Julius Wess

Jonathan Bagger

Supersymmetry and Supergravity

SECOND EDITION

REVISED AND EXPANDED

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Supersymmetry
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by

Julius Wess

and

Jonathan Bagger

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PREFACE TO THE SECOND EDITION

Since these lectures were given, supersymmetric particle phenomenology has been the subject of extensive study. Many models have been proposed, including some that make essential use of the supergravity multiplet. A variety of experimental searches have been carried out, and more are planned for the future.

Given this state of affairs, we felt that the second edition of this book should go substantially beyond the first. The second edition contains a total of six new chapters and five new appendixes. The new chapters are primarily devoted to deriving the component form of the most general supersymmetric gauge theory coupled to supergravity. The resulting Lagrangian, presented in Chapter XXV and Appendix G, is the starting point for all phenomenological studies of supergravity theories. Model-builders can use the Lagrangian without having to read the rest of the book.

The new appendixes contain introductions to Kähler geometry, isometries, and nonlinear realizations of symmetries. The material is essential for understanding the derivations in the book, but it is also of more general interest. In Chapter XXVI the techniques of nonlinear realizations are applied to supersymmetric gauge theories. The results pave the way for a model-independent approach to supersymmetry phenomenology, in the spirit of chiral dynamics.

The new additions have broadened the scope of the book so that it should appeal to physicists of formal and phenomenological interests. In its present form, the book provides a theoretical basis for further phenomenological studies of supersymmetric theories.

We would like to thank the Gottfried Wilhelm Leibnitz Program of the DFG and the Alfred P. Sloan Foundation for financial support during the preparation of the second edition.

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February 1991

PREFACE

The strong interest with which these lectures on supersymmetry and supergravity were received at Princeton University encouraged me to make their contents accessible to a larger audience. They are not a systematic review of the subject. Instead, they offer an introduction to the approach followed by Bruno Zumino and myself in our attempt to develop and understand the structure of supersymmetry and supergravity.

This book consists of two parts. The first develops a formalism which allows us to construct supersymmetric gauge theories. The second part extends this formalism to local supersymmetry transformations.

At the end of each chapter, two papers are cited which I recommend to the reader. I am aware that this selection does not do justice to many authors who have contributed to the subject. However, I would like to draw attention to the more complete lists of references found in P. Fayet and S. Ferrara, *Supersymmetry*, Physics Reports 32C, No. 5, 1977, and P. Van Nieuwenhuizen, *Supergravity*, Physics Reports 68C, No. 4, 1981.

Throughout the text, important equations are numbered in boldface. They are collected at the end of each chapter. Exercises are also included along with each chapter; many of them contain information essential to a deeper understanding of the subject.

This book was prepared in collaboration with Jonathan Bagger, without whom it would never have been written. Both Jon and I would like to thank Winnie Waring for her devoted assistance in the preparation of the manuscript. As a tribute to her high standards, we have tried our best to avoid errors in factors and signs. Many people have helped eliminate these errors. In particular, we would like to thank Martin Müller for his assistance with the second half of the book.

I wish to express my gratitude to the Federal Republic of Germany for the grant which made possible my stay at The Institute for Advanced Study as an Albert Einstein Visiting Professor, and Jon would like to express his appreciation to the U.S. National Science Foundation for his Graduate Fellowship at Princeton University.

In conclusion, I would like to thank Stephen Adler and the Members of the Institute for Advanced Study, as well as David Gross and the Department of Physics at Princeton University, for their most encouraging and critical interest in these lectures.

JULIUS WESS
UNIVERSITY OF KARLSRUHE
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Supersymmetry
and
Supergravity

I. WHY SUPERSYMMETRY?

Supersymmetry is a subject of considerable interest among physicists and mathematicians. Not only is it fascinating in its own right, but there is also a growing belief that it may play a fundamental role in particle physics. This belief is based on an important result of Haag, Sohnius, and Lopuszanski, who proved that the supersymmetry algebra is the only graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory. In this chapter, we shall discuss their theorem and its proof. (Readers specifically interested in supersymmetric theories might prefer to start directly with Chapter II or III.)

Before we begin, however, we first present the supersymmetry algebra:

$$
\begin{aligned}
\{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\}_+ &= 2\sigma_{\alpha\dot\beta}{}^m P_m\, \delta^A{}_B \\
\{Q_\alpha{}^A, Q_\beta{}^B\}_+ &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\}_+ = 0 \\
[P_m, Q_\alpha{}^A]_- &= [P_m, \bar{Q}_{\dot\alpha A}]_- = 0 \\
[P_m, P_n]_- &= 0.
\end{aligned}
\tag{I}
$$

The Greek indices $(\alpha, \beta, \ldots, \dot\alpha, \dot\beta, \ldots)$ run from one to two and denote two-component Weyl spinors. The Latin indices $(m, n, \ldots)$ run from one to four and identify Lorentz four-vectors. The capital indices $(A, B, \ldots)$ refer to an internal space; they run from 1 to some number $N \geq 1$. The algebra with $N = 1$ is called the supersymmetry algebra, while those with $N > 1$ are called extended supersymmetry algebras. All the notation and conventions used throughout this book are summarized in Appendix A.

We are now ready to consider the theorem. Of all the graded Lie algebras, only the supersymmetry algebras (together with their extensions to include central charges, which we shall discuss at the end of the chapter) generate symmetries of the S-matrix consistent with relativistic quantum field theory. The proof of this statement is based on the Coleman-Mandula theorem, the most precise and powerful in a series of no-go theorems about the possible symmetries of the S-matrix.

The Coleman-Mandula theorem starts from the following assumptions:

(1) the S-matrix is based on a local, relativistic quantum field theory in four-dimensional spacetime;
(2) there are only a finite number of different particles associated with one-particle states of a given mass; and
(3) there is an energy gap between the vacuum and the one particle states.

The theorem concludes that the most general Lie algebra of symmetries of the S-matrix contains the energy-momentum operator P_m, the Lorentz rotation generator M_{mn}, and a finite number of Lorentz scalar operators B_ℓ. The theorem further asserts that the B_ℓ must belong to the Lie algebra of a compact Lie group.

Supersymmetries avoid the restrictions of the Coleman-Mandula theorem by relaxing one condition. They generalize the notion of a Lie algebra to include algebraic systems whose defining relations involve anticommutators as well as commutators. These new algebras are called superalgebras or graded Lie algebras. Schematically, they take the following form:

$$\{Q,Q'\}_+ = X \qquad [X,X']_- = X'' \qquad [Q,X]_- = Q''. \tag{1.2}$$

Here Q, Q', and Q'' represent the odd (anticommuting) part of the algebra, and X, X', and X'' the even (commuting) part.

The operators X are determined by the Coleman-Mandula theorem. They are either elements of the Poincaré algebra $\mathscr{P} = \{P_m, M_{mn}\}$ or elements of a Lorentz-invariant compact Lie algebra $\mathscr{A}$. The algebra $\mathscr{A}$ is a direct sum of a semisimple algebra $\mathscr{A}_1$ and an Abelian algebra $\mathscr{A}_2$, $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

The generators Q may be decomposed into a sum of representations irreducible under the homogeneous Lorentz group $\mathscr{L}$:

$$Q = \sum Q_{\underline{\alpha_1 \cdots \alpha_a}, \underline{\dot{\alpha}_1 \cdots \dot{\alpha}_b}}. \tag{1.3}$$

The $Q_{\underline{\alpha_1 \cdots \alpha_a}, \underline{\dot{\alpha}_1 \cdots \dot{\alpha}_b}}$ are symmetric with respect to the underlined indices $\alpha_1 \cdots \alpha_a$ and $\dot{\alpha}_1 \cdots \dot{\alpha}_b$. They belong to irreducible spin-$\frac{1}{2}(a+b)$ representations of $\mathscr{L}$. Since the Q's anticommute, the connection between spin and statistics tells us that $a + b$ must be odd.

We shall now invoke two additional assumptions to prove that $a + b = 1$. These assumptions are:

(1) the operators Q act in a Hilbert space with positive definite metric; and
(2) both Q and its hermitian conjugate $\bar{Q}$ belong to the algebra.

We start by considering the anticommutator

$$\{Q_{\underline{\alpha_1\cdots\alpha_a},\underline{\dot\alpha_1\cdots\dot\alpha_b}},\bar{Q}_{\underline{\dot\beta_1\cdots\dot\beta_a},\underline{\beta_1\cdots\beta_b}}\}, \tag{1.4}$$

where all the indices are assigned the value 1. The product

$$Q_{\underbrace{1\cdots 1}_{a},\underbrace{\dot 1\cdots\dot 1}_{b}}\ \bar{Q}_{\underbrace{\dot 1\cdots\dot 1}_{a},\underbrace{1\cdots 1}_{b}} \tag{1.5}$$

belongs to a spin-$(a + b)$ representation of $\mathscr{L}$, so

$$\left\{Q_{\underbrace{1\cdots 1}_{a},\underbrace{\dot 1\cdots\dot 1}_{b}},\bar{Q}_{\underbrace{\dot 1\cdots\dot 1}_{a},\underbrace{1\cdots 1}_{b}}\right\} \tag{1.6}$$

must close into an even element of the algebra with spin $(a + b)$. From the Coleman-Mandula theorem, we know that this element is either zero or a component of P_m. For $a + b > 1$, it must be zero.

The anticommutator (1.6) is a positive definite operator in a Hilbert space with a positive definite metric. This tells us that $Q_{\underbrace{1\cdots 1}_{a},\underbrace{\dot 1\cdots\dot 1}_{b}} = 0$ for $a + b > 1$. Since the $Q_{\underline{\alpha_1\cdots\alpha_a},\underline{\dot\alpha_1\cdots\dot\alpha_b}}$ are irreducible under $\mathscr{L}$, they all must vanish for $a + b > 1$. From this we conclude that the odd part of the supersymmetry algebra is composed entirely of the spin-$\frac{1}{2}$ operators $Q_\alpha{}^L$ and $\bar{Q}_{\dot\alpha M}$.

The anticommutator of $Q_\alpha{}^L$ and $\bar{Q}_{\dot\alpha M}$ closes into $P_{\alpha\dot\alpha}$,

$$\{Q_\alpha{}^L,\bar{Q}_{\dot\alpha M}\} = P_{\alpha\dot\alpha}C^L{}_M, \tag{1.7}$$

where $P_{\alpha\dot\alpha} = \sigma_{\alpha\dot\alpha}{}^m P_m$. In Exercise 1 we show that the finite-dimensional matrix $C^L{}_M$ is hermitian. It may therefore be diagonalized by a unitary transformation. Since $\{Q_1{}^L,\bar{Q}_{\dot 1 L}\}$ is positive definite, the matrix $C^L{}_M$ has positive definite eigenvalues. This lets us choose a basis in the odd part of the algebra such that

$$\{Q_\alpha{}^L,\bar{Q}_{\dot\alpha M}\} = 2P_{\alpha\dot\alpha}\delta^L{}_M. \tag{1.8}$$

We now turn our attention to the anticommutator of two odd elements, both with undotted indices. The right-hand side of this expression may be decomposed into symmetric and antisymmetric parts. The symmetric part has spin 1. From the Coleman-Mandula theorem, the only possible candidate is the Lorentz generator $M_{\underline{\alpha\beta}}$:

$$\{Q_\alpha{}^L,Q_\beta{}^M\} = \varepsilon_{\alpha\beta}X^{\widehat{LM}} + M_{\underline{\alpha\beta}}Y^{\underline{LM}}. \tag{1.9}$$

From the fact that P_m commutes with $Q_\alpha{}^L$ (see Exercise 2), we find that the $Y^{\underline{LM}}$ must vanish. This lets us write the commutator (1.9) as follows:

$$\{Q_\alpha{}^L, Q_\beta{}^M\} = \varepsilon_{\alpha\beta} a^{\ell,\underset{\smile}{LM}} B_\ell. \tag{1.10}$$

Here B_ℓ is a hermitian element of $\mathscr{A}_1 \oplus \mathscr{A}_2$ and $a^{\ell,\underset{\smile}{LM}}$ is antisymmetric in L and M. With this result, the supersymmetry algebra takes the following form:

$$\begin{aligned}
\{Q_\alpha{}^L, \bar{Q}_{\dot\beta M}\} &= 2\sigma_{\alpha\dot\beta}{}^m P_m \delta^L{}_M \\
[P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot\beta M}] = 0 \\
\{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} a^{\ell,\underset{\smile}{LM}} B_\ell = \varepsilon_{\alpha\beta} X^{\underset{\smile}{LM}} \\
\{\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}\} &= \varepsilon_{\dot\alpha\dot\beta} a^*{}_{\ell,\underset{\smile}{LM}} B^\ell = \varepsilon_{\dot\alpha\dot\beta} X^+{}_{\underset{\smile}{LM}} \\
[Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\
[B^\ell, \bar{Q}_{\dot\alpha L}] &= S^{*\ell}{}_L{}^M \bar{Q}_{\dot\alpha M} \\
[B_\ell, B_m] &= i c_{\ell m}{}^k B_k.
\end{aligned} \tag{1.11}$$

We shall now use the Jacobi identities to further restrict the coefficients $a^{\ell,\underset{\smile}{LM}}$ and $S_\ell{}^L{}_M$ in (1.11). The ordinary Jacobi identity may be easily extended to include anticommutators, as is done in Exercise 3:

$$\{A, \{B, C]] \pm \{B, \{C, A]] \pm \{C, \{A, B]] = 0. \tag{1.12}$$

The bracket structure { ,] signifies either commutator or anticommutator, according to the even or odd character of A, B, and C. The signs are determined by the odd elements. If the odd elements are in a cyclic permutation of the first term, the sign is positive; if not, it is negative. By exploring the Jacobi identities in a certain order, we shall arrive at our results as quickly as possible.

We first consider the identity

$$[B_\ell, \{Q_\alpha{}^L, \bar{Q}_{\dot\beta M}\}] + \{Q_\alpha{}^L, [\bar{Q}_{\dot\beta M}, B_\ell]\} - \{\bar{Q}_{\dot\beta M}, [B_\ell, Q_\alpha{}^L]\} = 0. \tag{1.13}$$

The first term vanishes because B_ℓ and P_m commute. The second and third terms give

$$-\{Q_\alpha{}^L, \bar{Q}_{\dot\beta K}\} S^{*\ell}{}_M{}^K + \{\bar{Q}_{\dot\beta M}, Q_\alpha{}^K\} S_\ell{}^L{}_K = 0, \tag{1.14}$$

or

$$2P_{\alpha\dot\beta}[S^{*\ell}{}_M{}^L - S_\ell{}^L{}_M] = 0. \tag{1.15}$$

Equation (1.15) is true only if

$$S^{*\ell}{}_M{}^L = S_\ell{}^L{}_M, \tag{1.16}$$

so $S_\ell{}^L{}_M$ is hermitian.

Next we use the identity

$$[B_\ell, \{Q_\alpha{}^L, Q_\beta{}^M\}] + \{Q_\alpha{}^L, [Q_\beta{}^A, B_\ell]\} - \{Q_\beta{}^M [B_\ell, Q_\alpha{}^L]\} = 0 \tag{1.17}$$

to prove that the generators $X^{\underbrace{LM}} = a^{\ell,\underbrace{LM}} B_\ell$ form an invariant subalgebra of $\mathscr{A}_1 \oplus \mathscr{A}_2$. Evaluating (1.17) with the help of (1.11), we find

$$\varepsilon_{\alpha\beta}\{[B_\ell, X^{\underbrace{LM}}] + S_\ell{}^M{}_K X^{\underbrace{LK}} - S_\ell{}^L{}_K X^{\underbrace{MK}}\} = 0. \tag{1.18}$$

This shows that the commutator of B_ℓ with $X^{\underbrace{LM}}$ closes into the set of generators $X^{\underbrace{LM}}$. The $X^{\underbrace{LM}}$ are linear combinations of the B_ℓ, so we conclude that the $X^{\underbrace{LM}}$ form an invariant subalgebra of $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

We now use the identity

$$[Q_\alpha{}^L, \{Q_\beta{}^M, \bar{Q}_{\dot{\gamma}K}\}] + [Q_\beta{}^M, \{\bar{Q}_{\dot{\gamma}K}, Q_\alpha{}^L\}] + [\bar{Q}_{\dot{\gamma}K}, \{Q_\alpha{}^L, Q_\beta{}^M\}] = 0 \tag{1.19}$$

to show that the generators $X^{\underbrace{LM}}$ commute with all the generators of $\mathscr{A}$. Combining (1.19) with (1.11), we find

$$\varepsilon_{\alpha\beta}[\bar{Q}_{\dot{\gamma}K}, X^{\underbrace{LM}}] = 0, \tag{1.20}$$

so

$$[X^{\underbrace{KN}}, X^{\underbrace{LM}}] = \frac{1}{2}\varepsilon^{\beta\alpha}[\{Q_\alpha{}^K, Q_\beta{}^N\}, X^{\underbrace{LM}}] = 0. \tag{1.21}$$

This implies that the $X^{\underbrace{LM}}$ form an Abelian (invariant) subalgebra of $\mathscr{A}$. Since $\mathscr{A}_1$ is semisimple, the $X^{\underbrace{LM}}$ are elements of $\mathscr{A}_2$ and commute with all the generators of $\mathscr{A}$:

$$[X^{\underbrace{LM}}, B_\ell] = 0. \tag{1.22}$$

For this reason, they are called central charges. Inserting (1.22) into (1.18),

$$S_\ell{}^M{}_K X^{\underbrace{LK}} - S_\ell{}^L{}_K X^{\underbrace{MK}} = 0, \tag{1.23}$$

and substituting $X^{\underbrace{MK}} = a^{\ell,\underbrace{MK}} B_\ell$, we find

$$S_\ell{}^M{}_K a^{k,\underbrace{LK}} - S_\ell{}^L{}_K a^{k,\underbrace{MK}} = 0. \tag{1.24}$$

From the fact that $S_\ell{}^M{}_K$ is hermitian and $a_k^{\underbrace{MK}}$ antisymmetric, we conclude

$$S_\ell{}^M{}_K a^{k,\underbrace{KL}} = -a^{k,\underbrace{MK}} S^{*\ell}{}_K{}^L. \qquad \textbf{(1.25)}$$

In Exercise 4 we show that the $S_\ell{}^M{}_K$ form a representation of $A_1 \oplus A_2$. Equation (1.25) tells us that the matrices a_k intertwine the representation S_ℓ with its complex conjugate $-S_\ell{}^*$. Central charges exist only if the algebra $A_1 \oplus A_2$ permits such intertwiners. A trivial example is given by $S_\ell^{\underbrace{MK}} = 0$. Another is provided by orthogonal groups, where $S_\ell = -S_\ell{}^*$. A third example is given in Exercise 5.

No further restrictions follow from the other Jacobi identities, as may be proven by checking them all. We have therefore found the most general supersymmetry algebra:

$$\begin{aligned}
[P_m, P_n] &= 0 \\
[P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot{\alpha}L}] = 0 \\
[P_m, B_\ell] &= [P_m, X^{\underbrace{LM}}] = 0 \\
\{Q_\alpha{}^L, \bar{Q}_{\dot{\alpha}M}\} &= 2\sigma_{\alpha\dot{\alpha}}{}^m P_m \delta^L{}_M \\
\{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} X^{\underbrace{LM}} \\
\{\bar{Q}_{\dot{\alpha}L}, \bar{Q}_{\dot{\beta}M}\} &= \varepsilon_{\dot{\alpha}\dot{\beta}} X^+{}_{\underbrace{LM}} \\
[X^{\underbrace{LM}}, \bar{Q}_{\dot{\alpha}K}] &= [X^{\underbrace{LM}}, Q_\alpha{}^K] = 0 \\
[X^{\underbrace{LM}}, X^{\underbrace{KN}}] &= [X^{\underbrace{LM}}, B_\ell] = 0 \\
[B_\ell, B_m] &= ic_{\ell m}{}^k B_k \\
[Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\
[\bar{Q}_{\dot{\alpha}L}, B^\ell] &= -S^{*\ell}{}_L{}^M \bar{Q}_{\dot{\alpha}M} \\
X^{\underbrace{LM}} &= a^{\ell,\underbrace{LM}} B_\ell.
\end{aligned} \qquad \textbf{(1.26)}$$

This is the most general graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory. If central charges exist, they must be of the form $X^{\underbrace{LM}} = a^{\ell,\underbrace{LM}} B_\ell$, where a^ℓ intertwines the representations S_ℓ and $-S^{*\ell}$.

References

S. Coleman and J. Mandula, *Phys. Rev. 159*, 1251 (1967).
R. Haag, J. Lopuszanski, and M. Sohnius, *Nucl. Phys. B88*, 257 (1975).

Equations

$$\begin{aligned}
\{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\}_+ &= 2\sigma_{\alpha\dot\beta}{}^m P_m \delta^A{}_B \\
\{Q_\alpha{}^A, Q_\beta{}^B\}_+ &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\}_+ = 0 \\
[P_m, Q_\alpha{}^A]_- &= [P_m, \bar{Q}_{\dot\alpha A}]_- = 0 \\
[P_m, P_n]_- &= 0.
\end{aligned} \tag{I}$$

$$\{A, \{B, C]] \pm \{B, \{C, A]] \pm \{C, \{A, B]] = 0. \tag{1.12}$$

$$S^{*\ell}{}_M{}^L = S_\ell{}^L{}_M. \tag{1.16}$$

$$S_\ell{}^M{}_K a^{k,\underbrace{KL}} = -a^{k,\underbrace{MK}} S^{*\ell}{}_K{}^L. \tag{1.25}$$

$$\begin{aligned}
[P_m, P_n] &= 0 \\
[P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot\alpha L}] = 0 \\
[P_m, B_\ell] &= [P_m, X^{\underbrace{LM}}] = 0 \\
\{Q_\alpha{}^L, \bar{Q}_{\dot\alpha M}\} &= 2\sigma_{\alpha\dot\alpha}{}^m P_m \delta^L{}_M \\
\{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} X^{\underbrace{LM}} \\
\{\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}\} &= \varepsilon_{\dot\alpha\dot\beta} X^+{}_{\underbrace{LM}} \\
[X^{\underbrace{LM}}, \bar{Q}_{\dot\alpha K}] &= [X^{\underbrace{LM}}, Q_\alpha{}^K] = 0 \\
[X^{\underbrace{LM}}, X^{\underbrace{KN}}] &= [X^{\underbrace{LM}}, B_\ell] = 0 \\
[B_\ell, B_m] &= i c_{\ell m}{}^k B_k \\
[Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\
[\bar{Q}_{\dot\alpha L}, B^\ell] &= -S^{*\ell}{}_L{}^M \bar{Q}_{\dot\alpha M} \\
X^{\underbrace{LM}} &= a^{\ell,\underbrace{LM}} B_\ell.
\end{aligned} \tag{1.26}$$

Exercises

(1) Prove that $C^L{}_M$ in (1.7) is hermitian by comparing the anticommutator (1.7) with its hermitian conjugate.

(2) Show that $[Q_\alpha, P_m] = 0$. Start from the fact that there are no spin -$\frac{3}{2}$ generators. Deduce that $[P_{\alpha\dot\alpha}, Q_\gamma{}^L] = Z^L{}_M \varepsilon_{\alpha\gamma} \bar{Q}_{\dot\alpha}{}^M$, where the $Z^L{}_M$ are some set of numbers. Use the Jacobi identity for $[P_{\beta\dot\beta}, [P_{\alpha\dot\alpha}, Q_\gamma{}^L]]$ to prove that all the $Z^L{}_M$ vanish. This shows that the $Q_\alpha{}^L$ are translationally invariant.

(3) Prove the Jacobi identity (1.12). In particular, verify

$$[B_1, [B_2, B_3]] + [B_2, [B_3, B_1]] + [B_3, [B_1, B_2]] = 0$$
$$[Q_1, [B_2, B_3]] + [B_2, [B_3, Q_1]] + [B_3, [Q_1, B_2]] = 0$$
$$[B_1, \{Q_2, Q_3\}] + \{Q_2, [Q_3, B_1]\} - \{Q_3, [B_1, Q_2]\} = 0$$
$$[Q_1, \{Q_2, Q_3\}] + [Q_2, \{Q_3, Q_1\}] + [Q_3, \{Q_1, Q_2\}] = 0.$$

(4) Use the identity

$$[B_\ell, [B_m, Q_\alpha{}^L]] + [B_m, [Q_\alpha{}^L, B_\ell]] + [Q_\alpha{}^L, [B_\ell, B_m]] = 0$$

to prove

$$[S_m, S_\ell] = ic_{m\ell}{}^k S_k.$$

(The matrix S_ℓ has elements $S_\ell{}^K{}_M$.) Show that $-S^*_\ell$ satisfies the same commutation relations.

(5) The Pauli matrices $\boldsymbol{\sigma}$ and their conjugates $-\boldsymbol{\sigma}^*$ both form representations of SU(2). Show that ε is an intertwiner between these representations. Verify that the commutator

$$\{Q_\alpha{}^L, Q_\beta{}^M\} = \varepsilon_{\alpha\beta}\varepsilon^{LM}(c_1 Z_1 + ic_2 Z_2)$$

is consistent with the Jacobi identities if Z_1 and Z_2 are central charges.

II. REPRESENTATIONS OF THE SUPERSYMMETRY ALGEBRA

An exciting feature of the supersymmetry algebra is that there exist quantum field theories in which the supersymmetry generators Q_α may be represented in terms of conserved currents $J_\alpha{}^m$:

$$Q_\alpha = \int d^3\mathbf{x} J_\alpha{}^0$$
$$\frac{\partial}{\partial x^m} J_\alpha{}^m = 0. \tag{2.1}$$

The currents $J_\alpha{}^m$ are local expressions of the field operators. The algebra (I) is satisfied because of the canonical equal-time commutation relations, and the Hilbert space spans a representation of the supersymmetry algebra. In this chapter we shall study the supersymmetry representations of one-particle states.

The energy-momentum four-vector P_m commutes with the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$. The mass operator P^2 is a Casimir operator, so irreducible representations of the supersymmetry algebra are of equal mass. We shall construct these irreducible representations by the method of induced representations, considering fixed time-like ($P^2 < 0$) and light-like ($P^2 = 0$) momenta.

Before we do this, however, we shall first prove that every representation of the supersymmetry algebra contains an equal number of bosonic and fermionic states. We begin by introducing a fermion number operator N_F, such that $(-)^{N_F}$ has eigenvalue $+1$ on bosonic states that -1 on fermionic states. It follows immediately that

$$(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F}. \tag{2.2}$$

For any finite-dimensional representation of the algebra (such that the trace is well-defined), we find

$$\begin{aligned} \mathrm{Tr}[(-)^{N_F}\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\}] &= \mathrm{Tr}[(-)^{N_F}(Q_\alpha{}^A \bar{Q}_{\dot{\beta}B} + \bar{Q}_{\dot{\beta}B} Q_\alpha{}^A)] \\ &= \mathrm{Tr}[-Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B} + Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B}] \\ &= 0. \end{aligned} \tag{2.3}$$

Here we have used (2.2) and the cyclic property of the trace. Substituting

$$\{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\} = 2\sigma_{\alpha\dot\beta}{}^m P_m \, \delta^A{}_B \tag{2.4}$$

from the supersymmetry algebra (I), we conclude

$$\begin{aligned} \mathrm{Tr}[(-)^{N_F}\{Q_\alpha{}^A, \bar{Q}_{\dot\beta}{}^B\}] &= 2\sigma_{\alpha\dot\beta}{}^m \delta^A{}_B \, \mathrm{Tr}[(-)^{N_F} P_m] \\ &= 0. \end{aligned} \tag{2.5}$$

For fixed non-zero momentum P_m, this reduces to

$$\mathrm{Tr}(-)^{N_F} = 0, \tag{2.6}$$

proving that supersymmetry representations contain equal numbers of bosonic and fermionic states.

We are now ready to construct the representations of the supersymmetry algebra corresponding to massive, one-particle states, $P^2 = -M^2$. We first boost to the rest frame, where $P_m = (-M, 0, 0, 0)$. In this frame, the algebra (I) takes the following form:

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\} &= 2M \, \delta_{\alpha\dot\beta} \, \delta^A{}_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\} &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} = 0. \end{aligned} \tag{2.7}$$

The indices A and B run from 1 to N. The generators Q may be rescaled

$$\begin{aligned} a_\alpha{}^A &= \frac{1}{\sqrt{2M}} Q_\alpha{}^A \\ (a_\alpha{}^A)^+ &= \frac{1}{\sqrt{2M}} \bar{Q}_{\dot\alpha A} \end{aligned} \tag{2.8}$$

to show that (2.7) is isomorphic to the algebra of $2N$ fermionic creation and annihilation operators, $(a_\alpha{}^A)^+$ and $a_\alpha{}^A$:

$$\begin{aligned} \{a_\alpha{}^A, (a_\beta{}^B)^+\} &= \delta_\alpha{}^\beta \, \delta^A{}_B \\ \{a_\alpha{}^A, a_\beta{}^B\} &= \{(a_\alpha{}^A)^+, (a_\beta{}^B)^+\} = 0. \end{aligned} \tag{2.9}$$

The representations of this algebra are well known. They are constructed from a Clifford "vacuum" Ω. The Clifford vacuum is defined through the condition

$$a_\alpha{}^A \Omega = 0, \tag{2.10}$$

where, in contrast to the usual case, $P^2\,\Omega = -M^2\,\Omega$. The states are built by applying the creation operators $(a_\alpha{}^A)^+$ to Ω:

$$\Omega^{(n)\alpha_1}{}_{A_1} \cdots {}^{\alpha_n}{}_{A_n} = \frac{1}{\sqrt{n!}} (a_{\alpha_1}{}^{A_1})^+ \cdots (a_{\alpha_n}{}^{A_n})^+ \Omega. \tag{2.11}$$

Because the $(a_\alpha{}^A)^+$ anticommute, $\Omega^{(n)}$ is antisymmetric under the exchange of two pairs of indices $\alpha_i A_i$, $\alpha_j A_j$. Each pair of indices takes $2N$ different values, so n must be less than or equal to $2N$. For any given n, there are $\binom{2N}{n}$ different states. Summing over all n gives the dimension of the representation (2.11):

$$d = \sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N}. \tag{2.12}$$

If the vacuum Ω is not degenerate, we call (2.11) the fundamental irreducible massive multiplet. It has dimension 2^{2N}, with 2^{2N-1} bosonic and 2^{2N-1} fermionic states. The state with the highest spin is obtained by symmetrizing in as many spinor indices as possible. Because we must simultaneously antisymmetrize in the second index, we may only symmetrize in N spinor indices. This leads to spin-$\frac{1}{2}N$. The highest spin in the fundamental multiplet is $\frac{1}{2}N$; it occurs exactly once.

All other massive multiplets are based on vacuua Ω which are not invariant under the stability group. Their representations are found by composing the representation of Ω with that of the fundamental multiplet.

We now list a few examples. In the case $N = 1$, the fundamental representation consists of the states

$$\begin{gathered} \Omega \\ (a_\alpha)^+\Omega \\ \frac{1}{\sqrt{2}} (a_\alpha)^+ (a_\beta)^+ \Omega = -\frac{1}{2\sqrt{2}} \varepsilon^{\alpha\beta} (a^\gamma)^+ (a_\gamma)^+ \Omega. \end{gathered} \tag{2.13}$$

It has two states of spin 0 and one of spin $\frac{1}{2}$. When the vacuum Ω_j has spin j, with $j > 0$, it belongs to a $(2j + 1)$-dimensional representation of the stability group SU(2). This leads to a multiplet with spins $(j, j + \frac{1}{2}, j - \frac{1}{2}, j)$. These results are summarized in the following tables for

$N = 1, 2, 3$, and 4:

$N = 1$

Spin	Ω_0	$\Omega_{\frac{1}{2}}$	Ω_1	$\Omega_{\frac{3}{2}}$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1

$N = 2$

Spin	Ω_0	$\Omega_{\frac{1}{2}}$	Ω_1
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$\frac{3}{2}$		1	4
2			1

$N = 3$

Spin	Ω_0	$\Omega_{\frac{1}{2}}$
0	14	14
$\frac{1}{2}$	14	20
1	6	15
$\frac{3}{2}$	1	6
2		1

$N = 4$

Spin	Ω_0
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1

The representation space (2.11) of the algebra (2.9) also spans a representation space of the invariance group of the algebra. It is obvious from

(2.9) that SU(2) $\otimes$ U(N) is a possible invariance group. However, SO($4N$) is a larger invariance group of (2.9). It contains SU(2) $\otimes$ U(N) and SU(2) $\otimes$ USp($2N$) as subgroups. To make the SO($4N$) symmetry manifest it is convenient to write (2.9) as a Clifford algebra. To do this we define the operators

$$\begin{aligned}
\Gamma^{\ell} &= \frac{1}{\sqrt{2}}\left[a_1{}^{\ell} + (a_1{}^{\ell})^+\right] \\
\Gamma^{N+\ell} &= \frac{1}{\sqrt{2}}\left[a_2{}^{\ell} + (a_2{}^{\ell})^+\right] \\
\Gamma^{2N+\ell} &= \frac{i}{\sqrt{2}}\left[a_1{}^{\ell} - (a_1{}^{\ell})^+\right] \\
\Gamma^{3N+\ell} &= \frac{i}{\sqrt{2}}\left[a_2{}^{\ell} - (a_2{}^{\ell})^+\right].
\end{aligned} \tag{2.14}$$

The indices 1 and 2 refer to the SU(2) spinor indices and the index ℓ runs from 1 to N. By definition, the $2N$ operators (2.14) are hermitian. In addition, they obey the following anticommutation relations:

$$\{\Gamma^r,\Gamma^s\} = \delta^{rs}, \tag{2.15}$$

where $r,s = 1, \ldots, 4N$. This is a Clifford algebra with an SO($4N$) invariance group. The 2^{2N} states of the fundamental representation span a spinorial representation of SO($4N$). This spinorial representation contains two irreducible representations, each of dimension 2^{2N-1}, corresponding to the bosonic and fermionic states.

The algebra (2.9) may also be cast in a form which exhibits the SU(2) $\otimes$ USp($2N$) symmetry. This is done by defining a new set of operators

$$\begin{aligned}
q_\alpha{}^{\ell} &= a_\alpha{}^{\ell} \\
q_\alpha{}^{N+\ell} &= \sum_{\beta=1}^{2} \varepsilon_{\alpha\beta}(a_\beta{}^{\ell})^+,
\end{aligned} \tag{2.16}$$

where $\ell = 1, \ldots, N$. These operators transform as follows under hermitian conjugation:

$$\begin{aligned}
(q_\alpha{}^{\ell})^+ &= (a_\alpha{}^{\ell})^+ = \varepsilon^{\alpha\beta} q_\beta{}^{N+\ell} \\
(q_\alpha{}^{N+\ell})^+ &= -\varepsilon^{\alpha\beta} a_\beta{}^{\ell} = -\varepsilon^{\alpha\beta} q_\beta{}^{\ell}.
\end{aligned} \tag{2.17}$$

Equation (2.17) may be written in a more compact form

$$(q_\alpha{}^r)^+ = \varepsilon^{\alpha\beta}\Lambda^{rt}q^t{}_\beta, \tag{2.18}$$

where r,t run from 1 to $2N$ and Λ is the following symplectic matrix:

$$\Lambda = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right). \tag{2.19}$$

The anticommutation relations of the operators q

$$\{q_\alpha{}^r, q_\beta{}^t\} = -\varepsilon_{\alpha\beta}\Lambda^{rt} \tag{2.20}$$

exhibit the $\mathrm{SU}(2) \otimes \mathrm{USp}(2N)$ invariance. This invariance group is useful because states of a given spin transform irreducibly under $\mathrm{USp}(2N)$.

We shall now analyze the massless case, $P^2 = 0$. We begin by boosting to a fixed light-like reference frame, where $P_m = (-E, 0, 0, E)$. In this frame, the algebra (I) becomes

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\} &= 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}\delta^A{}_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\} &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} = 0. \end{aligned} \tag{2.21}$$

Rescaling the Q's

$$\begin{aligned} a^A &= \frac{1}{2\sqrt{E}}\, Q_1{}^A \\ a^+{}_A &= \frac{1}{2\sqrt{E}}\, \bar{Q}_{\dot 1}{}^A = (a^A)^+, \end{aligned} \tag{2.22}$$

we find that the algebra (2.21) consists of N creation and annihilation operators, $a^+{}_A$ and a^A:

$$\begin{aligned} \{a^A, a^+{}_B\} &= \delta^A{}_B \\ \{a^A, a^B\} &= \{a^+{}_A, a^+{}_B\} = 0. \end{aligned} \tag{2.23}$$

The operators $Q_2{}^A$ and $\bar{Q}_{\dot 2 A}$ are totally anticommuting and must therefore be represented by zero.

The operators $a^+{}_A$ and a^A raise and lower the helicity of a state by $\frac{1}{2}$. Consequently, a^A annihilates the state of lowest helicity, say, $\underline{\lambda}$:

$$a^A\Omega_{\underline{\lambda}} = 0. \tag{2.24}$$

The states

$$\Omega^{(n)}{}_{\underline{\lambda}+\frac{1}{2}n,\,A_1\cdots A_n} = \frac{1}{\sqrt{n!}}\, a^+{}_{A_n} \cdots a^+{}_{A_1}\Omega_{\underline{\lambda}} \tag{2.25}$$

are built by applying the creation operators $a^+{}_A$ on the Clifford vacuum $\Omega_{\underline{\lambda}}$. The states $\Omega^{(n)}_{\underline{\lambda}+n/2,\,A_1\cdots A_n}$ have helicity $\underline{\lambda} + \frac{1}{2}n$. They are antisymmetric in $A_1 \cdots A_n$ and $\binom{N}{n}$-times degenerate. The state with highest helicity in this representation has helicity $\overline{\lambda} = \underline{\lambda} + \frac{1}{2}N$, so the representation (2.25) has dimension 2^N. From this we see that one massive representation splits into 2^N massless representations.

We summarize these results in tables for $N = 1$, 2, 3, and 4:

$N = 1$

hel. \ $\underline{\lambda}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
2								1
$\frac{3}{2}$							1	1
1						1	1	
$\frac{1}{2}$					1	1		
0				1	1			
$-\frac{1}{2}$			1	1				
-1		1	1					
$-\frac{3}{2}$	1	1						
-2	1							

$N = 2$

hel. \ $\underline{\lambda}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
2							1
$\frac{3}{2}$						1	2
1					1	2	1
$\frac{1}{2}$				1	2	1	
0			1	2	1		
$-\frac{1}{2}$		1	2	1			
-1	1	2	1				
$-\frac{3}{2}$	2	1					
-2	1						

$N = 3$

hel. \ λ	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$
2						1
$\frac{3}{2}$					1	3
1				1	3	3
$\frac{1}{2}$			1	3	3	1
0		1	3	3	1	
$-\frac{1}{2}$	1	3	3	1		
-1	3	3	1			
$-\frac{3}{2}$	3	1				
-2	1					

$N = 4$

hel. \ λ	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
2					1
$\frac{3}{2}$				1	4
1			1	4	6
$\frac{1}{2}$		1	4	6	4
0	1	4	6	4	1
$-\frac{1}{2}$	4	6	4	1	
-1	6	4	1		
$-\frac{3}{2}$	4	1			
-2	1				

In CPT-invariant theories, the number of states must in general be doubled, for CPT reverses the sign of the helicity. Note, however, that the $N = 2$, $\lambda = -\frac{1}{2}$; $N = 4$, $\lambda = -1$; and $N = 8$, $\lambda = -2$ multiplets are automatically CPT complete.

To conclude this chapter, we consider the supersymmetry algebra (1.26) with central charges. We assume that $P^2 = -M^2$ and study the algebra in the rest frame:

$$\begin{aligned} \{Q_\alpha{}^L, (Q_\beta{}^M)^+\} &= 2M\,\delta_\alpha{}^\beta\,\delta^L{}_M \\ \{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} Z^{LM} \\ \{(Q_\alpha{}^L)^+, (Q_\beta{}^M)^+\} &= \varepsilon^{\alpha\beta} Z^*{}_{LM} \\ Z^{LM} &= -Z^{ML}. \end{aligned} \tag{2.26}$$

The central charges Z^{LM} commute with all the generators, so we may choose a basis in which the central charges are diagonal with eigenvalues Z^{LM}. These eigenvalues form an antisymmetric $N \times N$ matrix. Any such matrix may be rotated into a standard form by a unitary transformation:

$$\tilde{Z}^{LM} = U^L{}_K U^M{}_N Z^{KN}. \tag{2.27}$$

The standard form is given by

$$\begin{aligned} \tilde{Z} &= \varepsilon \otimes D && (N \text{ even}) \\ \tilde{Z} &= \begin{pmatrix} \varepsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix} && (N \text{ odd}) \end{aligned} \tag{2.28}$$

where D is diagonal with positive real eigenvalues Z_m and ε is the 2×2 antisymmetric matrix with $\varepsilon^{12} = 1$.

We shall study the case with N even. (The case with N odd is analogous.) We start by decomposing the indices L and M in accord with (2.28),

$$L = (a,m), \qquad M = (b,n), \tag{2.29}$$

where $a,b = 1,2$ and $m,n = 1, \ldots, \frac{1}{2}N$. We then perform a unitary transformation on the $Q_\alpha{}^L$,

$$\tilde{Q}_\alpha{}^L = U^L{}_K Q_\alpha{}^K. \tag{2.30}$$

This allows us to write the algebra (2.26) in the following form:

$$\begin{aligned} \{\tilde{Q}_\alpha{}^{am}, (\tilde{Q}_\beta{}^{bn})^+\} &= 2M\, \delta_\alpha{}^\beta\, \delta^a{}_b\, \delta^m{}_n \\ \{\tilde{Q}_\alpha{}^{am}, \tilde{Q}_\beta{}^{bn}\} &= \varepsilon_{\alpha\beta}\varepsilon^{ab}\, \delta^{mn} Z_n \\ \{(\tilde{Q}_\alpha{}^{am})^+, (\tilde{Q}_\beta{}^{bn})^+\} &= \varepsilon^{\alpha\beta}\varepsilon_{ab}\, \delta_{mn} Z_n. \end{aligned} \tag{2.31}$$

The operators $\tilde{Q}_\alpha{}^{am}$ and $(\tilde{Q}_\alpha{}^{am})^+$ may all be expressed as linear combinations of

$$\begin{aligned} a_\alpha{}^m &= \frac{1}{\sqrt{2}} [\tilde{Q}_\alpha{}^{1m} + \varepsilon_{\alpha\rho}(\tilde{Q}_\rho{}^{2m})^+] \\ b_\alpha{}^m &= \frac{1}{\sqrt{2}} [\tilde{Q}_\alpha{}^{1m} - \varepsilon_{\alpha\rho}(\tilde{Q}_\rho{}^{2m})^+] \end{aligned} \tag{2.32}$$

and their conjugates $(a_\alpha{}^m)^+$ and $(b_\alpha{}^m)^+$. The operators a and b satisfy the following algebra:

$$\begin{aligned}\{a_\alpha{}^n, a_\beta{}^m\} &= \{b_\alpha{}^n, b_\beta{}^m\} = \{a_\alpha{}^n, b_\beta{}^m\} = 0\\ \{a_\alpha{}^n, (a_\beta{}^m)^+\} &= \delta_{\alpha\beta}\delta^{mn}(2M + Z_n)\\ \{b_\alpha{}^n, (b_\beta{}^m)^+\} &= \delta_{\alpha\beta}\delta^{mn}(2M - Z_n).\end{aligned} \tag{2.33}$$

From these relations we see that $Z_n \leq 2M$ for all n. If a set of $Z_i = 2M$, with $i = 1, \ldots, r$, the corresponding operators b_i must vanish. This leaves us with a Clifford algebra of $2(N - r)$ creation and annihilation operators. The representations of this algebra have been studied before.

References

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Exercises

(1) Show that there are equal numbers of bosonic and fermionic states in the representation (2.11). Assign the number $+1$ to each bosonic state and the number -1 to each fermionic state. Then compute the sum

$$\sum_{n=0}^{2N} (-1)^n \binom{2N}{n}.$$

(2) Prove that the highest spin in the fundamental multiplet occurs exactly once. Construct the state with the highest spin in the z-direction. Verify that this state is unique.

(3) Show that $\varepsilon_{\alpha\beta}(\chi_\beta)^+$ transforms like χ_β under SU(2) transformations. This shows that complex conjugation raises and lowers SU(2) indices. (In particular, lower dotted indices of SL(2,C) transform as upper indices under the SU(2) rotation subgroup.)

III. COMPONENT FIELDS

To formulate a supersymmetric field theory we must first represent the supersymmetry algebra (I) in terms of fields not restricted by any mass-shell conditions. Anticommuting parameters $\xi^\alpha, \bar{\xi}_{\dot{\alpha}}$ simplify the task:

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \cdots = [P_m, \xi^\alpha] = 0. \tag{3.1}$$

These parameters allow us to express the supersymmetry algebra entirely in terms of commutators:

$$\begin{aligned} [\xi Q, \bar{\xi}\bar{Q}] &= 2\,\xi\sigma^m\bar{\xi}\,P_m \\ [\xi Q, \xi Q] &= [\bar{\xi}\bar{Q}, \bar{\xi}\bar{Q}] = 0 \\ [P^m, \xi Q] &= [P^m, \bar{\xi}\bar{Q}] = 0. \end{aligned} \tag{3.2}$$

Here we use the summation convention outlined in Appendix A:

$$\xi Q = \xi^\alpha Q_\alpha, \qquad \bar{\xi}\bar{Q} = \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}.$$

A component multiplet is a set of fields $(A, \psi, \ldots)$ on which we define the infinitesimal transformation δ_ξ:

$$\begin{aligned} \delta_\xi A &= (\xi Q + \bar{\xi}\bar{Q}) \times A, \\ \delta_\xi \psi &= (\xi Q + \bar{\xi}\bar{Q}) \times \psi \\ &\cdots \end{aligned} \tag{3.3}$$

The transformation δ_ξ satisfies

$$\begin{aligned} (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)A &= 2(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})P_m A \\ &= -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\,\partial_m A \\ &\cdots \end{aligned} \tag{3.4}$$

in accord with (3.2). This supersymmetry transformation maps tensor fields into spinor fields and vice versa. From the algebra (I) we see that Q has mass dimension $\frac{1}{2}$. Therefore, fields of dimension ℓ transform into fields of dimension $\ell + \frac{1}{2}$ or into derivatives of fields of lower dimension.

Starting with the scalar field A, we define the spinor ψ as the field into which A transforms:

$$\delta_\xi A = \sqrt{2}\xi\psi. \tag{3.5}$$

The field ψ transforms into a tensor field of higher dimension and into the derivative of A itself:

$$\delta_\xi\psi = i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F. \tag{3.6}$$

The coefficient of $\partial_m A$ is chosen to guarantee that the commutator of

$$\delta_\eta \delta_\xi A = 2i\xi\sigma^m\bar{\eta}\,\partial_m A + 2\xi\eta F \tag{3.7}$$

closes in the sense of (3.4). The same commutator acting on the field ψ yields

$$\begin{aligned}(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\psi = &-2i(\eta\sigma^n\bar{\xi} - \xi\sigma^n\bar{\eta})\,\partial_n\psi \\ &- i\sigma^n\bar{\sigma}^m\,\partial_m\psi[\eta\sigma^n\bar{\xi} - \xi\sigma^n\bar{\eta}] + \sqrt{2}(\xi\,\delta_\eta F - \eta\,\delta_\xi F).\end{aligned} \tag{3.8}$$

This closes if

$$\delta_\xi F = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\,\partial_m\psi. \tag{3.9}$$

It follows from (3.6) that the commutator on F closes as well.

If we had been willing to use the field equations, $-i\bar{\sigma}^n\partial_n\psi = m\bar{\psi}$, Eq. (3.9) could have been satisfied by $F = -mA^*$. In this case we would have said that the transformations (3.5) and (3.6) close through the field equations. In extended supersymmetry we are sometimes forced to close the commutators through the field equations because we do not yet know the full multiplet structure of the theory.

The component multiplet which we have constructed is called the chiral or scalar multiplet:

$$\begin{aligned}\delta_\xi A &= \sqrt{2}\xi\psi \\ \delta_\xi\psi &= i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\psi.\end{aligned} \tag{3.10}$$

These fields form a linear representation of the supersymmetry algebra (I). If A has dimension 1, then ψ has dimension $\frac{3}{2}$, while F has dimension 2 and must assume the role of auxiliary field.

From Eq. (3.10) we see that F transforms into a space derivative under δ_ξ. This will always be the case for the component of highest dimension in any given multiplet.

To construct an invariant action it is sufficient to find combinations of fields which transform into space derivatives. Such combinations are given by

$$\mathscr{L}_0 = i\,\partial_n\bar{\psi}\bar{\sigma}^n\psi + A^*\Box A + F^*F \tag{3.11}$$

and

$$\mathscr{L}_m = AF + A^*F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar{\psi}\bar{\psi}. \tag{3.12}$$

From the complete Lagrangian

$$\mathscr{L} = \mathscr{L}_0 + m\mathscr{L}_m \tag{3.13}$$

we determine the field equations:

$$\begin{aligned} i\bar{\sigma}^n\,\partial_n\psi + m\bar{\psi} &= 0 \\ F + mA^* &= 0 \\ \Box A + mF^* &= 0. \end{aligned} \tag{3.14}$$

They describe a Weyl spinor ψ and a complex scalar A, both of mass m.
The Lagrangian (3.13) has the curious property

$$\mathscr{L} = :\mathscr{L}:, \tag{3.15}$$

where : : denotes normal ordering. This simply reflects the fact that supersymmetric theories must contain an equal number of bosonic and fermionic degrees of freedom for a given mass. Equation (3.15) holds as long as supersymmetry remains unbroken. We may also expect that the vacuum expectation value of the energy-momentum tensor T^{mn} vanishes in an unbroken supersymmetric theory. This may be seen by considering $J_\alpha{}^m$, the local current of the supersymmetry charge Q_α,

$$Q_\alpha = \int d^3\mathbf{x} J_\alpha{}^0. \tag{3.16}$$

The supersymmetry algebra (I) yields the energy-momentum tensor T^{mn} as an anticommutator

$$\{\bar{Q}_{\dot{\alpha}}, J^n_\alpha\} = 2\sigma_{\alpha\dot{\alpha}}{}^m T_m{}^n + \text{S.T.} \tag{3.17}$$

The additional Schwinger terms have zero vacuum expectation value.

Therefore, $\langle 0|T_m{}^n|0\rangle = 0$ as long as $\bar{Q}_{\dot\alpha}|0\rangle = 0$. Bruno Zumino was the first to realize that this might account for a vanishing cosmological constant of the observable universe.

References

J. Wess and B. Zumino, *Nucl. Phys. B70*, 39 (1974).
B. Zumino, *Nucl. Phys. B89*, 535 (1975).

Equations

$$\begin{aligned}(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)A &= 2(\eta\sigma^m\bar\xi - \xi\sigma^m\bar\eta)P_m A \\ &= -2i(\eta\sigma^m\bar\xi - \xi\sigma^m\bar\eta)\,\partial_m A \end{aligned} \tag{3.4}$$

. . .

$$\begin{aligned}\delta_\xi A &= \sqrt{2}\xi\psi \\ \delta_\xi\psi &= i\sqrt{2}\sigma^m\bar\xi\partial_m A + \sqrt{2}\xi F \\ \delta_\xi F &= i\sqrt{2}\bar\xi\bar\sigma^m\partial_m\psi .\end{aligned} \tag{3.10}$$

$$\mathscr{L}_0 = i\,\partial_n\bar\psi\bar\sigma^n\psi + A^*\Box A + F^*F. \tag{3.11}$$

$$\mathscr{L}_m = AF + A^*F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar\psi\bar\psi. \tag{3.12}$$

Exercises

(1) Show $\psi\chi = \chi\psi$, $\chi\sigma^n\bar\psi = -\bar\psi\bar\sigma^n\chi$, $(\chi\sigma^m\bar\psi)^+ = \psi\sigma^m\bar\chi$, and $(\chi\sigma^m\bar\sigma^n\psi)^+ = \bar\psi\bar\sigma^n\sigma^m\bar\chi$.

(2) Prove the Fierz rearrangement formula

$$(\psi\phi)\bar\chi_{\dot\beta} = -\frac{1}{2}(\phi\sigma^m\bar\chi)(\psi\sigma^m)_{\dot\beta}.$$

(3) Use (3.5) and (3.6) to calculate

$$\delta_\eta\delta_\xi\psi_\alpha = -2i\eta\sigma^m\bar\xi\,\partial_m\psi_\alpha - i[\sigma^n\bar\sigma^m\,\partial_m\psi]_\alpha(\eta\sigma^n\bar\xi) + \sqrt{2}\xi_\alpha\delta_\eta F.$$

(4) Eliminate the auxiliary field F from the Lagrangian (3.13) to obtain

$$\mathscr{L} = i\partial_n\bar\psi\bar\sigma^n\psi - \tfrac{1}{2}m(\psi\psi + \bar\psi\bar\psi) + A^*\Box A - m^2A^*A.$$

(5) Show that $\delta_\xi(AF - \frac{1}{2}\psi\psi) = i\sqrt{2}\bar\xi\bar\sigma^n\partial_n(A\psi)$.

IV. SUPERFIELDS

Superfields provide an elegant and compact description of supersymmetry representations. They simplify the addition and multiplication of representations and are very useful in the construction of interacting Lagrangians. We shall show that superfields may always be constructed from component representations. Component fields may always be recovered from superfields by power series expansion.

We begin with the observation that the supersymmetry algebra may be viewed as a Lie algebra with anticommuting parameters [Eq. (3.2)]. This motivates us to define a corresponding group element:

$$G(x,\theta,\bar{\theta}) = e^{i(-x^m P_m + \theta Q + \bar{\theta}\bar{Q})}. \tag{4.1}$$

It is easy to multiply two group elements using Hausdorff's formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ because all higher commutators vanish. We find

$$G(0,\xi,\bar{\xi})G(x^m,\theta,\bar{\theta}) = G(x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}). \tag{4.2}$$

As usual, multiplication of group elements induces a motion in the parameter space,

$$g(\xi,\bar{\xi})\colon (x^m,\theta,\bar{\theta}) \to (x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}). \tag{4.3}$$

This motion may be generated by the differential operators Q and $\bar{Q}$:

$$\xi Q + \bar{\xi}\bar{Q} = \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\,\partial_m\right) + \bar{\xi}_{\dot{\alpha}}\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\beta}}{}^m\varepsilon^{\dot{\beta}\dot{\alpha}}\,\partial_m\right). \tag{4.4}$$

Here we use the same letters $Q,\bar{Q}$ for the differential operators as for the group generators because the differential operators do indeed represent the infinitesimal group action on the parameter space:

$$\begin{aligned} \{Q_\alpha,\bar{Q}_{\dot{\alpha}}\} &= 2i\sigma_{\alpha\dot{\alpha}}{}^m\,\partial_m \\ \{Q_\alpha,Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}},\bar{Q}_{\dot{\beta}}\} = 0. \end{aligned} \tag{4.5}$$

Note, however, the change in sign, $P_m = -i\,\partial_m$. This stems from the fact that the product of successive group elements corresponds to a motion with the order of multiplication reversed. For example, $G(0,\xi_1,\bar{\xi}_1)G(0,\xi_2,\bar{\xi}_2)$ induces the motion $g(\xi_2,\bar{\xi}_2)g(\xi_1,\bar{\xi}_1)$.

We could have studied right multiplication instead of left multiplication. We would then have found the induced motion generated by the differential operators D and $\bar{D}$,

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\,\partial_m \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\,\partial_m. \end{aligned} \tag{4.6}$$

By their very definition, D and $\bar{D}$ satisfy the following anticommutation relations

$$\begin{aligned} \{D_\alpha,\bar{D}_{\dot{\alpha}}\} &= -2i\sigma_{\alpha\dot{\alpha}}{}^m\,\partial_m \\ \{D_\alpha,D_\beta\} &= \{\bar{D}_{\dot{\alpha}},\bar{D}_{\dot{\beta}}\} = 0, \end{aligned} \tag{4.7}$$

while D and Q anticommute

$$\{D_\alpha,Q_\beta\} = \{D_\alpha,\bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}},Q_\beta\} = \{\bar{D}_{\dot{\alpha}},\bar{Q}_{\dot{\beta}}\} = 0. \tag{4.8}$$

We are now ready to introduce superfields and superspace. Elements of superspace are labeled by $z = (x, \theta, \bar{\theta})$. Superfields are functions of superspace which should be understood in terms of their power series expansions in θ and $\bar{\theta}$,

$$\begin{aligned} F(x,\theta,\bar{\theta}) = {} & f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) \\ & + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^m\bar{\theta}v_m(x) \\ & + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x). \end{aligned} \tag{4.9}$$

All higher powers of $\theta,\bar{\theta}$ vanish. The transformation law for superfields is defined as follows:

$$\begin{aligned} \delta_\xi F(x,\theta,\bar{\theta}) = {} & \delta_\xi f(x) + \theta\,\delta_\xi\phi(x) + \bar{\theta}\,\delta_\xi\bar{\chi}(x) \\ & + \theta\theta\,\delta_\xi m(x) + \bar{\theta}\bar{\theta}\,\delta_\xi n(x) + \theta\sigma^m\bar{\theta}\,\delta_\xi v_m(x) \\ & + \theta\theta\bar{\theta}\,\delta_\xi\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\,\delta_\xi\psi(x) + \theta\theta\bar{\theta}\bar{\theta}\,\delta_\xi d(x) \\ \equiv {} & (\xi Q + \bar{\xi}\bar{Q})F, \end{aligned} \tag{4.10}$$

where Q and $\bar{Q}$ are the differential operators (4.4). The transformation laws for the component fields $(f, \phi, \bar{\chi}, \ldots)$ may be found from (4.10) by matching appropriate powers of $\theta,\bar{\theta}$. The commutator of these transformations satisfies (3.4) as a consequence of (4.5).

It is easy to verify that linear combinations of superfields are again superfields. Similarly, products of superfields are again superfields because Q and $\bar{Q}$ are linear differential operators.

Thus we see that superfields form linear representations of the supersymmetry algebra. In general, however, the representations are highly reducible. We may eliminate the extra component fields by imposing covariant constraints, such as $\bar{D}F = 0$ or $F = F^+$. Superfields shift the problem of finding supersymmetry representations to that of finding appropriate constraints. Note that we must reduce superfields without restricting their x-dependence through differential equations in x-space.

Superfields satisfying the condition $\bar{D}\Phi = 0$ are called chiral or scalar superfields. This constraint does not yield a differential equation in x-space. Extra conditions, however, often give differential equations. For example, $DD\Phi = \bar{D}\Phi = 0$ yields massless field equations, while $D\Phi = \bar{D}\Phi = 0$ implies $\Phi = a =$ constant.

Vector superfields are defined to satisfy $V = V^+$. It is possible to construct all supersymmetric renormalizable Lagrangians in terms of vector and scalar superfields. We shall treat both vector and scalar superfields in great detail in the coming chapters.

It is always possible to construct a superfield from a component multiplet. We start with any component of the multiplet, say A, and apply the operator $\exp(\theta Q + \bar{\theta}\bar{Q})$, whose action is defined through (3.3). This yields a function of $x,\theta,\bar{\theta}$ which transforms like a superfield

$$F(x,\theta,\bar{\theta}) = e^{(\theta Q + \bar{\theta}\bar{Q})} \times A = A + \delta_\theta A + \cdots \tag{4.11}$$

We define the function $\delta_\xi F(x,\theta,\bar{\theta})$ to be the power series in $\theta,\bar{\theta}$ whose coefficients represent the transformed component fields,

$$\delta_\xi F(x,\theta,\bar{\theta}) = (\xi Q + \bar{\xi}\bar{Q}) \times F. \tag{4.12}$$

The multiplication $\times$ is defined in Eq. (3.3). It acts on the component fields and commutes with the parameters θ and $\bar{\theta}$. From Hausdorff's formula, we find

$$\begin{aligned}
\xi^\alpha \frac{\partial}{\partial\theta^\alpha} e^{(\theta Q + \bar{\theta}\bar{Q})} \times &= \xi^\alpha \frac{\partial}{\partial\theta^\alpha} e^{\theta Q} e^{\bar{\theta}\bar{Q}} e^{-\theta\sigma^m\bar{\theta}P_m} \times \\
&= (\xi Q - \xi\sigma^m\bar{\theta}P_m) \times e^{(\theta Q + \bar{\theta}\bar{Q})} \times \\
\bar{\xi}_{\dot{\alpha}} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} e^{(\theta Q + \bar{\theta}\bar{Q})} \times &= (\bar{\xi}\bar{Q} + \theta\sigma^m\bar{\xi}P_m) \times e^{(\theta Q + \bar{\theta}\bar{Q})} \times .
\end{aligned} \tag{4.13}$$

This shows that the action of $\xi Q \times$ and $\bar{\xi}\bar{Q} \times$ on $\exp(\theta Q + \bar{\theta}\bar{Q})$ may be represented by the differential operators ξQ and $\bar{\xi}\bar{Q}$:

$$\delta_\xi F(x,\theta,\bar{\theta}) = (\xi Q + \bar{\xi}\bar{Q})F. \tag{4.14}$$

Comparing with (4.10), we see that $F(x,\theta,\bar{\theta}) = e^{(\theta Q + \bar{\theta}\bar{Q})} \times A$ does indeed transform as a superfield under δ_ξ.

To obtain the superfield whose components correspond directly to a given set of component fields, the superfield must be constructed from the component field of lowest dimension. If there are several fields of lowest dimension, each will give rise to its own superfield, but these superfields will be related by constraint equations. We shall encounter this problem when we discuss gauge fields.

References

A. Salam and J. Strathdee, *Nucl. Phys. B76*, 477 (1974).
S. Ferrara, J. Wess, and B. Zumino, *Phys. Lett. 51B*, 239 (1974).

Equations

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\partial_m \\ \bar{Q}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\partial_m. \end{aligned} \tag{4.4}$$

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2i\sigma_{\alpha\dot{\alpha}}{}^m\partial_m \\ \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \end{aligned} \tag{4.5}$$

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\partial_m \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\partial_m. \end{aligned} \tag{4.6}$$

$$\begin{aligned} \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= -2i\sigma_{\alpha\dot{\alpha}}{}^m\partial_m \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \end{aligned} \tag{4.7}$$

Exercises

(1) Show

$$\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha}.$$

(2) Verify

$$\varepsilon^{\alpha\beta} \frac{\partial}{\partial\theta^\alpha} \frac{\partial}{\partial\theta^\beta} \theta\theta = 4$$

and

$$\varepsilon_{\dot\alpha\dot\beta} \frac{\partial}{\partial\bar\theta_{\dot\alpha}} \frac{\partial}{\partial\bar\theta_{\dot\beta}} \bar\theta\bar\theta = 4.$$

(3) Use Hausdorff's formula to show

$$e^{i(\xi Q + \bar\xi\bar Q)} e^{i\{-x^m P_m + \theta Q + \bar\theta\bar Q\}} = e^{i\{-P_m(x^m - i\xi\sigma^m\bar\theta + i\theta\sigma^m\bar\xi) + (\theta+\xi)Q + (\bar\theta+\bar\xi)\bar Q\}}.$$

(4) Given $G(x'^m,\theta',\bar\theta') = G(0,\xi_2,0)G(0,0,\bar\xi_1)G(x^m,\theta,\bar\theta)$, use Hausdorff's formula and (4.1) to demonstrate

$$\begin{aligned} x'^m &= x^m + i\theta\sigma^m\bar\xi_1 - i\xi_2\sigma^m(\bar\theta + \bar\xi_1) \\ \theta' &= \theta + \xi_2 \\ \bar\theta' &= \bar\theta + \bar\xi_1 . \end{aligned}$$

Show that this corresponds to the induced motion

$$g(0,\bar\xi_1)g(\xi_2,0)\colon (x^m,\theta,\bar\theta) \to (x'^m,\theta',\bar\theta')$$

where g is defined in (4.3).

(5) Evaluate $\{D_\alpha,\bar D_{\dot\alpha}\}$ using the definitions of $D,\bar D$ as differential operators.

(6) Compute $\bar D_{\dot\alpha}F(x,\theta,\bar\theta)$ where $\bar D$ is given in (4.6) and F in (4.9). Note that $\bar D_{\dot\alpha}F = 0$ yields a constraint rather than a field equation.

(7) Show that $\bar D_{\dot\alpha}F = D_\alpha F = 0$ implies $F = a =$ constant. Demonstrate that $\bar D_{\dot\alpha}F = 0$ and $D^\alpha D_\alpha F = 4mF^+$ yield massive field equations for the components of F.

(8) Construct the superfield whose lowest component is F, rather than A. Compare this to the superfield $DD\Phi$.

V. CHIRAL SUPERFIELDS

Chiral superfields are characterized by the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{5.1}$$

They correspond to the chiral multiplets of Chapter III.

The above constraint is easy to solve in terms of $y^m = x^m + i\theta\sigma^m\bar{\theta}$ and θ, for

$$\bar{D}_{\dot{\alpha}}(x^m + i\theta\sigma^m\bar{\theta}) = 0, \qquad \text{and} \qquad \bar{D}_{\dot{\alpha}}\theta = 0. \tag{5.2}$$

Any function of these variables satisfies (5.1):

$$\begin{aligned}\Phi &= A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\\ &= A(x) + i\theta\sigma^m\bar{\theta}\,\partial_m A(x) + \frac{1}{4}\,\theta\theta\bar{\theta}\bar{\theta}\Box A(x)\\ &\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\,\theta\theta\,\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x).\end{aligned} \tag{5.3}$$

This is the most general solution to (5.1), as may be seen from the expressions for D and $\bar{D}$ in terms of y, θ, and $\bar{\theta}$:

$$\begin{aligned}D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\,\frac{\partial}{\partial y^m}\\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}.\end{aligned} \tag{5.4}$$

The superfield Φ^+ satisfies the constraint $D_\alpha\Phi^+ = 0$. Φ^+ is a natural function of $y^{+m} = x^m - i\theta\sigma^m\bar{\theta}$ and $\bar{\theta}$; its power series expansion is obtained from (5.3) by conjugation:

$$\begin{aligned}\Phi^+ &= A^*(y^+) + \sqrt{2}\bar{\theta}\bar{\psi}(y^+) + \bar{\theta}\bar{\theta}F^*(y^+)\\ &= A^*(x) - i\theta\sigma^m\bar{\theta}\,\partial_m A^*(x) + \frac{1}{4}\,\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x)\\ &\quad + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\,\bar{\theta}\bar{\theta}\theta\sigma^m\,\partial_m\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x).\end{aligned} \tag{5.5}$$

Writing D and $\bar{D}$ in terms of y^+, θ, and $\bar{\theta}$,

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - 2i\theta^\alpha \sigma_{\alpha\dot{\alpha}}{}^m \frac{\partial}{\partial y^{+m}}, \end{aligned} \tag{5.6}$$

we see that (5.5) is the most general solution to $D_\alpha \Phi^+ = 0$.

It is easy to verify that the transformation laws for A, ψ, and F, derived through (4.10), are exactly those for the component multiplet (3.10). The computation is simplified when the differential operators $Q, \bar{Q}$ are expressed in terms of the variable y.

The highest components of Φ and Φ^+ are, respectively, F and F^*. All higher powers in $\theta, \bar{\theta}$ are spacetime derivatives. Thus the F or F^* component of a scalar superfield always transforms into a spacetime derivative.

Products of chiral superfields $\Phi_1 \Phi_2 \cdots \Phi_\ell$ are again chiral superfields, and likewise for their conjugates:

$$\begin{aligned} \Phi_i \Phi_j = {} & A_i(y) A_j(y) + \sqrt{2}\theta[\psi_i(y) A_j(y) + A_i(y)\psi_j(y)] \\ & + \theta\theta[A_i(y) F_j(y) + A_j(y) F_i(y) - \psi_i(y)\psi_j(y)]. \end{aligned} \tag{5.7}$$

$$\begin{aligned} \Phi_i \Phi_j \Phi_k = {} & A_i(y) A_j(y) A_k(y) \\ & + \sqrt{2}\theta[\psi_i A_j A_k + \psi_j A_k A_i + \psi_k A_i A_j] \\ & + \theta\theta[F_i A_j A_k + F_j A_k A_i + F_k A_i A_j \\ & - \psi_i \psi_j A_k - \psi_j \psi_k A_i - \psi_k \psi_i A_j]. \end{aligned} \tag{5.8}$$

The product $\Phi^+\Phi$, however, is not a chiral superfield:

$$\begin{aligned} \Phi_i^+ \Phi_j = {} & A_i^*(x) A_j(x) + \sqrt{2}\theta\psi_j(x) A_i^*(x) \\ & + \sqrt{2}\bar{\theta}\bar{\psi}_i(x) A_j(x) + \theta\theta A_i^*(x) F_j(x) + \bar{\theta}\bar{\theta} F_i^*(x) A_j(x) \\ & + \theta^\alpha \bar{\theta}^{\dot{\alpha}}[i\sigma_{\alpha\dot{\alpha}}{}^m (A_i^* \partial_m A_j - \partial_m A_i^* A_j) - 2\bar{\psi}_{i\dot{\alpha}} \psi_{j\alpha}] \\ & + \theta\theta\bar{\theta}^{\dot{\alpha}}\left[\frac{i}{\sqrt{2}} \sigma_{\alpha\dot{\alpha}}{}^m (A_i^* \partial_m \psi_j^\alpha - \partial_m A_i^* \psi_j^\alpha) - \sqrt{2} F_j \bar{\psi}_{i\dot{\alpha}}\right] \\ & + \bar{\theta}\bar{\theta}\theta^\alpha\left[-\frac{i}{\sqrt{2}} \sigma_{\alpha\dot{\alpha}}{}^m (\bar{\psi}_i^{\dot{\alpha}} \partial_m A_j - \partial_m \bar{\psi}_i^{\dot{\alpha}} A_j) + \sqrt{2} F_i^* \psi_{j\alpha}\right] \\ & + \theta\theta\bar{\theta}\bar{\theta}\left[F_i^* F_j + \frac{1}{4} A_i^* \Box A_j + \frac{1}{4} \Box A_i^* A_j - \frac{1}{2} \partial_m A_i^* \partial^m A_j \right. \\ & \left. + \frac{i}{2} \partial_m \bar{\psi}_i \bar{\sigma}^m \psi_j - \frac{i}{2} \bar{\psi}_i \bar{\sigma}^m \partial_m \psi_j \right]. \end{aligned} \tag{5.9}$$

In this product the $\theta\theta\bar{\theta}\bar{\theta}$ component transforms into a spacetime derivative.

We are now ready to write the most general supersymmetric renormalizable Lagrangian involving only chiral superfields:

$$\mathscr{L} = \Phi_i^+\Phi_i\big|_{\theta\theta\bar{\theta}\bar{\theta}\ \text{component}} + \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i\right)\bigg|_{\theta\theta\ \text{component}} + \text{h.c.}\right]. \qquad \textbf{(5.10)}$$

The couplings m_{ij} and g_{ijk} are symmetric in their indices. Note that changing the basis from y to x does not change $\mathscr{L}$.

In terms of component fields, $\mathscr{L}$ becomes

$$\begin{aligned}\mathscr{L} = {} & i\,\partial_m\bar{\psi}_i\bar{\sigma}^m\psi_i + A_i^*\Box A_i + F_i^*F_i \\ & + \left[m_{ij}\left(A_iF_j - \frac{1}{2}\psi_i\psi_j\right)\right. \\ & \left. + g_{ijk}(A_iA_jF_k - \psi_i\psi_jA_k) + \lambda_iF_i + \text{h.c.}\right],\end{aligned} \qquad \textbf{(5.11)}$$

where we have dropped all total derivatives. The auxiliary fields F_i may be eliminated through their Euler equations:

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial F_k^*} &= F_k + \lambda_k^* + m_{ik}^*A_i^* + g_{ijk}^*A_i^*A_j^* = 0 \\ \frac{\partial\mathscr{L}}{\partial F_k} &= F_k^* + \lambda_k + m_{ik}A_i + g_{ijk}A_iA_j = 0.\end{aligned} \qquad (5.12)$$

This gives $\mathscr{L}$ solely in terms of the dynamical fields A_i and ψ_i:

$$\begin{aligned}\mathscr{L} = {} & i\,\partial_m\bar{\psi}_i\bar{\sigma}^m\psi_i + A_i^*\Box A_i - \frac{1}{2}m_{ik}\psi_i\psi_k - \frac{1}{2}m_{ik}^*\bar{\psi}_i\bar{\psi}_k \\ & - g_{ijk}\psi_i\psi_jA_k - g_{ijk}^*\bar{\psi}_i\bar{\psi}_jA_k^* - \mathscr{V}(A_i,A_j^*).\end{aligned} \qquad (5.13)$$

In (5.13), the potential $\mathscr{V}$ takes the form

$$\mathscr{V} = F_k^*F_k, \qquad (5.14)$$

where F,F^* are solutions to (5.12). This potential is always greater than or equal to zero, a consequence of supersymmetry. Points where $F_k = 0$ are absolute minima of the potential.

Note that constants are chiral superfields, for $\Phi = a$ is the solution to the equations $\bar{D}_{\dot{\alpha}}\Phi = D_\alpha\Phi = 0$. Thus from any supersymmetric Lagrangian we may always obtain another by making the shift $\Phi_i \to \Phi_i + a_i$. The new Lagrangian has parameters:

$$\begin{aligned} \lambda_i' &= \lambda_i + m_{ij}a_j + g_{ijk}a_j a_k \\ m_{ij}' &= m_{ij} + 2g_{ijk}a_k \\ g_{ijk}' &= g_{ijk}. \end{aligned} \tag{5.15}$$

If the old potential had a minimum at $\Phi_i = -a_i$, the new potential has a minimum at the origin. The new potential belongs to a supersymmetric Lagrangian with parameters given by (5.15).

The class of renormalizable Lagrangians may be restricted by R-invariance. R acts on chiral multiplets as follows:

$$\begin{aligned} \mathrm{R}\Phi(\theta,x) &= e^{2in\alpha}\Phi(e^{-i\alpha}\theta,x) \\ \mathrm{R}\Phi^+(\bar{\theta},x) &= e^{-2in\alpha}\Phi^+(e^{i\alpha}\bar{\theta},x). \end{aligned} \tag{5.16}$$

Here n is called the R-character of the superfield. For the components, (5.16) implies

$$\begin{aligned} \mathrm{R}: A &\to e^{2in\alpha}A \\ \psi &\to e^{2i(n-\frac{1}{2})\alpha}\psi \\ F &\to e^{2i(n-1)\alpha}F. \end{aligned} \tag{5.17}$$

Mass terms or potentials are R-invariant only if the R-characters of their respective superfields add up to one.

References

J. Wess and B. Zumino, *Phys. Lett. 49B*, 52 (1974).
L. O'Raifeartaigh, *Nucl. Phys. B96*, 331 (1975).

Equations

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{5.1}$$

$$\begin{aligned} \Phi = A(x) &+ i\theta\sigma^m\bar{\theta}\,\partial_m A(x) + \frac{1}{4}\,\theta\theta\bar{\theta}\bar{\theta}\Box A(x) \\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\,\theta\theta\,\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x). \end{aligned} \tag{5.3}$$

$$\Phi^{+} = A^*(x) - i\theta\sigma^m\bar\theta\,\partial_m A^*(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box A^*(x)$$

$$+ \sqrt{2}\bar\theta\bar\psi(x) + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^m\partial_m\bar\psi(x) + \bar\theta\bar\theta F^*(x). \qquad \textbf{(5.5)}$$

$$\begin{aligned}\Phi_i\Phi_j = {} & A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i(y)A_j(y) + A_i(y)\psi_j(y)] \\ & + \theta\theta[A_i(y)F_j(y) + A_j(y)F_i(y) - \psi_i(y)\psi_j(y)]. \end{aligned} \qquad \textbf{(5.7)}$$

$$\begin{aligned}\Phi_i\Phi_j\Phi_k = {} & A_i(y)A_j(y)A_k(y) \\ & + \sqrt{2}\theta[\psi_iA_jA_k + \psi_jA_kA_i + \psi_kA_iA_j] \\ & + \theta\theta[F_iA_jA_k + F_jA_kA_i + F_kA_iA_j \\ & - \psi_i\psi_jA_k - \psi_j\psi_kA_i - \psi_k\psi_iA_j]. \end{aligned} \qquad \textbf{(5.8)}$$

$$\begin{aligned}\Phi_i^+\Phi_j = {} & A_i^*(x)A_j(x) + \sqrt{2}\theta\psi_j(x)A_i^*(x) \\ & + \sqrt{2}\bar\theta\bar\psi_i(x)A_j(x) + \theta\theta A_i^*(x)F_j(x) + \bar\theta\bar\theta F_i^*(x)A_j(x) \\ & + \theta^\alpha\bar\theta^{\dot\alpha}[i\sigma_{\alpha\dot\alpha}{}^m(A_i^*\partial_mA_j - \partial_mA_i^*A_j) - 2\bar\psi_{i\dot\alpha}\psi_{j\alpha}] \\ & + \theta\theta\bar\theta^{\dot\alpha}\left[\frac{i}{\sqrt{2}}\sigma_{\alpha\dot\alpha}{}^m(A_i^*\partial_m\psi_j{}^\alpha - \partial_mA_i^*\psi_j{}^\alpha) - \sqrt{2}F_j\bar\psi_{i\dot\alpha}\right] \\ & + \bar\theta\bar\theta\theta^\alpha\left[-\frac{i}{\sqrt{2}}\sigma_{\alpha\dot\alpha}{}^m(\bar\psi_i^{\dot\alpha}\partial_mA_j - \partial_m\bar\psi_i^{\dot\alpha}A_j) + \sqrt{2}F_i^*\psi_{j\alpha}\right] \\ & + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j + \frac{1}{4}A_i^*\Box A_j + \frac{1}{4}\Box A_i^*A_j\right. \\ & \left. - \frac{1}{2}\partial_mA_i^*\partial^mA_j + \frac{i}{2}\partial_m\bar\psi_i\bar\sigma^m\psi_j - \frac{i}{2}\bar\psi_i\bar\sigma^m\partial_m\psi_j\right]. \end{aligned} \qquad \textbf{(5.9)}$$

$$\begin{aligned}\mathscr{L} = {} & \Phi_i^+\Phi_i\big|_{\theta\theta\bar\theta\bar\theta\ \text{component}} + \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j\right.\right. \\ & \left.\left. + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i\right)\Big|_{\theta\theta\ \text{component}} + \text{h.c.}\right]. \end{aligned} \qquad \textbf{(5.10)}$$

$$\begin{aligned}\mathscr{L} = {} & i\,\partial_m\bar\psi_i\bar\sigma^m\psi_i + A_i^*\Box A_i + F_i^*F_i \\ & + \left[m_{ij}\left(A_iF_j - \frac{1}{2}\psi_i\psi_j\right) + g_{ijk}(A_iA_jF_k - \psi_i\psi_jA_k) + \lambda_iF_i + \text{h.c.}\right]. \end{aligned} \qquad \textbf{(5.11)}$$

$$\mathrm{R}\Phi(\theta,x) = e^{2in\alpha}\Phi(e^{-i\alpha}\theta,x), \qquad \mathrm{R}\Phi^+(\bar\theta,x) = e^{-2in\alpha}\Phi^+(e^{i\alpha}\bar\theta,x). \qquad \textbf{(5.16)}$$

$$\begin{aligned} \mathrm{R}: A &\to e^{2in\alpha}A \\ \psi &\to e^{2i(n-\frac{1}{2})\alpha}\psi \\ F &\to e^{2i(n-1)\alpha}F. \end{aligned} \tag{5.17}$$

Exercises

(1) Compute Q_α and $\bar{Q}_{\dot{\alpha}}$ in terms of $y^m,\theta,\bar{\theta}$:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\frac{\partial}{\partial y^m}.$$

(2) Compute D_α, $\bar{D}_{\dot{\alpha}}$, Q_α, and $\bar{Q}_{\dot{\alpha}}$ in terms of $y^{+m},\theta,\bar{\theta}$.

(3) Derive the transformation laws (3.10) using $Q,\bar{Q}$ and (5.3) expressed in terms of the variables y^m, θ, and $\bar{\theta}$.

(4) Define the components of a chiral superfield ($\bar{D}_{\dot{\alpha}}\Phi = 0$) as follows:

$$\begin{aligned} \mathscr{A} &= \Phi|_{\theta=\bar{\theta}=0} \\ \Psi_\alpha &= D_\alpha\Phi|_{\theta=\bar{\theta}=0} \\ \mathscr{F} &= DD\Phi|_{\theta=\bar{\theta}=0}. \end{aligned}$$

Express these components in terms of the component fields A,ψ,F of (5.3). Compute the transformation laws for $\mathscr{A}$, Ψ, and $\mathscr{F}$ using using Q and $\bar{Q}$ in the following form:

$$Q_\alpha = D_\alpha - 2i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m}$$

$$\bar{Q}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} + 2i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\frac{\partial}{\partial x^m}.$$

(5) Show that $\Phi = \bar{D}\bar{D}U$ is chiral for any superfield U. Relate the components of U to those of Φ.

(6) Show that the mass term $\frac{1}{2}m\Phi\Phi$ for a single superfield Φ is R-invariant if and only if Φ has R-character $\frac{1}{2}$. Note this condition excludes linear and trilinear terms from the Lagrangian.

VI. VECTOR SUPERFIELDS

Vector superfields satisfy the condition

$$V = V^{+}. \tag{6.1}$$

As usual, they should be understood in terms of their power series expansion in θ and $\bar{\theta}$:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\ & + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\ & - \theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^m\partial_m\chi(x)\right] \\ & - i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\Box C(x)\right]. \end{aligned} \tag{6.2}$$

The component fields C,D,M,N, and v_m must all be real for (6.2) to satisfy (6.1). The vector field v_m lends its name to the entire multiplet.

We have chosen very particular combinations of fields as coefficients of the $\theta\theta\bar{\theta}$, $\bar{\theta}\bar{\theta}\theta$, and $\theta\theta\bar{\theta}\bar{\theta}$ components of V. Our choice was dictated by the hermitian field $\Phi + \Phi^{+}$, where Φ and Φ^{+} are chiral fields:

$$\begin{aligned} \Phi + \Phi^{+} = {} & A + A^* + \sqrt{2}(\theta\psi + \bar{\theta}\bar{\psi}) + \theta\theta F + \bar{\theta}\bar{\theta}F^* \\ & + i\theta\sigma^m\bar{\theta}\,\partial_m(A - A^*) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^m\partial_m\psi \\ & + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box(A + A^*). \end{aligned} \tag{6.3}$$

This combination has the gradient $i\partial_m(A - A^*)$ as coefficient of $\theta\sigma^m\bar{\theta}$, motivating us to define the following supersymmetric generalization of a gauge transformation:

$$V \to V + \Phi + \Phi^+. \tag{6.4}$$

Under this transformation,

$$\begin{aligned} C &\to C + A + A^* \\ \chi &\to \chi - i\sqrt{2}\psi \\ M + iN &\to M + iN - 2iF \\ v_m &\to v_m - i\,\partial_m(A - A^*) \\ \lambda &\to \lambda \\ D &\to D. \end{aligned} \tag{6.5}$$

The choice of components in (6.2) renders λ and D gauge invariant.

From (6.5) we see that there is a special gauge* in which C, χ, M, and N are all zero. Fixing this gauge breaks supersymmetry but still allows the usual gauge transformations $v_m \to v_m + \partial_m a$. It is very easy to compute powers of V in this gauge:

$$\begin{aligned} V &= -\theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \\ V^2 &= -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}v_m v^m \\ V^3 &= 0. \end{aligned} \tag{6.6}$$

Thus we may view the vector field V as the supersymmetric generalization of the Yang-Mills potential. To construct the corresponding supersymmetric field strength, we observe that λ_α and $\bar{\lambda}_{\dot{\alpha}}$ are the lowest-dimensional gauge invariant component fields in V. They are also the lowest-dimensional component fields in

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \\ \bar{W}_{\dot{\alpha}} &= -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V. \end{aligned} \tag{6.7}$$

* In the literature this gauge is often called the Wess- Zumino or WZ gauge.

These superfields are chiral and gauge invariant. Chirality follows immediately from (6.7),

$$\bar{D}_{\dot{\beta}} W_\alpha = 0$$
$$D_\beta \bar{W}_{\dot{\alpha}} = 0, \tag{6.8}$$

while $\bar{D}\Phi = D\Phi^+ = 0$ must be used to prove gauge invariance:

$$W_\alpha \to -\frac{1}{4}\bar{D}\bar{D}D_\alpha(V + \Phi + \Phi^+) = W_\alpha - \frac{1}{4}\bar{D}\{\bar{D},D_\alpha\}\Phi = W_\alpha. \tag{6.9}$$

It is easy to compute the components of W_α in the special gauge (6.6). The computation is further simplified by use of the variables $y = x + i\theta\sigma\bar{\theta}$ or $y^+ = x - i\theta\sigma\bar{\theta}$:

$$\begin{aligned} V = &-\theta\sigma^m\bar{\theta}v_m(y) + i\theta\theta\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\theta\lambda(y) \\ &+ \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y) - i\,\partial_m v^m(y)] \\ = &-\theta\sigma^m\bar{\theta}v_m(y^+) - i\bar{\theta}\bar{\theta}\theta\lambda(y^+) + i\theta\theta\bar{\theta}\bar{\lambda}(y^+) \\ &+ \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y^+) + i\,\partial_m v^m(y^+)]. \end{aligned} \tag{6.10}$$

The result is

$$\begin{aligned} W_\alpha = &-i\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta D(y) - \frac{i}{2}(\sigma^m\bar{\sigma}^n)_\alpha{}^\beta(\partial_m v_n(y) - \partial_n v_m(y))\right]\theta_\beta \\ &+ \theta\theta\sigma_{\alpha\dot{\alpha}}{}^m\partial_m\bar{\lambda}^{\dot{\alpha}}(y), \\ \bar{W}_{\dot{\alpha}} = &\ i\bar{\lambda}_{\dot{\alpha}}(y^+) + \left[\varepsilon_{\dot{\alpha}\dot{\beta}}D(y^+) + \frac{i}{2}\varepsilon_{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}^m\sigma^n)^{\dot{\gamma}}{}_{\dot{\beta}}(\partial_m v_n(y^+) - \partial_n v_m(y^+))\right]\bar{\theta}^{\dot{\beta}} \\ &- \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\bar{\sigma}^{m\dot{\beta}\alpha}\,\partial_m\lambda_\alpha(y^+). \end{aligned} \tag{6.11}$$

The superfields $W_\alpha, \bar{W}_{\dot{\alpha}}$ contain only the gauge invariant fields D, λ_α, and $v_{mn} = \partial_m v_n - \partial_n v_m$. Furthermore, they are chiral and satisfy the additional constraint equation

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}. \tag{6.12}$$

For $\theta = \bar{\theta} = 0$, this relation simply expresses the fact that the component field D is real. Equation (6.12) may be verified component-by-component from (6.11) or directly from the definition (6.7). It may be shown that (6.11) represents the most general solution to the chirality conditions (6.8) and the constraint (6.12).

The superfields W_α and $\bar{W}_{\dot{\alpha}}$ are examples of representations which have the component fields λ_α and $\bar{\lambda}_{\dot{\alpha}}$ as fields of lowest dimension. We could have constructed W_α and $\bar{W}_{\dot{\alpha}}$ from λ_α and $\bar{\lambda}_{\dot{\alpha}}$ by applying the operator $\exp(\theta Q + \bar{\theta}\bar{Q}) \times$, as described in Chapter IV. We would then have found W_α and $\bar{W}_{\dot{\alpha}}$ to be related through Eq. (6.12).

Since W_α is chiral, the $\theta\theta$ component of $W^\alpha W_\alpha$,

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^m\partial_m\bar{\lambda} - \frac{1}{2}v^{mn}v_{mn} + D^2 + \frac{i}{4}v^{mn}v^{\ell k}\varepsilon_{mn\ell k}, \quad \mathbf{(6.13)}$$

transforms into a space derivative. Note that $W^\alpha W_\alpha$ may also be written as

$$W^\alpha W_\alpha = -\frac{1}{4}\bar{D}\bar{D}W^\alpha D_\alpha V. \quad (6.14)$$

From (6.13) we see that

$$\mathscr{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}) \quad \mathbf{(6.15)}$$

is the supersymmetric gauge invariant generalization of the Lagrangian for a free vector field. After some partial integration, this reduces to

$$\int d^4x\mathscr{L} = \int d^4x\left\{\frac{1}{2}D^2 - \frac{1}{4}v^{mn}v_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda}\right\}. \quad \mathbf{(6.16)}$$

This Lagrangian may also be obtained as a $\theta\theta\bar{\theta}\bar{\theta}$ component:

$$\int d^4x\mathscr{L} = \int d^4x\,\frac{1}{4}\{W^\alpha D_\alpha V + \bar{W}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}V\}\Big|_{\theta\theta\bar{\theta}\bar{\theta}}. \quad (6.17)$$

Equation (6.17) is equivalent to (6.15) because of (6.14) and the fact that $\bar{D}$ and $\partial/\partial\bar{\theta}$ differ only by an x-space derivative.

We can always add the mass term m^2V^2 to the Lagrangian (6.17). This term is not gauge invariant and cannot be computed in the WZ gauge. Starting from (6.2), we find

$$\begin{aligned} V^2|_{\theta\theta\bar{\theta}\bar{\theta}} = &-\frac{1}{2}v_m v^m - \chi\lambda - \bar{\chi}\bar{\lambda} + \frac{1}{2}(M^2 + N^2) \\ &-\frac{i}{2}\chi\sigma^m\partial_m\bar{\chi} - \frac{i}{2}\bar{\chi}\bar{\sigma}^m\partial_m\chi \\ &+\frac{1}{2}C\Box C + CD. \end{aligned} \quad (6.18)$$

It is interesting to note that this term not only gives mass to the vector field v_m but also introduces the additional degrees of freedom C and χ required for a massive multiplet. The Lagrangian (6.17) together with (6.18) describes one vector field, two spin-$\frac{1}{2}$ fields, and one scalar field, all of equal mass.

References

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J. Wess, *Acta Physica Austriaca*, Suppl. XV, 475 (1976).

Equations

$$V = V^+. \tag{6.1}$$

$$\begin{aligned} V(x,\theta,\bar\theta) = {} & C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) \\ & + \frac{i}{2}\theta\theta\left[M(x) + iN(x)\right] - \frac{i}{2}\bar\theta\bar\theta\left[M(x) - iN(x)\right] \\ & - \theta\sigma^m\bar\theta v_m(x) + i\theta\theta\bar\theta\left[\bar\lambda(x) + \frac{i}{2}\bar\sigma^m\partial_m\chi(x)\right] \\ & - i\bar\theta\bar\theta\theta\left[\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar\chi(x)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) + \frac{1}{2}\Box C(x)\right]. \end{aligned} \tag{6.2}$$

$$V \to V + \Phi + \Phi^+. \tag{6.4}$$

$$\begin{aligned} C &\to C + A + A^* \\ \chi &\to \chi - i\sqrt{2}\psi \\ M + iN &\to M + iN - 2iF \\ v_m &\to v_m - i\partial_m(A - A^*) \\ \lambda &\to \lambda \\ D &\to D. \end{aligned} \tag{6.5}$$

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar D\bar D D_\alpha V \\ \bar W_{\dot\alpha} &= -\frac{1}{4} D D \bar D_{\dot\alpha} V. \end{aligned} \tag{6.7}$$

$$\bar{D}_{\dot\beta} W_\alpha = 0$$
$$D_\beta \bar{W}_{\dot\alpha} = 0. \tag{6.8}$$

$$W_\alpha = -i\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta(\partial_m v_n(y) - \partial_n v_m(y))\right]\theta_\beta + \theta\theta\sigma_{\alpha\dot\alpha}{}^m\,\partial_m\bar\lambda^{\dot\alpha}(y)$$

$$\bar W_{\dot\alpha} = i\bar\lambda_{\dot\alpha}(y^+) + \left[\varepsilon_{\dot\alpha\dot\beta}D(y^+) + \frac{i}{2}\varepsilon_{\dot\alpha\dot\gamma}(\bar\sigma^m\sigma^n)^{\dot\gamma}{}_{\dot\beta}(\partial_m v_n(y^+) - \partial_n v_m(y^+))\right]\bar\theta^{\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta\bar\sigma^{m\dot\beta\alpha}\,\partial_m\lambda_\alpha(y^+). \tag{6.11}$$

$$D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha}. \tag{6.12}$$

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^m\partial_m\bar\lambda - \frac{1}{2}v^{mn}v_{mn} + D^2 + \frac{i}{4}v^{mn}v^{\ell k}\varepsilon_{mn\ell k}. \tag{6.13}$$

$$\mathscr{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar W_{\dot\alpha}\bar W^{\dot\alpha}|_{\bar\theta\bar\theta}). \tag{6.15}$$

$$\int d^4x\,\mathscr{L} = \int d^4x\left\{\frac{1}{2}D^2 - \frac{1}{4}v^{mn}v_{mn} - i\lambda\sigma^m\partial_m\bar\lambda\right\}. \tag{6.16}$$

Exercises

(1) Prove

$$[\bar D_{\dot\alpha}, \{\bar D_{\dot\beta}, D_\gamma\}] = 0,$$

$$[D_\alpha, \bar D_{\dot\beta}\bar D^{\dot\beta}] = -4i\sigma_{\alpha\dot\beta}{}^m\partial_m\bar D^{\dot\beta},$$

and

$$\frac{1}{8}[D^2, \bar D^2] = -iD^\alpha\sigma_{\alpha\dot\alpha}{}^n\bar D^{\dot\alpha}\,\partial_n - 2\Box$$
$$= i\bar D^{\dot\alpha}\sigma_{\alpha\dot\alpha}{}^n D^\alpha\,\partial_n + 2\Box.$$

(2) Compute the $\bar\theta\bar\theta$ component of $D_\alpha V$ in the general gauge (6.2) and the special gauge (6.6).

(3) Compute e^V in the WZ gauge.

(4) Show that the constraints (6.8) and (6.12) yield the following equations for the component fields:

$$D = D^*,$$
$$v^{mn} = v^{*mn},$$
$$\varepsilon^{m\ell kn}\,\partial_\ell v_{kn} = 0.$$

(5) Compute

$$DD\theta\theta = -4e^{-i\theta\sigma^m\bar{\theta}\frac{\partial}{\partial x^m}}$$
$$\bar{D}\bar{D}\bar{\theta}\bar{\theta} = -4e^{i\theta\sigma^m\bar{\theta}\frac{\partial}{\partial x^m}}.$$

(6) Use the definitions of W_α and $\bar{W}_{\dot{\alpha}}$ to verify (6.12).

(7) Derive the Euler-Lagrange equations for the Lagrangian (6.17) + (6.18).

(8) Use (6.13) to show

$$\int d^4x\,\frac{1}{4}\,(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}) = \frac{1}{2}\int d^4x W^\alpha W_\alpha|_{\theta\theta}.$$

(9) Find the supersymmetry transformations for the gauge invariant fields in the vector multiplet:

$$\begin{aligned}
\delta_\xi v_{mn} &= i[(\xi\sigma^n\,\partial_m\bar{\lambda} + \bar{\xi}\bar{\sigma}^n\,\partial_m\lambda) - (n \leftrightarrow m)]\\
\delta_\xi\lambda &= i\xi D + \sigma^{mn}\xi v_{mn}\\
\delta_\xi D &= \bar{\xi}\bar{\sigma}^m\,\partial_m\lambda - \xi\sigma^m\,\partial_m\bar{\lambda}.
\end{aligned}$$

VII. GAUGE INVARIANT INTERACTIONS

In this chapter we discuss the gauge invariant interactions of chiral and vector multiplets. We start with the U(1) case and later generalize our results to non-Abelian gauge groups.

Chiral superfields Φ_ℓ transform by a phase under global U(1) rotations,

$$\Phi'_\ell = e^{-it_\ell\lambda}\Phi_\ell. \tag{7.1}$$

The t_ℓ are the U(1) charges appropriate to the Φ_ℓ, and λ is the rigid U(1) rotation angle. The t_ℓ and λ are real constants. Constants are chiral superfields, satisfying the constraint equations $D_\alpha\lambda = \bar{D}_{\dot\alpha}\lambda = 0$. From (7.1), we see immediately that the Φ'_ℓ are chiral superfields as well.

It is easy to construct a Lagrangian invariant under (7.1) for constant parameters λ:

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_{K.E.} + \mathscr{L}_{P.E.} \\
\mathscr{L}_{K.E.} &= \Phi_\ell^+\Phi_\ell\big|_{\theta\theta\bar\theta\bar\theta} \\
\mathscr{L}_{P.E.} &= \left[\frac{1}{2}\, m_{ij}\Phi_i\Phi_j + \frac{1}{3}\, g_{ijk}\Phi_i\Phi_j\Phi_k\right]\Bigg|_{\theta\theta} + \text{h.c.}
\end{aligned} \tag{7.2}$$

Note that U(1) invariance requires m_{ij} or $g_{ijk} = 0$ whenever $t_i + t_j$ or $t_i + t_j + t_k \neq 0$. In the literature, the term $\mathscr{L}_{P.E.}$ is often called the superpotential.

Equation (7.1) takes one chiral superfield into another when λ is a constant chiral superfield. When λ depends on x, the situation is slightly more complicated. In this case, λ must be promoted to a full chiral multiplet:

$$\begin{aligned}
\Phi'_\ell &= e^{-it\ell\Lambda}\,\Phi_\ell, \qquad & \bar{D}_{\dot\alpha}\Lambda &= 0 \\
\Phi'^{+}_\ell &= e^{it\ell\Lambda^+}\,\Phi_\ell{}^+, \qquad & D_\alpha\Lambda^+ &= 0.
\end{aligned} \tag{7.3}$$

Only then do the Φ'_ℓ remain chiral superfields.

The Lagrangian (7.2) is not invariant under such local transformations. In particular, $\mathscr{L}_{P.E.}$ remains invariant, but $\mathscr{L}_{K.E.}$ does not:

$$\Phi_\ell'^{+}\Phi_\ell' = \Phi_\ell^{+}\Phi_\ell e^{it_\ell(\Lambda^+ - \Lambda)}. \tag{7.4}$$

It is easy to see that $\mathscr{L}_{K.E.}$ may be rendered invariant by introducing the vector superfield V with its transformation law (6.4):

$$V' = V + i(\Lambda - \Lambda^+). \tag{7.5}$$

With this addition, the full Lagrangian

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{4}(W^\alpha W_\alpha|_{\bar\theta\bar\theta} + \bar W_{\dot\alpha}\bar W^{\dot\alpha}|_{\bar\theta\bar\theta}) + \Phi_\ell^{+} e^{t_\ell V}\Phi_\ell|_{\theta\theta\bar\theta\bar\theta} \\ & + \left[\left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} g_{ijk}\Phi_i\Phi_j\Phi_k\right)\bigg|_{\theta\theta} + \text{h.c.}\right]\end{aligned} \tag{7.6}$$

becomes invariant under local U(1) gauge transformations.

At first, (7.6) looks non-renormalizable. It may, however, be evaluated in the WZ gauge, where $V^3 = 0$:

$$\begin{aligned}\Phi^+ e^{tV}\Phi|_{\theta\theta\bar\theta\bar\theta} = {} & FF^* + A\Box A^* + i\,\partial_n\bar\psi\bar\sigma^n\psi \\ & + tv^n\left(\frac{1}{2}\bar\psi\bar\sigma^n\psi + \frac{i}{2}A^*\partial_n A - \frac{i}{2}\partial_n A^* A\right) \\ & - \frac{i}{\sqrt{2}}t(A\bar\lambda\bar\psi - A^*\lambda\psi) + \frac{1}{2}\left(tD - \frac{1}{2}t^2 v_n v^n\right)A^*A.\end{aligned} \tag{7.7}$$

In this gauge, the Lagrangian contains no terms of dimension higher than four.

The supersymmetric extension of electrodynamics is constructed in terms of two chiral superfields:

$$\Phi_+' = e^{-ie\Lambda}\Phi_+, \qquad \Phi_-' = e^{ie\Lambda}\Phi_-. \tag{7.8}$$

In components, the Lagrangian

$$\begin{aligned}\mathscr{L}_{QED} = {} & \frac{1}{4}(WW|_{\theta\theta} + \bar W\bar W|_{\bar\theta\bar\theta}) + \Phi_+^{+}e^{eV}\Phi_+|_{\theta\theta\bar\theta\bar\theta} + \Phi_-^{+}e^{-eV}\Phi_-|_{\theta\theta\bar\theta\bar\theta} \\ & + m(\Phi_+\Phi_-|_{\theta\theta} + \Phi_+^{+}\Phi_-^{+}|_{\bar\theta\bar\theta})\end{aligned} \tag{7.9}$$

becomes

$$
\begin{aligned}
\mathscr{L}_{QED} = {} & \frac{1}{2} D^2 - \frac{1}{4} v_{mn} v^{mn} - i\lambda\sigma^n \partial_n \bar{\lambda} \\
& + F_+ F_+^* + F_- F_-^* + A_+^* \Box A_+ + A_-^* \Box A_- \\
& + i(\partial_n \bar{\psi}_+ \bar{\sigma}^n \psi_+ + \partial_n \bar{\psi}_- \bar{\sigma}^n \psi_-) + e v^n \bigg[\frac{1}{2} \bar{\psi}_+ \bar{\sigma}^n \psi_+ - \frac{1}{2} \bar{\psi}_- \bar{\sigma}^n \psi_- \\
& + \frac{i}{2} A_+^* \partial_n A_+ - \frac{i}{2} \partial_n A_+^* A_+ - \frac{i}{2} A_-^* \partial_n A_- + \frac{i}{2} \partial_n A_-^* A_- \bigg] \\
& - \frac{ie}{\sqrt{2}} (A_+ \bar{\psi}_+ \bar{\lambda} - A_+^* \psi_+ \lambda - A_- \bar{\psi}_- \bar{\lambda} + A_-^* \psi_- \lambda) \\
& + \frac{e}{2} D[A_+^* A_+ - A_-^* A_-] - \frac{1}{4} e^2 v_n v^n (A_+^* A_+ + A_-^* A_-) \\
& + m[A_+ F_- + A_- F_+ - \psi_+ \psi_- - \bar{\psi}_+ \bar{\psi}_- + A_+^* F_-^* + A_-^* F_+^*].
\end{aligned} \tag{7.10}
$$

From (7.10) we see that the two Weyl spinors ψ_+, ψ_- combine to form one massive Dirac spinor, the electron.

It is straightforward to generalize the transformation law (7.1) to non-Abelian compact groups:

$$\Phi' = e^{-i\Lambda}\Phi, \qquad \Phi'^+ = \Phi^+ e^{i\Lambda^+}. \tag{7.11}$$

In (7.11), Λ is a matrix:

$$\Lambda_{ij} = T_{ij}^a \Lambda_a. \tag{7.12}$$

The matrices T^a are the hermitian generators of the gauge group in the representation defined by the chiral field Φ. In the adjoint representation, we normalize our generators as follows:

$$\mathrm{Tr}\, T^a T^b = k\delta^{ab}, \qquad k > 0. \tag{7.13}$$

With this convention, the structure constants t^{abc}

$$[T^a, T^b] = i t^{abc} T^c \tag{7.14}$$

are completely antisymmetric.

The Lagrangian (7.6) is invariant under non-Abelian gauge transformations, provided we extend the transformation law (7.5):

$$e^{V'} = e^{-i\Lambda^+} e^V e^{i\Lambda}. \qquad \textbf{(7.15)}$$

In (7.15), both Λ and V are matrices:

$$\Lambda_{ij} = T^a_{ij}\Lambda_a, \qquad V_{ij} = T^a_{ij}V_a. \qquad \textbf{(7.16)}$$

With Hausdorff's formula, we encounter only commutators of group generators in computing the product of exponentials in (7.15). Evaluating the commutators by the group commutation relations (7.14) allows us to express V' in the following form:

$$V' = T^a V_a'. \qquad (7.17)$$

This shows that the transformation law (7.15) is independent of any specific representation for the generators T^a. Furthermore, the transformation law starts with a term independent of V,

$$V' = V + i(\Lambda - \Lambda^+) + \cdots, \qquad (7.18)$$

so non-Abelian theories also allow a WZ gauge where $V^3 = 0$.

Equation (7.15) may be evaluated for infinitesimal gauge transformations with the following form of Hausdorff's formula:

$$e^A e^B = e^{A + £_{A/2} \cdot [B + \coth(£_{A/2}) \cdot B] + \cdots}. \qquad (7.19)$$

This expression contains all terms linear in B. The Lie derivative $£_{A/2} \cdot B$ is given by $[\frac{1}{2}A,B]$. The hyperbolic cotangent in (7.19) must be understood in terms of its power series expansion, where

$$c_n(£_{A/2})^n \cdot B \equiv c_n\left[\frac{A}{2},\left[\frac{A}{2},\left[\ldots,\left[\frac{A}{2},B\right]\cdots\right]\right]\right], \qquad (7.20)$$

with n factors $\frac{1}{2}A$. Using (7.19) to evaluate (7.15) yields

$$\delta V = V' - V = i£_{V/2} \cdot [(\Lambda + \Lambda^+) + \coth(£_{V/2}) \cdot (\Lambda - \Lambda^+)]. \qquad (7.21)$$

The supersymmetric field strength W^α [Eq. (6.7)] may be readily generalized to the non-Abelian case:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V. \tag{7.22}$$

In (7.22), the vector superfields V are matrices, as in (7.16), with the generators in the adjoint representation of the gauge group. It is easy to verify that

$$W_\alpha \to W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda} \tag{7.23}$$

under non-Abelian gauge transformations. The proof is left to the reader as an exercise.

We are now ready to write down the most general Lagrangian for the supersymmetric renormalizable interaction of scalar, spinor, and vector fields:

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{16kg^2}\,\mathrm{Tr}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) + \Phi^+ e^V \Phi|_{\theta\theta\bar\theta\bar\theta} \\ & + \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\right)\bigg|_{\theta\theta} + \text{h.c.}\right].\end{aligned} \tag{7.24}$$

Gauge invariance requires the mass matrix m_{ij} and the coupling constants g_{ijk} to be totally symmetric invariant tensors with respect to the internal symmetry group. The normalization of the gauge-field kinetic term is chosen to recover the canonical normalization for the component action after scaling $V \to 2gV$ (see Exercise 7).

References

J. Wess and B. Zumino, *Nucl. Phys. B78*, 1 (1974).
S. Ferrara and B. Zumino, *Nucl. Phys. B79*, 413 (1974).

Equations

$$\begin{aligned}\Phi'_\ell &= e^{-it_\ell\Lambda}\Phi_\ell, & \bar{D}_{\dot\alpha}\Lambda &= 0 \\ \Phi'^{+}_\ell &= e^{it_\ell\Lambda^+}\Phi_\ell^{+}, & D_\alpha\Lambda^+ &= 0.\end{aligned} \tag{7.3}$$

$$V' = V + i(\Lambda - \Lambda^+). \tag{7.5}$$

$$\mathscr{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) + \Phi_\ell^+ e^{t_\ell V}\Phi_\ell|_{\theta\theta\bar\theta\bar\theta}$$
$$+ \left[\left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} g_{ijk}\Phi_i\Phi_j\Phi_k\right)\Big|_{\theta\theta} + \text{h.c.}\right]. \qquad \textbf{(7.6)}$$

$$\Phi^+ e^{tV}\Phi|_{\theta\theta\bar\theta\bar\theta} = FF^* + A\Box A^* + i\partial_n\bar\psi\bar\sigma^n\psi$$
$$+ tv^n\left(\frac{1}{2}\bar\psi\bar\sigma^n\psi + \frac{i}{2}A^*\partial_n A - \frac{i}{2}\partial_n A^* A\right)$$
$$- \frac{i}{\sqrt{2}}t(A\bar\lambda\bar\psi - A^*\lambda\psi) + \frac{1}{2}\left(tD - \frac{1}{2}t^2 v_n v^n\right)A^*A. \qquad \textbf{(7.7)}$$

$$\mathscr{L}_{QED} = \frac{1}{4}(WW|_{\theta\theta} + \bar{W}\bar{W}|_{\bar\theta\bar\theta}) + \Phi_+^+ e^{eV}\Phi_+|_{\theta\theta\bar\theta\bar\theta} + \Phi_-^+ e^{-eV}\Phi_-|_{\theta\theta\bar\theta\bar\theta}$$
$$+ m(\Phi_+\Phi_-|_{\theta\theta} + \Phi_+^+\Phi_-^+|_{\bar\theta\bar\theta}). \qquad \textbf{(7.9)}$$

$$\mathscr{L}_{QED} = \frac{1}{2}D^2 - \frac{1}{4}v_{mn}v^{mn} - i\lambda\sigma^n\partial_n\bar\lambda + F_+F_+^* + F_-F_-^* + A_+^*\Box A_+ + A_-^*\Box A_-$$
$$+ i(\partial_n\bar\psi_+\bar\sigma^n\psi_+ + \partial_n\bar\psi_-\bar\sigma^n\psi_-) + ev^n\left[\frac{1}{2}\bar\psi_+\bar\sigma^n\psi_+ - \frac{1}{2}\bar\psi_-\bar\sigma^n\psi_-\right.$$
$$\left. + \frac{i}{2}A_+^*\partial_n A_+ - \frac{i}{2}\partial_n A_+^* A_+ - \frac{i}{2}A_-^*\partial_n A_- + \frac{i}{2}\partial_n A_-^* A_-\right]$$
$$- \frac{ie}{\sqrt{2}}(A_+\bar\psi_+\bar\lambda - A_+^*\psi_+\lambda - A_-\bar\psi_-\bar\lambda + A_-^*\psi_-\lambda)$$
$$+ \frac{e}{2}D[A_+^*A_+ - A_-^*A_-] - \frac{1}{4}e^2 v_n v^n(A_+^*A_+ + A_-^*A_-)$$
$$+ m[A_+F_- + A_-F_+ - \psi_+\psi_- - \bar\psi_+\bar\psi_- + A_+^*F_-^* + A_-^*F_+^*]. \qquad \textbf{(7.10)}$$

$$\text{Tr } T^a T^b = k\delta^{ab}, \qquad k > 0. \qquad \textbf{(7.13)}$$

$$[T^a, T^b] = it^{abc}T^c. \qquad \textbf{(7.14)}$$

$$e^{V'} = e^{-i\Lambda^+}e^V e^{i\Lambda}. \qquad \textbf{(7.15)}$$

$$\Lambda_{ij} = T^a_{ij}\Lambda_a, \qquad V_{ij} = T^a_{ij}V_a. \qquad \textbf{(7.16)}$$

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V. \tag{7.22}$$

$$W_\alpha \to W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}. \tag{7.23}$$

$$\mathscr{L} = \frac{1}{16kg^2}\operatorname{Tr}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) + \Phi^+ e^V \Phi|_{\theta\theta\bar\theta\bar\theta}$$
$$+ \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\right)\bigg|_{\theta\theta} + \text{h.c.}\right]. \tag{7.24}$$

Exercises

(1) Show that $\mathscr{L}_{P.E.}$ of Eqs. (7.6) and (7.24) is also invariant under symmetry transformations (7.11) with complex parameters.

(2) Show that $e^{-V}D_\alpha e^V = D_\alpha V - \frac{1}{2}[V,D_\alpha V]$ in the special gauge (6.6).

(3) Demonstrate that $W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}$. Use the fact that $D_\alpha\Lambda^+ = 0$ and $\bar{D}_{\dot\alpha}\Lambda = 0$:

$$W'_\alpha = -\frac{1}{4}\bar{D}\bar{D}[e^{-i\Lambda}e^{-V}e^{i\Lambda^+}D_\alpha e^{-i\Lambda^+}e^V e^{i\Lambda}]$$
$$= e^{-i\Lambda}W_\alpha e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\{\bar{D},D_\alpha\}e^{i\Lambda}$$
$$= e^{-i\Lambda}W_\alpha e^{i\Lambda}.$$

(4) Construct an SO(3) invariant interaction using three chiral vector fields Φ_ℓ^A.

(5) Use the multiplication properties of the Pauli $\boldsymbol{\sigma}$-matrices $(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{m}\cdot\boldsymbol{\sigma}) = \mathbf{n}\cdot\mathbf{m} + i(\mathbf{n}\times\mathbf{m})\cdot\boldsymbol{\sigma}$ to show that for infinitesimal values of b,

$$e^{ia\mathbf{n}\cdot\boldsymbol{\sigma}}e^{ib\mathbf{m}\cdot\boldsymbol{\sigma}} = (\cos a + i\boldsymbol{\sigma}\cdot\mathbf{n}\sin a)(1 + ib\mathbf{m}\cdot\boldsymbol{\sigma})$$
$$= \cos[a + (\mathbf{n}\cdot\mathbf{m})b] + i\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\sin[a + (\mathbf{n}\cdot\mathbf{m})b],$$

where $\boldsymbol{\mu} = \mathbf{n} + b[(\mathbf{m} - (\mathbf{n}\cdot\mathbf{m})\mathbf{n})\cot a - \mathbf{n}\times\mathbf{m}]$, and $\mathbf{n},\mathbf{m}$ are unit vectors.

(6) Use the result of Exercise (5) and $[\mathbf{n}\cdot\boldsymbol{\sigma}, \mathbf{m}\cdot\boldsymbol{\sigma}] = 2i(\mathbf{n}\times\mathbf{m})\cdot\boldsymbol{\sigma}$ to verify Eq. (7.19) for the special case $e^{ia\mathbf{n}\cdot\boldsymbol{\sigma}}e^{ib\mathbf{m}\cdot\boldsymbol{\sigma}}$. Remember that b is infinitesimal.

(7) Expand

$$\mathscr{L} = \frac{1}{16kg^2}\,\mathrm{Tr}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) + \Phi^+ e^V \Phi|_{\theta\theta\bar\theta\bar\theta}$$

in the WZ gauge. Assume the gauge group is non-Abelian, and restore the coupling g by rescaling $V \to 2gV$,

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4} v_{mn}^{(a)} v_{mn}^{(a)} - i\bar\lambda^{(a)}\bar\sigma^m \mathscr{D}_m \lambda^{(a)} + \frac{1}{2} D^{(a)} D^{(a)} - \mathscr{D}_m A^+ \mathscr{D}_m A \\ &- i\bar\psi\bar\sigma^m \mathscr{D}_m \psi + F^+ F + i\sqrt{2}g(A^+ T^{(a)} \psi \lambda^{(a)} - \bar\lambda^{(a)} T^{(a)} A \bar\psi) \\ &+ gD^{(a)} A^+ T^{(a)} A,\end{aligned}$$

where

$$\begin{aligned}\mathscr{D}_m A &= \partial_m A + igv_m^{(a)} T^{(a)} A \\ \mathscr{D}_m \psi &= \partial_m \psi + igv_m^{(a)} T^{(a)} \psi \\ \mathscr{D}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} - gt^{abc} v_m^{(b)} \lambda^{(c)} \\ v_{mn}^{(a)} &= \partial_m v_n^{(a)} - \partial_n v_m^{(a)} - gt^{abc} v_m^{(b)} v_n^{(c)}.\end{aligned}$$

(8) Compute the transformation laws for A, ψ, F, $v_m^{(a)}$, $\lambda^{(a)}$ and $D^{(a)}$ in the WZ gauge. Use them to verify that the result of Exercise (7) is supersymmetric:

$$\begin{aligned}\delta_\xi A &= \sqrt{2}\xi\psi \\ \delta_\xi \psi &= i\sqrt{2}\sigma^m \bar\xi \mathscr{D}_m A + \sqrt{2}\xi F \\ \delta_\xi F &= i\sqrt{2}\bar\xi\bar\sigma^m \mathscr{D}_m \psi + i2gT^{(a)} A \bar\xi \bar\lambda^{(a)} \\ \delta_\xi v_m^{(a)} &= -i\bar\lambda^{(a)}\bar\sigma^m \xi + i\bar\xi\bar\sigma^m \lambda^{(a)} \\ \delta_\xi \lambda^{(a)} &= \sigma^{mn}\xi v_{mn}^{(a)} + i\xi D^{(a)} \\ \delta_\xi D^{(a)} &= -\xi\sigma^m \mathscr{D}_m \bar\lambda^{(a)} - \mathscr{D}_m \lambda^{(a)} \sigma^m \bar\xi.\end{aligned}$$

VIII. SPONTANEOUS SYMMETRY BREAKING

If supersymmetric gauge theories are to find realistic application in high energy physics, both supersymmetry and gauge symmetry must be broken spontaneously. The spontaneous breaking of ordinary gauge symmetry is well understood, but supersymmetry imposes additional conditions which need further discussion. These restrictions rest on the property

$$H = \frac{1}{4}(\bar{Q}_1 Q_1 + Q_1 \bar{Q}_1 + \bar{Q}_2 Q_2 + Q_2 \bar{Q}_2), \tag{8.1}$$

derived from the algebra (I). Equation (8.1) tells us that $\langle \Psi | H | \Psi \rangle \geq 0$ for every state $|\Psi\rangle$. Furthermore, it tells us that states with vanishing energy density are supersymmetric ground states of the theory. Such states are ground states because the expectation value of H may never be negative; they are supersymmetric because $\langle 0 | H | 0 \rangle = 0$ implies $Q|0\rangle = \bar{Q}|0\rangle = 0$. Ground states of zero energy preserve supersymmetry, while those of positive energy break it spontaneously. This situation is sketched in Figure 8.1.

In this chapter we shall discuss three models which exhibit the general properties of spontaneous symmetry breaking in supersymmetric theories. We first consider a supersymmetric model, constructed from chiral superfields, in which the ground state breaks supersymmetry. We know from Eq. (5.14) that the potential energy in such models takes the form $\mathscr{V} = F_k^* F_k$, where F_k is given by

$$F_k{}^* = -(\lambda_k + m_{ik} A_i + g_{ijk} A_i A_j). \tag{8.2}$$

Vacuum expectation values a_i of A_i for which $F_k = 0$ signal supersymmetric minima of the potential. To break supersymmetry, we must choose special values for the parameters λ_k, m_{ik}, and g_{ijk} such that the equation

$$0 = \lambda_k + m_{ik} a_i + g_{ijk} a_i a_j \tag{8.3}$$

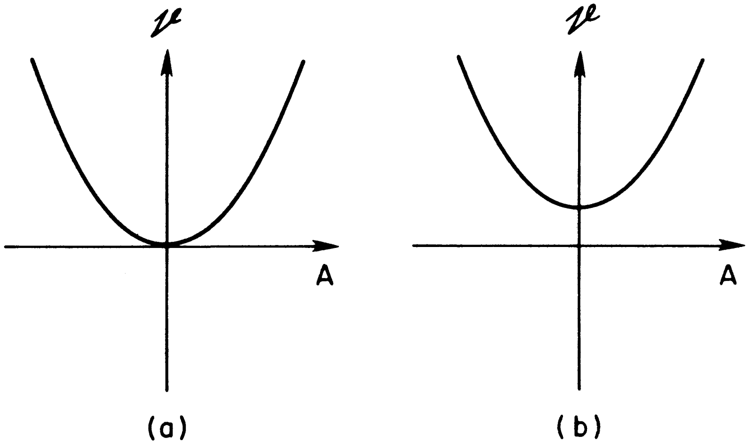

FIGURE 8.1. The ground state of (a) preserves supersymmetry, while the ground state of (b) breaks it spontaneously.

has no solution in a_i. Such models have been constructed by O'Raifeartaigh in the paper cited at the end of Chapter V. He found that three chiral superfields are required to break supersymmetry, the simplest model being given by

$$\mathscr{L}_{P.E.} = \{[\lambda\Phi_0 + m\Phi_1\Phi_2 + g\Phi_0\Phi_1\Phi_1] + \text{h.c.}\}. \tag{8.4}$$

Fayet and Iliopoulos have shown how to spontaneously break supersymmetry in gauge theories with Abelian gauge groups. They observe that the $\theta\theta\bar{\theta}\bar{\theta}$ component of the vector superfield is both supersymmetric and gauge invariant. They add this term to the Lagrangian (7.9) and find that it spontaneously breaks supersymmetry:

$$\begin{aligned}\mathscr{L} = \frac{1}{4}(W^\alpha W_\alpha + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}) + \Phi_1^+ e^{eV}\Phi_1 + \Phi_2^+ e^{-eV}\Phi_2 \\ + m(\Phi_1\Phi_2 + \Phi_1^+\Phi_2^+) + 2\kappa V.\end{aligned} \tag{8.5}$$

In this model, the potential is given by

$$\mathscr{V} = \frac{1}{2}D^2 + F_1F_1^* + F_2F_2^*, \tag{8.6}$$

where D, F_1, and F_2 are solutions to the Euler equations:

$$
\begin{aligned}
D + \kappa + \frac{e}{2}(A_1^* A_1 - A_2^* A_2) &= 0 \\
F_1 + mA_2^* &= 0 \\
F_2 + mA_1^* &= 0.
\end{aligned}
\tag{8.7}
$$

There is no solution to (8.7) which leaves $\mathscr{V} = 0$, so supersymmetry is broken spontaneously.

Let us examine the potential (8.6) in more detail. Substituting for the auxiliary fields, the potential $\mathscr{V}$ becomes

$$
\begin{aligned}
\mathscr{V} = \frac{1}{2}\kappa^2 &+ \left(m^2 + \frac{1}{2}e\kappa\right)A_1^* A_1 + \left(m^2 - \frac{1}{2}e\kappa\right)A_2^* A_2 \\
&+ \frac{1}{8}e^2(A_1^* A_1 - A_2^* A_2)^2.
\end{aligned}
\tag{8.8}
$$

We must distinguish between the two cases $m^2 > \frac{1}{2}e\kappa$ and $m^2 < \frac{1}{2}e\kappa$.

When $m^2 > \frac{1}{2}e\kappa$, both A_1 and A_2 have real masses. The model describes two complex scalar fields, one of mass $m_1{}^2 = m^2 + \frac{1}{2}e\kappa$, the other of mass $m_2{}^2 = m^2 - \frac{1}{2}e\kappa$, as well as three spinor fields ψ_1, ψ_2, λ, and one vector field v_m. The masses of the spinor and vector fields are unchanged by the symmetry breaking. In particular, the field ψ_i retains its mass m, while λ and v_m remain massless. Note that $m_1{}^2 + m_2{}^2 = 2m^2$.

The vector field v_m plays the role of gauge field for the unbroken U(1) symmetry group, and λ is the Goldstone fermion arising from spontaneously broken supersymmetry. From the transformation law for λ (Exercise 6.9),

$$
\delta_\xi \lambda = i\xi D + \sigma^{mn}\xi v_{mn}, \tag{8.9}
$$

we see that λ transforms inhomogeneously as soon as D acquires a vacuum expectation value:

$$
\delta_\xi \lambda = -i\xi\kappa + \cdots. \tag{8.10}
$$

This identifies λ as the Goldstone fermion. Non-zero vacuum expectation values of auxiliary fields induce the spontaneous breakdown of supersymmetry.

When $m^2 < \frac{1}{2}e\kappa$, $A_1 = A_2 = 0$ no longer minimizes the potential (8.8). To find the minimum, we must solve the equations

$$\begin{aligned}\frac{\partial \mathscr{V}}{\partial A_1^*} &= \left(m^2 + \frac{1}{2}e\kappa\right)A_1 + \frac{e^2}{4}(A_1^*A_1 - A_2^*A_2)A_1 = 0 \\ \frac{\partial \mathscr{V}}{\partial A_2^*} &= \left(m^2 - \frac{1}{2}e\kappa\right)A_2 - \frac{e^2}{4}(A_1^*A_1 - A_2^*A_2)A_2 = 0.\end{aligned} \tag{8.11}$$

This gives a minimum at $A_1 = 0, A_2 = v$, where $\frac{1}{4}e^2v^2 + (m^2 - \frac{1}{2}e\kappa) = 0$. By a gauge transformation, v may be chosen to be real. Expanding the potential around its minimum spontaneously breaks the U(1) symmetry. In terms of $A = A_1$, $\tilde{A} = A_2 - v$, the potential becomes:

$$\begin{aligned}\mathscr{V} &= \frac{2m^2}{e^2}(e\kappa - m^2) + 2m^2A^*A \\ &\quad + \frac{1}{2}\left(\frac{1}{2}e^2v^2\right)\left(\frac{1}{\sqrt{2}}[\tilde{A} + \tilde{A}^*]\right)^2 + \frac{1}{2}\left(\frac{1}{2}e^2v^2\right)v_mv^m + \cdots\end{aligned} \tag{8.12}$$

The constant

$$2\frac{m^2}{e^2}(e\kappa - m^2)$$

is positive; both supersymmetry and gauge symmetry are broken spontaneously. The vector field v_m acquires a mass by eating the Goldstone boson field $(\tilde{A} - \tilde{A}^*)/\sqrt{2}$, leaving the total number of degrees of freedom unchanged. The symmetry breaking also modifies the spinor mass terms:

$$-m(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2) + \frac{iev}{\sqrt{2}}(\bar{\psi}_2\bar{\lambda} - \psi_2\lambda). \tag{8.13}$$

With the following linear combinations,

$$\begin{aligned}\psi &= \psi_2 \\ \tilde{\psi} &= \frac{1}{\sqrt{m^2 + \frac{1}{2}e^2v^2}}\left(m\psi_1 + \frac{iev}{\sqrt{2}}\lambda\right) \\ \tilde{\lambda} &= \frac{1}{\sqrt{m^2 + \frac{1}{2}e^2v^2}}\left(m\lambda + \frac{iev}{\sqrt{2}}\psi_1\right),\end{aligned} \tag{8.14}$$

the mass terms become diagonal:

$$-\sqrt{m^2 + \frac{1}{2} e^2 v^2}(\psi\tilde{\psi} + \bar{\psi}\bar{\tilde{\psi}}). \tag{8.15}$$

The Goldstone spinor $\tilde{\lambda}$ remains massless. Note that $\tilde{\lambda}$ transforms inhomogeneously,

$$\delta_\xi \tilde{\lambda} = -2i \frac{m}{e} \xi \sqrt{e\kappa - m^2} + \cdots, \tag{8.16}$$

as expected for a Goldstone field.

This model describes two spinor fields of mass $\sqrt{m^2 + \frac{1}{2}e^2v^2}$, one vector field and one scalar field, each of mass $\sqrt{\frac{1}{2}e^2v^2}$, one complex scalar field of mass $\sqrt{2m^2}$, and one massless Goldstone spinor. Note that the sum of the masses squared weighted by the number of degrees of freedom is identical for the bosonic and fermionic modes:

$$2 \cdot 2m^2 + 4 \cdot \frac{1}{2} e^2 v^2 = 4\left(m^2 + \frac{1}{2} e^2 v^2\right). \tag{8.17}$$

This is also true for the U(1) symmetric case described earlier. In fact, such relationships between bosonic and fermionic masses are common in supersymmetric theories.

The situation for the Fayet-Iliopoulos model is sketched in Figure 8.2. Non-vanishing vacuum expectation values of auxiliary fields induce

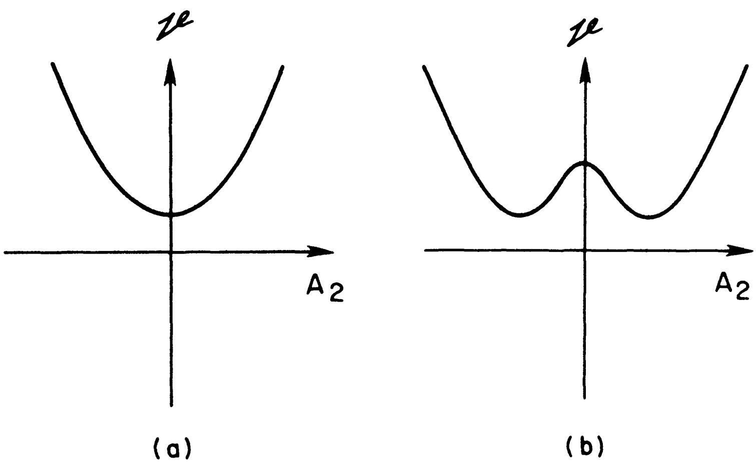

FIGURE 8.2. (a) When $m^2 > \frac{1}{2} e\kappa$, supersymmetry alone is broken.
(b) When $m^2 < \frac{1}{2}e\kappa$, both gauge symmetry and supersymmetry are broken.

supersymmetry breaking, while non-zero vacuum expectation values of dynamical scalar fields lead to the breaking of gauge symmetry.

After having seen a model in which supersymmetry and gauge symmetry are broken spontaneously, one might wish to construct a model in which only the gauge symmetry is broken. We first discuss such models with chiral superfields. In this case we must find a solution a_i to (8.3) which is not left invariant under the internal symmetry group. As a simple example, we consider the group U(1) with three chiral superfields: one neutral, one positive, and one negative. The Lagrangian

$$\mathscr{L}_{P.E.} = \frac{1}{2} m\Phi^2 + \mu\Phi_+\Phi_- + \lambda\Phi + g\Phi\Phi_+\Phi_- + \text{h.c.} \tag{8.18}$$

is U(1) invariant. The Eqs. (8.3) become

$$\begin{aligned} \lambda + ma + ga_+a_- &= 0 \\ a_-(\mu + ga) &= 0 \\ a_+(\mu + ga) &= 0. \end{aligned} \tag{8.19}$$

This set of equations has two solutions:

$$\begin{aligned} &(1) \quad a_+ = a_- = 0, \qquad a = -\frac{\lambda}{m} \\ &(2) \quad a_+a_- = -\frac{1}{g}\left(\lambda - \frac{m\mu}{g}\right), \qquad a = -\frac{\mu}{g}. \end{aligned} \tag{8.20}$$

The first does not break the U(1) symmetry, but the second does. In the second solution, only the product a_+a_- is determined. This stems from the fact that $\mathscr{L}_{P.E.}$ is invariant not only under the U(1) group, but also under its complex extension. For any solution a_+,a_- to (8.19), there exists an entire class of solutions, $e^{\lambda}a_+,e^{-\lambda}a_-$, for arbitrary complex λ. The ground state has a larger degeneracy than required by the initial symmetry group.

If we gauge the Lagrangian (8.18), we must introduce the vector superfield V, coupling to Φ_+ and Φ_- as in (8.5). This results in the following trilinear coupling between the scalar fields $A_\pm$ and the vector multiplet V:

$$eV(A_+^*A_+ - A_-^*A_-). \tag{8.21}$$

For the symmetry breaking solution, this contributes a piece to the D-term:

$$eV\left[a_+^* a_+ - a_-^* a_- + 2\frac{\kappa}{e}\right]. \tag{8.22}$$

Such a term would ordinarily break supersymmetry. Because of the degeneracy $a_\pm \to e^{\pm\lambda} a_\pm$, however, it is possible to transform away this term for any choice of κ. In this model, D-terms do not induce the spontaneous breakdown of supersymmetry.

The mass term associated with (8.20) is given by

$$\frac{1}{2} e^2 (a_+^* a_+ + a_-^* a_-) V^2. \tag{8.23}$$

It cannot be transformed away. It gives a mass to the vector field v_m. Comparing (8.23) with (6.18), we see that spontaneous gauge symmetry breaking in supersymmetric theories gives rise to an entire massive vector multiplet. This is the supersymmetric extension of the Higgs-Kibble mechanism.

These models are easily extended to non-Abelian symmetry groups. Supersymmetric solutions require

$$F_k^* = -\lambda_k - m_{ik} a_i - g_{ijk} a_i a_j = 0. \tag{8.24}$$

The parameters λ, m, and g are restricted by the internal symmetry group. In gauge theories, supersymmetric minima must also satisfy

$$D^\ell = a_i^+ T_{ik}^\ell a_k = 0. \tag{8.25}$$

The Fayet-Iliopoulos D-term is not gauge invariant and cannot appear in the non-Abelian sector of supersymmetric models.

In the remainder of this chapter we shall show that (8.24) determines the supersymmetry breaking of non-Abelian theories. That is, if (8.24) has a solution a_i, then it is always possible to find a solution $\hat{a}_i$ which satisfies (8.25) as well. We shall demonstrate this for the case of a semi-simple gauge group G.

To begin, let us suppose we have found a solution a_i such that $F_k^*(a_i) = 0$. We may then compute

$$d^\ell = a_i^+ T_{ik}^\ell a_k. \tag{8.26}$$

The vector d^ℓ specifies a certain direction in the regular representation. There is always a group element which transforms this vector into a linear combination of vectors in the Cartan subalgebra. Because (8.24) is invariant under G, this transformation rotates the a_i into another solution $\tilde{a}_i$. The vector d^ℓ transforms into a vector $\tilde{d}^\ell$ whose non-vanishing components lie in the Cartan subalgebra. We may now perform a linear transformation within the Cartan subalgebra such that the direction $\tilde{d}^\ell$ defines a single generator with eigenvalues μ_i. In this basis, the only non-vanishing component of $\tilde{d}^\ell$ is $\tilde{d}$:

$$\tilde{d} = \tilde{a}_i^+ \mu_i \tilde{a}_i. \tag{8.27}$$

The equations (8.24) are also invariant under gauge transformations with *complex* group parameters. This is because the complex conjugate representations of the scalar fields never enter F^*. We are free to perform such a transformation in the direction $\tilde{d}$:

$$\hat{a}_i = \exp(\mu_i \eta)\tilde{a}_i. \tag{8.28}$$

The parameters $\hat{a}_i$ solve (8.24) for all values of η. Taking η real, we find

$$\hat{d} = \tilde{a}_i^+ \mu_i e^{2\mu_i \eta}\tilde{a}_i. \tag{8.29}$$

We now distinguish two cases. In the first case, all the μ_i (for which $\tilde{a}_i \neq 0$) are of the same sign, say positive. We then let $\eta \to -\infty$ to find $\hat{d} = 0$. In the second case, the μ_i take both signs. We shall show that there is still a value of η where $\hat{d} = 0$. In particular, we note that

$$\hat{d} = \frac{1}{2}\frac{\partial}{\partial \eta}\, \tilde{a}_i^+ e^{2\mu_i \eta}\tilde{a}_i. \tag{8.30}$$

Considering $\tilde{a}_i^+ e^{2\mu_i \eta}\tilde{a}_i$ as a function of η, we see that it tends to $+\infty$ as $\eta \to \pm\infty$. Therefore it has a minimum for some value of η. At this point the derivative vanishes and $\hat{d} = 0$.

This completes the proof. We have shown that spontaneous supersymmetry breaking in non-Abelian models is controlled by F-terms. Supersymmetry is spontaneously broken if and only if the equations $F_k^* = 0$ have no solution. This is the O'Raifeartaigh mechanism for supersymmetry breaking.

References

P. Fayet and J. Iliopoulos, *Phys. Lett. 51B*, 461 (1974).
E. Witten, *Nucl. Phys. B188*, 513 (1981).

Exercises

(1) Show that $A_1 = A_2 = 0$ is a minimum of the O'Raifeartaigh model (8.4) when $m^2 > 2\lambda g$. The value of the potential at the minimum is λ^2, independent of A_0.

(2) Compute the boson and fermion masses in the O'Raifeartaigh model:

Real scalar masses:	$0, 0, m^2, m^2, m^2 \pm 2g\lambda$
Spinor masses:	$0, 2m$.

Massless scalars and spinors are general features of O'Raifeartaigh models with spontaneous supersymmetry breaking. Also compute $\partial F_k^*/\partial A_i$ in this model, and show that

$$\det\left(\frac{\partial F_k^*}{\partial A_i}\right) \equiv 0.$$

This is another general feature of O'Raifeartaigh models.

(3) Consider three chiral superfields, Φ_0, Φ_1, Φ_2, with R-characters $n_0 = 1, n_1 = 0, n_2 = 1$. Construct the most general renormalizable, supersymmetric, R-invariant Lagrangian also invariant under the following discrete transformation:

$$\Phi_0 \to \Phi_0$$
$$\Phi_1 \to -\Phi_1$$
$$\Phi_2 \to -\Phi_2 .$$

Show that this determines the O'Raifeartaigh model.

(4) Show that $\tilde{\psi}$ of (8.14) does not shift under a supersymmetry transformation.

(5) Add a D-term $2\kappa V$ to the Lagrangian (8.18). Determine the values of a_+ and a_- at the minimum of the potential.

(6) Consider three triplets of chiral superfields: $\mathbf{\Phi}_i$, $i = 1,2,3$. Find the minimum of the potential for the Lagrangian

$$\mathscr{L}_{P.E.} = m\,\mathbf{\Phi}_i \cdot \mathbf{\Phi}_i + \frac{1}{3}\,g\varepsilon_{ijk}\mathbf{\Phi}_i \cdot (\mathbf{\Phi}_j \times \mathbf{\Phi}_k).$$

(7) Show that the minimum of the potential of Exercise 6 is invariant under the rotation group with complex parameters.

(8) Gauge the model of Exercise 6 and show that an arbitrary D-term may always be eliminated. Supersymmetry may never be broken by the Fayet-Iliopoulos mechanism in this model.

IX. SUPERFIELD PROPAGATORS

In previous chapters we have found superfields very useful for the construction of supersymmetry representations and invariant Lagrangians. In this chapter we shall see that they also simplify the calculation of radiative corrections in quantized supersymmetric theories. The Feynman rules for supersymmetric theories may be stated in terms of superfield vertices and propagators. Many component-field Feynman diagrams are contained in one superfield diagram, so many miraculous cancellations between component diagrams are manifest in one superfield diagram. For this reason alone one would like to find a superfield formulation of supersymmetric theories.

To derive superfield propagators we must first introduce the concept of integration in superspace. An indefinite integral over a Grassmann variable η is defined as follows:

$$\int d\eta = 0, \qquad \int \eta\, d\eta = 1. \tag{9.1}$$

Any function of η is polynomial, $f(\eta) = c + \Delta\eta$, so definition (9.1) extends immediately to arbitrary functions of Grassmann variables:

$$\begin{aligned} f(\eta) &= c + \Delta\eta \\ \int f(\eta)\, d\eta &= \Delta \\ \int f(\eta)\eta\, d\eta &= c. \end{aligned} \tag{9.2}$$

Since

$$\int \frac{\partial}{\partial\eta} f(\eta)\, d\eta = 0,$$

partial integration is always possible. Note that integration and differentiation give the same result on functions of Grassmann variables.

Delta functions are defined by the integral

$$\int f(\eta)\,\delta(\eta)\,d\eta = f(0). \tag{9.3}$$

From (9.2), it follows that

$$\delta(\eta) = \eta, \tag{9.4}$$

so consequently,

$$\delta(\eta)\,\delta(\eta) = 0. \tag{9.5}$$

Defining volume elements in superspace,

$$\begin{aligned} d^2\theta &= -\frac{1}{4}\, d\theta^\alpha\, d\theta^\beta \varepsilon_{\alpha\beta} \\ d^2\bar\theta &= -\frac{1}{4}\, d\bar\theta_{\dot\alpha}\, d\bar\theta_{\dot\beta} \varepsilon^{\dot\alpha\dot\beta} \\ d^4\theta &= d^2\theta\, d^2\bar\theta, \end{aligned} \tag{9.6}$$

we find

$$\int \theta\theta\, d^2\theta = 1, \qquad \int \bar\theta\bar\theta\, d^2\bar\theta = 1. \tag{9.7}$$

This allows us to write the Lagrangian (5.10) as an integral over superspace:

$$\begin{aligned} \mathscr{L} = \int \Big\{ &\Phi_i^+ \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j \, \delta(\bar\theta) \\ &+ \frac{1}{2} m_{ij}^* \Phi_i^+ \Phi_j^+ \, \delta(\theta) + \frac{1}{3} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \, \delta(\bar\theta) \\ &+ \frac{1}{3} \lambda_{ijk}^* \Phi_i^+ \Phi_j^+ \Phi_k^+ \, \delta(\theta) \Big\} d^2\theta\, d^2\bar\theta. \end{aligned} \tag{9.8}$$

Perturbation theory in superspace may be developed as a direct extension of ordinary perturbation theory. In particular, one would like to calculate superfield Green's functions,

$$\begin{aligned} \langle 0 | \mathrm{T}\{ &\Phi(x^1,\theta^1,\bar\theta^1) \cdots \Phi(x^\ell,\theta^\ell,\bar\theta^\ell) \Phi^+(x^{\ell+1},\theta^{\ell+1},\bar\theta^{\ell+1}) \\ &\cdots \Phi^+(x^r,\theta^r,\bar\theta^r)\} | 0 \rangle. \end{aligned} \tag{9.9}$$

From these one recovers the component-field Green's functions by power series expansion in $\theta^1, \bar{\theta}^1 \cdots \theta^r, \bar{\theta}^r$.

As with any field theory, we begin our analysis by evaluating the free-field two-point functions, the propagators. For chiral fields, these are derived from the free-field part of Lagrangian (9.8):

$$\begin{aligned}\mathscr{L}_0 &= \int \left\{\Phi^+\Phi + \frac{1}{2} m\Phi\Phi\,\delta(\bar{\theta}) + \frac{1}{2} m^*\Phi^+\Phi^+\,\delta(\theta)\right\} d^2\theta\, d^2\bar{\theta} \\ &= A^*\Box A + i\,\partial_m \bar{\psi}\bar{\sigma}^m \psi + F^*F + m\left(AF + A^*F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar{\psi}\bar{\psi}\right). \qquad \textbf{(9.10)}\end{aligned}$$

In components, we find:

$$\begin{aligned}\langle 0|\mathrm{T}\{A(x)A^*(x')\}|0\rangle &= i\,\Delta_F(x - x') \\ \langle 0|\mathrm{T}\{A(x)F(x')\}|0\rangle &= \langle 0|T\{A^*(x)F^*(x')\}|0\rangle \\ &= -im\,\Delta_F(x - x') \\ \langle 0|\mathrm{T}\{F(x)F^*(x')\}|0\rangle &= i\Box\,\Delta_F(x - x') \qquad \textbf{(9.11)} \\ \langle 0|\mathrm{T}\{\psi_\alpha(x)\psi^\beta(x')\}|0\rangle &= i\,\delta_\alpha{}^\beta m\,\Delta_F(x - x') \\ \langle 0|\mathrm{T}\{\bar{\psi}^{\dot{\alpha}}(x)\bar{\psi}_{\dot{\beta}}(x')\}|0\rangle &= i\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} m\,\Delta_F(x - x') \\ \langle 0|\mathrm{T}\{\psi_\alpha(x)\bar{\psi}_{\dot{\beta}}(x')\}|0\rangle &= \sigma_{\alpha\dot{\beta}}{}^m\,\partial_m\,\Delta_F(x - x'),\end{aligned}$$

where

$$\Delta_F(x) = \frac{1}{\Box - m^2}.$$

All other two-point functions vanish. We may use these component propagators to construct the superfield propagators. For example,

$$\begin{aligned}\langle 0|\mathrm{T}\{\Phi(y,\theta)\Phi(y',\theta')\}|0\rangle &= \langle 0|\mathrm{T}\{[A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)] \\ &\qquad \times [A(y') + \sqrt{2}\theta'\psi(y') + \theta'\theta' F(y')]\}|0\rangle \\ &= \theta'\theta'\langle 0|\mathrm{T}\{A(y)F(y')\}|0\rangle \\ &\quad + \theta\theta\langle 0|\mathrm{T}\{F(y)A(y')\}|0\rangle \\ &\quad + 2\theta'^\beta\theta^\alpha\langle 0|\mathrm{T}\{\psi_\alpha(y)\psi_\beta(y')\}|0\rangle \\ &= -im(\theta - \theta')^2\,\Delta_F(y - y'). \qquad (9.12)\end{aligned}$$

From the definitions $y = x + i\theta\sigma\bar{\theta}$, $y^+ = x - i\theta\sigma\bar{\theta}$, we see that this propagator and the $\Phi^+\Phi^+$ propagator have the following x,x' dependence:

$$\begin{aligned}
&\langle 0|\mathrm{T}\{\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')\}|0\rangle \\
&\quad = -im\delta(\theta-\theta')\exp[i(\theta\sigma^m\bar{\theta}-\theta'\sigma^m\bar{\theta}')\,\partial_m{}^x]\,\Delta_F(x-x'), \\
&\langle 0|\mathrm{T}\{\Phi^+(x,\theta,\bar{\theta})\Phi^+(x',\theta',\bar{\theta}')\}|0\rangle \\
&\quad = +im\delta(\bar{\theta}-\bar{\theta}')\exp[-i(\theta\sigma^m\bar{\theta}-\theta'\sigma^m\bar{\theta}')\,\partial_m{}^x]\,\Delta_F(x-x').
\end{aligned} \tag{9.13}$$

Following exactly the same procedure we can construct the $\Phi\Phi^+$ propagator:

$$\begin{aligned}
&\langle 0|\mathrm{T}\{\Phi(x,\theta,\bar{\theta})\Phi^+(x',\theta',\bar{\theta}')\}|0\rangle \\
&\quad = i\exp[i(\theta\sigma^m\bar{\theta} + \theta'\sigma^m\bar{\theta}' - 2\theta\sigma^m\bar{\theta}')\,\partial_m{}^x]\,\Delta_F(x-x').
\end{aligned} \tag{9.14}$$

With these propagators we may evaluate the superfield Green's functions (9.9) to any order in perturbation theory. We start by writing the n-th order contribution in terms of free superfields:

$$\begin{aligned}
&\left\langle 0\left|T\left\{\Phi(x^1,\theta^1,\bar{\theta}^1)\cdots\Phi^+(x^{\ell+1},\theta^{\ell+1},\bar{\theta}^{\ell+1})\right.\right.\right. \\
&\quad\left.\left.\left.\cdots\int\mathcal{L}_{int}(x_1')\,d^4x_1'\cdots\int\mathcal{L}_{int}(x_n')\,d^4x_n'\right\}\right|0\right\rangle \\
&\quad = \left\langle 0\left|T\left\{\Phi\cdots\Phi^+\cdots\right.\right.\right. \\
&\qquad \int\frac{1}{3}\,[g\Phi^3(x_1',\theta_1,\bar{\theta}_1)\,\delta(\bar{\theta}_1) + g^*\Phi^{+3}(x_1',\theta_1,\bar{\theta}_1)\,\delta(\theta_1)] \\
&\qquad \left.\left.\left. d^4x_1'\,d^2\theta_1\,d^2\bar{\theta}_1\cdots\right\}\right|0\right\rangle.
\end{aligned} \tag{9.15}$$

Using Wick's theorem, we then reduce these expressions to the usual Feynman diagrams.

As a sample calculation, let us consider the one-loop corrections to the superfield two-point functions. These are illustrated in Figure 9.1. Diagram 9.1(a) is proportional to $\delta^2(\theta - \theta') = \delta(0) = 0$, while 9.1(b) goes as $\delta^2(\bar{\theta} - \bar{\theta}') = \delta(0) = 0$. This shows that all contributions to mass renormalization, both finite and infinite, cancel between the various com-

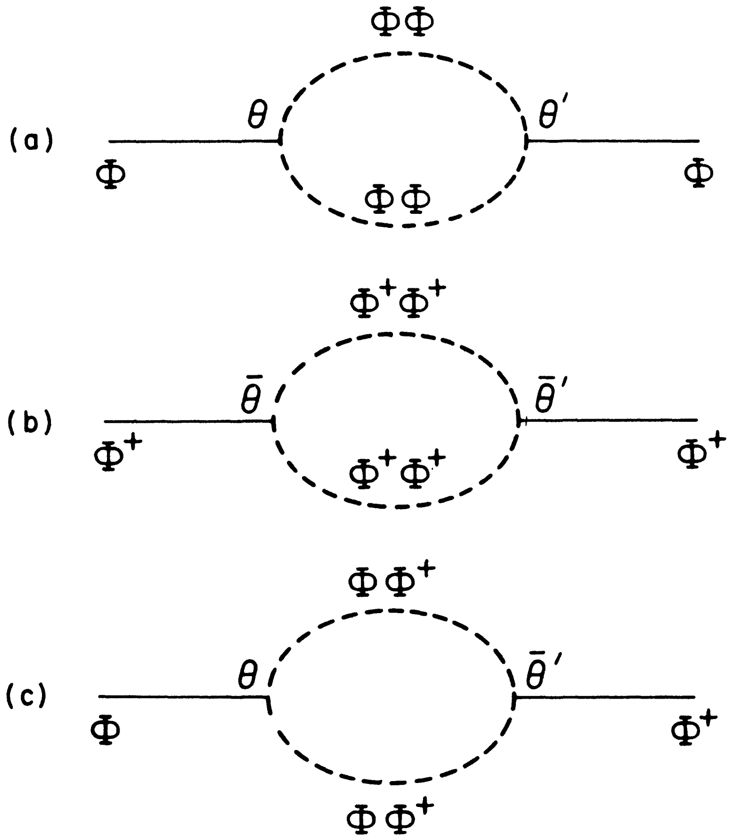

FIGURE 9.1. One-loop corrections to the (a) $\Phi\Phi$, (b) $\Phi^+\Phi^+$, and (c) $\Phi\Phi^+$ superfield propagators.

ponent fields. The final diagram, Figure 9.1(c), is proportional to

$$\begin{aligned}
&\int d^4x\, d^4x'\, d^2\theta\, d^2\theta'\, d^2\bar{\theta}\, d^2\bar{\theta}'\, \delta(\bar{\theta})\, \delta(\theta') \\
&\qquad \times \Phi(x,\theta,\bar{\theta}) \exp\left[i(\theta\sigma^m\bar{\theta} + \theta'\sigma^m\bar{\theta}' - 2\theta\sigma^m\bar{\theta}')\,\partial_m{}^x\right] \\
&\qquad \times \Delta_F(x - x') \exp\left[i(\theta\sigma^m\bar{\theta} + \theta'\sigma^m\bar{\theta}' - 2\theta\sigma^m\bar{\theta}')\,\partial_m{}^x\right] \Delta_F(x - x') \\
&\qquad \times \Phi^+(x',\theta',\bar{\theta}') \\
= &\int d^4x d^4x' d^2\theta d^2\bar{\theta} \Delta_F{}^2(x - x') \Phi(x,\theta,0) \exp\left[-2i\theta\sigma^m\bar{\theta}\,\partial_m{}^{x'}\right] \Phi^+(x',0,\bar{\theta}).
\end{aligned} \tag{9.16}$$

To obtain this result we have integrated over the δ-functions, replaced $\bar{\theta}'$ by $\bar{\theta}$, and integrated by parts. The $\Delta_F{}^2$ in the expression above leads

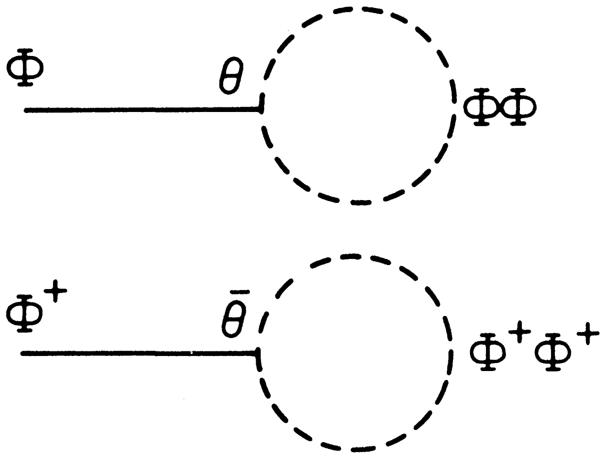

FIGURE 9.2. All tadpole graphs vanish identically.

to a logarithmic divergence which may be absorbed into a logarithmically divergent wave function renormalization.

It is easy to see that all closed-loop diagrams vanish when they contain only $\Phi\Phi$ or $\Phi^+\Phi^+$ propagators. This follows immediately from the fact that they are proportional to $\delta(0)$ in $\theta,\bar{\theta}$ space. In particular, there are no non-vanishing tadpole graphs in this theory (Figure 9.2). Similarly, there are no finite nor infinite contributions to the coupling constant renormalization (Figure 9.3).

The superfield propagators (9.13) and (9.14) may be obtained directly as superspace Green's functions for the free-field equations. To see this, we write the free-field Lagrangian (9.10) in the following form:

$$\begin{aligned}\mathscr{L}_0 &= \int \left\{\Phi^+\Phi - \frac{1}{8}m\left(\Phi\frac{DD}{\Box}\Phi + \Phi^+\frac{\bar{D}\bar{D}}{\Box}\Phi^+\right)\right\} d^4x\, d^2\theta\, d^2\bar{\theta} \\ &= \int \frac{1}{2}(\Phi,\Phi^+)\mathscr{M}\begin{pmatrix}\Phi \\ \Phi^+\end{pmatrix} d^4x\, d^2\theta\, d^2\bar{\theta}, \end{aligned} \tag{9.17}$$

where

$$\mathscr{M} = \begin{pmatrix} -\frac{1}{4}\frac{m}{\Box}DD & 1 \\ 1 & -\frac{1}{4}\frac{m}{\Box}\bar{D}\bar{D} \end{pmatrix}.$$

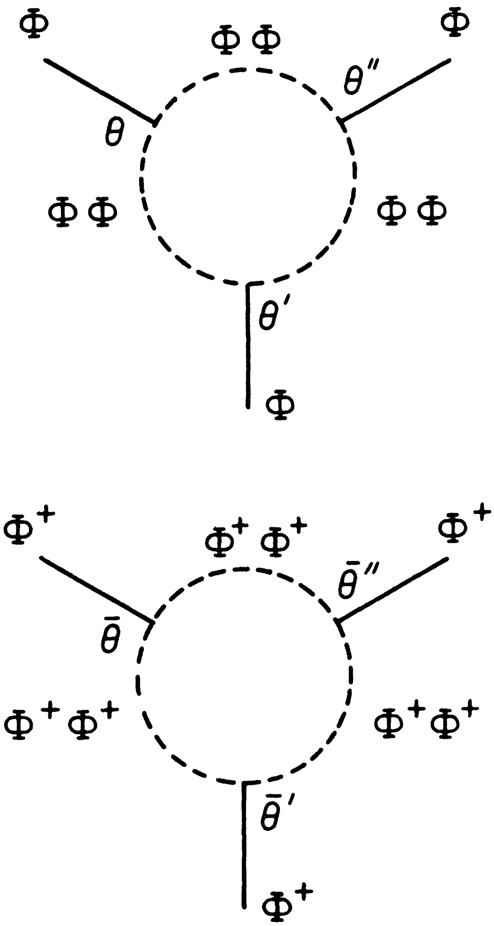

FIGURE 9.3. There are no non-vanishing corrections to the Φ^3 and Φ^{+3} couplings.

This expression is valid because $d^2\bar{\theta}$ is equivalent to $-\frac{1}{4}\bar{D}\bar{D}$ under an x-integration, and because $\frac{1}{16}(\bar{D}\bar{D}DD/\Box)$ is a projection operator on chiral fields:

$$\begin{aligned} &\frac{1}{16}\frac{\bar{D}\bar{D}DD}{\Box}\Phi = \Phi \qquad \text{if } \bar{D}\Phi = 0 \\ &\bar{D}\frac{1}{16}\frac{\bar{D}\bar{D}DD}{\Box} = 0. \end{aligned} \tag{9.18}$$

If we wish to derive field equations from (9.17) by a variational principle, we must take into account the fact that the chiral fields Φ and Φ^+ are subject to constraints. We do this by varying Φ and Φ^+ in the y and y^+ bases:

$$\frac{\delta}{\delta\Phi(y,\theta)}\Phi(y',\theta') = \delta(y - y')\,\delta(\theta - \theta'). \tag{9.19}$$

In these bases the field variations automatically remain chiral. We may use this result to find the variations of Φ and Φ^+ under full superspace integrations:

$$\begin{aligned}
&\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\int \Phi(x',\theta',\bar\theta')F(x',\theta',\bar\theta')\,d^4x'\,d^2\theta'\,d^2\bar\theta' \\
&= \frac{\delta}{\delta\Phi(y,\theta)}\int \Phi(y',\theta')F(y' - i\theta'\sigma\bar\theta', \theta', \bar\theta')\,d^4y'\,d^2\theta'\,d^2\bar\theta' \\
&= \int \delta(y - y')\,\delta(\theta - \theta')F(y' - i\theta'\sigma\bar\theta, \theta', \bar\theta)\,d^4y'\,d^2\theta'\,d^2\bar\theta \\
&= \int F(y - i\theta\sigma\bar\theta, \theta, \bar\theta)\,d^2\bar\theta \\
&= -\frac{1}{4}\bar D\bar D F(x,\theta,\bar\theta).
\end{aligned} \tag{9.20}$$

Here $\Phi(x,\theta,\bar\theta) = \Phi(y,\theta)$, where $y = x + i\theta\sigma\bar\theta$. Equation (9.20) may be summarized by a formal rule:

$$\frac{\delta}{\delta\Phi(x, \theta, \bar\theta)}\Phi(x',\theta',\bar\theta') = -\frac{1}{4}\bar D\bar D\,\delta(\theta - \theta')\,\delta(\bar\theta - \bar\theta')\,\delta(x - x'). \tag{9.21}$$

This rule reproduces (9.20):

$$\begin{aligned}
&\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\int \Phi(x',\theta',\bar\theta')F(x',\theta',\bar\theta')\,d^4x'\,d^2\theta'\,d^2\bar\theta' \\
&= \int -\frac{1}{4}\bar D\bar D\,\delta(\theta - \theta')\,\delta(\bar\theta - \bar\theta')\,\delta(x - x')F(x',\theta',\bar\theta')\,d^4x'\,d^2\theta'\,d^2\bar\theta' \\
&= -\frac{1}{4}\bar D\bar D F(x,\theta,\bar\theta).
\end{aligned} \tag{9.22}$$

Here we have integrated by parts to obtain the final result.

The free-field Euler-Lagrange equations are found by varying (9.17) according to (9.21):

$$-\frac{1}{4}\begin{pmatrix}\bar{D}^2 & 0\\ 0 & D^2\end{pmatrix}\mathscr{M}\begin{pmatrix}\Phi\\ \Phi^+\end{pmatrix} = 0. \tag{9.23}$$

These equations may be simplified with the help of (9.18):

$$\begin{aligned} m\Phi - \frac{1}{4}\bar{D}\bar{D}\Phi^+ &= 0\\ m\Phi^+ - \frac{1}{4}DD\Phi &= 0. \end{aligned} \tag{9.24}$$

Here we recognize the field equations for a massive chiral multiplet, first encountered in Chapter IV, Exercise 7.

We may always couple chiral superfields to classical external sources. For chiral sources,

$$\bar{D}_{\dot{\alpha}}J = D_\alpha J^+ = 0, \tag{9.25}$$

we find the following Lagrangian

$$\mathscr{L} = \int\left\{\frac{1}{2}(\Phi,\Phi^+)\mathscr{M}\begin{pmatrix}\Phi\\ \Phi^+\end{pmatrix} + (\Phi,\Phi^+)\begin{pmatrix}-\frac{1}{4}\frac{D^2}{\Box} & 0\\ 0 & -\frac{1}{4}\frac{\bar{D}^2}{\Box}\end{pmatrix}\begin{pmatrix}J\\ J^+\end{pmatrix}\right\}d^4x\,d^2\theta\,d^2\bar{\theta} \tag{9.26}$$

and field equations:

$$\frac{1}{4}\begin{pmatrix}\bar{D}^2 & 0\\ 0 & D^2\end{pmatrix}\mathscr{M}\begin{pmatrix}\Phi\\ \Phi^+\end{pmatrix} = \begin{pmatrix}J\\ J^+\end{pmatrix}. \tag{9.27}$$

The superfield Green's function is defined in analogy to (9.27):

$$\frac{1}{4}\begin{pmatrix}\bar{D}^2 & 0\\ 0 & D^2\end{pmatrix}\mathscr{M}\Delta = -\begin{pmatrix}-\frac{1}{4}\bar{D}^2 & 0\\ 0 & -\frac{1}{4}D^2\end{pmatrix}\delta(\theta-\theta')\,\delta(\bar{\theta}-\bar{\theta}')\,\delta(x-x'). \tag{9.28}$$

The δ-functions are multiplied by the operators $-\frac{1}{4}D^2$ and $-\frac{1}{4}\bar{D}^2$ because (9.27) has a solution only for chiral sources. The superfield Green's function Δ gives Φ and Φ^+ in the presence of J and J^+:

$$\begin{pmatrix} \Phi \\ \Phi^+ \end{pmatrix} = -\int \Delta(x,\theta,\bar{\theta};x',\theta',\bar{\theta}') \begin{pmatrix} -\frac{1}{4}\frac{D^2}{\Box} & 0 \\ 0 & -\frac{1}{4}\frac{\bar{D}^2}{\Box} \end{pmatrix} \begin{pmatrix} J \\ J^+ \end{pmatrix} d^4x'\, d^2\theta'\, d^2\bar{\theta}'. \tag{9.29}$$

In order to solve for the Green's functions, we exploit the algebraic properties of the chiral projection operators

$$P_1 = \frac{1}{16}\frac{D^2\bar{D}^2}{\Box}, \qquad P_2 = \frac{1}{16}\frac{\bar{D}^2 D^2}{\Box} \tag{9.30}$$

by introducing three additional operators:

$$P_+ = \frac{D^2}{4\Box^{\frac{1}{2}}} \qquad P_- = \frac{\bar{D}^2}{4\Box^{\frac{1}{2}}} \qquad P_T = -\frac{1}{8\Box} D\bar{D}^2 D. \tag{9.31}$$

After a short calculation, one may quickly confirm

$$P_1 + P_2 + P_T = 1, \tag{9.32}$$

as well as the following multiplication table:

	P_1	P_2	P_+	P_-	P_T
P_1	P_1	0	P_+	0	0
P_2	0	P_2	0	P_-	0
P_+	0	P_+	0	P_1	0
P_-	P_-	0	P_2	0	0
P_T	0	0	0	0	P_T

(9.33)

With this multiplication table, we may readily express the differential operator of Eq. (9.28) in the following form:

$$\frac{1}{4}\begin{pmatrix} \bar{D}^2 & 0 \\ 0 & D^2 \end{pmatrix} \mathscr{M} = \frac{1}{4}\begin{pmatrix} \bar{D}^2 & 0 \\ 0 & D^2 \end{pmatrix} \mathscr{M} \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix}. \tag{9.34}$$

Using (9.34), it is easy to show that

$$\Delta = \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \mathscr{M}^{-1}\, \delta(\theta - \theta')\,\delta(\bar{\theta} - \bar{\theta}')\,\delta(x - x') \tag{9.35}$$

is the Green's function for (9.28). The additional projection operators insure that Δ is the propagator for chiral superfields.

To find Δ, we must first invert $\mathscr{M}$. This is easiest if we expand it in terms of the P-operators:

$$\mathscr{M} = \begin{pmatrix} -\dfrac{m}{\Box^{\frac{1}{2}}} & 0 \\ 0 & 0 \end{pmatrix} P_+ + \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{m}{\Box^{\frac{1}{2}}} \end{pmatrix} P_- + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \{P_1 + P_2 + P_T\}. \tag{9.36}$$

The inverse of any operator of the type

$$X = AP_1 + DP_2 + BP_+ + CP_- + EP_T \tag{9.37}$$

is given by

$$\begin{aligned} X^{-1} = {} & [A - BD^{-1}C]^{-1}P_1 + [D - CA^{-1}B]^{-1}P_2 \\ & - A^{-1}B[D - CA^{-1}B]^{-1}P_+ \\ & - D^{-1}C[A - BD^{-1}C]^{-1}P_- + E^{-1}P_T, \end{aligned} \tag{9.38}$$

provided A,D,E are all invertible. This may be shown by direct multiplication or by use of the P-operator representation given in Exercise 7. With this result, we find

$$\begin{aligned} \mathscr{M}^{-1} = {} & \begin{pmatrix} 0 & 1 \\ \dfrac{\Box}{\Box - m^2} & 0 \end{pmatrix} P_1 + \begin{pmatrix} 0 & \dfrac{\Box}{\Box - m^2} \\ 1 & 0 \end{pmatrix} P_2 \\ & + \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} \end{pmatrix} P_+ + \begin{pmatrix} \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} & 0 \\ 0 & 0 \end{pmatrix} P_- + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_T \\ = {} & \begin{pmatrix} \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} P_- & P_1 + \dfrac{\Box}{\Box - m^2} P_2 + P_T \\ \dfrac{\Box}{\Box - m^2} P_1 + P_2 + P_T & \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} P_+ \end{pmatrix}. \end{aligned} \tag{9.39}$$

According to (9.35) and the multiplication table (9.33), the propagator Δ becomes

$$\Delta = \begin{pmatrix} \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} P_- & \dfrac{\Box}{\Box - m^2} P_2 \\ \dfrac{\Box}{\Box - m^2} P_1 & \dfrac{m}{\Box^{\frac{1}{2}}} \dfrac{\Box}{\Box - m^2} P_+ \end{pmatrix} \delta(x - x')\delta(\theta - \theta')\delta(\bar\theta - \bar\theta')$$

$$= \frac{1}{\Box - m^2} \begin{pmatrix} \dfrac{m}{4} \bar D^2 & \dfrac{1}{16} \bar D^2 D^2 \\ \dfrac{1}{16} D^2 \bar D^2 & \dfrac{m}{4} D^2 \end{pmatrix} \delta(x - x')\delta(\theta - \theta')\delta(\bar\theta - \bar\theta'). \tag{9.40}$$

To compare this result with the previous propagators (9.13) and (9.14), we must compute the spinor derivatives of the δ-functions:

$$\begin{aligned} D_1{}^2(\theta_1 - \theta_2)^2 &= -4 \exp[-i(\theta_1 - \theta_2)\sigma^n\bar\theta_1 \,\partial_n{}^1] \\ \bar D_1{}^2(\bar\theta_1 - \bar\theta_2)^2 &= -4 \exp[i\theta_1\sigma^n(\bar\theta_1 - \bar\theta_2)\,\partial_n{}^1] \\ \bar D_1{}^2 D_1{}^2(\theta_1 - \theta_2)^2(\bar\theta_1 - \bar\theta_2)^2 &= 16 \exp[i(\theta_1\sigma^n\bar\theta_1 + \theta_2\sigma^n\bar\theta_2 - 2\theta_1\sigma^n\bar\theta_2)\,\partial_n{}^1] \\ D_1{}^2 \bar D_1{}^2(\theta_1 - \theta_2)^2(\bar\theta_1 - \bar\theta_2)^2 &= 16 \exp[-i(\theta_1\sigma^n\bar\theta_1 + \theta_2\sigma^n\bar\theta_2 - 2\theta_2\sigma^n\bar\theta_1)\,\partial_n{}^1]. \end{aligned} \tag{9.41}$$

The proofs of these relations are left to the reader as exercises in straight differentiation. Substituting (9.41) into (9.40), we find

$$\frac{1}{\Box - m^2} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \delta(x - x'), \tag{9.42}$$

where

$$\begin{aligned} \Delta_{11} &= -m\delta(\theta - \theta') \exp[i(\theta\sigma^n\bar\theta - \theta'\sigma^n\bar\theta')\,\partial_n] \\ \Delta_{12} &= \exp[i(\theta\sigma^n\bar\theta + \theta'\sigma^n\bar\theta' - 2\theta\sigma^n\bar\theta')\,\partial_n] \\ \Delta_{21} &= \exp[-i(\theta\sigma^n\bar\theta + \theta'\sigma^n\bar\theta' - 2\theta'\sigma^n\bar\theta)\,\partial_n] \\ \Delta_{22} &= -m\delta(\bar\theta - \bar\theta') \exp[-i(\theta\sigma^n\bar\theta - \theta'\sigma^n\bar\theta')\,\partial_n]. \end{aligned} \tag{9.43}$$

This result is identical to (9.13) and (9.14). In (9.42) we replaced θ [$\bar\theta$] by θ' [$\bar\theta'$] whenever it was multiplied by $\delta(\theta - \theta')$ [$\delta(\bar\theta - \bar\theta')$].

Having gained some experience with superfield methods, we shall now compute the propagator for vector superfields. We start with the usual Lagrangian,

$$\mathscr{L} = \frac{1}{4} W^\alpha W_\alpha|_{\theta\theta} + \frac{1}{4} \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta} + m^2 V^2|_{\theta\theta\bar\theta\bar\theta}, \tag{9.44}$$

as outlined in Chapter VI. To this we add the gauge fixing term $-\frac{1}{8}\xi(\bar{D}^2 V)(D^2 V)$. This term yields a piece proportional to $(\partial_m v^m)^2$ in the component Lagrangian.

To find the propagator, we write the action as an integral over superspace:

$$\begin{aligned}
\mathscr{L} &= \int \left\{\frac{1}{4} W^\alpha W_\alpha \delta(\bar\theta) + \frac{1}{4} \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha} \delta(\theta) + m^2 V^2 - \frac{\xi}{8} (\bar{D}^2 V)(D^2 V)\right\} d^4x\, d^2\theta\, d^2\bar\theta \\
&= \int \left\{\frac{1}{8} V D\bar{D}^2 D V + m^2 VV - \frac{\xi}{16} V(D^2\bar{D}^2 + \bar{D}^2 D^2)V\right\} d^4x\, d^2\theta\, d^2\bar\theta \\
&= \int \{V[-\Box P_T + m^2(P_1 + P_2 + P_T) - \xi(P_1 + P_2)\Box]V\} d^4x\, d^2\theta\, d^2\bar\theta \\
&= \int V\mathscr{N} V\, d^4x\, d^2\theta\, d^2\bar\theta.
\end{aligned} \tag{9.45}$$

The Euler-Lagrange equations are found from a variational principle,

$$\mathscr{N} V = 0, \tag{9.46}$$

and the superfield Green's function is defined in the usual way:

$$\mathscr{N}\Delta = \delta(x - x')\delta(\theta - \theta')\delta(\bar\theta - \bar\theta'). \tag{9.47}$$

Note that we choose to invert $\mathscr{N}$ and not $2\mathscr{N}$ as might be expected from the Lagrangian (9.45). This normalization of the superfield propagator leads to the usual normalizations for the Green's functions of the component fields. Solving for Δ,

$$\Delta = \mathscr{N}^{-1}\delta(x - x')\delta(\theta - \theta')\delta(\bar\theta - \bar\theta'), \tag{9.48}$$

inverting $\mathscr{N}$,

$$\mathscr{N}^{-1} = \frac{1}{-\Box + m^2} P_T + \frac{1}{m^2 - \xi\Box}(P_1 + P_2), \tag{9.49}$$

and using (9.41),

$$(P_1+P_2)\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)$$
$$=\frac{1}{16\Box}(\bar D_1{}^2 D_1{}^2+D_1{}^2\bar D_1{}^2)\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)$$
$$=\frac{1}{2\Box}\exp[i(\theta_2\sigma^n\bar\theta_1-\theta_1\sigma^n\bar\theta_2)\,\partial_n][4+\Box(\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2]\,\delta(x-x'),$$

$$P_T\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)$$
$$=(1-P_1-P_2)\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)$$
$$=-\frac{1}{2\Box}\exp[i(\theta_2\sigma^n\bar\theta_1-\theta_1\sigma^n\bar\theta_2)\,\partial_n]$$
$$\times[4-\Box(\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2]\,\delta(x-x'), \tag{9.50}$$

we find the propagator for the vector superfield:

$$\Delta = -\frac{1}{2\Box}\exp[i(\theta_2\sigma^n\bar\theta_1 - \theta_1\sigma^n\bar\theta_2)\,\partial_n]$$
$$\times\left\{\frac{1}{-\Box+m^2}[4-\Box\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)]\right.$$
$$\left.-\frac{1}{-\xi\Box+m^2}[4+\Box\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)]\right\}\delta(x_1-x^2). \tag{9.51}$$

References

K. Fujikawa and W. Lang, *Nucl. Phys. B88*, 61 (1975).
S. Ferrara and P. Piguet, *Nucl. Phys. B93*, 261 (1975).

Equations

$$\mathscr{L}_0 = \int\left\{\Phi^+\Phi + \frac{1}{2}m\Phi\Phi\,\delta(\bar\theta) + \frac{1}{2}m^*\Phi^+\Phi^+\,\delta(\theta)\right\}d^2\theta\,d^2\bar\theta$$
$$= A^*\Box A + i\,\partial_m\bar\psi\bar\sigma^m\psi + F^*F$$
$$+\, m\left(AF + A^*F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar\psi\bar\psi\right). \tag{9.10}$$

$$\begin{aligned}
\langle 0|\mathrm{T}\{A(x)A^*(x')\}|0\rangle &= i\,\Delta_F(x-x') \\
\langle 0|\mathrm{T}\{A(x)F(x')\}|0\rangle &= \langle 0|\mathrm{T}\{A^*(x)F^*(x')\}|0\rangle \\
&= -im\,\Delta_F(x-x') \\
\langle 0|\mathrm{T}\{F(x)F^*(x')\}|0\rangle &= i\Box\,\Delta_F(x-x') \\
\langle 0|\mathrm{T}\{\psi_\alpha(x)\psi^\beta(x')\}|0\rangle &= i\,\delta_\alpha{}^\beta m\,\Delta_F(x-x') \\
\langle 0|\mathrm{T}\{\bar\psi^{\dot\alpha}(x)\bar\psi_{\dot\beta}(x')\}|0\rangle &= i\delta^{\dot\alpha}{}_{\dot\beta} m\,\Delta_F(x-x') \\
\langle 0|\mathrm{T}\{\psi_\alpha(x)\bar\psi_{\dot\beta}(x')\}|0\rangle &= \sigma_{\alpha\dot\beta}{}^m\,\partial_m\,\Delta_F(x-x') \\
\Delta_F(x) &= \frac{1}{\Box - m^2}.
\end{aligned} \tag{9.11}$$

$$\begin{aligned}
&\langle 0|\mathrm{T}\{\Phi(x,\theta,\bar\theta)\Phi(x',\theta',\bar\theta')\}|0\rangle \\
&\quad = -im\,\delta(\theta-\theta')\exp\left[i(\theta\sigma^m\bar\theta - \theta'\sigma^m\bar\theta')\,\partial_m{}^x\right]\Delta_F(x-x') \\
&\langle 0|\mathrm{T}\{\Phi^+(x,\theta,\bar\theta)\Phi^+(x',\theta',\bar\theta')\}|0\rangle \\
&\quad = +im\,\delta(\bar\theta-\bar\theta')\exp\left[-i(\theta\sigma^m\bar\theta - \theta'\sigma^m\bar\theta')\,\partial_m{}^x\right]\Delta_F(x-x').
\end{aligned} \tag{9.13}$$

$$\begin{aligned}
&\langle 0|\mathrm{T}\{\Phi(x,\theta,\bar\theta)\Phi^+(x',\theta',\bar\theta')\}|0\rangle \\
&\quad = i\exp\left[i(\theta\sigma^m\bar\theta + \theta'\sigma^m\bar\theta' - 2\theta\sigma^m\bar\theta')\,\partial_m{}^x\right]\Delta_F(x-x').
\end{aligned} \tag{9.14}$$

$$\begin{aligned}
\mathscr{L}_0 &= \int \left\{\Phi^+\Phi - \frac{1}{8}m\left(\Phi\frac{DD}{\Box}\Phi + \Phi^+\frac{\bar D\bar D}{\Box}\Phi^+\right)\right\} d^4x\,d^2\theta\,d^2\bar\theta \\
&= \int \frac{1}{2}(\Phi,\Phi^+)\mathscr{M}\begin{pmatrix}\Phi \\ \Phi^+\end{pmatrix} d^4x\,d^2\theta\,d^2\bar\theta, \\
\mathscr{M} &= \begin{bmatrix} -\dfrac{1}{4}\dfrac{m}{\Box}DD & 1 \\ 1 & -\dfrac{1}{4}\dfrac{m}{\Box}\bar D\bar D \end{bmatrix}.
\end{aligned} \tag{9.17}$$

$$\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\Phi(x',\theta',\bar\theta') = -\frac{1}{4}\bar D\bar D\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta')\,\delta(x-x'). \tag{9.21}$$

$$\frac{1}{4}\begin{pmatrix}\bar D^2 & 0 \\ 0 & D^2\end{pmatrix}\mathscr{M}\Delta = -\begin{pmatrix} -\dfrac{1}{4}\bar D^2 & 0 \\ 0 & -\dfrac{1}{4}D^2\end{pmatrix}\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta')\,\delta(x-x'). \tag{9.28}$$

$$P_1 = \frac{1}{16}\frac{D^2\bar{D}^2}{\Box}, \qquad P_2 = \frac{1}{16}\frac{\bar{D}^2 D^2}{\Box}. \tag{9.30}$$

$$P_+ = \frac{D^2}{4\Box^{\frac{1}{2}}} \qquad P_- = \frac{\bar{D}^2}{4\Box^{\frac{1}{2}}} \qquad P_T = -\frac{1}{8\Box} D\bar{D}^2 D. \tag{9.31}$$

$$\Delta = \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \mathscr{M}^{-1}\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta')\,\delta(x-x'). \tag{9.35}$$

$$\Delta = \frac{1}{\Box - m^2}\begin{pmatrix} \dfrac{m}{4}\bar{D}^2 & \dfrac{1}{16}\bar{D}^2 D^2 \\ \dfrac{1}{16}D^2\bar{D}^2 & \dfrac{m}{4}D^2 \end{pmatrix} \delta(x-x')\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta'). \tag{9.40}$$

$$\begin{aligned}\mathscr{L} &= \int \left\{\frac{1}{4} W^\alpha W_\alpha \delta(\bar\theta) + \frac{1}{4}\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\delta(\theta) + m^2 V^2 - \frac{\xi}{8}(\bar{D}^2 V)(D^2 V)\right\} d^4x\, d^2\theta\, d^2\bar\theta \\ &= \int \{V[-\Box P_T + m^2(P_1+P_2+P_T) - \xi(P_1+P_2)\Box]V\}\, d^4x\, d^2\theta\, d^2\bar\theta \\ &= \int V\mathscr{N} V\, d^4x\, d^2\theta\, d^2\bar\theta.\end{aligned} \tag{9.45}$$

$$\mathscr{N}\Delta = \delta(x-x')\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta'). \tag{9.47}$$

$$\begin{aligned}\Delta = &-\frac{1}{2\Box}\exp[i(\theta_2\sigma^n\bar\theta_1 - \theta_1\sigma^n\bar\theta_2)\,\partial_n] \\ &\times\left\{\frac{1}{-\Box+m^2}[4-\Box\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)]\right. \\ &\left.-\frac{1}{-\xi\Box+m^2}[4+\Box\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2)]\right\}\delta(x_1-x_2).\end{aligned} \tag{9.51}$$

Exercises

(1) Use definition (9.1) to show that $\eta' = a\eta$ implies $d\eta' = a^{-1}\,d\eta$.

(2) Check that $\int f(\eta)\,\delta(\eta-\rho)\,d\eta = f(\rho)$.

(3) The bosonic part of Lagrangian (9.10) may be written as

$$(A^+,F)\mathscr{M}\begin{pmatrix} A \\ F^+ \end{pmatrix}, \qquad \text{where } \mathscr{M} = \begin{pmatrix} \Box & m \\ m & 1 \end{pmatrix}.$$

Show that the AA^*, AF, A^*F^*, and FF^* propagators are given by the inverse of this operator.

(4) Compute the $\Phi^+\Phi^+$ and $\Phi\Phi^+$ propagators.

(5) Verify that the kinetic part of the chiral Lagrangian may be written as follows:

$$\int \Phi(x,\theta,\bar\theta)\Phi^+(x,\theta,\bar\theta)\, d^4x\, d^2\theta\, d^2\bar\theta$$
$$= \int \Phi(x,\theta,0)e^{-2i\theta\sigma^m\bar\theta\partial_m}\Phi^+(x,0,\bar\theta)\, d^4x\, d^2\theta\, d^2\bar\theta.$$

(6) Show

$$\int F(x,\theta,\bar\theta)\, d^4x\, d^2\theta\, d^2\bar\theta = \int \left(-\frac{1}{4}\right)\bar D\bar D F(x,\theta,\bar\theta)\, d^4x\, d^2\theta$$
$$= \int \left(-\frac{1}{4}\right) DDF(x,\theta,\bar\theta)\, d^4x\, d^2\bar\theta.$$

(7) Confirm the multiplication table (9.33) and show that these operators have the following matrix representation:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(8) Prove (9.38) by direct multiplication, using (9.32) and (9.33), or by using the P-operator representation given in Exercise 7.

(9) For any matrix X,

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

show that the inverse matrix X^{-1} is given by

$$X^{-1} = \begin{pmatrix} Z & Y \\ U & V \end{pmatrix},$$

where

$$\begin{aligned} Z &= (A - BD^{-1}C)^{-1} \\ U &= -D^{-1}CZ \\ V &= (D - CA^{-1}B)^{-1} \\ Y &= -A^{-1}BV, \end{aligned}$$

provided A^{-1} and D^{-1} exist. Compute the inverse if only B^{-1} and C^{-1} exist.

X. FEYNMAN RULES FOR SUPERGRAPHS

In this lecture we shall derive the Feynman rules for the supersymmetric Φ^3 model,

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, \Phi^+\Phi + \left\{\int d^2\theta \left[\frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3\right] + \text{h.c.}\right\}. \quad \textbf{(10.1)}$$

These rules may be applied to all chiral models and extended to supersymmetric gauge theories as well. We shall find that the effective action may be expressed in terms of one $d^4\theta = d^2\theta\, d^2\bar{\theta}$ integration of the following form:

$$\int d^4\theta \int d^4x_1 \cdots d^4x_n F_1(x_1,\theta,\bar{\theta}) \cdots F_n(x_n,\theta,\bar{\theta}) G(x_1, \ldots, x_n). \quad (10.2)$$

The function $G(x_1, \ldots, x_n)$ is translationally invariant and the F's are products of superfields and their derivatives. No factors of $\Box^{-1}$ appear in the F's, so for chiral operators the $d^4\theta$ integration cannot be converted into a $d^2\theta$ integration without introducing spacetime derivatives (see Exercise 2). This leads to the surprising result that mass and coupling terms of the $d^2\theta$ form are not renormalized in supersymmetric theories. Furthermore, no higher-dimensional momentum-independent chiral operators are induced in the effective superpotential to any order in perturbation theory. Equation (10.2) also implies that all vacuum-to-vacuum diagrams vanish. This is because expressions of the type (10.2) without any superfields are immediately annihilated by the $d^4\theta$ integration.

Before deriving the Feynman rules we will give a short derivation of the generating functional

$$\begin{aligned} Z[J,J^+] &= \left\langle 0 \middle| \mathrm{T} \exp i \int d^4\theta\, d^4x \left[J(z)\left(-\frac{1}{4}\frac{D^2}{\Box}\right)\Phi(z) \right.\right. \\ &\qquad \left.\left. + J^+(z)\left(-\frac{1}{4}\frac{\bar{D}^2}{\Box}\right)\Phi^+(z)\right] \middle| 0 \right\rangle \\ &= e^{i\int d^4x \mathscr{L}_{\text{int}}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta J^+}\right)} Z_0[J,J^+] \end{aligned} \quad \textbf{(10.3)}$$

for superfield Green's functions

$$G^{(N)}(z^1, \ldots, z^M; z^{M+1}, \ldots, z^N)$$
$$= (-i)^N \frac{\delta}{\delta J(z^1)} \cdots \frac{\delta}{\delta J(z^M)} \frac{\delta}{\delta J^+(z^{M+1})} \cdots \frac{\delta}{\delta J^+(z^N)} Z[J,J^+]\bigg|_{J=J^+=0}. \tag{10.4}$$

Here $Z_0[J,J^+]$ is the generating functional for free superfield Green's functions and $z^i = (x^i,\theta^i,\bar{\theta}^i)$ is an element of superspace. Equation (10.3) may be verified explicitly in terms of component fields. We shall take another tack and derive it directly with superfields and superspace techniques.

In the previous lecture we calculated the free-field two-point functions:

$$\begin{aligned}
\langle 0|T\Phi_0(z)\Phi_0(z')|0\rangle &= \frac{i}{\Box - m^2}\,\Delta_{11}(z,z')\,\delta(x - x') \\
\langle 0|T\Phi_0^+(z)\Phi_0^+(z')|0\rangle &= \frac{i}{\Box - m^2}\,\Delta_{22}(z,z')\,\delta(x - x') \\
\langle 0|T\Phi_0(z)\Phi_0^+(z')|0\rangle &= \frac{i}{\Box - m^2}\,\Delta_{12}(z,z')\,\delta(x - x') \\
\langle 0|T\Phi_0^+(z)\Phi_0(z')|0\rangle &= \frac{i}{\Box - m^2}\,\Delta_{21}(z,z')\,\delta(x - x').
\end{aligned} \tag{10.5}$$

The right-hand side of the equation includes the matrix elements Δ_{ij} of Eq. (9.43). These two-point functions may all be obtained from the free generating functional

$$\begin{aligned}
Z_0[J,J^+] &= \left\langle 0\middle|T \exp i \int d^4\theta\, d^4x\left[J(z)\left(-\frac{1}{4}\frac{D^2}{\Box}\right)\Phi_0(z) + J^+(z)\left(-\frac{1}{4}\frac{\bar{D}^2}{\Box}\right)\Phi_0^+(z)\right]\middle|0\right\rangle \\
&= \exp - \frac{1}{2}\int d^4\theta\, d^4x\, d^4\theta'\, d^4x'\, \frac{i}{\Box - m^2} \\
&\quad \times \left\{(J(z),J^+(z))\begin{pmatrix} -\frac{1}{4}\frac{D^2}{\Box} & 0 \\ 0 & -\frac{1}{4}\frac{\bar{D}^2}{\Box}\end{pmatrix}\begin{pmatrix}\Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22}\end{pmatrix}\delta(x - x')\right. \\
&\quad \times \left.\begin{pmatrix} -\frac{1}{4}\frac{D'^2}{\Box'} & 0 \\ 0 & -\frac{1}{4}\frac{\bar{D}'^2}{\Box'}\end{pmatrix}\begin{pmatrix} J(z') \\ J^+(z')\end{pmatrix}\right\}.
\end{aligned} \tag{10.6}$$

This functional generates all free-field Green's functions as sums of products of two-point functions Δ. With the help of (9.40), (9.18), and a few integrations by parts, we find

$$Z_0[J,J^+] = \exp -\frac{i}{2}\int d^4x\, d^4\theta\, d^4x'\, d^4\theta'\, (J(z), J^+(z))\, \Delta_{\mathrm{GRS}}(z,z') \begin{pmatrix} J(z') \\ J^+(z') \end{pmatrix}, \tag{10.7}$$

where Δ_{GRS} is the propagator introduced by Grisaru, Roček, and Siegel:

$$\Delta_{\mathrm{GRS}}(z,z') = \frac{1}{\Box - m^2} \begin{pmatrix} \dfrac{m}{4}\dfrac{D^2}{\Box} & 1 \\ 1 & \dfrac{m}{4}\dfrac{\bar{D}^2}{\Box} \end{pmatrix} \delta(z - z'). \tag{10.8}$$

We may differentiate Z_0 with respect to J and J^+ using the rule (9.21):

$$\frac{1}{i}\begin{pmatrix} \dfrac{\delta}{\delta J(z)} \\ \dfrac{\delta}{\delta J^+(z)} \end{pmatrix} Z_0 = -\int d^4x'\, d^4\theta'\, \frac{1}{\Box - m^2} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \delta(x - x')$$
$$\times \left(-\frac{1}{4\Box}\right) \begin{pmatrix} D^2 J(z) \\ \bar{D}^2 J^+(z) \end{pmatrix} Z_0. \tag{10.9}$$

From this we find a functional equation for Z_0

$$\frac{1}{4}\begin{pmatrix} \bar{D}^2 & 0 \\ 0 & D^2 \end{pmatrix} \mathscr{M} \frac{1}{i} \begin{pmatrix} \dfrac{\delta}{\delta J(z)} \\ \dfrac{\delta}{\delta J^+(z)} \end{pmatrix} Z_0 = \begin{pmatrix} J(z) \\ J^+(z) \end{pmatrix} Z_0. \tag{10.10}$$

Here we have used (9.28) and (9.18).

We may generalize this equation to the case of interacting fields. For the Φ^3 model, the field equations

$$\frac{1}{4}\begin{pmatrix} \bar{D}^2 & 0 \\ 0 & D^2 \end{pmatrix} \mathscr{M} \begin{pmatrix} \Phi \\ \Phi^+ \end{pmatrix} - g \begin{pmatrix} \Phi^2 \\ \Phi^{+2} \end{pmatrix} = \begin{pmatrix} J \\ J^+ \end{pmatrix} \tag{10.11}$$

lead to the following equation for Z:

$$\frac{1}{4}\begin{pmatrix}\bar{D}^2 & 0\\ 0 & D^2\end{pmatrix}\mathscr{M}\frac{1}{i}\begin{pmatrix}\dfrac{\delta}{\delta J(z)}\\[2ex] \dfrac{\delta}{\delta J^+(z)}\end{pmatrix}Z = \left\{\begin{pmatrix}J(z)\\ J^+(z)\end{pmatrix} + g\begin{pmatrix}\left(P_2\dfrac{1}{i}\dfrac{\delta}{\delta J(z)}\right)^2\\[2ex] \left(P_1\dfrac{1}{i}\dfrac{\delta}{\delta J^+(z)}\right)^2\end{pmatrix}\right\}Z. \tag{10.12}$$

Note that we have introduced projection operators in (10.12). We could have done this in (10.11), but there it is obvious that Φ is chiral

$$P_2\Phi = \Phi, \qquad P_1\Phi^+ = \Phi^+. \tag{10.13}$$

The chirality of the functional derivative is less explicit, so we choose to keep P_1 and P_2 in (10.12).

To solve for Z, we first compute the commutator

$$\begin{aligned}\left[\left(P_2\frac{\delta}{\delta J(z)}\right)^3, J(z')\right] &= 3\left(-\frac{1}{4}\right)\left(P_2\frac{\delta}{\delta J(z)}\right)^2\bar{D}^2\,\delta(z-z')\\ &= 3\left(-\frac{1}{4}\right)\bar{D}^2\left\{\left(P_2\frac{\delta}{\delta J(z)}\right)^2\delta(z-z')\right\}.\end{aligned} \tag{10.14}$$

The last step is possible because $\bar{D}P_2 = 0$. Integrating over $d^2\theta\,d^4x$,

$$\begin{aligned}\left[\int d^2\theta\,d^4x\left(P_2\frac{\delta}{\delta J(z)}\right)^3, J(z')\right] &= 3\int d^4\theta\,d^4x\,\delta(z-z')\left(P_2\frac{\delta}{\delta J(z)}\right)^2\\ &= 3\left(P_2\frac{\delta}{\delta J(z')}\right)^2,\end{aligned} \tag{10.15}$$

and using

$$\begin{aligned}&e^{i\frac{g}{3}\int d^4x'\,d^2\theta'\left(P_2\frac{1}{i}\frac{\delta}{\delta J(z')}\right)^3}J(z)e^{-i\frac{g}{3}\int d^4x'\,d^2\theta'\left(P_2\frac{1}{i}\frac{\delta}{\delta J(z')}\right)^3}\\ &\qquad = J(z) + g\left(P_2\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^2,\end{aligned} \tag{10.16}$$

we cast Eq. (10.12) in the following form:

$$\frac{1}{4}\begin{pmatrix} \bar{D}^2 & 0 \\ 0 & D^2 \end{pmatrix} \mathcal{M} \frac{1}{i} \begin{pmatrix} \dfrac{\delta}{\delta J(z)} \\ \dfrac{\delta}{\delta J^+(z)} \end{pmatrix} Z = e^{i\int d^4x' \mathcal{L}_{int}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta J^+}\right)} \begin{pmatrix} J(z) \\ J^+(z) \end{pmatrix}$$
$$\times\, e^{-i\int d^4x' \mathcal{L}_{int}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta J^+}\right)} Z. \tag{10.17}$$

This shows that

$$Z_0[J,J^+] = e^{-i\int d^4x' \mathcal{L}_{int}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta J^+}\right)} Z[J,J^+], \tag{10.18}$$

since the right-hand side satisfies the free equation (10.10). No normalization factor is needed because of the fact that all vacuum-to-vacuum diagrams vanish. With (10.18) we have proven (10.3) and solved for the generating functional of an interacting chiral supersymmetric theory.

Having found the generating functional, we shall now derive the Feynman rules. We begin by recalling the relation between the Green's functions and the generating functional:

$$\begin{aligned} G^{(N)}(z^1, \ldots, z^M; z^{M+1}, \ldots, z^N) \\ = (-i)^N \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_M} \frac{\delta}{\delta J^+{}_{M+1}} \cdots \frac{\delta}{\delta J^+{}_N} \sum_{K=0}^{\infty} \frac{(i)^K}{K!} \\ \times \int d^4x^K \mathcal{L}_{int}\left(\frac{\delta}{J_K}, \frac{\delta}{J_K{}^+}\right) Z_0[J,J^+]\Big|_{J=J^+=0}. \end{aligned} \tag{10.19}$$

The factors of

$$\int d^4x^K \mathcal{L}_{int}\left(\frac{\delta}{J_K}, \frac{\delta}{J_K{}^+}\right)$$

generate vertices at z^K. The derivatives in $\mathcal{L}_{int}$ act on previous derivatives and on Z_0 itself. Each derivative acting on Z_0 creates a new propagator at z^K. Each derivative acting on a previous derivative connects an existing propagator to z^K. In this way every new vertex is completely saturated with propagators.

As an explicit example, we consider a term in Φ^3 theory in which two new propagators are created at the point z:

$$
\begin{aligned}
& i\int d^4x\, d^2\theta \left\{\frac{1}{i} P_2 \frac{\delta}{\delta J(z)}\right\}^3 Z_0 \\
&= i\int d^4x\, d^2\theta \left\{\frac{1}{i} P_2 \frac{\delta}{\delta J(z)}\right\}\left[\left(-\frac{1}{4}\bar{D}^2\right)\int d^4x'\, d^4\theta' \frac{-i}{\Box - m^2}\right. \\
&\quad \times \left.\left(\frac{m}{4}\frac{D^2}{\Box}\delta(z-z')J(z') + \delta(z-z')J^+(z')\right)\right] \\
&\quad \times \left[\left(-\frac{1}{4}\bar{D}^2\right)\int d^4x''\, d^4\theta'' \frac{-i}{\Box - m^2}\right. \\
&\quad \times \left.\left(\frac{m}{4}\frac{D^2}{\Box}\delta(z-z'')J(z'') + \delta(z-z'')J^+(z'')\right)\right] Z_0 \\
&= i\int d^4x\, d^4\theta \left\{\frac{1}{i} P_2 \frac{\delta}{\delta J(z)}\right\} \\
&\quad \times \left[\int d^4x'\, d^4\theta' \frac{-i}{\Box - m^2}\left(\frac{m}{4}\frac{D^2}{\Box}\delta(z-z')J(z')\right.\right. \\
&\quad + \left.\left.\delta(z-z')J^+(z')\right)\right]\left[\left(-\frac{1}{4}\bar{D}^2\right)\int d^4x''\, d^4\theta'' \frac{-i}{\Box - m^2}\right. \\
&\quad \times \left.\left(\frac{m}{4}\frac{D^2}{\Box}\delta(z-z'')J(z'') + \delta(z-z'')J^+(z'')\right)\right] Z_0. \qquad (10.20)
\end{aligned}
$$

Here we have used (10.7) and (10.8) for Z_0. The last step (changing the $d^2\theta$ to a $d^4\theta$) was possible because of the chirality property of each factor. Note that we also used the fact that

$$\frac{\delta}{\delta J(z)}\int d^4x'\, d^4\theta' \frac{i}{\Box - m^2}\frac{m}{16}\frac{\bar{D}^2 D^2}{\Box}\delta(z-z')J(z') = 0. \qquad (10.21)$$

Such a piece corresponds to a closed Φ tadpole in a Feynman diagram. The proof that (10.21) indeed vanishes is left to the reader as Exercise 7.

The effective action is computed from the one particle irreducible (1PI) Green's functions. In general, 1PI diagrams have at least two internal lines leaving every vertex. The external legs of the 1PI diagrams are amputated with inverse propagators. They are then multiplied by the superfield amplitudes $\Phi(z)$ or $\Phi^+(z)$. This leads to the following Feynman rules:

(1) For each external line, write a chiral superfield $\Phi(z)$, $\Phi^+(z)$.

(2) At each Φ^3 vertex with two [three] internal lines, include factors of $-\frac{1}{4}\bar{D}^2$ acting on one [two] internal propagators. At each Φ^{+3} vertex, include similar factors of $-\frac{1}{4}D^2$.

(3) Write a factor of $\frac{1}{3}g$ for each vertex, and integrate $\int d^4x\, d^4\theta$ over each vertex.

(4) Use Grisaru-Roček-Siegel propagators for $\Phi\Phi$, $\Phi^+\Phi^+$, and $\Phi^+\Phi$ internal lines. These are given in (10.8).

(5) Compute the usual combinatoric factors for an A^3 theory. This is most easily done directly from (10.3) and (10.4).

Let us now use these rules to follow the θ-integrations around an arbitrary closed loop. The Feynman rules and the GRS propagators combine to give an expression of the following form:

$$(D_1{}^2)^{\ell_1}(\bar{D}_1{}^2)^{k_1}\,\delta(12)(D_2{}^2)^{\ell_2}(\bar{D}_2{}^2)^{k_2}\,\delta(23)\cdots(D_n{}^2)^{\ell_n}(\bar{D}_n{}^2)^{k_n}\,\delta(n1). \qquad (10.22)$$

The exponents ℓ_i,k_i are either zero or one, and $\delta(12)=\delta(\theta_1-\theta_2)\,\delta(\bar{\theta}_1-\bar{\theta}_2)$. For a general loop, the D and $\bar{D}$ factors might appear in the opposite order. However, any higher powers of D^2 and $\bar{D}^2$ may be reduced to the above form, up to powers of $\Box$:

$$\begin{aligned}\bar{D}^2D^2\bar{D}^2 &= 16\Box\bar{D}^2\\ D^2\bar{D}^2D^2 &= 16\Box D^2.\end{aligned} \qquad (10.23)$$

Of course, for the effective action, the above expression is multiplied by superfields for external legs and GRS propagators for adjoining closed loops. It is also integrated over $d^4x_1\,d^4\theta_1\cdots d^4x_n\,d^4\theta_n$. The final expression is evaluated by removing the D and $\bar{D}$ derivatives from one δ-function after another by partial integration. This introduces new derivatives on the lines that leave the loop. It also introduces a certain number of derivatives on the last δ-function, say $\delta(n1)$. All but one of the θ-integrations may be performed with the aid of the δ-functions $\delta(12),\ldots,$ $\delta([n-1]n)$. This leaves a factor of

$$\int d^4\theta_n(D^2)^{\ell}(\bar{D}^2)^k\,\delta(\theta_n-\theta_1)\,\delta(\bar{\theta}_n-\bar{\theta}_1)\Big|_{\substack{\theta_1=\theta_n\\ \bar{\theta}_1=\bar{\theta}_n}} \qquad (10.24)$$

or

$$\int d^4\theta_n(\bar{D}^2)^k(D^2)^{\ell}\,\delta(\theta_n-\theta_1)\,\delta(\bar{\theta}_n-\bar{\theta}_1)\Big|_{\substack{\theta_1=\theta_n\\ \bar{\theta}_1=\bar{\theta}_n}}. \qquad (10.25)$$

These expressions vanish unless $k = \ell = 1$. In this case, we find

$$\begin{aligned}\int d^4\theta_n D^2\bar{D}^2\,\delta(\theta_n - \theta_1)\,\delta(\bar{\theta}_n - \bar{\theta}_1)\Big|_{\substack{\theta_1=\theta_n\\ \bar{\theta}_1=\bar{\theta}_n}} &= \int d^4\theta_n \bar{D}^2 D^2\,\delta(\theta_n - \theta_1)\,\delta(\bar{\theta}_n - \bar{\theta}_1)\Big|_{\substack{\theta_1=\theta_n\\ \bar{\theta}_1=\bar{\theta}_n}}\\ &= 16\int d^4\theta_n, \end{aligned}\tag{10.26}$$

as follows from (9.41). The whole loop in θ-space has shrunk to one $d^4\theta$ integration. This process can now be carried over to the next loop and we finally arrive at the result (10.2). Note that (10.2) is true for each diagram, and as a consequence, for any particular sum of diagrams as well.

References

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B. A. Ovrut and J. Wess, *Phys. Rev. D25*, 409 (1982).

Equations

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\,\Phi^+\Phi + \left\{\int d^2\theta\left[\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right] + \text{h.c.}\right\}. \tag{10.1}$$

$$\begin{aligned}Z[J,J^+] &= \left\langle 0\middle| \mathrm{T}\exp i\int d^4\theta\, d^4x\left[J(z)\left(-\frac{1}{4}\frac{D^2}{\Box}\right)\Phi(z)\right.\right.\\ &\qquad \left.\left. + J^+(z)\left(-\frac{1}{4}\frac{\bar{D}^2}{\Box}\right)\Phi^+(z)\right]\middle| 0\right\rangle\\ &= e^{i\int d^4x\,\mathscr{L}_{\text{int}}\left(\frac{\delta}{\delta J},\frac{\delta}{\delta J^+}\right)}Z_0[J,J^+]. \end{aligned}\tag{10.3}$$

$$\begin{aligned}&G^{(N)}(z^1,\ldots,z^M;z^{M+1},\ldots,z^N)\\ &\quad = (-i)^N\frac{\delta}{\delta J(z^1)}\cdots\frac{\delta}{\delta J(z^M)}\frac{\delta}{\delta J^+(z^{M+1})}\cdots\frac{\delta}{\delta J^+(z^N)}Z[J,J^+]\Big|_{J=J^+=0}\end{aligned}\tag{10.4}$$

$$Z_0[J,J^+] = \exp -\frac{i}{2}\int d^4x\, d^4\theta\, d^4x'\, d^4\theta'(J(z),J^+(z))\,\Delta_{\mathrm{GRS}}(z,z')\begin{pmatrix}J(z')\\ J^+(z')\end{pmatrix}, \tag{10.7}$$

$$\Delta_{\mathbf{GRS}}(z,z') = \frac{1}{\Box - m^2}\begin{pmatrix} \frac{m}{4}\frac{D^2}{\Box} & 1 \\ 1 & \frac{m}{4}\frac{\bar{D}^2}{\Box} \end{pmatrix} \delta(z - z'). \qquad \textbf{(10.8)}$$

Exercises

(1) Show that (10.2) is supersymmetric.

(2) Use (9.18) to show that

$$\int d^4x\, d^4\theta\, \Phi\left(-\frac{1}{4}D^2\right) J = \int d^4x\, d^2\theta\, \Phi \Box J$$

for Φ and J chiral.

(3) Demonstrate that the generating functional (10.6) gives the two-point functions (10.5).

(4) Compute $\left[\left(P_1 \frac{\delta}{\delta J^+}\right)^n, J^+\right]$.

(5) Verify (10.16).

(6) Compute $\frac{\delta}{\delta J(z)} Z_0$. Use this to check (10.20).

(7) Prove that (10.21) does indeed vanish.

(8) Use the Feynman rules to calculate the diagrams of Figure 9.1. Show that (a) and (b) vanish and that (c) leads to a wave function renormalization.

XI. NONLINEAR REALIZATIONS

In this chapter we shall study a nonlinear realization of the supersymmetry algebra. This will introduce us to superspace differentials and provide a natural transition to differential forms. It will also demonstrate that supersymmetry may be realized entirely in terms of fermion fields. In fact, we shall construct a local supersymmetric Lagrangian from a single fermion field. We shall see that this nonlinear Lagrangian is highly non-renormalizable. It does not, therefore, change the pattern of Bose-Fermi symmetry in renormalizable supersymmetric field theories.

Nonlinear transformations for fermion fields are reminiscent of the nonlinear transformations for the Goldstone spinors in Chapter VIII. We shall see that the nonlinear Lagrangian gives rise to spontaneous supersymmetry breaking and that the fermion field is indeed a Goldstone spinor. The nonlinear Lagrangian is quite useful for studying the supersymmetric Higgs effect in supergravity theory. The supersymmetric Higgs effect occurs when the spin-$\frac{1}{2}$ Goldstone fermion combines with the spin-$\frac{3}{2}$ partner of the gravitational field to form one massive spin-$\frac{3}{2}$ field.

To derive the nonlinear transformation law, we first consider the supersymmetry transformation (4.3):

$$\begin{aligned} x' &= x + i(\theta\sigma\bar{\xi} - \xi\sigma\bar{\theta}) \\ \theta' &= \theta + \xi \\ \bar{\theta}' &= \bar{\theta} + \bar{\xi}. \end{aligned} \tag{11.1}$$

This transformation induces a nonlinear realization on the spinors θ and $\bar{\theta}$. We shall generalize this transformation to arbitrary spinor fields $\lambda(x)$ by drawing an analogy between θ and λ, $\theta = \kappa\lambda$:

$$\begin{aligned} \lambda'(x') &= \lambda(x) + \frac{1}{\kappa}\,\xi \\ \bar{\lambda}'(x') &= \bar{\lambda}(x) + \frac{1}{\kappa}\,\bar{\xi}. \end{aligned} \tag{11.2}$$

From (11.2) we may compute the changes of the fields at the same space-time point:

$$\begin{aligned} \delta_\xi \lambda^\alpha &= \lambda'^\alpha(x) - \lambda^\alpha(x) = \frac{1}{\kappa}\xi^\alpha - i\kappa(\lambda\sigma^m\bar{\xi} - \xi\sigma^m\bar{\lambda})\,\partial_m\lambda^\alpha \\ \delta_\xi \bar{\lambda}_{\dot{\alpha}} &= \bar{\lambda}'_{\dot{\alpha}}(x) - \bar{\lambda}_{\dot{\alpha}}(x) = \frac{1}{\kappa}\bar{\xi}_{\dot{\alpha}} - i\kappa(\lambda\sigma^m\bar{\xi} - \xi\sigma^m\bar{\lambda})\,\partial_m\bar{\lambda}_{\dot{\alpha}}. \end{aligned} \tag{11.3}$$

After some algebra, which we leave to the reader as Exercise 1, we find

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\lambda^\alpha = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\,\partial_m\lambda^\alpha. \tag{11.4}$$

This verifies that (11.3) does indeed realize the supersymmetry algebra (I).

Before constructing an invariant Lagrangian, we first examine the differentials dx, $d\theta$, and $d\bar{\theta}$. These transform as follows under general coordinate transformations in superspace:

$$\begin{aligned} x'^m &= x'^m(x, \theta, \bar{\theta}) \\ \theta'^\mu &= \theta'^\mu(x, \theta, \bar{\theta}) \\ \bar{\theta}'_{\dot{\mu}} &= \bar{\theta}'_{\dot{\mu}}(x, \theta, \bar{\theta}) \end{aligned}$$

$$\begin{aligned} dx'^m &= dx^n \frac{\partial x'^m}{\partial x^n} + d\theta^\nu \frac{\partial x'^m}{\partial \theta^\nu} + d\bar{\theta}_{\dot{\nu}} \frac{\partial x'^m}{\partial \bar{\theta}_{\dot{\nu}}} \\ d\theta'^\mu &= dx^n \frac{\partial \theta'^\mu}{\partial x^n} + d\theta^\nu \frac{\partial \theta'^\mu}{\partial \theta^\nu} + d\bar{\theta}_{\dot{\nu}} \frac{\partial \theta'^\mu}{\partial \bar{\theta}_{\dot{\nu}}} \\ d\bar{\theta}'_{\dot{\mu}} &= dx^n \frac{\partial \bar{\theta}'_{\dot{\mu}}}{\partial x^n} + d\theta^\nu \frac{\partial \bar{\theta}'_{\dot{\mu}}}{\partial \theta^\nu} + d\bar{\theta}_{\dot{\nu}} \frac{\partial \bar{\theta}'_{\dot{\mu}}}{\partial \bar{\theta}_{\dot{\nu}}}. \end{aligned} \tag{11.5}$$

Here one should note the summation convention for the spinor indices and the placement of the differentials to the left of their coefficients. For (11.1), this becomes

$$\begin{aligned} dx'^m &= dx^m + i\,d\theta\sigma^m\bar{\xi} - i\xi\sigma^m\,d\bar{\theta} \\ d\theta'^\mu &= d\theta^\mu \\ d\bar{\theta}'_{\dot{\mu}} &= d\bar{\theta}_{\dot{\mu}}. \end{aligned} \tag{11.6}$$

It is easy to find a combination of differentials

$$\begin{aligned} e^a &= dx^a - i\,d\theta\sigma^a\bar{\theta} + i\theta\sigma^a d\bar{\theta} \\ e^\alpha &= d\theta^\alpha \\ e_{\dot{\alpha}} &= d\bar{\theta}_{\dot{\alpha}} \end{aligned} \tag{11.7}$$

which is invariant under (11.1) and (11.6):

$$\begin{aligned} e'^a &= dx'^a - i\,d\theta'\sigma^a\bar{\theta}' + i\theta'\sigma^a d\bar{\theta}' \\ &= dx^a + i\,d\theta\sigma^a\bar{\xi} - i\xi\sigma^a d\bar{\theta} - i\,d\theta\sigma^a\bar{\theta} - i\,d\theta\sigma^a\bar{\xi} + i\theta\sigma^a d\bar{\theta} + i\xi\sigma^a d\bar{\theta} \\ &= e^a. \end{aligned} \tag{11.8}$$

Substituting $\theta = \kappa\lambda$ and $d\theta = \kappa(\partial\lambda/\partial x^m)\,dx^m$ into (11.7), we find

$$e^a \rightarrow dx^m[\delta_m{}^a - i\kappa^2\,\partial_m\lambda\sigma^a\bar{\lambda} + i\kappa^2\lambda\sigma^a\,\partial_m\bar{\lambda}] = dx^m A_m{}^a. \tag{11.9}$$

A short calculation shows

$$\delta_\xi A_m{}^a = i\kappa(\xi\sigma^n\,\partial_m\bar{\lambda} - \partial_m\lambda\sigma^n\bar{\xi})A_n{}^a - i\kappa(\lambda\sigma^n\bar{\xi} - \xi\sigma^n\bar{\lambda})\,\partial_n A_m{}^a. \tag{11.10}$$

With this transformation, the Lagrangian

$$\mathscr{L} = -\frac{1}{2\kappa^2}\det A \tag{11.11}$$

yields an invariant action

$$\begin{aligned} \delta_\xi \det A &= \det A \operatorname{Tr} \delta_\xi A A^{-1} \\ &= -i\kappa\,\partial_m[(\lambda\sigma^m\bar{\xi} - \xi\sigma^m\bar{\lambda})\det A]. \end{aligned} \tag{11.12}$$

From (11.9), we see that $\mathscr{L}$ describes one massless spinor λ:

$$\mathscr{L} = -\frac{1}{2\kappa^2} - \frac{i}{2}(\lambda\sigma^m\,\partial_m\bar{\lambda} - \partial_m\lambda\sigma^m\bar{\lambda}) + [\text{interaction terms}]. \tag{11.13}$$

The constant κ spontaneously breaks the supersymmetry. It also leads to a non-vanishing vacuum expectation value for the Lagrangian. This gives rise to a cosmological constant when (11.11) interacts with a gravitational field.

References

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S. Deser and B. Zumino, *Phys. Rev. Lett. 38*, 1433 (1977).

Exercises

(1) Use (11.3) to compute

$$\begin{aligned}\delta_\eta \delta_\xi \lambda = & -i(\eta\sigma^m\overline{\xi} - \xi\sigma^m\overline{\eta})\,\partial_m\lambda \\ & - \kappa^2\{(\lambda\sigma^m\overline{\xi} - \xi\sigma^m\overline{\lambda})(\lambda\sigma^n\overline{\eta} - \eta\sigma^n\overline{\lambda})\,\partial_m\partial_n\lambda \\ & + (\lambda\sigma^m\overline{\xi} - \xi\sigma^m\overline{\lambda})(\partial_m\lambda\sigma^n\overline{\eta} - \eta\sigma^n\,\partial_m\overline{\lambda})\,\partial_n\lambda \\ & + (\lambda\sigma^n\overline{\eta} - \eta\sigma^n\overline{\lambda})(\partial_n\lambda\sigma^m\overline{\xi} - \xi\sigma^m\,\partial_n\overline{\lambda})\,\partial_m\lambda\}.\end{aligned}$$

Note that the terms in (11.3) quadratic in λ come from a shift in the argument x. Verify the closure relation (11.4).

(2) General transformations in superspace

$$\begin{aligned} x^m &= x^m(x', \theta', \overline{\theta}') \\ \theta^\mu &= \theta^\mu(x', \theta', \overline{\theta}') \\ \overline{\theta}_{\dot\mu} &= \overline{\theta}_{\dot\mu}(x', \theta', \overline{\theta}') \end{aligned}$$

induce the following transformations on the partial derivatives:

$$\begin{aligned} \frac{\partial}{\partial x'^m} &= \frac{\partial x^n}{\partial x'^m}\frac{\partial}{\partial x^n} + \frac{\partial \theta^\nu}{\partial x'^m}\frac{\partial}{\partial \theta^\nu} + \frac{\partial \overline{\theta}_{\dot\nu}}{\partial x'^m}\frac{\partial}{\partial \overline{\theta}_{\dot\nu}} \\ \frac{\partial}{\partial \theta'^\mu} &= \frac{\partial x^n}{\partial \theta'^\mu}\frac{\partial}{\partial x^n} + \frac{\partial \theta^\nu}{\partial \theta'^\mu}\frac{\partial}{\partial \theta^\nu} + \frac{\partial \overline{\theta}_{\dot\nu}}{\partial \theta'^\mu}\frac{\partial}{\partial \overline{\theta}_{\dot\nu}} \\ \frac{\partial}{\partial \overline{\theta}'_{\dot\mu}} &= \frac{\partial x^n}{\partial \overline{\theta}'_{\dot\mu}}\frac{\partial}{\partial x^n} + \frac{\partial \theta^\nu}{\partial \overline{\theta}'_{\dot\mu}}\frac{\partial}{\partial \theta^\nu} + \frac{\partial \overline{\theta}_{\dot\nu}}{\partial \overline{\theta}'_{\dot\mu}}\frac{\partial}{\partial \overline{\theta}_{\dot\nu}}. \end{aligned}$$

Show that this, together with (11.5), gives

$$dx^m\frac{\partial}{\partial x^m} + d\theta^\mu\frac{\partial}{\partial \theta^\mu} + d\overline{\theta}_{\dot\mu}\frac{\partial}{\partial \overline{\theta}_{\dot\mu}} = dx'^m\frac{\partial}{\partial x'^m} + d\theta'^\mu\frac{\partial}{\partial \theta'^\mu} + d\overline{\theta}'_{\dot\mu}\frac{\partial}{\partial \overline{\theta}'_{\dot\mu}}.$$

(3) Write $e^a, e^\alpha, e_{\dot\alpha}$ of Eq. (11.7) in the following form:

$$\begin{aligned} e^a &= dx^m e_m{}^a + d\theta^\mu e_\mu{}^a + d\bar\theta_{\dot\mu} e^{\dot\mu a} \\ e^\alpha &= dx^m e_m{}^\alpha + d\theta^\mu e_\mu{}^\alpha + d\bar\theta_{\dot\mu} e^{\dot\mu\alpha} \\ e_{\dot\alpha} &= dx^m e_{m\dot\alpha} + d\theta^\mu e_{\mu\dot\alpha} + d\bar\theta_{\dot\mu} e^{\dot\mu}{}_{\dot\alpha}. \end{aligned}$$

Show that

$$\begin{aligned} e_m{}^a &= \delta_m{}^a, & e_\mu{}^a &= -i\sigma_{\mu\dot\mu}{}^a\bar\theta^{\dot\mu}, & e^{\dot\mu a} &= -i\bar\sigma^{a\dot\mu\mu}\theta_\mu \\ e_m{}^\alpha &= 0, & e_\mu{}^\alpha &= \delta_\mu{}^\alpha, & e^{\dot\mu\alpha} &= 0 \\ e_{m\dot\alpha} &= 0, & e_{\mu\dot\alpha} &= 0, & e^{\dot\mu}{}_{\dot\alpha} &= \delta^{\dot\mu}{}_{\dot\alpha}. \end{aligned}$$

(4) Compute

$$\begin{aligned} \delta_\xi A_b{}^a = {} & i\kappa(\xi\sigma^a\,\partial_b\bar\lambda - \partial_b\lambda\sigma^a\bar\xi) \\ & + \kappa^3(\lambda\sigma^m\bar\xi - \xi\sigma^m\bar\lambda)\,\partial_m(\lambda\sigma^a\,\partial_b\bar\lambda - \partial_b\lambda\sigma^a\bar\lambda) \\ & + \kappa^3(\partial_b\lambda\sigma^m\bar\xi - \xi\sigma^m\,\partial_b\bar\lambda)(\lambda\sigma^a\,\partial_m\bar\lambda - \partial_m\lambda\sigma^a\bar\lambda). \end{aligned}$$

Use this to prove (11.10).

(5) Use $\partial/\partial x^m \det A = \det A \operatorname{Tr} \partial_m A A^{-1}$ to verify (11.12).

XII. DIFFERENTIAL FORMS IN SUPERSPACE

Supergravity theories have been successfully formulated in terms of differential forms in superspace. This is not surprising, for supersymmetry transformations are among the general coordinate transformations of superspace. It is natural, therefore, to introduce supergravity in a way which is manifestly covariant under such coordinate transformations. This leads us to extend the concept of differential forms to superspace.

The elements of superspace are denoted by

$$z^M \sim (x^m, \theta^\mu, \overline{\theta}_{\dot{\mu}}). \tag{12.1}$$

The capital letter M represents the four-vector index m as well as the spinor indices μ and $\dot{\mu}$. M, m, and μ are all upper indices, while $\dot{\mu}$ is a lower index. Elements of superspace obey the following multiplication law:

$$z^M z^N = (-)^{nm} z^N z^M. \tag{12.2}$$

Here n is a function of N and m is a function of M. These functions take the values zero or one, depending on whether N and M are vector or spinor indices.

Exterior products in superspace are defined in complete analogy to ordinary space:

$$\begin{aligned} dz^M \wedge dz^N &= -(-)^{nm} dz^N \wedge dz^M \\ dz^M z^N &= (-)^{nm} z^N dz^M. \end{aligned} \tag{12.3}$$

With this definition, differential forms have an obvious extension to superspace:

$$\Omega = dz^{M_1} \wedge \cdots \wedge dz^{M_p} W_{M_p \cdots M_1}(z). \tag{12.4}$$

The differentials are written to the left of the coefficient function and the indices are labeled in such a way that there is always an even number of indices between those being summed. From now on we shall drop the symbol $\wedge$ for exterior multiplication. This is not ambiguous because we know no other way to multiply forms.

Functions of the superspace variable z are called zero-forms:

$$F(z). \tag{12.5}$$

One-forms are written as

$$\Lambda = dz^M W_M(z) = dx^m W_m(z) + d\theta^\mu W_\mu(z) + d\bar{\theta}_{\dot{\mu}} \bar{W}^{\dot{\mu}}(z), \tag{12.6}$$

while Ω in Eq. (12.4) is a p-form. Note that the definition (12.3) leads to coefficient functions of mixed symmetry. Thus, in contrast to the usual case, there is no value of p above which all forms vanish.

We shall always assume that coefficient functions with an odd number of spinorial indices are fermionic in character, and that those with an even number of spinorial indices are bosonic. These assignments reproduce the familiar rules for the multiplication of forms:

$$\begin{aligned} (c_1\Lambda_1 + c_2\Lambda_2)\Omega &= c_1\Lambda_1\Omega + c_2\Lambda_2\Omega \\ \Lambda\Omega &= (-)^{pq}\Omega\Lambda \\ \Lambda(\Omega\Xi) &= (\Lambda\Omega)\Xi. \end{aligned} \tag{12.7}$$

Here we have assumed that Λ is a p-form and Ω a q-form.

Having defined superspace forms, we must also introduce exterior derivatives. Exterior derivatives map zero-forms into one-forms,

$$dF = dz^M \frac{\partial}{\partial z^M} F = dz^M \partial_M F, \tag{12.8}$$

and p-forms into $(p + 1)$-forms,

$$\begin{aligned} \Omega &= dz^{M_1} \cdots dz^{M_p} W_{M_p \cdots M_1}(z) \\ d\Omega &= dz^{M_1} \cdots dz^{M_p} dz^N \frac{\partial}{\partial z^N} W_{M_p \cdots M_1}(z). \end{aligned} \tag{12.9}$$

In general, exterior derivatives have the following properties:

$$\begin{aligned} d(\Omega + \Sigma) &= d\Omega + d\Sigma \\ d(\Omega\Sigma) &= \Omega\, d\Sigma + (-)^q\, d\Omega\Sigma \\ dd &= 0, \end{aligned} \tag{12.10}$$

where Σ is a q-form. Equations (12.8) and (12.10) follow immediately from (12.3), (12.4), and (12.9). Alternatively, it is possible to define exterior derivatives through (12.8) and (12.10). This is done in Exercise 5.

Equations written in terms of differential forms and exterior derivatives are covariant under coordinate changes. To see this, let us assume that y and z represent two sets of superspace coordinates:

$$y^M = y^M(z). \tag{12.11}$$

Functions of y have a natural mapping into functions of z:

$$F(y) = F(y(z)) = \phi^* F(z). \tag{12.12}$$

If we maintain that y and z label the same point in superspace, the definition of $\phi^* F$ in (12.12) guarantees that a certain quantity takes the same value at the same point, independent of labeling scheme. In a similar fashion, ϕ^* induces a natural mapping between p-forms in the two coordinate systems

$$\begin{aligned} \Omega(y) &= dy^{M_1} \cdots dy^{M_p} W_{M_p \cdots M_1}(y) \\ &= \left(dz^{N_1} \frac{\partial y^{M_1}}{\partial z^{N_1}} \cdots dz^{N_p} \frac{\partial y^{M_p}}{\partial z^{N_p}} \right) W_{M_p \cdots M_1}(y(z)) \\ &= dz^{N_1} \cdots dz^{N_p} \phi^* W_{N_p \cdots N_1}(z) \\ &= \phi^* \Omega(z). \end{aligned} \tag{12.13}$$

The map ϕ^* enjoys the following properties:

$$\begin{aligned} &(1)\ \phi^*(\Omega + \Sigma) = \phi^*\Omega + \phi^*\Sigma \\ &(2)\ \phi^*(\Omega\Sigma) = (\phi^*\Omega)(\phi^*\Sigma) \\ &(3)\ d(\phi^*\Omega) = \phi^*(d\Omega). \end{aligned} \tag{12.14}$$

The proofs of (1) and (2) are straightforward. The proof of (3) is left as Exercise 10. These properties make a formalism based on differential forms and exterior derivatives automatically covariant under coordinate changes.

The mappings (12.12) and (12.13) simplify for infinitesimal coordinate transformations:

$$z^M = y^M + \xi^M. \tag{12.15}$$

In particular, we find

$$\begin{aligned} \delta F(z) &= \phi^* F(z) - F(z) \\ &= -\xi^L \partial_L F(z) \end{aligned} \tag{12.16}$$

for zero-forms and

$$\begin{aligned} \delta W_M(z) &= \phi^* W_M(z) - W_M(z) \\ &= -\xi^L \partial_L W_M(z) - \frac{\partial \xi^L}{\partial z^M} W_L(z) \end{aligned} \tag{12.17}$$

for one-forms. These expressions may be easily generalized for arbitrary p-forms Ω.

Gauge theories are not only covariant under general coordinate transformations. They are also covariant under a local structure group. This is a compact Lie group for Yang-Mills theories and the Lorentz group for gravity theories. In general, differential forms span a representation of this group:

$$\begin{aligned} \Omega'^a &= \Omega^b X_b{}^a(z) \\ \Omega' &= \Omega X . \end{aligned} \tag{12.18}$$

The index a runs from 1 to L, where L is the dimension of the representation X of the group.

Objects which transform linearly under a representation of the structure group are called tensors. Note that exterior derivatives do not map tensors into tensors:

$$d\Omega' = \Omega\, dX + d\Omega X . \tag{12.19}$$

A connection must be introduced to compensate for the inhomogeneous term $\Omega\, dX$. Connections are Lie algebra valued one-forms

$$\phi = dz^M \phi_M{}^r(z) i T^r \tag{12.20}$$

with the following transformation law:

$$\phi' = X^{-1}\phi X - X^{-1}\, dX . \tag{12.21}$$

In (12.20), the matrices T are the hermitian generators of the structure group, and r runs over the dimension of the algebra.

Connections allow us to define covariant derivatives,

$$\mathscr{D}\Omega = d\Omega + \Omega\phi, \tag{12.22}$$

or, more explicitly,

$$\begin{aligned}\mathscr{D}\Omega &= dz^N \mathscr{D}_N \Omega \\ &= dz^{M_1} \cdots dz^{M_p} dz^N \frac{\partial}{\partial z^N} W_{M_p \cdots M_1}(z) \\ &\quad + dz^{M_1} \cdots dz^{M_p} dz^N \phi_N{}^r W_{M_p \cdots M_1}(z) iT^r \end{aligned} \tag{12.23}$$

for Ω a p-form. Covariant derivatives map p-forms into $(p + 1)$-forms and tensors into tensors:

$$\begin{aligned}\mathscr{D}\Omega' &= d\Omega' + \Omega'\phi' \\ &= \Omega\, dX + d\Omega X + \Omega X(X^{-1}\phi X - X^{-1}\, dX) \\ &= (d\Omega + \Omega\phi)X \\ &= (\mathscr{D}\Omega)X. \end{aligned} \tag{12.24}$$

There is one tensor which can be constructed from the connection and its derivatives. It is called the curvature tensor:

$$F = d\phi + \phi\phi. \tag{12.25}$$

The curvature tensor is a Lie algebra valued two-form:

$$\begin{gathered} F = \frac{1}{2}\, dz^M\, dz^N F_{NM}(z) \\ F_{NM}(z) = F_{NM}{}^r(z) iT^r. \end{gathered} \tag{12.26}$$

Its transformation law is computed in Exercise 8:

$$F' = X^{-1}FX. \tag{12.27}$$

The curvature form and the covariant derivative of a tensor are, in general, the only tensorial quantities which may be constructed by taking derivatives. Higher derivatives lead to identities (and not to new tensors) because of the fact that $dd = 0$. These identities are called Bianchi identities.

Bianchi identities of the first type are found from the covariant derivative:

$$\begin{aligned} d\mathscr{D}\Omega &= \Omega\, d\phi - d\Omega\phi \\ &= \Omega(F - \phi\phi) - (\mathscr{D}\Omega - \Omega\phi)\phi \\ &= \Omega F - \mathscr{D}\Omega\phi. \end{aligned} \tag{12.28}$$

These may be written as follows:

$$\mathscr{D}\mathscr{D}\Omega = \Omega F$$

$$dz^M\, dz^N \mathscr{D}_N \mathscr{D}_M \Omega = \frac{1}{2}\, dz^M\, dz^N F_{NM}{}^r \Omega i T^r. \tag{\textbf{12.29}}$$

Bianchi identities of the second type are found from the curvature form (12.25):

$$\begin{aligned} dF &= \phi\, d\phi - d\phi\phi \\ &= \phi(F - \phi\phi) - (F - \phi\phi)\phi \\ &= \phi F - F\phi. \end{aligned} \tag{12.30}$$

These tell us

$$\mathscr{D}F = 0, \tag{\textbf{12.31}}$$

or, in terms of the coefficient functions,

$$dz^M\, dz^N\, dz^L\, \mathscr{D}_L F_{NM} = 0. \tag{\textbf{12.32}}$$

Summing over all permutations of the indices, and using the fact that $F_{NM} = -(-)^{mn} F_{MN}$, we find

$$\mathscr{D}_L F_{NM} + (-)^{\ell(n+m)} \mathscr{D}_N F_{ML} + (-)^{m(n+\ell)} \mathscr{D}_M F_{LN} = 0. \tag{12.33}$$

This is the superspace generalization of the usual cyclic identity on the curvature.

References

H. Flanders, *Differential Forms*, New York, Academic Press (1963).
F. A. Berezin, *Sov. J. Nucl. Phys. 30*, 605 (1979).

EQUATIONS

$$z^M \sim (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}). \tag{12.1}$$

$$z^M z^N = (-)^{nm} z^N z^M. \tag{12.2}$$

$$\begin{aligned} dz^M \wedge dz^N &= -(-)^{nm} dz^N \wedge dz^M \\ dz^M z^N &= (-)^{nm} z^N dz^M. \end{aligned} \tag{12.3}$$

$$\Omega = dz^{M_1} \wedge \cdots \wedge dz^{M_p} W_{M_p \cdots M_1}(z). \tag{12.4}$$

$$\begin{aligned} (c_1\Lambda_1 + c_2\Lambda_2)\Omega &= c_1\Lambda_1\Omega + c_2\Lambda_2\Omega \\ \Lambda\Omega &= (-)^{pq}\Omega\Lambda \\ \Lambda(\Omega\Xi) &= (\Lambda\Omega)\Xi. \end{aligned} \tag{12.7}$$

$$d\Omega = dz^{M_1} \cdots dz^{M_p} dz^N \frac{\partial}{\partial z^N} W_{M_p \cdots M_1}(z). \tag{12.9}$$

$$\begin{aligned} d(\Omega + \Sigma) &= d\Omega + d\Sigma \\ d(\Omega\Sigma) &= \Omega\, d\Sigma + (-)^q\, d\Omega\Sigma \\ dd &= 0. \end{aligned} \tag{12.10}$$

$$\Omega' = \Omega X. \tag{12.18}$$

$$\phi' = X^{-1}\phi X - X^{-1}\, dX. \tag{12.21}$$

$$\mathscr{D}\Omega = d\Omega + \Omega\phi. \tag{12.22}$$

$$\mathscr{D}\Omega' = (\mathscr{D}\Omega)X. \tag{12.24}$$

$$F = d\phi + \phi\phi. \tag{12.25}$$

$$F' = X^{-1}FX. \tag{12.27}$$

$$\begin{aligned} \mathscr{D}\mathscr{D}\Omega &= \Omega F \\ dz^M dz^N \mathscr{D}_N \mathscr{D}_M \Omega &= \frac{1}{2} dz^M dz^N F_{NM}{}^r \Omega i T^r. \end{aligned} \tag{12.29}$$

$$\mathscr{D}F = 0, \tag{12.31}$$

$$dz^M\, dz^N\, dz^L \mathscr{D}_L F_{NM} = 0. \tag{12.32}$$

Exercises

(1) Show that the p-forms on an ordinary n dimensional manifold span an $\binom{n}{p}$ dimensional linear space.

(2) In three dimensions, show that $dd = 0$ implies $\nabla \times \nabla \cdot = 0$ and $\nabla \cdot \nabla \times = 0$.

(3) Verify

$$\frac{\partial}{\partial z^M} z^N = (-)^{nm} z^N \frac{\partial}{\partial z^M} + \delta_M{}^N$$

$$\frac{\partial}{\partial z^M} \frac{\partial}{\partial z^N} = (-)^{nm} \frac{\partial}{\partial z^N} \frac{\partial}{\partial z^M}.$$

(4) Use Exercise 3 to show that $dd = 0$ holds for forms (12.9) in superspace.

(5) Demonstrate that (12.8) and (12.10) define $d\Omega$ as in (12.9).

(6) Check that the connection ϕ remains Lie algebra valued under the transformation (12.21).

(7) Show that if ϕ is Lie algebra valued, (12.25) implies that F is Lie algebra valued as well.

(8) Prove that the curvature F transforms like a tensor (12.27) under the structure group.

(9) Show that $\mathscr{D}\mathscr{D}\Omega = \Omega F$ gives

$$(\mathscr{D}_N \mathscr{D}_M - (-)^{nm} \mathscr{D}_M \mathscr{D}_N) W(z) = W F_{NM}$$

for W a zero-form.

(10) Compute $\phi^*\Omega$, $d(\phi^*\Omega)$, $d\Omega$, and $\phi^*(d\Omega)$ for an arbitrary p-form Ω. Verify that $d(\phi^*\Omega) = \phi^*(d\Omega)$.

XIII. GAUGE THEORIES REVISITED

Before beginning our study of supergravity, we shall examine supersymmetric gauge theories in the language of differential forms. We will reproduce our previous results and gain confidence in geometrical methods. Whenever possible, we will follow the steps we later take in formulating supergravity theories. In this way we will treat supersymmetric gauge theories as a model for supergravity.

In the previous chapter we introduced differential forms and exterior derivatives in superspace. We used the superspace differentials dz^M as a natural basis. We could, however, have chosen any other basis,

$$dz^M E_M{}^A(z). \tag{13.1}$$

Here $E_M{}^A(z)$ is an arbitrary invertible function of superspace,

$$\begin{aligned} E_M{}^A(z) E_A{}^N(z) &= \delta_M{}^N \\ E_A{}^N(z) E_N{}^B(z) &= \delta_A{}^B, \end{aligned} \tag{13.2}$$

where

$$\delta_M{}^N = \begin{pmatrix} \delta_m{}^n & 0 & 0 \\ 0 & \delta_\mu{}^\nu & 0 \\ 0 & 0 & \delta^{\dot\mu}{}_{\dot\nu} \end{pmatrix}. \tag{13.3}$$

In (13.3) it is important to note the position of the dotted indices.

The dz basis is not particularly useful for supersymmetry because the exterior derivative

$$dz^M \frac{\partial}{\partial z^M} \tag{13.4}$$

does not map superfields into superfields. This is because the differential operator $\partial/\partial z$ does not commute with the supersymmetry generators (4.4).

A more natural basis is defined by the supersymmetry covariant derivatives

$$
\begin{aligned}
D_a &= \frac{\partial}{\partial x^a} \\
D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}{}^m \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^m} \\
\bar{D}^{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i\theta^\alpha \sigma_{\alpha\dot{\beta}}{}^m \varepsilon^{\dot{\beta}\dot{\alpha}} \frac{\partial}{\partial x^m}.
\end{aligned}
\tag{13.5}
$$

These differential operators commute with the supersymmetry generators

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}^{\dot{\beta}}\} = \{\bar{D}^{\dot{\alpha}}, Q_\beta\} = \{\bar{D}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0 \tag{13.6}$$

and map superfields into superfields.

The exterior derivative may be written in terms of the differential operators (13.6) if we introduce a new basis

$$e^A(z) = dz^M e_M{}^A(z) \tag{13.7}$$

such that

$$dz^M \frac{\partial}{\partial z^M} = e^A D_A = dz^M e_M{}^A e_A{}^N \frac{\partial}{\partial z^N}, \tag{13.8}$$

where

$$D_A = e_A{}^N \frac{\partial}{\partial z^N}. \tag{13.9}$$

The matrix $e_A{}^M$ follows directly from (13.5):

$$
e_A{}^M = \begin{pmatrix}
e_a{}^m = \delta_a{}^m & e_a{}^\mu = 0 & e_{a\dot{\mu}} = 0 \\
e_\alpha{}^m = i\sigma_{\alpha\dot{\alpha}}{}^m \bar{\theta}^{\dot{\alpha}} & e_\alpha{}^\mu = \delta_\alpha{}^\mu & e_{\alpha\dot{\mu}} = 0 \\
e^{\dot{\alpha}m} = i\theta^\alpha \sigma_{\alpha\dot{\beta}}{}^m \varepsilon^{\dot{\beta}\dot{\alpha}} & e^{\dot{\alpha}\mu} = 0 & e^{\dot{\alpha}}{}_{\dot{\mu}} = \delta^{\dot{\alpha}}{}_{\dot{\mu}}
\end{pmatrix}. \tag{13.10}
$$

Its inverse is given by

$$
e_M{}^A = \begin{pmatrix}
e_m{}^a = \delta_m{}^a & e_m{}^\alpha = 0 & e_{m\dot{\alpha}} = 0 \\
e_\mu{}^a = -i\sigma_{\mu\dot{\mu}}{}^a \bar{\theta}^{\dot{\mu}} & e_\mu{}^\alpha = \delta_\mu{}^\alpha & e_{\mu\dot{\alpha}} = 0 \\
e^{\dot{\mu}a} = -i\theta^\rho \sigma_{\rho\dot{\nu}}{}^a \varepsilon^{\dot{\nu}\dot{\mu}} & e^{\dot{\mu}\alpha} = 0 & e^{\dot{\mu}}{}_{\dot{\alpha}} = \delta^{\dot{\mu}}{}_{\dot{\alpha}}
\end{pmatrix}. \tag{13.11}
$$

These matrices define supersymmetric flat space. Note, however, that the exterior derivatives of the basis forms do not vanish:

$$\begin{aligned} de^A &= dz^M\, dz^N \frac{\partial}{\partial z^N} e_M{}^A(z) \\ de^a &= -2ie^\alpha \sigma_{\alpha\dot{\alpha}}{}^a e^{\dot{\alpha}} \\ de^\alpha &= 0 \\ de_{\dot{\alpha}} &= 0. \end{aligned} \tag{13.12}$$

This is the price that must be paid in the flat space basis.

To discuss gauge theories, we must introduce a connection ϕ. As usual, the connection is a Lie algebra valued one-form:

$$\begin{aligned} \phi &= dz^M \phi_M = e^A \phi_A \\ \phi_A &= \phi_A{}^r iT^r. \end{aligned} \tag{13.13}$$

We shall make contact with ordinary gauge theories by demanding

$$\phi_m{}^r|_{\theta=\bar{\theta}=0} = v_m{}^r. \tag{13.14}$$

The field v_m is the familiar Yang-Mills vector potential.

The curvature two-form is defined as in (12.25):

$$\begin{aligned} F &= d\phi + \phi\phi \\ &= \frac{1}{2} dz^M\, dz^N F_{NM} \\ &= \frac{1}{2} e^A e^B F_{BA}. \end{aligned} \tag{13.15}$$

In the flat space basis, this becomes:

$$\begin{aligned} F &= e^A e^B D_B \phi_A + de^A \phi_A + e^A \phi_A e^B \phi_B \\ &= de^A \phi_A + \frac{1}{2} e^A e^B [D_B \phi_A - (-)^{ab} D_A \phi_B - \phi_B \phi_A + (-)^{ab} \phi_A \phi_B]. \end{aligned} \tag{13.16}$$

The coefficient function F_{BA} may be decomposed into its Lorentz-covariant components:

$$\begin{aligned}
F_{ba} &= \partial_b\phi_a - \partial_a\phi_b - [\phi_b,\phi_a] \\
F_{b\alpha} &= \partial_b\phi_\alpha - D_\alpha\phi_b - [\phi_b,\phi_\alpha] \\
F_{b\dot\alpha} &= \partial_b\phi_{\dot\alpha} - \bar D_{\dot\alpha}\phi_b - [\phi_b,\phi_{\dot\alpha}] \\
F_{\beta\alpha} &= D_\beta\phi_\alpha + D_\alpha\phi_\beta - \{\phi_\beta,\phi_\alpha\} \\
F_{\dot\beta\dot\alpha} &= \bar D_{\dot\beta}\phi_{\dot\alpha} + \bar D_{\dot\alpha}\phi_{\dot\beta} - \{\phi_{\dot\beta},\phi_{\dot\alpha}\} \\
F_{\beta\dot\alpha} &= D_\beta\phi_{\dot\alpha} + \bar D_{\dot\alpha}\phi_\beta - \{\phi_\beta,\phi_{\dot\alpha}\} + 2i\sigma_{\beta\dot\alpha}{}^a\phi_a.
\end{aligned} \tag{13.17}$$

Note that

$$F_{ba}|_{\theta=\bar\theta=0} = v_{ba}{}^r iT^r. \tag{13.18}$$

The Bianchi identities may also be decomposed into their Lorentz-covariant components. In particular,

$$\begin{aligned}
\mathscr{D}F &= 0 \\
\mathscr{D}F &= \frac{1}{2}e^Ae^Be^C\mathscr{D}_CF_{BA} + \frac{1}{2}e^A\,de^BF_{BA} - \frac{1}{2}de^Ae^BF_{BA}
\end{aligned} \tag{13.19}$$

gives

$$\begin{aligned}
&(1)\ \mathscr{D}_cF_{ba} + \mathscr{D}_bF_{ac} + \mathscr{D}_aF_{cb} = 0 \\
&(2)\ \mathscr{D}_\alpha F_{bc} + \mathscr{D}_bF_{c\alpha} + \mathscr{D}_cF_{\alpha b} = 0 \\
&(3)\ \bar{\mathscr{D}}_{\dot\alpha}F_{bc} + \mathscr{D}_bF_{c\dot\alpha} + \mathscr{D}_cF_{\dot\alpha b} = 0 \\
&(4)\ \mathscr{D}_cF_{\beta\alpha} + \mathscr{D}_\beta F_{\alpha c} - \mathscr{D}_\alpha F_{c\beta} = 0 \\
&(5)\ \mathscr{D}_cF_{\dot\beta\dot\alpha} + \bar{\mathscr{D}}_{\dot\beta}F_{\dot\alpha c} - \bar{\mathscr{D}}_{\dot\alpha}F_{c\dot\beta} = 0 \\
&(6)\ \mathscr{D}_cF_{\dot\beta\alpha} + \bar{\mathscr{D}}_{\dot\beta}F_{\alpha c} - \mathscr{D}_\alpha F_{c\dot\beta} + 2i\sigma_{\alpha\dot\beta}{}^aF_{ac} = 0 \\
&(7)\ \mathscr{D}_\gamma F_{\beta\alpha} + \mathscr{D}_\beta F_{\alpha\gamma} + \mathscr{D}_\alpha F_{\gamma\beta} = 0 \\
&(8)\ \bar{\mathscr{D}}_{\dot\gamma}F_{\beta\alpha} + \mathscr{D}_\beta F_{\alpha\dot\gamma} + \mathscr{D}_\alpha F_{\dot\gamma\beta} + 2i\sigma_{\beta\dot\gamma}{}^aF_{a\alpha} + 2i\sigma_{\alpha\dot\gamma}{}^aF_{a\beta} = 0 \\
&(9)\ \mathscr{D}_\gamma F_{\dot\beta\dot\alpha} + \bar{\mathscr{D}}_{\dot\beta}F_{\dot\alpha\gamma} + \bar{\mathscr{D}}_{\dot\alpha}F_{\gamma\dot\beta} + 2i\sigma_{\gamma\dot\beta}{}^aF_{a\dot\alpha} + 2i\sigma_{\gamma\dot\alpha}{}^aF_{a\dot\beta} = 0 \\
&(10)\ \bar{\mathscr{D}}_{\dot\gamma}F_{\dot\beta\dot\alpha} + \bar{\mathscr{D}}_{\dot\beta}F_{\dot\alpha\dot\gamma} + \bar{\mathscr{D}}_{\dot\alpha}F_{\dot\gamma\dot\beta} = 0.
\end{aligned} \tag{13.20}$$

In (13.20), the derivatives $\mathscr{D}$ are the full gauge-covariant derivatives.

Each tensor component of F represents a full superfield multiplet. These multiplets contain a large number of component fields. Most of the component fields are superfluous and must be eliminated through constraint equations. The constraint equations must be gauge covariant, Lorentz covariant, and supersymmetric. In addition, they should not restrict the x-dependence of the component fields.

Finding the proper set of constraints is not easy. It turns out that

$$F_{\alpha\beta} = F_{\dot\alpha\dot\beta} = F_{\alpha\dot\beta} = 0 \tag{13.21}$$

gives the right results. We shall solve the Bianchi identities subject to these constraints. Without them, we would have found (13.17) as the most general solution.

Identitites (7) and (10) in (13.20) are automatically satisfied because of the constraints. Identity (8), however, yields a further restriction on F:

$$\sigma_{\alpha\dot\beta}{}^{a}F_{a\beta} + \sigma_{\beta\dot\beta}{}^{a}F_{a\alpha} = 0. \tag{13.22}$$

The vector-spinor $F_{a\alpha}$ has spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ components. Equation (13.22) tells us that the spin-$\frac{3}{2}$ component vanishes:

$$\begin{aligned} F_{a\alpha} &= -i\sigma_{a\alpha\dot\beta}\bar W^{\dot\beta} \\ \bar W^{\dot\alpha} &= -\frac{i}{4}\bar\sigma^{a\dot\alpha\alpha}F_{a\alpha}. \end{aligned} \tag{13.23}$$

Identity (9) gives a similar result,

$$\begin{aligned} F_{a\dot\alpha} &= -iW^{\beta}\sigma_{a\beta\dot\alpha} \\ W^{\alpha} &= -\frac{i}{4}F_{a\dot\alpha}\bar\sigma^{a\dot\alpha\alpha}, \end{aligned} \tag{13.24}$$

while identity (6) allows us to express F_{ab} in terms of W and $\bar W$:

$$\begin{aligned} F_{ab} &= -\frac{i}{4}\bar\sigma_a{}^{\dot\beta\alpha}(\bar{\mathscr{D}}_{\dot\beta}F_{\alpha b} + \mathscr{D}_{\alpha}F_{\dot\beta b}) \\ &= \frac{1}{4}(\bar{\mathscr{D}}\bar\sigma_a\sigma_b\bar W - \mathscr{D}\sigma_a\bar\sigma_b W). \end{aligned} \tag{13.25}$$

Exploiting the antisymmetry of F_{ab}, we find

$$\bar{\mathscr{D}}\bar W - \mathscr{D}W = 0, \tag{13.26}$$

so

$$F_{ab} = \frac{1}{2}(\bar{\mathscr{D}}\bar{\sigma}_{ab}\bar{W} - \mathscr{D}\sigma_{ab}W). \tag{13.27}$$

Identity (5) leads to another restriction on W:

$$(\sigma_{\beta\dot{\alpha}}{}^{c}\bar{\mathscr{D}}_{\dot{\beta}} + \sigma_{\beta\dot{\beta}}{}^{c}\bar{\mathscr{D}}_{\dot{\alpha}})W^{\beta} = 0. \tag{13.28}$$

Contracting with $\bar{\sigma}^{c\dot{\sigma}\sigma}$ and using (A.12), we have

$$(\delta_{\dot{\alpha}}{}^{\dot{\sigma}}\bar{\mathscr{D}}_{\dot{\beta}} + \delta_{\dot{\beta}}{}^{\dot{\sigma}}\bar{\mathscr{D}}_{\dot{\alpha}})W^{\sigma} = 0. \tag{13.29}$$

Summing over $\dot{\alpha}$ and $\dot{\sigma}$ yields

$$\bar{\mathscr{D}}_{\dot{\alpha}}W_{\sigma} = 0. \tag{13.30}$$

An analogous result follows from (4):

$$\mathscr{D}_{\alpha}\bar{W}_{\dot{\sigma}} = 0. \tag{13.31}$$

Identities (1), (2), and (3) do not lead to any new results.

Identities (1), (2), and (3) are consequences of the other identities even without the constraints (13.21). To show this would require some tedious work which we shall omit here. Features like this are quite common in supersymmetric geometries. In general, part of the covariant curvature tensor may be expressed in terms of the other parts, and not all the Bianchi identities are independent. The technical reason for this stems from the fact that the derivatives of the basis forms E^A always contain a piece proportional to $\sigma_{\alpha\dot{\alpha}}{}^{a}$. We shall encounter this again (albeit in a much more complex form) in supergravity theories.

To conclude this chapter, we shall summarize our solution to the Bianchi identities, subject to the constraints (13.21). We discovered that the Bianchi identities are satisfied by two superfields, W_α and $\bar{W}^{\dot{\alpha}}$. These superfields obey the following constraint equations:

$$\begin{aligned} \bar{\mathscr{D}}_{\dot{\alpha}}W_{\alpha} &= 0 \\ \mathscr{D}_{\alpha}\bar{W}_{\dot{\alpha}} &= 0 \\ \mathscr{D}^{\alpha}W_{\alpha} - \bar{\mathscr{D}}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} &= 0. \end{aligned} \tag{13.32}$$

In the Abelian case, we recognize the conditions (6.8) and (6.12). These equations have (6.11) as their most general solution. In the non-Abelian case, it may be shown that Eqs. (13.32) have (7.22) as their most general solution.

References

J. Wess, in *Topics in Quantum Field Theory*, J. A. de Azcárraga, ed., Salamanca (1977); Lecture Notes in Physics 77, New York, Springer-Verlag (1978).

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Exercises

(1) Show that (13.11) is the inverse of (13.10).

(2) Compute e^a, e^α, and $e_{\dot\alpha}$ explicitly. Compare the result to (11.7).

(3) Decompose $F_{a\alpha}$ into its spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ parts:

$$F_{a\alpha} \to \sigma_{\beta\dot\beta}{}^{a} F_{a\alpha} \equiv F_{\beta\alpha\dot\beta}$$

$$F_{\beta\alpha\dot\beta} = \frac{1}{2} F_{(\beta\alpha)\dot\beta} + \frac{1}{2} F_{[\beta\alpha]\dot\beta}$$

$$\frac{1}{2} F_{(\beta\alpha)\dot\beta} = \frac{1}{2} \left[F_{\beta\alpha\dot\beta} + F_{\alpha\beta\dot\beta}\right] \qquad (\text{spin-}\tfrac{3}{2})$$

$$\frac{1}{2} F_{[\beta\alpha]\dot\beta} = \frac{1}{2} \left[F_{\beta\alpha\dot\beta} - F_{\alpha\beta\dot\beta}\right] \qquad (\text{spin-}\tfrac{1}{2}).$$

Show that

$$F_{a\alpha} = -\frac{1}{4} \sigma_{a\alpha\dot\kappa} \bar\sigma^{b\dot\kappa\beta} F_{b\beta}$$

if the spin-$\frac{3}{2}$ part of $F_{a\alpha}$ vanishes.

(4) Verify that (13.23) satisfies (13.22).

(5) Derive the explicit form for $\mathscr{D}_c F_{\alpha\beta}$ and $\mathscr{D}_\gamma F_{\alpha\beta}$.

(6) Extract the Yang-Mills field v_{ab} from the superfield W_α in (6.11). Compare the result to (13.25).

(7) Show that $DW - \bar D \bar W = 0$ implies

$$\partial_a v_{bc} + \partial_b v_{ca} + \partial_c v_{ab} = 0$$

in the Abelian case.

(8) Demonstrate, again in the Abelian case, that identity (1) is a consequence of the other identities and the constraints (13.21). Use

$$F_{ac} = -\frac{i}{4}\bar{\sigma}_a{}^{\dot{\beta}\alpha}(\bar{D}_{\dot{\beta}}F_{\alpha c} + D_\alpha F_{\dot{\beta}c})$$

and

$$\partial_b F_{ac} = -\frac{i}{4}\bar{\sigma}_a{}^{\dot{\beta}\alpha}(\bar{D}_{\dot{\beta}}\,\partial_b F_{\alpha c} + D_\alpha\,\partial_b F_{\dot{\beta}c})$$

$$= \frac{i}{4}\bar{\sigma}_a{}^{\dot{\beta}\alpha}\left[\bar{D}_{\dot{\beta}}(D_\alpha F_{bc} + \partial_c F_{\alpha b}) + D_\alpha(\bar{D}_{\dot{\beta}}F_{bc} + \partial_c F_{\dot{\beta}b})\right].$$

(9) Verify that $\phi_\beta = -e^V D_\beta e^{-V}$ is a solution to $F_{\alpha\beta} = 0$.

(10) Compute the coefficient functions of the identity (12.29), $\mathscr{D}\mathscr{D}\Omega = \Omega F$, in the e^A basis.

XIV. VIELBEIN, TORSION, AND CURVATURE

In previous lectures we considered theories invariant under rigid supersymmetry transformations. We now wish to gauge these transformations. In particular, we would like to construct theories invariant under x-dependent supersymmetry transformations. As in Chapter IV, such transformations induce motions in superspace:

$$
\begin{aligned}
x^m &\rightarrow x^m + i(\theta\sigma^m\bar\xi(x) - \xi(x)\sigma^m\bar\theta) \\
\theta^\mu &\rightarrow \theta^\mu + \xi^\mu(x) \\
\bar\theta_{\dot\mu} &\rightarrow \bar\theta_{\dot\mu} + \bar\xi_{\dot\mu}(x).
\end{aligned}
\tag{14.1}
$$

These motions generate certain coordinate transformations:

$$
z^M \rightarrow z'^M = z^M - \xi^M(z). \tag{14.2}
$$

Thus it is natural to express our theories in the language of differential forms. This formalism is automatically covariant under coordinate transformations, as was shown in Chapter XII.

Our basic dynamic variables shall be the vielbein and the connection. These superfields contain a large number of component fields. Some will be eliminated through covariant constraint conditions. Others will be gauged away with (14.2). In this way we shall arrive at a theory with the minimum number of component fields.

The vielbein forms $E^A(z)$ define a local reference frame:

$$
E^A = dz^M E_M{}^A(z). \tag{14.3}
$$

They are manifestly coordinate independent. The vielbein fields $E_M{}^A$ are the coefficient functions of the vielbein forms. The vielbein fields change with the coordinates:

$$
\phi^* E_M{}^A(z') = E'_M{}^A(z') = \frac{\partial z^N}{\partial z'^M} E_N{}^A(z). \tag{14.4}
$$

In the infinitesimal case, this becomes

$$\begin{aligned} z'^M &= z^M - \xi^M(z) \\ \delta E_M{}^A &= E'_M{}^A(z) - E_M{}^A(z) \\ &= -\xi^L \partial_L E_M{}^A - (\partial_M \xi^L) E_L{}^A \end{aligned} \tag{14.5}$$

in accord with (12.17). Note that only the lower index M enters the above transformation. It is an Einstein index. Einstein indices take part in coordinate transformations. They will be denoted by letters from the middle of the alphabet.

The upper index A is reserved for the structure group. We shall take the Lorentz group as our structure group. This is because we would like to recover supersymmetric flat space (13.11) as a solution to our dynamical theory. With this choice, the reference frame defined by the vielbein is locally Lorentz covariant:

$$\begin{aligned} \delta E^A &= E^B L_B{}^A(z) \\ \delta E_M{}^A &= E_M{}^B L_B{}^A(z). \end{aligned} \tag{14.6}$$

In general, indices transforming under the structure group will be taken from the beginning of the alphabet. They will be called Lorentz indices. Note that the Lorentz generators $L_B{}^A$ have three irreducible components:

$$L_b{}^a \qquad L_\beta{}^\alpha \qquad L^{\dot\beta}{}_{\dot\alpha}. \tag{14.7}$$

These components are related through the σ-matrices,

$$\sigma_{\alpha\dot\alpha}{}^a \sigma_{\beta\dot\beta}{}^b L_{ab} = -2\varepsilon_{\alpha\beta} L_{\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta} L_{\alpha\beta}, \tag{14.8}$$

as may be seen from (A.13).

The vielbein and its inverse

$$\begin{aligned} E_M{}^A E_A{}^N &= \delta_M{}^N \\ E_A{}^M E_M{}^B &= \delta_A{}^B \end{aligned} \tag{14.9}$$

connect the two types of indices:

$$\begin{aligned} V_M &= E_M{}^A V_A \\ V_A &= E_A{}^M V_M. \end{aligned} \tag{14.10}$$

Wherever possible, we shall write physical quantities in terms of Lorentz indices. They then have simple transformation properties. In addition, they may be fully decomposed into components irreducible under the Lorentz group. As an example, the vielbein forms $E^a = dz^M E_M{}^a$, $E^\alpha = dz^M E_M{}^\alpha$, and $E_{\dot\alpha} = dz^M E_{M\dot\alpha}$ are coordinate-independent irreducible Lorentz tensors.

To formulate covariant derivatives we must introduce a connection form

$$\phi = dz^M \phi_M, \qquad \phi_M = \phi_{MA}{}^B, \tag{14.11}$$

transforming as follows under the structure group:

$$\delta\phi = \phi L - L\phi - dL. \tag{\textbf{14.12}}$$

The connection is the second dynamical variable in our theory. Note that $\phi_{MA}{}^B$ is Lie algebra valued in its two Lorentz indices:

$$\phi_{MAB} = -(-)^{ab}\phi_{MBA}. \tag{14.13}$$

Its third index M is an Einstein index.

The covariant derivative of the vielbein is called torsion:

$$\begin{aligned} T^A &= dE^A + E^B \phi_B{}^A \\ &= \frac{1}{2}\, dz^M\, dz^N T_{NM}{}^A \\ &= \frac{1}{2}\, E^C E^B T_{BC}{}^A. \end{aligned} \tag{\textbf{14.14}}$$

Explicitly, this becomes

$$\begin{aligned} T_{NM}{}^A = {}& \partial_N E_M{}^A - (-)^{nm}\partial_M E_N{}^A \\ &+ (-)^{n(b+m)} E_M{}^B \phi_{NB}{}^A - (-)^{mb} E_N{}^B \phi_{MB}{}^A. \end{aligned} \tag{\textbf{14.15}}$$

The Lorentz tensor $T_{BC}{}^A$ is obtained from $T_{NM}{}^A$ with the help of the inverse vielbein:

$$T_{BC}{}^A = (-)^{b(m+c)} E_C{}^M E_B{}^N T_{NM}{}^A. \tag{14.16}$$

The sign factor may be derived from (14.14) and the definition of E^A. It simply expresses the fact that the summation over M is carried through the index B.

In flat space it is possible to transform the vielbein into the global reference frame (13.11):

$$E^A = e^A. \tag{14.17}$$

It is defined up to rigid Lorentz transformations. In this frame the connection vanishes:

$$\phi = 0. \tag{14.18}$$

The torsion, however, is non-zero:

$$T_{\alpha\dot{\beta}}{}^c = T_{\dot{\beta}\alpha}{}^c = 2i\sigma_{\alpha\dot{\beta}}{}^c. \tag{14.19}$$

All other torsion components vanish.

The curvature tensor is defined in terms of the connection:

$$R = d\phi + \phi\phi. \tag{14.20}$$

As usual, it is a Lie algebra valued two-form:

$$\begin{aligned} R_A{}^B &= \frac{1}{2}\, dz^M\, dz^N R_{NMA}{}^B \\ &= \frac{1}{2}\, E^C E^D R_{DCA}{}^B \\ &= dz^M\, dz^N\, \partial_N \phi_{MA}{}^B + dz^M \phi_{MA}{}^C\, dz^N \phi_{NC}{}^B. \end{aligned} \tag{14.21}$$

From (14.21) we may read off the coefficient function $R_{NMA}{}^B$:

$$\begin{aligned} R_{NMA}{}^B = \partial_N \phi_{MA}{}^B &- (-)^{nm}\, \partial_M \phi_{NA}{}^B \\ &+ (-)^{n(m+a+c)} \phi_{MA}{}^C \phi_{NC}{}^B \\ &- (-)^{m(a+c)} \phi_{NA}{}^C \phi_{MC}{}^B. \end{aligned} \tag{14.22}$$

Since R is a two-form, we have

$$R_{NMA}{}^B = -(-)^{mn} R_{MNA}{}^B, \tag{14.23}$$

and since it is Lie algebra valued, we find

$$\begin{aligned} R_{NMab} &= -R_{NMba} \\ R_{NM\alpha\beta} &= R_{NM\beta\alpha} \\ R_{NM}{}^{\dot\alpha\dot\beta} &= R_{NM}{}^{\dot\beta\dot\alpha} \\ \sigma_{\alpha\dot\alpha}{}^{a}\sigma_{\beta\dot\beta}{}^{b}R_{NMab} &= -2\varepsilon_{\alpha\beta}R_{NM\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta}R_{NM\alpha\beta}. \end{aligned} \tag{14.24}$$

The last relation follows from (14.8). All other components of R vanish.

The torsion and the curvature are the only covariant tensors which may be constructed from the vielbein and the connection. We must now find constraints in terms of these covariant quantities which reduce the number of component fields as much as possible. There are, unfortunately, no general recipes to indicate the proper constraints. Instead, one must examine the consequences of various choices. For example, it is impossible to set all torsion components to zero, for that would exclude supersymmetric flat space as a solution to our theory. Similarly, Eq. (14.19) allows only supersymmetric flat space as its solution. It turns out that

$$\begin{aligned} T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = 0 \qquad T_{\alpha\beta}{}^{c} &= T_{\dot\alpha\dot\beta}{}^{c} = 0 \\ T_{\alpha\dot\beta}{}^{c} &= T_{\dot\beta\alpha}{}^{c} = 2i\sigma_{\alpha\dot\beta}{}^{c} \\ T_{\underline{\alpha}b}{}^{c} &= T_{a\underline{\beta}}{}^{c} = 0 \\ T_{ab}{}^{c} &= 0 \end{aligned} \tag{14.25}$$

are the proper constraints. Here $\underline{\alpha}$ denotes either α or $\dot\alpha$.

In the next chapter we shall solve the Bianchi identities subject to these constraints. As with gauge theories, we will find that they considerably restrict the number of independent superfields. In fact, we will find that (14.25) yields the minimum number of independent component fields. These are the graviton, $e_m{}^a(x)$, the gravitino, $\psi_m{}^\alpha(x)$, $\bar\psi_{m\dot\alpha}(x)$, and the auxiliary fields, $M(x)$ and $b_a(x) = b_a{}^*(x)$. These fields are not restricted by any differential equations in x-space. The spin-2 graviton couples to the energy-momentum tensor, while the spin-$\frac{3}{2}$ gravitino couples to the spin-$\frac{3}{2}$ supercurrent. The auxiliary fields are just enough to equalize the number of bosonic and fermionic degrees of freedom off mass shell.

References

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J. Wess and B. Zumino, *Phys. Lett. 66B*, 361 (1977).

EQUATIONS

$$\begin{aligned} \delta E_M{}^A &= E'_M{}^A(z) - E_M{}^A(z) \\ &= -\xi^L \partial_L E_M{}^A - (\partial_M \xi^L) E_L{}^A. \end{aligned} \tag{14.5}$$

$$\delta E_M{}^A = E_M{}^B L_B{}^A(z). \tag{14.6}$$

$$\sigma_{\alpha\dot{\alpha}}{}^a \sigma_{\beta\dot{\beta}}{}^b L_{ab} = -2\varepsilon_{\alpha\beta} L_{\dot{\alpha}\dot{\beta}} + 2\varepsilon_{\dot{\alpha}\dot{\beta}} L_{\alpha\beta}. \tag{14.8}$$

$$\delta\phi = \phi L - L\phi - dL. \tag{14.12}$$

$$\begin{aligned} T^A &= dE^A + E^B \phi_B{}^A \\ &= \frac{1}{2} dz^M dz^N T_{NM}{}^A. \end{aligned} \tag{14.14}$$

$$\begin{aligned} T_{NM}{}^A = {} & \partial_N E_M{}^A - (-)^{nm} \partial_M E_N{}^A \\ & + (-)^{n(b+m)} E_M{}^B \phi_{NB}{}^A - (-)^{mb} E_N{}^B \phi_{MB}{}^A. \end{aligned} \tag{14.15}$$

$$R = d\phi + \phi\phi. \tag{14.20}$$

$$\begin{aligned} R_A{}^B &= \frac{1}{2} dz^M dz^N R_{NMA}{}^B \\ &= dz^M dz^N \partial_N \phi_{MA}{}^B + dz^M \phi_{MA}{}^C dz^N \phi_{NC}{}^B. \end{aligned} \tag{14.21}$$

$$\begin{aligned} R_{NMA}{}^B = {} & \partial_N \phi_{MA}{}^B - (-)^{nm} \partial_M \phi_{NA}{}^B \\ & + (-)^{n(m+a+c)} \phi_{MA}{}^C \phi_{NC}{}^B \\ & - (-)^{m(a+c)} \phi_{NA}{}^C \phi_{MC}{}^B. \end{aligned} \tag{14.22}$$

$$\begin{aligned} & T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = 0 \qquad T_{\alpha\beta}{}^c = T_{\dot{\alpha}\dot{\beta}}{}^c = 0 \\ & T_{\alpha\dot{\beta}}{}^c = T_{\dot{\beta}\alpha}{}^c = 2i\sigma_{\alpha\dot{\beta}}{}^c \\ & T_{\underline{\alpha}b}{}^c = T_{a\underline{\beta}}{}^c = 0 \\ & T_{ab}{}^c = 0. \end{aligned} \tag{14.25}$$

EXERCISES

(1) Compute dx'^M and $\partial/\partial x'^M$ under the transformation (14.2). Show

$$dx'^M \frac{\partial}{\partial x'^M} = dx^M \frac{\partial}{\partial x^M}.$$

(2) Compute $\delta\Omega_{MN} = \Omega'_{MN}(z) - \Omega_{MN}(z)$ for Ω a two-form.

(3) Show that $\sigma_{\alpha\dot{\alpha}}{}^{m}$ transforms as a four-vector under (14.8).

(4) Explicitly evaluate the covariant derivatives of the covariant and contravariant Lorentz vectors X_A and X^A:

$$\begin{aligned}
\mathscr{D}_M X^A &= \partial_M X^A + (-)^{mb} X^B \phi_{MB}{}^A \\
\mathscr{D}_M X_A &= \partial_M X_A - \phi_{MA}{}^B X_B \\
\mathscr{D}_B X^A &= E_B{}^M \mathscr{D}_M X^A \\
\mathscr{D}_B X_A &= E_B{}^M \mathscr{D}_M X_A .
\end{aligned}$$

(5) Use the covariant derivative of a Lorentz vector to define the covariant derivative of an Einstein vector:

$$\begin{aligned}
\nabla_N X_M &= (-)^{n(a+m)} E_M{}^A E_N{}^B \mathscr{D}_B X_A \\
&= \partial_N X_M + \Gamma_{NM}{}^R X_R \\
\Gamma_{NM}{}^R &= (-)^{n(a+m)} E_M{}^A (\mathscr{D}_N E_A{}^R).
\end{aligned}$$

(6) Show that ∇ in Exercise 5 reduces to the usual symmetric connection in torsion-free ordinary space.

(7) Decompose $T_{BC}{}^A$ into its Lorentz-irreducible tensors.

(8) Linearize $T_{BC}{}^A$ about supersymmetric flat space:

$$E_N{}^A = e_N{}^A + \kappa e_N{}^B H_B{}^A .$$

(9) Assume that $E_m{}^a = \delta_m{}^a + \cdots$ has mass dimension zero. Give the dimensions of $E_\mu{}^a$, $E_m{}^\alpha$, and $E_\mu{}^\alpha$, as well as $T_{ab}{}^c$, $T_{\alpha\beta}{}^c$, $T_{ab}{}^\gamma$, and $R_{abc}{}^d$, $R_{ab\gamma}{}^\delta$:

$$\begin{aligned}
[x] &= -1 & [\theta] &= -\frac{1}{2} \\
[E_m{}^a] &= 0 & [E_\mu{}^a] &= -\frac{1}{2} \\
[E_m{}^\alpha] &= \frac{1}{2} & [E_\mu{}^\alpha] &= 0
\end{aligned}$$

$$[T_{ab}{}^{c}] = 1 \qquad [T_{a\beta}{}^{c}] = \frac{1}{2}$$

$$[T_{ab}{}^{\gamma}] = \frac{3}{2}$$

$$[R_{abc}{}^{d}] = 2 \qquad [R_{ab\gamma}{}^{\delta}] = 2$$

XV. BIANCHI IDENTITIES

We are now ready to solve the supergravity Bianchi identities subject to the constraints (14.25). We will proceed in analogy to Chapter XIII, where we solved the Bianchi identities for supersymmetric gauge theories. We will find that the Bianchi identities reduce the number of independent superfields contained in $E_M{}^A(z)$ and $\phi_{MA}{}^B(z)$ to one complex chiral superfield R, one hermitian vector superfield $G_{\alpha\dot\beta}$, and one chiral superfield $W_{\alpha\beta\gamma}$, totally symmetric in its indices. The torsion and the curvature may both be expressed in terms of these three superfields.

We shall summarize our results at the end of this lecture. These formulae will be used frequently in the coming chapters. It is not necessary, however, to work through the details presented here to understand the rest of the book.

We begin by stating the Bianchi identities for the torsion and curvature. These follow directly from (12.29) and (14.14):

$$\begin{aligned} \mathscr{D}\mathscr{D}E^A &= E^B R_B{}^A \\ \mathscr{D}T^A &= E^B R_B{}^A. \end{aligned} \tag{15.1}$$

Here $R_B{}^A$ denotes the superspace curvature and T^A the torsion. We wish to break this equation into its Lorentz-irreducible components, so we compute $\mathscr{D}T^A$ in the basis defined by the vielbein forms:

$$\begin{aligned} \mathscr{D}T^A &= \frac{1}{2}\mathscr{D}(E^B E^C T_{CB}{}^A) \\ &= \frac{1}{2}E^B E^C \mathscr{D}T_{CB}{}^A + \frac{1}{2}E^B T^C T_{CB}{}^A - \frac{1}{2}T^B E^C T_{CB}{}^A. \end{aligned} \tag{15.2}$$

Substituting this in (15.1), we find

$$E^B E^C E^D(\mathscr{D}_D T_{CB}{}^A - R_{DCB}{}^A + T_{DC}{}^F T_{FB}{}^A) = 0. \tag{15.3}$$

This identity contains thirty Lorentz-covariant components. Some of them, however, are related by complex conjugation, and others are automatically satisfied because of the constraints (14.25). In all, there are thirteen independent components:

$$
\begin{aligned}
&(1)\ R_{\beta\delta\gamma\alpha} + R_{\delta\gamma\beta\alpha} + R_{\gamma\beta\delta\alpha} = 0\\
&(2)\ R_{\delta\dot\gamma\beta\alpha} + R_{\dot\gamma\beta\delta\alpha} = -2i\sigma_{\delta\dot\gamma}{}^{f} T_{\beta f\alpha} - 2i\sigma_{\beta\dot\gamma}{}^{f} T_{\delta f\alpha}\\
&(3)\ R_{\beta\delta\dot\gamma\dot\alpha} = -2i\sigma_{\delta\dot\gamma}{}^{f} T_{\beta f\dot\alpha} - 2i\sigma_{\beta\dot\gamma}{}^{f} T_{\delta f\dot\alpha}\\
&(4)\ R_{\delta b\gamma\alpha} + R_{\gamma b\delta\alpha} = -\mathscr{D}_{\gamma} T_{\delta b\alpha} - \mathscr{D}_{\delta} T_{\gamma b\alpha}\\
&(5)\ R_{b\delta\dot\gamma\dot\alpha} = \mathscr{D}_{\delta} T_{\dot\gamma b\dot\alpha} + \bar{\mathscr{D}}_{\dot\gamma} T_{\delta b\dot\alpha} + 2i\sigma_{\delta\dot\gamma}{}^{f} T_{fb\dot\alpha}\\
&(6)\ \bar{\mathscr{D}}_{\dot\gamma} T_{\dot\delta b\alpha} + \bar{\mathscr{D}}_{\dot\delta} T_{\dot\gamma b\alpha} = 0\\
&(7)\ R_{\dot\beta\dot\delta ca} = -2i\sigma_{a\phi\dot\delta} T_{\dot\beta c}{}^{\phi} - 2i\sigma_{a\phi\dot\beta} T_{\dot\delta c}{}^{\phi}\\
&(8)\ R_{\beta\dot\delta ca} = -2i\sigma_{a\phi\dot\delta} T_{\beta c}{}^{\phi} + 2i\sigma_{a\beta\dot\phi} T_{\dot\delta c}{}^{\dot\phi}\\
&(9)\ R_{bd\dot\gamma\dot\alpha} = \mathscr{D}_{b} T_{d\dot\gamma\dot\alpha} + \mathscr{D}_{d} T_{\dot\gamma b\dot\alpha} + \bar{\mathscr{D}}_{\dot\gamma} T_{bd\dot\alpha} + T_{d\dot\gamma}{}^{\underline{\phi}} T_{\underline{\phi}b\dot\alpha} + T_{\dot\gamma b}{}^{\underline{\phi}} T_{\underline{\phi}d\dot\alpha}\\
&(10)\ \mathscr{D}_{b} T_{d\dot\gamma\alpha} + \mathscr{D}_{d} T_{\dot\gamma b\alpha} + \bar{\mathscr{D}}_{\dot\gamma} T_{bd\alpha} + T_{d\dot\gamma}{}^{\underline{\phi}} T_{\underline{\phi}b\alpha} + T_{\dot\gamma b}{}^{\underline{\phi}} T_{\underline{\phi}d\alpha} = 0\\
&(11)\ R_{\beta dca} + R_{c\beta da} + 2i\sigma_{a\beta\dot\phi} T_{dc}{}^{\dot\phi} = 0\\
&(12)\ \mathscr{D}_{b} T_{dc\alpha} + \mathscr{D}_{d} T_{cb\alpha} + \mathscr{D}_{c} T_{bd\alpha} + T_{bd}{}^{\underline{\phi}} T_{\underline{\phi}c\alpha} + T_{dc}{}^{\underline{\phi}} T_{\underline{\phi}b\alpha} + T_{cb}{}^{\underline{\phi}} T_{\underline{\phi}d\alpha} = 0\\
&(13)\ R_{bdca} + R_{dcba} + R_{cbda} = 0.
\end{aligned}
\tag{15.4}
$$

The underlined index $\underline{\phi}$ is summed over both ϕ and $\dot\phi$.

We shall first solve the identities which are linear and without derivatives. These are Eqs. (1), (2), (3), (7), (8), (11), and (13). We start by converting (7)

$$R_{\dot\beta\dot\delta ca} = -2i\sigma_{a\phi\dot\delta} T_{\dot\beta c}{}^{\phi} - 2i\sigma_{a\phi\dot\beta} T_{\dot\delta c}{}^{\phi} \tag{15.5}$$

to spinor notation:

$$
\begin{aligned}
R_{\dot\beta\dot\delta\gamma\dot\gamma\alpha\dot\alpha} &= \sigma_{\gamma\dot\gamma}{}^{c}\sigma_{\alpha\dot\alpha}{}^{a} R_{\dot\beta\dot\delta ca}\\
T_{\dot\delta\gamma\dot\gamma\phi} &= \sigma_{\gamma\dot\gamma}{}^{c} T_{\dot\delta c\phi}.
\end{aligned}
\tag{15.6}
$$

Since R is Lie algebra valued, we have

$$R_{\dot\beta\dot\delta\gamma\dot\gamma\alpha\dot\alpha} = -2\varepsilon_{\gamma\alpha} R_{\dot\beta\dot\delta\dot\gamma\dot\alpha} + 2\varepsilon_{\dot\gamma\dot\alpha} R_{\dot\beta\dot\delta\gamma\alpha}, \tag{15.7}$$

where $R_{\dot\beta\dot\delta\gamma\alpha}$ and $R_{\dot\beta\dot\delta\dot\gamma\dot\alpha}$ are symmetric in $\gamma\alpha$ and $\dot\gamma\dot\alpha$ respectively. Since R is a two-form, $R_{\dot\beta\dot\delta\gamma\alpha}$ and $R_{\dot\beta\dot\delta\dot\gamma\dot\alpha}$ are also symmetric in $\dot\beta\dot\delta$. With these expressions for R and T, Eq. (15.5) becomes

$$\varepsilon_{\dot\gamma\dot\alpha} R_{\dot\beta\dot\delta\gamma\alpha} - \varepsilon_{\gamma\alpha} R_{\dot\beta\dot\delta\dot\gamma\dot\alpha} = 2i(\varepsilon_{\dot\alpha\dot\beta} T_{\dot\delta\gamma\dot\gamma\alpha} + \varepsilon_{\dot\alpha\dot\delta} T_{\dot\beta\gamma\dot\gamma\alpha}). \tag{15.8}$$

The tensor $T_{\dot\delta\gamma\dot\gamma\alpha}$ may be decomposed into components with definite symmetry properties:

$$T_{\dot\delta\gamma\dot\gamma\alpha} = \varepsilon_{\dot\delta\dot\gamma}\varepsilon_{\gamma\alpha}T + \varepsilon_{\dot\delta\dot\gamma}T_{\underline{\gamma\alpha}} + \varepsilon_{\gamma\alpha}T_{\underline{\dot\delta\dot\gamma}} + T_{\underline{\gamma\alpha}\,\underline{\dot\delta\dot\gamma}}. \tag{15.9}$$

In (15.9) and in what follows, tensors are symmetric with respect to underlined indices. Equation (15.8) now splits into several symmetry classes. We first consider the part which is antisymmetric in both $\gamma\alpha$ and $\dot\gamma\dot\alpha$. This may be projected out with $\varepsilon^{\alpha\gamma}$ and $\varepsilon^{\dot\alpha\dot\gamma}$:

$$T_{\dot\delta}{}^{\alpha}{}_{\dot\beta\alpha} + T_{\dot\beta}{}^{\alpha}{}_{\dot\delta\alpha} = 0. \tag{15.10}$$

Note that the curvature R drops out of this expression. Substituting (15.9) into (15.10), we discover

$$T_{\underline{\dot\beta\dot\delta}} = 0. \tag{15.11}$$

We next consider the part of (15.8) which is symmetric in $\gamma\alpha$:

$$\begin{aligned}\varepsilon_{\dot\gamma\dot\alpha}R_{\dot\beta\dot\delta\gamma\alpha} = {} & 2i(\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\dot\delta\dot\gamma} + \varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma})T_{\underline{\gamma\alpha}} \\ & + 2i(\varepsilon_{\dot\alpha\dot\beta}T_{\underline{\gamma\alpha}\,\underline{\dot\delta\dot\gamma}} + \varepsilon_{\dot\alpha\dot\delta}T_{\underline{\gamma\alpha}\,\underline{\dot\beta\dot\gamma}}).\end{aligned} \tag{15.12}$$

If we multiply this by $\varepsilon^{\dot\alpha\dot\beta}$, we obtain

$$R_{\dot\gamma\dot\delta\gamma\alpha} = 6i\varepsilon_{\dot\gamma\dot\delta}T_{\underline{\gamma\alpha}} - 6iT_{\underline{\gamma\alpha}\,\underline{\dot\delta\dot\gamma}}. \tag{15.13}$$

However, $R_{\dot\gamma\dot\delta\gamma\alpha}$ is symmetric in $\dot\gamma\dot\delta$, so

$$T_{\underline{\gamma\alpha}} = 0 \tag{15.14}$$

and

$$R_{\dot\gamma\dot\delta\gamma\alpha} = -6iT_{\underline{\gamma\alpha}\,\underline{\dot\delta\dot\gamma}}. \tag{15.15}$$

If we multiply (15.12) by $\varepsilon^{\dot\alpha\dot\gamma}$, we find

$$R_{\dot\beta\dot\delta\gamma\alpha} = -2iT_{\underline{\gamma\alpha}\,\underline{\dot\beta\dot\delta}}. \tag{15.16}$$

Equations (15.15) and (15.16) are consistent if and only if

$$\begin{aligned} T_{\underline{\gamma\alpha}\,\underline{\dot\beta\dot\delta}} &= 0. \\ R_{\dot\beta\dot\delta\gamma\alpha} &= 0. \end{aligned} \tag{15.17}$$

Only one term remains in the decomposition (15.9); we call it R, where $T = -2iR$:

$$T_{\dot{\delta}\gamma\dot{\gamma}\alpha} = -2i\varepsilon_{\dot{\delta}\dot{\gamma}}\varepsilon_{\gamma\alpha}R. \tag{15.18}$$

From (15.8), we see immediately that

$$R_{\dot{\beta}\dot{\delta}\dot{\gamma}\dot{\alpha}} = 4(\varepsilon_{\dot{\beta}\dot{\alpha}}\varepsilon_{\dot{\delta}\dot{\gamma}} + \varepsilon_{\dot{\delta}\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\gamma}})R. \tag{15.19}$$

With these results, we have found the most general solution to identity (7). We have also learned that $R_{\dot{\beta}\dot{\delta}ca}$ and $T_{\dot{\delta}c\phi}$ may be expressed in terms of a single superfield R. Similarly, we may write $R_{\beta\delta ca}$ and $T_{\delta c\dot{\phi}}$ in terms of R^+. The above expressions satisfy identities (1) and (3) as well.

We now consider identities (2) and (8). The computation is quite similar to what we have done, so we merely list the results:

$$\begin{aligned}
R_{\beta\dot{\delta}\gamma\alpha} &= \varepsilon_{\gamma\beta}G_{\alpha\dot{\delta}} + \varepsilon_{\alpha\beta}G_{\gamma\dot{\delta}} \\
T_{\beta\gamma\dot{\gamma}\phi} &= \frac{i}{4}(\varepsilon_{\gamma\phi}G_{\beta\dot{\gamma}} - 3\varepsilon_{\beta\gamma}G_{\phi\dot{\gamma}} - 3\varepsilon_{\beta\phi}G_{\gamma\dot{\gamma}}) \\
G^{+}{}_{\alpha\dot{\alpha}} &= G_{\alpha\dot{\alpha}}.
\end{aligned} \tag{15.20}$$

We leave the details of this calculation to the reader as Exercise 1.

Identity (11) gives $R_{\beta dca}$ in terms of the torsion. Since $R_{\beta dca}$ is antisymmetric with respect to c and a, we find

$$R_{\beta dca} = i(\sigma_{d\beta\dot{\phi}}T_{ca}{}^{\dot{\phi}} - \sigma_{a\beta\dot{\phi}}T_{dc}{}^{\dot{\phi}} - \sigma_{c\beta\dot{\phi}}T_{ad}{}^{\dot{\phi}}). \tag{15.21}$$

We have now solved all the derivative-free linear identities except (13). Identity (13), however, is just the usual cyclic identity on the curvature in four-dimensional space. It is familiar from ordinary gravity theory and its consequences are well known. In spinor notation, the symmetry properties of

$$R_{\gamma\dot{\gamma}\delta\dot{\delta}\beta\dot{\beta}\alpha\dot{\alpha}} = \sigma_{\gamma\dot{\gamma}}{}^{c}\sigma_{\delta\dot{\delta}}{}^{d}\sigma_{\beta\dot{\beta}}{}^{b}\sigma_{\alpha\dot{\alpha}}{}^{a}R_{cdba} \tag{15.22}$$

lead to the following decomposition:

$$\begin{aligned}
R_{\gamma\dot{\gamma}\delta\dot{\delta}\beta\dot{\beta}\alpha\dot{\alpha}} = {} & 4\varepsilon_{\gamma\delta}\varepsilon_{\beta\alpha}\bar{\mathrm{X}}_{\underline{\dot{\gamma}\dot{\delta}}\,\underline{\dot{\beta}\dot{\alpha}}} \\
& - 4\varepsilon_{\gamma\delta}\varepsilon_{\dot{\beta}\dot{\alpha}}\bar{\Psi}_{\underline{\dot{\gamma}\dot{\delta}}\,\underline{\beta\alpha}} - 4\varepsilon_{\dot{\gamma}\dot{\delta}}\varepsilon_{\beta\alpha}\Psi_{\underline{\gamma\delta}\,\underline{\dot{\beta}\dot{\alpha}}} \\
& + 4\varepsilon_{\dot{\gamma}\dot{\delta}}\varepsilon_{\dot{\beta}\dot{\alpha}}\mathrm{X}_{\underline{\gamma\delta}\,\underline{\beta\alpha}}.
\end{aligned} \tag{15.23}$$

Identity (13) is satisfied if and only if

$$\begin{aligned} \Psi_{\underline{\gamma\delta}\,\underline{\dot\beta\dot\alpha}} &= \bar{\Psi}_{\underline{\dot\beta\dot\alpha}\,\underline{\gamma\delta}} \\ X_{\alpha\beta\gamma}{}^{\beta} &= \varepsilon_{\alpha\gamma}\Lambda, \end{aligned} \tag{15.24}$$

where Λ is real.

We shall now proceed to solve the identities which contain derivatives but remain linear. These are identities (4), (5), and (6). We begin by inserting (15.18) into identity (6). This yields

$$\varepsilon_{\dot\delta\dot\beta}\varepsilon_{\beta\alpha}\bar{\mathscr{D}}_{\dot\gamma}R + \varepsilon_{\dot\gamma\dot\beta}\varepsilon_{\beta\alpha}\bar{\mathscr{D}}_{\dot\delta}R = 0. \tag{15.25}$$

Contracting with $\varepsilon^{\dot\delta\dot\beta}$, we find

$$\bar{\mathscr{D}}_{\dot\gamma}R = 0, \tag{15.26}$$

so the superfields R and R^+

$$\mathscr{D}_{\gamma}R^+ = 0 \tag{15.27}$$

are chiral.

Evaluating identity (4) is tedious. We must make use of the fact that $R_{\delta cab}$ is Lie algebra valued:

$$\begin{aligned} R_{\delta\gamma\dot\gamma\alpha\dot\alpha\beta\dot\beta} &= \sigma_{\gamma\dot\gamma}{}^{c}\sigma_{\alpha\dot\alpha}{}^{a}\sigma_{\beta\dot\beta}{}^{b}R_{\delta cab} \\ &= -2\varepsilon_{\alpha\beta}R_{\delta\gamma\dot\gamma\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta}R_{\delta\gamma\dot\gamma\alpha\beta}. \end{aligned} \tag{15.28}$$

The component $R_{\delta cab}$ is related to the torsion through (15.21), where $T_{dc\dot\phi}$ has the following decomposition:

$$T_{\delta\dot\delta\gamma\dot\gamma\dot\phi} = -2\varepsilon_{\delta\gamma}(\bar{W}_{\underline{\dot\delta\dot\gamma\dot\phi}} + \varepsilon_{\dot\phi\dot\gamma}\bar{W}_{\dot\delta} + \varepsilon_{\dot\phi\dot\delta}\bar{W}_{\dot\gamma}) + 2\varepsilon_{\dot\delta\dot\gamma}W_{\underline{\delta\gamma\dot\phi}}. \tag{15.29}$$

Combining (15.21), (15.28), (15.29), and identity (4), we find

$$\begin{aligned} T_{\delta\dot\delta\gamma\dot\gamma\dot\alpha} = &-2\varepsilon_{\delta\gamma}\bar{W}_{\underline{\dot\delta\dot\gamma\dot\alpha}} \\ &-\frac{1}{2}\varepsilon_{\delta\gamma}(\varepsilon_{\dot\alpha\dot\gamma}\mathscr{D}^{\beta}G_{\beta\dot\delta} + \varepsilon_{\dot\alpha\dot\delta}\mathscr{D}^{\beta}G_{\beta\dot\gamma}) \\ &+\frac{1}{2}\varepsilon_{\dot\delta\dot\gamma}(\mathscr{D}_{\delta}G_{\gamma\dot\alpha} + \mathscr{D}_{\gamma}G_{\delta\dot\alpha}) \end{aligned} \tag{15.30}$$

and

$$
\begin{aligned}
R_{\beta\delta\dot{\delta}\gamma\alpha} &= \frac{1}{2}\, i(\varepsilon_{\beta\delta}\mathscr{D}_{\gamma} + \varepsilon_{\beta\gamma}\mathscr{D}_{\delta})G_{\alpha\dot{\delta}} \\
&\quad + \frac{1}{2}\, i(\varepsilon_{\beta\delta}\mathscr{D}_{\alpha} + \varepsilon_{\beta\alpha}\mathscr{D}_{\delta})G_{\gamma\dot{\delta}} \\
&\quad + i(\varepsilon_{\alpha\beta}\varepsilon_{\delta\gamma} + \varepsilon_{\gamma\beta}\varepsilon_{\delta\alpha})\mathscr{D}^{\varepsilon}G_{\varepsilon\dot{\delta}} \\
R_{\beta\delta\dot{\delta}\dot{\gamma}\dot{\alpha}} &= 4i\varepsilon_{\beta\delta}\bar{W}_{\underline{\dot{\gamma}\dot{\alpha}\dot{\delta}}} + \frac{1}{2}\, i(\varepsilon_{\dot{\delta}\dot{\gamma}}\mathscr{D}_{\beta}G_{\delta\dot{\alpha}} + \varepsilon_{\dot{\delta}\dot{\alpha}}\mathscr{D}_{\beta}G_{\delta\dot{\gamma}}).
\end{aligned}
\tag{15.31}
$$

The symmetric tensor $W_{\underline{\alpha\beta\gamma}}$ drops out of the identity, so it remains undetermined. The tensors $\bar{W}_{\dot{\delta}}$ and $W_{\underline{\delta\gamma\dot{\phi}}}$ are related to G.

Identity (5) gives another relation between the same curvature and torsion components. It yields

$$\mathscr{D}^{\beta}G_{\beta\dot{\alpha}} = \bar{\mathscr{D}}_{\dot{\alpha}}R^{+} \tag{15.32}$$

as a consequence of (15.30) and (15.31).

We have now solved all the linear identities. The nonlinear identities either define components of the curvature and torsion as nonlinear expressions in G and R or they may be reduced to linear equations through the commutation relations of the covariant derivatives. For example, identity (9) expresses $R_{cd\,\dot{\gamma}\dot{\alpha}}$ and therefore $\bar{X}_{\underline{\dot{\gamma}\delta}\,\underline{\dot{\varepsilon}\dot{\alpha}}}$ and $\Psi_{\underline{\gamma\delta}\,\underline{\dot{\varepsilon}\dot{\alpha}}}$ in terms of torsion components. These, in turn, may be expressed in terms of W, G, and R:

$$\bar{X}_{\underline{\dot{\beta}\dot{\delta}}\,\underline{\dot{\gamma}\dot{\alpha}}} = \frac{1}{4}\left[-\mathscr{D}_{\beta\dot{\beta}}T_{\dot{\gamma}}{}^{\beta}{}_{\dot{\delta}\dot{\alpha}} - \mathscr{D}_{\beta\dot{\delta}}T_{\dot{\gamma}}{}^{\beta}{}_{\dot{\beta}\dot{\alpha}} + \bar{\mathscr{D}}_{\dot{\gamma}}T_{\beta\dot{\beta}}{}^{\beta}{}_{\dot{\delta}\dot{\alpha}} - T_{\dot{\gamma}}{}^{\beta}{}_{\dot{\delta}}{}^{\underline{\phi}}T_{\underline{\phi}\beta\dot{\beta}\dot{\alpha}} + T_{\dot{\gamma}\beta\dot{\beta}}{}^{\underline{\phi}}T_{\underline{\phi}}{}^{\beta}{}_{\dot{\delta}\dot{\alpha}}\right]. \tag{15.33}$$

Because of the symmetry properties of $\bar{X}_{\underline{\dot{\beta}\dot{\delta}}\,\underline{\dot{\gamma}\dot{\alpha}}}$ and because of the relation

$$
\begin{aligned}
\bar{\mathscr{D}}_{\dot{\delta}}\mathscr{D}^{\alpha}G_{\dot{\beta}\alpha} + \bar{\mathscr{D}}_{\dot{\beta}}\mathscr{D}^{\alpha}G_{\dot{\delta}\alpha} &= (\bar{\mathscr{D}}_{\dot{\delta}}\bar{\mathscr{D}}_{\dot{\beta}} + \bar{\mathscr{D}}_{\dot{\beta}}\bar{\mathscr{D}}_{\dot{\delta}})R^{+} \\
&= 0,
\end{aligned}
\tag{15.34}
$$

we find

$$\bar{\mathscr{D}}^{\dot{\alpha}}\bar{W}_{\underline{\dot{\alpha}\dot{\beta}\dot{\delta}}} + \frac{1}{2}\, i(\mathscr{D}_{\beta\dot{\beta}}G^{\beta}{}_{\dot{\delta}} + \mathscr{D}_{\beta\dot{\delta}}G^{\beta}{}_{\dot{\beta}}) = 0 \tag{15.35}$$

from (15.33).

Finally, we examine identity (10). The torsion terms may be expressed in terms of W, G, and R. All but one contain an ε-tensor, so symmetrization in all indices yields

$$\mathscr{D}_\alpha \bar{W}_{\underline{\dot\beta\dot\gamma\dot\delta}} = 0. \tag{15.36}$$

The symmetric tensor $\bar{W}_{\underline{\dot\beta\dot\gamma\dot\delta}}$ is a chiral superfield.

We have now solved the Bianchi identities (15.4), subject to the constraints (14.25). We have learned that all the components of the torsion and the curvature may be expressed in terms of the superfields R, $G_{\alpha\dot\alpha}$, and $W_{\alpha\beta\gamma}$. These superfields are subject to the following conditions:

$$\begin{aligned}
&(1)\ \bar{\mathscr{D}}_{\dot\alpha} R = 0\\
&(2)\ \mathscr{D}^\alpha G_{\alpha\dot\beta} = \bar{\mathscr{D}}_{\dot\beta} R^+ \qquad \bar{\mathscr{D}}^{\dot\beta} G_{\alpha\dot\beta} = \mathscr{D}_\alpha R\\
&(3)\ \bar{\mathscr{D}}_{\dot\alpha} W_{\beta\gamma\delta} = 0 \qquad \mathscr{D}_\alpha \bar{W}_{\dot\beta\dot\gamma\dot\delta} = 0\\
&(4)\ \bar{\mathscr{D}}^{\dot\alpha} \bar{W}_{\dot\alpha\dot\beta\dot\delta} + \frac{1}{2} i(\mathscr{D}_{\beta\dot\beta} G^\beta{}_{\dot\delta} + \mathscr{D}_{\beta\dot\delta} G^\beta{}_{\dot\beta}) = 0\\
&\qquad \mathscr{D}^\alpha W_{\alpha\beta\delta} + \frac{1}{2} i(\mathscr{D}_{\beta\dot\beta} G_\delta{}^{\dot\beta} + \mathscr{D}_{\delta\dot\beta} G_\beta{}^{\dot\beta}) = 0\\
&(5)\ (G_{\alpha\dot\alpha})^+ = G_{\alpha\dot\alpha}\\
&(6)\ (W_{\alpha\beta\gamma})^+ = \bar{W}_{\dot\alpha\dot\beta\dot\gamma}.
\end{aligned} \tag{15.37}$$

The superfield $W_{\alpha\beta\gamma}$ is completely symmetric in its indices.

For future reference, we collect our results below.

Torsion:

$$\begin{aligned}
&(1)\ T_{\gamma\dot\varepsilon}{}^c = T_{\dot\varepsilon\gamma}{}^c = 2i\sigma_{\gamma\dot\varepsilon}{}^c\\
&(2)\ T_{\dot\delta e}{}^\alpha = -T_{e\dot\delta}{}^\alpha = -\frac{1}{2}\bar\sigma_e{}^{\dot\varepsilon\varepsilon} T_{\dot\delta\varepsilon\dot\varepsilon}{}^\alpha\\
&\qquad T_{\dot\delta\varepsilon\dot\varepsilon\alpha} = -2i\varepsilon_{\dot\delta\dot\varepsilon}\varepsilon_{\varepsilon\alpha} R\\
&(3)\ T_{\delta e}{}^{\dot\alpha} = -T_{e\delta}{}^{\dot\alpha} = -\frac{1}{2}\bar\sigma_e{}^{\dot\varepsilon\varepsilon} T_{\delta\varepsilon\dot\varepsilon}{}^{\dot\alpha}\\
&\qquad T_{\delta\varepsilon\dot\varepsilon\dot\alpha} = -2i\varepsilon_{\delta\varepsilon}\varepsilon_{\dot\varepsilon\dot\alpha} R^+\\
&(4)\ T_{\delta e}{}^\alpha = -T_{e\delta}{}^\alpha = -\frac{1}{2}\bar\sigma_e{}^{\dot\varepsilon\varepsilon} T_{\delta\varepsilon\dot\varepsilon}{}^\alpha\\
&\qquad T_{\delta\varepsilon\dot\varepsilon\alpha} = \frac{i}{4}(\varepsilon_{\varepsilon\alpha} G_{\delta\dot\varepsilon} - 3\varepsilon_{\delta\alpha} G_{\varepsilon\dot\varepsilon} - 3\varepsilon_{\delta\varepsilon} G_{\alpha\dot\varepsilon})
\end{aligned}$$

$$(5)\quad T_{\dot\delta e}{}^{\dot\alpha} = -T_{e\dot\delta}{}^{\dot\alpha} = -\frac{1}{2}\bar\sigma_e{}^{\dot\varepsilon\varepsilon}T_{\dot\delta\varepsilon\dot\varepsilon}{}^{\dot\alpha}$$

$$T_{\dot\delta\varepsilon\dot\varepsilon\dot\alpha} = \frac{i}{4}(\varepsilon_{\dot\varepsilon\dot\alpha}G_{\varepsilon\dot\delta} - 3\varepsilon_{\dot\delta\dot\alpha}G_{\varepsilon\dot\varepsilon} - 3\varepsilon_{\dot\delta\dot\varepsilon}G_{\varepsilon\dot\alpha})$$

$$(6)\quad T_{dc}{}^{\alpha} = -T_{cd}{}^{\alpha} = \frac{1}{4}\bar\sigma_d{}^{\dot\delta\delta}\bar\sigma_c{}^{\dot\gamma\gamma}T_{\delta\dot\delta\gamma\dot\gamma}{}^{\alpha}$$

$$\begin{aligned} T_{\delta\dot\delta\gamma\dot\gamma\alpha} = {} & -2\varepsilon_{\dot\delta\dot\gamma}W_{\delta\gamma\alpha} \\ & - \frac{1}{2}\varepsilon_{\dot\delta\dot\gamma}(\varepsilon_{\delta\alpha}\bar{\mathscr{D}}_{\dot\phi}G_{\gamma}{}^{\dot\phi} + \varepsilon_{\gamma\alpha}\bar{\mathscr{D}}_{\dot\phi}G_{\delta}{}^{\dot\phi}) \\ & + \frac{1}{2}\varepsilon_{\delta\gamma}(\bar{\mathscr{D}}_{\dot\delta}G_{\alpha\dot\gamma} + \bar{\mathscr{D}}_{\dot\gamma}G_{\alpha\dot\delta}) \end{aligned}$$

$$(7)\quad T_{dc}{}^{\dot\alpha} = -T_{cd}{}^{\dot\alpha} = \frac{1}{4}\bar\sigma_d{}^{\dot\delta\delta}\bar\sigma_c{}^{\dot\gamma\gamma}T_{\delta\dot\delta\gamma\dot\gamma}{}^{\dot\alpha}$$

$$\begin{aligned} T_{\delta\dot\delta\gamma\dot\gamma\dot\alpha} = {} & -2\varepsilon_{\delta\gamma}\bar W_{\dot\delta\dot\gamma\dot\alpha} \\ & - \frac{1}{2}\varepsilon_{\delta\gamma}(\varepsilon_{\dot\delta\dot\alpha}\mathscr{D}_{\phi}G^{\phi}{}_{\dot\gamma} + \varepsilon_{\dot\gamma\dot\alpha}\mathscr{D}_{\phi}G^{\phi}{}_{\dot\delta}) \\ & + \frac{1}{2}\varepsilon_{\dot\delta\dot\gamma}(\mathscr{D}_{\delta}G_{\gamma\dot\alpha} + \mathscr{D}_{\gamma}G_{\delta\dot\alpha}). \end{aligned} \tag{15.38}$$

Curvature:

$$(1)\quad R_{\delta\gamma\varepsilon\alpha} = 4(\varepsilon_{\delta\varepsilon}\varepsilon_{\gamma\alpha} + \varepsilon_{\gamma\varepsilon}\varepsilon_{\delta\alpha})R^{+}$$

$$R_{\dot\delta\dot\gamma\dot\varepsilon\dot\alpha} = 4(\varepsilon_{\dot\delta\dot\varepsilon}\varepsilon_{\dot\gamma\dot\alpha} + \varepsilon_{\dot\gamma\dot\varepsilon}\varepsilon_{\dot\delta\dot\alpha})R$$

$$(2)\quad R_{\delta\gamma\dot\varepsilon\dot\alpha} = R_{\dot\delta\dot\gamma\varepsilon\alpha} = 0$$

$$(3)\quad R_{\delta\dot\gamma\varepsilon\alpha} = R_{\dot\gamma\delta\varepsilon\alpha} = -(\varepsilon_{\delta\varepsilon}G_{\alpha\dot\gamma} + \varepsilon_{\delta\alpha}G_{\varepsilon\dot\gamma})$$

$$R_{\delta\dot\gamma\dot\varepsilon\dot\alpha} = R_{\dot\gamma\delta\dot\varepsilon\dot\alpha} = -(\varepsilon_{\dot\gamma\dot\varepsilon}G_{\delta\dot\alpha} + \varepsilon_{\dot\gamma\dot\alpha}G_{\delta\dot\varepsilon})$$

$$(4)\quad R_{\varepsilon c\delta\alpha} = -R_{c\varepsilon\delta\alpha} = -\frac{1}{2}\bar\sigma_c{}^{\dot\gamma\gamma}R_{\varepsilon\gamma\dot\gamma\delta\alpha}$$

$$\begin{aligned} R_{\varepsilon\gamma\dot\gamma\delta\alpha} = {} & i(\varepsilon_{\varepsilon\delta}\varepsilon_{\gamma\alpha} + \varepsilon_{\varepsilon\alpha}\varepsilon_{\gamma\delta})\mathscr{D}_{\phi}G^{\phi}{}_{\dot\gamma} + \frac{i}{2}(\varepsilon_{\varepsilon\gamma}\mathscr{D}_{\delta} + \varepsilon_{\varepsilon\delta}\mathscr{D}_{\gamma})G_{\alpha\dot\gamma} \\ & + \frac{i}{2}(\varepsilon_{\varepsilon\gamma}\mathscr{D}_{\alpha} + \varepsilon_{\varepsilon\alpha}\mathscr{D}_{\gamma})G_{\delta\dot\gamma} \end{aligned}$$

(5) $R_{\varepsilon c\dot{\delta}\dot{\alpha}} = -R_{c\varepsilon\dot{\delta}\dot{\alpha}} = -\frac{1}{2}\bar{\sigma}_c{}^{\dot{\gamma}\gamma}R_{\varepsilon\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}}$

$$R_{\varepsilon\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}} = 4i\varepsilon_{\varepsilon\gamma}\bar{W}_{\dot{\gamma}\dot{\delta}\dot{\alpha}} + \frac{i}{2}(\varepsilon_{\dot{\gamma}\dot{\delta}}\mathscr{D}_\varepsilon G_{\gamma\dot{\alpha}} + \varepsilon_{\dot{\gamma}\dot{\alpha}}\mathscr{D}_\varepsilon G_{\gamma\dot{\delta}})$$

(6) $R_{\dot{\varepsilon}c\delta\alpha} = -R_{c\dot{\varepsilon}\delta\alpha} = -\frac{1}{2}\bar{\sigma}_c{}^{\dot{\gamma}\gamma}R_{\dot{\varepsilon}\gamma\dot{\gamma}\delta\alpha}$

$$R_{\dot{\varepsilon}\gamma\dot{\gamma}\delta\alpha} = 4i\varepsilon_{\dot{\varepsilon}\dot{\gamma}}W_{\delta\gamma\alpha} + \frac{i}{2}(\varepsilon_{\gamma\delta}\bar{\mathscr{D}}_{\dot{\varepsilon}}G_{\alpha\dot{\gamma}} + \varepsilon_{\gamma\alpha}\bar{\mathscr{D}}_{\dot{\varepsilon}}G_{\delta\dot{\gamma}})$$

(7) $R_{\dot{\varepsilon}c\dot{\delta}\dot{\alpha}} = -R_{c\dot{\varepsilon}\dot{\delta}\dot{\alpha}} = -\frac{1}{2}\bar{\sigma}_c{}^{\dot{\gamma}\gamma}R_{\dot{\varepsilon}\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}}$

$$\begin{aligned} R_{\dot{\varepsilon}\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}} &= i(\varepsilon_{\dot{\varepsilon}\dot{\delta}}\varepsilon_{\dot{\gamma}\dot{\alpha}} + \varepsilon_{\dot{\varepsilon}\dot{\alpha}}\varepsilon_{\dot{\gamma}\dot{\delta}})\bar{\mathscr{D}}_{\dot{\phi}}G_\gamma{}^{\dot{\phi}} \\ &\quad + \frac{i}{2}(\varepsilon_{\dot{\varepsilon}\dot{\gamma}}\bar{\mathscr{D}}_{\dot{\delta}} + \varepsilon_{\dot{\varepsilon}\dot{\delta}}\bar{\mathscr{D}}_{\dot{\gamma}})G_{\gamma\dot{\alpha}} + \frac{i}{2}(\varepsilon_{\dot{\varepsilon}\dot{\gamma}}\bar{\mathscr{D}}_{\dot{\alpha}} + \varepsilon_{\dot{\varepsilon}\dot{\alpha}}\bar{\mathscr{D}}_{\dot{\gamma}})G_{\gamma\dot{\delta}} \end{aligned}$$

(8) $R_{ed\gamma\alpha} = \frac{1}{4}\bar{\sigma}_e{}^{\dot{\varepsilon}\varepsilon}\bar{\sigma}_d{}^{\dot{\delta}\delta}R_{\varepsilon\dot{\varepsilon}\delta\dot{\delta}\gamma\alpha}$

$$R_{ed\dot{\gamma}\dot{\alpha}} = \frac{1}{4}\bar{\sigma}_e{}^{\dot{\varepsilon}\varepsilon}\bar{\sigma}_d{}^{\dot{\delta}\delta}R_{\varepsilon\dot{\varepsilon}\delta\dot{\delta}\dot{\gamma}\dot{\alpha}}$$

$$R_{\varepsilon\dot{\varepsilon}\delta\dot{\delta}\dot{\gamma}\dot{\alpha}} = -2\varepsilon_{\varepsilon\delta}\bar{\mathrm{X}}_{\dot{\varepsilon}\dot{\delta}\dot{\gamma}\dot{\alpha}} + 2\varepsilon_{\dot{\varepsilon}\dot{\delta}}\Psi_{\varepsilon\delta\dot{\gamma}\dot{\alpha}}$$

$$R_{\varepsilon\dot{\varepsilon}\delta\dot{\delta}\gamma\alpha} = 2\varepsilon_{\dot{\varepsilon}\dot{\delta}}\mathrm{X}_{\varepsilon\delta\gamma\alpha} - 2\varepsilon_{\varepsilon\delta}\bar{\Psi}_{\dot{\varepsilon}\dot{\delta}\gamma\alpha}$$

$$\begin{aligned} \mathrm{X}_{\gamma\delta\varepsilon\alpha} &= -\frac{1}{4}(\mathscr{D}_\gamma W_{\delta\varepsilon\alpha} + \mathscr{D}_\delta W_{\varepsilon\alpha\gamma} + \mathscr{D}_\varepsilon W_{\alpha\gamma\delta} + \mathscr{D}_\alpha W_{\gamma\delta\varepsilon}) \\ &\quad + (\varepsilon_{\gamma\alpha}\varepsilon_{\varepsilon\delta} + \varepsilon_{\delta\alpha}\varepsilon_{\varepsilon\gamma})\left\{-2RR^+ + \frac{1}{8}G_{\rho\dot{\rho}}G^{\rho\dot{\rho}}\right. \\ &\quad \left. + \frac{1}{16}(\bar{\mathscr{D}}_{\dot{\rho}}\bar{\mathscr{D}}^{\dot{\rho}}R^+ + \mathscr{D}^\rho\mathscr{D}_\rho R)\right\} \end{aligned}$$

$$\begin{aligned} \Psi_{\varepsilon\alpha\dot{\gamma}\dot{\delta}} &= \bar{\Psi}_{\dot{\gamma}\dot{\delta}\varepsilon\alpha} \\ &= \frac{1}{4}(G_{\varepsilon\dot{\delta}}G_{\alpha\dot{\gamma}} + G_{\alpha\dot{\delta}}G_{\varepsilon\dot{\gamma}}) \\ &\quad + \frac{i}{8}(\mathscr{D}_{\alpha\dot{\gamma}}G_{\varepsilon\dot{\delta}} + \mathscr{D}_{\varepsilon\dot{\gamma}}G_{\alpha\dot{\delta}} + \mathscr{D}_{\alpha\dot{\delta}}G_{\varepsilon\dot{\gamma}} + \mathscr{D}_{\varepsilon\dot{\delta}}G_{\alpha\dot{\gamma}}) \\ &\quad + \frac{1}{8}(\bar{\mathscr{D}}_{\dot{\gamma}}\mathscr{D}_\alpha G_{\varepsilon\dot{\delta}} + \bar{\mathscr{D}}_{\dot{\gamma}}\mathscr{D}_\varepsilon G_{\alpha\dot{\delta}} + \bar{\mathscr{D}}_{\dot{\delta}}\mathscr{D}_\alpha G_{\varepsilon\dot{\gamma}} + \bar{\mathscr{D}}_{\dot{\delta}}\mathscr{D}_\varepsilon G_{\alpha\dot{\gamma}}). \quad (15.39) \end{aligned}$$

All other components vanish.

It is quite remarkable that these relations also solve the Bianchi identities arising from the curvature (12.31):

$$\begin{gathered} \mathscr{D}R = 0 \\ E^C E^D E^E [\mathscr{D}_E R_{DCA}{}^B + T_{ED}{}^F R_{FCA}{}^B] = 0. \end{gathered} \tag{15.40}$$

References

R. Grimm, J. Wess, and B. Zumino, *Nucl. Phys. B152*, 255 (1979).
N. Dragon, *Z. Phys. C2*, 29 (1979).

Exercises

(1) Show that (15.20) is the solution to identities (2) and (8).

(2) Derive the conditions (15.24) from identity (13).

(3) Show that identity (5) implies (15.32).

(4) Verify that (15.38) and (15.39) satisfy (15.40).

XVI. SUPERGAUGE TRANSFORMATIONS

In the past few chapters we have considered the general coordinate transformations of superspace

$$z'^M = z^M - \xi^M(z). \tag{16.1}$$

We have also introduced a structure group and explored its transformation laws (12.18). In this chapter we shall define supergauge transformations. Supergauge transformations are constructed from the general coordinate and structure group transformations of superspace. They amount to a convenient reparametrization of these transformations. Supergauge transformations map Lorentz tensors into Lorentz tensors and reduce to supersymmetry transformations in the limit of flat space.

The parameter ξ characterizes infinitesimal changes in coordinates. It may be written with either an Einstein or a Lorentz index:

$$\xi^A = \xi^M E_M{}^A. \tag{16.2}$$

Note that either ξ^A or ξ^M may be chosen as the field-independent transformation parameter. Its companion then depends on the fields through the vielbein. Since we would like Lorentz tensors to transform into Lorentz tensors, we shall choose ξ^A to be field-independent.

We must now write the transformation properties of tensor superfields

$$\delta V^A = -\xi^M \partial_M V^A + V^B L_B{}^A \tag{16.3}$$

in terms of the parameter ξ^A. In (16.3), V^A represents a general tensor field, and the representation L of the Lorentz group corresponds to the tensor structure of V. For scalar fields, we have

$$\delta V = -\xi^M \partial_M V = -\xi^A E_A{}^M \partial_M V = -\xi^A \mathscr{D}_A V, \tag{16.4}$$

while for tensor fields, we find

$$\delta V^A = -\xi^B E_B{}^M \partial_M V^A + V^B L_B{}^A. \tag{16.5}$$

As it stands, Eq. (16.5) is not covariant under Lorentz transformations. The derivative in (16.5) must be replaced by a covariant derivative:

$$\begin{aligned} \mathscr{D}_M V^A &= \partial_M V^A + (-)^{mb} V^B \phi_{MB}{}^A \\ \mathscr{D}_B V^A &= E_B{}^M \mathscr{D}_M V^A. \end{aligned} \tag{16.6}$$

Substituting (16.6) into (16.5), we obtain

$$\delta V^A = -\xi^B \mathscr{D}_B V^A + V^B \xi^C \phi_{CB}{}^A + V^B L_B{}^A. \tag{16.7}$$

The connection $\phi_{CB}{}^A$ is Lie algebra valued, so $\xi^C \phi_{CB}{}^A$ acts like a field-dependent Lorentz transformation on V^B. If we set

$$L_B{}^A = -\xi^C \phi_{CB}{}^A, \tag{16.8}$$

we find

$$\delta_\xi V^A = -\xi^C \mathscr{D}_C V^A \tag{16.9}$$

for any tensor superfield V^A. Equation (16.9) is manifestly covariant under Lorentz transformations.

The condition (16.8) defines supergauge transformations. Supergauge transformations consist of a general coordinate transformation with field-independent parameter ξ^A followed by a structure group Lorentz transformation with field-dependent parameter $L_B{}^A = -\xi^C \phi_{CB}{}^A$. It is among this restricted class of transformations that we shall find the gauged supersymmetry transformations.

Let us now compute the commutator of two supergauge transformations. Since ξ^A is field-independent, we have

$$\delta_\eta \delta_\xi V^A = -\xi^C \delta_\eta \mathscr{D}_C V^A = \xi^C \eta^B \mathscr{D}_B \mathscr{D}_C V^A, \tag{16.10}$$

so

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta) V^A = \xi^C \eta^B (\mathscr{D}_B \mathscr{D}_C - (-)^{bc} \mathscr{D}_C \mathscr{D}_B) V^A. \tag{16.11}$$

This expression is easily evaluated with the help of the Bianchi identities (12.29):

$$\mathscr{D}\mathscr{D} V^A = V^B R_B{}^A. \tag{16.12}$$

Here $R_B{}^A$ is the Lie algebra valued curvature two-form and V^A is a tensor zero-form. As in (15.2), we write

$$\begin{aligned}\mathscr{D}\mathscr{D}V^A &= \mathscr{D}(E^B\mathscr{D}_B V^A)\\ &= E^B\mathscr{D}(\mathscr{D}_B V^A) + (\mathscr{D}E^B)\mathscr{D}_B V^A\\ &= E^B E^C \mathscr{D}_C\mathscr{D}_B V^A + T^B\mathscr{D}_B V^A.\end{aligned} \tag{16.13}$$

Substituting (16.12), we find

$$E^B E^C \mathscr{D}_C\mathscr{D}_B V^A = V^D R_D{}^A - T^D\mathscr{D}_D V^A, \tag{16.14}$$

or, for the coefficient functions,

$$(\mathscr{D}_C\mathscr{D}_B - (-)^{bc}\mathscr{D}_B\mathscr{D}_C)V^A = (-)^{d(c+b)}V^D R_{CBD}{}^A - T_{CB}{}^D\mathscr{D}_D V^A. \tag{16.15}$$

This tells us that the commutator (16.11) closes into a field-dependent Lorentz transformation and a field-dependent transformation of the type (16.9):

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)V^A = V^D\xi^C\eta^B R_{BCD}{}^A - \xi^C\eta^B T_{BC}{}^D\mathscr{D}_D V^A. \tag{16.16}$$

In flat superspace, where the curvature vanishes and the torsion is proportional to the σ-matrices, Eq. (16.16) reduces to a familiar form:

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)V^A = -2i(\eta\sigma^m\bar\xi - \xi\sigma^m\bar\eta)\,\partial_m V^A. \tag{16.17}$$

The $\theta = \bar\theta = 0$ components of ξ and η give the commutator of two supersymmetry transformations (3.4), so (16.9) indeed includes gauged supersymmetry transformations.

We conclude this chapter by computing the changes in the vielbein and the connection under supergauge transformations. In general, the transformation properties of the vielbein are given by (14.5) and (14.6):

$$\begin{aligned}\delta E_M{}^A &= -\xi^L\partial_L E_M{}^A - \partial_M\xi^L E_L{}^A + E_M{}^B L_B{}^A\\ &= -\xi^L(\partial_L E_M{}^A - (-)^{\ell m}\partial_M E_L{}^A) - \partial_M\xi^A + E_M{}^B L_B{}^A\\ &= -\partial_M\xi^A - \xi^L(T_{LM}{}^A - \phi_{LM}{}^A + (-)^{m\ell}\phi_{ML}{}^A) + E_M{}^B L_B{}^A.\end{aligned} \tag{16.18}$$

Here we have used the definition of the torsion (14.15). The connection $\phi_{ML}{}^A$ combines with $\partial_M\xi^A$ to make a covariant derivative. Substituting the special Lorentz transformation (16.8), we find the following supergauge transformation law:

$$\delta_\xi E_M{}^A = -\mathscr{D}_M \xi^A - \xi^B T_{BM}{}^A. \tag{16.19}$$

We proceed similarly for the connection:

$$\begin{aligned}\delta\phi_{MA}{}^B = & -\xi^L \partial_L \phi_{MA}{}^B - (\partial_M \xi^L)\phi_{LA}{}^B \\ & + \phi_{MA}{}^C L_C{}^B - (-)^{m(a+c)} L_A{}^C \phi_{MC}{}^B \\ & - \partial_M L_A{}^B.\end{aligned} \tag{16.20}$$

For a supergauge transformation, this becomes

$$\delta_\xi \phi_{MA}{}^B = -\xi^C R_{CMA}{}^B. \tag{16.21}$$

The proof of this relation is left to the reader as Exercise 3.

The transformation laws (16.9), (16.19), and (16.21) allow us to compute the transformation properties of all the independent supergravity component fields. This we shall do in the following lectures.

References

J. Wess and B. Zumino, *Phys. Lett. 79B*, 394 (1978).

J. Wess, in *Quantum Flavordynamics*, *Quantum Chromodynamics*, *and Unified Theories*, K. T. Mahanthappa and J. Randa, eds., New York, Plenum (1980).

Equations

$$\delta V^A = -\xi^B E_B{}^M \partial_M V^A + V^B L_B{}^A. \tag{16.5}$$

$$\delta V^A = -\xi^B \mathscr{D}_B V^A + V^B \xi^C \phi_{CB}{}^A + V^B L_B{}^A. \tag{16.7}$$

$$\delta_\xi V^A = -\xi^C \mathscr{D}_C V^A. \tag{16.9}$$

$$(\mathscr{D}_C \mathscr{D}_B - (-)^{bc} \mathscr{D}_B \mathscr{D}_C) V^A = (-)^{d(c+b)} V^D R_{CBD}{}^A - T_{CB}{}^D \mathscr{D}_D V^A. \tag{16.15}$$

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta) V^A = V^D \xi^C \eta^B R_{BCD}{}^A - \xi^C \eta^B T_{BC}{}^D \mathscr{D}_D V^A. \tag{16.16}$$

$$\delta_\xi E_M{}^A = -\mathscr{D}_M \xi^A - \xi^B T_{BM}{}^A. \tag{16.19}$$

$$\delta_\xi \phi_{MA}{}^B = -\xi^C R_{CMA}{}^B. \tag{16.21}$$

Exercises

(1) Show that (16.15) may be written

$$(\mathscr{D}_C\mathscr{D}_B - (-)^{bc}\mathscr{D}_B\mathscr{D}_C)V^A = -(-)^d R_{CB}{}^A{}_D V^D - T_{CB}{}^D\mathscr{D}_D V^A$$

for contravariant vectors V^A.

(2) Use the definition of the covariant derivative of a covariant vector

$$\mathscr{D}_M V_A = \partial_M V_A - \phi_{MA}{}^B V_B$$

to derive the analog of Exercise 1 for covariant vectors V_B.

(3) Prove (16.21) using (14.22).

XVII. THE $\theta = \bar{\theta} = 0$ COMPONENTS OF THE VIELBEIN, CONNECTION, TORSION, AND CURVATURE

In Chapter XIV we defined the torsion and the curvature in terms of the vielbein and the connection, the dynamical variables of supergravity. By construction, all are superfields, whose expansion coefficients are x-dependent component fields. In this chapter we will see that the components of the torsion and the curvature can be expressed in terms of the lowest components of R, G, and the vielbein. The same holds true for the vielbein and the connection. This implies that the lowest components of R, G, and E are the physical supergravity degrees of freedom. The remaining degrees of freedom are pure gauge, and can be transformed away.

The transformation parameters ξ^A and L_{ab} are functions of superspace. Their lowest components characterize general coordinate transformations in four-dimensional x-space $[\xi^a(x)]$, gauged supersymmetry transformations $[\xi^\alpha(x), \bar{\xi}_{\dot{\alpha}}(x)]$, and local Lorentz transformations $[L_{ab}(x)]$. We will use their higher components to transform away certain $\theta = \bar{\theta} = 0$ components of the vielbein and the connection.

We first consider the vielbein. Its transformation law (16.18) may be written as a supergauge transformation (16.19) together with an additional Lorentz transformation $L_B{}^A$:

$$\delta E_M{}^A = -\mathscr{D}_M \xi^A - \xi^B T_{BM}{}^A + E_M{}^B L_B{}^A. \tag{17.1}$$

The lowest component of this equation gives the transformation property of $E_M{}^A|_{\theta=\bar{\theta}=0}$. Higher components of ξ^A enter $\delta E_M{}^A|$ through the covariant derivatives $\mathscr{D}_\mu \xi^A$ and $\bar{\mathscr{D}}^{\dot{\mu}} \xi^A$. We may use these higher components to transform $E_M{}^A|$ to the following form (see Exercise 1):

$$E_M{}^A(z)|_{\theta=\bar{\theta}=0} = \begin{pmatrix} e_m{}^a(x) & \frac{1}{2}\psi_m{}^\alpha(x) & \frac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix}. \tag{17.2}$$

The fields $e_m{}^a$, $\psi_m{}^\alpha$, and $\bar{\psi}_{m\dot{\alpha}}$ cannot be gauged away. They describe the spin-2 graviton and the spin-$\frac{3}{2}$ gravitino. The inverse vielbein $E_A{}^M|$ has a similar structure,

$$E_A{}^M(z)\big|_{\theta=\bar\theta=0} = \begin{pmatrix} e_a{}^m(x) & -\frac{1}{2}\,\psi_a{}^\mu(x) & -\frac{1}{2}\,\bar\psi_{a\dot\mu}(x) \\ 0 & \delta_\alpha{}^\mu & 0 \\ 0 & 0 & \delta^{\dot\alpha}{}_{\dot\mu} \end{pmatrix}, \tag{17.3}$$

with

$$\begin{aligned} e_a{}^m e_m{}^b &= \delta_a{}^b \\ \psi_a{}^\mu &= e_a{}^m \psi_m{}^\alpha \delta_\alpha{}^\mu \\ \bar\psi_{a\dot\mu} &= e_a{}^m \bar\psi_{m\dot\alpha} \delta^{\dot\alpha}{}_{\dot\mu}. \end{aligned} \tag{17.4}$$

We now consider the connection. Its transformation law (16.20) may also be written as a combined supergauge and Lorentz transformation:

$$\begin{aligned} \delta\phi_{MA}{}^B = &-\xi^C R_{CMA}{}^B + \phi_{MA}{}^C L_C{}^B \\ &- (-)^{m(a+c)} L_A{}^C \phi_{MC}{}^B - \partial_M L_A{}^B. \end{aligned} \tag{17.5}$$

We may use the higher components of $L_A{}^B$ to transform away $\phi_{\mu A}{}^B|$ and $\phi^{\dot\mu}{}_A{}^B|$. This is possible because $\phi_{MA}{}^B$ is Lie algebra valued:

$$\begin{aligned} \phi_{mA}{}^B(z)\big|_{\theta=\bar\theta=0} &= \omega_{mA}{}^B(x) \\ \phi_{\mu A}{}^B(z)\big|_{\theta=\bar\theta=0} &= \phi^{\dot\mu}{}_A{}^B(z)\big|_{\theta=\bar\theta=0} = 0. \end{aligned} \tag{17.6}$$

No further components of $E|$ and $\phi|$ may be gauged away.

In ordinary relativity it is possible to express the connection in terms of the vierbein. This follows from the fact that the torsion is constrained to vanish (see Exercise 2). In supergravity we also have constraints on the torsion (14.25). These constraints allow us to express the connection in terms of e and ψ.

To proceed systematically, we start from Eq. (14.15). The $T_{mn}{}^A$ components of this equation contain no θ or $\bar\theta$ derivatives. They relate $T_{mn}{}^A$ to the lowest components of E and ϕ:

$$\begin{aligned} T_{nm}{}^a| &= \partial_n e_m{}^a - \partial_m e_n{}^a + \omega_{nm}{}^a - \omega_{mn}{}^a \\ T_{nm}{}^\alpha| &= \frac{1}{2}(\partial_n\psi_m{}^\alpha - \partial_m\psi_n{}^\alpha) + \frac{1}{2}(\psi_m{}^\beta\omega_{n\beta}{}^\alpha - \psi_n{}^\beta\omega_{m\beta}{}^\alpha) \\ &= \frac{1}{2}(\tilde{\mathscr{D}}_n\psi_m{}^\alpha - \tilde{\mathscr{D}}_m\psi_n{}^\alpha) = \frac{1}{2}\,\psi_{nm}{}^\alpha(x) \\ T_{nm\dot\alpha}| &= \frac{1}{2}\,\bar\psi_{nm\dot\alpha}(x). \end{aligned} \tag{17.7}$$

Here we have used the following definitions:

$$\begin{aligned} \omega_{nm}{}^{a} &= e_{m}{}^{b}\omega_{nb}{}^{a} \\ \tilde{\mathscr{D}}_{n}\psi_{m}{}^{\alpha} &= \partial_{n}\psi_{m}{}^{\alpha} + \psi_{m}{}^{\beta}\omega_{n\beta}{}^{\alpha} \\ \psi_{nm}{}^{\alpha} &= \tilde{\mathscr{D}}_{n}\psi_{m}{}^{\alpha} - \tilde{\mathscr{D}}_{m}\psi_{n}{}^{\alpha}. \end{aligned} \tag{17.8}$$

To apply the constraints, we must relate $T_{NM}{}^{A}$ to $T_{CB}{}^{A}$ through the vielbein

$$T_{NM}{}^{A} = E_{M}{}^{B}E_{N}{}^{C}T_{CB}{}^{A}(-)^{n(m+b)}. \tag{17.9}$$

Taking the $\theta = \bar{\theta} = 0$ component of (17.9) and applying the constraints, we find

$$\begin{aligned} T_{nm}{}^{a}| &= E_{m}{}^{\beta}E_{n\dot{\gamma}}T^{\dot{\gamma}}{}_{\beta}{}^{a}| + E_{m\dot{\beta}}E_{n}{}^{\gamma}T_{\gamma}{}^{\dot{\beta}a}| \\ &= -\frac{i}{2}(\psi_{m}\sigma^{a}\bar{\psi}_{n} - \psi_{n}\sigma^{a}\bar{\psi}_{m}) \end{aligned} \tag{17.10}$$

and

$$\begin{aligned} T_{nm}{}^{\alpha}| = {} & E_{m}{}^{b}E_{n}{}^{c}T_{cb}{}^{\alpha}| + E_{m}{}^{b}E_{n}{}^{\gamma}T_{\gamma b}{}^{\alpha}| \\ & + E_{m}{}^{\beta}E_{n}{}^{c}T_{c\beta}{}^{\alpha}| + E_{m}{}^{b}E_{n\dot{\gamma}}T^{\dot{\gamma}}{}_{b}{}^{\alpha}| \\ & + E_{m\dot{\beta}}E_{n}{}^{c}T_{c}{}^{\dot{\beta}\alpha}|. \end{aligned} \tag{17.11}$$

Combining (17.10) and (17.7) gives the connection in terms of e, ψ, and $\bar{\psi}$:

$$\begin{aligned} \omega_{nm\ell} = \frac{1}{2}\Big\{ & -\frac{i}{2}e_{\ell a}(\psi_{m}\sigma^{a}\bar{\psi}_{n} - \psi_{n}\sigma^{a}\bar{\psi}_{m}) - \frac{i}{2}e_{ma}(\psi_{n}\sigma^{a}\bar{\psi}_{\ell} - \psi_{\ell}\sigma^{a}\bar{\psi}_{n}) \\ & + \frac{i}{2}e_{na}(\psi_{\ell}\sigma^{a}\bar{\psi}_{m} - \psi_{m}\sigma^{a}\bar{\psi}_{\ell}) - e_{\ell a}(\partial_{n}e_{m}{}^{a} - \partial_{m}e_{n}{}^{a}) \\ & - e_{ma}(\partial_{\ell}e_{n}{}^{a} - \partial_{n}e_{\ell}{}^{a}) + e_{na}(\partial_{m}e_{\ell}{}^{a} - \partial_{\ell}e_{m}{}^{a})\Big\}. \end{aligned} \tag{17.12}$$

We must now evaluate Eq. (17.11). The torsion components in this equation were computed in Chapter XV. Equation (15.38.6) relates $T_{cb}{}^{\alpha}$ to $W_{\alpha\beta\gamma}$ and $\bar{\mathscr{D}}_{\dot{\alpha}}G_{\beta\dot{\gamma}}$. Equations (15.38.2) and (15.38.4) relate $T_{\dot{\gamma}b}{}^{\alpha}$ and $T_{\gamma b}{}^{\alpha}$ to R and $G_{\alpha\dot{\beta}}$. We may use these expressions, along with (17.7) and (17.11), to compute $W_{\alpha\beta\gamma}|$ and $\bar{\mathscr{D}}_{\dot{\alpha}}G_{\beta\dot{\gamma}}|$ in terms of $e_{m}{}^{a}$, $\psi_{m}{}^{\alpha}$, $\bar{\psi}_{m\dot{\alpha}}$, and the lowest components of R and $G_{\alpha\dot{\beta}}$. This is done in Exercises 5 and 7.

The lowest components of R and $G_{\alpha\dot{\beta}}$ cannot be expressed in terms of $e_m{}^a$, $\psi_m{}^\alpha$, and $\bar{\psi}_{m\dot{\alpha}}$. Nor may they be gauged away:

$$\begin{aligned} \delta R &= -\xi^C \mathscr{D}_C R \\ \delta G_a &= -\xi^C \mathscr{D}_C G_a + G_b L^b{}_a. \end{aligned} \tag{17.13}$$

This forces us to introduce two new component fields:

$$\begin{aligned} R(z)\big|_{\theta=\bar{\theta}=0} &= -\frac{1}{6} M(x) \\ G_a(z)\big|_{\theta=\bar{\theta}=0} &= -\frac{1}{3} b_a(x). \end{aligned} \tag{17.14}$$

These fields equalize the number of bosonic and fermionic degrees of freedom within the supergravity multiplet M, b, ψ, and e. We shall see that the supergravity multiplet forms a complete set of dynamical fields.

To conclude this chapter, we follow the same procedure with $R_{nmA}{}^B$, the only tensor we have not yet discussed. Taking the $\theta = \bar{\theta} = 0$ component of (14.22), we find

$$\begin{aligned} R_{nma}{}^b\big| &= \partial_n \omega_{ma}{}^b - \partial_m \omega_{na}{}^b + \omega_{ma}{}^c \omega_{nc}{}^b - \omega_{na}{}^c \omega_{mc}{}^b \\ &\equiv \mathscr{R}_{nma}{}^b. \end{aligned} \tag{17.15}$$

This equation defines the Riemann curvature $\mathscr{R}_{nma}{}^b$ in terms of the connection $\omega_{ma}{}^b$. In analogy with (17.11), we relate $R_{nma}{}^b$ to the Lorentz-covariant tensor $R_{CDa}{}^b$:

$$\begin{aligned} R_{nma}{}^b &= E_n{}^C E_m{}^D R_{CDa}{}^b (-)^{cd} \\ &= E_n{}^c E_m{}^d R_{cda}{}^b + E_n{}^{\underline{\gamma}} E_m{}^d R_{\underline{\gamma} da}{}^b \\ &\quad + E_n{}^c E_m{}^{\underline{\delta}} R_{c\underline{\delta} a}{}^b - E_n{}^{\underline{\gamma}} E_m{}^{\underline{\delta}} R_{\underline{\gamma}\underline{\delta} a}{}^b. \end{aligned} \tag{17.16}$$

The underlined spinor indices are summed over dotted and undotted indices. Comparing with the solutions to the Bianchi identities (15.39.8), we see that $R_{cda}{}^b$ is related to the second derivatives of R and G and the first derivative of W. Similarly, $R_{\underline{\gamma} da}{}^b$ and $R_{\underline{\gamma}\underline{\delta} a}{}^b$ are related to R, G, W, and the first derivative of G. Combining (17.15) and (17.16) allows us to solve for the second derivatives of R and G and the first derivative of W in terms of the supergravity multiplet M, b, ψ, and e.

All we have left to compute are the first derivative of R and the second derivative of W. The first derivative of R is related to the first derivative of G through the Bianchi identities. It is computed in Exercise 8. The second derivative of W is outlined in Exercise 10.

With the results of this chapter, we have what we need to compute the torsion and the curvature. The first step is to find the components of R, G, and W in terms of the supergravity multiplet. From this, we can then derive the torsion and curvature through the solutions to the Bianchi identities. Those components of R, G, and W that we will need are collected below.

References

B. Zumino, in *Recent Developments in Gravitation*, M. Levy and S. Deser, eds. (Cargèse 1978), New York, Plenum (1979).

K. S. Stelle and P. C. West, *Phys. Lett. 74B*, 330 (1978).

Equations

$$\delta E_M{}^A = -\mathscr{D}_M \xi^A - \xi^B T_{BM}{}^A + E_M{}^B L_B{}^A. \tag{17.1}$$

$$E_M{}^A(z)\big|_{\theta=\bar{\theta}=0} = \begin{pmatrix} e_m{}^a(x) & \frac{1}{2}\psi_m{}^\alpha(x) & \frac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix}. \tag{17.2}$$

$$E_A{}^M(z)\big|_{\theta=\bar{\theta}=0} = \begin{pmatrix} e_a{}^m(x) & -\frac{1}{2}\psi_a{}^\mu(x) & -\frac{1}{2}\bar{\psi}_{a\dot{\mu}}(x) \\ 0 & \delta_\alpha{}^\mu & 0 \\ 0 & 0 & \delta^{\dot{\alpha}}{}_{\dot{\mu}} \end{pmatrix}. \tag{17.3}$$

$$\begin{aligned} \delta\phi_{MA}{}^B = {} & -\xi^C R_{CMA}{}^B + \phi_{MA}{}^C L_C{}^B \\ & - (-)^{m(a+c)} L_A{}^C \phi_{MC}{}^B - \partial_M L_A{}^B. \end{aligned} \tag{17.5}$$

$$\begin{aligned} \phi_{mA}{}^B(z)\big|_{\theta=\bar{\theta}=0} &= \omega_{mA}{}^B(x) \\ \phi_{\mu A}{}^B(z)\big|_{\theta=\bar{\theta}=0} &= \phi^{\dot{\mu}}{}_A{}^B(z)\big|_{\theta=\bar{\theta}=0} = 0. \end{aligned} \tag{17.6}$$

$$\begin{aligned} \tilde{\mathscr{D}}_n \psi_m{}^\alpha &= \partial_n \psi_m{}^\alpha + \psi_m{}^\beta \omega_{n\beta}{}^\alpha \\ \psi_{nm}{}^\alpha &= \tilde{\mathscr{D}}_n \psi_m{}^\alpha - \tilde{\mathscr{D}}_m \psi_n{}^\alpha. \end{aligned} \tag{17.8}$$

$$\omega_{nm\ell} = \frac{1}{2}\Big\{ -\frac{i}{2}\, e_{\ell a}(\psi_m\sigma^a\bar{\psi}_n - \psi_n\sigma^a\bar{\psi}_m) - \frac{i}{2}\, e_{ma}(\psi_n\sigma^a\bar{\psi}_\ell - \psi_\ell\sigma^a\bar{\psi}_n)$$
$$+ \frac{i}{2}\, e_{na}(\psi_\ell\sigma^a\bar{\psi}_m - \psi_m\sigma^a\bar{\psi}_\ell) - e_{\ell a}(\partial_n e_m{}^a - \partial_m e_n{}^a)$$
$$- e_{ma}(\partial_\ell e_n{}^a - \partial_n e_\ell{}^a) + e_{na}(\partial_m e_\ell{}^a - \partial_\ell e_m{}^a)\Big\}. \qquad (17.12)$$

$$\delta R = -\xi^C \mathscr{D}_C R$$
$$\delta G_a = -\xi^C \mathscr{D}_C G_a + G_b L^b{}_a. \qquad (17.13)$$

$$R(z)\big|_{\theta=\bar{\theta}=0} = -\frac{1}{6} M(x)$$
$$G_a(z)\big|_{\theta=\bar{\theta}=0} = -\frac{1}{3} b_a(x). \qquad (17.14)$$

$$R_{nma}{}^b\big| = \partial_n\omega_{ma}{}^b - \partial_m\omega_{na}{}^b + \omega_{ma}{}^c\omega_{nc}{}^b - \omega_{na}{}^c\omega_{mc}{}^b$$
$$\equiv \mathscr{R}_{nma}{}^b. \qquad (17.15)$$

$$W_{\delta\gamma\alpha}\big| = \frac{1}{2\cdot 4!}\sum_{P(\delta\gamma\alpha)} (\psi_{\delta\dot{\delta}\gamma}{}^{\dot{\delta}}{}_\alpha + i\psi_{\delta\dot{\gamma}\gamma} b_\alpha{}^{\dot{\gamma}}). \qquad (5)$$

$$\mathscr{D}_\delta G_{\alpha\dot{\alpha}}\big| = \frac{1}{4}\bar{\psi}_\alpha{}^{\dot{\gamma}}{}_{\delta\dot{\gamma}\dot{\alpha}} + \frac{1}{12}\varepsilon_{\delta\alpha}\bar{\psi}^{\gamma\dot{\gamma}}{}_{\gamma\dot{\alpha}\dot{\gamma}} - \frac{i}{6}\psi_{\alpha\dot{\alpha}\delta}M^*$$
$$+ \frac{i}{12}(\bar{\psi}_{\alpha\dot{\rho}}{}^{\dot{\rho}} b_{\delta\dot{\alpha}} + \bar{\psi}_{\delta\dot{\rho}}{}^{\dot{\rho}} b_{\alpha\dot{\alpha}} - \bar{\psi}_\delta{}^{\dot{\rho}}{}_{\dot{\alpha}} b_{\alpha\dot{\rho}})$$

$$\bar{\mathscr{D}}_{\dot{\delta}} G_{\alpha\dot{\alpha}}\big| = \frac{1}{4}\psi^\gamma{}_{\dot{\delta}\gamma\dot{\alpha}\alpha} + \frac{1}{12}\varepsilon_{\dot{\delta}\dot{\alpha}}\psi_\alpha{}^{\dot{\gamma}\gamma}{}_{\dot{\gamma}\gamma} + \frac{i}{6}\bar{\psi}_{\alpha\dot{\alpha}\dot{\delta}}M$$
$$- \frac{i}{12}(\psi_{\rho\dot{\alpha}}{}^\rho b_{\alpha\dot{\delta}} + \psi_{\rho\dot{\delta}}{}^\rho b_{\alpha\dot{\alpha}} - \psi^\rho{}_{\dot{\delta}\alpha} b_{\rho\dot{\alpha}}). \qquad (7)$$

$$\mathscr{D}_\alpha R\big| = -\frac{1}{3}(\sigma^{ab})_\alpha{}^\beta\psi_{ab\beta} + \frac{i}{6}(\sigma^a\bar{\psi}_a)_\alpha M - \frac{i}{6}\psi_{a\alpha}b^a$$
$$\bar{\mathscr{D}}^{\dot{\alpha}} R^+\big| = -\frac{1}{3}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}_{ab}{}^{\dot{\beta}} + \frac{i}{6}(\bar{\sigma}^a\psi_a)^{\dot{\alpha}}M^* + \frac{i}{6}\bar{\psi}_a{}^{\dot{\alpha}}b^a. \qquad (8)$$

Exercises

(1) Show that $E_M{}^A$ may be gauged into the form (17.2). Use the freedom available in the higher components of ξ:

$$\begin{aligned}
\xi^A &= \xi^{(0,0)A}(x) + \theta^\mu \xi_\mu{}^{(1,0)A}(x) + \bar{\theta}_{\dot{\mu}} \bar{\xi}^{\dot{\mu}(0,1)A}(x) + \cdots \\
\mathscr{D}_\mu \xi^A &= \frac{\partial}{\partial \theta^\mu} \xi^A + \cdots \\
\bar{\mathscr{D}}^{\dot{\mu}} \xi^A &= \frac{\partial}{\partial \bar{\theta}_{\dot{\mu}}} \xi^A + \cdots
\end{aligned}$$

(2) In ordinary four-dimensional relativity, the torsion takes the form

$$T_{nm}{}^a = \partial_n e_m{}^a - \partial_m e_n{}^a + \omega_{nm}{}^a - \omega_{mn}{}^a.$$

Impose the constraint $T_{nm}{}^a = 0$ and solve for ω in terms of e.

(3) Solve (17.7) and (17.10) for $\omega_{mn\ell}$. Use the fact that

$$\omega_{mn\ell} = e_{\ell a} e_n{}^b \omega_{mb}{}^a = -\omega_{m\ell n}.$$

(4) Use the definitions

$$\begin{aligned}
\psi_{\delta\dot{\gamma}\gamma} &= \sigma_{\delta\dot{\gamma}}{}^a e_a{}^m \psi_{m\gamma} \\
\psi_{\delta\dot{\delta}\gamma\dot{\gamma}\alpha} &= \sigma_{\delta\dot{\delta}}{}^d \sigma_{\gamma\dot{\gamma}}{}^c e_d{}^n e_c{}^m \psi_{nm\alpha}
\end{aligned}$$

along with (17.7) and (17.11) to verify

$$\begin{aligned}
T_{\delta\dot{\delta}\gamma\dot{\gamma}\alpha}| &= \frac{1}{2}\psi_{\delta\dot{\delta}\gamma\dot{\gamma}\alpha} - \frac{1}{2}(\psi_{\delta\dot{\delta}}{}^\rho T_{\rho\gamma\dot{\gamma}\alpha}| - \psi_{\gamma\dot{\gamma}}{}^\rho T_{\rho\delta\dot{\delta}\alpha}|) \\
&\quad + \frac{1}{2}(\bar{\psi}_{\delta\dot{\delta}}{}^{\dot{\rho}} T_{\dot{\rho}\gamma\dot{\gamma}\alpha}| - \bar{\psi}_{\gamma\dot{\gamma}}{}^{\dot{\rho}} T_{\dot{\rho}\delta\dot{\delta}\alpha}|).
\end{aligned}$$

(5) Show that Exercise 4 and the solutions to the Bianchi identities give

$$W_{\delta\gamma\alpha}| = \frac{1}{2 \cdot 4!} \sum_{P(\delta\gamma\alpha)} (\psi_{\delta\dot{\delta}\gamma}{}^{\dot{\delta}}{}_\alpha + i\psi_{\delta\dot{\gamma}\gamma} b_\alpha{}^{\dot{\gamma}}).$$

(6) Use the solutions to the Bianchi identities to show that

$$\mathscr{D}_\delta G_{\alpha\dot{\alpha}} = \frac{1}{2}\varepsilon^{\dot{\gamma}\dot{\delta}}T_{\alpha\dot{\delta}\delta\dot{\gamma}\dot{\alpha}} + \frac{1}{6}\varepsilon_{\delta\alpha}\varepsilon^{\gamma\lambda}\varepsilon^{\dot{\lambda}\dot{\delta}}T_{\lambda\dot{\delta}\gamma\dot{\alpha}\dot{\lambda}}$$

$$\bar{\mathscr{D}}_{\dot{\delta}}G_{\alpha\dot{\alpha}} = \frac{1}{2}\varepsilon^{\gamma\delta}T_{\delta\dot{\delta}\gamma\dot{\alpha}\alpha} - \frac{1}{6}\varepsilon_{\dot{\delta}\dot{\alpha}}\varepsilon^{\dot{\lambda}\dot{\gamma}}\varepsilon^{\delta\gamma}T_{\gamma\dot{\lambda}\alpha\dot{\gamma}\delta}.$$

(7) Compute $\mathscr{D}_\delta G_{\alpha\dot{\alpha}}|$ and $\bar{\mathscr{D}}_{\dot{\delta}}G_{\alpha\dot{\alpha}}|$ in terms of the supergravity multiplet:

$$\mathscr{D}_\delta G_{\alpha\dot{\alpha}}| = \frac{1}{4}\bar{\psi}_\alpha{}^{\dot{\gamma}}{}_{\delta\dot{\gamma}\dot{\alpha}} + \frac{1}{12}\varepsilon_{\delta\alpha}\bar{\psi}^{\gamma\dot{\gamma}}{}_{\gamma\dot{\alpha}\dot{\gamma}} - \frac{i}{6}\psi_{\alpha\dot{\alpha}\delta}M^*$$

$$+ \frac{i}{12}(\bar{\psi}_{\alpha\rho}{}^{\dot{\rho}}b_{\delta\dot{\alpha}} + \bar{\psi}_{\delta\rho}{}^{\dot{\rho}}b_{\alpha\dot{\alpha}} - \bar{\psi}_\delta{}^{\dot{\rho}}{}_{\dot{\alpha}}b_{\alpha\dot{\rho}})$$

$$\bar{\mathscr{D}}_{\dot{\delta}}G_{\alpha\dot{\alpha}}| = \frac{1}{4}\psi^\gamma{}_{\dot{\delta}\gamma\dot{\alpha}\alpha} + \frac{1}{12}\varepsilon_{\dot{\delta}\dot{\alpha}}\psi_\alpha{}^{\dot{\gamma}\gamma}{}_{\dot{\gamma}\gamma} + \frac{i}{6}\bar{\psi}_{\alpha\dot{\alpha}\dot{\delta}}M$$

$$- \frac{i}{12}(\psi_{\rho\dot{\alpha}}{}^\rho b_{\alpha\dot{\delta}} + \psi_{\rho\dot{\delta}}{}^\rho b_{\alpha\dot{\alpha}} - \psi^\rho{}_{\dot{\delta}\alpha}b_{\rho\dot{\alpha}}).$$

(8) Use (15.37.2) and the results of Exercise 7 to compute $\mathscr{D}_\alpha R|$ and $\bar{\mathscr{D}}^{\dot{\alpha}}R^+|$ in terms of the supergravity multiplet:

$$\mathscr{D}_\alpha R| = -\frac{1}{3}(\sigma^{ab})_\alpha{}^\beta\psi_{ab\beta} + \frac{i}{6}(\sigma^a\bar{\psi}_a)_\alpha M - \frac{i}{6}\psi_{a\alpha}b^a$$

$$\bar{\mathscr{D}}^{\dot{\alpha}}R^+| = -\frac{1}{3}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}_{ab}{}^{\dot{\beta}} + \frac{i}{6}(\bar{\sigma}^a\psi_a)^{\dot{\alpha}}M^* + \frac{i}{6}\bar{\psi}_a{}^{\dot{\alpha}}b^a.$$

(9) Denote F by R and write the Bianchi identities (12.31) in the following form:

$$E^C E^D E^E\{\mathscr{D}_E R_{DCA}{}^B + T_{ED}{}^F R_{FCA}{}^B\} = 0.$$

Show that this implies

$$\begin{aligned}&\bar{\mathscr{D}}_{\dot{\varepsilon}}R_{dc\alpha}{}^\beta + \mathscr{D}_d R_{c\dot{\varepsilon}\alpha}{}^\beta + \mathscr{D}_c R_{\dot{\varepsilon}d\alpha}{}^\beta \\ &\quad + T_{\dot{\varepsilon}d}{}^F R_{Fc\alpha}{}^\beta + T_{dc}{}^F R_{F\dot{\varepsilon}\alpha}{}^\beta + T_{c\dot{\varepsilon}}{}^F R_{Fd\alpha}{}^\beta = 0.\end{aligned}$$

(10) Use the solutions of the Bianchi identities and the result of the previous exercise to show that $\mathscr{D}\mathscr{D}W$ may be computed. *Warning*: the actual calculation is tedious!

XVIII. THE SUPERGRAVITY MULTIPLET

We are now ready to derive the transformation law of the supergravity multiplet $e_m{}^a$, $\psi_m{}^\alpha$, $\bar\psi_{m\dot\alpha}$, b_a, and M. We start with the general transformation law of the vielbein

$$\delta E_M{}^A = -\mathscr{D}_M \xi^A - \xi^B T_{BM}{}^A + E_M{}^B L_B{}^A \tag{18.1}$$

and evaluate its lowest component. The $\theta = \bar\theta = 0$ components of ξ^α and $\bar\xi_{\dot\alpha}$ parametrize gauged supersymmetry transformations. We shall focus on these by setting

$$\begin{aligned} \xi^a(z)\big|_{\theta=\bar\theta=0} &= 0 \\ \xi^\alpha(z)\big|_{\theta=\bar\theta=0} &= \zeta^\alpha(x) \\ \bar\xi_{\dot\alpha}(z)\big|_{\theta=\bar\theta=0} &= \bar\zeta_{\dot\alpha}(x) \\ L_{AB}(z)\big|_{\theta=\bar\theta=0} &= 0. \end{aligned} \tag{18.2}$$

Higher components of ξ^A and L_{AB} will be chosen to preserve the gauge (17.2) and (17.6).

To preserve (17.2) we must require

$$\delta E_\mu{}^A\big| = \delta E^{\dot\mu A}\big| = 0. \tag{18.3}$$

From (18.1) and the constraints (14.25), we find

$$\begin{aligned} \delta E_\mu{}^a\big| &= \left(-\frac{\partial}{\partial\theta^\mu}\xi^a - \xi^b \phi_{\mu b}{}^a - \bar\xi_{\dot\beta} T^{\dot\beta}{}_\mu{}^a + E_\mu{}^b L_b{}^a\right)\Bigg| \\ &= -\frac{\partial}{\partial\theta^\mu}\xi^a\bigg| + 2i\sigma_{\mu\dot\beta}{}^a \bar\zeta^{\dot\beta} = 0 \\ \delta E^{\dot\mu a}\big| &= -\frac{\partial}{\partial\bar\theta_{\dot\mu}}\xi^a\bigg| + 2i\zeta^\beta \sigma_{\beta\dot\kappa}{}^a \varepsilon^{\dot\kappa\dot\mu} = 0. \end{aligned} \tag{18.4}$$

Equations (18.4) are satisfied if

$$\xi^a = 2i(\theta\sigma^a\bar{\zeta} - \zeta\sigma^a\bar{\theta}). \tag{18.5}$$

No further conditions follow from (18.3).

To preserve (17.6) we must demand

$$\delta\phi_{\mu A}{}^{B}| = \delta\phi^{\dot\mu}{}_{A}{}^{B}| = 0. \tag{18.6}$$

From (16.20), we have

$$\delta\phi_{\mu\alpha}{}^{\beta}| = -\xi^C R_{C\mu\alpha}{}^{\beta}| - \partial_\mu L_\alpha{}^\beta|. \tag{18.7}$$

The curvature term does not vanish. From the solutions (15.39) of the Bianchi identities, we know that it contains M and b:

$$\begin{aligned}
R_{\gamma\mu\alpha\beta}| &= 4(\varepsilon_{\gamma\alpha}\varepsilon_{\mu\beta} + \varepsilon_{\mu\alpha}\varepsilon_{\gamma\beta})R^{+}|. \\
&= -\frac{2}{3}(\varepsilon_{\gamma\alpha}\varepsilon_{\mu\beta} + \varepsilon_{\mu\alpha}\varepsilon_{\gamma\beta})M^* \\
R_{\dot\gamma\mu\alpha\beta}| &= -(\varepsilon_{\mu\alpha}G_{\beta\dot\gamma} + \varepsilon_{\mu\beta}G_{\alpha\dot\gamma})| \\
&= \frac{1}{3}(\varepsilon_{\mu\alpha}b_{\beta\dot\gamma} + \varepsilon_{\mu\beta}b_{\alpha\dot\gamma}).
\end{aligned} \tag{18.8}$$

Substituting (18.8) into (18.7), and imposing (18.6), we find

$$\begin{aligned}
\partial_\mu L_{\alpha\beta} = &-\frac{2}{3}(\zeta_\alpha\varepsilon_{\mu\beta} + \zeta_\beta\varepsilon_{\mu\alpha})M^* \\
&+ \frac{1}{3}(\varepsilon_{\mu\alpha}b_{\beta\dot\gamma} + \varepsilon_{\mu\beta}b_{\alpha\dot\gamma})\bar{\zeta}^{\dot\gamma}.
\end{aligned} \tag{18.9}$$

This tells us that a gauged supersymmetry transformation ζ_α must be accompanied by a field-dependent Lorentz transformation

$$\begin{aligned}
L_{\alpha\beta} &= \frac{1}{3}[\theta_\alpha(2\zeta_\beta M^* - b_{\beta\dot\gamma}\bar{\zeta}^{\dot\gamma}) + \theta_\beta(2\zeta_\alpha M^* - b_{\alpha\dot\gamma}\bar{\zeta}^{\dot\gamma})] \\
L_{\dot\alpha\dot\beta} &= \frac{1}{3}[\bar{\theta}_{\dot\alpha}(2\bar{\zeta}_{\dot\beta}M - \zeta^\gamma b_{\gamma\dot\beta}) + \bar{\theta}_{\dot\beta}(2\bar{\zeta}_{\dot\alpha}M - \zeta^\gamma b_{\gamma\dot\alpha})]
\end{aligned} \tag{18.10}$$

to preserve the gauge (17.6). Equations (18.5) and (18.10) are the only conditions that follow from the gauge fixing.

To summarize, we have found a set of transformations, parametrized by ζ, which include gauged supersymmetry transformations and preserve the gauge (17.2), (17.6). We shall call these the supergravity transformations:

$$\begin{aligned}
\xi^\alpha(z) &= \zeta^\alpha(x) \qquad \bar{\xi}_{\dot\alpha}(z) = \bar{\zeta}_{\dot\alpha}(x) \\
\xi^a(z) &= 2i[\theta\sigma^a\bar{\zeta}(x) - \zeta(x)\sigma^a\bar{\theta}] \\
L_{\alpha\beta}(z) &= \frac{1}{3}\{\theta_\alpha[2\zeta_\beta(x)M^*(x) - b_{\beta\dot\gamma}(x)\bar{\zeta}^{\dot\gamma}(x)] \\
&\qquad + \theta_\beta[2\zeta_\alpha(x)M^*(x) - b_{\alpha\dot\gamma}(x)\bar{\zeta}^{\dot\gamma}(x)]\} \\
L_{\dot\alpha\dot\beta}(z) &= \frac{1}{3}\{\bar{\theta}_{\dot\alpha}[2\bar{\zeta}_{\dot\beta}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot\beta}(x)] \\
&\qquad + \bar{\theta}_{\dot\beta}[2\bar{\zeta}_{\dot\alpha}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot\alpha}(x)]\} \\
L_{ab} &= \frac{1}{2}(\bar{\sigma}_a\sigma_b\varepsilon)^{\dot\alpha\dot\beta}L_{\dot\alpha\dot\beta} - \frac{1}{2}(\varepsilon\sigma_a\bar{\sigma}_b)^{\alpha\beta}L_{\alpha\beta}.
\end{aligned} \qquad \mathbf{(18.11)}$$

We are now ready to compute the transformation laws of the component fields. We start with the vierbein. From (18.1) and (18.11), we find

$$\begin{aligned}
\delta e_m{}^a = \delta E_m{}^a| &= -\mathcal{D}_m\xi^a| - \xi^B T_{Bm}{}^a| \\
&= -\xi^\beta T_{\beta m}{}^a| - \bar{\xi}_{\dot\beta}T^{\dot\beta}{}_m{}^a|.
\end{aligned} \qquad (18.12)$$

The terms proportional to ξ^a do not contribute for $\theta = \bar{\theta} = 0$. The torsion terms may be evaluated with the help of the constraints:

$$\begin{aligned}
T_{\beta m}{}^a &= E_m{}^C T_{\beta C}{}^a(-)^{b(m+c)} \\
&= 2iE_m{}^{\dot\gamma}\sigma_{\beta\dot\gamma}{}^a \\
T_{\dot\beta m}{}^a &= E_m{}^C T_{\dot\beta C}{}^a(-)^{b(m+c)} \\
&= -2iE_m{}^\gamma\sigma_{\gamma\dot\beta}{}^a.
\end{aligned} \qquad (18.13)$$

Their $\theta = \bar{\theta} = 0$ components are specified through (17.2):

$$\begin{aligned}
T_{\beta m}{}^a| &= i\sigma_{\beta\dot\beta}{}^a\bar{\psi}_m{}^{\dot\beta} \\
T_{\dot\beta m}{}^a| &= -i\psi_m{}^\beta\sigma_{\beta\dot\beta}{}^a.
\end{aligned} \qquad (18.14)$$

Inserting (18.14) into (18.12) gives the transformation law of the vierbein:

$$\delta e_m{}^a(x) = i(\psi_m\sigma^a\bar{\zeta} - \zeta\sigma^a\bar{\psi}_m). \qquad (18.15)$$

We now turn to the gravitino:

$$\frac{1}{2}\,\delta\psi_m{}^\alpha = \delta E_m{}^\alpha\big| = -\mathscr{D}_m\xi^\alpha\big| - \xi^B T_{Bm}{}^\alpha\big|. \tag{18.16}$$

As before, we write

$$\begin{aligned} T_{\dot\beta m}{}^\alpha &= E_m{}^C T_{\dot\beta C}{}^\alpha(-)^{b(m+c)} \\ T_{\beta m}{}^\alpha &= E_m{}^C T_{\beta C}{}^\alpha(-)^{b(m+c)}. \end{aligned} \tag{18.17}$$

The solutions (15.38.2) and (15.38.4) of the Bianchi identities give $T_{\dot\beta c}{}^\alpha$ and $T_{\beta c}{}^\alpha$ in terms of R and G_a:

$$\begin{aligned} T^{\dot\beta}{}_c{}^\alpha &= -i\bar\sigma^{\dot\beta\alpha}{}_c R \\ T_{\beta c}{}^\alpha &= \frac{i}{8}\,\bar\sigma^{\dot\varepsilon\varepsilon}{}_c\{\delta_\varepsilon{}^\alpha G_{\beta\dot\varepsilon} - 3\delta_\beta{}^\alpha G_{\varepsilon\dot\varepsilon} + 3\varepsilon_{\beta\varepsilon}G^\alpha{}_{\dot\varepsilon}\}. \end{aligned} \tag{18.18}$$

Restricting to $\theta = \bar\theta = 0$, we find

$$\begin{aligned} T_{\dot\beta c}{}^\alpha\big| &= \frac{i}{6}\,(\varepsilon\bar\sigma_c)_{\dot\beta}{}^\alpha M \\ T_{\beta c}{}^\alpha\big| &= -\frac{i}{2}\left\{\frac{1}{3}\,(\sigma_d\bar\sigma_c)_\beta{}^\alpha + \delta_\beta{}^\alpha\eta_{cd}\right\}b^d. \end{aligned} \tag{18.19}$$

Substituting (18.19) in (18.16) gives the transformation law for the gravitino:

$$\begin{aligned} \delta\psi_m{}^\alpha = {} & -2\mathscr{D}_m\zeta^\alpha + \frac{i}{3}\,e_m{}^a(\varepsilon\sigma_a\bar\zeta)^\alpha M \\ & + ie_m{}^b\zeta^\beta\left(\delta_\beta{}^\alpha\eta_{bc} + \frac{1}{3}\,(\sigma_c\bar\sigma_b)_\beta{}^\alpha\right)b^c. \end{aligned} \tag{18.20}$$

A similar calculation holds for $\bar\psi$:

$$\begin{aligned} \delta\bar\psi_{m\dot\alpha} = {} & -2\mathscr{D}_m\bar\zeta_{\dot\alpha} - \frac{i}{3}\,e_m{}^a\zeta^\beta\sigma_{a\beta\dot\alpha}M^* \\ & - ie_m{}^b\bar\zeta_{\dot\beta}\left(\delta^{\dot\beta}{}_{\dot\alpha}\eta_{bc} + \frac{1}{3}\,(\bar\sigma_c\sigma_b)^{\dot\beta}{}_{\dot\alpha}\right)b^c. \end{aligned} \tag{18.21}$$

The transformation laws for M and b_a follow from (16.9):

$$
\begin{aligned}
-\frac{1}{6}\,\delta M &= \delta R| = -\xi^\alpha \mathcal{D}_\alpha R| \\
-\frac{1}{3}\,\delta b_a &= \delta G_a| = -(\xi^\alpha \mathcal{D}_\alpha - \bar{\xi}^{\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha}) G_a|.
\end{aligned}
\tag{18.22}
$$

The term proportional to $\bar{\xi}_{\dot\alpha}$ in δM drops out because R is chiral. The Lorentz transformation (18.10) does not contribute to δG_a for $\theta = \bar\theta = 0$. In Chapter XVII, Exercises 7 and 8, we computed $\mathcal{D}_\alpha R|$ and $\mathcal{D}_\alpha G|$ in terms of the supergravity multiplet. From here it is only a short calculation to find δM and δb_a.

In conclusion, we collect our results for future reference:

$$
\begin{aligned}
\delta e_m{}^a &= i(\psi_m \sigma^a \bar\zeta - \zeta \sigma^a \bar\psi_m) \\
\delta \psi_m{}^\alpha &= -2\mathcal{D}_m \zeta^\alpha + i e_m{}^c \left\{ \frac{1}{3} M (\varepsilon \sigma_c \bar\zeta)^\alpha + b_c \zeta^\alpha + \frac{1}{3} b^d (\zeta \sigma_d \bar\sigma_c)^\alpha \right\} \\
\delta \bar\psi_{m\dot\alpha} &= -2\mathcal{D}_m \bar\zeta_{\dot\alpha} - i e_m{}^c \left\{ \frac{1}{3} M^* (\zeta \sigma_c)_{\dot\alpha} + b_c \bar\zeta_{\dot\alpha} - \frac{1}{3} b^d (\bar\sigma_c \sigma_d \bar\zeta)_{\dot\alpha} \right\} \\
\delta M &= -\zeta(\sigma^a \bar\sigma^b \psi_{ab} + i b^a \psi_a - i \sigma^a \bar\psi_a M) \\
\delta b_{\alpha\dot\alpha} &= \zeta^\delta \left\{ \frac{3}{4} \bar\psi_\alpha{}^{\dot\gamma}{}_{\delta\dot\gamma\dot\alpha} + \frac{1}{4} \varepsilon_{\delta\alpha} \bar\psi^{\gamma\dot\gamma}{}_{\gamma\dot\alpha\dot\gamma} - \frac{i}{2} M^* \psi_{\alpha\dot\alpha\delta} + \frac{i}{4} (\bar\psi_{\alpha\dot\rho}{}^{\dot\rho} b_{\delta\dot\alpha} \right. \\
&\quad \left. + \bar\psi_{\delta\dot\rho}{}^{\dot\rho} b_{\alpha\dot\alpha} - \bar\psi_\delta{}^{\dot\rho}{}_{\dot\alpha} b_{\alpha\dot\rho}) \right\} - \bar\zeta^{\dot\delta} \left\{ \frac{3}{4} \psi^\gamma{}_{\dot\delta\gamma\dot\alpha\alpha} + \frac{1}{4} \varepsilon_{\dot\delta\dot\alpha} \psi_\alpha{}^{\dot\gamma\gamma}{}_{\dot\gamma\gamma} \right. \\
&\quad \left. + \frac{i}{2} M \bar\psi_{\alpha\dot\alpha\dot\delta} - \frac{i}{4} (\psi_{\rho\dot\alpha}{}^\rho b_{\alpha\dot\delta} + \psi_{\rho\dot\delta}{}^\rho b_{\alpha\dot\alpha} - \psi^\rho{}_{\dot\delta\alpha} b_{\rho\dot\alpha}) \right\}.
\end{aligned}
\tag{18.23}
$$

References

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S. Ferrara, F. Gliozzi, J. Scherk, and P. van Nieuwenhuizen, *Nucl. Phys. B117*, 333 (1976).

Equations

$$
\xi^\alpha(z) = \zeta^\alpha(x) \qquad \bar\xi_{\dot\alpha}(z) = \bar\zeta_{\dot\alpha}(x)
$$

$$
\xi^a(z) = 2i[\theta \sigma^a \bar\zeta(x) - \zeta(x) \sigma^a \bar\theta]
$$

$$L_{\alpha\beta}(z) = \frac{1}{3}\{\theta_\alpha[2\zeta_\beta(x)M^*(x) - b_{\beta\dot\gamma}(x)\bar\zeta^{\dot\gamma}(x)] + \theta_\beta[2\zeta_\alpha(x)M^*(x) - b_{\alpha\dot\gamma}(x)\bar\zeta^{\dot\gamma}(x)]\}$$

$$L_{\dot\alpha\dot\beta}(z) = \frac{1}{3}\{\bar\theta_{\dot\alpha}[2\bar\zeta_{\dot\beta}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot\beta}(x)] + \bar\theta_{\dot\beta}[2\bar\zeta_{\dot\alpha}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot\alpha}(x)]\}$$

$$L_{ab} = \frac{1}{2}(\bar\sigma_a\sigma_b\varepsilon)^{\dot\alpha\dot\beta}L_{\dot\alpha\dot\beta} - \frac{1}{2}(\varepsilon\sigma_a\bar\sigma_b)^{\alpha\beta}L_{\alpha\beta}. \qquad \textbf{(18.11)}$$

$$\begin{aligned}
\delta e_m{}^a &= i(\psi_m\sigma^a\bar\zeta - \zeta\sigma^a\bar\psi_m)\\
\delta\psi_m{}^\alpha &= -2\mathscr{D}_m\zeta^\alpha + ie_m{}^c\left\{\frac{1}{3}M(\varepsilon\sigma_c\bar\zeta)^\alpha + b_c\zeta^\alpha + \frac{1}{3}b^d(\zeta\sigma_d\bar\sigma_c)^\alpha\right\}\\
\delta\bar\psi_{m\dot\alpha} &= -2\mathscr{D}_m\bar\zeta_{\dot\alpha} - ie_m{}^c\left\{\frac{1}{3}M^*(\zeta\sigma_c)_{\dot\alpha} + b_c\bar\zeta_{\dot\alpha} - \frac{1}{3}b^d(\bar\sigma_c\sigma_d\bar\zeta)_{\dot\alpha}\right\}\\
\delta M &= -\zeta(\sigma^a\bar\sigma^b\psi_{ab} + ib^a\psi_a - i\sigma^a\bar\psi_a M)\\
\delta b_{\alpha\dot\alpha} &= \zeta^\delta\left\{\frac{3}{4}\bar\psi_\alpha{}^{\dot\gamma}{}_{\delta\dot\gamma\dot\alpha} + \frac{1}{4}\varepsilon_{\delta\alpha}\bar\psi^{\gamma\dot\gamma}{}_{\gamma\dot\alpha\dot\gamma} - \frac{i}{2}M^*\psi_{\alpha\dot\alpha\delta} + \frac{i}{4}(\bar\psi_{\alpha\dot\rho}{}^{\dot\rho}b_{\delta\dot\alpha}\right.\\
&\quad\left. + \bar\psi_{\delta\dot\rho}{}^{\dot\rho}b_{\alpha\dot\alpha} - \bar\psi_\delta{}^{\dot\rho}{}_{\dot\alpha}b_{\alpha\dot\rho})\right\} - \bar\zeta^{\dot\delta}\left\{\frac{3}{4}\psi^\gamma{}_{\dot\delta\gamma\dot\alpha\alpha} + \frac{1}{4}\varepsilon_{\dot\delta\dot\alpha}\psi_\alpha{}^{\dot\gamma\gamma}{}_{\dot\gamma\gamma}\right.\\
&\quad\left. + \frac{i}{2}M\bar\psi_{\alpha\dot\alpha\dot\delta} - \frac{i}{4}(\psi_{\rho\dot\alpha}{}^\rho b_{\alpha\dot\delta} + \psi_{\rho\dot\delta}{}^\rho b_{\alpha\dot\alpha} - \psi^\rho{}_{\dot\delta\alpha}b_{\rho\dot\alpha})\right\}. \qquad \textbf{(18.23)}
\end{aligned}$$

Exercises

(1) Compute $\delta e_m{}^a$ for $\zeta^\alpha = \bar\zeta_{\dot\alpha} = L_{AB} = 0$, $\xi^a(z) = \xi^a(x)$. Compare this with a general coordinate transformation and a local Lorentz rotation in ordinary relativity.

(2) Compute $\delta\bar\psi_{m\dot\alpha}$ by conjugating $\delta\psi_m{}^\alpha$ in (18.20).

(3) Show that the supergravity transformations (18.11) can be augmented by terms higher-order in $(\theta,\bar\theta)$ and still preserve the transformations (18.23) of the component fields.

XIX. CHIRAL AND VECTOR SUPERFIELDS IN CURVED SPACE

In Chapter XXI we shall construct a Lagrangian, invariant under supergravity transformations, which reduces to (7.24) in the limit of flat space. Before we do this, however, we must define chiral and vector superfields in curved space.

We start with chiral superfields, which satisfy the covariant constraint condition

$$\bar{\mathscr{D}}_{\dot{\alpha}}\Phi = 0. \tag{19.1}$$

This reduces to $\bar{D}_{\dot{\alpha}}\,\Phi = 0$ in flat space.

Chiral superfields contain three component fields. We could define them as the coefficient functions of a power series expansion in θ and $\bar{\theta}$. This decomposition, however, is coordinate-dependent, for θ and $\bar{\theta}$ carry Einstein indices. It is much more convenient to define them in analogy to Exercise 4 of Chapter V:

$$\begin{aligned} A &= \Phi\big|_{\theta=\bar{\theta}=0} \\ \chi_\alpha &= \frac{1}{\sqrt{2}}\,\mathscr{D}_\alpha\Phi\big|_{\theta=\bar{\theta}=0} \\ F &= -\frac{1}{4}\,\mathscr{D}^\alpha\mathscr{D}_\alpha\Phi\big|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{19.2}$$

These components carry Lorentz indices. They are related to the $\theta,\bar{\theta}$ expansion coefficients through a transformation which depends on the supergravity multiplet.

The transformation laws of the component fields are found from the transformation law of the superfield Φ:

$$\delta\Phi = -\xi^A\mathscr{D}_A\Phi. \tag{19.3}$$

The parameters ξ^A are specified in (18.11). Since Φ is chiral, we have:

$$\delta\Phi = -\xi^a\mathscr{D}_a\Phi - \xi^\alpha\mathscr{D}_\alpha\Phi. \tag{19.4}$$

The change in A follows immediately:

$$\delta A = \delta\Phi| = -\xi^\alpha\mathscr{D}_\alpha\Phi| = -\sqrt{2}\zeta^\alpha\chi_\alpha. \tag{19.5}$$

The change in χ requires a little more work:

$$\begin{aligned}\delta\chi_\alpha &= -\frac{1}{\sqrt{2}}\xi^B\mathscr{D}_B\mathscr{D}_\alpha\Phi| \\ &= -\frac{1}{\sqrt{2}}(\xi^\beta\mathscr{D}_\beta - \bar{\xi}^{\dot\beta}\bar{\mathscr{D}}_{\dot\beta})\mathscr{D}_\alpha\Phi|.\end{aligned} \tag{19.6}$$

To proceed, we must evaluate $\mathscr{D}_\beta\mathscr{D}_\alpha\Phi|$ and $\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_\alpha\Phi|$. This may be done with (16.14):

$$(\mathscr{D}_C\mathscr{D}_B - (-)^{bc}\mathscr{D}_B\mathscr{D}_C)V^A = -T_{CB}{}^D\mathscr{D}_DV^A + (-)^{d(b+c)}V^DR_{CBD}{}^A. \tag{19.7}$$

For $\mathscr{D}_\beta\mathscr{D}_\alpha\Phi|$, Eq. (19.7) and the constraints (14.25) imply

$$\{\mathscr{D}_\beta,\mathscr{D}_\alpha\}\Phi = 0, \tag{19.8}$$

so

$$\begin{aligned}\mathscr{D}_\beta\mathscr{D}_\alpha\Phi &= \frac{1}{2}\varepsilon_{\beta\alpha}\mathscr{D}^\gamma\mathscr{D}_\gamma\Phi \\ \mathscr{D}_\beta\mathscr{D}_\alpha\Phi| &= -2\varepsilon_{\beta\alpha}F.\end{aligned} \tag{19.9}$$

For $\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_\alpha\Phi|$, we use (19.1), (19.7), and the constraints (14.25). These give

$$\begin{aligned}\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_\alpha\Phi &= \{\bar{\mathscr{D}}_{\dot\beta},\mathscr{D}_\alpha\}\Phi \\ &= -2i\sigma_{\alpha\dot\beta}{}^d\mathscr{D}_d\Phi.\end{aligned} \tag{19.10}$$

The derivative $\mathscr{D}_d\Phi$ is related to the components (19.2) through the definition (16.6) of the covariant derivative:

$$\begin{aligned}\mathscr{D}_a\Phi &= E_a{}^m\mathscr{D}_m\Phi + E_a{}^\mu\mathscr{D}_\mu\Phi + E_{a\dot\mu}\bar{\mathscr{D}}^{\dot\mu}\Phi \\ \mathscr{D}_\alpha\Phi &= E_\alpha{}^m\mathscr{D}_m\Phi + E_\alpha{}^\mu\mathscr{D}_\mu\Phi + E_{\alpha\dot\mu}\bar{\mathscr{D}}^{\dot\mu}\Phi \\ \bar{\mathscr{D}}^{\dot\alpha}\Phi &= E^{\dot\alpha m}\mathscr{D}_m\Phi + E^{\dot\alpha\mu}\mathscr{D}_\mu\Phi + E^{\dot\alpha}{}_{\dot\mu}\bar{\mathscr{D}}^{\dot\mu}\Phi.\end{aligned} \tag{19.11}$$

Restricting to $\theta = \bar{\theta} = 0$, we find

$$\begin{aligned}
\mathscr{D}_a\Phi| &= e_a{}^m\left(\mathscr{D}_m\Phi - \frac{1}{2}\psi_m{}^\mu\mathscr{D}_\mu\Phi\right)\Big| \\
&= e_a{}^m\left(\partial_m A - \frac{1}{\sqrt{2}}\psi_m{}^\mu\chi_\mu\right) \\
\mathscr{D}_\alpha\Phi| &= \delta_\alpha{}^\mu\mathscr{D}_\mu\Phi| = \sqrt{2}\chi_\alpha \\
\bar{\mathscr{D}}^{\dot\alpha}\Phi| &= \delta^{\dot\alpha}{}_{\dot\mu}\bar{\mathscr{D}}^{\dot\mu}\Phi| = 0.
\end{aligned} \tag{19.12}$$

From (19.11) we see that spacetime derivatives $e_a^m\mathscr{D}_m$ of the component fields are always accompanied by extra terms proportional to the gravitino field and higher components of the matter multiplet. We shall combine these terms into supercovariant derivatives $\hat{D}_a$. Equation (19.12) provides our first example

$$\hat{D}_a A \equiv e_a{}^m\left(\partial_m A - \frac{1}{\sqrt{2}}\psi_m{}^\mu\chi_\mu\right). \tag{19.13}$$

Combining the above results, we find the change in χ:

$$\delta\chi_\alpha = -\sqrt{2}\zeta_\alpha F - i\sqrt{2}\sigma_{\alpha\dot\beta}{}^a\bar{\zeta}^{\dot\beta}\hat{D}_a A. \tag{19.14}$$

All we have left is the change in F. We start from (16.9):

$$\delta F = \frac{1}{4}(\xi^\alpha\mathscr{D}_\alpha - \bar{\xi}^{\dot\alpha}\bar{\mathscr{D}}_{\dot\alpha})\mathscr{D}^\beta\mathscr{D}_\beta\Phi|. \tag{19.15}$$

In Exercise 2 we show that

$$\begin{aligned}
\mathscr{D}_\alpha\mathscr{D}^\beta\mathscr{D}_\beta\Phi &= \frac{2}{3}\{\mathscr{D}_\alpha,\mathscr{D}^\gamma\}\mathscr{D}_\gamma\Phi \\
&= -\frac{2}{3}R_\alpha{}^{\gamma\delta}{}_\gamma\mathscr{D}_\delta\Phi.
\end{aligned} \tag{19.16}$$

Inserting (15.39.1) for $R_{\alpha\beta\gamma\delta}$, we discover the very important result,

$$\begin{aligned}
\mathscr{D}_\alpha(\mathscr{D}^\gamma\mathscr{D}_\gamma - 8R^+)\Phi &= 0 \\
\bar{\mathscr{D}}_{\dot\alpha}(\bar{\mathscr{D}}_{\dot\gamma}\bar{\mathscr{D}}^{\dot\gamma} - 8R)\Phi^+ &= 0.
\end{aligned} \tag{19.17}$$

This tells us that $(\mathscr{D}^\gamma \mathscr{D}_\gamma - 8R^+)$ and $(\bar{\mathscr{D}}_{\dot\gamma} \bar{\mathscr{D}}^{\dot\gamma} - 8R)$ are the covariant generalizations of the chiral projection operators $D^\gamma D_\gamma$ and $\bar{D}_\gamma \bar{D}^\gamma$, provided the superfields on which they act carry no Lorentz indices. [If the superfields carry Lorentz indices, (19.17) changes because of the curvature term in (19.7).] The $\theta = \bar\theta = 0$ component of (19.17) gives the first term of (19.15):

$$\mathscr{D}_\alpha \mathscr{D}^\beta \mathscr{D}_\beta \Phi| = -\frac{4}{3}\sqrt{2}\chi_\alpha M^*. \tag{19.18}$$

The second term is computed in Exercise 3. Combining the two results, we have

$$\delta F = -\frac{1}{3}\sqrt{2}M^*\zeta^\alpha\chi_\alpha + \bar\zeta^{\dot\alpha}\left(\frac{1}{6}\sqrt{2}b_{\alpha\dot\alpha}\chi^\alpha - i\sqrt{2}\hat{D}_{\alpha\dot\alpha}\chi^\alpha\right). \tag{19.19}$$

The supercovariant derivative $\hat{D}_a\chi_\alpha$ is defined as follows:

$$\hat{D}_a\chi_\alpha = e_a{}^m\left(\mathscr{D}_m\chi_\alpha - \frac{1}{\sqrt{2}}\psi_{m\alpha}F - \frac{i}{\sqrt{2}}\bar\psi_m{}^{\dot\beta}\hat{D}_{\alpha\dot\beta}A\right), \tag{19.20}$$

where $D_m\chi_\alpha = \partial_m\chi_\alpha - \omega_{m\alpha}{}^\beta\chi_\beta$.

Equations (19.5), (19.14), and (19.19) give the transformation law of the chiral multiplet:

$$\begin{aligned}
\delta A &= -\sqrt{2}\zeta^\alpha\chi_\alpha \\
\delta\chi_\alpha &= -\sqrt{2}\zeta_\alpha F - i\sqrt{2}\sigma_{\alpha\dot\beta}{}^a\bar\zeta^{\dot\beta}\hat{D}_a A \\
\delta F &= -\frac{1}{3}\sqrt{2}M^*\zeta^\alpha\chi_\alpha \\
&\quad + \bar\zeta^{\dot\alpha}\left(\frac{1}{6}\sqrt{2}b_{\alpha\dot\alpha}\chi^\alpha - i\sqrt{2}\hat{D}_{\alpha\dot\alpha}\chi^\alpha\right).
\end{aligned} \tag{19.21}$$

Vector superfields in curved space obey the usual constraint,

$$V = V^+. \tag{19.22}$$

As with chiral superfields, their components may be defined through covariant derivatives:

$$
\begin{aligned}
C &= V| \\
\phi_\alpha &= -i\mathscr{D}_\alpha V| \qquad \bar{\phi}_{\dot\alpha} = i\bar{\mathscr{D}}_{\dot\alpha} V| \\
M &= \frac{i}{4}(\mathscr{D}^\alpha\mathscr{D}_\alpha - \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha})V| \qquad N = \frac{1}{4}(\mathscr{D}^\alpha\mathscr{D}_\alpha + \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha})V| \\
v_{\alpha\dot\alpha} &= -\frac{1}{2}[\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\alpha}]V| \\
\lambda_\alpha &= iW_\alpha| \qquad \bar{\lambda}_{\dot\alpha} = -i\bar{W}_{\dot\alpha}| \\
D &= -\frac{1}{2}\mathscr{D}^\alpha W_\alpha\Big| = -\frac{1}{2}\bar{\mathscr{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|.
\end{aligned}
\tag{19.23}
$$

Here we have used the superfields W_α and $\bar{W}_{\dot\alpha}$, where

$$
\begin{aligned}
W_\alpha &= -\frac{1}{4}(\bar{\mathscr{D}}_{\dot\beta}\bar{\mathscr{D}}^{\dot\beta} - 8R)\mathscr{D}_\alpha V \\
\bar{W}_{\dot\alpha} &= -\frac{1}{4}(\mathscr{D}^\beta\mathscr{D}_\beta - 8R^+)\bar{\mathscr{D}}_{\dot\alpha} V.
\end{aligned}
\tag{19.24}
$$

These superfields are chiral and gauge invariant. Chirality is proven in Exercise 4:

$$
\bar{\mathscr{D}}_{\dot\beta} W_\alpha = 0, \qquad \mathscr{D}_\beta \bar{W}_{\dot\alpha} = 0. \tag{19.25}
$$

Gauge invariance follows from (19.7) and (15.38.2):

$$
\begin{aligned}
\delta V &= \Lambda + \Lambda^+, \qquad \bar{\mathscr{D}}_{\dot\alpha}\Lambda = \mathscr{D}_\alpha\Lambda^+ = 0 \\
\delta W_\alpha &= -\frac{1}{4}(\bar{\mathscr{D}}_{\dot\beta}\bar{\mathscr{D}}^{\dot\beta} - 8R)\mathscr{D}_\alpha\Lambda \\
&= -\frac{1}{4}\bar{\mathscr{D}}_{\dot\beta}\{\bar{\mathscr{D}}^{\dot\beta}, \mathscr{D}_\alpha\}\Lambda + 2R\mathscr{D}_\alpha\Lambda = 0.
\end{aligned}
\tag{19.26}
$$

Since W_α is gauge invariant, we may compute its components in the WZ gauge:

$$
V| = \mathscr{D}_\alpha V| = \bar{\mathscr{D}}_{\dot\alpha} V| = \mathscr{D}_\alpha\mathscr{D}_\beta V| = \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta} V| = 0. \tag{19.27}
$$

Higher derivatives of V are computed in Exercise 7. These lead to the following results:

$$\begin{aligned} W_\alpha| &= -i\lambda_\alpha \\ \mathscr{D}^\alpha W_\alpha| &= -2D \\ (\mathscr{D}_\alpha W_\beta + \mathscr{D}_\beta W_\alpha)| &= -4i(\sigma^{ba}\varepsilon)_{\alpha\beta}\hat{D}_b v_a \\ \frac{1}{4}\mathscr{D}^\beta\mathscr{D}_\beta W_\alpha| &= -\sigma_{\alpha\dot{\beta}}{}^c\hat{D}_c\bar{\lambda}^{\dot{\beta}} + \frac{i}{2}(\lambda_\alpha M^* + b_\alpha{}^{\dot{\beta}}\bar{\lambda}_{\dot{\beta}}), \end{aligned} \tag{19.28}$$

where

$$\begin{aligned} \hat{D}_b v_{\alpha\dot{\alpha}} &= e_b{}^m\left\{\mathscr{D}_m v_{\alpha\dot{\alpha}} + i(\psi_{m\alpha}\bar{\lambda}_{\dot{\alpha}} + \bar{\psi}_{m\dot{\alpha}}\lambda_\alpha) + \frac{i}{2}\psi_m v\bar{\psi}_a\sigma_{\alpha\dot{\alpha}}{}^a\right\} \\ \hat{D}_a\bar{\lambda}^{\dot{\beta}} &= e_a{}^m\left\{\mathscr{D}_m\bar{\lambda}^{\dot{\beta}} + \frac{i}{2}\bar{\psi}_m{}^{\dot{\beta}}D - (\bar{\sigma}^{db})^{\dot{\beta}}{}_{\dot{\kappa}}\bar{\psi}_m{}^{\dot{\kappa}}\hat{D}_d v_b\right\}. \end{aligned} \tag{19.29}$$

Equation (19.28) gives all the components of W_α.

References

S. Ferrara and P. van Nieuwenhuizen, *Phys. Lett. 76B*, 404 (1978).

S. Ferrara, D. Z. Freedman, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi, and J. Scherk, *Phys. Rev. D15*, 1013 (1977).

Equations

$$\bar{\mathscr{D}}_{\dot{\alpha}}\Phi = 0. \tag{19.1}$$

$$\begin{aligned} A &= \Phi|_{\theta=\bar{\theta}=0} \\ \chi_\alpha &= \frac{1}{\sqrt{2}}\mathscr{D}_\alpha\Phi|_{\theta=\bar{\theta}=0} \\ F &= -\frac{1}{4}\mathscr{D}^\alpha\mathscr{D}_\alpha\Phi|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{19.2}$$

$$\delta\Phi = -\xi^A\mathscr{D}_A\Phi. \tag{19.3}$$

$$\hat{D}_a A \equiv e_a{}^m\left(\partial_m A - \frac{1}{\sqrt{2}}\psi_m{}^\mu\chi_\mu\right). \tag{19.13}$$

$$\hat{D}_a\chi_\alpha = e_a{}^m\left(\mathscr{D}_m\chi_\alpha - \frac{1}{\sqrt{2}}\psi_{m\alpha}F - \frac{i}{\sqrt{2}}\bar{\psi}_m{}^{\dot{\beta}}\hat{D}_{\alpha\dot{\beta}}A\right). \tag{19.20}$$

$$
\begin{aligned}
\delta A &= -\sqrt{2}\zeta^\alpha \chi_\alpha \\
\delta \chi_\alpha &= -\sqrt{2}\zeta_\alpha F - i\sqrt{2}\sigma_{\alpha\dot\beta}{}^a \bar\zeta^{\dot\beta}\hat D_a A \\
\delta F &= -\frac{1}{3}\sqrt{2}M^*\zeta^\alpha\chi_\alpha + \bar\zeta^{\dot\alpha}\left(\frac{1}{6}\sqrt{2}b_{\alpha\dot\alpha}\chi^\alpha - i\sqrt{2}\hat D_{\alpha\dot\alpha}\chi^\alpha\right).
\end{aligned}
\qquad \textbf{(19.21)}
$$

$$
V = V^+. \qquad \textbf{(19.22)}
$$

$$
\begin{gathered}
C = V| \\
\phi_\alpha = -i\mathscr{D}_\alpha V| \qquad \bar\phi_{\dot\alpha} = i\bar{\mathscr{D}}_{\dot\alpha} V| \\
M = \frac{i}{4}(\mathscr{D}^\alpha\mathscr{D}_\alpha - \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha})V| \qquad N = \frac{1}{4}(\mathscr{D}^\alpha\mathscr{D}_\alpha + \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha})V| \\
v_{\alpha\dot\alpha} = -\frac{1}{2}[\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\alpha}]V| \\
\lambda_\alpha = iW_\alpha| \qquad \bar\lambda_{\dot\alpha} = -i\bar W_{\dot\alpha}| \\
D = -\frac{1}{2}\mathscr{D}^\alpha W_\alpha| = -\frac{1}{2}\bar{\mathscr{D}}_{\dot\alpha}\bar W^{\dot\alpha}|.
\end{gathered}
\qquad \textbf{(19.23)}
$$

$$
W_\alpha = -\frac{1}{4}(\bar{\mathscr{D}}_{\dot\beta}\bar{\mathscr{D}}^{\dot\beta} - 8R)\mathscr{D}_\alpha V. \qquad \textbf{(19.24)}
$$

$$
\begin{aligned}
W_\alpha| &= -i\lambda_\alpha \\
\mathscr{D}^\alpha W_\alpha| &= -2D \\
(\mathscr{D}_\alpha W_\beta + \mathscr{D}_\beta W_\alpha)| &= -4i(\sigma^{ba}\varepsilon)_{\alpha\beta}\hat D_b v_a \\
\frac{1}{4}\mathscr{D}^\beta\mathscr{D}_\beta W_\alpha| &= -\sigma_{\alpha\dot\beta}{}^c\hat D_c\bar\lambda^{\dot\beta} + \frac{i}{2}(\lambda_\alpha M^* + b_\alpha{}^{\dot\beta}\bar\lambda_{\dot\beta}),
\end{aligned}
\qquad \textbf{(19.28)}
$$

$$
\begin{aligned}
\hat D_b v_{\alpha\dot\alpha} &= e_b{}^m\left\{\mathscr{D}_m v_{\alpha\dot\alpha} + i(\psi_{m\alpha}\bar\lambda_{\dot\alpha} + \bar\psi_{m\dot\alpha}\lambda_\alpha) + \frac{i}{2}\psi_m v\bar\psi_a\sigma_{\alpha\dot\alpha}{}^a\right\} \\
\hat D_a\bar\lambda^{\dot\beta} &= e_a{}^m\left\{\mathscr{D}_m\bar\lambda^{\dot\beta} + \frac{i}{2}\bar\psi_m{}^{\dot\beta}D - (\bar\sigma^{db})^{\dot\beta}{}_{\dot\kappa}\bar\psi_m{}^{\dot\kappa}\hat D_d v_b\right\}.
\end{aligned}
\qquad \textbf{(19.29)}
$$

Exercises

(1) Verify

$$
\sum_{P(\alpha\beta\gamma)}(-)^P\mathscr{D}_\alpha\mathscr{D}_\beta\mathscr{D}_\gamma\,\Phi = 0.
$$

Use (19.8) to write this in the following form:

$$
(\mathscr{D}_\alpha\mathscr{D}_\beta\mathscr{D}_\gamma + \mathscr{D}_\beta\mathscr{D}_\gamma\mathscr{D}_\alpha + \mathscr{D}_\gamma\mathscr{D}_\alpha\mathscr{D}_\beta)\Phi = 0.
$$

Show

$$\mathscr{D}_\alpha \mathscr{D}_\beta \mathscr{D}_\gamma \Phi = \frac{1}{3}(\{\mathscr{D}_\alpha, \mathscr{D}_\beta\}\mathscr{D}_\gamma - \{\mathscr{D}_\alpha, \mathscr{D}_\gamma\}\mathscr{D}_\beta)\Phi.$$

(2) Use (19.7) and Exercise 1 to prove (19.16).

(3) Show

$$\bar{\mathscr{D}}_{\dot\alpha} \mathscr{D}_\gamma \mathscr{D}_\alpha \Phi = \{\bar{\mathscr{D}}_{\dot\alpha}, \mathscr{D}_\gamma\}\mathscr{D}_\alpha \Phi - \mathscr{D}_\gamma \{\bar{\mathscr{D}}_{\dot\alpha}, \mathscr{D}_\alpha\}\Phi$$

when Φ is chiral. Use (19.7), the constraints(14.25), and the solutions of the Bianchi identities to confirm

$$\begin{aligned}
\mathscr{D}_\gamma \{\bar{\mathscr{D}}_{\dot\alpha}, \mathscr{D}_\alpha\}\Phi &= -2i\sigma_{\alpha\dot\alpha}{}^a \mathscr{D}_\gamma \mathscr{D}_a \Phi \\
&= -2i\sigma_{\alpha\dot\alpha}{}^a \mathscr{D}_a \mathscr{D}_\gamma \Phi - 2i\sigma_{\alpha\dot\alpha}{}^a [\mathscr{D}_\gamma, \mathscr{D}_a]\Phi \\
&= -2i\sigma_{\alpha\dot\alpha}{}^a \mathscr{D}_a \mathscr{D}_\gamma \Phi \\
&\quad + \frac{1}{2}[\varepsilon_{\alpha\delta} G_{\gamma\dot\alpha} - 3\varepsilon_{\gamma\delta} G_{\alpha\dot\alpha} - 3\varepsilon_{\gamma\alpha} G_{\delta\dot\alpha}]\mathscr{D}^\delta \Phi \\
\{\bar{\mathscr{D}}_{\dot\alpha}, \mathscr{D}_\gamma\}\mathscr{D}_\alpha \Phi &= -2i\sigma_{\gamma\dot\alpha}{}^a \mathscr{D}_a \mathscr{D}_\alpha \Phi - [\varepsilon_{\gamma\alpha} G_{\delta\dot\alpha} + \varepsilon_{\gamma\delta} G_{\alpha\dot\alpha}]\mathscr{D}^\delta \Phi.
\end{aligned}$$

Take the $\theta = \bar\theta = 0$ components of these expressions using (19.11) and (19.12). Combine these results with (19.18) to prove (19.19).

(4) Use (19.7) to check that W_α is chiral.

(5) Show that the transformation law for a chiral multiplet reduces to (3.10) in flat space.

(6) Use (17.12) to show that $(\mathscr{D}_\alpha W_\beta + \mathscr{D}_\beta W_\alpha)|$ in Eq. (19.28) is invariant under ordinary gauge transformations $v_a \to v_a + e_a{}^m \partial_m f(x)$.

(7) Prove as many of the following relations as you wish. (Be sure to work in the WZ gauge.)

(a) $\bar{\mathscr{D}}_{\dot\beta} \mathscr{D}_\alpha V| = -\mathscr{D}_\alpha \bar{\mathscr{D}}_{\dot\beta} V| = v_{\alpha\dot\beta}$

(b) $\mathscr{D}_\alpha \mathscr{D}_b V| = \mathscr{D}_b \mathscr{D}_\alpha V| = \frac{1}{2} v_{\alpha\dot\kappa} \bar\psi_b{}^{\dot\kappa}$

$\bar{\mathscr{D}}_{\dot\alpha} \mathscr{D}_b V| = \mathscr{D}_b \bar{\mathscr{D}}_{\dot\alpha} V| = \frac{1}{2} \psi_b{}^\kappa v_{\kappa\dot\alpha}$

(c) $\mathscr{D}_a \mathscr{D}_b V| = \mathscr{D}_b \mathscr{D}_a V|$

$$= -\frac{1}{4} v_{\kappa\dot\kappa}(\psi_a{}^\kappa \bar\psi_b{}^{\dot\kappa} + \psi_b{}^\kappa \bar\psi_a{}^{\dot\kappa})$$

(d) $\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_{\alpha}V| = -2i\varepsilon_{\dot\alpha\dot\beta}\lambda_{\alpha}$

$\mathscr{D}_{\alpha}\mathscr{D}_{\beta}\bar{\mathscr{D}}_{\dot\alpha}V| = -2i\varepsilon_{\alpha\beta}\bar\lambda_{\dot\alpha}$

(e) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{\alpha}\bar{\mathscr{D}}_{\dot\beta}V| = 2i\varepsilon_{\dot\alpha\dot\beta}\lambda_{\alpha} - i\sigma_{\alpha\dot\beta}{}^{c}\psi_{c}{}^{\delta}v_{\delta\dot\alpha}$

$\mathscr{D}_{\alpha}\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{\beta}V| = 2i\varepsilon_{\alpha\beta}\bar\lambda_{\dot\alpha} - i\sigma_{\beta\dot\alpha}{}^{c}v_{\alpha\dot\delta}\bar\psi_{c}{}^{\dot\delta}$

(f) $\mathscr{D}_{\alpha}\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}V| = -2i\varepsilon_{\dot\alpha\dot\beta}\left(\lambda_{\alpha} - \frac{1}{2}\sigma_{\alpha\dot\gamma}{}^{c}\psi_{c}{}^{\kappa}v_{\kappa}{}^{\dot\gamma}\right)$

$\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{\alpha}\mathscr{D}_{\beta}V| = -2i\varepsilon_{\alpha\beta}\left(\bar\lambda_{\dot\alpha} - \frac{1}{2}\sigma_{\gamma\dot\alpha}{}^{c}v^{\gamma}{}_{\dot\kappa}\bar\psi_{c}{}^{\dot\kappa}\right)$

(g) $\mathscr{D}_{\alpha}\mathscr{D}_{\beta}\mathscr{D}_{\gamma}V| = 0$

$\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\bar{\mathscr{D}}_{\dot\gamma}V| = 0$

(h) $\mathscr{D}_{a}\mathscr{D}_{\alpha}\mathscr{D}_{\beta}V| = -i\varepsilon_{\alpha\beta}\bar\psi_{a}{}^{\dot\alpha}\left(\bar\lambda_{\dot\alpha} - \frac{1}{2}\sigma_{\gamma\dot\alpha}{}^{c}v^{\gamma}{}_{\dot\kappa}\bar\psi_{c}{}^{\dot\kappa}\right)$

$\mathscr{D}_{a}\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}V| = i\varepsilon_{\dot\alpha\dot\beta}\psi_{a}{}^{\alpha}\left(\lambda_{\alpha} - \frac{1}{2}\sigma_{\alpha\dot\gamma}{}^{c}\psi_{c}{}^{\kappa}v_{\kappa}{}^{\dot\gamma}\right)$

(i) $\mathscr{D}_{\alpha}\mathscr{D}_{a}\mathscr{D}_{\beta}V| = -i\varepsilon_{\alpha\beta}\bar\psi_{a}{}^{\dot\alpha}\left(\bar\lambda_{\dot\alpha} - \frac{1}{2}\sigma_{\gamma\dot\alpha}{}^{c}v^{\gamma}{}_{\dot\kappa}\bar\psi_{c}{}^{\dot\kappa}\right) - \frac{i}{6}\sigma_{\alpha\dot\gamma a}v_{\beta}{}^{\dot\gamma}M^{*}$

$\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{a}\bar{\mathscr{D}}_{\dot\beta}V| = i\varepsilon_{\dot\alpha\dot\beta}\psi_{a}{}^{\alpha}\left(\lambda_{\alpha} - \frac{1}{2}\sigma_{\alpha\dot\gamma}{}^{c}\psi_{c}{}^{\kappa}v_{\kappa}{}^{\dot\gamma}\right) - \frac{i}{6}\sigma_{\gamma\dot\alpha a}v^{\gamma}{}_{\dot\beta}M$

(j) $\mathscr{D}_{\alpha}\mathscr{D}_{\beta}\mathscr{D}_{a}V| = -i\varepsilon_{\alpha\beta}\left(\bar\psi_{a}{}^{\dot\alpha}\bar\lambda_{\dot\alpha} - \frac{1}{2}\bar\psi_{a}{}^{\dot\alpha}\sigma^{\gamma}{}_{\dot\alpha}{}^{c}v^{\gamma}{}_{\dot\kappa}\bar\psi_{c}{}^{\dot\kappa} + \frac{1}{3}v_{a}M^{*}\right)$

$\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_{a}V| = i\varepsilon_{\dot\alpha\dot\beta}\left(\psi_{a}{}^{\alpha}\lambda_{\alpha} - \frac{1}{2}\psi_{a}{}^{\alpha}\sigma_{\alpha\dot\gamma}{}^{c}\psi_{c}{}^{\kappa}v_{\kappa}{}^{\dot\gamma} - \frac{1}{3}v_{a}M\right)$

(k) $\mathscr{D}_{a}\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{\alpha}V| = \hat{D}_{a}v_{\alpha\dot\alpha}$

$\mathscr{D}_{a}\mathscr{D}_{\alpha}\bar{\mathscr{D}}_{\dot\alpha}V| = -\hat{D}_{a}v_{\alpha\dot\alpha}{}^{+}$

(l) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{a}\mathscr{D}_{\alpha}V| = \hat{D}_{a}v_{\alpha\dot\alpha} - \frac{i}{24}\bar\sigma_{a}{}^{\dot\varepsilon\varepsilon}v_{\alpha}{}^{\dot\kappa}(\varepsilon_{\dot\varepsilon\dot\kappa}b_{\varepsilon\dot\alpha} - 3\varepsilon_{\dot\alpha\dot\kappa}b_{\varepsilon\dot\varepsilon} - 3\varepsilon_{\dot\alpha\dot\varepsilon}b_{\varepsilon\dot\kappa})$

$\mathscr{D}_{\alpha}\mathscr{D}_{a}\bar{\mathscr{D}}_{\dot\alpha}V| = -\hat{D}_{a}v_{\alpha\dot\alpha}{}^{+} - \frac{i}{24}\bar\sigma_{a}{}^{\dot\varepsilon\varepsilon}v^{\kappa}{}_{\dot\alpha}(\varepsilon_{\varepsilon\kappa}b_{\alpha\dot\varepsilon} - 3\varepsilon_{\alpha\kappa}b_{\varepsilon\dot\varepsilon} - 3\varepsilon_{\alpha\varepsilon}b_{\kappa\dot\varepsilon})$

(m) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{\alpha}\mathscr{D}_{a}V| = \hat{D}_{a}v_{\alpha\dot\alpha} + \frac{i}{6}\bar\sigma_{a}{}^{\dot\varepsilon\varepsilon}(v_{\varepsilon\dot\alpha}b_{\alpha\dot\varepsilon} - v_{\alpha\dot\varepsilon}b_{\varepsilon\dot\alpha})$

$\mathscr{D}_{\alpha}\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_{a}V| = -\hat{D}_{a}v_{\alpha\dot\alpha}{}^{+} + \frac{i}{6}\bar\sigma_{a}{}^{\dot\varepsilon\varepsilon}(v_{\alpha\dot\varepsilon}b_{\varepsilon\dot\alpha} - v_{\varepsilon\dot\alpha}b_{\alpha\dot\varepsilon}).$

XX. NEW Θ VARIABLES AND THE CHIRAL DENSITY

In the previous chapter we defined the covariant components (19.2) of chiral superfields. In this chapter we introduce new Θ variables. These new variables are defined such that the expansion coefficients of chiral superfields are precisely the covariant components (19.2):

$$\Phi = A(x) + \sqrt{2}\Theta^\alpha \chi_\alpha(x) + \Theta^\alpha \Theta_\alpha F(x). \qquad \textbf{(20.1)}$$

In this expression, the Θ variables carry local Lorentz indices rather than Einstein indices.

The transformation law for a chiral multiplet is given in (19.21). Our goal is to reproduce this law in the following form:

$$\delta\Phi = -\eta^M(x,\Theta)\,\partial_M \Phi. \qquad \textbf{(20.2)}$$

The differential operator ∂_M acts on the spacetime coordinates x^m and the new variables Θ^α. The new transformation parameters

$$\eta^M(x,\Theta) = \eta^M{}_{(0)}(x) + \Theta^\alpha \eta^M{}_{(1)\alpha}(x) + \Theta^\alpha \Theta_\alpha \eta^M{}_{(2)}(x) \qquad (20.3)$$

must be found in terms of the old parameters $\zeta(x)$ and $\bar{\zeta}(x)$. The ansatz (20.2) will be justified by the fact that (19.21) may indeed be written in the form (20.2). Because (20.2) involves a linear differential operator, a product of chiral superfields still transforms as a chiral superfield.

We shall now compute the parameters η. From (20.1) and (20.2), we see

$$\delta A = -\eta^m{}_{(0)}\,\partial_m A - \sqrt{2}\eta^\alpha{}_{(0)}\chi_\alpha. \qquad (20.4)$$

Comparing with (19.21),

$$\delta A = -\sqrt{2}\zeta^\alpha \chi_\alpha, \qquad (20.5)$$

we find

$$\begin{aligned} \eta^m{}_{(0)} &= 0 \\ \eta^\alpha{}_{(0)} &= \zeta^\alpha. \end{aligned} \tag{20.6}$$

Next we consider $\delta\chi$. From (20.1) and (20.2), we have

$$\begin{aligned} \sqrt{2}\delta\chi_\alpha = &-\eta^m{}_{(1)\alpha}\partial_m A - \sqrt{2}\eta^\beta{}_{(1)\alpha}\chi_\beta \\ &- \sqrt{2}\eta^m{}_{(0)}\partial_m\chi_\alpha - 2\eta_{(0)\alpha}F. \end{aligned} \tag{20.7}$$

Comparing with (19.21),

$$\delta\chi_\alpha = -\sqrt{2}\zeta_\alpha F - i\sqrt{2}\sigma_{\alpha\dot\beta}{}^a\bar\zeta^{\dot\beta}e_a{}^m\left(\partial_m A - \frac{1}{\sqrt{2}}\psi_m{}^\beta\chi_\beta\right), \tag{20.8}$$

and using $\eta^m{}_{(0)}$ and $\eta^\alpha{}_{(0)}$ from (20.6), we conclude:

$$\begin{aligned} \eta^m{}_{(1)\alpha} &= 2i\sigma_{\alpha\dot\beta}{}^a\bar\zeta^{\dot\beta}e_a{}^m \\ \eta^\beta{}_{(1)\alpha} &= -i\sigma_{\alpha\dot\beta}{}^a\bar\zeta^{\dot\beta}e_a{}^m\psi_m{}^\beta. \end{aligned} \tag{20.9}$$

The computation of δF is left as an exercise. All told, we find that (19.21) may be written in the form (20.2) with the following parameters η:

$$\begin{aligned} \eta^m &= 2i\Theta\sigma^m\bar\zeta + \Theta\Theta\bar\psi_n\bar\sigma^m\sigma^n\bar\zeta \\ \eta^\alpha &= \zeta^\alpha - i\Theta\sigma^m\bar\zeta\psi_m{}^\alpha \\ &\quad + \Theta\Theta\left\{\frac{1}{3}M^*\zeta^\alpha + \frac{1}{6}b_a(\varepsilon\sigma^a\bar\zeta)^\alpha - i\omega_m{}^{\alpha\beta}(\sigma^m\bar\zeta)_\beta \right. \\ &\quad\quad \left. - \frac{1}{2}\psi_n{}^\alpha(\bar\psi_m\bar\sigma^n\sigma^m\bar\zeta)\right\}. \end{aligned} \tag{20.10}$$

The variables Θ may be used to construct invariant actions. Before we do this, however, we must introduce the concept of a chiral density. Chiral densities are functions of superspace with the following transformation law:

$$\begin{aligned} \delta\Delta &= -\partial_M[\eta^M\Delta(-)^m] \\ &= -\eta^M\partial_M\Delta - (-)^m(\partial_M\eta^M)\Delta. \end{aligned} \tag{20.11}$$

This law is chosen so that the product of a chiral density and a chiral superfield is again a chiral density:

$$\begin{aligned} \delta\,\Delta\Phi &= -\partial_M[\eta^M\,\Delta(-)^m]\Phi - \Delta\eta^M\,\partial_M\Phi \\ &= -\partial_M[\eta^M\,\Delta\Phi(-)^m]. \end{aligned} \tag{20.12}$$

This fact allows us to construct invariant actions from chiral superfields:

$$\begin{aligned} \delta\mathscr{L} &= \delta\int d^4x\,d^2\Theta\,\Delta g(\Phi) \\ &= -\int d^4x\,d^2\Theta\,\partial_M[\eta^M\,\Delta g(\Phi)(-)^m] = 0. \end{aligned} \tag{20.13}$$

Here g is a chiral function of Φ.

Chiral densities may be decomposed in terms of component fields:

$$\Delta = a + \sqrt{2}\Theta\rho + \Theta\Theta f. \tag{\textbf{20.14}}$$

The transformation laws of the component fields follow from (20.10) and (20.11):

$$\begin{aligned} \delta a &= -\sqrt{2}\zeta\rho + ia\psi\sigma\bar{\zeta} \\ \delta\rho_\alpha &= -\sqrt{2}\zeta_\alpha f - i\sqrt{2}\mathscr{D}_m(\sigma^m\bar{\zeta}a)_\alpha + i\psi\sigma\bar{\zeta}\rho_\alpha \\ &\quad + i(\sigma^m\bar{\zeta})_\alpha\psi_m\rho - \frac{1}{3}\sqrt{2}\zeta_\alpha M^*a \\ &\quad - \frac{1}{6}\sqrt{2}a(\sigma^a\bar{\zeta})_\alpha b_a + \frac{1}{2}\sqrt{2}\psi_{n\alpha}\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\zeta}a \\ \delta f &= \partial_m[-a\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\zeta} + i\sqrt{2}\rho\sigma^m\bar{\zeta}]. \end{aligned} \tag{\textbf{20.15}}$$

The expression for δf shows again that $\int d^4x f$ is invariant.

There is a special chiral density $\mathscr{E}$ connected to the vielbein. We shall construct this density from its lowest component:

$$a = \frac{1}{2}e = \frac{1}{2}\det e_m{}^a. \tag{\textbf{20.16}}$$

The transformation law of $e_m{}^a$ was given in (18.23). From this it follows that

$$\begin{aligned} \delta e &= ee_a{}^m\,\delta e_m{}^a \\ &= iee_a{}^m(\psi_m\sigma^a\bar{\zeta} - \zeta\sigma^a\bar{\psi}_m). \end{aligned} \tag{20.17}$$

Comparing (20.17) with (20.15) gives the middle component:

$$\rho = \frac{i}{4}\sqrt{2}e\sigma^m\bar{\psi}_m. \qquad \textbf{(20.18)}$$

To find the remaining component, we need only compute the terms in $\delta\rho$ proportional to ζ:

$$\begin{aligned}\delta\rho_\alpha &= \frac{i}{4}\sqrt{2}\sigma_{\alpha\dot{\alpha}}{}^a\delta(ee_a{}^m\bar{\psi}_m{}^{\dot{\alpha}}) \\ &= -\sqrt{2}\zeta_\alpha f - \frac{1}{6}\sqrt{2}eM^*\zeta_\alpha + [\bar{\zeta}\text{ terms}]. \end{aligned} \qquad (20.19)$$

From (18.23) we have:

$$\begin{aligned}\delta\rho_\alpha = \frac{i}{4}\sqrt{2}\sigma_{\alpha\dot{\alpha}}{}^a e\Big\{ &-i(\zeta\sigma^b\bar{\psi}_n)(e_b{}^n e_a{}^m \\ &- e_a{}^n e_b{}^m)\bar{\psi}_m{}^{\dot{\alpha}} + \frac{i}{3}M^*(\zeta\sigma_a\varepsilon)^{\dot{\alpha}}\Big\} + [\bar{\zeta}\text{ terms}]. \end{aligned} \qquad (20.20)$$

Comparing the two results gives:

$$f = -\frac{1}{2}eM^* - \frac{1}{8}e\bar{\psi}_m(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)\bar{\psi}_n. \qquad \textbf{(20.21)}$$

It requires a lengthy calculation to show that (20.16), (20.18), and (20.21) transform as a chiral density under the full transformation law (18.23) of the supergravity multiplet.

In the next chapter we shall couple supersymmetric models to supergravity. We shall find that the chiral superfield R is the Lagrangian of the supergravity multiplet. In Chapter XVII we discovered how to compute the components of R. Here we shall use the transformation laws of the chiral and gravity multiplets to derive the same results. We start from the lowest component,

$$R| = -\frac{1}{6}M, \qquad \textbf{(20.22)}$$

and build the full superfield in analogy with (4.11). From (18.23) we know that

$$\delta R| = \frac{1}{6}\zeta(\sigma^a\bar{\sigma}^b\psi_{ab} + ib^a\psi_a - i\sigma^a\bar{\psi}_a M). \qquad (20.23)$$

From (19.5) it follows that

$$\delta R| = -\zeta^\alpha \mathscr{D}_\alpha R|, \tag{20.24}$$

so

$$\mathscr{D}_\alpha R| = -\frac{1}{6}(\sigma^a \bar{\sigma}^b \psi_{ab} + i b^a \psi_a - i\sigma^a \bar{\psi}_a M)_\alpha. \tag{20.25}$$

This agrees with Chapter XVII, Exercise 8. In a similar way, we find:

$$\begin{aligned}\mathscr{D}\mathscr{D}R| = &-\frac{1}{3} e_a{}^m e_b{}^n \mathscr{R}_{mn}{}^{ab} + \frac{2}{3} i\bar{\psi}^m \bar{\sigma}^n \psi_{mn} \\ &+ \frac{1}{12}\varepsilon^{k\ell mn}[\bar{\psi}_k \bar{\sigma}_\ell \psi_{mn} + \psi_k \sigma_\ell \bar{\psi}_{mn}] \\ &- \frac{2}{3} i e_a{}^m \mathscr{D}_m b^a + \frac{4}{9} MM^* + \frac{2}{9} b^a b_a \\ &+ \frac{1}{3}\bar{\psi}\bar{\psi}M - \frac{1}{3}\psi_m \sigma^m \bar{\psi}_n b^n. \end{aligned} \tag{20.26}$$

References

V. Ogievetsky and E. Sokatchev, *Phys. Lett. 79B*, 222 (1978).
J. Wess and B. Zumino, *Phys. Lett. 74B*, 51 (1978).

Equations

$$\Phi = A(x) + \sqrt{2}\Theta^\alpha \chi_\alpha(x) + \Theta^\alpha \Theta_\alpha F(x). \tag{20.1}$$

$$\delta\Phi = -\eta^M(x,\Theta)\,\delta_M \Phi. \tag{20.2}$$

$$\begin{aligned}\eta^m &= 2i\Theta\sigma^m \bar{\zeta} + \Theta\Theta\bar{\psi}_n \bar{\sigma}^m \sigma^n \bar{\zeta} \\ \eta^\alpha &= \zeta^\alpha - i\Theta\sigma^m \bar{\zeta}\psi_m{}^\alpha \\ &\quad + \Theta\Theta\left\{\frac{1}{3}M^*\zeta^\alpha + \frac{1}{6}b_a(\varepsilon\sigma^a\bar{\zeta})^\alpha - i\omega_m{}^{\alpha\beta}(\sigma^m\bar{\zeta})_\beta \right. \\ &\qquad \left. - \frac{1}{2}\psi_n{}^\alpha(\bar{\psi}_m \bar{\sigma}^n \sigma^m \bar{\zeta})\right\}. \end{aligned} \tag{20.10}$$

$$\begin{aligned}\delta\Delta &= -\partial_M[\eta^M \Delta(-)^m] \\ &= -\eta^M \partial_M \Delta - (-)^m(\partial_M \eta^M)\Delta.\end{aligned} \tag{20.11}$$

$$\Delta = a + \sqrt{2}\Theta\rho + \Theta\Theta f. \tag{20.14}$$

$$\begin{aligned}\delta a &= -\sqrt{2}\zeta\rho + ia\psi\sigma\bar{\zeta} \\ \delta\rho_\alpha &= -\sqrt{2}\zeta_\alpha f - i\sqrt{2}\mathscr{D}_m(\sigma^m\bar{\zeta}a)_\alpha + i\psi\sigma\bar{\zeta}\rho_\alpha \\ &\quad + i(\sigma^m\bar{\zeta})_\alpha\psi_m\rho - \frac{1}{3}\sqrt{2}\zeta_\alpha M^* a \\ &\quad - \frac{1}{6}\sqrt{2}a(\sigma^a\bar{\zeta})_\alpha b_a + \frac{1}{2}\sqrt{2}\psi_{n\alpha}\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\zeta}a \\ \delta f &= \partial_m[-a\bar{\psi}_n\bar{\sigma}^m\sigma^n\zeta + i\sqrt{2}\rho\sigma^m\bar{\zeta}].\end{aligned} \tag{20.15}$$

$$a = \frac{1}{2}e = \frac{1}{2}\det e_m{}^a. \tag{20.16}$$

$$\rho = \frac{i}{4}\sqrt{2}e\sigma^m\bar{\psi}_m. \tag{20.18}$$

$$f = -\frac{1}{2}eM^* - \frac{1}{8}e\bar{\psi}_m(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)\bar{\psi}_n. \tag{20.21}$$

$$R| = -\frac{1}{6}M. \tag{20.22}$$

$$\mathscr{D}_\alpha R| = -\frac{1}{6}(\sigma^a\bar{\sigma}^b\psi_{ab} + ib^a\psi_a - i\sigma^a\bar{\psi}_a M)_\alpha. \tag{20.25}$$

$$\begin{aligned}\mathscr{D}\mathscr{D}R| &= -\frac{1}{3}e_a{}^m e_b{}^n \mathscr{R}_{mn}{}^{ab} + \frac{2}{3}i\bar{\psi}^m\bar{\sigma}^n\psi_{mn} \\ &\quad + \frac{1}{12}\varepsilon^{k\ell mn}[\bar{\psi}_k\bar{\sigma}_\ell\psi_{mn} + \psi_k\sigma_\ell\bar{\psi}_{mn}] \\ &\quad - \frac{2}{3}ie_a{}^m\mathscr{D}_m b^a + \frac{4}{9}MM^* + \frac{2}{9}b^a b_a \\ &\quad + \frac{1}{3}\bar{\psi}\bar{\psi}M - \frac{1}{3}\psi_m\sigma^m\bar{\psi}_n b^n.\end{aligned} \tag{20.26}$$

Exercises

(1) Compute δF using (20.2) and (20.10). Compare the result with (19.20).

(2) Derive (20.21) from (20.19) and (20.20). [You may wish to use (A.17) to simplify the $\bar\psi\bar\psi$ terms.]

(3) Show

$$\begin{aligned}\mathscr{D}^\alpha \mathscr{D}_\alpha R - \bar{\mathscr{D}}_{\dot\alpha} \bar{\mathscr{D}}^{\dot\alpha} R^+ &= \{\mathscr{D}^\alpha, \bar{\mathscr{D}}^{\dot\beta}\} G_{\alpha\dot\beta} \\ &= 4i \mathscr{D}_a G^a .\end{aligned}$$

(4) Use the Bianchi identities to verify

$$\begin{aligned} e_a{}^m e_b{}^n R_{mn}{}^{ab}| = &-\frac{3}{2}(\mathscr{D}^\alpha \mathscr{D}_\alpha R + \bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha} R^+)| + 48\, RR^+| \\ &+ i[(\bar\psi_m \bar\sigma^m)^\alpha \mathscr{D}_\alpha R + (\psi_m \sigma^m)_{\dot\alpha} \bar{\mathscr{D}}^{\dot\alpha} R^+]| \\ &- 2[\bar\psi_m \bar\sigma^{mn} \bar\psi_n R + \psi_m \sigma^{mn} \psi_n R^+]| \\ &+ 6 G_a G^a| + 2i(\psi_a{}^\alpha \mathscr{D}_\alpha - \bar\psi_{a\dot\alpha} \bar{\mathscr{D}}^{\dot\alpha}) G^a| \\ &+ \frac{1}{2}[(\bar\psi_m \bar\sigma^m)^\alpha (\bar\sigma^n \psi_n)^{\dot\alpha} - (\bar\psi_n \bar\sigma^m)^\alpha (\bar\sigma^n \psi_m)^{\dot\alpha}] G_{\alpha\dot\alpha}| . \end{aligned}$$

(5) Use the results of Exercises 3 and 4 to reproduce (20.26). *Beware*: The calculation is tedious!

XXI. THE MINIMAL CHIRAL SUPERGRAVITY MODEL

We now have what we need to construct the supergravity matter couplings. The general case is rather involved, so we shall start here with a simpler example. We take the Lagrangian to be given by

$$\mathscr{L} = \int d^2\theta \, d^2\bar{\theta} \, \Phi_i{}^+ \Phi_i + \left[\int d^2\theta \left(a_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \right) + \text{h.c.} \right]. \quad (21.1)$$

This Lagrangian was first introduced in Chapter V; it is the most general renormalizable supersymmetric Lagrangian involving only chiral superfields. In what follows, we will extend (21.1) to curved space. The techniques we introduce in analyzing this model will prove useful in discussing the general case in later chapters. Since the result we derive reduces to (21.1) in the limit of flat space, we call it the minimal chiral supergravity model.

We start our construction by writing down an invariant action for the supergravity multiplet,

$$\mathscr{L}_{S.G.} = -\frac{6}{\kappa^2} \int d^2\Theta \, \mathscr{E} R + \text{h.c.} \quad \mathbf{(21.2)}$$

Here $\kappa^2 = 8\pi G_N$ is the gravitational coupling, which we set equal to one. The chiral density $\mathscr{E}$ and superspace curvature R were computed in Chapter XX. Their Θ expansions are listed later in this chapter. Inserting these expressions into (21.2) gives $\mathscr{L}_{S.G.}$ in terms of the supergravity multiplet:

$$\mathscr{L}_{S.G.} = -\frac{1}{2} e\mathscr{R} - \frac{1}{3} eM^*M + \frac{1}{3} eb^a b_a + \frac{1}{2} e\varepsilon^{k\ell mn} (\bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathscr{D}}_m \psi_n - \psi_k \sigma_\ell \tilde{\mathscr{D}}_m \bar{\psi}_n). \quad \mathbf{(21.3)}$$

The curvature $\mathscr{R}$ was introduced in (17.15),

$$\mathscr{R} = e_a{}^n e_b{}^m(\partial_n \omega_m{}^{ab} - \partial_m \omega_n{}^{ab} + \omega_m{}^{ac}\omega_{nc}{}^b - \omega_n{}^{ac}\omega_{mc}{}^b), \tag{21.4}$$

while the covariant derivative $\tilde{\mathscr{D}}_m\psi_n$ was defined in (17.8). From (21.3) we see that $\mathscr{L}_{S.G.}$ contains the Einstein action for the gravitational field. It also contains the Rarita-Schwinger action for the spin-$\frac{3}{2}$ gravitino. The fields M and b_a do not propagate; they are the auxiliary fields of the supergravity multiplet. Note that they enter (21.3) with opposite signs.

The Lagrangian (21.1) is easily extended to curved superspace. We first write it in chiral form,

$$\begin{aligned}\mathscr{L} = \int d^2\theta\bigg[& -\frac{1}{8}\bar{D}\bar{D}\Phi_i{}^+\Phi_i + a_i\Phi_i \\ & + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\bigg] + \text{h.c.},\end{aligned} \tag{21.5}$$

as outlined in Exercise 6 of Chapter IX. We then add the supergravity action (21.2), and replace $\theta \to \Theta$, $d^2\theta \to d^2\Theta\, 2\mathscr{E}$, and $-\frac{1}{4}\bar{D}\bar{D} \to -\frac{1}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)$. This gives the action (21.1) in curved superspace:

$$\begin{aligned}\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\bigg[& -3R - \frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\Phi_i{}^+\Phi_i \\ & - \frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)[c_i\Phi_i + \bar{c}_i\Phi_i{}^+] + d + a_i\Phi_i \\ & + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\bigg] + \text{h.c.}\end{aligned} \tag{21.6}$$

This Lagrangian describes the minimal chiral model. It reduces to (21.1) in flat space. The c and d terms are included because they arise from shifts in the superfields Φ_i. They vanish in flat space. Note that gauge invariance restricts $c_i = 0$ unless Φ_i is neutral.

Equation (21.6) contains two types of terms: those with the chiral projector $(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)$, and those without. The terms with projector are curved-space generalizations of the chiral kinetic energy. Those without are curved-space extensions of the usual superspace potential. We will emphasize this distinction by writing (21.6) in the following form,

$$\mathscr{L} = \int d^2\Theta \, 2\mathscr{E}\left[-\frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\Omega(\Phi,\Phi^+) + P(\Phi)\right] + \text{h.c.}, \quad (21.7)$$

where $\Omega(\Phi,\Phi^+) = \Phi_i{}^+\Phi_i + c_i\Phi_i + \bar{c}_i\Phi_i{}^+ - 3$ is the superspace kinetic energy, and $P(\Phi) = d + a_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k$ is the superspace potential. In Chapter XXIII we shall see that this distinction is preserved in the general case, where Ω and P are arbitrary functions of their respective superfields.

The Lagrangian (21.7) has a long expansion in terms of component fields. To find it, we need the Θ expansions of Φ_i, $\mathscr{E}$, R:

$$\begin{aligned}
\Phi_i &= A_i + \sqrt{2}\Theta\chi_i + \Theta\Theta F_i \\
2\mathscr{E} &= e\{1 + i\Theta\sigma^a\bar{\psi}_a - \Theta\Theta[M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b]\} \\
R &= -\frac{1}{6}\Big\{M + \Theta[\sigma^a\bar{\sigma}^b\psi_{ab} - i\sigma^a\bar{\psi}_a M + i\psi_a b^a] \\
&\quad + \Theta\Theta\Big[-\frac{1}{2}\mathscr{R} + i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} + \frac{2}{3}MM^* \\
&\quad + \frac{1}{3}b^a b_a - ie_a{}^m\mathscr{D}_m b^a + \frac{1}{2}\bar{\psi}\bar{\psi}M - \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c \\
&\quad + \frac{1}{8}\varepsilon^{abcd}[\bar{\psi}_a\bar{\sigma}_b\psi_{cd} + \psi_a\sigma_b\bar{\psi}_{cd}]\Big]\Big\}. \qquad \textbf{(21.8)}
\end{aligned}$$

The components of $\Xi_i = (\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\Phi_i{}^+$ can be computed with the help of (19.7):

$$\begin{aligned}
\Xi_i &= (\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)\Phi_i{}^+ \\
&= (\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)\Phi_i{}^+| + \Theta^\alpha\mathscr{D}_\alpha(\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)\Phi_i{}^+| \\
&\quad - \frac{1}{4}\Theta\Theta\mathscr{D}^\alpha\mathscr{D}_\alpha(\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)\Phi_i{}^+|. \qquad (21.9)
\end{aligned}$$

The necessary ingredients are given in Exercise 1. We find:

$$\begin{aligned}
\Xi_i = & -4F_i^* + \frac{4}{3}MA_i^* + \Theta\Big\{-4i\sqrt{2}\sigma^c\hat{D}_c\bar{\chi}_i - \frac{2}{3}\sqrt{2}\sigma^a b_a\bar{\chi}_i \\
& + \frac{4}{3}A_i^*(2\sigma^{ab}\psi_{ab} - i\sigma^a\bar{\psi}_a M + i\psi_a b^a)\Big\} \\
& + \Theta\Theta\Big\{-4e_a{}^m\mathscr{D}_m\hat{D}^a A_i^* - \frac{8}{3}ib_a\hat{D}^a A_i^* \\
& - \frac{2}{3}\sqrt{2}\bar{\psi}_{ab}\bar{\sigma}^{ab}\bar{\chi}_i + 2\sqrt{2}\bar{\psi}_a\hat{D}^a\bar{\chi}_i - \frac{8}{3}M^*F_i^* \\
& - \frac{2}{3}i\sqrt{2}\bar{\psi}_a\bar{\chi}_i b^a + \frac{1}{3}i\sqrt{2}\bar{\psi}_a\bar{\sigma}^a\sigma^c\bar{\chi}_i b_c \\
& + \frac{4}{3}A_i^*\Big[-\frac{1}{2}\mathscr{R} + i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} - ie_a{}^m\mathscr{D}_m b^a \\
& + \frac{2}{3}M^*M + \frac{1}{3}b_a b^a + \frac{1}{2}\bar{\psi}\bar{\psi}M - \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c \\
& + \frac{1}{8}\varepsilon^{abcd}(\bar{\psi}_a\bar{\sigma}_b\psi_{cd} + \psi_a\sigma_b\bar{\psi}_{cd})\Big]\Big\}. \qquad \textbf{(21.10)}
\end{aligned}$$

Here we have used the following supercovariant derivatives:

$$\begin{aligned}
\hat{D}_a A_i^* &= e_a{}^m\partial_m A_i^* - \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\kappa}}\bar{\chi}_i^{\dot{\kappa}} \\
\hat{D}_a\bar{\chi}_{i\dot{\alpha}} &= e_a{}^m\mathscr{D}_m\bar{\chi}_{i\dot{\alpha}} + \frac{i}{2}\sqrt{2}\psi_a{}^\beta\sigma_{\beta\dot{\alpha}}{}^b\hat{D}_b A_i^* - \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\alpha}}F_i^*, \qquad \textbf{(21.11)}
\end{aligned}$$

where $\mathscr{D}_m\bar{\chi}_{i\dot{\alpha}} = \partial_m\bar{\chi}_{i\dot{\alpha}} + \bar{\chi}_{i\dot{\beta}}\omega_m{}^{\dot{\beta}}{}_{\dot{\alpha}}$. The superfields $\mathscr{E}$, R, Φ_i, and Ξ_i allow us to compute any supergravity Lagrangian involving only chiral fields.

The expansion of the Lagrangian (21.7) contains kinetic terms for the physical fields A_i, χ_i, $e_m{}^a$, and $\psi_m{}^\alpha$, as well as terms involving the auxiliary fields M, b_a, and F_i. The Lagrangian also has higher-order interaction terms, such as nonrenormalizable four-fermion couplings, which are suppressed by powers of Newton's constant. For ease of exposition, we will write the full Lagrangian as follows,

$$\mathscr{L} = \mathscr{L}_{kin} + \mathscr{L}_{aux} + \mathscr{L}_{quartic}, \qquad (21.12)$$

where

$$
\begin{aligned}
\mathcal{L}_{kin} = \frac{1}{6} e\Omega\mathcal{R} &- e\,\partial_m A_i\,\partial^m A_i^* \\
&- \frac{i}{2} e[\chi_i\sigma^m\mathcal{D}_m\bar{\chi}_i + \bar{\chi}_i\bar{\sigma}^m\mathcal{D}_m\chi_i] \\
&- \frac{1}{12} e\Omega\varepsilon^{k\ell mn}[\bar{\psi}_k\bar{\sigma}_\ell\psi_{mn} - \psi_k\sigma_\ell\bar{\psi}_{mn}] \\
&+ \frac{1}{4} e\varepsilon^{k\ell mn}(\Omega_i\,\partial_k A_i - \Omega_{i^*}\,\partial_k A_i^*)\psi_\ell\sigma_m\bar{\psi}_n \\
&- \frac{1}{2}\sqrt{2}e[\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\chi}_i\,\partial_m A_i + \psi_n\sigma^m\bar{\sigma}^n\chi_i\,\partial_m A_i^*] \\
&+ \frac{1}{3}\sqrt{2}e[\Omega_i\chi_i\sigma^{mn}\psi_{mn} + \Omega_{i^*}\bar{\chi}_i\bar{\sigma}^{mn}\bar{\psi}_{mn}] \\
&- \frac{i}{2}\sqrt{2}eP_i\chi_i\sigma^a\bar{\psi}_a - \frac{i}{2}\sqrt{2}eP_{i^*}^*\bar{\chi}_i\bar{\sigma}^a\psi_a \\
&- \frac{1}{2} eP_{ij}\chi_i\chi_j - \frac{1}{2} eP_{i^*j^*}^*\bar{\chi}_i\bar{\chi}_j \\
&- eP\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b - eP^*\psi_a\sigma^{ab}\psi_b
\end{aligned}
\tag{21.13}
$$

is the kinetic part of the Lagrangian,

$$
\begin{aligned}
\mathcal{L}_{aux} = \frac{1}{9} e\Omega|M - 3(\log\Omega)_{i^*}F_i^*|^2 &+ e\Omega(\log\Omega)_{ij^*}F_iF_j^* \\
&- \frac{1}{9} e\Omega b_a b^a - \frac{i}{3} e(\Omega_i\,\partial_m A_i - \Omega_{i^*}\,\partial_m A_i^*)b^m \\
&- \frac{1}{6} e\chi_i\sigma^a\bar{\chi}_i b_a + \frac{i}{6}\sqrt{2}e(\Omega_i\chi_i\psi_m - \Omega_{i^*}\bar{\chi}_i\bar{\psi}_m)b^m \\
&- ePM^* - eP^*M + eP_iF_i + eP_{i^*}^*F_i^*
\end{aligned}
\tag{21.14}
$$

is the auxiliary field contribution, and $\mathcal{L}_{quartic}$ contains four-fermi terms that we ignore for the moment. In (21.13) and (21.14), Ω and P are the same as before, except that the superfields Φ_i and Φ_i^+ are replaced by their lowest components A_i and A_i^*. The subscripts on Ω and P denote derivatives with respect to the scalar fields. For example, $P_i = (\partial/\partial A_i)P$ and $\Omega_{i^*} = (\partial/\partial A_i^*)\Omega$.

To proceed further, we must eliminate the auxiliary fields from $\mathscr{L}_{aux}$. This is most readily done by shifting, $N = M - 3(\log\Omega)_{i*}F_i^*$. The shift decouples N and F_i, and allows the equations of motion to be easily solved:

$$\begin{aligned} N &= 9P\Omega^{-1} \\ \Omega(\log\Omega)_{ij*}F_i &= -P_{j*}^* + 3P^*(\log\Omega)_{j*} \\ b_a &= -\frac{3}{2}i(\Omega_i\,\partial_a A_i - \Omega_{i*}\,\partial_a A_i^*)\Omega^{-1} - \frac{3}{4}\chi_i\sigma_a\bar{\chi}_i\Omega^{-1} \\ &\quad + \frac{3}{4}\sqrt{2}i(\Omega_i\chi_i\psi_a - \Omega_{i*}\bar{\chi}_i\bar{\psi}_a)\Omega^{-1}. \end{aligned} \tag{21.15}$$

Substituting (21.15) into (21.14), we find

$$\begin{aligned} \mathscr{L}_{aux} = &-9ePP^*\Omega^{-1} \\ &- e(\log\Omega)_{ij*}^{-1}[P_i - 3P(\log\Omega)_i][P_{j*}^* - 3P^*(\log\Omega)_{j*}]\Omega^{-1} \\ &- \frac{1}{4}e[\Omega_i\,\partial_a A_i - \Omega_{i*}\,\partial_a A_i^*][(\Omega_j\,\partial^a A_j - \Omega_{j*}\,\partial^a A_j^*) \\ &- i\chi_j\sigma^a\bar{\chi}_j - \sqrt{2}(\Omega_j\chi_j\psi^a - \Omega_{j*}\bar{\chi}_j\bar{\psi}^a)]\Omega^{-1} + \cdots. \end{aligned} \tag{21.16}$$

The dots denote additional four-fermi terms that we absorb in $\mathscr{L}_{quartic}$. Equation (21.16) contains derivative terms, fermion masses, Yukawa couplings, and the scalar potential, which we shall call $\mathscr{V}(A,A^*)$.

The above expressions are not quite ready for model building. We must still check the normalizations of the physical fields. From (21.13), we see that the gravitational action has an unconventional Brans-Dicke form. This normalization can be fixed by performing a field-dependent Weyl rescaling of the gravitational field:

$$e_n{}^a \to e_n{}^a \exp(\lambda), \tag{21.17}$$

where

$$\exp(2\lambda) = -\frac{3}{\Omega}. \tag{21.18}$$

This transformation restores the canonical normalization (21.3) for the Einstein action. The matter-field normalizations can be restored through a field-dependent redefinition of the spinors,

$$\begin{aligned} \chi_i &\to \exp(-\lambda/2)\,\chi_i \\ \psi_m &\to \exp(\lambda/2)\,\psi_m, \end{aligned} \tag{21.19}$$

followed by an additional shift of the gravitino,

$$\psi_m \to \psi_m + i\sqrt{2}\sigma_m \bar{\chi}_i \lambda_{i^*}. \tag{21.20}$$

Adding the four-fermi terms from $\mathscr{L}_{quartic}$, and performing the transformations (21.17)–(21.20), we find our final result for the supergravity matter coupling:

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{2}e\mathscr{R} - eK_{ij^*}\partial_m A_i \partial^m A_j^* \\
& - \frac{i}{2}eK_{ij^*}[\chi_i\sigma^m\mathscr{D}_m\bar{\chi}_j + \bar{\chi}_j\bar{\sigma}^m\mathscr{D}_m\chi_i] \\
& + \frac{1}{2}e\varepsilon^{k\ell mn}[\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathscr{D}}_m\psi_n - \psi_k\sigma_\ell\tilde{\mathscr{D}}_m\bar{\psi}_n] \\
& + \frac{1}{4}e\varepsilon^{k\ell mn}(K_i\partial_k A_i - K_{i^*}\partial_k A_i^*)\psi_\ell\sigma_m\bar{\psi}_n \\
& - \frac{i}{4}e[K_{ij^*}(K_k\partial_m A_k - K_{k^*}\partial_m A_k^*) \\
& - 2(K_{ij^*k}\partial_m A_k - K_{ij^*k^*}\partial_m A_k^*)]\chi_i\sigma^m\bar{\chi}_j \\
& - \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_j^*\chi_i\sigma^m\bar{\sigma}^n\psi_m - \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_i\bar{\chi}_j\bar{\sigma}^m\sigma^n\bar{\psi}_m \\
& + \frac{1}{4}eK_{ij^*}[i\varepsilon^{k\ell mn}\psi_k\sigma_\ell\bar{\psi}_m + \psi_m\sigma^n\bar{\psi}^m]\chi_i\sigma_n\bar{\chi}_j \\
& + \frac{1}{16}e[K_{ij^*}K_{k\ell^*} - 2K_{ij^*k\ell^*} + 2K^{rs^*}K_{iks^*}K_{j^*\ell^*r}]\chi_i\sigma_m\bar{\chi}_j\chi_k\sigma^m\bar{\chi}_\ell \\
& + e\exp(K/2)\Big\{-P^*\psi_a\sigma^{ab}\psi_b - P\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b \\
& - \frac{i}{2}\sqrt{2}D_iP\chi_i\sigma^a\bar{\psi}_a - \frac{i}{2}\sqrt{2}D_{i^*}P^*\bar{\chi}_i\bar{\sigma}^a\psi_a \\
& - \frac{1}{2}[P_{ij} + K_{ij}P + K_iD_jP + K_jD_iP - K_iK_jP \\
& - K^{k\ell^*}K_{ij\ell^*}D_kP]\chi_i\chi_j - \frac{1}{2}[P^*_{i^*j^*} + K_{i^*j^*}P^* + K_{i^*}D_{j^*}P^* \\
& + K_{j^*}D_{i^*}P^* - K_{i^*}K_{j^*}P^* - K^{k^*\ell}K_{i^*j^*\ell}D_{k^*}P^*]\bar{\chi}_i\bar{\chi}_j\Big\} \\
& - e\exp(K)[K^{ij^*}(D_iP)(D_jP)^* - 3P^*P],
\end{aligned} \tag{21.21}$$

where $D_iP = P_i + K_iP$, $K(A,A^*) = -3\log(-\Omega/3)$, and $K^{ij^*} = K^{-1}_{ij^*}$. In this expression, $\Omega(A,A^*) = A_i^*A_i + c_i(A_i^* + A_i) - 3$ and $P(A) = d + a_iA_i + \frac{1}{2}m_{ij}A_iA_j + \frac{1}{3}g_{ijk}A_iA_jA_k$. In Chapter XXIII we will derive a similar result for the general chiral coupling.

Equation (21.21) gives the full supergravity coupling of the minimal chiral model. It has properly normalized kinetic energies for all physical fields, and the full set of four-fermi terms is included. Equation (21.21) is automatically invariant under supergravity transformations (up to total derivatives) because it was derived from a superspace formalism. It also has the correct flat-space limit.

From (21.21) we see that the supergravity scalar potential emerges in a form that will turn out to be quite general:

$$\mathscr{V}(A_i,A_j^*) = \exp(K)[K^{ij^*}(D_iP)(D_jP)^* - 3P^*P]. \qquad \textbf{(21.22)}$$

Note that this expression is not positive definite, so the connection between the potential and supersymmetry breaking is more subtle than before. In Chapter XXIII we shall see that the signal for spontaneous supersymmetry breaking is $\langle D_iP\rangle \neq 0$. Equation (21.22) shows that supersymmetry can be spontaneously broken with *zero* vacuum energy.

The preceding expressions are all written in terms of the real function $K(A,A^*)$. In the coming chapters, we shall see that this function is called a *Kähler potential*, and that (21.21) and (21.22) have a natural interpretation in the language of complex geometry.

References

S. Deser and B. Zumino, *Phys. Lett. 62B*, 335 (1976).

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Equations

$$\mathscr{L}_{S.G.} = -\frac{6}{\kappa^2}\int d^2\Theta\,\mathscr{E}R + \text{h.c.} \qquad \textbf{(21.2)}$$

$$\mathscr{L}_{S.G.} = -\frac{1}{2}e\mathscr{R} - \frac{1}{3}eM^*M + \frac{1}{3}eb^ab_a + \frac{1}{2}e\varepsilon^{k\ell mn}(\bar\psi_k\bar\sigma_\ell\tilde{\mathscr{D}}_m\psi_n - \psi_k\sigma_\ell\tilde{\mathscr{D}}_m\bar\psi_n). \qquad \textbf{(21.3)}$$

$$\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\Bigg[-3R - \frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\Phi_i{}^+\Phi_i - \frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)[c_i\Phi_i + \bar{c}_i\Phi_i{}^+] + d + a_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\Bigg] + \text{h.c.} \qquad \textbf{(21.6)}$$

$$\Phi_i = A_i + \sqrt{2}\Theta\chi_i + \Theta\Theta F_i$$

$$2\mathscr{E} = e\{1 + i\Theta\sigma^a\bar{\psi}_a - \Theta\Theta[M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b]\}$$

$$R = -\frac{1}{6}\Bigg\{M + \Theta[\sigma^a\bar{\sigma}^b\psi_{ab} - i\sigma^a\bar{\psi}_a M + i\psi_a b^a] + \Theta\Theta\Bigg[-\frac{1}{2}\mathscr{R} + i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} + \frac{2}{3}MM^* + \frac{1}{3}b^a b_a - ie_a{}^m\mathscr{D}_m b^a + \frac{1}{2}\bar{\psi}\bar{\psi}M - \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c + \frac{1}{8}\varepsilon^{abcd}[\bar{\psi}_a\bar{\sigma}_b\psi_{cd} + \psi_a\sigma_b\bar{\psi}_{cd}]\Bigg]\Bigg\}. \qquad \textbf{(21.8)}$$

$$\Xi_i = -4F_i^* + \frac{4}{3}MA_i^* + \Theta\Bigg\{-4i\sqrt{2}\sigma^c\hat{D}_c\bar{\chi}_i - \frac{2}{3}\sqrt{2}\sigma^a b_a\bar{\chi}_i + \frac{4}{3}A_i^*(2\sigma^{ab}\psi_{ab} - i\sigma^a\bar{\psi}_a M + i\psi_a b^a)\Bigg\}$$
$$+ \Theta\Theta\Bigg\{-4e_a{}^m\mathscr{D}_m\hat{D}^a A_i^* - \frac{8}{3}ib_a\hat{D}^a A_i^* - \frac{2}{3}\sqrt{2}\bar{\psi}_{ab}\bar{\sigma}^{ab}\bar{\chi}_i + 2\sqrt{2}\bar{\psi}_a\hat{D}^a\bar{\chi}_i - \frac{8}{3}M^*F_i^* - \frac{2}{3}i\sqrt{2}\bar{\psi}_a\bar{\chi}_i b^a + \frac{1}{3}i\sqrt{2}\bar{\psi}_a\bar{\sigma}^a\sigma^c\bar{\chi}_i b_c$$
$$+ \frac{4}{3}A_i^*\Bigg[-\frac{1}{2}\mathscr{R} + i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} - ie_a{}^m\mathscr{D}_m b^a + \frac{2}{3}M^*M + \frac{1}{3}b_a b^a + \frac{1}{2}\bar{\psi}\bar{\psi}M - \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c + \frac{1}{8}\varepsilon^{abcd}(\bar{\psi}_a\bar{\sigma}_b\psi_{cd} + \psi_a\sigma_b\bar{\psi}_{cd})\Bigg]\Bigg\}. \qquad \textbf{(21.10)}$$

$$\hat{D}_a A_i^* = e_a{}^m \partial_m A_i^* - \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\kappa}}\bar{\chi}_i^{\dot{\kappa}}$$

$$\hat{D}_a \bar{\chi}_{i\dot{\alpha}} = e_a{}^m \mathscr{D}_m \bar{\chi}_{i\dot{\alpha}} + \frac{i}{2}\sqrt{2}\psi_a{}^\beta \sigma_{\beta\dot{\alpha}}{}^b \hat{D}_b A_i^* - \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\alpha}} F_i^*. \tag{21.11}$$

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{2} e\mathscr{R} - eK_{ij^*}\partial_m A_i \partial^m A_j^* \\
& - \frac{i}{2} eK_{ij^*}[\chi_i \sigma^m \mathscr{D}_m \bar{\chi}_j + \bar{\chi}_j \bar{\sigma}^m \mathscr{D}_m \chi_i] \\
& + \frac{1}{2} e\varepsilon^{k\ell mn}[\bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathscr{D}}_m \psi_n - \psi_k \sigma_\ell \tilde{\mathscr{D}}_m \bar{\psi}_n] \\
& + \frac{1}{4} e\varepsilon^{k\ell mn}(K_i \partial_k A_i - K_{i^*}\partial_k A_i^*)\psi_\ell \sigma_m \bar{\psi}_n \\
& - \frac{i}{4} e[K_{ij^*}(K_k \partial_m A_k - K_{k^*}\partial_m A_k^*) \\
& - 2(K_{ij^*k}\partial_m A_k - K_{ij^*k^*}\partial_m A_k^*)]\chi_i \sigma^m \bar{\chi}_j \\
& - \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_j^* \chi_i \sigma^m \bar{\sigma}^n \psi_m - \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_i \bar{\chi}_j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
& + \frac{1}{4} eK_{ij^*}[i\varepsilon^{k\ell mn}\psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m]\chi_i \sigma_n \bar{\chi}_j \\
& + \frac{1}{16} e[K_{ij^*}K_{k\ell^*} - 2K_{ij^*k\ell^*} + 2K^{rs^*}K_{iks^*}K_{j^*\ell^*r}]\chi_i \sigma_m \bar{\chi}_j \chi_k \sigma^m \bar{\chi}_\ell \\
& + e\exp(K/2)\Big\{ -P^*\psi_a \sigma^{ab}\psi_b - P\bar{\psi}_a \bar{\sigma}^{ab}\bar{\psi}_b \\
& - \frac{i}{2}\sqrt{2}D_i P \chi_i \sigma^a \bar{\psi}_a - \frac{i}{2}\sqrt{2}D_{i^*}P^* \bar{\chi}_i \bar{\sigma}^a \psi_a \\
& - \frac{1}{2}[P_{ij} + K_{ij}P + K_i D_j P + K_j D_i P - K_i K_j P \\
& - K^{k\ell^*}K_{ij\ell^*}D_k P]\chi_i \chi_j - \frac{1}{2}[P_{i^*j^*}^* + K_{i^*j^*}P^* + K_{i^*}D_{j^*}P^* \\
& + K_{j^*}D_{i^*}P^* - K_{i^*}K_{j^*}P^* - K^{k^*\ell}K_{i^*j^*\ell}D_{k^*}P^*]\bar{\chi}_i \bar{\chi}_j \Big\} \\
& - e\exp(K)[K^{ij^*}(D_i P)(D_j P)^* - 3P^*P].
\end{aligned} \tag{21.21}$$

$$\mathscr{V}(A_i,A_j^*) = \exp(K)[K^{ij^*}(D_iP)(D_jP)^* - 3P^*P]. \qquad \textbf{(21.22)}$$

Exercises

(1) Verify as many of the following relations as you wish:

(a) $\Phi^+| = A^*$

(b) $\mathscr{D}_\beta\Phi^+| = 0$

(c) $\bar{\mathscr{D}}_{\dot\alpha}\Phi^+| = \sqrt{2}\bar\chi_{\dot\alpha}$

(d) $\mathscr{D}_\alpha\mathscr{D}_\beta\Phi^+| = 0$

(e) $\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\Phi^+| = 2\varepsilon_{\dot\alpha\dot\beta}F^*$

(f) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_\beta\Phi^+| = 0$

(g) $\mathscr{D}_\beta\bar{\mathscr{D}}_{\dot\alpha}\Phi^+| = -2i\sigma_{\beta\dot\alpha}{}^a\hat{D}_aA^*$

(h) $\mathscr{D}_\alpha\mathscr{D}_\beta\mathscr{D}_\gamma\Phi^+| = 0$

(i) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_\alpha\mathscr{D}_\beta\Phi^+| = 0$

(j) $\mathscr{D}_\alpha\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_\beta\Phi^+| = 0$

(k) $\mathscr{D}_\alpha\mathscr{D}_\beta\bar{\mathscr{D}}_{\dot\alpha}\Phi^+| = -\frac{2}{3}\sqrt{2}\varepsilon_{\alpha\beta}\bar\chi_{\dot\alpha}M^*$

(l) $\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\mathscr{D}_\alpha\Phi^+| = 0$

(m) $$\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_\alpha\bar{\mathscr{D}}_{\dot\beta}\Phi^+| = -i2\sqrt{2}\sigma_{\alpha\dot\beta}{}^c\hat{D}_c\bar\chi_{\dot\alpha} + \frac{1}{6}\sqrt{2}(\bar\chi_{\dot\beta}b_{\alpha\dot\alpha} - 3\bar\chi_{\dot\alpha}b_{\alpha\dot\beta} - 3\varepsilon_{\dot\alpha\dot\beta}b_{\alpha\dot\gamma}\bar\chi^{\dot\gamma})$$

(n) $\mathscr{D}_\alpha\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}_{\dot\beta}\Phi^+| = i2\sqrt{2}\varepsilon_{\dot\alpha\dot\beta}\sigma_{\alpha\dot\gamma}{}^c\hat{D}_c\bar\chi^{\dot\gamma} + \frac{1}{3}\sqrt{2}\varepsilon_{\dot\alpha\dot\beta}b_{\alpha\dot\gamma}\bar\chi^{\dot\gamma}$

(o) $\mathscr{D}_a\Phi^+| = \hat{D}_aA^* = e_a{}^m\mathscr{D}_mA^* - \frac{1}{\sqrt{2}}\bar\psi_{a\dot\alpha}\bar\chi^{\dot\alpha}$

(p) $\mathscr{D}_a\mathscr{D}_\beta\Phi^+| = 0$

(q) $\mathscr{D}_\alpha\mathscr{D}_c\Phi^+| = -\frac{i}{6}\sqrt{2}\sigma_{c\alpha\dot\beta}\bar\chi^{\dot\beta}M^*$

(r) $\mathscr{D}_c\bar{\mathscr{D}}_{\dot\alpha}\Phi^+| = \sqrt{2}\hat{D}_c\bar{\chi}_{\dot\alpha}$

$$= \sqrt{2}\left(e_c{}^m\mathscr{D}_m\bar{\chi}_{\dot\alpha} + \frac{i}{\sqrt{2}}\psi_c{}^\beta\sigma_{\beta\dot\alpha}{}^a\hat{D}_aA^* - \frac{1}{\sqrt{2}}\bar{\psi}_{c\dot\alpha}F^*\right)$$

(s) $\bar{\mathscr{D}}_{\dot\alpha}\mathscr{D}_c\Phi^+| = \sqrt{2}\left\{\hat{D}_c\bar{\chi}_{\dot\alpha} - \frac{i}{24}\bar{\sigma}_c{}^{\dot\beta\beta}(\bar{\chi}_{\dot\beta}b_{\beta\dot\alpha} - 3\bar{\chi}_{\dot\alpha}b_{\beta\dot\beta} - 3\varepsilon_{\dot\alpha\dot\beta}b_{\beta\dot\kappa}\bar{\chi}^{\dot\kappa})\right\}$

(t) $\mathscr{D}_a\mathscr{D}_b\Phi^+| = e_a{}^m\mathscr{D}_m\hat{D}_bA^* + \frac{i}{12}\sqrt{2}\psi_a{}^\alpha\sigma_{b\alpha\dot\beta}\bar{\chi}^{\dot\beta}M^*$

$$+ \frac{1}{2}\sqrt{2}\bar{\psi}_a{}^{\dot\alpha}\left\{\hat{D}_b\bar{\chi}_{\dot\alpha} - \frac{i}{24}\bar{\sigma}_b{}^{\dot\beta\beta}(\bar{\chi}_{\dot\beta}b_{\beta\dot\alpha}\right.$$

$$\left. - 3\bar{\chi}_{\dot\alpha}b_{\beta\dot\beta} - 3\varepsilon_{\dot\alpha\dot\beta}b_{\beta\dot\kappa}\bar{\chi}^{\dot\kappa})\right\}$$

(u) $\mathscr{D}_c\mathscr{D}_\alpha\mathscr{D}_\beta\Phi^+| = 0$

(v) $\mathscr{D}_\alpha\mathscr{D}_c\mathscr{D}_\beta\Phi^+| = 0$

(w) $\mathscr{D}_\alpha\mathscr{D}_\beta\mathscr{D}_c\Phi^+| = i\sqrt{2}\sigma_{c\beta\dot\beta}\bar{\chi}^{\dot\beta}\mathscr{D}_\alpha R^+| - \frac{1}{3}(\sigma^a\bar{\sigma}^c\varepsilon)_{\alpha\beta}M^*\hat{D}_aA^*$

(x) $\mathscr{D}^\alpha\mathscr{D}_\alpha\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha}\Phi^+| = 16e_a{}^m\mathscr{D}_m\hat{D}_aA^* + \frac{32}{3}ib^a\hat{D}_aA^*$

$$- 8\sqrt{2}\bar{\psi}_a\hat{D}^a\bar{\chi} + \frac{32}{3}M^*F^* + \frac{8}{3}\sqrt{2}\bar{\psi}_{mn}\bar{\sigma}^{mn}\bar{\chi}$$

$$+ \frac{8}{3}i\sqrt{2}\bar{\psi}_a\bar{\chi}b^a - \frac{4}{3}i\sqrt{2}\bar{\psi}_a\bar{\sigma}^a\sigma^c\bar{\chi}b_c.$$

(2) Use the above relations to derive (21.10).

(3) Verify that $\mathscr{L}_{kin}$ and $\mathscr{L}_{aux}$ are given by (21.13) and (21.14), respectively.

(4) Show that the Weyl rescaling (21.17) takes

$$\frac{1}{6}e\Omega\mathscr{R} \to -\frac{1}{2}e\mathscr{R} - \frac{3}{4}e\Omega^{-2}\partial_m\Omega\,\partial^m\Omega + \frac{3}{2}\partial_m[eg^{mn}\Omega^{-1}\partial_n\Omega].$$

(5) Use the result of Exercise 4 to show that (21.17)–(21.20) restore the proper kinetic energies in (21.21).

(6) Use (21.14) and (21.17)–(21.20) to check that the potential (21.22) is indeed correct.

(7) Show that for $\Omega(A,A^*) = A_i^*A_i - 3$, the field redefinition,

$$A'_1 = \frac{A_1 + a}{1 + \frac{1}{3}a^*A_1}, \qquad A'_i = \frac{A_i\sqrt{1 - \frac{1}{3}|a|^2}}{1 + \frac{1}{3}a^*A_1} \quad (i \neq 1),$$

induces the Kähler transformation,

$$K(A',A'^*) = K(A,A^*) - 3\log\frac{1 - \frac{1}{3}|a|^2}{|1 + \frac{1}{3}a^*A_1|^2}.$$

As discussed in Appendix C, this is an isometry transformation—it leaves the Kähler geometry invariant. Note that after such a transformation, the supergravity potential $\mathscr{V}$ can again be related to a superpotential P of third order. This feature characterizes the minimal chiral supergravity model.

(8) Let $\Omega = A^*A - 3$ and $P = \mu(1 + \frac{1}{3}\sqrt{3}A)^3$. Show that the potential $\mathscr{V}$ vanishes. Since the potential does not determine the expectation value $\langle A\rangle$, the field A is known as a "sliding singlet." Show that the gravitino mass slides as well:

$$m_{3/2} = \mu\left(\frac{1 + a}{\sqrt{1 - |a|^2}}\right)^3,$$

where $\langle A\rangle = \sqrt{3}a$.

XXII. CHIRAL MODELS AND KÄHLER GEOMETRY

In the previous chapter, we constructed the minimal coupling of chiral superfields to supergravity. We found that the resulting Lagrangian could be written in terms of a Kähler potential and its derivatives. In what follows, we will begin to explore the relation between matter couplings and Kähler geometry. We will work in flat space, where the connection first appears, leaving the curved-space generalization until Chapter XXIII. A brief introduction to Kähler geometry is given in Appendix C.

We start by studying the most general Lagrangian that can be built from chiral superfields Φ^i, for $i = 1, \ldots, n$. This Lagrangian takes a very simple form,

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^i,\Phi^{+j}) + \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right]. \tag{22.1}$$

Here K and P are superfields, with power series expansions in terms of the chiral superfields Φ^i,

$$\begin{aligned} K(\Phi,\Phi^+) &= \sum c_{i_1 \cdots i_N, j_1 \cdots j_M} \Phi^{i_1} \cdots \Phi^{i_N} \Phi^{+j_1} \cdots \Phi^{+j_M} \\ P(\Phi) &= \sum g_{i_1 \ldots i_N} \Phi^{i_1} \cdots \Phi^{i_N}. \end{aligned} \tag{22.2}$$

To find the component Lagrangian, we must expand K and P in terms of the θ variables. The expansions of the individual fields were given in Eqs. (5.3) and (5.5). For the superpotential P, Eqs. (5.7) and (5.8) immediately extend to

$$\begin{aligned} P(\Phi) = P(A) &+ \sqrt{2}\theta\chi^i \frac{\partial P(A)}{\partial A^i} \\ &+ \theta\theta \left\{ F^i \frac{\partial P(A)}{\partial A^i} - \frac{1}{2} \chi^i \chi^j \frac{\partial^2 P(A)}{\partial A^i \partial A^j} \right\}. \end{aligned} \tag{22.3}$$

Here all component fields are functions of $y^m = x^m + i\theta\sigma^m\bar\theta$. The conjugate superpotential P^+ has an analogous expansion,

$$P^+(\Phi^+) = P^*(A^*) + \sqrt{2}\bar\theta\bar\chi^i \frac{\partial P^*(A^*)}{\partial A^{*i}} + \bar\theta\bar\theta\left\{F^{*i}\frac{\partial P^*(A^*)}{\partial A^{*i}} - \frac{1}{2}\bar\chi^i\bar\chi^j\frac{\partial^2 P^*(A^*)}{\partial A^{*i}\partial A^{*j}}\right\}, \qquad (22.4)$$

where the fields now depend on y^+.

The θ expansion of $K(\Phi,\Phi^+)$ can be computed starting from the monomial

$$K_{NM} = \Phi^{i_1}\cdots\Phi^{i_N}\Phi^{+j_1}\cdots\Phi^{+j_M}. \qquad (22.5)$$

Its $\theta\theta\bar\theta\bar\theta$-component may be found with the help of (5.9) and an appropriate interpretation of (22.3) and (22.4),

$$\begin{aligned} K_{NM} = \cdots + \theta\theta\bar\theta\bar\theta\Bigg\{&\left[F^k\frac{\partial}{\partial A^k}(A^{i_1}\cdots A^{i_N}) - \frac{1}{2}\chi^k\chi^\ell\frac{\partial^2(A^{i_1}\cdots A^{i_N})}{\partial A^k\partial A^\ell}\right] \\ &\times\left[F^{*k}\frac{\partial}{\partial A^{*k}}(A^{*j_1}\cdots A^{*j_M}) - \frac{1}{2}\bar\chi^k\bar\chi^\ell\frac{\partial^2(A^{*j_1}\cdots A^{*j_M})}{\partial A^{*k}\partial A^{*\ell}}\right] \\ &- \partial_m(A^{i_1}\cdots A^{i_N})\,\partial^m(A^{*j_1}\cdots A^{*j_M}) \\ &- i\frac{\partial}{\partial A^{*k}}(A^{*j_1}\cdots A^{*j_M})\,\bar\chi^k\bar\sigma^m\partial_m\left(\chi^\ell\frac{\partial(A^{i_1}\cdots A^{i_N})}{\partial A^\ell}\right)\Bigg\}, \end{aligned} \qquad (22.6)$$

where we have used partial integration. This result can be rewritten more elegantly in terms of derivatives of $K_{NM}|$, the lowest component of the superfield (22.5),

$$\begin{aligned} K_{NM} = \cdots + \theta\theta\bar\theta\bar\theta\Bigg\{&\frac{\partial^2 K_{NM}|}{\partial A^i\partial A^{*j}}F^iF^{*j} \\ &- \frac{1}{2}\frac{\partial^3 K_{NM}|}{\partial A^i\partial A^{*j}\partial A^{*k}}F^i\bar\chi^j\bar\chi^k - \frac{1}{2}\frac{\partial^3 K_{NM}|}{\partial A^{*i}\partial A^j\partial A^k}F^{*i}\chi^j\chi^k \\ &+ \frac{1}{4}\frac{\partial^4 K_{NM}|}{\partial A^i\partial A^j\partial A^{*k}\partial A^{*\ell}}\chi^i\chi^j\bar\chi^k\bar\chi^\ell \\ &- \frac{\partial^2 K_{NM}|}{\partial A^i\partial A^{*j}}\partial_m A^i\partial^m A^{*j} - i\frac{\partial^2 K_{NM}|}{\partial A^i\partial A^{*j}}\bar\chi^j\bar\sigma^m\partial_m\chi^i \\ &- i\frac{\partial^3 K_{NM}|}{\partial A^i\partial A^j\partial A^{*k}}\bar\chi^k\bar\sigma^m\chi^i\partial_m A^j\Bigg\}. \end{aligned} \qquad (22.7)$$

Equation (22.7) is also true for the full polynomial K. The expression simplifies in the notation of a Kähler manifold, where

$$
\begin{aligned}
g_{ij^*} &= \frac{\partial}{\partial A^i}\frac{\partial}{\partial A^{*j}} K| \\
g_{ij^*,k} &= \frac{\partial}{\partial A^k} g_{ij^*} = g_{mj^*}\Gamma^m_{ik} \\
g_{ij^*,k^*} &= \frac{\partial}{\partial A^{*k}} g_{ij^*} = g_{im^*}\Gamma^{m^*}_{j^*k^*}.
\end{aligned} \tag{22.8}
$$

One finds

$$
\begin{aligned}
K = \cdots + \theta\theta\bar\theta\bar\theta \Big\{ & g_{ij^*}F^iF^{*j} - \frac{1}{2} g_{im^*}\Gamma^{m^*}_{j^*k^*}F^i\bar\chi^j\bar\chi^k \\
& - \frac{1}{2} g_{mi^*}\Gamma^m_{jk}F^{*i}\chi^j\chi^k + \frac{1}{4} g_{ij^*,k\ell^*}\chi^i\chi^k\bar\chi^j\bar\chi^\ell \\
& - g_{ij^*}\,\partial_m A^i\,\partial^m A^{*j} - i g_{ij^*}\bar\chi^j\bar\sigma^m\,\partial_m\chi^i \\
& - i g_{mk^*}\Gamma^m_{ij}\bar\chi^k\bar\sigma^m\chi^i\,\partial_m A^j \Big\}.
\end{aligned} \tag{22.9}
$$

We now have all we need to write the full Lagrangian in terms of component fields. Substituting (22.3) and (22.9) into (22.1), we find

$$
\begin{aligned}
\mathscr{L} = {} & g_{ij^*}F^iF^{*j} + \frac{1}{4} g_{ij^*,k\ell^*}\chi^i\chi^k\bar\chi^j\bar\chi^\ell \\
& - F^i\left\{\frac{1}{2} g_{im^*}\Gamma^{m^*}_{j^*k^*}\bar\chi^j\bar\chi^k - \frac{\partial P}{\partial A^i}\right\} \\
& - F^{*i}\left\{\frac{1}{2} g_{mi^*}\Gamma^m_{jk}\chi^j\chi^k - \frac{\partial P^*}{\partial A^{*i}}\right\} \\
& - g_{ij^*}\,\partial_m A^i\,\partial^m A^{*j} - i g_{ij^*}\bar\chi^j\bar\sigma^m D_m\chi^i \\
& - \frac{1}{2}\frac{\partial^2 P}{\partial A^i\partial A^j}\chi^i\chi^j - \frac{1}{2}\frac{\partial^2 P^*}{\partial A^{*i}\partial A^{*j}}\bar\chi^i\bar\chi^j.
\end{aligned} \tag{22.10}
$$

Here $D_m\chi^i = \partial_m\chi^i + \Gamma^i_{jk}\,\partial_m A^j\chi^k$ is a covariant spacetime derivative, assuming χ^i transforms like a contravariant vector under the transformations (C.1) on a Kähler manifold.

The auxiliary fields in this expression may be eliminated by their Euler equations,

$$g_{ij^*}F^i - \frac{1}{2}\,g_{kj^*}\Gamma^k_{m\ell}\chi^m\chi^\ell + \frac{\partial P^*}{\partial A^{*j}} = 0. \qquad (22.11)$$

Substituting into (22.10), we obtain the final form of the component Lagrangian,

$$\begin{aligned}\mathscr{L} = &-g_{ij^*}\,\partial_m A^i\,\partial^m A^{*j} - i g_{ij^*}\bar{\chi}^j\bar{\sigma}^m D_m\chi^i \\ &+ \frac{1}{4}\,R_{ij^*k\ell^*}\chi^i\chi^k\bar{\chi}^j\bar{\chi}^\ell \\ &- \frac{1}{2}\,D_iD_jP\chi^i\chi^j - \frac{1}{2}\,D_{i^*}D_{j^*}P^*\bar{\chi}^i\bar{\chi}^j \\ &- g^{ij^*}D_iPD_{j^*}P^*, \end{aligned} \qquad \mathbf{(22.12)}$$

where

$$\begin{aligned} D_iP &= \frac{\partial}{\partial A^i}\,P \\ D_iD_jP &= \frac{\partial^2}{\partial A^i\partial A^j}\,P - \Gamma^k_{ij}\,\frac{\partial}{\partial A^k}\,P. \end{aligned} \qquad \mathbf{(22.13)}$$

Equation (22.12) describes the most general supersymmetric coupling of chiral multiplets. We have used Kähler notation to illustrate the geometrical nature of the result. Invariance under Kähler transformations is manifest.

Each term in the Lagrangian (22.12) has a natural interpretation in the language of Kähler geometry. The scalar fields should be thought of as the coordinates of a Kähler manifold, and the fermions as tensors in the tangent space. The Lagrangian (22.12) is a supersymmetric version of the sigma model, expressed in geometrical form.

In superspace notation, the appearance of the Kähler geometry can be traced to the invariance of the Lagrangian (22.1) under the superfield Kähler transformation:

$$K(\Phi,\Phi^+) \to K(\Phi,\Phi^+) + F(\Phi) + F^+(\Phi^+). \qquad (22.14)$$

This invariance will play an important role in what follows.

References

B. Zumino, *Phys. Lett. 87B*, 203 (1979).
D. Z. Freedman and L. Alvarez-Gaumé, *Comm. Math. Phys. 80*, 443 (1981).

Equations

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^i, \Phi^{+j}) + \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right]. \quad \textbf{(22.1)}$$

$$\begin{aligned}\mathscr{L} = &- g_{ij^*}\,\partial_m A^i\,\partial^m A^{*j} - i g_{ij^*}\bar{\chi}^j \bar{\sigma}^m D_m \chi^i \\ &+ \frac{1}{4} R_{ij^*k\ell^*}\chi^i\chi^k\bar{\chi}^j\bar{\chi}^\ell \\ &- \frac{1}{2} D_i D_j P \chi^i \chi^j - \frac{1}{2} D_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \\ &- g^{ij^*} D_i P D_{j^*} P^*. \end{aligned} \quad \textbf{(22.12)}$$

$$\begin{aligned} D_i P &= \frac{\partial}{\partial A^i} P \\ D_i D_j P &= \frac{\partial^2}{\partial A^i \partial A^j} P - \Gamma^k_{ij} \frac{\partial}{\partial A^k} P. \end{aligned} \quad \textbf{(22.13)}$$

Exercises

(1) Check that the Lagrangian (22.12) is invariant (up to a total derivative) under the following supersymmetry transformations:

$$\begin{aligned} \delta_\xi A^i &= \sqrt{2}\xi\chi \\ \delta_\xi \chi^i &= i\sqrt{2}\sigma^m \bar{\xi}\,\partial_m A^i - \Gamma^i_{jk}\,\delta_\xi A^j \chi^k - \sqrt{2} g^{ij^*} \frac{\partial P^*}{\partial A^{*j}}\,\xi. \end{aligned}$$

(2) Let $K = A^{*i}A^i$ and $P = \lambda_i A^i + \frac{1}{2} m_{ij} A^i A^j + \frac{1}{3} g_{ijk} A^i A^j A^k$. Show that (22.12) reduces to the renormalizable Lagrangian given in (5.13).

XXIII. GENERAL CHIRAL SUPERGRAVITY MODELS

Having discussed the role of Kähler geometry in flat space, we will now compute the most general coupling of chiral superfields to supergravity. As in flat space, we will find that the component Lagrangian has a natural interpretation in the language of Kähler geometry.

Motivated by our discussion in Chapter XXI, we take our superspace Lagrangian to be

$$\mathscr{L} = \frac{1}{\kappa^2}\int d^2\Theta\, 2\mathscr{E}\left[\frac{3}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{\kappa^2}{3}K(\Phi,\Phi^+)\right\} + \kappa^2 P(\Phi)\right] + \text{h.c.}, \tag{23.1}$$

where $K(\Phi,\Phi^+)$ is a hermitian function of the superfields Φ^i and Φ^{+j}, and $P(\Phi)$ is the superpotential. The exponential form is suggested by the relation between K and Ω below (21.21). Expanding in κ^2, we see that K is the flat-space Kähler potential,

$$\mathscr{L} = -\frac{6}{\kappa^2}\int d^2\Theta\, \mathscr{E}R + \int d^2\Theta\, 2\mathscr{E}\left[-\frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)K(\Phi,\Phi^+) + P(\Phi)\right] + \cdots + \text{h.c.} \tag{23.2}$$

In this chapter, we will find that K is a Kähler potential in curved space as well.

The Lagrangian (23.1) is manifestly invariant under supergravity transformations. Its component form can be found using the techniques introduced for the minimal case in Chapter XXI. The steps are virtually identical; there are just a few extra terms that follow from the general nature of K. At the end of the computation, one finds precisely Eq. (21.21), where K is now an arbitrary real function of the scalar fields A^i, the lowest component of the superfield $K(\Phi,\Phi^+)$.

Equation (21.21) gives the component Lagrangian in terms of K and its derivatives. It can be written more compactly if we use g_{ij^*} and $R_{ij^*k\ell^*}$, the metric and curvature of a Kähler manifold. In this form the geometric invariance of the Lagrangian is manifest. Comparing (C.10), (C.18), and (21.21), we find

$$
\begin{aligned}
\mathscr{L} = & -\frac{1}{2} e\mathscr{R} - e g_{ij^*} \partial_m A^i \partial^m A^{*j} \\
& - i e g_{ij^*} \bar{\chi}^j \bar{\sigma}^m \mathscr{D}_m \chi^i + e \varepsilon^{k\ell mn} \bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathscr{D}}_m \psi_n \\
& - \frac{1}{2} \sqrt{2} e g_{ij^*} \partial_n A^{*j} \chi^i \sigma^m \bar{\sigma}^n \psi_m - \frac{1}{2} \sqrt{2} e g_{ij^*} \partial_n A^i \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
& + \frac{1}{4} e g_{ij^*} [i \varepsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m] \chi^i \sigma_n \bar{\chi}^j \\
& - \frac{1}{8} e [g_{ij^*} g_{k\ell^*} - 2 R_{ij^*k\ell^*}] \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\
& - e \exp(K/2) \Big\{ P^* \psi_a \sigma^{ab} \psi_b + P \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
& + \frac{i}{2} \sqrt{2} D_i P \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} D_{i^*} P^* \bar{\chi}^i \bar{\sigma}^a \psi_a \\
& + \frac{1}{2} \mathscr{D}_i D_j P \chi^i \chi^j + \frac{1}{2} \mathscr{D}_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \Big\} \\
& - e \exp(K) [g^{ij^*} (D_i P)(D_j P)^* - 3 P^* P] . \qquad \textbf{(23.3)}
\end{aligned}
$$

The covariant derivatives are defined as follows:

$$
\begin{aligned}
\mathscr{D}_m \chi^i &= \partial_m \chi^i + \chi^i \omega_m + \Gamma^i_{jk} \partial_m A^j \chi^k - \frac{1}{4} (K_j \partial_m A^j - K_{j^*} \partial_m A^{*j}) \chi^i \\
\tilde{\mathscr{D}}_m \psi_n &= \partial_m \psi_n + \psi_n \omega_m + \frac{1}{4} (K_j \partial_m A^j - K_{j^*} \partial_m A^{*j}) \psi_n \\
D_i P &= P_i + K_i P \\
\mathscr{D}_i D_j P &= P_{ij} + K_{ij} P + K_i D_j P + K_j D_i P - K_i K_j P - \Gamma^k_{ij} D_k P . \qquad \textbf{(23.4)}
\end{aligned}
$$

The covariant derivatives contain the Christoffel symbols for the Kähler geometry, and the spin connection (17.12) for spacetime. Note that they also contain a U(1) connection proportional to $\mathrm{Im}(K_j \partial_m A^j)$. The meaning of the U(1) connection will become clear as we proceed.

The Lagrangian (23.3) is invariant under supergravity transformations because it was derived from a superspace formalism. It is useful, however, to verify the invariance directly, using the following supergravity transformations:

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta\sigma^a\bar\psi_m + \bar\zeta\bar\sigma^a\psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\zeta\chi^i \\
\delta_\zeta\chi^i &= i\sqrt{2}\sigma^m\bar\zeta\hat D_m A^i - \Gamma^i_{jk}\,\delta_\zeta A^j\chi^k \\
&\quad + \frac{1}{4}(K_j\,\delta_\zeta A^j - K_{j^*}\,\delta_\zeta A^{*j})\chi^i - \sqrt{2}e^{K/2}g^{ij^*}D_{j^*}P^*\zeta \\
\delta_\zeta\psi_m &= 2\mathscr{D}_m\zeta - \frac{1}{4}(K_j\,\delta_\zeta A^j - K_{j^*}\,\delta_\zeta A^{*j})\psi_m \\
&\quad - \frac{i}{2}\sigma_{mn}\zeta\; g_{ij^*}\chi^i\sigma^n\bar\chi^j + ie^{K/2}P\sigma_m\bar\zeta,
\end{aligned} \tag{23.5}
$$

where $\mathscr{D}_m\zeta$ includes the U(1) connection,

$$
\mathscr{D}_m\zeta = \partial_m\zeta + \zeta\omega_m + \frac{1}{4}(K_j\,\partial_m A^j - K_{j^*}\,\partial_m A^{*j})\zeta. \tag{23.6}
$$

Note that the transformation for the field χ^i indicates that supersymmetry is spontaneously broken whenever $\langle D_iP\rangle \neq 0$. In this case, χ^i shifts by a constant and plays the role of the Goldstone fermion.

To check the Kähler invariance, let us first examine the component Lagrangian (23.3). Under a Kähler transformation,

$$
K(A,A^*) \to K(A,A^*) + F(A) + F^*(A^*), \tag{23.7}
$$

the metric, Christoffel symbols and curvature terms are all invariant. The U(1) connection is not:

$$
\begin{aligned}
\mathscr{D}_m\chi^i &\to \mathscr{D}_m\chi^i - \frac{i}{2}\partial_m(\mathrm{Im}\,F)\chi^i \\
\tilde{\mathscr{D}}_m\psi_n &\to \tilde{\mathscr{D}}_m\psi_n + \frac{i}{2}\partial_m(\mathrm{Im}\,F)\psi_n.
\end{aligned} \tag{23.8}
$$

The Kähler invariance is restored if Kähler transformations are accompanied by Weyl rotations of the spinor fields,

$$\chi^i \to \exp +\frac{i}{2}(\mathrm{Im}\, F)\, \chi^i$$

$$\psi_n \to \exp -\frac{i}{2}(\mathrm{Im}\, F)\, \psi_n. \tag{23.9}$$

With this rule, the kinetic terms in (23.3) are invariant under the combined Kähler-Weyl transformations. The Kähler-Weyl invariance insures that the kinetic terms are invariant under field redefinitions (such as isometries) that induce Kähler transformations of the Kähler potential K.

The superpotential contributions to the component Lagrangian contain explicit factors of K, so their invariance is not automatic under the Kähler-Weyl transformations. For example, the scalar potential

$$\mathscr{V} = e^K[g^{ij^*}(D_iP)(D_jP)^* - 3P^*P] \tag{23.10}$$

is not invariant unless

$$P \to e^{-F}P \tag{23.11}$$

as well. With this choice, the D_iP transform covariantly,

$$D_iP \to e^{-F}D_iP, \tag{23.12}$$

and the full Lagrangian is invariant. Note that (23.11) does not in general preserve a polynomial structure in the superpotential (see Exercise 21.7).

In mathematical language, the transformations (23.9) and (23.11) imply that the spinors and the superpotential are not ordinary functions, but rather sections of appropriate line bundles over the Kähler manifold. If the manifold is nontrivial, the combined Kähler-Weyl invariance is necessary for the Lagrangian to be globally well defined. Locally, however, we can simply think of the geometrical notation as giving a convenient shorthand that is useful for describing the full set of supergravity couplings.

The Kähler-Weyl invariance of the matter couplings can also be seen from the superspace Lagrangian (23.1). Now, however, the superfield

Kähler transformation

$$K(\Phi,\Phi^+) \to K(\Phi,\Phi^+) + F(\Phi) + F^+(\Phi^+) \tag{23.13}$$

must be accompanied by a super-Weyl transformation of the vielbein.

A super-Weyl transformation is defined to be a superfield rescaling of the vielbein, consistent with the torsion constraints (14.25). In the exercises, it is shown that the most general such transformation is of the form

$$\begin{aligned} \delta E_M{}^a &= (\Sigma + \bar{\Sigma})E_M{}^a \\ \delta E_M{}^\alpha &= (2\bar{\Sigma} - \Sigma)E_M{}^\alpha + \frac{i}{2} E_M{}^b(\varepsilon\sigma_b)^\alpha{}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}}\bar{\Sigma}, \end{aligned} \tag{23.14}$$

where Σ and $\bar{\Sigma}$ are chiral superfields,

$$\mathscr{D}_\alpha\bar{\Sigma} = \bar{\mathscr{D}}_{\dot{\alpha}}\Sigma = 0. \tag{23.15}$$

This implies

$$\begin{aligned} \delta R &= -2(2\Sigma - \bar{\Sigma})R - \frac{1}{4}\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}}\bar{\Sigma} \\ \delta G_{\alpha\dot{\alpha}} &= -(\Sigma + \bar{\Sigma})G_{\alpha\dot{\alpha}} + i\mathscr{D}_{\alpha\dot{\alpha}}(\bar{\Sigma} - \Sigma). \end{aligned} \tag{23.16}$$

The transformations (23.14) and (23.16) determine the super-Weyl transformations of the supergravity multiplet.

The super-Weyl transformations of the matter fields are also parametrized by Σ and $\bar{\Sigma}$. They are defined in such a way as to preserve the appropriate constraints. For example, a super-Weyl transformation of a chiral superfield is given by

$$\delta\Phi = w\Sigma\Phi, \tag{23.17}$$

while that of a hermitian vector superfield is just

$$\delta V = w'(\Sigma + \bar{\Sigma})V. \tag{23.18}$$

In these expressions, w and w' are called the Weyl weights of the respective superfields.

Equations (23.14)–(23.18) allow one to find the super-Weyl scalings of the various component fields. These, in turn, can be written as variations of superfields in the new Θ variables. After a small computation, one finds

$$\delta\mathscr{E} = 6\Sigma\mathscr{E} + \frac{\partial}{\partial\Theta^\alpha}(S^\alpha\mathscr{E})$$

$$\delta\Phi = w\Sigma\Phi - S^\alpha\frac{\partial}{\partial\Theta^\alpha}\Phi$$

$$\begin{aligned}\delta(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)U &= (\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)[(w' - 4)\Sigma + (w' + 2)\bar{\Sigma}]U \\ &\quad - S^\alpha\frac{\partial}{\partial\Theta^\alpha}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)U, \qquad (23.19)\end{aligned}$$

where

$$S^\alpha = \Theta^\alpha(2\bar{\Sigma} - \Sigma)| + \Theta\Theta\mathscr{D}^\alpha\Sigma|, \qquad (23.20)$$

and U is an arbitrary hermitian superfield of weight w'.

The transformations (23.19) induce a variation of the superspace Lagrangian. For $w = 0$, we find

$$\delta\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\left[\frac{3}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)(\Sigma + \bar{\Sigma})e^{-K/3} + 6\Sigma P\right] + \text{h.c.} \qquad (23.21)$$

This is precisely a Kähler transformation,

$$\delta\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\left[-\frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)(F + F^*)e^{-K/3} - FP\right] + \text{h.c.}, \qquad (23.22)$$

where P is scaled to $e^{-F}P$ in accord with (23.11). Comparing the two transformations, we see that (23.21) cancels (23.22) if $F = 6\Sigma$. With this choice, the superspace Lagrangian is invariant under combined Kähler-Weyl transformations. It is a useful exercise to show that superspace Kähler-Weyl transformations induce local Weyl rotations (23.9) of the component fields.

An arbitrary super-Weyl transformation can be used to change the form of the Lagrangian (23.1). In particular, a super-Weyl transformation with a finite parameter $-6\Sigma = \log P$ simplifies that component expression by rescaling the superpotential to one. Of course, this is an allowed transformation only if the expectation value $\langle P \rangle$ is nonzero. This is not an innocent assumption: it gives a nonvanishing contribution to the cosmological constant. This contribution can be canceled only if supersymmetry is spontaneously broken.

The transformation with $-6\Sigma = \log P$ changes the Kähler potential K to a new potential $G = K + \log P + \log P^+$. Since this is a Kähler transformation, the geometry is left invariant. Therefore, to find the new Lagrangian, we simply replace P by 1, K by G, D_iP by G_i, and $\mathscr{D}_iD_jP$ by $G_{ij} + G_iG_j - \Gamma^k_{ij}G_k$. This gives

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{2} e\mathscr{R} - eg_{ij^*}\partial_m A^i \partial^m A^{*j} \\
& - ieg_{ij^*}\bar{\chi}^j\bar{\sigma}^m\mathscr{D}_m\chi^i + e\varepsilon^{k\ell mn}\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathscr{D}}_m\psi_n \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\partial_n A^{*j}\chi^i\sigma^m\bar{\sigma}^n\psi_m - \frac{1}{2}\sqrt{2}eg_{ij^*}\partial_n A^i\bar{\chi}^j\bar{\sigma}^m\sigma^n\bar{\psi}_m \\
& + \frac{1}{4}eg_{ij^*}[i\varepsilon^{k\ell mn}\psi_k\sigma_\ell\bar{\psi}_m + \psi_m\sigma^n\bar{\psi}^m]\chi^i\sigma_n\bar{\chi}^j \\
& - \frac{1}{8}e[g_{ij^*}g_{k\ell^*} - 2R_{ij^*k\ell^*}]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^\ell \\
& - ee^{G/2}\Big\{\psi_a\sigma^{ab}\psi_b + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b \\
& + \frac{i}{2}\sqrt{2}G_i\chi^i\sigma^a\bar{\psi}_a + \frac{i}{2}\sqrt{2}G_{i^*}\bar{\chi}^i\bar{\sigma}^a\psi_a \\
& + \frac{1}{2}[G_{ij} + G_iG_j - \Gamma^k_{ij}G_k]\chi^i\chi^j \\
& + \frac{1}{2}[G_{i^*j^*} + G_{i^*}G_{j^*} - \Gamma^{k^*}_{i^*j^*}G_{k^*}]\bar{\chi}^i\bar{\chi}^j\Big\} \\
& - ee^G[g^{ij^*}G_iG_{j^*} - 3].
\end{aligned} \tag{23.23}$$

Let us now examine the physical content of (23.23). We first note that the kinetic terms are properly normalized if $g_{ij^*} = \delta_{ij^*} + \cdots$. We shall always assume this to be true. We then remark that the potential

$$\mathscr{V} = e^G[g^{ij^*}G_iG_{j^*} - 3] \tag{23.24}$$

is extremized if $\langle \partial \mathscr{V}/\partial A^i \rangle = 0$, and that the resulting cosmological constant is zero if $\langle \mathscr{V} \rangle = 0$. Taken together, these conditions are satisfied if

$$\begin{aligned} \langle G^i G_i \rangle &= 3 \\ \langle G^i \nabla_k G_i + G_k \rangle &= 0, \end{aligned} \tag{23.25}$$

where $G^i = g^{ij^*} G_{j^*}$ and $\nabla_k G_i = \partial_k G_i - \Gamma^j_{ki} G_j$.

The scalar mass matrix is found from the second variation of $\mathscr{V}$. It is of the form

$$\begin{pmatrix} M^2_{ij^*} & M^2_{ij} \\ M^2_{i^*j^*} & M^2_{i^*j} \end{pmatrix}, \tag{23.26}$$

with

$$\begin{aligned} M^2_{ij^*} &= \langle [\nabla_i G_k \nabla_{j^*} G^k - R_{ij^*k\ell^*} G^k G^{\ell^*} + g_{ij^*}] e^G \rangle \\ M^2_{ij} &= \langle [\nabla_i G_j + \nabla_j G_i] e^G \rangle, \end{aligned} \tag{23.27}$$

where we have repeatedly used (23.25).

The spinor mass matrix can be found from (23.23) as well. Focusing on the quadratic terms, we find

$$\begin{aligned} - \langle e^{G/2} \rangle \Big\{ & \psi_a \sigma^{ab} \psi_b + \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\ & + \frac{i}{2} \sqrt{2} \langle G_i \rangle \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} \langle G_{i^*} \rangle \bar{\chi}^i \bar{\sigma}^a \psi_a \\ & + \frac{1}{2} \langle \nabla_i G_j + G_i G_j \rangle \chi^i \chi^j \\ & + \frac{1}{2} \langle \nabla_{i^*} G_{j^*} + G_{i^*} G_{j^*} \rangle \bar{\chi}^i \bar{\chi}^j \Big\}. \end{aligned} \tag{23.28}$$

The mass of the gravitino is easily seen to be $\langle e^{G/2} \rangle$.

The masses of the spinors χ^i are a little more subtle because of the mixing between the gravitino $\sigma_n \bar{\psi}_n$ and the spinor $G_i \chi^i$. When $\langle G_i \rangle \neq 0$, this mixing must be removed to find the physical mass matrix. Of course, it is always possible to diagonalize the coupling by redefining the fields. It is more instructive, however, to recognize that the mixing has an important physical origin.

To see this, let us consider the supergravity transformation of the field $\eta = G_i\chi^i$, given by (23.5) with the appropriate substitutions,

$$\begin{aligned}\delta\eta &= -\sqrt{2}e^{G/2}G^iG_i\zeta + \cdots \\ &= -3\sqrt{2}m_\psi\zeta + \cdots,\end{aligned} \tag{23.29}$$

where we have used the fact that the cosmological constant is zero, $\langle G^iG_i\rangle = 3$. We see that η transforms by a shift. This indicates that η is a Goldstone fermion, and supersymmetry is spontaneously broken.

Exactly as in ordinary gauge theory, the Goldstone fermion can be gauged away through a supersymmetric analog of the Higgs effect. In this "unitary gauge," all terms proportional to $G_i\chi^i$ vanish identically. This removes the gravitino-Goldstino mixing, and allows one to read off the mass matrix for the spinors χ^i, subject to the constraint $G_i\chi^i = 0$.

Of course, it is also possible to find the spinor mass matrix in a gauge-independent manner, by diagonalizing the terms quadratic in the fermion fields. The necessary field redefinition is suggested by the above arguments. We find

$$\tilde{\psi}_n = \psi_n + \frac{1}{3}\sqrt{2}m_\psi^{-1}\,\partial_n\eta + \frac{i}{6}\sqrt{2}\sigma_n\bar{\eta}. \tag{23.30}$$

With this choice, the mixings are eliminated and the mass terms are diagonal:

$$\begin{aligned}-\,m_\psi\Big\{&\tilde{\psi}_a\sigma^{ab}\tilde{\psi}_b + \bar{\tilde{\psi}}_a\bar{\sigma}^{ab}\bar{\tilde{\psi}}_b \\ &+ \frac{1}{2}\langle\nabla_iG_j + \frac{1}{3}G_iG_j\rangle\chi^i\chi^j \\ &+ \frac{1}{2}\langle\nabla_{i^*}G_{j^*} + \frac{1}{3}G_{i^*}G_{j^*}\rangle\bar{\chi}^i\bar{\chi}^j\Big\}.\end{aligned} \tag{23.31}$$

The spinor mass matrix is just

$$m_{ij} = \left\langle \nabla_iG_j + \frac{1}{3}G_iG_j\right\rangle m_\psi. \tag{23.32}$$

Squaring, we have

$$\begin{aligned}m^2_{ij^*} &= g^{k\ell^*}\left\langle\nabla_iG_k + \frac{1}{3}G_iG_k\right\rangle\left\langle\nabla_{j^*}G_{\ell^*} + \frac{1}{3}G_{j^*}G_{\ell^*}\right\rangle m_\psi^2 \\ &= \left\langle\nabla_iG_k\nabla_{j^*}G^k - \frac{1}{3}G_iG_{j^*}\right\rangle m_\psi^2.\end{aligned} \tag{23.33}$$

We are now in a position to derive a mass sum rule for the physical fields. Combining (23.26), (23.27), and (23.33), we find

$$\begin{aligned}\mathrm{Str}\, M^2 &= \sum_{\text{spins } J} (-1)^{2J}(2J+1)\,\mathrm{Tr}\, M^2 \\ &= \langle 2g^{ij^*}M^2_{ij^*} - 2g^{ij^*}m^2_{ij^*} - 4e^G\rangle \\ &= 2(n-1)m_\psi^2 - 2\langle R_{ij^*}G^iG^{j^*}\rangle m_\psi^2, \end{aligned} \qquad \textbf{(23.34)}$$

for n scalar fields A^i. From the mass sum rule, we see that the boson-fermion mass splittings are proportional to m_ψ. When the cosmological constant is zero, the gravitino mass serves as the order parameter for the spontaneous breaking of supergravity.

References

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Equations

$$\mathscr{L} = \frac{1}{\kappa^2}\int d^2\Theta\, 2\mathscr{E}\left[\frac{3}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{\kappa^2}{3}K(\Phi,\Phi^+)\right\} + \kappa^2 P(\Phi)\right] + \text{h.c.} \qquad \textbf{(23.1)}$$

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}e\mathscr{R} - eg_{ij^*}\partial_m A^i\partial^m A^{*j} \\ &- ieg_{ij^*}\bar\chi^j\bar\sigma^m\mathscr{D}_m\chi^i + e\varepsilon^{k\ell mn}\bar\psi_k\bar\sigma_\ell\tilde{\mathscr{D}}_m\psi_n \\ &- \frac{1}{2}\sqrt{2}eg_{ij^*}\partial_n A^{*j}\chi^i\sigma^m\bar\sigma^n\psi_m - \frac{1}{2}\sqrt{2}eg_{ij^*}\partial_n A^i\bar\chi^j\bar\sigma^m\sigma^n\bar\psi_m \\ &+ \frac{1}{4}eg_{ij^*}[i\varepsilon^{k\ell mn}\psi_k\sigma_\ell\bar\psi_m + \psi_m\sigma^n\bar\psi^m]\chi^i\sigma_n\bar\chi^j \\ &- \frac{1}{8}e[g_{ij^*}g_{k\ell^*} - 2R_{ij^*k\ell^*}]\chi^i\chi^k\bar\chi^j\bar\chi^\ell \\ &- e\exp(K/2)\Big\{P^*\psi_a\sigma^{ab}\psi_b + P\bar\psi_a\bar\sigma^{ab}\bar\psi_b \\ &+ \frac{i}{2}\sqrt{2}D_iP\chi^i\sigma^a\bar\psi_a + \frac{i}{2}\sqrt{2}D_{i^*}P^*\bar\chi^i\bar\sigma^a\psi_a \\ &+ \frac{1}{2}\mathscr{D}_iD_jP\chi^i\chi^j + \frac{1}{2}\mathscr{D}_{i^*}D_{j^*}P^*\bar\chi^i\bar\chi^j\Big\} \\ &- e\exp(K)[g^{ij^*}(D_iP)(D_jP)^* - 3P^*P]. \end{aligned} \qquad \textbf{(23.3)}$$

$$\mathscr{D}_m\chi^i = \partial_m\chi^i + \chi^i\omega_m + \Gamma^i_{jk}\partial_m A^j\chi^k - \frac{1}{4}(K_j\partial_m A^j - K_{j*}\partial_m A^{*j})\chi^i$$

$$\tilde{\mathscr{D}}_m\psi_n = \partial_m\psi_n + \psi_n\omega_m + \frac{1}{4}(K_j\partial_m A^j - K_{j*}\partial_m A^{*j})\psi_n$$

$$D_iP = P_i + K_iP$$

$$\mathscr{D}_iD_jP = P_{ij} + K_{ij}P + K_iD_jP + K_jD_iP - K_iK_jP - \Gamma^k_{ij}D_kP. \quad \textbf{(23.4)}$$

$$\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta\sigma^a\bar{\psi}_m + \bar{\zeta}\bar{\sigma}^a\psi_m)\\
\delta_\zeta A^i &= \sqrt{2}\zeta\chi^i\\
\delta_\zeta\chi^i &= i\sqrt{2}\sigma^m\bar{\zeta}\hat{D}_m A^i - \Gamma^i_{jk}\delta_\zeta A^j\chi^k\\
&\quad + \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\chi^i - \sqrt{2}e^{K/2}g^{ij*}D_{j*}P^*\zeta\\
\delta_\zeta\psi_m &= 2\mathscr{D}_m\zeta - \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\psi_m\\
&\quad - \frac{i}{2}\sigma_{mn}\zeta\, g_{ij*}\chi^i\sigma^n\bar{\chi}^j + ie^{K/2}P\sigma_m\bar{\zeta}.
\end{aligned} \quad \textbf{(23.5)}$$

$$\begin{aligned}
\chi^i &\to \exp +\frac{i}{2}(\mathrm{Im}\,F)\,\chi^i\\
\psi_n &\to \exp -\frac{i}{2}(\mathrm{Im}\,F)\,\psi_n.
\end{aligned} \quad \textbf{(23.9)}$$

$$\mathscr{V} = e^K[g^{ij*}(D_iP)(D_jP)^* - 3P^*P]. \quad \textbf{(23.10)}$$

$$\begin{aligned}
\mathrm{Str}\,M^2 &= \sum_{\text{spins } J}(-1)^{2J}(2J+1)\,\mathrm{Tr}\,M^2\\
&= 2(n-1)m_\psi^2 - 2\langle R_{ij*}G^iG^{j*}\rangle m_\psi^2.
\end{aligned} \quad \textbf{(23.34)}$$

Exercises

(1) Given variations $\delta E_M{}^A$ and $\delta\phi_{MB}{}^A$ of the vielbein and the connection, show that the most general variation of the torsion $T_{CB}{}^A$ is given by

$$\begin{aligned}\delta T_{CB}{}^{A} = {} & \mathscr{D}_C H_B{}^A - (-)^{bc}\mathscr{D}_B H_C{}^A \\ & + \Omega_{CB}{}^A - (-)^{bc}\Omega_{BC}{}^A \\ & + T_{CB}{}^D H_D{}^A - H_C{}^D T_{DB}{}^A + (-)^{bc} H_B{}^D T_{DC}{}^A,\end{aligned}$$

where $H_A{}^B = E_A{}^M \, \delta E_M{}^B$ and $\Omega_{CB}{}^A = E_C{}^M \, \delta\phi_{MB}{}^A$.

(2) Use the results of Exercise 1 to show that the most general Weyl rescaling of the vielbein, consistent with the torsion constraints, is of the form (23.14).

(3) Find the Weyl rescalings of R and $G_{\alpha\dot{\alpha}}$. Check your results against (23.16).

(4) Show that the conditions (23.25) imply that the scalar potential (23.24) is extremized with vanishing cosmological constant.

(5) Compute the scalar mass matrix (23.27).

(6) For infinitesimal η, show that $\zeta = \eta\sqrt{2}/6m_\psi$ transforms η to zero.

(7) Show that the matrix (23.32) has a zero eigenvalue, with eigenvector proportional to G^i.

(8) Verify the mass sum rule (23.34).

(9) Show that $\mathrm{Str}\, M^2 = 0$ for the minimal chiral model, where $G = -3\log(1 - \frac{1}{3}A_i{}^* A_i)$. This is an important property of the model because most radiative corrections are proportional to $\mathrm{Str}\, M^2$.

XXIV. GAUGE INVARIANT MODELS

In Chapter XXII we studied the most general coupling of chiral superfields in flat space,

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^i,\Phi^{+j}) + \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right]. \tag{24.1}$$

We found that K has a natural interpretation as the Kähler potential for a Kähler manifold $\mathscr{M}$. We also noted that the action (24.1) is invariant under the Kähler transformations,

$$K(\Phi^i,\Phi^{+j}) \rightarrow K(\Phi^i,\Phi^{+j}) + F(\Phi^i) + F^{+}(\Phi^{+j}), \tag{24.2}$$

where F is an analytic function of the superfields Φ^i. In this chapter we will gauge the analytic isometries of the Kähler geometry, and in this way generalize (24.1) to include vector fields. We will take advantage of the fact that K transforms by a Kähler transformation under each of the analytic isometries of $\mathscr{M}$.

The analytic isometries of a Kähler manifold are generated by holomorphic Killing vectors,

$$\begin{aligned} X^{(b)} &= X^{i(b)}(a^j)\,\frac{\partial}{\partial a^i} \\ X^{*(b)} &= X^{*i(b)}(a^{*j})\,\frac{\partial}{\partial a^{*i}}, \end{aligned} \tag{24.3}$$

where the index (b) runs over the dimension d of the isometry group G. As shown in Appendix D, this implies that Killing's equation reduces to the statement that there exist d real scalar functions $D^{(a)}(a,a^*)$, such that

$$\begin{aligned} g_{ij^*}X^{*j(a)} &= i\,\frac{\partial}{\partial a^i}\,D^{(a)} \\ g_{ij^*}X^{i(a)} &= -i\,\frac{\partial}{\partial a^{*j}}\,D^{(a)}. \end{aligned} \tag{24.4}$$

The $D^{(a)}$ are known as Killing potentials. They are defined up to constants $c^{(a)}$, $D^{(a)} \to D^{(a)} + c^{(a)}$. In what follows, we shall see that the freedom to redefine the potentials is related to the Fayet-Iliopoulos D term introduced in Chapter VIII.

The Killing vectors $X^{(a)}$ and $X^{*(a)}$ generate independent representations of the isometry group G. They obey the Lie bracket relations

$$\begin{aligned} [X^{(a)}, X^{(b)}] &= -f^{abc} X^{(c)} \\ [X^{*(a)}, X^{*(b)}] &= -f^{abc} X^{*(c)} \\ [X^{(a)}, X^{*(b)}] &= 0, \end{aligned} \tag{24.5}$$

where the f^{abc} are the structure constants of G. In Appendix D it is shown that the Killing potentials $D^{(a)}$ can be chosen to transform in the adjoint representation,

$$\left[X^{i(a)} \frac{\partial}{\partial a^i} + X^{*i(a)} \frac{\partial}{\partial a^{*i}} \right] D^{(b)} = -f^{abc} D^{(c)}. \tag{24.6}$$

This fixes the constants $c^{(a)}$ for non-Abelian groups. For each U(1) factor, however, there is an undetermined constant c.

Under an isometry in G, the variations of K and P are determined by the Killing vectors $X^{(a)}$,

$$\begin{aligned} \delta K &= [\varepsilon^{(a)} X^{(a)} + \varepsilon^{*(a)} X^{*(a)}] K \\ \delta P &= \varepsilon^{(a)} X^{(a)} P. \end{aligned} \tag{24.7}$$

The variation of the superpotential must vanish for the action to be invariant. The variation of the Kähler potential, however, does not need to vanish. As shown in Appendix D, it can be cast in the following form:

$$\delta K = \varepsilon^{(a)} F^{(a)} + \varepsilon^{*(a)} F^{*(a)} - i(\varepsilon^{(a)} - \varepsilon^{*(a)}) D^{(a)}, \tag{24.8}$$

where $F^{(a)} = X^{(a)} K + i D^{(a)}$ is an analytic function of the coordinates. For real parameters $\varepsilon^{(a)}$, (24.8) is just a Kähler transformation. For complex $\varepsilon^{(a)}$, it is not of Kähler form; there is a change in K proportional to the Killing potential $D^{(a)}$.

The fact that (24.8) reduces to a Kähler transformation for real $\varepsilon^{(a)}$ implies that the action (24.1) is invariant under the rigid isometries of the manifold $\mathscr{M}$. For local motions, however, the story is more complicated. This is because the parameter $\varepsilon^{(a)}$ must be promoted to a chiral superfield in superspace. In this case, the variation of the action (24.1) is

$$\begin{aligned}\delta\mathscr{L} &= \int d^2\theta\, d^2\bar\theta\, \delta K \\ &= \int d^2\theta\, d^2\bar\theta \left[\Lambda^{(a)}F^{(a)} + \Lambda^{+(a)}F^{+(a)} - i(\Lambda^{(a)} - \Lambda^{+(a)})D^{(a)}\right] \\ &= -i\int d^2\theta\, d^2\bar\theta (\Lambda^{(a)} - \Lambda^{+(a)})D^{(a)},\end{aligned} \tag{24.9}$$

where $\Lambda^{(a)}$ is a chiral superfield with lowest component $\varepsilon^{(a)}$, and the $D^{(a)}$ are hermitian functions of the chiral superfields Φ^i and Φ^{+j}.

In the rest of this chapter, we will see how to construct a supersymmetric gauge theory, invariant under the isometries parametrized by $\Lambda^{(a)}$. We will add a term to the action whose variation exactly cancels (24.9), using the formalism developed in Appendixes E and F. We will find that the counterterm involves the vector superfield $V = V^{(a)}T^{(a)}$, where the $T^{(a)}$ are the hermitian generators of the isometry group G.

Since $\varepsilon^{(a)}$ is complex, we must study the complexification of G, which we call $\mathscr{G}$. An arbitrary element of $\mathscr{G}$ can be written in the form

$$g = e^{\frac{1}{2}v^{(a)}T^{(a)}}e^{-\frac{i}{2}u^{(a)}T^{(a)}}, \tag{24.10}$$

where $u^{(a)}$ and $v^{(a)}$ are real, and as above, the $T^{(a)}$ are the hermitian generators of G. Equation (24.10) splits g into the product of a hermitian and a unitary matrix, which can always be done.

Given the complexification $\mathscr{G}$ of G, the space $\mathscr{G}/G$ is constructed by identifying elements g and $g' \in \mathscr{G}$ if $g = g'u'$, for some $u' \in G$. Thus a point of the coset can be represented by

$$v = e^{\frac{1}{2}v^{(a)}T^{(a)}}. \tag{24.11}$$

The matrix v is an element of $\mathscr{G}$, and the $v^{(a)}$ are coordinates of $\mathscr{G}/G$.

The group $\mathscr{G}$ acts naturally on the cosets $\mathscr{G}/G$ by *left* multiplication on v: $v' = g_0 v$. To find the transformation of v, it is useful to examine two cases, the first with $g_0 = u_0 \in G$, and the second with $g_0 = v_0 \in \mathscr{G}$ (but not in G). For a transformation parametrized by u_0, we have

$$v \to u_0 v = u_0 v u_0^{-1} u_0 \equiv v'u', \tag{24.12}$$

where $v' = u_0 v u_0^{-1}$ and $u' = u_0$. In terms of the coordinates $v^{(a)}$, this implies

$$e^{\frac{1}{2}v'^{(a)}T^{(a)}} = u_0 e^{\frac{1}{2}v^{(a)}T^{(a)}} u_0^{-1}, \tag{24.13}$$

and we see that the $v^{(a)}$ transform linearly under elements $u_0 \in G$. In contrast, for a transformation v_0, we have

$$v \to v_0 v \equiv v' u'. \tag{24.14}$$

Taking the hermitian conjugate, we find

$$v \to v v_0 \equiv u'^{\dagger} v'. \tag{24.15}$$

Combining the two expressions, we see that

$$v'^2 = v_0 v^2 v_0. \tag{24.16}$$

In terms of the coordinates $v^{(a)}$, this implies

$$e^{v'^{(a)}T^{(a)}} = v_0 e^{v^{(a)}T^{(a)}} v_0, \tag{24.17}$$

a manifestly nonlinear transformation law. Note that the $v^{(a)}$ can be transformed to zero if we take $v_0 = e^{-\frac{1}{2}v^{(a)}T^{(a)}}$.

For infinitesimal variations, the transformations (24.13) and (24.17) can be combined to give

$$\begin{aligned} \delta \exp(v^{(a)}T^{(a)}) &= -\frac{i}{2}[u_0^{(b)} + iv_0^{(b)}]T^{(b)} \exp(v^{(a)}T^{(a)}) \\ &\quad + \frac{i}{2}\exp(v^{(a)}T^{(a)})[u_0^{(b)} - iv_0^{(b)}]T^{(b)} \\ &= -i\varepsilon^* e^{v^{(a)}T^{(a)}} + i e^{v^{(a)}T^{(a)}}\varepsilon, \end{aligned} \tag{24.18}$$

where we have set $\varepsilon = \varepsilon^{(b)}T^{(b)}$, with $\varepsilon^{(b)} = \frac{1}{2}(u_0^{(b)} - iv_0^{(b)})$. If we identify $\varepsilon^{(b)}$ with the lowest component of a chiral superfield $\Lambda^{(b)}$, and $v^{(a)}$ with the lowest component of vector superfield $V^{(a)}$, we see that the transformation (24.18) is precisely the lowest component of the gauge transformation (7.15):

$$\delta e^V = -i\Lambda^+ e^V + i e^V \Lambda. \tag{24.19}$$

Comparing (24.13) and (24.17) with (E.17) and (E.23), we see that the transformation law of a vector superfield is just a nonlinear realization, corresponding to the coset $\mathcal{G}/G$. As in Appendix F, we can exploit this fact to construct a fully gauge invariant theory. Recall that previously we found

$$\delta \mathcal{L} = -i \int d^2\theta\, d^2\bar{\theta}\, (\Lambda^{(a)} - \Lambda^{+(a)}) D^{(a)}. \tag{24.20}$$

To cancel this variation, we need to find a function $\Gamma(\Phi^i,\Phi^{+j},V^{(a)})$ such that

$$\delta\Gamma = i[\Lambda^{(a)} - \Lambda^{+(a)}]D^{(a)}. \tag{24.21}$$

Then

$$\begin{aligned}\mathscr{L} = &\int d^2\theta\, d^2\bar{\theta}\,[K(\Phi^i,\Phi^{+j}) + \Gamma(\Phi^i,\Phi^{+j},V^{(a)})] \\ &+ \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right]\end{aligned} \tag{24.22}$$

will be a fully gauge invariant action.

To find the counterterm Γ, we first restrict to its lowest component $\Gamma(a^i,a^{*j},v^{(a)})$. We then write the variation $\delta\Gamma$ in terms of differential operators,

$$\begin{aligned}\delta\Gamma &= \varepsilon^{(a)}X^{(a)}\Gamma + \varepsilon^{*(a)}X^{*(a)}\Gamma + \delta v^{(a)}\frac{\delta\Gamma}{\delta v^{(a)}} \\ &\equiv \frac{1}{2}(\varepsilon^{(a)} + \varepsilon^{*(a)})\mathscr{P}^{(a)}\Gamma + \frac{1}{2}(\varepsilon^{(a)} - \varepsilon^{*(a)})\mathscr{O}^{(a)}\Gamma,\end{aligned} \tag{24.23}$$

where $\mathscr{P}^{(a)}$ and $\mathscr{O}^{(a)}$ include the variations of the coordinates a^i and a^{*j}, as well as the appropriate variations of the $v^{(a)}$. For (24.23) to agree with (24.21), we must demand

$$\begin{aligned}\mathscr{P}^{(a)}\Gamma &= 0 \\ \mathscr{O}^{(a)}\Gamma &= 2iD^{(a)}.\end{aligned} \tag{24.24}$$

Furthermore, we also require that Γ satisfy the boundary condition

$$\Gamma(a^i,a^{*j},0) = 0. \tag{24.25}$$

With these ingredients, it is not hard to integrate (24.24). Following the steps of Appendix F, we find

$$\begin{aligned}\Gamma &= \frac{e^{\frac{i}{2}v^{(a)}O^{(a)}} - 1}{\frac{i}{2}v^{(b)}O^{(b)}}\, v^{(c)}D^{(c)} \\ &= \int_0^1 d\alpha\, e^{\frac{i}{2}\alpha v^{(a)}O^{(a)}} v^{(c)}D^{(c)}.\end{aligned} \tag{24.26}$$

In this expression, the operator $O^{(a)}$ is the same as $\mathcal{O}^{(a)}$ but without the variations of the $v^{(a)}$:

$$O^{(a)} = X^{(a)} - X^{*(a)}. \tag{24.27}$$

It is a useful exercise to check that Γ indeed obeys (24.24).

Having found the counterterm Γ, we are now ready to write the gauge invariant action in superspace. We first promote Γ to a superfield, replacing a^i, a^{*j} and $v^{(a)}$ by superfields Φ^i, Φ^{+j} and $V^{(a)}$. In a symbolic notation, we have

$$\Gamma(\Phi^i,\Phi^{+j},V^{(a)}) = \int_0^1 d\alpha\, e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}} V^{(b)}D^{(b)}, \tag{24.28}$$

where the differentiations $O^{(a)}$ are performed *before* the fields are replaced by superfields. Substituting this expression into (24.22), we obtain the complete gauge invariant action in superspace:

$$\begin{aligned}\mathscr{L} = \int d^2\theta\, d^2\bar\theta\, K(\Phi^i,\Phi^{+j}) + \int_0^1 d\alpha \int d^2\theta\, d^2\bar\theta\, e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}} V^{(b)}D^{(b)} \\ + \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right].\end{aligned} \tag{24.29}$$

The action (24.29) is manifestly supersymmetric because it is written in superspace form. By construction, it is also invariant under the local isometries in G:

$$\begin{aligned}\delta\Phi^i &= \Lambda^{(a)}X^{i(a)}(\Phi^j) \\ \delta e^V &= -i\Lambda^{+(a)}T^{(a)}e^V + ie^V\Lambda^{(a)}T^{(a)}.\end{aligned} \tag{24.30}$$

Note that the explicit appearance of the Killing potentials in (24.29) implies that their global existence is necessary for gauging of the isometry group G.

To write this action in components, we add the kinetic term for the vector multiplet,

$$\mathscr{L} = \frac{1}{16kg^2}\int d^2\theta\, \mathrm{Tr}\, WW + \text{h.c.}, \tag{24.31}$$

and then pass to the WZ gauge. It is a straightforward exercise to eliminate the auxiliary fields and cast the remaining terms into geometrical form. We find

$$\begin{aligned}
\mathscr{L} = & -g_{ij^*}\mathscr{D}_m A^i \mathscr{D}^m A^{*j} - i\lambda^{(a)}\sigma^m \mathscr{D}_m \bar{\lambda}^{(a)} - \frac{1}{2} g^2 D^{(a)2} \\
& - i g_{ij^*}\chi^i \sigma^m \mathscr{D}_m \bar{\chi}^j - \frac{1}{4} F_{mn}{}^{(a)} F^{mn(a)} \\
& + g\sqrt{2} g_{ij^*}[X^{i(a)}\bar{\chi}^j \bar{\lambda}^{(a)} + X^{*j(a)}\chi^i \lambda^{(a)}] \\
& - \frac{1}{2} D_i D_j P \chi^i \chi^j - \frac{1}{2} D_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \\
& - g^{ij^*} D_i P D_{j^*} P^* \\
& + \frac{1}{4} R_{ij^*k\ell^*}\chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell,
\end{aligned} \tag{24.32}$$

where

$$\begin{aligned}
\mathscr{D}_m A^i &= \partial_m A^i - g v_m{}^{(a)} X^{i(a)} \\
\mathscr{D}_m \chi^i &= \partial_m \chi^i + \Gamma^i_{jk} \mathscr{D}_m A^j \chi^k - g v_m{}^{(a)} \frac{\partial X^{i(a)}}{\partial A^j}\chi^j \\
\mathscr{D}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} - g f^{abc} v_m{}^{(b)} \lambda^{(c)} \\
D_i P &= \frac{\partial P}{\partial A^i} \\
D_i D_j P &= \frac{\partial^2 P}{\partial A^i \partial A^j} - \Gamma^k_{ij} \frac{\partial P}{\partial A^k},
\end{aligned} \tag{24.33}$$

and we have rescaled $V \to 2gV$. The action (24.32) is invariant under the following gauge transformations:

$$\begin{aligned}
\delta A^i &= \varepsilon^{(a)} X^{i(a)} \\
\delta \chi^i &= \varepsilon^{(a)} \frac{\partial X^{i(a)}}{\partial A^j}\chi^j \\
\delta \lambda^{(a)} &= f^{abc}\varepsilon^{(b)}\lambda^{(c)} \\
\delta v_m{}^{(a)} &= g^{-1}\,\partial_m \varepsilon^{(a)} + f^{abc}\varepsilon^{(b)} v_m{}^{(c)}.
\end{aligned} \tag{24.34}$$

The covariant derivatives (24.33) are fully gauge covariant, as is evident from the transformations (24.34).

The Lagrangian (24.32) includes the following scalar potential:

$$\mathscr{V} = \frac{1}{2} g^2 D^{(a)2} + g^{ij^*} D_i P D_{j^*} P^* . \qquad \mathbf{(24.35)}$$

The first term is the sigma-model generalization of the "D-term" introduced in Chapter VII. Equation (24.35) implies that supersymmetry is spontaneously broken if either $\langle D^{(a)} \rangle \neq 0$ or $\langle D_i P \rangle \neq 0$, for some value of a or i.

For U(1) factors in the gauge group G, the relations (24.4) and (24.6) do not completely determine the Killing potentials. They leave the D's undetermined up to additive constants c,

$$D \to D + c. \qquad (24.36)$$

Therefore, by choosing the constants appropriately, it is always possible to arrange for supersymmetry to be spontaneously broken. This is the sigma-model version of the Fayet-Iliopoulos mechanism for supersymmetry breaking.

To illustrate the generality of the formalism developed above, we conclude this chapter with two examples. We first consider $\mathbb{C}^n$, and gauge the U(n) rotations about the origin. We take $K = a^{*i} a^i + d$, so $g_{ij^*} = \delta_{ij^*}$ and $R_{ij^*k\ell^*} = 0$. The Killing vectors $X^{i(a)}$ are simply $-iT^{(a)i}{}_j a^j$; the Killing potentials are $D^{(a)} = a^{*i} T^{(a)i}{}_j a^j$. Promoting a^i and a^{*j} to superfields Φ^i and Φ^{+j}, we find

$$\Gamma(\Phi^i, \Phi^{+j}, V^{(a)}) = \int d^2\theta \, d^2\bar{\theta} \, \Phi^+ [e^V - 1] \Phi. \qquad (24.37)$$

Using this result, it is obvious that (24.29) reduces to the usual superspace Lagrangian for a U(n) gauge theory.

For our second example, we consider $CP^1 = S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. This is a Kähler manifold as well as a homogeneous space. For simplicity, we use projective coordinates a and a^*. In these coordinates, we take $K = \log(1 + aa^*)$ and $P = 0$ (see Exercise 6 of Appendix D). We choose to gauge the entire isometry group $G = \mathrm{SU}(2)$, so the functions $D^{(a)}$ are as follows:

$$D^{(1)} = \frac{1}{2} \frac{a + a^*}{(1 + a^* a)}, \qquad D^{(2)} = -\frac{i}{2} \frac{a - a^*}{(1 + a^* a)}, \qquad D^{(3)} = -\frac{1}{2} \left(\frac{1 - a^* a}{1 + a^* a} \right). \qquad (24.38)$$

From here one can work out the Lagrangian, in superspace and in components. We shall work in components, starting with the Lagrangian (24.32). Because we have gauged the full SU(2), we can go to the "unitary gauge" where $a = a^* = 0$. This gauge exhibits the particle content of the theory:

$$\mathscr{L} = -\frac{1}{4} F_{mn}{}^{(a)} F^{mn(a)} - i\lambda^{(a)}\sigma^m \mathscr{D}_m \bar{\lambda}^{(a)} - i\chi\sigma^m\mathscr{D}_m\bar{\chi}$$
$$- \frac{1}{2} g^2 v_m^+ v^{-m} - \frac{1}{8} g^2 - ig(\chi\lambda_- - \bar{\chi}\bar{\lambda}_-) - \frac{1}{2}\chi\chi\bar{\chi}\bar{\chi}, \quad (24.39)$$

where

$$v_m^{\pm} = \frac{1}{2}\sqrt{2}(v_m{}^{(1)} \pm iv_m{}^{(2)})$$
$$\lambda_{\pm} = \frac{1}{2}\sqrt{2}(\lambda^{(1)} \pm i\lambda^{(2)})$$
$$\mathscr{D}_m\chi = \partial_m\chi - igv_m{}^{(3)}\chi. \quad (24.40)$$

The SU(2) symmetry implies that $D^{(a)2}$ is a constant. The constant is positive, so supersymmetry is spontaneously broken. The mass spectrum is as follows. The charged vector mesons $v_m^{\pm}$ are massive; they have eaten the scalars a and a^*. The massless vector meson $v_m{}^{(3)}$ is the gauge field corresponding to the unbroken U(1) symmetry. Its supersymmetry partner is the massless Goldstone spinor $\lambda^{(3)}$. The Majorana spinors χ and λ_- are massive; they have combined to form one massive Dirac spinor. Finally, λ_+ is both massless and charged. The CP^1 model has spontaneously broken supersymmetry, no leftover Higgs, and a massless Weyl spinor in a complex representation of the unbroken gauge group. This model is remarkable because the particle spins (as well as their masses) violate supersymmetry. No model with unbroken supersymmetry has the same spin spectrum. Nevertheless, the numbers of bosonic and fermionic degrees of freedom balance on mass shell.

References

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Equations

$$g_{ij^*}X^{*j(a)} = i\frac{\partial}{\partial a^i}D^{(a)}$$

$$g_{ij^*}X^{i(a)} = -i\frac{\partial}{\partial a^{*j}}D^{(a)}. \tag{24.4}$$

$$[X^{(a)},X^{(b)}] = -f^{abc}X^{(c)}$$
$$[X^{*(a)},X^{*(b)}] = -f^{abc}X^{*(c)}$$
$$[X^{(a)},X^{*(b)}] = 0. \tag{24.5}$$

$$\left[X^{i(a)}\frac{\partial}{\partial a^i} + X^{*i(a)}\frac{\partial}{\partial a^{*i}}\right]D^{(b)} = -f^{abc}D^{(c)}. \tag{24.6}$$

$$\delta K = \varepsilon^{(a)}F^{(a)} + \varepsilon^{*(a)}F^{*(a)} - i(\varepsilon^{(a)} - \varepsilon^{*(a)})D^{(a)}. \tag{24.8}$$

$$e^{\frac{1}{2}v'^{(a)}T^{(a)}} = u_0 e^{\frac{1}{2}v^{(a)}T^{(a)}}u_0^{-1}. \tag{24.13}$$

$$e^{v'^{(a)}T^{(a)}} = v_0 e^{v^{(a)}T^{(a)}}v_0. \tag{24.17}$$

$$\Gamma = \frac{e^{\frac{i}{2}v^{(a)}O^{(a)}} - 1}{\frac{i}{2}v^{(b)}O^{(b)}}v^{(c)}D^{(c)}$$
$$= \int_0^1 d\alpha\, e^{\frac{i}{2}\alpha v^{(a)}O^{(a)}}v^{(c)}D^{(c)}. \tag{24.26}$$

$$O^{(a)} = X^{(a)} - X^{*(a)}. \tag{24.27}$$

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^i,\Phi^{+j}) + \int_0^1 d\alpha \int d^2\theta\, d^2\bar{\theta}\, e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}}V^{(b)}D^{(b)}$$
$$+ \left[\int d^2\theta\, P(\Phi^i) + \text{h.c.}\right]. \tag{24.29}$$

$$\delta\Phi^i = \Lambda^{(a)}X^{i(a)}(\Phi^j)$$
$$\delta e^V = -i\Lambda^{+(a)}T^{(a)}e^V + ie^V\Lambda^{(a)}T^{(a)}. \tag{24.30}$$

$$\begin{aligned}
\mathscr{L} = & -g_{ij^*}\mathscr{D}_m A^i \mathscr{D}^m A^{*j} - i\lambda^{(a)}\sigma^m \mathscr{D}_m \bar{\lambda}^{(a)} - \frac{1}{2} g^2 D^{(a)2} \\
& - i g_{ij^*}\chi^i \sigma^m \mathscr{D}_m \bar{\chi}^j - \frac{1}{4} F_{mn}{}^{(a)} F^{mn(a)} \\
& + g\sqrt{2} g_{ij^*}[X^{i(a)}\bar{\chi}^j \bar{\lambda}^{(a)} + X^{*j(a)}\chi^i \lambda^{(a)}] \\
& - \frac{1}{2} D_i D_j P \chi^i \chi^j - \frac{1}{2} D_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \\
& - g^{ij^*} D_i P D_{j^*} P^* \\
& + \frac{1}{4} R_{ij^*k\ell^*}\chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell .
\end{aligned} \tag{24.32}$$

$$\begin{aligned}
\mathscr{D}_m A^i &= \partial_m A^i - g v_m{}^{(a)} X^{i(a)} \\
\mathscr{D}_m \chi^i &= \partial_m \chi^i + \Gamma^i_{jk}\mathscr{D}_m A^j \chi^k - g v_m{}^{(a)} \frac{\partial X^{i(a)}}{\partial A^j} \chi^j \\
\mathscr{D}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} - g f^{abc} v_m{}^{(b)} \lambda^{(c)} \\
D_i P &= \frac{\partial P}{\partial A^i} \\
D_i D_j P &= \frac{\partial^2 P}{\partial A^i \partial A^j} - \Gamma^k_{ij} \frac{\partial P}{\partial A^k} .
\end{aligned} \tag{24.33}$$

$$\begin{aligned}
\delta A^i &= \varepsilon^{(a)} X^{i(a)} \\
\delta \chi^i &= \varepsilon^{(a)} \frac{\partial X^{i(a)}}{\partial A^j} \chi^j \\
\delta \lambda^{(a)} &= f^{abc} \varepsilon^{(b)} \lambda^{(c)} \\
\delta v_m{}^{(a)} &= g^{-1} \partial_m \varepsilon^{(a)} + f^{abc} \varepsilon^{(b)} v_m{}^{(c)} .
\end{aligned} \tag{24.34}$$

$$\mathscr{V} = \frac{1}{2} g^2 D^{(a)2} + g^{ij^*} D_i P D_{j^*} P^* . \tag{24.35}$$

Exercises

(1) Prove that the Killing potentials can always be chosen to satisfy (24.6). This can be done by first differentiating the left-hand side with respect to a^i, and then using the relations introduced above to obtain

the a^i derivative of the right-hand side of (24.6). The proof can be completed by repeating the procedure, this time differentiating with respect to a^{*i}.

(2) Show that

$$X^{i(a)} \frac{\partial}{\partial a^i} D^{(b)} + X^{*i(b)} \frac{\partial}{\partial a^{*i}} D^{(a)} = 0.$$

(3) Verify that the differential operators,

$$X^{(a)} = -ia^j T^{(a)k}{}_j \frac{\partial}{\partial a^k}$$

$$X^{*(a)} = ia^{*j} T^{(a)j}{}_k \frac{\partial}{\partial a^{*k}},$$

are indeed Killing vectors, where the commutation relations of the $T^{(a)}{}_j{}^k$ are given in (7.14). Show that their Lie brackets close into (24.5).

(4) Let $\mathscr{M}$ be the complex plane. In this exercise we will gauge translations in the y-direction on $\mathscr{M}$. (Note that one could have chosen to gauge translations in the x-direction, but because of (24.6), one cannot gauge both simultaneously.) As above, take $K = a^*a + d$, so $g_{aa^*} = 1$ and $R_{aa^*aa^*} = 0$. For D take the Killing potential $D = m(a + a^*)$. Find the Lagrangian and the mass spectrum in unitary gauge.

(5) Show that (24.29) reduces to (24.32) in the WZ gauge.

XXV. GAUGE INVARIANT SUPERGRAVITY MODELS

Having discussed the geometrical interpretation of supersymmetric theories, we are now ready to write down the general coupling of matter fields to supergravity. The Lagrangian we derive is the starting point for the phenomenological study of supergravity theories. We present the Lagrangian in superspace (25.1), in two-component spinor notation (25.12), and as a service to the reader, in a more conventional form with four-component spinors (25.24). Readers interested only in the results should feel free to skip to the relevant part of the chapter.

The supergravity extension of the gauge invariant superspace Lagrangian is easy to find using the material from the previous chapters. As in (24.17), one first adds the counterterm Γ to the Kähler potential K. Then, as in (23.1), one exponentiates the result to find

$$\begin{aligned}\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\Bigg[&\frac{3}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{1}{3}\left[K(\Phi,\Phi^+) + \Gamma(\Phi,\Phi^+,V)\right]\right\} \\ &+ \frac{1}{16g^2}H_{(ab)}(\Phi)W^{(a)}W^{(b)} + P(\Phi)\Bigg] + \text{h.c.},\end{aligned} \qquad \textbf{(25.1)}$$

where $\kappa^2 = 1$, and

$$W_\alpha \equiv W_\alpha^{(a)}T^{(a)} = -\frac{1}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)e^{-V}\mathscr{D}_\alpha e^V \qquad \textbf{(25.2)}$$

is the curved-space generalization of the supersymmetric Yang-Mills field strength. In this expression, K is an arbitrary hermitian function of the superfields Φ^i and Φ^{+j}, P is the superpotential, and Γ is the counterterm (24.22), which is necessary for gauge invariance, as we will see below. The analytic function $H_{(ab)}$ is included for generality. Under a gauge transformation, it must transform as required to render (25.1) invariant. In what follows, we shall take $H_{(ab)} = \delta_{ab}$; the Lagrangian with nontrivial $H_{(ab)}$ is presented in Appendix G.

The supergravity invariance of (25.1) is manifest because of the superspace formalism. The gauge invariance, however, is a little more subtle.

To check it, let us recall the gauge transformations for K, Γ, and P,

$$\begin{aligned}\delta K &= \Lambda^{(a)}F^{(a)} + \Lambda^{+(a)}F^{+(a)} - i[\Lambda^{(a)} - \Lambda^{+(a)}]D^{(a)}\\ \delta\Gamma &= i[\Lambda^{(a)} - \Lambda^{+(a)}]D^{(a)}\\ \delta P &= \Lambda^{(a)}X^{(a)}P,\end{aligned} \tag{25.3}$$

as given in Chapter XXIV. Here

$$F^{(a)} = X^{(a)}K + iD^{(a)} \tag{25.4}$$

is an analytic function of the Φ^i, and $\Lambda^{(a)}$ is the superfield gauge parameter. When applied to the Lagrangian (25.1), the transformations (25.3) induce a variation of the following form:

$$\begin{aligned}\delta\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\bigg[&-\frac{1}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)[\Lambda^{(a)}F^{(a)} + \Lambda^{+(a)}F^{+(a)}]e^{-(K+\Gamma)/3}\\ &+ \Lambda^{(a)}X^{(a)}P\bigg] + \text{h.c.}\end{aligned} \tag{25.5}$$

In Chapter XXIII, such a variation is canceled by a super-Weyl transformation, where the Weyl weight of Φ^i is taken to be zero. Setting the weight of $V^{(a)}$ to be zero as well, we find

$$\begin{aligned}\delta\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\bigg[&\frac{3}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)[\Sigma + \bar{\Sigma}]e^{-(K+\Gamma)/3}\\ &+ 6\Sigma P\bigg] + \text{h.c.}\end{aligned} \tag{25.6}$$

under a super-Weyl transformation with superfield parameter Σ. Comparing (25.5) to (25.6), we see that the variation is canceled if

$$\begin{aligned}\Sigma &= \frac{1}{6}\Lambda^{(a)}F^{(a)}\\ \delta P &= \Lambda^{(a)}X^{(a)}P = -\Lambda^{(a)}F^{(a)}P.\end{aligned} \tag{25.7}$$

The condition on δP is a nontrivial condition on the superpotential that is necessary for the gauge invariance of the theory.

The superspace Lagrangian presented above can be expressed in components using the techniques developed in the previous chapters. One first passes to the WZ gauge, where

$$\Gamma = V^{(a)}D^{(a)} + \frac{1}{2}g_{ij^*}X^{i(a)}X^{*j(b)}V^{(a)}V^{(b)} \tag{25.8}$$

and

$$W_\alpha = -\frac{1}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\left\{\mathscr{D}_\alpha V - \frac{1}{2}[V,\mathscr{D}_\alpha V]\right\}. \tag{25.9}$$

One then works out the Θ expansions for

$$(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{1}{3}[K + \Gamma]\right\} \tag{25.10}$$

and

$$W^{(a)}W^{(a)}. \tag{25.11}$$

After eliminating the auxiliary fields, and rescaling and redefining the other fields as in Chapters XXI and XXIII, one finds the component Lagrangian in terms of the physical fields. This Lagrangian is the starting point for phenomenological studies of supergravity theories:

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{2}e\mathscr{R} - eg_{ij^*}\tilde{\mathscr{D}}_m A^i\tilde{\mathscr{D}}^m A^{*j} - \frac{1}{2}eg^2D^{(a)2} \\
& - \frac{1}{4}eF_{mn}{}^{(a)}F^{mn(a)} - ie\bar{\lambda}^{(a)}\bar{\sigma}^m\tilde{\mathscr{D}}_m\lambda^{(a)} \\
& - ieg_{ij^*}\bar{\chi}^j\bar{\sigma}^m\tilde{\mathscr{D}}_m\chi^i + e\varepsilon^{k\ell mn}\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathscr{D}}_m\psi_n \\
& + \sqrt{2}egg_{ij^*}X^{*j(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^{i(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{1}{2}egD^{(a)}\psi_m\sigma^m\bar{\lambda}^{(a)} + \frac{1}{2}egD^{(a)}\bar{\psi}_m\bar{\sigma}^m\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\tilde{\mathscr{D}}_n A^{*j}\chi^i\sigma^m\bar{\sigma}^n\psi_m - \frac{1}{2}\sqrt{2}eg_{ij^*}\tilde{\mathscr{D}}_n A^i\bar{\chi}^j\bar{\sigma}^m\sigma^n\bar{\psi}_m \\
& + \frac{i}{4}e[\psi_m\sigma^{ab}\sigma^m\bar{\lambda}^{(a)} + \bar{\psi}_m\bar{\sigma}^{ab}\bar{\sigma}^m\lambda^{(a)}][F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)}] \\
& + \frac{1}{4}eg_{ij^*}[i\varepsilon^{k\ell mn}\psi_k\sigma_\ell\bar{\psi}_m + \psi_m\sigma^n\bar{\psi}^m]\chi^i\sigma_n\bar{\chi}^j
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8} e[g_{ij^*}g_{k\ell^*} - 2R_{ij^*k\ell^*}]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^\ell \\
&+ \frac{1}{8} eg_{ij^*}\bar{\chi}^j\bar{\sigma}^m\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_m\lambda^{(a)} - \frac{3}{16} e\lambda^{(a)}\sigma^m\bar{\lambda}^{(a)}\lambda^{(b)}\sigma_m\bar{\lambda}^{(b)} \\
&- e\exp(K/2)\Big\{P^*\psi_a\sigma^{ab}\psi_b + P\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b \\
&+ \frac{i}{2}\sqrt{2}D_iP\chi^i\sigma^a\bar{\psi}_a + \frac{i}{2}\sqrt{2}D_{i^*}P^*\bar{\chi}^i\bar{\sigma}^a\psi_a \\
&+ \frac{1}{2}\mathscr{D}_iD_jP\chi^i\chi^j + \frac{1}{2}\mathscr{D}_{i^*}D_{j^*}P^*\bar{\chi}^i\bar{\chi}^j\Big\} \\
&- e\exp(K)[g^{ij^*}(D_iP)(D_jP)^* - 3P^*P]. \qquad \textbf{(25.12)}
\end{aligned}
$$

In this expression, the scalars A^i and the spinors χ^i and $\lambda^{(a)}$ are matter fields, while the vectors $v_m{}^{(a)}$ are the gauge fields for the gauge group G. The field ψ_m is the gravitino, and $e_m{}^a$ is the graviton. In (25.12), K, P, and D are functions of the scalar fields. As before, the metric g_{ij^*} is Kähler.

The Lagrangian (25.12) contains derivatives covariant with respect to gauge transformations, as well as spacetime and Kähler coordinate transformations:

$$
\begin{aligned}
\tilde{\mathscr{D}}_mA^i &= \partial_mA^i - gv_m{}^{(a)}X^{i(a)} \\
\tilde{\mathscr{D}}_m\chi^i &= \partial_m\chi^i + \chi^i\omega_m + \Gamma^i_{jk}\tilde{\mathscr{D}}_mA^j\chi^k - gv_m{}^{(a)}\frac{\partial X^{i(a)}}{A^j}\chi^j \\
&\quad - \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\chi^i - \frac{i}{2}gv_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\chi^i \\
\tilde{\mathscr{D}}_m\lambda^{(a)} &= \partial_m\lambda^{(a)} + \lambda^{(a)}\omega_m - gf^{abc}v_m{}^{(b)}\lambda^{(c)} \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\lambda^{(a)} + \frac{i}{2}gv_m{}^{(b)}\,\mathrm{Im}\,F^{(b)}\lambda^{(a)} \\
\tilde{\mathscr{D}}_m\psi_n &= \partial_m\psi_n + \psi_n\omega_m \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\psi_n + \frac{i}{2}gv_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\psi_n \\
D_iP &= P_i + K_iP \\
\mathscr{D}_iD_jP &= P_{ij} + K_{ij}P + K_iD_jP + K_jD_iP - K_iK_jP - \Gamma^k_{ij}D_kP. \qquad \textbf{(25.13)}
\end{aligned}
$$

The covariant derivatives contain the Christoffel symbols for the Kähler geometry and the spin connection (17.12) for spacetime. They also contain the vector potential $v_m{}^{(a)}$. Note that the covariant derivatives contain a coupling between Im $F^{(a)}$ and the vector potential. This is a reflection of the fact that gauge transformations are accompanied by super-Weyl rotations of the component fields.

The above Lagrangian is invariant under the gauge group G. The gauge transformations of the component fields are given by

$$\begin{aligned}
\delta A^i &= \varepsilon^{(a)} X^{i(a)} \\
\delta \chi^i &= \varepsilon^{(a)} \frac{\partial X^{i(a)}}{A^j} \chi^j + \frac{i}{2} \varepsilon^{(a)} \operatorname{Im} F^{(a)} \chi^i \\
\delta \lambda^{(a)} &= f^{abc} \varepsilon^{(b)} \lambda^{(c)} - \frac{i}{2} \varepsilon^{(b)} \operatorname{Im} F^{(b)} \lambda^{(a)} \\
\delta v_m{}^{(a)} &= g^{-1} \partial_m \varepsilon^{(a)} + f^{abc} \varepsilon^{(b)} v_m{}^{(c)} \\
\delta \psi_n &= -\frac{i}{2} \varepsilon^{(a)} \operatorname{Im} F^{(a)} \psi_n .
\end{aligned} \tag{25.14}$$

It is automatically invariant under supergravity transformations because it was derived from a superspace formalism. It is instructive, however, to verify the invariance directly, using the following transformations laws,

$$\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta \sigma^a \bar{\psi}_m + \bar{\zeta} \bar{\sigma}^a \psi_m) \\
\delta_\zeta A^i &= \sqrt{2} \zeta \chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2} \sigma^m \bar{\zeta} \hat{\tilde{D}}_m A^i - \Gamma^i_{jk} \delta_\zeta A^j \chi^k \\
&\quad + \frac{1}{4} (K_j \delta_\zeta A^j - K_{j^*} \delta_\zeta A^{*j}) \chi^i - \sqrt{2} e^{K/2} g^{ij^*} D_{j^*} P^* \zeta \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta \sigma_m \bar{\lambda}^{(a)} + \bar{\zeta} \bar{\sigma}_m \lambda^{(a)}) \\
\delta_\zeta \lambda^{(a)} &= \hat{F}_{ab}{}^{(a)} \sigma^{ab} \zeta - \frac{1}{4} (K_j \delta_\zeta A^j - K_{j^*} \delta_\zeta A^{*j}) \lambda^{(a)} - i g D^{(a)} \zeta \\
\delta_\zeta \psi_m &= 2 \tilde{\mathscr{D}}_m \zeta - \frac{i}{2} \sigma_{mn} \zeta\, g_{ij^*} \chi^i \sigma^n \bar{\chi}^j + \frac{i}{2} (g_{mn} + \sigma_{mn}) \zeta\, \lambda^{(a)} \sigma^n \bar{\lambda}^{(a)} \\
&\quad - \frac{1}{4} (K_j \delta_\zeta A^j - K_{j^*} \delta_\zeta A^{*j}) \psi_m + i e^{K/2} P \sigma_m \bar{\zeta} .
\end{aligned} \tag{25.15}$$

Here $\tilde{\mathscr{D}}_m \zeta$ is defined to be

$$\tilde{\mathscr{D}}_m \zeta = \partial_m \zeta + \zeta \omega_m + \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j^*} \tilde{\mathscr{D}}_m A^{*j})\zeta, \qquad (25.16)$$

while the supercovariant expressions $\hat{\tilde{D}}_m A^i$ and $\hat{F}_{mn}$ are given by

$$\begin{aligned} \hat{\tilde{D}}_m A^i &= \hat{D}_m A^i - g v_m{}^{(a)} X^{i(a)} \\ &= \tilde{\mathscr{D}}_m A^i - \frac{1}{2}\sqrt{2}\, \psi_m \chi^i \\ \hat{F}_{mn}{}^{(a)} &= \hat{D}_m v_n{}^{(a)} - \hat{D}_n v_m{}^{(a)} \\ &= F_{mn}{}^{(a)} - \frac{i}{2}[\psi_m \sigma_n \bar{\lambda}^{(a)} + \bar{\psi}_m \bar{\sigma}_n \lambda^{(a)} \\ &\quad - \psi_n \sigma_m \bar{\lambda}^{(a)} - \bar{\psi}_n \bar{\sigma}_m \lambda^{(a)}]. \end{aligned} \qquad (25.17)$$

The action (25.12) differs from that of Chapter XXIII by the addition of the gauge supermultiplets. The additional fields change the form of the scalar potential from (23.10) to

$$\mathscr{V} = \frac{1}{2} g^2 D^{(a)2} + e^K [g^{ij^*}(D_i P)(D_j P)^* - 3P^* P]. \qquad \mathbf{(25.18)}$$

They also change the trace formula from (23.34) to

$$\begin{aligned} \operatorname{Str} M^2 &= \sum_{\text{spins } J} (-1)^{2J}(2J+1) \operatorname{Tr} M^2 \\ &= (n-1)[2m_\psi^2 - g^2 \langle D^{(a)2} \rangle] \\ &\quad + 2g^2 \langle g^{ij^*} D^{(a)}_{ij^*} D^{(a)} \rangle - 2\langle R_{ij^*} G^i G^{j^*} \rangle m_\psi^2. \end{aligned} \qquad \mathbf{(25.19)}$$

From the form of the transformation laws, we see the condition for spontaneous supersymmetry breaking is either

$$\langle D_i P \rangle \neq 0 \qquad (25.20)$$

or

$$\langle D^{(a)} \rangle \neq 0, \qquad (25.21)$$

for some value of i or (a). Depending on the relative magnitudes of (25.20) and (25.21), an appropriate linear combination of χ^i and $\lambda^{(a)}$ plays the role of the Goldstone fermion.

Note that the Lagrangian (25.12) explicitly contains the Killing potentials $D^{(a)}$. Their existence is both necessary and sufficient to gauge the group G. If the group G contains a U(1) factor, we know from previous arguments that the $D^{(a)}$ are not uniquely defined. There is an arbitrary integration constant associated with each U(1) factor,

$$D \to D + \xi. \tag{25.22}$$

In the globally supersymmetric case, shifts of these constants give rise to the Fayet-Iliopoulos D-term $\mathscr{L}_{\mathrm{FI}}$. The same is true in supergravity. By shifting the functions D, we find the gauge invariant supergravity version of $\mathscr{L}_{\mathrm{FI}}$:

$$\mathscr{L}_{\mathrm{FI}} = -\frac{1}{2} eg^2\xi^2 - eg^2\xi D - \frac{1}{2} eg\xi(\psi_m\sigma^m\bar{\lambda} - \bar{\psi}_m\bar{\sigma}^m\lambda). \tag{25.23}$$

Note that the shift (25.22) changes the spinor covariant derivatives as well as the transformation laws (25.15). New terms proportional to ξ are induced in all expressions involving the Killing potentials D.

In the rest of this chapter, we will present the Lagrangian (25.12) in four-component notation, following the conventions described in Appendix A. Care should be exercised in comparing this formula to those in the references; conventions vary throughout the literature. With this said, we write the Lagrangian as follows:

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2} e\mathscr{R} - eg_{ij^*}\tilde{\mathscr{D}}_m A^i\tilde{\mathscr{D}}^m A^{*j} - \frac{1}{2} eg^2 D^{(a)2} \\
&- \frac{1}{4} eF_{mn}{}^{(a)}F^{mn(a)} - ie\bar{\lambda}_L^{(a)}\gamma^m\tilde{\mathscr{D}}_m\lambda_L^{(a)} \\
&- ieg_{ij^*}\bar{\chi}_L^j\gamma^m\tilde{\mathscr{D}}_m\chi_L^i + e\varepsilon^{k\ell mn}\bar{\psi}_{Lk}\gamma_\ell\tilde{\mathscr{D}}_m\psi_{Ln} \\
&+ eg\sqrt{2}g_{ij^*}X^{*j(a)}\bar{\chi}_R^i\lambda_L^{(a)} + eg\sqrt{2}g_{ij^*}X^{i(a)}\bar{\chi}_L^j\lambda_R^{(a)} \\
&+ \frac{1}{2} egD^{(a)}\bar{\psi}_{Lm}\gamma^m\lambda_L^{(a)} - \frac{1}{2} egD^{(a)}\bar{\psi}_{Rm}\gamma^m\lambda_R^{(a)} \\
&- \frac{1}{2}\sqrt{2}eg_{ij^*}\tilde{\mathscr{D}}_n A^{*j}\bar{\chi}_R^i\gamma^m\gamma^n\psi_{Lm} - \frac{1}{2}\sqrt{2}eg_{ij^*}\tilde{\mathscr{D}}_n A^i\bar{\chi}_L^j\gamma^m\gamma^n\psi_{Rm}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{i}{4} e[\bar{\psi}_{Lm}\sigma^{ab}\gamma^m\lambda_L^{(a)} + \bar{\psi}_{Rm}\sigma^{ab}\gamma^m\lambda_R^{(a)}][F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)}] \\
&+ \frac{1}{4} e g_{ij^*}[i\varepsilon^{k\ell mn}\bar{\psi}_{Lk}\gamma_\ell\psi_{Lm} - \bar{\psi}_{Lm}\gamma^n\psi_L^m]\bar{\chi}^i_R\gamma_n\chi^j_R \\
&- \frac{1}{8} e[g_{ij^*}g_{k\ell^*} - 2R_{ij^*k\ell^*}]\bar{\chi}^i_R\chi^k_L\bar{\chi}^j_L\chi^\ell_R \\
&- \frac{1}{8} e g_{ij^*}\bar{\chi}^j_L\gamma^m\chi^i_L\bar{\lambda}^{(a)}_R\gamma_m\lambda^{(a)}_R + \frac{3}{16} e\bar{\lambda}^{(a)}_L\gamma^m\lambda^{(a)}_L\bar{\lambda}^{(b)}_R\gamma_m\lambda^{(b)}_R \\
&- e\exp(K/2)\Big\{P^*\psi_{Ra}\sigma^{ab}\psi_{Lb} + P\bar{\psi}_{La}\sigma^{ab}\psi_{Rb} \\
&+ \frac{i}{2}\sqrt{2}D_iP\bar{\chi}^i_R\gamma^a\psi_{Ra} + \frac{i}{2}\sqrt{2}D_{i^*}P^*\bar{\chi}^i_L\gamma^a\psi_{La} \\
&+ \frac{1}{2}\mathscr{D}_iD_jP\bar{\chi}^i_R\chi^j_L + \frac{1}{2}\mathscr{D}_{i^*}D_{j^*}P^*\bar{\chi}^i_L\chi^j_R\Big\} \\
&- e\exp(K)[g^{ij^*}(D_iP)(D_jP)^* - 3P^*P],
\end{aligned} \tag{25.24}
$$

where $\chi^i_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\chi^i$, and similarly for $\lambda^{(a)}$ and ψ_m. The covariant derivatives are defined as follows:

$$
\begin{aligned}
\tilde{\mathscr{D}}_mA^i &= \partial_mA^i - gv_m{}^{(a)}X^{i(a)} \\
\tilde{\mathscr{D}}_m\chi^i_L &= \partial_m\chi^i_L - \omega_m\chi^i_L + \Gamma^i_{jk}\tilde{\mathscr{D}}_mA^j\chi^k_L - gv_m{}^{(a)}\frac{\partial X^{i(a)}}{A^j}\chi^j_L \\
&\quad - \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\chi^i_L - \frac{i}{2}gv_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\chi^i_L \\
\tilde{\mathscr{D}}_m\lambda^{(a)}_L &= \partial_m\lambda^{(a)}_L - \omega_m\lambda^{(a)}_L - gf^{abc}v_m{}^{(b)}\lambda^{(c)}_L \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\lambda^{(a)}_L + \frac{i}{2}gv_m{}^{(b)}\,\mathrm{Im}\,F^{(b)}\lambda^{(a)}_L \\
\tilde{\mathscr{D}}_m\psi_{Ln} &= \partial_m\psi_{Ln} - \omega_m\psi_{Ln} \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\psi_{Ln} + \frac{i}{2}gv_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\psi_{Ln} \\
D_iP &= P_i + K_iP \\
\mathscr{D}_iD_jP &= P_{ij} + K_{ij}P + K_iD_jP + K_jD_iP - K_iK_jP - \Gamma^k_{ij}D_kP.
\end{aligned} \tag{25.25}
$$

The Lagrangian (25.24) is invariant under the supergravity transformations,

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\bar{\zeta}_L \gamma^a \psi_{Lm} + \bar{\zeta}_R \gamma^a \psi_{Rm}) \\
\delta_\zeta A^i &= \sqrt{2}\bar{\zeta}_R \chi_L^i \\
\delta_\zeta \chi_L^i &= i\sqrt{2}\gamma^m \zeta_R \hat{\tilde{D}}_m A^i - \Gamma^i_{jk}\,\delta_\zeta A^j \chi_L^k \\
&\quad + \frac{1}{4}(K_j \delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\chi_L^i - \sqrt{2}e^{K/2} g^{ij*} D_{j*} P^* \zeta_L \\
\delta_\zeta v_m{}^{(a)} &= i(\bar{\zeta}_L \gamma_m \lambda_L^{(a)} + \bar{\zeta}_R \gamma_m \lambda_R^{(a)}) \\
\delta_\zeta \lambda_L^{(a)} &= \hat{F}_{ab}{}^{(a)} \sigma^{ab} \zeta_L - \frac{1}{4}(K_j \delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\lambda_L^{(a)} - igD^{(a)}\zeta_L \\
\delta_\zeta \psi_{Lm} &= 2\tilde{\mathscr{D}}_m \zeta_L - \frac{i}{2}\sigma_{mn}\zeta_L\, g_{ij*}\bar{\chi}^i_R \gamma^n \chi^j_R + \frac{i}{2}(g_{mn} + \sigma_{mn})\zeta_L\, \bar{\lambda}_R^{(a)}\gamma^n \lambda_R^{(a)} \\
&\quad - \frac{1}{4}(K_j \delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\psi_{Lm} + ie^{K/2} P \gamma_m \zeta_R,
\end{aligned}
\tag{25.26}
$$

where $\tilde{\mathscr{D}}_m \zeta_L$ is given by

$$
\tilde{\mathscr{D}}_m \zeta_L = \partial_m \zeta_L - \omega_m \zeta_L + \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j*}\tilde{\mathscr{D}}_m A^{*j})\zeta_L. \tag{25.27}
$$

References

J. A. Bagger, *Nucl. Phys. B211*, 302 (1983).

E. Cremmer, S. Ferrara, L. Girardello, and A. van Proeyen, *Nucl. Phys. B212*, 413 (1983).

Equations

$$
\begin{aligned}
\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\Bigg[&\frac{3}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{1}{3}[K(\Phi,\Phi^+) + \Gamma(\Phi,\Phi^+,V)]\right\} \\
&+ \frac{1}{16g^2} H_{(ab)}(\Phi) W^{(a)} W^{(b)} + P(\Phi)\Bigg] + \text{h.c.}
\end{aligned}
\tag{25.1}
$$

$$
W_\alpha \equiv W_\alpha^{(a)} T^{(a)} = -\frac{1}{4}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)e^{-V}\mathscr{D}_\alpha e^V. \tag{25.2}
$$

$$
\begin{aligned}
\mathscr{L} = &-\frac{1}{2} e\mathscr{R} - eg_{ij^*}\tilde{\mathscr{D}}_m A^i \tilde{\mathscr{D}}^m A^{*j} - \frac{1}{2} eg^2 D^{(a)2} \\
&- \frac{1}{4} eF_{mn}{}^{(a)} F^{mn(a)} - ie\bar{\lambda}^{(a)}\bar{\sigma}^m \tilde{\mathscr{D}}_m \lambda^{(a)} \\
&- ieg_{ij^*}\bar{\chi}^j \bar{\sigma}^m \tilde{\mathscr{D}}_m \chi^i + e\varepsilon^{k\ell mn}\bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathscr{D}}_m \psi_n \\
&+ \sqrt{2} egg_{ij^*} X^{*j(a)} \chi^i \lambda^{(a)} + \sqrt{2} egg_{ij^*} \chi^{i(a)} \bar{\chi}^j \bar{\lambda}^{(a)} \\
&- \frac{1}{2} egD^{(a)} \psi_m \sigma^m \bar{\lambda}^{(a)} + \frac{1}{2} egD^{(a)} \bar{\psi}_m \bar{\sigma}^m \lambda^{(a)} \\
&- \frac{1}{2}\sqrt{2} eg_{ij^*}\tilde{\mathscr{D}}_n A^{*j} \chi^i \sigma^m \bar{\sigma}^n \psi_m - \frac{1}{2}\sqrt{2} eg_{ij^*}\tilde{\mathscr{D}}_n A^i \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
&+ \frac{i}{4} e[\psi_m \sigma^{ab} \sigma^m \bar{\lambda}^{(a)} + \bar{\psi}_m \bar{\sigma}^{ab} \bar{\sigma}^m \lambda^{(a)}][F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)}] \\
&+ \frac{1}{4} eg_{ij^*}[i\varepsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m] \chi^i \sigma_n \bar{\chi}^j \\
&- \frac{1}{8} e[g_{ij^*} g_{k\ell^*} - 2R_{ij^*k\ell^*}] \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\
&+ \frac{1}{8} eg_{ij^*} \bar{\chi}^j \bar{\sigma}^m \chi^i \bar{\lambda}^{(a)} \bar{\sigma}_m \lambda^{(a)} - \frac{3}{16} e\lambda^{(a)} \sigma^m \bar{\lambda}^{(a)} \lambda^{(b)} \sigma_m \bar{\lambda}^{(b)} \\
&- e \exp(K/2) \Big\{ P^* \psi_a \sigma^{ab} \psi_b + P \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
&+ \frac{i}{2} \sqrt{2} D_i P \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} D_{i^*} P^* \bar{\chi}^i \bar{\sigma}^a \psi_a \\
&+ \frac{1}{2} \mathscr{D}_i D_j P \chi^i \chi^j + \frac{1}{2} \mathscr{D}_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \Big\} \\
&- e \exp(K)[g^{ij^*}(D_i P)(D_j P)^* - 3P^* P]. \qquad \textbf{(25.12)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathscr{D}}_m A^i &= \partial_m A^i - gv_m{}^{(a)} X^{i(a)} \\
\tilde{\mathscr{D}}_m \chi^i &= \partial_m \chi^i + \chi^i \omega_m + \Gamma^i_{jk} \tilde{\mathscr{D}}_m A^j \chi^k - gv_m{}^{(a)} \frac{\partial X^{i(a)}}{A^j} \chi^j \\
&\quad - \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j^*} \tilde{\mathscr{D}}_m A^{*j}) \chi^i - \frac{i}{2} gv_m{}^{(a)} \operatorname{Im} F^{(a)} \chi^i
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathscr{D}}_m\lambda^{(a)} &= \partial_m\lambda^{(a)} + \lambda^{(a)}\omega_m - gf^{abc}v_m{}^{(b)}\lambda^{(c)} \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\lambda^{(a)} + \frac{i}{2}gv_m{}^{(b)}\,\mathrm{Im}\,F^{(b)}\lambda^{(a)} \\
\tilde{\mathscr{D}}_m\psi_n &= \partial_m\psi_n + \psi_n\omega_m \\
&\quad + \frac{1}{4}(K_j\tilde{\mathscr{D}}_mA^j - K_{j^*}\tilde{\mathscr{D}}_mA^{*j})\psi_n + \frac{i}{2}gv_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\psi_n \\
D_iP &= P_i + K_iP \\
\mathscr{D}_iD_jP &= P_{ij} + K_{ij}P + K_iD_jP + K_jD_iP - K_iK_jP - \Gamma^k_{ij}D_kP.
\end{aligned}
\tag{25.13}
$$

$$
\begin{aligned}
\delta A^i &= \varepsilon^{(a)}X^{i(a)} \\
\delta\chi^i &= \varepsilon^{(a)}\frac{\partial X^{i(a)}}{A^j}\chi^j + \frac{i}{2}\varepsilon^{(a)}\,\mathrm{Im}\,F^{(a)}\chi^i \\
\delta\lambda^{(a)} &= f^{abc}\varepsilon^{(b)}\lambda^{(c)} - \frac{i}{2}\varepsilon^{(b)}\,\mathrm{Im}\,F^{(b)}\lambda^{(a)} \\
\delta v_m{}^{(a)} &= g^{-1}\,\partial_m\varepsilon^{(a)} + f^{abc}\varepsilon^{(b)}v_m{}^{(c)} \\
\delta\psi_n &= -\frac{i}{2}\varepsilon^{(a)}\,\mathrm{Im}\,F^{(a)}\psi_n.
\end{aligned}
\tag{25.14}
$$

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta\sigma^a\bar{\psi}_m + \bar{\zeta}\bar{\sigma}^a\psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\zeta\chi^i \\
\delta_\zeta\chi^i &= i\sqrt{2}\sigma^m\bar{\zeta}\hat{\tilde{D}}_mA^i - \Gamma^i_{jk}\,\delta_\zeta A^j\chi^k \\
&\quad + \frac{1}{4}(K_j\,\delta_\zeta A^j - K_{j^*}\,\delta_\zeta A^{*j})\chi^i - \sqrt{2}e^{K/2}g^{ij^*}D_{j^*}P^*\zeta \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta\sigma_m\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_m\lambda^{(a)}) \\
\delta_\zeta\lambda^{(a)} &= \hat{F}_{ab}{}^{(a)}\sigma^{ab}\zeta - \frac{1}{4}(K_j\,\delta_\zeta A^j - K_{j^*}\,\delta_\zeta A^{*j})\lambda^{(a)} - igD^{(a)}\zeta \\
\delta_\zeta\psi_m &= 2\tilde{\mathscr{D}}_m\zeta - \frac{1}{2}\sigma_{mn}\zeta\,g_{ij^*}\chi^i\sigma^n\bar{\chi}^j + \frac{i}{2}(g_{mn} + \sigma_{mn})\zeta\,\lambda^{(a)}\sigma^n\bar{\lambda}^{(a)} \\
&\quad - \frac{1}{4}(K_j\,\delta_\zeta A^j - K_{j^*}\,\delta_\zeta A^{*j})\psi_m + ie^{K/2}P\sigma_m\bar{\zeta}.
\end{aligned}
\tag{25.15}
$$

$$\mathscr{V} = \frac{1}{2} g^2 D^{(a)2} + e^K [g^{ij^*}(D_i P)(D_j P)^* - 3P^* P]. \tag{25.18}$$

$$\begin{aligned} \mathrm{Str}\, M^2 &= \sum_{\text{spins } J} (-1)^{2J}(2J + 1)\mathrm{Tr}\, M^2 \\ &= (n - 1)[2m_\psi^2 - g^2\langle D^{(a)2}\rangle] \\ &\quad + 2g^2\langle g^{ij^*} D^{(a)}_{ij^*} D^{(a)}\rangle - 2\langle R_{ij^*} G^i G^{j^*}\rangle m_\psi^2. \end{aligned} \tag{25.19}$$

Exercises

(1) Show that

$$W_\alpha = -\frac{1}{4}(\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)(e^{-V}\mathscr{D}_\alpha e^V)$$

is gauge covariant under the following non-Abelian gauge transformation:

$$e^{V'} = e^{-i\Lambda^+} e^V e^{i\Lambda},$$

where

$$\bar{\mathscr{D}}_{\dot{\alpha}}\Lambda = \mathscr{D}_\alpha \Lambda^+ = 0.$$

(2) Verify that

$$e^{-V}\mathscr{D}_\alpha e^V = \mathscr{D}_\alpha V - \frac{1}{2}[V, \mathscr{D}_\alpha V]$$

in the WZ gauge.

(3) For an Abelian group, the components of W were given in (19.28). Use the results of Exercise 7 in Chapter XIX to show

$$\begin{aligned} (\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)[V,\mathscr{D}_\alpha V]| &= 0 \\ \mathscr{D}_\beta(\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)[V,\mathscr{D}_\alpha V]| &= 8(\sigma^{ab}\varepsilon)_{\beta\alpha} v_a v_b \\ \mathscr{D}^\beta\mathscr{D}_\beta(\bar{\mathscr{D}}_{\dot{\alpha}}\bar{\mathscr{D}}^{\dot{\alpha}} - 8R)[V,\mathscr{D}_\alpha V]| &= -16i[v_{\alpha\dot{\alpha}},\bar{\lambda}^{\dot{\alpha}}] - 8i[v^c, v_{\alpha\dot{\alpha}}\bar{\psi}_c{}^{\dot{\alpha}}]. \end{aligned}$$

Then find all the components of W for a non-Abelian group G.

(4) Compute

$$
\begin{aligned}
(\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha} - 8R)\Phi^+ V = {} & 2\Theta\{iA^*(2\lambda + \sigma^c\bar\sigma^a\psi_c v_a) - \sqrt{2}\sigma^a\bar\chi v_a\} \\
& + \Theta\Theta\{i2\sqrt{2}\bar\chi\bar\lambda + 4i\hat{D}_c A^* v^c \\
& + A^*[-2D + 2ie_c{}^m\mathscr{D}_m v^c - \frac{4}{3}v_a b^a \\
& + \bar\psi_c\bar\sigma^c\lambda - \bar\lambda\bar\sigma^c\psi_c - \psi^d\sigma^c\bar\psi_d v_c]\}
\end{aligned}
$$

and

$$(\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha} - 8R)\Phi^+ V^2 = 2\Theta\Theta A^* v_a v^a.$$

(5) Check that

$$(\bar{\mathscr{D}}_{\dot\alpha}\bar{\mathscr{D}}^{\dot\alpha} - 8R)\Phi^+ V^3 = 0$$

in the WZ gauge.

XXVI. LOW-ENERGY THEOREMS

In the study of chiral dynamics, nonlinear realizations of chiral symmetries have proven to be useful tools for constructing low-energy effective Lagrangians. In this chapter we shall see that similar techniques can be used to describe the low-energy effects of spontaneously broken supersymmetry. The resulting low-energy theorems describe the effective couplings of Goldstone and matter fields at energies far below the scale of the symmetry breaking.

The fact that the low-energy theorems hold for supersymmetry might seem surprising, for the usual proofs in chiral dynamics rely on the finite volume of a compact group. For the case of supersymmetry, the anti-commuting nature of the group parameters makes such volumes vanish. Nevertheless, we shall see that alternative proofs can be supplied which validate the supersymmetric versions of the low-energy theorems.

In chiral dynamics, the low-energy theorems apply when a group G is spontaneously broken to a subgroup H. The subgroup H is linearly represented on the physical fields, while the remaining generators of G are realized nonlinearly in terms of the coset parameters for G/H. The coset parameters can be interpreted as Goldstone bosons associated with the spontaneous breaking of G down to H.

The nonlinear realizations of G are determined up to field redefinitions. They are often parametrized in certain canonical forms known as *standard realizations*. These realizations linearize on the subgroup H. Any linear representation of H can be promoted to a standard realization of G. Conversely, any realization of G that linearizes on H can be decomposed into a set standard realizations and Goldstone fields.

For the case of supersymmetry, the Lorentz group plays the role of the subgroup H. The remaining generators generate pure supersymmetry transformations. In Chapter XI we used this construction to find a nonlinear realization for the Goldstone fermion λ,

$$\begin{aligned} \delta_\xi \lambda_\alpha(x) &= \frac{1}{\kappa}\,\xi_\alpha - i v_\xi^m(x)\,\partial_m \lambda_\alpha(x) \\ \delta_\xi \bar{\lambda}_{\dot{\alpha}}(x) &= \frac{1}{\kappa}\,\bar{\xi}_{\dot{\alpha}} - i v_\xi^m(x)\,\partial_m \bar{\lambda}_{\dot{\alpha}}(x) \\ v_\xi^m(x) &= \kappa[\lambda(x)\sigma^m\bar{\xi} - \xi\sigma^m\bar{\lambda}(x)], \end{aligned} \tag{26.1}$$

where κ is a constant that parametrizes the supersymmetry breaking scale, analogous to f_π in chiral dynamics. These transformations can be lifted to superfield form using the techniques introduced in Chapter IV. The relevant construction is given in (4.11); for the case at hand, it gives a superfield Λ whose lowest component is the Goldstino λ:

$$\begin{aligned} \Lambda_\alpha(x,\theta,\bar\theta) &= \exp(\theta Q + \bar\theta\bar Q) \times \lambda_\alpha(x) \\ \bar\Lambda_{\dot\alpha}(x,\theta,\bar\theta) &= \exp(\theta Q + \bar\theta\bar Q) \times \bar\lambda_{\dot\alpha}(x). \end{aligned} \tag{26.2}$$

The superfield Λ is built out of λ, its derivatives, and the constant κ:

$$\begin{aligned} \Lambda_\alpha(x,\theta,\bar\theta) &= \lambda_\alpha(x) + \frac{1}{\kappa}\,\theta_\alpha + \cdots \\ \bar\Lambda_{\dot\alpha}(x,\theta,\bar\theta) &= \bar\lambda_{\dot\alpha}(x) + \frac{1}{\kappa}\,\bar\theta_{\dot\alpha} + \cdots. \end{aligned} \tag{26.3}$$

It is a short exercise to show that the transformations (4.10) reduce to (26.1) when applied to the lowest component of Λ.

The Goldstone superfield Λ can also be defined as the solution to a certain set of constraints. These conditions can be found with the help of the identity,

$$\begin{aligned} D_\alpha \exp(\theta Q + \bar\theta\bar Q) \times &= \exp(\theta Q + \bar\theta\bar Q)\, Q_\alpha \times \\ \bar D_{\dot\alpha} \exp(\theta Q + \bar\theta\bar Q) \times &= \exp(\theta Q + \bar\theta\bar Q)\, \bar Q_{\dot\alpha} \times. \end{aligned} \tag{26.4}$$

Applying (26.4) to (26.2), and using (26.1), we find

$$\begin{aligned} D_\beta\Lambda_\alpha &= \frac{1}{\kappa}\,\varepsilon_{\alpha\beta} + i\kappa\sigma_{\beta\dot\beta}{}^m\bar\Lambda^{\dot\beta}\,\partial_m\Lambda_\alpha \\ \bar D_{\dot\beta}\Lambda_\alpha &= -i\kappa\Lambda^\beta\sigma_{\beta\dot\beta}{}^m\,\partial_m\Lambda_\alpha. \end{aligned} \tag{26.5}$$

These constraints are consistent with the D algebra (4.7). Their solution is the superfield Λ as defined in (26.2).

To derive the low-energy theorems, we need the supersymmetric analogs of standard realizations. We shall define a standard realization of supersymmetry to have the following transformation law:

$$\delta_\xi f(x) = -iv_\xi^n(x)\,\frac{\partial}{\partial x^n}\,f(x), \tag{26.6}$$

where v_ξ^n is given in (26.1). In the exercises, you will show that (26.6) closes into the supersymmetry algebra. The field f is free to carry an arbitrary set of Lorentz or internal symmetry indices.

As with the Goldstone fermion λ, we would like to promote f to a superfield F whose variation reduces to (26.6) when restricted to its lowest component. Using the construction of Chapter IV, we find

$$\begin{aligned} F(x,\theta,\bar\theta) &= \exp(\theta Q + \bar\theta\bar Q) \times f(x) \\ &= f(x) - iv_\theta^n(x)\frac{\partial}{\partial x^n}f(x) + \cdots . \end{aligned} \tag{26.7}$$

In (26.7), the superfield F carries the same indices as f. Its component fields are built out of λ, f, and their derivatives. It is also possible to derive (26.7) from the constraint equations,

$$\begin{aligned} D_\alpha F &= i\kappa(\sigma^m\bar\Lambda)_\alpha\,\partial_m F \\ \bar D_{\dot\alpha} F &= -i\kappa(\Lambda\sigma^m)_{\dot\alpha}\,\partial_m F. \end{aligned} \tag{26.8}$$

In the case of chiral dynamics, it is well known how to convert any nonlinear realization into a standard realization. As shown in Appendix E, one simply applies a finite group transformation with the field-dependent parameter that would transform the Goldstone fields to zero. This procedure also works for supersymmetry. To see this, let $\tilde f$ be an arbitrary nonlinear realization of supersymmetry, and let $\tilde F$ be its superfield extension. A standard realization F' is obtained by taking

$$F'(x,\theta,\bar\theta,\lambda) = e^{\xi Q + \bar\xi\bar Q}\tilde F(x,\theta,\bar\theta)|_{\xi = -\kappa\lambda}, \tag{26.9}$$

where the Q's are the differential operators (4.4), and the substitution $\xi = -\kappa\lambda$ is made after all the differentiations are performed. We can also write (26.9) in a more explicit form, avoiding derivatives on λ, by changing arguments as follows:

$$\begin{aligned} F'(x,\theta,\bar\theta,\lambda) &= \exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta} + \bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right] \\ &\quad\times \tilde F(x,\theta,\bar\theta)|_{x=y}. \end{aligned} \tag{26.10}$$

In this expression, we are able to separate the exponents because of the fact that

$$\left[\left(\lambda(y)\frac{\partial}{\partial\theta} + \bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right), v_\theta^m(y)\right] = 0. \tag{26.11}$$

To show that f' is a standard realization, we must compute the change in F' from a supersymmetry transformation. This is most easily done using (26.10). The variation of v_θ^m follows from (26.1):

$$\delta_\xi v_\theta^m(y) = \xi\sigma^m\bar\theta - \theta\sigma^m\bar\xi - iv_\xi^n(y)\frac{\partial}{\partial y^n}v_\theta^m(y). \tag{26.12}$$

Using (26.1), (26.12), and (4.4), we can then compute the change in F',

$$\begin{aligned}\delta_\xi F'(x,\theta,\bar\theta,\lambda) = &\left\{\left[i(\xi\sigma^m\bar\theta - \theta\sigma^m\bar\xi)\frac{\partial}{\partial x^m} - iv_\xi^n(y)\frac{\partial}{\partial y^n}\right]\exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\right\}\\ &\times\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta} + \bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right]\tilde F(x,\theta,\bar\theta)\\ &+\exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\left\{\left[-\xi\frac{\partial}{\partial\theta} - \bar\xi\frac{\partial}{\partial\bar\theta} - iv_\xi^n(y)\frac{\partial}{\partial y^n}\right]\right.\\ &\left.\times\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta} + \bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right]\right\}\tilde F(x,\theta,\bar\theta)\\ &+\exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta} + \bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right]\\ &\times\left\{\xi\frac{\partial}{\partial\theta} + \bar\xi\frac{\partial}{\partial\bar\theta} - i(\xi\sigma^m\bar\theta - \theta\sigma^m\bar\xi)\frac{\partial}{\partial x^m}\right\}\tilde F(x,\theta,\bar\theta)\Big|_{x=y}\\ = &-iv_\xi^n(x)\frac{\partial}{\partial x^n}F'(x,\theta,\bar\theta,\lambda).\end{aligned} \tag{26.13}$$

Taking the lowest component, we see that f' indeed transform as a standard realization,

$$\delta_\xi f'(x,\lambda) = -iv_\xi^n(x)\frac{\partial}{\partial x^n}f'(x,\lambda). \tag{26.14}$$

As above, the fields f' and F' can carry any Lorentz or internal symmetry indices.

With these results, we are now in a position to supersymmetrize any Lorentz invariant Lagrangian. The first step is to find a Lagrangian for the Goldstone spinor Λ. Two obvious choices are

$$\mathscr{L}_0 = -\frac{\kappa^2}{2}\int d^2\theta\, d^2\bar\theta\,\Lambda^2\bar\Lambda^2 \tag{26.15}$$

and

$$\mathscr{L}_1 = -\frac{1}{2}\int d^2\theta\, d^2\bar{\theta}(\Lambda^2 + \bar{\Lambda}^2). \tag{26.16}$$

It is not hard to show that the highest component of $\Lambda^2 + \bar{\Lambda}^2$ is a total spacetime derivative, so (26.16) is unsuitable for a supersymmetric action. In contrast, (26.15) is perfectly fine, and coincides with (11.11) when expanded in terms of component fields. We shall take it to be the Lagrangian for the Goldstone fermion.

The next step is to construct the matter superfields. We start with the original matter fields, which have well-defined transformations with respect to the Lorentz and internal symmetry groups. We assign the fields supersymmetry transformations via (26.6), and promote them to superfields via (26.7). In this way we build a superfield out of each matter field in the original theory.

The final step is to construct the supersymmetric matter coupling. We start with the original Lagrangian $\tilde{\mathscr{L}}$, and replace all the matter fields by their corresponding superfields. This gives a superfield Lagrangian whose lowest component is the original Lagrangian. We then turn this lowest component into a highest component by multiplying the superfield expression by $\Lambda^2\bar{\Lambda}^2$,

$$\Lambda^2\bar{\Lambda}^2 = \frac{1}{\kappa^4}\,\theta^2\bar{\theta}^2 + \cdots. \tag{26.17}$$

This gives a fully supersymmetric Lagrangian,

$$\mathscr{L} = \kappa^4 \int d^2\theta\, d^2\bar{\theta}\,\Lambda^2\bar{\Lambda}^2\tilde{\mathscr{L}}, \tag{\textbf{26.18}}$$

whose λ-independent part is just the original Lagrangian $\tilde{\mathscr{L}}$.

As usual in the theory of nonlinear realizations, it is always possible to include higher-derivative terms in the effective action. For example, a contribution of the form

$$\mathscr{L}_2 = \int d^2\theta\, d^2\bar{\theta}\,(\bar{D}^2\Lambda^2)(D^2\bar{\Lambda}^2) \sim (\partial_m\lambda\sigma^{mn}\,\partial_n\lambda)(\partial_k\bar{\lambda}\bar{\sigma}^{k\ell}\,\partial_\ell\bar{\lambda}) + \cdots \tag{26.19}$$

adds a higher-derivative interaction to $\mathscr{L}$. The coefficients of such terms are not determined by symmetry, and must be regarded as parameters of the theory. The leading term in the derivative expansion is the only term that is unique. At high energies, where the higher-order terms become important, the predictive power breaks down.

The Lagrangian (26.18) describes the low-energy interactions in a theory where supersymmetry is spontaneously broken at some scale much greater than the energies involved in the low-energy effective theory. For example, the formalism would apply to the situation where all the supersymmetric partners of the physical fields are very heavy (except for the Goldstino). In this case, the low-energy scattering amplitudes are determined by the effective theory. The only signals of supersymmetry are the nonlinear couplings of the Goldstino to the physical fields.

To illustrate this construction, let us consider the case of a free scalar field $a(x)$ and a free spinor field $\psi(x)$,

$$\tilde{\mathscr{L}} = -\frac{1}{2}\,\partial_m a\,\partial^m a - \frac{1}{2}\,m^2 a^2 - i\psi\sigma^m\,\partial_m\bar{\psi} - \frac{1}{2}\,\mu(\psi^2 + \bar{\psi}^2). \quad (26.20)$$

We supersymmetrize the Lagrangian by assigning transformations to a and ψ via (26.6), and lifting them to superfields A and Ψ that satisfy the constraints (26.8). We then replace the fields in (26.20) by A and Ψ, to find the superfield Lagrangian $\mathscr{L}$,

$$\mathscr{L} = \mathscr{L}_0 + \int d^2\theta\, d^2\bar{\theta}\,\Lambda^2\bar{\Lambda}^2\bigg[-\frac{1}{2}\,\partial_m A\,\partial^m A - \frac{1}{2}\,m^2 A^2 \\ -i\Psi\sigma^m\,\partial_m\bar{\Psi} - \frac{1}{2}\,\mu(\Psi^2 + \bar{\Psi}^2)\bigg]. \quad (26.21)$$

The Lagrangian (26.21) should be expanded in terms of the Goldstone spinor λ. A helpful trick is to replace $d^2\theta\, d^2\bar{\theta}$ by $D^2\bar{D}^2/16$ and use the constraints (26.5) and (26.8) to compute the D and $\bar{D}$ derivatives. To second order in λ, the resulting Lagrangian is of the form,

$$\mathscr{L} = -\frac{1}{2\kappa^2} - i\lambda\sigma^m\,\partial_m\bar{\lambda} + \frac{i}{\kappa^2}\,\lambda\sigma^m\bar{\lambda}\,\partial_m\tilde{\mathscr{L}} \\ + \frac{i}{\kappa^2}\,(\lambda\sigma^m\,\partial^n\bar{\lambda} - \partial^n\lambda\sigma^m\bar{\lambda})T_{mn} + \cdots. \quad \mathbf{(26.22)}$$

At low energies, the Goldstino couples to the energy-momentum tensor T_{mn}, independent of the details of the symmetry breaking. This is the low-energy theorem for supersymmetry.

References

E. A. Ivanov and A. A. Kapustnikov, *J. Phys. A11*, 2375 (1978).
S. Samuel and J. Wess, *Nucl. Phys. B221*, 153 (1983).

Equations

$$\begin{aligned}\delta_\xi \lambda_\alpha(x) &= \frac{1}{\kappa}\,\xi_\alpha - iv_\xi^m(x)\,\partial_m\lambda_\alpha(x)\\ \delta_\xi \bar\lambda_{\dot\alpha}(x) &= \frac{1}{\kappa}\,\bar\xi_{\dot\alpha} - iv_\xi^m(x)\,\partial_m\bar\lambda_{\dot\alpha}(x)\\ v_\xi^m(x) &= \kappa[\lambda(x)\sigma^m\bar\xi - \xi\sigma^m\bar\lambda(x)].\end{aligned} \tag{26.1}$$

$$\begin{aligned}D_\beta\Lambda_\alpha &= \frac{1}{\kappa}\,\varepsilon_{\alpha\beta} + i\kappa\sigma_{\alpha\dot\beta}{}^m\bar\Lambda^{\dot\beta}\,\partial_m\Lambda_\alpha\\ \bar D_{\dot\beta}\Lambda_\alpha &= -i\kappa\Lambda^\beta\sigma_{\beta\dot\beta}{}^m\,\partial_m\Lambda_\alpha.\end{aligned} \tag{26.5}$$

$$\delta_\xi f(x) = -iv_\xi^n(x)\,\frac{\partial}{\partial x^n}f(x). \tag{26.6}$$

$$\begin{aligned}D_\alpha F &= i\kappa(\sigma^m\bar\Lambda)_\alpha\,\partial_m F\\ \bar D_{\dot\alpha}F &= -i\kappa(\Lambda\sigma^m)_{\dot\alpha}\,\partial_m F.\end{aligned} \tag{26.8}$$

$$\mathscr{L}_0 = -\frac{\kappa^2}{2}\int d^2\theta\, d^2\bar\theta\,\Lambda^2\bar\Lambda^2. \tag{26.15}$$

$$\mathscr{L} = \kappa^4\int d^2\theta\, d^2\bar\theta\,\Lambda^2\bar\Lambda^2\tilde{\mathscr{L}}. \tag{26.18}$$

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2\kappa^2} - i\lambda\sigma^m\,\partial_m\bar\lambda + \frac{i}{\kappa^2}\,\lambda\sigma^m\bar\lambda\,\partial_m\tilde{\mathscr{L}}\\ &+ \frac{i}{\kappa^2}\,(\lambda\sigma^m\,\partial^n\bar\lambda - \partial^n\lambda\sigma^m\bar\lambda)T_{mn} + \cdots.\end{aligned} \tag{26.22}$$

Exercises

(1) Show that (26.2) satisfies the constraints (26.5).

(2) Check that the transformation law (26.6) for a standard realization closes into the supersymmetry algebra.

(3) Verify that (26.7) is a solution to the constraints (26.8).

(4) Show that (26.15) coincides with (11.11) when expanded in terms of component fields.

(5) The Lagrangian (26.21) is supersymmetric because the derivative of a superfield is still a superfield. However, the derivative of a standard realization is not a standard realization. Use the techniques introduced here and in Appendix E to find a "covariant derivative" Δ that preserves the transformation properties of a standard realization. The Lagrangian

$$\mathscr{L} = \int d^2\theta\, d^2\bar{\theta}\, \Lambda^2\bar{\Lambda}^2 \bigg[-\frac{1}{2}\Delta_m A\, \Delta^m A - \frac{1}{2} m^2 A^2 - i\Psi\sigma^m \Delta_m \bar{\Psi} - \frac{1}{2}\mu(\Psi^2 + \bar{\Psi}^2) \bigg]$$

is another possible extension of (26.20). It differs from (26.21) by higher-order terms in λ. The derivative Δ is natural to use when gauging an internal symmetry if the vector superfields belong to a standard realization.

(6) Show that (26.21) reduces to (26.22) in terms of component fields.

APPENDIX A
NOTATION AND SPINOR ALGEBRA

We use the metric $\eta_{mn} \sim (-1,1,1,1)$ throughout these lectures. Furthermore, we work with Weyl spinors in the Van der Waerden notation.

To begin, we define M to be a two-by-two matrix of determinant one: $M \in \mathrm{SL(2,C)}$. The matrix M, its complex conjugate M^*, its transpose inverse $(M^T)^{-1}$, and its hermitian conjugate inverse $(M^+)^{-1}$ all represent SL(2,C). They represent the action of the Lorentz group on two-component Weyl spinors.

Two-component spinors with upper or lower dotted or undotted indices transform as follows under M:

$$\begin{aligned} \psi'_\alpha &= M_\alpha{}^\beta \psi_\beta & \bar{\psi}'_{\dot{\alpha}} &= M^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \\ \psi'^\alpha &= M^{-1}{}_\beta{}^\alpha \psi^\beta & \bar{\psi}'^{\dot{\alpha}} &= (M^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}}. \end{aligned} \tag{A.1}$$

Spinors are denoted by Greek indices. Those with dotted indices transform under the $(0,\frac{1}{2})$ representation of the Lorentz group, while those with undotted indices transform under the $(\frac{1}{2},0)$ conjugate representation.

The connection between SL(2,C) and the Lorentz group is established through the σ-matrices

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{A.2}$$

in complete analogy to the relation between SU(2) and the rotation group. These matrices form a basis for two-by-two complex matrices:

$$P = P_m \sigma^m = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix}. \tag{A.3}$$

Any hermitian matrix may be expanded with the P_m real.

From any hermitian matrix P, we may always obtain another by the following transformation:

$$P' = MPM^{+}. \tag{A.4}$$

Both P and P' have expansions in σ,

$$\sigma^m P'_m = M\sigma^m P_m M^{+}. \tag{A.5}$$

Since M is unimodular ($\det M = 1$), the coefficients P_m and P'_m are connected by a Lorentz transformation:

$$\det[\sigma^m P'_m] = \det[\sigma^m P_m] = P_0'^2 - \mathbf{P}'^2 = P_0^2 - \mathbf{P}^2. \tag{A.6}$$

Vectors and tensors are distinguished from spinors by their Latin indices.

From (A.1) and (A.5), we see that σ^m has the following index structure:

$$\sigma_{\alpha\dot{\alpha}}{}^{m}. \tag{A.7}$$

With these conventions, $\psi^\alpha\psi_\alpha$, $\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$, and $\psi^\alpha\sigma_{\alpha\dot{\alpha}}{}^{m}\,\partial_m\bar{\psi}^{\dot{\alpha}}$ are all Lorentz scalars.

Since M is unimodular, the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ ($\varepsilon_{21} = \varepsilon^{12} = 1, \varepsilon_{12} = \varepsilon^{21} = -1, \varepsilon_{11} = \varepsilon_{22} = 0$) are invariant under Lorentz transformations:

$$\begin{aligned} \varepsilon_{\alpha\beta} &= M_\alpha{}^\gamma M_\beta{}^\delta \varepsilon_{\gamma\delta} \\ \varepsilon^{\alpha\beta} &= \varepsilon^{\gamma\delta} M_\gamma{}^\alpha M_\delta{}^\beta. \end{aligned} \tag{A.8}$$

Spinors with upper and lower indices are related through the ε-tensor:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta. \tag{A.9}$$

Note that we have defined $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ such that $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$. An analogous treatment holds for the ε-tensor with dotted indices.

The ε-tensor may also be used to raise the indices of the σ-matrices:

$$\bar{\sigma}^{m\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^m{}_{\beta\dot{\beta}}. \tag{A.10}$$

From the definition of the σ-matrices, we find

$$\begin{aligned} (\sigma^m\bar{\sigma}^n + \sigma^n\bar{\sigma}^m)_\alpha{}^\beta &= -2\eta^{mn}\,\delta_\alpha{}^\beta \\ (\bar{\sigma}^m\sigma^n + \bar{\sigma}^n\sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} &= -2\eta^{mn}\,\delta^{\dot{\alpha}}{}_{\dot{\beta}}, \end{aligned} \tag{A.11}$$

as well as the following completeness relations:

$$\mathrm{Tr}\, \sigma^m \bar{\sigma}^n = -2\eta^{mn}$$
$$\sigma_{\alpha\dot{\alpha}}{}^m \bar{\sigma}_m{}^{\dot{\beta}\beta} = -2\delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}}. \tag{A.12}$$

These relations may be used to convert a vector to a bispinor and vice versa:

$$v_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}{}^m v_m, \qquad v^m = -\frac{1}{2}\bar{\sigma}^{m\dot{\alpha}\alpha} v_{\alpha\dot{\alpha}}. \tag{A.13}$$

The generators of the Lorentz group in the spinor representation are given by

$$\sigma^{nm}{}_\alpha{}^\beta = \frac{1}{4}(\sigma_{\alpha\dot{\alpha}}{}^n \bar{\sigma}^{m\dot{\alpha}\beta} - \sigma_{\alpha\dot{\alpha}}{}^m \bar{\sigma}^{n\dot{\alpha}\beta})$$
$$\bar{\sigma}^{nm\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}(\bar{\sigma}^{n\dot{\alpha}\alpha}\sigma_{\alpha\dot{\beta}}{}^m - \bar{\sigma}^{m\dot{\alpha}\alpha}\sigma_{\alpha\dot{\beta}}{}^n). \tag{A.14}$$

Other useful relations involving the σ-matrices are

$$\bar{\sigma}^a \sigma^b \bar{\sigma}^c - \bar{\sigma}^c \sigma^b \bar{\sigma}^a = -2i\varepsilon^{abcd}\bar{\sigma}_d$$
$$\sigma^a \bar{\sigma}^b \sigma^c - \sigma^c \bar{\sigma}^b \sigma^a = 2i\varepsilon^{abcd}\sigma_d, \tag{A.15}$$

where $\varepsilon_{0123} = -1$, as well as

$$\sigma^a \bar{\sigma}^b \sigma^c + \sigma^c \bar{\sigma}^b \sigma^a = 2(\eta^{ac}\sigma^b - \eta^{bc}\sigma^a - \eta^{ab}\sigma^c)$$
$$\bar{\sigma}^a \sigma^b \bar{\sigma}^c + \bar{\sigma}^c \sigma^b \bar{\sigma}^a = 2(\eta^{ac}\bar{\sigma}^b - \eta^{bc}\bar{\sigma}^a - \eta^{ab}\bar{\sigma}^c), \tag{A.16}$$

and

$$\sigma_{\alpha\dot{\alpha}}{}^n \sigma_{\beta\dot{\beta}}{}^m - \sigma_{\alpha\dot{\alpha}}{}^m \sigma_{\beta\dot{\beta}}{}^n = 2[(\sigma^{nm}\varepsilon)_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + (\varepsilon\bar{\sigma}^{nm})_{\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta}]$$
$$\sigma_{\alpha\dot{\alpha}}{}^n \sigma_{\beta\dot{\beta}}{}^m + \sigma_{\alpha\dot{\alpha}}{}^m \sigma_{\beta\dot{\beta}}{}^n = -\eta^{nm}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + 4(\sigma^{\ell n}\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}^{\ell m})_{\dot{\alpha}\dot{\beta}}. \tag{A.17}$$

The equations (A.11) make it easy to relate two-component to four-component spinors. This is done through the following realization of the Dirac γ-matrices:

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}. \tag{A.18}$$

We shall call this the Weyl basis. In this basis, Dirac spinors contain two Weyl spinors,

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.19}$$

while Majorana spinors contain only one:

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.20}$$

Throughout these lectures we shall use the following spinor summation convention:

$$\begin{aligned} \psi\chi &= \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \end{aligned} \tag{A.21}$$

Here we have assumed, as always, that spinors anticommute. The definition of $\bar{\psi}\bar{\chi}$ is chosen in such a way that

$$(\chi\psi)^+ = (\chi^\alpha\psi_\alpha)^+ = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}. \tag{A.22}$$

Note that conjugation reverses the order of the spinors.

References

E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations*, London, Blackie and Son (1953).

W. Thirring, *Supplemento del Nuovo Cimento 14*, no. 2, 415 (1959).

Exercises

(1) Compute P'_m in Eq. (A.5) for $M = \exp(\frac{1}{2}i\phi\sigma_3)$ and $M = \exp(\frac{1}{2}\chi\sigma_3)$.

(2) Show that M, $(M^T)^{-1}$ form equivalent representations of SL(2,C).

(3) Demonstrate:

$$\bar{\sigma}^0 = \sigma^0$$
$$\bar{\sigma}^{1,2,3} = -\sigma^{1,2,3}.$$

(4) Verify Eqs. (A.11) and (A.12).

(5) Show:

$$\sigma^{mn\ \alpha}_{\ \ \alpha} = 0$$

$$\sigma^{mn\ \beta}_{\ \ \alpha}\varepsilon_{\beta\gamma} = \sigma^{mn\ \beta}_{\ \ \gamma}\varepsilon_{\beta\alpha}.$$

(6) Verify:

$$\varepsilon^{abcd}\sigma_{cd} = -2i\sigma^{ab}$$

$$\varepsilon^{abcd}\bar{\sigma}_{cd} = 2i\bar{\sigma}^{ab}.$$

(7) Demonstrate:

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

in the Weyl basis.

(8) Show that the canonical basis for the Dirac γ-matrices,

$$\gamma_C^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_C^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

is related to the Weyl basis (A.18) by the following similarity transformation:

$$\Gamma_W = X\Gamma_C X^{-1}, \qquad X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Also show that the Majorana basis, in which $\gamma_M^{n*} = -\gamma_M^n$,

$$\gamma_M^0 = \begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \qquad \gamma_M^1 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}$$

$$\gamma_M^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \gamma_M^3 = \begin{pmatrix} 0 & -i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix},$$

is related to the Weyl basis by a further similarity transformation:

$$\Gamma_W = Y\Gamma_M Y^{-1} \qquad Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ \varepsilon & -i\varepsilon \end{pmatrix}.$$

(9) Let Ψ_M denote a Majorana spinor in the Weyl basis,

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

Convert this to the Majorana basis of Exercise 8:

$$\Psi_M \to \sqrt{2} \begin{pmatrix} \mathrm{Re}\ \chi_\alpha \\ \mathrm{Im}\ \chi_\alpha \end{pmatrix}.$$

(10) Prove the following relations:

$$\theta^\alpha \theta^\beta = -\frac{1}{2} \varepsilon^{\alpha\beta} \theta\theta$$

$$\theta_\alpha \theta_\beta = \frac{1}{2} \varepsilon_{\alpha\beta} \theta\theta$$

$$\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}\bar{\theta}$$

$$\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = -\frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}\bar{\theta}$$

$$\theta\sigma^m\bar{\theta}\theta\sigma^n\bar{\theta} = -\frac{1}{2} \theta\theta\bar{\theta}\bar{\theta}\eta^{mn}.$$

(11) Use Exercise 10 to show

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta),$$

$$(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\frac{1}{2}(\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}).$$

(12) Verify:

$$\mathrm{Tr}\ \sigma^{mn}\sigma^{k\ell} = -\frac{1}{2}(\eta^{mk}\eta^{n\ell} - \eta^{m\ell}\eta^{nk}) - \frac{i}{2}\varepsilon^{mnk\ell},$$

where $\varepsilon_{0123} = -1$.

(13) Rewrite the supersymmetry algebra (I) in terms of four-component Majorana spinors:

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^m_{\alpha\beta} P_m$$
$$[Q_\alpha, P_m] = [\bar{Q}_\alpha, P_m] = 0.$$

APPENDIX B
RESULTS IN SPINOR ALGEBRA

Conventions:

$$\eta_{mn} \sim (-1,1,1,1)$$

$$\varepsilon_{21} = \varepsilon^{12} = 1, \qquad \varepsilon_{12} = \varepsilon^{21} = -1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0$$

$$\varepsilon_{0123} = -1$$

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\psi_{\beta}, \qquad \psi_{\alpha} = \varepsilon_{\alpha\beta}\psi^{\beta}$$

$$\psi\chi = \psi^{\alpha}\chi_{\alpha} = -\psi_{\alpha}\chi^{\alpha} = \chi^{\alpha}\psi_{\alpha} = \chi\psi$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}$$

$$(\chi\psi)^{+} = (\chi^{\alpha}\psi_{\alpha})^{+} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$$

$$\Psi_{D} = \begin{pmatrix} \chi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

$$\gamma^{m} = \begin{pmatrix} 0 & \sigma^{m} \\ \bar{\sigma}^{m} & 0 \end{pmatrix}$$

$$\gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{B.1}$$

Sigma Matrices:

$$\sigma^{0} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2}$$

$$\bar{\sigma}^{m\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma_{\beta\dot{\beta}}{}^{m}$$
$$\bar{\sigma}^{0} = \sigma^{0} \tag{B.3}$$
$$\bar{\sigma}^{1,2,3} = -\sigma^{1,2,3}.$$

$$\begin{aligned} \mathrm{Tr}\,\sigma^m\bar\sigma^n &= -2\eta^{mn} \\ \sigma_{\alpha\dot\alpha}{}^m\bar\sigma_m{}^{\dot\beta\beta} &= -2\delta_\alpha{}^\beta\,\delta_{\dot\alpha}{}^{\dot\beta}. \end{aligned} \tag{B.4}$$

$$\begin{aligned} (\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m)_\alpha{}^\beta &= -2\eta^{mn}\,\delta_\alpha{}^\beta \\ (\bar\sigma^m\sigma^n + \bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta} &= -2\eta^{mn}\,\delta^{\dot\alpha}{}_{\dot\beta}. \end{aligned} \tag{B.5}$$

$$\begin{aligned} \sigma^{nm}{}_\alpha{}^\beta &= \frac{1}{4}\,(\sigma_{\alpha\dot\alpha}{}^n\bar\sigma^{m\dot\alpha\beta} - \sigma_{\alpha\dot\alpha}{}^m\bar\sigma^{n\dot\alpha\beta}) \\ \bar\sigma^{nm\dot\alpha}{}_{\dot\beta} &= \frac{1}{4}\,(\bar\sigma^{n\dot\alpha\alpha}\sigma_{\alpha\dot\beta}{}^m - \bar\sigma^{m\dot\alpha\alpha}\sigma_{\alpha\dot\beta}{}^n). \end{aligned} \tag{B.6}$$

$$\begin{aligned} \sigma^{mn}{}_\alpha{}^\alpha &= 0 \\ \sigma^{mn}{}_\alpha{}^\beta\varepsilon_{\beta\gamma} &= \sigma^{mn}{}_\gamma{}^\beta\varepsilon_{\beta\alpha}. \end{aligned} \tag{B.7}$$

$$\begin{aligned} \varepsilon^{abcd}\sigma_{cd} &= -2i\sigma^{ab} \\ \varepsilon^{abcd}\bar\sigma_{cd} &= 2i\bar\sigma^{ab}. \end{aligned} \tag{B.8}$$

$$\begin{aligned} \sigma_{\alpha\dot\alpha}{}^n\sigma_{\beta\dot\beta}{}^m - \sigma_{\alpha\dot\alpha}{}^m\sigma_{\beta\dot\beta}{}^n &= 2\big[(\sigma^{nm}\varepsilon)_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + (\varepsilon\bar\sigma^{nm})_{\dot\alpha\dot\beta}\varepsilon_{\alpha\beta}\big] \\ \sigma_{\alpha\dot\alpha}{}^n\sigma_{\beta\dot\beta}{}^m + \sigma_{\alpha\dot\alpha}{}^m\sigma_{\beta\dot\beta}{}^n &= -\eta^{nm}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + 4(\sigma^{\ell n}\varepsilon)_{\alpha\beta}(\varepsilon\bar\sigma^{\ell m})_{\dot\alpha\dot\beta}. \end{aligned} \tag{B.9}$$

$$\mathrm{Tr}\,\sigma^{mn}\sigma^{k\ell} = -\frac{1}{2}\,(\eta^{mk}\eta^{n\ell} - \eta^{m\ell}\eta^{nk}) - \frac{i}{2}\,\varepsilon^{mnk\ell}. \tag{B.10}$$

$$\begin{aligned} \sigma^a\bar\sigma^b\sigma^c + \sigma^c\bar\sigma^b\sigma^a &= 2(\eta^{ac}\sigma^b - \eta^{bc}\sigma^a - \eta^{ab}\sigma^c) \\ \bar\sigma^a\sigma^b\bar\sigma^c + \bar\sigma^c\sigma^b\bar\sigma^a &= 2(\eta^{ac}\bar\sigma^b - \eta^{bc}\bar\sigma^a - \eta^{ab}\bar\sigma^c). \end{aligned} \tag{B.11}$$

$$\begin{aligned} \sigma^a\bar\sigma^b\sigma^c - \sigma^c\bar\sigma^b\sigma^a &= 2i\varepsilon^{abcd}\sigma_d. \\ \bar\sigma^a\sigma^b\bar\sigma^c - \bar\sigma^c\sigma^b\bar\sigma^a &= -2i\varepsilon^{abcd}\bar\sigma_d \end{aligned} \tag{B.12}$$

Spinor Algebra:

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\,\varepsilon^{\alpha\beta}\theta\theta \\ \theta_\alpha\theta_\beta &= \frac{1}{2}\,\varepsilon_{\alpha\beta}\theta\theta \\ \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} &= \frac{1}{2}\,\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \\ \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} &= -\frac{1}{2}\,\varepsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta. \end{aligned} \tag{B.13}$$

$$\theta\sigma^m\bar{\theta}\theta\sigma^n\bar{\theta} = -\frac{1}{2}\,\theta\theta\bar{\theta}\bar{\theta}\eta^{mn}. \tag{B.14}$$

$$\begin{aligned}(\theta\phi)(\theta\psi) &= -\frac{1}{2}(\phi\psi)(\theta\theta)\\ (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) &= -\frac{1}{2}(\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}).\end{aligned} \tag{B.15}$$

$$\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha}. \tag{B.16}$$

$$\begin{aligned}\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\,\theta\theta &= 4\\ \varepsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\,\bar{\theta}\bar{\theta} &= 4.\end{aligned} \tag{B.17}$$

$$\begin{aligned}\chi\sigma^n\bar{\psi} &= -\bar{\psi}\bar{\sigma}^n\chi, \qquad (\chi\sigma^m\bar{\psi})^+ = \psi\sigma^m\bar{\chi}\\ \chi\sigma^m\bar{\sigma}^n\psi &= \psi\sigma^n\bar{\sigma}^m\chi, \qquad (\chi\sigma^m\bar{\sigma}^n\psi)^+ = \bar{\psi}\bar{\sigma}^n\sigma^m\bar{\chi}.\end{aligned} \tag{B.18}$$

$$(\psi\phi)\bar{\chi}_{\dot{\beta}} = -\frac{1}{2}(\phi\sigma^m\bar{\chi})(\psi\sigma^m)_{\dot{\beta}}. \tag{B.19}$$

APPENDIX C
KÄHLER GEOMETRY

The matter couplings of chiral multiplets are conveniently described in the language of Kähler geometry. It is useful, therefore, to introduce the notion of a Kähler manifold. A Kähler manifold is a special type of analytic Riemann manifold, subject to certain conditions that we will discuss below. Since the manifold is analytic, it can be parametrized in terms of complex coordinates a^i and a^{*i}, where $i = 1, \ldots, n$. Under an analytic coordinate transformation,

$$a^i = a^i(a') \qquad a^{*i} = a^{*i}(a^{*\prime}), \tag{C.1}$$

the differentials and derivatives transform as follows:

$$\begin{aligned} da'^i &= \frac{\partial a'^i}{\partial a^j}\, da^j \qquad & da^{*\prime i} &= \frac{\partial a^{*\prime i}}{\partial a^{*j}}\, da^{*j} \\ \frac{\partial}{\partial a'^i} &= \frac{\partial a^j}{\partial a'^i}\frac{\partial}{\partial a^j} \qquad & \frac{\partial}{\partial a^{*\prime i}} &= \frac{\partial a^{*j}}{\partial a^{*\prime i}}\frac{\partial}{\partial a^{*j}}. \end{aligned} \tag{C.2}$$

These transformations preserve the analytic nature of the coordinates. They also define the transformations of covariant and contravariant vector fields,

$$\begin{aligned} V'_i(a',a^{*\prime}) &= \frac{\partial a^j}{\partial a'^i}\, V_j(a,a^*) \\ V'^i(a',a^{*\prime}) &= \frac{\partial a'^i}{\partial a^j}\, V^j(a,a^*) \\ V'_{i^*}(a',a^{*\prime}) &= \frac{\partial a^{*j}}{\partial a^{*\prime i}}\, V_{j^*}(a,a^*) \\ V'^{i^*}(a',a^{*\prime}) &= \frac{\partial a^{*\prime i}}{\partial a^{*j}}\, V^{j^*}(a,a^*). \end{aligned} \tag{C.3}$$

The first condition on a Kähler manifold is that it be endowed with a hermitian metric g_{ij^*}. The metric must be positive definite and invertible, which allows us to raise and lower the indices i and j^*:

$$V_i = g_{ij^*} V^{j^*} \qquad V_{j^*} = g_{ij^*} V^i$$
$$V^i = g^{ij^*} V_{j^*} \qquad V^{j^*} = g^{ij^*} V_i. \tag{C.4}$$

The second requirement is that the covariant derivative must respect the analytic structure. This implies that $\Gamma^{k^*}_{ij} = \Gamma^{k^*}_{i^*j} = 0$, so the covariant derivative is of the following form:

$$\nabla_i V_j = \frac{\partial}{\partial a^i} V_j - \Gamma^k_{ij} V_k$$
$$\nabla_{i^*} V_j = \frac{\partial}{\partial a^{*i}} V_j - \Gamma^k_{i^*j} V_k. \tag{C.5}$$

The third condition is that the connection be compatible with the hermitian metric. This imposes the additional restriction,

$$\nabla_k g_{ij^*} = 0 \qquad \nabla_{k^*} g_{ij^*} = 0. \tag{C.6}$$

The transformation law for the connection is chosen to assure that covariant derivatives of tensors transform as tensors. This implies

$$\Gamma'^k_{ij} = \frac{\partial a^\ell}{\partial a'^i} \frac{\partial a^m}{\partial a'^j} \frac{\partial a'^k}{\partial a^n} \Gamma^n_{\ell m} + \frac{\partial^2 a^n}{\partial a'^i \partial a'^j} \frac{\partial a'^k}{\partial a^n}$$
$$\Gamma'^k_{i^*j} = \frac{\partial a^{*\ell}}{\partial a^{*\prime i}} \frac{\partial a^m}{\partial a'^j} \frac{\partial a'^k}{\partial a^n} \Gamma^n_{\ell^* m}. \tag{C.7}$$

The first of the equations (C.7) tells us that it is consistent to set the torsion to zero, leaving only the symmetric part of the connection.

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{C.8}$$

The second equation implies that it is also permissible to demand

$$\Gamma^k_{i^*j} = 0. \tag{C.9}$$

Equations (C.8) and (C.9) are the two remaining postulates that define a Kähler manifold.

On a Kähler manifold, the conditions discussed above imply that the only nonvanishing components of the connection are Γ^i_{jk} and its complex conjugate Γ^{i*}_{j*k*}. Equation (C.6) can be solved to give

$$\Gamma^k_{ij} = g^{k\ell*}\frac{\partial}{\partial a^i}g_{j\ell*}. \tag{C.10}$$

Since

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \tag{C.11}$$

the metric must obey the following integrability condition:

$$\frac{\partial}{\partial a^k}g_{ij*} = \frac{\partial}{\partial a^i}g_{kj*}. \tag{C.12}$$

A similar relation holds for the conjugate derivatives,

$$\frac{\partial}{\partial a^{*k}}g_{ij*} = \frac{\partial}{\partial a^{*j}}g_{ik*}. \tag{C.13}$$

Equations (C.12) and (C.13) imply that the metric is the derivative of a scalar function K,

$$g_{ij*} = \frac{\partial}{\partial a^i}\frac{\partial}{\partial a^{*j}}K(a,a^*). \tag{C.14}$$

The function K is called the Kähler potential; its derivatives determine the metric and the connection. Kähler manifolds are often *defined* through (C.14), in which case the conditions on the connection are then deduced.

The Kähler potential completely specifies the Kähler geometry. Note that the metric g_{ij*} is invariant under analytic shifts of K,

$$K(a,a^*) \to K(a,a^*) + F(a) + F^*(a^*). \tag{C.15}$$

Such a shift is called a Kähler transformation of the Kähler potential.

The curvature of a Kähler manifold can be defined as the commutator of two covariant derivatives:

$$\begin{aligned} [\nabla_i,\nabla_j]V_k &= R^\ell_{ijk}V_\ell \\ [\nabla_i,\nabla_{j*}]V_k &= R^\ell_{ij*k}V_\ell. \end{aligned} \tag{C.16}$$

The upper index on R can be lowered with the help of the metric, giving

$$\begin{aligned} R_{ijk\ell*} &= g_{m\ell*}R^m_{ijk} \\ R_{ij*k\ell*} &= g_{m\ell*}R^m_{ij*k}. \end{aligned} \tag{C.17}$$

In Exercise 3 we will see that only $R_{ij^*k\ell^*}$ and its complex conjugate are nonvanishing. From the definition of the covariant derivative, we find

$$\begin{aligned} R_{ij^*k\ell^*} &= g_{m\ell^*} \frac{\partial}{\partial a^{*j}} \Gamma^m_{ik} \\ &= \frac{\partial}{\partial a^i} \frac{\partial}{\partial a^{*j}} g_{k\ell^*} - g^{mn^*} \left(\frac{\partial}{\partial a^{*j}} g_{m\ell^*} \right) \left(\frac{\partial}{\partial a^i} g_{kn^*} \right). \end{aligned} \tag{C.18}$$

Using (C.16), (C.17), and (C.18), it is not hard to show that the curvature obeys the following symmetries:

$$R_{ij^*k\ell^*} = -R_{ij^*\ell^*k} = -R_{j^*ik\ell^*} = R_{j^*i\ell^*k}. \tag{C.19}$$

This is all the Kähler geometry we need to discuss the general couplings of chiral fields.

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Exercises

(1) Verify the transformation law (C.7) for the connection Γ.

(2) Impose the Kähler conditions (C.8) and (C.9), and solve for the connection in terms of the metric.

(3) Show that $R_{ij^*k\ell^*}$ is the only nonvanishing component of the curvature on a Kähler manifold, and solve for the curvature in terms of the metric.

(4) Compute the curvature, Ricci tensor, and curvature scalar for the manifold with Kähler potential $K = -3\log(1 - \frac{1}{3}a^{*i}a^i)$.

(5) Show that in the language of differential forms, the Kähler condition (C.14) is equivalent to the statement that the fundamental form

$$\Omega = \frac{i}{2} g_{ij^*}\, da^i\, da^{*j}$$

is closed,

$$d\Omega = 0.$$

APPENDIX D
ISOMETRIES AND KÄHLER GEOMETRY

In this appendix we will discuss the isometries of Kähler manifolds. The techniques we introduce will prove useful in constructing gauge invariant matter couplings in flat and curved space. Before specializing to Kähler manifolds, however, we first define the general notion of an isometry group. Consider, therefore, an arbitrary differentiable manifold $\mathscr{M}$, and a set of parametrized curves that fill the manifold without intersecting. Then construct the map ϕ_t: $\mathscr{M} \to \mathscr{M}$ which takes each point $p \in \mathscr{M}$ a parameter distance t along the unique curve that passes through p. This map also induces a map on the tangent space. If the induced map leaves the metric invariant, ϕ_t is said to be an *isometry* of the manifold $\mathscr{M}$. The set of isometries forms a group, called the *isometry group* of $\mathscr{M}$.

Curves and vectors are closely related geometrical objects. Consider a curve λ, described by real coordinates $x^i = x^i(t)$, and a differentiable function f: $\mathscr{M} \to \mathbb{R}$. Then the *directional derivative* of f along the curve λ is given by

$$\frac{df}{dt} = \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} \equiv Xf, \tag{D.1}$$

and the operator

$$X = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} \tag{D.2}$$

maps any function f to its directional derivative along λ. In mathematical language, X is called a vector, and the dx^i/dt are its components. The operator X is the natural generalization of a tangent vector to curved space.

This definition of a vector can be applied to a space-filling set of curves as well. The components dx^i/dt become functions on $\mathscr{M}$, and $X \equiv (dx^i/dt)\, \partial/\partial x^i$ is known as a *vector field.*

Alternatively, given a set of continuous functions X^i on $\mathscr{M}$, it is always possible to define an associated set of *integral curves* $x^i(t)$ as solutions to the differential equations,

$$\frac{dx^i}{dt} = X^i. \tag{D.3}$$

The corresponding vector field is just $X = X^i \partial/\partial x^i$. Locally, such curves can never cross because the solutions to (D.3) are unique. They are also globally well defined because (D.3) holds at each point of the manifold $\mathscr{M}$.

Thus we have seen that sets of space-filling curves are in one-one correspondence with vector fields X. The map ϕ_t defines a motion along the integral curve defined by X.

As with any map of a manifold onto itself, ϕ_t induces a map between vectors in the tangent space. The induced map allows us to compare vectors at different points along integral curves. To construct it explicitly, let x^i denote the coordinates at p, and x'^i the coordinates at p'. Then let Y be a vector field, with components $Y^i(x)$ at p. The components $Y^i(x)$ at p can be mapped to components $\tilde{Y}^i(x')$ at p' as follows,

$$\tilde{Y}^i(x') = \frac{\partial x'^i}{\partial x^j} Y^j(x(x')). \tag{D.4}$$

Equation (D.4) defines a map of vectors at p onto vectors at p'. It is sometimes called *Lie transport*. The Lie-transported vector field,

$$\tilde{Y} \equiv \tilde{Y}^i(x') \frac{\partial}{\partial x'^i}, \tag{D.5}$$

is defined for all points p' along the integral curve. Infinitesimally, if $x'^i = x^i + X^i \delta t$, (D.4) reduces to

$$\tilde{Y}^i(x') = Y^i(x') + \frac{\partial X^i}{\partial x^j} Y^j \delta t - \frac{\partial Y^i}{\partial x^j} X^j \delta t, \tag{D.6}$$

and the new field $\tilde{Y}$ is infinitesimally close to Y.

Since Y and $\tilde{Y}$ are both defined at the same points, it makes sense to take their difference and construct the *Lie derivative* of Y with respect to X:

$$\begin{aligned}
(\pounds_X Y)^i &\equiv \lim_{\delta t \to 0} \frac{Y^i(x') - \tilde{Y}^i(x')}{\delta t} \\
&= X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i \\
&\equiv [X,Y]^i.
\end{aligned} \tag{D.7}$$

The Lie derivative of two vector fields gives a third vector field on the manifold $\mathscr{M}$.

Using similar logic, the definition of the Lie derivative can be generalized to any other tensor field. For example, an expression analogous to (D.4) implies that the Lie derivative of a covariant vector field is as follows:

$$(\pounds_X Y)_i = X^j \frac{\partial}{\partial x^j} Y_i + Y_j \frac{\partial}{\partial x^i} X^j. \tag{D.8}$$

In a similar fashion, the Lie derivative of the metric is given by

$$\begin{aligned}(\pounds_X g)_{ij} &= X^k \frac{\partial}{\partial x^k} g_{ij} + g_{ik} \frac{\partial}{\partial x^j} X^k + g_{jk} \frac{\partial}{\partial x^i} X^k \\ &= \nabla_i X_j + \nabla_j X_i,\end{aligned} \tag{D.9}$$

where $X_i = g_{ij}X^j$ and $\nabla_i X_j = \partial_i X_j - \Gamma^k_{ij} X_k$ contains the torsion-free connection compatible with the metric g_{ij}.

A field is *invariant* under Lie transport if it has a vanishing Lie derivative. If the metric is invariant, then

$$(\pounds_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = 0 \tag{D.10}$$

for some vector field X. In this case, X generates an *isometry* of the manifold $\mathscr{M}$. It is called a *Killing vector field*, and (D.10) is known as Killing's equation.

The Killing vectors generate the continuous symmetries of a manifold. These symmetries close into the isometry group. Indeed, it is not hard to show that the Lie bracket of two Killing vectors gives another,

$$[X^{(a)}, X^{(b)}] = -f^{abc} X^{(c)}, \tag{D.11}$$

where the f^{abc} are the *structure constants* of the isometry group G.

Let us now assume that our manifold is Kähler, with metric g_{ij^*} and complex coordinates a^i and a^{*i}. We shall focus our attention on the analytic isometries, those that preserve the analytic structure of the manifold. This requires that the associated Killing vectors be holomorphic vector fields,

$$\begin{aligned}X^{(b)} &= X^{i(b)}(a) \frac{\partial}{\partial a^i} \\ X^{*(b)} &= X^{*i(b)}(a^*) \frac{\partial}{\partial a^{*i}}.\end{aligned} \tag{D.12}$$

The index (b) labels the Killing vectors and runs over the dimension d of the isometry group G.

Because the $X^{(a)}$ are holomorphic, Killing's equation (D.10) reduces to the following form:

$$\begin{aligned} \nabla_i X_j^{(a)} + \nabla_j X_i^{(a)} &= 0 \\ \nabla_{i^*} X_j^{(a)} + \nabla_j X_i^{*(a)} &= 0. \end{aligned} \tag{D.13}$$

On a Kähler manifold, the first equation is automatically satisfied because of the definition of the covariant derivative. The second is an integrability condition; it is locally equivalent to the statement that there exist d real scalar functions $D^{(a)}(a,a^*)$, such that

$$\begin{aligned} g_{ij^*} X^{*j(a)} &= i \frac{\partial}{\partial a^i} D^{(a)} \\ g_{ij^*} X^{i(a)} &= -i \frac{\partial}{\partial a^{*j}} D^{(a)}. \end{aligned} \tag{D.14}$$

The $D^{(a)}$ are known as Killing potentials and defined up to constants $c^{(a)}$, $D^{(a)} \to D^{(a)} + c^{(a)}$. In Chapter XXIV we show that the freedom to redefine the potentials is related to the Fayet-Iliopoulos D term for Abelian groups.

The relations (D.14) can be inverted to give the Killing vectors in terms of the Killing potentials,

$$\begin{aligned} X^{i(a)} &= -ig^{ij^*} \frac{\partial}{\partial a^{*j}} D^{(a)} \\ X^{*j(a)} &= ig^{ij^*} \frac{\partial}{\partial a^i} D^{(a)}. \end{aligned} \tag{D.15}$$

The requirement that the fields $X^{i(a)}$ be holomorphic places a constraint on the $D^{(a)}$. Solving this constraint is equivalent to solving (D.13). In general, it may be difficult to find the Killing potentials on a given Kähler manifold.

Because of the holomorphic structure, the Killing vectors $X^{(a)}$ and $X^{*(a)}$ generate independent representations of the isometry group G. They obey the Lie bracket relations,

$$\begin{aligned} [X^{(a)}, X^{(b)}] &= -f^{abc} X^{(c)} \\ [X^{*(a)}, X^{*(b)}] &= -f^{abc} X^{*(c)} \\ [X^{(a)}, X^{*(b)}] &= 0, \end{aligned} \tag{D.16}$$

where the f^{abc} are the structure constants of G. The Killing potentials $D^{(a)}$ also transform under the isometry group. As shown in Exercise 3, they can be chosen to transform in the adjoint representation,

$$\left[X^{i(a)}\frac{\partial}{\partial a^i} + X^{*i(a)}\frac{\partial}{\partial a^{*i}}\right]D^{(b)} = -f^{abc}D^{(c)}. \tag{D.17}$$

This fixes the constants $c^{(a)}$ for non-Abelian groups. For each U(1) factor, however, there is an undetermined constant c.

Let us now turn our attention to the variation of the Kähler potential under an isometry in G. Such an isometry is generated by the Killing vectors $X^{(a)}$ and $X^{*(a)}$:

$$\delta K = (\varepsilon^{(a)}X^{(a)} + \varepsilon^{*(a)}X^{*(a)})K. \tag{D.18}$$

Note that we have used a complex parameter $\varepsilon^{(a)}$, and that the hermitian nature of the Kähler potential is preserved. It is straightforward to show that (D.18) can be rewritten as follows:

$$\delta K = \varepsilon^{(a)}F^{(a)} + \varepsilon^{*(a)}F^{*(a)} - i(\varepsilon^{(a)} - \varepsilon^{*(a)})D^{(a)}, \tag{D.19}$$

where the $F^{(a)} = X^{(a)}K + iD^{(a)}$ are analytic functions of the coordinates,

$$\frac{\partial F^{(a)}}{\partial a^{*j}} = g_{ij^*}X^{i(a)} + i\frac{\partial D^{(a)}}{\partial a^{*j}} = 0, \tag{D.20}$$

and we have used (D.14). For real parameters $\varepsilon^{(a)}$, (D.19) reduces to a Kähler transformation. For complex parameters, however, it is not of Kähler form; there is a change in K proportional to the Killing potential $D^{(a)}$. In Chapter XXIV this plays an important role in the construction of gauge-invariant actions.

References

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Exercises

(1) Show that the Lie bracket of two Killing vectors gives another.

(2) Demonstrate that the first equation in (D.13) is automatically satisfied on a Kähler manifold, and that the second is locally equivalent to (D.14).

(3) Prove that the Killing potentials can always be chosen to satisfy (D.17). This can be done by first differentiating the left-hand side with respect to a^i, and then using the relations introduced above to obtain the a^i derivative of the right-hand side of (D.17). The proof can be completed by repeating the procedure, this time differentiating with respect to a^{*i}.

(4) Show that

$$X^{i(a)} \frac{\partial}{\partial a^i} D^{(b)} + X^{*i(b)} \frac{\partial}{\partial a^{*i}} D^{(a)} = 0.$$

(5) Consider the manifold with Kähler potential $K = a^{*i}a^i$. Verify that the differential operators

$$X^{(a)} = -ia^j T^{(a)k}{}_j \frac{\partial}{\partial a^k}$$

$$X^{*(a)} = ia^{*j} T^{(a)j}{}_k \frac{\partial}{\partial a^{*k}}$$

are indeed Killing vectors, where the $T^{(a)k}{}_j$ are given in (7.14). Show that their Lie brackets close into (D.16).

(6) Given the Kähler potential $K = \log(1 + aa^*)$, and the Killing potentials

$$D^{(1)} = \frac{1}{2}\frac{a + a^*}{(1 + a^*a)},\ D^{(2)} = -\frac{i}{2}\frac{a - a^*}{(1 + a^*a)},\ D^{(3)} = -\frac{1}{2}\left(\frac{1 - a^*a}{1 + a^*a}\right),$$

find the Killing vectors $X^{(a)}$ using (D.15). Compute their commutators and identify the isometry group G.

APPENDIX E
NONLINEAR REALIZATIONS

Nonlinear realizations play an important role in theories with spontaneously broken symmetries. They were first studied in the context of chiral dynamics, where they were used to describe the pion and its interactions. They can also be applied to theories with spontaneously broken supersymmetry, where they are used to derive low-energy theorems for the Goldstone fermion. In this appendix we will develop the necessary formalism for the case of compact, connected, semisimple Lie groups. This will serve as a guide for our study of spontaneously broken supersymmetry, where similar results can be proved using different techniques.

We start by assuming that we have a manifold $\mathscr{M}$ and a group G of transformations that act on $\mathscr{M}$,

$$x' = g \cdot x, \tag{E.1}$$

where $g \in G$, and x, x' are points of $\mathscr{M}$. These transformations induce a *realization* of G on the coordinates in each neighborhood of $\mathscr{M}$. Such realizations clearly include the case of linear representations, but they also include more general realizations that cannot be reduced to linear transformations by appropriate coordinates on $\mathscr{M}$.

Given a particular realization, one would like to know whether or not it can be reduced to a linear transformation. For the case of compact, connected, semisimple Lie groups, there is a simple answer: a realization can be linearized (in a given coordinate patch) if and only if it leaves a point in the patch invariant.

Now, a linear transformation always leaves the origin invariant, so the first direction is trivial. The other direction, however, is a little less obvious. Therefore, let us assume that we have a point $x_0 \in \mathscr{M}$ that is invariant under all the transformations in G,

$$g \cdot x_0 = x_0. \tag{E.2}$$

We will explicitly construct a set of coordinates that linearize the transformation (E.1) in the neighborhood of x_0. Since x_0 is invariant, we assign it the coordinate $\vec{0}$. Away from x_0, we choose an arbitrary set of coordinates, denoting the coordinates of x by $\vec{x}$. In terms of these parameters, the transformation (E.1) has a power series expansion,

$$\vec{x}' = g \cdot \vec{x} \equiv D(g)\vec{x} + O(\vec{x}^2), \tag{E.3}$$

where $D(g)$ is a matrix expression. [In Exercise 1 you will show that $D(g)$ is a matrix representation of G.] The constant term is absent because the origin is invariant. We now introduce new coordinates $\vec{y}$ at the point x as follows:

$$\vec{y} = \int d\mu(g) D^{-1}(g) g \cdot \vec{x}. \tag{E.4}$$

The integration is over the group G, and is well defined for compact groups. The measure $d\mu(g)$ can be chosen to be left- and right-invariant,

$$d\mu(g_0 g) = d\mu(g g_0) = d\mu(g), \tag{E.5}$$

and normalized so that

$$\int d\mu(g) = 1. \tag{E.6}$$

With these conventions, it is easy to see that

$$\vec{y} = \vec{x} + O(\vec{x}^2), \tag{E.7}$$

so (E.4) is an allowed change of coordinates.

Let us now study the action of G on the coordinates $\vec{y}$. We find

$$\begin{aligned} g_0 \cdot \vec{y} &= \int d\mu(g) D^{-1}(g) g \cdot g_0 \cdot \vec{x} \\ &= \int d\mu(g g_0) D(g_0) D^{-1}(g_0) D^{-1}(g) g \cdot g_0 \cdot \vec{x} \\ &= D(g_0) \int d\mu(g) D^{-1}(g) g \cdot \vec{x} \\ &= D(g_0)\vec{y}, \end{aligned} \tag{E.8}$$

which demonstrates that the coordinates $\vec{y}$ do indeed linearize the transformation (E.1).

This construction relies heavily on the properties of group integration. Curiously enough, similar results hold even when group integration cannot be properly defined. For example, in Chapter XXVI we study the case of supersymmetry, in which a supergroup of transformations acts on superspace. The linearization condition still holds, even though the group volume is formally zero.

Given an arbitrary point $x_0 \in \mathscr{M}$, the transformations that leave the point invariant close into a group H, called the *stability group* of x_0. In general, H is a proper subgroup of G. We have just seen that the transformations in the stability group are precisely those that can be realized linearly in the neighborhood of the point x_0.

In preparation for what follows, let us now shift our attention to the submanifold $\mathscr{N}$ of $\mathscr{M}$ that can be reached by group transformations acting on the point x_0,

$$x = g \cdot x_0. \tag{E.9}$$

Clearly, the points in $\mathscr{N}$ are in one-one correspondence with the coset space G/H. This space has a natural parametrization in terms of the group parameters. An arbitrary element of G can be written in the form,

$$g = e^{-i\vec{\xi}\cdot\vec{X}} e^{-i\vec{u}\cdot\vec{T}}, \tag{E.10}$$

where the parameters $\vec{u}$ and $\vec{\xi}$ are real. In this expression, the $\vec{T}$ are the (hermitian) generators of H, while the $\vec{X}$ are the generators of G in the orthogonal complement of H. Two elements g and $g' \in G$ correspond to the same point of G/H if they are related by a *right* H transformation: $g \sim g'$ if $g = g'u'$, for some u' of the form

$$u' = e^{-i\vec{u}'\cdot\vec{T}}. \tag{E.11}$$

This implies that the cosets can be parametrized by the group elements,

$$v = e^{-i\vec{\xi}\cdot\vec{X}} \tag{E.12}$$

and that the $\vec{\xi}$ are coordinates of the space G/H.

With these conventions, an element $g_0 \in G$ acts on the cosets by *left* multiplication,

$$g_0 e^{-i\vec{\xi}\cdot\vec{X}} = e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}}. \tag{E.13}$$

The coordinates $\vec{\xi}'$ are completely determined in terms of $\vec{\xi}$ and g_0,

$$g_0\colon \vec{\xi} \to \vec{\xi}'(\vec{\xi}, g_0). \tag{E.14}$$

The parameters $\vec{u}'$ can be computed as well; they too depend on g_0 and the coset parameters $\vec{\xi}$:

$$g_0: \vec{u} \to \vec{u}'(\vec{\xi}, g_0). \tag{E.15}$$

For elements $g_0 = u_0 \in H$, the transformations (E.14) and (E.15) can always be written in closed form. Then

$$v \to u_0 v = u_0 v u_0^{-1} u_0 \equiv v'u', \tag{E.16}$$

where $v' = u_0 v u_0^{-1}$ and $u' = u_0$. In terms of the coordinates $\vec{\xi}$, this implies

$$e^{-i\vec{\xi}' \cdot \vec{X}} = u_0 e^{-i\vec{\xi} \cdot \vec{X}} u_0^{-1}, \tag{E.17}$$

where $v' = u_0 v u_0^{-1}$ and $u' = u_0$. In terms of the coordinates $\vec{\xi}$, this implies

$$u_0: \vec{\xi} \to \vec{\xi}' = D(u_0)\vec{\xi}. \tag{E.18}$$

For transformations $g_0 \in G$ that are not in H, however, Eqs. (E.14) and (E.15) cannot generally be written in closed form.

There is a special case, however, where these transformations can be made more explicit. This is when the structure relations of G admit the automorphism,

$$\begin{aligned} \vec{T} &\to \vec{T} \\ \vec{X} &\to -\vec{X}, \end{aligned} \tag{E.19}$$

in which case G/H is called a symmetric space. To see how this works, consider a transformation v_0,

$$v \to v_0 v \equiv v'u'. \tag{E.20}$$

This can be rewritten by first taking the inverse and then applying the automorphism (E.19),

$$v \to v v_0 \equiv u'^{-1} v'. \tag{E.21}$$

Combining the two expressions, we find

$$\begin{aligned} v'^2 &= v_0 v^2 v_0 \\ u' &= v'^{-1} v_0 v. \end{aligned} \tag{E.22}$$

In terms of the coordinates $\vec{\xi}$, this implies

$$e^{-2i\vec{\xi}'\cdot\vec{X}} = v_0 e^{-2i\vec{\xi}\cdot\vec{X}} v_0. \tag{E.23}$$

This is a manifestly nonlinear transformation law. Note that $\vec{\xi}$ can be transformed to zero if we take $v_0 = e^{i\vec{\xi}\cdot\vec{X}}$

We will now show that we can use these results to promote any representation of H to a realization of G, with the help of the coset parameters $\vec{\xi}$. We start with a representation $\tilde{D}$, which acts linearly on a vector space spanned by $\vec{\psi}$,

$$u_0\colon \vec{\psi} \to \tilde{D}(u_0)\vec{\psi}, \tag{E.24}$$

for $u_0 \in H$. Then, using (E.15), this transformation can immediately be extended to a realization of G:

$$g_0\colon \vec{\psi} \to \tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\vec{\psi}. \tag{E.25}$$

The variables $\vec{u}'$ parametrize an element of H, but they are functions of $\vec{\xi}$ and g_0. To show that (E.25) is indeed a realization of G, we compute

$$\begin{aligned} g_1 e^{-i\vec{\xi}\cdot\vec{X}} &= e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}} \\ g_2 e^{-i\vec{\xi}'\cdot\vec{X}} &= e^{-i\vec{\xi}''\cdot\vec{X}} e^{-i\vec{u}''\cdot\vec{T}} \\ g_2 g_1 e^{-i\vec{\xi}\cdot\vec{X}} &= e^{-i\vec{\xi}'''\cdot\vec{X}} e^{-i\vec{u}'''\cdot\vec{T}} \\ &= e^{-i\vec{\xi}''\vec{X}} e^{-i\vec{u}''\cdot\vec{T}} e^{-i\vec{u}'\cdot\vec{T}}. \end{aligned} \tag{E.26}$$

From this we see that

$$e^{-i\vec{u}'''\cdot\vec{T}} = e^{-i\vec{u}''\cdot\vec{T}} e^{-i\vec{u}'\cdot\vec{T}}, \tag{E.27}$$

which implies

$$\tilde{D}(e^{-i\vec{u}'''\cdot\vec{T}}) = \tilde{D}(e^{-i\vec{u}''\cdot\vec{T}})\tilde{D}(e^{-i\vec{u}'\cdot\vec{T}}), \tag{E.28}$$

since $\tilde{D}$ is a representation of H. In this way we can realize the group G on the space spanned by the vectors $\vec{\psi}$.

The transformation (E.25) plays a special role in the study of nonlinear realizations. It defines what is known as a *standard realization* of the group G. The realization is standard because *any* realization of G that linearizes on H can be reduced to this form with the help of the coset parameters $\vec{\xi}$.

To see this, we assume as before that we have a manifold $\mathscr{M}$ and a group G that acts on $\mathscr{M}$ as a group of transformations. We also assume that we can choose coordinates $(\vec{\xi},\vec{\chi})$ in some neighborhood $\mathscr{U}$ of $\mathscr{M}$, where the coordinates $\vec{\xi}$ parametrize the points in $\mathscr{U}$ that can be reached from $(\vec{0},\vec{\chi})$ by the action of G. Because of the construction (E.8), the parameters $\vec{\chi}$ can be chosen to transform linearly under H. The transformations of the $\vec{\xi}$ are completely determined by (E.14). Therefore, under an H-transformation, we have

$$u_0 \cdot (\vec{\xi},\vec{\chi}) = (D(u_0)\vec{\xi},\tilde{D}(u_0)\vec{\chi}), \tag{E.29}$$

where $\vec{\xi}$ and $\vec{\chi}$ transform in the representations D and $\tilde{D}$, respectively.

Now, among the full set of G transformations, there is one that transforms $\vec{\xi}$ to zero,

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\chi}) = (\vec{0},\vec{\psi}). \tag{E.30}$$

This transformation also takes $\vec{\chi}$ to $\vec{\psi}$, which can be computed because we know the action of G on the manifold $\mathscr{M}$. The parameters $\vec{\psi}$ transform in the representation $\tilde{D}$ under H, as follows from (E.29).

We shall now construct a new coordinate system on $\mathscr{M}$ as follows. Start at a point x, parametrized by the coordinates $(\vec{\xi},\vec{\chi})$, and map it to the point $(\vec{0},\vec{\psi})$ as in (E.30):

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\chi}) = (\vec{0},\vec{\psi}). \tag{E.31}$$

Then take the new coordinates at the original point x to be given by $(\vec{\xi},\vec{\psi})$. This defines an acceptable coordinate transformation on the manifold $\mathscr{M}$ because the Jacobian of the transformation $(\vec{\xi},\vec{\chi}) \to (\vec{\xi},\vec{\psi})$ is nonvanishing near the origin. In terms of the new coordinates, the transformation (E.31) can be written as $e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\psi}) = (\vec{0},\vec{\psi})$. This allows us to show that the new coordinates $\vec{\psi}$ transform as a standard realization:

$$\begin{aligned} g \cdot (\vec{\xi},\vec{\psi}) &= g e^{-i\vec{\xi}\cdot\vec{X}} \cdot (\vec{0},\vec{\psi}) \\ &= e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}} \cdot (\vec{0},\vec{\psi}) \\ &= e^{-i\vec{\xi}'\cdot\vec{X}} \cdot (\vec{0},\tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\vec{\psi}) \\ &= (\vec{\xi}',\tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\vec{\psi}). \end{aligned} \tag{E.32}$$

Together with $\vec{\xi}$, they are the natural coordinates on $\mathscr{M}$ adapted to the action of G.

In physical applications, the coordinates $\vec{\xi}$ and $\vec{\psi}$ are x^m-dependent fields. The coset coordinates $\vec{\xi}$ play the role of the Goldstone bosons that arise from spontaneously breaking G to H. The standard realizations $\vec{\psi}$ describe the other fields that transform in representations of the unbroken group H. In this appendix, we have seen that any representation of H can be extended to a realization of G with the help of the Goldstone bosons $\vec{\xi}$.

To write down invariant Lagrangians we would like to have covariant derivatives that transform as standard realizations. Our general arguments tell us that such derivatives must exist. Constructing them provides a straightforward application of what we have just learned, as well as a nice illustration.

To find the covariant derivatives, we start from the manifold parametrized by $(\vec{\xi},\vec{\psi},\partial_m\vec{\xi},\partial_m\vec{\psi})$. As above, we apply a group transformation with $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$. This gives

$$e^{i\vec{\xi}\cdot\vec{X}}\cdot(\vec{\xi},\vec{\psi},\partial_m\vec{\xi},\partial_m\vec{\psi}) \equiv (\vec{0},\vec{\psi},\Delta_m\vec{\xi},\Delta_m\vec{\psi}), \tag{E.33}$$

and from our general prescription we know that $\Delta_m\vec{\xi}$ and $\Delta_m\vec{\psi}$ are covariant derivatives that transform as standard realizations.

To compute $\Delta_m\vec{\xi}$, we start from the formula (E.13),

$$g_0 e^{-i\vec{\xi}\cdot\vec{X}} = e^{-i\vec{\xi}'\cdot\vec{X}}e^{-i\vec{u}'\cdot\vec{T}}, \tag{E.34}$$

and differentiate with respect to x^m,

$$g_0\partial_m e^{-i\vec{\xi}\cdot\vec{X}} = (\partial_m e^{-i\vec{\xi}'\cdot\vec{X}})e^{-i\vec{u}'\cdot\vec{T}} + e^{-i\vec{\xi}'\cdot\vec{X}}(\partial_m e^{-i\vec{u}'\cdot\vec{T}}). \tag{E.35}$$

As before, the parameters $\vec{\xi}'$ and $\vec{u}'$ depend on x^m through $\vec{\xi}$. We now choose $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$, which transforms $\vec{\xi}'$ and $\vec{u}'$ to zero at the point x^m. This gives

$$\begin{aligned} e^{i\vec{\xi}\cdot\vec{X}}\,\partial_m e^{-i\vec{\xi}\cdot\vec{X}} &= \partial_m e^{-i\vec{\xi}'\cdot\vec{X}} + \partial_m e^{-i\vec{u}'\cdot\vec{T}}\big|_{\vec{\xi}'=\vec{u}'=0} \\ &= i\,\partial_m\vec{\xi}'\cdot\vec{X} - i\,\partial_m\vec{u}'\cdot\vec{T}\big|_{\vec{\xi}'=\vec{u}'=0} \\ &\equiv -i\,\Delta_m\vec{\xi}\cdot\vec{X} - i\vec{V}_m\cdot\vec{T}. \end{aligned} \tag{E.36}$$

Equation (E.36) allows us to compute $\Delta_m\vec{\xi}$ as a function of the parameters $\vec{\xi}$. In Exercise 3 we will see that $\Delta_m\vec{\xi}$ indeed transforms as a standard realization.

Similar techniques can be used to find $\Delta_m\vec{\psi}$. One starts by differentiating (E.25),

$$g_0\cdot\partial_m\vec{\psi} = \partial_m\tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\vec{\psi} + \tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\,\partial_m\vec{\psi}. \tag{E.37}$$

As above, one then takes $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$, to find

$$e^{i\vec{\xi}\cdot\vec{X}}\cdot\partial_m\vec{\psi} = -i(\partial_m\vec{u}'\cdot\vec{T})\vec{\psi} + \partial_m\vec{\psi}|_{\vec{\xi}'=\vec{u}'=0}. \tag{E.38}$$

Comparing (E.38) with (E.36), we find

$$\Delta_m\vec{\psi} = \partial_m\vec{\psi} - i(\vec{V}_m\cdot\vec{T})\vec{\psi}. \tag{E.39}$$

In Exercise 4 one is asked to show that $\Delta_m\vec{\psi}$ transforms as a standard realization.

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Exercises

(1) Show that the matrices D, defined in (E.3), form a representation of the group G.

(2) Demonstrate that $\Delta_m\vec{\psi}$ transforms as a standard realization. Start by eliminating g_0 between (E.34) and (E.35):

$$e^{-i\vec{\xi}'\cdot\vec{X}}e^{-i\vec{u}'\cdot\vec{T}}e^{i\vec{\xi}\cdot\vec{X}}(\partial_m e^{-i\vec{\xi}\cdot\vec{X}}) = (\partial_m e^{-i\vec{\xi}'\cdot\vec{X}})e^{-i\vec{u}'\cdot\vec{T}} + e^{-i\vec{\xi}'\cdot\vec{X}}(\partial_m e^{-i\vec{u}'\cdot\vec{T}}).$$

Then multiply on the left by $e^{i\vec{\xi}'\cdot\vec{X}}$ and $e^{i\vec{u}'\cdot\vec{T}}$, to find

$$e^{i\vec{\xi}\cdot\vec{X}}\partial_m e^{-i\vec{\xi}\cdot\vec{X}} = e^{i\vec{u}'\cdot\vec{T}}(e^{i\vec{\xi}'\cdot\vec{X}}\partial_m e^{-i\vec{\xi}'\cdot\vec{X}})e^{-i\vec{u}'\cdot\vec{T}} + e^{i\vec{u}'\cdot\vec{T}}\partial_m e^{-i\vec{u}'\cdot\vec{T}}.$$

This shows that

$$(\Delta_m\vec{\xi}\cdot\vec{X})' = e^{-i\vec{u}'\cdot\vec{T}}(\Delta_m\vec{\xi}\cdot\vec{X})e^{i\vec{u}'\cdot\vec{T}},$$

and

$$(\vec{V}_m\cdot\vec{T})' = e^{-i\vec{u}'\cdot\vec{T}}(\vec{V}_m\cdot\vec{T})e^{i\vec{u}'\cdot\vec{T}} + e^{-i\vec{u}'\cdot\vec{T}}\partial_m e^{i\vec{u}'\cdot\vec{T}}.$$

(4) Use the transformation law for $\vec{V}_m$ to show that $\Delta_m\vec{\psi}$ transforms as a standard realization.

APPENDIX F
NONLINEAR REALIZATIONS AND INVARIANT ACTIONS

In this appendix we will continue our study of nonlinear realizations. We will use the methods introduced in Appendix E to show that an action invariant under a group H can be promoted to a new action invariant under a larger group $G \supset H$. The results derived here are used in Chapter XXIV to construct the gauge invariant matter couplings in superspace.

We start by assuming we have a Lagrangian $\mathscr{L}_H$ which is a function of certain fields A^i. The Lagrangian is invariant under a symmetry group H. The fields A^i are arbitrary, except that they have well-defined transformations under a group $G \supset H$. The Lagrangian $\mathscr{L}_H$, however, is not invariant under the full group G. Instead, it has a variation $\delta\mathscr{L}_H \neq 0$.

In this appendix we will construct a counterterm $\mathscr{L}_{CT}$ whose variation precisely cancels that of $\mathscr{L}_H$. We will build the counterterm out of the fields A^i, together with fields $\xi^{(\alpha)}$ that parametrize the coset G/H. We impose the condition that $\mathscr{L}_{CT}$ must vanish when $\xi^{(\alpha)} = 0$. In this way the Lagrangian

$$\mathscr{L}_G = \mathscr{L}_H + \mathscr{L}_{CT} \tag{F.1}$$

is invariant under the full group G, and reduces to $\mathscr{L}_H$ for $\xi^{(\alpha)} = 0$.

As in Appendix E, let us split the transformations in G into two classes, those in H and those not. Under a transformation $u_0 \in H$, the Lagrangian $\mathscr{L}_H$ is assumed to be invariant:

$$\delta_H\mathscr{L}_H \equiv -iu_0^{(a)}\hat{T}^{(a)}\mathscr{L}_H = 0, \tag{F.2}$$

where the $\hat{T}^{(a)}$ are differential operators that act on the fields A^i and generate the transformations in H. Under a transformation $v_0 \in G$, $\mathscr{L}_H$ has an infinitesimal variation of the form

$$\delta_{G/H}\mathscr{L}_H \equiv -iv_0^{(\alpha)}\hat{X}^{(\alpha)}\mathscr{L}_H \equiv -v_0^{(\alpha)}D^{(\alpha)}, \tag{F.3}$$

where the operators $\hat{X}^{(\alpha)}$ generate the transformations in G that are in the orthogonal complement of H. We see that we need to find a function

$\mathscr{L}_{CT}(A^i,\xi^{(\alpha)})$, such that

$$
\begin{aligned}
u_0^{(a)}\hat{T}^{(a)}\mathscr{L}_{CT} &= 0 \\
v_0^{(\alpha)}\hat{X}^{(\alpha)}\mathscr{L}_{CT} &= iv_0^{(\alpha)}D^{(\alpha)},
\end{aligned}
\tag{F.4}
$$

subject to the boundary condition

$$\mathscr{L}_{CT}(A^i,0) = 0. \tag{F.5}$$

In these expressions, the operators $\hat{T}^{(a)}$ and $\hat{X}^{(\alpha)}$ act on the fields A^i and on the parameters $\xi^{(\alpha)}$ in the counterterm Lagrangian.

We shall now find $\mathscr{L}_{CT}$ as follows. We first compute

$$(-iv_0^{(\alpha)}\hat{X}^{(\alpha)})^n\mathscr{L}_{CT} = (-iv_0^{(\alpha)}\hat{X}^{(\alpha)})^{n-1}(v_0^{(\gamma)}D^{(\gamma)}). \tag{F.6}$$

This can be exponentiated to give

$$e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}}\mathscr{L}_{CT} = \mathscr{L}_{CT} + \frac{e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}} - 1}{-iv_0^{(\beta)}\hat{X}^{(\beta)}}(v_0^{(\gamma)}D^{(\gamma)}), \tag{F.7}$$

where, on the right-hand side, the differential operators $\hat{X}^{(\alpha)}$ reduce to operators $\delta^{(\alpha)}A_i(\delta/\delta A_i)$ because the $D^{(\alpha)}$ do not contain the fields $\xi^{(\alpha)}$. We can now solve for $\mathscr{L}_{CT}$ by noting that $e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}}$ transforms $\mathscr{L}_{CT}$ with parameter $v_0^{(\alpha)}$. In Appendix E we showed that such a transformation with parameter $v_0^{(\alpha)} = -\xi^{(\alpha)}$ maps $\xi^{(\alpha)}$ to zero. Therefore, in conjunction with the boundary condition (F.5), this implies

$$
\begin{aligned}
\mathscr{L}_{CT} &= \frac{e^{i\xi^{(\alpha)}\hat{X}^{(\alpha)}} - 1}{i\xi^{(\beta)}\hat{X}^{(\beta)}}\,\xi^{(\gamma)}D^{(\gamma)} \\
&= \int_0^1 d\alpha\,\exp(i\alpha\xi^{(\alpha)}\hat{X}^{(\alpha)})\xi^{(\gamma)}D^{(\gamma)},
\end{aligned}
\tag{F.8}
$$

where the derivatives in $\hat{X}^{(\alpha)}$ do not act on the fields $\xi^{(\gamma)}$. It is a useful exercise to check that (F.8) satisfies (F.4), following the steps outlined in Exercises 2 and 3.

References

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Exercises

(1) Exponentiate (F.6) to find (F.7).

(2) Derive the conditions on $D^{(\alpha)}$ that follow from applying the group commutators on $\mathscr{L}_{CT}$,

$$\begin{aligned} \hat{X}^{(\alpha)}D^{(\beta)} - \hat{X}^{(\beta)}D^{(\alpha)} &= if^{\alpha\beta\gamma}D^{(\gamma)} \\ \hat{T}^{(a)}D^{(\beta)} &= if^{a\beta\gamma}D^{(\gamma)}. \end{aligned}$$

(3) Use the relations of Exercise 2 to show that (F.8) obeys (F.4).

APPENDIX G
GAUGE INVARIANT SUPERGRAVITY MODELS

In this Appendix, we will write the most general gauge invariant supergravity model in terms of component fields. We start with the superspace Lagrangian, as given in Chapter XXV:

$$\mathscr{L} = \int d^2\Theta\, 2\mathscr{E}\left[\frac{3}{8}(\bar{\mathscr{D}}\bar{\mathscr{D}} - 8R)\exp\left\{-\frac{1}{3}\left[K(\Phi,\Phi^+) + \Gamma(\Phi,\Phi^+,V)\right]\right\}\right.$$
$$\left. + \frac{1}{16g^2}H_{(ab)}(\Phi)W^{(a)}W^{(b)} + P(\Phi)\right] + \text{h.c.} \tag{G.1}$$

Then, using the techniques developed in Chapters XXI through XXV, we expand this Lagrangian in terms of component fields. This gives

$$\begin{aligned}
\mathscr{L} = & -\frac{1}{2}e\mathscr{R} - eg_{ij*}\tilde{\mathscr{D}}_m A^i \tilde{\mathscr{D}}^m A^{*j} - \frac{1}{2}eg^2 D_{(a)}D^{(a)} \\
& - ieg_{ij*}\bar{\chi}^j\bar{\sigma}^m\tilde{\mathscr{D}}_m\chi^i + e\varepsilon^{k\ell mn}\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathscr{D}}_m\psi_n \\
& - \frac{1}{4}eh^R{}_{(ab)}F_{mn}{}^{(a)}F^{mn(b)} + \frac{1}{8}eh^I{}_{(ab)}\varepsilon^{mnk\ell}F_{mn}{}^{(a)}F_{k\ell}{}^{(b)} \\
& - \frac{i}{2}e[\lambda_{(a)}\sigma^m\tilde{\mathscr{D}}_m\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^m\tilde{\mathscr{D}}_m\lambda^{(a)}] + \frac{1}{2}h^I{}_{(ab)}\tilde{\mathscr{D}}_m[e\lambda^{(a)}\sigma^m\bar{\lambda}^{(b)}] \\
& + \sqrt{2}egg_{ij*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{i}{4}\sqrt{2}eg\,\partial_i h_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{i}{4}\sqrt{2}eg\,\partial_{i*}h^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\,\partial_i h_{(ab)}\chi^i\sigma^{mn}\lambda^{(a)}F_{mn}{}^{(b)} - \frac{1}{4}\sqrt{2}e\,\partial_{i*}h^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{mn}\bar{\lambda}^{(a)}F_{mn}{}^{(b)} \\
& - \frac{1}{2}egD_{(a)}\psi_m\sigma^m\bar{\lambda}^{(a)} + \frac{1}{2}egD_{(a)}\bar{\psi}_m\bar{\sigma}^m\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij*}\tilde{\mathscr{D}}_n A^{*j}\chi^i\sigma^m\bar{\sigma}^n\psi_m - \frac{1}{2}\sqrt{2}eg_{ij*}\tilde{\mathscr{D}}_n A^i\bar{\chi}^j\bar{\sigma}^m\sigma^n\bar{\psi}_m
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{i}{4} e[\psi_m \sigma^{ab} \sigma^m \bar{\lambda}_{(a)} + \bar{\psi}_m \bar{\sigma}^{ab} \bar{\sigma}^m \lambda_{(a)}][F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)}] \\
&+ \frac{1}{4} e g_{ij^*}[i\varepsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m] \chi^i \sigma_n \bar{\chi}^j \\
&- \frac{1}{8} e[g_{ij^*} g_{k\ell^*} - 2R_{ij^*k\ell^*}] \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\
&+ \frac{1}{16} e[2g_{ij^*} h^R{}_{(ab)} + h^{R(cd)-1}\, \partial_i h_{(bc)}\, \partial_{j^*} h^*_{(ad)}] \bar{\chi}^j \bar{\sigma}^m \chi^i \bar{\lambda}^{(a)} \bar{\sigma}_m \lambda^{(b)} \\
&+ \frac{1}{8} e \nabla_i\, \partial_j h_{(ab)} \chi^i \chi^j \lambda^{(a)} \lambda^{(b)} + \frac{1}{8} e \nabla_{i^*}\, \partial_{j^*} h^*_{(ab)} \bar{\chi}^i \bar{\chi}^j \bar{\lambda}^{(a)} \bar{\lambda}^{(b)} \\
&+ \frac{1}{16} e h^{R(cd)-1}\, \partial_i h_{(ac)}\, \partial_j h_{(bd)} \chi^i \lambda^{(a)} \chi^j \lambda^{(b)} \\
&+ \frac{1}{16} e h^{R(cd)-1}\, \partial_{i^*} h^*_{(ac)}\, \partial_{j^*} h^*_{(bd)} \bar{\chi}^i \bar{\lambda}^{(a)} \bar{\chi}^j \bar{\lambda}^{(b)} \\
&- \frac{1}{16} e g^{ij^*}\, \partial_i h_{(ab)}\, \partial_{j^*} h^*_{(cd)} \lambda^{(a)} \lambda^{(b)} \bar{\lambda}^{(c)} \bar{\lambda}^{(d)} \\
&- \frac{3}{16} e \lambda_{(a)} \sigma^m \bar{\lambda}^{(a)} \lambda_{(b)} \sigma_m \bar{\lambda}^{(b)} \\
&+ \frac{i}{4} \sqrt{2} e\, \partial_i h_{(ab)} \left[\chi^i \sigma^{mn} \lambda^{(a)} \psi_m \sigma_n \bar{\lambda}^{(b)} + \frac{1}{4} \bar{\psi}_m \bar{\sigma}^m \chi^i \lambda^{(a)} \lambda^{(b)} \right] \\
&+ \frac{i}{4} \sqrt{2} e\, \partial_{i^*} h^*_{(ab)} \left[\bar{\chi}^i \bar{\sigma}^{mn} \bar{\lambda}^{(a)} \bar{\psi}_m \bar{\sigma}_n \lambda^{(b)} + \frac{1}{4} \psi_m \sigma^m \bar{\chi}^i \bar{\lambda}^{(a)} \bar{\lambda}^{(b)} \right] \\
&- e \exp(K/2) \Big\{ P^* \psi_a \sigma^{ab} \psi_b + P \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
&+ \frac{i}{2} \sqrt{2} D_i P \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} D_{i^*} P^* \bar{\chi}^i \bar{\sigma}^a \psi_a \\
&+ \frac{1}{2} \mathscr{D}_i D_j P \chi^i \chi^j + \frac{1}{2} \mathscr{D}_{i^*} D_{j^*} P^* \bar{\chi}^i \bar{\chi}^j \\
&- \frac{1}{4} g^{ij^*} D_{j^*} P^*\, \partial_i h_{(ab)} \lambda^{(a)} \lambda^{(b)} - \frac{1}{4} g^{ij^*} D_i P\, \partial_{j^*} h^*_{(ab)} \bar{\lambda}^{(a)} \bar{\lambda}^{(b)} \Big\} \\
&- e \exp(K)[g^{ij^*}(D_i P)(D_j P)^* - 3P^* P], \qquad \text{(G.2)}
\end{aligned}
$$

where $h^R{}_{(ab)} = \mathrm{Re}\, H_{(ab)}|$ and $h^I{}_{(ab)} = \mathrm{Im}\, H_{(ab)}|$. The covariant derivatives are given by

$$
\begin{aligned}
\tilde{\mathscr{D}}_m A^i &= \partial_m A^i - g v_m{}^{(a)} X^i_{(a)} \\
\tilde{\mathscr{D}}_m \chi^i &= \partial_m \chi^i + \chi^i \omega_m + \Gamma^i_{jk} \tilde{\mathscr{D}}_m A^j \chi^k - g v_m{}^{(a)} \frac{\partial X^i_{(a)}}{A^j} \chi^j \\
&\qquad - \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j*} \tilde{\mathscr{D}}_m A^{*j}) \chi^i - \frac{i}{2} g v_m{}^{(a)} \,\mathrm{Im}\, F_{(a)} \chi^i \\
\tilde{\mathscr{D}}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} + \lambda^{(a)} \omega_m - g f^{abc} v_m{}^{(b)} \lambda^{(c)} \\
&\qquad + \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j*} \tilde{\mathscr{D}}_m A^{*j}) \lambda^{(a)} + \frac{i}{2} g v_m{}^{(b)} \,\mathrm{Im}\, F_{(b)} \lambda^{(a)} \\
\tilde{\mathscr{D}}_m \psi_n &= \partial_m \psi_n + \psi_n \omega_m \\
&\qquad + \frac{1}{4}(K_j \tilde{\mathscr{D}}_m A^j - K_{j*} \tilde{\mathscr{D}}_m A^{*j}) \psi_n + \frac{i}{2} g v_m{}^{(a)} \,\mathrm{Im}\, F_{(a)} \psi_n \\
D_i P &= P_i + K_i P \\
\mathscr{D}_i D_j P &= P_{ij} + K_{ij} P + K_i D_j P + K_j D_i P - K_i K_j P - \Gamma^k_{ij} D_k P. \qquad \text{(G.3)}
\end{aligned}
$$

In these expressions, the fields in the vector multiplet are defined to have *upper* gauge indices, such as $v_m{}^{(a)}$ and $\lambda^{(a)}$. The Killing vectors and Killing potentials have *lower* gauge indices, $X^i_{(a)}$ and $D_{(a)}$. These indices can be raised and lowered with $h^R{}_{(ab)}$ and its inverse. Using these conventions, one can check that the Lagrangian (G.2) is invariant under the following set of supergravity transformations:

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta \sigma^a \bar{\psi}_m + \bar{\zeta} \bar{\sigma}^a \psi_m) \\
\delta_\zeta A^i &= \sqrt{2} \zeta \chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2} \sigma^m \bar{\zeta} \hat{\tilde{\mathscr{D}}}_m A^i - \Gamma^i_{jk} \delta_\zeta A^j \chi^k \\
&\qquad + \frac{1}{4}(K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j}) \chi^i - \sqrt{2} e^{K/2} g^{ij*} D_{j*} P^* \zeta \\
&\qquad + \frac{1}{4} \sqrt{2} \zeta g^{ij*} \partial_{j*} h^*_{(ab)} \bar{\lambda}^{(a)} \bar{\lambda}^{(b)} \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta \sigma_m \bar{\lambda}^{(a)} + \bar{\zeta} \bar{\sigma}_m \lambda^{(a)})
\end{aligned}
$$

$$\delta_\zeta \lambda^{(a)} = \hat{F}_{ab}{}^{(a)}\sigma^{ab}\zeta - \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\lambda^{(a)} - igD^{(a)}\zeta$$

$$+ \frac{1}{4}\sqrt{2}\zeta h^{R(ab)-1}\,\partial_i h_{(bc)}\chi^i\lambda^{(c)} - \frac{1}{4}\sqrt{2}\zeta h^{R(ab)-1}\,\partial_{i*}h^*_{(bc)}\bar{\chi}^i\bar{\lambda}^{(c)}$$

$$\delta_\zeta \psi_m = 2\tilde{\mathscr{D}}_m\zeta - \frac{i}{2}\sigma_{mn}\zeta\, g_{ij*}\chi^i\sigma^n\bar{\chi}^j + \frac{i}{2}(g_{mn} + \sigma_{mn})\zeta\,\lambda_{(a)}\sigma^n\bar{\lambda}^{(a)}$$

$$- \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\psi_m + ie^{K/2}P\sigma_m\bar{\zeta}. \tag{G.4}$$

The Lagrangian (G.2) is the starting point for phenomenological studies of supergravity theories.

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