D^{(n-1)v} + \ldots + a_n D^0\right)\left[y(t)\right] = x(t)$), the coefficients a_1, $a_2 \ldots, a_n$ are remaining constants and so the equation is a linear fractional differential equation. We can also have a non-linear fractional differential equation, such as:

$$\left(p_0(t) D^{nv} + p_1(t) D^{(n-1)v} + \ldots + p_n(t) D^0\right)\left[y(t)\right] = x(t) \tag{7.58}$$

With $p_i(t)$ as a function of t, say defined as $p_i(t) = a_i t^{(n-i)v}$, $i = 0,1,\ldots n$. As examples, we take $t\left(D^{1/2}\left[y(t)\right]\right) - y(t) = 0$ or $\left(tD^0 - \sqrt{\pi} D^{-1/2}\right)\left[y(t)\right] = 0$. This we have solved in an earlier chapter (Section-6.2) with the Laplace transform tricks and using an alternate analytical method.

7.4.2 The indicial polynomial corresponding to the fractional differential equation

For convenience, we introduce the indicial polynomial $P(x)$ corresponding to FDE represented as (7.56), i.e. $\left(D^{nv} + a_1 D^{(n-1)v} + \ldots + a_n D^0\right)\left[y(t)\right] = x(t)$, as follows:

$$P(x) = x^n + a_1 x^{n-1} + \ldots + a_n \tag{7.59}$$

Then, $P\left(D^v\right)$ is a fractional differential operator as shown in:

$$P\left(D^v\right) \equiv D^{nv} + a_1 D^{(n-1)v} + \ldots + a_n D^0 \tag{7.60}$$

With this (7.60), the FDE is then compactly described as $P\left(D^v\right) y(t) = x(t)$

7.4.3 The order of the fractional differential equation and the number of linearly independent solutions

The order of this fractional differential equation (7.56) is of (n,q), and is of constant coefficients. If $q = 1$, then $v = 1$, and the equation is simply an ordinary differential equation of order n with constant coefficients. The definition in (7.56) is for a non-homogeneous fractional differential equation with constant coefficients. With $x(t) = 0$, the homogeneous FDE with constant coefficients vanishes as the source term of the equation. First, this homogeneous FDE has to be solved by obtaining Green's function, and then for the source term, the standard convolution of this Green's function using a source function will yield a solution.

Consider a simple FDE of order $(4,3)$ as $D^{4/3}\left[y(t)\right] = 0$. Then if C_1 and C_2 are arbitrary constants:

$$y(t) = C_1 t^{1/3} + C_2 t^{-2/3} \tag{7.61}$$

is a solution. The first solution is $y_1(t) = C_1 t^{1/3}$. Putting this $y_1(t)$ in the FDE, i.e. $D^{4/3}\left[y_1(t)\right]$, the following is obtained (noting that $\lim_{x \downarrow 0} \Gamma(x) = \infty$):

$$D^{4/3}\left[y_1(t)\right]=D\left[C_1t^{1/3}\right]=C_1\frac{\Gamma\left(\frac{1}{3}+1\right)t^{\frac{1}{3}-\frac{4}{3}}}{\Gamma\left(\frac{1}{3}-\frac{4}{3}+1\right)}=C_1\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma(0)}t^{-1}=0 \quad (7.62)$$

This (7.62) satisfies one solution of $D^{4/3}\left[y(t)\right]=0$. A derivative of the first solution, i.e. $y_1(t)$, shall be the second that follows from an ODE. This is given as follows:

$$D\left[y_1(t)\right]=D\left[t^{1/3}\right]=C_2t^{-2/3} \quad (7.63)$$

which is the solution of $D^{4/3}\left[y(t)\right]=0$. The combination of these two gives us a solution for FDE as mentioned above (7.61).

One may draw an inference parallel to the ODE. Consider $D^1\left[y(t)\right]=0$; it has one solution, namely $y(t)=C$, a constant. Take $D^2\left[y(t)\right]=0$ as a second example; it has two solutions, namely $y_1(t)=C_1t$ and $y_2(t)=C_2$. Here $D\left[y_2(t)\right]=y_1(t)$ and the composite solution is $y(t)=C_1t+C_2$. Similarly, $D^3\left[y(t)\right]=0$ has three solutions, and the combined solution is $y(t)=C_1t^2+C_2t+C_3$. A general n order ODE has n linearly independent solutions. The FDE (i.e. $D^{1.333}\left[y(t)\right]=0$) has two solutions and not 1.333.. solutions. Here $nv=\frac{4}{3}=1.333..$ must logically have an integer number of solutions, $N\geq nv$ in this FDE which is two (7.61), i.e. $N=2>\frac{4}{3}$.

7.5 Finding a solution to a homogeneous fractional differential equation

7.5.1 An ordinary homogeneous classical differential equation and solution in terms of an exponential function

An ordinary homogeneous differential equation, with constant coefficients of order n is:

$$\left({}_0D^n+(a_1)\,{}_0D^{(n-1)}+...(a_n)\,{}_0D^{(0)}\right)\left[y(t)\right]=0 \quad (7.64)$$

We write the indicial polynomial for this (7.64) as:

$$P(x)=x^n+a_1x^{n-1}+...+a_n \quad (7.65)$$

We write the differential operator as $P\left(D\right)\equiv\left({}_0D^n+(a_1)\,{}_0D^{n-1}+...(a_n)\,{}_0D^0\right)$. Therefore, in a compact way, we write the differential equation as $\left(P(D)\right)\left[y(t)\right]=0$. Let us try $y(t)=e^{ct}$ as a solution; placing this in the differential equation, we obtain:

$$\left(P(D)\right)\left[e^{ct}\right]=\left(c^n+a_1c^{n-1}+...a_n\right)e^{ct}=\left(P(c)\right)e^{ct} \quad (7.66)$$

If c is the root of an indicial polynomial $P(x)=x^n+a_1x^{n-1}+...a_n$, that is, $P(c)=0$, then $y(t)=e^{ct}$ is one solution. This is because ${}_0D_t^n\left[e^{ct}\right]=c^ne^{ct}$ for a n positive integer returns the exponential function itself, without altering its form.

7.5.2 A fractional differential equation and solution candidate in terms of a higher transcendental function

We have seen that the fractional order derivative (say order u) of an exponential function is not an exponential function. That is ${}_0D_t^u\left[e^{ct}\right]\neq c^ue^{ct}$. Therefore, this exponential function is not a choice candidate function for the solution of a fractional differential equation. We have seen in an earlier chapter (Section-2.9) that ${}_0D_t^u\left[e^{ct}\right]=E_t\left(-u,c\right)$, where E_t represents the Miller-Ross function (refer to Appendix-A) in a series representation (that is $E_t\left(-u,c\right)=t^{-u}\sum_{k=0}^{\infty}\frac{(ct)^k}{\Gamma(k-u+1)}$). This is a variant of the Mittag-Leffler function. We represent this Miller-Ross function also as $E_t\left(-u,c\right)=t^{-u}E_{1,1-u}(ct)$, in the form of a two-parameter Mittag-Leffler function (see Appendix-A). In addition, we have $E_t\left(0,c\right)=e^{ct}$ and $E_t\left(0,ic\right)=\cos(ct)+i\sin(ct)$.

Interestingly, the fractional derivative of the Miller-Ross function is ${}_0D_t^u\left[E_t(w,c)\right]=E_t\left(w-u,c\right)$, i.e. retaining its form. Therefore this Miller-Ross function (a variant of the Mittag-Leffler function), is one possible choice as a candidate function to be a solution to the fractional differential equation. We write the following identities for a Miller-Ross function and the exponential function (see Appendix-A):

$$\begin{aligned}
&{}_0D_t^u\left[E_t(w,c)\right]=E_t\left(w-u,c\right) \qquad D_t\left[e^{ct}\right]=ce^{ct}\\
&{}_0D_t^u\left[t\left(E_t(w,c)\right)\right]=t\left(E_t(w-u,c)\right)+u\left(E_t(w-u+1,c)\right)\\
&D_t\left[t\left(e^{ct}\right)\right]=cte^{ct}+e^{ct}
\end{aligned} \tag{7.67}$$

We wrote that $E_t\left(0,c\right)=e^{ct}$, from the above (7.67), and we write ${}_0D_t^u\left[E_t(0,c)\right]={}_0D_t^u\left[e^{ct}\right]=E_t\left(-u,c\right)$ (see Appendix-A). Therefore, we can possibly try functions of type $E_t\left(kv,c\right)$, where k is an integer, as the candidate solution for a fractional differential equation (FDE).

7.5.3 A direct approach using the Miller-Ross function as a solution for a fractional differential equation (FDE)

Let us now go for a 'direct approach' to an FDE as represented in the following expression:

$$\left({}_0D_t^1+(a)\,{}_0D_t^{1/2}+b\right)\left[y(t)\right]=0 \tag{7.68}$$

In this case (7.68), we have $n=2$, $q=2$ and $v=\left(\frac{1}{q}\right)=\frac{1}{2}$, so we can change the FDE to our (n,q) notation and write with our standard definition as noted earlier (i.e. $\left(D^{nv}+a_1D^{(n-1)v}+....+a_nD^0\right)\left[y(t)\right]=0$) the following equations:

$$\left({}_0D_t+(a)\,{}_0D_t^{1/2}+b\right)\left[y(t)\right]=0 \qquad \left({}_0D_t^{2\left(\frac{1}{2}\right)}+(a)\,{}_0D_t^{1\left(\frac{1}{2}\right)}+b\right)\left[y(t)\right]=0 \tag{7.69}$$

In (7.69) we have $(n,q)\equiv(2,2)$. In this case, $P\left(D^v\right)\equiv P\left(D^{1/2}\right)=D+aD^{1/2}+bD^0$. The indicial polynomial is $P(x)=x^2+ax+b$.

With our hypothesis of using $E_t(kv,c)$ as a candidate, it seems reasonable to construct linear combinations of $E_t\left(0,c\right)$, $E_t\left(-\frac{1}{2},c\right)$, and $E_t\left(\frac{1}{2},c\right)$ as a solution of our given FDE. The expression of the Miller-Ross function relates to $E_t\left(\frac{1}{2},c\right)$ and $E_t\left(-\frac{1}{2},c\right)$ as $E_t\left(-\frac{1}{2},c\right)=cE_t\left(\frac{1}{2},c\right)+\frac{t^{-1/2}}{\sqrt{\pi}}$ (see Appendix-A). This renders the term $E_t\left(\frac{1}{2},c\right)$ as an extra term. We will use this conjecture now.

We note here that for an ordinary differential equation (ODE) of order $(2,1)$ with $v=\left(\frac{1}{q}\right)=1$, the indicial polynomial is the same as for our FDE example of order $(2,2)$. For ODE, we have $v=\left(\frac{1}{q}\right)=1$, then the two solutions, namely $y_1(t)=e^{ct}$ and $y_2(t)=D_t^v\left[y_1(t)\right]=D_t^{(1)}\left[y_1(t)\right]$, that is $y_2(t)=ce^{ct}$. For the FDE case with $v=\left(\frac{1}{q}\right)=\frac{1}{2}$, we try in a similar way, $y_1(t)=e^{ct}=E_t(0,c)$, and $y_2(t)={}_0D_t^v\left[y_1(t)\right]$, that is, ${}_0D_t^{1/2}\left[e^{ct}\right]=E_t\left(-\frac{1}{2},c\right)$. With this similarity, we try one solution and call it:

$$\psi_1(t)=AE_t\left(0,c\right)+E_t\left(-\tfrac{1}{2},c\right) \tag{7.70}$$

With $P\left(D^{1/2}\right)\equiv D^1+aD^{1/2}+bD^0$, we now operate this differential operator on (7.70) (i.e. $\psi_1(t)$), and write the following:

$$\begin{aligned}\left(P\left(D^{1/2}\right)\right)\left[\psi_1(t)\right] &= \left(D^1 + aD^{1/2} + bD^0\right)\left[AE_t(0,c)\right] \\ &\quad + \left(D^1 + aD^{1/2} + bD^0\right)\left[E_t\left(-\tfrac{1}{2},c\right)\right] \\ &= \left(AcE_t\left(0,c\right) + aAE_t\left(-\tfrac{1}{2},c\right) + bAE_t\left(0,c\right)\right) \\ &\quad + \left(E_t\left(-\tfrac{3}{2},c\right) + aE_t\left(-1,c\right) + bE_t\left(-\tfrac{1}{2},c\right)\right) \\ &= \left(cA + ac + bA\right)E_t\left(0,c\right) + \left(c + aA + b\right)E_t\left(-\tfrac{1}{2},c\right) + \frac{t^{-3/2}}{\Gamma\left(-\tfrac{1}{2}\right)}\end{aligned} \tag{7.71}$$

In the above steps (7.71), a property of the derivative of the Miller-Ross function (see Appendix-A) is used, along with other properties (i.e. $E_t(-1,c) = cE_t(0,c)$ and $E_t(\upsilon,c) = cE_t(\upsilon+1,c) + \frac{t^{-\upsilon}}{\Gamma(\upsilon+1)}$), with $\upsilon = -\frac{3}{2}$; and with a simplification using these, we arrive at the result (7.71).

Now, let $A = \lambda$ and $c = \lambda^2$. Then from the indicial polynomial, which is $P(x) = x^2 + ax + b$, we can write $P\left(\lambda\right) = \lambda^2 + a\lambda + b = c + aA + b$. Then the above result becomes:

$$\left(P\left(D^{1/2}\right)\right)\left[\psi_1(t)\right] = \lambda P(\lambda)E_t\left(0,\lambda^2\right) + P(\lambda)E_t\left(-\tfrac{1}{2},\lambda^2\right) + \frac{t^{-3/2}}{\Gamma\left(-\tfrac{1}{2}\right)} \tag{7.72}$$

If λ is the root of the indicial polynomial $P(x)$, then we have $P(\lambda) = 0$. From the above (7.72), we find that $\left(P\left(D^{1/2}\right)\right)\left[\psi_1(t)\right] = \frac{t^{-3/2}}{\Gamma\left(-\frac{1}{2}\right)}$, meaning that $\left({}_0D_t + (a)\,{}_0D_t^{1/2} + b\right)\left[\psi_1(t)\right] \neq 0$, and implying that $\psi_1(t)$ which we constructed via the Miller-Ross functions is not a solution. So, what is the solution?

7.5.4 Getting a solution from the roots of an indicial polynomial corresponding to the FDE using the Miller-Ross function

Let us indicate that α and β are the roots of the indicial polynomial $P(x) = x^2 + ax + b$. Let the two functions be formed for these two roots as described below:

$$\psi_1(t) = \alpha E_t\left(0,\alpha^2\right) + E_t\left(-\tfrac{1}{2},\alpha^2\right), \qquad \psi_2(t) = \beta E_t\left(0,\beta^2\right) + E_t\left(-\tfrac{1}{2},\beta^2\right) \tag{7.73}$$

These two functions will follow $\left(P\left(D^{1/2}\right)\right)\left[\psi_1(t)\right] = \frac{t^{-3/2}}{\Gamma\left(-\frac{1}{2}\right)} = \left(P\left(D^{1/2}\right)\right)\left[\psi_1(t)\right]$. This derivation we have described in a previous section (Section-7.5.3). Obviously it states that neither $\psi_1(t)$ nor $\psi_2(t)$ is a solution. Thus, let $\psi(t) = \psi_1(t) - \psi_2(t)$, meaning that $\psi(t)$ is a solution of the given FDE, which is:

$$\psi(t) = \alpha E_t\left(0,\alpha^2\right) - \beta E_t\left(0,\beta^2\right) + E_t\left(-\tfrac{1}{2},\alpha^2\right) - E_t\left(-\tfrac{1}{2},\beta^2\right) \tag{7.74}$$

Here we observe that $\psi(0) = \alpha - \beta$, ${}_0D_t^{-1/2}\psi(t)\Big|_{t=0} = 0$ ${}_0D_t^{1/2}\psi(t)\Big|_{t=0} = \infty$ and ${}_0D_t^{1}\psi(t)\Big|_{t=0} = \infty$. As we have just seen, $\psi(t)$ given by the above expression (7.74), is a solution of the given FDE (i.e. $\left({}_0D_t + (a)\,{}_0D_t^{1/2} + b\right)\left[y(t)\right] = 0$), if the roots α and β of the indicial equation $P(x) = 0$ are not equal. When they are equal, say, $\alpha = \beta$ (as we have seen for the ODE solution with $e^{\alpha t}$ and $te^{\alpha t}$ of the two distinct solutions); of $\left[P(D)\right]y(t) = 0$. So referring to this discussion, it appears that a linear combination of terms of the form $E_t\left(0,\alpha^2\right)$, $E_t\left(-\frac{1}{2},\alpha^2\right)$, $t\left(E_t\left(0,\alpha^2\right)\right)$, $t\left(E_t\left(-\frac{1}{2},\alpha^2\right)\right)$, $E_t\left(\frac{1}{2},\alpha^2\right)$ and $t\left(E_t\left(\frac{1}{2},\alpha^2\right)\right)$ is a likely candidate for the solution when roots are equal.

If we make this assumption, then proceeding as above, we may write:

$$\psi(t) = \left(1 + 2\alpha^2 t\right)E_t\left(0,\alpha^2\right) + \alpha E_t\left(\tfrac{1}{2},\alpha^2\right) + 2\alpha t\left(E_t\left(-\tfrac{1}{2},\alpha^2\right)\right) \tag{7.75}$$

We note here that if the ODE of order $(2,1)$ has the root λ for an indicial polynomial $P(x)$, then the FDE of order $(2,2)$, which has the same indicial polynomial $P(x)$ and has the roots $\lambda^{1/2}$ or $\sqrt[2]{\lambda}$. We can thus write the indicial polynomial of FDE of order (n,q), which will have the roots λ^{v} or $\sqrt[q]{\lambda}$, where $q=1/v$. The solution to FDE with a direct method that we described yields a similarity with the ODE. The basis functions of course in the case of the FDE is of higher transcendental functions.

7.6 Motivation for the Laplace transform technique

7.6.1 The indicial polynomial is the same as we get from the Laplace transform of a fractional differential equation

Let us have the same homogeneous FDE (7.68), that is, $\left({}_0D_t+(a)\,{}_0D_t^{\frac{1}{2}}+(b)\,{}_0D^0\right)\left[y(t)\right]=0$. Note that this FDE is composed via an RL-fractional derivative of order half. Applying a generalized Laplace transform formula (as in Section-5.14), to the FDE, we write the following:

$$\left(sY(s)-y(0)\right)+a\left(s^{\frac{1}{2}}Y(s)-{}_0D_t^{-\frac{1}{2}}y(0)\right)+bY(s)=0$$

$$Y(s)=\frac{y(0)+{}_0D_t^{-\frac{1}{2}}y(0)}{s+as^{\frac{1}{2}}+b}=\frac{C}{P\left(s^{\frac{1}{2}}\right)} \tag{7.76}$$

$$C=y(0)+{}_0D_t^{-\frac{1}{2}}y(0) \qquad P(x)=x^2+ax+b$$

We have used $\mathcal{L}\left\{{}_0D_t y(t)\right\}=sY(s)-y(0)$ and $\mathcal{L}\left\{{}_0D_t^{\frac{1}{2}}y(t)\right\}=s^{\frac{1}{2}}Y(s)-{}_0D_t^{-\frac{1}{2}}y(0)$, noting that ${}_0D_t^{-\frac{1}{2}}y(0)\equiv {}_0D_t^{-\frac{1}{2}}y(t)\Big|_{t=0}$ and $\mathcal{L}\left\{y(t)\right\}=Y(s)$.

Here we discuss two issues. The first one is how we know that $C=y(0)+{}_0D_t^{-\frac{1}{2}}y(0)$ is finite. If C is not finite then the problem is a serious one, and this approach becomes meaningless. Therefore, we must assume that C is finite. The second issue is how to find an inverse Laplace transform that is $\mathcal{L}^{-1}\left\{\frac{1}{P(\sqrt{s})}\right\}$. We will resolve this second issue.

7.6.2 A solution with distinct roots in an indicial polynomial

Expand $\frac{1}{P(x)}$ as partial fractions, assuming two distinct roots α and β for the indicial polynomial $P(x)$, and write it as:

$$\frac{1}{P(x)}=\frac{1}{x^2+ax+b}=\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right) \tag{7.77}$$

Put $x=s^{\frac{1}{2}}$ in the above (7.77) to get the following:

$$\frac{1}{P\left(s^{\frac{1}{2}}\right)}=\frac{1}{s+as^{\frac{1}{2}}+b}=\frac{1}{\alpha-\beta}\left(\frac{1}{s^{\frac{1}{2}}-\alpha}-\frac{1}{s^{\frac{1}{2}}-\beta}\right) \tag{7.78}$$

We therefore need to find now that $\mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{1}{2}}-c}\right\}$. From the tables of the Laplace transform of a higher-transcendental function, we find that $\mathcal{L}^{-1}\left\{\frac{1}{s^{\upsilon}-c}\right\}=F_{\upsilon}(c,t)$, where $F_{\upsilon}(c,t)$ is the Robotnov-Hartley function (see Appendix-A), defined in series form as $F_{\upsilon}(c,t)=\sum_{n=0}^{\infty}\frac{c^n t^{(n+1)\upsilon-1}}{\Gamma((n+1)\upsilon)}$. Therefore, we write the solution in terms of the Robotnov-Hartley function (another variant of the Mittag-Leffler function), as:

$$y(t)=\mathcal{L}^{-1}\left\{Y(s)\right\}=\frac{C}{\alpha-\beta}\left(F_{\frac{1}{2}}(\alpha,t)-F_{\frac{1}{2}}(\beta,t)\right) \tag{7.79}$$

However, with a direct approach we have found the solution in terms of the Miller-Ross function, but is it the same solution? Let us find out. We manipulate $\frac{1}{\sqrt{s}-c}$, which appears in partial fractions (7.78), as follows:

$$\begin{aligned}\frac{1}{s^{1/2}-c}&=\frac{s^{1/2}+c}{s-c^2}\\&=\frac{s^{1/2}\left(1+cs^{-1/2}\right)}{s-c^2}=\frac{1}{s-c^2}\left(\frac{1+cs^{-1/2}}{s^{-1/2}}\right)\\&=\frac{1}{s^{-1/2}(s-c^2)}+\frac{c}{s-c^2}\end{aligned}\tag{7.80}$$

From tables of the Laplace transform for the Miller-Ross function (see Appendix-A), we write:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}-c}\right\}&=\mathcal{L}^{-1}\left\{\frac{1}{s^{-1/2}(s-c^2)}+\frac{c}{(s-c^2)}\right\}\\&=E_t\left(-\tfrac{1}{2},c^2\right)+cE_t\left(0,c^2\right)\end{aligned}\tag{7.81}$$

With the above derivation, we write:

$$\begin{aligned}y(t)=\mathcal{L}^{-1}\left\{Y(s)\right\}&=\mathcal{L}^{-1}\left\{\frac{C}{P\left(s^{1/2}\right)}\right\}\\&=\frac{C}{\alpha-\beta}\begin{pmatrix}\alpha E_t\left(0,\alpha^2\right)-\beta E_t\left(0,\beta^2\right)\\+E_t\left(-\tfrac{1}{2},\alpha^2\right)-E_t\left(-\tfrac{1}{2},\beta^2\right)\end{pmatrix}\end{aligned}\tag{7.82}$$

The solution in (7.82) is a constant $\frac{C}{\alpha-\beta}$ multiplied by an obtained $\psi(t)$ using the direct method.

7.6.3 The solution with equal roots for indicial polynomials

When the roots are equal ($\alpha=\beta$), we write $Y(s)=C\left(s^{1/2}-\alpha\right)^{-2}$, and carry out the following manipulations:

$$\begin{aligned}Y(s)&=\frac{C}{\left(s^{1/2}-\alpha\right)^2}\\\frac{1}{\left(s^{1/2}-\alpha\right)^2}&=\left(\frac{1}{s^{-1/2}\left(s-\alpha^2\right)}+\frac{\alpha}{\left(s-\alpha^2\right)}\right)^2\\&=\frac{1}{s^{-1}\left(s-\alpha^2\right)^2}+\frac{\alpha^2}{\left(s-\alpha^2\right)^2}+2\frac{\alpha}{s^{-1/2}\left(s-\alpha^2\right)^2}\end{aligned}\tag{7.83}$$

We use the Miller-Ross function and its Laplace transform identity (see Appendix-A), which is:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\upsilon}(s-a)^2}\right\}=t\left(E_t\left(\upsilon,a\right)\right)-\upsilon E_t\left(\upsilon+1,a\right)\tag{7.84}$$

and write the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^{1/2}-\alpha^2\right)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^{-1}\left(s-\alpha^2\right)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{\alpha^2}{\left(s-\alpha^2\right)^2}\right\}$$
$$+2\mathcal{L}^{-1}\left\{\frac{\alpha}{s^{-1/2}\left(s-\alpha^2\right)^2}\right\} \tag{7.85}$$
$$=\left(t\left(E_t\left(-1,\alpha^2\right)\right)+E_t\left(-1+1,\alpha^2\right)\right)+\left(\alpha^2t^0E_t\left(0,\alpha^2\right)-\alpha^2\left(0\right)E_t\left(0+1,\alpha^2\right)\right)$$
$$+\left(2\alpha t\left(E_t\left(-\tfrac{1}{2},\alpha^2\right)\right)+2\alpha\left(\tfrac{1}{2}\right)E_t\left(-\tfrac{1}{2},\alpha^2\right)\right)$$
$$=\left(t\left(E_t\left(-1,\alpha^2\right)\right)+E_t\left(0,\alpha^2\right)\right)+\left(\alpha^2E_t\left(0,\alpha^2\right)\right)$$
$$+\left(2\alpha t\left(E_t\left(-\tfrac{1}{2},\alpha^2\right)\right)+\alpha E_t\left(-\tfrac{1}{2},\alpha^2\right)\right)$$

We then use the identity $E_t\left(-1,a\right)=aE_t\left(0,a\right)$ (see Appendix-A) to get the following:

$$y(t)=C\begin{pmatrix}\left(1+2\alpha^2t\right)E_t\left(0,\alpha^2\right)+\alpha E_t\left(\tfrac{1}{2},\alpha^2\right)\\ +2\alpha t\left(E_t\left(-\tfrac{1}{2},\alpha^2\right)\right)\end{pmatrix} \tag{7.86}$$

We obtained the above solution in terms of the Miller-Ross function. Here, we can write a solution in terms of the Robotnov-Hartley function as a convolution (represented as $*$):

$$y(t)=\mathcal{L}^{-1}\left\{\frac{C}{\left(s^{1/2}-\alpha^2\right)^2}\right\}$$
$$=C\left(\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}-\alpha^2}\right\}\right)*\left(\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}-\alpha^2}\right\}\right) \tag{7.87}$$
$$=C\left(F_{1/2}\left(\alpha,t\right)*F_{1/2}\left(\alpha,t\right)\right)$$
$$=C\int_0^t\left(F_{1/2}\left(\alpha,t-\tau\right)\right)\left(F_{1/2}\left(\alpha,\tau\right)\right)\mathrm{d}\tau$$

The starting point of the convolution integral is from 0 , as our FDE has a fractional differentiation operator with zero as its starting point, i.e. ${}_0D_t^{\upsilon}$.

7.7 A linearly independent solution of the fractional differential equation (FDE)

Ordinary differential equations (ODEs) have a set of linearly independent solutions, from which Green's function is obtained via a Wronskian determinant. This Green's function is a solution of the homogeneous ODE and is utilized to obtain a solution of a non-homogeneous ODE. The ODE is of the form $D^2y(t)+aDy(t)+by(t)=0$ with an indicial polynomial as $P(x)=x^2+ax+b$. We have simplified the symbol from ${}_0D_t^n\left[y(t)\right]$ to $D^ny(t)$.

Let a zero (root) of this indicial polynomial $P(x)$ be denoted as α . Then $y_1(t)=e^{\alpha t}$ and $Dy_1(t)=\alpha e^{\alpha t}$ will be solutions, as will other higher derivatives. However, they are not linearly-independent. If β is the other zero (root) of the indicial polynomial, then $y_2(t)=e^{\beta t}$ is the other solution along with its higher derivatives. Again, they are not linearly independent. The combination $y(t)=y_1(t)+y_2(t)=e^{\alpha t}+e^{\beta t}$ is the solution.

Here, $y(t)$ and $Dy(t)=\alpha e^{\alpha t}+\beta e^{\beta t}$ are obviously solutions and are too linearly independent .Therefore, the fundamental solution corresponding to the roots is as follows:

$$y_1(t)=\frac{\beta y(t)-Dy(t)}{\beta-\alpha}\qquad\qquad y_2(t)=\frac{Dy(t)-\alpha y(t)}{\beta-\alpha} \tag{7.88}$$

For $\alpha = \beta$, we have $y_2(t) = t\left(e^{\alpha t}\right)$.

Let us take an example of the homogeneous FDE as:

$$\left(D^{3/2} - 2D - D^{1/2} + 2D^0\right)\left[y(t)\right] = 0 \quad (7.89)$$

Herein (7.89), $n = 3$ and $v = \frac{1}{2}$ so $q = 2$. This is a fractional differential equation (FDE) of the order $(3,2)$. Taking the Laplace transform of the FDE (7.89), we get the following:

$$\begin{aligned}\left(s^{3/2}Y(s) - D^{1/2}y(0) - sD^{-1/2}y(0)\right) - 2\left(sY(s) - y(0)\right)\\ - \left(s^{1/2}Y(s) - D^{-1/2}y(0)\right) + 2\left(Y(s)\right) = 0\end{aligned} \quad (7.90)$$

giving the relationships (7.91) :

$$\begin{aligned}&Y(s) = \frac{C}{s^{3/2} - 2s - s^{1/2} + 2} \qquad C = D^{1/2}y(0) + sD^{-1/2}y(0) - 2y(0) - D^{-1/2}y(0)\\ &C = A + Bs \qquad A = D^{1/2}y(0) - 2y(0) - D^{-1/2}y(0) \qquad B = D^{-1/2}y(0)\end{aligned} \quad (7.91)$$

We can write the indicial polynomial as $P(x) = x^3 - 2x^2 - x + 2 = (x-2)(x^2-1)$ with $x = s^{1/2}$, and we have $P\left(s^{1/2}\right) = s^{3/2} - 2s - s^{1/2} + 2$. Therefore, we write:

$$Y(s) = \frac{C}{s^{3/2} - 2s - s^{1/2} + 2} = \frac{A}{P\left(s^{1/2}\right)} + \frac{Bs}{P\left(s^{1/2}\right)} = AY_1(s) + BY_2(s) \quad (7.92)$$

The solution is an inverse Laplace transform of the above (7.92), obtained through $Y(s)$, that is:

$$\begin{aligned}y(t) &= A\mathcal{L}^{-1}\left\{Y_1(s)\right\} + B\mathcal{L}^{-1}\left\{Y_2(s)\right\}\\ &= A\mathcal{L}^{-1}\left\{\frac{1}{P\left(s^{1/2}\right)}\right\} + B\mathcal{L}^{-1}\left\{\frac{s}{P\left(s^{1/2}\right)}\right\}\\ &= Ay_1(t) + By_2(t)\end{aligned} \quad (7.93)$$

$$y_1(t) = \mathcal{L}^{-1}\left\{Y_1(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{P\left(s^{1/2}\right)}\right\}$$

$$y_2(t) = \mathcal{L}^{-1}\left\{Y_2(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s}{P\left(s^{1/2}\right)}\right\}$$

Using the identity $\mathcal{L}\left\{Dy_1(t)\right\} = sY_1(s) - y(0)$, we get $\mathcal{L}^{-1}\left\{sY_1(s)\right\} = Dy_1(t) + y(0)\delta(t)$. Therefore, $y_1(t)$ and its one-whole derivative $y_2(t) = Dy_1(t) + y_1(0)\delta(t)$ are linearly independent solutions, and their linear combination provides the total solution.

Looking at $P(x) = x^3 - 2x^2 - x + 2$, as a cubic equation, it has the roots $x = 2$, $x = 1$ and $x = -1$. The partial fraction is:

$$\frac{1}{P(x)} = \frac{1}{x^3 - 2x^2 - x + 2} = \frac{1}{3(x-2)} + \frac{1}{6(x+1)} - \frac{1}{2(x-1)} \quad (7.94)$$

As demonstrated earlier, we write $x = s^{1/2}$ and re-write the following:

$$\frac{1}{P\left(s^{1/2}\right)} = \frac{1}{3\left(s^{1/2} - 2\right)} + \frac{1}{6\left(s^{1/2} + 1\right)} - \frac{1}{2\left(s^{1/2} - 1\right)} \quad (7.95)$$

We will carry out the inverse Laplace transform using identities of the Miller-Ross function as we demonstrated in an earlier part, to write:

$$y_1(t) = \tfrac{1}{3}\left(-E_t\left(\tfrac{1}{2}, 1\right) + 4E_t\left(\tfrac{1}{2}, 4\right) - 2E_t\left(0, 1\right) + 2E_t(0, 4)\right) \quad (7.96)$$

We note that $y_1(0)=0$; and from the above (7.96), we carry out one whole derivative and write $y_2(t)$:

$$y_2(t)=Dy_1(t)=\tfrac{1}{3}\begin{pmatrix}-E_t\left(\tfrac{1}{2},1\right)+16E_t\left(\tfrac{1}{2},4\right)\\ -2E_t\left(0,1\right)+8E_t(0,4)\end{pmatrix}+\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} \tag{7.97}$$

Note that $y_2(0)=\infty$; although it is a solution of the given FDE. We have $y_1(t)$ and $y_2(t)$, which are linearly independent. Therefore, with this, we write $y(t)=Ay_1(t)+By_2(t)$ as a solution of the given FDE.

7.8 The explicit solution for a homogeneous fractional differential equation (FDE)

From the observation of the previous sections, we now write the following statement for the FDE $\left(D^{nv}+a_1D^{(n-1)v}+.....+a_nD^0\right)\left[y(t)\right]=0$ of order (n,q), with $q=\frac{1}{v}$. Note that mathematicians write the RHS as zero, while physicists and engineers will write it as $\delta(t)$ (i.e. the Delta-function). The fundamental solution to this homogeneous system is also termed as an 'impulse response function'. The indicial polynomial is $P(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_n$, meaning that the fundamental solution is:

$$y_1(t)=\mathcal{L}^{-1}\left\{\frac{1}{P(s^v)}\right\}=\mathcal{L}^{-1}\left\{P^{-1}(s^v)\right\} \tag{7.98}$$

The integer N is greater than or equal to nv, that is $N-1<nv\le N$, then N numbers of a linearly independent solution, such as $y_1(t),y_2(t),......,y_j(t),....y_N(t)$, where $y_{m+1}(t)=D^my_1(t)$, with $m=0,1,2,........,(N-1)$.

As an example, let us take the FDE of order $(2,q)$ with $q=\frac{1}{v}$ as:

$$\left(D^{2v}+a_1D^v+a_2\right)\left[y(t)\right]=0 \tag{7.99}$$

In addition, the indicial polynomial is $P(x)=x^2+a_1x+a_2$, and it can be factorized and re-written as $P(x)=(x-\alpha_1)(x-\alpha_2)$. Therefore, we have $y_1(t)=\mathcal{L}^{-1}\left\{P^{-1}(s^v)\right\}$. For $\alpha_1\ne\alpha_2$, we write:

$$P^{-1}(s^v)=\frac{1}{(s^v-\alpha_1)(s^v-\alpha_2)}=\frac{1}{\alpha_1-\alpha_2}\left\{\frac{1}{s^v-\alpha_1}-\frac{1}{s^v-\alpha_2}\right\} \tag{7.100}$$

and for $\alpha_1=\alpha_2$, we write $P^{-1}(s^v)=(s^v-\alpha_1)^{-2}$. Let us write the Robotnov-Hartley function and the summation of the Miller-Ross function (see Appendix-A) as:

$$\begin{aligned}F_v(\alpha_i,t)&=e_i(t)\\&=\sum_{k=0}^{q-1}\alpha_i^{q-k-1}E_t(-kv,\alpha_i^2)=\mathcal{L}^{-1}\left\{\frac{1}{s^v-\alpha_i}\right\};\qquad i=1,2\end{aligned} \tag{7.101}$$

Then, $y_1(t)=A\left(e_1(t)-e_2(t)\right)$ for $\alpha_1\ne\alpha_2$, and $y_1(t)=A\left(e_1(t)*e_1(t)\right)$ for $\alpha_1=\alpha_2$, where the symbol ($*$) means convolution. In addition, we have $y_1(0)=y_2(0)=........y_{N-1}(0)=0$ and $y_N(0)=1$ or a constant, with $N=nv$ and when $N>nv$ then $y_N(0)=\infty$. We elaborate on this statement with a few simpler examples.

Let us take an FDE of order $(3,q)$, with $q=\frac{1}{v}$, as:

$$\left(D^{3v}+a_1D^{2v}+a_2D^v+a_3D^0\right)\left[y(t)\right]=0 \tag{7.102}$$

The indicial polynomial is $P(x)=x^3+a_1x^2+a_2x+a_3=(x-\alpha_1)(x-\alpha_2)^2$ obtained in its factored form. Therefore, we write the following partial fraction:

$$\frac{1}{P(s^v)}=\frac{B_1}{(s^v-\alpha_1)}+\frac{B_2}{(s^v-\alpha_2)}+\frac{C_1}{(s^v-\alpha_2)^2} \tag{7.103}$$

With $B_1=\frac{1}{(\alpha_2-\alpha_1)^2}$, $B_2=-\frac{1}{(\alpha_2-\alpha_1)^2}$ and $C_2=\frac{1}{(\alpha_2-\alpha_1)}$. The fundamental solution from the above equation when taking an inverse Laplace transform of (7.103) is $y_1(t)=B_1e_1(t)+B_2e_2(t)+C_1\left(e_2(t)*e_2(t)\right)$, i.e. using:

$$\mathcal{L}^{-1}\left\{(s^{v}-\alpha_i)^{-1}\right\}=F_v(\alpha_i,t)$$
$$=e_i(t)=\sum\nolimits_{k=0}^{q-1}\alpha_i^{q-k-1}E_t(-kv,\alpha_i^2)$$
$$\mathcal{L}^{-1}\left\{(s^{v}-\alpha_i)^{-2}\right\}=e_i(t)*e_i(t)=F_v(\alpha_i,t)*F_v(\alpha_i,t)$$
$$=\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\alpha_i^{2q-j-k-2}\left(t\left(E_t\left(-(j+k)v,\alpha_i^q\right)\right)+(j+k)v\left(E_t\left(1-(j+k)v,\alpha_i^q\right)\right)\right) \tag{7.104}$$

We have given the explicit solution as (7.104) above. The other linearly independent solution is obtained via differentiation of this $y_1(t)$.

7.9 The non homogeneous fractional differential equation and its solution

A type $\left(D^{nv}+a_1D^{(n-1)v}+a_2D^{(n-2)v}+.....+a_nD^0\right)\left[y(t)\right]=x(t)$, represents a non-homogeneous FDE, with conditions $D^m y(0)=0$ for $m=0,1,2,...........(N-1)$. The indicial polynomial is $P(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_n$. The fundamental solution to the corresponding homogeneous FDE (that is, with $x(t)=0$) gives Green's function shown by $k(t)=\mathcal{L}^{-1}\left\{P^{-1}(s^v)\right\}$, which we earlier termed $y_1(t)$. With this Green's function, we find the solution of a non-homogeneous FDE as the following in the convolution integral (it is also termed as a 'Particular-Integral'):

$$y(t)=k(t)*x(t)=\int_0^t\left(k(t-\xi)\right)\left(x(\xi)\right)\mathrm{d}\xi \tag{7.105}$$

For example, we have:

$$\left(D^{2v}-aD^{v}\right)\left[y(t)\right]=x(t) \tag{7.106}$$

a non-homogeneous FDE of order $(n,q)\equiv(2,4)$, $v=\frac{1}{4}$, $q=4$; with the condition given as $y(0)=0$. The indicial polynomial is $P(x)=x^2-ax=x(x-a)$. The fundamental solution or Green's function is (as obtained from the Miller-Ross function; see Appendix-A):

$$k(t)=\mathcal{L}^{-1}\{P^{-1}(s^v)\}=\mathcal{L}^{-1}\{s^{-1}(s^v-a)^{-1}\}$$
$$=\sum_{j=0}^{3}a^jE_t\left((j-2)v,a^4\right) \tag{7.107}$$

Therefore, we write the solution to the non-homogeneous system as follows:

$$y(t)=\int_0^t\left(k(t-\tau)\right)\left(x(\tau)\right)\mathrm{d}\tau$$
$$=\int_0^t\left(\sum_{j=0}^{3}a^jE_{t-\tau}\left((j-2)v,a^4\right)\right)\left(x(\tau)\right)\mathrm{d}\tau \tag{7.108}$$

We extend further the equation (7.106), i.e. $\left(D^{2v}-aD^{v}\right)\left[y(t)\right]=x(t)$, with $y(0)=0$, and $x(t)=\sin bt$. The Laplace transform of this is $X(s)=\frac{b}{(s^2+b^2)}=\mathcal{L}\left\{\sin bt\right\}$. Taking the Laplace transform of the entire equation (7.106), after rearranging, we get the following:

$$Y(s)=\frac{X(s)}{P(s^v)}$$
$$=\frac{b}{s^v(s^v-a)(s^2+b^2)} \tag{7.109}$$

After partial fractions and applying an inverse Laplace transform of the Miller-Ross function and its corresponding cosine functions (see Appendix-A), we get:

$$y(t)=\frac{1}{a^8+b^2}\sum_{k=1}^{4}a^{k-1}\begin{pmatrix}bE_t\left((k+1)v-1,a^4\right)-bC_t\left((k+1)v-1,b\right)\\-a^4S_t\left((k+1)v-1,b\right)\end{pmatrix} \tag{7.110}$$

With E_t as the Miller-Ross function, C_t and S_t are higher trigonometric cosine and sine functions corresponding to the Miller-Ross function, as detailed in Appendix-A.

Let us take another example:

$$\left(D^{6v}+D^{v}\right)\left[y(t)\right]=x(t) \tag{7.111}$$

Here, $v=\frac{1}{6}$ $n=6$, and the indicial polynomial is $P(x)=x(x^5+1)$. With $x(t)=A\left(55t+\frac{36}{\Gamma(5v)}t^{11v}\right)$, we have $X(s)=\frac{55A}{s^2}\left(1-\frac{1}{s^{5v}}\right)$. With these, we write:

$$Y(s)=\frac{X(s)}{P(s^v)}=\frac{55A(s^{5v}-1)}{s^{5v+2}}=\frac{55A}{s^3} \tag{7.112}$$

With an inverse Laplace transformation of (7.112), we obtain a solution as shown:

$$y(t)=\frac{55A}{2}t^2 \tag{7.113}$$

7.10 Fractional integral equations and their solution

7.10.1 Describing a fractional integral equation

The Laplace transform of an equation of the type $\left(D^0+b_1D^{-v}+b_2D^{-2v}+..........+b_mD^{-mv}\right)\left[y(t)\right]=x(t)$, with $m>0$ and $q>0$, and with a forcing function at the RHS of an exponential order, is:

$$\left(1+b_1s^{-v}+b_2s^{-2v}+..........+b_ms^{-mv}\right)Y(s)=X(s) \tag{7.114}$$

We can re-write the above (7.114) as:

$$\left(s^{mv}+b_1s^{(m-1)v}+....................+b_m\right)Y(s)=s^{mv}X(s) \tag{7.115}$$

Thus, from this (7.115), we have $Y(s)=s^{mv}\left(\frac{X(s)}{R(s^v)}\right)$, with $R(x)=x^m+b_1x^{m-1}+b_2x^{m-2}+.......+b_m$. Then the solution of the fractional integral equation is $y(t)=\mathcal{L}^{-1}\left\{s^{mv}\left(\frac{X(s)}{R(s^v)}\right)\right\}$.

If we wish to be more explicit and express $y(t)$ in terms of $x(t)$, we should make further assumptions. With $R(D^v)$ as a fractional differential operator of order (m,q) and if $k(t)$ is the Green's function associated with the homogeneous FDE (that is, $R(D^v)\left[y(t)\right]=0$), that is, a fundamental solution, then $k(t)=\mathcal{L}^{-1}\left\{R^{-1}(s^v)\right\}$.

Now, let M be an integer just greater than mv (i.e. $(M-1)<mv\le M$). The Laplace transform of a fractional derivative of the RL type is $\mathcal{L}\left\{D^\alpha x(t)\right\}=s^\alpha X(s)-\sum_{n=0}^{M-1}s^{M-n-1}\left(D^{n-M+\alpha}x(t)\Big|_{t=0}\right)$ (see Section-5.14 for further information). With $\alpha=mv$, and with this rearrangement, we get the following, with an understanding of symbolism (i.e. $D^{n-M+\alpha}x(t)\Big|_{t=0}\equiv D^{n-M+\alpha}x(0)$):

$$s^{mv}X(s)=\mathcal{L}\left\{D^{mv}x(t)\right\}+\sum_{n=0}^{M-1}s^{M-n-1}\left(D^{n-M+mv}x(0)\right) \tag{7.116}$$

In our earlier description (7.115), we obtained $Y(s)=\left(\frac{s^{mv}X(s)}{R(s^v)}\right)$. By placing this in the above expression (7.116), we get the following:

$$Y(s)=\frac{s^{mv}X(s)}{R(s^v)}=\frac{\mathcal{L}\left\{D^{mv}x(t)\right\}}{R(s^v)}+\sum_{n=0}^{M-1}\left(D^{n-M+mv}x(0)\right)\left(\frac{s^{M-n-1}}{R(s^v)}\right) \tag{7.117}$$

7.10.2 Extension of the final value theorem of integral calculus

Let us now revise the final value theorem of integral calculus, before slightly extending it. Let $M-1 < mv \le M$, that is, let M be an integer just greater than mv. Also, let us take p as an integer from 0 to $(M-1)$. We should also assume that $D^p k(t)$ is piecewise continuous for values upwards of $p = 0,1,2,...(M-1)$. We write

$$\mathcal{L}\left\{D^p k(t)\right\} = s^p \mathcal{L}\left\{k(t)\right\} - \sum_{j=0}^{p-1}\left(s^{p-1-j}\right)\left(D^j k(0)\right) \\ = s^p k(s) \tag{7.118}$$

The above expression is true for $D^j k(0) = 0$ with $j = 0,1,........,(M-2)$. This is true since $y_1(t) = k(t)$ is a fundamental solution of the homogeneous differential equation, $R(D^v) = 0$ and since the other linearly independent solutions are the derivatives of $y_1(t)$, namely $y_{j+1} = D^j y_1(t)$ We noted in an earlier section that we have $y_1(0) = y_2(0) = ... = y_{M-2}(0) = 0$. With this observation, we therefore have $\mathcal{L}\left\{D^p k(t)\right\} = s^p \mathcal{L}\left\{k(t)\right\}$ for $p = 0,1,2,..........,(M-2)$. This will be proven by the extension of a final value theorem.

Extension of the final value theorem states that if $\lim_{s\uparrow\infty}\left(s^{v+1}\mathcal{L}\left\{f(t)\right\}\right) = 0$, $\lim_{t\downarrow 0}\left(D^v f(0)\right) = 0$. We test this by setting $v = 0$, so we have for $\lim_{s\uparrow\infty}\left(s\mathcal{L}\left\{f(t)\right\}\right) = 0$, implying $\lim_{t\downarrow 0}\left(f(0)\right) = 0$ -this is the original final value theorem.

We have $\mathcal{L}\left\{k(t)\right\} = K(s) = \frac{1}{R(s^v)} = \frac{1}{s^{mv} + b_1 s^{(m-1)v} + b_2 s^{(m-2)v} + ...b_m}$, and we write the following:

$$\lim_{s\uparrow\infty}\left(s^p K(s)\right) \\ = s^p\left(\frac{1}{s^{mv} + b_1 s^{(m-1)v} + b_2 s^{(m-2)v} + .. + b_m}\right) = \frac{1}{\infty} = 0 \tag{7.119}$$

where, $p = (M-2) < mv$. The above derivation (7.119) implies that $\lim_{t\downarrow 0}\left(D^p k(t)\right) = 0$. So, for $j = 0,1,2,......,(M-2)$, we state that $D^j k(0) = 0$ .Using $\left(\frac{s^{M-n-1}}{R(s^v)}\right) = \mathcal{L}\left\{D^{M-n-1}k(t)\right\}$, we write the following (that is, because $\mathcal{L}\left\{D^p k(t)\right\} = s^p \mathcal{L}\left\{k(t)\right\} = s^p K(s)$ for $p = 0,1,2,............,(M-2)$):

$$Y(s) = \frac{\mathcal{L}\left\{D^{mv}x(t)\right\}}{R(s^v)} + \sum_{n=0}^{M-1}\left(D^{n-M+mv}x(0)\right)\left(s^{M-n-1}\right)\left(\frac{1}{R(s^v)}\right)$$

$$\begin{aligned} Y(s) &= \frac{\mathcal{L}\left\{D^{mv}x(t)\right\}}{R(s^v)} + \sum_{n=0}^{M-1}\left(D^{(n-M+mv)}x(0)\right)\mathcal{L}\left\{D^{M-n-1}k(t)\right\} \\ &= \left(\frac{1}{R(s^v)}\right)\left(\mathcal{L}\left\{D^{mv}x(t)\right\}\right) + \sum_{n=0}^{M-1}\left(D^{(n-M+mv)}x(0)\right)\mathcal{L}\left\{D^{M-n-1}k(t)\right\} \\ &= \left(\mathcal{L}\left\{k(t)\right\}\right)\left(\mathcal{L}\left\{D^{mv}x(t)\right\}\right) \\ &\quad + \sum_{n=0}^{M-1}\left(D^{(n-M+mv)}x(0)\right)\mathcal{L}\left\{D^{M-n-1}k(t)\right\} \end{aligned} \tag{7.120}$$

Getting an inverse Laplace transform of (7.120), we write:

$$y(t) = \int_0^t \left(k(t-\tau)\right)\left(D^{mv}x(\tau)\right)\mathrm{d}\tau \\ + \sum_{n=0}^{M-1}\left(D^{n-M+mv}x(0)\right)\left(D^{M-n-1}k(t)\right) \tag{7.121}$$

We will simplify (7.121) (i.e. $y(t)=\int_0^t\left(k(t-\tau)\right)\left(D^{mv}x(\tau)\right)\mathrm{d}\tau+\sum_{n=0}^{M-1}\left(D^{n-M+mv}x(0)\right)\left(D^{M-n-1}k(t)\right)$) with $M\geq mv$. For $n=0$ and $M>mv$, the term $D^{-M+mv}x(t)$ is a fractional integral. This integral, if written in the RL formula (that is, ${}_aD_t^{-\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\left(f(\tau)\right)\mathrm{d}\tau$) becomes zero, that is:

$$ {}_0D_0^{-(M-mv)}\left[x(0)\right]=\frac{1}{\Gamma(M-mv)}\int_0^0(t-\tau)^{(M-mv)-1}\left(x(\tau)\right)\mathrm{d}\tau=0 \tag{7.122}$$

7.10.3 The solution to fractional integral equations

Thus, for $M>mv$, when rewriting the indexes of the summation in the expression (7.121) of $y(t)$, we get:

$$y(t)=\int_0^t\left(k(t-\tau)\right)\left(D^{mv}x(\tau)\right)\mathrm{d}\tau+\sum_{j=0}^{M-2}\left(D^{mv-1-j}x(0)\right)\left(D^jk(t)\right) \tag{7.123}$$

For $n=0$ and $M=mv$, we write that $D^{-M+mv}x(0)=D^0x(0)=x(0)$. Again, for $M=mv$ and rearranging the indexes of summation in the expression (7.121) of $y(t)$ we get:

$$y(t)=\int_0^t\left(k(t-\tau)\right)\left(D^Mx(\tau)\right)\mathrm{d}\tau+\sum_{j=0}^{M-1}\left(D^jx(0)\right)\left(D^jk(t)\right) \tag{7.124}$$

In short, we get the following solutions:

$$\begin{aligned} &M=1 \qquad M>mv, \qquad y(t)=\int_0^t\left(k(t-\tau)\right)\left(D^{mv}x(\tau)\right)\mathrm{d}\tau \\ &M=1 \qquad M=mv, \qquad y(t)=\int_0^t\left(k(t-\tau)\right)\left(Dx(\tau)\right)\mathrm{d}\tau+x(0)k(t) \end{aligned} \tag{7.125}$$

7.11 Examples of fractional integral equations with explicit solutions

We now take explicit examples of the fractional integral equation for a solution. As an example, we take the following equation:

$$\left(D^0+bD^{-2v}\right)\left[y(t)\right]=x(t) \tag{7.126}$$

Taking the Laplace transform of (7.126), we obtain $\left(1+s^{-2v}b\right)Y(s)=X(s)$ or, after a rearrangement, we write $\left(s^{2v}+b\right)Y(s)=s^{2v}X(s)$. This gives us the following:

$$Y(s)=\frac{s^{2v}X(s)}{\left(s^{2v}+b\right)} \tag{7.127}$$

The equation above shows $R(x)=x^2+b$, with $k_q(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^{2v}+b}\right\}$. Note here that we have added a sub-script in the earlier notation of $k(t)$, which is the fundamental solution of $R\left(D^v\right)\left[y(t)\right]=0$, the homogeneous FDE.

Drawing from the discussions in the previous section (7.125) we write, for $M=1$ and $M>mv$, that the solution is $y(t)=\int_0^t\left(k_q(t-\tau)\right)\left(D^{mv}x(\tau)\right)\mathrm{d}\tau$. For $M=1=mv$, the solution is as below:

$$y(t)=\int_0^t\left(k_q(t-\tau)\right)\left(Dx(\tau)\right)\mathrm{d}\tau+x(0)k_q(t) \tag{7.128}$$

With this (7.128), we give an explicit solution for $\left(D^0+bD^{-2v}\right)\left[y(t)\right]=x(t)$. With say, $q=2$, $v=\frac{1}{q}=\frac{1}{2}$ and $m=2$, $M=mv=1$, we write, using $\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}=e^{-bt}=E_t(0,-b)$, the following solutions:

$$\begin{aligned} k_2(t)&=\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}=E_t(0,-b)=e^{-bt} \\ y(t)&=\int_0^t e^{-b(t-\tau)}\left(Dx(\tau)\right)\mathrm{d}\tau+x(0)e^{-bt} \end{aligned} \tag{7.129}$$

For $q=3$, $v=\frac{1}{3}$, $m=2$, and $M=1$ with $M>mv$, we write the following solutions; with the use of the Miller-Ross fractional sine and cosine functions (see Appendix-A)using the following inverse Laplace transform:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{2v}+b}\right\} &= C_t\left(-v,-b^{q/2}\right)+b^{-1/2}S_t\left(-2v,-b^{q/2}\right)\\ &\quad -b^{1/2}S_t\left(0,-b^{q/2}\right)\end{aligned} \tag{7.130}$$

For (7.130), where $q=\left(1/v\right)$ the solution is:

$$\begin{aligned}k_3(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^{2/3}+b}\right\}=C_t\left(-\tfrac{1}{3},-b^{3/2}\right)+b^{-1/2}S_t\left(-\tfrac{2}{3},-b^{3/2}\right)-b^{1/2}S_t\left(0,-b^{3/2}\right)\\ C_t(v,a) &= C_t(v,-a) \qquad S_t(v,a)=-S_t(v,-a)\\ k_3(t) &= C_t\left(-\tfrac{1}{3},b^{3/2}\right)-b^{-1/2}S_t\left(-\tfrac{2}{3},b^{3/2}\right)+b^{1/2}S_t\left(0,b^{3/2}\right)\\ y(t) &= \int_0^t\left(C_{t-\tau}\left(-\tfrac{1}{3},b^{3/2}\right)-b^{-1/2}S_{t-\tau}\left(-\tfrac{2}{3},b^{3/2}\right)+b^{1/2}S_{t-\tau}\left(0,b^{3/2}\right)\right)\left(D^{2/3}x(\tau)\right)\mathrm{d}\tau\end{aligned} \tag{7.131}$$

For $q=4$, $v=\frac{1}{3}$, $m=2, M=1$ with $M>mv$, we write the solution of $\left(D^0+bD^{-2v}\right)\left[y(t)\right]=x(t)$, that is, $\left(D^0+bD^{-1/2}\right)\left[y(t)\right]=x(t)$:

$$\begin{aligned}k_4(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}+b}\right\}=E_t\left(-\tfrac{1}{2},b^2\right)-bE_t\left(0,b^2\right)\\ y(t) &= \int_0^t\left(E_{t-\tau}\left(-\tfrac{1}{2},b^2\right)-bE_{t-\tau}\left(0,b^2\right)\right)\left(D^{1/2}x(\tau)\right)\mathrm{d}\tau\end{aligned} \tag{7.132}$$

With all these discussions we note that the solution should be obtained by taking the inverse Laplace transform of $Y(s)$. For example, when the forcing function of the fractional integral equation $\left(D^0+bD^{-2v}\right)\left[y(t)\right]=x(t)$, is $x(t)=t^{\lambda}$, with the Laplace transform set at $X(s)=\frac{\Gamma(\lambda+1)}{s^{\lambda+1}}$, with $\lambda>-1$, we get the following:

$$\begin{aligned}Y(s) &= \frac{s^{2v}}{s^{2v}+b}X(s)\\ &= \frac{\Gamma(\lambda+1)}{s^{\lambda+1-2v}(s^{2v}+b)}\end{aligned} \tag{7.133}$$

For $q=2$ and $v=\frac{1}{2}$, we have $Y(s)=\frac{\Gamma(\lambda+1)}{s^{\lambda}(s+b)}$, which gives $y(t)=\left(\Gamma(\lambda+1)\right)E_t(\lambda,-b)$. For $q=3$ and $v=\frac{1}{3}$, we have $Y(s)=\left(\Gamma(\lambda+1)\right)\left(\frac{s}{s^{\lambda}(s^2+b^3)}+\frac{b}{s^{\lambda+v}(s^2+b^3)}+\frac{b}{s^{\lambda-v}(s^2+b^3)}\right)$ as an inverse Laplace transform of this solution i.e.

$$y(t)=\left(\Gamma(\lambda+1)\right)\begin{pmatrix}C_t\left(\lambda,b^{3/2}\right)+b^{1/2}S_t\left(\lambda+v,b^{3/2}\right)\\ -b^{1/2}S_t\left(\lambda-v,b^{3/2}\right)\end{pmatrix} \tag{7.134}$$

For $q=4$ and $v=\frac{1}{4}$, we have $Y(s)=\left(\Gamma(\lambda+1)\right)\left(\frac{1}{s^{\lambda}(s-b^2)}-\frac{b}{s^{\lambda+\frac{1}{2}}(s-b^2)}\right)$ The solution is:

$$y(t)=\left(\Gamma(\lambda+1)\right)\left(E_t\left(\lambda,b^2\right)-bE_t\left(\lambda+\tfrac{1}{2},b^2\right)\right) \tag{7.135}$$

For the inverse Laplace transform and the use of the Miller-Ross function, refer to Appendix-A.

7.12 Sequential fractional derivative of the Miller-Ross type ${}_a\mathscr{D}_x^{k\alpha}$ and sequential fractional differential equations (SFDE)

7.12.1 The sequential fractional differential equation (SFDE)

A sequential linear fractional differential equation of the order $n\alpha$ and $n \in \mathbb{N}$ is represented as:

$$\begin{gathered} b_0(x)y(x) + b_1(x)\left({}_a\mathscr{D}_x^{\alpha}\left[y(x)\right]\right) + b_2(x)\left({}_a\mathscr{D}_x^{2\alpha}\left[y(x)\right]\right) + \ldots \\ \ldots + b_n(x)\left({}_a\mathscr{D}_x^{n\alpha}\left[y(x)\right]\right) = f(x) \end{gathered} \tag{7.136}$$

Where ${}_a\mathscr{D}_x^{k\alpha}$ is a fractional sequential derivative operator, of commensurate order α, ${}_a\mathscr{D}_x^{\alpha}\left[y(x)\right] = {}_a^*D_x^{\alpha}\left[y(x)\right]$, where ${}_a^*D_x^{\alpha}$ is an RL or Caputo operator (denoted via *). In addition, ${}_a\mathscr{D}_x^{k\alpha}\left[y(x)\right] = {}_a\mathscr{D}_x^{\alpha}\left[{}_a\mathscr{D}_x^{(k-1)\alpha}\left[y(x)\right]\right]$; $k = 2, 3, \ldots$. For $k = 2$ and $0 < \alpha < \frac{1}{2}$, we have:

$${}_a\mathscr{D}_x^{2\alpha}\left[y(x)\right] = {}_a^*D_x^{\alpha}\left[{}_a^*D_x^{\alpha}\left[y(x)\right]\right] \neq {}_a^*D_x^{2\alpha}\left[y(x)\right] \tag{7.137}$$

The above (7.137) comes from the fact that for $\alpha > 0, \beta > 0$, generally the composition $\left({}^*D^{\alpha}\right)\left({}^*D^{\beta}\right)f(x) \neq {}^*D^{\alpha+\beta}f(x)$, that is, ${}_a^*D_x^{\alpha}\left[{}_a^*D_x^{\alpha}\left[y(x)\right]\right] \neq {}_a^*D_x^{2\alpha}\left[y(x)\right]$. We will look at the detailed proof of this in Chapter-9, while discussing the composition rules, and those of fractional derivatives and integrals.

7.12.2 The matrix form representation of SFDE

When representing the SFDE in matrix form, we write it as:

$$\mathscr{D}^{n\alpha}\left[y(x)\right] + \sum_{k=0}^{(n-1)} a_k(x)\mathscr{D}^{k\alpha}\left[y(x)\right] = f(x); \quad n \in \mathbb{N} \tag{7.138}$$

(7.138) reduces to:

$${}^*D^{\alpha}\mathrm{Y}(x) = \mathrm{A}(x)\mathrm{Y}(x) + \mathrm{B}(x) \tag{7.139}$$

by changing the variables with $y_1(x) = y(x)$; ${}^*D^{\alpha}y_j(x) = y_{j+1}(x)$; $j = 1, 2, \ldots, (n-1)$, where, in (7.139), we have:

$$\mathrm{A}(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{pmatrix}, \quad \mathrm{B}(x) = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ f(x) \end{pmatrix}; \quad \mathrm{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \ldots \\ \ldots \\ y_n(x) \end{pmatrix} \tag{7.140}$$

As an example, we take the following fractional differential equation:

$$y''(t) + 3\left({}_0D_t^{3/2}\left[y(t)\right]\right) + y(t) = f(t) \tag{7.141}$$

Setting $y''(t) = {}_0\mathscr{D}_t^{4(\alpha)}\left[y(t)\right]$ with $\alpha = \frac{1}{2}$ gives SFDE as the following:

$${}_0\mathscr{D}_t^{4\alpha}y(t) + 3\left({}_0\mathscr{D}_t^{3\alpha}y(t)\right) + y(t) = f(t); \quad \alpha = \tfrac{1}{2} \tag{7.142}$$

which is reduced to:

$${}_0^*D_t^{\alpha}\mathrm{Y}(t) = \mathrm{AY}(t) + \mathrm{B}(t) \tag{7.143}$$

with:

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -3 \end{pmatrix}; \quad \mathrm{B}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{pmatrix}; \quad \mathrm{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} \tag{7.144}$$

The matrix notation then becomes the following:

$$\left({}_0^*D_t^{\alpha}\right)\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{pmatrix} \tag{7.145}$$

From the above (7.145), we get the systems of equations as the following sets:

$$\begin{gathered} {}_0^*D_t^\alpha y_1(t)=y_2\,,\quad {}_0^*D_t^\alpha y_2(t)=y_3\,,\quad {}_0^*D_t^\alpha y_3(t)=y_4\,, \\ {}_0^*D_t^\alpha y_4(t)=-y_1(t)-3y_4(t)+f(t) \end{gathered} \tag{7.146}$$

Starting from the last one, and replacing the first three, we obtain the following:

$${}_0^*D_t^\alpha\,{}_0^*D_t^\alpha\,{}_0^*D_t^\alpha\,{}_0^*D_t^\alpha\left[y_1(t)\right]+3\left({}_0^*D_t^\alpha\,{}_0^*D_t^\alpha\,{}_0^*D_t^\alpha\left[y_1(t)\right]\right)+y_1(t)=f(t) \tag{7.147}$$

By writing $y_1(t)$ as $y(t)$, and with a defined sequential fractional derivative ${}_a\mathscr{D}_x^{k\alpha}$, we attain our original system, that is:

$${}_0\mathscr{D}_t^{4\alpha}y(t)+3\left({}_0\mathscr{D}_t^{3\alpha}y(t)\right)+y(t)=f(t);\quad \alpha=\tfrac{1}{2} \tag{7.148}$$

The solution of this SFDE is similar to what we discussed in earlier sections of the book; we will also use this formulation in later sections.

7.13 Solution of the ordinary differential equation using state transition matrices -a review

7.13.1 Origin of the state transition matrix in a solution to systems of linear differential equations

This is also a study of systems of linear differential equations. Let us take a general (classical) differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)\right]+a(t)x(t)=bu(t) \tag{7.149}$$

Here, we point out that $u(t)$ is not a Heaviside step function, but a general forcing function. We also set an initial condition as $x(t_0)=x_0$. By multiplying both sides of (7.149) by $e^{\int a(t)\mathrm{d}t}$, we get the following:

$$\begin{gathered} e^{\int a(t)\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)\right]+e^{\int a(t)\mathrm{d}t}\left(a(t)x(t)\right)=e^{\int a(t)\mathrm{d}t}bu(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t}x(t)\right]=e^{\int a(t)\mathrm{d}t}bu(t) \end{gathered} \tag{7.150}$$

and then integrate (7.150) as demonstrated below:

$$\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(e^{\int a(\tau)\mathrm{d}\tau}\right)x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}\left(e^{\int a(\tau)\mathrm{d}\tau}\right)b\left(u(\tau)\right)\mathrm{d}\tau \tag{7.151}$$

Using the notation $\Phi(t)=e^{\int a(t)\mathrm{d}t}$, we write the following steps:

$$\begin{gathered} \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\Phi(\tau)x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}\Phi(\tau)b\left(u(\tau)\right)\mathrm{d}\tau \\ \Phi(t)x(t)-\Phi(t_0)x(t_0)=\int_{t_0}^{t}\Phi(\tau)b\left(u(\tau)\right)\mathrm{d}\tau \\ x(t)=[\Phi(t)]^{-1}\Phi(t_0)x_0+[\Phi(t)]^{-1}\int_{t_0}^{t}\Phi(\tau)b\left(u(\tau)\right)\mathrm{d}\tau \end{gathered} \tag{7.152}$$

For a constant $a(t)=a$, we have $\Phi(t)=e^{at}$ and $\Phi(t_0)=e^{at_0}$. As such, we have:

$$\begin{aligned} x(t)&=e^{-at}e^{at_0}x_0+e^{-at}\int_{t_0}^{t}e^{a\tau}b\left(u(\tau)\right)\mathrm{d}\tau \\ &=e^{a(t_0-t)}x_0+\int_{t_0}^{t}e^{a(\tau-t)}b\left(u(\tau)\right)u(\tau)\mathrm{d}\tau \\ &=\Phi(t_0-t)x_0+\int_{t_0}^{t}\Phi(\tau-t)b\left(u(\tau)\right)\mathrm{d}\tau \\ &=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)b\left(u(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{7.153}$$

7.13.2 The state transition matrix as Green's function is a solution to the homogeneous system of differential equations

The integer order system-solution and the classical multivariate system are defined as Ω below

$$\begin{gathered}\Omega: \quad (D_t x)(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t) \qquad y(t) = \mathrm{C}(t)x(t) + \mathrm{D}(t)u(t) \\ x(t) \in \mathbb{R}^{n\times 1}; \quad u(t) \in \mathbb{R}^{p\times 1}; \quad y(t) \in \mathbb{R}^{q\times 1}; \quad \mathrm{A} \in \mathbb{R}^{n\times n}; \quad \mathrm{B} \in \mathbb{R}^{n\times p}; \\ \mathrm{C} \in \mathbb{R}^{q\times n}; \quad \mathrm{D} \in \mathbb{R}^{q\times q}\end{gathered} \tag{7.154}$$

whose entries are a continuous function of time. The operator is $D_t \equiv \frac{\mathrm{d}}{\mathrm{d}t}$. The solution to (7.154) is as follows:

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau \\ &= \Phi(t,t_0)\left(x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau \right) \\ y(t) &= \mathrm{C}(t)\Phi(t,t_0)\left(x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau \right) + \mathrm{D}(t)u(t) \end{aligned} \tag{7.155}$$

Where $\Phi(t_0,t) = e^{\int_{t_0}^{t} \mathrm{A}(\tau)\mathrm{d}\tau}$ is the state transition matrix (Green's function) of the homogeneous system, that is, $(D_t x)(t) = \mathrm{A}(t)x(t)$:

For a linear time invariant (LTI) system $\Phi(t_0,t) = \Phi(t-t_0) = e^{\mathrm{A}(t-t_0)} = G(t-t_0)$, the homogeneous solution is given by $G(t-t_0) = \Phi(t-t_0) = e^{\mathrm{A}(t-t_0)}$. This particular solution is given as a convolution integral (the starting point of the convolution integral is t_0) depicted as follows:

$$x_p(t) = (G *^{t_0} \mathrm{B}u)(t) = \int_{t_0}^{t} G(t-\tau)\mathrm{B}u(\tau)\mathrm{d}\tau = \int_{t_0}^{t} e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau \tag{7.156}$$

We get a state trajectory and output trajectory as solutions to Ω, as depicted in the following expressions:

$$\begin{aligned} x(t) &= e^{\mathrm{A}t}x_0 + \int_{0}^{t} e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau = e^{\mathrm{A}t}\left(x_0 + \int_{0}^{t} e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau \right) \\ y(t) &= \mathrm{C}e^{\mathrm{A}t}\left(x_0 + \int_{0}^{t} e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau \right) + \mathrm{D}u(t) \end{aligned} \tag{7.157}$$

For a constant $a(t) = a$, we have $\Phi(t) = e^{at}$ and $\Phi(t_0) = e^{at_0}$, and thus we have:

$$\begin{aligned} x(t) &= e^{-at}e^{at_0}x_0 + e^{-at}\int_{t_0}^{t} e^{a\tau}bu(\tau)\mathrm{d}\tau = e^{a(t_0-t)}x_0 + \int_{t_0}^{t} e^{a(\tau-t)}bu(\tau)\mathrm{d}\tau \\ &= \Phi(t_0-t)x_0 + \int_{t_0}^{t} \Phi(\tau-t)bu(\tau)\mathrm{d}\tau \\ &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)bu(\tau)\mathrm{d}\tau \end{aligned} \tag{7.158}$$

If $a(t)$ is a matrix $\mathrm{A}(t) \in \mathbb{R}^{n\times n}$, there are several ways to represent $e^{\int \mathrm{A}(t)\mathrm{d}t}$, which will be elucidated in this section using examples.

7.13.3 A demonstration of the ways to represent the state transition matrix and its usage in multivariate dynamic systems

Here, we define a state transition matrix as $\Phi(t_0 - t) = e^{\int_{t_0}^{t} a(\xi)\mathrm{d}\xi} = \Phi(t, t_0)$ which is also a Green's function of the homogeneous part of the system of a differential equation. One way of representing the matrix exponential, as it is for constant A, is as follows:

$$e^{\int \mathrm{A}(t)\mathrm{d}t} = e^{\mathrm{A}t} \cong \mathrm{I} + \mathrm{A}t + \frac{\mathrm{A}^2 t^2}{2} + \frac{\mathrm{A}^3 t^3}{3!} + \ldots \tag{7.159}$$

The other way is via an inverse Laplace transform, which is:

$$e^{\mathrm{A}t} = \mathcal{L}^{-1}\left\{(s\mathrm{I} - \mathrm{A})^{-1}\right\} \tag{7.160}$$

where I is the identity matrix.

For example, for illustration's sake, take the following linear time variant (LTV) system, represented as the following in its state space form:

$$\begin{gathered}\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ \mathrm{A} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x(1) = x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{gathered} \tag{7.161}$$

Given an excitation such as $u(t) = 5.8384e^{-2t} - 0.3026e^{t}$, we need to find $\overline{x}(t)$. Calculation of the state transition matrix Φ for this linear time variant (LTV) system is demonstrated as follows from the two sets of coupled differential equations, for which the states are:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1 = x_1 + e^{-t} x_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} x_2 = -x_2 + u(t) \tag{7.162}$$

Let us take the second state of (7.162) and solve the 'homogeneous' system, that is $x_2^{(1)} = -x_2$. To solve this, let us take a general differential equation, $x^{(1)}(t) + a(t)x(t) = y(t)$. Multiplying both sides by $e^{\int a(t)\mathrm{d}t}$, and with a manipulation as in (7.150), we get the form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t} x\right] = e^{\int a(t)\mathrm{d}t} y \tag{7.163}$$

In our case, that is $x^{(1)} + x = 0$, we have $a(t) = 1$ and $y = 0$, so $e^{\int a(t)\mathrm{d}t} = e^{t}$. We write the above equation, and integrate it from t_0 to t (for our case for x_2 in place of x in (7.163)), as follows:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{t} x_2\right] = e^{t} y \\ \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{\theta} x_2(\theta)\right]\mathrm{d}\theta = \int_{t_0}^{t} e^{\theta} y(\theta)\mathrm{d}\theta \\ e^{t} x_2(t) - e^{t_0} x_2(t_0) = \int_{t_0}^{t} e^{\theta} y(\theta)\mathrm{d}\theta\end{gathered} \tag{7.164}$$

This gives us the following:

$$\begin{gathered}x_2(t) = e^{-t} e^{t_0} x_2(t_0) + e^{-t}\int_{t_0}^{t} e^{\theta} y(\theta)\mathrm{d}\theta = e^{-(t-t_0)} x_2(t_0) \\ y(\theta) = 0\end{gathered} \tag{7.165}$$

Putting the solution (7.165) of the second state's homogeneous equation into the first state equation in (7.162), we obtain the following:

$$\begin{gathered}x_1^{(1)} = x_1 + e^{-t} e^{-(t-t_0)} x_2(t_0) = x_1 + e^{-2t}\left(e^{t_0} x_2(t_0)\right) \\ x_1^{(1)} - x_1 = e^{-2t}\left(e^{t_0} x_2(t_0)\right)\end{gathered} \tag{7.166}$$

Comparing (7.166) to $x^{(1)}+a(t)x=y$, we get $a(t)=-1$, $e^{\int a(t)\mathrm{d}t}=e^{-t}$ and $y(t)=e^{-2t}\left(e^{t_0}x_2(t_0)\right)$. Using the same procedure as state x_2, we write the following similar expressions for state x_1:

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t}x_1\right]=e^{-t}y \\
\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{-\theta}x_1(\theta)\right]\mathrm{d}\theta=\int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta \\
e^{-t}x_1(t)-e^{-t_0}x_1(t_0)=\int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta
\end{gathered} \tag{7.167}$$

From (7.167), we get the following:

$$\begin{aligned}
x_1(t)&=e^{t-t_0}x_1(t_0)+e^{t}\int_{t_0}^{t}e^{-\theta}\left(e^{-2\theta}e^{t_0}x_2(t_0)\right)\mathrm{d}\theta \\
&=e^{(t-t_0)}x_1(t_0)+\left(\tfrac{1}{3}e^{(t-2t_0)}-\tfrac{1}{3}e^{(t_0-2t)}\right)x_2(t_0)
\end{aligned} \tag{7.168}$$

In matrix form, the solution of a homogeneous system, like we obtained for the two state variables, is expressed as follows:

$$\begin{aligned}
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}&=\begin{pmatrix} e^{(t-t_0)} & \left(\frac{1}{3}e^{(t-2t_0)}-\frac{1}{3}e^{(t_0-2t)}\right) \\ 0 & e^{-(t-t_0)} \end{pmatrix}\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} \\
&=\Phi(t,t_0)\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}
\end{aligned} \tag{7.169}$$

Thus, from the homogeneous system's solution we obtain the state transition matrix as:

$$\Phi(t,\tau)=\begin{pmatrix} e^{t-\tau} & \frac{1}{3}\left(e^{t-2\tau}-e^{-2t+\tau}\right) \\ 0 & e^{-t+\tau} \end{pmatrix},\qquad \Phi(0,\tau)=\begin{pmatrix} e^{-\tau} & \frac{1}{3}\left(e^{-2\tau}-e^{\tau}\right) \\ 0 & e^{\tau} \end{pmatrix} \tag{7.170}$$

The state trajectory due to control input $u(t)=5.8384e^{-2t}-0.3026e^{t}$ is as follows:

$$\begin{aligned}
\bar{x}(t)&=\begin{pmatrix} e^{t} & \frac{1}{3}\left(e^{t}-e^{-2t}\right) \\ 0 & e^{-t} \end{pmatrix}\left[\int_0^t\begin{pmatrix} \frac{1}{3}\left(e^{-2\tau}-e^{\tau}\right) \\ e^{\tau} \end{pmatrix}\left(5.8384e^{-2t}-0.3026e^{t}\right)\mathrm{d}\tau\right] \\
&=\begin{pmatrix} 0.3856e^{t}-0.1513-1.9966e^{-2t}+1.4596e^{-2t} \\ -0.1513e^{t}+5.9897e^{-t}-5.8384e^{-2t} \end{pmatrix}
\end{aligned} \tag{7.171}$$

The above example elucidates in detail the calculation of the state transition matrix for an LTV system. When the system is a linear time invariant with matrix A (a constant matrix), the simpler method is invoked via an inverse Laplace transform to get to $\Phi(t)$.

The equation $x^{(1)}=\mathrm{A}x+\mathrm{B}u$ has a fundamental solution for the homogeneous system $x^{(1)}(t)=\mathrm{A}x(t)$ as $\Phi(t)=e^{\mathrm{A}t}=\mathcal{L}^{-1}\left\{(s\mathrm{I}-\mathrm{A})^{-1}\right\}$; where s is a complex frequency (the Laplace variable) and I is the 'identity matrix'.

Let us elaborate upon this with the following example, where an LTI system matrix is $\mathrm{A}=\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$.

$$\Phi(t)=\mathcal{L}^{-1}\left\{(s\mathrm{I}-\mathrm{A})^{-1}\right\}=\mathcal{L}^{-1}\left\{\left(\begin{bmatrix} s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix} -3 & -2\\ 1 & 0\end{bmatrix}\right)^{-1}\right\}$$
$$=\mathcal{L}^{-1}\left\{\begin{pmatrix} s+3 & 2\\ -1 & s\end{pmatrix}^{-1}\right\}$$
$$=\mathcal{L}^{-1}\left[\frac{1}{s(s+3)+2}\begin{pmatrix} s & -2\\ 1 & s+3\end{pmatrix}\right] \tag{7.172}$$
$$=\mathcal{L}^{-1}\begin{pmatrix} \frac{2}{s+2}-\frac{1}{s+1} & \frac{2}{s+2}-\frac{2}{s+1}\\ -\frac{1}{s+2}+\frac{1}{s+1} & -\frac{1}{s+2}+\frac{2}{s+1}\end{pmatrix}$$
$$=\begin{pmatrix} 2e^{-2t}-e^{-t} & 2e^{-2t}-2e^{-t}\\ -e^{-2t}+e^{-t} & -e^{-2t}+2e^{-t}\end{pmatrix}$$

This method will be used in the following section to find the state transition matrix for the fractional order differential equation system as $\Phi_\alpha(t)=\mathcal{L}^{-1}\left\{(s^\alpha\mathrm{I}-\mathrm{A})^{-1}\right\}$ and $0<\alpha<1$.

7.14 'Alpha-exponential functions' as eigen-functions for the Riemann-Liouville (RL) and Caputo fractional derivative operators

7.14.1 The alpha-exponential functions (1 & 2) for fractional derivative operators are similar to the exponential function as an eigen-function for a classical derivative

In integer order calculus, the function $e^{\lambda t}$ plays an important role in the solution of ordinary differential equation LTI systems, as it satisfies $\frac{\mathrm{d}e^{\lambda t}}{\mathrm{d}t}=\lambda e^{\lambda t}$. We have seen that ${}_0D_t^\alpha\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right]=\lambda t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ and ${}_0^CD_t^\alpha\left[E_\alpha(\lambda t^\alpha)\right]=E_\alpha(\lambda t^\alpha)$. Thus, $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ and $E_\alpha(\lambda t^\alpha)$ could be the candidate functions for fractional differential equations. We call them alpha-exponential functions 1 and 2. This alpha exponential function-1 (i.e. $x(t)=e_\alpha^{\lambda(t-a)}$) satisfies ${}_aD_t^\alpha x(t)=\lambda x(t)$, with an RL derivative. The alpha exponential function-2 (i.e. $x(t)=\tilde{e}_\alpha^{\lambda(t-a)}$) satisfies the Caputo derivative, ${}_a^CD_t^\alpha x(t)=\lambda x(t)$. These are similar to an exponential function that satisfies $D_t^1x(t)=\lambda x(t)$; that is, they are eigen-functions for the two types of fractional derivative operators. We have seen this concept in Section-4.14 of the book. We will further develop it in this part.

We used the notation Φ as a 'state transition matrix' (an associated Green's function) which is $\Phi(t)=e^{\mathrm{A}t}$ for an LTI system of an integer order differential equation system. For a fractional order system, we can (similarly) define the state transition matrix as $\Phi_\alpha(t)=e_\alpha^{\mathrm{A}t}$ and $\tilde{\Phi}_\alpha(t)=\tilde{e}_\alpha^{\mathrm{A}t}$. In this case, the notation, $e_\alpha^{\mathrm{A}t}$ is an alpha-exponential function-1, and $\tilde{e}_\alpha^{\mathrm{A}t}$ is an alpha-exponential function-2. These are also Green's functions and eigenvectors for the RL and Caputo derivatives (respectively) based on homogeneous linear differential equations.

7.14.2 The alpha-exponential functions $\Phi_\alpha(t)=e_\alpha^{\mathrm{A}t}$ and $\tilde{\Phi}_\alpha(t)=\tilde{e}_\alpha^{\mathrm{A}t}$ as related to the Mittag-Leffler function

The alpha-exponential functions follow on from the higher transcendental functions of basic types of the Mittag-Leffler function. The two parameters of the Mittag-Leffler function are defined by the series as $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$, with $\alpha>0$, $\beta>0$ and $z\in\mathbb{C}$ (see Appendix-A). For the matrix $\mathrm{A}\in\mathbb{R}^{n\times n}$, the $E_{\alpha,\beta}(z)$ is extended as follows:

$$E_{\alpha,\beta}(\mathrm{A}t^\alpha)=\sum_{k=0}^{\infty}\mathrm{A}^k\frac{t^{k\alpha}}{\Gamma(k\alpha+\beta)} \tag{7.173}$$

By adding $\beta = \alpha$ to the above (7.173), we can define the alpha-exponential function-1 as:

$$
\begin{aligned}
e_\alpha^{\mathrm{A}t} &= (t^{\alpha-1})E_{\alpha,\alpha}(\mathrm{A}t^\alpha) = (t^{\alpha-1})\sum_{k=0}^{\infty}\mathrm{A}^k \frac{t^{k\alpha}}{\Gamma\big((k+1)\alpha\big)} \\
&= \sum_{k=0}^{\infty}\mathrm{A}^k \frac{t^{(k+1)\alpha-1}}{\Gamma\big((k+1)\alpha\big)}
\end{aligned}
\tag{7.174}
$$

For $\alpha = 1$, we have $E_1(\mathrm{A}t) = e_1^{\mathrm{A}t} = e^{\mathrm{A}t} = \Phi(t)$. This is a state transition matrix of an integer order LTI system with the initial point $t_0 = 0$, as described in the previous section. By adding $\beta = 1$ to the two-parameter Mittag-Leffler function above, we get a single parameter Mittag-Leffler function, as $E_{\alpha,1}(\mathrm{A}t^\alpha) = E_\alpha(\mathrm{A}t^\alpha) = \sum_{k=0}^{\infty}\mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$.

We define the second alpha-exponential function-2 as:

$$
\tilde{e}_\alpha^{\mathrm{A}t} = E_{\alpha,1}(\mathrm{A}t^\alpha) = \sum_{k=0}^{\infty}\mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{7.175}
$$

The alpha-exponential function-1; $e_\alpha^{\mathrm{A}t}$ is useful for solving the sequential fractional differential equations (SFDEs) with the Riemann-Liouville (RL) fractional derivative, while the alpha-exponential-2; $\tilde{e}_\alpha^{\mathrm{A}t}$ is a useful solution for a SFDE with a Caputo derivative.

Since $e_\alpha^{\mathrm{A}t} = t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{A}t^\alpha)$ and each Mittag-Leffler function $E_{\alpha,\alpha}(\lambda z^\alpha)$, $\lambda > 0$ $z \in \mathbb{C}$ is an entire function on the complex plane (see Appendix-A). As such, we can have a uniquely determined function $f(t) = t^{1-\alpha}F(t)$, such that $e_\alpha^{\mathrm{A}t} f(t) = E_{\alpha,\alpha}(\mathrm{A}t^\alpha)F(t) = \mathrm{I}$ for $t \neq 0$ and $\lim_{t\downarrow 0} e_\alpha^{\mathrm{A}t} f(t) = \mathrm{I}$, which is depicted as:

$$
\begin{aligned}
&E_{\alpha,\alpha}(\mathrm{A}t^\alpha)F(t) = \mathrm{I}, \quad \left(\sum_{k=0}^{\infty}\frac{\mathrm{A}^k t^{k\alpha}}{\Gamma\big((k+1)\alpha\big)}\right)F(t) = \mathrm{I} \\
&\left(\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)} + \ldots\right)F(t) = \mathrm{I}
\end{aligned}
\tag{7.176}
$$

Therefore, we have the following limit:

$$
\begin{aligned}
&\lim_{t\downarrow 0} e_\alpha^{\mathrm{A}t} f(t) = \lim_{t\downarrow 0}\left(\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)} + \ldots\right)F(t) = \mathrm{I} \\
&\frac{F(t)}{\Gamma(\alpha)} = \mathrm{I}
\end{aligned}
\tag{7.177}
$$

We have $F(t) = \Gamma(\alpha)$ or $f(t) = t^{1-\alpha}\big(\Gamma(\alpha)\big)$. Therefore for this particular case $f(t) = t^{1-\alpha}\big(\Gamma(\alpha)\big)$, so that the identity condition (i.e. $\lim_{t\downarrow 0} e_\alpha^{\mathrm{A}t} f(t) = \mathrm{I}$) is satisfied. In this way, we can describe the inverse function $f(t) = t^{1-\alpha}\big(\Gamma(\alpha)\big)$ for $e_\alpha^{\mathrm{A}t}$ as $t \downarrow 0$. We have already documented this discussion in Section-7.3.4.

7.14.3 Defining the alpha-exponential functions (1 & 2) via a kernel of convolution as the power law functions and their Laplace transforms

Now let us define a convolution kernel as a power law function and its Laplace transform depicted as:

$$
k_\alpha(t) \stackrel{\text{def}}{=} \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \qquad \alpha > 0 \quad , \quad \mathcal{L}\{k_\alpha(t)\} = \frac{1}{s^\alpha} \qquad \mathrm{Re}[s] > 0 \tag{7.178}
$$

When $\alpha \to 1$, then $k_\alpha(t) \to u(t)$ is the Heaviside unit step function; as $\mathcal{L}\{k_\alpha(t)\} \to s^{-1}$; and, when $\alpha \downarrow 0$, then $k_\alpha(t) \to \delta(t)$ is the Dirac's delta function, as $\mathcal{L}\{k_\alpha(t)\} \to 1$. The alpha-exponential functions can then, via this kernel, be restructured as:

$$e_{\alpha}^{\lambda t}=(t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^{\alpha})=t_{+}^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\lambda t_{+}^{\alpha})^{n}}{\Gamma\left((1+n)\alpha\right)}=\sum_{n=0}^{\infty}\lambda^{n}\left(k_{(1+n)\alpha}(t)\right)$$

$$k_{(1+n)\alpha}(t)=\frac{t^{\left((n+1)\alpha-1\right)}}{\Gamma\left((n+1)\alpha\right)} \tag{7.179}$$

$$\mathcal{L}\left\{e_{\alpha}^{\lambda t}\right\}=\left(s^{\alpha}-\lambda\right)^{-1}$$

The proof of the Laplace transform as described in (7.179) is described next.

The alpha-exponential function-1 above is the same as the Robotnov-Hartley function (i.e. $F_{\alpha}(\lambda,t)$). The proof of the Laplace transform of the alpha-exponential function-1 is the expansion of $e_{\alpha}^{\lambda t}=\sum_{n=0}^{\infty}\lambda^{n}\left(k_{(1+n)\alpha}(t)\right)$, through which we get the following (it should be noted that we are dropping t_{+}, and writing t as a variable for the defined kernel in the following expansion):

$$\begin{aligned} e_{\alpha}^{\lambda t} &= k_{\alpha}(t)+\lambda k_{2\alpha}(t)+\lambda^{2}k_{3\alpha}(t)+\ldots \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda t^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{\lambda^{2}t^{3\alpha-1}}{\Gamma(3\alpha)}+\ldots \end{aligned} \tag{7.180}$$

Using the known identity (i.e. $\mathcal{L}\left\{k_{\alpha}(t)\right\}=\mathcal{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}=\frac{1}{s^{\alpha}}$), we get the following:

$$\begin{aligned} \mathcal{L}\left\{e_{\alpha}^{\lambda t}\right\} &= \mathcal{L}\left\{k_{\alpha}(t)\right\}+\mathcal{L}\left\{\lambda k_{2\alpha}(t)\right\}+\mathcal{L}\left\{\lambda^{2}k_{3\alpha}(t)\right\}+\ldots \\ &= \mathcal{L}\left\{\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}+\lambda\left(\mathcal{L}\left\{\tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}\right\}\right)+\lambda^{2}\left(\mathcal{L}\left\{\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha)}\right\}\right)+\ldots \\ &= s^{-\alpha}+\lambda s^{-2\alpha}+\lambda^{2}s^{-3\alpha}+\ldots.. \\ &= s^{-\alpha}\left(1+\lambda s^{-\alpha}+\lambda^{2}s^{-2\alpha}+\ldots\right) \\ &= s^{-\alpha}\left(1-\lambda s^{-\alpha}\right)^{-1}=\frac{1}{s^{\alpha}\left(1-\lambda s^{-\alpha}\right)} \\ &= \frac{1}{s^{\alpha}-\lambda} \end{aligned} \tag{7.181}$$

We have used in the above derivation (7.181) the series expansion formula $1+x+x^{2}+x^{3}+\ldots\ldots=(1-x)^{-1}$. For the alpha-exponential function-2, we define the following relationship:

$$\tilde{e}_{\alpha}^{\lambda t}=E_{\alpha}(\lambda t^{\alpha})=\sum_{n=0}^{\infty}\frac{(\lambda t_{+}^{\alpha})^{n}}{\Gamma(1+n\alpha)}=\sum_{n=0}^{\infty}\lambda^{n}\left(k_{(1+n\alpha)}(t)\right)$$

$$k_{(1+n\alpha)}(t)=\frac{t_{+}^{(n\alpha+1)-1}}{\Gamma(n\alpha+1)} \tag{7.182}$$

$$\mathcal{L}\left\{\tilde{e}_{\alpha}^{\lambda t}\right\}=s^{\alpha-1}\left(s^{\alpha}-\lambda\right)^{-1}$$

We derive the Laplace transform shown in (7.182) as we did for an earlier case (7.181). Here we use $k_{1}(t)\equiv u(t)$, with $u(t)$ as a Heaviside unit step function with $\mathcal{L}\left\{u(t)\right\}=s^{-1}$. The steps are shown below:

$$\begin{aligned}
\mathcal{L}\left\{\tilde{e}_{\alpha}^{\lambda t}\right\} &= \mathcal{L}\left\{k_1(t)\right\} + \mathcal{L}\left\{\lambda k_{1+\alpha}(t)\right\} + \mathcal{L}\left\{\lambda^2 k_{1+2\alpha}(t)\right\} + \ldots \\
&= \mathcal{L}\left\{u(t)\right\} + \lambda\left(\mathcal{L}\left\{\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right\}\right) + \lambda^2\left(\mathcal{L}\left\{\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right\}\right) + \ldots \\
&= s^{-1} + \lambda s^{-(1+\alpha)} + \lambda^2 s^{-(1+2\alpha)} + \ldots.. \\
&= s^{-1}\left(1 + \lambda s^{-\alpha} + \lambda^2 s^{-2\alpha} + \ldots\right) \\
&= s^{-1}\left(1 - \lambda s^{-\alpha}\right)^{-1} = \frac{1}{s\left(1-\lambda s^{-\alpha}\right)} = \frac{1}{ss^{-\alpha}\left(s^{\alpha}-\lambda\right)} \\
&= \frac{1}{s^{1-\alpha}\left(s^{\alpha}-\lambda\right)} = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda}
\end{aligned} \tag{7.183}$$

7.14.4 The alpha-exponential functions (1 & 2) are related via the convolution relationship

The alpha-exponential function-2 is a one-parameter Mittag-Leffler function. We use the symbol $(*)$ for the convolution operation, and find it to be an interesting convolution link between these two alpha-exponential functions:

$$\begin{aligned}
\tilde{e}_{\alpha}^{\lambda t} = E_{\alpha}(\lambda t^{\alpha}) &= \left(k_{(1-\alpha)}(t)\right) * \left((t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^{\alpha})\right) \\
&= \left(k_{(1-\alpha)}(t)\right) * \left(e_{\alpha}^{\lambda t}\right)
\end{aligned} \tag{7.184}$$

The two alpha-exponential functions are the same for $\alpha = 1$, which is an exponential function:

$$\tilde{e}_1^{\lambda t} = E_1(\lambda t) = \left(e_1^{\lambda t}\right) * \left(k_0(t)\right) = e_1^{\lambda t} = e^{\lambda t}; \qquad k_0(t) = \delta(t) \tag{7.185}$$

7.15 Fractional derivatives of the 'alpha-exponential functions (1 & 2)'

7.15.1 The Caputo derivative of the alpha-exponential function-2 $\tilde{\Phi}_{\alpha}(t-t_0) = \tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}$

We find the Caputo derivative of $\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})$ as follows using the Euler expression of the Caputo derivative (and the RL derivative) of the power function, denoted as the following (it should be noted that the Caputo derivative of the constant function is zero, but is not for the RL derivative):

$${}_{t_0+}^{\ C}D_t^{\alpha}(t-t_0)^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha} & \beta \neq 0 \\ 0 & \beta = 0 \end{cases} \tag{7.186}$$

We apply the above (7.186) to the series-expression and get:

$$\begin{aligned}
{}_{t_0+}^{\ C}D_t^{\alpha}\left[\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}\right] &= {}_{t_0+}^{\ C}D_t^{\alpha}\left[E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})\right] = {}_{t_0+}^{\ C}D_t^{\alpha}\left[\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{(t-t_0)^{\alpha k}}{\Gamma(k\alpha+1)}\right] \\
&= \sum_{k=1}^{\infty}\mathrm{A}^k\,\frac{(t-t_0)^{(k-1)\alpha}}{\Gamma\left((k-1)\alpha+1\right)} = \mathrm{A}E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha}) \\
&= \mathrm{A}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}
\end{aligned} \tag{7.187}$$

Therefore, we have a useful relationship, from the calculations (i.e. ${}_{t_0+}^{\ C}D_t^{\alpha}\left[\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}\right] = \mathrm{A}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}$). This is similar to an exponential function in an integer order differential equation where we have the relationship $\frac{\mathrm{d}e^{\lambda t}}{\mathrm{d}t} = \lambda e^{\lambda t}$. This alpha-exponential $\tilde{e}_{\alpha}^{\mathrm{A}t}$ function-2, therefore, is useful in solving a fractional differential equation with the Caputo derivative formulation. It follows that ${}_{0+}^{\ C}D_t^{\alpha}[\tilde{e}_{\alpha}^{\lambda t}] = \lambda\tilde{e}_{\alpha}^{\lambda t}$ (that is $y(t) = \tilde{e}_{\alpha}^{\lambda t}$) is a fundamental solution (the eigen-function) for the Caputo system, i.e. ${}_{0+}^{\ C}D_t^{\alpha}\left[y(t)\right] = \lambda y(t)$.

7.15.2 The Riemann-Liouville derivative of the alpha-exponential function-1 $\Phi_\alpha(t-t_0)=e_\alpha^{\mathrm{A}(t-t_0)}$

The RL derivative of $e_\alpha^{\mathrm{A}(t-t_0)}$ is evaluated by applying the Euler formula term by term, and using $\lim_{\alpha\downarrow 0}\frac{1}{\Gamma(\alpha)}=0$, for the series expression. As such, we get the following steps:

$$\begin{aligned}
{}_{t_0+}D_t^\alpha\left[e_\alpha^{\mathrm{A}(t-t_0)}\right] &= {}_{t_0+}D_t^\alpha\left[\left((t-t_0)^{\alpha-1}\right)E_{\alpha,\alpha}\left(\mathrm{A}(t-t_0)^\alpha\right)\right]\\
&= {}_{t_0+}D_t^\alpha\left[\left((t-t_0)^{\alpha-1}\right)\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{k\alpha}}{\Gamma\left((k+1)\alpha\right)}\right]\\
&= {}_{t_0+}D_t^\alpha\left[\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma\left((k+1)\alpha\right)}\right]\\
&=\left({}_{t_0+}D_t^\alpha\left[\mathrm{I}\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}+\mathrm{A}\frac{(t-t_0)^{2\alpha-1}}{\Gamma(2\alpha)}+\ldots\right]\right)\\
&= \mathrm{I}\frac{\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(0)}(t-t_0)^{-1}+\mathrm{A}\frac{\Gamma(2\alpha)}{\Gamma(2\alpha)\Gamma(\alpha)}(t-t_0)^{\alpha-1}+\ldots\\
&=\sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{k\alpha-1}}{\Gamma(k\alpha)}=\sum_{m=0}^{\infty}\mathrm{A}^{(m+1)}\frac{(t-t_0)^{(m+1)\alpha-1}}{\Gamma\left((m+1)\alpha\right)}\\
&= \mathrm{A}\left(\sum_{m=0}^{\infty}\mathrm{A}^m\frac{(t-t_0)^{(m+1)\alpha-1}}{\Gamma\left((m+1)\alpha\right)}\right)\\
&= \mathrm{A}e_\alpha^{\mathrm{A}(t-t_0)}
\end{aligned}\tag{7.188}$$

7.15.3 The backward Riemann-Liouville derivative of the alpha-exponential function-1 $\Phi_\alpha(T-t)=e_\alpha^{\mathrm{A}(T-t)}$

Using derivative formulas such as ${}_tD_{T-}^\alpha\left[(T-t)^\gamma\right]=\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\right)(T-t)^{\gamma-\alpha}$ which we described in Section-4.3, we write the following steps:

$$\begin{aligned}
&{}_tD_{T-}^\alpha\left[(T-t)^{\beta-1}\right]=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(T-t)^{\beta-\alpha-1}\\
&\lim_{\beta\to\alpha}\left({}_tD_{T-}^\beta\left[\frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)}\right]\right)=\frac{\Gamma(\alpha)(T-t)^{\alpha-1-\beta}}{\Gamma(\beta-\alpha)\Gamma(\alpha)}=0
\end{aligned}\tag{7.189}$$

We get a useful eigenvector for the Riemann-Liouville operator as depicted below:

$$\begin{aligned}
{}_tD_{T-}^\alpha\left[e_\alpha^{\mathrm{A}(T-t)}\right] &= {}_tD_{T-}^\alpha\left[\left(\mathrm{I}\frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)}+\mathrm{A}\frac{(T-t)^{2\alpha-1}}{\Gamma(2\alpha)}+\ldots\right)\right]\\
&= \mathrm{A}\frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)}+\mathrm{A}^2\frac{(T-t)^{2\alpha-1}}{\Gamma(2\alpha)}+\ldots\ldots\ldots\ldots..\\
&=(T-t)^{\alpha-1}\mathrm{A}\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(T-t)^{k\alpha}}{\Gamma\left((k+1)\alpha\right)}\\
&= \mathrm{A}\left[e_\alpha^{\mathrm{A}(T-t)}\right]
\end{aligned}\tag{7.190}$$

Thus, we have the useful property ${}_{t_0+}D_t^\alpha\left[e_\alpha^{\mathrm{A}(t-t_0)}\right]=\mathrm{A}e_\alpha^{\mathrm{A}(t-t_0)}$, which states that a fractional differential equation involving an RL derivative has a solution in terms of the alpha-exponential function-1, that is, $e_\alpha^{\mathrm{A}t}$. This alpha-exponential $e_\alpha^{\mathrm{A}t}$ function-1, therefore, is useful in solving the fractional differential equation with the Riemann-

Liouville derivative formulation. It follows that $D^{\alpha}[e_{\alpha}^{\lambda t}] = \lambda[e_{\alpha}^{\lambda t}]$ or ${}_{0+}D_t^{\alpha}[e_{\alpha}^{\lambda t}] = \lambda e_{\alpha}^{\lambda t}$ (that is, $y(t) = e_{\alpha}^{\lambda t}$) is a fundamental solution (eigen-function) for the RL system, i.e. ${}_{0+}D_t^{\alpha} y(t) = \lambda y(t)$.

7.15.4 The relationship between the state transition matrices $\Phi_{\alpha}(t)$ and $\tilde{\Phi}_{\alpha}(t)$

Below, we define an interesting relationship between the two alpha-exponential functions:

$$\begin{aligned}\int_{t_0}^{t} \mathrm{A} e_{\alpha}^{\mathrm{A}(t-\tau)} \mathrm{d}\tau &= \int_{t_0}^{t} \sum_{k=0}^{\infty} \mathrm{A}^{k+1} \frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma\left((k+1)\alpha\right)} \mathrm{d}\tau \\ &= \sum_{k=1}^{\infty} \mathrm{A}^{k} \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} = E_{\alpha}\left(\mathrm{A}(t-t_0)^{\alpha}\right) - \mathrm{I} \\ &= \tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} - \mathrm{I}\end{aligned} \tag{7.191}$$

The relationship from (7.191) for the two state transition matrices $\tilde{\Phi}_{\alpha}(t-t_0)$ and $\Phi_{\alpha}(t-\tau)$ is:

$$\begin{aligned}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} &= \mathrm{I} + \int_{t_0}^{t} \mathrm{A} e_{\alpha}^{\mathrm{A}(t-\tau)} \mathrm{d}\tau \\ \tilde{\Phi}_{\alpha}(t-t_0) &= \mathrm{I} + \int_{t_0}^{t} \mathrm{A}\left(\Phi_{\alpha}(t-\tau)\right) \mathrm{d}\tau\end{aligned} \tag{7.192}$$

Denoting $\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t) = \tilde{e}_{\alpha}^{\mathrm{A}t}$ as state transition matrices or as the Green's function for homogeneous fractional multivariate dynamics with the Riemann-Liouville and Caputo derivative formulations respectively, we have the useful expression as described above (7.192).

7.16 The general solution to a sequential fractional differential equation using 'alpha-exponential functions'

7.16.1 The homogeneous SFDE and its characteristic equation (or indicial polynomial)

We now write the sequential fractional order differential equation which we described in Section-7.12 as a homogeneous SFDE, as follows:

$$\begin{aligned}&\left({}_a\mathscr{D}_t^{n\alpha} + \sum\nolimits_{k=0}^{n-1} a_k(t)\, {}_a\mathscr{D}_t^{k\alpha}\right)\left[x(t)\right] = 0 \qquad a \in \mathbb{R} \qquad a_k(t) \in C[a,b] \\ &{}_a\mathscr{D}_t^{k\alpha} = {}_a\mathscr{D}_t^{\alpha}\left({}_a\mathscr{D}_t^{(k-1)\alpha}\right); \quad k = 2,3,.....n\end{aligned} \tag{7.193}$$

For the constant coefficients $a_k(t) \equiv a_k$, we get the LTI system, and the above system (7.193) exhibits an indicial polynomial (or characteristic polynomial) as $P(\lambda) = \lambda^n + \sum\nolimits_{k=0}^{n-1} a_k \lambda^k$. In addition, we reiterate that $\mathscr{D}_t^{\alpha}$ can be of a RL fractional derivative D_t^{α}, or a Caputo type fractional derivative ${}^{C}D_t^{\alpha}$. This is one important fact about SFDEs, which shows that the indicial polynomials are integer order polynomials, just as we get indicial polynomials for an integer order differential equation (linear, quadratic, cubic, etc.). We assume here that $x_1(t), x_2(t), ...x_n(t)$ are n functions, defined in $[a,b]$, as they are seen to be linearly dependent in $[a,b]$ if the constants $c_1, c_2, ..., c_n$ exist and are not zero simultaneously, such that $c_1 x_1(t) + c_2 x_2(t) + ... + c_n x_n(t) = 0$, for $a \le t \le b$. Otherwise, $x_1(t), x_2(t), ...x_n(t)$ will be defined as linearly independent in $[a,b]$.

7.16.2 A generalized Wronskian

To check the linear dependence in a fractional calculus context, we use a generalized Wronskian, which is defined as follows:

$$\mathrm{W}_\alpha(t) = \mathrm{W}_\alpha(x_1, x_2, \ldots x_n)$$
$$= \det\begin{pmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ {}_aD_t^\alpha\left[x_1(t)\right] & {}_aD_t^\alpha\left[x_2(t)\right] & \ldots & {}_aD_t^\alpha\left[x_n(t)\right] \\ \ldots & \ldots & \ldots & \ldots \\ {}_aD_t^{(n-1)\alpha}\left[x_1(t)\right] & {}_aD_t^{(n-1)\alpha}\left[x_2(t)\right] & \ldots & {}_aD_t^{(n-1)\alpha}\left[x_n(t)\right] \end{pmatrix} \quad (7.194)$$

7.16.3 Linearly independent solutions of SFDEs

Let us assume that the fractional derivatives in SFDEs are of the RL type. If $x_1(t), x_2(t), \ldots x_n(t)$ were a solution to our SFDE, then we would have:

$${}_aD_t^\alpha\left[\mathrm{W}_\alpha(t)\right] + (a_{n-1})\mathrm{W}_\alpha(t) = 0 \qquad a \le t \le b \quad (7.195)$$

The general solution of (7.195) is thus:

$$\mathrm{W}_\alpha(t) = ce_\alpha^{-a_{n-1}(t-a)} = c(t-a)^{\alpha-1}E_{\alpha,\alpha}\left(-a_{n-1}(t-a)^\alpha\right) \quad (7.196)$$

with c as a constant. We have obtained this from the previous sections, which have explained the eigen-value of the RL fractional derivative operator. Therefore, the solutions $x_1(t), x_2(t), \ldots x_n(t)$ of the SFDE are linearly dependent in $[a,b]$, if and only if there is a $t_0 \in [a,b]$, for which $\mathrm{W}_\alpha(t_0) = 0$.

If the indicial polynomial $p(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k\lambda^k$ has n different $\lambda_1, \lambda_2, \ldots\lambda_n$ simple roots, then the corresponding SFDE with the RL formulation will have $x_1(t) = e_\alpha^{\lambda_1(t-a)}$ $x_2(t) = e_\alpha^{\lambda_2(t-a)}$ .... $x_n(t) = e_\alpha^{\lambda_n(t-a)}$ and the general solution to the SFDE is given as:

$$x(t) = c_1e_\alpha^{\lambda_1(t-a)} + c_2e_\alpha^{\lambda_2(t-a)} + \ldots\ldots + c_ne_\alpha^{\lambda_n(t-a)} \quad (7.197)$$

The $c_1, c_2, \ldots c_n$ are arbitrary constants in (7.197).

If the indicial polynomial $P(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k\lambda^k$ has repeated the roots with multiplicity $l, (1 < l \le n)$, then $e_\alpha^{\lambda(t-a)}, \frac{\partial}{\partial\lambda}\left[e_\alpha^{\lambda(t-a)}\right], \frac{\partial^2}{\partial\lambda^2}\left[e_\alpha^{\lambda(t-a)}\right], \ldots\ldots\ldots\ldots, \frac{\partial^{l-1}}{\partial\lambda^{l-1}}\left[e_\alpha^{\lambda(t-a)}\right]$ are l linearly independent solutions of SFDE.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be different roots with the multiplicities $l_1, l_2, \ldots., l_k$ respectively, with $l_1 + l_2 + \ldots + l_n = n$, meaning that the general solution of the SFDE is the linear combination of the following fundamental solutions:

$$\begin{array}{ccccc} e_\alpha^{\lambda_1(t-\alpha)}, & \frac{\partial}{\partial\lambda_1}e_\alpha^{\lambda_1(t-\alpha)}, & \frac{\partial^2}{\partial{\lambda_1}^2}e_\alpha^{\lambda_1(t-\alpha)}, & \ldots & \frac{\partial^{l_1-1}}{\partial{\lambda_1}^{l_1-1}}e_\alpha^{\lambda_1(t-\alpha)} \\ e_\alpha^{\lambda_2(t-a)}, & \frac{\partial}{\partial\lambda_2}e_\alpha^{\lambda_2(t-a)}, & \frac{\partial^2}{\partial{\lambda_2}^2}e_\alpha^{\lambda_2(t-a)}, & \ldots & \frac{\partial^{l_2-1}}{\partial{\lambda_2}^{l_2-1}}e_\alpha^{\lambda_2(t-a)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_\alpha^{\lambda_k(t-a)} & \frac{\partial}{\partial\lambda_k}e_\alpha^{\lambda_k(t-a)} & \frac{\partial^2}{\partial{\lambda_k}^2}e_\alpha^{\lambda_k(t-a)} & \ldots & \frac{\partial^{l_k-1}}{\partial{\lambda_k}^{l_k-1}}e_\alpha^{\lambda_k(t-a)} \end{array} \quad (7.198)$$

If the SFDE is formed with the Caputo derivative then we get similar solutions to the above cases with the alpha exponential function-2 (that is, $\tilde{e}_\alpha^{\lambda(t-a)} = E_{\alpha,1}(\lambda(t-a)^\alpha)$).

7.16.4 A demonstration of linearly independent solutions of the SFDE in an equation of motion with fractional order damping

To illustrate this, let us take an example of the equation of motion with a fractional damping term as follows:

$$\ddot{x}(t) + \mu\left({}_0D_t^\alpha\left[x(t)\right]\right) + x(t) = 0 \qquad \mu > 0 \qquad \alpha = 1/2 \quad \text{or} \quad \alpha = 3/2 \quad (7.199)$$

Clearly, this equation (7.199) can be cast into the SFDE as follows:

$$
\begin{aligned}
&{}_0\mathscr{D}_t^{4\alpha}\left[x(t)\right]+\mu\left({}_0\mathscr{D}_t^{\alpha}\left[x(t)\right]\right)+x(t)=0 \qquad \alpha=\tfrac{1}{2}\\
&{}_0\mathscr{D}_t^{4\beta}\left[x(t)\right]+\mu\left({}_0\mathscr{D}_t^{3\beta}\left[x(t)\right]\right)+x(t)=0 \qquad \beta=\frac{1}{2}=\frac{\alpha}{3} \qquad \alpha=\frac{3}{2}
\end{aligned}
\tag{7.200}
$$

The indicial polynomial reads as $P(\lambda)=\lambda^4+\mu\lambda^k+1$ with $k=1$ for $\alpha=\tfrac{1}{2}$ and $k=3$ for $\alpha=\tfrac{3}{2}$. If we take the case for $k=1$, then $P(\lambda)=0=\lambda^4+\mu\lambda+1$, which has four roots (real or complex, distinct or repeated) depending on the value of μ. For a specific case of value μ, we have repeated the real roots (say $\lambda_1=\lambda_2$) and a pair of the complex-conjugate (which we call $\lambda_{3,4}$). For this combination, we write the general solution as:

$$
x(t)=c_1x_1(t)+c_2x_2(t)+c_3x_3(t)+\bar{c}_3\bar{x}_3(t) \tag{7.201}
$$

with $c_1,c_2\in\mathbb{R}$ and $c_3\in\mathbb{C}$ as arbitrary constants; for the case of SFDE (7.200) with $\alpha=\tfrac{1}{2}$.

7.16.4.(a) SFDE with a Caputo derivative

With the Caputo derivative $\mathscr{D}^{\alpha}\equiv {}^{C}D^{\alpha}; \alpha=\tfrac{1}{2}$, we take $x_i(t)=\tilde{e}_{(1/2)}^{\lambda_i t}$; $i=1,3$, that is:

$$
\begin{aligned}
x_1(t)=\tilde{e}_{(1/2)}^{\lambda_1 t}&=\sum_{k=0}^{\infty}\frac{\left(\lambda_1 t^{1/2}\right)^k}{\Gamma\left(\frac{k}{2}+1\right)}\\
&=1+\frac{\lambda_1 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{\lambda_1^2 t}{\Gamma(2)}+\frac{\lambda_1^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{\lambda_1^4 t^2}{\Gamma\left(3\right)}+\ldots
\end{aligned}
\tag{7.202}
$$

Then we have the following:

$$
\begin{aligned}
x_2(t)&=\frac{\partial}{\partial\lambda_1}\left[\tilde{e}_{1/2}^{\lambda_1 t}\right]=\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{2\lambda_1 t}{\Gamma(2)}+\frac{3\lambda_1^2 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{4\lambda_1^3 t^2}{\Gamma(3)}+\ldots\\
x_3(t)&=\tilde{e}_{(1/2)}^{\lambda_1 t}=\sum_{k=0}^{\infty}\frac{\left(\lambda_3 t^{1/2}\right)^k}{\Gamma\left(\frac{k}{2}+1\right)}=1+\frac{\lambda_3 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{\lambda_3^2 t}{\Gamma(2)}+\frac{\lambda_3^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{\lambda_3^4 t^2}{\Gamma(3)}+\ldots\\
x_4(t)&=\bar{x}_3(t)=1+\frac{\bar{\lambda}_3 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{\left(\bar{\lambda}_3\right)^2 t}{\Gamma(2)}+\frac{\left(\bar{\lambda}_3\right)^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{\left(\bar{\lambda}_3\right)^4 t^2}{\Gamma(3)}+
\end{aligned}
\tag{7.203}
$$

To find the general solution $x(t)$ for the SFDE of the above formulated system (7.200), it is required that we study the asymptotic behavior of $x(t)$ as $t\downarrow 0$. For the above systems, it holds that for practical cases (first and second derivatives of a position function) i.e. $\left|x^{(1)}(0)\right|<\infty$ and $\left|x^{(2)}(0)\right|<\infty$, depicts the magnitude of velocity and acceleration respectively. That is a magnitude of velocity and acceleration, which should be bound at the start point of time, $t=0$. Equivalently, the terms involving $t^{1/2}$ and $t^{3/2}$, etc., in (7.202) and (7.203) vanish in the solution $x(t)$; that we describe below:

The $x(t)$ is therefore:

$$
\begin{aligned}
x(t)=&\,c_1\left(1+\frac{\lambda_1 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{\lambda_1^2 t}{\Gamma(2)}+\frac{\lambda_1^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{\lambda_1^4 t^2}{\Gamma(3)}+\ldots\right)\\
&+c_2\left(\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{2\lambda_1 t}{\Gamma(2)}+\frac{3\lambda_1^2 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{4\lambda_1^3 t^2}{\Gamma(3)}+\ldots\right)\\
&+c_3\left(1+\frac{\lambda_3 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{\lambda_3^2 t}{\Gamma(2)}+\frac{\lambda_3^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{\lambda_3^4 t^2}{\Gamma(3)}+\ldots\right)\\
&+\bar{c}_3\left(1+\frac{\bar{\lambda}_3 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+\frac{(\bar{\lambda}_3)^2 t}{\Gamma(2)}+\frac{(\bar{\lambda}_3)^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{(\bar{\lambda}_3)^4 t^2}{\Gamma(3)}+\right)
\end{aligned}
\tag{7.204}
$$

For the condition $\left|x^{(1)}(0)\right| < \infty$ to be bounded at the start, $t = 0$, we need $t^{1/2}$ in the above $x(t)$ to vanish, or else $x^{(1)}(t) \sim t^{-1/2}$ will blow up at $t \downarrow 0$. This means we have a condition $c_1\lambda_1 + c_2 + c_3\lambda_3 + \bar{c}_3\bar{\lambda}_3 = 0$. Similarly, for $\left|x^{(2)}(0)\right| < \infty$ to be bounded, we need $t^{3/2}$ terms in $x(t)$ to vanish, or otherwise $x^{(2)}(t) \sim t^{-1/2}$ terms will blow up at $t \downarrow 0$. That gives us the condition $c_1\lambda_1^3 + 3c_2\lambda_1^2 + c_3\lambda_3^3 + \bar{c}_3\bar{\lambda}_3^3 = 0$; both are summarized below:

$$
\begin{gathered}
c_1\lambda_1 + c_2 + c_3\lambda_3 + \bar{c}_3\bar{\lambda}_3 = 0 \\
c_1\lambda_1^3 + 3c_2\lambda_1^2 + c_3\lambda_3^3 + \bar{c}_3\bar{\lambda}_3^3 = 0
\end{gathered}
\tag{7.205}
$$

With such a complex number c_3, the arbitrary real constants c_1, c_2 in the general solution $x(t)$ of the SFDE equation are determined by an initial condition of $x(0) = x_0$; $x^{(1)}(0) = v_0$ and $x^{(2)}(0) = a_0$.

7.16.4.(b) SFDE with a Riemann-Liouville (RL) derivative

For case of SFDE with $\alpha = \frac{1}{2}$ and with the RL derivative, $\mathscr{D}^\alpha \equiv D^\alpha; \alpha = \frac{1}{2}$, we take $x_i(t) = e_{1/2}^{\lambda_i t}$, $i = 1, 3$. Then:

$$
\begin{aligned}
x_1(t) &= e_{(1/2)}^{\lambda_1 t} = t^{(1/2)-1}\sum_{k=0}^{\infty}\frac{\left(\lambda_1 t^{1/2}\right)^k}{\Gamma\left(\frac{1}{2}(k+1)\right)} = \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\lambda_1}{\Gamma(1)} + \frac{\lambda_1^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{\lambda_1^3 t}{\Gamma(2)} + \dots \\
x_2(t) &= \frac{\partial}{\partial\lambda_1} e_{1/2}^{\lambda_1 t} = \frac{1}{\Gamma(1)} + \frac{2\lambda_1 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{3\lambda_1^2 t}{\Gamma(2)} + \frac{4\lambda_1^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + \dots \\
x_3(t) &= e_{(1/2)}^{\lambda_3 t} = t^{-1/2}\sum_{k=0}^{\infty}\frac{\left(\lambda_3 t^{1/2}\right)^k}{\Gamma\left(\frac{1}{2}(k+1)\right)} = \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\lambda_3}{\Gamma(1)} + \frac{\lambda_3^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{\lambda_3^3 t}{\Gamma(2)} + \dots \\
x_4(t) &= \bar{x}_3(t) = \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\bar{\lambda}_3}{\Gamma(1)} + \frac{(\bar{\lambda}_3)^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{(\bar{\lambda}_3)^3 t}{\Gamma(2)} + \dots
\end{aligned}
\tag{7.206}
$$

Thus, $x(t)$ is:

$$
\begin{aligned}
x(t) = c_1&\left(\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\lambda_1}{\Gamma(1)} + \frac{\lambda_1^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{\lambda_1^3 t}{\Gamma(1)} + \dots\right) \\
&+ c_2\left(\frac{1}{\Gamma(1)} + \frac{2\lambda_1 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{3\lambda_1^2 t}{\Gamma(2)} + \frac{4\lambda_1^3 t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + \dots\right) \\
&+ c_3\left(\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\lambda_3}{\Gamma(1)} + \frac{\lambda_3^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{\lambda_3^3 t}{\Gamma(2)} + \dots\right) \\
&+ \bar{c}_3\left(\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\bar{\lambda}_3}{\Gamma(1)} + \frac{(\bar{\lambda}_3)^2 t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{(\bar{\lambda}_3)^3 t}{\Gamma(2)} + \dots\right)
\end{aligned}
\tag{7.207}
$$

To find the general solution $x(t)$ for the SFDE of the system in (7.200), we must study the asymptotic behavior of $x(t)$ as $t \downarrow 0$. In order that $\left|x(0)\right| < \infty$ and $\left|x^{(1)}(0)\right| < \infty$, those are magnitudes of the position and the velocity at the initial point of time ($t = 0$) to be bounded. Therefore, two terms appear in $x(t)$ (7.207), which has multiplying terms, as $t^{-1/2}$ and $t^{1/2}$ are $(c_1 + c_3 + \bar{c}_3)\left(\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}\right)$ and $\left(c_1\lambda_1^2 + 2c_2\lambda_1 + c_3\lambda_3^2 + \bar{c}_3\bar{\lambda}_3^2\right)\left(\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}\right)$, respectively, should be zero. The reasoning is similar to that which we discussed above for the Caputo case. Therefore, we have:

$$
\begin{gathered}
c_1 + c_3 + \bar{c}_3 = 0 \\
c_1\lambda_1^2 + 2c_2\lambda_1 + c_3\lambda_3^2 + \bar{c}_3\bar{\lambda}_3^2 = 0
\end{gathered}
\tag{7.208}
$$

With such a complex number as c_3, the arbitrary real constants c_1, c_2 in the general solution $x(t)$ of the above system SFDE equation (7.200) are determined by the initial condition $x(0) = x_0$ and $x^{(1)}(0) = v_0$.

7.17 The solution of a multivariate system of a fractional order differential equation with the RL Caputo derivative using a state transition matrix of the 'alpha-exponential functions -1 & 2'

7.17.1 The multivariate system with an RL derivative and its solution with the state transition matrix

We take a fractional differential equation with an RL fractional derivative as follows:

$$0 < \alpha \le 1 \qquad (D^{\alpha}\mathrm{Y})(t) = \mathrm{AY}(t) + \mathrm{B}(t) \qquad \mathrm{Y}_0 = \mathrm{Y}(t_0) \tag{7.209}$$

We write the general solution of (7.209) as follows:

$$\begin{aligned} \mathrm{Y}(t) &= e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Y}_0 + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\left(\mathrm{B}(\tau)\right)\mathrm{d}\tau \\ &= \Phi_{\alpha}(t-t_0) + \int_{t_0}^{t} \Phi_{\alpha}(t-\tau)\left(\mathrm{B}(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{7.210}$$

Where the fundamental solution or the Green's function (RL-Case) is as follows:

$$G_{\alpha}(t-\tau) = \Phi_{\alpha}(t-\tau) = e_{\alpha}^{\mathrm{A}(t-\tau)} \tag{7.211}$$

We write the particular integral as the following convolution (with a start point of integration as t_0):

$$\mathrm{Y}_p(t) = (G_{\alpha} *^{t_0} \mathrm{B})(t) = \int_{t_0}^{t} G_{\alpha}(t-\tau)\left(\mathrm{B}(\tau)\right)\mathrm{d}\tau \tag{7.212}$$

7.17.2 A Multivariate system with the Caputo derivative and its solution with state transition matrices

For a fractional differential equation with the Caputo derivative for $0 < \alpha \le 1$, the homogeneous one is:

$$\left({}_{t_0+}^{\;C}D_t^{\alpha}\mathrm{Y}\right)(t) = \mathrm{AY}(t) \tag{7.213}$$

While $\mathrm{Y}(t_0) = b$ has a general solution as follows:

$$\mathrm{Y}(t) = b + \int_{t_0}^{t}\left(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b\right)\mathrm{d}\tau = b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,\mathrm{d}\tau\right) \tag{7.214}$$

The RL-Caputo relationship for $0 < \alpha \le 1$ is $\left({}_{t_0}^{C}D_t^{\alpha} f\right)(t) = {}_{t_0}D_t^{\alpha}\left[f(t) - f(t_0)\right]$ (Section-3.10). Using this, we get the following:

$$\left({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y}\right)(t) = \mathrm{AY}(t) \qquad {}_{t_0}D_t^{\alpha}\left[\mathrm{Y}(t) - b\right] = \mathrm{AY}(t) \tag{7.215}$$

By putting $\mathrm{Y}(t) = \mathrm{Z}(t) + b$ in (7.215), we get $\mathrm{Z}(t_0) = 0 = \mathrm{Z}_0$. Thus, the equivalent equation in the RL derivative based on a fractional differential equation is:

$$\left({}_{t_0}D_t^{\alpha}\mathrm{Z}\right)(t) = \mathrm{A}\left(\mathrm{Z}(t) + b\right) = \mathrm{AZ}(t) + \mathrm{A}b \tag{7.216}$$

The solution is as follows:

$$\mathrm{Z}(t) = e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Z}_0 + \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau = \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau = \mathrm{Y}(t) - b \tag{7.217}$$

We have thus the below result:

$$\mathrm{Y}(t) = b + \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau = b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,\mathrm{d}\tau\right) \tag{7.218}$$

Similarly for a system as described, it is as follows:

$$\left({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y}\right)(t) = \mathrm{AY}(t) + \mathrm{B}(t) \tag{7.219}$$

Setting $\mathrm{Y}(t_0) = b$ as a constant, we write the solution below:

$$\begin{aligned}\mathrm{Y}(t) &= b + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)} \big(\mathrm{B}(\tau) + \mathrm{A}b\big)\mathrm{d}\tau \\ &= b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)} \mathrm{A}\,\mathrm{d}\tau\right) + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)} \mathrm{B}(\tau)\mathrm{d}\tau\end{aligned} \tag{7.220}$$

which is also:

$$\mathrm{Y}(t) = b\ \tilde{\Phi}_{\alpha}(t) + \int_{t_0}^{t} \big(\Phi_{\alpha}(t-\tau)\big)\big(\mathrm{B}(\tau)\big)\,\mathrm{d}\tau \tag{7.221}$$

where state transition matrices are $\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t) = \tilde{e}_{\alpha}^{\mathrm{A}t}$. The solution of SFDE with the Caputo's formulation requires thus two Green's functions (or state transition matrices).

$$ {}_{0+}^{\ C}D_t^{\alpha}\big[x(t)\big] = u(t) \qquad 0 < \alpha < 1; \quad x(0) = a \in \mathbb{R}; \quad x(T) = b \in \mathbb{R}; \quad T > 0 \tag{7.222}$$

In terms of a system matrix equation, we have:

$$ {}^{C}D^{\alpha}\big[x(t)\big] = \mathrm{A}x(t) + \mathrm{B}u(t) \tag{7.223}$$

in this case, $\mathrm{A} = 0;\quad \mathrm{B} = 1$. Here, $u(t)$ is a general input function and not a Heaviside step, and:

$$\begin{aligned}\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t} &= t^{\alpha-1}\left(\sum_{k=0}^{\infty} \frac{\mathrm{A}^{k} t^{k\alpha}}{\Gamma\big((k+1)\alpha\big)}\right) \\ &= t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^{\alpha}}{\Gamma(2\alpha)} + \ldots\right) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \qquad ;\mathrm{A} = 0\end{aligned} \tag{7.224}$$

For $\mathrm{A} = 0$, we have $\tilde{\Phi}_{\alpha}(t) = \mathrm{I} + \int_{0}^{t} e_{\alpha}^{\mathrm{A}\tau} \mathrm{A}\,\mathrm{d}\tau = \mathrm{I} = 1$. Therefore, the state trajectory of the system for the solution is:

$$\begin{aligned}x(t)\Big|_0^T &= x(0)\tilde{\Phi}_{\alpha}(t) + \int_{0}^{T} \big(\Phi_{\alpha}(t-\tau)\big)\big(u(\tau)\big)\,\mathrm{d}\tau \\ &= a + \frac{1}{\Gamma(\alpha)} \int_{0}^{T} (T-t)^{\alpha-1}\big(u(t)\big)\mathrm{d}t\end{aligned} \tag{7.225}$$

7.17.3 Formalizing the multivariate problem to get a state trajectory as a solution with the defined state transition matrices

Now we formalize what we conducted in the above discussions, and use the results, with alpha-exponential functions. Consider a linear time invariant system as denoted by $\sum$ of the fractional commensurate order α, where $0 < \alpha \le 1$.

$$\sum: \qquad {}_{0+}^{\ C}D_t^{\alpha}\big[x(t)\big] = \mathrm{A}x(t) + \mathrm{B}u(t) \qquad y(t) = \mathrm{C}x(t) \tag{7.226}$$

Considering what a commensurate order means, for each component the same fractional order of α is used. For a function $x:\big[0,T\big] \to \mathbb{R}^{n\times 1}$:

$$ {}_{0+}^{\ C}D_t^{\alpha}\big[x(t)\big] = {}_{0+}^{\ C}D_t^{\alpha}\begin{pmatrix} x_1(t) \\ \cdot \\ \cdot \\ x_n(t)\end{pmatrix} = \begin{pmatrix} {}_{0+}^{\ C}D_t^{\alpha} x_1(t) \\ \cdot \\ \cdot \\ {}_{0+}^{\ C}D_t^{\alpha} x_n(t)\end{pmatrix} \tag{7.227}$$

Where, $x(t) \in \mathbb{R}^{n\times 1}$ $u(t) \in \mathbb{R}^{m\times 1}$ matrix $\mathrm{A} \in \mathbb{R}^{n\times n}, \mathrm{B} \in \mathbb{R}^{n\times m}, \mathrm{C} \in \mathbb{R}^{p\times n}$. The Caputo fractional derivative ${}^{C}D_t^{\alpha}$ is used here. The control is $u(t) \in \mathbb{R}^{m\times 1}$ (also called forcing input); the state is $x(t) \in \mathbb{R}^{n\times 1}$, while the output (or observation) is $y(t) \in \mathbb{R}^{p\times 1}$.

The forward trajectory of the system $\sum$ above starting at $t_0 = 0$ and evaluated at $t \geq 0$ is the initial value problem of a fractional differential equation ${}_{0+}^{C}D_t^{\alpha}\left[x(t)\right] = \mathrm{A}x(t) + \mathrm{B}u(t)$, with $x(0) = a$, where $a \in \mathbb{R}^{n\times 1}$

The solution $x(t)$ of (7.226) is expressed as a state trajectory as follows:

$$x(t) = \left(\mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau\right)a + \int_0^t \left(\Phi_\alpha(t-\tau)\right)\mathrm{B}\left(u(\tau)\right)\mathrm{d}\tau \tag{7.228}$$

In the above expression (7.228), we have $\Phi_\alpha(t) = e_\alpha^{\mathrm{A}t}$. Another way to write this expression is as below:

$$\begin{aligned} x(t) &= a\left(\tilde{\Phi}_\alpha(t)\right) + \int_0^t \Phi_\alpha(t-\tau)\mathrm{B}\left(u(\tau)\right)\mathrm{d}\tau \\ \tilde{\Phi}_\alpha(t) &= E_\alpha(\mathrm{A}t^\alpha) = \mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau \\ y(t) &= \mathrm{C}x(t) = \mathrm{C}a\left(\mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau\right) + \mathrm{C}\int_0^t \Phi_\alpha(t-\tau)\mathrm{B}\left(u(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{7.229}$$

7.17.4 An application to obtain the state trajectory solution for a multivariate fractional differential equation system with a given initial condition and a forcing function

Let system $\sum 1$ be defined as:

$$\sum 1: \quad \begin{cases} {}^{C}D_{0+}^{1/2}\left[x_1(t)\right] = x_2(t) \\ {}^{C}D_{0+}^{1/2}\left[x_2(t)\right] = u(t) \end{cases} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.230}$$

Take the initial point $a = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, and the final point as origin $b = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$.

$$\begin{aligned} \Phi_\alpha(t) &= \mathcal{L}^{-1}\left\{\left(s^{1/2}[\mathrm{I}] - [\mathrm{A}]\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{s} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\begin{pmatrix} \sqrt{s} & -1 \\ 0 & \sqrt{s} \end{pmatrix}^{-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\begin{pmatrix} \sqrt{s} & 1 \\ 0 & \sqrt{s} \end{pmatrix}\right\} = \mathcal{L}^{-1}\begin{pmatrix} \frac{1}{\sqrt{s}} & \frac{1}{s} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} \end{aligned} \tag{7.231}$$

Using $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$; $\quad \mathcal{L}^{-1}\left\{s^{-1/2}\right\} = \frac{1}{\sqrt{\pi t}}$ $\quad \mathcal{L}\left\{1\right\} = \frac{1}{s}$, $t > 0$, we write:

$$\begin{aligned} \Phi_\alpha(t) &= \begin{pmatrix} \frac{1}{\sqrt{\pi t}} & 1 \\ 0 & \frac{1}{\sqrt{\pi t}} \end{pmatrix} \\ \tilde{\Phi}_\alpha(t) &= \mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi\tau}} & 1 \\ 0 & \frac{1}{\sqrt{\pi\tau}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathrm{d}\tau \\ &= \begin{pmatrix} 1 & 2\sqrt{\frac{t}{\pi}} \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{7.232}$$

Using the formulas as obtained above in (7.228) and (7.229), we get the following state trajectory:

$$x(t) = \begin{pmatrix} 1 & \frac{2\sqrt{t}}{\sqrt{\pi}} \\ 0 & 1 \end{pmatrix}a + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi(t-\tau)}} & 1 \\ 0 & \frac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix}\mathrm{B}\left(u(\tau)\right)\mathrm{d}\tau \tag{7.233}$$

Taking $u(t) \equiv 1$, then for a given $a = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$; $\mathrm{B} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ (7.230), we get $x(t) = \begin{pmatrix} (1+t) & \left(2\sqrt{\frac{t}{\pi}}\right) \end{pmatrix}^T$ as derived in the following steps:

$$\begin{aligned} x(t) &= \begin{pmatrix} 1 & \frac{2\sqrt{t}}{\sqrt{\pi}} \\ 0 & 1 \end{pmatrix} a + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi(t-\tau)}} & 1 \\ 0 & \frac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix} \mathrm{B}\left(u(\tau)\right) \mathrm{d}\tau \\ &= \begin{pmatrix} 1 & \frac{2\sqrt{t}}{\sqrt{\pi}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi(t-\tau)}} & 1 \\ 0 & \frac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix} \mathrm{d}\tau = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 2\sqrt{\frac{t}{\pi}} \end{pmatrix} \\ &= \begin{pmatrix} (1+t) \\ 2\sqrt{\frac{t}{\pi}} \end{pmatrix} \end{aligned} \tag{7.234}$$

7.17.5 An application to obtain the state trajectory solution for a multivariate fractional differential equation with matrix A as a skew-symmetric system with a given initial condition and forcing function

For $0 < \alpha < 1$, a system in $\mathbb{R}^3$, that is, $\sum 2$, is described as:

$$\sum 2: \quad \begin{cases} {}^C D_{0+}^{\alpha}\left[x_1(t)\right] = x_2(t) \\ {}^C D_{0+}^{\alpha}\left[x_2(t)\right] = -x_1(t) + u(t) \end{cases}$$
$$\mathrm{A}^0 = \mathrm{I} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathrm{A}^2 = -\mathrm{I} \tag{7.235}$$

The system matrix A is skew symmetric; hence $\mathrm{A}^k = \mathrm{I}$ for $k = 0, 2, 4, 8, ..,$ $\mathrm{A}^k = \mathrm{A}$ for $k = 1, 5, 9, ...$ and $\mathrm{A}^k = -\mathrm{A}$ for $k = 3, 7, 11, ...$

We write (7.235) as follows:

$$\begin{aligned} &{}^C D_{0+}^{\alpha}\left[x_1(t)\right] = x_2(t) \\ &{}^C D_{0+}^{\alpha}\left[x_2(t)\right] = -x_1(t) + u(t) \\ &{}^C D_{0+}^{\alpha} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &{}^C D_{0+}^{\alpha}\left[x(t)\right] = \mathrm{A}\left[x(t)\right] + \mathrm{B}u(t) \end{aligned} \tag{7.236}$$

We have for $\Phi_\alpha(t)$ the following series:

$$\Phi_\alpha(t) = t^{\alpha-1}\left(\mathrm{A}^0 \frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)} + \mathrm{A}^2 \frac{t^{2\alpha}}{\Gamma(3\alpha)} + \mathrm{A}^3 \frac{t^{3\alpha}}{\Gamma(4\alpha)} + ... \right) \tag{7.237}$$

Now with the property of this skew symmetric matrix A, we get the following:

$$\begin{aligned} \Phi_\alpha(t) &= t^{\alpha-1}\left(\mathrm{I}\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)} - \mathrm{I}\frac{t^{2\alpha}}{\Gamma(3\alpha)} - \mathrm{A}\frac{t^{3\alpha}}{\Gamma(4\alpha)} + ... \right) \\ &= \mathrm{I}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + .. \right) + \mathrm{A}\left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{4\alpha-1}}{\Gamma(4\alpha)} + ... \right) \end{aligned} \tag{7.238}$$

Use the following notation, to simplify the above expression (7.237):

$$\mathrm{s}_\alpha(t) = \left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{4\alpha-1}}{\Gamma(4\alpha)} + ... \right); \qquad \mathrm{c}_\alpha(t) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + ... \right) \tag{7.239}$$

we get the state transition matrix $\Phi_\alpha(t)$ in this case as follows:

$$\Phi_\alpha(t) = \mathrm{I}\left(\mathrm{c}_\alpha(t)\right) + \mathrm{A}\left(\mathrm{s}_\alpha(t)\right) = \begin{pmatrix} \mathrm{c}_\alpha(t) & \mathrm{s}_\alpha(t) \\ -\mathrm{s}_\alpha(t) & \mathrm{c}_\alpha(t) \end{pmatrix} \tag{7.240}$$

$$\begin{aligned} \tilde{\Phi}_\alpha(t) &= \mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\mathrm{d}\tau \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \mathrm{c}_\alpha(\tau) & \mathrm{s}_\alpha(\tau) \\ -\mathrm{s}_\alpha(\tau) & \mathrm{c}_\alpha(\tau) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} -\mathrm{s}_\alpha(\tau) & \mathrm{c}_\alpha(\tau) \\ \mathrm{s}_\alpha(\tau) & \mathrm{c}_\alpha(\tau) \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} 1 - \int_0^t \mathrm{s}_\alpha(\tau)\mathrm{d}\tau & \int_0^t \mathrm{c}_\alpha(\tau)\mathrm{d}\tau \\ \int_0^t \mathrm{s}_\alpha(\tau)\mathrm{d}\tau & 1 + \int_0^t \mathrm{c}_\alpha(\tau)\mathrm{d}\tau \end{pmatrix} \end{aligned} \tag{7.241}$$

Using (7.240) and (7.241), we use the formula $x(t) = a\left(\tilde{\Phi}_\alpha(t)\right) + \int_0^t \Phi_\alpha(t-\tau)\mathrm{B}\left(u(\tau)\right)\mathrm{d}\tau$ in (7.228) and (7.229) to attain the state $x(t) = \left(x_1(t) \quad x_2(t)\right)^T$ with the trajectory for any given a acting as the initial point, and the forcing input $u(t)$.

7.18 The solution to a fractional differential equation of type $\sigma + b\left({}^*D^\alpha\right)\sigma = \mathrm{E}_0\varepsilon + \mathrm{E}_1\left({}^*D^\alpha\right)\varepsilon$ with RL or Caputo formulations

Let us take the following system of visco-elasticity, as we described and analyzed in Chapter-6, that is following

$$\begin{gathered} \sigma(t) + b\left({}^*D^\alpha\sigma\right)(t) = \mathrm{E}_0\varepsilon(t) + \mathrm{E}_1\left({}^*D^\alpha\varepsilon\right)(t) \\ 0 < \alpha \le 1, \quad b \ge 0, \quad \mathrm{E}_0 \ge 0, \quad \mathrm{E}_1 > 0, \quad b \le (\mathrm{E}_1/\mathrm{E}_0) \end{gathered} \tag{7.242}$$

7.18.1 Using an RL-derivative formulation for a solution of ε with a known σ input

We will obtain a solution for $\varepsilon(t)$, for a known input function $\sigma(t)$, and then a solution for $\sigma(t)$, for a known $\varepsilon(t)$. The solution for a known input function $\sigma(t)$ is:

$$\left({}^*D^\alpha z\right)(t) + \lambda z(t) = X(t) \tag{7.243}$$

With $z(t) = \left(\frac{b}{\mathrm{E}_1}\right)\sigma(t) - \varepsilon(t)$, $\lambda = \frac{\mathrm{E}_0}{\mathrm{E}_1}$, $X(t) = A\sigma(t)$ and $A = \frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}$. The general solution of the above (7.243) with the RL derivative ${}^*D^\alpha \equiv {}_aD^\alpha_{t,+}$ and with $\sigma(t)$ is continuous and integrable in $[a,t]$, and is:

$$\begin{aligned} z(t) &= Ce_\alpha^{-\lambda(t-a)} + \left(e_\alpha^{-\lambda(t-a)}\right) *^a \left(X(t)\right) \\ &= Ce_\alpha^{-\lambda(t-a)} + \int_a^t \left(e_\alpha^{-\lambda(t-\xi)}\right)\left(X(\xi)\right)\mathrm{d}\xi \\ &= Ce_\alpha^{-\lambda(t-a)} + A\int_a^t e_\alpha^{-\lambda(t-\xi)}\left(\sigma(\xi)\right)\mathrm{d}\xi \end{aligned} \tag{7.244}$$

where C is an arbitrary real constant. Therefore, for the general solution to the above model in (7.242) and (7.243) (with a change of variables from the original one), we have:

$$\varepsilon(t) = \frac{b}{\mathrm{E}_1}\sigma(t) - Ce_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-a)} - \left(\frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}\right)\int_a^t e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-\xi)}\left(\sigma(\xi)\right)\mathrm{d}\xi \tag{7.245}$$

A $t=a$, we have $\varepsilon(a)=\frac{b}{\mathrm{E}_1}\sigma(a)-C\Phi_\alpha(0)$, where $\Phi_\alpha(t)=e_\alpha^{-\lambda t}=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ and $\Phi_\alpha(0)=\infty$. From here, we obtain $C=\left(\Phi_\alpha^{-1}(0)\right)\left(\frac{b}{\mathrm{E}_1}\sigma(a)-\varepsilon(a)\right)$. However, as per our earlier argument (in Section-7.3 and 7.14), $\lim_{t\downarrow 0} f(t)\Phi_\alpha(t)=1$ gives us an inverse function for $\Phi_\alpha(t)$ at origin, that is, $f(t)=t^{1-\alpha}\left(F(t)\right)$ with $F(t)=\Gamma(\alpha)$. Therefore, we have $\lim_{t\downarrow 0}\Phi_\alpha^{-1}(t)=t^{1-\alpha}\left(\Gamma(\alpha)\right)$. Using this concept of an inverse function of $\Phi_\alpha(t)=e_\alpha^{-\lambda t}$, we write in the limiting case the following value of C at $t=a$:

$$\begin{aligned}C&=\lim_{t\to a}\left(\Phi_\alpha^{-1}(t-a)\right)\left(\tfrac{b}{\mathrm{E}_1}\sigma(t)-\varepsilon(t)\right)\\&=\lim_{t\to a}\left((t-a)^{1-\alpha}\left(\Gamma(\alpha)\right)\right)\left(\tfrac{b}{\mathrm{E}_1}\sigma(t)-\varepsilon(t)\right)\\&=\left(\Gamma(\alpha)\right)\left(\tfrac{b}{\mathrm{E}_1}\lim_{t\to a}(t-a)^{1-\alpha}\sigma(t)-\lim_{t\to a}\varepsilon(t)\right)\end{aligned}\tag{7.246}$$

Thus, for the case noted where the initial conditions are specified as $\lim_{t\to a+}(t-a)^{1-\alpha}\sigma(t)=k_1$, and $\lim_{t\to a+}(t-a)^{1-\alpha}\varepsilon(t)=k_2$, the solution is:

$$\begin{aligned}&\varepsilon(t)=\left(\tfrac{b}{\mathrm{E}_1}\right)\sigma(t)-Ce_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-a)}-\left(\tfrac{b\mathrm{E}_0-1}{\mathrm{E}_1^2}\right)\int_a^t e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-\xi)}\left(\sigma(\xi)\right)\mathrm{d}\xi\\&C=\left(\tfrac{b}{\mathrm{E}_1}k_1-k_2\right)\Gamma(\alpha)\end{aligned}\tag{7.247}$$

Moreover, if $C=0$ means that $z(a)=0$, i.e. $\varepsilon(a)=\left(\frac{b}{\mathrm{E}_1}\right)\sigma(a)$, in particular $\varepsilon(a)=\sigma(a)=0$, then we write:

$$\varepsilon(t)=\left(\tfrac{b}{\mathrm{E}_1}\right)\sigma(t)-\left(\tfrac{b\mathrm{E}_0-1}{\mathrm{E}_1^2}\right)\int_a^t e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-\xi)}\left(\sigma(\xi)\right)\mathrm{d}\xi\tag{7.248}$$

7.18.2 Using the Caputo derivative formulation for a solution of ε with a known σ input

In addition, we can write the general solution of the equation $\left({}^*D^\alpha z\right)(t)+\lambda z(t)=X(t)$, with the Caputo derivative ${}^*D^\alpha z\equiv {}_a^C D_{t,+}^\alpha z$, from the derived cases as in the previous sections, giving:

$$z(t)=k+\int_a^t e_\alpha^{-\lambda(t-\xi)}\left(A\sigma(\xi)-\lambda k\right)\mathrm{d}\xi\qquad z(a)=k\tag{7.249}$$

The corresponding solution to the following system with a Caputo derivative:

$$\begin{aligned}&\sigma(t)+b\left({}^C D^\alpha\sigma\right)(t)=\mathrm{E}_0\varepsilon(t)+\mathrm{E}_1\left({}^C D^\alpha\varepsilon\right)(t)\\&0<\alpha\le 1,\quad b\ge 0,\quad \mathrm{E}_0\ge 0,\quad \mathrm{E}_1>0,\quad b\le\frac{\mathrm{E}_1}{\mathrm{E}_0}\end{aligned}\tag{7.250}$$

is:

$$\varepsilon(t)=k+\frac{b}{\mathrm{E}_1}\sigma(t)-\int_a^t e_\alpha^{-\lambda(t-\xi)}\left(\frac{b\mathrm{E}_0-1}{\mathrm{E}_1^2}\sigma(\xi)-\lambda k\right)\mathrm{d}\xi\tag{7.251}$$

7.18.3 Using the RL-derivative formulation for a solution of σ with a known ε input

The solution for a known function $\varepsilon(t)$ is:

$$\begin{aligned}&\left({}^*D^\alpha y\right)(t)+\gamma y(t)=g(t)\\&y(t)=\frac{\mathrm{E}_1}{b}\varepsilon(t)-\sigma(t);\quad \gamma=\frac{1}{b};\quad g(t)=B\varepsilon(t);\quad B=\frac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\end{aligned}\tag{7.252}$$

If $\varepsilon(t)$ is continuous and integrable in $[a,t]$ then the general solution of the above with an RL derivative ${}^*D^\alpha\equiv {}_aD_{t,+}^\alpha$ is:

$$y(t) = Ce_\alpha^{-\gamma(t-a)} + B\int_a^t e_\alpha^{-\gamma(t-\xi)}\left(\varepsilon(\xi)\right)\mathrm{d}\xi \tag{7.253}$$

Therefore, the general solution to equation (7.242) with all RL derivatives is:

$$\begin{gathered}\sigma(t) + b\left(D^\alpha\sigma\right)(t) = \mathrm{E}_0\varepsilon(t) + \mathrm{E}_1\left(D^\alpha\varepsilon\right)(t) \\ 0<\alpha\le 1, \quad b\ge 0, \quad \mathrm{E}_0\ge 0, \quad \mathrm{E}_1>0, \quad b\le\frac{\mathrm{E}_1}{\mathrm{E}_0}\end{gathered} \tag{7.254}$$

and then the following:

$$\sigma(t) = \left(\tfrac{\mathrm{E}_1}{b}\right)\varepsilon(t) - Ce_\alpha^{-\frac{1}{b}(t-a)} - \left(\tfrac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\right)\int_a^t e_\alpha^{-\frac{1}{b}(t-\xi)}\left(\varepsilon(\xi)\right)\mathrm{d}\xi \tag{7.255}$$

While the initial conditions are given as $\lim_{t\to a+}(t-a)^{1-\alpha}\sigma(t) = k_1$ and $\lim_{t\to a+}(t-a)^{1-\alpha}\varepsilon(t) = k_2$, the solution is:

$$\begin{aligned}&\sigma(t) = \left(\tfrac{\mathrm{E}_1}{b}\right)\varepsilon(t) - Ce_\alpha^{-\frac{1}{b}(t-a)} - \left(\tfrac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\right)\int_a^t e_\alpha^{-\frac{1}{b}(t-\xi)}\varepsilon(\xi)\mathrm{d}\xi \\ &C = \left(\tfrac{\mathrm{E}_1}{b}k_2 - k_1\right)\Gamma(a)\end{aligned} \tag{7.256}$$

For $C=0$ at $t=a$, i.e. when $\sigma(a)=\varepsilon(a)=0$, we have:

$$\sigma(t) = \left(\tfrac{\mathrm{E}_1}{b}\right)\varepsilon(t) - \left(\tfrac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\right)\int_a^t e_\alpha^{-\frac{1}{b}(t-\xi)}\left(\varepsilon(\xi)\right)\mathrm{d}\xi \tag{7.257}$$

7.18.4 Using the Caputo derivative formulation for a solution of σ with a known ε input

For the case of using the Caputo derivative for $\left({}^*D^\alpha y\right)(t) + \gamma y(t) = g(t)$ with $D^\alpha y \equiv {}_a^C D_{t,+}^\alpha y$, we write the solution as:

$$y(t) = k + \int_a^t e_\alpha^{-\gamma(t-\xi)}\left(B\varepsilon(\xi) - \gamma k\right)\mathrm{d}\xi \qquad y(a) = k \tag{7.258}$$

The corresponding solution to the equation with the Caputo formulation below:

$$\begin{gathered}\sigma(t) + b\left({}^C D^\alpha\sigma\right)(t) = \mathrm{E}_0\varepsilon(t) + \mathrm{E}_1\left({}^C D^\alpha\varepsilon\right)(t) \\ 0<\alpha\le 1, \quad b\ge 0, \quad \mathrm{E}_0\ge 0, \quad \mathrm{E}_1>0, \quad b\le\frac{\mathrm{E}_1}{\mathrm{E}_0}\end{gathered} \tag{7.259}$$

is:

$$\sigma(t) = k + \left(\tfrac{\mathrm{E}_1}{b}\right)\varepsilon(t) - \int_a^t e_\alpha^{-\gamma(t-\xi)}\left(\left(\tfrac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\right)\varepsilon(\xi) - \gamma k\right)\mathrm{d}\xi \tag{7.260}$$

7.19 A generalization of a fractional differential equation with a sequential fractional derivative

Case-I: We take the following fractional differential equation:

$$\left(D^\alpha z\right)(t) + \lambda z(t) = X(t) \tag{7.261}$$

We generalize this as the following SFDE:

$$\left(\mathscr{D}^{n\alpha}z\right)(t) + \sum_{j=0}^{n-1}\lambda_j\left(\mathscr{D}^{j\alpha}z\right)(t) = K\sigma(t) \qquad z(t) = \beta\sigma(t) - \varepsilon(t) \tag{7.262}$$

with parameters α, K, β, and $\lambda_j \in \mathbb{R}$ $(j = 1,2,3\ldots n-1)$, or as the following SFDE:

$$\left(\mathscr{D}^{n\alpha}z\right)(t) + \sum_{j=0}^{n-1}\eta_j\left(\mathscr{D}^{j\alpha}z\right)(t) = \mathrm{E}\varepsilon(t) \qquad z(t) = \gamma\varepsilon(t) - \sigma(t) \tag{7.263}$$

with parameters α, E, γ and $\eta_j \in \mathbb{R}$, $(j = 1,2,3\ldots n-1)$. The model shown in (7.262) in (7.263) with a sequential fractional derivative can be expressed as follows in a multivariate fractional differential equation formulation:

$$\left({}^{*}D^{\alpha}\bar{\mathrm{Z}}\right)(t)=\mathrm{A}\bar{\mathrm{Z}}(t)+\bar{\mathrm{B}}(t) \tag{7.264}$$

Where ${}^{*}D^{\alpha}$ is an RL derivative or Caputo derivative, and with the following representations for A, $\bar{\mathrm{B}}(t)$ and $\bar{\mathrm{Z}}(t)$ in matrix form:

$$\mathrm{A}=\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 \\ \ldots & . & . & . & . & . & . & . \\ \ldots & . & . & . & . & . & . & . \\ \ldots & . & . & . & . & . & . & . \\ \ldots & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ -\lambda_0 & -\lambda_1 & -\lambda_2 & . & . & . & -\lambda_{n-2} & -\lambda_{n-1} \end{pmatrix}$$

$$\bar{\mathrm{B}}(t)=K\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ 0 \\ \sigma(t) \end{pmatrix} \qquad \bar{\mathrm{Z}}(t)=\begin{pmatrix} z_1(t) \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ z_n(t) \end{pmatrix} \tag{7.265}$$

In (7.265), we have $z_j(t)=\left({}_aD_t^{(j-1)\alpha}z\right)(t)$, $j=1,2,\ldots n$ or $z_j(t)=\left({}_a^CD_t^{(j-1)\alpha}z\right)(t)$, $j=(1,2,\ldots,n)$. The general solution is the following:

$$\bar{\mathrm{Z}}(t)=\bar{\mathrm{C}}e_{\alpha}^{\mathrm{A}(t-a)}+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\left(\bar{\mathrm{B}}(\xi)\right)\mathrm{d}\xi \tag{7.266}$$

where $z_1(t)=z(t)=\beta\sigma(t)-\varepsilon(t)$, $\beta\in\mathbb{R}$, and $\bar{\mathrm{C}}$ is an arbitrary constant vector.

Case-II: Take the following multivariate system of a fractional differential equation:

$$\left({}_aD_{t,+}^{\alpha}\bar{\mathrm{Z}}\right)(t)=\mathrm{A}\bar{\mathrm{Z}}(t)+\bar{\mathrm{B}}(t) \tag{7.267}$$

where:

$$\mathrm{A}=\left(a_{ij}\right)_{i,j=1,\ldots n};\quad \bar{\mathrm{B}}(t)=\begin{pmatrix} \sigma_1(t) \\ \ldots \\ \ldots \\ \sigma_n(t) \end{pmatrix};\quad \bar{Z}(t)=\begin{pmatrix} z_1(t) \\ \ldots \\ \ldots \\ z_2(t) \end{pmatrix} \tag{7.268}$$

$$z_j(t)=\left({}_aD_t^{(j-1)\alpha}z\right)(t);\quad j=(1,2,\ldots,n-1)$$

The solution is:

$$\bar{\mathrm{Z}}(t)=\bar{\mathrm{C}}e_{\alpha}^{\mathrm{A}(t-a)}+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\bar{\mathrm{B}}(\xi)\mathrm{d}\xi \tag{7.269}$$

Case-III: Take the following multivariate system of a fractional differential equation:

$$\left({}_a^CD_{t,+}^{\alpha}\bar{\mathrm{Z}}\right)(t)=\mathrm{A}\bar{\mathrm{Z}}(t)+\bar{\mathrm{B}}(t) \tag{7.270}$$

with the same parameters as used in Case-II. The solution is:

$$\bar{\mathrm{Z}}(t)=\bar{\mathrm{Z}}(a)+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\left(\bar{\mathrm{B}}(\xi)+\mathrm{A}\bar{\mathrm{Z}}(a)\right)\mathrm{d}\xi \tag{7.271}$$

In particular, if $\bar{\mathrm{Z}}(a)=\bar{b}$ in (7.271), then we have:

$$\bar{\mathrm{Z}}(t)=\bar{b}+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\left(\bar{\mathrm{B}}(\xi)+\mathrm{A}\bar{b}\right)\mathrm{d}\xi \tag{7.272}$$

Here we saw a generalization method that could be used to provide a response to a fractional order model comprising the fractional differential operators of the Caputo or RL type, using an SFDE.

7.20 Short summary

This chapter offered insights into how fractional differential equations and fractional integral equations are formalized. It showed the rigor of solving fractional differential and integral equations, and their conjugation with classical calculus. The topic is perhaps as rigorous as its counterpart in classical calculus. We saw how the solution matters if the formulation of a fractional differential equation is carried out via a Riemann-Liouville fractional derivative or by a Caputo derivative. The use of the Laplace transform technique we have used vigorously is also shown in this chapter, as is the concept of obtaining linearly independent solutions. We may ask ourselves which type of fractional derivative (the RL or the Caputo) is the one that best follows the fractional dynamics of a physical system. This is a good topic for further investigation.

7.21 References

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The above works are furnished with further detail and listed in alphabetical order in the Bibliography section.

CHAPTER EIGHT

FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

8.1 Introduction

This chapter is a continuation of the application of the Laplace and Fourier Transform tricks used to solve the fractional partial differential equation. Here we have taken the classical partial differential equation of wave propagation and diffusion, and then have developed their fractional counterparts. We have evolved different sets of functions that are similar to what was discussed in earlier chapters like Mittag-Leffler, Miller-Ross, and the Robotnov-Hartley functions in which we used them. Similarly, we generalised the Wright function and called it an M-Wright function with several auxiliary functions of this type. This M- stands for Prof. Mairandi; this is developed by him and Prof. M. Caputo. We will be using their development and elaborating upon all the derivations, step-wise. We see that these new sets of functions are a generalisation of the Gaussian function that appears in the solution of the classical partial differential equations. We have taken the 'time fractional diffusion wave equation' (TFDWE) as developed in this chapter, similar to the integer order counterparts that are studied in details in classical calculus books.

8.2 Time fractional diffusion wave equation (TFDWE): Cauchy and the signalling problem

The standard partial differential equation (PDE) governing the known phenomena of diffusion and the wave propagation are governed through the Fourier Diffusion Equation (popularly called Fick's law) and D'Alembert's wave equation: $\frac{\partial\, u(x,t)}{\partial\, t}=\mathbb{D}\frac{\partial^2 u(x,t)}{\partial x^2}$ and $\frac{\partial^2 u(x,t)}{\partial\, t^2}=c^2\,\frac{\partial^2 u(x,t)}{\partial\, x^2}$ respectively, where $u(x,t)$ was denoting a response variable, with $\mathbb{D}$ as a diffusivity constant and c as a characteristic velocity. These above equations are studied in various books, as examples of the standard PDE. We denote the time fractional diffusion wave equation (TFDWE) as follows:

$$\frac{\partial^\beta}{\partial\, t^\beta}u(x,t)=a\frac{\partial^2}{\partial\, x^2}u(x,t)\qquad 0<\beta\le 2 \tag{8.1}$$

The fractional order β is the order of the fractional derivative of the Caputo type ${}_0^C D_t^\beta$, i.e. a fractional derivative w.r.t. time. We define these derivatives for $0<\beta\le 1$ and $1<\beta\le 2$ as follows:

$$\frac{\partial^\beta u}{\partial\, t^\beta}=\begin{cases}\dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t\left[\frac{\partial u(x,\tau)}{\partial\tau}\right]\frac{\mathrm{d}\tau}{(t-\tau)^\beta}={}_0I_t^{1-\beta}\left({}_0D_t^1\left[u(x,t)\right]\right) & 0<\beta<1\\[2ex] \dfrac{\partial\, u}{\partial\, t} & \beta=1\end{cases} \tag{8.2}$$

$$\frac{\partial^\beta u}{\partial\, t^\beta}=\begin{cases}\dfrac{1}{\Gamma(2-\beta)}\displaystyle\int_0^t\left[\frac{\partial^2 u(x,\tau)}{\partial\,\tau^2}\right]\frac{\mathrm{d}\tau}{(t-\tau)^{\beta-1}}={}_0I_t^{2-\beta}\left({}_0D_t^2\left[u(x,t)\right]\right) & 1<\beta<2\\[2ex] \dfrac{\partial^2 u}{\partial\, t^2} & \beta=2\end{cases} \tag{8.3}$$

For $1<\beta<2$, we integrate the Caputo fractional derivative, by order β and get the following:

$$\begin{aligned}{}_0I_t^\beta\left({}_0^C D_t^\beta\left[u(x,t)\right]\right)&={}_0I_t^\beta\left({}_0I_t^{2-\beta}\left({}_0D_t^2\right)\right)u(x,t)\\&={}_0I_t^2\left({}_0D_t^2\left[u(x,t)\right]\right)=u(x,t)-u(x,0^+)-t\left(u_t^{(1)}(x,0^+)\right)\end{aligned} \tag{8.4}$$

$$u_t^{(1)}(x,0^+)=\left.\frac{\partial\, u(x,t)}{\partial\, t}\right|_{t=0^+}$$

For $0<\beta<1$, we integrate the Caputo fractional derivative, by order β and get the following:

$$
\begin{aligned}
{}_0I_t^{\beta}\left({}_0^{C}D_t^{\beta}u(x,t)\right) &= {}_0I_t^{\beta}\left({}_0I_t^{1-\beta}\left(D_t^{1}\right)\right)u(x,t) \\
&= {}_0I_t^{1}\left(D_t^{1}\left[u(x,t)\right]\right) = u(x,t)-u(x,0^{+})
\end{aligned} \tag{8.5}
$$

Integrating by order β the LHS and RHS of TFDWE (8.1) and by using the above (8.5) expression of fractional integration for the Caputo fractional derivative, for $0<\beta<1$, we write this by using a definition of fractional integration to RHS of (8.1), given by the following fractional integral equation:

$$
{}_0I_t^{\beta}\left[\frac{\partial^{\beta}}{\partial t^{\beta}}\right]u(x,t) = {}_0I_t^{\beta}\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 0<\beta<1
$$

$$
u(x,t)-u(x,0^{+}) = \frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{1}{(t-\tau)^{1-\beta}}\right)\mathrm{d}\tau \tag{8.6}
$$

$$
u(x,t) = u(x,0^{+}) + \frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{1}{(t-\tau)^{1-\beta}}\right)\mathrm{d}\tau
$$

Integrating by order β the LHS and RHS of TFDWE (8.1) and by using the above (8.4) expression of fractional integration of the Caputo fractional derivative, for $1<\beta<2$, we write by applying a definition for the fractional integration to RHS of (8.1), the following fractional integral equation:

$$
{}_0I_t^{\beta}\left[\frac{\partial^{\beta}}{\partial t^{\beta}}\right]u(x,t) = {}_0I_t^{\beta}\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 1<\beta<2
$$

$$
u(x,t)-u(x,0^{+})-t\left(u_t^{(1)}(x,0^{+})\right) = \frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{1}{(t-\tau)^{1-\beta}}\right)\mathrm{d}\tau \tag{8.7}
$$

$$
u(x,t) = u(x,0^{+})+t\left(u_t^{(1)}(x,0^{+})\right) + \frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{1}{(t-\tau)^{1-\beta}}\right)\mathrm{d}\tau
$$

Now we set $\beta=2\upsilon$, to write TFDWE (8.1) as the following (8.8) with the constant a replaced by $\mathbb{D}$ as a diffusivity constant or being proportional to the wave propagation constant (velocity):

$$
\frac{\partial^{2\upsilon}u}{\partial t^{2\upsilon}} = \mathbb{D}\frac{\partial^2 u}{\partial x^2} \qquad 0<\upsilon\le 1 \qquad \mathbb{D}>0 \tag{8.8}
$$

The dimension of $\mathbb{D}$ is $L^2/T^{2\upsilon} \equiv \mathrm{cm}^2/\mathrm{sec}^{2\upsilon}$, and the $u(x,t)$ is a 'field' variable which is assumed to be a causal function of time, i.e. $u(x,t)=0$ for $t<0$. The above (8.8) is a one-dimensional time fractional diffusion wave equation (TFDWE). For $\upsilon=\frac{1}{2}$, the classical integer order diffusion equation is recovered, that is $\frac{\partial u}{\partial t}=\mathbb{D}\frac{\partial^2 u}{\partial x^2}$. For $\upsilon=1$, the classical integer order classical wave equation is recovered i.e. $\frac{\partial^2 u}{\partial t^2}=\mathbb{D}\frac{\partial^2 u}{\partial x^2}$. With the fractional order υ, as in the above (8.8), the meaning of $\mathbb{D}$ is different-i.e. having fractional units.

The 'Cauchy problem' is the Initial Value Problem (IVP) where the initial values are given as:

$$
\begin{aligned}
&u(x,0^{+}) = g(x) \quad \text{for} \quad -\infty<x<\infty; \\
&u(\mp\infty,t) = 0 \quad \text{for} \quad t>0
\end{aligned} \tag{8.9}
$$

The 'Signalling problem' is a Boundary Value Problem (BVP) where the boundary conditions are given as:

$$
\begin{aligned}
&u(x,0^{+}) = 0 \quad \text{for} \quad x>0; \quad u(0^{+},t)=h(t) \quad \text{for} \quad t>0 \\
&u(+\infty,t) = 0 \quad \text{for} \quad t>0
\end{aligned} \tag{8.10}
$$

However, to ensure the continuous dependence of our solution on the 'fractional parameter' υ, (8.8) where $0<\upsilon<1$ is also on the transition for $\upsilon=\left(\frac{1}{2}\right)^{-}$ to $\upsilon=\left(\frac{1}{2}\right)^{+}$, we assume $u_t^{(1)}(x,0^{+})=\left.\frac{\partial u}{\partial t}\right|_{t=0^{+}}=0$. The reason is that for $\upsilon<\frac{1}{2}$, $u_t^{(1)}(x,0^{+})$ a condition is not required, and that is zero, and that it is continued for $\upsilon>\frac{1}{2}$, in order to have continuity vis-à-vis υ at $\frac{1}{2}$.

8.3 Green's function

The two Green's functions are represented by the symbol $G_*^{**}(x,t)$ for the Cauchy and signalling problem. The first is $G_c(x,t)$ for the Cauchy problem with $g(x)=\delta(x)$ and the second is $G_s(x,t)$ for the signalling problem with $h(t)=\delta(t)$. The solutions to the above (8.8) problems in terms of the respective Green's function is therefore:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} \left(G_c(\xi,t)\right)\left(g(x-\xi)\right) \mathrm{d}\xi \\ u(x,t) &= \int_0^t \left(G_s(x,\tau)\right)\left(h(t-\tau)\right) \mathrm{d}\tau \end{aligned} \tag{8.11}$$

The above (8.11) expressions are a convolution of the Green's function with forcing functions. We also point out that $G_c(x,t)=G_c\left(|x|,t\right)$, since the Green's function turns out to be even a function of x.

8.3.1 Green's function for the diffusion equation with auxiliary functions as defined by a similarity variable

The Green's Function for a Diffusion Equation we now discuss. For $\upsilon=\frac{1}{2}$ we get the integer order diffusion equation and its Green's function is as follows, (given with the corresponding Fourier transform and Laplace transform respectively):

$$\begin{aligned} G_c^d(x,t) &= \frac{1}{2\sqrt{\pi \mathbb{D} t}} e^{-\frac{x^2}{4\mathbb{D}t}} \qquad \mathcal{F}\left\{G_c^d(x,t)\right\} = e^{-\mathbb{D}tk^2} \\ G_s^d(x,t) &= \frac{x}{2\sqrt{\pi \mathbb{D} t^3}} e^{-\frac{x^2}{4\mathbb{D}t}} \qquad \mathcal{L}\left\{G_s^d(x,t)\right\} = e^{-x\sqrt{\frac{s}{\mathbb{D}}}} \end{aligned} \tag{8.12}$$

We define the similarity variable as $z=\frac{|x|}{\sqrt{\mathbb{D}t}}$ for $x>0$, and we get the following expression:

$$x\left(G_c^d(x,t)\right) = t\left(G_s^d(x,t)\right) = F^d(z) = \frac{z}{2} M^d(z) \tag{8.13}$$

We define $M^d(z)$ in (8.13) as $M^d(z)=\frac{1}{\sqrt{\pi}}e^{-z^2/4}$. We call $M^d(z)$ the auxiliary function for the diffusion equation. It provides the fundamental solution satisfying the condition as $\int_0^{+\infty} M^d(z)\mathrm{d}z=1$.

It is easy to see that $\int_0^{\infty} M^d(z)\mathrm{d}z=1$. By putting $\left(\frac{z}{2}\right)=y$, $\mathrm{d}z=(2\mathrm{d}y)$, we have the following derivation, by using a known definite integral (refer Chapter-1) i.e. $\int_{-\infty}^{+\infty} e^{-y^2}\mathrm{d}y=\sqrt{\pi}$:

$$\begin{aligned} \int_0^{\infty} M^d(z)\mathrm{d}z &= \int_0^{\infty} \frac{1}{\sqrt{\pi}} e^{-\left(z^2/4\right)} \mathrm{d}z = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(z^2/4\right)} \mathrm{d}z \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}(2\mathrm{d}y) = 1 \end{aligned} \tag{8.14}$$

In the above (8.14) the superscript d denotes the diffusion process.

8.3.2 Green's function for the wave equation with auxiliary functions as defined by the similarity variable

Green's function for the Wave Equation we describe here. For $\upsilon=1$ in (8.8), we obtain an ordinary wave equation as mentioned above. That is $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$ making $c=\sqrt{\mathbb{D}}$, then for the Cauchy problem with $g(x)=\delta(x)$, we obtain the following along with the Fourier transform expression:

$$
\begin{aligned}
&G_c^w(x,t) = \frac{1}{2}\delta\left(x - t\sqrt{\mathbb{D}}\right) + \frac{1}{2}\delta\left(x + t\sqrt{\mathbb{D}}\right) \\
&\mathcal{F}\left\{G_c^w(x,t)\right\} = \frac{1}{2}e^{+itk\sqrt{\mathbb{D}}} + \frac{1}{2}e^{-itk\sqrt{\mathbb{D}}} \\
&-\infty < x < \infty \qquad c = \sqrt{\mathbb{D}}
\end{aligned}
\tag{8.15}
$$

A signalling problem with $h(t) = \delta(t)$, for $x > 0$ we obtain the following with $c = \sqrt{\mathbb{D}}$ along with the Laplace transform expression:

$$
G_s^w(x,t) = \delta\left(t - x/\sqrt{\mathbb{D}}\right) \qquad \mathcal{L}\left\{G_s^w(x,t)\right\} = e^{-\left(x/\sqrt{\mathbb{D}}\right)s} \qquad x > 0 \tag{8.16}
$$

From the above (8.16) explicit relationship we recognise the reciprocal relationship between the two Green functions (wave and signalling) for $x > 0$ and $t > 0$:

$$
\begin{aligned}
&2x\left(G_c^w(x,t)\right) = t\left(G_s^w(x,t)\right) = F^w(z) = zM^w(z) \\
&M^w(z) = \delta(1 - z) \qquad z = \frac{x}{\sqrt{\mathbb{D}t}}
\end{aligned}
\tag{8.17}
$$

However, in the case where $\upsilon = 1$, the similarity variable $z = \frac{|x|}{\sqrt{\mathbb{D}t}}$ is not defined, but for a general case of $0 < \upsilon < 1$, we can have:

$$
z = \frac{|x|}{\sqrt{\mathbb{D}t^{2\upsilon}}} = \frac{|x|}{t^{\upsilon}\sqrt{\mathbb{D}}}; \quad 0 < \upsilon < 1 \tag{8.18}
$$

Then for the generalised case of TFDWE for $x > 0$ a certain auxiliary function in $M_\upsilon(z)$ may exist such that the following condition holds:

$$
\begin{aligned}
&x\left(G_{c,\upsilon}(x,t)\right) = \frac{t}{2\upsilon}G_{s,\upsilon}(x,t) = \frac{z}{2}M_\upsilon(z); \quad 0 < \upsilon < 1 \\
&\int_0^\infty M_\upsilon(z)\mathrm{d}z = 1
\end{aligned}
\tag{8.19}
$$

This auxiliary function i.e. $M_\upsilon(z)$ in (8.19) we will use in further study in this chapter:

8.4 Solution of TFDWE via the Laplace transform for the Green's function for Cauchy and signalling problems

We will get $\tilde{G}_c(x,s)$ and $\tilde{G}_s(x,s)$ for $0 < \upsilon < 1$ by solving an ordinary differential equation of the second order in x and then by an inverse Laplace transformation we will get $G_c(x,t)$ and $G_s(x,t)$.

8.4.1 The Cauchy problem for a TFDWE solution in the Laplace domain

The Cauchy problem exists with a condition at the initial time i.e. $t = 0$ we have $g(x) = \delta(x) = u(x,0)$ then $u(x,t) = G_c(x,t)$. We have $\frac{\partial^{2\upsilon}u}{\partial t^{2\upsilon}} = \mathbb{D}\frac{\partial^2 u}{\partial x^2}$ and we use $\tilde{u}(x,s) = \tilde{G}_c(x,s) = \mathcal{L}\left\{u(x,t)\right\}$. The following is the Laplace transformed equations, by using a Laplace transform of the Caputo derivative (Section-5.16) i.e. $\mathcal{L}\left\{{}_0^C D_x^\alpha f(x)\right\} = s^\alpha F(s) - s^{\alpha-1}f(0)$:

$$
\begin{aligned}
s^{2\upsilon}\tilde{u}(x,s) - s^{2\upsilon-1}u(x,0^+) &= \mathbb{D}\frac{\mathrm{d}^2\tilde{u}(x,s)}{\mathrm{d}x^2} \\
s^{2\upsilon}\tilde{u}(x,s) - s^{2\upsilon-1}g(x) &= \mathbb{D}\frac{\mathrm{d}^2\tilde{u}(x,s)}{\mathrm{d}x^2}
\end{aligned}
\tag{8.20}
$$

For $g(x) = \delta(x)$ we write:

$$
s^{2\upsilon}\tilde{G}_c(x,s) - s^{2\upsilon-1}\delta(x) = \mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2} \tag{8.21}
$$

There is a singular term $\delta(x)$ at $x=0$ in (8.21). We therefore need to consider a solution to the above (8.21) for $x>0$ and $x<0$ while imposing a boundary condition at $x=\pm\infty$, and matching the condition of continuity at $x=0^{\pm}$. Look at the following, which is obtained via a rearrangement of the above (8.21) expression i.e.

$$\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}-\left(\frac{s^{2\upsilon}}{\mathbb{D}}\right)\tilde{G}_c(x,s)=-\frac{s^{2\upsilon-1}}{\mathbb{D}}\delta(x) \tag{8.22}$$

The indicial polynomial for the above (8.22) second order differential equation with roots is the following:

$$p^2-\frac{s^{2\upsilon}}{\mathbb{D}}=0 \quad , \quad p=\pm\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}} \tag{8.23}$$

We expect the solution as follows:

$$\tilde{G}_c(x,s)=\begin{cases} c_1(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}+c_2(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x>0 \\ c_3(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}+c_4(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x<0 \end{cases} \tag{8.24}$$

Since at $x=\pm\infty$, $\tilde{G}_c(x,s)=0$, we have $c_2(s)=c_1(s)=0$, ensuring the solution vanishes at $|x|\uparrow\infty$. The following is a solution to the Green's function in the Laplace (s-frequency) domain:

$$\tilde{G}_c(x,s)=\begin{cases} c_1(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x>0 \\ c_4(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x<0 \end{cases} \tag{8.25}$$

Continuity at $x=0^+$ and $x=0^-$ states that: $\tilde{G}_c(0^+,s)-\tilde{G}_c(0^-,s)=0$. That is , $\tilde{G}_c(0^+,s)=c_1(s)=\tilde{G}_c(0^-,s)=c_4(s)$, making $c_1(s)=c_4(s)$. The first derivative at $x=0^{\pm}$, is as follows:

$$\begin{aligned} \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} &=-\big(c_1(s)\big)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)e^{-(s^{\upsilon}/\sqrt{\mathbb{D}})x}=-\big(c_1(s)\big)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right) \\ \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} &=+\big(c_4(s)\big)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)e^{+(s^{\upsilon}/\sqrt{\mathbb{D}})x}=+\big(c_4(s)\big)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right) \end{aligned} \tag{8.26}$$

From the above (8.26) we write the following, as $c_1(s)=c_4(s)$:

$$\begin{aligned} \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+}-\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} &=-\big(c_1(s)+c_4(s)\big)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right) \\ &=-2\big(c_1(s)\big)\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\neq 0 \end{aligned} \tag{8.27}$$

Integrating $s^{2\upsilon}\tilde{G}_c(x,s)-s^{2\upsilon-1}\delta(x)=\mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}$, from $x=0^-$ to $x=0^+$ we get the following:

$$\begin{aligned} &\int_{0^-}^{0^+}(\mathrm{d}x)s^{2\upsilon}\tilde{G}_c(x,s)-\int_{0^-}^{0^+}(\mathrm{d}x)s^{2\upsilon-1}\delta(x)=\int_{0^-}^{0^+}(\mathrm{d}x)\left(\mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}\right) \\ &s^{2\upsilon}\left(\int_{0^-}^{0^+}\big(c_1(s)\big)e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}}\mathrm{d}x+\int_{0^-}^{0^+}\big(c_4(s)\big)e^{+(x/\sqrt{\mathbb{D}})s^{\upsilon}}\mathrm{d}x\right)-s^{2\upsilon-1} \\ &=\mathbb{D}\left(\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+}-\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-}\right) \end{aligned} \tag{8.28}$$

The first term for the LHS of the above (8.28) expression is zero since the function $\tilde{G}_c(x,s)$ is continuous at $x=0$, i.e. the infinitesimal area under this from 0^- to 0^+ equals zero. The second term in (8.28) is the integral of the delta function which is unity, thus we get the $s^{2\upsilon-1}$ (a constant function w.r.t. the variable x). The RHS of (8.28) is expressing a discontinuity in the first derivative of $\tilde{G}_c(x,s)$ w.r.t. x. Therefore, we have the following:

$$-\frac{s^{2\upsilon-1}}{\mathbb{D}} = \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} \tag{8.29}$$

Substituting the value of RHS as obtained above (8.27) in (8.29) we get, with $c_1(s) = c_4(s)$, the following:

$$-\frac{s^{2\upsilon-1}}{\mathbb{D}} = \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-}$$

$$= -\left(c_1(s) + c_4(s)\right)\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}} = -2\left(c_1(s)\right)\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}} \tag{8.30}$$

$$c_1(s) = \frac{s^{\upsilon-1}}{2\sqrt{\mathbb{D}}}$$

We have $c_1(s) = c_4(s) = \frac{s^{\upsilon-1}}{2\sqrt{\mathbb{D}}} = \frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}$. We therefore write the following:

$$\tilde{G}_c(x,s) = \begin{cases} \dfrac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x > 0 \\ \dfrac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x < 0 \end{cases} = \frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}|x|}, \quad -\infty < x < \infty \tag{8.31}$$

8.4.2 Signalling problem for TFDWE in Laplace domain

For a signalling problem we apply the Laplace Transform to $\frac{\partial^{2\upsilon}u(x,t)}{\partial t^{2\upsilon}} = \mathbb{D}\frac{\partial^2 u(x,t)}{\partial x^2}$, with $0 < \upsilon < 1$, and $\mathbb{D} > 0$, to get the following expression:

$$s^{2\upsilon}\tilde{u}(x,s) - s^{2\upsilon-1}u(x,0^+) = \mathbb{D}\frac{\partial^2\tilde{u}(x,s)}{\partial x^2} \tag{8.32}$$

For a signalling problem, the condition is $u(x,0^+) = 0$, for $x > 0$ and $u(0^+,t) = h(t) = \delta(t)$ with $u(+\infty,t) = 0$ for $t > 0$. We therefore write the solution of the above (8.32) problem as Green's function $\tilde{G}_s(x,s)$, and get the following:

$$\mathbb{D}\frac{\mathrm{d}^2\tilde{G}_s(x,s)}{\mathrm{d}x^2} - s^{2\upsilon}\tilde{G}_s(x,s) = 0 \tag{8.33}$$

The corresponding indicial polynomial is $p^2 - \frac{s^{2\upsilon}}{\mathbb{D}} = 0$ and gives roots such as $p = \pm\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}$. The solution is:

$$\tilde{G}_s(x,s) = c_1(s)e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}} + c_2(s)e^{+\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \tag{8.34}$$

Clearly, we must put $c_2(s) = 0$, to ensure that the solution vanishes as $x \uparrow +\infty$. Therefore for $x > 0$ we have the Green's function in the Laplace domain as $\tilde{G}_s(x,s) = \left(c_1(s)\right)e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}}$. At an initial position that is near the origin $x = 0^+$, we have $\tilde{G}_s(0^+,s) = c_1(s)$. Also, we have the condition $u(0^+,t) = \delta(t)$, therefore we get $\tilde{u}(0^+,s) = \mathcal{L}\left\{\delta(t)\right\} = 1 = \tilde{G}_s(0^+,s) = c_1(s)$. This gives a value of $c_1(s) = 1$. Therefore the Green's function in the Laplace domain for a signalling problem for $x > 0$ is $\tilde{G}_s(x,s) = e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}}$.

8.4.3 The solution of Cauchy and the signalling problems of TFDWE in the time domain by inverting the Laplace domains solution by using auxiliary functions F_υ and M_υ

The transformed solutions for Cauchy and signalling are the following:

$$\tilde{G}_c(x,s) = \frac{1}{2\sqrt{\mathbb{D}}s^{1-\upsilon}} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \quad ; \quad \tilde{G}_s(x,s) = e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \tag{8.35}$$

The above (8.35) expressions are transformed and the solution must be inverted to provide for the Green's function in space-time's domain. For this we require some transformed pair of Wright type function that we write as follows for $0 < \upsilon < 1$:

$$\begin{aligned} &\frac{1}{\upsilon}F_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)=\frac{1}{t^{\upsilon}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right) \qquad \mathcal{L}\left\{\tfrac{1}{\upsilon}F_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\right\}=\mathcal{L}\left\{\tfrac{1}{t^{\upsilon}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\right\}=\frac{e^{-s^{\upsilon}}}{s^{1-\upsilon}} \\ &\frac{1}{t}F_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)=\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right) \qquad \mathcal{L}\left\{\tfrac{1}{t}F_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\right\}=\mathcal{L}\left\{\tfrac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\right\}=e^{-s^{\upsilon}} \end{aligned} \tag{8.36}$$

We note that $F_{\upsilon}\left(x\right)$ used here is not the Robotnov-Hartley function, which is represented as $F_{\upsilon}(a,x)$ referred to as Appendix-A. The above (8.36) formulas can be used to invert the transformed Green's function. Then introducing the similarity variable for $x>0$ and $t>0$, that is $z=\frac{x}{\sqrt{\mathbb{D}}t^{\upsilon}}>0$ then by using the rules of scale change of the Laplace transform pair i.e. for $\mathcal{L}\left\{f(t)\right\}=\tilde{f}(s)$, we have $\mathcal{L}\left\{f(at)\right\}=\frac{1}{a}\tilde{f}\left(\frac{s}{a}\right)$ and $\mathcal{L}\left\{\frac{1}{a}f\left(\frac{t}{a}\right)\right\}=\tilde{f}(as)$ where $a>0$, and after some manipulations we obtain the Green's function in variable x and t, with space-time as the following:

$$\begin{aligned} G_c(x,t)&=\frac{1}{2\upsilon x}F_{\upsilon}(z)=\frac{1}{2\sqrt{\mathbb{D}}t^{\upsilon}}M_{\upsilon}(z) \\ G_s(x,t)&=\frac{1}{t}F_{\upsilon}(z)=\frac{\upsilon x}{\sqrt{\mathbb{D}}t^{1+\upsilon}}M_{\upsilon}(z) \end{aligned} \tag{8.37}$$

We get the reciprocal relation of the Green's function as described below:

$$2\upsilon x\left(G_c(x,t)\right)=t\left(G_s(x,t)\right)=F_{\upsilon}(z)=\upsilon z\left(M_{\upsilon}(z)\right) \tag{8.38}$$

Where the $F_{\upsilon}(z)$ and $M_{\upsilon}(z)$ are auxiliary functions for a general case $0<\upsilon\le 1$. This generalisation those for standard diffusions given for $\upsilon=\frac{1}{2}$ by the following:

$$\begin{aligned} &x\left(G_c^d(x,t)\right)=t\left(G_s^d(x,t)\right)=F^d(z)=\frac{1}{2}z\left(M^d(z)\right) \\ &M^d(z)=\frac{1}{\sqrt{\pi}}e^{-\left(z^2/4\right)}; \quad z=\frac{x}{\sqrt{\mathbb{D}}t^{(1/2)}}>0 \end{aligned} \tag{8.39}$$

and for a standard wave equation for $\upsilon=1$ given as:

$$\begin{aligned} &2x\left(G_c^w(x,t)\right)=t\left(G_s^w(x,t)\right)=F^w(z)=z\left(M^w(z)\right) \\ &M^w(z)=\delta(1-z); \quad z=\frac{x}{\sqrt{\mathbb{D}}t}>0 \end{aligned} \tag{8.40}$$

In fact for $\upsilon=\frac{1}{2}$ and $\upsilon=1$ we recover the expressions of the auxiliary functions of classical cases (8.13), (8.17) as follows:

$$\begin{aligned} &M^d(z)=\frac{1}{\sqrt{\pi}}e^{-\left(z^2/4\right)}; \quad z=\frac{x}{\sqrt{\mathbb{D}}t^{(1/2)}}>0 \quad \text{for} \quad \upsilon=\frac{1}{2} \\ &M^w(z)=\delta(1-z); \quad z=\frac{x}{\sqrt{\mathbb{D}}t}>0 \quad \text{for} \quad \upsilon=1 \end{aligned} \tag{8.41}$$

8.5 Reciprocal relation between Green's function of Cauchy and signalling problem

We got the transformed Green's function (in the Laplace domain) for the Cauchy problem for $-\infty<x<+\infty$ as $\tilde{G}_c(x,s)=\frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\left(\left|x/\sqrt{\mathbb{D}}\right|\right)s^{\upsilon}}$. For a signalling problem the transformed Green's function for $x>0$ is $\tilde{G}_s(x,s)=e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}}$. Taking the derivative w.r.t. s for these expressions we get the following:

$$\begin{aligned} \frac{\mathrm{d}\tilde{G}_s(x,s)}{\mathrm{d}s}&=\upsilon s^{\upsilon-1}\left(-\frac{x}{\sqrt{\mathbb{D}}}\right)e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \\ &=-2\upsilon x\left(\frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}}\right) \\ &=-2\upsilon x\left(\tilde{G}_c(x,s)\right) \end{aligned} \tag{8.42}$$

Therefore the relationship of the two expressions in the Laplace (frequency) domain is:

$$x\left(\tilde{G}_c(x,s)\right) = -\frac{1}{2\upsilon}\frac{\mathrm{d}\tilde{G}_s(x,s)}{\mathrm{d}s} \tag{8.43}$$

With the inverse Laplace transform applied to the above (8.43) we obtain the following relationship:

$$x\left(G_c(x,t)\right) = \left(\frac{t}{2\upsilon}\right)G_s(x,t) \tag{8.44}$$

We have used in (8.43) the Laplace transform relationship i.e. $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n\left(\frac{\mathrm{d}^n\tilde{f}(s)}{\mathrm{d}s^n}\right)$ that gives $\mathcal{L}^{-1}\left\{\frac{\mathrm{d}\tilde{f}(s)}{\mathrm{d}s}\right\} = (-t)f(t)$, where $\mathcal{L}^{-1}\left\{\tilde{f}(s)\right\} = f(t)$, and finally we obtain (8.44).

8.6 The origin of the auxiliary function from the Green's function of the Cauchy problem

8.6.1 An inverse Laplace transform of the Green's function: $G_c(x,t)$ in terms of an auxiliary function $M_\upsilon(z)$ with z as a similarity variable

We have the transformed Green's function for the Cauchy problem (8.35) and we then modify it as follows:

$$\tilde{G}_c(x,s) = \frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^\upsilon}, \qquad |x|\,\tilde{G}_c(x,s) = \frac{|x|}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^\upsilon} \tag{8.45}$$

Apply the inverse Laplace transform to the above (8.45) and write the following steps:

$$\begin{aligned}
|x|\,G_c(x,t) &= \mathcal{L}^{-1}\left\{|x|\,\tilde{G}_c(x,s)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{|x|}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^\upsilon}\right\} \\
&= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{st}e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^\upsilon}\frac{\mathrm{d}s}{s^{1-\upsilon}}\right] \\
&= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{st-\left(|x|/\sqrt{\mathbb{D}}\right)s^\upsilon}\frac{\mathrm{d}s}{s^{1-\upsilon}}\right]
\end{aligned} \tag{8.46}$$

The Br Bromwich path on which the Laplace inversion integration is done is a line on the complex plane from $s=\gamma-i\infty$ to $s=\gamma+i\infty$, we have used this in Chapter-6 (Appendix-G). Substitute in the above (8.46) $\sigma = st$ and $\mathrm{d}s = \frac{(\mathrm{d}\sigma)}{t}$ to get the following steps:

$$\begin{aligned}
|x|\,G_C(x,t) &= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(|x|/\sqrt{\mathbb{D}}\right)\left(\sigma^\upsilon/t^\upsilon\right)}\frac{1}{\left(\sigma^{1-\upsilon}/t^{1-\upsilon}\right)}\left(\frac{\mathrm{d}\sigma}{t}\right)\right] \\
&= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(\frac{|x|}{(\sqrt{\mathbb{D}})t^\upsilon}\right)\sigma^\upsilon}\left(\frac{1}{t^\upsilon}\right)\left(\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\right] \\
&= \frac{|x|}{2(\sqrt{\mathbb{D}})t^\upsilon}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(\frac{|x|}{(\sqrt{\mathbb{D}})t^\upsilon}\right)\sigma^\upsilon}\left(\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\right]
\end{aligned} \tag{8.47}$$

Using a similarity variable i.e. $z = \frac{|x|}{\left(\sqrt{\mathbb{D}}\right)t^\upsilon}$ and substituting in the above (8.47) expression we get the following formula:

$$|x|\,G_c(x,t) = \left(\frac{z}{2}\right)M_\upsilon(z) \tag{8.48}$$

With the function $M_\upsilon(z)$ defined in the following way:

$$M_\upsilon(z) = \frac{1}{2\pi i}\int_{Br} e^{\sigma-z\sigma^\upsilon}\left(\frac{1}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma \tag{8.49}$$

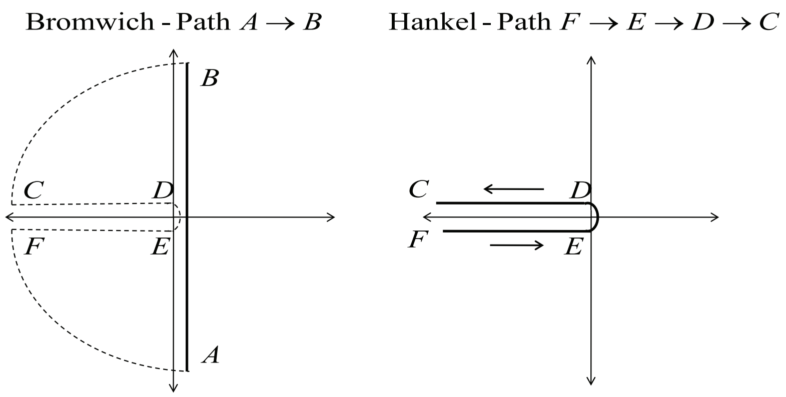
for $z > 0$, and $0 < \upsilon < 1$. Therefore $M_\upsilon(z)$ is an auxiliary function that has the following integral representation:

$$M_\upsilon(z) = \frac{1}{2\pi i}\int_{Br} e^{\sigma - z\sigma^\upsilon} \frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}, \quad z > 0, \quad 0 < \upsilon < 1 \tag{8.50}$$

The representation Br is the Bromwich path on which the Laplace inversion integration is done is a line on the complex plane ($\mathbb{C}$) from $\sigma = \gamma - i\infty$ to $\sigma = \gamma + i\infty$, (Appendix-G). This above (8.50) integral representation of $M_\upsilon(z)$ (Bromwich representation) can be analytically continued for all z in complex-plane i.e. $\mathbb{C}$, by adopting a suitable integral and a 'series-representations' valid in all of $\mathbb{C}$ i.e. the entire complex-plane For this purpose, let us deform the Bromwich path Br into the Hankel path Ha. Refer to Appendix-G for detailing inverse Laplace transforms.

8.6.2 The description of the Hankel contour from the Bromwich path integral used for an integral representation of the auxiliary function $M_\upsilon(z)$

The Hankel contour that begins at $\sigma = -\infty - ia$, $a > 0$, encircles the branch cut that lies along the negative real axis, and ends up at $\sigma = -\infty + ib$, $b > 0$ with $a \downarrow 0$ and $b \downarrow 0$ which is equivalent to the original path of Bromwich. This is shown in Figure-8.1, the complex plane σ :



$$\int_{\text{Bromwich}} F(\sigma)\mathrm{d}\sigma = \int_{\text{Hankel}} F(\sigma)\mathrm{d}\sigma$$

Figure-8.1: Showing the Bromwich path and the Hankel path

The inverse Laplace transform is carried out as an integral on the Bromwich path which is A to B as indicated in Figure-8.1. Notionally we represent it as the following ($F(\sigma)$ is inclusive of the reciprocal of $2\pi i$ and the exponential kernel in an inverse Laplace transform formula) i.e. $f(t) = \int_{A\to B} F(\sigma)\mathrm{d}\sigma$. We make a closed contour across the branch cut line (i.e. the negative real axis) and call the closed contour Ω as A, B, C, D, E, F, A in an anti-clockwise direction. Assuming the contour made does not include any poles, meaning the Residue is zero, and then we have the following:

$$\int_\Omega F(\sigma)\mathrm{d}\sigma = \int_{A\to B\to C\to D\to E\to F\to A} F(\sigma)\mathrm{d}\sigma = 0$$

$$\int_{A\to B} F(\sigma)\mathrm{d}\sigma + \int_{B\to C} F(\sigma)\mathrm{d}\sigma + \int_{C\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to F} F(\sigma)\mathrm{d}\sigma + \int_{F\to A} F(\sigma)\mathrm{d}\sigma = 0 \tag{8.51}$$

As the radius of the arcs BC and FA grows to infinity, the function $F(\sigma) \downarrow 0$, for a well behaved function to have an inverse Laplace transform, this condition is satisfied (as per Jordan Lemma Appendix-G). Therefore, the integrals on these arcs vanish. Therefore, we are left with the following paths on which we do the integration:

$$\int_{A\to B} F(\sigma)\mathrm{d}\sigma + \int_{C\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to F} F(\sigma)\mathrm{d}\sigma = 0$$

$$\begin{aligned}\int_{\text{Bromwich}} F(\sigma)\mathrm{d}\sigma &= \int_{A\to B} F(\sigma)\mathrm{d}\sigma \\ &= -\left(\int_{C\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to F} F(\sigma)\mathrm{d}\sigma\right) \\ &= \int_{D\to C} F(\sigma)\mathrm{d}\sigma + \int_{E\to D} F(\sigma)\mathrm{d}\sigma + \int_{F\to E} F(\sigma)\mathrm{d}\sigma \\ &= \int_{F\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to C} F(\sigma)\mathrm{d}\sigma = \int_{\text{Hankel}} F(\sigma)\mathrm{d}\sigma\end{aligned} \tag{8.52}$$

In addition, if the contour $ABCDEFA$ encloses poles, we write:

$$\int_{\text{Bromwich}} F(\sigma)\mathrm{d}\sigma = \int_{\text{Hankel}} F(\sigma)\mathrm{d}\sigma + 2\pi i \sum \text{Re sidues of poles} \tag{8.53}$$

Recall that we have applied this technique to find the inverse Laplace transform in an earlier chapter (Section-6.6), but here we elaborated upon the Bromwich and Hankel paths.

We have described Hankel's formula with a contour defined by Hankel for an integral representation of the Gamma function, and the reciprocal Gamma function, in Chapter-1 (Section 1.10.6, 1.10.11), that is:

$$\Gamma(z) = \frac{-1}{2i\sin(\pi z)}\int_{\mathrm{C}} (-t)^{z-1} e^{-t}\mathrm{d}t \qquad \frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_{\infty}^{(0+)} (-t)^{-z} e^{-t}\mathrm{d}t \tag{8.54}$$

As defined during the derivation (Section 1.10.6 and 1.10.11) of the above (8.54) formulas the integral is on the contour that is a path starting at $+\infty$ just above the positive real axis, encircling 0 in a positive (anti-clockwise) sense, and returning to $+\infty$ just below the positive real axis, respecting the branch cut along the positive real axis. This was the original contour chosen by Hankel for the derivation of the above (8.54) formula, which is the opposite of what is described in Figure-8.1. We take the formula of the reciprocal Gamma function, and do the substitution $-t = \tau$ in order to write the following steps:

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \frac{i}{2\pi}\int_{\infty}^{(0+)} (-t)^{-z} e^{-t}\mathrm{d}t \\ &= \frac{i}{2\pi}\int_{-\infty}^{(0-)} \tau^{-z} e^{\tau}\left(-\mathrm{d}\tau\right) = \frac{-i}{2\pi}\int_{-\infty}^{(0-)} \tau^{-z} e^{\tau}\mathrm{d}\tau \\ &= \frac{1}{2\pi i}\int_{-\infty}^{(0-)} \tau^{-z} e^{\tau}\mathrm{d}\tau\end{aligned} \tag{8.55}$$

Here we have achieved the contour integration as $\int_{-\infty}^{(0-)}$ meaning this contour starts at $-\infty$, just below on the negative real axis encircling 0 in a positive sense (anti-clockwise) and returning to $-\infty$ just above the negative real axis, respecting the branch cut along the negative real axis. This is as per the Hankel path of Figure-8.1. Thus $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha} \tau^{-z} e^{\tau}\mathrm{d}\tau$ is a Hankel representation of the reciprocal Gamma function.

We will now demonstrate by use of the inverse Laplace transform formula for the reciprocal of the Gamma function as the Hankel representation obtained (8.55) above. We take the basic definition of the Gamma function in an integral representation, i.e. $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\mathrm{d}t$ (Section-1.10). Substituting $t = su$ i.e. $\mathrm{d}t = s\left(\mathrm{d}u\right)$ into the above formula gives the following steps:

$$\begin{aligned}\Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t}\mathrm{d}t \\ &= \int_0^{\infty} (su)^{z-1} e^{-su}\left(s\mathrm{d}u\right) \\ &= s^z \int_0^{\infty} u^{z-1} e^{-su}\mathrm{d}u\end{aligned} \tag{8.56}$$

Define $F(s)=\frac{\Gamma(z)}{s^z}$ and use the definition of the Laplace Transform (Appendix-G) i.e. $F(s)=\mathcal{L}\left\{f(u)\right\}=\int_0^\infty f(u)e^{-su}\mathrm{d}u$ and use the following to write, with $f(u)=u^{z-1}$:

$$\mathcal{L}\left\{u^{z-1}\right\}=\int_0^\infty u^{z-1}e^{-su}\mathrm{d}u=\frac{\Gamma(z)}{s^z}=F(s) \tag{8.57}$$

We write the inverse Laplace transform equation (Appendix-G) knowing that $f(u)=\mathcal{L}^{-1}\left\{F(s)\right\}=\frac{1}{2\pi i}\int_{\mathrm{C}}F(s)e^{su}\mathrm{d}s$, as the following expression:

$$u^{z-1}=\mathcal{L}^{-1}\left\{F(s)\right\}=\frac{1}{2\pi i}\int_{\mathrm{C}}F(s)e^{su}\mathrm{d}s \tag{8.58}$$

The path C is any deformed Bromwich contour such that C winds around the negative real-axis in an anti-clockwise sense as depicted in Figure-8.1, i.e. the Hankel path:

$$\begin{aligned}u^{z-1}&=\mathcal{L}^{-1}\left\{F(s)\right\}=\frac{1}{2\pi i}\int_{\mathrm{C}}F(s)e^{su}\mathrm{d}s\\&=\frac{1}{2\pi i}\int_{\mathrm{C}}\left(\frac{\Gamma(z)}{s^z}\right)e^{su}\mathrm{d}s=\frac{\Gamma(z)}{2\pi i}\int_{\mathrm{C}}s^{-z}e^{su}\mathrm{d}s\end{aligned} \tag{8.59}$$

Put $su=\tau$ in (8.59), we get $\mathrm{d}s=\left(\frac{1}{u}\right)\mathrm{d}\tau$, and rearranging the above (8.59) we get the following:

$$\begin{aligned}\frac{u^{z-1}}{\Gamma(z)}&=\frac{1}{2\pi i}\int_{\mathrm{C}}s^{-z}e^{su}\mathrm{d}s; \qquad su=\tau\\&=\frac{1}{2\pi i}\int_{\mathrm{C}}\left(\frac{\tau}{u}\right)^{-z}e^{\tau}\left(\frac{\mathrm{d}\tau}{u}\right)=\frac{u^{z-1}}{2\pi i}\int_{\mathrm{C}}\tau^{-z}e^{\tau}\mathrm{d}\tau\end{aligned} \tag{8.60}$$

From the above (8.60) we get the expression:

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{\mathrm{C}}\tau^{-z}e^{\tau}\mathrm{d}\tau \tag{8.61}$$

The function τ^{-z} has a branch cut on a negative real axis $(-\infty,0)$ (Appendix-E) and is analytic everywhere. Hence the contour C can be chosen for an Inverse Laplace transform as with the Hankel path (Ha) and as per the discussion above. With this description, we can represent the formula (8.50) i.e. $M_\upsilon(z)=\frac{1}{2\pi i}\int_{Br}e^{\sigma-z\sigma^\upsilon}\left(\frac{1}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma$, $z>0$, $0<\upsilon<1$ with the Hankel path integral as:

$$M_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\left(\frac{1}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma; \qquad 0<\upsilon<1 \tag{8.62}$$

8.7 The integral and series representation of auxiliary functions

8.7.1 Getting a series representation of the auxiliary function $M_\upsilon(z)$ from its integral representation

Therefore, this equivalence of both the Bromwich path and Hankel path redefines the auxiliary function on the Hankel path as $M_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\left(\frac{1}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma$, $0<\upsilon<1$ (8.62). This representation is an integral representation of the auxiliary function. This auxiliary function as originated earlier is also called M-Wright's function, as it is very similar to Wright's function (*M*- for Mainardi). We study this auxiliary function $M_\upsilon(z)$, and get its series representation as described in the following steps:

$$\begin{aligned}
M_\upsilon(z) &= \frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^\upsilon}\,\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(e^{-z\sigma^\upsilon}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \qquad \text{use} \qquad e^{-z\sigma^\upsilon} = \sum_{n=0}^{\infty}\frac{(-1)^n (z\sigma^\upsilon)^n}{n!} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\sigma^{\upsilon n}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\sigma^{\upsilon n+\upsilon-1}\mathrm{d}\sigma\right]
\end{aligned} \tag{8.63}$$

Use $\quad \dfrac{1}{\Gamma(z)} := \dfrac{1}{2\pi i}\displaystyle\int_{Ha} e^x x^{-z}\mathrm{d}x$

$$M_\upsilon(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)}$$

We have used, in the above (8.63) expansion, the Hankel representation of the reciprocal Gamma function i.e. $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha} e^x x^{-z}\mathrm{d}x$ (8.61). Therefore, we have a series representation of this auxiliary function $M_\upsilon(z)$ as mentioned below:

$$M_\upsilon(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)} \qquad 0 < \upsilon < 1 \tag{8.64}$$

Using the reflection formula $\frac{1}{\Gamma(z)} = \frac{1}{\pi}\Gamma(1-z)\left(\sin \pi z\right)$, (Section-1.10.9) we write the auxiliary function in the following form:

$$\begin{aligned}
M_\upsilon(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)} \\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(\Gamma\left(\upsilon(n+1)\right)\right)\left(\sin\left(\pi\upsilon(n+1)\right)\right)\frac{z^n}{n!} \\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\left(\Gamma\left(\upsilon(n+1)\right)\right)\left(\sin\left(\pi\upsilon(n+1)\right)\right) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\left(\Gamma(\upsilon n)\right)\left(\sin(\pi\upsilon n)\right)
\end{aligned} \tag{8.65}$$

8.7.2 The relationship of Wright's function to M-Wright's function or the auxiliary function $M_\upsilon(z)$

The Wright's function (Appendix-A) is defined as follows:

$$\begin{aligned}
W(z;\lambda,\mu) = W_{\lambda,\mu}(z) &\overset{\text{def}}{=} \sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^\mu}; \quad \lambda > -1, \quad \mu > 0
\end{aligned} \tag{8.66}$$

Therefore the auxiliary function in terms of the Wright function is:

$$M_\upsilon(z) = W(-z; -\upsilon, 1-\upsilon) = W_{-\upsilon,1-\upsilon}(-z) \qquad 0 < \upsilon < 1 \tag{8.67}$$

For $\upsilon = \frac{1}{2}$, we get an integer order diffusion problem, and earlier we wrote, for this case, the following function:

$$M_{1/2}(z) = M^d(z) = \frac{1}{\sqrt{\pi}}e^{-\left(z^2/4\right)} \tag{8.68}$$

and also wrote the reciprocity relationship as $x\left(G_c^d(x,t)\right) = t\left(G_s^d(x,t)\right) = \frac{z}{2}M^d(z)$. In addition, we showed that $\int_0^\infty M^d(z)\mathrm{d}z = 1$.

8.7.3 Proof of the property of the auxiliary function i.e. $\int_0^\infty M_\upsilon(z)\mathrm{d}z = 1$

Here we try to show that we get $\int_0^\infty M_\upsilon(z)\mathrm{d}z = 1$ via the following derivation:

$$\begin{aligned}\int_0^\infty M_\upsilon(z)\mathrm{d}z &= \int_0^\infty \frac{1}{2\pi i}\int_{Ha}\left(e^{\sigma-\sigma^\upsilon z}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{Ha} e^\sigma\left(\int_0^\infty e^{-\sigma^\upsilon z}\mathrm{d}z\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\ &= \frac{1}{2\pi i}\int_{Ha} e^\sigma\left(\frac{e^{-\sigma^\upsilon z}}{-\sigma^\upsilon}\right)_{z=0}^{z=\infty}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\ &= \frac{1}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma}\mathrm{d}\sigma\end{aligned} \tag{8.69}$$

Now we need to evaluate the integral on the Hankel path for the function $F(\sigma) = \frac{e^\sigma}{\sigma}$. The first leg is section FE (Figure-8.1). Here we write $\sigma = re^{-i\pi} = -r$, $\mathrm{d}\sigma = -\mathrm{d}r$. The r varies from ∞ to 0^+ so the integration on FE is $\int_{F\to E} e^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma = \int_\infty^{0^+} e^{-r}\left(\frac{1}{r}\right)\mathrm{d}r$. Similarly the second leg of the Hankel path is DC with $\sigma = re^{i\pi} = -r$; $\mathrm{d}\sigma = -\mathrm{d}r$ and with r varying from 0^+ to ∞, yields the integration on path DC (Figure-8.1) as $\int_{D\to C} e^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma = \int_{0^+}^\infty e^{-r}\left(\frac{1}{r}\right)\mathrm{d}r$. Summing these two above, the integrations on the legs of the Hankel path FE and DC gives zero.

We are left with a small circle on the Hankel path as ED, where we write $\sigma = \in e^{i\theta}$, with $\in$ a small constant such that $\in\downarrow 0$ and θ varying from $-\pi$ to $+\pi$. With this we have $\mathrm{d}\sigma = \in ie^{i\theta}\mathrm{d}\theta$, and the integration on this small circle is:

$$\begin{aligned}\int_{ED}\frac{e^\sigma}{\sigma}\mathrm{d}\sigma &= \lim_{\in\downarrow 0}\int_{\theta=-\pi}^{\theta=\pi}\frac{e^{\in\exp(i\theta)}}{\in e^{i\theta}}\in ie^{i\theta}\mathrm{d}\theta \\ &= \lim_{\in\downarrow 0}\int_{-\pi}^{\pi} e^{\in\exp(i\theta)}i\mathrm{d}\theta = i\int_{-\pi}^{\pi}\mathrm{d}\theta = 2\pi i\end{aligned} \tag{8.70}$$

Therefore, the total integration on the Hankel path gives the following result:

$$\int_0^\infty M_\upsilon(z)\mathrm{d}z = \frac{1}{2\pi i}\int_{Ha} e^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma = 1 \tag{8.71}$$

8.7.4 Moments of the M-Wright function or the auxiliary function $M_\upsilon(z)$

A very important deduction is as follows that is the absolute moment of this M-Wright function that is derived for an α order moment with z in $\mathbb{R}^+$, for $M_\upsilon(z)$ with $\alpha > -1$ and $0 \le \upsilon < 1$:

$$
\begin{aligned}
\int_0^\infty z^\alpha M_\upsilon(z)\mathrm{d}z &= \int_0^\infty z^\alpha\left(\frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(\int_0^\infty e^{-z\sigma^\upsilon} z^\alpha \mathrm{d}z\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\
&\text{Use}\quad \int_0^\infty e^{-z\sigma^\upsilon} z^\alpha \mathrm{d}z = \frac{\Gamma(\alpha+1)}{(\sigma^\upsilon)^{\alpha+1}} \\
\int_0^\infty z^\alpha M_\upsilon(z)\mathrm{d}z &= \frac{\Gamma(\alpha+1)}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{\upsilon\alpha+1}}\mathrm{d}\sigma \\
&\text{Use}\quad \frac{1}{\Gamma(\upsilon\alpha+1)} = \frac{1}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{\upsilon\alpha+1}}\mathrm{d}\sigma \\
\int_0^\infty z^\alpha M_\upsilon(z)\mathrm{d}z &= \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)} \qquad \alpha > -1;\quad 0 \le \upsilon < 1
\end{aligned}
\tag{8.72}
$$

Therefore, we have a generalised moment expression for the auxiliary function as follows:

$$
\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)} \qquad x \in \mathbb{R}^+;\quad \alpha > -1,\quad 0 \le \upsilon < 1 \tag{8.73}
$$

8.8 The auxiliary functions $M_\upsilon(z)$ and $F_\upsilon(z)$ as a fractional generalisation of the Gaussian function

8.8.1 The series representation of the Wright function derived from its integral representation formula

The Wright function we denote by $W_{\lambda,\mu}(z)$, the function is defined by the series representation (Appendix-A) and is convergent in the entire $z \in \mathbb{C}$ complex plane:

$$
W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)} \tag{8.74}
$$

With $\lambda > -1$ and $\mu \in \mathbb{C}$. The integral representation of the Wright's function reads, with $\lambda > -1$ and $\mu \in \mathbb{C}$ as:

$$
W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\left(\frac{1}{\sigma^\mu}\right)\mathrm{d}\sigma \tag{8.75}
$$

The origin of the above (8.75) can be related to as being similar to that which was obtained in the previous section for $M_\upsilon(z)$. Using the Hankel representation of the reciprocal of the Gamma function that is $\frac{1}{\Gamma(z)} = \int_{Ha} e^u u^{-z}\mathrm{d}u$ with $z \in \mathbb{C}$, (8.61) we obtain the following series representation:

$$
\begin{aligned}
W_{\lambda,\mu}(z) &= \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^\mu} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\sigma^{-\lambda n}\right)\frac{\mathrm{d}\sigma}{\sigma^\mu} \\
&= \sum_{n=0}^{\infty}\frac{z^n}{n!}\left(\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\sigma^{-\lambda n-\mu}\mathrm{d}\sigma\right) \\
&= \sum_{n-0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)}
\end{aligned}
\tag{8.76}
$$

For $\lambda = 0$, we get $W_{0,\mu}(z) = \frac{e^z}{\Gamma(\mu)}$ provided that $\mu \ne 0, -1, -2, \ldots$.

The auxiliary functions we introduced as in the previous sections are for $0 < \upsilon < 1$ are $M_\upsilon(z) = W_{-\upsilon,1-\upsilon}(-z)$ and $F_\upsilon(\upsilon) = W_{-\upsilon,0}(-z)$. These two auxiliary functions are related by $F_\upsilon(z) = \upsilon z\left(M_\upsilon(z)\right)$. As a matter of fact $F_\upsilon(z)$ and $M_\upsilon(z)$ are particular cases of the Wright function $W_{\lambda,\mu}(z)$ by setting $\lambda = -\upsilon$ for both $\mu = 0$ and $\mu = 1$ respectively.

8.8.2 A further series representation of auxiliary functions $F_\upsilon(z)$ and $M_\upsilon(z)$

Using the identities of the Gamma function $z\Gamma(z) = \Gamma(z+1)$ (Section-1.10) we get the series expression of $F_\upsilon(z)$, by using the obtained series expression of $M_\upsilon(z)$ (8.64) as follows:

$$\begin{aligned}
F_\upsilon(z) = \upsilon z M_\upsilon(z) &= \upsilon z \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!\Gamma\left(-\upsilon m + (1-\upsilon)\right)} = \upsilon \sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{m!\Gamma\left(-\upsilon m + (1-\upsilon)\right)} \\
&= \upsilon \sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{m!\Gamma\left(-\upsilon m - \upsilon + 1\right)} = \upsilon \sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{m!\left((-\upsilon)(m+1)\right)\left(\Gamma\left(-\upsilon m - \upsilon\right)\right)} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{m+1}}{m!(m+1)\Gamma\left(-\upsilon(m+1)\right)} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{m+1}}{(m+1)!\Gamma\left(-\upsilon(m+1)\right)}, \quad m+1 = n \\
&= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\upsilon n)}
\end{aligned} \tag{8.77}$$

Using $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ (Section-1.10.9), we get $\frac{1}{\Gamma(-\upsilon n)} = \left(\frac{\sin(-\pi\upsilon n)}{\pi}\right)\Gamma\left(1-(-\upsilon n)\right)$ and placing this in (8.77) we get following formulas for $F_\upsilon(z)$:

$$\begin{aligned}
F_\upsilon(z) &= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\upsilon n)}, \qquad \frac{1}{\Gamma(-\upsilon n)} = \left(\frac{\sin(-\pi\upsilon n)}{\pi}\right)\Gamma\left(1-(-\upsilon n)\right) \\
&= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!}\left(\frac{1}{\pi}\left(\Gamma(1+\upsilon n)\right)\left(\sin(-\pi\upsilon n)\right)\right) \\
&= \sum_{n=1}^{\infty} -\frac{(-z)^n}{n!}\left(\frac{1}{\pi}\left(\Gamma(1+\upsilon n)\right)\left(\sin(\pi\upsilon n)\right)\right) \\
&= -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^n}{n!}\left(\Gamma(\upsilon n+1)\right)\left(\sin(\pi\upsilon n)\right)
\end{aligned} \tag{8.78}$$

Using the series formula for $M_\upsilon(z)$ (8.64) and using $\frac{1}{\Gamma(-z+1)} = \frac{\sin \pi z}{\pi}\Gamma(z)$ with $z = \upsilon(n+1)$ we write the following:

$$\begin{aligned}
M_\upsilon(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma\left(-\upsilon n + (1-\upsilon)\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma\left(-\upsilon(n+1)+1\right)} \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^n}{n!}\right) \frac{\sin\left(\pi\upsilon(n+1)\right)\Gamma\left(\upsilon(n+1)\right)}{\pi} \\
&= \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\left(\Gamma\left(\upsilon(n+1)\right)\right)\left(\sin\left(\pi\upsilon(n+1)\right)\right)
\end{aligned} \tag{8.79}$$

Continuing from (8.79) by changing the index of summation from n to $n-1$ we write the following steps:

$$\begin{aligned}
M_\upsilon(z) &= \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\left(\Gamma\left(\upsilon(n+1)\right)\right)\left(\sin\left(\pi\upsilon(n+1)\right)\right) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\left(\Gamma(\upsilon n)\right)\left(\sin(\pi\upsilon n)\right)
\end{aligned} \tag{8.80}$$

8.8.3 The relationship between auxiliary functions $M_\upsilon(z)$ and $F_\upsilon(z)$

Furthermore $F_\upsilon(0)=0$ and $M_\upsilon(0)=\frac{1}{\Gamma(1-\upsilon)}$, so the relationship between $F_\upsilon(z)$ and $M_\upsilon(z)$ is expressed as follows (we use the formula $\Gamma(z+1)=z\big(\Gamma(z)\big)$ in the following derivation):

$$\begin{aligned}
F_\upsilon(z) &= \sum_{n=1}^{\infty}\frac{(-z)^n}{n!\Gamma(-\upsilon n)} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}\big(\Gamma(\upsilon n+1)\big)\big(\sin(\pi\upsilon n)\big)\\
&= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}(\upsilon n)\big(\Gamma(\upsilon n)\big)\big(\sin(\pi\upsilon n)\big)\\
&= -\upsilon\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)(-z)^{n-1}}{n!}n\big(\Gamma(\upsilon n)\big)\big(\sin(\pi\upsilon n)\big)\\
&= \upsilon z\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\big(\Gamma(\upsilon n)\big)\big(\sin(\pi\upsilon n)\big)\\
&= \upsilon z\big(M_\upsilon(z)\big)
\end{aligned}\tag{8.81}$$

Therefore, we get the relationship:

$$F_\upsilon(z)=\upsilon z\big(M_\upsilon(z)\big)\tag{8.82}$$

However, we have used (8.82) in deriving (8.77).

The integral representation for the auxiliary functions is $F_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\mathrm{d}\sigma$ and $M_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\left(\frac{1}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma$. The interrelationship between them can be derived from this integral representation of $M_\upsilon(z)$ (8.62) as depicted below:

$$\begin{aligned}
M_\upsilon(z) &= \frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\\
&= \frac{1}{2\pi i}\int_{Ha}e^{\sigma}\left(\frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma\\
&= \frac{1}{2\pi i}\int_{Ha}e^{\sigma}\left(-\frac{1}{\upsilon z}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma
\end{aligned}\tag{8.83}$$

The obtained result (8.82) is due to the following, which we have used to get the above (8.83), expression:

$$\begin{aligned}
\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma} &= -z(\upsilon\sigma^{\upsilon-1})e^{-z\sigma^\upsilon} = -z\upsilon\frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}}\\
\frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}} &= -\frac{1}{z\upsilon}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}
\end{aligned}\tag{8.84}$$

Now we do integration by parts for the expression (8.83) that is by formula i.e. $\int\big(f(u)\big)\big(g(u)\big)\mathrm{d}u=f(u)\int g(u)\mathrm{d}u-\int\Big(f^{(1)}(u)\Big(\int g(u)\mathrm{d}u\Big)\Big)\mathrm{d}u$ for the integral expression of $M_\upsilon(z)$ as depicted in the following derivation:

$$\begin{aligned}
M_\upsilon(z) &= \frac{1}{2\pi i}\int_{Ha} e^\sigma \left(-\frac{1}{\upsilon z}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma \\
&= \frac{1}{2\pi i}\left(\frac{1}{\upsilon z}\right)\int_{Ha} e^\sigma \left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma \\
&= \frac{1}{2\pi i}\left(\frac{1}{\upsilon z}\right)\left[e^\sigma \int_{Ha}\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma - \int_{Ha}\left\{\int\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma\right\}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}e^\sigma\right)\mathrm{d}\sigma\right] \\
&= \left(\frac{1}{\upsilon z}\right)\left(\frac{1}{2\pi i}\left(\int_{Ha}(e^{-z\sigma^\upsilon})(e^\sigma)\mathrm{d}\sigma\right)\right) \\
&= \frac{1}{\upsilon z}\left(\frac{1}{2\pi i}\left(\int_{Ha} e^{\sigma - z\sigma^\upsilon}\mathrm{d}\sigma\right)\right) \\
&= \frac{1}{\upsilon z}F_\upsilon(z)
\end{aligned} \tag{8.85}$$

Which gives $M_\upsilon(z) = \left(\frac{1}{\upsilon z}\right)F_\upsilon(z)$ or $F_\upsilon(z) = \upsilon z\left(M_\upsilon(z)\right)$. In the above (8.85) derivation we used $\int_{Ha}\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma = 0$ as this integral along the closed contour is not enclosing any singularity (as the Hankel path from $-\infty$ to the origin encircling it and going again to $-\infty$). We may write this as $\left[-e^{-z\sigma^\upsilon}\right]_{\sigma=-\infty-i\in}^{\sigma=-\infty+i\in} \simeq 0$ with limit $\in\downarrow 0$. The other steps are obvious. For special values say for $\upsilon = \frac{1}{2}$, we obtain the following identity by using the notation $(m)_n = m(m+1)(m+2)..(m+n-1)$ with $(m)_0 = 1$.

8.8.4 An auxiliary function $M_\upsilon(z)$ for υ at $\frac{1}{2}, \frac{1}{3}$ and 0

$$\begin{aligned}
M_{\frac{1}{2}}(z) &= \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\,\Gamma\left(\frac{1-n}{2}\right)} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{z}{\Gamma(0)} + \frac{z^2}{2!\,\Gamma\left(-\frac{1}{2}\right)} - \frac{z^3}{3!\,\Gamma(-1)} + \frac{z^4}{4!\,\Gamma\left(-\frac{3}{2}\right)} + \ldots \\
&= \frac{1}{\sqrt{\pi}} - 0 + \frac{z^2}{2!(-2)\sqrt{\pi}} - 0 + \frac{z^4}{4!\left(\frac{4}{3}\right)\sqrt{\pi}} + \ldots \\
&= \frac{1}{\sqrt{\pi}}\left(1 + \left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)} + \left(\frac{3}{4}\right)\frac{(+1)z^4}{4!} + \ldots\right) \\
&= \frac{1}{\sqrt{\pi}}\left(1 + \left(\frac{1}{2}\right)\frac{(-1)^1(z)^{2\times 1}}{(2\times 1)!} + \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\frac{(-1)^2(z)^{2\times 2}}{(2\times 2)!} + \ldots\right) \\
&= \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)_n\frac{(z)^{2n}}{(2n)!}
\end{aligned} \tag{8.86}$$

We have a rising truncated factorial as $(m)_n = m(m+1)(m+2)..(m+n-1)$; $(m)_0 = 1$. The notation $(m)_n$ is also called a Pochhammer number (Section-1.9.5) which is also $(m)_n = \frac{\Gamma(m+n)}{\Gamma(m)}$:

$$
\begin{aligned}
M_{\frac{1}{2}}(z) &= \frac{1}{\sqrt{\pi}}\left(1+\left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)}+\left(\frac{3}{4}\right)\frac{(+1)z^4}{4!}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(1+\frac{(-z^2)}{2\times 2\times 1}+\frac{3}{4}\frac{(-z^2)^2}{(4\times 3\times 2\times 1)}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(1+\frac{\left(-\frac{z^2}{4}\right)}{1!}+\frac{\left(-\frac{z^2}{4}\right)^2}{2!}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^n}{n!}\right)\\
&= \frac{1}{\sqrt{\pi}}e^{\left(-z^2/4\right)}
\end{aligned}
\tag{8.87}
$$

Therefore, we write:

$$
M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)_n\left(\frac{z^{2n}}{(2n)!}\right) = \frac{1}{\sqrt{\pi}}e^{\left(-z^2/4\right)} \tag{8.88}
$$

For $\upsilon = \frac{1}{3}$, we have the following formula:

$$
\begin{aligned}
M_{\frac{1}{3}}(z) &= \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\,\Gamma\left(\frac{2-n}{3}\right)}\\
&= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)_n\frac{z^{3n}}{(3n)!}\\
&= \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)_n\frac{z^{3n+1}}{(3n+1)!}
\end{aligned}
\tag{8.89}
$$

In the above derivations (8.86) and (8.89) the formula $M_{\upsilon}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma(-\upsilon n+(1-\upsilon))}$ (8.63) is used. From this series representation, we get $M_0(z) = e^{-z}$ for $\upsilon = 0$, also by use of $F_{\upsilon}(z) = \upsilon z\left(M_{\upsilon}(z)\right)$ (8.82) we get $F_0(z) \equiv 0$.

We know that the Mittag-Leffler function is a (fractional) generalisation of the exponential function; similarly, we may call this $M_{\upsilon}(z)$ a fractional generalisation of the Gaussian function, which is the solution of an integer order diffusion equation.

8.9 Laplace and Fourier transforms of the auxiliary functions

We now state some of the relevant properties of the $M_{\upsilon}(z)$ and $F_{\upsilon}(z)$ in view of their role in the TFDWE processes. The Laplace transform pair for $0 < \upsilon < 1$ is as follows:

$$
\begin{aligned}
&\frac{1}{t}F_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right) = \frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right) \qquad \mathcal{L}\left\{\frac{1}{t}F_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)\right\} = \mathcal{L}\left\{\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)\right\} = e^{-s^{\upsilon}}\\
&\frac{1}{\upsilon}F_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right) = \frac{1}{t^{\upsilon}}M_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right) \qquad \mathcal{L}\left\{\frac{1}{\upsilon}F_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)\right\} = \mathcal{L}\left\{\frac{1}{t^{\upsilon}}M_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)\right\} = \frac{e^{-s^{\upsilon}}}{s^{1-\upsilon}}
\end{aligned}
\tag{8.90}
$$

8.9.1 The inverse Laplace transform of function $X(s)=e^{-s^{\upsilon}}$ giving $F_{\upsilon}(t^{-\upsilon})$ and $M_{\upsilon}(t^{-\upsilon})$

We calculate by following steps for an inverse Laplace transform of $X(s)=e^{-s^{\upsilon}}$:

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-s^{\upsilon}}\right\}&=\frac{1}{2\pi i}\int_{Br}(\mathrm{d}s)e^{st}\left(e^{-s^{\upsilon}}\right)\\&=\frac{1}{2\pi i}\int_{Ha}e^{st-s^{\upsilon}}\mathrm{d}s \qquad \text{put} \qquad \sigma=st;\quad \frac{\mathrm{d}\sigma}{t}=\mathrm{d}s;\quad s=\frac{\sigma}{t}\\&=\left(\frac{1}{t}\right)\frac{1}{2\pi i}\int_{Ha}e^{\sigma-\left(\frac{\sigma}{t}\right)^{\upsilon}}\mathrm{d}\sigma\\&=\left(\frac{1}{t}\right)F_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)=\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\end{aligned}\tag{8.91}$$

We used $F_{\upsilon}(z)=\upsilon z\left(M_{\upsilon}(z)\right)$, with $z\equiv\left(\frac{1}{t^{\upsilon}}\right)$. The above (8.91) result is also obtained via expansion in the power-series, and the Laplace transformed expression, and through inverting it term by term as follows:

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-s^{\upsilon}}\right\}&=\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{(-1)^{n}s^{\upsilon n}}{n!}\right\}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\mathcal{L}^{-1}\left\{s^{\upsilon n}\right\}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{t^{-\upsilon n-1}}{\Gamma(-\upsilon n)}=\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^{n}t^{-\upsilon n}}{n!\Gamma(-\upsilon n)}\\&=\frac{1}{t}F_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)=\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\frac{1}{t^{\upsilon}}\right)\end{aligned}\tag{8.92}$$

Furthermore, we have $\mathcal{L}\left\{t^{-\upsilon}M_{\upsilon}(t^{-\upsilon})\right\}=\mathcal{L}\left\{\upsilon^{-1}F_{\upsilon}(t^{-\upsilon})\right\}=s^{\upsilon-1}e^{-s^{\upsilon}}$, as shown in the following derivation:

$$\begin{aligned}&\mathcal{L}^{-1}\left\{e^{-s^{\upsilon}}\right\}=\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}\left(\tfrac{1}{t^{\upsilon}}\right)\\&\mathcal{L}\left\{\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon+1}}\right\}=\frac{e^{-s^{\upsilon}}}{\upsilon}\\&\mathcal{L}\left\{t\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon+1}}\right\}=(-1)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^{-s^{\upsilon}}}{\upsilon}\right)=(-1)(--1\upsilon s^{\upsilon-1})\left(\frac{e^{-s^{\upsilon}}}{\upsilon}\right)\\&\mathcal{L}\left\{\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon}}\right\}=s^{\upsilon}e^{-s^{\upsilon}}\end{aligned}\tag{8.93}$$

8.9.2 The Mittag-Leffler function as an integral representation on Hankel's path

The Mittag-Leffler function (Appendix-A) is defined for any z in the complex $\mathbb{C}$-plane, for any $\upsilon\geq 0$, by the integral representation $E_{\upsilon}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\upsilon n+1)}=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^{x}}{x^{\upsilon}-z}\mathrm{d}x$, $\upsilon>0$ and $z\in\mathbb{C}$. This comes from the Laplace transform pair of a Mittag-Leffler function which is $\mathcal{L}\left\{E_{\upsilon}(at^{\upsilon})\right\}=\frac{s^{\upsilon-1}}{s^{\upsilon}-a}$. From this pair we write an inverse Laplace transform integral as:

$$E_\upsilon(at^\upsilon) = \frac{1}{2\pi i}\int_{Br}\frac{s^{\upsilon-1}}{s^\upsilon - a}e^{st}\mathrm{d}s \qquad \text{put} \quad st = x, \quad \mathrm{d}s = \frac{\mathrm{d}x}{t}, \quad s = \frac{x}{t}$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{Ha}\frac{\left(\frac{x^{\upsilon-1}}{t^{\upsilon-1}}\right)}{\left(\frac{x^\upsilon}{t^\upsilon}\right) - a}e^x\frac{\mathrm{d}x}{t} \\
&= \frac{1}{2\pi i}\int_{Ha}\left(\frac{x^{\upsilon-1}}{t^{\upsilon-1}}\right)\left(\frac{t^\upsilon}{x^\upsilon - at^\upsilon}\right)\left(\frac{e^x}{t}\right)\mathrm{d}x \\
&= \frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^x}{x^\upsilon - at^\upsilon}\mathrm{d}x \qquad \text{put} \quad at^\upsilon = z
\end{aligned} \tag{8.94}$$

$$E_\upsilon(z) = \frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^x}{x^\upsilon - z}\mathrm{d}x$$

Similarly following the above steps, and with the use of $\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(at^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha - a}$, we get the following representation of a two-parameter Mittag-Leffler function i.e. $E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{Ha}\frac{x^{\alpha-\beta}}{x^\alpha - z}e^x\mathrm{d}x$ (Appendix-F).

8.9.3 The Laplace transform of an auxiliary function $M_\upsilon(t)$ as the Mittag-Leffler function

We now write a Laplace transform pair for the M-Wright function, that is $\mathcal{L}\left\{M_\upsilon(t)\right\} = E_\upsilon(-s)$ for $0 < \upsilon < 1$, which we derive from the following steps:

$$\mathcal{L}\left\{M_\upsilon(t)\right\} = \int_0^\infty e^{-st}M_\upsilon(t)\mathrm{d}t$$

$$\begin{aligned}
&= \int_0^\infty e^{-st}\left(\frac{1}{2\pi i}\int_{Ha}e^{\sigma - t\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\mathrm{d}t = \frac{1}{2\pi i}\int_0^\infty e^{-st}\left(\int_{Ha}e^{\sigma - t\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{1-\upsilon}}\left(\int_0^\infty e^{-t(s+\sigma^\upsilon)}\mathrm{d}t\right)\mathrm{d}\sigma \\
&= \frac{1}{2\pi i}\int_{Ha}e^\sigma\sigma^{\upsilon-1}\left[-\frac{e^{-t(s+\sigma^\upsilon)}}{s+\sigma^\upsilon}\right]_{t=0}^{t=\infty}\mathrm{d}\sigma \\
&= \frac{1}{2\pi i}\int_{Ha}e^\sigma\sigma^{\upsilon-1}\left(\frac{1}{s+\sigma^\upsilon}\right)\mathrm{d}\sigma \\
&= \frac{1}{2\pi i}\int_{Ha}\frac{\sigma^{\upsilon-1}e^\sigma}{\sigma^\upsilon - (-s)}\mathrm{d}\sigma \\
&= E_\upsilon(-s)
\end{aligned} \tag{8.95}$$

In the above (8.95) derivation we used the interchangeability of integrals in view of the analyticity property of the involved functions, and we used the derived integral representation of the one-parameter Mittag-Leffler function. In the second approach we expand the series for an exponential kernel e^{-st} and use the absolute moments of $M_\upsilon(z)$ that are obtained earlier, which are $\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)}$ (8.73), and then use the series definition of the one-parameter Mittag-Leffler function as follows:

$$\begin{aligned}
\mathcal{L}\left\{M_\upsilon(t)\right\} &= \int_0^\infty e^{-st} M_\upsilon(t)\mathrm{d}t \\
&= \int_0^\infty \left(\sum_{n=0}^\infty \frac{(-st)^n}{n!}\right) M_\upsilon(t)\mathrm{d}t \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!} \int_0^\infty t^n M_\upsilon(t)\mathrm{d}t \qquad \text{use} \qquad \int_0^\infty t^n M_\upsilon(t)\mathrm{d}t = \frac{\Gamma(n+1)}{\Gamma(\upsilon n+1)} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!} \frac{\Gamma(n+1)}{\Gamma(\upsilon n+1)} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!} \frac{n!}{\Gamma(\upsilon n+1)} \qquad \text{use} \qquad \Gamma(n+1) = n! \qquad \text{for} \quad n \in \mathbb{N} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma(\upsilon n+1)} = E_\upsilon(-s) \qquad \text{by using} \qquad E_\upsilon(z) \overset{\text{def}}{=} \sum_{n=0}^\infty \frac{z^n}{\Gamma(\upsilon n+1)}
\end{aligned} \tag{8.96}$$

The other way of getting the above (8.96) result is to use the series definition of $M_\upsilon(z)$, and the Laplace transform expression, $\int_0^\infty e^{-st}(-t)^n \mathrm{d}t = \mathcal{L}\left\{(-t)^n\right\} = (-1)^n \mathcal{L}\left\{(t)^n\right\} = (-1)^n \left(\frac{n!}{s^{n+1}}\right)$ as indicated below:

$$\begin{aligned}
\mathcal{L}\left\{M_\upsilon(t)\right\} &= \int_0^\infty e^{-st} M_\upsilon(t)\mathrm{d}t \\
&= \int_0^\infty e^{-st} \sum_{n=0}^\infty \frac{(-t)^n}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)}\mathrm{d}t \\
&= \sum_{n=0}^\infty \frac{\int_0^\infty e^{-st}(-t)^n \mathrm{d}t}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)} \\
&= \sum_{n=0}^\infty \frac{1}{n!\,\Gamma\left(-\upsilon n + (1-\upsilon)\right)} \left(\frac{(-1)^n n!}{s^{n+1}}\right) \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma\left(-\upsilon n + (1-\upsilon)\right)} \left(\frac{1}{s^{n+1}}\right) \qquad \text{put} \qquad n+1 = m \\
&= \sum_{m=1}^\infty \frac{(-1)^{m-1}}{\Gamma(1-\upsilon m)} \left(\frac{1}{s^m}\right) = \left(\frac{s^{-1}}{\Gamma(1-\upsilon)} + \dots\right) \sim E_\upsilon(-s)
\end{aligned} \tag{8.97}$$

The expression $\frac{s^{-1}}{\Gamma(1-\upsilon)}$ is an asymptotic expansion of the Mittag-Leffler function for large s, i.e. for limit $s \uparrow \infty$ therefore we have $\left(s^{-1}/\Gamma(1-\upsilon)\right)$ as $E_\upsilon(-s)$.

8.9.4 The Fourier Transform of the auxiliary function $M_\upsilon(x)$

Now we write the Fourier transform of the symmetric M-Wright function, where it should be extended to the negative real axis as an even function. The Fourier transform is then related to the Mittag-Leffler function via a pair given for $0 < \upsilon < 1$, $\mathcal{F}\left\{M_\upsilon\left(|x|\right)\right\} = 2E_{2\upsilon}(-k^2)$. We now do the one-sided cosine Fourier integration as demonstrated below, where we used the formula for generalised absolute moments for $M_\upsilon(z)$ (8.63) and made use of the series definition of $E_\upsilon(z)$.

$$\cos kx = \sum_{n=0}^{\infty}(-1)^n \frac{(kx)^{2n}}{(2n)!}$$

$$\int_0^{\infty}\left(\cos(kx)\right)\left(M_{\upsilon}(x)\right)\mathrm{d}x = \int_0^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n \frac{(kx)^{2n}}{(2n)!}\right)\left(M_{\upsilon}(x)\right)\mathrm{d}x\,,$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{k^{2n}}{(2n)!}\int_0^{\infty} x^{2n}\left(M_{\upsilon}(x)\right)\mathrm{d}x$$

$$\text{Use} \quad \int_0^{\infty} x^{2n} M_{\upsilon}(x)\mathrm{d}x = \frac{\Gamma(2n+1)}{\Gamma(2\upsilon n+1)} \tag{8.98}$$

$$\int_0^{\infty}\left(\cos(kx)\right)\left(M_{\upsilon}(x)\right)\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \frac{k^{2n}}{(2n)!}\left(\frac{(2n)!}{\Gamma(2n\upsilon+1)}\right) \qquad \Gamma(2n+1)=(2n)!$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(k^2)^n}{\Gamma\left((2\upsilon)n+1\right)} = E_{\upsilon}(-k^2)$$

From this we get the two-sided Fourier integral and the Fourier pair for $M_{\upsilon}\left(|x|\right)$ and write the following useful identity:

$$\mathcal{F}\left\{M_{\upsilon}\left(|x|\right)\right\} = 2\int_0^{\infty}\left(\cos(kx)\right)\left(M_{\upsilon}(x)\right)\mathrm{d}x = 2E_{2\upsilon}(k^2) \tag{8.99}$$

8.10 A graphical representation of the M-Wright or auxiliary function $M_{\upsilon}(x)$

We describe the graphs $M_{\upsilon}(x)$ by schematic diagrams. First, we take the case for $0<\upsilon<\frac{1}{2}$, depicted schematically in Figure-8.2. For $\upsilon=\frac{1}{2}$ we have seen $M_{1/2}(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}$. This is the standard Gaussian bell shaped curve centred at $x=0$ (when extended to a negative axis by reflecting about $x=0$ line, shown by a dotted extension in Figure-8.2). The other extreme is at $\upsilon=0$ where $M_0(x)=e^{-x}$. Shown in the Figure-8.2 which is a curve for $\upsilon=\frac{1}{4}$. The observation points while υ is much less than $\frac{1}{2}$, say near zero, indicate that $M_{\upsilon}(x)$ starts from $x=0$ and has a steep fall as x increases, causing the curve to go to zero. While υ is nearing $\frac{1}{2}$, the curve becomes Gaussian in nature, with a slow initial fall followed by a sudden fall and then eventually heading to zero. For the case of υ, which is varying in nature from 0 to $\frac{1}{2}$, we observe the maximum value of $M_{\upsilon}(x)$ is at $x=0$ and accordingly, a value of υ as the curve of $M_{\upsilon}(x)$ decays and goes to zero at say $x=5$ i.e. for a large x.

Now we take the case of $M_{\upsilon}(x)$ for υ varying from $\frac{1}{2}$ to 1, which is depicted in the schematic of Figure-8.3. At the value of $\upsilon=\frac{1}{2}$, the curve $M_{1/2}(x)$ is a Gaussian curve. As υ increases from the value of $\frac{1}{2}$ the peak value of the curve shifts from $x=0$ to $x>0$, and after attaining the peak value the curve falls towards zero. This is as shown for the case $\upsilon=\frac{5}{8}$, in Figure-8.3. Still increasing the value of υ towards 1, it makes the peak of the curve shift towards $x=1$, making it steeper and sharper. At the value $\upsilon=1$, it is expected to take the form of the delta function, as $M_1(x)=\delta(x-1)$. For the negative x the function $M_{\upsilon}(x)$ is plotted by reflection of theses curves about the line $x=0$ as $M_{\upsilon}(x)$ is a symmetric function.

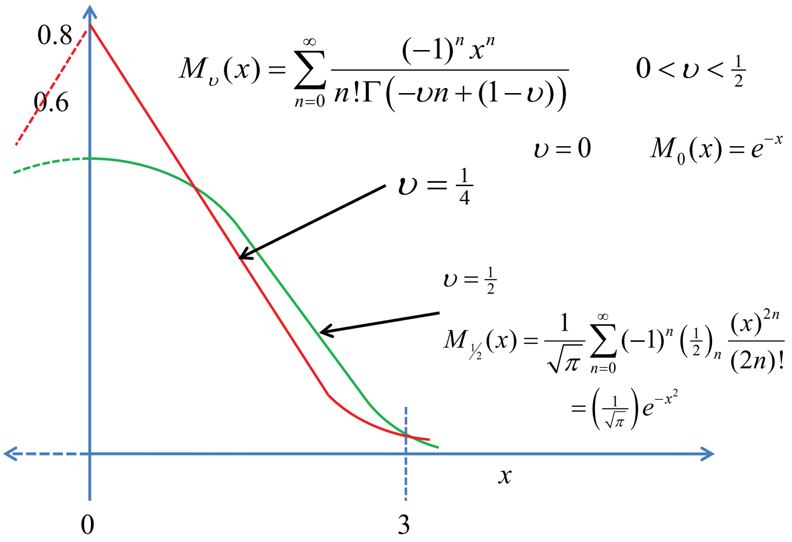


Figure-8.2: Plot of the symmetrical M-Wright function for υ = zero to 0.5

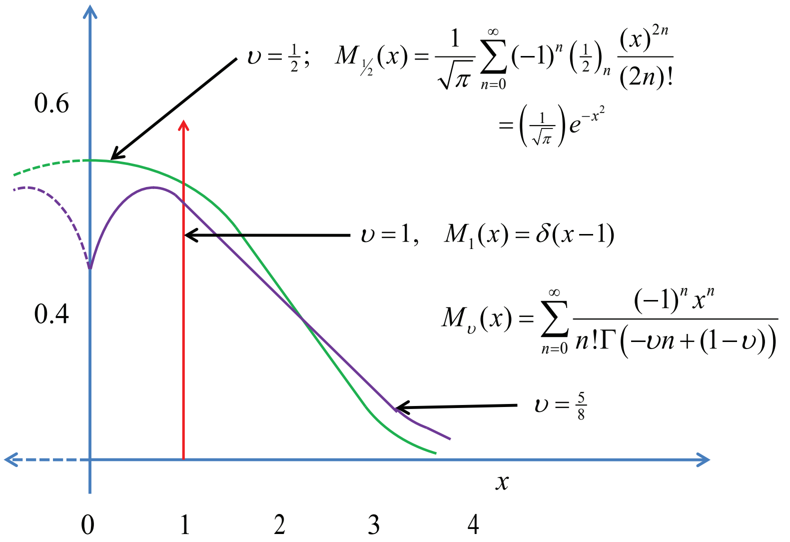


Figure-8.3: Plot of the symmetrical M-Wright function for υ = 0.5 to 1.0

Since the function $M_\upsilon(x)$ is a symmetrical function, for $\upsilon=0$, the function $M_0\left(|x|\right)$ is $e^{-|x|}$ and is transformed into $\left(\frac{1}{\sqrt{\pi}}\right)e^{-x^2}$ for $\upsilon=\frac{1}{2}$, as depicted in the Figure-8.2. In the Figure-8.3 the $\left(\frac{1}{\sqrt{\pi}}\right)e^{-x^2}$ gets transformed into $\delta(x\pm 1)$ for $\upsilon=1$.

8.11 Auxiliary function in two variables $M_\upsilon(x,t)$ and its Laplace & Fourier transforms

8.11.1 Defining $M_\upsilon(x,t)$ in relation to $M_\upsilon(xt^{-\upsilon})$

For $0<\upsilon<1$ where $x,t\in\mathbb{R}^+$ M-Wright functions in two variables $M_\upsilon(x,t)$ introduced as follows in (8.100):

$$M_\upsilon(x,t)=t^{-\upsilon}M_\upsilon(xt^{-\upsilon}) \tag{8.100}$$

This above (8.100) function defines the probability density function (pdf) in space x, evolving with time t, with exponent υ. This exponent is also called self-similarity exponent or Hurst parameter, or the Hurst index $H=\upsilon$ (Section 1.24). Obviously we should consider a symmetrical version, obtained from the above (8.100) definition and by multiplying by $\frac{1}{2}$ and replacing x by $|x|$ for $x\in\mathbb{R}$.

8.11.2 Laplace Transform of $M_\upsilon(x,t)$

From the Laplace transform pair i.e. $\mathcal{L}\left\{\upsilon^{-1}F_\upsilon(t^{-\upsilon})\right\} = \mathcal{L}\left\{(t^{-\upsilon})M_\upsilon(t^{-\upsilon})\right\} = s^{\upsilon-1}e^{-s^\upsilon}$ we derive the Laplace transform of $M_\upsilon(x,t)$ w.r.t. $t \in \mathbb{R}^+$ as:

$$\mathcal{L}\left\{M_\upsilon(x,t); t \to s\right\} = \mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon}); t \to s\right\} = s^{\upsilon-1}e^{-xs^\upsilon} \tag{8.101}$$

The derivation is as follows:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{e^{-xs^\upsilon}; s \to t\right\} &= \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n s^{\upsilon n}}{n!}\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\mathcal{L}^{-1}\left\{s^{\upsilon n}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\frac{t^{-\upsilon n-1}}{\Gamma(-\upsilon n)} \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n x^n t^{-\upsilon n}}{n!\,\Gamma(-\upsilon n)} = \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n (xt^{-\upsilon})^n}{n!\,\Gamma(-\upsilon n)} \\
&= \frac{1}{t}F_\upsilon(xt^{-\upsilon}) = \frac{\upsilon x}{t^{\upsilon+1}}M_\upsilon\left(\tfrac{1}{t^\upsilon}\right);
\end{aligned} \tag{8.102}$$

$$F_\upsilon(z) = \upsilon z M_\upsilon(z); \quad z = xt^{-\upsilon}$$

We have the following:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{e^{-xs^\upsilon}\right\} &= \frac{1}{t}F_\upsilon(xt^{-\upsilon}) = \frac{\upsilon x}{t^{\upsilon+1}}M_\upsilon(xt^{-\upsilon}) \\
\mathcal{L}\left\{\frac{1}{t^{\upsilon+1}}M_\upsilon(xt^{-\upsilon})\right\} &= \frac{1}{\upsilon x}e^{-xs^\upsilon}
\end{aligned} \tag{8.103}$$

We get now the following steps:

$$\begin{aligned}
\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon}); t \to s\right\} &= \mathcal{L}\left\{t\left(\frac{1}{t^{\upsilon+1}}M_\upsilon(-1xt^{-\upsilon})\right)\right\} \\
&= (-1)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\upsilon x}e^{-xs^\upsilon}\right) \\
&= \frac{(-1)}{\upsilon x}(-x\upsilon s^\upsilon)e^{-xs^\upsilon} \\
&= s^{\upsilon-1}e^{-xs^\upsilon}
\end{aligned} \tag{8.104}$$

Again from the Laplace transform pair i.e. $M_\upsilon(x,t) = t^{-\upsilon}M_\upsilon(xt^{-\upsilon}) \overset{\mathcal{L}}{\leftrightarrow} E_\upsilon(-s); \quad x \to s$, we derive the Laplace transform of $M_\upsilon(x,t)$ w.r.t. $x \in \mathbb{R}^+$ as:

$$\begin{aligned}
\mathcal{L}\left\{M_\upsilon(x,t); x \to s\right\} &= \mathcal{L}\left\{t^{-\upsilon}M_\upsilon(t^{-\upsilon}x); x \to s\right\} \\
&= \int_0^\infty e^{-sx}(t^{-\upsilon})\left(M_\upsilon(t^{-\upsilon}x)\right)\mathrm{d}x \\
&= \int_0^\infty\left(\sum_{n=0}^{\infty}\frac{(-sx)^n}{n!}\right)(t^{-\upsilon})\left(M_\upsilon(t^{-\upsilon}x)\right)\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}(t^{-\upsilon})\int_0^\infty x^n\left(M_\upsilon(xt^{-\upsilon})\right)\mathrm{d}x
\end{aligned} \tag{8.105}$$

Now put $xt^{-\upsilon} = y$, $\mathrm{d}x = t^\upsilon \mathrm{d}y$, $x = t^\upsilon y$ and we get the following steps:

$$\mathcal{L}\left\{M_\upsilon(x,t);x\to s\right\}=\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}(t^{-\upsilon})\int_0^\infty (yt^\upsilon)^n\left(M_\upsilon(y)\right)t^\upsilon \mathrm{d}y$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(s)^n(t^\upsilon)^n}{n!}\int_0^\infty y^n\left(M_\upsilon(y)\right)\mathrm{d}y$$

Use $$\int_0^\infty y^n\left(M_\upsilon(y)\right)\mathrm{d}y=\frac{n!}{\Gamma(\upsilon n+1)} \tag{8.106}$$

$$\mathcal{L}\left\{M_\upsilon(x,t);x\to s\right\}=\sum_{n=0}^{\infty}\frac{(-1)^n(st^\upsilon)^n}{n!}\frac{n!}{\Gamma(\upsilon n+1)}$$

$$=\sum_{n=0}^{\infty}\frac{(-st^\upsilon)^n}{\Gamma(\upsilon n+1)}=E_\upsilon(-st^\upsilon)\quad \text{by using}\quad E_\upsilon(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\upsilon n+1)}$$

Therefore, we write:

$$\mathcal{L}\left\{M_\upsilon(x,t);x\to s\right\}=\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon});x\to s\right\}=E_\upsilon(-st^\upsilon) \tag{8.107}$$

8.11.3 Fourier transform of $M_\upsilon(x,t)$

Note the variable x of representing a quantity position in $M_\upsilon(x,t)$, then the frequency s, of spatial frequency generally represented as a pure complex number ik (signifying the steady state response). In that case we may write $\mathcal{L}\left\{M_\upsilon(x,t);x\to ik\right\}=\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon});x\to ik\right\}=E_\upsilon(-ikt^\upsilon)$, with $x\in\mathbb{R}^+$ and $k\in\mathbb{R}$. From $\mathcal{F}\left\{M_\upsilon\left(|x|\right)\right\}=2E_{2\upsilon}(-k^2)$, we write Fourier of $M_\upsilon\left(|x|,t\right)$ as:

$$\mathcal{F}\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2E_{2\upsilon}(-k^2t^{2\upsilon}) \tag{8.108}$$

We derive this by the following steps:

$$\mathcal{F}\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2\int_0^\infty\left(\cos(kx)\right)\left(t^{-\upsilon}M_\upsilon(xt^{-\upsilon})\right)\mathrm{d}x$$

$$=2\int_0^\infty\left(\sum_{n=0}^{\infty}\frac{(-1)^n(kx)^{2n}}{(2n)!}\right)\left(t^{-\upsilon}M_\upsilon(xt^{-\upsilon})\right)\mathrm{d}x \tag{8.109}$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(x^{2n})(t^{-\upsilon})M_\upsilon(xt^{-\upsilon})\mathrm{d}x$$

Put $xt^{-\upsilon}=y$, $\mathrm{d}x=t^\upsilon\,\mathrm{d}y$ to get the following steps:

$$\mathcal{F}\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2\sum_{n=0}^{\infty}\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(x^{2n})(t^{-\upsilon})\left(M_\upsilon(xt^{-\upsilon})\right)\mathrm{d}x$$
$$=2\sum_{n=0}^{\infty}\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(yt^{\upsilon})^{2n}(t^{-\upsilon})\left(M_\upsilon(y)\right)(t^{\upsilon})\mathrm{d}y$$
$$=2\sum_{n=0}^{\infty}\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(yt^{\upsilon})^{2n}\left(M_\upsilon(y)\right)\mathrm{d}y$$
$$=2\sum_{n=0}^{\infty}\frac{(-1)^n(kt^{\upsilon})^{2n}}{(2n)!}\int_0^\infty(y)^{2n}\left(M_\upsilon(y)\right)\mathrm{d}y \tag{8.110}$$

Use $\int_0^\infty y^m\left(M_\upsilon(y)\right)\mathrm{d}y=\frac{m!}{\Gamma(\upsilon m+1)}$

$$\mathcal{F}\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2\sum_{n=0}^{\infty}\left(\frac{(-1)^n(k^2t^{2\upsilon})^n}{(2n)!}\right)\left(\frac{(2n)!}{\Gamma(2\upsilon n+1)}\right)$$
$$=2E_{2\upsilon}(-k^2t^{2\upsilon})$$

8.11.4 Special cases for $M_\upsilon(x,t)$ for υ as $\frac{1}{2}$ and one

As a special case for $\upsilon=\frac{1}{2}$, we get:

$$\frac{1}{2}M_{\frac{1}{2}}\left(|x|,t\right)=\frac{1}{2}t^{-\frac{1}{2}}M_{\frac{1}{2}}\left(xt^{-\frac{1}{2}}\right)$$
$$=\frac{1}{2\sqrt{t}}\frac{1}{\sqrt{\pi}}e^{-\left(xt^{-1/2}\right)^2/4}\qquad\text{use}\qquad M_{\frac{1}{2}}(z)=\frac{1}{\sqrt{\pi}}e^{-z^2/4} \tag{8.111}$$
$$=\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

For limiting the case $\upsilon=1$ we obtain:

$$\left(\tfrac{1}{2}\right)M_1(x,t)=\left(\tfrac{1}{2}\right)\delta(x-t)+\left(\tfrac{1}{2}\right)\delta(x+t) \tag{8.112}$$

8.12 The use of the two variable auxiliary functions in TFDWE

8.12.1 Time fractional diffusion equation and its solution with $M_\upsilon(x,t)$

Let us have a time fractional diffusion equation (with the Caputo fractional derivative of order β):

$$\frac{\partial^\beta}{\partial t^\beta}\left[u(x,t)\right]=a\frac{\partial^2u(x,t)}{\partial x^2}\qquad 0<\beta<1 \tag{8.113}$$

Integrating fractionally by order β on both sides of (8.113), we obtain the following steps:

$$
\begin{aligned}
&{}_0I_t^{\beta}\left[\frac{\partial^{\beta}}{\partial\, t^{\beta}}u(x,t)\right]={}_0I_t^{\beta}\left[a\frac{\partial^2 u(x,t)}{\partial\, x^2}\right] \qquad 0<\beta<1\\
&u(x,t)-u(x,0^+)=\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u(x,\tau)}{\partial\, x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}\\
&u(x,t)=u(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u(x,\tau)}{\partial\, x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}\\
&u(x,t)=u(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u(x,\tau)}{\partial\, x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}\\
&\text{Use}\quad {}_0I_t^{\beta}\left[f(t)\right]=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f(\tau)\mathrm{d}\tau;\quad a=1\\
&u(x,t)=u(x,0^+)+{}_0I_t^{\beta}\left[\frac{\partial^2 u(x,t)}{\partial\, t^2}\right]
\end{aligned}
\tag{8.114}
$$

With $u(x,0^+)=\delta(x)=G_\beta(x,0^+)$ therefore we obtain the fundamental solution $u(x,t)=G_\beta(x,t)$ and that we write as:

$$G_\beta(x,t)=\delta(x)+{}_0I_t^{\beta}\frac{\partial^2}{\partial\, x^2}G_\beta(x,t) \tag{8.115}$$

We take first Fourier transform of the above (8.115) equation, by using ik for operator $\frac{\partial}{\partial\, x}$ and $\mathcal{F}\left\{\delta(x)\right\}=1$ then write the following transformed equation:

$$\hat{G}_\beta(k,t)=1+{}_0I_t^{\beta}\left[(-k^2)\hat{G}_\beta(x,t)\right] \tag{8.116}$$

Now we apply the Laplace transform to the above (8.116) by using $s^{-\beta}$ for ${}_0I_t^{\beta}$ (Section-5.14) and, $\mathcal{L}\left\{1\right\}=\frac{1}{s}$, and write the following steps:

$$\tilde{\hat{G}}_\beta(k,s)=\left(\tfrac{1}{s}\right)-k^2s^{-\beta}\left(\tilde{\hat{G}}_\beta(k,s)\right);\quad \tilde{\hat{G}}_\beta(k,s)=\frac{s^{\beta-1}}{s^{\beta}+k^2} \tag{8.117}$$

We can have two methods, for inverting this (8.117). First, we do an inverse Fourier transform then we do the inverse Laplace transform. That is recalling the Fourier transform pair (where $a,b>0$) as follows:

$$\mathcal{F}^{-1}\left\{\frac{2|b|}{b^2+\omega^2}\right\}=e^{-|b|t},\qquad \mathcal{F}^{-1}\left\{\frac{a}{b+k^2}\right\}=\frac{a}{2\sqrt{b}}e^{-|x|\sqrt{b}} \tag{8.118}$$

We set $a=s^{\beta-1}$ and $b=s^{\beta}$, and apply the above (8.118) transform pair to get from (8.117) to the following:

$$\tilde{G}_\beta(x,s)=\frac{s^{\beta-1}}{2\sqrt{s^{\beta}}}e^{-|x|\sqrt{s^{\beta}}}=\frac{1}{2}s^{(\beta/2)-1}e^{-|x|s^{\beta/2}} \tag{8.119}$$

Now we do an inverse Laplace transform by using $\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(x,t)\right\}=s^{\upsilon-1}e^{-xs^{\upsilon}}$, here we set $\upsilon=\frac{\beta}{2}$ to get:

$$G_\beta(x,t)=\frac{1}{2}t^{-(\beta/2)}M_{\beta/2}\left(|x|t^{\left(-\beta/2\right)}\right) \tag{8.120}$$

By second method first we do an inverse Laplace transform of $\tilde{\hat{G}}_\beta(k,s)$ and then apply the inverse Fourier transform. That is the first recalling Laplace transform pair $\mathcal{L}^{-1}\left\{\frac{s^{\beta-1}}{s^{\beta}+c}\right\}=E_\beta(-ct^{\beta})$, with $c>0$ (Appendix-A) and then by setting $c=k^2$, we get $\hat{G}_\beta(k,t)=E_\beta(-k^2t^{\beta})$. Use $\mathcal{F}\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2E_{2\upsilon}(-k^2t^{2\upsilon})$ to have an inverse Fourier transformation $\tilde{\hat{G}}_\beta(k,s)=s^{\beta-1}(s^{\beta}+k^2)^{-1}$ (8.117), with $2\upsilon=\beta$; $\upsilon=\frac{\beta}{2}$, to get the following:

$$G_\beta(x,t)=\frac{1}{2}M_{\beta/2}(x,t)=\frac{1}{2}t^{-\left(\beta/2\right)}M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right) \tag{8.121}$$

This is the same as we obtained in (8.120).

8.12.2 A time fractional drift equation and its solution with $M_\upsilon(x,t)$

We shall now demonstrate the use of the M-Wright function as a solution to time fractional drift equation that is as follows:

$$\frac{\partial^\beta}{\partial\, t^\beta}u(x,t)=-\frac{\partial}{\partial\, x}u(x,t);\qquad -\infty<x<\infty,\quad t\ge 0 \tag{8.122}$$
$$\text{with}\quad u(x,0^+)=\delta(x)$$

Applying the Laplace transform on (8.122) we get the following:

$$\mathcal{L}\left\{\frac{\partial^\beta}{\partial\, t^\beta}u(x,t)\right\}=-\mathcal{L}\left\{\frac{\partial}{\partial\, x}u(x,t)\right\}$$
$$s^\beta\tilde{u}(x,s)-s^{\beta-1}\left(u(x,t)\big|_{t=0^+}\right)=-\frac{\partial}{\partial\, x}u(x,s) \tag{8.123}$$

Now use $u(x,t)\big|_{t=0^+}=\delta(x)$ to get the following:

$$s^\beta\tilde{u}(x,s)-s^{\beta-1}u(x,0^+)=-\frac{\partial}{\partial x}\tilde{u}(x,s)$$
$$u(x,0^+)=\delta(x)\qquad\text{then}\qquad u(x,t)=G_\beta(x,t) \tag{8.124}$$
$$s^\beta\tilde{G}_\beta(x,s)-s^{\beta-1}\delta(x)=-\frac{\partial}{\partial x}\tilde{G}_\beta(x,s)$$

Now we do the Fourier transform of (8.124), with $\mathcal{F}\{\delta(x)\}=1$, and for $\frac{\partial}{\partial\, x}$ we write ik, to have the following:

$$s^\beta\hat{\tilde{G}}_\beta(k,s)-s^{\beta-1}(1)=-ik\hat{\tilde{G}}_\beta(k,s)$$
$$\hat{\tilde{G}}_\beta(k,s)=\frac{s^{\beta-1}}{s^\beta+ik} \tag{8.125}$$

We first do the inverse Fourier transform of the above (8.125) by recalling:

$$\mathcal{F}^{-1}\left\{\frac{1}{|b|+i\omega}\right\}=e^{-|b|t};\qquad \mathcal{F}^{-1}\left\{\frac{a}{b+ik}\right\}=ae^{-bx}$$
$$a,b>0;\quad x>0 \tag{8.126}$$

We set $a=s^{\beta-1}$ and $b=s^\beta$, so that we obtain the following:

$$\hat{G}_\beta(x,s)=s^{\beta-1}e^{-xs^\beta} \tag{8.127}$$

Now we get an inverse Laplace transform by recalling $\mathcal{L}\{M_\upsilon(x,t);t\to s\}=s^{\upsilon-1}e^{-xs^\upsilon}$, getting the following expression:

$$G_\beta(x,t)=M_\beta(x,t)=t^{-\beta}M_\beta(xt^{-\beta}) \tag{8.128}$$

We apply the second method of first using an inverse Laplace transform of $\hat{\tilde{G}}_\beta(k,s)=\frac{s^{\beta-1}}{s^\beta+ik}$, by using the Laplace transform pair $\mathcal{L}^{-1}\left\{\frac{s^{\beta-1}}{s^\beta+c}\right\}=E_\beta(-ct^{-\beta})$ for $c>0$. Set $c=ik$, to get:

$$\hat{G}_\beta(k,t)=E_\beta(-ikt^\beta) \tag{8.129}$$

Now we transform with $ik\to x$ to get from the obtained Laplace transform pair (8.107) that is:

$$\mathcal{L}\{M_\upsilon(x,t);x\to ik\}=\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon});x\to ik\right\}=E_\upsilon(-ikt^\upsilon)$$
$$x\in\mathbb{R}^+,\quad k\in\mathbb{R} \tag{8.130}$$

The space x is transformed into a complex spatial frequency ik which is the same as in the Fourier case. Therefore, above (8.130) the Laplace transform pair we use as Fourier transform pair, and we write the following expression from (8.129):

$$G_\beta(x,t)=M_\beta(x,t)=t^{-\beta}M_\beta\left(\frac{x}{t^\beta}\right) \tag{8.131}$$

8.13 Reviewing time fractional diffusion equations with several manifestations

8.13.1 Classical diffusion equation

We consider a variety of diffusion equations including the standard integer order diffusion equation whose fundamental solution, the Green's function, is expressed in terms of the M-Wright function depending on space or time variables. The two variables however are related through a self-similarity condition (Hurst exponent $H \in (0,1)$) which we discussed in Chapter-1 (Section 1.24).

The standard integer order diffusion equation for the field variable $u(x,t)$ with initial condition $u(x,0) = u_0(x)$ is

$$\frac{\partial u}{\partial t} = K_1 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty; \quad t > 0 \tag{8.132}$$

Where K_1 is a suitable diffusion coefficient with dimensions of $\mathrm{cm}^2/\mathrm{sec}$. This is an initial value problem, and we perform integration of both sides to write the solution in terms of Volterra-Integral Equation i.e.

$$\int_0^t \frac{\partial u}{\partial \tau} \mathrm{d}\tau = K_1 \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} \mathrm{d}\tau \qquad u(x,t) = u_0(x) + K_1 \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} \mathrm{d}\tau \tag{8.133}$$

It is well known that the fundamental solution is also called the Green's function corresponding to $u_0(x) = \delta(x)$ and is the Gaussian probability density evolving in time with the mean square displacement proportional to time (that is a variance or second moment). We write the Green's function as $G_1(x,t) = \frac{1}{2\sqrt{\pi K_1 t}} e^{-x^2/(4K_1 t)}$, having a variance as $\sigma_1^2(t) = \int_{-\infty}^{+\infty} x^2 G_1(x,t) \mathrm{d}x = 2K_1 t$.

8.13.2 The Green's function for a classical diffusion equation and its moment

The Green's function in terms of the M-Wright function is $G_1(x,t) = \frac{1}{2}\left(\frac{1}{\sqrt{K_1 t^{1/2}}}\right) M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t}}\right)$. Let us find the moment of this Green's function $G_1(x,t)$ that is:

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{2\sqrt{K_1 t}} M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t}}\right) \mathrm{d}x \tag{8.134}$$

Put $\frac{|x|}{\sqrt{K_1 t}} = y; \quad x^2 = y^2 K_1 t; \quad \mathrm{d}x = \mathrm{d}y\sqrt{K_1 t}$, in the above (8.134) expression to get the following:

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} \frac{y^2 K_1 t}{2\sqrt{K_1 t}} \left(M_{1/2}(y)\right) \sqrt{K_1 t}\left(\mathrm{d}y\right) \\
&= \frac{K_1 t}{2} \int_{-\infty}^{\infty} y^2 \left(M_{1/2}(y)\right) \mathrm{d}y \\
&= \left(\frac{K_1 t}{2}\right)\left(2\int_0^{\infty} y^2 \left(M_{1/2}(y)\right) \mathrm{d}y\right) = K_1 t \int_0^{\infty} y^2 \left(M_{1/2}(y)\right) \mathrm{d}y
\end{aligned} \tag{8.135}$$

We use an already derived expression about α-moments of the M-Wright function (8.63) that is $\int_0^{\infty} x^{\alpha} M_{\upsilon}(x) \mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)}$. In this general expression put $\alpha = 2, \quad \upsilon = \frac{1}{2}$, so we have the following:

$$\int_0^{\infty} x^2 \left(M_{1/2}(x)\right) \mathrm{d}x = \frac{\Gamma(2+1)}{\Gamma\left(\frac{1}{2} \times 2 + 1\right)} = \frac{\Gamma(3)}{\Gamma(2)} = \frac{2!}{1!} = 2 \tag{8.136}$$

Use of the (8.136) above derived expression gives the variance as proportional to time:

$$\sigma^2 = K_1 t \int_0^{\infty} y^2 \left(M_{1/2}(y)\right) \mathrm{d}y = 2K_1 t \tag{8.137}$$

From the self-similarity of the Green's function, we write the following:

$$G_1(x,t)=\frac{1}{2\sqrt{\pi K_1 t^{1/2}}}e^{-x^2/(4K_1t)} \qquad G_1(x,t)=\frac{1}{2\sqrt{K_1 t^{1/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t^{1/2}}}\right) \tag{8.138}$$

Furthermore, we can write:

$$G_1(x,t)=\frac{1}{\sqrt{K_1 t^H}}G_1(z) \qquad z=\frac{|x|}{\sqrt{K_1 t^H}} \tag{8.139}$$

where $H=\frac{1}{2}$ is the self-similarity (or Hurst) exponent (this Hurst exponent is a self-similarity we have described in Chapter-1, Section 1.24). The variable $z=\frac{|x|}{\sqrt{K_1 t^H}}$ acts as the similarity variable, with $G_1(z)$ as one variable from the 'reduced' Green's function. Note that the above derived variance (8.137) law $\propto t$ characterises normal diffusion as it emerges from a Brownian motion approach. With $H\neq\frac{1}{2}$, $H\in(0,1)$ the case is of a fractional Brownian motion with a persistent motion $H\in\left(\frac{1}{2},1\right)$ and anti-persistent with $H\in\left(0,\frac{1}{2}\right)$, giving us the case for anomalous diffusion; we will not detail this aspect here.

8.13.3 A classical diffusion equation with a stretched time variable in its Green's function and moment

Now we stretch the time variable in the integer order diffusion equation by replacing t with t^α where $\alpha\in(0,2)$, and we write the modified equation as:

$$\frac{\partial u}{\partial(t^\alpha)}=K_\alpha\frac{\partial^2 u}{\partial x^2} \qquad -\infty<x<+\infty;\quad t\geq 0 \tag{8.140}$$

We call K_α a stretched diffusion coefficient having the dimension $\text{cm}^2/\sec^\alpha$. We will use $\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\alpha\right]=\alpha(t^{\alpha-1})$ in the following discussion. We recognise easily that the above diffusion equation is akin to the standard diffusion equation, but with a diffusion coefficient depending on time $K_1(t)=\alpha\left(t^{\alpha-1}\right)K_\alpha$, which follows from the following derivation:

$$\frac{\partial u}{\partial t^\alpha}=\frac{\partial u}{\alpha\left(t^{\alpha-1}\right)\partial t}=K_\alpha\frac{\partial^2 u}{\partial x^2} \qquad \text{use} \qquad \mathrm{d}(t^\alpha)=\alpha\left(t^{\alpha-1}\right)\mathrm{d}t$$

$$\frac{\partial u}{\partial t}=\alpha\left(t^{\alpha-1}\right)K_\alpha\frac{\partial^2 u}{\partial x^2}=K_1(t)\frac{\partial^2 u}{\partial x^2} \tag{8.141}$$

$$K_1(t)=\alpha\left(t^{\alpha-1}\right)K_\alpha$$

The Volterra-integral corresponding to the time stretched case is obtained by taking the integration of both the LHS and RHS of the stretched time diffusion equation $\frac{\partial u}{\partial t}=\left(\alpha t^{\alpha-1}K_\alpha\right)\frac{\partial^2 u}{\partial x^2}$ (8.141) to get the following:

$$u(x,t)=u_0(x)+\alpha\left(K_\alpha\right)\int_0^t \tau^{\alpha-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \tag{8.142}$$

We write the corresponding fundamental solution that is a stretched-time Gaussian function:

$$G_\alpha(x,t)=\frac{1}{2\sqrt{\pi K_\alpha t^{\alpha/2}}}e^{-x^2/\left(4K_\alpha t^\alpha\right)}=\frac{1}{2\sqrt{K_\alpha t^{\alpha/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_\alpha t^{\alpha/2}}}\right) \tag{8.143}$$

The above (8.143) comes from $G_1(x,t)=\frac{1}{2\sqrt{K_1 t^{1/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t^{1/2}}}\right)$, by placing t^α instead of t and placing K_α instead of K_1.

We can derive the corresponding variance (via the method described earlier) as:

$$\sigma_\alpha^2=\int_{-\infty}^{\infty}x^2\left(G_\alpha(x,t)\right)\mathrm{d}x=2K_\alpha t^\alpha \tag{8.144}$$

This comes from $\sigma^2=2K_1t$ by placing t^α instead t and K_α in place of K_1. This is characteristic of a general process of anomalous diffusion, precisely slow diffusion for $0<\alpha<1$, and fast diffusion for $1<\alpha<2$.

8.13.4 Time fractional diffusion equation

Now we take a time fractional diffusion equation and write that for $\beta \in (0,1)$ as the following equivalent fractional differential equations (with the fractional Caputo derivative):

$$\begin{aligned}
&\frac{\partial u}{\partial t} = K_\beta \left({}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right) \qquad \beta \in (0,1) \qquad \text{apply } {}_0D_t^{\beta-1} \\
&\left({}_0D_t^{\beta-1} \left(\frac{\partial u}{\partial t} \right) \right) = K_\beta \left({}_0D_t^{\beta} \left({}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right) \right) \\
&\left({}_0D_t^{\beta}\, {}_0D_t^{-1} \frac{\partial u}{\partial t} \right) = K_\beta \left(\frac{\partial^2 u}{\partial x^2} \right) \qquad {}_0D_t^{-1} = {}_0I_t^{1} \\
&{}_0D_t^{\beta} u = K_\beta \frac{\partial^2 u}{\partial x^2}
\end{aligned} \tag{8.145}$$

Therefore, we have the following equivalent time fractional diffusion equations:

$$\frac{\partial u}{\partial t} = K_\beta \left({}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right) \quad ; \quad {}_0D_t^{\beta} u = K_\beta \frac{\partial^2 u}{\partial x^2} \tag{8.146}$$

In this time fractional diffusion equation K_β is a sort of fractional diffusion coefficient having a dimension of $\mathrm{cm}^2 / \mathrm{sec}^\beta$. Integrating LHS and RHS of $\frac{\partial u}{\partial t} = K_\beta \left({}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right)$ (8.146) we obtain the following:

$$\begin{aligned}
u(x,t) - u_0(x) &= K_\beta \left({}_0I_t^{1} \circ {}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right) \\
&= K_\beta \left({}_0I_t^{\beta} \frac{\partial^2 u}{\partial x^2} \right) \qquad {}_0D_t^{-\beta} = {}_0I_t^{\beta} \\
&= \frac{K_\beta}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \frac{\partial^2 u(x,\tau)}{\partial x^2} \mathrm{d}\tau
\end{aligned} \tag{8.147}$$

In (8.147) the last expression is obtained by applying the RL Fractional integration formula for ${}_0I_t^{\beta} f(t)$ operator (Chapter-2). Like the diffusion equation of an integer order we consider the equivalent Volterra integral equation corresponding to the fractional diffusion equation of the above (8.147) as by taking the β order fractional integration for both the sides of the LHS and RHS of ${}_0D_t^{\beta} u = K_\beta \frac{\partial^2 u}{\partial x^2}$ (8.146) as demonstrated below:

$$\begin{aligned}
{}_0D_t^{\beta} u &= K_\beta \frac{\partial^2 u}{\partial x^2} \\
{}_0I_t^{\beta} \left({}_0D_t^{\beta} u \right) &= K_\beta \left({}_0I_t^{\beta} \frac{\partial^2 u}{\partial x^2} \right) \\
u(x,t) - u_0(x) &= K_\beta \left({}_0I_t^{\beta} \frac{\partial^2 u}{\partial x^2} \right)
\end{aligned} \tag{8.148}$$

Finally, we write the following by applying the RL fractional integration formula ${}_0I_t^{\beta} f(t)$:

$$u(x,t) = u_0(x) + \frac{K_\beta}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \frac{\partial^2 u(x,\tau)}{\partial x^2} \mathrm{d}\tau \tag{8.149}$$

8.13.5 The Green's function for a time fractional diffusion equation and its moments

The Green's function, $G_\beta(x,t)$ for this time a fractional diffusion equation can be expressed in terms of the M-Wright function as:

$$G_\beta(x,t) = \frac{1}{2\sqrt{K_\beta t^{\beta/2}}} M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta t^{\beta/2}}}\right) \tag{8.150}$$

The above (8.150) we have derived in the previous section. Let us find the moments of the above (8.150) Green's function:

$$\sigma_\beta^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{2\sqrt{K_\beta t^{\beta/2}}}\left(M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta t^{\beta/2}}}\right)\right)\mathrm{d}x \tag{8.151}$$

Put $\frac{|x|}{\sqrt{K_\beta t^\beta}} = y$; $x^2 = y^2 K_\beta t^\beta$; $\mathrm{d}x = \mathrm{d}y\sqrt{K_1 t^\beta}$, in the above (8.150) expression to get the following:

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} \frac{y^2 K_\beta t^\beta}{2\sqrt{K_\beta t^\beta}}\left(M_{\beta/2}(y)\right)\left(\sqrt{K_\beta t^\beta}\right)(\mathrm{d}y) \\
&= \frac{K_\beta t^\beta}{2}\int_{-\infty}^{\infty} y^2\left(M_{\beta/2}(y)\right)\mathrm{d}y \\
&= \left(\frac{K_\beta t^\beta}{2}\right)\left(2\int_0^\infty y^2\left(M_{\beta/2}(y)\right)\mathrm{d}y\right) = K_\beta t\int_0^\infty y^2\left(M_{\beta/2}(y)\right)\mathrm{d}y
\end{aligned} \tag{8.152}$$

We use an already derived expression about the α moments of the M-Wright function (8.63) that is $\int_0^\infty x^\alpha\left(M_\upsilon(x)\right)\mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)}$. In this general expression put $\alpha = 2$, $\upsilon = \frac{\beta}{2}$, so we have the following:

$$\int_0^\infty x^2 M_{\beta/2}(x)\mathrm{d}x = \frac{\Gamma(2+1)}{\Gamma\left(\frac{\beta}{2}\times 2+1\right)} = \frac{\Gamma(3)}{\Gamma(\beta+1)} = \frac{2}{\Gamma(\beta+1)} \tag{8.153}$$

Use of the above (8.153) derived expression gives the variance as proportional to time as follows:

$$\sigma_\beta^2 = K_\beta t^\beta\int_0^\infty y^2\left(M_{\beta/2}(y)\right)\mathrm{d}y = \frac{2K_\beta t^\beta}{\Gamma(\beta+1)} \tag{8.154}$$

The corresponding variance we write as we obtained from the absolute moments of the M-Wright function as:

$$\sigma_\beta^2(t) = \int_{-\infty}^{+\infty} x^2\left(G_\beta(x,t)\right)\mathrm{d}x = \frac{2}{\Gamma(\beta+1)}K_\beta t^\beta \tag{8.155}$$

Consequently, for $0<\beta<1$ the variance is consistent with a process of slow diffusion with a similarity exponent $H = \frac{\beta}{2}$.

8.13.6 A time fractional diffusion equation with a stretched time variable

Now we make one more step as advancement, that is, a stretched one with the time-variable for a fractional diffusion equation, by stretching the time through replacing t with $t^{\alpha/\beta}$, where $0<\alpha<2$ and $0<\beta\le 1$, so we get the following (with the Caputo fractional derivative):

$$\frac{\partial u}{\partial(t^{\alpha/\beta})} = K_{\alpha\beta}\left({}_0D_{t^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u}{\partial x^2}\right) \tag{8.156}$$

Where $K_{\alpha\beta}$ is a sort of stretched diffusion coefficient of dimension $\mathrm{cm}^2/\sec^\alpha$ that reduces to K_α for $\beta=1$ and to K_β for $\alpha=\beta$. From the above (8.156) we get the following equivalent fractional diffusion equation:

$$\frac{\partial u}{\partial t} = \left(\frac{\alpha}{\beta}\right)t^{\left(\frac{\alpha}{\beta}\right)-1}K_{\alpha\beta}\left({}_0D_{t^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u}{\partial x^2}\right) \tag{8.157}$$

The steps to get the above (8.157) expression are as follows:

$$\frac{\partial\ u}{\partial(t^{\alpha/\beta})}=\frac{\partial\ u}{\left(\frac{\alpha}{\beta}\right)t^{\left(\frac{\alpha}{\beta}\right)-1}\partial\ t}=K_{\alpha\beta}\left({}_{0}D_{t^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u}{\partial\ x^2}\right)$$

$$\text{use}\qquad \mathrm{d}(t^{\alpha/\beta})=\tfrac{\alpha}{\beta}t^{\left(\frac{\alpha}{\beta}\right)-1}\mathrm{d}t \tag{8.158}$$

$$\frac{\partial\ u}{\partial\ t}=\left(\frac{\alpha}{\beta}\right)t^{\left(\frac{\alpha}{\beta}\right)-1}K_{\alpha\beta}\left({}_{0}D_{t^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u}{\partial\ x^2}\right)$$

Now we integrate both sides of the LHS and RHS of the above (8.157) to obtain the following expression:

$$u(x,t)-u_0(x)=\left(\frac{\alpha}{\beta}\right)K_{\alpha\beta}\int_{\tau=0}^{\tau=t}\tau^{\left(\frac{\alpha}{\beta}\right)-1}\left({}_{0}D_{\tau^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u(x,\tau)}{\partial\ x^2}\right)\mathrm{d}\tau \tag{8.159}$$

Let $\tau^{\alpha/\beta}=\xi$, $\tau=\xi^{\beta/\alpha}$ for $\tau=0$, $\xi=0$ and for $\tau=t$ we have $\xi=t^{\alpha/\beta}$. Furthermore, we have $(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau=\mathrm{d}\xi$. Substituting all these, we have the following steps:

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)K_{\alpha\beta}\int_{\tau=0}^{\tau=t}\tau^{\left(\frac{\alpha}{\beta}\right)-1}\left({}_{0}D_{\tau^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u(x,\tau)}{\partial\ x^2}\right)\mathrm{d}\tau\\
&\quad=\left(\frac{\alpha}{\beta}\right)K_{\alpha\beta}\int_{\xi=0}^{\xi=t^{\alpha/\beta}}\tau^{\left(\frac{\alpha}{\beta}\right)-1}\left({}_{0}D_{\xi}^{1-\beta}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial\ x^2}\right)\left(\frac{\mathrm{d}\xi}{\left(\frac{\alpha}{\beta}\right)\tau^{(\alpha/\beta)-1}}\right)\\
&\quad=K_{\alpha\beta}\int_{\xi=0}^{\xi=t^{\alpha/\beta}}\left({}_{0}D_{\xi}^{1-\beta}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial\ x^2}\right)\mathrm{d}\xi\\
&\quad=K_{\alpha\beta}\left({}_{0}I_{\xi}^{1}\,{}_{0}D_{\xi}^{1-\beta}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial\ x^2}\right)\\
&\quad=K_{\alpha\beta}\left({}_{0}I_{\xi}^{1}\,{}_{0}D_{\xi}^{1}\,{}_{0}D_{\xi}^{-\beta}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial\ x^2}\right)\\
&\quad=K_{\alpha\beta}\left({}_{0}D_{\xi}^{-\beta}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial\ x^2}\right)
\end{aligned}$$

$$\text{Using}\quad {}_{0}D_{\xi}^{-\beta}f(\xi)={}_{0}I_{\xi}^{\beta}f(\xi)=\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\varsigma)^{\beta-1}f(\varsigma)\mathrm{d}\varsigma$$

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)K_{\alpha\beta}\int_{\tau=0}^{\tau=t}\tau^{\left(\frac{\alpha}{\beta}\right)-1}\left({}_{0}D_{\tau^{\alpha/\beta}}^{1-\beta}\frac{\partial^2 u(x,\tau)}{\partial\ x^2}\right)\mathrm{d}\tau\\
&\quad=K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\varsigma)^{\beta-1}\frac{\partial^2 u(x,\varsigma^{\beta/\alpha})}{\partial\ x^2}\mathrm{d}\varsigma\qquad\qquad \varsigma\equiv\xi
\end{aligned}\tag{8.160}$$

We have $\tau^{\alpha/\beta}=\xi$, $\mathrm{d}\xi=(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau$. Take $\tau^{\alpha/\beta}=\varsigma$ and $(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau=\mathrm{d}\varsigma$, to have the following expression:

$$\begin{aligned}
&K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\varsigma)^{\beta-1}\frac{\partial^2 u(x,\varsigma^{\beta/\alpha})}{\partial\ x^2}\mathrm{d}\varsigma\\
&\quad=K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}\left(\xi-\tau^{\alpha/\beta}\right)^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial\ x^2}\mathrm{d}\tau\left(\frac{\alpha}{\beta}\tau^{\left(\frac{\alpha}{\beta}\right)-1}\right)
\end{aligned}\tag{8.161}$$

Put $\xi=t^{\alpha/\beta}$, so we have $\varsigma\equiv\xi=\tau^{\alpha/\beta}$, thus for $\varsigma=t^{\alpha/\beta}$, we have the upper limit $\tau=t$:

$$K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau\left(\frac{\alpha}{\beta}\tau^{(\alpha/\beta)-1}\right)$$

$$=K_{\alpha\beta}\left(\frac{\alpha}{\beta}\right)\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=t^{\alpha/\beta}}\tau^{\frac{\alpha}{\beta}-1}(t^{\alpha/\beta}-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau;\quad \varsigma\equiv\xi=\tau^{\alpha/\beta} \tag{8.162}$$

$$=K_{\alpha\beta}\left(\frac{\alpha}{\beta}\right)\frac{1}{\Gamma(\beta)}\int_{\tau=0}^{\tau=t}\tau^{\frac{\alpha}{\beta}-1}(t^{\alpha/\beta}-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

Therefore, our integral equation becomes:

$$u(x,t)=u_0(x)+K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\left(\frac{\alpha}{\beta}\right)\int_0^t\tau^{\left(\frac{\alpha}{\beta}\right)-1}\left(t^{\alpha/\beta}-\tau^{\alpha/\beta}\right)^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \tag{8.163}$$

8.13.7 The Green's function for a time fractional diffusion equation with a stretched time variable

The equation (8.156) has the following Green's function:

$$G_{\alpha\beta}(x,t)=\frac{1}{2}\frac{1}{\sqrt{K_{\alpha\beta}t^{\alpha/2}}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_{\alpha\beta}}t^{(\alpha/2)}}\right) \tag{8.164}$$

which comes from the expression:

$$G_{\beta}(x,t)=\frac{1}{2\sqrt{K_{\beta}t^{\beta/2}}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_{\beta}}t^{\beta/2}}\right) \tag{8.165}$$

by placing $t^{\alpha/\beta}$ in place of t and changing K_β to $K_{\alpha\beta}$. The variance is:

$$\sigma^2_{\alpha\beta}=\int_{-\infty}^{+\infty}x^2\left(G_{\alpha\beta}(x,t)\right)\mathrm{d}x=\frac{2}{\Gamma(\beta+1)}K_{\alpha\beta}t^{\alpha} \tag{8.166}$$

This comes from $\sigma^2_\beta(t)=\int_{-\infty}^{+\infty}x^2\left(G_\beta(x,t)\right)\mathrm{d}x=\frac{2}{\Gamma(\beta+1)}K_\beta t^\beta$, by placing $t^{\alpha/\beta}$ in place of t and $K_{\alpha\beta}$ in place of K_β. Consequently, the resulting diffusion process turns out to be self-similar having Hurst exponent i.e. $H=\frac{\alpha}{2}$ (Section-1.24) and the variance law consistent both with slow diffusion $(0<\alpha<1)$ and fast diffusion $(1<\alpha<2)$. We note that the parameter β explicitly enters the variance expression only to modify the proportionality constant. It is straight forward to note that the evolution equations of this process reduce to those for fractional-time diffusion if $\alpha=\beta<1$, for stretched diffusion if $\alpha\neq 1$ and $\beta=1$, and finally to the integer order diffusion equation if $\alpha=\beta=1$.

8.14 Short summary

This chapter extends the lessons of earlier chapters of fractional calculus. Here we have seen how a time fractional partial differential equation, especially a diffusion-wave equation evolves, and after the application of the Laplace Fourier transform tricks we get the solution. The evolution of the M-Wright function and other auxiliary functions is a new idea of Mainardi, as we saw these are a generalisation of a classical Gaussian function, which helps to obtain a solution to a time fractional partial differential equation. We have several derivations regarding these new sets of functions. We conclude with the observation that the treatment is as rigorous as in a classical partial differential equation.

8.15 References

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The details of the above literature are found in the bibliography section, placed in alphabetical order.

Chapter Nine

Modified Fractional Calculus in conjugation with Classical Calculus

9.1 Introduction

We will start the chapter by generalising the concept of term-by-term differentiation as obtained for classical calculus, for the fractional differ-integration context, and then we will formalise the composition rules for a generalised differ-integral operator. Here we try to simplify operators of fractional calculus and try for modifications of the classical Riemann-Liouville (RL) fractional integral, and of the RL and Caputo's fractional derivatives. We merge the two concepts and developed that which is the RL derivative and the Caputo derivative, and call the modified Riemann-Liouville derivative, of Jumarie Type, defined for continuous but non-differentiable functions. We will make the fractional derivative of a constant function zero, by this new modified RL fractional derivative. Thereby, we derive a fractional Taylor series, and define the modified RL fractional derivative for a continuous but non-differentiable function. We have seen in the previous chapters how complicated and difficult the Chain rule, Leibniz's product rule, and Integration by parts formulas are. Here we shall see how this modified RL fractional derivative can make these formulas as a parallel to the classical formulas, by using approximations of fractional differential increments obtained by a fractional Taylor series-for non-differentiable (but α -differentiable) functions. In addition, we shall introduce the notion that by using the ('fractal') Mittag-Leffler function, one can construct fractional integral transforms like the fractional Laplace transforms and the fractional order Gamma function and many others. This new development gives a conjugation to classical calculus for a non-differentiable function or 'fractal functions', developed by G. Jumarie; that we will elaborate upon. We will also further generalise the notion of fractional differ-integration when the order instead of the non-integer number (that we discussed throughout) is a 'continuous distribution function'; and we will try to give a solution to such a differential equation. These are very modern developments and are still under review and while controversial presently, are not necessarily wrong.

9.2 Condition of term-by-term differ-integration of a series

Here we will generalise the notion of term-by-term differ-integration as we have used in earlier chapters (in Chapters-2, 3, 4, 6). We generalise the condition of its applicability or convergence for classical integer order integration and differentiation, and extend this to a fractional calculus context. We will use short form symbol that is $D^{\alpha} f$ instead of ${}_aD_x^{\alpha}\left[f(x)\right]$ in this discussion.

9.2.1 Linearity and homogeneity of fractional differ-integral operators

We note that the fractional differ-integration is a linear operator meaning that we have following relationship:

$$ {}_aD_x^q\left[f_1(x)+f_2(x)\right] = {}_aD_x^q\left[f_1(x)\right] + {}_aD_x^q\left[f_2(x)\right] \tag{9.1}$$

In short, we write $D^q\left[f_1+f_2\right] = D^q f_1 + D^q f_2$. This comes from the direct application of the formulas for fractional differentiation and integration. In addition, we re-write that the fractional differ-integration operation has the property of homogeneity. That is:

$$ {}_aD_x^q\left[Cf(x)\right] = C\left({}_aD_x^q\left[f(x)\right]\right) \tag{9.2}$$

Here C is a constant. With these two (9.1), (9.2) obvious properties we write the following:

$$ {}_aD_x^q\left[\sum_{j=0}^{n} f_j\right] = \sum_{j=0}^{n}\left({}_aD_x^q\left[f_j\right]\right) \tag{9.3}$$

This is called term-by-term differ-integration, which we have used several times in previous chapters (in Chapters-2, 3, 4, 6).

We want to discuss the conditions that permit the term-by-term differ-integration of an infinite series of functions. Our major objective is to establish the term-by-term differ-integrability of the differ-integrable series that we noted earlier, which is $f(y) = (y-a)^p \sum_{j=0}^{\infty} a_j (y-a)^{j/n}$, with $p > -1$ and $a_0 \neq 0$ (Section 2.22).

We shall make use of classical results on differentiation and integration of an infinite series term-by-term, that we have discussed in an earlier chapter (Section 1.18). In order to use the results of classical differentiation and integration, we must ensure that the terms f_j of the series (9.3) are either continuous or continuously differentiable functions. If we restrict our attention to the summands f_j that are from differ-integrable series, the form of such a series that is of $f(y)=(y-a)^p\sum_{j=0}^{\infty}a_j(y-a)^{j/n}$, with $a_0\neq 0$ and $p>-1$. Its term-by-term derivative shows that the requisite continuity assumptions are valid away from the lower terminal (i.e. $x=a$) that is also a start point for differ-integration

Henceforth we shall consider the infinite sums of differ-integrable series and establish results on the term-by-term differ-integrability of such sums, which in general are valid on open intervals such as $a<x<(a+X)$, (where X is the radius of convergence of the differ-integrable series). We note here that the classical results are stated for closed intervals rather than open ones; thus, in this case of generalisation, we will use term convergence in an open interval (meaning convergence in every closed subinterval of the open interval).

9.2.2 Convergence of a differ-integration of series

We now study some of the facts about convergence. Consider an ordinary power series $\phi=\sum_{j=0}^{\infty}a_j(x-a)^j$, with $a_j=\frac{\phi^{(j)}(a)}{j!}$; it is convergent for $0\le|x-a|<X$. From here, can we say anything about the series that is formed through term-by-term differ-integration of the stated power series? We apply the operator ${}_aD_x^q$ term-by-term to the series of ϕ to get ${}_aD_x^q\phi(x)$ as the following:

$$\begin{aligned}{}_aD_x^q\phi(x)&=\sum_{j=0}^{\infty}a_j\frac{\Gamma(j+1)}{\Gamma(j+1-q)}(x-a)^{j-q}\\&=\sum_{j=0}^{\infty}\left(\frac{\phi^{(j)}(a)}{j!}\right)\frac{\Gamma(j+1)}{\Gamma(j+1-q)}(x-a)^{j-q}\\&=(x-a)^{-q}\sum_{j=0}^{\infty}\frac{\phi^{(j)}(a)}{\Gamma(j+1-q)}(x-a)^{j}\end{aligned}\tag{9.4}$$

We know that the series $\phi(x)=\sum_{j=0}^{\infty}a_j(x-a)^j$ converges for $|x-a|<X$, whereby through the ratio test we have the following condition:

$$X\equiv\lim_{j\uparrow\infty}\left|\frac{a_j}{a_{j+1}}\right|=\lim_{j\uparrow\infty}\left|\frac{(j+1)\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|\tag{9.5}$$

While differ-integrable series will converge for:

$$\begin{aligned}|x-a|&<\lim_{j\uparrow\infty}\left|\frac{\left(\phi^{(j)}(a)\right)\Gamma(j-q+2)}{\left(\phi^{(j+1)}(a)\right)\Gamma(j-q+1)}\right|=\lim_{j\uparrow\infty}\left|\frac{(j+1-q)\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|\\&\ge\left(\lim_{j\uparrow\infty}\left|\frac{(j+1)\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|\right)-q\left(\lim_{j\uparrow\infty}\left|\frac{\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|\right)=X-qA\end{aligned}\tag{9.6}$$

where $A\equiv\lim_{j\uparrow\infty}\left|\frac{\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|$. We may examine and write that the possibility is that A is invariably negligible in comparison with X, so the differ-integrated series converges in an open interval $0<|x-a|<X$.

Furthermore, the same result is valid for the differ-integrable series whose j-th term is $a_j(x-a)^{j+p}$; since this series has the same radius of convergence as an analytical part i.e. $\sum a_j(x-a)^j$ and thus is valid for a general differ-integrable series. That is if the differ-integrable series f, which is a finite sum of functions, that is each represented as:

$$(x-a)^p\sum_{j_1=0}^{\infty}a_{j_1}(x-a)^{j_1}+(x-a)^{\frac{(np+1)}{n}}\sum_{j_2=0}^{\infty}a_{j_2}(x-a)^{j_2}+ \\ \ldots+(x-a)^{\frac{(np+n-1)}{n}}\sum_{j_n}^{\infty}a_{j_n}(x-a)^{j_n} \tag{9.7}$$

converges for $|x-a|<X$, then so does the series obtained by differ-integrating each unit term-by-term, (except possibly at the start point at $x=a$). This is a very important point, which we will follow.

9.2.3 Discussion on convergence by application of the Riemann-Liouville fractional integration formula from term-by-term to a differ-integrable series

Let f be any differ-integrable series, since f may be decomposed as a finite sum of a differ-integrable series units such as:

$$f_\upsilon=(x-a)^p\sum_{j=0}^{\infty}a_j(x-a)^j \qquad p>-1 \qquad a_0\neq 0 \tag{9.8}$$

The term-by-term differ-integrability of f will follow from that of f_υ. We will establish the following for all q i.e.

$${}_aD_x^q\left[(x-a)^p\sum\nolimits_{j=0}^{\infty}a_j(x-a)^j\right]=\sum_{j=0}^{\infty}a_j\left({}_aD_x^q\left[(x-a)^{p+j}\right]\right) \tag{9.9}$$

We will prove the validity of the above (9.9) expression inside the region (interval) of convergence for the differ-integrable series $\sum a_j(x-a)^{p+j}$.

For $q\le 0$, we extended the result of the integration in classical calculus for term-by-term integration. Suppose the infinite series of differ-integrable functions $\sum f_i$ converges uniformly in $0<|x-a|<X$, then:

$${}_aD_x^q\left[\sum\nolimits_{j=0}^{\infty}f_i\right]=\sum_{j=0}^{\infty}{}_aD_x^q\left[f_i\right] \qquad q\le 0 \tag{9.10}$$

and the RHS series (9.10) also converges uniformly in $0<|x-a|<X$.

Let us demonstrate this by taking $f\equiv\sum\nolimits_{j=0}^{\infty}f_j$ and let $S_N\equiv\sum\nolimits_{j=0}^{N}f_j$. Since $q<0$, and the Riemann-Liouville formula which is following:

$${}_aD_x^q f(x)=\frac{1}{\Gamma(-q)}\int_a^x\frac{f(y)\mathrm{d}y}{(x-y)^{q+1}}, \qquad {}_aD_x^q f_j(x)=\frac{1}{\Gamma(-q)}\int_a^x\frac{f_j(y)\mathrm{d}y}{(x-y)^{q+1}} \tag{9.11}$$

is valid, we can write it as the following expression:

$${}_aD_x^q f-{}_aD_x^q S_N=\frac{1}{\Gamma(-q)}\int_a^x\frac{\left(f(y)-S_N(y)\right)\mathrm{d}y}{(x-y)^{q+1}} \tag{9.12}$$

The assumption of uniform convergence means that, given $\varepsilon>0$, there is an integer $N=N(\varepsilon)$ such that $|f(y)-S_N(y)|<\varepsilon$ for $n>N$ and for all y in the interval $a\le y\le x$ with $|x-a|<X$. Then we write the following:

$$\begin{aligned}\left|{}_aD_x^q f-\sum\nolimits_{j=0}^{\infty}\left({}_aD_x^q f_j\right)\right|&=\frac{1}{\Gamma(-q)}\left|\int_a^x\frac{\left(f(y)-S_N(y)\right)\mathrm{d}y}{(x-y)^{q+1}}\right|\\&\le\frac{1}{\Gamma(-q)}\int_a^x\frac{|f(y)-S_N(y)|\mathrm{d}y}{(x-y)^{q+1}}<\frac{\varepsilon}{\Gamma(-q)}\int_a^x(x-y)^{-q-1}\mathrm{d}y\\&=\frac{\varepsilon(x-a)^{-q}}{q\left(\Gamma(-q)\right)}\end{aligned} \tag{9.13}$$

which can be made to be small independently of x in the interval $0<|x-a|<X$. This shows that $\sum {}_aD_x^q f_j$ converges uniformly to ${}_aD_x^q f$ in the interval $0<|x-a|<X$. The above (9.13) result shows that the equation ${}_aD_x^q\left[(x-a)^p\sum_{j=0}^{\infty}a_j(x-a)^j\right]=\sum_{j=0}^{\infty}a_j\left({}_aD_x^q\left[(x-a)^{p+j}\right]\right)$ is valid for $q\le 0$ and thus if any f is a differ-integrable series the operator ${}_aD_x^q$ may be distributed through the several infinite series that define f as long as $q\le 0$.

We apply the formula ${}_aD_x^q\left[(x-a)^p\right]=\frac{\Gamma(p+1)}{\Gamma(p+1-q)}(x-a)^{p-q}$, with $p>-1$ and write the following:

$$ {}_aD_x^q f_\upsilon=\sum_{j=0}^{\infty}a_j\frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q} \qquad q\le 0 \tag{9.14}$$

The equation

${}_aD_x^q\left[(x-a)^p\sum_{j=0}^{\infty}a_j(x-a)^j\right]=\sum_{j=0}^{\infty}a_j\left({}_aD_x^q\left[(x-a)^{p+j}\right]\right)$ and the above (9.14)are also valid for $q>0$, which we will discuss now.

9.2.4 Discussion on convergence by application of the Euler fractional derivative formula for power functions term-by-term into differ-integrable series

We decompose the series for f_υ into two parts that are:

$$ f_\upsilon=\sum_{j=0}^{\infty}a_j(x-a)^{p+j}=\sum_{j=J_1}a_j(x-a)^{p+j}+\sum_{j=J_2}a_j(x-a)^{p+j} \tag{9.15}$$

Where J_1 is a set of non-negative integers j for which $\Gamma(p-q+j+1)=\infty$ and J_2 is a set of all non-negative integers not in J_1, here $\Gamma(p-q+j+1)<\infty$ means finite. For a fixed, q the properties of the Gamma function ensure that the set J_1 has only a finite number of elements, (the Gamma function is infinite for zero and has negative integer points). With this observation, we write:

$$\begin{aligned} {}_aD_x^q f_\upsilon &= {}_aD_x^q\left[\sum_{j=J_1}a_j(x-a)^{p+j}\right]+{}_aD_x^q\left[\sum_{j=J_2}a_j(x-a)^{p+j}\right]\\ &=\sum_{j=J_1}a_j\left({}_aD_x^q\left[(x-a)^{p+j}\right]\right)+{}_aD_x^q\left[\sum_{j=J_2}a_j(x-a)^{p+j}\right] \end{aligned} \tag{9.16}$$

In the above (9.16) expression we have made use of the linearity of the operator ${}_aD_x^q$. The first term of (9.16) will be zero since in this set i.e. J_2, we have the gamma function expression $\Gamma(p-q+j+1)=\infty$, which will appear in the denominator.

Now we give proof for the validity of ${}_aD_x^q f_\upsilon=\sum_{j=0}^{\infty}a_j\frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q}$ in the case of $q>0$, by using the following steps, for $q>0$:

$$\begin{aligned} {}_aD_x^q\left[\sum_{j=J_2}a_j(x-a)^{p+j}\right]&=\sum_{j=J_2}a_j\left({}_aD_x^q\left[(x-a)^{p+j}\right]\right)\\ &=\sum_{j=J_2}a_j\frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q} \end{aligned} \tag{9.17}$$

Here we assume the series for f_υ converges uniformly in $0<|x-a|<X$, as will the series on the RHS of the above (9.17) expression, as we have proved at the start. Thus, the operator ${}_aD_x^{-1}$ may be distributed through the terms of this series to give the following steps:

$$
\begin{aligned}
{}_aD_x^{-1}&\left[\sum_{j=J_2} a_j \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q}\right]\\
&=\sum_{j=J_2} {}_aD_x^{-1}\left[a_j \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q}\right]\\
&=\sum_{j=J_2} a_j \frac{\Gamma(p+j+1)\Gamma(p-q+j+1)}{\Gamma(p-q+j+1)\Gamma(p-q+j+2)}(x-a)^{p+j-q+1}\\
&=\sum_{j=J_2} a_j \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+2)}(x-a)^{p+j-q+1}\\
&=\sum_{j=J_2} a_j\left({}_aD_x^{q-1}\left[(x-a)^{p+j}\right]\right)
\end{aligned}
\tag{9.18}
$$

The last series in the above (9.18) steps converge uniformly in $0<|x-a|<X$, as does the series obtained from it by differentiating each term. We apply a classical result on a term-by-term differentiation as we had already discussed in the previous chapter (Section 1.18), to the series $\sum a_j\left({}_0D_x^{q-1}\left[(x-a)^{p+j}\right]\right)$ that gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{j=J_2} a_j\left({}_aD_x^{q-1}\left[(x-a)^{p+j}\right]\right)\right]=\sum_{j=J_2} a_j\left({}_aD_x^{q}\left[(x-a)^{p+j}\right]\right) \tag{9.19}$$

Similarly, we can have, for every positive integer, n the following:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sum_{j=J_2} a_j\left({}_aD_x^{q-n}\left[(x-a)^{p+j}\right]\right)\right]=\sum_{j=J_2} a_j\left({}_aD_x^{q}\left[(x-a)^{p+j}\right]\right) \tag{9.20}$$

Again choosing $n>q$ permits us to use ${}_aD_x^{\alpha} f_\upsilon=\sum_{j=0}^{\infty} a_j \frac{\Gamma(p+j+1)}{\Gamma(p-\alpha+j+1)}(x-a)^{p+j-\alpha}$, for $\alpha\le 0$ as discussed in (Section 2.14), to give the following result, with $\alpha=q-n$:

$$\sum_{j=J_2} a_j\left({}_aD_x^{q-n}\left[(x-a)^{p+j}\right]\right)={}_aD_x^{q-n}\left[\sum_{j=J_2} a_j(x-a)^{p+j}\right] \tag{9.21}$$

We differentiate both sides of (9.21) n times and we write:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sum_{j=J_2} a_j\left({}_aD_x^{q-n}\left[(x-a)^{p+j}\right]\right)={}_aD_x^{q}\left[\sum_{j=J_2} a_j(x-a)^{p+j}\right] \tag{9.22}$$

Utilising $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sum_{j=J_2} a_j\left({}_aD_x^{q-n}\left[(x-a)^{p+j}\right]\right)=\sum_{j=J_2} a_j\left({}_aD_x^{q}\left[(x-a)^{p+j}\right]\right)$ we finally write:

$${}_aD_x^{q}\left[\sum_{j=J_2} a_j(x-a)^{p+j}\right]=\sum_{j=J_2} a_j\left({}_aD_x^{q}\left[(x-a)^{p+j}\right]\right) \qquad q>0 \tag{9.23}$$

Therefore the expression that is described earlier (9.17):

$$
\begin{aligned}
{}_aD_x^{q}\left[\sum_{j=J_2} a_j(x-a)^{p+j}\right]&=\sum_{j=J_2} a_j\left({}_aD_x^{q}\left[(x-a)^{p+j}\right]\right)\\
&=\sum_{j=J_2} a_j \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q}
\end{aligned}
\tag{9.24}
$$

is valid for $q>0$ hence for any arbitrary order q .

We have thus proved the term-by-term fractional differ-integration of the arbitrary differ-integrable series. In the above discussions while proving this we established a generalisation (of ${}_aD_x^{q}$ for any $q\le 0$) of the classical calculus result on a term-by-term integration valid for the uniform convergent series. Can this classical result of integration be generalised for $q>0$? The answer is yes.

If the infinite series for $\sum f_j$ as well as the series $\sum {}_aD_x^{q}\left[f_j\right]$ converge uniformly in $0<|x-a|<X$, then ${}_aD_x^{q}\left[\sum_{j=0}^{\infty} f_\upsilon\right]=\sum_{j=0}^{\infty}\left({}_aD_x^{q}\left[f_\upsilon\right]\right)$ for $q>0$ and for $0<|x-a|<X$ is valid. We note here that, this is a very

natural extension of the classical result on a term-by-term differentiation, which also requires an addition to the convergence of $\sum f_j$ which requires the uniform convergence of $\sum \frac{\mathrm{d}}{\mathrm{d}x}\left[f_j\right]$.

We use this derivation for a differ-integration of the analytical function $\varphi(x)$ represented by series say $\varphi = \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(a)}{\Gamma(j+1)}(x-a)^j$ as:

$$ {}_aD_x^{\alpha}\left[\varphi\right] = \sum_{j=0}^{\infty} \varphi^{(j)}(a)\frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}(x-a)^{j-\alpha} \tag{9.25}$$

The series above converges uniformly in $0 < \left|x-a\right| < X$, with X as the radius of convergence.

9.3 Composition rule in fractional calculus

We will discuss the validity between the operations ${}_aD_x^{q}\left({}_aD_x^{Q}\right)\left[f(x)\right]$ and ${}_aD_x^{q+Q}\left[f(x)\right]$. For meaningful discussions, we assume that $f(x)$ is a differ-integrable function along with ${}_aD_x^{Q}\left[f(x)\right]$ as well. We will use the short form of symbol that is $D^{\alpha}f$ instead of ${}_aD_x^{\alpha}\left[f(x)\right]$ in this discussion. We restrict our discussion to the differ-integrable series.

We have written that the most general non-zero differ-integrable series is a finite sum of differ-integrable units each having a form:

$$ f_{\upsilon}(x) = (x-a)^p \sum_{j=0}^{\infty} a_j (x-a)^j \qquad p > -1 \qquad a_0 \neq 0 \tag{9.26}$$

We will find out that the composition rule may be valid for some units of $f(x)$ but possibly not for others.

9.3.1 Condition for the inverse operation $f = D^{-Q}D^{Q}f$ to be satisfied

It follows from the linearity of differ-integral operators that $D^qD^Qf = D^{q+Q}f$ if $D^qD^Qf_{\upsilon} = D^{q+Q}f_{\upsilon}$, for every unit f_{υ} of f. First, we see the validity of the composition $D^qD^Qf_{\upsilon} = D^{q+Q}f_{\upsilon}$, for a differ-integrable series unit function f_{υ}. Obviously if $f_{\upsilon} = 0$ then $D^Qf_{\upsilon} = 0$ for every Q, and we write $D^qD^Q\left[0\right] = D^{q+Q}\left[0\right] = 0$. While this composition is trivially satisfied for the differ-integrable function $f_{\upsilon} = 0$ we will see that the possibility when $f_{\upsilon} \neq 0$ but $D^Qf_{\upsilon} = 0$ which is exactly the condition that will negate the composition $D^qD^Qf_{\upsilon} = D^{q+Q}f_{\upsilon}$. Now take $f_{\upsilon} \neq 0$ and evaluate D^Qf_{υ} as follows:

$$ D^Qf_{\upsilon} = \sum_{j=0}^{\infty} a_jD^Q\left[(x-a)^{p+j}\right] = \sum_{j=0}^{\infty} a_j\frac{\Gamma(p+j+1)}{\Gamma(p+j+1-Q)}(x-a)^{p+j-Q} \tag{9.27}$$

Here we note that since $p > -1$ we have $p+j > -1$. This makes $\Gamma(p+j+1)$ always finite and non-zero. The individual terms of the series D^Qf_{υ} will vanish only when the coefficients a_j are zero or the denominator comprising of $\Gamma(p+j+1-Q) = \infty$ is infinite. Therefore, we write a necessary and sufficient condition for $D^Qf_{\upsilon} \neq 0$ when for j from 0 to ∞, i.e. we have $\Gamma(p+j+1-Q) < \infty$, that is finite for which $a_j \neq 0$. We apply D^{-Q} to the obtained D^Qf_{υ} (9.27) and write:

$$\begin{aligned} D^{-Q}D^Qf_{\upsilon} &= \sum_{j=0}^{\infty} a_j\frac{\Gamma(p+j+1)}{\Gamma(p+j+1-Q)}D^{-Q}\left[(x-a)^{p+j-Q}\right] \\ &= \sum_{j=0}^{\infty} a_j\frac{\Gamma(p+j+1)}{\Gamma(p+j+1-Q)}\left(\frac{\Gamma(p+j-Q+1)}{\Gamma(p+j-Q+1+Q)}\right)(x-a)^{p+j-Q+Q} \\ &= \sum_{j=0}^{\infty} a_j(x-a)^{p-j} = f_{\upsilon} \end{aligned} \tag{9.28}$$

The cancellation of $\Gamma(p+j+1)$ and $\Gamma(p+j+1-Q)$ is due to these being finite. This condition may be shown to be equivalent to:

$$f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0 \tag{9.29}$$

meaning that the condition of the differ-integrable units f_υ be re-generated upon this inverse operation, that is first doing D^Q and then following it with D^{-Q}.

9.3.2 Condition $D^qD^Qf = D^{q+Q}f$ to be satisfied

Assume that we can apply the operator D^q on the obtained D^Qf_υ that is the above series (9.27).We will get the following:

$$D^qD^Qf_\upsilon = \sum_{j=0}^{\infty} a_j \frac{\Gamma(p+j+1)\Gamma(p+j+1-Q)}{\Gamma(p+j+1-Q)\Gamma(p+j+1-Q-q)}(x-a)^{p+j-Q-q} \tag{9.30}$$

The above (9.30) is obtained by application of ${}_aD_x^q\left[f_\upsilon\right] = \sum_{j=0}^{\infty} a_j \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}(x-a)^{p+j-q}$, $q \le 0$ is valid since D^Qf_υ is assumed to be differ-integrable series. With the condition $f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$, that means $\Gamma(p+j+1-Q) < \infty$ (finite) in effect we may cancel the $\Gamma(p+j-Q+1)$ and re-write (9.30) as:

$$D^qD^Qf_\upsilon = \sum_{j=0}^{\infty} a_j \frac{\Gamma(p+j+1)}{\Gamma(p+j+1-Q-q)}(x-a)^{p+j-Q-q} \tag{9.31}$$

On the other hand, the same steps show that:

$$\begin{aligned} D^{q+Q}f_\upsilon &= \sum_{j=0}^{\infty} a_j D^{q+Q}f_\upsilon \\ &= \sum_{j=0}^{\infty} a_j \frac{\Gamma(p+j+1)}{\Gamma(p+j+1-Q-q)}(x-a)^{p+j-Q-q} = D^qD^Qf_\upsilon \end{aligned} \tag{9.32}$$

Thus the composition of $D^qD^Qf_\upsilon = D^{q+Q}f_\upsilon$ is satisfied as long as $f_\upsilon - D^{-Q}D^{q}f_\upsilon = 0$ is satisfied or in other terms $\Gamma(p+j+1-Q) < \infty$ i.e. finite.

This condition implies that they are invariably satisfied for $Q < 0$. Take $Q = -M$ for any $M > 0$, and as $p > -1$ is our condition for differ-integrable units f_υ. Then $p+j+1+M > -1$ and the $\Gamma(p+j+1+M) < \infty$ will be finite. However, when the condition $f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$ is violated, $D^Qf_\upsilon = 0$ so that $D^qD^Qf_\upsilon = 0$. On the other hand it is not necessary that the case $D^{q+Q}f_\upsilon = 0$.

For example for $f_\upsilon = x^{-\frac{1}{2}}$, $a = 0$, $Q = \frac{1}{2}$, and $q = -\frac{1}{2}$, then we observe the following:

$$\begin{aligned}
&D^Qf_\upsilon = D^{1/2}\left[x^{-\left(\frac{1}{2}\right)}\right] = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(0)}x^{-1} = 0 \\
&f_\upsilon - D^{-Q}D^{Q}f_\upsilon = x^{-1/2} - D^{-1/2}D^{1/2}\left[x^{-1/2}\right] = x^{-1/2} - D^{-1/2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(0)}\right]x^{-1} = x^{-1/2} \ne 0 \\
&D^qD^Qf_\upsilon = D^q\left[0\right] = 0 \\
&D^{q+Q}f_\upsilon = D^{-1/2+1/2}f_\upsilon = D^0\left[x^{-1/2}\right] = x^{-1/2} \ne 0 \\
&D^qD^Qf_\upsilon \ne D^{q+Q}f_\upsilon \\
&D^{q+Q}f_\upsilon - D^{q+Q}\left[f_\upsilon - D^{-Q}D^{Q}f_\upsilon\right] = D^{-1/2+1/2}\left[x^{-1/2}\right] - D^{-1/2+1/2}\left[x^{-1/2}\right] = 0 \\
&0 = D^qD^Qf_\upsilon = D^{q+Q}f_\upsilon - D^{q+Q}\left[f_\upsilon - D^{-Q}D^{Q}f_\upsilon\right]
\end{aligned} \tag{9.33}$$

So the condition $f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$ is certainly violated in the above (9.33) example. Therefore, we have $D^{Q}f_\upsilon = 0$ and $D^{q}D^{Q}f_\upsilon = 0$ while $D^{q+Q}f_\upsilon \neq 0$. Generalising this observation, we easily see the relationship between $D^{q}D^{Q}f_\upsilon$ and $D^{q+Q}f_\upsilon$ in the case $f_\upsilon - D^{-Q}D^{Q}f_\upsilon \neq 0$, to be:

$$0 = D^{q}D^{Q}f_\upsilon = D^{q+Q}f_\upsilon - D^{q+Q}\left[f_\upsilon - D^{-Q}D^{Q}f_\upsilon\right] \tag{9.34}$$

The condition of composition regarding differ-integrable units f_υ is listed in Table 9.1

Condition	$f_\upsilon = 0$	$f_\upsilon \neq 0$
$D^{Q}f_\upsilon = 0$	$f_\upsilon - D^{-Q}D^{Q}f_\upsilon$ $= 0$ $D^{q}D^{Q}f_\upsilon$ $= D^{q+Q}f_\upsilon = 0$	$f_\upsilon - D^{-Q}D^{Q}f_\upsilon \neq 0$ $0 = D^{q}D^{Q}f_\upsilon$ $= D^{q+Q}f_\upsilon - D^{q+Q}\left[f_\upsilon - D^{-Q}D^{Q}f_\upsilon\right]$
$D^{Q}f_\upsilon \neq 0$	Not possible	$f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$ $D^{q}D^{Q}f_\upsilon = D^{q+Q}f_\upsilon$

Table 9.1: Composition rule for differ-integrable units. The requirement for this rule is that f_υ and $D^{Q}f_\upsilon$ are both differ-integrable

While the above (9.34) expression is trivial identity for differ-integrable units f_υ, we shall see its usefulness in general differ-integrable series. As the equation $D^{q}D^{Q}f = D^{q+Q}f$ is valid for a general differ-integrable series if and only if $D^{q}D^{Q}f_\upsilon = D^{q+Q}f_\upsilon$ is valid for every differ-integrable unit f_υ of f. It is straightforward to apply the theory just discussed for units f_υ, to write the composition rule for general f. The only difference is that; while the condition $f_\upsilon \neq 0$ and $f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$ for units guaranteed that $D^{Q}f_\upsilon \neq 0$, this is no longer true for arbitrary case. The reason of course is that some units of f may satisfy the condition $f_\upsilon - D^{-Q}D^{Q}f_\upsilon = 0$ for $f_\upsilon \neq 0$ while other units may not. This will make it possible to violate the composition rule that is $D^{q}D^{Q}f = D^{q+Q}f$ even though $f \neq 0$ and $D^{Q}f \neq 0$.

The condition $f - D^{-Q}D^{Q}f = 0$ for a general differ-integrable series, f is however still necessary and sufficient to guarantee $D^{q}D^{Q}f = D^{q+Q}f$. We mention here that for a general differ-integrable f the condition is $D^{q}D^{Q}f = D^{q+Q}f$ (as was the case for differ-integrable units f_υ) is valid at least when $Q < 0$ and even when $Q < 1$ for a function bounded at $x = a$. For a general differ-integrable series, f we summarise this as in Table 9.2

Condition	$f = 0$	$f \neq 0$
$D^{Q} f = 0$	$f - D^{-Q} D^{Q} f$ $= 0$ $D^{q} D^{Q} f$ $= D^{q+Q} f = 0$	$f - D^{-Q} D^{Q} f \neq 0$ $0 = D^{q} D^{Q} f = D^{q+Q} f$ $- D^{q+Q}\left[f - D^{-Q} D^{Q} f\right]$
$D^{Q} f \neq 0$	Not possible	If $f - D^{-Q} D^{Q} f = 0$ then $D^{q} D^{Q} f = D^{q+Q} f$ If $f - D^{-Q} D^{Q} f \neq 0$ then $D^{q} D^{Q} f = D^{q+Q} f - D^{q+Q}\left[f - D^{-Q} D^{Q} f\right]$

Table9.2: The composition rule for an arbitrary differ-integrable function f. The requirement for this rule is f and $D^{Q} f$ both of which are differ-integrable

We have noted just earlier that where the composition rule is violated the equation $D^{q} D^{Q} f = D^{q+Q} f - D^{q+Q}\left[f - D^{-Q} D^{Q} f\right]$ relates $D^{q} D^{Q} f$ to $D^{q+Q} f$. The utility of this expression is there for calculating the $D^{q} D^{Q} f$ i.e. the LHS in marginal general cases.

9.3.3 Condition for $D^{N} D^{q} f = D^{q} D^{N} f$, N as positive integer

Let us take one marginal general case with $Q = N$ as a positive integer. Indeed, we may then use the following equations that we have developed for a classical case in the earlier chapter (Section 1.18):

$$ {}_aD_x^{N}\left[{}_aD_x^{-n} f\right] = {}_aD_x^{N-n} f = {}_aD_x^{-n}\left[{}_aD_x^{N} f\right] + \sum_{k=n-N}^{n-1} \frac{(x-a)^k}{k!} f^{(k+N-n)}(a) \tag{9.35} $$

$$ {}_aD_x^{q}\left[(x-a)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p-q+1)}(x-a)^{p-q} \qquad p > -1 $$

In the above (9.35), with the change of index numbers as $k + N - n = m$ we write the following:

$$ {}_aD_x^{N}\left[{}_aD_x^{-n} f\right] = {}_aD_x^{N-n} f = {}_aD_x^{-n}\left[{}_aD_x^{N} f\right] + \sum_{m=0}^{N-1} \frac{(x-a)^{m-N+n}}{(m-N+n)!} f^{(m)}(a) \tag{9.36} $$

Put $n = -q$ and we get:

$$ {}_aD_x^{N}\left[{}_aD_x^{q} f\right] = {}_aD_x^{N+q} f = {}_aD_x^{q}\left[{}_aD_x^{N} f\right] + \sum_{m=0}^{N-1} \frac{(x-a)^{m-N-q}}{(m-N-q)!} f^{(m)}(a) \tag{9.37} $$

Writing $(m-N-q)! = \Gamma(m-q-N+1)$ and rearranging the above (9.37) expression we get the following:

$$ {}_aD_x^{q}\left[{}_aD_x^{N} f\right] = {}_aD_x^{N+q} f - \sum_{m=0}^{N-1} \frac{(x-a)^{m-N-q}}{(m-N-q)!} f^{(m)}(a) \tag{9.38} $$

With the above (9.38) expression and with $D^{q} D^{N} f = D^{q+N} f - D^{q+N}\left[f - D^{-N} D^{N} f\right]$ we write the desired expression as follows:

$$
\begin{aligned}
D^q D^N f &= D^{q+N} f - D^{q+N}\left[f - D^{-N} D^N f\right] \\
&= D^{q+N} f - \sum_{m=0}^{N-1} \frac{(x-a)^{m-q-N} f^{(m)}(a)}{\Gamma(m-q-N+1)}
\end{aligned} \tag{9.39}
$$

For $q=\frac{1}{2}$ and $q=-\frac{1}{2}$ with $a=0$, for $N=1,2,3,...$ and $f(0)\neq\infty$ we have the following:

$$
\begin{aligned}
\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}x^{\frac{1}{2}}}\left[\frac{\mathrm{d}^N f}{\mathrm{d}x^N}\right] &= \frac{\mathrm{d}^{N+\frac{1}{2}} f}{\mathrm{d}x^{N+\frac{1}{2}}} - \sum_{m=0}^{N-1} \frac{x^{m-N-\frac{1}{2}} f^{(m)}(0)}{\Gamma\left(m-N+\frac{1}{2}\right)} \\
\frac{\mathrm{d}^{-\frac{1}{2}}}{\mathrm{d}x^{-\frac{1}{2}}}\left[\frac{\mathrm{d}^N f}{\mathrm{d}x^N}\right] &= \frac{\mathrm{d}^{N-\frac{1}{2}} f}{\mathrm{d}x^{N-\frac{1}{2}}} - \sum_{m=0}^{N-1} \frac{x^{m-N+\frac{1}{2}} f^{(m)}(0)}{\Gamma\left(m-N+\frac{3}{2}\right)}
\end{aligned} \tag{9.40}
$$

With $N=1$, $f(0)\neq\infty$ with $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$, and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ we have the following compositions:

$$
\begin{aligned}
\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}x^{\frac{1}{2}}}\left[\frac{\mathrm{d}[f]}{\mathrm{d}x}\right] &= \frac{\mathrm{d}^{\frac{3}{2}} f}{\mathrm{d}x^{\frac{3}{2}}} - \frac{x^{-\frac{3}{2}} f(0)}{\Gamma\left(-\frac{1}{2}\right)} = \frac{\mathrm{d}^{\frac{3}{2}} f}{\mathrm{d}x^{\frac{3}{2}}} + \frac{x^{-\frac{3}{2}} f(0)}{2\sqrt{\pi}} \\
\frac{\mathrm{d}^{-\frac{1}{2}}}{\mathrm{d}x^{-\frac{1}{2}}}\left[\frac{\mathrm{d}[f]}{\mathrm{d}x}\right] &= \frac{\mathrm{d}^{\frac{1}{2}} f}{\mathrm{d}x^{\frac{1}{2}}} - \frac{x^{-\frac{1}{2}} f(0)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\mathrm{d}^{\frac{1}{2}} f}{\mathrm{d}x^{\frac{1}{2}}} - \frac{x^{-\frac{1}{2}} f(0)}{\sqrt{\pi}}
\end{aligned} \tag{9.41}
$$

The above (9.40), (9.41) derived equation can be established under the relaxed assumption of differentiability. For instance, functions such as x^p, with $p\leq -1$ are differentiable. On differentiation, we get px^{p-1}; but this is not fractionally differ-integrable. Thus, we can have f be N fold differentiable and that $f^{(m)}(a)$ be finite, for $m=0,1,2,...(N-1)$. If all these $f(a)$, $f^{(1)}(a)$, $f^{(2)}(a)$, … $f^{(N-1)}(a)$ are zero, that is the values at the start point, then $D^q D^N = D^{q+N}$ are valid.

The Table 9.3 gives a summary of the above discussion, the composition rule for the functions that are, N fold differentiable, and its N th derivative are differ-integrable.

Condition	$f=0$	$f\neq 0$
$D^N f=0$	$f-D^{-N}D^N f$ $=0$ $D^q D^N f$ $=D^{q+N} f=0$	$f-D^{-N}D^N f\neq 0$ $0=D^q D^N f = D^{q+N} f - \sum_{m=0}^{N-1} \frac{(x-a)^{k-q-N} f^{(m)}(a)}{\Gamma(m-q-N+1)}$
$D^N f\neq 0$	Not possible	If $f-D^{-N}D^N f=0$ then $D^q D^N f = D^{q+N} f$ If $f-D^{-N}D^N f\neq 0$ then $D^q D^N f = D^{q+N} f - \sum_{m=0}^{N-1} \frac{(x-a)^{m-q-N} f^{(m)}(a)}{\Gamma(m-q-N+1)}$

Table 9.3: The composition rule for the differentiable but not essentially the differ-integrable function f. The requirement for this rule is f is N times differentiable and $D^N f$ is differ-integrable.

The general rule was just discussed regarding composing D^Q with D^q, which makes it clear that while the operators D^Q and D^{-Q} are 'usually' inverse to each other, this is not always the case.

Let us choose $f_\upsilon = x$, then $D^2 f_\upsilon = 0$ and $D^{-2}D^2 f_\upsilon = 0$. So certainly in this case D^{-2} and D^2 are not the inverse of each other. This difficulty is due to the condition $f_\upsilon - D^{-Q}D^Q f_\upsilon = 0$ which is violated for the differ-integrable unit i.e. $f_\upsilon = x$. Nor is this problem restricted to integer q and Q. In fact by choosing $Q=\frac{3}{2}$, $q=-\frac{3}{2}$, $a=0$ and $f_\upsilon = x^{1/2}$, we see that $D^Q f_\upsilon = D^{3/2}x^{1/2}=0$. So that $D^q D^Q f_\upsilon = D^{-3/2}D^{3/2}x^{1/2}=0$, and $f_\upsilon - D^{-Q}D^Q f_\upsilon = x^{1/2}\neq 0$. However, we know that $D^{q+Q}f_\upsilon = D^0 f_\upsilon = f_\upsilon = x^{1/2}$. We used ${}_aD_x^q\left[(x-a)^p\right]=\frac{\Gamma(p+1)}{\Gamma(p-q+1)}(x-a)^{p-q}$, $p>-1$. Here in this case, too, we chose $f_\upsilon \neq 0$ but $D^Q f_\upsilon = 0$, which guarantees the violation of $f_\upsilon - D^{-Q}D^Q f_\upsilon = 0$. The composition rule fails even if $f\neq 0$ and $D^Q f \neq 0$.

For example, choose $f(x)=x^{1/2}+1$, $N=1$, $a=0$ and arbitrary q. Here f is the sum of two differ-integrable units and $D^1 f = \frac{1}{2}x^{-1/2}$, $D^{-1}D^1 f = x^{1/2}$, $f - D^{-1}D^1 f = 1$ yet:

$$D^q D^1 f = \frac{\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(-q-\frac{1}{2}\right)}x^{-q-\left(\frac{1}{2}\right)} \tag{9.42}$$

and:

$$\begin{aligned} D^{q+1}f &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-q-\frac{1}{2}\right)}x^{-q-\left(\frac{1}{2}\right)} + \frac{1}{\Gamma(-q)}x^{-q-1} \\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(-q-\frac{1}{2}\right)}x^{-q-\left(\frac{1}{2}\right)} + \frac{1}{\Gamma(-q)}x^{-q-1} = D^q D^1 f + \frac{1}{\Gamma(-q)}x^{-q-1} \end{aligned} \tag{9.43}$$

The condition $f - D^{-Q}D^Q f = 0$ is violated in this example as we demonstrated above that is we get in this case $f - D^{-1}D^1 f = 1$ and thus the equation:

$$\begin{aligned} D^q D^N f &= D^{q+N} f - \sum_{m=0}^{N-1}\frac{(x-a)^{m-q-N} f^{(m)}(a)}{\Gamma(m-q-N+1)} \\ D^q D^1 f &= D^{q+1} f - \frac{1}{\Gamma(-q)}x^{-q-1} \end{aligned} \tag{9.44}$$

correctly relates $D^q D^1 f$ and $D^{q+1} f$. Realising $f(0)=1$, we may write from the above (9.44) $D^q D^1 f = D^{q+1} f - \frac{f(0)}{\Gamma(-q)}x^{-q-1}$. Putting $q=\frac{1}{2}$, we have the following:

$$\begin{aligned} \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{\mathrm{d}[f]}{\mathrm{d}x}\right] &= \frac{\mathrm{d}^{3/2}[f]}{\mathrm{d}x^{3/2}} - \frac{f(0)}{\Gamma\left(-\frac{1}{2}\right)}x^{-(1/2)-1} \qquad \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi} \\ &= \frac{\mathrm{d}^{3/2}[f]}{\mathrm{d}x^{3/2}} + \frac{f(0)}{2\sqrt{\pi}}x^{-(3/2)} \end{aligned} \tag{9.45}$$

Now take $q=-\frac{1}{2}$, and we get the following

$$\begin{aligned} \frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left[\frac{\mathrm{d}[f]}{\mathrm{d}x}\right] &= \frac{\mathrm{d}^{1/2}[f]}{\mathrm{d}x^{1/2}} - \frac{f(0)}{\Gamma\left(\frac{1}{2}\right)}x^{-(-1/2)-1} \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \\ &= \frac{\mathrm{d}^{1/2}[f]}{\mathrm{d}x^{1/2}} - \frac{f(0)}{\sqrt{\pi x}} \end{aligned} \tag{9.46}$$

The LHS of the above (9.46) expression is the Caputo's fractional derivative and we have earlier seen the relationship between the RL and Caputo derivative (Section 3.10, 5.13). When the order of the Caputo derivative is half we get from the above (9.46) by properly re-writing the expression ${}_0D_x^{1/2}\left[f(x)\right] = {}_0^CD_x^{1/2}\left[f(x)\right] + \frac{f(0)}{\sqrt{\pi x}}$.

9.3.4 Generalising the composition $D^q D^Q f$ to $D^{q+Q} f$ by initial values of f

We generalise the composition of (9.44) by the constants $\frac{f^{(m)}(a)}{\Gamma(m-q-N+1)}$ as C_m and get the following:

$$D^q D^N f = D^{q+N} f - \sum_{m=0}^{N-1} C_m (x-a)^{m-q-N} \tag{9.47}$$

Say N is chosen as an arbitrary non-integer i.e. Q, with $N-1<Q<N$, where N is a positive integer, then we have the following expressions:

$$\begin{aligned} D^q D^Q f &= D^{q+Q} f - \sum_{m=0}^{N-1} \frac{(x-a)^{m-q-N} f^{(m)}(a)}{\Gamma(m-q-N+1)} \\ D^q D^Q f &= D^{q+Q} f - \sum_{m=0}^{N-1} C_m (x-a)^{m-q-N} \end{aligned} \tag{9.48}$$

Say $q=Q=\frac{1}{2}$, then we have $N=1$, so we get the composition:

$$D^{\frac{1}{2}} D^{\frac{1}{2}} f = D^{\frac{1}{2}+\frac{1}{2}} f - \sum_{m=0}^{1-1} C_m (x-a)^{0-\left(\frac{1}{2}\right)-1} = D^1 f - C_0 (x-a)^{-\frac{3}{2}} \tag{9.49}$$

For $q=Q=\alpha$ and $\alpha<1$, we have $N=1$, then the sequential derivative $\mathscr{D}^{2\alpha}$ is:

$$\mathscr{D}^{2\alpha} f = D^\alpha D^\alpha f = D^{2\alpha} f - \frac{(x-a)^{-\alpha-1} f(a)}{\Gamma(-\alpha)} \tag{9.50}$$

This above (9.50) argument we used in an earlier chapter (Section 7.12).In addition, we note that if $f(a)=0$ that is a function value at the start point of differ-integration in the above (9.50) case, then $D^\alpha D^\alpha = D^{2\alpha}$ holds.

9.4 The reversibility of a differ-integral operator $D^Q f = g$ to $f = D^{-Q} g$

The discussion of the above sections reveals the danger when one asks about the commutability of differ integral operators $D^q D^Q = D^Q D^q$ or the irreversibility of differ-integration that is if $D^Q f = g$ then $f = D^{-Q} g$. Of course, this reversibility comes into play while solving fractional differential equations. The above results of various discussions may be used to derive conditions under which commutability holds, and as always, the fractional differ-integral operators are applied to a suitable class of functions.

Consider the simplest fractional differential equation $D^Q f = F$, where Q is arbitrary. The function F is a known one, and f is an unknown function. Here we choose the lower limit of differ-integration as zero that is $a=0$. We apply inversion and write the following as $f = D^{-Q} F$. Nevertheless, we have used this in previous chapters (Section 5.8), to get the solutions of fractional differential equations. With this we get perhaps a solution, but unfortunately this is not the most general solution. In fact, referring to the previous section regarding the composition we say that it is precisely the condition $f - D^{-Q} D^Q f = 0$, which guarantees the adherence to the compositional rule for a general differ-integrable series. In general, the difference $f - D^{-Q} D^Q f$ will not be zero, but will contain of those parts of the differ-integrable series units f_υ in f that are sent to zero under the operation D^Q.

For example in the case of $f = Cx^0 + x^{\frac{1}{2}}$, with $Q=N=1$, the differ-integrable unit that is Cx^0 was sent to zero by operation D^1, but appeared in $f - D^{-1} D^1 f = Cx^0$, in the example that was shown earlier (9.33). In general, we can write:

$$f - D^{-N} D^N f = C_1 x^{N-1} + C_2 x^{N-2} + \ldots . C_m x^{N-m} \tag{9.51}$$

with $(m-1)$ terms which goes to zero upon operation D^N. We decompose f into differ-integrable series units $f_{\upsilon,i}$ where:

$$f_{\upsilon,i} = x^{p_1} \sum_{j=0}^{\infty} a_{ij} x^j \qquad p_i > -1 \qquad a_{i0} \neq 0 \qquad i = 1,2,\ldots n \tag{9.52}$$

We now investigate the condition on $f_{\upsilon,i}$ that is required to give $f - D^{-Q} D^Q f \neq 0$. We have conditions in the earlier section i.e. $\Gamma(p+j+1-Q) < \infty$ i.e. finite for each j for which $a_j \neq 0$ was equivalent to $f_\upsilon - D^{-Q} D^Q f_\upsilon = 0$. This tells us that $f - D^{-Q} D^Q f \neq 0$ obtains i in the range of $1 \le i \le n$, for which the term $\Gamma(p_i - Q + 1) = \infty$, that is not finite (which in turns makes $D^Q f_{\upsilon,i} = 0$). This condition can occur only when

$p_i = (Q-1),(Q-2),...,(Q-m)$ that is the Gamma function values at zero, and negative integer points. Putting these facts together, we say the most general case is:

$$f - D^{-Q}D^{Q}f = C_1x^{Q-1} + C_2x^{Q-2} + ... + C_mx^{Q-m} \tag{9.53}$$

Where $C_1,....,C_m$ are arbitrary constants and $0 < Q \le m < Q+1$; and for $Q \le 0$ the RHS members of the above (9.53) equation is zero. Thus, we have the following (remembering that $D^{Q}f = F$):

$$f - C_1x^{Q-1} - C_2x^{Q-2} - ..... - C_mx^{Q-m} = D^{-Q}D^{Q}f = D^{-Q}F \tag{9.54}$$

Therefore, we write the most general solution as:

$$f = D^{-Q}F + C_1x^{Q-1} + C_2x^{Q-2} + ... + C_mx^{Q-m} \tag{9.55}$$

We note the presence in this general solution (9.55) of m arbitrary constants where $0 < Q \le m < Q+1$ or $m = 0$ for $Q \le 0$. For $Q = 1$, we have $f = D^{-1}F + C_1x^{0}$, where constant C_1 is the integration constant, we recover our classical calculus result. With $Q = 2$, we obtain the solution as $f = D^{-2}F + C_1x + C_2x^{0}$, the same as in classical calculus.

Let us take the example of a simple equation $D^{3/2}f = x^{5}$, whose general solution as per (9.55) is:

$$f = D^{-3/2}\left[x^{5}\right] + C_1x^{1/2} + C_2x^{-1/2} = \frac{\Gamma(6)}{\Gamma\left(\frac{15}{2}\right)}x^{13/2} + C_1x^{1/2} + C_2x^{-1/2} \tag{9.56}$$

containing two arbitrary constants namely C_1 and C_2. Now consider $D^{Q}f + b\left(D^{Q-1}f\right) = F(x)$, where Q is arbitrary and b is a known constant, and F is a known function. We apply the operator $D^{-(Q-1)}$ that is D^{1-Q} to this given equation, which thereafter yields the following steps:

$$\begin{aligned}
&D^{-(Q-1)}D^{Q}f + bD^{-(Q-1)}D^{(Q-1)}f = D^{-(Q-1)}F \\
&D^{1}f + b\left(f - B_1x^{(Q-1)-1} - B_2x^{(Q-1)-2} - ....B_mx^{(Q-1)-m}\right) = D^{1-Q}F \\
&D^{1}f + bf = D^{1-Q}F + C_1x^{Q-2} + C_2x^{Q-3} + ... + C_mx^{Q-m-1} \\
&\frac{\mathrm{d}\ f(x)}{\mathrm{d}x} + bf(x) = \frac{\mathrm{d}^{1-Q}F(x)}{\mathrm{d}x^{1-Q}} + C_1x^{Q-2} + C_2x^{Q-3} + ... + C_mx^{Q-m-1}
\end{aligned} \tag{9.57}$$

We got the first order ODE equation, and we can now invoke the standard methods of classical calculus.

The above two (9.56), (9.57) examples are quite general. The solution of an even slightly more general equation say $D^{q}f + D^{Q}f = F$ will encounter great difficulty with this approach, unless the difference $q - Q$ is an integer or half-integer.

In the above (9.57) example we have demonstrated that when $q - Q = n$, that is $D^{Q}f + aD^{Q-1}f = F$, $q \equiv Q$ and $Q \equiv Q-1$ which leads to an n-th order ODE. In our example (9.57), we have $n = 1$. We extend our above (9.57) logic to the following FDE:

$$D^{1/2}f + f = 0 \tag{9.58}$$

Applying in (9.58) $D^{1/2}$ to both sides we get $D^{1/2}D^{1/2}f + D^{1/2}f = 0$. Now apply the composition rule that is $D^{1/2}D^{1/2}f = D^{1}f - C_0(x-a)^{-3/2}$, which we have derived in an earlier i.e. (9.49).Here as per (9.49) we have $a = 0$ and call C_0 of (9.49) as C_1 to write $D^{1/2}D^{1/2}f = D^{1}f - C_1x^{-\frac{3}{2}}$. We now get $D^{1}f - C_1x^{-3/2} + D^{1/2}f = 0$. From this, we subtract the original one $D^{1/2}f + f = 0$, and write the ODE as:

$$D^{1}f - f = C_1x^{-3/2} \qquad \frac{\mathrm{d}\ f(x)}{\mathrm{d}x} - f(x) = C_1x^{-3/2} \tag{9.59}$$

The above equation (9.59) has a standard solution i.e. $f(x) = Ae^{x} + e^{x}\int_0^x e^{-y}C_1y^{-3/2}\mathrm{d}y$.

9.5 An alternate representation for fractional differ-integration for real analytic functions

9.5.1 A representation using the fractional differ-integral formula of the Riemann-Liouville

To some extent we have discussed this concept in an earlier chapter (Section 2.11). The purpose here is to discuss alternate representations for ${}_aD_x^q$; for real analytical functions φ that have a convergent power series expansion in the interval $a \le y \le x$. We restrict here the RL definition with $q < 0$ that is ${}_aD_x^q f = \frac{1}{\Gamma(-q)}\int_a^x (x-y)^{-q-1} f(y)\mathrm{d}y$. Therefore, we write (with $\upsilon = x - y$) the following:

$$ {}_aD_x^q \varphi = \frac{1}{\Gamma(-q)}\int_a^x \frac{\varphi(y)\mathrm{d}y}{(x-y)^{q+1}} = \frac{1}{\Gamma(-q)}\int_0^{x-a} \frac{\varphi(x-\upsilon)\mathrm{d}\upsilon}{\upsilon^{q+1}} \tag{9.60} $$

Using the Taylor expansion that is , $f(x+y) = f(x) + y\left(f^{(1)}(x)\right) + \frac{y^2}{2!}\left(f^{(2)}(x)\right) + ...$ we write the expansion for $\varphi(x-\upsilon)$ as follows:

$$ \varphi(x-\upsilon) = \varphi - \upsilon\varphi^{(1)}(x) + \frac{\upsilon^2}{2!}\varphi^{(2)}(x) - ...... = \sum_{k=0}^{\infty} \frac{(-1)^k \upsilon^k \varphi^{(k)}(x)}{k!} \tag{9.61} $$

The above (9.61) expansion has no remainder term since we have assumed φ to have a convergent power series expansion and since such an expansion is unique. We insert this in the expression of ${}_aD_x^q\varphi$ as in the RL definition and then write after doing term-by-term integration as shown in the following steps:

$$ \begin{aligned} {}_aD_x^q \varphi &= \frac{1}{\Gamma(-q)}\int_0^{x-a} \frac{\varphi(x-\upsilon)\mathrm{d}\upsilon}{\upsilon^{q+1}} = \frac{1}{\Gamma(-q)}\int_0^{x-a} \sum_{k=0}^{\infty} \frac{(-1)^k \upsilon^k \varphi^{(k)}}{k!}\frac{1}{\upsilon^{q+1}}\mathrm{d}\upsilon \\ &= \frac{1}{\Gamma(-q)}\sum_{k=0}^{\infty}\frac{(-1)^k\varphi^{(k)}}{k!}\int_0^{x-a}\upsilon^{k-q-1}\mathrm{d}\upsilon \\ {}_aD_x^q\varphi &= \sum_{k=0}^{\infty}\frac{(-1)^k (x-a)^{k-q}\varphi^{(k)}}{\left(\Gamma(-q)\right)(k-q)k!} \end{aligned} \tag{9.62} $$

Analyticity in the q argument that we have noted in earlier chapters too (Section 2.11, 2.14), may be used to establish the formula ${}_aD_x^q\varphi = \sum_{k-0}^{\infty}\frac{(-1)^k (x-a)^{k-q}\varphi^{(k)}}{(\Gamma(-q))(k-q)k!}$ for all q, though it was derived for $q<0$, that is the RL fractional integration formula.

9.5.2 Representation using the Grunwald-Letnikov formula for differ-integration

We start another formulation of this ${}_aD_x^q\varphi$ by starting with the following derived definition for any arbitrary q (Section 3.14):

$$ \frac{\mathrm{d}^q f}{\left[\mathrm{d}(x-a)\right]^q} = \lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x-a}{N}\right)\right)\right) \tag{9.63} $$

Then write the following by using $\varphi(x+jy) = \sum_{m=0}^{j} G_m(\varphi,x,y)\left({}^jC_m\right)$ and $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = {}^{j-q-1}C_j$, $\sum_{j=0}^{n}\left({}^{j-q-1}C_j\right)\left({}^jC_m\right) = \left({}^{m-q-1}C_m\right)\left({}^{n-q}C_{n-m}\right)$. These we have discussed in an earlier chapter (Section 1.14) while discussing Power numbers, (and Stirling numbers), that we again use below:

$$\begin{aligned}
{}_aD_x^q\varphi &= (x-a)^{-q}\lim_{N\uparrow\infty}\left(N^q\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\varphi\left(x+j\left(\tfrac{a-x}{N}\right)\right)\right)\\
&= (x-a)^{-q}\lim_{N\uparrow\infty}\left(N^q\sum_{j=0}^{N-1}\binom{j-q-1}{j}\sum_{m=0}^{\infty}\binom{j}{m}G_m\left(\varphi,x,\tfrac{a-x}{N}\right)\right)\\
&= (x-a)^{-q}\lim_{N\uparrow\infty}\left(N^q\sum_{m=0}^{\infty}G_m\left(\varphi,x,\tfrac{a-x}{N}\right)\sum_{j=0}^{N-1}\binom{j-q-1}{j}\binom{j}{m}\right)\\
&= (x-a)^{-q}\lim_{N\uparrow\infty}\left(N^q\sum_{m=0}^{\infty}\binom{m-q-1}{m}\binom{N-q-1}{N-m-1}G_m\left(\varphi,x,\tfrac{a-x}{N}\right)\right)
\end{aligned}\tag{9.64}$$

Now we use the expression $G_m(\varphi,x,y)=\sum_{k=0}^{\infty}\frac{m!}{k!}S_k^{[m]}y^k\varphi^{(k)}(x)$ as discussed earlier (Section 1.14, while discussing Power numbers, Stirling numbers), to write the following:

$$\begin{aligned}
{}_aD_x^q\varphi &= (x-a)^{-q}\lim_{N\uparrow\infty}\left(N^q\sum_{m=0}^{\infty}m!\binom{m-q-1}{m}\binom{N-q-1}{N-m-1}\sum_{k=m}^{\infty}\left(\frac{a-x}{N}\right)^k S_k^{[m]}\frac{\varphi^{(k)}(x)}{k!}\right)\\
&= \sum_{m=0}^{\infty}\frac{\Gamma(m-q)}{\Gamma(-q)}\sum_{k=m}^{\infty}(-1)^k\frac{(x-a)^{k-q}}{k!}S_k^{[m]}\varphi^{(k)}(x)\left(\lim_{N\uparrow\infty}\left(N^{q-k}\binom{N-q-1}{N-m-1}\right)\right)
\end{aligned}\tag{9.65}$$

The summation on k involves only one non-zero term by virtue of the following:

$$\lim_{N\uparrow\infty}\left(N^{q-1}\binom{N-q-1}{N-m-1}\right)=\begin{cases}\frac{1}{\Gamma(m-q+1)}, & k=m\\ 0, & k>m\end{cases}\tag{9.66}$$

Which follows from the following identity described previously (Section 1.11 and 1.12):

$$\lim_{j\uparrow\infty}\left(j^{c+q+1}\frac{\Gamma(j+k-q)}{\Gamma(j+k+1}\right)=\begin{cases}+\infty, & c>0\\ 1, & c=0\\ 0, & c<0\end{cases}\tag{9.67}$$

This results in the following form:

$$\begin{aligned}
{}_aD_x^q\varphi &= \sum_{m=0}^{\infty}\frac{\Gamma(m-q)}{\Gamma(-q)}\frac{(-1)^m(x-a)^{m-q}}{m!}S_m^{[m]}\frac{\varphi^{(m)}(x)}{\Gamma(m-q+1)}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(x-a)^{m-q}\varphi^{(m)}(x)}{\left(\Gamma(-q)\right)\left((m-q)m!\right)}
\end{aligned}\tag{9.68}$$

which we have obtained by use of $S_j^{[j]}=1$ for all j and the recurrence relationship i.e. $\Gamma(x+1)=x\Gamma(x)$. The above (9.68) expression is equivalent to ${}_aD_x^q\varphi=\sum_{k=0}^{\infty}\frac{(-1)^k(x-a)^{k-q}\varphi^{(k)}}{\Gamma(-q)(k-q)k!}$ (9.62). This alternate derivation is to show how is it possible to handle the definition that is:

$${}_aD_x^q\left[f(x)\right]=\lim_{N\uparrow\infty}\left(\frac{\left(\frac{x-a}{N}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\left(\tfrac{x-a}{N}\right)\right)\right)\tag{9.69}$$

In general, however, as with this example, the manipulations required us to use the above (9.62) formulation for an approach that employs the RL formula that is ${}_aD_x^qf=\frac{1}{\Gamma(-q)}\int_a^x(x-y)^{-q-1}\left(f(y)\right)\mathrm{d}y$ and which is then followed by an analytic continuation argument (which we have described in an earlier chapter, Section 2.14). However, the above (9.69) formula we have written in a more concise form i.e. following in Section 2.14:

$$\begin{aligned}
{}_aD_x^q\varphi &= \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(1+k-q)}{\left(\Gamma(-q)\right)(k-q)k!}\left({}_aD_x^{q-k}\left[1\right]\right)\varphi^{(k)}\\
{}_aD_x^q\varphi &= \sum_{k=0}^{\infty}\binom{q}{k}\left({}_aD_x^{q-k}\left[1\right]\right)\left({}_aD_x^k\varphi\right)
\end{aligned}\tag{9.70}$$

9.6 The fractional derivative for non-differentiable functions is defined

9.6.1 Defining a quotient i.e. $\Delta x / \left(\Delta t\right)^{\alpha}$ for a fractional derivative such that the fractional derivative of the constant is zero

From now on we will consider the order of the fractional derivative α, between 0 and 1, and develop the modified fractional derivative; that is local in nature as it is parallel to the classical calculus. We will consider all functions primarily to be non-differentiable until Section-9.22 called 'fractal' functions $f_{\text{fract}}(x)$ (Section 1.24). These nowhere differentiable functions are undoubtedly impossible to draw. We cannot graphically draw a 'fractal Mittag-Leffler function' which we will consider is nowhere differentiable, by defining it in a suitable way. Here we develop the concept of a local fractional derivative for α-differentiable functions in conjugation with classical calculus. In the earlier chapters, we dealt with non-local fractional derivatives. This is a still evolving concept and is controversial too. However, we give the development of the very modern methods that we discuss now onwards until Section 9.22.

A function $x(t)$ is said to have a fractional derivative $x^{(\alpha)}(t)$ of the order α with $0<\alpha<1$ whenever the limit:

$$x^{(\alpha)}(t) \propto \lim_{\Delta t \downarrow 0} \frac{\Delta x(t)}{\left(\Delta t\right)^{\alpha}} \tag{9.71}$$

exists and is finite and where, $\Delta x = x(t) - x(0)$. Shortly afterwards, this amounts to us writing the equality in terms of differentials $\mathrm{d}x = b(\mathrm{d}t)^{\alpha}$. As a direct result of this (9.71) definition, when $x(t)$ is a constant function one gets $\mathrm{d}\left(x(t)\right) = 0$. Thus, the fractional derivative of a constant function should be zero.

In a likewise manner, when $x(t)$ is differentiable we have $\mathrm{d}x = \left(x^{(1)}(t)\right)\mathrm{d}t$ and as a result the fractional derivative is zero i.e.

$$x^{(\alpha)}(t) \propto \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\left(\Delta t\right)^{\alpha}} = \lim_{\Delta t \downarrow 0} \frac{\left(x^{(1)}(t)\right)\left(\Delta t\right)}{\left(\Delta t\right)^{\alpha}} = \lim_{\Delta t \downarrow 0} \left(x^{(1)}(t)\right)\left(\Delta t\right)^{1-\alpha} = 0 \tag{9.72}$$

Thus for a differentiable function we have the fractional derivative $x^{(\alpha)}(t) = 0$ from (9.72). Clearly, the definition (9.71) deals with non-differentiable functions.

9.6.2 Corollary to the definition of a fractional derivative

The above (9.71) definition indicates that the function $x(t)$ is $\alpha-$th differentiable, with $0<\alpha<1$. Corollary for a fractional derivative of a non-differentiable function are as follows:

(1) Assume that $x(t) = g(t) + f(t)$ where $f(t)$ is differentiable and $g(t)$ is $\alpha-$th differentiable, then one has $\Delta x = b_0\left(\Delta t\right)^{\alpha} + b_1\left(\Delta t\right)$. Therefore, one can still take the limit condition of definition $x^{(\alpha)}(t) \propto \lim_{\Delta t \downarrow 0}\left(\frac{\Delta x(t)}{\Delta t^{\alpha}}\right)$. It follows that, exactly how a derivative defines a function up to an arbitrary additive constant, a function of which only the fractional derivative is known, is defined up to an additive differentiable function. This is exemplified in (9.73) and (9.74), below.

(2) The function $x(t)$ is self-similar with the Hurst exponent (Section 1.24) $\alpha > 0$ whenever there is a positive constant λ such that we have $x(\lambda t) \propto \lambda^{\alpha} x(t)$ as a result, these yields $x(t) \propto t^{\alpha} x(1)$. This we get by doing changes $1 \leftarrow t$ and $t \leftarrow \lambda$ in $x(\lambda t) \propto \lambda^{\alpha} x(t)$ (Section 1.24). This expression $x(t) \propto t^{\alpha} x(1)$ suggests, as $x(t)$ is self-similar, we have $x(t) = 0$. We write two remarks.

(i) According to $\mathrm{d}x = b(\mathrm{d}t)^{\alpha}$, that is from the definition (9.71) considering the limit exists and is equal b, for small t one has $x(t) - x(0) \cong bt^{\alpha}$. In addition, we write $x(\lambda t) - x(0) \cong \lambda^{\alpha}\left(x(t) - x(0)\right)$. In other words, if $x(t)$ is $\alpha-$differentiable, then $x(t) - x(0)$ is self-similar with the Hurst exponent as α, or again when $x(t)$ is locally self-similar at $t = 0$

(ii) The converse of this is if $x(t)$ is self-similar, then $x(0) = 0$, and according to $x(t) \propto t^{\alpha} x(1)$ one has $(\mathrm{d}x) \propto x(1)(\mathrm{d}t)^{\alpha}$ in such a manner that $x(t)$ is locally $\alpha-$th differentiable. In the definition, $x\left(\lambda t\right) \propto \lambda^{\alpha} x(t)$ one has $x(0) = 0$ but $x(0)$ may have any value.

(3) For (9.71), for the limit existing, say it equals b, then we have $dx = b(dt)^{\alpha}$. Accordingly, fractional the derivative of the constant is zero.

(4) For the differentiable case we have following:

$$\begin{aligned} x(t) &= f(t) + C \\ \Delta x &= b_1 \Delta t + 0 \\ \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\Delta t} &= b_1 = x^{(1)}(t) \end{aligned} \tag{9.73}$$

(5) For a fractionally differentiable case we write the following:

$$\begin{aligned} x(t) &= g(t) + f(t) \\ \Delta x &= b_0 \left(\Delta t\right)^{\alpha} + b_1 \left(\Delta t\right), \quad 0 < \alpha < 1 \\ \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\left(\Delta t\right)^{\alpha}} &= \lim_{\Delta t \downarrow 0} b_0 + \lim_{\Delta t \downarrow 0} b_1 \left(\Delta t\right)^{1-\alpha} = b_0 = x^{(\alpha)}(t) \end{aligned} \tag{9.74}$$

More generally, if one has equality for $\Delta x = b_0 \left(\Delta t\right)^{\alpha} + b_1 \left(\Delta t\right)^{\beta}$, $\alpha < \beta$ then the function $x(t)$ is $\alpha -$th differentiable at $t = 0$.

9.7 Utilising the Mittag-Leffler function to get suitable fractional derivatives as a conjugation to the classical derivative

The Mittag-Leffler function $E_{\alpha}(y)$, $y = t^{\alpha}$ is defined by a series $E_{\alpha}(y) = \sum_{k=0}^{\infty} \frac{y^k}{(\alpha k)!}$. Exactly like the exponential function e^t is a solution to the differential equation $D^1\left(y(t)\right) = y(t)$, $y(0) = 1$ we would like to have a 'suitable' fractional derivative $\mathbf{D}^{\alpha}$ which should have $E_{\alpha}(t^{\alpha})$ as a solution (eigenfunction) to the fractional differential equation i.e. $\mathbf{D}^{\alpha} E_{\alpha}(t^{\alpha}) = E_{\alpha}(t^{\alpha})$. This way we would be in a position to expand the fractional calculus in conjugation or parallel with classical calculus, by substituting $E_{\alpha}(\lambda t^{\alpha})$ for $e^{\lambda t}$ almost everywhere. We caution here for the function $E_{\alpha}(t^{\alpha})$, we say $t \geq 0$, as for a negative t, we may have t^{α} as a complex-quantity. The negative value for t we will see later is valid for α as $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ not for all α.

On substituting $y(t) = E_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{(\alpha k)!}$ into, $\mathbf{D}^{\alpha} y(t) = y(t)$ we get:

$$\begin{aligned} &\mathbf{D}^{\alpha} E_{\alpha}(t^{\alpha}) = E_{\alpha}(t^{\alpha}) \\ &\mathbf{D}^{\alpha} \left(1 + \tfrac{t^{\alpha}}{\alpha!} + \tfrac{t^{2\alpha}}{(2\alpha)!} + \tfrac{t^{3\alpha}}{(3\alpha)!} + \ldots\right) \\ &\quad = \mathbf{D}^{\alpha}\left(1\right) + \mathbf{D}^{\alpha}\left(\tfrac{t^{\alpha}}{\alpha!}\right) + \mathbf{D}^{\alpha}\left(\tfrac{t^{2\alpha}}{(2\alpha)!}\right) + \mathbf{D}^{\alpha}\left(\tfrac{t^{3\alpha}}{(3\alpha)!}\right) + \ldots \\ &\quad = \left(1 + \tfrac{t^{\alpha}}{\alpha!} + \tfrac{t^{2\alpha}}{(2\alpha)!} + \tfrac{t^{3\alpha}}{(3\alpha)!} + \ldots\right) \end{aligned} \tag{9.75}$$

If in (9.75) we let $\mathbf{D}^{\alpha}\left(1\right) = 0$, $\mathbf{D}^{\alpha}\left(\frac{t^{\alpha}}{\alpha!}\right) = 1$ and $\mathbf{D}^{\alpha}\left(\frac{t^{2\alpha}}{(2\alpha)!}\right) = \frac{t^{\alpha}}{\alpha!}\ldots$ then it is fine, and we are at our goal of the definition for a suitable $\mathbf{D}^{\alpha}$, that gives $\mathbf{D}^{\alpha} E_{\alpha}(t^{\alpha}) = E_{\alpha}(t^{\alpha})$. Therefore, we arrive at the following 'so called' definition $\mathbf{D}^{\alpha}$ for a suitable fractional derivative of order $\alpha > 0$ i.e.

$$\mathbf{D}^{\alpha}\left(C\right) = 0 \qquad\qquad \mathbf{D}^{\alpha} x^{\alpha k} = \frac{\left(\alpha k\right)!}{\left(\alpha k - \alpha\right)!} x^{\alpha k - \alpha} \tag{9.76}$$

Where C is a constant. Obviously this fractional derivative provides a customary derivative when $\alpha = 1$. This modification looks like conjugation to a classical derivative.

Using (9.76), we apply to $y(t) = E_{\alpha}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^{\alpha} t^{\alpha k}}{(\alpha k)!}$; $t \geq 0$ and see $\mathbf{D}^{\alpha} y(t) = \lambda E_{\alpha}(\lambda t^{\alpha})$.

This gives $\mathbf{D}^{\alpha} y(t)\Big|_{t=0} = \lambda$, $\mathbf{D}^{\alpha} y(t)\Big|_{t=a} = \lambda E_{\alpha}(\lambda a^{\alpha})$. This is in conjugation with $y^{(1)}(t) = \lambda e^{\lambda t}$ as in the classical calculus with $y(t) = e^{\lambda t}$, with values $y^{(1)}(0) = \lambda$, $y^{(1)}(a) = \lambda e^{\lambda a}$.

We define a suitable higher transcendental function, by use of the Mittag-Leffler function and find the conjugation with a classical calculus of this new way to have a fractional derivative.

$$\begin{aligned}
&E_{\alpha}\left(x\right) = \cosh_{\alpha}\left(x\right) + \sinh_{\alpha}\left(x\right)\\
&\qquad \cosh_{\alpha}\left(x\right) \overset{\text{def}}{=} \tfrac{1}{2}\left(E_{\alpha}(x) + E_{\alpha}(-x)\right) \qquad \sinh_{\alpha}\left(x\right) \overset{\text{def}}{=} \tfrac{1}{2}\left(E_{\alpha}(x) - E_{\alpha}(-x)\right)\\
&E_{\alpha}\left(ix\right) = \cos_{\alpha}\left(x\right) + i\sin_{\alpha}\left(x\right)\\
&\qquad \cos_{\alpha}\left(x\right) \overset{\text{def}}{=} \tfrac{1}{2}\left(E_{\alpha}(ix) + E_{\alpha}(-ix)\right) \qquad \sin_{\alpha}\left(x\right) \overset{\text{def}}{=} \tfrac{1}{2i}\left(E_{\alpha}(ix) - E_{\alpha}(-ix)\right) \qquad (9.77)\\
&\mathbf{D}^{\alpha} E_{\alpha}\left(\lambda x^{\alpha}\right) = \lambda E_{\alpha}\left(\lambda x^{\alpha}\right)\\
&\mathbf{D}^{\alpha} \cos_{\alpha} x^{\alpha} = -\sin_{\alpha} x^{\alpha} \qquad \mathbf{D}^{\alpha} \sin_{\alpha} x^{\alpha} = \cos_{\alpha} x^{\alpha}\\
&\mathbf{D}^{\alpha} \cosh_{\alpha} x^{\alpha} = -\sinh_{\alpha} x^{\alpha} \qquad \mathbf{D}^{\alpha} \sinh_{\alpha} x^{\alpha} = \cosh_{\alpha} x^{\alpha}
\end{aligned}$$

We find an interesting parallel to the usual exponential, hyperbolic, and trigonometric functions. Only with a modified derivative as we described in (9.76), with $\mathbf{D}^{\alpha}\left(C\right) = 0$ and $\mathbf{D}^{\alpha} x^{\alpha k} = \frac{(\alpha k)!}{(\alpha k - \alpha)!} x^{\alpha k - \alpha}$. Now we have to formally define $\mathbf{D}^{\alpha}$ in conjugation with classical calculus.

9.8 The fractional derivative via fractional difference and its Laplace Transform

9.8.1 Defining the fractional derivative $f^{(\alpha)}$ via the Forward Shift Operator

Let $f : \mathbb{R} \to \mathbb{R}$, $t \to f(t)$, a continuous (but not necessarily differentiable) function. Let $h > 0$ represent a constant discretisation element. We call a forward shift operator $\mathrm{E}_{\mathrm{F}h}$ defined as $\mathrm{E}_{\mathrm{F}h}\left[f(t)\right] = f(t+h)$. Then one can write the finite difference $\Delta f(t)$ of $f(t)$ in the form:

$$\Delta f(t) = f(t+h) - f(t) = \left(\mathrm{E}_{\mathrm{F}h} - 1\right) f(t) \qquad (9.78)$$

In quite a natural way, this (9.78) suggests that we define the fractional difference of order α, $0 < \alpha < 1$ of $f(t)$ by the following expression:

$$\Delta^{\alpha} f(t) \overset{\text{def}}{=} \left(\mathrm{E}_{\mathrm{F}h} - 1\right)^{\alpha} f(t) = \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} f\left(t + (\alpha - k)h\right) \qquad (9.79)$$

with the notation $\mathrm{E}_{\mathrm{F}h}^{\mu} f(t) = f(t + \mu h)$. This (9.79) is by using the expansion in the series $(\mathrm{x}+1)^{\alpha}$ as done in Section 3.13, Section 3.14. In Chapter-3 (Section 3.13, 3.14) we have used the backward shift operator i.e. E_h, when operated on $f(t)$, to give $\mathrm{E}_h\left[f(t)\right] = f(t-h)$. Thus, we have $\mathrm{E}_h = \mathrm{E}_{\mathrm{F}h}^{-1}$ and with this backward shift operator E_h we write the fractional difference as $\Delta_{(-)}^{\alpha} f(t) = \left(1 - \mathrm{E}_h\right)^{\alpha} f(t) = \left(1 - \mathrm{E}_{\mathrm{F}h}^{-1}\right)^{\alpha} f(t)$. As done in Section 3.13, 3.14 expanding on this we have $\Delta_{(-)}^{\alpha} f(t) = \sum_{k=0}^{\infty} (-1)^{k} \left({}^{\alpha}C_{k}\right) f(t - kh)$. We also observe from these discussions $\Delta_{(-)}^{\alpha} f(t) = \mathrm{E}_{\mathrm{F}h}^{-\alpha}\left[\Delta^{\alpha} f(t)\right]$, that it is a relationship between the backward fractional differential ($\Delta_{(-)}^{\alpha} f$) and the forward fractional differential ($\Delta^{\alpha} f$). We will consider (9.79) in our further discussions.

With these above notations and definitions in a quite natural way, we define the fractional derivative of order α as the limit:

$$f^{(\alpha)}(t) = \lim_{h \downarrow 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}} \qquad (9.80)$$

According to the definition (that we described in the previous section of this chapter for a non-differentiable function) i.e. $f^{(\alpha)}(t) \propto \lim_{\Delta t \downarrow 0} \frac{\Delta f(t)}{\Delta t^{\alpha}}$ (9.71) and via the previous argument (9.75) of the eigenfunction i.e. $y(t) = E_{\alpha}(t^{\alpha})$ be the solution of $\mathbf{D}^{\alpha} y(t) = y(t)$ we have seen the fractional derivative of a constant should be zero. However, we have to examine that if $f^{(\alpha)}(t) = \lim_{h\downarrow 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}}$, does definition (9.80) give a zero for the constant function? While we saw, the definition (9.71) i.e. $f^{(\alpha)}(t) \propto \lim_{\Delta t\downarrow 0} \frac{\Delta f(t)}{\Delta t^{\alpha}}$ gives zero for a constant function.

9.8.2 Getting the fractional derivative of the constant as a non-zero by the Laplace transform technique

According to the (9.80) definition of $f^{(\alpha)}$ i.e. $f^{(\alpha)}(t) = \lim_{h\downarrow 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}}$ the Laplace transform is $\mathcal{L}\left\{f^{(\alpha)}(t)\right\} = s^{\alpha} F(s)$. Accordingly, the constant function $f(t) = C$ gives following steps:

$$\mathcal{L}\left\{f^{(\alpha)}(t)\right\} = s^{\alpha} F(s), \quad f(t) = C$$

$$\mathcal{L}\left\{f^{(\alpha)}(t)\right\} = s^{\alpha} \mathcal{L}\left\{C\right\} = s^{\alpha}\left(\frac{C}{s}\right) = s^{\alpha-1} C \tag{9.81}$$

$$\mathcal{L}^{-1}\left\{s^{\alpha-1} C\right\} = \frac{t^{-\alpha}}{(-\alpha)!} C \qquad f^{(\alpha)}(t) = \frac{t^{-\alpha}}{(-\alpha)!} C \neq 0$$

That is the fractional derivative of the constant function $f(t) = C$, which is not zero. However, what if we wrote it as $\mathcal{L}\left\{\lim_{h\downarrow 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}}\right\} = s^{\alpha} F(s)$? The proof is in following section.

9.8.3 Getting the Laplace Transform $\mathcal{L}\left\{\Delta^{\alpha} f(t)\right\}$

We use the definition of the Laplace transform i.e. $X(s) = \mathcal{L}\left\{x(t)\right\} = \int_0^{\infty} e^{-st}\left(x(t)\right) \mathrm{d}t$ (Appendix-G) and write the following steps:

$$\begin{aligned}
\mathcal{L}\left\{f(t+h)\right\} &= \int_0^{\infty} e^{-st}\left(f(t+h)\right)\mathrm{d}t \qquad t+h=\tau; \quad \mathrm{d}t = \mathrm{d}\tau \\
&= \int_h^{\infty} e^{-s(\tau-h)}\left(f(\tau)\right)\mathrm{d}\tau \\
&= e^{sh}\int_h^{\infty} e^{-s\tau}\left(f(\tau)\right)\mathrm{d}\tau = e^{sh}\int_0^{\infty} e^{-s\tau}\left(f(\tau)\right)\mathrm{d}\tau \\
&\qquad - e^{sh}\int_0^{h} e^{-s\tau}\left(f(\tau)\right)\mathrm{d}\tau \\
&= e^{sh}F(s) - e^{sh}\int_0^{h} e^{-s\tau}\left(f(\tau)\right)\mathrm{d}\tau
\end{aligned} \tag{9.82}$$

Examine the integral of the second term in the above (9.82) obtained expression i.e. $\int_0^{h} e^{-s\tau}\left(f(\tau)\right)\mathrm{d}\tau$. For small h and near $\tau = 0$ we can approximately write this expression as $\int_0^{h} e^{-s\tau} f(\tau)\mathrm{d}\tau \cong \left(e^{-s(0)} f(0)\right)h = h\left(f(0)\right)$, by considering $\mathrm{d}\tau \cong h$. This approximation is the area under the curve $e^{-s\tau} f(\tau)$ at $\tau = 0$.

With this derivation, we have $\mathcal{L}\left\{f(t+h)\right\} \cong e^{sh}F(s) - he^{sh} f(0)$. With the above (9.82) derived Laplace transform identity we apply to our fractional difference expression (9.79) i.e. $\Delta^{\alpha} f(t) = \sum_{k=0}^{\infty} (-1)^k \left({}^{\alpha}C_k\right) f\left(t + (\alpha - k)h\right)$, and try to write the expression for $\mathcal{L}\left\{\Delta^{\alpha} f(t)\right\}$, i.e. as described in the following steps:

$$\begin{aligned}
\mathcal{L}\left\{\Delta^{\alpha} f(t)\right\} &= \mathcal{L}\left\{\sum_{k=0}^{\infty}(-1)^{k}\left({}^{\alpha}C_{k}\right) f\left(t+(\alpha-k)h\right)\right\}; \qquad {}^{\alpha}C_{k}=\binom{\alpha}{k} \\
&= \mathcal{L}\left\{f\left(t+\alpha kh\right)-\alpha f\left(t+(\alpha-1)h\right)+\frac{\alpha(\alpha-1)}{2} f\left(t+(\alpha-2)h\right)+\ldots.\right\} \\
&= \mathcal{L}\left\{f\left(t+\alpha kh\right)\right\}-\mathcal{L}\left\{\alpha f\left(t+(\alpha-1)h\right)\right\} \\
&\quad +\mathcal{L}\left\{\frac{\alpha(\alpha-1)}{2} f\left(t+(\alpha-2)h\right)\right\}+\ldots. \\
&= \left(e^{\alpha hs}F(s)-\alpha h e^{\alpha hs} f(0)\right)-\left(\alpha e^{(\alpha-1)hs}F(s)-\alpha(\alpha-1)h e^{(\alpha-1)hs} f(0)\right) \\
&\quad +\left(\tfrac{\alpha(\alpha-1)}{2} e^{(\alpha-2)hs}F(s)-\tfrac{\alpha(\alpha-1)(\alpha-2)h}{2} e^{(\alpha-2)hs} f(0)\right)+\ldots. \\
&= \left(e^{\alpha hs}F(s)-\alpha e^{(\alpha-1)hs}F(s)+\tfrac{\alpha(\alpha-1)}{2} e^{(\alpha-2)hs}F(s)+\ldots\right) \\
&\quad -\left(\alpha h e^{\alpha hs} f(0)-\alpha(\alpha-1)h e^{(\alpha-1)hs} f(0)+\tfrac{\alpha(\alpha-1)(\alpha-2)h}{2} e^{(\alpha-2)hs} f(0)+\ldots\right)
\end{aligned} \tag{9.83}$$

Take the first term in bracket of the last line of (9.83) and we do the following manipulation and approximation for small h as depicted (9.84):

$$\begin{aligned}
&\left(e^{\alpha hs}F(s)-\alpha e^{(\alpha-1)hs}F(s)+\tfrac{\alpha(\alpha-1)}{2} e^{(\alpha-2)hs}F(s)+\ldots\right) \\
&= e^{\alpha hs}F(s)\left(1-\alpha e^{-hs}+\tfrac{\alpha(\alpha-1)}{2} e^{-2hs}-\ldots\right) \\
&= e^{\alpha hs}F(s)\left(1-e^{-hs}\right)^{\alpha}=\left(e^{hs}\left(1-e^{-hs}\right)\right)^{\alpha} F(s) \\
&= \left(e^{hs}-1\right)^{\alpha} F(s); \qquad e^{hs}\cong 1+hs \\
&= \left(1+hs-1\right)^{\alpha} F(s)=h^{\alpha}s^{\alpha}F(s)
\end{aligned} \tag{9.84}$$

Therefore, we have the following relationship:

$$\mathcal{L}\left\{\Delta^{\alpha} f(t)\right\}=h^{\alpha}s^{\alpha}F(s)-\begin{pmatrix}\alpha h e^{\alpha hs} f(0)-\alpha(\alpha-1)h e^{(\alpha-1)hs} f(0) \\ +\tfrac{\alpha(\alpha-1)(\alpha-2)h}{2} e^{(\alpha-2)hs} f(0)+\ldots\end{pmatrix} \tag{9.85}$$

9.8.4 Getting the Laplace Transform $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}$

Divide (9.85) by h^{α} to get the following expression:

$$\mathcal{L}\left\{\frac{\Delta^{\alpha} f(t)}{h^{\alpha}}\right\}=s^{\alpha}F(s)-\alpha h^{1-\alpha} e^{\alpha hs} f(0)\left(1-(\alpha-1)e^{-hs}+\tfrac{(\alpha-1)(\alpha-2)}{2} e^{-2hs}+\ldots\right) \tag{9.86}$$

Take the limit $h\downarrow 0$ and we get the following:

$$\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=\mathcal{L}\left\{\lim_{h\downarrow 0}\frac{\Delta^{\alpha} f(t)}{h^{\alpha}}\right\}=s^{\alpha}F(s) \tag{9.87}$$

With this fractional difference, we saw that $f^{(\alpha)}(t)\neq 0$, for $f(t)=C$ as we get $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=s^{\alpha-1}C\neq 0$. In other words, with this definition (9.80), (9.87) the fractional derivative of a constant would not be zero as that contradicts our assumption.

We remark that the classical Laplace transform formula for the first derivative is $\mathcal{L}\left\{f^{(1)}(t)\right\}=sF(s)-f(0)$ and that it is evident it is not equivalent to $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=\mathcal{L}\left\{\lim_{h\downarrow 0}\left(\frac{\Delta^{\alpha} f(t)}{h^{\alpha}}\right)\right\}=s^{\alpha}F(s)$ when $\alpha=1$.

This gives rise to a query as to whether there is some mistake. First, in two expressions i.e. $\mathcal{L}\left\{f^{(1)}(t)\right\}$ and $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}$ the consistency is complete when $f(0)=0$, for $\alpha=1$. This $f(0)=0$ is a condition satisfied by self-similar functions (Section 1.24).

9.8.5 The definition of fractional differentiability is satisfactory for self-similar functions

Therefore, the definition (9.80) $f^{(\alpha)}(t)=\lim_{h\downarrow 0}\frac{\Delta^{\alpha}f(t)}{h^{\alpha}}$ is quite satisfactory for self-similar functions. According to our earlier discussion (Section 1.24), i.e. when we restrict ourselves to the class of functions that satisfies the condition i.e. $f(0)=0$ then, we can claim that:

(1) If the function is a fractional differentiable, then it is self-similar.
(2) Conversely, if it is self-similar then it is a fractional differentiable.

To cope with the problem due to a constant term $f(0)=C$ if we assert that $\mathbf{D}^{\alpha}C=0$, then we will have the required modification for the RL fractional derivative (that we represented D^{α}).This modification is called the Jumarie fractional derivative (that we represent by $\mathbf{D}^{\alpha}$).

9.9 The modified fractional derivative for a function with non-zero initial condition-Jumarie type

9.9.1 Defining the fractional derivative via the offset function construction to have a fractional derivative of the constant as a zero-Jumarie type

Let $f:\mathbb{R}\to\mathbb{R}$, $t\to f(t)$ denote a continuous (but not necessarily differentiable) function, and define $\tilde{f}(t)=f(t)-f(0)$ i.e. the offset function. Let $h>0$ denote a constant discretisation element (span). The fractional derivative of order α of $f(t)$ is defined as the limit:

$$f^{(\alpha)}(t)\overset{\text{def}}{=}\lim_{h\downarrow 0}\frac{\Delta^{\alpha}\tilde{f}(t)}{h^{\alpha}} \tag{9.88}$$

Alternatively assume $g(t)$ is self-similar with the Hurst index H with $0<H<1$ (Section 1.24). Then the fractional derivative of order H is defined by the expression:

$$g^{(H)}(t)=\lim_{h\downarrow 0}\frac{\Delta^{H}g(t)}{h^{H}} \tag{9.89}$$

According to this (9.88) definition, the fractional derivative of the constant is zero. For $f(t)=C$ we have the offset function as zero; $\tilde{f}(t)=0$. This (9.88) is similar to the Caputo fractional derivative but not the same. They are the same when $f(t)$ is differentiable.

We call it a modified R-L fractional derivative-Jumarie type. Note that the function $f(t)$ need not be differentiable, in this modified R-L fractional derivative definition (9.88). The definition (9.88) i.e. the definition of the fractional derivative via the forward fractional difference, is the right fractional derivative. This definition is in conjugation with the classical derivative $f^{(1)}(t)$. We say that $f^{(\alpha)}$ or $\mathbf{D}^{\alpha}f$ was defined as in (9.88) and is the 'local fractional derivative', unlike the classical Riemann-Liouville fractional derivative $D^{\alpha}f(t)$, which we have dealt with in all previous chapters which are 'non-local'.

By using the backward fractional difference, $\Delta^{\alpha}_{(-)}f(t)$ we have left the fractional derivative as $f^{(\alpha)}_{(-)}(t)=\lim_{h\downarrow 0}\left(\frac{\Delta^{\alpha}_{(-)}\tilde{f}(t)}{h^{\alpha}}\right)$, with $\tilde{f}(t)=f(t)-f(0)$. We place $\alpha=1$ in (9.88) to get $f^{(1)}(t)=\lim_{h\downarrow 0}\left(\frac{1}{h}\Delta\ \tilde{f}(t)\right)$. Where $\tilde{f}(t)=f(t)-f(0)$, and $\Delta\left(f(t)-f(0)\right)=f(t)$.

Therefore $f^{(1)}(t)=\lim_{h\downarrow 0}\left(\frac{1}{h}\Delta\tilde{f}(t)\right)=\lim_{h\downarrow 0}\left(\frac{1}{h}\Delta f(t)\right)$ is the classical result. We may infer that in the definition of a classical derivative $f^{(1)}$, which is a local quantity, $\tilde{f}(t)$ is inbuilt in the definition $f^{(1)}(t)=\lim_{h\downarrow 0}\left(\frac{\Delta\tilde{f}(t)}{h}\right)$, which gives $f^{(1)}(t)=\lim_{h\downarrow 0}\left(\frac{\Delta f(t)}{h}\right)$. That is if we subtract the initial value of the function $f(0)$ from $f(t)$ and then do one-whole differentiation, it is the same if we do not subtract the initial value. This brings us to the point that (9.88) is a definition of the local fractional derivative in conjugation with a classical derivative.

9.9.2 The Laplace Transform of the Modified RL fractional Derivative and its conjugation to a classical derivative

We find the Laplace transform of the modified R-L derivative and its conjugation with ordinary classical derivatives as demonstrated below:

$$f^{(\alpha)}(t)\overset{\text{def}}{=}\lim_{h\downarrow 0}\frac{\Delta^{\alpha}\tilde{f}(t)}{h^{\alpha}}$$

$$\begin{aligned}
\mathcal{L}\left\{f^{(\alpha)}(t)\right\}&=\mathcal{L}\left\{\lim_{h\downarrow 0}\frac{\Delta^{\alpha}\left(f(t)-f(0)\right)}{h^{\alpha}}\right\}\\
&=\mathcal{L}\left\{\lim_{h\downarrow 0}\frac{\Delta^{\alpha}f(t)}{h^{\alpha}}-\lim_{h\downarrow 0}\frac{\Delta^{\alpha}f(0)}{h^{\alpha}}\right\}\\
&=s^{\alpha}\mathcal{L}\left\{f(t)\right\}-s^{\alpha}\mathcal{L}\left\{f(0)\right\}=s^{\alpha}F(s)-s^{\alpha}\mathcal{L}\left\{f(0)\right\}\\
&=s^{\alpha}F(s)-s^{\alpha}\left(\frac{f(0)}{s}\right)\\
&=s^{\alpha}F(s)-s^{\alpha-1}f(0)
\end{aligned}\tag{9.90}$$

In (9.90) we used (9.87) i.e. $\mathcal{L}\left\{\lim_{h\downarrow 0}\frac{\Delta^{\alpha}f(t)}{h^{\alpha}}\right\}=s^{\alpha}\mathcal{L}\left\{f(t)\right\}$. Now for the case in (9.90) with $\alpha=1$ we get $\mathcal{L}\left\{f^{(1)}(t)\right\}=s\left(F(s)\right)-f(0)$ here the classical result is recovered. We have found a similarity of (9.90) with the classical derivative in the Laplace transform identity.

Now we define the integral form of this modified RL fractional derivative of Jumarie type (Section 9.10) However, before that we make the result of (9.90) for $f(t)=E_{\alpha}(\lambda t^{\alpha});\quad t\geq 0$, to obtain $f^{(\alpha)}(t)$.

With this formula (9.90) which is from the formula (9.88) we demonstrate that the local value of the fractional derivative can be obtained for $\alpha-$differentiable function $f(t)$ at $t=0$. For the demonstration of obtaining a local fractional derivative at the point by this modified RL method (Jumarie Type), we take the function $f(t)=E_{\alpha}(\lambda t^{\alpha})$. Let us apply the formula (9.90), to $f(t)=E_{\alpha}(\lambda t^{\alpha});\quad t\geq 0$ i.e. the Mittag-Leffler function with its Laplace transform as $F(s)=\mathcal{L}\left\{E_{\alpha}(\lambda t^{\alpha})\right\}=\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}$ having a non-zero initial condition i.e. $f(0)=1$ (Appendix-A). Putting these values in (9.90), we obtain $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=s^{\alpha}F(s)-s^{\alpha-1}f(0)=\lambda\left(\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\right)$. Now writing the inverse Laplace transform, we get $f^{(\alpha)}(t)=\lambda E_{\alpha}(\lambda t^{\alpha})$, and we have $f^{(\alpha)}(0)=\lambda$. This is in conjugation with classical calculus where we have $f(t)=e^{\lambda t}$ giving $f^{(1)}(t)=\lambda e^{\lambda t}$ and $f^{(1)}(0)=\lambda$.

We also have $\mathbf{D}^{\alpha}E_{\alpha}(\lambda t^{\alpha})=\lambda E_{\alpha}(\lambda t^{\alpha})$, that is in conjugation with $f^{(1)}(t)=\lambda f(t)$ when $f(t)=e^{\lambda t}$. Therefore, if the function is $f(t)=E_{\alpha}(\lambda t^{\alpha})$, we have the expression of conjugation to classical calculus i.e. $f^{(\alpha)}(t)=\left(f^{(\alpha)}(0)\right)f(t)$ similar to $f^{(1)}(t)=\left(f^{(1)}(0)\right)f(t)$, for $f(t)=e^{\lambda t}$, at $t=0$.

Therefore our fractional differential equation (with modified RL fractional derivative-Jumarie Type) is $f^{(\alpha)}(t)=\left(f^{(\alpha)}(0)\right)f(t)$ which has the solution $f(t)=E_{\alpha}(\lambda t^{\alpha})$, in conjugation with classical calculus $f^{(1)}(t)=\left(f^{(1)}(0)\right)f(t)$ where $f(t)=e^{\lambda t}$. As we can write $f(t)=e^{\left(f^{(1)}(0)\right)t}$ for $f^{(1)}(t)=\left(f^{(1)}(0)\right)f(t)$, for $t\geq 0$

in the same way we are tempted to write $f(t)=E_\alpha\left(\left(f^{(\alpha)}(0)\right)t^\alpha\right)$ as a solution of $f^{(\alpha)}(t)=\left(f^{(\alpha)}(0)\right)f(t)$ for $t\geq 0$.

9.10 Integral representation of modified RL fractional derivative of Jumarie type

Assume that $f(t)$ is a constant function i.e. $f(t)=C$, then fractional derivative of order α is

$$\begin{aligned}\mathbf{D}_t^\alpha C &= \frac{C}{\Gamma(1-\alpha)}t^{-\alpha} \qquad \alpha\leq 0\\ &= 0 \qquad \alpha>0\end{aligned} \tag{9.91}$$

Note that first expression of (9.91) is fractional integration $\alpha\leq 0$ of constant and second one $\alpha>0$ is fractional derivative.

When $f(t)$ is not a constant then we can write $f(t)=f(0)+\left(f(t)-f(0)\right)$ and fractional derivative is defined by the expression $f^{(\alpha)}(t)=\mathbf{D}_t^\alpha f(0)+\mathbf{D}_t^\alpha\left(f(t)-f(0)\right)$. For $\alpha>0$ we have $\mathbf{D}_t^\alpha f(0)=0$ (9.91). In (9.91) the negative α, is just simply the RL fractional integration i.e. $f^{(\alpha)}(t)=\mathbf{D}_t^\alpha\left(f(t)-f(0)\right)=\frac{1}{\Gamma(-\alpha)}\int_0^t(t-\tau)^{-\alpha-1}f(\tau)\mathrm{d}\tau;\quad \alpha\leq 0$.

From the derived expression (9.90) i.e. $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=s^\alpha F(s)-s^{\alpha-1}f(0);\quad \alpha>0$; we modify the RHS to write it as $\mathcal{L}\left\{f^{(\alpha)}(t)\right\}=ss^{\alpha-1}F(s)-ss^{\alpha-1}s^{-1}f(0)=s\left(s^{\alpha-1}\left(F(s)-\mathcal{L}\left\{f(0)\right\}\right)\right)$, where $\mathcal{L}\left\{f(0)\right\}=\frac{f(0)}{s}$. We recognise the Laplace operator $s^{\alpha-1}$ as a fractional integration of the order $1-\alpha$ i.e. ${}_0I_t^{1-\alpha}$ (Section 5.14) and the Laplace operator s as a one-order derivative, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}$.

Thus by using the definition of fractional integration of the order $1-\alpha$, for positive α, i.e. $\alpha>0$ we have thus modified the RL fractional derivative as
$f^{(\alpha)}(t)=\mathbf{D}_t^\alpha\left(f(t)-f(0)\right)=\mathcal{L}^{-1}\left\{s\left(s^{\alpha-1}\left(F(s)-\mathcal{L}\left\{f(0)\right\}\right)\right)\right\}$.

This we write as $f^{(\alpha)}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0I_t^{1-\alpha}\left(f(t)-f(0)\right)\right)=\left(\tilde{f}^{(\alpha-1)}(t)\right)^{(1)}$ where $\tilde{f}(t)=f(t)-f(0)$. Thus from these arguments, we write the following:

$$\begin{aligned}f^{(\alpha)}(t)&=\left(\tilde{f}^{(\alpha-1)}(t)\right)^{(1)},\qquad 0<\alpha<1\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{-\alpha}\left(f(\tau)-f(0)\right)\mathrm{d}\tau\end{aligned} \tag{9.92}$$

Thus, the above (9.92) is an integral representation of $f^{(\alpha)}(t)$ or $\mathbf{D}^\alpha f(t)$ of the modified RL derivative of the Jumarie type for $0<\alpha<1$.

However we have got $f^{(\alpha)}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0I_t^{1-\alpha}\left(f(t)-f(0)\right)\right)$ where if we place $\alpha=1$ we obtain $f^{(1)}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0I_t^{0}\left(f(t)-f(0)\right)\right)=\tilde{f}^{(1)}(t)$, with $\tilde{f}(t)=f(t)-f(0)$. Thus, we again infer that the offsetting of a function is built in for a classical derivative too, (this we discussed in Section 9.9.1). Therefore, the integral formula (9.92) is an analytical way to find the local fractional derivative for order $0<\alpha<1$, and then obtain $f^{(\alpha)}(t)$ which can be used to obtain values at the points of interest.

When $n-1\leq\alpha<n$, and $\alpha>1$, with n as positive integer, we generalise the above (9.92) as follows:

$$\begin{aligned}f^{(\alpha)}(t)&=\left(\tilde{f}^{(\alpha-n)}(t)\right)^{(n)},\qquad n-1\leq\alpha<n;\qquad n>1\\ &=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t(t-\tau)^{(n-\alpha)-1}\left(f(\tau)-f(0)\right)\mathrm{d}\tau\end{aligned} \tag{9.93}$$

The logic is to have an offset function i.e. $\tilde{f}(t) = f(t) - f(0)$ and then to take the usual RL fractional derivative of $\tilde{f}(t)$. We clarify the notation while $\mathbf{D}^\alpha$ stands for a modified RL fractional derivative of the Jumarie type i.e. we also represent this with $f^{(\alpha)}$, while D^α stands for the usual RL fractional derivative. We note that when $f(0) = 0$, then $D^\alpha f$ and $\mathbf{D}^\alpha f$ are the same. While for a constant function, $f(t) = C$ we have $D^\alpha C \neq 0$ whereas $\mathbf{D}^\alpha C = 0$; $\alpha > 0$ .In addition, we clarify that the fractional integration operation in a modified sense and conventional RL sense is the same, i.e. $D^{-\alpha} f(t) = \mathbf{D}^{-\alpha} f(t)$, for $\alpha > 0$ where $f(t)$ is unchanged.

Applying this logic to $f(t) = e^{\lambda t}$, $f(0) = 1$ in order to get $f^{(\alpha)}(t) = {}_0D_t^\alpha \left[f(t) - f(0) \right]$; $\alpha > 0$ where ${}_0D_t^\alpha$ is the usual RL fractional derivative operator. We have ${}_0D_t^\alpha e^{\lambda t} = t^{-\alpha} e^{\lambda t} \gamma^*(-\alpha, \lambda t)$, where $\gamma^*(-\alpha, \lambda t)$ is called Tricomi's Incomplete Gamma function (Section-4.6), and also we have ${}_0D_t^\alpha \left[1\right] = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$, using these results we write $f^{(\alpha)}(t) = \mathbf{D}^\alpha e^{\lambda t} = t^{-\alpha} \left(e^{\lambda t} \gamma^*(-\alpha, \lambda t) - \frac{1}{\Gamma(1-\alpha)} \right)$, that is a modified fractional derivative of the Jumarie type.

While taking the modified fractional derivative of order $\alpha > 0$ we have to take the conventional RL fractional derivative on a changed function called the offset function i.e. $\tilde{f}(t) = f(t) - f(0)$. The start point $t = 0$ could be any other point say $t = a$ then the lower limit of fractional integration will be non-zero. Then the offset function will read as $\tilde{f}(t) = f(t) - f(a)$. Now if at $t = 0$ or at start point, the function is undefined say $f(0) = \pm\infty$, then we consider the offset function as $\tilde{f}(t) = f(t) - f(0^+)$ or we consider the finite part of the offset function. We also note that ${}^C D_t^\alpha f(t) = D_t^\alpha \left(f(t) - f(0) \right)$ i.e. the Caputo derivative for $0 < \alpha < 1$, but here we are describing it as a modified RL fractional derivative, i.e. $\mathbf{D}^\alpha$.

We stress here that the modified RL derivative of the Jumarie type does not demand that the function be a differentiable, whereas the Caputo definition demands the condition of differentiability. These two fractional derivatives i.e. the modified RL (Jumarie) and the Caputo are the same if and only if the function being considered is differentiable. In the earlier chapter (Section 4.15, 4.16) we had obtained the RL fractional derivatives for the offset function for some examples.

In the above (9.92) modified definition we have used the left/causal fractional derivative. Similarly a right modified fractional derivative of a function f defined on the interval say $t \in \left[0, b\right]$ is:

$$f(t) = f(b) - \left(f(b) - f(t) \right); \quad \tilde{f}(t) = f(b) - f(t)$$

$$\mathbf{D}^\alpha \left[f(t) \right] = f^{(\alpha)}(t) = {}_tD_b^\alpha \left(f(b) - f(t) \right) = \left(\tilde{f}(t)^{(\alpha-1)} \right)^{(1)}, \qquad 0 < \alpha < 1$$

$$= \frac{(-1)}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (\tau - t)^{-\alpha} \left(f(b) - f(\tau) \right) \mathrm{d}\tau \tag{9.94}$$

$$f^{(\alpha)}(t) = \left(f^{(\alpha-n)}(t) \right)^{(n)}, \qquad n - 1 \le \alpha < n; \qquad n > 1$$

$$= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^b (\tau - t)^{(n-\alpha)-1} \left(f(b) - f(\tau) \right) \mathrm{d}\tau$$

9.11 The application of the modified fractional RL derivative of the Jumarie type to various functions

9.11.1 Modified RL derivative of order α for function $f(x) = E_\alpha(\lambda x^\alpha)$

We apply this concept of the modified RL fractional derivative to $f(x) = E_\alpha(\lambda x^\alpha)$, $x \ge 0$. Take the series form of the Mittag-Leffler function:

$$f(x) = E_\alpha(\lambda x^\alpha) = 1 + \frac{(\lambda x^\alpha)}{\alpha!} + \frac{(\lambda x^\alpha)^2}{(2\alpha)!} + \frac{(\lambda x^\alpha)^3}{(3\alpha)!} + \ldots \tag{9.95}$$

For a modified RL derivative, $f^{(\alpha)}(x)$, we take an offset function $\tilde{f}(x)$ and we get:

$$\begin{aligned}\tilde{f}(x) &= f(x) - f(0)\\ &= E_\alpha(\lambda x^\alpha) - 1 = \frac{(\lambda x^\alpha)}{(\alpha!)} + \frac{(\lambda x^\alpha)^2}{(2\alpha)!} + \frac{(\lambda x^\alpha)^3}{(3\alpha)!} + \ldots\end{aligned} \tag{9.96}$$

Then take RL fractional derivative D^α of the offset function $\tilde{f}(x)$ as depicted in the following steps:

$$\mathbf{D}^\alpha f(x) = f^{(\alpha)}(x) = D^\alpha\left(f(x) - f(0)\right) = D^\alpha\left(E_\alpha(\lambda x^\alpha) - 1\right)$$

$$\begin{aligned}\mathbf{D}^\alpha\left(E_\alpha(\lambda x^\alpha)\right) &= \left(E_\alpha(\lambda x^\alpha)\right)^{(\alpha)}\\ &= \frac{\lambda}{(\alpha!)} D^\alpha\left(x^\alpha\right) + \frac{\lambda^2}{(2\alpha)!} D^\alpha\left(x^{2\alpha}\right) + \frac{\lambda^3}{(3\alpha)!} D^\alpha\left(x^{3\alpha}\right) + \ldots\\ &= \frac{\lambda}{(\alpha!)}\left(\frac{(\alpha!)}{0!}\right) + \frac{\lambda^2}{(2\alpha)!}\left(\frac{(2\alpha)!}{(\alpha!)}\right)x^\alpha + \frac{\lambda^3}{(3\alpha)!}\left(\frac{(3\alpha)!}{(2\alpha)!}\right)x^{2\alpha} + \ldots\\ &= \lambda\left(1 + \frac{\lambda x^\alpha}{(\alpha!)} + \frac{\lambda^2 x^{2\alpha}}{(2\alpha)!} + \ldots\right) = \lambda E_\alpha(\lambda x^\alpha)\end{aligned} \tag{9.97}$$

Therefore, we verified that with the modified RL definition $y = E_\alpha(\lambda x^\alpha)$ it is a solution of $\mathbf{D}^\alpha y = \lambda y$. For a fractional integration, we are not taking an offset, as we demonstrate in the following steps:

$$\begin{aligned}\mathbf{D}^{-\alpha}\left(f(x)\right) &= D^{-\alpha}\left(E_\alpha(\lambda x^\alpha)\right); \quad \alpha \ge 0\\ &= D^{-\alpha}\left(1\right) + \frac{\lambda}{(\alpha!)} D^{-\alpha}\left(x^\alpha\right) + \frac{\lambda^2}{(2\alpha)!} D^{-\alpha}\left(x^{2\alpha}\right) + \frac{\lambda^3}{(3\alpha)!} D^{-\alpha}\left(x^{3\alpha}\right) + \ldots\\ &= \frac{1}{(\alpha)!}x^\alpha + \frac{\lambda}{(\alpha!)}\left(\frac{(\alpha!)}{(2\alpha)!}\right)x^{2\alpha} + \frac{\lambda^2}{(2\alpha)!}\left(\frac{(2\alpha)!}{(3\alpha)!}\right)x^{3\alpha} + \frac{\lambda^3}{(3\alpha)!}\left(\frac{(3\alpha)!}{(4\alpha)!}\right)x^{4\alpha} + \ldots\\ &= \left(\frac{1}{\lambda}\right)\left(\frac{\lambda}{(\alpha)!}x^\alpha + \frac{\lambda^2}{(2\alpha)!}x^{2\alpha} + \frac{\lambda^3}{(3\alpha)!}x^{3\alpha} + \frac{\lambda^4}{(4\alpha)!}x^{4\alpha} + \ldots\right)\\ &= \left(\frac{1}{\lambda}\right)\left(1 + \left(\frac{\lambda}{(\alpha)!}x^\alpha + \frac{\lambda^2}{(2\alpha)!}x^{2\alpha} + \frac{\lambda^3}{(3\alpha)!}x^{3\alpha} + \frac{\lambda^4}{(4\alpha)!}x^{4\alpha} + \ldots\right) - 1\right)\\ &= \lambda^{-1}\left(E_\alpha(\lambda x^\alpha) - 1\right)\end{aligned} \tag{9.98}$$

This shows that $y = E_\alpha(\lambda x^\alpha)$ is exactly similar to the exponential function $y = e^{\lambda x}$ and this modified RL fractional derivative is with conjugation to classical calculus.

9.11.2 A modified RL derivative of order $\beta > 0$ for function $f(x) = E_\alpha(\lambda x^\alpha)$

Now take the function $y(x) = E_\alpha(x^\alpha)$, and calculate the modified fractional derivative of order $\beta > 0$ that is $y^{(\beta)}(x) = \mathbf{D}^\beta y(x)$ as follows:

$$\begin{aligned}y^{(\beta)}(x) &= \left(E_\alpha(x^\alpha)\right)^{(\beta)} = \mathbf{D}^\beta y(x) = \mathbf{D}^\beta\left(E_\alpha(x^\alpha)\right)\\ &= \mathbf{D}^\beta\left(1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + ..\right)\\ &= 0 + \frac{x^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + \frac{x^{2\alpha-\beta}}{\Gamma(1+2\alpha-\beta)} + \frac{x^{3\alpha-\beta}}{\Gamma(1+3\alpha-\beta)} + \ldots\\ &= x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(x^\alpha)\end{aligned} \tag{9.99}$$

Where, we have a two-parameter Mittag-Leffler function defined as $E_{p,q}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(q+kp)}$ (Appendix-A).

9.11.3 The RL fractional derivative of order α for function $f(x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$

If the fractional derivative is a classical RL derivative then we have ${}^{RL}D^{\alpha}C \neq 0$, then the solution for ${}^{RL}_{\ 0}D^{\alpha}_{x}y = \lambda y$ is as follows by letting $e^{\lambda x}_{\alpha} = (x^{\alpha-1})E_{\alpha,\alpha}(\lambda x^{\alpha})$ we write:

$$e^{\lambda x}_{\alpha} = (x^{\alpha-1})E_{\alpha,\alpha}(\lambda x^{\alpha})$$

$$= (x^{\alpha-1})\sum_{k=0}^{\infty}\lambda^{k}\frac{x^{k\alpha}}{\Gamma\left((k+1)\alpha\right)} = \sum_{k=0}^{\infty}\lambda^{k}\frac{x^{(k+1)\alpha-1}}{\Gamma\left((k+1)\alpha\right)}$$

$${}^{RL}_{\ 0}D^{\alpha}_{x}\left[e^{\lambda(x)}_{\alpha}\right] = {}^{RL}_{\ 0}D^{\alpha}_{x}\left(\sum_{k=0}^{\infty}\lambda^{k}\frac{(x)^{(k+1)\alpha-1}}{\Gamma\left((k+1)\alpha\right)}\right) \tag{9.100}$$

$$= \sum_{k=1}^{\infty}\lambda^{k}\frac{(x)^{k\alpha-1}}{\Gamma(k\alpha)}$$

$$= \lambda e^{\lambda x}_{\alpha}$$

Thus we have $y = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$ as a solution to FDE with a classical RL derivative ${}^{RL}_{\ 0}D^{\alpha}_{x}y = \lambda y$ or eigenfunction for ${}^{RL}_{\ 0}D^{\alpha}_{x}y = \lambda y$ is $y = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$. This we have discussed earlier (Section 4.14, 7.3, 7.14, 7.18). In addition, we discussed that ${}^{C}_{0}D^{\alpha}_{x}y = \lambda y$ has a solution as $y = E_{\alpha,1}(\lambda x^{\alpha})$, (Section 4.14, 7.2, 7.14, 7.18). Two functions $x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$ and $E_{\alpha,1}(\lambda x^{\alpha})$ we termed then as the alpha-exponential functions 1 & 2, we had used them in an earlier chapter (Section 7.14).

9.11.4 The modified RL derivative of order α for function $f(x) = E_{\alpha}(\lambda x)$

With a modified RL derivative $\mathbf{D}^{\alpha}$, we will find $\mathbf{D}^{\alpha}E_{\alpha}(\lambda x)$ knowing the fact that $E_{\alpha}(\lambda x^{\alpha})$ is a solution of $\mathbf{D}^{\alpha}E_{\alpha}(\lambda x^{\alpha})$, i.e. we write it in the following form:

$$\mathbf{D}^{\alpha}E_{\alpha}(\lambda x^{\alpha}) = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left(E_{\alpha}(\lambda x^{\alpha})\right) = \lambda E_{\alpha}(\lambda x^{\alpha}) \tag{9.101}$$

Let $y = x^{\alpha}$ then $\mathrm{d}y = \alpha x^{\alpha-1}\mathrm{d}x$ and $\left(\mathrm{d}y\right)^{\alpha} = \left(\alpha x^{\alpha-1}\right)^{\alpha}\left(\mathrm{d}x\right)^{\alpha}$. With this substitution in (9.101) we write the following steps:

$$\frac{\mathrm{d}^{\alpha}\left(E_{\alpha}(\lambda y)\right)}{\mathrm{d}\ y^{\alpha}} = \frac{\mathrm{d}^{\alpha}\left(E_{\alpha}(\lambda y)\right)}{\left(\mathrm{d}y\right)^{\alpha}} = \frac{\mathrm{d}^{\alpha}\left(E_{\alpha}(\lambda y)\right)}{\left(\alpha x^{(\alpha-1)}\right)^{\alpha}\left(\mathrm{d}x\right)^{\alpha}}$$

$$= \frac{1}{\alpha^{\alpha}\left(y^{\left(\frac{1}{\alpha}\right)(\alpha-1)}\right)^{\alpha}}\left(\frac{\mathrm{d}^{\alpha}E_{\alpha}(\lambda x^{\alpha})}{\left(\mathrm{d}x\right)^{\alpha}}\right) \tag{9.102}$$

$$= \alpha^{-\alpha}y^{1-\alpha}\left(\lambda E_{\alpha}(\lambda x^{\alpha})\right)$$

$$= \lambda\alpha^{-\alpha}y^{1-\alpha}E_{\alpha}(\lambda y)$$

Therefore, we have an interesting relationship i.e. $\mathbf{D}^{\alpha}E_{\alpha}(\lambda x) = \lambda\alpha^{-\alpha}x^{1-\alpha}E_{\alpha}(\lambda x)$.

9.11.5 Applying the modified RL fractional derivative to a non-differentiable point of a function: $f(x) = |x|^{1/2} + b$

We now see what is the half derivative at the point $x = 0$ for function i.e. $f(x) = |x|^{1/2} + b$, where $f^{(1)}(0) = \infty$. So this is non-differentiable function at $x = 0$. We use this to calculate the following:

$$
\begin{aligned}
f^{(1/2)}(x) &= D^{1/2}\left[f(x) - f(0)\right] \\
&= D^{1/2}\left[|x|^{1/2}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma(1)}|x|^{0} = \left(\tfrac{1}{2}\right)!
\end{aligned}
\tag{9.103}
$$

We note that for $f(x) = |x|^{1/2} + b$ we have $f^{(1/2)}(x) = \mathbf{D}^{1/2}\left(|x|^{1/2} + b\right) = \mathbf{D}^{1/2}|x|^{1/2} + \mathbf{D}^{1/2}b$. With $\mathbf{D}^{1/2}$ as a modified RL derivative of the Jumarie type and with $\mathbf{D}^{1/2}b = 0$ we have $f^{(1/2)}(x) = \left(\frac{1}{2}\right)!$ We observe that the modified RL derivative Jumarie type i.e. $\mathbf{D}^{\alpha}$ is acting as a local operator like that of classical calculus.

9.11.6 Defining the critical order for a non-differentiable point by use of the modified RL fractional derivative

Now we write the fractional derivative α with $0 < \alpha < 1$ of $f(x) = |x|^{1/2} + b$ by use of the modified RL derivative and write via the above (9.103) steps:

$$
f^{(\alpha)}(x) = \frac{\left(\frac{1}{2}\right)!}{\left(\frac{1}{2} - \alpha\right)!}|x|^{(1/2)-\alpha}
\tag{9.104}
$$

We observe that $f^{(\alpha)}(x)\Big|_{x=0}$ has values of zero for $0 \le \alpha < \frac{1}{2}$, a constant value for $\alpha = \frac{1}{2}$ and infinity for $\frac{1}{2} < \alpha \le 1$, which we summarise below:

$$
f^{(\alpha)}(x)\Big|_{x=0} = \begin{cases} 0 & , \quad 0 \le \alpha < \frac{1}{2} \\ \left(\frac{1}{2}\right)! & , \quad \alpha = \frac{1}{2} \\ \infty & , \quad \frac{1}{2} < \alpha \le 1 \end{cases}
\tag{9.105}
$$

Therefore, we call $\alpha = \frac{1}{2}$ as the critical order for $f(x) = |x|^{1/2} + b$.

Now we take a function $f(x)$ that is defined on the interval $[0,1]$ as a set of points $[x_1, x_2, \ldots x_N]$ with $N \uparrow \infty$, $x_1 = 0$ and $x_N = N$. Let the function be described as $f(x) = \lim_{N\uparrow\infty} \sum_{k=1}^{N}\left(x - x_k\right)^{1/2} + b$, for $x \ge x_k$ and $f(x) = b$ for $x < x_k$, for all x_k, $k = 1, 2, 3 \ldots, N$, (Section 1.24). We note that it is a function that is not differentiable at $x = x_1, x_2, .., x_N$ in the interval $[0,1]$, but $f^{(1/2)}(x)$ exists at all those non-differentiable points with value $f^{(1/2)}(x)\Big|_{x=x_1, x_2, \ldots x_N} = \left(\frac{1}{2}\right)!$. This we will derive in Section- 9.18.2. Thus, a continuous but nowhere differentiable function can have fractional derivatives at all those non-differentiable points; in this case, the critical order of this is nowhere differentiable function is half.

9.12 Fractional Taylor series

We started this concept from Section 9.6 of the development of a modified RL fractional derivative Jumarie type for the non-differentiable functions. Until now, we have seen $f(x) = E_{\alpha}(\lambda x^{\alpha})$; $x \ge 0$ in the context of a modified RL fractional derivative Jumarie type is in conjugation with the function $e^{\lambda x}$ of classical calculus. We have seen that $f^{(\alpha)}(0) = \mathbf{D}^{\alpha}E_{\alpha}(\lambda x^{\alpha})\Big|_{x=0} = \lambda$ while the classical derivative is not defined as $x = 0$ i.e. $f^{(1)}(0) = \infty$, for $f(x) = E_{\alpha}(\lambda x^{\alpha})$. Thus, the function $f(x) = E_{\alpha}(\lambda x^{\alpha})$ is not differentiable at $x = 0$ but certainly differentiable at other points i.e. at $x > 0$. We want functions that are nowhere differentiable but continuous ones.

With $f(x) = E_{\alpha}(\lambda x^{\alpha})$ we make an everywhere non-differentiable function by considering the function $f(x)$ defined in the interval say $[0,1]$ composed of discrete points $[x_0, x_1, x_2, \ldots\ldots\ldots x_N]$ with $x_0 = 0$ and $x_N = 1$ as $f(x) = E_{\alpha}(\lambda x^{\alpha})$ if $x = x_0$, $x = x_1, \ldots\ldots\ldots x = x_N$ else $f(x) = 0$. When we have $N \uparrow \infty$, for the fractal-set

$\left[x_0, x_1, x_2, ...x_N\right]$ we get a function $f(x)$ which is continuous but nowhere differentiable. These types of functions are called 'fractal-functions'. This function satisfies $\mathbf{D}^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$ and is $\alpha-$differentiable at all points with values at $x_0, x_1, x_2, ...x_N$ where classically it is not differentiable at $x_0, x_1, x_2, ...x_N$ i.e. nowhere differentiable.

Similarly, we can have a function $f(x) = \lim_{N\uparrow\infty}\sum_{k=0}^{N} E_\alpha(\lambda(x-x_k)^\alpha)$ for $x \geq x_k$; and $f(x)=1$ for $x < x_k$. This is not differentiable at points $x_0, x_1, x_2, ...x_N$. We will show later in the Section-9.18.2 that this function is also having the property $f^{(\alpha)}(x) = \lambda f(x)$, thus is $\alpha-$differentiable at points $x_0, x_1, x_2, ...x_N$. In order to simplify the notations we will imply that the functions are considered that are nowhere differentiable. That is still what we represent by normal symbols like $x(t), f(x), y(x), E_\alpha(\lambda x^\alpha), \cos_\alpha(x^\alpha)$ etc., with an understanding that all these functions are nowhere differentiable functions-i.e. they are 'fractal functions' ($f_{\text{fract}}(x)$) as described in Section-1.24.

We started defining the fractional derivative (9.71) via defining a quotient i.e. $\Delta x/\left(\Delta t\right)^\alpha$. Then through further definition (9.88) we defined it as $\lim_{\Delta t\downarrow 0}\left(\Delta^\alpha \tilde{x}(t)/\left(\Delta t\right)^\alpha\right)$ via fractional difference of the zero-corrected function described as $\tilde{x}(t) = x(t) - x(0)$. What is the relationship between these two definitions (9.71), (9.88)? This will make our development and this approach complete.

With the above definitions of fractional derivatives it is easy to show a solution of $y^{(\alpha)}(t) = \lambda y(t)$, $y(0) = y_0$ is $y(t) = y_0 E_\alpha(\lambda t^\alpha)$. In addition, we have an operational equality as $\mathbf{D}_h^\alpha \mathrm{E}_{\mathrm{F}h} = \mathbf{D}_t^\alpha \mathrm{E}_{\mathrm{F}h}$, where $\mathrm{E}_{\mathrm{F}h}\left(x(t)\right) = x(t+h)$, is a forward shift operator. Now look at the following Laplace identity that we have derived earlier in this chapter (9.82), (9.87):

$$\begin{aligned}\mathcal{L}\left\{\mathbf{D}_h^\alpha f(t+h)\right\} &\cong s^\alpha\left(e^{st}F(s) - te^{st}f(0)\right)\\ \mathcal{L}\left\{\mathbf{D}_t^\alpha f(t+h)\right\} &\cong s^\alpha\left(e^{hs}F(s) - he^{hs}f(0)\right)\end{aligned} \tag{9.106}$$

Here we see for $h\downarrow 0$ and $t\downarrow 0$ that both are equal i.e. $\mathcal{L}\left\{\mathbf{D}_t^\alpha f(t+h)\right\} = \mathcal{L}\left\{\mathbf{D}_h^\alpha f(t+h)\right\}$. Therefore with the operational identity i.e. $\mathbf{D}_h^\alpha \mathrm{E}_{\mathrm{F}h} = \mathbf{D}_t^\alpha \mathrm{E}_{\mathrm{F}h}$ which may be considered as a fractional differential equation of type $\mathbf{D}_h^\alpha y(h) = \mathrm{K}\left(y(h)\right)$ and taking $\mathbf{D}_t^\alpha = \mathrm{K}$ as a constant and $y(h) = \mathrm{E}_{\mathrm{F}h}(h)$. We write the solution in terms of the Mittag-Leffler function, i.e. $y(h) = E_\alpha\left(\mathrm{K}h^\alpha\right)$ or $\mathrm{E}_{\mathrm{F}h} = E_\alpha\left(h^\alpha \mathbf{D}_t^\alpha\right)$. Here we have used the fact that with a modified RL derivative $\mathbf{D}^\alpha$ we have a solution to $\mathbf{D}_x^\alpha y = \lambda y$ as $y = E_\alpha(\lambda x^\alpha)$.

We obtained the operation identity as $\mathrm{E}_{\mathrm{F}h} = E_\alpha\left(h^\alpha \mathbf{D}_t^\alpha\right)$ and with this, we operate on $x(t)$ to write $\left(\mathrm{E}_{\mathrm{F}h}\right)x(t) = \left(E_\alpha\left(h^\alpha \mathbf{D}_t^\alpha\right)\right)x(t)$. After expanding the Mittag-Leffler function in series form (see Appendix A), and noting $\left(\mathrm{E}_{\mathrm{F}h}\right)x(t) = x(t+h)$, we get the following:

$$x(t+h) = \left(E_\alpha\left(h^\alpha \mathbf{D}_t^\alpha\right)\right)x(t) = \left(1 + \frac{h^\alpha}{\alpha!}\mathbf{D}_t^\alpha + \frac{h^{2\alpha}}{(2\alpha)!}\mathbf{D}_t^{2\alpha} + ...\right)x(t) \tag{9.107}$$

Writing $\mathbf{D}_t^\alpha x(t) = x^{(\alpha)}(t)$, $\mathbf{D}_t^{2\alpha} x(t) = x^{(2\alpha)}(t)$ we get the fractional Taylor series as:

$$\begin{aligned}x(t+h) &= x(t) + \frac{h^\alpha}{(\alpha)!}x^{(\alpha)}(t) + \frac{h^{2\alpha}}{(2\alpha)!}x^{(2\alpha)}(t) + ...\\ &= x(t) + \sum_{k=1}^{\infty}\frac{h^{\alpha k}}{(\alpha k)!}x^{(\alpha k)}(t)\end{aligned} \tag{9.108}$$

Making $0 \leftarrow t$, and $t \leftarrow h$ gives us the fractional Maclaurent series as follows

$$x(t) = x(0) + \sum_{k=1}^{\infty} \frac{t^{\alpha k}}{(\alpha k)!} x^{(\alpha k)}(0) \tag{9.109}$$

We note in (9.109) that we have the following notation:

$$x^{(\alpha k)}(t) = \underbrace{\mathbf{D}^{\alpha}\left(\mathbf{D}^{\alpha} \dots \mathbf{D}^{\alpha} x(t)\right)}_{k} \neq \mathbf{D}^{k\alpha} x(t)$$

that is $\mathbf{D}^{2\alpha} \neq \mathbf{D}^{\alpha+\alpha}$ but $\mathbf{D}^{2\alpha} = \mathbf{D}^{\alpha}\mathbf{D}^{\alpha}$, that we have discussed earlier in this chapter as the rules of composition (Section 9.3). For a small h we have the following approximation with the first two terms of the fractional Taylor series as follows:

$$\begin{aligned} x(t+h) &\simeq x(t) + \frac{h^{\alpha}}{(\alpha)!} x^{(\alpha)}(t) \\ \Delta x(t) &\simeq \left((\alpha!)\right)^{-1} x^{(\alpha)}(t) \left(\Delta t\right)^{\alpha} \qquad \alpha!\mathrm{d}x \simeq x^{(\alpha)}(t)\left(\mathrm{d}t\right)^{\alpha} \end{aligned} \tag{9.110}$$

We remark that the developed fractional Taylor series is suitable for a modified RL derivative of the Jumarie type, and is not valid for a conventional Riemann-Liouville fractional derivative. This fractional Taylor series is for non-differentiable fractal functions.

9.13 The use of fractional Taylor series

Here we will use the fractional Taylor series to expand a non-differential function at a non-differentiable point. In addition, by the fractional Taylor series, we will have conversion formulas to represent fractional differentials to classical differentials i.e. relating $\mathrm{d}^{\alpha} f$ to the differential $\mathrm{d}\ f$ and $(\mathrm{d}\ f)^{\alpha}$. Our fractional Maclaurent series is

$$f(x) = f(0) + \sum_{k=1}^{\infty} \frac{x^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(0) \tag{9.111}$$

The Mittag-Leffler function $f(x) = E_{\alpha}(x^{\alpha})$ with $0 < \alpha < 1$ is not-differentiable at $x = 0$, and its expansion at $x = 0$ (the non-differentiable point) is as follows:

$$\begin{aligned} f(x) &= f(0) + \frac{x^{\alpha}}{\alpha!} f^{(\alpha)}(0) + \frac{x^{2\alpha}}{(2\alpha)!} f^{(2\alpha)}(0) + \dots \\ E_{\alpha}(x^{\alpha}) &= E_{\alpha}(0) + \frac{x^{\alpha}}{\alpha!} \mathbf{D}^{\alpha}\left(E_{\alpha}(x^{\alpha})\right)\Big|_{x=0} + \frac{x^{2\alpha}}{(2\alpha)!} \mathbf{D}^{\alpha}\left(\mathbf{D}^{\alpha}\left(E_{\alpha}(x^{\alpha})\right)\right)\Big|_{x=0} + \dots\dots \\ &= 1 + \frac{x^{\alpha}}{\alpha!} E_{\alpha}(x^{\alpha})\Big|_{x=0} + \frac{x^{2\alpha}}{(2\alpha)!} \mathbf{D}^{\alpha}\left(E_{\alpha}(x^{\alpha})\right|_{x=0} + \dots\dots \\ &= 1 + \frac{x^{\alpha}}{\alpha!} + \frac{x^{2\alpha}}{(2\alpha)!} E_{\alpha}(x^{\alpha})\Big|_{x=0} + \dots \\ &= 1 + \frac{x^{\alpha}}{\alpha!} + \frac{x^{2\alpha}}{(2\alpha)!} + \dots \end{aligned} \tag{9.112}$$

We have obtained the series definition of $f(x) = E_{\alpha}(x^{\alpha})$. Thus, a series that defines the Mittag-Leffler function is nothing but a fractional Taylor series at point zero-where the function is not differentiable. We note that $\mathbf{D}^{\alpha}$ is a modified RL fractional derivative and we have used the above the relationship $\mathbf{D}_x^{\alpha} E_{\alpha}(x^{\alpha}) = E_{\alpha}(x^{\alpha})$ which we discussed earlier. We note that $E_{\alpha}(x^{\alpha})\Big|_{x=0} = E_{\alpha}(0) = 1$ i.e. used in (9.112).

There is in conjugation with the classical calculus, as for the function $f(x) = e^{x}$ we obtain a series representation such as $e^{x} = 1 + x + \frac{x^2}{2} + \dots$ i.e. for the exponential function via the classical Taylor series expansion at $x = 0$ for $f(x) = e^{x}$ via the classical Taylor series.

Let $f(x) = x^{1/2} + b$, which is not-differentiable at $x = 0$, its expansion at $x = 0$ is the following (via a fractional Taylor series):

$$f(x)=f(0)+\frac{x^{\alpha}}{\alpha!}f^{(\alpha)}(0)+\frac{x^{2\alpha}}{(2\alpha)!}f^{(2\alpha)}(0)+....;\quad \alpha=\frac{1}{2}$$

$$f(x)=b+\frac{x^{1/2}}{\left(\frac{1}{2}\right)!}\mathbf{D}^{1/2}\left(x^{1/2}+b\right)\Big|_{x=0}+\frac{x^{2\left(\frac{1}{2}\right)}}{\left(2\left(\frac{1}{2}\right)\right)!}\mathbf{D}^{1/2}\left(\mathbf{D}^{1/2}\left(x^{1/2}+b\right)\right)\Big|_{x=0}+..... \tag{9.113}$$

$$=b+\frac{x^{1/2}}{\left(\frac{1}{2}\right)!}\frac{\left(\frac{1}{2}\right)!}{0!}+x\mathbf{D}^{1/2}\left(\left(\frac{1}{2}\right)!\right)=b+x^{1/2}$$

We apply the normal Maclaurent series for a polynomial $f(x)=x^2+x+c$ and write the following:

$$f(x)=f(0)+x\left(f^{(1)}(0)\right)+\frac{x^2}{2}\left(f^{(2)}(0)\right)+...$$

$$=c+x\left(\left(2x+1\right)\Big|_{x=0}\right)+\frac{x^2}{2}\left(2\Big|_{x=0}\right)+0 \tag{9.114}$$

$$=c+x+x^2$$

We observe a parallel to classical calculus for the fractional Taylor series at a non-differentiable point, for a non-differentiable function vis-à-vis the classical Taylor series for differentiable functions. Intuitively, if one considers that in fractional calculus the Mittag-Leffler function plays a similar role as the exponential function in classical calculus, one is led to extend the Taylor series $f(x+h)=\left(\exp(h\mathbf{D}_x)\right)\left(f(x)\right)$ in the form of $f(x+h)=\left(E_{\alpha}\left(h^{\alpha}\mathbf{D}_x^{\alpha}\right)\right)\left(f(x)\right)$.

Here we clarify that $\exp(h\mathbf{D}_x)$ is an operator that operates on $f(x)$. We can write this as $f(x+h)=\left(e^{h\mathbf{D}_x}\right)\left(f(x)\right)$. It is the same as the operator $E_{\alpha}\left(h^{\alpha}\mathbf{D}_x^{\alpha}\right)$ that operates on $f(x)$. Of course, despite the simplicity of the formula this is not sufficient and needs to be supported by a sound derivation. Actually, the problem that appears is how to suitably define the concept of a fractional derivative (that we have tried to do in previous sections) in order that it is fully consistent with the Mittag-Leffler function which is considered as the solution of the fractional differential equation, i.e. $x^{(\alpha)}(t)=\lambda x(t)$.

9.14 Conversion formulas for fractional differentials

From the Fractional Taylor series, we have a conversion formula (approximate) that is:

$$\alpha!\mathrm{d}x\simeq\left(x^{(\alpha)}(t)\right)\left(\mathrm{d}t\right)^{\alpha} \tag{9.115}$$

From this, we can have:

$$\alpha!\mathrm{d}x\simeq\frac{\mathrm{d}^{\alpha}x}{\left(\mathrm{d}t\right)^{\alpha}}\left(\mathrm{d}t\right)^{\alpha}=\mathrm{d}^{\alpha}x \tag{9.116}$$

and then the relationship:

$$\mathrm{d}^{\alpha}x\simeq\alpha!\mathrm{d}x \tag{9.117}$$

From the formula of the fractional derivative, we have:

$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}x^{\alpha}}=\frac{1}{\left(1-\alpha\right)!}x^{1-\alpha} \tag{9.118}$$

that is the Euler formula $\mathbf{D}^{\alpha}x=\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha}$ which gives us another formula i.e.

$$\mathrm{d}^{\alpha}x=\left(\left(1-\alpha\right)!\right)^{-1}x^{1-\alpha}\left(\mathrm{d}x\right)^{\alpha} \tag{9.119}$$

These are approximate conversion formulas of differentials relating $\mathrm{d}^{\alpha}x$ with $\mathrm{d}x$ and $(\mathrm{d}x)^{\alpha}$. In the above expressions we have used factorial notations instead of Gamma functions.

9.15 Integration with respect to fractional differentials

9.15.1 Integration w.r.t. $(\mathrm{d}\xi)^{\alpha}$ and its relationship to the Riemann-Liouville fractional integration formula

In classical calculus the equation $x^{(1)}(t) = g(t)$ i.e. the following:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t); \qquad x(0) = 0 \tag{9.120}$$

is equivalent to $x(t) = \int_0^t g(\tau)\mathrm{d}\tau$. Here the RHS of (9.120) is considered as an anti-derivative, in the sense that it is defined by the following equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(\tau)\mathrm{d}\tau = g(t) \quad , \quad \mathrm{d}\left(\int_0^t g(\tau)\mathrm{d}\tau\right) = g(t)\mathrm{d}t \tag{9.121}$$

On applying the same point of view to the modified RL fractional derivative, we can write the following fractional differential quantity for the equation $y^{(\alpha)}(t) = g(t)$ via the definition (9.71) as:

$$\mathrm{d}y = g(t)\left(\mathrm{d}t\right)^{\alpha}; \qquad y(0) = 0 \qquad 0 < \alpha < 1 \tag{9.122}$$

This (9.122) is equivalent to the integral $y(t) = \int_0^t g(\tau)\left(\mathrm{d}\tau\right)^{\alpha}$. Here the RHS is a fractional anti-derivative of order α and is described (defined) by the following equality:

$$\mathrm{d}\left(\int_0^t g(\tau)\left(\mathrm{d}\tau\right)^{\alpha}\right) = g(t)\left(\mathrm{d}t\right)^{\alpha} \qquad \frac{\mathrm{d}}{\mathrm{d}t^{\alpha}}\int_0^t g(\tau)\left(\mathrm{d}\tau\right)^{\alpha} = g(t) \tag{9.123}$$

We have a formula (we will prove it subsequently) defined as:

$$\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} \overset{\text{def}}{=} \alpha\int_0^x (x-\xi)^{\alpha-1}\left(f(\xi)\right)\mathrm{d}\xi, \quad 0 < \alpha < 1 \tag{9.124}$$

relating the integral with respect to the element $\left(\mathrm{d}x\right)^{\alpha}$, i.e. LHS of (9.124) and the Riemann-Liouville fractional integral RHS of (9.124) (with the exception of the Gamma function).

The integral w.r.t. $\left(\mathrm{d}x\right)^{\alpha}$ is defined as a solution to the fractional differential equation, that is:

$$\mathrm{d}y = f(x)\left(\mathrm{d}x\right)^{\alpha}, \quad x \ge 0, \quad y(0) = 0 \tag{9.125}$$

that is:

$$y(x) = \int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} \tag{9.126}$$

Multiply both sides of above (9.125) expression i.e. $\mathrm{d}y = f(x)\left(\mathrm{d}x\right)^{\alpha}$ by $\alpha!$ and we get the following:

$$(\alpha!)\mathrm{d}y = (\alpha!)f(x)\left(\mathrm{d}x\right)^{\alpha} \tag{9.127}$$

The fractional Taylor series gives $\alpha!\mathrm{d}y \simeq y^{(\alpha)}(x)\left(\mathrm{d}x\right)^{\alpha}$ (9.115), using this expression in the above (9.127) obtained expression we get the following formulas with a modified RL fractional derivative:

$$y^{(\alpha)}(x) = \alpha!f(x) \qquad \mathbf{D}^{\alpha}\left(y(x)\right) = \alpha!f(x) \tag{9.128}$$

This provides the following important relationship:

$$\begin{aligned} y(x) = \alpha!\mathbf{D}^{-\alpha}f(x) &= \alpha!\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi\right) \\ &= \alpha!\left(\frac{1}{(\alpha+1)!}\int_0^x (x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi\right) \\ &= \alpha\int_0^x (x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi; \qquad y(x) = \int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} \end{aligned} \tag{9.129}$$

$$\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} = \alpha\int_0^x (x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi$$

We have proved our definition (9.124).

9.15.2 A demonstration of the integration w.r.t. $(\mathrm{d}\xi)^{\alpha}$ for $f(x) = x^{\gamma}$, $f(x) = 1$, $f(x) = \delta(x)$ and $f(x) = E_{\alpha}(\lambda x^{\alpha})$

On making $f(x) = x^{\gamma}$ we get the following from (9.124):

$$\begin{aligned}\int_0^x \xi^\gamma \left(\mathrm{d}\xi\right)^\alpha &= \alpha\int_0^x (x-\xi)^{\alpha-1}\xi^\gamma \mathrm{d}\xi \\ &= \alpha\left(\Gamma(\alpha)\right)\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}\xi^\gamma \mathrm{d}\xi\right) \\ &= \Gamma\left(\alpha+1\right)\left({}_0\mathbf{D}_x^{-\alpha}x^\gamma\right) \qquad (9.130) \\ &= \Gamma(\alpha+1)\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}x^{\gamma+\alpha} \\ &= \frac{\left(\alpha!\right)\left(\gamma!\right)}{\left(\alpha+\gamma\right)!}x^{\gamma+\alpha}, \quad 0<\alpha<1\end{aligned}$$

On making $f(x)=1$ we get (9.124) the following:

$$\int_0^x \left(\mathrm{d}\xi\right)^\alpha = x^\alpha, \quad 0<\alpha<1; \quad x\geq 0 \qquad (9.131)$$

In the physical or geometrical systems, the coordinates are drawn in a normal case via the process of integration of the differential element $\mathrm{d}x$ i.e. we write as $\int_0^x \mathrm{d}\xi = x$ on this coordinate we write/draw the function $y=f(x)$ w.r.t. this coordinate x. Thus with the fractional differential the drawn coordinate will be $\int_0^x \left(\mathrm{d}\xi\right)^\alpha = x^\alpha$ with a function mapped as $f(x^\alpha)$ w.r.t. the transformed coordinate x^α, for $x\geq 0$, and $0<\alpha<1$.

On making $f(x)=\delta(x)$ and by using the property of the delta-function i.e. $\int \left(f(x)\right)\left(\delta(x)\right)\mathrm{d}x = f(0)$ we obtain the following relationship by using (9.124):

$$\int_0^x \delta(\xi)\left(\mathrm{d}\xi\right)^\alpha = \alpha\int_0^x \left((x-\xi)^{\alpha-1}\right)\left(\delta(\xi)\right)\mathrm{d}\xi = \alpha x^{\alpha-1}, \quad 0<\alpha<1 \qquad (9.132)$$

On making $f(x)=E_\alpha(\lambda x^\alpha)$ we get from (9.124):

$$\begin{aligned}\int_0^x E_\alpha(\lambda\xi^\alpha)\left(\mathrm{d}\xi\right)^\alpha &= \alpha\left(\Gamma(\alpha)\right)\left(\mathbf{D}^{-\alpha}E_\alpha(\lambda x^\alpha)\right) \\ &= \alpha!\lambda^{-1}\left(E_\alpha(\lambda x^\alpha)-1\right)\end{aligned} \qquad (9.133)$$

9.15.3 Some useful identities for the integration w.r.t. $(\mathrm{d}\xi)^\alpha$

From our discussions, we have $y(x)=\alpha!\mathbf{D}^{-\alpha}f(x)$ as a result of one-whole integration of $\mathrm{d}y = f(x)\left(\mathrm{d}x\right)^\alpha$ giving $y(x)=\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^\alpha$ that is:

$$\begin{aligned}\mathbf{D}^{-\alpha}f(x) &= \frac{1}{\alpha!}\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^\alpha \qquad 0<\alpha<1 \\ f(x) &= \frac{1}{\alpha!}\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left(\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^\alpha\right)\end{aligned} \qquad (9.134)$$

We write another useful relationship i.e.

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left(\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^\alpha\right) = \alpha!f(x) \qquad (9.135)$$

As a direct consequence of the definition, i.e. $\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^\alpha = \alpha\int_0^x (x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi$ (9.124), we derive the following for $0<\alpha+\beta<1$:

$$\begin{aligned}\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha+\beta} &= (\alpha+\beta)\int_0^x (x-\xi)^{\alpha+\beta-1} f(\xi)\mathrm{d}\xi \\ &= (\alpha+\beta)\int_0^x (x-\xi)^{\beta}(x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi\end{aligned} \tag{9.136}$$

From the definition i.e. $\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} = \alpha\int_0^x (x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi$, we can write:

$$f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} = \alpha(x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi \tag{9.137}$$

Using the above (9.136) relationship we get the following identity:

$$\begin{aligned}\int_0^x f(\xi)\left(\mathrm{d}\xi\right)^{\alpha+\beta} &= \left(\frac{\alpha+\beta}{\alpha}\right)\int_0^x (x-\xi)^{\beta} f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} \\ &= \left(\frac{\alpha+\beta}{\alpha}\right)\int_0^x (x-\xi)^{\alpha} f(\xi)\left(\mathrm{d}\xi\right)^{\beta}, \qquad 0<\alpha+\beta<1\end{aligned} \tag{9.138}$$

9.16 Leibniz's rule for the product of two functions for a modified fractional derivative

From the fractional Taylor series we have (9.115) $\alpha!\mathrm{d}y = y^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}$. Let us write $y(x) = \left(u(x)\right)\left(v(x)\right)$ that is the product of two functions. We do the following manipulations:

$$\begin{aligned}&\alpha!\mathrm{d}\left(uv\right) = \left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha} \\ &\left(\alpha!\right)\left(u(x)\mathrm{d}v + v(x)\mathrm{d}u\right) = \left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha} \\ &\left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha} = u(x)\left(\alpha!\mathrm{d}v\right) + v(x)\left(\alpha!\mathrm{d}u\right) \\ &\left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha} = u(x)\left(v^{(\alpha)}(\mathrm{d}x)^{\alpha}\right) + v(x)\left(u^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)\end{aligned} \tag{9.139}$$

Dividing both sides by $\left(\mathrm{d}x\right)^{\alpha}$ we obtain the following formula:

$$\left(u(x)v(x)\right)^{(\alpha)} = u(x)v^{(\alpha)}(x) + v(x)u^{(\alpha)}(x) \tag{9.140}$$

For $\alpha = 1$ we recover the classical formula i.e.

$$\left(u(x)v(x)\right)^{(1)} = u(x)v^{(1)}(x) + v(x)u^{(1)}(x) \tag{9.141}$$

Let $u(x) = \delta(x)$ and $v(x) = f(x)$ with application of above (9.140), we have:

$$\begin{aligned}&\left(\delta(x)f(x)\right)^{(\alpha)} = \delta(x)f^{(\alpha)}(x) + f(x)\delta^{(\alpha)}(x) \\ &\left(\delta(x)f(x)\right)^{(\alpha)} = \left(f(0)\right)^{(\alpha)} = 0 \\ &\delta^{(\alpha)}\left(x\right)f(x) = -\delta(x)f^{(\alpha)}(x)\end{aligned} \tag{9.142}$$

In the above (9.142) derivation we have used $\delta(x)f(x) = f(0)$ and then $\mathbf{D}^{\alpha}f(0) = \left(f(0)\right)^{(\alpha)} = 0$; that is a modified fractional RL derivative of the constant i.e. $f(0) = C$ is zero. Now integrating (9.142) with the fractional differential, we obtain:

$$\begin{aligned}\int \delta^{(\alpha)}(\xi)f(\xi)\left(\mathrm{d}\xi\right)^{\alpha} &= -\int \delta(\xi)f^{(\alpha)}(\xi)\left(\mathrm{d}\xi\right)^{\alpha}, \quad 0<\alpha<1 \\ &= -\alpha\int (x-\xi)^{\alpha-1}\delta(\xi)f^{(\alpha)}(\xi)\mathrm{d}\xi \\ &= -\alpha x^{\alpha-1}f^{(\alpha)}(0)\end{aligned} \tag{9.143}$$

We have $\left(u(x)v(x)\right)^{(\alpha)} = u(x)v^{(\alpha)}(x) + v(x)u^{(\alpha)}(x)$ i.e. the product rule. First let $u(x) = x^a$ and $v(x) = x^b$, with $0<a,b<1$. Then using the product rule we get the following for $\mathbf{D}^{\alpha}\left(x^a x^b\right)$:

$$\begin{aligned}\mathbf{D}^{\alpha}\left(x^{a}x^{b}\right) &= x^{a}\mathbf{D}^{\alpha}\left(x^{b}\right)+x^{b}\mathbf{D}^{\alpha}\left(x^{a}\right)\\ &= x^{a}\left(\tfrac{b!}{(b-\alpha)!}x^{b-\alpha}\right)+x^{b}\left(\tfrac{a!}{(a-\alpha)!}x^{a-\alpha}\right)\\ &= \left(\frac{a!}{(a-\alpha)!}+\frac{b!}{(b-\alpha)!}\right)x^{a+b-\alpha}\end{aligned} \tag{9.144}$$

While using a direct formula i.e. $\mathbf{D}^{\alpha}x^{m}=\frac{m!}{(m-\alpha)!}x^{m-\alpha}$ we write the following:

$$\mathbf{D}^{\alpha}\left(x^{a}x^{b}\right)=\mathbf{D}_{x}^{\alpha}\left(x^{a+b}\right)=\frac{(a+b)!}{(a+b-\alpha)!}x^{a+b-\alpha} \tag{9.145}$$

They (9.144) (9.145) are not the same yet they are of the form $Kx^{a+b-\alpha}$, with K as a constant. Is there a contradiction? Do we observe that these two formulas are dealing with $x^{a}x^{b}$ and x^{a+b} respectively in (9.144) and (9.145)? We must conclude from the point of a fractional derivative that they are not the same in the sense that their fractional differentials or differential increments are not of the same value. The $(x^{a})^{(\alpha)}(\mathrm{d}x)^{\alpha}$, $(x^{b})^{(\alpha)}(\mathrm{d}x)^{\alpha}$ are required in the case-1 i.e. in (9.144) (i.e. using product rule) while $(x^{a+b})^{(\alpha)}(\mathrm{d}x)^{\alpha}$ are required in the case-2 i.e. in (9.145) (i.e. direct calculation). We write these two cases as follows:

$$\begin{aligned}&\mathbf{D}^{\alpha}\left(x^{a}x^{b}\right)=\frac{\mathrm{d}^{\alpha}\left(x^{a}x^{b}\right)}{\mathrm{d}x^{\alpha}};\\ &\mathrm{d}^{\alpha}\left(x^{a}x^{b}\right)=x^{a}\left((x^{b})^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)+x^{b}\left((x^{a})^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)\\ &\mathbf{D}^{\alpha}\left(x^{a}x^{b}\right)=\frac{\mathrm{d}^{\alpha}\left(x^{a+b}\right)}{\mathrm{d}x^{\alpha}};\quad \mathrm{d}^{\alpha}\left(x^{a}x^{b}\right)=(x^{a+b})^{(\alpha)}(\mathrm{d}x)^{\alpha}\end{aligned} \tag{9.146}$$

9.17 Integration by parts for a fractional differential

We have the formula (9.140) i.e. $\left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}=u(x)\left(v^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)+v(x)\left(u^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)$. Now integrating this with respect of fractional differential $(\mathrm{d}x)^{\alpha}$, we obtain the following:

$$\int\left(uv\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}=\int u(x)\left(v^{(\alpha)}(\mathrm{d}x)^{\alpha}\right)+\int v(x)\left(u^{(\alpha)}(\mathrm{d}x)^{\alpha}\right) \tag{9.147}$$

In addition, we have $\alpha!\left(\mathrm{d}y\right)=y^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}$, (9.115) integrating this on both sides we write:

$$\int_{0}^{y(x)}\alpha!\left(\mathrm{d}y\right)=\int_{0}^{x}y^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha} \tag{9.148}$$

Which gives us $\alpha!y(x)=\int_{0}^{x}y^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}$. Using this obtained relationship, we write:

$$\int\left(u(x)v(x)\right)^{(\alpha)}\left(\mathrm{d}x\right)^{\alpha}=\alpha!\left(u(x)v(x)\right) \tag{9.149}$$

With the use of a concept of fractional differentials, we write the following:

$$\begin{aligned}&\int_{a}^{b}u^{(\alpha)}(x)v(x)\left(\mathrm{d}x\right)^{\alpha}=\alpha!\left(u(x)v(x)\right)\Big|_{a}^{b}-\int_{a}^{b}u(x)v^{(\alpha)}(x)\left(\mathrm{d}x\right)^{\alpha}\\ &\int_{a}^{b}v\left(\mathrm{d}^{\alpha}u\right)=\alpha!\left(vu\right)\Big|_{a}^{b}-\int_{a}^{b}u\left(\mathrm{d}^{\alpha}v\right)\end{aligned} \tag{9.150}$$

It is like how in classical calculus we have the following, which is recovered when we place $\alpha=1$

$$\begin{aligned}&\int_{a}^{b}u^{(1)}(x)v(x)\left(\mathrm{d}x\right)=\left(u(x)v(x)\right)\Big|_{a}^{b}-\int_{a}^{b}u(x)v^{(1)}(x)\left(\mathrm{d}x\right)\\ &\int_{a}^{b}v\left(\mathrm{d}u\right)=(vu)\Big|_{a}^{b}-\int_{a}^{b}u\left(\mathrm{d}v\right)\end{aligned} \tag{9.151}$$

9.18 The chain rule for a modified fractional derivative

9.18.1 The derivation of three formulas for the chain rule with a modified fractional derivative

Assume that $f(u)$ is $\alpha-$this differentiable with respect to u and $u(x)$ is differentiable with respect to x, then we have the following:

$$\begin{aligned}\left(f\left(u(x)\right)\right)^{(\alpha)} &= \lim_{\Delta x\downarrow 0}\left(\frac{\Delta^{\alpha}\left(f\left(u(x)\right)\right)}{\left(\Delta x\right)^{\alpha}}\right)\\ &= \lim_{\Delta x\downarrow 0}\left(\frac{\Delta^{\alpha}\left(f\left(u(x)\right)\right)}{\left(\Delta u(x)\right)^{\alpha}}\right)\left(\frac{\left(\Delta u(x)\right)^{\alpha}}{\left(\Delta x\right)^{\alpha}}\right)\\ &= \left(\frac{\mathrm{d}^{\alpha} f(u)}{\mathrm{d}u^{\alpha}}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{\alpha}\end{aligned} \tag{9.152}$$

We write from the above (9.152), the following:

$$\left(f\left(u(x)\right)\right)^{(\alpha)} = \left(f_u^{(\alpha)}\left(u\right)\right)\left(u_x^{(1)}(x)\right)^{\alpha} \tag{9.153}$$

Now we assume that $f(u)$ is differentiable with respect to u and $u(x)$ is $\alpha-$th differentiable w.r.t. x. We have a fractional derivative of $f(u)=u$ as:

$$\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}u^{\alpha}} = \frac{1}{\Gamma\left(2-\alpha\right)}u^{1-\alpha} = \frac{u^{1-\alpha}}{\left(1-\alpha\right)!} \tag{9.154}$$

From the above (9.154) expression, we get a conversion formula such as:

$$\mathrm{d}^{\alpha}u = \left(\left(1-\alpha\right)!\right)^{-1}u^{1-\alpha}\left(\mathrm{d}u\right)^{\alpha} \tag{9.155}$$

Likewise (9.155), we also write:

$$\mathrm{d}^{\alpha} f = \left(\left(1-\alpha\right)!\right)^{-1} f^{1-\alpha}\left(\mathrm{d}\ f\right)^{\alpha} \tag{9.156}$$

This being the case one gets the following steps:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}} &= \frac{\mathrm{d}^{\alpha} f}{\mathrm{d}^{\alpha} u}\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}x^{\alpha}}\\ &= \frac{\left(\left(1-\alpha\right)!\right)f^{1-\alpha}\left(\mathrm{d}\ f\right)^{\alpha}}{\left(\left(1-\alpha\right)!\right)u^{1-\alpha}\left(\mathrm{d}u\right)^{\alpha}}\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}x^{\alpha}} = \left(\frac{f}{u}\right)^{1-\alpha}\left(f_u^{(1)}\right)^{\alpha}u_x^{(\alpha)}(x)\end{aligned} \tag{9.157}$$

For this case, we write the chain rule as the following expression:

$$\left(f\left(u(x)\right)\right)^{(\alpha)} = \left(\frac{f(u(x))}{u(x)}\right)^{1-\alpha}\left(f_u^{(1)}(u(x))\right)^{\alpha}u^{(\alpha)}(x) \tag{9.158}$$

Now we write the chain rule when both are $\alpha-$th differentiable, then we have the following:

$$\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}} = \frac{\mathrm{d}^{\alpha} f}{\mathrm{d}^{\alpha} u}\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}x^{\alpha}} \tag{9.159}$$

Use $\mathrm{d}^{\alpha}u = \left(\left(1-\alpha\right)!\right)^{-1}u^{1-\alpha}\left(\mathrm{d}u\right)^{\alpha}$ for the denominator on RHS of (9.159) and we get the following expression:

$$\begin{aligned}\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}} &= \frac{\mathrm{d}^{\alpha} f}{\mathrm{d}^{\alpha} u}\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}x^{\alpha}} = \frac{\mathrm{d}^{\alpha} f}{\left(\left(1-\alpha\right)!\right)^{-1}u^{1-\alpha}\left(\mathrm{d}u\right)^{\alpha}}\frac{\mathrm{d}^{\alpha} u}{\mathrm{d}x^{\alpha}}\\ &= \left(\left(1-\alpha\right)!\right)u^{\alpha-1}\left(\frac{\mathrm{d}^{\alpha} f}{\left(\mathrm{d}u\right)^{\alpha}}\right)\left(\frac{\mathrm{d}^{\alpha} u}{\left(\mathrm{d}x\right)^{\alpha}}\right)\\ &= \left(\left(1-\alpha\right)!\right)u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(\alpha)}(x)\right)\end{aligned} \tag{9.160}$$

Therefore, in this case we have a chain rule expressed as follows:

$$\left(f\left(u(x)\right)\right)^{(\alpha)}=\left((1-\alpha)!\right)u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(\alpha)}(x)\right)$$
$$f_x^{(\alpha)}\left(u(x)\right)=\left(\Gamma(2-\alpha)\right)u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(\alpha)}(x)\right) \tag{9.161}$$

As a corollary, the $\alpha-$th derivative of x provides a relationship for $\mathrm{d}^\alpha x=\left((1-\alpha)!\right)^{-1}x^{1-\alpha}\left(\mathrm{d}x\right)^\alpha$, or $\left((1-\alpha)!\right)x^{\alpha-1}\left(\mathrm{d}^\alpha x\right)=\left(\mathrm{d}x\right)^\alpha$ which allows us to write successively $\mathrm{d}^\alpha f=f_x^{(\alpha)}\left(\mathrm{d}x\right)^\alpha$. From this we get

$$\mathrm{d}^\alpha f=f_x^{(\alpha)}(x)\left((1-\alpha)!\right)x^{\alpha-1}\mathrm{d}^\alpha x \tag{9.162}$$

Compare this (9.162) with the above (9.161) chain rule.
Thus, we have three arrangements for chain rule formulas as listed below:

$$\text{(i)}\quad \left(f\left(u(x)\right)\right)^{(\alpha)}=\left(f_u^{(\alpha)}\left(u\right)\right)\left(u_x^{(1)}(x)\right)^\alpha$$
$$\text{(ii)}\quad \left(f\left(u(x)\right)\right)^{(\alpha)}=\left(\frac{f(u(x))}{u(x)}\right)^{1-\alpha}\left(f_u^{(1)}(u(x))\right)^\alpha u_x^{(\alpha)}(x) \tag{9.163}$$
$$\text{(iii)}\quad \left(f\left(u(x)\right)\right)^{(\alpha)}=\left((1-\alpha)!\right)u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(\alpha)}(x)\right)$$

Note as we put $\alpha=1$ all the above formulas obtain the form $\left(f\left(u(x)\right)\right)^{(1)}=\left(f_u^{(1)}(u)\right)\left(u_x^{(1)}(x)\right)$. The classical chain rule is therefore recovered.

9.18.2 The application of three formulas for the chain rule with a modified fractional derivative for different cases

The application of these three formulas must be handled with care. The first one (i) of (9.163) is valid when $u(x)$ is differentiable and $f(u)$ is $\alpha-$th differentiable. The second (ii) of (9.163) is applied while $f(u)$ is differentiable and $u(x)$ is $\alpha-$th differentiable. The third formula (iii) of (9.163) applies, when the functions $f(u)$ and $u(x)$ are both $\alpha-$th differentiable.

Let us have $f\left(u(x)\right)=E_\alpha(\lambda x^\alpha)=E_\alpha\left((\lambda^{1/\alpha}x)^\alpha\right)=E_\alpha(u^\alpha)$ with $u=\lambda^{1/\alpha}x$, and see the above chain rules. With a formula of (9.163), i.e. $\mathbf{D}^\alpha\left(f\left(u(x)\right)\right)=\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(1)}\right)^\alpha$ we have:

$$\mathbf{D}^\alpha\left(E_\alpha(\lambda x^\alpha)\right)=\left(E_\alpha(u^\alpha)\right)\left(\lambda^{1/\alpha}\right)^\alpha=\lambda E_\alpha\left(\lambda x^\alpha\right) \tag{9.164}$$

The above (9.164) is a known result. We take $E_\alpha(\lambda(x-x_k)^\alpha)$; $x\ge x_k$. Take $u=\lambda^{1/\alpha}(x-x_k)$ to get $u_x^{(1)}=\lambda^{1/\alpha}$ and $f(u)=E_\alpha(u^\alpha)$ $f_u^{(\alpha)}(u)=E_\alpha(u^\alpha)$. With the formula of (9.163), we write $\mathbf{D}^\alpha\left(E_\alpha(\lambda(x-x_k)^\alpha)\right)=\left(E_\alpha(u^\alpha)\right)\left(\lambda^{1/\alpha}\right)^\alpha=\lambda E_\alpha\left(\lambda(x-x_k)^\alpha\right)$. We construct a fractal function (Section 1.24) as $g(x)=\lim_{N\uparrow\infty}\sum_{k=0}^N E_\alpha(\lambda(x-x_k)^\alpha)$. Using this derived expression, we thus have the following, $\mathbf{D}^\alpha\left(\lim_{N\uparrow\infty}\sum_{k=0}^N E_\alpha(\lambda(x-x_k)^\alpha)\right)=\lambda\left(\lim_{N\uparrow\infty}\sum_{k=0}^N E_\alpha(\lambda(x-x_k)^\alpha)\right)$

That is, we get $g^{(\alpha)}(x)=\lambda g(x)$. This example we had in Section 9.12 is a fractal function of a continuous but nowhere differentiable, but everywhere $\alpha-$ differentiable.

With the formula for (9.163), i.e. $\mathbf{D}^\alpha\left(f\left(u(x)\right)\right)=\left(\frac{f}{u}\right)^{1-\alpha}\left(f_u^{(1)}(u)\right)^\alpha u_x^{(\alpha)}(x)$ we write:

$$\mathbf{D}^\alpha\left(E_\alpha(\lambda x^\alpha)\right)=\left(\frac{E_\alpha(\lambda x^\alpha)}{\lambda^{1/\alpha}x}\right)^{1-\alpha}\left(\mathbf{D}_u\left(E_\alpha(u^\alpha)\right)\right)^\alpha\left(\frac{\lambda^{1/\alpha}x^{1-\alpha}}{(1-\alpha)!}\right)$$
$$=\left(\frac{\lambda}{(1-\alpha)!}\right)\left(E_\alpha(\lambda x^\alpha)\right)^{1-\alpha}\left(\mathbf{D}_u\left(E_\alpha(u^\alpha)\right)\right)^\alpha \tag{9.165}$$

We note that the function $E_\alpha(u^\alpha)$ is not differentiable at $u=0$ w.r.t. u so it fails. (Write the function in series form, do normal differentiation term-by-term and observe that it blows up at $u=0$).

With the formula for (9.163) i.e. $\mathbf{D}^\alpha\left(f\left(u(x)\right)\right)=(1-\alpha)!u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_x^{(\alpha)}(x)\right)$ we write

$$\begin{aligned}\mathbf{D}^\alpha\left(E_\alpha(\lambda x^\alpha)\right)&=(1-\alpha)!\left(\lambda^{1/\alpha}x\right)^{\alpha-1}\left(E_\alpha(\lambda x^\alpha)\right)\left(\frac{\lambda^{1/\alpha}x^{1-\alpha}}{(1-\alpha)!}\right)\\&=\lambda E_\alpha(\lambda x^\alpha)\end{aligned}\tag{9.166}$$

Again, we get the known result. We note that $f(u)=E_\alpha(u^\alpha)$ is not differentiable and $u=\lambda^{1/\alpha}x$ is differentiable as well as $\alpha-$th differentiable.

Similarly this chain rule (9.163) can be applied for a function which is nowhere differentiable as defined in Section 9.11.6 i.e. $f(x)=\lim_{N\uparrow\infty}\sum_{k=1}^{N}\left(x-x_k\right)^{1/2}+b$, with $f(x)=b$ for $x<x_i$ and $f(x)=\sum_{k=1}^{N}\left(x-x_k\right)^{1/2}+b$ for $x\ge x_k$, in the interval $\left[0,1\right]\equiv\left[x_1,x_2,...x_N\right]$. Take $u=(x-x_k)$, this gives $u_x^{(1)}=1$, $f(u)=u^{1/2}$ giving $f_u^{(\alpha)}(u)=\frac{\left(\frac{1}{2}\right)!}{\left(\frac{1}{2}-\alpha\right)!}u^{\left(\frac{1}{2}\right)-(\alpha)}=\left(\frac{1}{2}\right)!$, for $\alpha=\frac{1}{2}$. Thus, we get from formula (i) of (9.163) the result i.e. $f^{(1/2)}(x)=\left(f_u^{(1/2)}(u)\right)\left(u_x^{(1)}\right)^{1/2}=f^{(1/2)}(x)\Big|_{x=x_1,x_2,...x_N}=\left(\frac{1}{2}\right)!$. Note here that $f^{(1/2)}b=0$.

Let $f\left(u(x)\right)=x^{ab}=(x^a)^b=u^b$, with $0<a,b<1$. Take $u=x^a$. According to the application of a standard RL formula, we write:

$$\mathbf{D}_x^\alpha(x^{ab})=\left(\frac{(ab)!}{(ab-\alpha)!}\right)x^{ab-\alpha}\tag{9.167}$$

Since the $f(0)=0$ for this function D^α is the same as $\mathbf{D}^\alpha$ the function is self-similar at $x=0$.
With the first chain rule formula of (9.163) we write:

$$\begin{aligned}\mathbf{D}^\alpha\left(f(u(x))\right)&=\frac{b!}{\left(b-\alpha\right)!}u^{b-\alpha}\left(ax^{a-1}\right)^\alpha\\&=\frac{b!}{\left(b-\alpha\right)!}x^{a(b-\alpha)}a^\alpha x^{a\alpha-\alpha}\\&=\frac{b!a^\alpha}{\left(b-\alpha\right)!}x^{ab-\alpha}\end{aligned}\tag{9.168}$$

With second chain rule formula of (9.163) we get:

$$\begin{aligned}\mathbf{D}^\alpha\left(f(u(x))\right)&=\left(\frac{u^b}{u}\right)^{1-\alpha}\left(bu^{b-1}\right)^\alpha\left(\frac{a!}{\left(a-\alpha\right)!}x^{a-\alpha}\right)\\&=\left(x^{ab-a}\right)^{1-\alpha}b^\alpha\left(x^{a(b-1)}\right)^\alpha\left(\frac{a!}{\left(a-\alpha\right)!}x^{a-\alpha}\right)\\&=\frac{b^a(a!)}{\left(a-\alpha\right)!}x^{ab-\alpha}\end{aligned}\tag{9.169}$$

With the third chain rule formula of (9.163) we write:

$$\begin{aligned}\mathbf{D}^\alpha\left(f(u(x))\right)&=(1-\alpha)!x^{a(\alpha-1)}\frac{b!}{\left(b-\alpha\right)!}(x^a)^{b-\alpha}\frac{a!}{\left(a-\alpha\right)!}x^{a-\alpha}\\&=\frac{(1-\alpha)!a!b!}{(a-\alpha)!(b-\alpha)!}x^{ab-\alpha}\end{aligned}\tag{9.170}$$

We note that x^a and u^b are both non-differentiable, hence the chain rule formula first and second are disqualified. Therefore, the fractional differentiation with the RL formula (Euler rule) and the third chain rule formula of (9.163) are applicable. The results in (9.167) and (9.170) are different since different differential increments are involved in these two functions, though they are the same.

9.19 Coarse grained system

9.19.1 Need for coarse graining

In classical calculus, everything happens when an elemental point (the differential element say Δt) is made infinitesimally small i.e. made zero. Thereby we take the rate for a function $x(t)$ with respect to this element, and classically the term for this rate as $\Delta x / \Delta t$, in the limit as $\Delta t \downarrow 0$; that is a normal differentiation $\mathrm{d}x / \mathrm{d}t$. In a coarse grained system (or phenomena), everything happens as if this elemental point or the differential element Δt is having some spread (or thickness), and is non-zero. This can be viewed as a fractional differential quantity $(\Delta t)^{\alpha}$ with $0 < \alpha < 1$; so $(\Delta t)^{\alpha} > \Delta t$ as we take the limit $\Delta t \downarrow 0$. This gives us the rate as per the unit of this fractional differential i.e. $\Delta x /(\Delta t)^{\alpha}$, suggesting that the usage of a fractional derivative i.e. $\mathrm{d}x /(\mathrm{d}t)^{\alpha}$.

We have coarse graining in a space variable x and or a time variable t, to have a fractional differential as $(\mathrm{d}x)^{\alpha}$ or $(\mathrm{d}t)^{\alpha}$, for $0 < \alpha < 1$.

9.19.2 The fractional velocity u_α with coarse grained time differential

With a coarse-grained time (CGT) we write velocity as the following relationship:

$$u_\alpha = \frac{\mathrm{d}x}{\left(\mathrm{d}t\right)^{\alpha}} \tag{9.171}$$

According to the fractional Taylor series approximation we have the following conversion expressions:

$$\mathrm{d}x = \frac{1}{\alpha!} x^{(\alpha)} (\mathrm{d}t)^{\alpha} \quad , \quad \mathrm{d}x = \left(\alpha!\right)^{-1} \frac{\mathrm{d}^{\alpha} x}{\left(\mathrm{d}t\right)^{\alpha}} \left(\mathrm{d}t\right)^{\alpha} \tag{9.172}$$

Therefore, for CGT we get the following:

$$\begin{aligned} u_\alpha(t) &= \frac{\mathrm{d}x}{\left(\mathrm{d}t\right)^{\alpha}} \\ &= \left(\alpha!\right)^{-1} \frac{\mathrm{d}^{\alpha} x}{\left(\mathrm{d}t\right)^{\alpha}} \\ &= \left(\alpha!\right)^{-1} \left(x^{(\alpha)}(t)\right) \qquad 0 < \alpha < 1, \quad \mathrm{d}t > 0 \end{aligned} \tag{9.173}$$

9.19.3 Fractional velocity v_α with a coarse grained space differential

For a coarse grained space (CGS) variable, we write:

$$v_\alpha(t) = \frac{\left(\mathrm{d}x\right)^{\alpha}}{\mathrm{d}t} \qquad \mathrm{d}x > 0, \quad 0 < \alpha < 1 \tag{9.174}$$

Given a function $y = f(x)$ and its inverse $x = g(y)$, their fractional derivative of order α, with $0 < \alpha < 1$ satisfy the following condition

$$y^{(\alpha)}(x) x^{(\alpha)}(y) = \left((1-\alpha)!\right)^{-2} \left(xy\right)^{1-\alpha} \tag{9.175}$$

The proof is given in the following steps:

$$\begin{aligned}
y^{(\alpha)}(x)x^{(\alpha)}(y) &= \left(\frac{\mathrm{d}^{\alpha} y}{\mathrm{d}x^{\alpha}}\right)\left(\frac{\mathrm{d}^{\alpha} x}{\mathrm{d}y^{\alpha}}\right) \\
&= \left(\frac{\mathrm{d}^{\alpha} y}{\mathrm{d}y^{\alpha}}\right)\left(\frac{\mathrm{d}^{\alpha} x}{\mathrm{d}x^{\alpha}}\right) \\
&= \left(\frac{1}{(1-\alpha)!} y^{1-\alpha}\right)\left(\frac{1}{(1-\alpha)!} x^{1-\alpha}\right) \\
&= \frac{\left(xy\right)^{1-\alpha}}{\left((1-\alpha)!\right)^{2}}
\end{aligned} \tag{9.176}$$

Therefore, for CGS we have:

$$\begin{aligned}
v_{\alpha}(t) &= \frac{\left(\mathrm{d}x\right)^{\alpha}}{\mathrm{d}t} = \frac{\alpha!\left(\mathrm{d}x\right)^{\alpha}}{\alpha!\left(\mathrm{d}t\right)} \\
&= \alpha!\frac{\left(\mathrm{d}x\right)^{\alpha}}{\mathrm{d}^{\alpha} t}, \qquad \left(t^{(\alpha)}(x)\right) = \frac{\mathrm{d}^{\alpha} t(x)}{\mathrm{d}x^{\alpha}} \\
&= \frac{\alpha!}{\left(t^{(\alpha)}(x)\right)}
\end{aligned} \tag{9.177}$$

The above (9.177) steps express the quantity i.e. $v_{\alpha}(t)$, in terms of the fractional derivative of the time variable i.e. $t^{(\alpha)}(x)$ w.r.t. space variable i.e.

$$t^{(\alpha)}(x) = \frac{\mathrm{d}^{\alpha} t(x)}{\mathrm{d}x^{\alpha}} \tag{9.178}$$

Using $y^{(\alpha)}(x)x^{(\alpha)}(y) = \left((1-\alpha)!\right)^{-2}\left(xy\right)^{1-\alpha}$ (9.175) and manipulating the above (9.177) steps we get the following:

$$\begin{aligned}
v_{\alpha}(t) &= \frac{\alpha!}{\left(t^{(\alpha)}(x)\right)} = \frac{\alpha!\left(x^{(\alpha)}(t)\right)}{\left(x^{(\alpha)}(t)\right)\left(t^{(\alpha)}(x)\right)} \\
&= \frac{\alpha!\left(x^{(\alpha)}(t)\right)}{\left((1-\alpha)!\right)^{-2}\left(xt\right)^{1-\alpha}} = \left(\alpha!\right)\left((1-\alpha)!\right)^{2}(xt)^{\alpha-1}\left(x^{(\alpha)}(t)\right) \\
&= k_{\alpha}\left(xt\right)^{\alpha-1}\left(x^{(\alpha)}(t)\right) \qquad k_{\alpha} = \alpha!\left((1-\alpha)!\right)^{2}
\end{aligned} \tag{9.179}$$

We have fractional velocities described as:

$$\begin{aligned}
u_{\alpha}(t) &= \left(\alpha!\right)^{-1}\left(x^{(\alpha)}(t)\right), \quad 0<\alpha<1, \quad \mathrm{d}t>0 \\
v_{\alpha}(t) &= k_{\alpha}\left(xt\right)^{\alpha-1}\left(x^{(\alpha)}(t)\right), \quad k_{\alpha} = \alpha!\left((1-\alpha)!\right)^{2}
\end{aligned} \tag{9.180}$$

for CGT and CGS cases respectively.

9.20 The solution for a fractional differential equation with a modified fractional derivative

9.20.1 Fractional differential equation defined with modified fractional derivative

We take the fractional differential equation:

$$x^{(\alpha)}(t) = b(t), \quad x(0) = x_0 \tag{9.181}$$

That is (9.181) with a modified RL fractional derivative:

$$\mathbf{D}_t^{\alpha} x(t) = \frac{\mathrm{d}^{\alpha} x}{\mathrm{d}t^{\alpha}} = b(t) \qquad \mathrm{d}^{\alpha} x = b(t)\left(\mathrm{d}t\right)^{\alpha} \tag{9.182}$$

One successively has the following:

$$\int_{x_0}^{x(t)} \mathrm{d}^{\alpha}\xi = x(t) - x_0 = \int_0^t b(\tau)\left(\mathrm{d}\tau\right)^{\alpha} \tag{9.183}$$

Therefore we write from the definition of integration with respect to a fractional differential (9.124) i.e. $\int_0^t b(\tau)\left(\mathrm{d}\tau\right)^\alpha = \alpha\int_0^t (t-\tau)^{\alpha-1} b(\tau)\mathrm{d}\tau$, the following solution:

$$x(t) = x_0 + \alpha\int_0^t (t-\tau)^{\alpha-1} b(\tau)\mathrm{d}\tau \tag{9.184}$$

We take the fractional differential equation with a modified RL fractional derivative as:

$$x^{(\alpha)}(t) = a(t)x(t) + b(t) \qquad x(0) = x_0 \tag{9.185}$$

We will proceed as we do in the case of a classical calculus in following section.

9.20.2 Defining the function $Ln_\alpha(x)$ in conjugation to $\ln x$ and obtaining the solution to a fractional differential equation with a modified fractional derivative

The solution to the homogeneous equation in classical calculus $x^{(1)}(t) = \left(a(t)\right)\left(x(t)\right)$ is given as $x(t) = C\exp\left(\int_0^t \left(a(\tau)\right)\mathrm{d}\tau\right)$. This comes from:

$$\begin{aligned}
&x^{(1)}(t) = a(t)x(t) \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x(t)\\
&\int_{x_0}^{x(t)}\left(\frac{\mathrm{d}\xi(t)}{\xi(t)}\right) = \int_0^t \left(a(\tau)\right)\mathrm{d}\tau\\
&\ln x(t) = \ln x_0 + \int_0^t \left(a(\tau)\right)\mathrm{d}\tau\\
&x(t) = x_0 \exp\int_0^t \left(a(\tau)\right)\mathrm{d}\tau
\end{aligned} \tag{9.186}$$

In conjugation with an exponential function, $\exp(x)$ we have seen that we have the Mittag-Leffler function $E_\alpha(x^\alpha)$ in fractional calculus. Similarly, the fractional logarithm function $Ln_\alpha x$ may be defined as follows:

$$\int\frac{\mathrm{d}^\alpha x}{x} = Ln_\alpha\left(\frac{x}{C}\right) \qquad x = E_\alpha\left(Ln_\alpha x\right) \qquad \left(\frac{x}{C}\right) > 0 \tag{9.187}$$

We note that $Ln_\alpha x$ should not be confused as a usual logarithm to the base α . With the modified RL derivative we have the property $\mathbf{D}^\alpha Ln_\alpha x = \alpha^\alpha\left((1-\alpha)!\right)^{-2} x^{-\alpha}$, for $0<\alpha<1$ (we prove this in Section 9.20.3). With the above (9.187) definition of $Ln_\alpha x$ we write the solution for $x^{(\alpha)}(t) = a(t)x(t)$ as done to the above changes to the following:

$$\begin{aligned}
&x^{(\alpha)}(t) = a(t)x(t) \qquad \frac{\mathrm{d}^\alpha x(t)}{(\mathrm{d}t)^\alpha} = a(t)x(t)\\
&\int_{x_0}^{x(t)}\left(\frac{\mathrm{d}^\alpha\xi(t)}{\xi(t)}\right) = \int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\\
&Ln_\alpha\left(\frac{x(t)}{x_0}\right) = \int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\\
&E_\alpha\left(Ln_\alpha\left(\tfrac{x(t)}{x_0}\right)\right) = E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\\
&\frac{x(t)}{x_0} = E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\\
&x(t) = x_0 E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)
\end{aligned} \tag{9.188}$$

Some properties of a fractional logarithm function $Ln_\alpha x$ are produced in Appendix D, where we can see by making $\alpha = 1$ we get the standard properties of a natural logarithm function. We note that as we said for $E_\alpha(x)$ that is a fractal function, the same is $Ln_\alpha x$ the non-differentiable type, where the modified RL fractional derivative Jumarie type applies.

The solution of the homogeneous equation $x^{(\alpha)}(t) = a(t)x(t)$ therefore could be as follows:

$$x(t) = C\left(E_\alpha \left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha \right)\right) \tag{9.189}$$

We now do an α derivative of $E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)$ by using the chain rule (9.163) number three (iii) that is as following:

$$\left(f\left(u(t)\right)\right)^{(\alpha)} = \left((1-\alpha)!\right)u^{\alpha-1}\left(f_u^{(\alpha)}(u)\right)\left(u_t^{(\alpha)}(t)\right)$$

$$\begin{aligned}&\left(E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{(\alpha)}\\ &= \left((1-\alpha)!\right)\left(\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{\alpha-1}\left(E_\alpha(u)\right)_u^{(\alpha)}\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)_t^{(\alpha)}\end{aligned} \tag{9.190}$$

We note that $\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)_t^{(\alpha)} = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left[\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right] = \alpha!\left(a(t)\right)$ and with the formula (9.102) obtained earlier i.e. $\left(E_\alpha(\lambda x)\right)_x^{(\alpha)} = \lambda\alpha^{-\alpha}x^{1-\alpha}E_\alpha\left(\lambda x\right)$ we proceed as follows:

$$\begin{aligned}x^{(\alpha)}(t) &= \left(CE_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{(\alpha)}\\
&= C\left(1-\alpha\right)!\left(\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{\alpha-1}\left(E_\alpha(u)\right)_u^{(\alpha)}\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)_t^{(\alpha)}\\
&= C\left(1-\alpha\right)!\left(\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{\alpha-1}\begin{pmatrix}\alpha^{-\alpha}\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)^{1-\alpha}\\ E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\end{pmatrix}\left(\alpha!a(t)\right)\\
&= C(1-\alpha)!\left(\alpha\right)^{-\alpha}\alpha!a(t)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\\
&= \left((1-\alpha)!\left(\alpha\right)^{-\alpha}\alpha!\right)a(t)CE_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\\
&= C_1a(t)x(t) \qquad C_1 = \left((1-\alpha)!\left(\alpha\right)^{-\alpha}\alpha!\right)\end{aligned} \tag{9.191}$$

The above (9.191) steps also indicate that $x(t) = C\left(E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)$ is a possible solution to the equation $x^{(\alpha)}(t) = a(t)x(t)$ (with adjustments of constants). This being the case let us look for a special solution of the complete equation in the form i.e.

$$x(t) = C(t)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \tag{9.192}$$

Use formula (9.140) i.e. $\left(u(x)v(x)\right)^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x)$ to write the following:

$$
\begin{aligned}
x^{(\alpha)}(t) &= \left(C^{(\alpha)}(t)\right)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \\
&\quad + \left(C_\alpha(t)\right)\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \\
&= \left(C^{(\alpha)}(t)\right)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \\
&\quad + \left(C_\alpha(t)\right)a(t)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \\
&= b(t) + a(t)x(t)
\end{aligned}
\tag{9.193}
$$

We write $C_\alpha = (1-\alpha)!\left(\alpha\right)^{-\alpha}\alpha!C(t)$ while doing $\left(C(t)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)^{(\alpha)}$ in the above (9.193) steps by using the chain rule. Use the above (9.193) expression in the given equation i.e. $x^{(\alpha)}(t) = a(t)x(t) + b(t)$ and write the following:

$$
\begin{aligned}
\left(C^{(\alpha)}(t)\right)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) &= b(t) \\
\left(C^{(\alpha)}(t)\right) &= \frac{b(t)}{E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)} \\
\left(C^{(\alpha)}(t)\right) &= \left(b(t)\right)\left(E_\alpha^{-1}\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right)\right)
\end{aligned}
\tag{9.194}
$$

We have used the following representation:

$$
\frac{1}{E_\alpha(x)} = E_\alpha^{-1}(x) \tag{9.195}
$$

Integrating (9.194) by recognizing $C^{(\alpha)}(t) = \frac{\mathrm{d}^\alpha C(t)}{(\mathrm{d}t)^\alpha}$ we have the following:

$$
C(t) = \int_0^t \left(b(u)\right)\left(E_\alpha^{-1}\left(\int_0^t a(\tau)(\mathrm{d}\tau)^\alpha\right)\right)\left(\mathrm{d}u\right)^\alpha \tag{9.196}
$$

The general solution we get is:

$$
\begin{aligned}
x(t) &= \left(x_0 + C(t)\right)E_\alpha\left(\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^\alpha\right) \\
C(t) &= \int_0^t \left(b(u)\right)E_\alpha^{-1}\left(\int_0^t a(\tau)(\mathrm{d}\tau)^\alpha\right)\left(\mathrm{d}u\right)^\alpha
\end{aligned}
\tag{9.197}
$$

A fractional differential equation with a constant coefficient we write it as:

$$
x^{(\alpha)}(t) = ax(t) + b \qquad x(0) = x_0 \tag{9.198}
$$

Using the above (9.197) result, we get:

$$
\begin{aligned}
x(t) &= \left(x_0 + \int_0^t bE_\alpha^{-1}\left(\int_0^t a\left(\mathrm{d}\tau\right)^\alpha\right)\left(\mathrm{d}u\right)^\alpha\right)E_\alpha\left(\int_0^t a\left(\mathrm{d}\tau\right)^\alpha\right) \\
&= \left(x_0 + b\int_0^t E_\alpha^{-1}\left(at^\alpha\right)\left(\mathrm{d}u\right)^\alpha\right)E_\alpha\left(at^\alpha\right) \\
&= x_0E_\alpha\left(at^\alpha\right) + bE_\alpha\left(at^\alpha\right)\int_0^t E_\alpha^{-1}\left(at^\alpha\right)\left(\mathrm{d}u\right)^\alpha
\end{aligned}
\tag{9.199}
$$

We simplify the second term of (9.199) as follows:

$$
\begin{aligned}
bE_\alpha\left(at^\alpha\right)\int_0^t E_\alpha^{-1}\left(a\tau^\alpha\right)\left(\mathrm{d}\tau\right)^\alpha &= bE_\alpha\left(at^\alpha\right)\int_0^t\left(\frac{1}{E_\alpha\left(a\tau^\alpha\right)}\right)\left(\mathrm{d}\tau\right)^\alpha \\
&= bE_\alpha\left(at^\alpha\right)\int_0^t\left(E_\alpha\left(a(-\tau^\alpha)\right)\right)\left(\mathrm{d}\tau\right)^\alpha \\
&= b\int_0^t E_\alpha\left(a(t-\tau)^\alpha\right)\left(\mathrm{d}\tau\right)^\alpha \\
&= -\frac{b}{a}\Big[E_\alpha\left(a(t-\tau)^\alpha\right)\Big]_{\tau=0}^{\tau=t} = \frac{b}{a}\left(E_\alpha\left(at^\alpha\right)-1\right)
\end{aligned}
\tag{9.200}
$$

In the above (9.200) we are assuming that $\left(E_\alpha(ax^\alpha)\right)\left(E_\alpha(ay^\alpha)\right)=E_\alpha\left(a(x+y)^\alpha\right)$ and its inverse as $E_\alpha^{-1}(ax^\alpha)=E_\alpha(a(-x)^\alpha)$. The proofs of these assumptions are mentioned in Section 9.20.3. Now combining them with the term $x_0E_\alpha\left(at^\alpha\right)$, we write it in compact form:

$$
x(t)=\left(x_0+\tfrac{b}{a}\right)E_\alpha\left(at^\alpha\right)-\left(\tfrac{b}{a}\right) \tag{9.201}
$$

9.20.3 Proof of the identity $\left(E_\alpha(ax^\alpha)\right)\left(E_\alpha(ay^\alpha)\right)\geq E_\alpha\left(a(x+y)^\alpha\right)$ in respect of a modified fractional derivative definition

We know from our classical calculus, that a differentiable solution of the functional equation $f(x+y)=f(x)f(y)$ is the exponential function i.e. $f(x)\equiv e^x$, and is a unique one. We have developed in this chapter fractional calculus via the fractional difference of non-differentiable functions. Therefore, the solution of the functional equation $f(x+y)=f(x)f(y)$ is exactly defined as the solution for a linear fractional differential equation.

Consider the functional equation $f(x+y)=f(x)f(y)$, for $x>0$ and $y>0$ where f is real valued function. If one assumes $f(x)$ is not differentiable but is $\alpha-$differentiable where $0<\alpha<1$ for every x, then it is the solution of $f^{(\alpha)}(x)=\left(f^{(\alpha)}(0)\right)f(x)$ for which the solution is $f(x)=E_\alpha\left(\left(f^{(\alpha)}(0)\right)x^\alpha\right)$. This we have described in Section 9.9.2. Assuming f is $\alpha-$differentiable, we apply the chain rule formula (9.163) (iii) to the functional equation $f(x+y)=f(x)f(y)$, with $x+y=u$. We perform a first fractional differentiation w.r.t x, and then w.r.t y for this functional equation. We write the following steps:

$$
\begin{aligned}
f^{(\alpha)}(u)(1)^\alpha &= f^{(\alpha)}(x)f(y); \quad u=x+y \\
f^{(\alpha)}(u)(1)^\alpha &= f(x)f^{(\alpha)}(y): \quad u=x+y
\end{aligned}
\tag{9.202}
$$

From the above (9.202) we have $f^{(\alpha)}(x)f(y)=f(x)f^{(\alpha)}(y)$, and we have the following ratio as a constant λ:

$$
\frac{f^{(\alpha)}(x)}{f(x)}=\frac{f^{(\alpha)}(y)}{f(y)}=\lambda \tag{9.203}
$$

From the above (9.203) we obtain a fractional differential equation $f^{(\alpha)}(x)=\lambda f(x)$, which is exactly $f(x)=E_\alpha\left(f^{(\alpha)}(0)x^\alpha\right)$. We note that $x=0$ is a non-differentiable point in $f(x)$. Thus, we have the function f as the Mittag-Leffler function i.e. $f\equiv E_\alpha$.

Therefore, we write $f(x)=E_\alpha(\lambda x^\alpha)$, $f(y)=E_\alpha(\lambda y^\alpha)$ thus $f(x+y)=E_\alpha\left(\lambda(x+y)^\alpha\right)$, and our identity in the conjugation to a classical exponential functions, for the non-differentiable function E_α as $E_\alpha\left(a(x+y)^\alpha\right)=E_\alpha(ax^\alpha)E_\alpha(ay^\alpha)$.

We may now prove that $f^{(\alpha)}(x)=\lambda x$ satisfies $f(x+y)=f(x)f(y)$. The equation $f^{(\alpha)}(x)=\lambda x$ provides us with the following:

$$\begin{aligned} f^{(2\alpha)}(x) &= f^{(\alpha)}\left(f^{(\alpha)}(x)\right), \quad f^{(\alpha)}(x) = \lambda f(x) \\ &= f^{(\alpha)}\left(\lambda f(x)\right) = \lambda f^{(\alpha)}(x) \\ &= \lambda^2 f(x) \end{aligned} \tag{9.204}$$

From (9.204) we have $f^{(k\alpha)}(x) = \lambda^k f(x)$; $k = 0,1,2,3,\dots$. The fractional Taylor series (9.108) for non-differentiable functions yields the following by making $h \leftarrow y$ in the expansion $f(x+h)$, considering the non-differentiability in x:

$$\begin{aligned} f(x+h) &= f(x) + \sum_{k=1}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad y \leftarrow h \\ f(x+y) &= f(x) + \sum_{k=1}^{\infty} \frac{y^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x); \qquad f^{(\alpha k)}(x) = \lambda^k f(x) \\ &= f(x) + \sum_{k=1}^{\infty} \frac{\lambda^k y^{\alpha k} f(x)}{(\alpha k)!} \\ &= f(x)\left(\sum_{k=0}^{\infty} \frac{\lambda^k y^{\alpha k}}{(\alpha k)!}\right) = f(x) f(y), \quad f(y) = \sum_{k=0}^{\infty} \frac{\lambda^k y^{\alpha k}}{(\alpha k)!} = E_\alpha(\lambda y^\alpha) \end{aligned} \tag{9.205}$$

Ina similar way with (9.205), we write $f(x+y) = f(y)\left(\sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{(\alpha k)!}\right) = f(y) f(x)$ by using the fractional Taylor series $f(x+h)$ with $x \leftarrow h$ and then using $f(x) = \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{(\alpha k)!} = E_\alpha(\lambda x^\alpha)$. This (9.205) says that $f(z)$ is $E_\alpha(\lambda z^\alpha)$ or the functional equation $f(x+y) = f(x) f(y)$. Thus we have $f(x) = E_\alpha(\lambda x^\alpha)$, $f(y) = E_\alpha(\lambda y^\alpha)$ and $f(x+y) = E_\alpha\left(\lambda (x+y)^\alpha\right)$.

Now we verify the identity of $E_\alpha\left(a(x+y)^\alpha\right) = E_\alpha(ax^\alpha) E_\alpha(ay^\alpha)$. We apply the fractional Taylor series to $E_\alpha\left(a(x+y)^\alpha\right)$ with $y \leftarrow h$ as follows:

$$\begin{aligned} f(x+h) &= f(x) + \sum_{k=1}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} \mathbf{D}^{\alpha k} f(x), \quad y \leftarrow h \\ E_\alpha\left(a(x+y)^\alpha\right) &= E_\alpha(ax^\alpha) + \sum_{k=1}^{\infty} \frac{y^{\alpha k}}{(\alpha k)!} \mathbf{D}^{\alpha k} E_\alpha(ax^\alpha) \qquad \mathbf{D}^{\alpha k} E_\alpha(ax^\alpha) = a^k E_\alpha(ax^\alpha) \\ &= E_\alpha(ax^\alpha) + \sum_{k=1}^{\infty} \frac{a^k y^{\alpha k} E_\alpha(ax^\alpha)}{(\alpha k)!} \\ &= E_\alpha(ax^\alpha)\left(\sum_{k=0}^{\infty} \frac{a^k y^{\alpha k}}{(\alpha k)!}\right) = E_\alpha(ax^\alpha) E_\alpha(ay^\alpha), \quad \sum_{k=0}^{\infty} \frac{a^k y^{\alpha k}}{(\alpha k)!} = E_\alpha(ay^\alpha) \end{aligned} \tag{9.206}$$

In our identity $E_\alpha\left(a(x+y)^\alpha\right) = E_\alpha(ax^\alpha) E_\alpha(ay^\alpha)$, we have considered positive values for x and y.

However x and y may have negative values with some conditions; as for general negative values x^α and y^α may be complex numbers depending on α e.g. for $\alpha = \frac{1}{2}$. In order to have this result $E_\alpha\left(a(x+y)^\alpha\right) = E_\alpha(ax^\alpha) E_\alpha(ay^\alpha)$ for negative values it is better that we restrict α to $1/3, 1/5, 1/7, 1/9, \dots$ i.e. $\alpha = \frac{1}{2n+1}$ with n as integers. Thus for these values, we can write $E_\alpha\left((-x)^\alpha\right) = E_\alpha(-x^\alpha)$ for $\alpha = 1/(2n+1)$; and as a result, we get:

$$
\begin{aligned}
&E_\alpha\left((x+y)^\alpha\right) = E_\alpha(x^\alpha)E_\alpha(y^\alpha); \quad y=-x \\
&E_\alpha\left((x-x)^\alpha\right) = E_\alpha(0) = 1 \\
&E_\alpha\left((x-x)^\alpha\right) = E_\alpha\left(\left(x+(-x)\right)^\alpha\right) \\
&\qquad = E_\alpha(x^\alpha)E_\alpha((-x)^\alpha); \quad \alpha = \frac{1}{2n+1} \\
&\qquad = E_\alpha(x^\alpha)E_\alpha(-x^\alpha)
\end{aligned} \tag{9.207}
$$

From (9.207) we obtain $E_\alpha(x^\alpha)E_\alpha(-x^\alpha)=1$ that gives:

$$
E_\alpha(-x^\alpha) = \frac{1}{E_\alpha(x^\alpha)} = E_\alpha^{-1}(x^\alpha); \quad \alpha = \frac{1}{2n+1} \tag{9.208}
$$

We note here that $f\left((x+y)^\alpha\right) = f(x^\alpha)f(y^\alpha)$ applies to non-differentiable functions only, (with $f(u^\alpha)=E_\alpha(u^\alpha)$) or to functions defined on a 'fractal set'. Otherwise the continuous set of definitions we would have is $f\left((x+y)^\alpha\right) \le f(x^\alpha)f(y^\alpha)$.

As with classical calculus we have an exponential function i.e. $y=f(x)=e^x$ corresponding to $x=g(y)=\ln y$; similarly for non-differentiable functions with the modified RL fractional derivative we have $y=E_\alpha(x^\alpha)=f(x)$ and the corresponding inverse function is $x=\left(Ln_\alpha y\right)^{1/\alpha}=g(y)$. From derivation (9.176), we have $y^{(\alpha)}(x)x^{\alpha}(y)=\frac{1}{((1-\alpha)!)^2}(xy)^{1-\alpha}$. Using this and $y=E_\alpha(x^\alpha)=f(x)$; and $x=\left(Ln_\alpha y\right)^{1/\alpha}=g(y)$, we may write the following:

$$
\begin{aligned}
&f^{(\alpha)}(x)g^{(\alpha)}(y) = \frac{1}{\left((1-\alpha)!\right)^2}(xy)^{1-\alpha} \\
&g^{(\alpha)}(y) = \frac{1}{\left((1-\alpha)!\right)^2}\frac{(xy)^{1-\alpha}}{f^{(\alpha)}(x)}; \qquad f^{(\alpha)}(x) = E_\alpha(x^\alpha) = y \\
&\qquad = \frac{1}{\left((1-\alpha)!\right)^2}\frac{(xy)^{1-\alpha}}{y} = \frac{1}{\left((1-\alpha)!\right)^2}\frac{y^{1-\alpha}x^{1-\alpha}}{y}, \quad x=\left(Ln_\alpha y\right)^{1/\alpha} \\
&\qquad = \frac{1}{\left((1-\alpha)!\right)^2}\frac{\left(Ln_\alpha y\right)^{(1-\alpha)/\alpha}}{y^\alpha}
\end{aligned} \tag{9.209}
$$

On the other hand, by use of the chain rule formula (ii) of (9.163) on $g(y)=\left(Ln_\alpha y\right)^{1/\alpha}$ i.e. $f_x^{(\alpha)}\left(u(x)\right)=\left(\frac{f}{u}\right)^{1-\alpha}\left(f_u^{(1)}(u)\right)^\alpha u_x^{(\alpha)}$, by taking $f(u)=u^{1/\alpha}$, $u=Ln_\alpha y$ we write the following expression

$$
\begin{aligned}
&g^{(\alpha)}(y) = \mathbf{D}^\alpha\left(Ln_\alpha y\right)^{1/\alpha} = \left(\frac{f(u)}{u}\right)^{1-\alpha}\left(f_u^{(1)}(u)\right)^\alpha u_y^{(\alpha)}, \qquad f(u)=u^{1/\alpha}, \quad u = Ln_\alpha y \\
&\qquad = \left(\frac{u^{1/\alpha}}{u}\right)^{1-\alpha}\left(\frac{1}{\alpha}u^{\left(\frac{1}{\alpha}\right)-1}\right)^\alpha\left(\mathbf{D}^\alpha Ln_\alpha y\right) = \frac{1}{\alpha^\alpha}\left(u^{\left(\frac{1}{\alpha}\right)-1}\right)^{1-\alpha}\left(u^{\left(\frac{1}{\alpha}\right)-1}\right)^\alpha\left(\mathbf{D}^\alpha Ln_\alpha y\right) \\
&\qquad = \frac{1}{\alpha^\alpha}u^{\left(\frac{1}{\alpha}\right)-1}\left(\mathbf{D}^\alpha Ln_\alpha y\right) = \frac{1}{\alpha^\alpha}\left(Ln_\alpha y\right)^{\left(\frac{1}{\alpha}\right)-1}\left(\mathbf{D}^\alpha Ln_\alpha y\right)
\end{aligned} \tag{9.210}
$$

Equating (9.209) and (9.210), we get:

$$\frac{1}{\alpha^{\alpha}}\left(Ln_{\alpha}y\right)^{\left(\frac{1}{\alpha}\right)-1}\left(\mathbf{D}^{\alpha}Ln_{\alpha}y\right)=\frac{1}{\left((1-\alpha)!\right)^{2}}\frac{\left(Ln_{\alpha}y\right)^{(1-\alpha)/\alpha}}{y^{\alpha}}$$
$$\mathbf{D}^{\alpha}Ln_{\alpha}y=\frac{\alpha^{\alpha}}{\left((1-\alpha)!\right)^{2}}y^{-\alpha};\qquad 0<\alpha<1 \tag{9.211}$$

The functional equation $f(x+y)=f(x)f(y)$ has an inverse equation as $g(xy)=g(x)+g(y)$, for $x>0$ and $y>0$. In classical calculus $g(u)=\ln u$ is the solution of this inverse functional equation $g(xy)=g(x)+g(y)$. Now we assume g is a non-differentiable but is $\alpha-$differentiable with $0<\alpha<1$; then $g(u)=Ln_{\alpha}u$.

Let us take an $\alpha-$derivative of $g(xy)=g(x)+g(y)$. With the chain rule formula (i) of (9.163), making $xy=u$ w.r.t. x we get $g^{(\alpha)}(u)y^{\alpha}=g^{(\alpha)}(x)$ and w.r.t y we get $g^{(\alpha)}(u)x^{\alpha}=g^{(\alpha)}(y)$. From these two expressions, we obtain $\left(g^{(\alpha)}(x)\right)y^{-\alpha}=\left(g^{(\alpha)}(y)\right)x^{-\alpha}=g^{(\alpha)}(u)$. This gives us the equation $\left(g^{(\alpha)}(x)\right)x^{\alpha}=\left(g^{(\alpha)}(y)\right)y^{\alpha}=c$, where c is a constant. We have modified the RL derivative of $Ln_{\alpha}x$ as $\mathbf{D}^{\alpha}Ln_{\alpha}x=\alpha^{\alpha}\left((1-\alpha)!\right)^{-2}x^{-\alpha}$ (9.211), comparing it with this we get $\left(g^{(\alpha)}(x)\right)x^{\alpha}=c$, $g(x)=Ln_{\alpha}x$.

We note here that $\left(g(xy)\right)^{1/\alpha}=\left(g(x)\right)^{1/\alpha}+\left(g(y)\right)^{1/\alpha}$ applies to non-differentiable functions only, (with $g(u)=Ln_{\alpha}u$) or to the functions defined on the 'fractal set'. Otherwise, through a continuous set of definitions we would have $\left(g(xy)\right)^{1/\alpha}\geq\left(g(x)\right)^{1/\alpha}+\left(g(y)\right)^{1/\alpha}$. We have expression for $g(x)=Ln_{\alpha}x$ as $\left(Ln_{\alpha}(xy)\right)^{1/\alpha}=\left(Ln_{\alpha}(x)\right)^{1/\alpha}+\left(Ln_{\alpha}(y)\right)^{1/\alpha}$ (Appendix D).

9.20.4 The fractional differential equation $x^{(\alpha)}(t)=\lambda\left(a(t)\right)x(t)+b(t)$ with a modified fractional derivative

For a fractional differential equation:

$$x^{(\alpha)}(t)=\lambda\left(a(t)\right)x(t)+b(t) \tag{9.212}$$

With $x(0)=x_0$ from the solution as obtained above (9.197) we write the following:

$$x(t)=\left(x_0+C(t)\right)E_{\alpha}\left(\lambda\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^{\alpha}\right)$$
$$C(t)=\int_0^t b(u)E_{\alpha}^{-1}\left(\lambda\int_0^t a(\tau)(\mathrm{d}\tau)^{\alpha}\right)\left(\mathrm{d}u\right)^{\alpha} \tag{9.213}$$

In this case with $b(t)=0$ and we have for:

$$x^{(\alpha)}(t)=\lambda\left(a(t)\right)x(t) \tag{9.214}$$

The solution is as:

$$x(t)=\left(x_0+\int_0^t b(u)E_{\alpha}^{-1}\left(\lambda\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^{\alpha}\right)\left(\mathrm{d}u\right)^{\alpha}\right)E_{\alpha}\left(\lambda\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^{\alpha}\right)$$
$$=x_0E_{\alpha}\left(\lambda\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^{\alpha}\right) \tag{9.215}$$

9.20.5 The fractional differential equation $x^{(\alpha)}(t)=\lambda t^{1-\alpha}x(t)$ with a modified fractional derivative

As a special case for (9.212) when $a(t)=t^{1-\alpha}$ we have the following:

$$\begin{aligned}\int_0^t a(\tau)\left(\mathrm{d}\tau\right)^{\alpha}=\int_0^t \tau^{1-\alpha}\left(\mathrm{d}\tau\right)^{\alpha}&=\alpha\int_0^t\left(t-\tau\right)^{\alpha-1}\tau^{1-\alpha}\mathrm{d}\tau\\&=\left(\alpha\right)\left(\Gamma(\alpha)\right)\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{1-\alpha}\mathrm{d}\tau\right)\\&=\left(\alpha\right)\left(\Gamma(\alpha)\right)\left({}_0\mathbf{D}_t^{-\alpha}\left[t^{1-\alpha}\right]\right)\\&=\left(\alpha\right)\left(\Gamma(\alpha)\right)\left(\frac{\Gamma(1-\alpha+1)}{\Gamma(1-\alpha+1+\alpha)}t^{1-\alpha+\alpha}\right)\\&=\left(\alpha\right)\left(\Gamma(\alpha)\right)\left(\Gamma(2-\alpha)\right)t\\&=\left(\Gamma(\alpha+1)\right)\left(\Gamma(2-\alpha)\right)t\\&=\left(\alpha!\right)\left((1-\alpha)!\right)t\end{aligned}\tag{9.216}$$

Therefore $x(t)=x_0E_\alpha\left(\lambda\alpha!(1-\alpha)!t\right)$ is a solution to $x^{(\alpha)}(t)=\lambda t^{1-\alpha}x(t)$.

9.20.6 The fractional differential equation: $a\left(x^{(2\alpha)}(t)\right)+b\left(x^{(\alpha)}(t)\right)+c=0$ with a modified fractional derivative

We take the fractional differential equation:

$$a\left(x^{(2\alpha)}(t)\right)+b\left(x^{(\alpha)}(t)\right)+c=0\tag{9.217}$$

This (9.217) can be written as:

$$a\left(\mathbf{D}^{\alpha}\mathbf{D}^{\alpha}x(t)\right)+b\left(\mathbf{D}^{\alpha}x(t)\right)+c=0\qquad\left(\mathbf{D}^{\alpha}-r_1\right)\left(\mathbf{D}^{\alpha}-r_2\right)=0\tag{9.218}$$

Where r_1 and r_2 are the roots (or zeros) of the characteristic polynomial (indicial polynomial) i.e. $ar^2+br+c=0$. We write the solutions. For $r_1\neq r_2$, we have:

$$x(t)=C_1E_\alpha\left(r_1t^{\alpha}\right)+C_2E_\alpha\left(r_2t^{\alpha}\right)\tag{9.219}$$

and for equal roots that is the $r_1=r_2=\hat{r}$ solution is:

$$x(t)=\left(C_1+C_2t^{\alpha}\right)E_\alpha\left(\hat{r}t^{\alpha}\right)\tag{9.220}$$

with constants C_1 and C_2 determined via given initial conditions.

9.21 The application to dynamics close to the equilibrium position are subjected to coarse graining

9.21.1 The dynamic system at equilibrium

Consider the dynamics given by the differentials $\mathrm{d}x$ and $\mathrm{d}t$ related as $\mathrm{d}x=f(x)\mathrm{d}t$, or the we say via a differential equation i.e. $x^{(1)}(t)=f(x)$; with the equilibrium point as x_0, that is $f(x_0)=0$. Close to this equilibrium we have a variation equation i.e. $\mathrm{d}\left(\Delta x\right)=\left(f_x^{(1)}(x_0)\right)\left(\Delta x\right)\mathrm{d}t$, where $f_x^{(1)}(x_0)=\frac{\mathrm{d}}{\mathrm{d}x}f(x)\Big|_{x=x_0}$. With $y_1\equiv\Delta x$ we have $\mathrm{d}y_1=f_x^{(1)}(x_0)y_1\mathrm{d}t$, which has a solution as $y_1(t)=y_1(0)\exp\left(f_x^{(1)}(x_0)t\right)$. This problem is the standard calculus problem, where fine-grain differentials are used. Now what happens when $x^{(1)}(t)=f(x)$ is subjected to the coarse graining in time (CGT) and the coarse graining in space (CGS)?

9.21.2 Dynamic system subjected to coarse graining in time differential $\mathrm{d}t$

As we have seen for a continuously differentiable deviation in the dynamic $x^{(1)}(t) = f(x)$ one has $y_1^{(1)}(t) = \left(f_x^{(1)}(x_0)\right) y_1(t)$. This being the case we assume $x^{(1)}(t) = f(x)$ is subjected to CGT, and we have the following:

$$u_\alpha(t) = \frac{\mathrm{d}x}{\left(\mathrm{d}t\right)^\alpha} = \left(\alpha!\right)^{-1} \frac{\mathrm{d}^\alpha x}{\left(\mathrm{d}t\right)^\alpha} = \left(\alpha!\right)^{-1} x^{(\alpha)}(t) \qquad 0 < \alpha < 1, \quad \mathrm{d}t > 0 \tag{9.221}$$

Using the above (9.221) in $\mathrm{d}x = f(x)\mathrm{d}t$ we have for the CGT case as $\mathrm{d}x = f(x)(\mathrm{d}t)^\alpha$; and by using the above (9.221) relationship we have:

$$x^{(\alpha)}(t) = \alpha!\, f(x) \tag{9.222}$$

Therefore the deviation equation for the CGT case is:

$$\begin{aligned}
&\mathbf{D}_t^\alpha (x_0 + \Delta x) - \mathbf{D}_t^\alpha x_0 = \alpha!\left(f(x_0 + \Delta x) - f(x_0)\right) \\
&\frac{\mathrm{d}^\alpha (x_0 + \Delta x)}{\mathrm{d}t^\alpha} - \frac{\mathrm{d}^\alpha x_0}{\mathrm{d}t^\alpha} = \alpha!\left(f(x_0 + \Delta x) - f(x_0)\right), \qquad \frac{\mathrm{d}^\alpha (x_0)}{\mathrm{d}t^\alpha} = 0 \\
&\frac{\mathrm{d}^\alpha (\Delta x)}{\mathrm{d}t^\alpha} = \alpha!\left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right)\left(\Delta x\right) = \alpha!\left(f_x^{(1)}(x_0)\right)\left(\Delta x\right) \qquad y_2 = \Delta x \\
&\frac{\mathrm{d}^\alpha y_2}{\mathrm{d}t^\alpha} = \alpha!\left(f_x^{(1)}(x_0) y_2(t)\right)
\end{aligned} \tag{9.223}$$

That gives:

$$y_2^{(\alpha)}(t) = \alpha!\, f_x^{(1)}(x_0) y_2(t) \tag{9.224}$$

The solution is $y_2(t) = y_2(0) E_\alpha\left(\alpha!\, f_x^{(1)}(x_0) t^\alpha\right)$. We have used the earlier obtained result that is $x(t) = x_0 E_\alpha(a t^\alpha)$ and which is a solution of $x^{(\alpha)}(t) = a\left(x(t)\right)$.

9.21.3 The dynamic system subjected to coarse graining in space differential $\mathrm{d}x$

Assume now $\mathrm{d}x = \left(f(x)\right)\mathrm{d}t$ subjected to CGS. For CGS we have the following from (9.179):

$$v_\alpha(t) = \frac{\left(\mathrm{d}x\right)^\alpha}{\mathrm{d}t} = k_\alpha \left(xt\right)^{\alpha-1} x^{(\alpha)}(t) \qquad k_\alpha = \alpha!\left((1-\alpha)!\right)^2 \tag{9.225}$$

Using the above (9.225) derivation and substituting in $(\mathrm{d}x)^\alpha = f(x)\mathrm{d}t$ we obtain for CGS the following:

$$x^{(\alpha)}(t) = \frac{\left(xt\right)^{1-\alpha}}{k_\alpha} f(x) \tag{9.226}$$

The deviation equation is done as in the following steps:

$$\begin{aligned}
&\mathbf{D}_t^\alpha (x_0 + \Delta x) - \mathbf{D}_t^\alpha x_0 = \frac{\left((x_0 + \Delta x)t\right)^{1-\alpha}}{k_\alpha} f(x_0 + \Delta x) - \frac{\left(x_0 t\right)^{1-\alpha}}{k_\alpha} f(x_0) \\
&\mathbf{D}_t^\alpha \left(\Delta x\right) = \frac{\left((x_0 + \Delta x)t\right)^{1-\alpha}}{k_\alpha} f(x_0 + \Delta x) - \frac{\left(x_0 t\right)^{1-\alpha}}{k_\alpha} f(x_0) \qquad \Delta x = y_3 \\
&y_3^{(\alpha)} = \frac{(x_0 + y_3)^{1-\alpha} t^{1-\alpha}}{k_\alpha} f(x_0 + y_3) - \frac{x_0^{1-\alpha} t^{1-\alpha}}{k_\alpha} f(x_0)
\end{aligned} \tag{9.227}$$

Write $f(x_0+y_3)=f(x_0)+f_x(x_0)y_3$ as it comes from the fact that $f_x^{(1)}(x)=\lim_{\Delta x\downarrow 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$, to get the following expression:

$$y_3^{(\alpha)}=\left(\frac{t^{1-\alpha}}{k_\alpha}\right)\left((x_0+y_3)^{1-\alpha}\left(f(x_0)+f_x^{(1)}(x_0)y_3\right)-x_0^{1-\alpha}f(x_0)\right) \tag{9.228}$$

Use the power series expansion i.e. $(x_0+y_3)^{1-\alpha}\cong x_0^{1-\alpha}+(1-\alpha)x_0^{-\alpha}y_3+.....$ and write:

$$y_3^{(\alpha)}=\left(\frac{t^{1-\alpha}}{k_\alpha}\right)\begin{pmatrix}x_0^{1-\alpha}f(x_0)+(1-\alpha)x_0^{-\alpha}f(x_0)y_3+x_0^{1-\alpha}f_x^{(1)}(x_0)y_3\\ +(1-\alpha)x_0^{-\alpha}f_x^{(1)}(x_0)y_3^2+.......-x_0^{1-\alpha}f(x_0)\end{pmatrix} \tag{9.229}$$

Now put $y_3^2\approx 0$ in the above (9.229) expression to get the final form as follows:

$$y_3^{(\alpha)}=\left(k_\alpha\right)^{-1}t^{1-\alpha}\left((1-\alpha)x_0^{-\alpha}f(x_0)+x_0^{1-\alpha}f_x^{(1)}(x_0)\right)y_3 \tag{9.230}$$

The above (9.230) expression suggests that $x_0=0$ is a special case in the CGS case. The solution to the CGS deviation equation is (for $x_0\neq 0$) the following:

$$y_3(t)=y_3(0)E_\alpha\left(\left((1-\alpha)!\right)^{-1}\left((1-\alpha)x_0^{-\alpha}f(x_0)+x_0^{1-\alpha}f_x^{(1)}(x_0)\right)(t)\right) \tag{9.231}$$

From the section of the fractional differential equation in Section- 9.20.5, we have used $x(t)=x_0E_\alpha\left(\lambda\alpha!\left((1-\alpha)!\right)t\right)$ as a solution of $x^{(\alpha)}(t)=\lambda t^{1-\alpha}x(t)$ as obtained therein; in (9.231).

9.21.4 The system when both differentials dx and dt are subjected to coarse graining

Now we discuss when both the variables x and t are subjected to a coarse-graining process for the same dynamic system $\mathrm{d}x=f(x)\mathrm{d}t$. The differentials $(\mathrm{d}x,\mathrm{d}t)$ are changed to $(\mathrm{d}x^\mu,\mathrm{d}t^\gamma)$ with $0<\mu<1$ and $0<\gamma<1$. We may write the system as:

$$\left(\mathrm{d}x\right)^\mu=f(x)\left(\mathrm{d}t\right)^\gamma \tag{9.232}$$

Depending on whether $\mu>\gamma$ or $\mu<\gamma$ we shall be getting the following:

$$\begin{aligned}\left(\mathrm{d}x\right)&=f^{1/\mu}(x)\left(\mathrm{d}t\right)^{\gamma/\mu} && \mu>\gamma\\ \left(\mathrm{d}x\right)^{\mu/\gamma}&=f^{1/\gamma}(x)\left(\mathrm{d}t\right) && \mu<\gamma\end{aligned} \tag{9.233}$$

The above is related to the CGT and CGS systems discussed earlier, represented as follows:

$$\begin{aligned}\left(\mathrm{d}x\right)&=g_1(x)\left(\mathrm{d}t\right)^\alpha && \alpha=\frac{\gamma}{\mu} && g_1(x)=\left(f(x)\right)^{1/\mu}\\ \left(\mathrm{d}x\right)^\beta&=g_2(x)\left(\mathrm{d}t\right) && \beta=\frac{\mu}{\gamma} && g_2(x)=\left(f(x)\right)^{1/\gamma}\end{aligned} \tag{9.234}$$

We may use a fractional derivative to deal with differentiable functions, but we ask the question whether this is essential, as far as function is differentiable. As far as the function is differentiable and the system is Markovian (i.e. system state is not affected by past), then its dynamic equation is quite well defined. If the dynamics are differentiable, strictly speaking, we should not need fractional derivative to analyse its behavior. We may ask, given a system which is defined by a differential equation, $x^{(1)}(t)=f(t)$ will a dynamic with a fractional derivative $x^{(\alpha)}(t)=g(t)$, contribute extra information?

Many system dynamics show memory based behavior where a non-Markovian approach is required, there the dynamics $x^{(\alpha)}(t)=g(t)$, with a classical fractional derivative of the RL or Caputo type have been employed as we described up to Chapter 8. However, if the function in the consideration is not differentiable, then this modified RL fractional derivative of the Jumarie type should be of use. The basic problem is then to find suitable techniques for modeling such functions, given that they are continuous everywhere but nowhere differentiable. One way is to have the function defined on a ‘fractal set’ and another developing approach (under development and research) is to randomise the problem and describe the non-differentiability by using Gaussian white-noise. We will however not deal with this aspect.

9.22 Using the Mittag-Leffler function in Integral Transform formulas

If the Mittag-Leffler function i.e. $E_\alpha(x^\alpha)$ has a conjugate role in fractional calculus can we use this function instead of the exponential function $\exp(x)$ in integral transform equations like Laplace Transforms and Fourier Transforms and call them α integral transforms. Then this type of new transforms will be applicable to non-differentiable dynamics, with fractional differential equations and functions, which are fractionally differentiable; and is a controversial but evolving topic.

9.22.1 Defining the fractional Laplace transform by using the Mittag-Leffler function

The classical Laplace Transform is one such integral transform given by:

$$\mathcal{L}\left\{f(t)\right\} = F(s) \overset{\text{def}}{=} \lim_{T\uparrow\infty}\int_0^T e^{-st}\left(f(t)\right)\mathrm{d}t \tag{9.235}$$

In conjugation with this (9.235), we have a fractional Laplace Transform defined as the following:

$$\mathcal{L}_\alpha\left\{f(t)\right\} = F_\alpha(s) \overset{\text{def}}{=} \lim_{T\uparrow\infty}\int_0^T \left(E_\alpha(-s^\alpha t^\alpha)\right)f(t)\left(\mathrm{d}t\right)^\alpha \tag{9.236}$$

We will have the following Table 9.4 depicting the operational identity for the fractional Laplace transform, along with listed standard classical Laplace identities. We are not proving them, as one can use all the above techniques to derive these.

Fractional Laplace Transform	Classical Laplace Transform
$\mathcal{L}_\alpha\left\{f(t)\right\} = F_\alpha(s)$ $= \int_0^\infty \left(E_\alpha(-s^\alpha t^\alpha)\right)f(t)\left(\mathrm{d}t\right)^\alpha$	$\mathcal{L}\left\{f(t)\right\} = F(s)$ $= \int_0^\infty e^{-st} f(t)\mathrm{d}t$
$\mathcal{L}_\alpha\left\{t^\alpha f(t)\right\} = -D_s^\alpha\left(F_\alpha(s)\right)$	$\mathcal{L}\left\{t\left(f(t)\right)\right\} = -D_s^{(1)}\left(F(s)\right)$
$\mathcal{L}_\alpha\left\{f(at)\right\} = \left(\frac{1}{a}\right)^\alpha F_\alpha\left(\frac{s}{a}\right)$	$\mathcal{L}\left\{f(at)\right\} = \left(\frac{1}{a}\right)F\left(\frac{s}{a}\right)$
$\mathcal{L}_\alpha\left\{f(t-\tau_d)\right\}$ $= \left(E_\alpha(-s^\alpha\tau_d^\alpha)\right)F_\alpha(s)$	$\mathcal{L}\left\{f(t-\tau_d)\right\}$ $= \left(\exp(-s\tau_d)\right)F(s)$
$\mathcal{L}_\alpha\left\{\left(E_\alpha(-a^\alpha t^\alpha\right)f(t)\right\} = F_\alpha(s+a)$	$\mathcal{L}\left\{\left(\exp(-at\right)f(t)\right\}$ $= F(s+a)$
$\mathcal{L}_\alpha\left\{\int_0^t f(u)\left(\mathrm{d}u\right)^\alpha\right\}$ $= \left(\alpha!\right)^{-1}s^{-\alpha}F_\alpha(s)$	$\mathcal{L}\left\{\int_0^t f(u)\left(\mathrm{d}u\right)\right\} = s^{-\alpha}F(s)$
$\mathcal{L}_\alpha\left\{f^{(\alpha)}(t)\right\} = s^\alpha F_\alpha(s) - \left(\alpha!\right)f(0)$	$\mathcal{L}\left\{f^{(1)}(t)\right\} = s\left(F(s)\right) - f(0)$

Table 9.4: Fractional Laplace transform identities

The above table is derived by Jumarie, and the use of this is still developing.

9.22.2 Defining a fractional convolution integration process by use of the fractional Laplace Transforms

As a fractional order convolution then we represent it as follows:

$$\begin{gathered}\left(f(t)*g(t)\right)_\alpha = \int_0^t f(t-\tau)g(\tau)\left(\mathrm{d}\tau\right)^\alpha \\ \mathcal{L}_\alpha\left\{\left(f(t)*g(t)\right)_\alpha\right\} = \left(F_\alpha(s)\right)\left(G_\alpha(s)\right)\end{gathered} \tag{9.237}$$

While the classical one is $\left(f(t)*g(t)\right) = \int_0^t f(t-\tau)g(\tau)\left(\mathrm{d}\tau\right)$ having the Laplace transform as $\mathcal{L}\left\{\left(f(t)*g(t)\right)\right\} = \left(F(s)\right)\left(G(s)\right)$.

9.22.3 Demonstrating fractional Laplace Transform for the Heaviside unit step function

We now give a demonstration for finding the fractional Laplace transform for the Heaviside unit step function defined as $u(t)=1$ for $t \geq 0$ and $u(t)=0$ for $t<0$. The standard classical Laplace transform is in the following steps:

$$\begin{aligned} U(s)=\mathcal{L}\{u(t)\} &= \int_0^{\infty} e^{-st}u(t)\mathrm{d}t \\ &= \int_0^{\infty} e^{-st}\mathrm{d}t = \left.\frac{e^{-st}}{(-s)}\right|_0^{\infty} = \frac{1}{s} \end{aligned} \tag{9.238}$$

We get a known result.
Now we calculate the fractional Laplace transform as follows:

$$\begin{aligned} U_\alpha(s)=\mathcal{L}_\alpha\{u(t)\} &= \int_0^{\infty} E_\alpha\left(-s^\alpha t^\alpha\right)u(t)\left(\mathrm{d}t\right)^\alpha \\ &= \int_0^{\infty} E_\alpha\left(-s^\alpha t^\alpha\right)\left(\mathrm{d}t\right)^\alpha \\ &= \alpha\int_0^{\infty}\left(t-\xi\right)^{\alpha-1} E_\alpha\left(-s^\alpha \xi^\alpha\right)\left(\mathrm{d}\xi\right) \\ &= \alpha\left(\Gamma(\alpha)\right)\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left(t-\xi\right)^{\alpha-1} E_\alpha\left(-s^\alpha \xi^\alpha\right)\left(\mathrm{d}\xi\right)\right) \\ &= \alpha\left(\Gamma(\alpha)\right)\left(\mathbf{D}_t^{-\alpha}\left(E_\alpha\left(-s^\alpha t^\alpha\right)\right)\Big|_0^{\infty}\right) \\ &= \alpha\left(\Gamma(\alpha)\right)\left(\left.\frac{E_\alpha\left(-s^\alpha t^\alpha\right)}{\left(-s^\alpha\right)}\right|_0^{\infty}\right) = \frac{\left(\Gamma\left(\alpha+1\right)\right)}{s^\alpha} = \frac{\alpha!}{s^\alpha} \end{aligned} \tag{9.239}$$

Therefore, we have a pair $\mathcal{L}_\alpha\{u(t)\}=\frac{\alpha!}{s^\alpha}$. Can we also then write $\mathcal{L}_\alpha^{-1}\left\{\frac{\alpha!}{s^\alpha}\right\}=u(t)=1$? Logically, yes. Now what about the inverse Laplace transforms in an integral representation? This representation is a slightly difficult one, which we will discuss shortly.

9.22.4 Defining the fractional delta distribution function $\delta_\alpha(x)$ and its fractional Fourier integral representation by use of the Mittag-Leffler function

First, let us discuss the Dirac Delta distribution of the fractional order. A normal classical Delta function $\delta(x)$ or delta distribution has the property i.e. $\int_{\mathbb{R}} f(x)\delta(x)\mathrm{d}x = f(0)$. Similarly, we define a fractional delta function with the following property:

$$\int_{\mathbb{R}} f(x)\delta_\alpha(x)\left(\mathrm{d}x\right)^\alpha = \alpha\left(f(0)\right) \tag{9.240}$$

Define the function in the following way:

$$\delta_\alpha(x,\varepsilon)=\begin{cases} 0, & x \notin [0,\varepsilon] \\ \varepsilon^{-\alpha}, & 0<x\leq\varepsilon \end{cases} \tag{9.241}$$

Then one has the limit i.e. $\lim_{\varepsilon\downarrow 0}\delta_\alpha(x,\varepsilon)=\delta_\alpha(x)$. A relationship with $\delta_\alpha(x)$ and the complex valued Mittag-Leffler is suitable to form fractional Fourier transform that is in conjugation with a standard Fourier integral that is $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(-\omega x)}\mathrm{d}\omega = \delta(x)$, is following:

$$\frac{\alpha}{\left(T_\alpha\right)^\alpha}\int_{-\infty}^{+\infty} E_\alpha\left(i(-\omega x)^\alpha\right)\left(\mathrm{d}\omega\right)^\alpha = \delta_\alpha(x) \tag{9.242}$$

Here $\left(T_\alpha\right)^\alpha$ is the 'period' of a complex valued Mittag-Leffler function $E_\alpha\left(i(T_\alpha)^\alpha\right)=1$ it is like the classical period i.e. $T=2\pi$ for $\exp\left(i\left(T\right)\right)=e^{2\pi i}=1$.

9.22.5 The inverse fractional Laplace transformation by the Mittag-Leffler Function

As we have an inverse Laplace transform defined as:

$$f(t)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\exp(st)\right)\left(F(s)\right)\mathrm{d}s \tag{9.243}$$

We attempt to write in conjugation with the above (9.243) inverse fractional Laplace transform using the following expression with theMittag-Leffler function:

$$f(t)=\mathcal{L}_{\alpha}^{-1}\left\{F_{\alpha}(s)\right\}=\frac{1}{i\left(T_{\alpha}\right)^{\alpha}}\int_{-i\infty}^{+i\infty}\left(E_{\alpha}\left(s^{\alpha}t^{\alpha}\right)\right)\left(F_{\alpha}(s)\right)\left(\mathrm{d}s\right)^{\alpha} \tag{9.244}$$

In a similar way, one can use the Mittag-Leffler function $E_{\alpha}(x^{\alpha})$ to have the fractional Fourier transform, and other integral transforms of the fractional order. As such, no such close form tables pertain to fractional Laplace and fractional Fourier transform exist.

9.22.6 The Mittag-Leffler function to define the fractional order Gamma function $\Gamma_{\alpha}(x)$

As we know, the integral representation of the classical Gamma function is:

$$\Gamma(x)=\int_{0}^{\infty}\left(\exp(-t)\right)\left(t^{(x-1)}\right)\left(\mathrm{d}t\right) \tag{9.245}$$

We can have a definition of the fractional order Gamma function as the following form:

$$\Gamma_{\alpha}(x)=\left(\alpha!\right)^{-1}\int_{0}^{\infty}\left(E_{\alpha}(-t^{\alpha})\right)\left(t^{(x-1)\alpha}\right)\left(\mathrm{d}t\right)^{\alpha} \tag{9.246}$$

Some interesting conjugation with the classical Gamma or Beta functions is following:

$$\begin{aligned}
&\Gamma_{\alpha}(x+1)=\left(\alpha!\right)\left(x\left(\Gamma_{\alpha}(x)\right)\right)\\
&\Gamma_{\alpha}(n+1)=\left(\alpha!\right)^{n}\left(n!\right)\\
&\mathrm{B}_{\alpha}(x,y)=\int_{0}^{1}(1-t)^{(x-1)\alpha}t^{(y-1)\alpha}\left(\mathrm{d}t\right)^{\alpha}\\
&\mathrm{B}_{\alpha}(x,y)=\frac{\Gamma_{\alpha}(x)\Gamma_{\alpha}(y)}{\Gamma_{\alpha}(x+y)}
\end{aligned} \tag{9.247}$$

These are very new developments, and further detailed research in this direction is progressing. Here we have utilised the concept developed for integration with respect of fractional differentials and used the Mittag-Leffler function of the form $E_{\alpha}(x^{\alpha})$ instead of the exponential function e^{x} to get several new types of fractional integral representations of transforms and functions for non-differentiable systems.

9.23 Derivatives with the order as a continuous distributed function

9.23.1 The fractional differential equation generalised from integer order to fractional order

Now we generalise the concept of fractional derivatives that we discussed until Chapter-8 the classical RL and Caputo types. Here we discuss considering the fractional order of the derivative as a continuous function (or a distribution function). The first order classical linear differential equation is following on with k_1 and k_0 as non-zero constants:

$$k_1\frac{\mathrm{d}x(t)}{\mathrm{d}t}+k_0x(t)=f(t) \tag{9.248}$$

The fractional differential equation is:

$$\begin{gathered}
k_0\frac{\mathrm{d}^{\alpha_0}x(t)}{\mathrm{d}t^{\alpha_0}}+k_1\frac{\mathrm{d}^{\alpha_1}x(t)}{\mathrm{d}t^{\alpha_1}}+\ldots..k_{m-1}\frac{\mathrm{d}^{\alpha_{m-1}}x(t)}{\mathrm{d}t^{\alpha_{m-1}}}=f(t)\\
0\le\alpha_n\le 1;\quad n=0,1,\ldots(m-1)
\end{gathered} \tag{9.249}$$

The (9.249) becomes (9.248) if $\alpha_0=0$, $\alpha_1=1$ and $k_2=k_3=\ldots=k_{m-1}=0$, $k_0\ne 0$, $k_1\ne 0$. We write (9.249) as the following:

$$\sum_{n=0}^{m-1}k_n\frac{\mathrm{d}^{\alpha_n}x(t)}{\mathrm{d}t^{\alpha_n}}=f(t) \tag{9.250}$$

From (9.250) we write the Laplace transformed equation as (9.251) with assuming all the initial values of $x(t)$ as zero, with $X(s)=\mathcal{L}\left\{x(t)\right\}$ and $F(s)=\mathcal{L}\left\{f(t)\right\}$; and by using $\mathcal{L}\left\{\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t)\right\}=s^\alpha X(s)$:

$$\left(\sum_{n=0}^{m-1} s^{\alpha_n}k_n\right)X(s)=F(s) \quad , \quad X(s)=\frac{F(s)}{\sum_{n=0}^{m-1}k_n s^{\alpha_n}} \tag{9.251}$$

The equation (9.251) is a general fractional differential equation in the Laplace domain, its solution is obtained via the inverse Laplace transform technique and we write:

$$x(t)=\mathcal{L}^{-1}\left\{\left(F(s)\right)/\left(\sum_{n=0}^{m-1}k_n s^{\alpha_n}\right)\right\} \tag{9.252}$$

The denominator term i.e. $\left(\sum_{n=0}^{m-1}s^{\alpha_n}k_n\right)X(s)=F(s)$ in (9.251) has orders of fractional numbers α_n, we need not be always positive. Thus, we can still generalise this (9.252) by including negative orders as following, with the coefficients k_n in (9.251) replaced by a_n and b_n:

$$\left(\sum_{n=0}^{m-1}s^{\alpha_n}a_n+\sum_{n=0}^{m-1}s^{-\beta_n}b_n\right)X(s)=F(s); \quad \alpha_n,\beta_n>0 \tag{9.253}$$

The summation on n from 0 to $m-1$ in (9.253) we can change in general terms from 0 to $+\infty$, and write the following:

$$\left(\sum_{n=0}^{\infty}s^{\alpha_n}a_n+\sum_{n=0}^{\infty}s^{-\beta_n}b_n\right)X(s)=F(s); \quad \alpha_n,\beta_n\ge 0 \tag{9.254}$$

This (9.254) gives a much more general expression for a generalised differential equation. The above (9.254) generalised equation says that the fractional orders are distributed in real line from $-\infty$ to ∞, with orders specified for each fractional differ-integral operator by a fractional number α_n for the positive order, and β_n for the negative order. The coefficients a_n and b_n are specifying respective weights of fractional orders α_n and β_n that participate in the generalised fractional differential equation of (9.254).

9.23.2 The concept of order distribution

We take the example of a mass spring damper system as represented below:

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+b_0\frac{\mathrm{d}x(t)}{\mathrm{d}t}+kx(t)=f(t) \tag{9.255}$$

By use of the Laplace transforms in (9.255), assuming initial conditions as zero gives the following:

$$(ms^2+b_0s+k)X(s)=F(s) \tag{9.256}$$

Using the property of delta function i.e. $\int\left(g(\alpha)\right)\delta(\alpha-\alpha_0)\mathrm{d}\alpha=g(\alpha_0)$ we write for (9.256) with $g(\alpha)\equiv s^\alpha$ the following expression:

$$\begin{aligned}&(ms^2+b_0s+k)X(s)=F(s)\\&\left(m\int s^\alpha\delta(\alpha-2)\mathrm{d}\alpha+b_0\int s^\alpha\delta(s-1)\mathrm{d}\alpha+k\int s^\alpha\delta(s-0)\mathrm{d}\alpha\right)X(s)=F(s)\\&\left(\int_0^\infty\left(m\delta(\alpha-2)+b_0\delta(\alpha-1)+k\delta(\alpha)\right)s^\alpha\mathrm{d}\alpha\right)X(s)=F(s)\end{aligned} \tag{9.257}$$

In (9.257) we have only integer orders, those are 2, 1 and 0 with weights m, b_0 and k respectively. This is depicted as the delta distributed order distribution in Figure-9.1a. We can generalise (9.257) by adding a few more terms called the fractional damping terms. Say we add extra fractional damping terms at fractional orders of $3/2$, $1/2$, with weights b_1 and b_2 respectively in addition to the existing integer orders; then we get the representation as follows (9.258). This is depicted in Figure-9.1b:

$$\begin{aligned}&(ms^2+b_1s^{3/2}+b_0s+b_2s^{1/2}+k)X(s)=F(s)\\&\left(\int_0^\infty\begin{pmatrix}m\delta\left(\alpha-2\right)+b_1\delta\left(\alpha-\frac{3}{2}\right)+b_0\delta\left(\alpha-1\right)\\+b_2\delta\left(\alpha-\frac{1}{2}\right)+k\delta\left(\alpha\right)\end{pmatrix}s^\alpha\mathrm{d}\alpha\right)X(s)=F(s)\end{aligned} \tag{9.258}$$

The corresponding differential equation for (9.258) is as follows, obtained by performing an inverse Laplace transform on (9.258):

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+b_1\frac{\mathrm{d}^{3/2}x(t)}{\mathrm{d}t^{3/2}}+b_0\frac{\mathrm{d}x(t)}{\mathrm{d}t}+b_2\frac{\mathrm{d}^{1/2}x(t)}{\mathrm{d}t^{1/2}}+kx(t)=f(t) \tag{9.259}$$

9.23.3 From the discrete delta distributed function to a continuous distribution function for describing fractional orders

We generalise (9.258) with several more delta functions with respective weights as follows:

$$\left(\int_{-\infty}^{\infty}\left(\sum_{n=0}^{N}k_n\delta\left(\alpha-\alpha_n\right)\right)s^{\alpha}\mathrm{d}\alpha\right)X(s)=F(s) \qquad \alpha_n\in\mathbb{R} \tag{9.260}$$

When the adjacent delta functions at α_n become close and dense we can write a continuous distribution function $k(\alpha)$. This $k(\alpha)$ is a distribution function for the order α, for (9.260).For the discrete delta distributed order, we write $k(\alpha)=\sum_{n=0}^{N}k_n\delta\left(\alpha-\alpha_n\right)$. We write the continuous order system as the following (9.261), and the continuous distributed order is depicted in Figure-9.1c:

$$\left(\int_{-\infty}^{\infty}\left(k(\alpha)\right)s^{\alpha}\mathrm{d}\alpha\right)X(s)=F(s) \quad , \quad X(s)=\frac{F(s)}{\int_{-\infty}^{\infty}s^{\alpha}\left(k(\alpha)\right)\mathrm{d}\alpha} \tag{9.261}$$

In order to get solution $x(t)$ we need to perform the following inverse Laplace Transform:

$$x(t)=\mathcal{L}^{-1}\left\{\left(F(s)\right)/\left(\int_a^b s^{\alpha}\left(k(\alpha)\right)\mathrm{d}\alpha\right)\right\} \tag{9.262}$$

Where $k(\alpha)$ is an order distribution function, which is a continuous function defined in the range $a\le\alpha\le b$. That derivative interval is $\alpha\in\left[a,b\right]$:

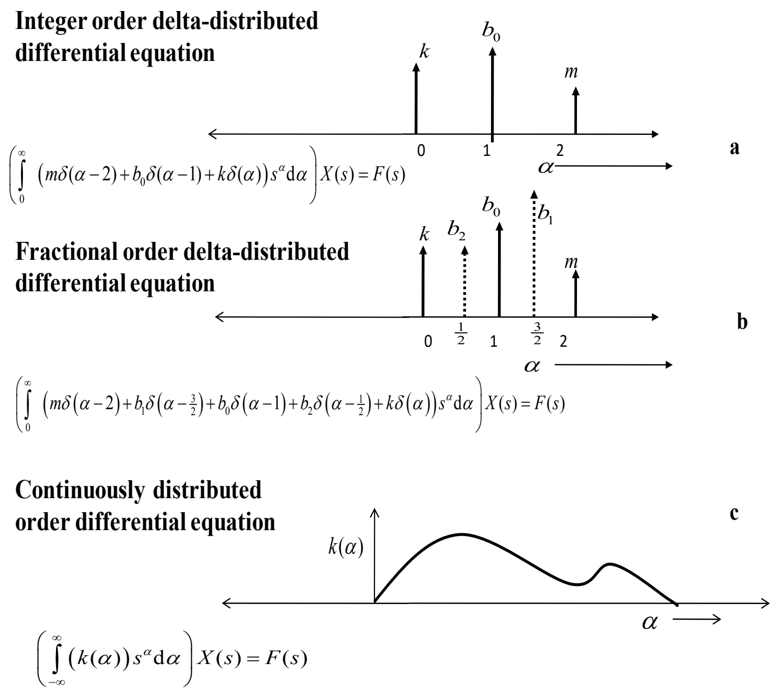

Figure 9.1: Integer order delta distributed, fractional order delta distributed and continuously distributed order differential equations

9.23.4 The continuous order distributed differential equation

From (9.261) with $k(\alpha)$ defined in the range $a\le\alpha\le b$, $\left(\int_a^b\left(k(\alpha)\right)s^{\alpha}\mathrm{d}\alpha\right)X(s)=F(s)$ we can write the following generalised differential equation as follows:

$$\int_a^b \left(k(\alpha)\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} \right)\mathrm{d}\alpha = f(t) \tag{9.263}$$

Substituting the integral with the function $k(\alpha)$ distributed and limited in the derivative interval $a \le \alpha \le b$ is the operation of a weighted averaging of fractional orders in a continuous interval. We consider $0 \le a < b \le 1$, and $k(\alpha)$ to have a uniform distribution between $a \le \alpha \le b$ as:

$$k(\alpha) = p; \quad a \le \alpha \le b; \quad 0 \le a < b \le 1 \tag{9.264}$$

Consider in (9.263) $f(t) = \delta(t)$ i.e. the driving function as a unit delta function at $t = 0$. Then we have $\int_a^b \left(k(\alpha)\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} \right)\mathrm{d}\alpha = \delta(t)$, as our homogeneous system. For $f(t) = \delta(t)$ we have $F(s) = 1$, with this we have $X(s)$ in the Laplace domain as in the following:

$$\begin{aligned}
X(s) &= \left\{ \frac{1}{\int_a^b ps^\alpha \mathrm{d}\alpha} \right\} = \frac{1}{\Phi(s)} \\
\Phi(s) &= \int_a^b ps^\alpha \mathrm{d}\alpha = p\int_a^b e^{\alpha \ln s}\mathrm{d}\alpha; \qquad e^{\alpha \ln s} = s^\alpha \\
& p\left[\frac{e^{\alpha \ln s}}{\ln s} \right]_{\alpha=a}^{\alpha=b} = p\frac{s^b - s^a}{\ln s}
\end{aligned} \tag{9.265}$$

The $\Phi(s)$ in the above (9.265) is similar to the indicial polynomial or the characteristic function for differential equation. Thus $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{\Phi(s)}\right\}$ is the solution of a homogeneous equation of (9.265). To get the function $x(t)$ we can resort to the Berberan-Santos technique as we discussed in Section 6.11.7, instead of going in for the contour integration technique.

The function $X(s) = \frac{\ln s}{ps^b - ps^a}$ seems to have a singularity at $s = 1$. Applying L'Hospital's rule we get $\lim_{s\to 1} = \frac{1}{p(b-a)}$. In addition, we note that $\lim_{s\uparrow\infty} \frac{\ln s}{ps^b - ps^a} = 0$. We chose $s = \sigma_0 + i\omega$ with $\sigma_0 = 0$. To the right of $s = 0$ i.e. $\mathrm{Re}\left[s\right] > 0$ we do not expect to have a singularity for $X(s)$. With this, we have following for $X(i\omega)$:

$$X(i\omega) = \frac{\begin{pmatrix} \left(\ln\omega\left(\omega^b \cos\frac{b\pi}{2} - \omega^a \cos\frac{a\pi}{2}\right) + \frac{\pi}{2}\left(\omega^b \sin\frac{b\pi}{2} - \omega^a \sin\frac{a\pi}{2}\right)\right) \\ -i\left(\ln\omega\left(\omega^b \sin\frac{b\pi}{2} - \omega^a \sin\frac{a\pi}{2}\right) - \frac{\pi}{2}\left(\omega^b \cos\frac{b\pi}{2} - \omega^a \cos\frac{a\pi}{2}\right)\right) \end{pmatrix}}{p\left(\omega^{2b} + \omega^{2a} - \omega^{a+b}\cos\left(\frac{(b-a)\pi}{2}\right)\right)} \tag{9.266}$$

This (9.266) gives $\mathrm{Re}\left[X(i\omega)\right]$ and $\mathrm{Im}\left[X(i\omega)\right]$ for use in the Berberan-Santos formula, giving the following:

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{X(s)\right\} = \mathcal{L}^{-1}\left\{\frac{\ln s}{ps^b - ps^a}\right\}, \quad \sigma_0 = 0 \\
x(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[X(\sigma_0 + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[X(\sigma_0 + i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left(\frac{\ln\omega\left(\omega^b\cos\frac{b\pi}{2} - \omega^a\cos\frac{a\pi}{2}\right) + \frac{\pi}{2}\left(\omega^b\sin\frac{b\pi}{2} - \omega^a\sin\frac{a\pi}{2}\right)}{p\left(\omega^{2b} + \omega^{2a} - \omega^{a+b}\cos\left(\frac{(b-a)\pi}{2}\right)\right)}\cos(\omega t) \right)\mathrm{d}\omega \\
&\quad + \frac{1}{\pi}\int_0^\infty \left(\frac{\ln\omega\left(\omega^b\sin\frac{b\pi}{2} - \omega^a\sin\frac{a\pi}{2}\right) - \frac{\pi}{2}\left(\omega^b\cos\frac{b\pi}{2} - \omega^a\cos\frac{a\pi}{2}\right)}{p\left(\omega^{2b} + \omega^{2a} - \omega^{a+b}\cos\left(\frac{(b-a)\pi}{2}\right)\right)}\sin(\omega t) \right)\mathrm{d}\omega
\end{aligned} \tag{9.267}$$

The continuous order distributed differential equations do have significance, but this type of system has not been developed. This is a developing field.

9.24 Short summary

We may use the fractional derivative to deal with differentiable functions, but we ask the question as to whether or not this is essential. As far as the function is differentiable and the system is Markovian (i.e. the present state of the system is not affected by past states), then it is a dynamic equation and is quite well defined and classical Newtonian calculus is sufficient. If the system is differentiable strictly speaking, we should not need a fractional derivative to analyse its behavior. We may ask if a given dynamic system, which is defined by a classical differential equation, then -if the same dynamics be described with the fractional differential equation should contribute any extra information? .Many system dynamics show memory based behavior where a non-Markovian approach is required; there the dynamics, with a classical fractional derivative of the RL or Caputo type have been employed. We started this chapter by formally deriving important rules for fractional differ-integration and its compositions. Though we had used these facts in earlier chapters, we derived them systematically from the theory of classical calculus by its generalisation. After that, we gave detailed treatment to modify the fractional calculus theory that we obtained in earlier chapters, to get parallel to the classical calculus, and called that the modified Riemann-Liouville fractional derivative of the Jumarie type; and obtained various formulas that are in conjugation with classical calculus, for non-differentiable functions. The fractional Taylor series, for non-differentiable functions that was developed using this modified concept gave several conversion formulas for fractional differentials that were used to develop and simplify the Leibniz product rule, integration by parts, and chain rule formulas. We showed how we can use this new developed concept and apply it to study the non-differentiable function, the ease with which we can solve the fractional differential equation, and formulate and solve problems with fractional differentials arising due to the coarse graining process. The new integral transform formulas (of fractional Laplace transforms, and fractional Fourier transforms) and thereby new functions (fractional Gamma, fractional Beta, fractional Delta functions) that are due to this modified fractional calculus by use of the fractal Mittag-Leffler function instead of the classical exponential function, give an immense similarity to classical calculus techniques. However, these concepts, especially with the modified fractional calculus of (Jumarie type) are a yet developing subject (and controversial too), and have immense potential in the research in mathematical sciences. Therefore, if the function in consideration is not differentiable, then this modified RL fractional derivative of the Jumarie type should be of use. The basic problem is then to find suitable techniques for modeling such functions, given that they are continuous everywhere but nowhere differentiable. One way is to have functions defined on the 'fractal set' and another developing approach is to randomise the problem and describe the non-differentiability by using Gaussian white noise; that is still under development. We also introduced the concept of generalising the differential equations composing derivatives where the order distribution is of continuous function type instead of discrete fractional numbers. This concept is a developing field. Nonetheless, the acceptability of this subject is still very little, and generally, most established scientists and engineers still turn a blind eye here. However, in this course we have at least learnt about the existence of this vast subject parallel to our classical calculus, and tried to appreciate this so called 'paradox'!

9.25 References

The motivation of this chapter is derived from the pioneering works which are:

Michelle Caputo, "Linear Models of Dissipation whose Q is almost Frequency Independent" (1967)

Jumarie G, "Laplace Transform of Fractional Order via Mittag-Leffler function and Modified Riemann-Liouville derivative" (2009); "Oscillations of non-linear systems close to equilibrium position in the presence of coarse graining time and space" (2009); "Tables of some basic fractional calculus formulae from modified Riemann-Liouville derivative for non-differentiable functions" (2009); "Cauchy Integral Formula via Modified Riemann-Liouville derivative for analytic functions of fractional order" (2010); "Modified Riemann-Liouville Derivative and Fractional Taylor Series of non-differentiable functions further results" (2006); "From self similarity to fractional derivative of non-differentiable functions via Mittag-Leffler function" (2008); "On the fractional solution of the equation $f(x+y)=f(x)f(y)$ and application to fractional Laplace transform" (2012)

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Mandelbrot B.B, "Self-Affine and Fractal Dimension" (1985)

Miller K S and Ross B, "An Introduction to Fractional Calculus and Fractional Differential Equations" (1993)

Oldham and Spanier, "Fractional calculus" (1974)

Munkhammar .J.D, "Fractional calculus and Taylor-Riemann series" (2005)

Massopust P, Zayed A, "On the validity of Fourier Series Expansion of Fractional Order," (2015)

The details of the above pioneering literature and work are listed in the bibliography section, in alphabetical order.

Appendix A

Higher Transcendental and Special Functions

This appendix presents a number of functions that are found to be useful in the solution of the problems of fractional calculus. The base function is the gamma function, which generalizes the factorial expression, used in multiple differentiations and repeated integrations, in integer order calculus. We have discussed this and its properties in Chapter One. The Mittag-Leffler function is the basis function of fractional calculus; as an exponential function it is an integer order (classical) calculus. Several modifications of the Mittag-Leffler functions, along with other variants, have been introduced and further developed since 1903, for the study of fractional calculus. These functions are called higher transcendental functions, and their use in solving fractional differential equations is similar to the use of transcendental functions for solving integer order differential equations. The use of some of these functions has been demonstrated in earlier chapters (Chapter Two to Chapter Nine).

A.1 Hyper-geometric functions

We have not used these hyper-geometric functions much in the book, but we will list them here, as these functions are the basis for all other functions that we describe. The hyper-geometric function and its generalization encompass an extensive class of analytical functions. Definition and symbolism are as follows:

$$\begin{aligned}\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right] &= \sum_{j=0}^{\infty} x^j \frac{\big((j+b_1)!\big)\big((j+b_2)!\big)...\big((j+b_K)!\big)}{\big((j+c_1)!\big)\big((j+c_2)!\big)...\big((j+c_L)!\big)} \\ &= \sum_{j=0}^{\infty} x^j \frac{\prod_{k=1}^{K}\Gamma(j+1+b_k)}{\prod_{l=1}^{L}\Gamma(j+1+c_l)}\end{aligned} \tag{A1}$$

We write from the above (A1), the expanded form below:

$$\begin{aligned}\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right] &= \frac{\Gamma(1+b_1)\Gamma(1+b_2)...\Gamma(1+b_K)}{\Gamma(1+c_1)\Gamma(1+c_2)...\Gamma(1+c_L)} \\ &+ x\frac{\Gamma(2+b_1)\Gamma(2+b_2)...\Gamma(2+b_K)}{\Gamma(2+c_1)\Gamma(2+c_2)...\Gamma(2+c_L)} \\ &+ x^2\frac{\Gamma(3+b_1)\Gamma(3+b_2)...\Gamma(3+b_K)}{\Gamma(3+c_1)\Gamma(3+c_2)...\Gamma(3+c_L)}+.... \\ &+ x^n\frac{\Gamma(n+1+b_1)\Gamma(n+1+b_2)...\Gamma(n+1+b_K)}{\Gamma(n+1+c_1)\Gamma(n+1+c_2)...\Gamma(n+1+c_L)}+...\end{aligned} \tag{A2}$$

We assume that parameter values and the values of the argument are such that they ensure convergence of the series; this will often imply restrictions $|x|<1$.

Say for $b_1=c_1=1$, and $b_2=b_3=....=b_K=c_2=.....=c_L=0$ in (A1) and (A2) then, we have the following known formula:

$$\left[x\,\frac{1}{1}\right]=\sum_{j=0}^{\infty}x^j=1+x+x^2+....=(1-x)^{-1} \tag{A3}$$

We shall call this as $\frac{K}{L}$ hyper-geometric. Note that the leading term (A1) within this summation will be zero if one term of the denominator parameter is a negative integer; if $c_l=-n$; terms in $x^0,x^1,x^2,.....,x^{n-1}$ are absent. Conversely, the leading terms (A1) will be infinite if one of the numerators, say b_1, is a negative integer, i.e. $-n$, due to the value of the gamma function at negative integer points being set at infinity. We consequently encounter this possibility only as the quotient, by representing that case as $\frac{1}{\Gamma(1-n)}\left[x\,\frac{-n,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right]$. While the gamma function at the negative integer point is infinity, the reciprocal and the ratio of the gamma function at negative integer points are finite, i.e. $\frac{\Gamma(-n)}{\Gamma(-N)}=\left(-N\right)\left(-N+1\right).....\left(-n-2\right)\left(-n-1\right)=\left(-1\right)^{N-n}\left(\frac{N!}{n!}\right)$.

On doing one whole derivative of the above (A2) series term by term, and by using $n=\frac{\Gamma(n+1)}{\Gamma(n)}$ and $\frac{1}{\Gamma(0)}=0$, we get the following steps:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right] &= 0+\frac{\Gamma(2+b_1)\Gamma(2+b_2)...\Gamma(2+b_K)}{\Gamma(2+c_1)\Gamma(2+c_2)...\Gamma(2+c_L)} \\
&\quad + 2x\frac{\Gamma(3+b_1)\Gamma(3+b_2)...\Gamma(3+b_K)}{\Gamma(3+c_1)\Gamma(3+c_2)...\Gamma(3+c_L)}+ \\
&\quad ...+nx^{n-1}\,\frac{\Gamma(n+1+b_1)\Gamma(n+1+b_2)...\Gamma(n+1+b_K)}{\Gamma(n+1+c_1)\Gamma(n+1+c_2)...\Gamma(n+1+c_L)}+........ \\
&= x^{-1}\begin{pmatrix} 0+\frac{\Gamma(2+b_1)\Gamma(2+b_2)...\Gamma(2+b_K)}{\Gamma(2+c_1)\Gamma(2+c_2)...\Gamma(2+c_L)}+2\,\frac{\Gamma(3+b_1)\Gamma(3+b_2)...\Gamma(3+b_K)}{\Gamma(3+c_1)\Gamma(3+c_2)...\Gamma(3+c_L)}+ \\ ...+nx^{n}\,\frac{\Gamma(n+1+b_1)\Gamma(n+1+b_2)...\Gamma(n+1+b_K)}{\Gamma(n+1+c_1)\Gamma(n+1+c_2)...\Gamma(n+1+c_L)}+........ \end{pmatrix} \\
&= x^{-1}\begin{pmatrix} \left(\frac{\Gamma(1+0)}{\Gamma(1-1)}\right)\frac{\Gamma(1+b_1)\Gamma(1+b_2)...\Gamma(1+b_K)}{\Gamma(1+c_1)\Gamma(1+c_2)...\Gamma(1+c_L)}+ \\ x\left(\frac{\Gamma(2+0)}{\Gamma(2-1)}\right)\frac{\Gamma(2+b_1)\Gamma(2+b_2)...\Gamma(2+b_K)}{\Gamma(2+c_1)\Gamma(2+c_2)...\Gamma(2+c_L)}+ \\ x^2\left(\frac{\Gamma(3+0)}{\Gamma(3-1)}\right)\frac{\Gamma(3+b_1)\Gamma(3+b_2)...\Gamma(3+b_K)}{\Gamma(3+c_1)\Gamma(3+c_2)...\Gamma(3+c_L)}+......... \\ ......+x^n\left(\frac{\Gamma(n+1)}{\Gamma(n+1-1)}\right)\frac{\Gamma(n+1+b_1)\Gamma(n+1+b_2)...\Gamma(n+1+b_K)}{\Gamma(n+1+c_1)\Gamma(n+1+c_2)...\Gamma(n+1+c_L)}+.......... \end{pmatrix} \\
&= x^{-1}\left[x\,\frac{0,b_1,b_2,.....,b_K}{-1,c_1,c_2,.....,c_L}\right]
\end{aligned}
\tag{A4}
$$

$$
\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right]=x^{-m}\left[x\,\frac{0,b_1,b_2,....,b_K}{-m,c_1,c_2,...,c_L}\right];\quad m=0,\pm1,\pm2,....
$$

We can derive from similar lines as done above (A4) and write the following formula, with a restriction on m removed. Then from the above (A4) is a fractional differ-integration of order α; for a generalized hyper-geometric function, given as:

$$
\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right] &= x^{-\alpha}\left[x\,\frac{0,b_1,b_2,.....,b_K}{-\alpha,c_1,c_2,.....,c_L}\right] \\
\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[x^\beta\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right] &= x^{\beta-\alpha}\left[x\,\frac{\beta,b_1,b_2,.....,b_K}{(\beta-\alpha),c_1,c_2,.....,c_L}\right]\qquad \beta>-1
\end{aligned}
\tag{A5}
$$

Representation of $\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right]$ in terms of the Gauss function $F\left(-,-;-;x\right)$ is the following:

$$
\left[x\,\frac{b_1,b_2,.....,b_K}{c_1,c_2,.....,c_L}\right]={}_{K+1}F_L\begin{pmatrix}1,1+b_1,1+b_2,.....,1+b_k; \\ 1+c_1,.....,1+c_L;x\end{pmatrix}\frac{\prod_{k=1}^{K}\Gamma(1+b_k)}{\prod_{l=1}^{L}\Gamma(1+c_l)}
\tag{A6}
$$

The Gauss function or Gauss series is the following (as defined in the *Handbook of Mathematical Functions*):

$$
\begin{aligned}
F\left(a,b;c;x\right) &= {}_2F_1\left(a,b;c;x\right)=F\left(b,a;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}
\end{aligned}
\tag{A7}
$$

for $\left|x\right|<1$.

We have used $(a)_n = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)}$ for $n>0$ Pochhammer number or the rising factorial (Section-1.9.5) and $F\left(a,b;c;1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ with $c \neq 0,-1,-2,...$ and $\mathrm{Re}\left[c-a-b\right] > 0$, in (A7)

We write the following useful identities for various hyper-geometric functions:

$$\begin{aligned}
&\left[x\frac{1}{1}\right] = \frac{1}{1-x} \qquad \left[x\frac{1}{0}\right] = e^x \qquad \left[x\frac{1}{c}\right] = e^x\gamma^*(c,x) \\
&\left[x\frac{b}{0}\right] = \frac{\Gamma(1+b)}{(1-x)^{1+b}} \\
&\left[x\frac{b}{c}\right] = \frac{\Gamma(1+b)}{\Gamma(c)}\frac{\mathrm{B}_x(c,1+b-c)}{x^c(1-x)^{1+b-c}} \\
&\left[x\frac{b}{0,c}\right] = \frac{\Gamma(1+b)}{\Gamma(1+c)}M(1+b,1+c,x) \\
&\left[x\frac{b_1,b_2}{0,c}\right] = \frac{\Gamma(1+b_1)\Gamma(1+b_2)}{\Gamma(1+c)}F(1+b_1,1+b_2;1+c;x)
\end{aligned} \tag{A8}$$

Where $\gamma^*\left(-,x\right)$, $\mathrm{B}_x\left(-,-\right)$ $M\left(-,-,x\right)$ and $F\left(-,-;-;x\right)$ denote Tricomi's incomplete gamma, incomplete beta, Kummer and Gauss functions respectively. Of special interest, let us take $\left[x\frac{b}{c}\right] = \frac{\Gamma(1+b)}{\Gamma(c)}\frac{\mathrm{B}_x(c,1+b-c)}{x^c(1-x)^{1+b-c}}$. From this we write:

$$\begin{aligned}
\frac{\mathrm{B}_x(c,1+b-c)}{(1-x)^{1+b-c}} &= \frac{x^c\Gamma(c)}{\Gamma(1+b)}\left(\left[x\frac{b}{c}\right]\right) \\
&= \frac{x^c\Gamma(c)}{\Gamma(1+b)}\left(\sum_{j=0}^{\infty}x^j\frac{\Gamma(j+1+b)}{\Gamma(j+1+c)}\right)
\end{aligned} \tag{A9}$$

Putting $c = p-q$ and $b = p$, we obtain the following:

$$\begin{aligned}
\frac{\mathrm{B}_x(p-q,1+q)}{(1-x)^{1+q}} &= \frac{x^{p-q}\Gamma(p-q)}{\Gamma(1+p)}\left(\left[x\frac{p}{p-q}\right]\right) \\
&= \frac{x^{p-q}\Gamma(p-q)}{\Gamma(1+p)}\left(\sum_{j=0}^{\infty}x^j\frac{\Gamma(j+1+p)}{\Gamma(j+1+p-q)}\right)
\end{aligned} \tag{A10}$$

The above identity (A10) is very useful in order to compactly write the summation in terms of the incomplete beta-function, i.e.:

$$\sum_{j=0}^{\infty}x^j\frac{\Gamma(j+1+b)}{\Gamma(j+1+c)} = \frac{\Gamma(1+b)}{x^c(1-x)^{1+b-c}\Gamma(c)}\mathrm{B}_x(c,1+b-c) \tag{A11}$$

The above identity is used to reduce the summation term as in the LHS of the above (A10).
The generalized hyper-geometric series is defined as the following:

$$\begin{aligned}
&{}_pF_q\left(a_1,...,a_p,b_1,...,b_q;z\right) \\
&\qquad = \frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)...\Gamma(a_p+k)}{\Gamma(b_1+k)...\Gamma(b_q+k)}\frac{z^k}{k!}
\end{aligned} \tag{A12}$$

provided that the arguments b_i are not non positive integers. The series converges for all z if $p \le q$, converges for $\left|z\right| < 1$ if $p = q+1$, and diverges for all nonzero z if $p > q+1$. If $p=2$, and $q=1$ then:

$${}_2F_1\left(a,b,c;z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!} \tag{A13}$$

and all its analytical continuations are called hyper-geometric functions. The series converges for all z with $\left|z\right| < 1$. If $\mathrm{Re}\left[c\right] > \mathrm{Re}\left[a\right] > 0$, the integral representation is:

$$ {}_2F_1\left(a,b,c;z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\,\mathrm{d}t \tag{A14} $$

In particular, if $\mathrm{Re}\left[c\right] > \mathrm{Re}\left[(a+b)\right]$ and c is not a non-positive integer, then:

$$ {}_2F_1\left(a,b,c;z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{A15} $$

The following are useful identities:

$$ \begin{aligned} {}_2F_1\left(a,b,c;z\right) &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,c;z) \\ {}_2F_1\left(a,b,c;z\right) &= (1-z)^{-a}\,{}_2F_1\left(a,c-b,c;\tfrac{z}{z-1}\right) \end{aligned} \tag{A16} $$

In particular, if $p = 1 = q$, then:

$$ {}_1F_1\left(a,c;z\right) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{z^k}{k!} \tag{A17} $$

is a confluent hyper-geometric function. It converges for all z provided that $c \neq 0,-1,-2,\ldots$. The name comes from the fact that it may be defined by the limit as ${}_1F_1\left(a,c;z\right) = \lim_{b\uparrow\infty} {}_2F_1\left(a,b,c;\frac{z}{b}\right)$.

If $\mathrm{Re}\left[c\right] > \mathrm{Re}\left[a\right] > 0$, the integral representation is:

$$ {}_1F_1\left(a,c;z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\,\mathrm{d}t \tag{A18} $$

The following are interesting identities:

$$ \begin{aligned} {}_1F_1\left(a,c;z\right) &= e^{z}\left({}_1F_1(c-a,c;z)\right) \\ D\left[{}_1F_1\left(a,c;z\right)\right] &= \left(\tfrac{a}{c}\right)\left({}_1F_1(a+1,c+1;z)\right) \end{aligned} \tag{A19} $$

The relationship of elementary functions to hyper-geometric functions is, for example described in the following expressions:

$$ \begin{aligned} &(1-z)^{-a} = {}_1F_0\left(a;z\right); \qquad \left|z\right| < 1 \\ &\ln\left(\frac{1+x}{1-x}\right) = 2x\left({}_2F_1\left(\tfrac{1}{2},1,\tfrac{3}{2};x^2\right)\right); \qquad 0 \le x < 1 \\ &\gamma^*(v,z) = \frac{1}{\Gamma(v+1)}\left({}_1F_1\left(v,v+1;z\right)\right) \\ &\qquad = \frac{1}{\Gamma(v+1)} e^{-z}\left({}_1F_1(1,v+1;z)\right) \\ &\mathrm{B}_\tau(x,y) = x^{-1}\tau^{x}\left({}_2F_1(x,1-y,x+1;\tau)\right) \\ &\qquad = x^{-1}\tau^{x}(1-\tau)^{y}\left({}_2F_1(x+y,1,x+1;\tau)\right) \end{aligned} \tag{A20} $$

A.2 Mittag-Leffler function

In the integer order (classical) calculus for the Ordinary Differential Equations (ODE), the exponential function $\exp(z)$ plays an important role. Similarly, in the fractional order calculus of the Mittag-Leffler function, it plays an important part. For this function (i.e. $E_q\left(az\right)$, $q > 0$), Mittag-Leffler (1903) considered that the parameter a be a complex number, as $a = \left|a\right|\exp(i\phi)$. As Mittag-Leffler studied this function it became apparent that this function is either stable (decays to zero), or unstable (goes to infinity) as z increases depending upon how the parameters a and q are chosen. The result was that the function remained bounded for increasing z if $\left|\phi\right| \ge \left(q\,\frac{\pi}{2}\right)$. We will describe the one parameter and two parameter Mittag-Leffler functions below.

A.2.1 One parameter Mittag-Leffler function

This function $E_\alpha(z)$ is defined as the following series form:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\qquad \alpha\in\mathbb{R}\qquad \alpha>0 \tag{A21}$$

We will discuss the case with $\alpha<0$ separately. The expanded form is the infinite series and looks as follows:

$$E_\alpha(z)=1+\frac{z}{\Gamma(\alpha+1)}+\frac{z^2}{\Gamma(2\alpha+1)}+\frac{z^3}{\Gamma(3\alpha+1)}+\ldots \tag{A22}$$

This function was introduced by Mittag-Leffler in 1903. The Mittag-Leffler function plays an important role in the study of fractional order systems, and is used to express several physical processes. With $\alpha=0$ and $\alpha=1$, we write an interesting relationship as follows:

$$\begin{aligned} E_0(z)&=1+\frac{z}{\Gamma(1)}+\frac{z^2}{\Gamma(1)}+\frac{z^3}{\Gamma(1)}+\ldots\\ &=1+z+z^2+\ldots.\\ &=\frac{1}{1-z}\qquad ;\qquad |z|<1\\ E_0(-z)&=\frac{1}{1+z}\qquad ;\qquad |z|<1 \end{aligned} \tag{A23}$$

The convergence of the above series is valid for $|z|<1$, when for $E_\alpha(z)$, we make the limit $\alpha\downarrow 0^+$. Therefore the series definition restricts the order $\alpha>0$, where $E_\alpha(z)$ is an entire function (analytic everywhere in the complex plane $\mathbb{C}$). With $\alpha=1$ we write the following:

$$\begin{aligned} E_1(z)&=1+\frac{z}{\Gamma(2)}+\frac{z^2}{\Gamma(3)}+\frac{z^3}{\Gamma(4)}+\ldots\\ &=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots.=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z\\ E_1(-z)&=e^{-z} \end{aligned} \tag{A24}$$

Therefore with $\alpha=0$, we get $E_0(-z)$ as the hyperbolic decay function, while with $\alpha=1$, we get $E_1(-z)$ as the pure exponential decay function e^{-z}. This function incorporates the initially a 'stretched exponentials' for small z; (i.e. in limit $z\downarrow 0$) which is expressed as $E_\alpha(-z^\alpha)\sim\exp\left(-\frac{z^\alpha}{\Gamma(1+\alpha)}\right)$ and for large z (i.e. in limit $z\uparrow\infty$) which is the asymptotic approximation of an 'inverse-power-law'; that is expressed as $E_\alpha(-z^\alpha)\sim\frac{z^{-\alpha}}{\Gamma(1-\alpha)}$. For $\alpha=1$, the function coincides with the ordinary exponential function. The following are some asymptotic expressions of the Mittag-Leffler function which are useful in system identification:

$$\Psi(x)=E_\alpha\left(-x^\alpha\right)\cong 1-\frac{x^\alpha}{\Gamma(\alpha+1)}\cong\exp\left(-\frac{x^\alpha}{\Gamma(\alpha+1)}\right);\qquad x\downarrow 0 \tag{A25}$$

for small x, that is $0\le x\ll 1$; and:

$$\Psi(x)=E_\alpha\left(-x^\alpha\right)\cong\left(\frac{\sin(\alpha\pi)}{\pi}\right)\left(\frac{\Gamma(\alpha)}{x^\alpha}\right);\qquad 0<\alpha<1;\qquad x\uparrow\infty \tag{A26}$$

for large x; it is a power-law asymptote. The asymptotic representation of the derivative of the Mittag-Leffler function is as follows:

$$\psi(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\left[E_\alpha\left(-x^\alpha\right)\right]\cong\frac{x^{-(1-\alpha)}}{\Gamma(\alpha)},\qquad x\downarrow 0 \tag{A27}$$

for small x, it is $0\le x\ll 1$; and for large x, it is following:

$$\psi(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\left[E_\alpha\left(-x^\alpha\right)\right]\cong\left(\frac{\sin(\alpha\pi)}{\pi}\right)\left(\frac{\Gamma(\alpha+1)}{x^{\alpha+1}}\right),\qquad 0<\alpha<1\qquad x\uparrow\infty \tag{A28}$$

We have derived an earlier integral representation of the Mittag-Leffler function, via the contour integral in Chapter-6 (where Ha represents Hankel's contour) as follows:

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{Ha} \frac{x^{\alpha-1}}{x^\alpha - z} e^x \mathrm{d}x \tag{A29}$$

The Hankel loop that we have described starts at $-\infty$ and encircles the circular disk $|x| \le |z|^{1/\alpha}$ in a positive sense (i.e. counter-clockwise) such that $-\pi \le \arg x \le \pi$. Some interesting asymptotic expansions of the series for $E_\alpha(z)$ are listed below for $z \uparrow \infty$; and $0 < \alpha < 2$:

$$\begin{aligned} E_\alpha(z) &\sim \frac{1}{\alpha}\exp\left(z^{1/\alpha}\right) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}; \quad |z| \uparrow \infty; \quad |\arg z| < \frac{\alpha\pi}{2} \\ E_\alpha(z) &\sim \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}; \quad |z| \uparrow \infty; \quad \frac{\alpha\pi}{2} < \arg z < \left(2\pi - \alpha\frac{\pi}{2}\right) \end{aligned} \tag{A30}$$

For the case of $\alpha \ge 2$, we have the following expression:

$$E_\alpha(z) \sim \frac{1}{\alpha}\sum_m \exp\left(z^{1/\alpha} e^{\left(2\pi i m/\alpha\right)}\right) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}; \quad |z| \uparrow \infty \tag{A31}$$

Where m takes all integer values such that $-\frac{\alpha\pi}{2} < \arg z + 2\pi m < \frac{\alpha\pi}{2}$, and $\arg z$ can assume any values between $-\pi$ and $+\pi$ inclusively.

A.2.2 The two parameter Mittag-Leffler function

The 'two parameter' Mittag-Leffler functions plays a very important role in fractional calculus. This function type was introduced by R P Agrawal and Erdelyi, in 1953-54. The two-parameter Mittag-Leffler function is defined in series form by use of the following expression:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; \quad \alpha > 0, \quad \beta > 0 \qquad \alpha, \beta \in \mathbb{R} \tag{A32}$$

We get the useful recursive relationship as $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\left(E_{\alpha,\alpha+\beta}(z)\right)$ from the above series definition as demonstrated below:

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{\Gamma(\beta)} + \frac{z}{\Gamma(\alpha+\beta)} + \frac{z^2}{\Gamma(2\alpha+\beta)} + \frac{z^3}{\Gamma(3\alpha+\beta)} + \dots \\ &= \frac{1}{\Gamma(\beta)} + z\left(\frac{1}{\Gamma(\alpha+\beta)} + \frac{z}{\Gamma(\alpha+(\alpha+\beta))} + \frac{z^2}{\Gamma(2\alpha+(\alpha+\beta))} + \dots\right) \\ &= \frac{1}{\Gamma(\beta)} + z\left(\sum_{m=0}^{\infty} \frac{z^m}{\Gamma\left(m\alpha+(\alpha+\beta)\right)}\right) = \frac{1}{\Gamma(\beta)} + z\left(E_{\alpha,(\alpha+\beta)}(z)\right) \end{aligned} \tag{A33}$$

The function $E_{\alpha,\beta}(z)$ has a similar integral representation on Hankel's contour as $E_\alpha(z)$, which is:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{Ha} \frac{x^{\alpha-\beta}}{x^\alpha - z} e^x \mathrm{d}x \tag{A34}$$

(A34) we will derive later in the Appendix, and will use it for extending the series definition for negative arguments of the Mittag-Leffler function, i.e. $\alpha < 0$ for $E_{\alpha,\beta}(z)$.

For $\beta = 1$, we have $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z)$, i.e. the one parameter Mittag-Leffler function. From the recursion expression (A33) i.e. $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\left(E_{\alpha,\alpha+\beta}(z)\right)$, we set $\beta = 1$, and write $E_\alpha(z) = 1 + z\left(E_{\alpha,\alpha+1}(z)\right)$.
The following identities were established from the definition of the Mittag-Leffler function:

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z} \qquad \text{(A35)}$$

$$E_{1,3}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{e^z-1-z}{z^2}$$

The above expressions (A35) have the general form $E_{1,m}(z)=\frac{1}{z^{m-1}}\left(e^z-\sum_{k=0}^{m-2}\frac{z^k}{k!}\right)$.

The trigonometric and hyperbolic ones are also manifestations of the two-parameter Mittag-Leffler function, which is indicated below:

$$E_{2,1}(z^2)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}=\cosh(z),$$

$$E_{2,2}(z^2)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}=\frac{\sinh(z)}{z} \qquad \text{(A36)}$$

In addition, we note the following relationships:

$$E_\alpha(-x)=E_{2\alpha}(x^2)-xE_{2\alpha,1+\alpha}(x^2)$$

$$E_{2\alpha}(x^2)=\frac{E_\alpha(x)+E_\alpha(-x)}{2}$$

$$E_\alpha(-ix)=E_{2\alpha}(-x^2)-ix\left(E_{2\alpha,1+\alpha}(-x^2)\right) \qquad \text{(A37)}$$

$$\mathrm{Re}\left[E_\alpha(-ix)\right]=E_{2\alpha}(-x^2)$$

A.2.3 Graphical representations of Mittag-Leffler function: $f(x)=E_{\alpha,\beta}(x)$

The general graphical representation of $E_{\alpha,\beta}(x)$ is depicted in Figures A1 to A3 for different conditions. The fact that $E_{\alpha,\beta}(x)$ is a complete monotonic function for $0<\alpha\le 1$ with $\beta\ge\alpha$ is shown in Figure-A1, plotted for $E_{1/2,3/2}(x)$. The general behavior of $E_{\alpha,\beta}(x)$ with $0<\alpha\le 1$ but with $\beta<\alpha$ is depicted in Figure-A2, plotted for $E_{1/2,1/8}(x)$. It is a smooth monotonic curve, as shown in Figure-A1 for $z\ge 0$, but shows an inflection point in the region of $x<0$. For $\alpha>1$, the graph of $E_{\alpha,\beta}(x)$ is a smooth curve as shown in Figure-A1, for region $x\ge 0$; but for $x<0$, the graph exhibits indefinite oscillations which continue in the region $x<0$. This is depicted in Figure-A3 plotted for $E_{2,2}(x)$. However, depending on the value of β; the magnitudes of oscillations decrease (or increase). This is shown in Figure-A3, in the negative x-direction.

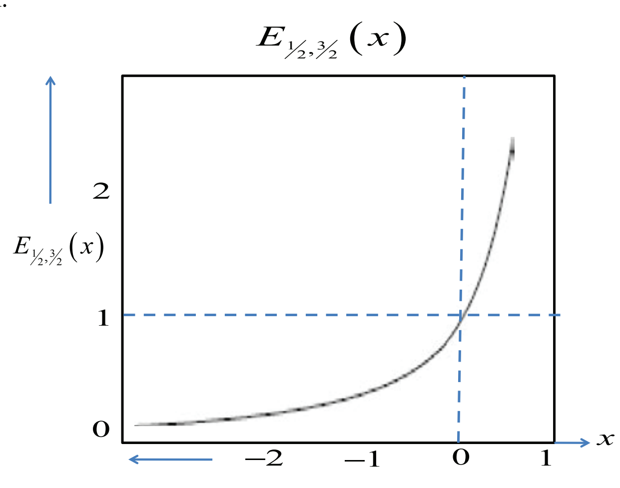

Figure-A1: Illustrating the general behavior of $E_{\alpha,\beta}(x)$ for $0<\alpha\le 1$, and $\alpha\le\beta$

All the graphs depicted in Figures A1 to A-3, when rotated about an ordinate axis, or rotate right, will provide figures for $E_{\alpha,\beta}(-x)$ which resemble the relaxation responses of process dynamics.

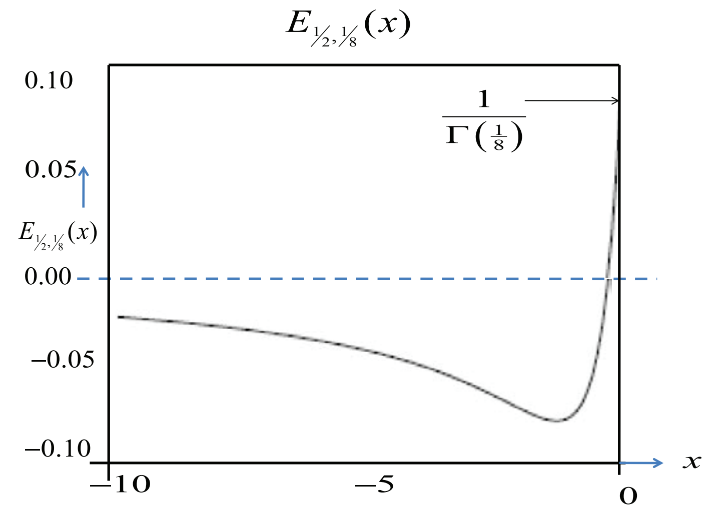

Figure-A2: Illustrating the general behavior of $E_{\alpha,\beta}(x)$ for $0 < \alpha \le 1$, and $\alpha > \beta$

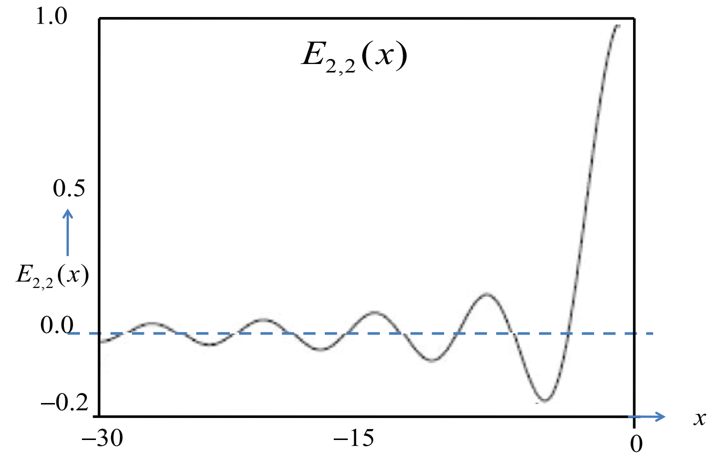

Figure-A3: Illustrating the general behavior of $E_{\alpha,\beta}(x)$ for $\alpha > 1$

A.2.4 Generalized Hyperbolic and Trigonometric Functions

The generalized hyperbolic function of order n, is represented below:

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1} E_{n,r}(z^n), \qquad r = 1,2,3,\ldots,n \tag{A38}$$

The generalized trigonometric function of order n, is represented below:

$$k_r(z,n) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{nm+r-1}}{(nm+r-1)!} = z^{r-1} E_{n,r}(-z^n), \qquad r = 1,2,3,\ldots,n \tag{A39}$$

A.3 The error function and its fractional generalization

The error function defined as

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\mathrm{d}t \tag{A40}$$

Note that $\operatorname{erf}(\infty)=1$. The error function is represented by the series as follows:

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)n!}=\frac{2}{\sqrt{\pi}}\left(z-\frac{z^3}{3}+\frac{z^5}{10}-\frac{z^7}{42}+\frac{z^9}{216}+...\right) \tag{A41}$$

The complementary error function is defined as:

$$\operatorname{erfc}(z)=1-\operatorname{erf}(z)=1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\mathrm{d}t=\frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}\mathrm{d}t \tag{A42}$$

The series asymptotic expansion of the complementary error function and its relationship with the Mittag-Leffler function is the following:

$$\begin{aligned}\operatorname{erfc}(z)&=\frac{e^{-z^2}}{z\sqrt{\pi}}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{1.3.5...(2n-1)}{(2z^2)^n}\right)\\&=\frac{e^{-z^2}}{z\sqrt{\pi}}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!(2z)^{2n}}\right)\\E_{1/2,1}(z)&=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)}=e^{z^2}\operatorname{erfc}(-z)\end{aligned} \tag{A43}$$

The error function in terms of the hyper-geometric function is expressed as the following:

$$\operatorname{erf}(x)=2\pi^{-(1/2)}xe^{-x^2}\left({}_1F_1\left(1,\tfrac{3}{2};x^2\right)\right)=2\pi^{-(1/2)}x\left({}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};-x^2\right)\right) \tag{A44}$$

Error function and Tricomi's incomplete-gamma function are related as follows:

$$\gamma^*\left(\tfrac{1}{2},at\right)=(at)^{(-1/2)}\left[\operatorname{erf}\left(\sqrt{at}\right)\right]\qquad \operatorname{erf}(z)=z\left[\gamma^*\left(\tfrac{1}{2},z^2\right)\right] \tag{A45}$$

This error function integral (A40) has no algebraic formula, so it is numerically evaluated and is tabulated in mathematical handbooks.

The fractional generalization of the error function is $\operatorname{serf}_\alpha(z)$. Where we define an α-stable error function $\operatorname{serf}_\alpha$ similar to the error function, call it a generalized error function; that is twice the integral of α's stable density for argument 0 to z defined as:

$$\operatorname{serf}_\alpha(z)=2\int_0^z s_\alpha(x)\mathrm{d}x \tag{A46}$$

Where $s_\alpha(x)=\exp\left(-|x|^\alpha\right)$ is the standard symmetric α stable density. The error function is thus a special case when the probability density function is normal (Gaussian) with $\alpha=2$, that is:

$$\operatorname{erf}(z)=\operatorname{serf}_2(2z) \tag{A47}$$

A.4 Variants of the Mittag-Leffler function

One variant of the Mittag-Leffler function is described as ξ_t and is the following:

$$\xi_t(\nu,a)=t^{\nu}\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(\nu+k+1)}=t^{\nu}E_{1,\nu+1}(at) \tag{A48}$$

This function (A48) is important for solving fractional differential equations. Another variant $\ni_\alpha$ is the following:

$$\ni_\alpha(\beta,t)=t^{\alpha}\sum_{k=0}^{\infty}\frac{\beta^k t^{k(\alpha+1)}}{\Gamma\left((k+1)(\alpha+1)\right)}=t^{\alpha}E_{\alpha+1,\alpha+1}\left(\beta t^{\alpha+1}\right) \tag{A49}$$

This function above (A49) is called the Robotnov function and one is a special variant too as Sc_α

$$Sc_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{(2-\alpha)n+1}}{\Gamma\left((2-\alpha)n+2\right)}=zE_{2-\alpha,2}\left(-z^{2-\alpha}\right) \tag{A50}$$

The above function (A50) is a fractional sine function form-I. The other is a special variant known as Cs_α which is:

$$Cs_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{(2-\alpha)n}}{\Gamma\left((2-\alpha)n+1\right)}=E_{2-\alpha,1}\left(-z^{2-\alpha}\right) \tag{A51}$$

The above function is a fractional cosine function of the form-I. The fractional sine and cosine function form-II are depicted below:

$$\begin{aligned}\cos_{\lambda,\mu}(z)&=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\Gamma\left(2\mu k+\mu-\lambda+1\right)}=E_{2\mu,\mu-\lambda+1}\left(-z^2\right)\\ \sin_{\lambda,\mu}(z)&=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{\Gamma\left(2\mu k+2\mu-\lambda+1\right)}=zE_{2\mu,2\mu-\lambda+1}\left(-z^2\right)\end{aligned} \tag{A52}$$

Generalization of the Mittag-Leffler function to two variables was suggested and was further extended by Srivastava to the following type of symmetric form.

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}\left(x,y\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m+\frac{\beta(\nu n+1)-1}{\alpha}}\,y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma\left(m\alpha+(\nu n+1)\beta\right)\Gamma\left(n\lambda+(\sigma m+1)\mu\right)} \tag{A53}$$

Several manifestations including several variable representations of the Mittag-Leffler have been made for multi-dimensional studies on the fractional calculus context.

A.5 The Laplace integral and its connection to the Mittag-Leffler function

The Mittag-Leffler function is connected to the Laplace integral by use of the following formula:

$$\int_0^\infty e^{-u}E_\alpha(u^\alpha z)\mathrm{d}u=\frac{1}{1-z},\qquad \int_0^\infty e^{-u}u^{\beta-1}E_{\alpha,\beta}(u^\alpha z)\mathrm{d}u=\frac{1}{1-z} \tag{A54}$$

The proof is as follows. Here we used $E_\alpha(u^\alpha z)=\sum_{k=0}^{\infty}\frac{(u^\alpha z)^k}{\Gamma(\alpha k+1)}$, $E_{\alpha,\beta}(u^\alpha z)=\sum_{k=0}^{\infty}\frac{(u^\alpha z)^k}{\Gamma(\alpha k+\beta)}$ and the integral representation of the gamma function (Section-1.10), i.e. $\Gamma(\upsilon)=\int_0^\infty e^{-u}u^{\upsilon-1}\mathrm{d}u$.

$$\begin{aligned}\int_0^\infty e^{-u}E_\alpha(u^\alpha z)\mathrm{d}u&=\int_0^\infty e^{-u}\left(1+\tfrac{u^\alpha z}{\Gamma(\alpha+1)}+\tfrac{u^{2\alpha}z^2}{\Gamma(2\alpha+1)}+....\right)\mathrm{d}u\\ &=\int_0^\infty e^{-u}\mathrm{d}u+\frac{z}{\Gamma\left(\alpha+1\right)}\int_0^\infty e^{-u}u^\alpha\mathrm{d}u\\ &\quad+\frac{z^2}{\Gamma\left(2\alpha+1\right)}\int_0^\infty e^{-u}u^{2\alpha}\mathrm{d}u+...\\ &=1+z\left(\tfrac{1}{\Gamma(\alpha+1)}\right)\int_0^\infty e^{-u}u^{(\alpha+1)-1}\mathrm{d}u\\ &\quad+z^2\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)\int_0^\infty e^{-u}u^{(2\alpha+1)-1}\mathrm{d}u+...\\ &=1+z\left(\tfrac{1}{\Gamma(\alpha+1)}\right)\Gamma\left(\alpha+1\right)+z^2\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)\Gamma\left(2\alpha+1\right)+...\\ &=1+z+z^2+....=\sum_{k=0}^{\infty}z^k\\ &=\frac{1}{1-z}\end{aligned} \tag{A55}$$

$$\int_0^\infty e^{-u}u^{\beta-1}E_{\alpha,\beta}(u^\alpha z)\mathrm{d}u$$

$$=\int_0^\infty e^{-u}u^{\beta-1}\left(\tfrac{1}{\Gamma(\beta)}+\tfrac{u^\alpha z}{\Gamma(\alpha+\beta)}+\tfrac{u^{2\alpha}z^2}{\Gamma(2\alpha+\beta)}+....\right)\mathrm{d}u$$

$$=\frac{1}{\Gamma\left(\beta\right)}\int_0^\infty e^{-u}u^{\beta-1}\mathrm{d}u+\frac{z}{\Gamma\left(\alpha+\beta\right)}\int_0^\infty e^{-u}u^{\beta-1}u^\alpha\mathrm{d}u$$

$$+\frac{z^2}{\Gamma\left(2\alpha+\beta\right)}\int_0^\infty e^{-u}u^{\beta-1}u^{2\alpha}\mathrm{d}u+..$$

$$=\tfrac{1}{\Gamma(\beta)}\int_0^\infty e^{-u}u^{\beta-1}\mathrm{d}u+z\left(\tfrac{1}{\Gamma(\alpha+\beta)}\right)\int_0^\infty e^{-u}u^{(\alpha+\beta)-1}\mathrm{d}u$$

$$+z^2\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)\int_0^\infty e^{-u}u^{(2\alpha+\beta)-1}\mathrm{d}u+...$$

$$=\tfrac{1}{\Gamma(\beta)}\Gamma\left(\beta\right)+z\left(\tfrac{1}{\Gamma(\alpha+\beta)}\right)\Gamma\left(\alpha+\beta\right)$$

$$+z^2\left(\tfrac{1}{\Gamma(2\alpha+1)}\right)\Gamma\left(2\alpha+\beta\right)+...$$

$$=1+z+z^2+....=\sum_{k=0}^{\infty}z^k$$

$$=\frac{1}{1-z} \tag{A56}$$

The above integrals are the fundamental formula for the evaluation of the Laplace transforms of $E_\alpha(-at^\alpha)$ and $E_{\alpha,\beta}(-at^\alpha)$, with $\alpha,\beta>0$ and $a\in\mathbb{C}$. Put $u^\alpha z=-at^\alpha$, and $u=st$, with $t\geq 0$, and $a\in\mathbb{C}$ in the above (A54) Laplace integral relationship to get the following steps by juxtaposition of the variable:

$$\int_0^\infty e^{-u}E_\alpha(u^\alpha z)\mathrm{d}u=\frac{1}{1-z}$$

$$\int_0^\infty e^{-st}E_\alpha(-at^\alpha)\left(s\right)\mathrm{d}t=\frac{1}{1-\left(-a\,\frac{t^\alpha}{u^\alpha}\right)}$$

$$s\int_0^\infty e^{-st}E_\alpha(-at^\alpha)\mathrm{d}t=\frac{1}{1+a\left(\frac{1}{s^\alpha}\right)} \tag{A57}$$

$$s\int_0^\infty e^{-st}E_\alpha(-at^\alpha)\mathrm{d}t=\frac{s^\alpha}{s^\alpha+a}$$

$$\int_0^\infty e^{-st}E_\alpha(-at^\alpha)\mathrm{d}t=\frac{s^{\alpha-1}}{s^\alpha+a}$$

Using the definition of the Laplace transform, i.e. $\mathcal{L}\left\{f(t)\right\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$, we write from the above derivation (A57) the following equation:

$$\mathcal{L}\left\{E_\alpha(-at^\alpha)\right\}=\frac{s^{\alpha-1}}{s^\alpha+a} \tag{A58}$$

Similarly as done in (A57) from relationship (A54), we carry out the following steps:

$$\int_0^\infty e^{-u}u^{\beta-1}E_{\alpha,\beta}(u^\alpha z)\mathrm{d}u=\frac{1}{1-z}$$

$$\int_0^\infty e^{-st}(st)^{\beta-1}E_{\alpha,\beta}(-at^\alpha)\left(s\right)\mathrm{d}t=\frac{1}{1-\left(-a\frac{t^\alpha}{u^\alpha}\right)}$$

$$s^\beta\int_0^\infty e^{-st}t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)\mathrm{d}t=\frac{1}{1+a\left(\frac{1}{s^\alpha}\right)} \tag{A59}$$

$$s^\beta\int_0^\infty e^{-st}t^{\beta-1}E_{\alpha}(-at^\alpha)\mathrm{d}t=\frac{s^\alpha}{s^\alpha+a}$$

$$\int_0^\infty e^{-st}\left(t^{\beta-1}E_{\alpha}(-at^\alpha)\right)\mathrm{d}t=\frac{s^{\alpha-\beta}}{s^\alpha+a}$$

We get the following useful Laplace identity:

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha+a} \tag{A60}$$

A.6 The Laplace transforms of the Mittag-Leffler function and several other variants

The following expressions give some identities for Laplace transforms pairs of Mittag-Leffler functions:

$$\begin{aligned}
&\mathcal{L}\left\{t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(at^\alpha)\right\}=\frac{s^{\alpha-\beta}k!}{(s^\alpha-a)^{k+1}};\quad \mathrm{Re}(s)>\left|a\right|^{(1/\alpha)}\\
&\mathcal{L}\left\{t^{\left(\frac{1}{2}\right)k-\left(\frac{1}{2}\right)}E^{(k)}_{(1/2),(1/2)}\left(a\sqrt{t}\right)\right\}=\frac{k!}{\left(\sqrt{s}-a\right)^{k+1}};\quad \alpha=\beta=\frac{1}{2}\\
&\mathcal{L}\left\{t^{-1/2}E_{1/2,1/2}\left(a\sqrt{t}\right)\right\}=\frac{1}{\sqrt{s}-a};\quad \alpha=\beta=\frac{1}{2},\quad k=0\\
&\mathcal{L}\left\{t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)\right\}=\frac{1}{s^\alpha-a};\quad \alpha=\beta,\quad k=0
\end{aligned} \tag{A61}$$

Here in (A61), $E^{(k)}_{\alpha,\beta}(at^\alpha)=\frac{\mathrm{d}^k}{\mathrm{d}t^k}E_{\alpha,\beta}(at^\alpha)$ means the $k-$th integer order derivative or integration. For $k>0$, the operation is the differentiation of the Mittag-Leffler function, and for $k<0$ the operation is the integration of the Mittag-Leffler function. The following lists some Laplace transform identities:

$$\beta = 1, \qquad k = 0, \qquad E_{\alpha,\beta}^{(k)}(at^{\alpha}) \equiv E_{\alpha,1}(at^{\alpha})$$

$$\mathcal{L}\left\{E_{\alpha}(at^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha} - a}$$

$$\mathcal{L}\left\{E_{\alpha}(-\lambda t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha} + \lambda} = \frac{s^{\alpha}}{s(s^{\alpha} + \lambda)}; \quad \alpha > 0$$

$$\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha} + 1} = \frac{s^{\alpha}}{s(s^{\alpha} + 1)}$$

$$\mathcal{L}\left\{E_{\alpha}(\lambda t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha} - \lambda} = \frac{s^{\alpha}}{s(s^{\alpha} - \lambda)}$$

$$\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha}(-\lambda t^{\alpha})\right\} = s\left(\mathcal{L}\left\{E_{\alpha}(-\lambda t^{\alpha})\right\}\right) = \frac{s^{\alpha}}{s^{\alpha} + \lambda}$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}} E_{\alpha}(-t^{\alpha})\right\} = s^{k}\left(\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}\right) = \frac{s^{\alpha+k-1}}{s^{\alpha} + 1}$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^{-1}}{\mathrm{d}t^{-1}} E_{\alpha}(-t^{\alpha})\right\} = s^{-1}\left(\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}\right) = \frac{1}{s}\frac{s^{\alpha-1}}{s^{\alpha} + 1} = \frac{s^{\alpha-2}}{s^{\alpha} + 1}$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^{-k}}{\mathrm{d}t^{-k}} E_{\alpha}(-t^{\alpha})\right\} = s^{-k}\left(\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}\right) = \frac{1}{s^{k}}\frac{s^{\alpha-1}}{s^{\alpha} + 1} = \frac{s^{\alpha-k-1}}{s^{\alpha} + 1}$$

$$\mathcal{L}\left\{\delta(t) - \frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha}(-\lambda t^{\alpha})\right\} = 1 - \frac{s^{\alpha}}{s^{\alpha} + \lambda} = \frac{\lambda}{s^{\alpha} + \lambda} \tag{A62}$$

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha} - 1} \qquad \alpha, \beta > 0$$

A.7 Agrawal Function

The Mittag-Leffler function $E_{\alpha}(z)$ was further generalized by R.P Agrawal in 1953. This function is particularly interesting to the fractional order system theory due to its Laplace transform as given by Agrawal. The function is defined as follows:

$$t^{\beta-1} E_{\alpha,\beta}(t^{\alpha}) = \sum_{k=0}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k + \beta)} \qquad \alpha, \beta > 0$$

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha} - 1}$$

$$\beta = 1 \qquad E_{\alpha,1}(t^{\alpha}) = E_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}; \tag{A63}$$

$$\mathcal{L}\left\{E_{\alpha,\beta}(t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha} - 1}$$

A.8 Erdelyi's Function

Erdelyi (1954) studied the generalization of the Mittag-Leffler function as follows:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k + \beta)}; \quad \alpha, \beta > 0$$

$$\mathcal{L}\left\{E_{\alpha,\beta}(t)\right\} = \sum_{k=0}^{\infty} \frac{k!}{s^{k+1}\Gamma(\alpha k + \beta)} \tag{A64}$$

Where the powers of t are integers, k is an integer number.

A.9 The Robotnov-Hartley function

To affect the direct solution of the fundamental linear fractional order differential equations, the following function was introduced by Robotnov and Hartley in 1998:

$$F_q(-a,t) = t^{q-1}\sum_{n=0}^{\infty}\frac{(-a)^n t^{nq}}{\Gamma(nq+q)};\quad q>0\quad \mathcal{L}\left\{F_q(-a,t)\right\} = \frac{1}{s^q+a}$$

$$F_q(-a,t) = t^{q-1}E_{q,q}(-at^q) = t^{q-1}\sum_{n=0}^{\infty}\frac{(a)^n t^{qn}}{\Gamma(qn+q)} \tag{A65}$$

This function is the 'impulse-response' of the fundamental fractional differential equation, and is used by control system analysis in order to obtain the forced or the initialized system reaction.

A.10 Miller-Ross function

Miller and Ross in 1993 introduced a function as the basis of the solution of a fractional order initial value problem. It is defined as the v-th integral of the exponential function, that is:

$$E_t(v,a) = \frac{\mathrm{d}^{-v}}{\mathrm{d}t^{-v}}\left[e^{at}\right] = t^v\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(v+k+1)};\quad \mathcal{L}\left\{E_t(v,a)\right\} = \frac{s^{-v}}{s-a} \tag{A66}$$

In (A66) we note $\mathrm{Re}[v]>1$

The Miller-Ross function in terms of a hyper-geometric function is:

$$E_t(v,a) = \frac{t^v}{\Gamma(v+1)}\,{}_1F_1(1,v+1;at) \tag{A67}$$

The one-parameter Mittag-Leffler function for $w \ge 0$ is $E_w(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(1+nw)}$. Expanding this $E_w(ct^w)$ series and with the grouping we get the expression as demonstrated in the following steps:

$$\begin{aligned}
E_w(ct^w) &= \sum_{n=0}^{\infty}\frac{(ct^w)^n}{\Gamma(1+nw)}\\
&= \sum_{n=0,q,2q,\ldots\ldots}^{\infty}\frac{(ct^w)^n}{\Gamma(1+nw)} + \sum_{n=1,q+1,2q+1,\ldots}^{\infty}\frac{(ct^w)^n}{\Gamma(1+nw)} + \ldots.\\
&\quad \ldots + \sum_{n=q-1,2q-1,3q-1,\ldots}^{\infty}\frac{(ct^w)^n}{\Gamma(1+nw)}\\
&= E_t(0,c^q) + cE_t(w,c^q) + c^2E_t(2w,c^q) + \ldots\\
&\quad \ldots.. + c^{q-1}E_t\left((q-1)w,c^q\right)
\end{aligned} \tag{A68}$$

Therefore:

$$E_w(ct^w) = \sum_{k-0}^{q-1}c^kE_t(kw,c^q) \tag{A69}$$

is the relationship between the one-parameter Mittag-Leffler and Miller-Ross functions.
In terms of Tricomi's incomplete gamma function, we define the Miller-Ross function as:

$$E_z(v,a) = z^v e^{az}\gamma^*(v,az) \tag{A70}$$

We have the following identity for a generalized cosine and sine function with the Miller-Ross formulation.

$$\begin{aligned}
C_z(v,a) &= \frac{\left(E_z(v,ia)+E_z(v,-ia)\right)}{2}\\
S_z(v,a) &= \frac{\left(E_z(v,ia)-E_z(v,-ia)\right)}{2i}
\end{aligned} \tag{A71}$$

where $C_z(v,a)$ and $S_z(v,a)$ are generalized cosine and sine functions respectively, with the Miller-Ross formulation. These identities of Miller-Ross expressions are similar to the exponential function and its polar representation is to get the cosine and sine trigonometric functions. These functions play a crucial role in fractional differential equation solutions, and are similar to the normal transcendental and trigonometric functions. In the above definition (A66) of the Miller-Ross function let a be replaced by a purely imaginary number as ia. The series form we can re-write by organizing even and odd terms together as the following:

$$E_t(v,ia) = t^v\left(\sum_{k=\text{even}}^{\infty}\frac{(-1)^{k/2}(at)^k}{\Gamma(v+k+1)} + i\sum_{k=\text{odd}}^{\infty}\frac{(-1)^{\frac{(k-1)}{2}}(at)^k}{\Gamma(v+k+1)}\right)$$

$$E_t(v,ia) = C_t(v,a) + iS_t(v,a) \qquad E_t(0,it) = e^{it} \tag{A72}$$

$$e^{it} = \cos t + i\sin t$$

The above (A72) identities indicate the similarity of the Miller-Ross function with an exponential function and trigonometric functions.

Integration of the Miller-Ross function is the following:

$$\int_0^t E_u(v,a)\mathrm{d}u = E_t(v+1,a) \qquad \mathrm{Re}\left[v\right] > -1$$

$$\int_0^t u^w E_{t-u}(v,a)\mathrm{d}u = \Gamma(w+1)E_t(v+w+1,a) \tag{A73}$$

$$\mathrm{Re}\left[w\right] > -1 \qquad \mathrm{Re}\left[v\right] > -1$$

$$D^{-p}\left[E_t(v,a)\right] = E_t(v+p,a) \qquad p = 1,2,3,... \qquad \mathrm{Re}\left[v\right] > -1$$

Differentiation of the Miller Ross Function is the following:

$$D\left[E_t(v,a)\right] = E_t(v-1,a)$$

$$D^p\left[E_t(v,a)\right] = E_t(v-p,a) = a^p E_t(v,a) + \sum_{k=0}^{p-1}\frac{a^k t^{v+k-p}}{\Gamma(v+k+1-p)}$$

$$p = 0,1,2,3...$$

$$D\left[tE_t(v,a)\right] = tE_t(v-1,a) + E_t(v,a) \tag{A74}$$

$$D\left[t^\mu E_t(v,a)\right] = t^\mu E_t(v-1,a) + \mu t^{\mu-1}E_t(v,a)$$

$$D^p\left[t^\mu E_t(v,a)\right] = \sum_{k=0}^{p}\binom{p}{k}\frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)}t^{\mu-k}E_t(v+k-p,a)$$

$$p = 0,1,2,..$$

Special values of the Miller-Ross function are the following:

$$E_t(0,a) = e^{at}$$

$$E_0(v,a) = 0 \qquad \mathrm{Re}\left[v\right] = 0$$

$$E_t(-1,a) = aE_t(0,a)$$

$$E_t(-p,a) = a^p E_t(0,a) \qquad p = 0,1,2,3,...$$

$$E_t(1,a) = \frac{E_t(0,a)-1}{a} \tag{A75}$$

$$E_t\left(\tfrac{1}{2},a\right) = a^{-1/2}e^{-at}\left(\mathrm{erf}\sqrt{at}\right)$$

$$E_t\left(-\tfrac{1}{2},a\right) = aE_t\left(\tfrac{1}{2},a\right) + \frac{1}{\sqrt{\pi t}}t^{-1/2}$$

$$E_t(v,0) = \frac{t^v}{\Gamma(v+1)}$$

Recursive relationships of the Miller-Ross functions are listed as follows:

$$E_t(v,a) = aE_t(v+1,a) + \frac{t^v}{\Gamma(v+1)}$$

$$E_t(v,a) = a^p E_t(v+p,a) + \sum_{k=0}^{p-1} \frac{t^{v+k}}{\Gamma(v+k+1)} \qquad p = 0,1,2,3,...$$

$$E_t(v,a) - E_t(v,b) = aE_t(v+1,a) - bE_t(v+1,b) \tag{A76}$$

$$E_t(v,a) - E_t(v,b) = a^p E_t(v+p,a) - b^p E_t(v+p,b) + \sum_{k=1}^{p-1} \frac{(a^k - b^k)t^{v+k}}{\Gamma(v+k+1)} \qquad p = 0,1,2...$$

Integral representation of the Miller-Ross function is the following:

$$E_t(v,a) = \frac{1}{\Gamma(v)} \int_0^t u^{v-1} e^{a(t-u)} \mathrm{d}u \qquad \mathrm{Re}\left[v\right] > 0 \tag{A77}$$

This Miller-Ross function $E_t(v,a)$ is the solution of the ordinary differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} - ay = \frac{t^{v-1}}{\Gamma(v)} \qquad v > 0 \tag{A78}$$

A.11 The generalized cosine and sine function in the Miller-Ross formulation

In the Miller-Ross modification above (A72), the generalized cosine function is defined as the following:

$$C_t(v,a) = t^v \sum_{k=\mathrm{even}}^{\infty} \frac{(-1)^{k/2} (at)^k}{\Gamma(v+k+1)} \qquad C_t(0,a) = \cos(at) \tag{A79}$$

The Laplace transform of the generalized cosine function is as below:

$$\mathcal{L}\left\{C_t(v,a)\right\} = \frac{s}{s^v(s^2+a^2)} \qquad \mathrm{Re}\left[v\right] > -1 \tag{A80}$$

In the Miller-Ross modification above (A72), the generalized sine function is defined as follows:

$$S_t(v,a) = t^v \sum_{k=\mathrm{odd}}^{\infty} \frac{(-1)^{\frac{(k-1)}{2}} (at)^k}{\Gamma(v+k+1)} \qquad S_t(0,a) = \sin(at) \tag{A81}$$

The Laplace transform of the generalized sine function is the following:

$$\mathcal{L}\left\{S_t(v,a)\right\} = \frac{a}{s^v(s^2+a^2)} \qquad \mathrm{Re}\left[v\right] > -2 \tag{A82}$$

Integral representations of the generalized cosine and sine functions are the following:

$$\begin{aligned} C_t(v,a) &= \frac{1}{\Gamma(v)} \int_0^t u^{v-1} \cos\left(a(t-u)\right) \mathrm{d}u \\ S_t(v,a) &= \frac{1}{\Gamma(v)} \int_0^t u^{v-1} \sin\left(a(t-u)\right) \mathrm{d}u \qquad \mathrm{Re}\left[v\right] > 0 \end{aligned} \tag{A83}$$

Special values of the generalized cosine and sine functions are:

$$\begin{aligned}
&C_t(0,a)=\cos(at) \qquad S_t(0,a)=\sin(at)\\
&C_0(v,a)=0 \qquad S_0(v,a)=0 \qquad \mathrm{Re}\left[v\right]>-1\\
&C_t(-1,a)=-a\sin(at) \qquad S_t(-1,a)=a\cos(at)\\
&C_t(-p,a)=(-1)^{p/2}a^p\cos(at) \qquad S_t(-p,a)=(-1)^{p/2}a^p\sin(at)\\
&p=0,2,4,...\\
&C_t(-p,a)=(-1)^{(p+1)/2}a^p\sin(at)\\
&S_t(-p,a)=(-1)^{(p-1)/2}a^p\cos(at) \qquad p=1,3,5,...\\
&C_t(1,a)=\tfrac{1}{a}\sin(at) \qquad S_t(1,a)=\tfrac{2}{a}\sin^2\left(\tfrac{1}{2}at\right)\\
&C_t\left(-\tfrac{1}{2},a\right)=\frac{1}{\sqrt{\pi t}}-aS_t\left(\tfrac{1}{2},a\right) \qquad S_t\left(-\tfrac{1}{2},a\right)=aC_t\left(\tfrac{1}{2},a\right)\\
&C_t(v,0)=\tfrac{1}{\Gamma(v+1)}t^v \qquad S_t(v,0)=0\\
&C_t\left(\tfrac{1}{2},a\right)=\sqrt{\frac{2}{a}}\left((\cos at)C(z)+(\sin at)S(z)\right)\\
&S_t\left(\tfrac{1}{2},a\right)=\sqrt{\frac{2}{a}}\left((\sin at)C(z)-(\cos at)S(z)\right)
\end{aligned} \tag{A84}$$

Where $z=\sqrt{\frac{2at}{\pi}}$ and $C(z)$, $S(z)$ are used in Fresnel integrals (A84), they are defined as follows:

$$C(z)=\int_0^z\cos\left(\tfrac{1}{2}\pi t^2\right)\mathrm{d}t \qquad S(z)=\int_0^z\sin\left(\tfrac{1}{2}\pi t^2\right)\mathrm{d}t \tag{A85}$$

The recursive relationship of the generalized cosine and sine functions are the following:

$$\begin{aligned}
&C_t(v-1,a)=-aS_t(v,a)+\frac{t^{v-1}}{\Gamma(v)} \qquad S_t(v-1,a)=aC_t(v,a)\\
&C_t(v-1,a)+a^2C_t(v+1,a)=\frac{t^{v-1}}{\Gamma(v)}\\
&S_t(v-1,a)+a^2S_t(v+1,a)=\frac{at^v}{\Gamma(v+1)}
\end{aligned} \tag{A86}$$

Integration of the generalized cosine and sine functions are the following:

$$\begin{aligned}
&\int_0^t C_u(v,a)\mathrm{d}u=C_t(v+1,a), \quad \mathrm{Re}\left[v\right]>-1\\
&\int_0^t S_u(v,a)\mathrm{d}u=S_t(v+1,a); \quad \mathrm{Re}\left[v\right]>-2
\end{aligned} \tag{A87}$$

The differentiation of the generalized cosine and sine functions are the following:

$$\begin{aligned}
&D\left[C_t(v,a)\right]=C_t(v-1,a) \qquad D\left[S_t(v,a)\right]=S_t(v-1,a)\\
&D^m\left[C_t(v,a)\right]=C_t(v-m,a) \qquad D^m\left[S_t(v,a)\right]=S_t(v-m,a)\\
&m=0,1,2,...\\
&D\left[tC_t(v,a)\right]=tC_t(v-1,a)+C_t(v,a)\\
&D\left[tS_t(v,a)\right]=tS_t(v-1,a)+S_t(v,a)\\
&D\left[t^\mu C_t(v,a)\right]=t^\mu C_t(v-1,a)+\mu t^{\mu-1}C_t(v,a)\\
&D\left[t^\mu S_t(v,a)\right]=t^\mu S_t(v-1,a)+\mu t^{\mu-1}S_t(v,a)
\end{aligned} \tag{A88}$$

This generalized cosine and sine function $S_t(v,a)$ or $aC_t(v+1,a)$ is the solution to the ordinary differential Equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a^2 y = a\frac{t^{v-1}}{\Gamma(v)} \qquad v > 0 \tag{A89}$$

This detailed description of Miller-Ross E_t and the generalized cosine C_t and sine S_t functions is given in order to draw attention towards their similarity with exponential, cosine and sine functions. These similarities in property encourage us to select these functions as a solution to fractional differential equations. Similarly, one can construct generalized hyperbolic cosine and generalized hyperbolic sine functions with similar properties as described above for generalized cosine and sine functions.

Generalized hyperbolic cosine and sine functions are constructed as follows:

$$\begin{aligned} HC_t(v,a) &= \frac{1}{\Gamma(v)}\int_0^t u^{v-1}\cosh\left(a(t-u)\right)\mathrm{d}u \\ HS_t(v,a) &= \frac{1}{\Gamma(v)}\int_0^t u^{v-1}\sinh\left(a(t-u)\right)\mathrm{d}u, \quad \mathrm{Re}\left[v\right] > -1 \end{aligned} \tag{A90}$$

A.12 Generalized *R* function and *G* function

It is of significant usefulness to develop a generalized function, which, when fractionally differentiated or integrated (differ-integrated) by any order, returns itself. Like exponential, trigonometric, hyperbolic functions of integer order calculus, the definitions of such generalized Mittag-Leffler functions are important in fractional calculus. In earlier sections, some variants of Mittag-Leffler were noted. Here a more generalized R function and G function is introduced as:

$$R_{q,v}\left(a,c,t\right) = \sum_{n=0}^{\infty}\frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\left((n+1)q-v\right)} \equiv R_{q,v}\left(a,t-c\right) \tag{A91}$$

In (A91), t is an independent variable where c is the lower limit of fractional differ-integration. Our interest in this function will be normal for the range $t > c$. The Laplace transforms of the R function are the following:

$$\begin{aligned} \mathcal{L}\left\{R_{q,v}(a,0,t)\right\} &= \frac{s^v}{s^q - a}; \quad \mathrm{Re}\left[(q-v)\right] > 0; \quad \mathrm{Re}\left[s\right] > 0 \\ \mathcal{L}\left\{R_{q,v}(a,c,t)\right\} &= \frac{e^{-cs}s^v}{s^q - a}; \quad c \geq 0 \quad \mathrm{Re}\left[(q-v)\right] > 0; \mathrm{Re}\left[s\right] > 0 \end{aligned} \tag{A92}$$

The relationship of the R function to elementary functions are following:

$$\begin{aligned} & R_{1,0}(a,0,t) = e^{at} \\ & aR_{2,0}(-a^2,0,t) = a\left(t - \frac{a^2t^3}{3!} + \frac{a^4t^5}{5!} - ...\right) = \sin(at) \\ & R_{2,1}(-a^2,0,t) = \left(1 - \frac{a^2t^2}{2!} + \frac{a^4t^4}{4!} - ...\right) = \cos(at) \\ & aR_{2,0}(a^2,0,t) = \sinh(at) \\ & R_{2,1}(a^2,0,t) = \cosh(at) \\ & R_{1,0}(a,0,x) = e^{ax} \end{aligned} \tag{A93}$$

The relationship of the R function to other generalized functions:
The Mittag-Leffler function

$$\begin{aligned} & \mathcal{L}\left\{E_q\left(-at^q\right)\right\} = \frac{1}{s}\left(\frac{s^q}{s^q + a}\right) = \frac{s^{q-1}}{s^q + a}, \qquad q > 0 \\ & \mathcal{L}\left\{E_q(-at^q)\right\} = \frac{s^{q-1}}{s^q + a} = \mathcal{L}\left\{R_{q,q-1}(-a,0,t)\right\}; \\ & {}_cD_t^{q-1}\left[R_{q,0}(-a,c,t)\right] = E_q\left(-a(t-c)^q\right) \end{aligned} \tag{A94}$$

The Agrawal function

$$\mathcal{L}\left\{E_{q,p}(t^q)\right\} = \frac{s^{q-p}}{s^q - 1} = \mathcal{L}\left\{R_{q,q-p}(1,0,t)\right\}$$
$$R_{q,q-p}(1,0,t) = E_{q,p}(t^q) \tag{A95}$$

Erdelyi's function

$$t^{1-\beta}E_{q,\beta}(t^q) = R_{q,q-\beta}(1,0,t) = t^{1-\beta}\sum_{n=0}^{\infty}\frac{t^{nq}}{\Gamma(nq+1)} \tag{A96}$$

The Robotnov and Hartley function

$$\mathcal{L}\left\{F_q(-a,t)\right\} = \frac{1}{s^q + a} = \mathcal{L}\left\{R_{q,0}(-a,0,t)\right\};$$
$$R_{q,0}(-a,0,t) = \sum_{n=0}^{\infty}\frac{(-a)^n t^{(n+1)q-1}}{\Gamma\left((n+1)q\right)} = F_q(-a,t)$$
$$\mathcal{L}\left\{F_q(a,t)\right\} = \frac{1}{s^q - a}, \qquad q > 0$$
$$\mathcal{L}\left\{E_q\left(-at^q\right)\right\} = \frac{1}{s}\left(s^q\mathcal{L}\left\{F_q(-a,t)\right\}\right)$$
$${}_0D_t^{q-1}\left[F_q(a,t)\right] = E_q(at^q)$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^q + a)}\right\} = \left(\frac{1}{a}\left(u(t) - E_q(-at^q)\right)\right) = {}_0D_t^{-q}\left[E_q(at^q)\right]$$
$$\mathcal{L}^{-1}\left\{\frac{s^q}{s^q + a}\right\} = {}_0D_t^{q}\left[F_q(-a,t)\right] = {}_0D_t^{1}\left[E_q(-at^q)\right]$$
$$= \mathcal{L}^{-1}\left\{1 - \frac{a}{s^q + a}\right\} = \delta(t) - aF_q(-a,t) \tag{A97}$$

Where $u(t)$ is the Heaviside unit step function at $t = 0$ and $\delta(t)$ is the Dirac delta function in (A97).
The Miller and Ross function:

$$\mathcal{L}\left\{E_t(v,a)\right\} = \frac{s^{-v}}{s-a} = \mathcal{L}\left\{R_{1,-v}(a,0,t)\right\},$$
$$R_{1,-v}(a,0,t) = \sum_{n=0}^{\infty}\frac{(a)^n t^{n+v}}{\Gamma(n+v+1)} = E_t(v,a) \tag{A98}$$

The further generalized function (G function)

$$G_{q,v,r}(a,t) = \sum_{j=0}^{\infty}\frac{\left((-r)(-1-r)...(1-j-r)\right)(-a)^j t^{(r+j)q-v-1}}{\Gamma(1+j)\Gamma\left((r+j)q-v\right)}$$
$$\mathcal{L}\left\{G_{q,v,r}(a,t)\right\} = \frac{s^v}{(s^q - a)}; \quad \mathrm{Re}\left[(qr-v)\right] > 0 \quad \mathrm{Re}\left[s\right] > 0 \quad \left|\tfrac{a}{s^q}\right| > 0 \tag{A99}$$

A.13-Bessel Function

Perhaps amongst all higher transcendental functions, the Bessel functions are the most ubiquitous; they appear very frequently in theoretical physics and engineering. The Bessel function, $J_v(z)$ of the first kind and order v is defined as an infinite series as the following:

$$J_v(z) = \left(\tfrac{1}{2}z\right)^v\sum_{k=0}^{\infty}\frac{(-1)^k\left(\tfrac{1}{2}z\right)^{2k}}{k!\,\Gamma(v+k+1)} \tag{A100}$$

The, $J_v(z)$ is an entire function of v. This is the solution of the Bessel equation, which is the following:

$$z^2 D^2\left[w\right] + zD\left[w\right] + (z^2 - v^2)w = 0 \tag{A101}$$

The Bessel function $Y_v(z)$ of the second kind and order v is also the solution of the Bessel equation, linearly independent of $J_v(z)$. One of the integral representations of the Bessel function is Poisson's formula, as follows:

$$J_v(z) = \frac{2}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(v+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^v \int_0^1 (1-t^2)^{v-(1/2)} \cos(zt)\mathrm{d}t; \quad \mathrm{Re}\left[v\right] > -\frac{1}{2} \tag{A102}$$

Another representation is Sonin's formula, as follows:

$$J_{\mu+v+1}(z) = \frac{z^{v+1}}{2^v \Gamma(v+1)} \int_0^{\left(\frac{1}{2}\right)\pi} J_\mu\left(z\sin\theta\right)\left(\sin^{\mu+1}\theta\right)\left(\cos^{2v+1}\theta\right)\mathrm{d}\theta$$
$$\mathrm{Re}\left[\mu\right], \mathrm{Re}\left[v\right] > 1 \tag{A103}$$

Some special relationships of the elementary function to the Bessel functions are:

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z) \quad , \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z) \tag{A104}$$

The modified Bessel function $I_v(z)$ of the first kind of order v is defined as the infinite series:

$$I_v(z) = \left(\frac{1}{2}z\right)^v \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(v+k+1)}\left(\frac{1}{2}z\right)^{2k} \tag{A105}$$

(A105) is the solution of the modified Bessel equation, that is:

$$z^2 D^2\left[w\right] + zD\left[w\right] - (z^2 + v^2)w = 0 \tag{A106}$$

The modified Bessel function $K_v(z)$ of the second kind is also the solution of the modified Bessel equation and is linearly independent of $I_v(z)$.

For $\mathrm{Re}\left[v\right] > -\frac{1}{2}$ and $\mathrm{Re}\left[z\right] > 0$, the integral representation is:

$$\begin{aligned} K_v(z) &= \frac{\pi^{(1/2)}\left(\frac{1}{2}z\right)^v}{\Gamma\left(v+\frac{1}{2}\right)} \int_0^\infty \left(\sinh\theta\right)^{2v} e^{-z\cosh\theta}\mathrm{d}\theta \\ &= \int_0^\infty \left(\cosh(v\theta)\right)\left(e^{-z\cosh\theta}\right)\mathrm{d}\theta \end{aligned} \tag{A107}$$

Particularly for a non-negative integer v, we may write the infinite series for $K_v(z)$. For example, we write the series for $v = 0$ as:

$$K_0(z) = -\left(\ln\left(\frac{1}{2}z\right)\right)I_0(z) + \sum_{n=0}^{\infty} \frac{\psi(n+1)}{(n!)^2}\left(\frac{1}{2}z\right)^{2n} \tag{A108}$$

Here $\psi(z)$ is the 'psi' function which is a derivative of the logarithm of the Gamma function that is $\psi(z) = D\ln\left(\Gamma(z)\right) = \frac{D(\Gamma(z))}{z}$, which we have discussed in Chapter-1.

Some relationships between special values in relation to the elementary functions are the following:

$$\begin{aligned} &I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh z, \quad I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cosh z \\ &K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \end{aligned} \tag{A109}$$

A.14 Wright Function

The Wright function, which we denote by $W_{\lambda,\mu}(z)$, is named after E. Maitland Wright (a British mathematician), who introduced this function in 1933. The function is defined by the series representation convergent in the entire z – complex plane:

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)},\quad \lambda>-1,\quad \mu\in\mathbb{C} \tag{A110}$$

The integral representation of the Wright's function reads as follows:

$$W_{\lambda,\mu}(z)=\frac{1}{2\pi i}\int_{Ha} e^{\sigma+z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^{\mu}}\quad ;\quad \lambda>-1,\quad \mu\in\mathbb{C} \tag{A111}$$

The origin of the above (A111) can be related in a similar way to those obtained for auxiliary function $M_{\upsilon}(z)$, in Chapter-8. Using the Hankel representation of the reciprocal of the gamma function, that is $\frac{1}{\Gamma(z)}=\int_{Ha} e^{u}u^{-z}\mathrm{d}u$ for $z\in\mathbb{C}$, we obtain the following series representation for $W_{\lambda,\mu}(z)$ as:

$$\begin{aligned}W_{\lambda,\mu}(z)&=\frac{1}{2\pi i}\int_{Ha} e^{\sigma+z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^{\mu}}\\&=\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\sigma^{-\lambda n}\right)\frac{\mathrm{d}\sigma}{\sigma^{\mu}}\\&=\sum_{n=0}^{\infty}\frac{z^n}{n!}\left(\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\sigma^{-\lambda n-\mu}\mathrm{d}\sigma\right)\\&=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)}\end{aligned} \tag{A112}$$

For $\lambda=0$, we get $W_{0,\mu}(z)=\frac{e^z}{\Gamma(\mu)}$ provided $\mu\neq 0,-1,-2,...$

A.15 Prabhakar Function

A-15.1 The three parameter Mittag-Leffler function $f(z)=E_{\alpha,\beta}^{\gamma}(z)$

This function is a modification of the two parameter Mittag-Leffler function carried out by T.R. Prabhakar in 1971, defined in series form as the following:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \tag{A113}$$

with $z\in\mathbb{C}$, $\alpha,\beta,\gamma\in\mathbb{C}$ and $\mathrm{Re}\left[\alpha\right]>0$. In (A113) we have $(\gamma)_k$ as the Pochhammer Number, the rising factorial (Section-1.9.5), which is $(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. This (A113) is the entire function.

The Laplace transformation of the Prabhakar function is the following:

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}^{\gamma}(-\omega t^{\alpha})\right\}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\omega)^{\gamma}};\quad \mathrm{Re}\left[s\right]>\left|\omega\right|^{1/\alpha} \tag{A114}$$

From (A113) and (A115), we note that $E_{\alpha,\beta}(z)=E_{\alpha,\beta}^{1}(z)$ and $E_{\alpha}(z)=E_{\alpha,1}^{1}(z)$.

A.15.2 Prabhakar Integral

Using the function $\kappa_{\alpha,\beta}^{\gamma}(t)=t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha})$ as a kernel of the convolution integral, we can define the Prabhakar integral as ${}_{a}\mathbf{E}_{(\alpha,\beta,\omega);t}^{\gamma}\left[f(t)\right]=\left(\kappa_{\alpha,\beta}^{\gamma}(t)\right)*\left(f(t)\right)$, described as the following expression:

$${}_{a}\mathbf{E}_{(\alpha,\beta,\omega);t}^{\gamma}\left[f(t)\right]=\int_{a}^{t}\left((t-\tau)^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega(t-\tau)^{\alpha})\right)\left(f(\tau)\right)\mathrm{d}\tau \tag{A115}$$

We note that the Prabhakar integral (A115) has a kernel of integration different from the power law function ($\sim t^{\upsilon-1}$) that we used as the kernel in the Riemann-Liouville fractional integration formula.

A.15.3 The Prabhakar integral as a series-sum of the Riemann-Liouville fractional integrals

We know the Riemann-Liouville fractional integral as ${}_aI_t^{\upsilon} f(t) = \frac{1}{\Gamma(\upsilon)}\int_a^t (t-\tau)^{\upsilon-1} f(\tau)\mathrm{d}\tau$. Let us expand (A115) by inserting (A113) as in the following steps:

$$
\begin{aligned}
&{}_a\mathbf{E}^{\gamma}_{(\alpha,\beta,\omega);t}\left[f(t)\right] = \int_a^t \Big((t-\tau)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega(t-\tau)^{\alpha})\Big)\big(f(\tau)\big)\mathrm{d}\tau \\
&E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \\
&{}_a\mathbf{E}^{\gamma}_{(\alpha,\beta,\omega);t}\left[f(t)\right] = \int_a^t \Big((t-\tau)^{\beta-1}\textstyle\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{(\omega(t-\tau)^{\alpha})^k}{k!}\Big)\big(f(\tau)\big)\mathrm{d}\tau \\
&{}_aI_t^{\upsilon} f(t) = \frac{1}{\Gamma(\upsilon)}\int_a^t (t-\tau)^{\upsilon-1}f(\tau)\mathrm{d}\tau \\
&{}_a\mathbf{E}^{\gamma}_{(\alpha,\beta,\omega);t}\left[f(t)\right] = \sum_{k=0}^{\infty}\left(\frac{(\gamma)_k\omega^k}{k!}\left(\frac{1}{\Gamma(\alpha k+\beta)}\right)\int_a^t (t-\tau)^{\alpha k+\beta-1}f(\tau)\mathrm{d}\tau\right) \\
&\qquad = \sum_{k=0}^{\infty}\frac{\big((\gamma)_k\big)\omega^k}{k!}\Big({}_aI_t^{\alpha k+\beta}f(t)\Big) = \textstyle\sum_{k=0}^{\infty}\frac{\omega^k\Gamma(\gamma+k)}{k!\Gamma(\gamma)}\Big({}_aI_t^{\alpha k+\beta}f(t)\Big)
\end{aligned}
\tag{A116}
$$

Using (A116) we can write the following in expanded form:

$$
\begin{aligned}
&{}_a\mathbf{E}^{\gamma}_{(\alpha,\beta,\omega);t}\left[f(t)\right] = \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!}\Big({}_aI^{\alpha k+\beta}f(t)\Big) \\
&(\gamma)_k = \frac{\Gamma(\upsilon+k)}{\Gamma(\upsilon)};\quad \Gamma(\upsilon+1) = \upsilon\Gamma(\upsilon) \\
&{}_a\mathbf{E}^{\gamma}_{(\alpha,\beta,\omega);t}\left[f(t)\right] = {}_aI_t^{\beta}f(t) + \gamma\omega\Big({}_aI_t^{\alpha+\beta}f(t)\Big) \\
&\qquad + \frac{\gamma(\gamma+1)\omega^2}{2!}\Big({}_aI_t^{2\alpha+\beta}f(t)\Big) \\
&\qquad + \frac{\gamma(\gamma+1)(\gamma+2)\omega^3}{3!}\Big({}_aI_t^{3\alpha+\beta}f(t)\Big) + \ldots
\end{aligned}
\tag{A117}
$$

In (A117) putting $\gamma = \beta = 1$, we have the following:

$$
\begin{aligned}
{}_a\mathbf{E}^{1}_{(\alpha,1,\omega);t}\left[f(t)\right] &= \int_a^t \Big(E_{\alpha}(\omega(t-\tau)^{\alpha})\Big)\big(f(\tau)\big)\mathrm{d}\tau \\
&= {}_aI_t^{1}f(t) + \omega\Big({}_aI_t^{\alpha+1}f(t)\Big) + \omega^2\Big({}_aI_t^{2\alpha+1}f(t)\Big) \\
&\quad + \omega^3\Big({}_aI_t^{3\alpha+1}f(t)\Big) + \ldots
\end{aligned}
\tag{A118}
$$

We note that the Prabhakar integral defined via the non-singular kernel $\kappa^1_{\alpha,1}(t) = E_{\alpha}(\omega t^{\alpha})$ in (A118) is the sum of the classical Riemann-Liouville fractional integration using the power law (singular) kernel.

APPENDIX B

LIST OF LAPLACE TRANSFORM PAIRS OF FUNCTIONS RELATED TO FRACTIONAL CALCULUS

Laplace Transform $F(s)=\mathcal{L}\left\{f(x)\right\}$	**Function** $f(x)=\mathcal{L}^{-1}\left\{F(s)\right\}$
$\frac{s^{\alpha-1}}{s^{\alpha}\mp\lambda},\quad \mathrm{Re}\left[s\right]>\left\lvert\lambda\right\rvert^{\left(1/\alpha\right)}$	$E_{\alpha,1}(\pm\lambda x^{\alpha})$
$\frac{k!\,s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{k+1}},\quad \mathrm{Re}\left[s\right]>\left\lvert\lambda\right\rvert^{\left(1/\alpha\right)}$	$x^{\alpha k+\beta-1}E_{\alpha,\beta}{}^{(k)}(\pm\lambda x^{\alpha})$
$\frac{k!}{(\sqrt{s}\mp\lambda)^{k+1}},\quad \mathrm{Re}\left[s\right]>\lambda^{2}$	$x^{\frac{k-1}{2}}E^{(k)}_{1/2,1/2}\left(\pm\lambda\sqrt{x}\right)$
$s^{-\alpha}$	$\frac{1}{\Gamma(\alpha)}x^{\alpha-1}$
$\arctan\left(\frac{k}{s}\right)$	$\left(\frac{1}{x}\right)\sin(kx)$
$\log\left(\frac{s^{2}-a^{2}}{s^{2}}\right)$	$\frac{2}{x}(1-\cosh ax)$
$\log\left(\frac{s^{2}+a^{2}}{s^{2}}\right)$	$\frac{2}{x}(1-\cos ax)$
$\log\left(\frac{s-a}{s-b}\right)$	$\frac{1}{x}(e^{bx}-e^{ax})$
$\frac{e^{-k\sqrt{s}}}{\sqrt{s}\left(a+\sqrt{s}\right)},\quad k\ge 0$	$e^{ak}e^{a^{2}x}\mathrm{erfc}\left(a\sqrt{x}+\frac{k}{2\sqrt{x}}\right)$
$\frac{ae^{-k\sqrt{s}}}{s\left(a+\sqrt{s}\right)},\quad k\ge 0$	$\mathrm{erfc}\left(\frac{k}{2\sqrt{x}}\right)-e^{ak}e^{a^{2}x}\mathrm{erfc}\left(a\sqrt{x}+\frac{k}{2\sqrt{x}}\right)$
$\frac{1}{s\sqrt{s}}e^{-k\sqrt{s}},\quad k\ge 0$	$\left(2\sqrt{\frac{x}{\pi}}e^{-(k^{2}/4x)}\right)-k\left(\mathrm{erfc}\left(\frac{k}{2\sqrt{x}}\right)\right)$
$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}},\quad k\ge 0$	$\frac{1}{\sqrt{\pi x}}e^{-\frac{k^{2}}{4x}}$
$\frac{1}{s}e^{-k\sqrt{s}},\quad k\ge 0$	$\mathrm{erfc}\left(\frac{k}{2\sqrt{x}}\right)$
$e^{-k\sqrt{s}},\quad k\ge 0$	$\frac{k}{2\sqrt{\pi x^{3}}}e^{-\frac{k^{2}}{4x}}$
$\frac{1}{s^{v}}e^{k/s},\quad v>0$	$\left(\frac{x}{k}\right)^{(v-1)/2}I_{v-1}\left(2\sqrt{kx}\right)$
$\frac{1}{s^{v}}e^{-k/s},\quad v>0$	$\left(\frac{x}{k}\right)^{(v-1)/2}J_{v-1}\left(2\sqrt{kx}\right)$
$\frac{1}{s\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi x}}\sinh 2\sqrt{kx}$
$\frac{1}{s\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi x}}\sin 2\sqrt{kx}$
$\frac{1}{\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi x}}\cosh 2\sqrt{kx}$
$\frac{1}{\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi x}}\cos 2\sqrt{kx}$
$\frac{1}{s}e^{-k/s}$	$J_{0}\left(2\sqrt{kx}\right)$
$\left(\frac{k}{s^{2}+k^{2}}\right)\coth\left(\frac{\pi s}{2k}\right)$	$\left\lvert\sin kx\right\rvert$

$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi x}}$
$\frac{1}{s\sqrt{s}}$	$2\sqrt{\frac{x}{\pi}}$
$\frac{1}{s^n\sqrt{s}}, \quad n=1,2,...$	$\frac{2^n x^{n-(1/2)}}{1.3.5...(2n-1)\sqrt{\pi}}$
$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi x}} e^{ax}(1+2ax)$
$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi x^3}}\left(e^{bx}-e^{ax}\right)$
$\frac{1}{\sqrt{s}+a}$	$\left(\frac{1}{\sqrt{\pi x}}\right)-ae^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
$\frac{\sqrt{s}}{s-a^2}$	$\left(\frac{1}{\sqrt{\pi x}}\right)+ae^{a^2x}\operatorname{erf}\left(a\sqrt{x}\right)$
$\frac{\sqrt{s}}{s+a^2}$	$\left(\frac{1}{\sqrt{\pi x}}\right)-\left(\frac{2a}{\sqrt{\pi}}\right)e^{-a^2x}\int_0^{a\sqrt{x}} e^{y^2}\,\mathrm{d}y$
$\frac{1}{\sqrt{s}\left(s-a^2\right)}$	$\frac{1}{a}e^{a^2x}\operatorname{erf}\left(a\sqrt{x}\right)$
$\frac{1}{\sqrt{s}\left(s+a^2\right)}$	$\left(\frac{2}{a\sqrt{\pi}}\right)e^{-a^2x}\int_0^{a\sqrt{x}} e^{y^2}\,\mathrm{d}y$
$\frac{b^2-a^2}{\left(s-a^2\right)\left(\sqrt{s}+b\right)}$	$e^{a^2x}\left[b-a\left(\operatorname{erf}\left(a\sqrt{x}\right)\right)\right]-be^{b^2x}\operatorname{erfc}\left(b\sqrt{x}\right)$
$\frac{1}{\sqrt{s}\left(\sqrt{s}+a\right)}$	$e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
$\frac{1}{\sqrt{s+b}(s+a)}$	$\frac{1}{\sqrt{b-a}}e^{-ax}\operatorname{erf}\left(\sqrt{b-a}\sqrt{x}\right)$
$\frac{b^2-a^2}{\sqrt{s}\left(s-a^2\right)\left(\sqrt{s}+b\right)}$	$e^{a^2x}\left[\frac{b}{a}\operatorname{erf}\left(a\sqrt{x}\right)-1\right]+e^{b^2x}\operatorname{erfc}\left(b\sqrt{x}\right)$
$\frac{(1-s)^n}{s^{n+(1/2)}}$	$\frac{n!}{(2n)!\sqrt{\pi x}}H_{2n}\left(\sqrt{x}\right), \quad H_n(x)=e^{x^2}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[e^{-x^2}\right]\right)$ H_n :Hermite Polynomial
$\frac{(1-s)^n}{s^{n+(3/2)}}$	$-\frac{n!}{(2n+1)!\sqrt{\pi x}}H_{2n+1}(\sqrt{x})$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s}}$	$ae^{-ax}\left[I_1(ax)+I_0(ax)\right], \quad I_n(x)=i^{-n}J_n(ix)$ J_n Bessel function of first kind
$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)x}I_0\left(\frac{a-b}{2}x\right)$
$\frac{\Gamma(k)}{(s+a)(s+b)}, \quad k>0$	$\sqrt{\pi}\left(\frac{x}{a-b}\right)^{k-(1/2)}e^{-\frac{1}{2}(a+b)x}I_{k-(1/2)}\left(\frac{a-b}{2}x\right)$
$\frac{1}{(s+a)^{1/2}(s+b)^{3/2}}$	$xe^{-\frac{1}{2}(a+b)x}\left[I_0\left(\frac{a-b}{2}x\right)+I_1\left(\frac{a-b}{2}x\right)\right]$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{x}e^{-ax}I_1(ax)$
$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}, \quad k>0$	$\frac{k}{x}e^{-\frac{1}{2}(a+b)x}I_k\left(\frac{a-b}{2}x\right)$
$\frac{1}{\sqrt{s}\sqrt{s+a}\left(\sqrt{s+a}+\sqrt{s}\right)^{2v}}, \quad k>0$	$\frac{1}{a^v}e^{-\frac{1}{2}ax}I_v\left(\frac{a}{2}x\right)$
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(ax)$

$\frac{1}{\sqrt{s^2-a^2}}$	$I_0(ax)$, Modified Bessel function of the first kind zero order.
$\frac{\left(\sqrt{s^2+a^2}-s\right)^v}{\sqrt{s^2+a^2}}, \quad v>-1$	$a^v J_v(ax)$
$\frac{1}{\left(\sqrt{s^2+a^2}\right)^k}, \quad k>0$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{x}{2a}\right)^{k-(1/2)} J_{k-(1/2)}(ax)$
$\left(\sqrt{s^2+a^2}-s\right)^k, \quad k>0$	$\frac{ka^k}{x} J_k(ax)$
$\frac{\left(s+\sqrt{s^2-a^2}\right)^v}{\sqrt{s^2-a^2}}, \quad v>-1$	$a^v I_v(ax)$
$\frac{1}{(s^2-a^2)^k}, \quad k>0$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{x}{2a}\right)^{k-(1/2)} I_{k-(1/2)}(ax)$
$\frac{1}{s\sqrt{s+1}}$	$\operatorname{erf}\left(\sqrt{x}\right)$
$\frac{1}{s+\sqrt{s^2+a^2}}$	$\frac{1}{ax} J_1(ax)$
$\frac{1}{\left(s+\sqrt{s^2+a^2}\right)^N}$	$\frac{N}{a^N x} J_N(ax)$
$\frac{1}{\sqrt{s^2+a^2}\left(s+\sqrt{s^2+a^2}\right)}$	$\frac{1}{a} J_1(ax)$
$\frac{1}{\sqrt{s^2+a^2}\left(s+\sqrt{s^2+a^2}\right)^N}$	$\frac{1}{a^N} J_N(ax)$

Table-B-1: Laplace transform pairs of some important functions related to fractional calculus

Function	Higher Transcendental Function $f(x)$ for $x \ge 0$	Laplace Transform $F(s)=\mathcal{L}\left\{f(x)\right\}$
Mittag-Leffler	$E_q(ax^q)=\sum_{n=0}^{\infty}\frac{a^n x^{nq}}{\Gamma(nq+1)}$	$\frac{s^q}{s(s^q-a)}$
Agarwal	$x^{\beta-1}E_{\alpha,\beta}(x^\alpha)=\sum_{m=0}^{\infty}\frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$	$\frac{s^{\alpha-\beta}}{s^\alpha-1}$
Erdelyi	$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{x^m}{\Gamma(\alpha m+\beta)}$	$\sum_{m=0}^{\infty}\frac{m!}{s^{m+1}\Gamma(\alpha m+\beta)}$
Robotnov-Hartley	$F_q(a,x)=\sum_{n=0}^{\infty}\frac{a^n x^{(n+1)q-1}}{\Gamma((n+1)q)}$	$\frac{1}{s^q-a}$
Miller-Ross	$E_x(v,a)=\sum_{k=0}^{\infty}\frac{a^k x^{k+v}}{\Gamma(v+k+1)}$	$\frac{s^{-v}}{s-a}$
Generalized Cosine (Miller-Ross)	$C_x(v,a)=x^v\sum_{k=\text{even}}^{\infty}\frac{(-1)^{k/2}(ax)^k}{\Gamma(v+k+1)}, \quad \operatorname{Re}\left[v\right]>-1$	$\frac{s}{s^v(s^2+a^2)}$
Generalized Sine (Miller-Ross)	$S_x(v,a)=x^v\sum_{k=\text{odd}}^{\infty}\frac{(-1)^{\frac{(k-1)}{2}}(ax)^k}{\Gamma(v+k+1)}; \quad \operatorname{Re}\left[v\right]>-2$	$\frac{a}{s^v(s^2+a^2)}$
Generalized *R*	$R_{q,v}(a,x)=\sum_{n=0}^{\infty}\frac{a^n x^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}$	$\frac{s^v}{s^q-a}$
Generalized *G*	$G_{q,v,r}(a,x)=\sum_{j=0}^{\infty}\frac{((-r)(-1-r)\dots(1-j-r))(-a)^j x^{(r+j)q-v-1}}{\Gamma(1+j)\Gamma((r+j)q-v)}$	$\frac{s^v}{(s^q-a)^r}$

Table-B-2: List of Laplace transforms and inverse Laplace transforms of higher transcendental functions related to fractional calculus

APPENDIX C

FRACTIONAL DERIVATIVES AND INTEGRALS OF SOME IMPORTANT FUNCTIONS

In Table-C1, we tabulate some semi-derivatives and semi-integrals of a few monomials

f	${}_0D_x^{1/2}\left[f\right]$	${}_0D_x^{-1/2}\left[f\right]$
$x^p;\ p>-1$	$\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{1}{2}\right)}x^{p-\left(\frac{1}{2}\right)}$	$\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}x^{p+\left(\frac{1}{2}\right)}$
0	0	0
C, any constant	$\frac{C}{\sqrt{\pi x}}$	$2C\sqrt{\frac{x}{\pi}}$
$x^{-\alpha}$, $\alpha=0.79195...$	$-x^{-\alpha-\left(\frac{1}{2}\right)}$	$\frac{x^{\left(\frac{1}{2}\right)-\alpha}}{\alpha-\left(\frac{1}{2}\right)}$
$\frac{1}{\sqrt{x}}$	0	$\sqrt{\pi}$
$x^0\equiv 1$	$\frac{1}{\sqrt{\pi x}}$	$2\sqrt{\frac{x}{\pi}}$
x^β, $\beta=0.22119$	$\left(\beta+\frac{1}{2}\right)x^{\beta-\left(\frac{1}{2}\right)}$	$x^{\beta+\left(\frac{1}{2}\right)}$
$\sqrt{x}$	$\frac{1}{2}\sqrt{\pi}$	$\frac{1}{2}\left(\sqrt{\pi}\right)x$
$x^{\beta+\left(\frac{1}{2}\right)}$	x^β	$\frac{x^{\beta+1}}{\beta+1}$
x	$2\sqrt{\frac{x}{\pi}}$	$\frac{4x^{\left(\frac{3}{2}\right)}}{3\sqrt{\pi}}$
$x^{\left(\frac{3}{2}\right)}$	$\frac{3}{4}\left(\sqrt{\pi}\right)x$	$\frac{3}{8}\left(\sqrt{\pi}\right)x^2$
x^2	$\frac{8x^{\left(\frac{3}{2}\right)}}{3\sqrt{\pi}}$	$\frac{16x^{\left(\frac{5}{2}\right)}}{15\sqrt{\pi}}$
x^n, $n=0,1,2,3...$	$\frac{(n!)^2(4x)^n}{(2n)!\sqrt{\pi x}}$	$\frac{(n!)^2(4x)^{n+\left(\frac{1}{2}\right)}}{((2n+1)!)^2\sqrt{\pi}}$
$x^{n+\left(\frac{1}{2}\right)}$, $n=0,1,2,3,...$	$\frac{(2n+1)!\sqrt{\pi}}{2(n!)^2}\left(\frac{x}{4}\right)^n$	$\frac{(2n+2)!\sqrt{\pi}}{((n+1)!)^2}\left(\frac{x}{4}\right)^{n+1}$

Table-C-1: List of semi-differentiation and semi-integration of monomials

Here in this Table-C2, examples of fractional integrals and derivatives are tabulated. These are useful functions for physics and engineering. For simplicity, we assume that all variables are real ($\mathbb{R}$) and that x is positive and $x>0$. The fractional order v is assumed to be arbitrary (positive, negative or zero) unless stated explicitly. The constants a, c, λ and μ are assumed to be unrestricted unless otherwise indicated. The section demonstrates the use of higher transcendental functions Γ, B_τ, ${}_2F_1$, E_t, $\psi(x)$, and others described earlier. The derivative in the table is the Riemann-Liouville type.

Function $f(x)$	**Fractional derivative/ integral expression** ${}_cD_x^{\pm v}\left[f(x)\right]$
$(x-a)^{\lambda}$	${}_cD_x^{v}\left[(x-a)^{\lambda}\right]=\frac{(c-a)^{\lambda}}{\Gamma(1-v)}(x-c)^{-v}\times{}_2F_1\left(-\lambda,1,1-v;-\frac{x-c}{c-a}\right)$ $x>c>a \qquad c\ge 0$
$(x-a)^{\lambda}$	${}_cD_x^{-v}\left[(x-a)^{\lambda}\right]=\frac{(x-a)^{\lambda+v}}{\Gamma(v)}\mathrm{B}_{\chi}(v,\lambda+1),\quad v>0$ $x>c>a \qquad c\ge 0$
$(x-a)^{\lambda}$	${}_cD_x^{v}\left[(x-a)^{\lambda}\right]=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}(x-a)^{\lambda-v};\quad \lambda>-1$ $x>c=a\ge 0$
x^{λ}	${}_0D_x^{v}\left[x^{\lambda}\right]=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}x^{\lambda-v};\quad \lambda>-1;\qquad x>c=a=0$
$f(x)=1$	${}_cD_x^{v}\left[1\right]=\frac{1}{\Gamma(1-v)}(x-c)^{-v};\qquad x>c\ge 0$
$f(x)=1$	${}_0D_x^{v}\left[1\right]=\frac{1}{\Gamma(1-v)}x^{-v};\qquad x>c=0$
$(a-x)^{\lambda}$	${}_cD_x^{v}\left[(a-x)^{\lambda}\right]=\frac{(a-c)^{\lambda}}{\Gamma(1-v)}(x-c)^{-v}\times{}_2F_1\left(-\lambda,1,1-v;\frac{x-c}{a-c}\right)$ $a>x>c\ge 0$
$(a-x)^{\lambda}$	${}_cD_x^{-v}\left[(a-x)^{\lambda}\right]=\frac{(a-x)^{\lambda+v}}{\Gamma(v)}\mathrm{B}_{\chi}\left(v,-(\lambda+v)\right)$ $a>x>c\ge 0,\qquad v>0$
$(a-x)^{-(1/2)}$	${}_0D_x^{-1/2}\left[(a-x)^{-(1/2)}\right]=\frac{1}{\sqrt{\pi}}\ln\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}},\qquad a>x$
$(a-x)^{-(1/2)}$	${}_0D_x^{1/2}\left[(a-x)^{-(1/2)}\right]=\sqrt{\frac{a}{\pi x}}\frac{1}{a-x},\qquad a>x$
$\exp(ax)$	${}_0D_x^{v}\left[e^{ax}\right]=E_x(-v,a)$
$\exp(ax)$	${}_0D_x^{-1/2}\left[e^{ax}\right]=a^{-1/2}e^{ax}\mathrm{erf}\left[\sqrt{ax}\right],\qquad a>0$
$x^{\lambda}\exp ax$	${}_0D_x^{v}\left[x^{\lambda}e^{ax}\right]=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}x^{\lambda-v}\times{}_1F_1(\lambda+1,\lambda-v+1;ax)$ $\lambda>-1$
$x\exp ax$	${}_0D_x^{v}\left[xe^{ax}\right]=xE_x(-v,a)+vE_x(1-v,a)$
$E_x(\mu,a)$	${}_0D_x^{v}\left[E_x(\mu,a)\right]=E_x(\mu-v,a),\qquad \mu>-1$
$x^{\lambda}E_x(\mu,a)$	${}_0D_x^{v}\left[x^{\lambda}E_x(\mu,a)\right]$ $=\frac{\Gamma(\lambda+\mu+1)x^{\lambda+\mu-v}}{\Gamma(\mu+1)\Gamma(\lambda+\mu-v+1)}\,{}_2F_2(\lambda+\mu+1,1,\mu+1,\lambda+\mu-v+1;ax)$ $\lambda+\mu>-1$
$xE_x(\mu,a)$	${}_0D_x^{v}\left[xE_x(\mu,a)\right]=xE_x(\mu-v,a)+vE_x(\mu-v+1,a)$ $\mu>-2$
$\cos ax$	${}_0D_x^{v}\left[\cos ax\right]=C_x(-v,a)$

$\cos ax$	${}_0D_x^{-1/2}\left[\cos ax\right] = \sqrt{\tfrac{2}{a}}\left(C(z)\cos ax + S(z)\sin ax\right); \qquad a>0$ $C(z)=\int_0^z \cos\tfrac{1}{2}\pi y^2 \mathrm{d}y$ and $S(z)=\int_0^z \sin\tfrac{1}{2}\pi y^2 \mathrm{d}y$ are Fresnel integrals
$\cos^2 ax$	${}_0D_x^{v}\left[\cos^2 ax\right] = \frac{x^{-v}}{2\Gamma(1-v)} + \frac{1}{2}C_x(-v,2a)$
$x\cos ax$	${}_0D_x^{v}\left[x\cos ax\right] = xC_x(-v,a) + vC_x(1-v,a)$
$C_x(\mu,a)$	${}_0D_x^{v}\left[C_x(\mu,a)\right] = C_x(\mu-v,a), \qquad \mu>-1$
$xC_x(\mu,a)$	${}_0D_x^{v}\left[xC_x(\mu,a)\right] = xC_x(\mu-v,a) + vC_x(\mu-v+1,a)$ $\mu>-2$
$x^{-1/2}\cos\sqrt{x}$	${}_0D_x^{v}\left[x^{-1/2}\cos\sqrt{x}\right] = \sqrt{\pi}\left(2\sqrt{x}\right)^{-(1/2)-v} J_{-(1/2)-v}\left(\sqrt{x}\right)$
$x^{-1/2}\cosh\sqrt{x}$	${}_0D_x^{v}\left[x^{-1/2}\cosh\sqrt{x}\right] = \sqrt{\pi}\left(2x^{1/2}\right)^{-(1/2)-v} I_{-(1/2)-v}\left(\sqrt{x}\right)$
$\sin ax$	${}_0D_x^{v}\left[\sin ax\right] = S_x(-v,a)$
$\sin ax$	${}_0D_x^{-1/2}\left[\sin ax\right] = \sqrt{\tfrac{2}{a}}\left(C(z)\sin ax - S(z)\cos ax\right), \quad a>0$ $C(z)=\int_0^z \cos\tfrac{1}{2}\pi y^2 \mathrm{d}y$, $S(z)=\int_0^z \sin\tfrac{1}{2}\pi y^2 \mathrm{d}y$
$\sin^2 ax$	${}_0D_x^{v}\left[\sin^2 ax\right] = aS_x(1-v,2a)$
$x\sin ax$	${}_0D_x^{v}\left[x\sin ax\right] = xS_x(-v,a) + vS_x(1-v,a)$
$S_x(\mu,a)$	${}_0D_x^{v}\left[S_x(\mu,a)\right] = S_x(\mu-v,a), \quad \mu>-2$
$xS_x(\mu,a)$	${}_0D_x^{v}\left[xS_x(\mu,a)\right] = xS_x(\mu-v,a) + vS_x(\mu-v+1,a)$ $\mu>-3$
$\sin\sqrt{x}$	${}_0D_x^{v}\left[\sin\sqrt{x}\right] = \frac{1}{2}\sqrt{\pi}\left(2x^{1/2}\right)^{(1/2)-v} J_{(1/2)-v}\left(\sqrt{x}\right)$
$\sinh\sqrt{x}$	${}_0D_x^{v}\left[\sinh\sqrt{x}\right] = \frac{1}{2}\sqrt{\pi}\left(2x^{1/2}\right)^{(1/2)-v} I_{(1/2)-v}\left(\sqrt{x}\right)$
$x^{\lambda}\ln x$	${}_0D_x^{v}\left[x^{\lambda}\ln x\right]$ $= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}x^{\lambda-v}\left(\ln x + \psi(\lambda+1) - \psi(\lambda-v+1)\right), \qquad \lambda>-1$
$\ln x$	${}_0D_x^{v}\left[\ln x\right] = \frac{x^{-v}}{\Gamma(1-v)}\left(\ln x - \gamma - \psi(1-v)\right)$
$x^{-1/2}\ln x$	${}_0D_x^{-1/2}\left[x^{-1/2}\ln x\right] = \sqrt{\pi}\,\ln\left(\tfrac{1}{4}x\right)$
$\ln x$	${}_0D_x^{-1/2}\left[\ln x\right] = 2\sqrt{\tfrac{x}{\pi}}\left(\ln 4x - 2\right)$

$x^{\lambda}\left(\ln x\right)^{2}$	${}_0D_x^{v}\left[x^{\lambda}\left(\ln x\right)^{2}\right]$ $=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}x^{\lambda-v}\begin{pmatrix}\left(\ln x+\psi(\lambda+1)-\psi(\lambda-v+1)\right)^{2}\\+D\left[\psi(\lambda+1)\right]-D\left[\psi(\lambda-v+1)\right]\end{pmatrix},\quad \lambda>-1$
$x^{\lambda/2}J_{\lambda}\left(\sqrt{x}\right)$	${}_0D_x^{v}\left[x^{\lambda/2}J_{\lambda}\left(\sqrt{x}\right)\right]=2^{-v}x^{(\lambda-v)/2}J_{\lambda-v}\left(\sqrt{x}\right),\qquad \lambda>-1$
$x^{\lambda/2}I_{\lambda}\left(\sqrt{x}\right)$	${}_0D_x^{v}\left[x^{\lambda/2}I_{\lambda}\left(\sqrt{x}\right)\right]=2^{-v}x^{(\lambda-v)/2}I_{\lambda-v}\left(\sqrt{x}\right),\qquad \lambda>-1$

Table-C2: List of fractional differ-integration of various functions

Appendix D

Formulas of Modified Fractional Derivative

The formulas of the modified fractional derivative of the Jumarie type are listed as follows for $0 < \alpha < 1$:

$$\begin{aligned}
&\mathbf{D}^{\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda E_{\alpha}(\lambda x^{\alpha}) \\
&\mathbf{D}^{\alpha} E_{\alpha}(\lambda x) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_{\alpha}(\lambda x) \\
&\mathbf{D}^{\alpha} E_{\alpha}\left(u^{\alpha}(x)\right) = \left(E_{\alpha}\left(u^{\alpha}(x)\right)\right)\left(u'(x)\right)^{\alpha} \\
&\int_0^x E_{\alpha}(\lambda \xi^{\alpha})\left(\mathrm{d}\xi\right)^{\alpha} = \lambda^{-1}\left(\alpha!\right)\left(E_{\alpha}(\lambda x^{\alpha}) - 1\right) \\
&(\mathrm{d}x)^{\alpha} = (\alpha!)^{-1} \mathrm{d}^{\alpha}(x^{\alpha}) \\
&\alpha!\left(1-\alpha\right)!\left(\mathrm{d}x\right) = x^{1-\alpha}\left(\mathrm{d}x\right)^{\alpha} \\
&\int \mathrm{d}^{\alpha} x = \left((1-\alpha)!\right)^{-1} \int x^{1-\alpha}\left(\mathrm{d}x\right)^{\alpha} = \alpha! \int \mathrm{d}x = \alpha!\, x \\
&\int f^{(\alpha)}(x)\left(\mathrm{d}x\right)^{\alpha} = \int \mathrm{d}^{\alpha} f = \alpha!\, f(x)
\end{aligned} \tag{D1}$$

The formulas regarding the Mittag-Leffler function and the application of the modified fractional derivative of the Jumarie type are listed as follows for $0 < \alpha < 1$:

$$\begin{aligned}
&E_{\alpha}\left(ix\right) = \cos_{\alpha} x + i \sin_{\alpha} x \\
&\mathbf{D}^{\alpha} \cos_{\alpha} x^{\alpha} = -\sin_{\alpha} x^{\alpha} \qquad \mathbf{D}^{\alpha} \sin_{\alpha} x^{\alpha} = \cos_{\alpha} x^{\alpha} \\
&E_{\alpha}(x) = \cosh_{\alpha} x + \sinh_{\alpha} x \\
&\mathbf{D}^{\alpha} \sinh_{\alpha} x^{\alpha} = \cosh_{\alpha} x^{\alpha} \qquad \mathbf{D}^{\alpha} \sinh_{\alpha} x^{\alpha} = \cosh_{\alpha} x^{\alpha}
\end{aligned} \tag{D2}$$

We have $E_{\alpha}\left((x+y)^{\alpha}\right) = E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha})$ for the modified RL fractional derivative Jumarie type for fractal functions E_{α}, which are continuous but non- differentiable. The inverse of $y = E_{\alpha}(u)$ we define as $u = Ln_{\alpha} y$. With this, we set $E_{\alpha}(x^{\alpha}) = u$, which gives $x = \left(Ln_{\alpha} u\right)^{1/\alpha}$ and $v = E_{\alpha}(y^{\alpha})$ gives $\left(Ln_{\alpha} v\right)^{1/\alpha} = y$. Using the derived expression, $E_{\alpha}\left((x+y)^{\alpha}\right) = E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha})$ we get $E_{\alpha}\left(((Ln_{\alpha} u)^{1/\alpha} + (Ln_{\alpha} v)^{1/\alpha})^{\alpha}\right) = uv$. From here we write $((Ln_{\alpha} u)^{1/\alpha} + (Ln_{\alpha} v)^{1/\alpha})^{\alpha} = Ln_{\alpha}(uv)$, which gives us the relationship of the $\alpha-$logarithm function as $(Ln_{\alpha} u)^{1/\alpha} + (Ln_{\alpha} v)^{1/\alpha} = \left(Ln_{\alpha}(uv)\right)^{1/\alpha}$ in conjugation to $\ln uv = \ln u + \ln v$ of classical calculus. Like we have $\mathbf{D}^{\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda E_{\alpha}(\lambda x^{\alpha})$; we have the expression $\mathbf{D}^{\alpha} Ln_{\alpha} x = \alpha^{\alpha}\left((1-\alpha)!\right)^{-2} x^{-\alpha}$. The formulas regarding the fractional logarithm function and the Mittag-Leffler functions are listed as follows for $0 < \alpha < 1$:

$$\int \frac{\mathrm{d}^{\alpha} x}{x} = Ln_{\alpha}\left(\tfrac{x}{C}\right) \qquad x = E_{\alpha}\left(Ln_{\alpha} x\right) \qquad \left(\frac{x}{C}\right) > 0$$

$$Ln_{\alpha}(x^{y}) = y^{\alpha} Ln_{\alpha} x$$

$$E_{\alpha}\left((xy)^{\alpha}\right) = E_{\alpha}(x^{\alpha} y^{\alpha}) = \left(E_{\alpha}(y^{\alpha})\right)^{x} = \left(E_{\alpha}(x^{\alpha})\right)^{y}$$

$$\left(Ln_{\alpha}\left(uv\right)\right)^{1/\alpha} = \left(Ln_{\alpha} u\right)^{1/\alpha} + \left(Ln_{\alpha} v\right)^{1/\alpha}$$

$$\frac{\mathrm{d}^{\alpha} Ln_{\alpha} x}{\mathrm{d}x^{\alpha}} = \frac{\alpha\,!}{(1-\alpha)\,!}\frac{1}{x^{\alpha}};$$

$$Ln_{\alpha} x = \frac{1}{(1-\alpha)\,!}\int_{0}^{x}\left(\frac{\mathrm{d}\xi}{\xi}\right)^{\alpha}, \qquad x = E_{\alpha}\left(\tfrac{1}{(1-\alpha)!}\int_{0}^{x}\left(\tfrac{\mathrm{d}\xi}{\xi}\right)^{\alpha}\right)$$

$$\int \frac{\left(\mathrm{d}x\right)^{\alpha}}{x} = \frac{x^{\alpha-1} - 1}{\left(\alpha - 1\right)\left((1-\alpha)\,!\right)}$$

$$\int_{\mathbb{R}} E_{\alpha}\left(-\left(\tfrac{x^{2}}{2\sigma^{2}}\right)^{\alpha}\right)\left(\mathrm{d}x\right)^{\alpha} = \left(2\sigma\right)^{\alpha}\left(\sqrt{\frac{\left(\alpha\,!\right)^{3}}{\left(2\alpha\right)!}}\right)\pi^{\alpha/2} \tag{D3}$$

Fractional order gamma and beta function are listed as the following for $0 < \alpha < 1$:

$$\Gamma_{\alpha}(x) = \left(\alpha\,!\right)^{-1}\int_{0}^{\infty}\left(E_{\alpha}(-y^{\alpha})\right) y^{(x-1)\alpha}\left(\mathrm{d}y\right)^{\alpha}$$

$$\Gamma_{\alpha}(x+1) = \left(\alpha\,!\right)\left(x\left(\Gamma_{\alpha}(x)\right)\right)$$

$$\Gamma_{\alpha}(n+1) = \left(\alpha\,!\right)^{n}\left(n!\right) \tag{D4}$$

$$\mathrm{B}_{\alpha}(x, y) = \int_{0}^{1}(1-t)^{(x-1)\alpha} t^{(y-1)\alpha}\left(\mathrm{d}t\right)^{\alpha}$$

$$\mathrm{B}_{\alpha}(x, y) = \frac{\Gamma_{\alpha}(x)\Gamma_{\alpha}(y)}{\Gamma_{\alpha}(x+y)}$$

The formulas derived by using $E_{\alpha}\left((x+y)^{\alpha}\right) = E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha}), \quad 0 < \alpha < 1$ are the following:

$$E_{\alpha}(1) = \left(\sum_{n=0}^{\infty}\tfrac{2^{\alpha n}}{(\alpha n)!}\right)\left(\sum_{n=0}^{\infty}\tfrac{1}{(\alpha n)!}\right)^{-1}$$

$$E_{\alpha}\left((1+x)^{\alpha}\right) = E_{\alpha}(1)E_{\alpha}(x^{\alpha})$$

$$E_{\alpha}\left((x+m)^{\alpha}\right) = E_{\alpha}^{m}(1)E_{\alpha}(x^{\alpha})$$

$$E_{\alpha}(n^{\alpha}) = E_{\alpha}\left((0+n)^{\alpha}\right) = E_{\alpha}^{n}(1)E_{\alpha}(0) \tag{D5}$$

$$E_{\alpha}\left((mx)^{\alpha}\right) = \left(E_{\alpha}(x^{\alpha})\right)^{m}$$

$$\sum_{k=0}^{\infty} {}^{\alpha n}C_{k} u^{k} = \sum_{k=0}^{\infty} {}^{\alpha n}C_{\alpha k} u^{\alpha k}; \quad {}^{n}C_{r} = \tfrac{n!}{r!(n-r)!}$$

$$(x+y)^{\alpha n} = x^{\alpha n}\sum_{k=0}^{\infty} {}^{\alpha n}C_{\alpha k} x^{-\alpha k} y^{\alpha k}$$

APPENDIX E

BRANCH-POINTS, BRANCH-CUTS AND RIEMANN SHEETS FOR MULTI-VALUED FUNCTIONS

E.1 Branch-points and branches in multi-valued function

A function of a complex variable $\omega = f(z)$ is viewed as a mapping of points in the $z-$plane to points in the $\omega-$plane. If to each value of the independent variable, z there is one and only one image point ω, the mapping is said to be single valued. In contrast, let us examine a multi-valued function. Consider a small circular path about a point z_0. The circular path is $z = z_0 + re^{i\theta}$ where $r > 0$ is a small constant and θ varies in a counter clockwise direction about the point z_0. If we have a function $\omega = f(z)$ such that $\omega = f(z_0 + re^{i\theta})$ takes on different values as θ increases by 2π, (i.e. one full rotation about z_0); then the point z_0 is called the 'branch point' of the function and the different values of ω are called 'branches' of the function.

By definition, a multi-valued function occurs if for each value of z here is more than one value for the dependent variable ω. Then several values of ω are said to be branches of the complex valued function. If it is possible to solve an equation of the form $F(z,\omega) = 0$, connecting the complex variable $z = x + iy$ and $\omega = u + iv$ to obtain a single valued function $\omega_1 = f_1(z)$, $\omega_2 = f_2(z)$, $\omega_3 = f_3(z)$, … then these functions are called branches of the function ω.

E.2 The branch point of order $(n-1)$ and branch-cut for $n-$valued function

A point, say z_0, satisfying the property that there is no neighborhood $|z - z_0| < \in$ in which the function $\omega = f(z)$ is single valued, then the point z_0 is called a branch point of $f(z)$. A branch point is said to be of order $(n-1)$ whenever a function $\omega = f(z)$ is a $n-$valued function in the neighbourhood $|z - z_0| < \in$. A line that connects two and only two branch points is called a branch cut (or branch line).

Consider a two-valued function as follows:

$$\omega_1 = f(z) = \frac{1}{\sqrt{(z - z_1)(z - z_2)....(z - z_n)}} \tag{E1}$$

This (E1) has singularities at the points $z_1, z_2, ..., z_n$ in the $z-$plane. Let z denote a variable point in the complex $z-$plane and construct straight lines from the point z to each point $z_1, z_2, ..., z_n$ and denote the length of these lines by $r_1, r_2, ..., r_n$. These straight lines make angles $\theta_1, \theta_2, ..., \theta_n$ respectively with horizontal lines through each of the points $z_1, z_2, ..., z_n$, as depicted in Figure-E1.

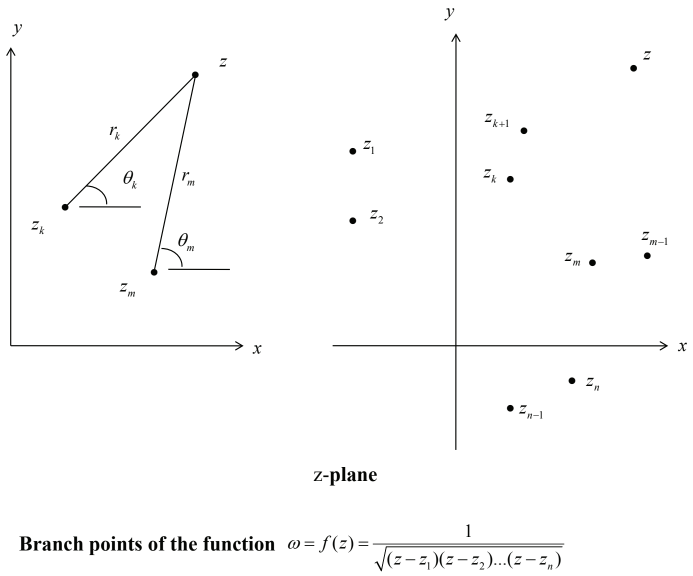

Figure-E1: Branch points for function $\left((z-z_1)(z-z_2)...(z-z_n)\right)^{-1/2}$

The variable point z can then be represented in terms of the modulus $r_1, r_2, ..., r_n$ and the arguments $\theta_1, \theta_2, ..., \theta_n$ by writing $z = z_1 + r_1 e^{i\theta_1}$, $z = z_2 + r_2 e^{i\theta_2}$,…., $z = z_n + r_n e^{i\theta_n}$, and we get the following:

$$\begin{aligned} \omega_1 &= f(z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)....(z-z_n)}} \\ z - z_1 &= r_1 e^{i\theta_1}, \quad z - z_2 = r_2 e^{i\theta_2} ..., \quad z - z_n = r_n e^{i\theta_n} \\ \omega_1 &= \frac{1}{\sqrt{r_1 r_2 .....r_n e^{i(\theta_1+\theta_2+...+\theta_n)}}} \end{aligned} \tag{E2}$$

After moving the point z in a small circle about the point z_m, m fixed, the values $r_i, i = 1, 2, ..., n$ return to their original values. The angle θ_m increases to the value $\theta_m + 2\pi$ and the other values of θ_i, $i \neq m$ return to their original values. The given equation becomes $\omega_2 = \omega_1 e^{-i\pi} = -\omega_1$. Observe that if we move the point z in a small circle about any one of the points $z_1, z_2, ..., z_n$, the same thing happens, and changes ω_1 to $\omega_2 = -\omega_1$.

In order to examine the behavior of the function ω_1 at the point $z = \infty$, we make the substitution $z = \frac{1}{z^*}$ and examine the behavior of ω_1 for z^* near the origin 0^* of the $z^* -$ plane, and write:

$$\begin{aligned} \omega_1 &= \frac{1}{\sqrt{\left(\frac{1}{z^*} - \frac{1}{z_1^*}\right)\left(\frac{1}{z^*} - \frac{1}{z_2^*}\right)....\left(\frac{1}{z^*} - \frac{1}{z_n^*}\right)}} \\ &= \frac{\sqrt{z_1^* z_2^* ....z_n^*}}{\sqrt{\left(z_1^* - z^*\right)\left(z_2^* - z^*\right)....\left(z_n^* - z^*\right)}} \left(z^*\right)^{n/2} \end{aligned} \tag{E3}$$

For point z^* near the origin 0^* let $z^* = re^{i\theta^*}$ and as $r \downarrow 0$, one obtains $\omega_2 = (re^{i(\theta^*)})^{n/2}$. Also as z^* moves about the origin 0^* the angle θ^* changes to $\theta^* + 2\pi$ and ω_1 changes to $\omega_2 = \omega_1 e^{in\pi}$. Therefore if n is even, ω_1 keeps the same value and if n is odd, then ω_1 becomes $\omega_2 = -\omega_1$.

This shows the function of the form:

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)}}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} \tag{E4}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)....(z-z_{2m})}}$$

The equations have respectively $2, 4, ..., 2m$ branch points but no branch point at infinity.

In contrast the functions with the form:

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)}} \tag{E5}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)....(z-z_{2m})(z-z_{2m+1})}}$$

have respectively $3, 5, ..., (2m+1)$ branch points with each function having a branch point at infinity.

In case of an even number of branch points, the branch points are connected in groups of any two pairs, where the connecting cuts do not cross one another. In the case of an odd number of branch points the cuts are made in groups of any two pairs, where connecting cuts do not cross one another. The remaining points are joined to infinity by a cut line, which does not cross the other cut lines.

E.3 Multiple Riemann sheets connected along the branch-cut for a multi-valued function

To avoid the problem that the same value of z corresponds to two or more values of ω for a multi-valued function $\omega = f(z)$ the $z-$plane is split into n parallel $z-$planes called sheets of Riemann surface (or simply Riemann-sheets); where n corresponds to the multiplicity of the function. These n sheets are separated by an infinitesimal distance (gap) and are connected along a branch-cut or along each of the branch cuts if more than one branch cut exists. In this way as z moves around the first sheet the image of ω is that of the first branch $\omega_1 = f_1(z)$. Then on, as z moves around the second sheet; the image of ω is that of a second branch $\omega_2 = f_2(z)$. In general, the value of z on the $i-$th sheet, for $i = 1, 2, ..., n$, produces a single valued function $\omega_i = f_i(z)$. As z moves around a sheet and crosses a branch-cut then there occurs a change in the branch of the function. All the sheets are connected along the branch-cut(s) and are to be regarded as a continuous surface, called the Riemann surface (or sheets).

E.4 An example elaborating branches and branch-cut for a two-valued function

We consider the function $\omega^2 = z$, which has a branch point at $z = 0$ and is a two-valued function. It has got two branches $\omega_1 = f_1(z) = +\sqrt{z}$ and $\omega_2 = f_2(z) = -\sqrt{z}$. Let $z = re^{i(\theta+2k\pi)}$, with $k = 0,1$. Write $\omega^2 = z$ in the form $\omega^2 = z = re^{i(\theta+2k\pi)}$. From here we write $\omega = z^{1/2} = r^{1/2}e^{i(\theta+2k\pi)/2}$ for $k = 0,1$. We obtain for $k = 0$, the first branch, i.e. $\omega_1 = f_1(z) = +\sqrt{z} = +\sqrt{r}e^{i\theta/2}$ and for $k = 1$ the second branch $\omega_2 = f_2(z) = -\sqrt{z} = +\sqrt{r}e^{i(\pi+(\theta/2))}$. Note that when $k = 3, 5, 7, ...$ we are on the first branch ω_1 and when $k = 4, 6, 8, ...$ we are back onto the second branch ω_2.

We desire to define a domain where these branches of the function are single-valued and analytic at each point of the domain. The derivative of the function, i.e. $\omega_1 = f_1(z) = r^{1/2}\cos\left(\frac{\theta}{2}\right) + ir^{1/2}\sin\left(\frac{\theta}{2}\right) = u(r,\theta) + iv(r,\theta)$ is obtained by partial derivatives, i.e. $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$, as described in the following expressions:

$$\frac{\partial u}{\partial r} = \frac{1}{2} r^{-1/2} \cos\left(\tfrac{\theta}{2}\right) \quad , \quad \frac{\partial v}{\partial r} = \frac{1}{2} r^{-1/2} \sin\left(\tfrac{\theta}{2}\right)$$
$$\frac{\partial u}{\partial \theta} = -\frac{1}{2} r^{1/2} \sin\left(\tfrac{\theta}{2}\right) \quad , \quad \frac{\partial v}{\partial \theta} = \frac{1}{2} r^{1/2} \sin\left(\tfrac{\theta}{2}\right) \tag{E6}$$

The total complex derivative is:

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}z} = f_1^{(1)}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\sqrt{z} = \frac{1}{2} z^{-1/2} = \frac{1}{2} r^{-1/2} e^{-i\theta/2}$$
$$\frac{\mathrm{d}\omega_2}{\mathrm{d}z} = f_2^{(1)}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(-\sqrt{z}\right) = \frac{1}{2} r^{-1/2} e^{-i\left(\pi-(\theta/2)\right)} \tag{E7}$$

The functions ω_1 and ω_2 fail to be analytic at point $z=0$ and $z=\infty$. The points $z=0$ and $z=\infty$ are singular points associated with the function $\omega = z^{1/2}$. Let us examine the behavior of the function ω_1 as we move around the singular point $z=0$. If we hold r to be constant and let θ vary from θ to $\theta+2\pi$, we find that $\omega_1 = r^{1/2} e^{i\theta/2}$ changes to $r^{1/2} e^{i(\theta+2\pi)/2} = e^{i\pi} r^{1/2} e^{i\theta/2} = -\omega_2$.

Similarly, for the function ω_2 as we move around $z=0$; we have $\omega_2 = -r^{1/2} e^{i(\theta/2)}$, which changes to $-r^{1/2} e^{i(\theta+2\pi)/2} = -e^{i\pi} r^{1/2} e^{i\theta/2} = \omega_1$. This shows that as θ increases by 2π the functions ω_1 and ω_2 change into each other.

If we construct a branch cut from 0 to ∞ along the negative real axis in the $z-$plane; and require that z not to be allowed to cross the branch-cut, then the functions ω_1 and ω_2 become single-valued and analytic when defined as follows:

$$\omega_1 = f_1(z) = r^{1/2} e^{i\theta/2} \qquad r>0 \qquad -\pi<\theta\le\pi; \quad \theta = \arg[z]; \quad \arg[\omega_1] = \tfrac{\theta}{2}$$
$$\omega_2 = f_2(z) = r^{1/2} e^{i(\theta+2\pi)/2} \qquad r>0 \qquad -\pi<\theta\le\pi; \quad \theta = \arg[z]; \quad \arg[\omega_2] = \tfrac{\theta}{2}+\pi \tag{E8}$$

Note that at each point on the branch cut, there occurs a discontinuity in the functions ω_1 and ω_2. The branch cut is a way of preventing these discontinuities from occurring and keeping the function (in this case the square-root function) single-valued. Figure-E2 depicts the branch cut, and two Riemann-sheets.

For Sheet-1 in Figure-E2, as we move in a circle near $z=0$ with $-\pi < \arg[z] < \pi$, in the complex $z-$plane we have its image moving in Branch-1 i.e. ω_1 where $-\pi/2 < \arg[\omega_1] < \pi/2$, in the complex $\omega-$plane. In Sheet-2 of Figure-E2, as we go around point $z=0$, with $-\pi < \arg[z] < \pi$, in the Complex $z-$plane, we have its image moving to Branch-2, i.e. ω_2, where the image of z moves as $\pi/2 < \arg[\omega_1] < 3\pi/2$, in the complex $\omega-$plane.

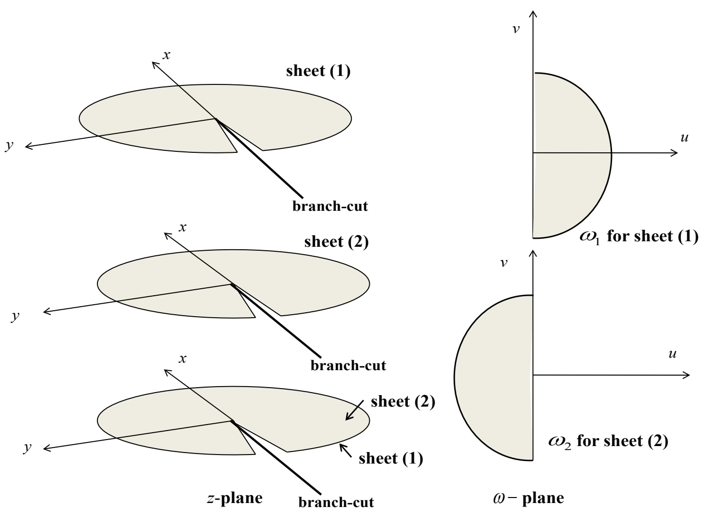

Figure-E2: Showing branch-cut at negative real axis and two Riemann-sheets

E.5 Choosing the contour with branch-cut for a function having branch-point

For the evaluation of contour integration $\int_C \left((\omega - z)^{-n-1}\right) f(\omega)\mathrm{d}\omega$; where C describes a closed contour surrounding the point z in the complex ω-plane; and enclosing a region of analyticity of the function i.e. $f(\omega)$. When the positive integer as in the integrand (i.e. n) is replaced by a non-integer say $\alpha > 0$; then $(\omega - z)^{-\alpha-1}$ no longer has a pole at $\omega = z$ but a branch point.

Then one is no longer free to deform (or squeeze) the contour C surrounding the point z at will; since the integral will depend on the location of the point at which C crosses the branch line (branch-cut) for $(\omega - z)^{-\alpha-1}$. In Figure-E3, this point is chosen to be 0 (the origin of ω-plane). The branch cut (branch-line) then is to be a straight line joining $\omega = 0$ and $\omega = z$, and continuing thereafter indefinitely in the third quadrant i.e. $\mathrm{Re}\left[\omega\right] \le 0$ and $\mathrm{Im}\left[\omega\right] \le 0$; as depicted in Figure-E3.

Then for when α is not a negative integer (i.e. $\alpha > 0$) we have the integral as $\int_C \left((\omega - z)^{-\alpha-1}\right) f(\omega)\mathrm{d}\omega$, where the contour C begins and ends at $\omega = 0$ enclosing z once in a counter clockwise direction (positive sense). To uniquely specify the denominator in the above expression of the integrand, we define $(\omega - z)^{\alpha+1} = e^{(\alpha+1)\ln(\omega - z)}$ where $\ln(\omega - z)$ is real when $(\omega - z) > 0$.

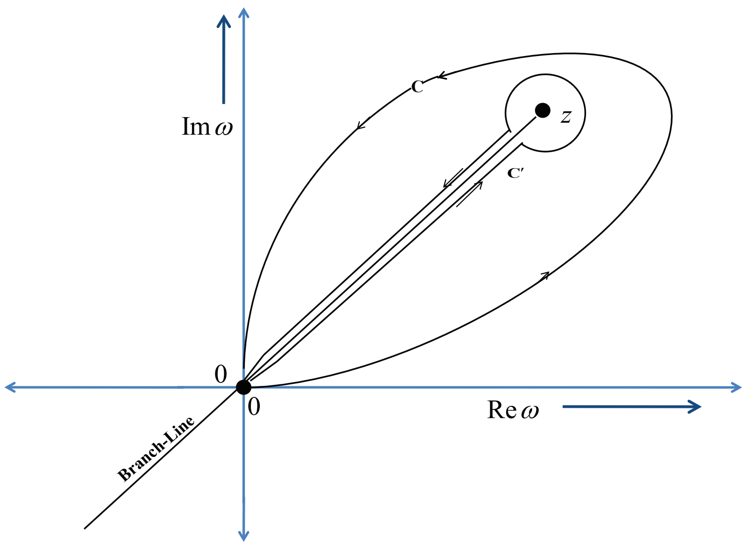

Figure-E3: The Branch line and the contour C is deformed into C'

We now deform the contour C into contour C' lying on both sides of the branch line. We write $\int_{C'} \left((\omega - z)^{-\alpha-1}\right) f(\omega)\mathrm{d}\omega$; also represented in this case as $\int_0^{(z+)} \left((\omega - z)^{-\alpha-1}\right) f(\omega)\mathrm{d}\omega$. We have used this concept in earlier chapters (Chapters-1, 2, and 6), where point z was on a real axis, i.e. $z = x$, and point 0 was at some other point than zero at c and we evaluated the integral by the contour integration method i.e. $\int_c^{(x+)} \left((\omega - z)^{-\alpha-1}\right) f(\omega)\mathrm{d}\omega$.

That is, we employed this method of complex analysis in the evaluation of the integrals $\int_c^x (y - z)^{-\alpha-1} f(y)\mathrm{d}y$.

APPENDIX F

MITTAG-LEFFLER FUNCTION WITH NEGATIVE ORDER

F.1 Restriction in Series Representation of the Mittag-Leffler Function

We recall the series definition of the 'two parameter' Mittag-Leffler function, that is

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)};\qquad \alpha,\beta\in\mathbb{R};\qquad \alpha,\beta>0 \tag{F1}$$

While $E_{\alpha,\beta}(z)$ is an entire function (e.g. analytic everywhere in complex plane $\mathbb{C}$), with order $1/\alpha$; here we will consider the function on real z. Furthermore the domain of $E_{\alpha,\beta}(z)$ can be expanded to include a complex α and β; with $\mathrm{Re}\left[\alpha\right]>0$ and $\mathrm{Re}\left[\beta\right]>0$. Here we will discuss the Mittag-Leffler $E_{\alpha,\beta}(z)$ function for case $\alpha\le 0$. Let us examine why the domain of α excludes both $\alpha=0$ and $\alpha<0$. In the limit as $\alpha\downarrow 0^+$, from the series definition as described above (F1), we find it reduces to a geometric series, that is:

$$E_{0,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta)}=\frac{1}{\left(\Gamma(\beta)\right)\left(1-z\right)};\qquad \left|z\right|<1 \tag{F2}$$

Keeping the condition $\beta>0$; the above (F2) expression for $E_{0,\beta}(z)$ converges; provided the geometric series $\sum_{k=0}^{\infty}z^k=\frac{1}{1-z}$ converges, i.e. for $\left|z\right|<1$. Consequently, since z must be restricted in the region $\left|z\right|<1$, the analyticity of $E_{\alpha,\beta}(z)$ in the entire complex plane is lost when $\alpha=0$. Thus, the restriction that $\alpha\neq 0$ is written in the series representation (F1); so that $E_{\alpha,\beta}(z)$ remains the entire function. In addition, we note that $E_{\alpha,1}(z)=E_{\alpha}(z)$; for $\beta=1$, that is the one-parameter Mittag-Leffler function with a positive order α.

The infinite series as defined in the series definition (F1) diverges in the domain of the negative α. We demonstrate this by setting $\beta=0$, thus getting $E_{\alpha,0}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k)}$. Placing α as a negative alpha by $-\alpha$, we write the ratio (say) r for $(k+1)$ and the k-th term in the following expression:

$$r=z\frac{\Gamma\left(-\alpha k\right)}{\Gamma\left(-\alpha(k+1)\right)}=z(-1)^{\alpha}\frac{\left(\alpha(k+1)\right)!}{\left(\alpha k\right)!} \tag{F3}$$

We used the relationship of the ratio of the gamma function with negative arguments, i.e. $\frac{\Gamma(-n)}{\Gamma(-N)}=(-1)^{N-n}\frac{N!}{n!}$; with n and N positive numbers; described earlier (in Chapter-1). From the above ratio (F3), it is seen that $\left|r\right|>1$ for $\left|z\right|>1$. For $\left|z\right|<1$, we can have a value of k, beyond which the $\left|r\right|>1$ and thus the series diverges. Therefore, the Mittag-Leffler function is defined by the series that is restricted to $\alpha>0$. We will construct the case for $E_{-\alpha,\beta}(z)$, with the notation α set to always be positive, so $-\alpha$ represents a negative order.

F.2 The Integral Representation of the two-parameter Mittag-Leffler function

Let us derive the integral representation of $E_{\alpha,\beta}(z)$ with a known Laplace transform identity of the Agrawal function, i.e. $\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right\}=\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$. Now we write the following steps as we have learnt previously for a one-parameter Mittag-Leffler function:

$$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})=\frac{1}{2\pi i}\int_{Br}\frac{s^{\alpha-\beta}}{s^{\alpha}-a}e^{st}\mathrm{d}s$$

$$\text{Put}\quad st=x,\quad \mathrm{d}s=\frac{\mathrm{d}x}{t},\quad s=\frac{x}{t}$$

$$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})=\frac{1}{2\pi i}\int_{Ha}\frac{\left(\frac{x^{\alpha-\beta}}{t^{\alpha-\beta}}\right)}{\left(\frac{x^{\alpha}}{t^{\alpha}}\right)-a}e^{x}\frac{\mathrm{d}x}{t}$$

$$=\frac{1}{2\pi i}\int_{Ha}\left(\frac{x^{\alpha-\beta}}{t^{\alpha-\beta}}\right)\left(\frac{t^{\alpha}}{x^{\alpha}-at^{\alpha}}\right)\left(\frac{e^{x}}{t}\right)\mathrm{d}x$$

$$=\frac{1}{2\pi i}\int_{Ha}t^{\beta-1}\frac{x^{\alpha-\beta}e^{x}}{x^{\alpha}-at^{\alpha}}\mathrm{d}x$$

$$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})=\frac{t^{\beta-1}}{2\pi i}\int_{Ha}\left(\frac{x^{\alpha-\beta}e^{x}}{x^{\alpha}-at^{\alpha}}\right)\mathrm{d}x\qquad\text{put}\qquad at^{\upsilon}=z$$

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{Ha}\left(\frac{x^{\alpha-\beta}}{x^{\alpha}-z}e^{x}\right)\mathrm{d}x \tag{F4}$$

Therefore, we have an integral representation as follows:

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{Ha}\left(\frac{t^{\alpha-\beta}e^{t}}{t^{\alpha}-z}\right)\mathrm{d}t \tag{F5}$$

Where the path of integration Ha is Hankel's contour (i.e. a loop starting and ending at $-\infty$) and encircling the circular disk $|t|\le|z|^{1/\alpha}$, in a positive (counter clockwise) sense. We write the above integral representation (F5) with a modification as follows:

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{Ha}\left(\frac{e^{t}}{t^{\beta}-zt^{\beta-\alpha}}\right)\mathrm{d}t \tag{F6}$$

We expand the part of the integrand in the above expression in a partial fraction as follows:

$$\frac{1}{t^{\beta}-zt^{\beta-\alpha}}=\frac{1}{t^{\beta}}-\frac{1}{t^{\beta}-z^{-1}t^{\alpha+\beta}} \tag{F7}$$

Using the above (F7), we write expression for $E_{\alpha,\beta}(z)$ as:

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{Ha}\left(\frac{e^{t}}{t^{\beta}}\right)\mathrm{d}t-\frac{1}{2\pi i}\int_{Ha}\left(\frac{e^{t}}{t^{\beta}-z^{-1}t^{\alpha+\beta}}\right)\mathrm{d}t \tag{F8}$$

From our derivation in Chapter-1, we have a relationship for the reciprocal gamma function in an integral form as $\frac{1}{\Gamma(\upsilon)}=\frac{1}{2\pi i}\int_{Ha}(x)^{-\upsilon}e^{x}\mathrm{d}x$. Using this formula, we say that the first term of the above expression is $\frac{1}{\Gamma(\beta)}$.

F.3 The formula for a two-parameter Mittag-Leffler function with a negative order

The second term of (F8), i.e. $\frac{1}{2\pi i}\int_{Ha}(t^{\beta}-z^{-1}t^{\alpha+\beta})^{-1}e^{t}\mathrm{d}t$ we compare with the obtained expression (F6) for $E_{\alpha,\beta}(z)$, i.e. $E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{Ha}(t^{\beta}-zt^{\beta-\alpha})^{-1}e^{t}\mathrm{d}t$. We note that the second expression in (F8) is the same as the integral representation with z changed to z^{-1} and α changed to $-\alpha$, so we have:

$$E_{-\alpha,\beta}(z^{-1})=\frac{1}{2\pi i}\int_{Ha}\left(\frac{e^{t}}{t^{\beta}-z^{-1}t^{\beta+\alpha}}\right)\mathrm{d}t \tag{F9}$$

With these observations we arrive at the following expression:

$$E_{\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}-E_{-\alpha,\beta}(z^{-1}) \tag{F10}$$

Replacing z by z^{-1}, we get the desired expression for $E_{-\alpha,\beta}(z)$, that is:

$$E_{-\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}-E_{\alpha,\beta}\left(\tfrac{1}{z}\right) \tag{F11}$$

Applying the recursion formula (Appendix-A), i.e. $E_{\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}+zE_{\alpha,\alpha+\beta}(z)$, to the above (F11) we obtain:

$$E_{-\alpha,\beta}(z)=-\frac{1}{z}E_{\alpha,\alpha+\beta}\left(\tfrac{1}{z}\right) \tag{F12}$$

The above-obtained (F12) formulas are for $E_{-\alpha,\beta}(z)$.

We note here that for $\beta=1$, we write $E_{-\alpha,1}(z)=E_{-\alpha}(z)$, the one-parameter Mittag-Leffler function with the negative order.

F.4 The series representation of the two-parameter Mittag-Leffler function with a negative order

In the formula (F11), i.e. $E_{-\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}-E_{\alpha,\beta}\left(\frac{1}{z}\right)$, using the series definition of $E_{\alpha,\beta}\left(z\right)$ (F1), i.e. $E_{\alpha,\beta}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$, we write the following useful series formula for $E_{-\alpha,\beta}(z)$:

$$\begin{aligned}E_{-\alpha,\beta}(z)&=\frac{1}{\Gamma\left(\beta\right)}-E_{\alpha,\beta}\left(\frac{1}{z}\right)=\frac{1}{\Gamma\left(\beta\right)}-\left(\sum_{k=0}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\alpha k+\beta\right)}\right)\\&=\frac{1}{\Gamma\left(\beta\right)}-\left(\frac{1}{\Gamma\left(\beta\right)}+\sum_{k=1}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\alpha k+\beta\right)}\right)\\&=-\sum_{k=1}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\alpha k+\beta\right)}\end{aligned} \tag{F13}$$

The series representation of $E_{-\alpha,\beta}(z)$ therefore is:

$$E_{-\alpha,\beta}(z)=-\sum_{k=1}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\alpha k+\beta\right)};\qquad E_{-\alpha}(z)=-\sum_{k=1}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\alpha k+1\right)} \tag{F14}$$

Using the above series formula for $E_{-\alpha,\beta}(z)$ we observe that $\lim_{z\uparrow\infty}E_{-\alpha,\beta}(z)=0$ approaches from below the positive z-axis, and $\lim_{z\downarrow 0^+}E_{-\alpha,\beta}(z)=-\infty$. Similarly we observe $\lim_{z\downarrow-\infty}E_{-\alpha,\beta}(z)=0$ approaching from the above negative z-axis.

Now substitute $\alpha=0$ in $E_{-\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}-E_{\alpha,\beta}\left(\frac{1}{z}\right)$, yielding:

$$E_{0,\beta}(z)=\frac{1}{\Gamma\left(\beta\right)}-E_{0,\beta}\left(\frac{1}{z}\right)=\frac{1}{\Gamma\left(\beta\right)}-\sum_{k=0}^{\infty}\frac{\left(1/z\right)^k}{\Gamma\left(\beta\right)} \tag{F15}$$

Summing the result in the above (F15) geometric series, and assuming $\left|z\right|<1$ we have the following:

$$\begin{aligned}E_{0,\beta}(z)&=\frac{1}{\Gamma\left(\beta\right)}-\frac{1}{\Gamma\left(\beta\right)}\left(\frac{1}{1-\left(1/z\right)}\right)\\&=\frac{1}{\left(\Gamma(\beta)\right)\left(1-z\right)}\end{aligned} \tag{F16}$$

We find from the above (F16), that the result is the same as what we obtained for $\lim_{\alpha\downarrow 0^+}E_{\alpha,\beta}(z)=\frac{1}{(\Gamma(\beta))(1-z)}$, i.e. for positive α. This confirms that the Mittag-Leffler functions defined for positive and negative order (α) are equivalent for $\alpha=0$ and $\left|z\right|<1$.

F.5 The graphical representation of the Mittag-Leffler function with the negative order

Now we see the graphical representation for $E_{-\alpha,\beta}(x)$, depicted in Figures F1 to F3; for various ranges of α and various conditions for β.

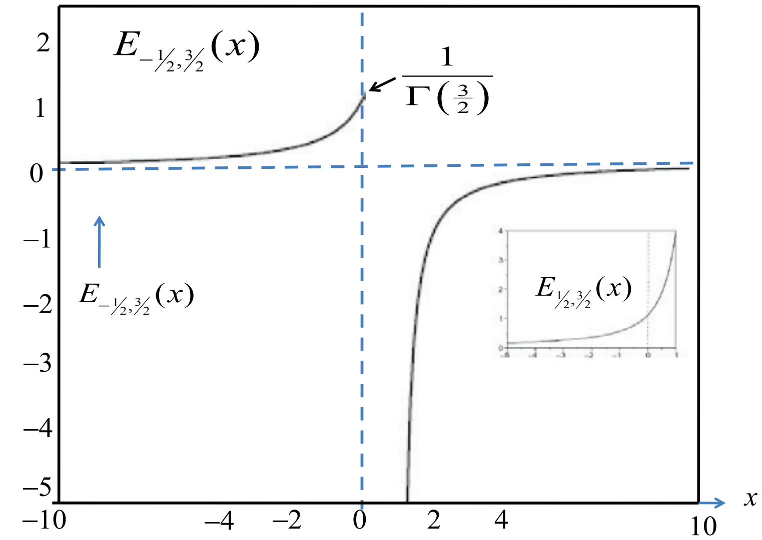

Figure-F1: General pattern of $E_{-\alpha,\beta}(x)$ for $0<\alpha\le 1$ and for $\beta \ge \alpha$

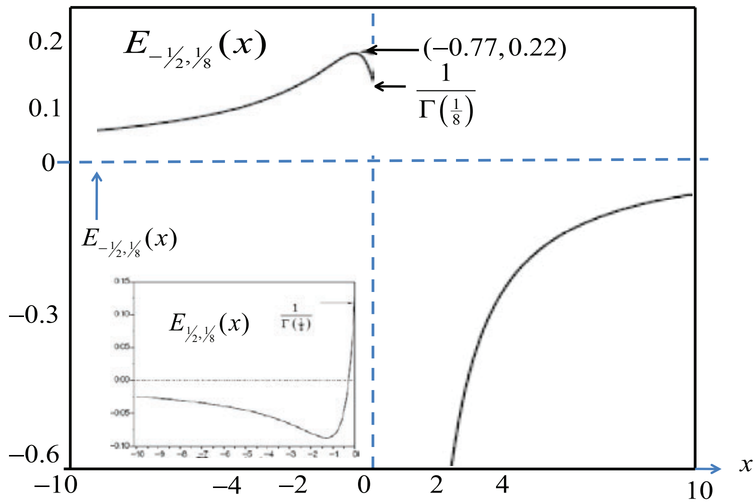

Figure-F2: General pattern of $E_{-\alpha,\beta}(x)$ for $0<\alpha\le 1$ and for $\beta<\alpha$

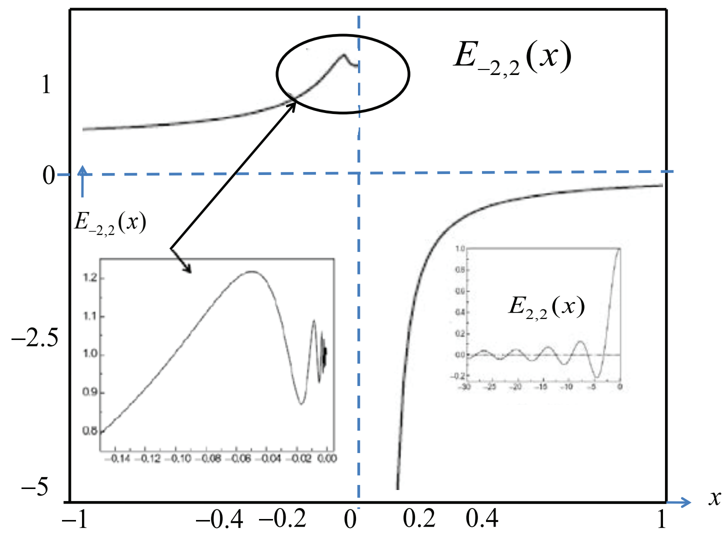

Figure-F3: General pattern of $E_{-\alpha,\beta}(x)$ for $\alpha > 1$

As the value of $x \uparrow 0^-$ we observe from Figure-F1 that $\lim_{x\uparrow 0^-} E_{-\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)}$, where $E_{-\frac{1}{2},\frac{3}{2}}(x)$ is pictured, for a general pattern with $0 < \alpha \le 1$ and $\beta \ge \alpha$. For comparison in Figure-F1, the inset picture shows the graph of $E_{\frac{1}{2},\frac{3}{2}}(x)$. Figure-F2 shows the general pattern of $E_{-\alpha,\beta}(x)$, with $0 < \alpha \le 1$ and $\beta < \alpha$; the plotted function is $E_{-\frac{1}{2},\frac{1}{8}}(x)$ Here a maximum occurs at the negative x-axis point other than $x = 0$. For comparison the graph of $E_{\frac{1}{2},\frac{1}{8}}(x)$ is given as an inset in Figure-F2. Figure-F3 shows the case of $E_{-\alpha,\beta}(x)$ for $\alpha > 1$. The plot in Figure-F3 is for $E_{-2,2}(x)$; and the expanded portion as marked by arrows, and shows very rapid oscillations close to $x = 0$ at a negative x-axis. The inset in Figure-F3 also depicts the nature of $E_{\alpha,\beta}(x)$ for the case $\alpha > 1$, the plotted function is $E_{2,2}(x)$. This Mittag-Leffler function with a negative order function was studied extensively by John W Hannekan and Narahari Achar; some results of their study is presented here.

F.6 Comparisons of the functional relationship of $E_{\alpha,\beta}(x)$ and $E_{-\alpha,\beta}(x)$

Now we list the various functional relationships for $E_{\alpha,\beta}(x)$ and $E_{-\alpha,\beta}(x)$ for comparison, in Table-F1.

Functional relations for $E_{\alpha,\beta}(x)$	**Functional relations for $E_{-\alpha,\beta}(x)$**
$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} x^k$	$E_{-\alpha,\beta}(x) = -\sum_{k=1}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \left(\frac{1}{x}\right)^k$
$E_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + xE_{\alpha,\alpha+\beta}(x)$	$E_{-\alpha,\beta}(x) = \frac{-1}{x(\Gamma(\alpha+\beta))} + \frac{1}{x} E_{-\alpha,\alpha+\beta}(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\beta-1}E_{\alpha,\beta}(x^\alpha)\right] = x^{\beta-2}E_{\alpha,\beta-1}(x^\alpha)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^{\beta-1}} E_{-\alpha,\beta}(x^\alpha)\right] = -\frac{1}{x^\beta} E_{-\alpha,\beta}(x^\alpha)$
$E_{\alpha,\beta}(x) = \beta E_{\alpha,\beta+1}(x) + ax\frac{\mathrm{d}}{\mathrm{d}x}\left[E_{\alpha,\beta+1}(x)\right]$	$E_{-\alpha,\beta}(x) = \beta E_{-\alpha,\beta+1}(x) + ax\frac{\mathrm{d}}{\mathrm{d}x}\left[E_{-\alpha,\beta+1}(x)\right]$
$\int_0^x y^{\beta-1}E_{\alpha,\beta}(y^\alpha)\mathrm{d}y = x^\beta E_{\alpha,\beta+1}(x^\alpha)$	$\int_0^x y^{-\beta-1}E_{-\alpha,\beta}(y^\alpha)\mathrm{d}y = -x^{-\beta}E_{-\alpha,\beta+1}(x^\alpha)$
$\mathcal{L}\left\{x^{\beta-1}E_{\alpha,\beta}(\pm ax^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}$	$\mathcal{L}\left\{x^{\beta-1}E_{-\alpha,\beta}\left(\pm\frac{1}{ax^\alpha}\right)\right\} = \frac{\mp a}{s^\beta(s^\alpha \mp a)}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left[E_\alpha(x)\right]=\frac{1}{\alpha}E_{\alpha,\alpha}(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[E_{-\alpha}(x)\right]=-\frac{1}{\Gamma\left(\alpha+1\right)}+\frac{1}{\alpha}E_{-\alpha,\alpha}(x)$
$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[E_n(x^n)\right]=E_n(x^n);\quad n\in\mathbb{Z}^+$	$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[E_{-n}(x^{-n})\right]=E_{-n}(x^{-n});\quad n\in\mathbb{Z}^+$
$E_{\alpha,\beta}(x)+E_{\alpha,\beta}(-x)=2E_{2\alpha,\beta}(x^2)$	$E_{-\alpha,\beta}(x)+E_{-\alpha,\beta}(-x)=2E_{-2\alpha,\beta}(x^2)$
$E_{n\alpha,\beta}(x^n)=\frac{1}{n}\sum_{k=0}^{n-1}E_{\alpha,\beta}\left(xe^{i2\pi k/n}\right)$ $n\in\mathbb{Z}^+$	$E_{-n\alpha,\beta}(x^n)=\frac{1}{n}\sum_{k=0}^{n-1}E_{-\alpha,\beta}\left(xe^{-i2\pi k/n}\right)$ $n\in\mathbb{Z}^+$
$E_{\alpha,\beta}(x)=x^{-n}E_{\alpha,\beta-\alpha n}(x)$ $-\sum_{k=1}^{n}\frac{x^{-k}}{\Gamma(\beta-\alpha k)};\quad n\in\mathbb{Z}^+$	$E_{-\alpha,\beta}(x)=x^nE_{-\alpha,\beta-\alpha n}(x)$ $+\sum_{k=1}^{n-1}\frac{x^k}{\Gamma(\beta-\alpha k)};\quad n\in\mathbb{Z}^+$
$E_\alpha(-x)=E_{2\alpha}(x^2)-xE_{2\alpha,\alpha+1}(x^2)$	$E_{-\alpha}(-x)=E_{-2\alpha}(x^2)-xE_{-2\alpha,\alpha+1}(x^2)$

Table-F1: Several functional relationships for a positive and negative order Mittag-Leffler function

APPENDIX-G

INVERSE LAPLACE TRANSFORM BY CONTOUR INTEGRATION

G.1 Revising basics of Laplace transforms

For a function $x(t)$, that is zero for $t < 0$, and is defined for $t \geq 0$; the Laplace transform is defined by the following (G1) integral. This is also termed as the Laplace integral; which we have used many times:

$$X(s) = \int_0^{\infty} e^{-st} x(t) \mathrm{d}t \tag{G1}$$

Here we have a complex variable $s = \sigma + i\omega$; with this, we have the following step:

$$X(s) = \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) \mathrm{d}t \tag{G2}$$

The inverse problem is stated as how we can get $x(t)$ from $X(s)$.

The Fourier transform of $x(t)$ is $\hat{x}(\omega)$ defined as $\hat{x}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} x(t) \mathrm{d}t$, and the inverse Fourier transform is given by the following integral:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{x}(\omega) \mathrm{d}\omega \tag{G3}$$

In the Laplace transform expression (i.e. $X(s) = \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) \mathrm{d}t$), as obtained above (G2), let us make the following substitution:

$$\begin{aligned} \phi(t) &= e^{-\sigma t} x(t); \quad t \geq 0 \\ &= 0 \qquad\qquad\; t < 0 \end{aligned} \tag{G4}$$

where σ is a constant in (G4). Taking the Fourier transform of $\phi(t)$ we write the following:

$$\begin{aligned} \hat{\phi}(\omega) &= \int_0^{\infty} e^{-i\omega t} \phi(t) \mathrm{d}t \\ &= \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) \mathrm{d}t \\ &= \int_0^{\infty} e^{-(\sigma + i\omega)t} x(t) \mathrm{d}t = X(\sigma + i\omega) \end{aligned} \tag{G5}$$

Now we take the inverse Fourier transform of $X(\sigma + i\omega)$ as follows:

$$\phi(t) = e^{-\sigma t} x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega t} X(\sigma + i\omega) \mathrm{d}\omega \tag{G6}$$

Therefore, in the above (G6) we obtain $x(t) = e^{\sigma t} \phi(t)$ and we write the following expression:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{\sigma t} \left(e^{i\omega t} X(\sigma + i\omega) \right) \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{(\sigma + i\omega)t} X(\sigma + i\omega) \mathrm{d}\omega \end{aligned} \tag{G7}$$

Letting $s = \sigma + i\omega$, we have $\mathrm{d}s = i\mathrm{d}\omega$, and get the following from the above (G7):

$$\begin{aligned} x(t) &= \frac{1}{2\pi i} \int_{s=\sigma - i\omega}^{s=\sigma + i\omega} e^{st} X(s) \mathrm{d}s; \quad t \geq 0 \\ &= 0; \quad t < 0 \end{aligned} \tag{G8}$$

G.2-The inverse Laplace transform via contour integration

The formula that we derived (G8) that is as follows, called the inverse Laplace transform:

$$x(t) = \frac{1}{2\pi i} \int_{s=\sigma-i\omega}^{s=\sigma+i\omega} e^{st} X(s)\mathrm{d}s \tag{G9}$$

We ask the following questions on this expression (G9)

1). How do we choose the real part of s , i.e. σ ?

2). How do we calculate/evaluate the above integral (G9) in the complex domain?

We already know that $x(t) = 0$ for $t < 0$; that will help to answer (1).

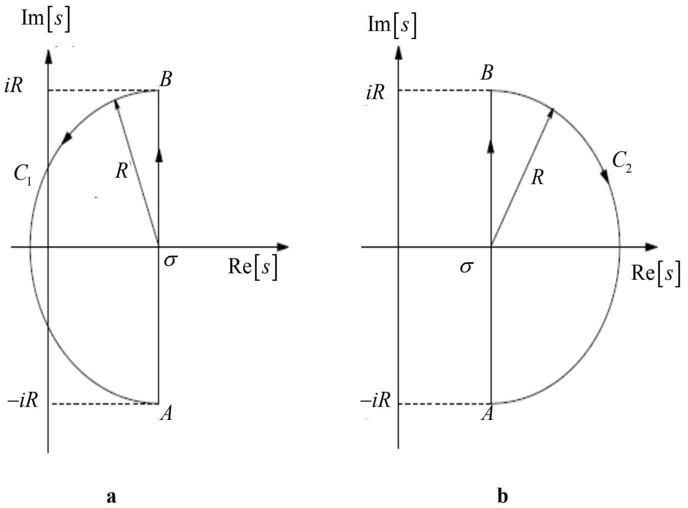

Figure-G1: Contour in the complex-plane to evaluate the inverse Laplace transforms

Consider Figure-G1a, for the closed contour is $A \to B \to C_1$. We write the contour integration as per the residue theorem of complex analysis (Section-1.7) as follows:

$$\begin{aligned}\int_{A\to B\to C_1} e^{st} X(s)\mathrm{d}s &= \int_{A\to B} e^{st} X(s)\mathrm{d}s + \int_{C_1} e^{st} X(s)\mathrm{d}s \\ &= \int_{\sigma-iR}^{\sigma+iR} e^{st} X(s)\mathrm{d}s + \int_{C_1} e^{st} X(s)\mathrm{d}s \\ &= 2\pi i \sum_{\text{At poles}} \text{Residues}\left[e^{st} X(s)\right]\end{aligned} \tag{G10}$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_1$ of Figure-G1a. Now as $R \uparrow \infty$ the integral $\int_{\sigma-iR}^{\sigma+iR} e^{st} X(s)\mathrm{d}s$ is the integral of interest that we require to evaluate the inverse Laplace transform formula (G9). We note that the integral on the line $A \to B$ is the Bromwich integral, and is used for finding the inverse Laplace transform. We will use Jordan's lemma (described shortly) which says that for $t > 0$, $\lim_{R\uparrow\infty} \int_{C_1} e^{st} X(s)\mathrm{d}s = 0$.

Therefore for $t > 0$, we write the following, as $R \uparrow \infty$

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} X(s)\mathrm{d}s = \sum_{\text{Poles at Left of } \sigma} \text{Residues}\left[e^{st} X(s)\right] \tag{G11}$$

Consider Figure-G1b, the closed contour is $A \to B \to C_2$; we write the contour integration as the following:

$$\int_{A\to B\to C_2} e^{st}X(s)\mathrm{d}s = \int_{A\to B} e^{st}X(s)\mathrm{d}s + \int_{C_2} e^{st}X(s)\mathrm{d}s$$
$$= \int_{\sigma-iR}^{\sigma+iR} e^{st}X(s)\mathrm{d}s + \int_{C_2} e^{st}X(s)\mathrm{d}s \qquad \text{(G12)}$$
$$= -2\pi i \sum_{\text{At Poles}} \text{Residues}\left[e^{st}X(s)\right]$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_2$ of Figure-G1b. The negative sign in (G12) indicates that the contour is taken in a clock-wise direction. We will use Jordan lemma's which says for $t<0$, $\lim_{R\uparrow\infty}\int_{C_2} e^{st}X(s)\mathrm{d}s = 0$. Thus we write for $t<0$ the following:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)\mathrm{d}s = -\sum_{\text{Poles at Right of }\sigma} \text{Residues}\left[e^{st}X(s)\right] \qquad \text{(G13)}$$

We know that this above integral (G13) must be zero, since for $t<0$, we have $x(t)=0$. This implies that we have $x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)\mathrm{d}s = 0;\quad t<0$. Thus the LHS of (G13) is zero, making $\sum_{\text{Poles at Right of }\sigma}\text{Residues}\left[e^{st}X(s)\right]=0$. This says that there are no poles at the right side of the line $\text{Re}\left[s\right]=\sigma$ (Figure-G1b).

Therefore, the $\text{Re}\left[s\right]=\sigma$, the line AB (Figure-G1), or Bromwich line, must be chosen such that the contour of Figure-1b, i.e. $A \to B \to C_2$ does not contain any poles of $e^{st}X(s)$ as $R\uparrow\infty$. Thus the contour of Figure-G1a, i.e. $A \to B \to C_1$ must have all poles of $e^{st}X(s)$. This gives the answer to point (1) above. In addition, we note that since e^{st} is analytic everywhere (i.e. it has no poles in the entire Complex-s plane), the poles of $e^{st}X(s)$ are the same as that of $X(s)$. This gives a reply to point (2) as posed above. Thus, we apply residue calculus of the complex analysis and write the following:

$$x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)\mathrm{d}s = \sum_{\text{all poles of }X(s)} \text{Residues}\left[e^{st}X(s)\right] \qquad \text{(G14)}$$

This is how we must evaluate the integral for obtaining an inverse Laplace transform.

G.3 Jordan's Lemma

While we discussed the inverse Laplace transform using contour integration and residue calculus, we also stated that we need a condition that is an integral on arc C_1 in Figure-G1a; as follows:

$$\lim_{R\uparrow\infty}\int_{C_1} e^{st}X(s)\mathrm{d}s = 0, \qquad t>0 \qquad \text{(G15)}$$

While for $t>0$, the points on this arc C_1 are given as:

$$s=\sigma+Re^{i\theta};\quad \frac{\pi}{2}\le\theta\le\frac{3\pi}{2} \qquad \text{(G16)}$$

Examining the standard Laplace transform tables we observe most functions which satisfy $\lim_{|s|\uparrow\infty}X(s)=0$. For example, $X(s)=\frac{1}{s}$, $X(s)=\frac{1}{s+a}$, $X(s)=\frac{a}{s^2+a^2}$ etc. Therefore, in those cases where $R\uparrow\infty$ we have $X(s)\downarrow 0$. We will take a case of $X(s)=\sqrt{s-a}$ shortly where $\lim_{|s|\uparrow\infty}X(s)=\infty$, and still the inverse Laplace transform is obtained.

This condition (i.e. $\lim_{|s|\uparrow\infty}X(s)=0$) means that for any $M_R>0$ a radius R can be found such that $\left|X(s)\right|=\left|X(\sigma+Re^{i\theta})\right|<M_R$. By using the inequality (i.e. $\left|\int_C f(s)\mathrm{d}s\right|\le\int_C\left|f(s)\right|\mathrm{d}s$) for this R we have the following expressions:

$$\begin{aligned}\left|\int_{C_1} e^{st} X(s)\mathrm{d}s\right| &\le \int_{C_1} \left|e^{st} X(s)\right| \mathrm{d}s \\ \left|\int_{C_1} e^{st} X(s)\mathrm{d}s\right| &\le M_R \int_{C_1} \left|e^{st}\right| \mathrm{d}s\end{aligned} \tag{G17}$$

On the arc C_1 for $t > 0$, $s = \sigma + Re^{i\theta}$; $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ and $\mathrm{d}s = iRe^{i\theta}\mathrm{d}\theta$. With these we write the following steps:

$$\begin{aligned}\left|e^{st}\right| = \left|e^{(\sigma+R\mathrm{e}^{i\theta})t}\right| &= \left|e^{(\sigma+R\cos\theta+iR\sin\theta)t}\right| \\ &= \left|e^{(\sigma+R\cos\theta)t} e^{iRt\sin\theta}\right|, \quad \left|e^{i(Rt\sin\theta)}\right| = 1 \\ &= \left|e^{(\sigma+R\cos\theta)t}\right|, \quad e^{(\sigma t+Rt\cos\theta)} > 0 \\ &= e^{\sigma t} e^{Rt\cos\theta}\end{aligned} \tag{G18}$$

Further, we write the steps from (G17) and (G18), with the observation that $e^{Rt\cos\theta}$ is an 'even-function' of θ as the following:

$$\begin{aligned}M_R \int_{C_1} \left|e^{st}\mathrm{d}s\right| &= M_R \int_{\pi/2}^{3\pi/2} \left|e^{\sigma t} e^{Rt\cos\theta} iRe^{i\theta}\mathrm{d}\theta\right| \\ &\le M_R Re^{\sigma t} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\mathrm{d}\theta = 2M_R Re^{\sigma t} \int_{\pi/2}^{\pi} e^{Rt\cos\theta}\mathrm{d}\theta\end{aligned} \tag{G19}$$

By changing the variable $\theta = \xi + \frac{\pi}{2}$, we obtain from (G19) the following expression:

$$\begin{aligned}M_R \int_{C_1} \left|e^{st}\mathrm{d}s\right| &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{Rt\cos\left(\xi+\frac{\pi}{2}\right)}\mathrm{d}\xi \\ &= 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\sin\xi}\mathrm{d}\xi\end{aligned} \tag{G20}$$

Plotting the graphs of $y = \sin\xi$ and a straight line (i.e. $y = \frac{2}{\pi}\xi$), it is observed that in the region $0 \le \xi \le \frac{\pi}{2}$, we see $\sin\xi \ge \frac{2}{\pi}\xi$. With this observation we write the following from (G20):

$$\begin{aligned}M_R \int_{C_1} \left|e^{st}\mathrm{d}s\right| &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\sin\xi}\mathrm{d}\xi \\ &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\left(\frac{2}{\pi}\xi\right)}\mathrm{d}\xi = \frac{M_R \pi e^{\sigma t}}{t}\left(1 - e^{-Rt}\right)\end{aligned} \tag{G21}$$

Therefore, for any $t > 0$ as $R \uparrow \infty$, we have $M_R \downarrow 0$, meaning the above quantity in (G21) tends to zero. This proves our case (i.e. $\lim_{R\uparrow\infty} \int_{C_1} e^{st} X(s)\mathrm{d}s = 0$, for $t > 0$).

G.4 The application of Residue Calculus to get an Inverse Laplace Transform of $X(s) = 2e^{-2s}/(s^2+4)$ by contour integration

We would like to evaluate the inverse Laplace transform of $X(s) = \frac{2e^{-2s}}{s^2+4}$. The Bromwich path integral for the inverse Laplace transform is the following:

$$\begin{aligned}x(t) &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} \frac{2e^{-2s}}{s^2+4}\mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{2e^{s(t-2)}}{s^2+4}\mathrm{d}s\end{aligned} \tag{G22}$$

We observe that the given function i.e. $X(s) = \frac{2e^{-2s}}{s^2+4}$ has two simple poles at $s = 2i$ and $s = -2i$ with both having $\mathrm{Re}\left[s\right] = 0$. In Figure-G1a, we choose $\sigma = 0$. Rather we take an arbitrary (positive) small σ close to zero. We can distinguish the two cases (i) $t < 2$ and (ii) $t > 2$.

For case (i), the exponent $s(t-2)$ has a negative real part if $\mathrm{Re}\left[s\right] > 0$.

We note that $e^{s(t-2)} = e^{(\mathrm{Re}[s]+i\,\mathrm{Im}[s])(t-2)} = e^{(t-2)\mathrm{Re}[s]} e^{i(t-2)\mathrm{Im}[s]}$. Therefore, it is the part $e^{(t-2)\mathrm{Re}[s]}$ which determines the function's behavior at infinity, since $\left|e^{i(t-2)\mathrm{Im}[s]}\right| = 1$. Thus, as $\mathrm{Re}[s] \uparrow \infty$, the function $e^{s(t-2)}$ tends to zero. At the same time, the denominator s^2+4 diverges as $\mathrm{Re}[s] \uparrow \infty$. This means the term $\frac{1}{s^2+4}$ that multiplies $e^{s(t-2)}$ along the path C_2 (Figure-G1b), for $R \uparrow \infty$, and reaches zero. Therefore, we have $\lim_{R\uparrow\infty} \int_{C_2} \frac{2e^{-2s}}{s^2+4} e^{st} \mathrm{d}s = 0$, i.e. an integral on the curve C_2 as $R \uparrow \infty$; as zero. We can calculate the Bromwich path integral by considering $A \to B \to C_2$, but since this closed contour does not have any poles we say that residues are zero, resulting in $x(t)$ as zero.

For case (ii), the function (i.e. $e^{s(t-2)}$) tends to zero as $\mathrm{Re}[s]$ tends to $-\infty$. That means the $\lim_{R\uparrow\infty} \int_{C_1} \frac{2e^{s(t-2)}}{s^2+4} \mathrm{d}s = 0$ along the curve C_1 (Figure G1 a). For the residue theorem, this integral on the closed path $A \to B \to C_1$ is given by the sum of residues of the function $e^{st} X(s) = \frac{2e^{s(t-2)}}{s^2+4}$ at all the poles namely $\pm 2i$; and we write following steps:

$$\begin{aligned} x(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{2e^{-2s}}{s^2+4} e^{st} \mathrm{d}s &= \mathrm{Residue}_{\text{at } s=2i}\left(\frac{2e^{-2s}}{s^2+4} e^{st}\right) \\ &\quad + \mathrm{Residue}_{\text{at } s=-2i}\left(\frac{2e^{-2s}}{s^2+4} e^{st}\right) \\ &= \lim_{s\to 2i}\left((s-2i)\frac{2e^{s(t-2)}}{s^2+4}\right) \\ &\quad + \lim_{s\to -2i}\left((s+2i)\frac{2e^{s(t-2)}}{s^2+4}\right) \\ &= \lim_{s\to 2i} \frac{2e^{s(t-2)}}{s+2i} + \lim_{s\to -2i} \frac{2e^{s(t-2)}}{s-2i} \\ &= \frac{e^{2i(t-2)}}{2i} - \frac{e^{-2i(t-2)}}{2i} \\ &= \sin\left(2(t-2)\right) \end{aligned} \tag{G23}$$

Thus, we write the following result:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\} = \begin{cases} \sin\left(2(t-2)\right) & t > 2 \\ 0 & t < 2 \end{cases} \tag{G24}$$

G.5 The application of residue calculus to get the inverse Laplace transform of $X(s) = \sqrt{s-a}$ by contour integration on the branch cut in a complex plane

This is for a multi-valued function; we take a function $X(s) = \sqrt{s-a}$, with $a \in \mathbb{R}$. The inverse Laplace transform is the following:

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\sqrt{s-a}\,\mathrm{d}s \tag{G25}$$

The function $e^{st}\sqrt{s-a}$ has no poles but the function $\sqrt{z}$ (taking $z = s-a$) is a multi-valued function in the complex plane. Therefore we see a branch point at $z = 0$, namely at $s = a$. This is the only singularity of our function $X(s)e^{st}$ and therefore to evaluate the Bromwich path integral, we have to take σ larger than a. Thus, the integral will be the following:

$$
\begin{aligned}
x(t) = \mathcal{L}^{-1}\left\{\sqrt{s-a}\right\} &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\sqrt{s-a}\mathrm{d}s\,; \quad s-a=z \\
&= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{(a+z)t}\sqrt{z}\mathrm{d}z\,; \quad \lambda=\sigma-a \\
&= \frac{e^{at}}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{zt}\sqrt{z}\mathrm{d}z
\end{aligned}
\tag{G26}
$$

In this case, the branch point is zero. Therefore, the $\mathrm{Re}\left[z\right]=\lambda$ can be arbitrarily small (but is always larger than zero). Since $z=0$ is a branch point of the function to integrate, we have to introduce a branch cut to evaluate the integral. Although what we have taken so far is the positive real axis as a branch cut, we have said that this choice is arbitrary and to make the function $\sqrt{z}$ into a single value, it is enough that closed curves are not allowed to enclose the origin. We can therefore take a branch cut as the negative real axis. In Figure-G2, we indicate the contour used to integrate the given function.

Since the closed contour $A \to B \to C \to D \to E \to F \to G \to A$ does not enclose any singularity, its integral is zero. To evaluate the Bromwich path integral, (namely along $A \to B$), we have to calculate the integral along the arcs C, G along the straight lines D, F and along the circumference on the small circle E (Figure-G2).

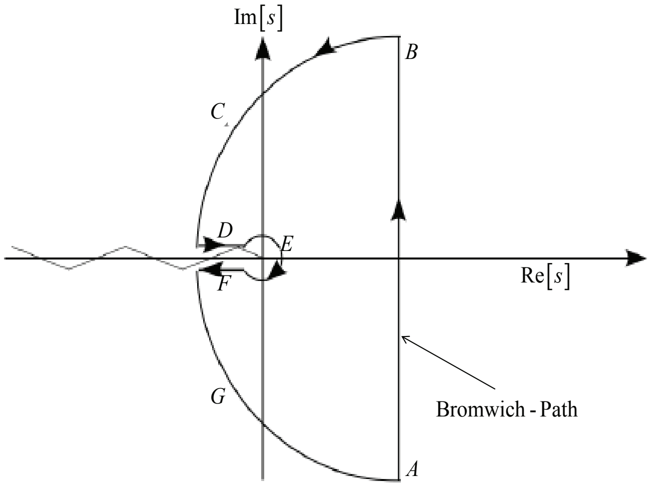

Figure-G2: Contour of integration with branch cut in complex plane

The function $e^{zt}\sqrt{z}$ goes to zero, for $\mathrm{Re}\left[z\right]$ tends towards $-\infty$ (the term $\sqrt{z}$ cannot match the exponential decay of e^{zt} for $t>0$ as always). Thus, the integral along the arcs C, G is zero and disappears as the radius of these arcs grows. To evaluate the integral along a small circle E we take $z = \in e^{i\theta}$ in the interval $\left[\pi, -\pi\right]$ and we take the limit $\in \downarrow 0$. With this we have the following:

$$
\int_E e^{zt}\sqrt{z}\mathrm{d}z = \int_{\pi}^{-\pi} e^{\in t e^{i\theta}}\sqrt{\in}e^{i\left(\frac{\theta}{2}\right)} i \in e^{i\theta}\mathrm{d}\theta \tag{G27}
$$

The above expression (G27) integrating the function clearly tends to zero for a limit $\in \downarrow 0$. Hence, there is no contribution from the integration over the circumference of E.

Along the straight lines D, F, we can assume that the arguments of the complex variables lying on them are π (along D) and $-\pi$ (along F) and that their imaginary parts are close to zero. Therefore we have $z = re^{i\pi}$ (for D) and $z = re^{-i\pi}$ (for F). Consequently, we have $\mathrm{d}z = e^{i\pi}\mathrm{d}r$ for D and $\mathrm{d}z = e^{-i\pi}\mathrm{d}r$ for F. We note $e^{i\pi} = e^{-i\pi} = -1$. The parameter r runs between $+\infty$ and 0 for line D and between 0 and $+\infty$ for line F. These integrals are given in the following expressions:

$$\begin{aligned}\int_D \sqrt{z}e^{zt}\mathrm{d}z &= \int_\infty^0 \sqrt{r}e^{i\left(\frac{\pi}{2}\right)}e^{tre^{i\pi}}e^{i\pi}\mathrm{d}r \\ &= \int_\infty^0 \sqrt{r}(i)(e^{-rt})(-1)\mathrm{d}r = i\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r \\ \int_F \sqrt{z}e^{zt}\mathrm{d}z &= \int_0^\infty \sqrt{r}e^{i\left(-\frac{\pi}{2}\right)}e^{tre^{-i\pi}}e^{-i\pi}\mathrm{d}r \\ &= \int_0^\infty \sqrt{r}(-i)(e^{-rt})(-1)\mathrm{d}r = i\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r\end{aligned} \tag{G28}$$

From the above calculations of contour integrations on C, D, E, F, G and with the residue theorem, we get the following (with an observation that the closed contour Figure-G2 A, B, C, D, E, F, G, A does not enclose any poles of $X(s)$, so the residue is zero):

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\left\{X(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\sqrt{s-a}\right\} \\ &= \frac{1}{2\pi i}\int_{A\to B} e^{st}\sqrt{s-a}\mathrm{d}s = -\frac{e^{at}}{2\pi i}\int_{D+E}\sqrt{z}e^{zt}\mathrm{d}z \\ &= -\frac{e^{at}}{\pi}\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r\end{aligned} \tag{G29}$$

In order to evaluate $\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r$, put $rt = \tau^2$ and $\mathrm{d}r = \frac{2\tau\mathrm{d}\tau}{t}$, then write the following:

$$\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r = \frac{1}{t^{3/2}}\int_0^\infty \tau e^{-\tau^2}2\tau\mathrm{d}\tau \tag{G30}$$

We observe that $\frac{\mathrm{d}}{\mathrm{d}\tau}e^{-\tau^2} = -2\tau e^{-\tau^2}$, with this we carry out integration by parts for the above RHS and write the following:

$$\begin{aligned}\int_0^\infty \sqrt{r}e^{-rt}\mathrm{d}r &= -\frac{1}{t^{3/2}}\left(\left[\tau e^{-\tau^2}\right]_0^\infty - \int_0^\infty e^{-\tau^2}\mathrm{d}\tau\right) \\ &= \frac{\sqrt{\pi}}{2t^{3/2}}\end{aligned} \tag{G31}$$

We have used the known result (i.e. $\int_0^\infty e^{-\tau^2}\mathrm{d}\tau = \frac{\sqrt{\pi}}{2}$) in the above derivation. Now compactly we write the result as following:

$$\mathcal{L}^{-1}\left\{\sqrt{s-a}\right\} = -\frac{e^{at}}{2\sqrt{\pi t^3}} \tag{G32}$$

We can write the above derivation $\mathcal{L}^{-1}\left\{-\sqrt{s-a}\right\} = \frac{\mathrm{e}^{at}}{2\sqrt{\pi t^3}}$, and thus we have the following useful Laplace transform identity:

$$\mathcal{L}^{-1}\left\{\sqrt{s-a} - \sqrt{s-b}\right\} = \frac{e^{bt} - e^{at}}{2\sqrt{\pi t^3}} \tag{G33}$$

G.6 The application of residue calculus to get the inverse Laplace transform of $X(s) = \frac{\ln s}{\mathrm{K}(s^b - s^a)}$ by contour integration on the branch cut in a complex plane

Let us find $x(t) = \mathcal{L}^{-1}\left\{\left(\ln s\right)/\left(\mathrm{K}(s^b - s^a)\right)\right\}$; $b > a > 0$. The behavior of the function $X(s) = \frac{\ln s}{\mathrm{K}(s^b - s^a)}$ in limit as $s \uparrow \infty$ we note by the L'Hospital rule in the following step:

$$\lim_{s\uparrow\infty} X(s) = \lim_{s\uparrow\infty}\frac{\frac{1}{s}}{\mathrm{K}(bs^{b-1} - as^{a-1})} = 0 \tag{G34}$$

We may feel $s=1$ is a singular point of $X(s)$, but by L'Hospital rule, we see it is not, as follows:

$$\begin{aligned}\lim_{s\to 1} X(s) &= \lim_{s\to 1} \frac{\frac{1}{s}}{\mathrm{K}(bs^{b-1}-as^{a-1})}\\ &= \frac{1}{\mathrm{K}(b-a)}\end{aligned} \tag{G35}$$

So we find $s=1$ is not a singular point. The point $s=0$ is a singularity in the $X(s)$.

Due to the multi-valued nature of the function $\ln s$ we call the singular point a branch point and not a pole. The contour shown in Figure-G2 excludes the branch point (zero), and the branch-cut is taken as a standard branch-cut i.e. the negative real axis. The inverse Laplace transform integral is the following:

$$\begin{aligned}x(t) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{\text{Bromwich}} e^{st}X(s)\mathrm{d}s\end{aligned} \tag{G36}$$

The above integration is the Bromwich path integration on the line AB as in Figure-G2. The $\mathrm{Re}\left[s\right]=\sigma$ is the line, to right of which we do not have any singularity, or the singularity is contained in the left side of the line AB (Figure-G2). So we select $\sigma=0^{+}$ close to the imaginary axis as shown in Figure-G2. By use of Residue Calculus, we get the following:

$$\int_{A\to B\to C\to D\to E\to F\to G\to A} e^{st}X(s)\mathrm{d}s = 2\pi i\sum \text{Residues of}\left[e^{st}X(s)\right] \tag{G37}$$

The discussion about $X(s)=\frac{\ln(s)}{\mathrm{K}(s^b-s^a)}$ indicates that there are no singularities in the path enclosed by the contour of integration A,B,C,D,E,F,G,A. Thus, the residues are zero, and we write the following:

$$\begin{aligned}&\int_{A\to B\to C\to D\to E\to F\to G\to A} e^{st}X(s)\mathrm{d}s = 0\\ &\int_{A\to B} e^{st}X(s)\mathrm{d}s = -\int_C e^{st}X(s)\mathrm{d}s - \int_D e^{st}X(s)\mathrm{d}s\\ &\qquad -\int_E e^{st}X(s)\mathrm{d}s - \int_F e^{st}X(s)\mathrm{d}s\\ &\qquad -\int_G e^{st}X(s)\mathrm{d}s\end{aligned} \tag{G38}$$

That is we need to evaluate the integral on big arcs C and G; and on lines just above and the below the negative real axis D and F, and also on a small circle at origin, i.e. E.

We take $X(s)=\frac{\ln s}{s^2-1}$, that is $b=2$, $a=0$, $\mathrm{K}=1$, in our function $X(s)=\frac{\ln(s)}{\mathrm{K}(s^b-s^a)}$ and do the calculations.

Let the big arcs be represented by $C: Re^{i\theta}$; $\frac{\pi}{2}\le\theta\le-\pi$ and $G: Re^{i\theta}$; $-\pi\le\theta\le-\frac{\pi}{2}$, then the integrals are:

$$\begin{aligned}\int_C e^{st}X(s)\mathrm{d}s &= \lim_{R\uparrow\infty}\int e^{Re^{i\theta}t}X(Re^{i\theta})e^{i\theta}\mathrm{d}R;\quad s=Re^{i\theta}\\ &= \lim_{R\uparrow\infty}\int e^{Re^{i\theta}t}\left(\frac{\ln(Re^{i\theta})}{R^2e^{i2\theta}-1}\right)e^{i\theta}\mathrm{d}R\\ &= \lim_{R\uparrow\infty}\int e^{Re^{i\theta}t}\left(\frac{\ln(R)+i\theta}{R^2e^{i2\theta}-1}\right)e^{i\theta}\mathrm{d}R\end{aligned} \tag{G39}$$

It may be noted that the function $X(s)=\frac{\ln s}{s^2-1}$ vanishes at large values of s; hence the integral on the large arcs C and G are zero (we discussed this in the section on Jordan's Lemma):

$$\int_C e^{st}X(s)\mathrm{d}s = \int_G e^{st}X(s)\mathrm{d}s = 0 \tag{G40}$$

Now we do an integration on the small circle E with $s = \in e^{i\theta}$ encircling the origin, with $\in \downarrow 0$ in the interval $\left[\pi, -\pi\right]$ for θ that is the integration in a positive sense (anti-clockwise). Therefore, we have the following:

$$\begin{aligned}\int_E e^{st} X(s)\mathrm{d}s &= \lim_{\in\downarrow 0} \int_{\pi}^{-\pi} e^{\in e^{i\theta} t} \left(\frac{\ln \in + i\theta}{\in^2 e^{i2\theta} - 1} \right) i \in e^{i\theta} \mathrm{d}\theta \\ &\sim \lim_{\in\downarrow 0} \int_{\pi}^{-\pi} (1) \left(\frac{\ln \in + i\theta}{-1} \right) i \in e^{i\theta} \mathrm{d}\theta \\ &= \lim_{\in\downarrow 0, M\uparrow\infty} \int_{\pi}^{-\pi} \left(M - i\theta\right)\left(i \in e^{i\theta}\right) \mathrm{d}\theta = 0\end{aligned} \tag{G41}$$

In the above steps (G41), we took for $\in^2 e^{i\theta} - 1 \sim -1$; $e^{\in t e^{i\theta}} \sim 1$ and $\ln \in \sim -M$ with $M \uparrow \infty$ as we have for the limit, i.e. $\in \downarrow 0$. The expression (G41) when multiplied by $\in$ gives zero, for $\in \downarrow 0$; thus we have an integration on the small circle which is zero. We note that $\lim_{s\downarrow 0} X(s) = \frac{\ln s}{s^2 - 1} = \infty$, but the integration on a small circle encircling $s = 0$ with $\lim_{\in\downarrow 0} \int_{s = \in e^{i\theta}} e^{st} X(s)\mathrm{d}s = 0$.

Thus we are left with integration on D where we take $s = ze^{i\pi}$; $\mathrm{d}s = e^{i\pi}\mathrm{d}z$ with z varying from ∞ to zero; and on line F with $s = ze^{-i\pi}$, $\mathrm{d}s = e^{-i\pi}\mathrm{d}z$ and with z varying from zero to ∞ We note that we are on a negative real axis in both cases where z is positive and $e^{i\pi} = e^{-i\pi} = -1$. Thus the integration on line D is:

$$\begin{aligned}\int_D e^{st} X(s)\mathrm{d}s &= \int_{\infty}^{0} e^{ze^{i\pi} t} \left(\frac{\ln\left(ze^{i\pi}\right)}{z^2 e^{i(2\pi)} - 1} \right) e^{i\pi}\mathrm{d}z \\ &= \int_{\infty}^{0} e^{-zt} \left(\frac{\ln z + i\pi}{z^2 - 1} \right)(-\mathrm{d}z) = \int_{0}^{\infty} e^{-zt} \left(\frac{\ln z + i\pi}{z^2 - 1} \right)\mathrm{d}z\end{aligned} \tag{G42}$$

Similarly, we write the integration on F as follows:

$$\begin{aligned}\int_F e^{st} X(s)\mathrm{d}s &= \int_{0}^{\infty} e^{ze^{-i\pi} t} \left(\frac{\ln\left(ze^{-i\pi}\right)}{z^2 e^{i(-2\pi)} - 1} \right) e^{-i\pi}\mathrm{d}z \\ &= \int_{0}^{\infty} e^{-zt} \left(\frac{\ln z - i\pi}{z^2 - 1} \right)(-\mathrm{d}z)\end{aligned} \tag{G43}$$

Adding the two (G42) and (G43), we get the following expression:

$$\begin{aligned}\int_{D+F} e^{st} X(s)\mathrm{d}s &= \int_{0}^{\infty} e^{-zt} \left(\frac{\ln z + i\pi}{z^2 - 1} \right)\mathrm{d}z - \int_{0}^{\infty} e^{-zt} \left(\frac{\ln z - i\pi}{z^2 - 1} \right)\mathrm{d}z \\ &= \int_{0}^{\infty} e^{-zt} \left(\frac{2i\pi}{z^2 - 1} \right)\mathrm{d}z\end{aligned} \tag{G44}$$

The inverse Laplace transform integral on the Bromwich path is thus:

$$\begin{aligned}x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} &= \frac{1}{2\pi i} \int_{\text{Bromwich}} e^{st} X(s)\mathrm{d}s \\ &= -\frac{1}{2\pi i} \int_{D+F} e^{st} X(s)\mathrm{d}s \\ &= -\frac{1}{2\pi i} \int_{0}^{\infty} e^{-zt} \left(\frac{2\pi i}{z^2 - 1} \right)\mathrm{d}z \\ &= -\int_{0}^{\infty} e^{-zt} \left(\frac{1}{z^2 - 1} \right)\mathrm{d}z \\ &= \int_{0}^{\infty} e^{-zt} \left(\frac{1}{1 - z^2} \right)\mathrm{d}z\end{aligned} \tag{G45}$$

Therefore, we have obtained the following:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{\ln s}{s^2-1}\right\} = \int_0^\infty e^{-zt}\left(\frac{1}{1-z^2}\right)\mathrm{d}z \tag{G46}$$

The observation is for $X(s) = \frac{\ln s}{s^b - s^a}$; $b > a \geq 0$; that if $a \neq 0$ then the integration on the small circle E blows up, and the method fails. However, we can resort to Berberan-Santos method that we discussed in Chapter-6.

G.7 The application of residue calculus to get the inverse Laplace transform of $X(s) = \frac{s^b - s^a}{\ln s}$ by contour integration on branch cut in the complex plane

Here we need to do an inverse Laplace transform of $X(s) = \frac{s^b - s^a}{\ln s}$, $b > a$. We note here that $\lim_{s\uparrow\infty} X(s) = \infty$. Thus, if we modify $X(s)$ as $\bar{X}(s) = \frac{s^b - s^a}{s\ln s}$, then we get $\lim_{s\uparrow\infty} \bar{X}(s) = 0$, and we can comfortably apply the formulas for inverse Laplace transforms.

Let us take $b = 2$ and $a = 0$, then via the contour integration method as we did in earlier Section-G.6, we see-an integral on the big arcs C and G are zero, as $\bar{X}(s) = \frac{s^2-1}{s\ln s}$ goes to zero for large R with $s = Re^{i\theta}$. For the small circle $s = \in e^{i\theta}$ in the interval $\left[\pi, -\pi\right]$ for θ (i.e. E) (Figure-G2) enclosing the origin we have the following:

$$\begin{aligned}
\int_E e^{st}\bar{X}(s)\mathrm{d}s &= \lim_{\in\downarrow 0}\int_\pi^{-\pi} e^{st}\frac{s^2-1}{s\ln s}\mathrm{d}s\Bigg|_{s=\in e^{i\theta}} \\
&= \lim_{\in\downarrow 0}\int_\pi^{-\pi} e^{\in e^{i\theta}}\frac{\in^2 e^{i2\theta}-1}{\in e^{i\theta}\left(\ln\in + i\theta\right)}\in ie^{i\theta}\mathrm{d}\theta \\
&= \int_\pi^{-\pi}(1)\lim_{M\uparrow\infty}\frac{(-1)}{\left(-M+i\theta\right)}i\mathrm{d}\theta \\
&= \int_\pi^{-\pi}\lim_{M\uparrow\infty}\frac{1}{\left(\sqrt{M^2+\theta^2}\right)\tan^{-1}\left(-\frac{\theta}{M}\right)}i\mathrm{d}\theta = 0
\end{aligned} \tag{G47}$$

Thus we are left with integrals on the lines D with $s = ze^{i\pi}$, $\infty > z > 0$ and F with $s = ze^{-i\pi}$, $0 < z < \infty$ and $\mathrm{d}s = -\mathrm{d}z$. So we have the following:

$$\begin{aligned}
\int_{D+F} e^{st}\bar{X}(s)\mathrm{d}s &= \int_\infty^0 e^{-zt}\frac{z^2-1}{(-z)\left(\ln z + i\pi\right)}\left(-\mathrm{d}z\right) \\
&\quad + \int_0^\infty e^{-zt}\frac{z^2-1}{(-z)\left(\ln z - i\pi\right)}\left(-\mathrm{d}z\right) \\
&= \int_0^\infty e^{-zt}\left(\frac{z^2-1}{(-z)}\right)\left(\frac{1}{\ln z + i\pi} - \frac{1}{\ln z - i\pi}\right)\mathrm{d}z \\
&= \int_0^\infty e^{-zt}\left(\frac{z^2-1}{z}\right)\left(\frac{2i\pi}{(\ln z)^2 + \pi^2}\right)\mathrm{d}z
\end{aligned} \tag{G48}$$

As we have seen, the inverse Laplace transform is an integration on the Bromwich path we have from the residue calculus, and contour integration the following, noting that there are no poles enclosed in the contour of Figure-G2;

$$\begin{aligned}
\bar{x}(t) = \mathcal{L}^{-1}\left\{\bar{X}(s)\right\} &= \frac{1}{2\pi i}\int_{A\to B} e^{st}\bar{X}(s)\mathrm{d}s \\
&= -\frac{1}{2\pi i}\int_{D+F} e^{st}\bar{X}(s)\mathrm{d}s \\
&= \int_0^\infty e^{-zt}\left(\frac{1-z^2}{z\left((\ln z)^2+\pi^2\right)}\right)\mathrm{d}z
\end{aligned} \tag{G49}$$

We started with $X(s)$ conditioned it to $\bar{X}(s)$ by dividing by s, and then obtained the inverse Laplace transform. To get $x(t)=\mathcal{L}^{-1}\left\{X(s)\right\}$ in this example of $X(s)=\frac{s^b-s^a}{\ln s}$, $b>a$ we have to perform a derivative of the obtained $\bar{x}(t)$, i.e. $x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t)$. This, we write as follows:

$$x(t)=\mathcal{L}^{-1}\left\{\frac{s^2-1}{\ln s}\right\}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \frac{(1-z^2)e^{-zt}}{z\left((\ln z)^2+\pi^2\right)}\mathrm{d}z\right) \tag{G50}$$

We come across these types of 'characteristic functions' (transfer functions) discussed in Sections-G.6 and G.7 for a "continuous order distributed differential equation" discussed in Chapter-9, where we used the Berberan-Santos method to get an inverse Laplace transform to work out the time responses of these systems.

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…I shall always remain a student of this beautiful subject…

There is a lot more to learn

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We must develop it further and further in order to know which derivative nature functions best

…Thus, it is a small beginning