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Karl H. Hofmann, Sidney A. Morris

THE STRUCTURE OF COMPACT GROUPS

A PRIMER FOR THE STUDENT - A HANDBOOK FOR THE EXPERT

STUDIES IN MATHEMATICS 25

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This book is dedicated to

Isolde Hofmann and Elizabeth Morris

Preface to the Fourth Edition

In this preface we intend to give the reader a glimpse of those portions and aspects of this book which have been added to its existing body in this present, fourth edition. However, the reader wishing to get an impression of the spirit permeating the book as a whole should also peruse the prefaces of the first, second and third edition. Indeed it might be a good idea to read them in this order. The first edition of this book appeared in 1998, the second in 2006 and the third in 2013. All three were well received by reviewers and frequently quoted by researchers in a variety of areas. Many developments happened in the 20 years since the appearance of the first edition. The present one includes new material amounting to doubling of the size of one of the previous chapters and to the addition of one new appendix. Also there are numerous augmentations clear across the text.

Nothing of the body of the text that accumulated through the years was ever eliminated. In the preface to the second and third edition we have already emphasized an important characteristic of the book and reiterate it now: the original internal numbering system has been retained through all editions. Accordingly all citations of items identified by the internal numbers in any of the previous editions remain intact throughout.

In the past, the Tannaka Duality Theorem for compact groups was not included. Edwin Hewitt and Kenneth Ross [148] referred in 1969 to this result as presented by Tannaka and Krein by writing "Although these theorems were published in 1938 and 1949, respectively, mathematicians have used them very little, and they have not contributed to harmonic analysis on compact non-Abelian groups as the Pontryagin-van Kampen theorem has done for LCA groups." Fifty years later we can say that they have not contributed to the knowledge on the structure of compact groups—the subject of this book, and that this was primarily the reason why we did not include Tannaka Duality in the earlier editions. So why do we include it now?

In the new Appendix 7 we focus on the class $\mathcal{V}$ of vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and on the class $\mathcal{W}$ of weakly complete topological $\mathbb{K}$-vector spaces. Here a topological $\mathbb{K}$-vector space is called *weakly complete* if and only if, as a topological $\mathbb{K}$-vector space, it is isomorphic to $\mathbb{K}^J$, for some set J. These two categories possess two features that deserve to be mentioned at once: Firstly, the dual $V^* = \mathrm{Hom}_{\mathcal{V}}(V, \mathbb{K})$ of a $\mathcal{V}$-object V is a $\mathcal{W}$-object, and the dual $W' = \mathrm{Hom}_{\mathcal{W}}(W, \mathbb{K})$ of a $\mathcal{W}$-object is a $\mathcal{V}$-object and, moreover, there are natural isomorphisms $V \cong (V^*)'$ and $W \cong (W')^*$ establishing a rather elementary duality between $\mathcal{V}$ and $\mathcal{W}$. For $\mathbb{K} = \mathbb{R}$, we have also isomorphisms $\mathrm{Hom}_{\mathcal{V}}(V, \mathbb{R}) \cong \mathrm{Hom}(V, \mathbb{R}/\mathbb{Z}) = \widehat{V}$ and $\mathrm{Hom}_{\mathcal{W}}(W, \mathbb{R}) \cong \mathrm{Hom}(W, \mathbb{R}/\mathbb{Z}) = \widehat{W}$, so that the duality between $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{R}$ is a part of the Pontryagin Duality as we shall discuss in Appendix 7. Indeed their

Pontryagin Duality had been the subject of Chapter 7 already in all preceding editions.

The second noteworthy feature of the dual categories $\mathcal{V}$ and $\mathcal{W}$ is the compatibility of their respective tensor products with duality: $(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^*$ (and dually). These tensor products make $\mathcal{V}$ and $\mathcal{W}$ what in Appendix 3, following a general practice, we call *symmetric monoidal categories*. Therefore both of them have algebras and Hopf algebras.

In particular, we introduce in a natural fashion the class of weakly complete topological algebras A and show that for each of them the set of invertible elements A^{-1} is a topological group, indeed a pro-Lie group. So there is a functor $A \mapsto A^{-1}$ from the category of all weakly complete topological algebras to the category of all topological groups, each with their morphisms. The Adjoint Functor Existence Theorem applies and secures the existence of a left adjoint $\mathbb{K}[-]$ to this functor, which takes a topological group G to the weakly compact topological algebra $\mathbb{K}[G]$. This leads us to study in the new Part 3 of Chapter 3 the *group algebras of compact groups*. If G is a compact group, then the isomorphic copy of G in $\mathbb{R}[G]^{-1}$ can be characterized in terms of the Hopf algebra structure of $\mathbb{R}[G]$. Indeed we shall be able to characterize precisely those real weakly complete symmetric cocommutative Hopf algebras which occur (up to isomorphism) as group Hopf algebras $\mathbb{R}[G]$ for compact groups G. We shall call them *compactlike*. That is, our approach yields the folllowing equivalence theorem: *There is a precise categorical equivalence between the category of compact groups and the category of weakly complete compactlike real symmetric Hopf algebras.*

The relevance of this context for the traditional theory of compact groups is this: The duality between $\mathcal{V}$ and $\mathcal{W}$ implements in a straightforward fashion a duality between weakly complete cocomplete real symmetric Hopf algebras and (abstract) commutative real symmetric Hopf algebras. The (abstract) real symmetric Hopf algebras appearing as dual objects of the weakly complete compactlike real symmetric Hopf algebras (namely, the $\mathbb{R}[G]$ with compact G) are called *reduced Hopf algebras* (following G. Hochschild in [155]). Thus the equivalence theorem above yields the following duality theorem: *The category of compact groups is dual to the category of reduced real Hopf algebras.* This is the Tannaka-Hochschild Duality Theorem. It is now filled with additional significance due to the fact which we establish in Part 3, namely, that the dual $\mathbb{R}[G]'$ of the weakly complete real symmetric group Hopf algebra of a compact group is naturally isomorphic to the real symmetric Hopf algebra $R(G, \mathbb{R})$ of representative functions of the compact group G.

Aside from the innovation regarding the weakly complete group algebras of compact groups, we implemented numerous smaller local improvements of material present in earlier editions. An example is Theorem 6.55 in which it is now clearly formulated that for a compact Lie group G every element of the commutator algebra $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of G is itself a commutator. Another significant improvement of an earlier result is Theorem A1.32 concerning the general theory of divisibility in abelian groups. This material benefitted from the developments of the recent monograph [144] by Herfort, Hofmann and Russo. Among the cardinal numbers naturally attached to mainly compact groups we expanded notably the presentation of *density.* The presentaton of these complementary results begins with the definition of the logarithm of arbitrary cardinal numbers in Definition 12.16a and finally culminates in our systematic comparison of density and weight for arbitrary compact groups in 12.31a, from which we conclude that the density of a closed subgroup of a compact group never exceeds the density of the latter. By a theorem of Itzkowitz [216], a closed subgroup of a separable compact group is separable which now emerges as a special case of the general situation. The Appendix 5 on Measures on Compact Groups is complemented by a subsection on infinite compact groups G dealing with the existence of subgroups of G failing to be measurable with respect to Haar measure of G. This issue sounds simple but leads into complications of set theory and logic.

Selected references for the Preface of the Fourth Edition

[∗] The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras, J. of Lie Theory 28 (2019), 413–455 (by Rafael Dahmen and Karl H. Hofmann).

[†] On Weakly Complete Group Algebras of Compact Groups, J. of Lie Theory **30** (2020), 407–424 (by Karl H. Hofmann and Linus Kramer).

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The first author also thanks Rafael Dahmen and Linus Kramer for their coauthoring a portion of this added material. He also thanks the Mathematisches Forschungsinstitut Oberwolfach for providing the hospitality of its Program “Research in Pairs” extended to Linus Kramer and him on several occasions.

We also thank the Editors of the Series of De Gruyter Studies in Mathematics, Niels Jacobs and Karl-Hermann Neeb for their continued encouragement to prepare this edition; in his capacity as Managing Editor of the Journal of Lie Theory the latter also rendered his support to one major augmentation of this edition. We thank Nadja Schedensack and André Horn at De Gruyter in Berlin for their constructive help in finalizing the printing of the present edition.

The authors express their gratitude to the academic institutions with which they are associated: The Technical University of Darmstadt and Tulane University for Karl Hofmann and La Trobe University and Federation University Australia for Sidney Morris.

K. H. Hofmann — Darmstadt and New Orleans
S. A. Morris — Ballarat and Melbourne
January, 2020

Preface to the Second and Third Editions

In the first edition of this book, which appeared in 1998, we endeavored to produce a self-contained exposition of the structure theory of compact groups. The focus was indeed on structure theory and not on representation theory or abstract harmonic analysis, yet the book does contain not only an introduction to, but a wealth of material on, these topics. Indeed, the first four chapters of the book provide a more than adequate introduction to the basics of representation theory of compact groups and the judiciously pedestrian style of these chapters is meant to serve graduate students. Because of our aim for the book to be self-contained it also included material such as an introduction to linear Lie groups, to abstract abelian groups, and to category theory. A prominent feature of the book is the derivation of the structure theory of arbitrary compact groups using an extension of Lie theory, unfettered by dimension restrictions. We record again our apt comments in the preface to the first edition: "It is generally believed that the approximation of arbitrary compact groups by Lie groups settles any issue on the structure of compact groups as soon as it is resolved for Lie groups. ...Our book will go a long way to shed light on this paradigm.... In fact the structure theory presented in this book, notably in Chapter 9, in most cases removes the necessity to use projective limit arguments at all. The structure theorems presented in this book are richer and often more precise than information gleaned directly from approximation arguments."

The driving forces behind the writing of the second edition, which was published in 2006, and this third edition were new material discovered by the authors and others since the first edition appeared, various questions about the structure of compact groups put to the authors by our readers over the ensuing years, and our wish to clarify some aspects of the book which we feel needed improvement. However, in writing the second and third editions we were cognizant of the fact that the book was already sizable, and our commitment to self-containment was not feasible if the book were to remain at less than a thousand pages. The second and third editions also provided us with opportunities to correct many typographical errors which inevitably occur in such a large book, and a small number of mathematical errors which were easily corrected and not of serious consequence.

The list of references has increased as we include recent publications which are most pertinent to the specific content of the book. We still do not claim completeness in these additional references any more than we did in the first edition, otherwise we would have produced an unreasonably bulky list of references.

Both the second and third editions, however, remain unchanged in one important way: The numbering system of the first edition for Definitions, Lemmas, Remarks, Propositions, Theorems, and Main Theorems remains completely intact in the subsequent editions. Therefore all earlier citations and references made to them by numbers remain valid for users of later editions. This cannot hold for references to page numbers, since, due to the augmentation of the text, they have changed. We were able to accomodate all additions organically in the text. Frequently, we added new pieces of information to existing propositions and theorems. In two cases we added a second portion to an existing theorem and numbered it Theorem 8.36bis and Theorem 9.76bis. Sometimes we added a subsection at the end of a chapter, thereby avoiding any recasting of existing numbering.

As already indicated, one of our mathematical philosophies is the emphasis and application of *Lie theory* pervading the book from Chapter 5. By that we mean a consistent use of not necessarily finite dimensional Lie algebras and the associated exponential function wherever it is feasible and advances the structural insight. In this sense we should alert readers to our publications [∗], [†], and [‡], which may be considered as an extension of the Lie theoretical aspects of this book in the direction of a wide-ranging Lie theory for the class of pro-Lie groups: Indeed the book [∗] may be considered a sequel to this book. A pro-Lie group is a limit of a projective system of finite dimensional real Lie groups in which the kernels of the bonding morphisms are not necessarily compact. If we call a topological group G *almost connected* whenever the group G/G_0 of connected components is compact, then the structure and Lie theory of pro-Lie groups covers that of locally compact almost connected groups which includes all connected locally compact groups and all compact groups. However, not all pro-Lie groups are locally compact—as even infinite products $\mathbb{R}^X$ illustrate. It is shown in [∗] that a topological group is a pro-Lie group if and only if it is isomorphic to a closed subgroup of a product of finite dimensional real Lie groups; so this example is already representative.

Judging the significance of recent developments for further research and application is tricky. However, with this reservation in mind, we mention a few among the numerous topics which were added to the content of the second and third editions.

In the second edition, we clarified the ambiguity in the common terminology surrounding the concept of a simple compact connected Lie group. In abstract group theory a simple group is a group without nonsingleton proper normal subgroups. However, in classical Lie group theory a connected Lie group is called "simple" if its Lie algebra is simple, that is, has no nonzero proper ideals. This is equivalent to saying that every nonsingleton closed proper normal subgroup is discrete or, equivalently, that it is locally isomorphic to a compact connected Lie group without nonsingleton proper *closed* normal subgroups. But is a compact group of the latter type *simple* in the sense of abstract group theory? Could it perhaps contain some nonclosed nontrival normal subgroups? The answer is that a compact group having no nontrivial closed normal subgroups has no nontrivial normal subgroups at all and thus is simple in the sense of abstract group theory. A proof is surprisingly nontrivial in so far as it requires Yamabe's Theorem that an arcwise connected subgroup of a Lie group is an analytic subgroup. We recorded these matters around Theorem 9.90. Having clarified the fact that a compact algebraically simple group is either finite or connected, we call a compact group *weakly reductive* if it is isomorphic to a cartesian product of simple compact groups. This allowed us to record also our *Countable Layer Theorem* 9.91 which says that *every compact group G contains a canonical finite or countably infinite descending sequence:*

$$G = N_0 \supset N_1 \supset N_2 \supset \cdots$$

of closed normal subgroups such that, firstly, each quotient group N_j/N_{j+1} *is strictly reductive and, secondly, that* $\bigcap_j N_j$ *is the identity component of the center of the identity component* G_0. It is quite surprising that countability should occur without the hypothesis of metrizability. The Countable Layer Theorem confirms the intuition that compact groups, no matter how large their weight is, are "wide" rather than "deep". We use the Countable Layer Theorem in our proof of Theorem 10.40 that every compact group is dyadic.

It reasonable to ask what is the probability that two randomly chosen elements commute in a compact group. A so-called FC-*group* is a group with finitely many conjugacy classes. In this third edition, from Definition 9.92 through Example 9.100 we discuss the structure of compact FC-groups. It is probably not surprising that from the viewpoint of topological groups these are "almost abelian," and what this means is said in Theorem 9.99. This gives rise to the subsequent discussion of the *commutativity degree* $d(G)$ of a compact group, that is, the probability that a pair (x, y) of randomly picked elements x and y satisfies $xy = yx$. (Thus $d(G)$ is the Haar measure in $G \times G$ of the set $\{(x, y)|xy = yx\}$.) The nature of $d(G)$ is completely clarified in Theorem 9.102. This question lies in the realm of topological combinatorial group theory.

Hilbert's Fifth Problem can be cast in the following form: Is a locally euclidean group a Lie group? This was answered affirmatively in the 1950s by Gleason, Montgomery and Zippin [263]. In 1974, Szenthe [347] formulated a much used transformation group theory extension saying, in the context of a compact group G, that a transitive action of G on a space X causes X to be a real analytic manifold provided X is locally contractible. Our Chapter 10 on actions of compact groups is now augmented by a rather detailed and important discussion of transitive actions of a compact group G on a space X. From Definition 10.60 through Corollary 10.93 we discuss what happens if X is rationally and mod 2 acyclic. Subsequently we consider the situation when X has some open subset contractible to a point in X. In this latter case, X is a compact manifold. All locally contractible spaces fall into this category. The first of these two topics emerged around 1965 in the context of compact monoid theory but has attracted renewed and recent interest by researchers indicating a need for alerting the audience of this book to this aspect of compact transformation group theory. The second consolidates Szenthe's theory. Renewed interest in this issue was kindled by Antonyan's discovery [9] of a serious gap in Szenthe's original proof. Several recent publications provide alternative proofs (see [11], [120], [172]), thereby closing the gap; our presentation in this book is close to [172], but not identical with it.

Chapter 12 on cardinality invariants of compact groups has been expanded and revised in several places so as to include Theorem 12.31 saying that for any compact group G of weight $w(G)$ and every infinite cardinal $\aleph \leq w(G)$ there is a closed subgroup $H \subseteq G$ such that $w(H) = \aleph$.

We mention, finally, that Appendix 5 and Appendix 6 were added. Appendix 5 discusses, from scratch, the compact semigroup $P(G)$ of all probability measures on a compact group G under convolution. The fact that $P(G)$ has a zero element is a classical result of Wendel, which for us secures the existence and uniqueness of Haar measure on a compact group. In previous editions that information was packed into a long drawn-out exercise in Chapter 2. So, in this one particular aspect providing a proof of the existence and uniqueness of normalized Haar measure on a compact group, the present third edition is even more self-contained than the earlier editions were.

Appendix 6 reports very briefly on a technique, which was introduced in the second edition of Pontryagin's famed book [295] on topological groups, namely, the representation of compact groups in terms of projective limits of certain well-ordered inverse systems. This technique yields itself to the application of transfinite induction and therefore has been used in several recent publications as well. In this appendix we mention, in particular, the theorem that the underlying space of every compact group is supercompact. Appendix 6 also contains the theorem that a compact group can be isomorphic to the full homeomorphism group of a completely regular Hausdorff space only if it is profinite.

Selected references for the Preface of the Third Edition

[∗] The Lie Theory of Connected Pro-Lie Groups—A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups, European Mathematical Society Publishing House, 2006, xii + 663pp.

[†] *The Structure of Almost Connected Pro-Lie Groups*, J. of Lie Theory **21** (2011), 347–383.

[‡] *Local Splitting of Locally Compact Groups and Pro-Lie Groups*, J. of Group Theory **14** (2011), 931–935.

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K. H. Hofmann — Darmstadt and New Orleans
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Preface to the First Edition

The territory of compact groups seems boundless. How vast it is we realized in the course of teaching the subject off and on for thirty years, after pursuing joint research in the area for eighteen years, and by writing this book. We cover a lot of material in it, but we remain in awe of the enormity of those topics on compact groups we felt we must leave out.

Therefore, we must indicate the drift of the contents and explain our strategy. The theme of this book is the *Structure Theory* of compact groups. One cannot speak about compact groups without dealing with representation theory, and indeed various topics which belong to abstract harmonic analysis. While this book is neither on the representation theory nor on the harmonic analysis of compact groups, it does contain in its early chapters the elements of representation theory of compact groups (and some in great generality). But a large volume of material in the research and textbook literature referring to technical aspects of representation theory and harmonic analysis of compact groups remains outside the purview of this text.

One cannot speak about compact groups without, at some time, examining Lie group and Lie algebra theory seriously. This book contains a completely self-contained introduction to linear Lie groups and a substantial body of material on compact Lie groups. Our approach is distinctive in so far as we define a linear Lie group as a particular subgroup of the multiplicative group of a Banach algebra. Compact Lie groups are recognized at an early stage as being linear Lie groups. This approach avoids the use of machinery on manifolds.

There are quite a number of excellent and accessible sources dealing with such matters as the classification of complex simple Lie algebras and, equivalently, compact simple Lie algebras; we do not have to reserve space for them here.

Two of the results on compact groups best known among the educated mathematical public are that, firstly, a compact group is a limit of compact Lie groups, and, secondly, compact Lie groups are compact groups of matrices. Of course we will prove both of these facts and use them extensively. But we hasten to point out a common misconception even among mathematicians who are reasonably well informed on the subject. It is generally believed that the approximation of arbitrary compact groups by Lie groups settles any issue on the structure of compact groups as soon as it is resolved for Lie groups. There is veracity to this legend, as most legends are founded in reality—somewhere, but this myth is far from reflecting the whole truth. Our book will go a long way to shed light on this paradigm on the structure theory of compact groups. In fact the structure theory presented in this book, notably in Chapter 9, in most cases removes the necessity to use projective limit arguments at all. The structure theorems presented in this book are richer and often more precise than information gleaned directly from approximation arguments. In this spirit we present the structure theory with a goal to be free, in the end, of all dimensional restrictions, in particular, of the manifold aspect of Lie groups.

Finally, one cannot speak about compact groups without talking about totally disconnected compact groups; according to the principle of approximating compact groups by compact Lie groups, totally disconnected groups are approximated by finite groups and are therefore also known as profinite groups. We will, by necessity, discuss this subject—even early on, but the general theory of profinite groups has a strong arithmetic flavor; this is not a main thrust of this book. In contrast with that direction we emphasize the strong interplay between the algebraic and the topological structures underlying a compact group. In a simplified fashion one might say that this is, in the first place, a book on the structure of *connected* compact groups and, in the second, on the various ways that general compact groups are composed of connected compact groups and totally disconnected ones.

The table of contents, fairly detailed as it is, serves the reader as a first vantage point for an overview of the topics covered in the book. The first four chapters are devoted to the basics and to the fundamental facts of linear representation of compact groups. After this we step back and take a fresh approach to another core piece, Chapters 5 and 6, in which we deal with the requisite Lie theory. From there on out, it is all general structure theory of compact groups without dimensional restrictions. It will emerge as one of the lead motives, that so much of the structure theory of compact groups is understood, once *commutative* compact groups are elucidated; the inner and technical reasons will emerge as we progress into the subject. But it is for this reason that we begin emphasizing the abelian group aspect from the first chapter onwards. Compact abelian groups have a territory of their own, called *duality theory*; some of it can be dealt with in the first chapter on a very elementary level—and we do that; some requires more information on characters and we shall have sufficient information in the second chapter to get, at this early stage, a proof of the PONTRYAGIN–VAN KAMPEN Duality Theorem for compact abelian groups. Yet the finer aspects of duality and a fuller exploitation is deferred to Chapters 7 and 8. There are very good reasons why, in the context of abelian topological groups, we do not restrict ourselves to compact groups but cover at least locally compact ones, and indeed develop a certain amount of duality even beyond these. The theory of compact abelian groups will lead us deeply into aspects of topology and even set theory and logic. Armed with adequate knowledge of compact abelian groups we finally deal in earnest with the structure of compact groups in Chapter 9. In Chapter 10 we broach the topic of compact transformation groups; part of this material is so basic that it could have been presented in the first chapter. However for the applications of compact transformations group theory that we need, more sophisticated results are required. We can present these only after we completed certain parts of the structure theory such as compact Lie group theory. Therefore we opt for keeping material on transformation groups in one place. The later chapters then discuss a variety of special topics pointing up additional ramifications of the structure of compact groups of large infinite dimension; much of this material reflects some of the authors' own research interests.

Most works in the general area of group theory, topological algebra, and functional analysis will consider compact groups a "classical topic," having its roots in the second and third decade of this century. (For some discussion of the history of Lie groups and topological groups see e.g. [41] (notably p. 287–305), [70], [138], [292].) Nevertheless, interested readers will observe novel aspects in the way we approach the subject and place our emphasis. In the first four chapters it was our aim to progress into the theory with as few prerequisites as possible and with the largest pay-offs possible at the same time. Thus our approach to the basic representation theory, while still reflecting the classical approach by Peter and Weyl has our particular stamp on it. Later on, one of our main motives, emerging from Chapter 5 on, is a very general theory of the exponential function far beyond the more traditional domain of finite dimensional Lie groups; and even there our direct approach places much heavier emphasis on the exponential function as the essential feature of Lie group theory than authors commonly do. But we are certainly not bucking a trend in doing so. In many respects, in the end, one might consider our approach as a *general Lie theory for compact groups*, irrespective of dimension.

Strategies for using the book. We were employing certain strategies ourselves: the first and foremost being to make this a source book which is as self-contained as possible. This caused us to present rather fully some source material one needs in separate appendices of which there are four. Dealing with an advanced topic like this we do not find it always possible to abstain from citing other sources. We make an effort to state the prerequisites at the beginning of each chapter and warn the reader about those rare points where we have to invoke outside source material. In lieu of lengthy introductions to either the book or to the individual chapters we have provided, for an orientation of the reader, a postscript at the end of each chapter with commentaries on the material that was covered.

With regard to our efforts to make this a self-contained source on the structure of compact groups, we think it to be quite justified to call the book a *primer for the student.* The initial chapters should be accessible for the beginning graduate student having had basic analysis, algebra, elementary functional analysis up through the elements of Banach spaces and Banach algebras, and having acquired a small body of background information about point set topology; later chapters will require more background knowledge, including some algebraic topology.

For reasons we have indicated, one might argue with us whether it is right to call the book a *handbook for the expert.* Yet correspondence with mathematicians working in various fields of specialisation who asked us about information regarding the structure of compact groups has convinced us that there is a large body of structural information at hand which is not accessible, or, at least, not *easily and readily* accessible in the textbook and handbook literature, and we hope that some of the material presented here justifies, to some extent, the designation "handbook" as well. We have made an attempt to compile a fairly detailed table of contents and a large alphabetical index. The list of references is substantial but is by no means exhaustive.

The book contains material for several separate courses or seminars. In Chapters 1 through 4 one has the body of an introductory course on compact groups of two semesters. Chapter 5 contains more than enough material for a one-semester introductory course on the general theory of (linear) Lie groups. Chapter 6 has up to two semesters worth of additional material on the structure of compact Lie groups. Appendix 1 makes up for a one semester course in basic abelian group theory, starting from the most basic facts reaching up to rather sophisticated logical aspects of modern algebraic theory of groups. Appendix 3 is an introductory course into category theory for the working mathematician (to borrow a title from MacLane) replete with examples from numerous mathematical endeavors not immediately related to compact group theory. Such courses have been taught by us through the years in one form or another and our lecture notes became the bases for some of the material in the book. Other portions of the book are more likely to lend themselves to seminars or specialized advanced courses rather than to basic courses.

Background and Acknowledgements. Much material that was developed in courses given by the authors in their respective environments made its way into this monograph. The first author has taught courses on Lie group theory at the University of Tübingen, Germany, at Tulane University in New Orleans, and the Technische Hochschule Darmstadt now Technische Universität Darmstadt) from the early sixties through recent years. He has given courses on compact groups at Tulane University repeatedly, beginning 1966, at the Université de Paris VI in the fall of 1973, at the Université Catholique de Louvain-la-Neuve in March 1988, at Technische Hochschule Darmstadt from 1989 on. The Lecture Notes from these courses formed the nucleus of the first four chapters of this book. At the Technische Hochschule Darmstadt, in the context of these courses, he taught courses on abelian groups and on category theory; the lecture notes from these provided the basis material for parts of Chapter 8 and of the Appendices 1 and 3. The second author has had a history of courses of general topology and of topological groups which strongly influenced certain pedagogical aspects of our presentation. From the year 1979 on, both authors cooperated on a research project involving compact groups. Morris brought to this project his expertise on topological groups, varieties of topological groups, and his interest in free topological groups, and Hofmann contributed his knowledge of compact groups and Lie groups. Basically, the project concerned the study of free compact groups and is reflected in many parts of the book from Chapter 8 on, notably in Chapters 11 and 12.

We owe a debt of gratitude to people and institutions. First and foremost to our families. They have shown much appreciated tolerance for the stress caused to them in the process of our research and our writing of this book. Without their encouragement the project would hardly have materialized.

We have had the active support in many concrete ways through our colleagues. Laszlo Fuchs at Tulane University graciously allowed us to use his course notes for our treatment of the Whitehead Problem in Appendix 1; without his assistance in the matter this section could not have been written. Karl-Hermann Neeb

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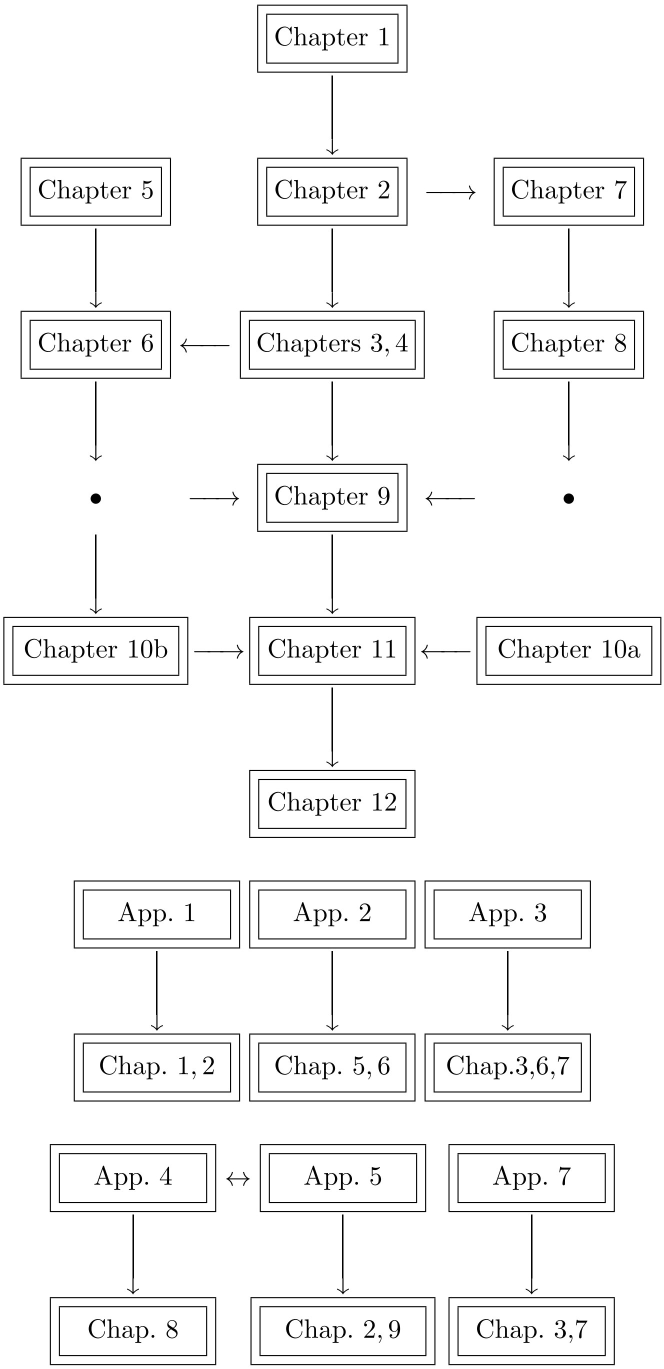
K. H. Hofmann and S. A. Morris Darmstadt and Adelaide, 1998

The Logical Dependence of the Contents



Contents

Chapter 1

Basic Topics and Examples

In this chapter we introduce the notion of a topological group, and in particular a compact (topological) group. We discover that we are surrounded by topological groups, and indeed by compact groups. Concrete examples are the linear (or matrix) groups. But these are more than just interesting examples, we shall see that they play a central role in the theory.

It is very easy to build new compact groups from given ones—by forming closed subgroups, quotient groups or even arbitrary products. The last of these shows that arbitrarily large compact groups are at hand. Another operation of central importance is that of projective limit. Some of the basic results on the structure of compact groups require that we familiarize ourselves with this concept at an early stage.

Pride of place in the theory of compact abelian groups goes to the Pontryagin Duality Theorem. To give the flavor of this, we need to know that if G is a compact abelian group then its character group, $\widehat{G}$, is an abelian group, and that if A is an abelian group, then $\widehat{A}$ is a compact abelian group. The Pontryagin Duality Theorem says that the character group $\widehat{\widehat{G}}$ of the character group of G is isomorphic as a topological group to G and that $\widehat{\widehat{A}}$ is isomorphic as a group to A. We are familiar with duality theory from vector space theory, but here we see the surprising fact that in going from a compact abelian group G to its character group, the abelian group, $\widehat{G}$ absolutely no information is lost, since we can retrieve the compact group G from its character group $\widehat{G}$ simply by taking its character group $\widehat{\widehat{G}}$. So the study of compact abelian groups is "reduced" to the study of abelian groups. In this chapter we prove half of the Pontryagin Duality Theorem, using projective limits. The key to the proof is observing that every abelian group is a directed union of its finitely generated subgroups, and analyzing what this means for the character groups. This necessitates the study of projective limits which are also used to show that compact totally disconnected groups are profinite.

Prerequisites. This chapter requires some basic knowledge of linear algebra, point set topology and abelian group theory (such as the structure of finitely generated abelian groups). In order to keep the book sufficiently self-contained, we shall present basic aspects of the theory of abelian groups in Appendix 1. Some references to these basic subject matter areas are given at the end of the chapter.

Definitions and Elementary Examples

Definitions 1.1. (i) A *topological group* is a group G together with a topology such that multiplication $(x, y) \mapsto xy: G \times G \to G$ and inversion $x \mapsto x^{-1}: G \to G$ are continuous functions.

(ii) A *compact group* is a topological group whose topology is compact Hausdorff.

(iii) A *locally compact group* is a topological group whose topology is a Hausdorff space in which the identity has a compact neighborhood. □

The following remark is immediate:

Remark 1.2. If G is a topological group and H a subgroup, then H is a topological group with respect to the induced topology. If H is a compact subspace, then H is a compact group. □

Examples 1.3. (i) The additive group $\mathbb{R}$ of real numbers with the usual topology is a Hausdorff topological group which is not compact. The multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers with the induced topology is a topological group. The subgroup $\mathbb{S}^0 \stackrel{\text{def}}{=} \{1, -1\}$ is a compact (and discrete) subgroup.

(ii) The multiplicative group $\mathbb{C}^\times \stackrel{\text{def}}{=} \mathbb{C} \setminus \{0\}$ of nonzero complex numbers with the induced topology is a topological group. Its subgroup, $\mathbb{S}^1 \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid |z| = 1\}$, consisting of all complex numbers of absolute value 1 is a compact group on the 1-sphere, often called the *circle group*.

(iii) The multiplicative group $\mathbb{H}^\times \stackrel{\text{def}}{=} \mathbb{H} \setminus \{0\}$ of nonzero quaternions with the usual topology is a topological group. Also, the subgroup $\mathbb{S}^3 \stackrel{\text{def}}{=} \{q \in \mathbb{H} \mid |q| = 1\}$ of unit quaternions is a compact group, called the 3-*sphere group*.

(iv) All finite groups are compact groups with their discrete topology. □

The last example appears rather trivial. Some basic construction processes, however, will allow us to take them as a point of departure for the construction of rather complicated groups. Examples 1.3(i), (ii) and (iii) seem to indicate that spheres tend to have compact group structures. Yet the spheres in dimensions 0, 1, and 3 are the only ones which carry topological group structures; this is by no means obvious at this stage.

One of the most prevalent sources of topological groups is the class of groups of automorphisms of finite-dimensional vector spaces over the field of real numbers or of complex numbers, or, equivalently, the class of groups of real or complex matrices. These are the so-called *linear groups* or *matrix groups*. We approach this subject in greater generality. Generality is not a goal in itself, but on a level of greater generality concepts and proofs actually become simpler and more lucid.

We shall write $\mathbb{K}$ for the field $\mathbb{R}$ or the field $\mathbb{C}$ of complex numbers.

An *algebra over* $\mathbb{K}$ is a vector space over $\mathbb{K}$ which is also a ring, in such a way that for all ring elements x, y and scalars α, we have $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

A *Banach algebra over* $\mathbb{K}$ is an algebra A with identity endowed with a norm which makes the underlying vector space a Banach space and satisfies the inequality $\|xy\| \le \|x\|\|y\|$. (Banach algebras without identity are interesting, too, but not to us.) An element $u \in A$ is called a *unit* if it is invertible, that is if there is an element u' with $uu' = u'u = 1$. The set of all units will be denoted A^{-1}. It is the largest subgroup containing 1 in the multiplicative semigroup of A.

Proposition 1.4. *The group A^{-1} of units of any Banach algebra A is a topological group. It is an open subset of A; that is, every point of A^{-1} has a neighborhood which is an open ball in A.*

Proof. Continuity of multiplication: If a, $b \in A$ and $\|y-b\| \le 1$ then $\|y\| \le \|b\|+1$ and we obtain the estimate $\|xy - ab\| \le \|xy - ay\| + \|ay - ab\| \le \|x - a\|\|y\| + \|a\|\|y - b\| \le \|x - a\|(\|b\| + 1) + \|a\|\|y - b\|$. This number is small when x is close to a and y is close to b. (In fact we have shown the continuity of multiplication in the multiplicative semigroup A.)

Continuity of inversion: If a is a unit then $\|x^{-1} - a^{-1}\| = \|(x^{-1}a - 1)a^{-1}\| \le \|\big((a^{-1}x)^{-1} - 1\big)\|\|a^{-1}\|$. If x is close to a then $a^{-1}x$ is close to 1 by the continuity of multiplication. Thus it suffices to show continuity of inversion at the identity. But if $\|h\| < 1$, then $\|(1-h)^{-1}-1\| = \|h+h^2+\cdots\| \le \|h\|(1+\|h\|+\|h\|^2+\cdots) = \|h\|(1 - \|h\|)^{-1}$. We can make this number as small as we like by choosing $\|h\|$ close to 0. This proves continuity of inversion at 1. So A^{-1} is a topological group.

The set A^{-1} of units is open: If $\|h\| < 1$, then $1-h$ has the inverse $1+h+h^2+\cdots$ and thus the open ball B of radius 1 around 1 is contained in A^{-1}. If a is a unit, then, in view of the continuity of multiplication, the function $\lambda_a\colon A \to A$ given by $\lambda_a(x) = ax$ is continuous and has a continuous inverse λ_a^{-1} given by $\lambda_a^{-1}(x) = a^{-1}x$. Thus λ_a is a homeomorphism. Hence $\lambda_a(B)$ is an open neighborhood of $\lambda_a(1) = a$ in A. But $\lambda_a(B) = aB$ is contained in A^{-1}. This shows that A^{-1} is open in A. □

If V is an arbitrary Banach space over $\mathbb{K}$ then $\mathcal{L}(V) = \mathrm{Hom}(V, V)$, the algebra of all continuous linear operators of V, is a Banach algebra with respect to the operator norm defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| \le 1\}$. Its group of units, the group of all invertible continuous operators is called $\mathrm{Gl}(V)$, the *general linear group on* V. If $V = \mathbb{K}^n$, then $\mathrm{Gl}(V)$ is also denoted $\mathrm{Gl}(n, \mathbb{K})$ or $\mathrm{Gl}_n(\mathbb{K})$. This group may be identified with the group of all invertible $n \times n$-matrices over $\mathbb{K}$, and it is called the *general linear group of degree n*. (Recall from the theory of topological vector spaces that a finite-dimensional vector space such as $\mathrm{Hom}(V, V)$ with $\dim V < \infty$ supports only one vector space topology! The topology of $\mathrm{Gl}(n, \mathbb{K})$ is therefore the topology induced from the unique vector space topology of $\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n)$ which, as a vector space, is isomorphic to $\mathbb{K}^{n^2}$.)

Corollary 1.5. *The general linear group* $\mathrm{Gl}(V)$ *on a Banach space V is a topological group when it is given the topology induced by the operator norm topology.*

In particular, the general linear group $\mathrm{Gl}(n,\mathbb{K})$ *of degree* n *over* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{K}=\mathbb{C}$ *is a topological group. All subgroups of these groups are topological groups.*

Proof. This is immediate from Proposition 1.4 and the preceding remarks! □

Examples 1.6. (i) The group $\mathrm{Sl}(n,\mathbb{K}) \overset{\mathrm{def}}{=} \{g\in \mathrm{Gl}(n,\mathbb{K}) \mid \det g = 1\}$ is a topological group, called the *special linear group of degree* n over $\mathbb{K}$.

(ii) Let $\mathcal{H}$ denote a Hilbert space over $\mathbb{K}$ with scalar product $(\cdot \mid \cdot)$. The set of all $g\in \mathrm{Gl}(\mathcal{H})$ satisfying $(gx \mid gx) = (x \mid x)$ for all $x\in\mathcal{H}$ is a subgroup $\mathrm{U}(\mathcal{H})$, called the *unitary group on* $\mathcal{H}$.
This gives rise to two important special cases:

(a) If $\mathcal{H}$ is $\mathbb{R}^n$ with the standard scalar product $(x \mid y) = x_1y_1+\cdots+x_ny_n$, then the group $\mathrm{U}(\mathcal{H})$ is written $\mathrm{O}(n)$ and is called *the orthogonal group of degree* n. The group $\mathrm{SO}(n) = \mathrm{O}(n)\cap \mathrm{Sl}(n,\mathbb{R})$ is called the *special orthogonal group of degree* n.

(b) If $\mathcal{H}$ is $\mathbb{C}^n$ with the standard scalar product $(x \mid y) = x_1\overline{y_1}+\cdots+x_n\overline{y_n}$, then the group $\mathrm{U}(\mathcal{H})$ is written $\mathrm{U}(n)$ and is called *the unitary group of degree* n. The group $\mathrm{SU}(n) = \mathrm{U}(n)\cap \mathrm{Sl}(n,\mathbb{C})$ is called the *special unitary group of degree* n.

All groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ *are compact groups for* $n = 1,\ldots$.

(c) The set $\mathbb{H}^n$ is a vector space under componentwise addition and scalar multiplication on the right: $(x_1,\ldots,x_n)\cdot h = (x_1h,\ldots,x_nh)$. An endomorphism of $\mathbb{H}^n$ is then a morphism $g\colon \mathbb{H}^n\to\mathbb{H}^n$ of the underlying addition groups satisfying $g(x\cdot h) = (gx)\cdot h$. For $x\in\mathbb{H}$, $x = a_1+ia_2+ja_3+ka_4$, we define $\overline{x} = a_1-ia_2-ja_3-ka_4$ and endow $\mathbb{H}^n$ with an inner product given by $(x \mid y) = \mathrm{Re}(\overline{y_1}x_1+\cdots+\overline{y_n}x_n)$. Thus we make $\mathbb{H}^n$ into a real Hilbert space $\mathcal{H}$. Now we define the *symplectic group of degree* n, written $\mathrm{Sp}(n)$, to be the group of all endomorphisms g of the $\mathbb{H}$-vector space $\mathbb{H}^n$ that satisfy $(gx \mid gy) = (x \mid y)$, i.e. that are contained in $\mathrm{O}(\mathcal{H})$. These are certainly invertible. Thus $\mathrm{Sp}(n) = \mathrm{Gl}(n,\mathbb{H})\cap \mathrm{O}(\mathcal{H})$. □

Exercise E1.1. (i) Show that $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$ are compact.

(ii) Let A be an algebra over $\mathbb{K}$. An *involution on* A is a self-map ${}^*\colon A\to A$ with $a^{**}=a$, $(c\cdot a)^* = \overline{c}\cdot a^*$ (where $\overline{c}$ is the complex conjugate of c if $c\in\mathbb{C}$ and is c if $c\in\mathbb{R}$), $(a+b)^* = a^*+b^*$, $(ab)^* = b^*a^*$. An element $u\in A$ is called *unitary* if $uu^* = u^*u = 1$, that is if $u^{-1} = u^*$. Show that the set $U(A)$ of all unitary elements is a group. A Banach algebra A with an involution satisfying $\|a^*a\| = \|a\|^2$ is called a C*-*algebra*. Show that for a Hilbert space $\mathcal{H}$ the algebra $\mathcal{L}(\mathcal{H})$ is a C*-algebra with respect to the forming of the adjoint operator $T\mapsto T^*$ given by $(Tx \mid y) = (x \mid T^*y)$. □

We shall return much more systematically to groups of invertible elements in Banach algebras when we treat *linear Lie groups* (Chapter 5).

Definitions 1.7. A *morphism of topological groups* is a continuous function $f: G \to H$ between topological groups that is also a group homomorphism. It is called an *isomorphism of topological groups* if it has an inverse morphism of topological groups. If G and H are isomorphic as topological groups, then we write $G \cong H$. □

We do not have to dwell here on the fact that the concepts introduced in Definition 1.7 are of a category theoretical nature; the context will make it clear that a *morphism of compact groups* $f: G \to H$ is a morphism of topological groups between compact groups, and that an *isomorphism of compact groups* is an isomorphism of topological groups between compact groups.

Remark 1.8. If G is a compact group and H a Hausdorff topological group, and if $f: G \to H$ is an injective morphism of topological groups then the corestriction $f': G \to f(G)$, $f'(g) = f(g)$ is an isomorphism of compact groups.

Proof. Since G is compact, f is continuous and H is Hausdorff, the image $f(G)$ is a compact group, and f maps closed, hence compact subsets of G onto compact, hence closed subsets of H. Then f', being a bijective continuous and closed map is a homeomorphism, and thus f'^{-1} is a morphism of compact groups. □

Exercise E1.2. (i) Recall (and verify) that there is an injective morphism of real algebras from $\mathbb{H}$ into the algebra $M_2(\mathbb{C})$ of complex 2×2 matrices given by

$$r{\cdot}1 + x{\cdot}i + y{\cdot}j + z{\cdot}k \mapsto \begin{pmatrix} r + x{\cdot}i & y + z{\cdot}i \\ -y + z{\cdot}i & r - x{\cdot}i \end{pmatrix}.$$

Show that this morphism induces an isomorphism of compact groups

$$f: \mathbb{S}^3 \to \mathrm{SU}(2).$$

(ii) Identify $\mathbb{R}^3$ with the subspace $\mathbb{R}{\cdot}i + \mathbb{R}{\cdot}j + \mathbb{R}{\cdot}k$ of $\mathbb{H}$. Show that $\mathbb{R}^3$ is invariant under all inner automorphisms $z \mapsto qzq^{-1}$, $q \in \mathbb{H}$. Show that unit quaternions $q \in \mathbb{S}^3$ induce orthogonal maps of $\mathbb{R}^3$. Show that in this fashion one defines a morphism of compact groups $p: \mathbb{S}^3 \to \mathrm{SO}(3)$ whose kernel is $\mathbb{S}^0 = \{-1, 1\}$. (To be continued in Exercise E1.3(iv) below where it is shown that p is surjective. In Appendix A2.29 it is proved that $\mathbb{S}^3$ and $\mathrm{SO}(3)$ are, up to isomorphism, the only connected topological groups locally isomorphic to $\mathrm{SO}(3)$.)

(iii) Show that $\mathrm{O}(n)$ is isomorphic to a subgroup of $\mathrm{U}(n)$ and that $\mathrm{U}(n)$ is isomorphic to a subgroup of $\mathrm{O}(2n)$ (in the sense of compact groups).

(iv) Show that an algebraic homomorphism between topological groups is continuous if and only if it is continuous at the identity element. □

Actions, Subgroups, Quotient Spaces

We recall a few facts from group theory. If G is a group and X a set, we say that G *operates* or *acts on* X if there is a function $(g, x) \mapsto gx: G \times X \to X$ such that

$1x = x$ and $g(hx) = (gh)x$. We say that the action is *transitive* if $Gx = X$ for one (hence all) $x \in X$. For each $x \in X$ the set $G_x = \{g \in G \mid gx = x\}$ is a subgroup called the *stability subgroup* or *isotropy subgroup* (German: *Standuntergruppe*) of G at x. If G acts on X and Y, then a function $f\colon X \to Y$ is called *equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

Any subgroup H of a group G gives rise to a partition of G into a set G/H of cosets gH, $g \in G$. The group G acts transitively on the set G/H via $(g, g'H) \mapsto gg'H\colon G \times G/H \to G/H$ and the stability group G_H of G at H is H.

If G acts on X, then for each $x \in X$ there is an equivariant bijection $f_x\colon G/G_x \to Gx$ given unambiguously by $f_x(gG_x) = gx$ and the function $g \mapsto gx\colon G \to Gx$ decomposes into the composition of the quotient map $q = (g \mapsto gG_x)\colon G \to G/G_x$ and f_x. We have a commutative diagram of equivariant functions

$$\begin{array}{ccc} G & \xrightarrow{g \mapsto gx} & X \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \mathrm{incl}} \\ G/G_x & \xrightarrow[f_x]{} & Gx. \end{array}$$

If H is a subgroup of G acting transitively on X and containing the stability group G_x then $H = G$.

Exercise E1.3. (i) Verify the preceding assertion.

(ii) Show that $\mathrm{SO}(n)$ acts transitively on the $n-1$-sphere $\mathbb{S}^{n-1}$, $n = 1, \ldots$. In fact, much more is true: $\mathrm{SO}(n)$ acts transitively on the set X of oriented orthonormal n-tuples $(e_1, \ldots, e_n) \in (\mathbb{R}^n)^n$.

(iii) The stability subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{S}^{n-1}$ at $(0, \ldots, 0, 1) \in \mathbb{S}^{n-1}$ may be identified with $\mathrm{SO}(n-1)$. If G is a subgroup of $\mathrm{SO}(n)$ acting transitively on $\mathbb{S}^{n-1}$ and containing $\mathrm{SO}(n-1)$, then $G = \mathrm{SO}(n)$.

(iv) Apply (iii) with $n = 3$ and the subgroup $p(\mathbb{S}^3)$ with p from Exercise E1.2(ii) and show that p is surjective. □

If N is a normal subgroup of G, that is, satisfies $gN = Ng$ or, equivalently, $gNg^{-1} = N$ for all $g \in G$, then G/N is a group with the multiplication $(gN, hN) \mapsto ghN\colon G/N \times G/N \to G/N$. The quotient map $q\colon G \to G/N$, $q(g) = gN$ is a surjective morphism of groups with kernel N. Conversely, if $f\colon G \to H$ is a morphism of groups we have *the canonical decomposition of morphisms* indicated by the following diagram involving the well-defined isomorphism of groups $f'\colon G/\ker f \to f(G)$, $f'\big(g(\ker f)\big) = f(g)$:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \mathrm{quot}}\downarrow & & \uparrow{\scriptstyle \mathrm{incl}} \\ G/\ker f & \xrightarrow[f']{} & f(G). \end{array}$$

Each of the statements in this discussion has its counterpart for topological groups and topological spaces. In order to secure terminology we formulate the relevant definitions and add a few remarks special to the topological situation:

Definitions 1.9. (i) We say that a topological group G *acts on* a topological space X if there is a continuous function $(g,x) \mapsto gx: G \times X \to X$ which implements a group action.

(ii) If H is a subgroup of a topological group G, then the set G/H of cosets gH, $g \in G$ is a topological space with respect to the quotient topology, called *the quotient space of G modulo H* or also the *homogeneous space of G modulo H*. If N is a normal subgroup of G, then the group G/N with the quotient topology is called the *quotient group of G modulo N*. The quotient group $\mathbb{R}/\mathbb{Z}$ will be written $\mathbb{T}$. □

Just for the record, we recall that the quotient topology on G/H is the finest topology making the quotient map $q: G \to G/H$, $q(g) = gH$ continuous. Therefore a subset V of G/H is open if and only if $q^{-1}(V)$ is open in G, that is if and only if there is an open subset U of G that is *saturated* in the sense that $UH = U$ and that $V = \{uH: u \in U\}$.

We now proceed to establish some basic facts:

Proposition 1.10. (i) *If the topological group G acts on the Hausdorff topological space X then each stability subgroup G_x is closed. The bijective function $f_x: G/G_x \to Gx$ arising from the canonical decomposition of the function $g \mapsto gx: G \to Gx$ is continuous. If G is compact, it is a homeomorphism. In particular, if G is compact and acts transitively on a Hausdorff space X, then for any $x \in X$, the spaces G/G_x and X are naturally homeomorphic and X may be considered as a homogeneous space of G (modulo the stability group G_x).*

(ii) *If H is a subgroup of G, then the quotient map $q: G \to G/H$ is open, and G/H is a Hausdorff space if and only if H is closed. If N is a normal subgroup of G, then G/N is a topological group. If G is a compact group and N a closed normal subgroup, then G/N is a compact group.*

(iii) *Assume that G is a group with a topology such that all left translations λ_g, $\lambda_g x = gx$, are continuous. Then every open subgroup is closed. This applies, in particular, to every topological group G.*

(iv) *If $f: G \to H$ is a morphism of topological groups, then the bijective map $f': G/\ker f \to f(G)$ arising from the canonical decomposition of f is continuous. If $G/\ker f$ is compact and H is Hausdorff then f' is an isomorphism of compact groups. The quotient group G/N of a topological group modulo a closed normal subgroup is compact if there is a compact subset C with $CN = G$. This is certainly the case if G is compact.*

Proof. Exercise E1.4. □

Exercise E1.4. Prove Proposition 1.10. □

Exercise E1.5. Show that the sphere $\mathbb{S}^{n-1}$ may be identified with a homogeneous space of $\mathrm{SO}(n)$ modulo a subgroup isomorphic to $\mathrm{SO}(n-1)$. □

The following exercise will show that the circle group has several natural manifestations: multiplicative ones, namely, $\mathbb{S}^1 \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$, and an additive one, namely, $\mathbb{T}$. It will also illustrate that the action of the group $\mathrm{SO}(2)$ of planar rotations on the euclidean plane is the basis of an exact formulation of the elementary concept of an angle.

Exercise E1.6. Prove the following statements.

(i) *The function* $f: \mathbb{R} \to \mathbb{C}^\times$ *given by* $f(t) = e^{2\pi i t}$ *is a morphism of topological groups inducing an isomorphism*

$$f': \mathbb{T} \to \mathbb{S}^1, \qquad f'(t + \mathbb{Z}) = e^{2\pi i t}$$

of compact abelian groups.

(ii) $\mathbb{S}^1$ *may be identified with* $\mathrm{U}(1)$ *if one identifies* $\mathbb{C}^\times$ *and* $\mathrm{Gl}(1, \mathbb{C})$ *in the obvious way.*

(iii) *The isomorphism* $t \mapsto 2\pi t: \mathbb{R} \to \mathbb{R}$ *induces an isomorphism* $t + \mathbb{Z} \mapsto 2\pi t + 2\pi\mathbb{Z}: \mathbb{T} \to \mathbb{R}/2\pi\mathbb{Z}$.

$$R: \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{SO}(2), \qquad R(t + 2\pi\mathbb{Z}) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is an isomorphism of compact groups.

(iv) Let $E = \mathbb{R}^2$ denote the euclidean plane with the scalar product $(X \mid Y) = x_1 y_1 + x_2 y_2$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$. The group $\mathrm{SO}(2)$ acts on E by matrix multiplication, i.e. such that $R(t + 2\pi\mathbb{Z}) \cdot (x_1, x_2) = (x_1', x_2')$, where

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

A *half-line* is a set of points of the form $H = \mathbb{R}^+ \cdot X$, $0 \neq X \in E$, $\mathbb{R}^+ = [0, \infty[$. A *line* is a set of the form $L = \mathbb{R} \cdot X$, $0 \neq X \in E$. By abuse of notation we also consider cos and sin as functions on $\mathbb{R}/2\pi\mathbb{Z}$ via $\sin(t + 2\pi\mathbb{Z}) = \sin t$ and $\cos(t + 2\pi\mathbb{Z}) = \cos t$.

If (H_1, H_2) is an ordered pair of half-lines in E we write $H_j = \mathbb{R}^+ \cdot X_j$ with $\|X_1\| = \|X_2\|$; then there is a unique $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $X_2 = R(\theta) \cdot X_1$. The element θ is called the *oriented angle* $\mathrm{ang}(H_1, H_2)$ *between the half-lines* H_1 *and* H_2 (in this order!). If (L_1, L_2) is an ordered pair of lines, write $L_j = \mathbb{R} \cdot X_j$ with $\|X_1\| = \|X_2\|$. Let $p: \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/\pi\mathbb{Z}$ denote the ("double covering") morphism given by $p(r + 2\pi\mathbb{Z}) = r + \pi\mathbb{Z}$. Then $p\big(\mathrm{ang}(\mathbb{R}^+ \cdot X_1, \mathbb{R}^+ \cdot X_2)\big) = p\big(\mathrm{ang}(\mathbb{R}^+ \cdot X_1, -\mathbb{R}^+ \cdot X_2)\big)$ and this element of $\mathbb{R}/\pi \cdot Z$ is called the *oriented angle* $\mathrm{ang}(L_1, L_2)$ *between the lines* L_1 *and* L_2 (in this order!). In order to avoid an inflation of notation we use the same functional symbol for the oriented angle between half-lines and between lines; confusion is impossible if one looks at the arguments of the function $\mathrm{ang}(\cdot, \cdot)$.

If X_1 *and* X_2 *are nonzero elements of* E, *then*

$$(X_1 \mid X_2) = \|X_1\| \cdot \|X_2\| \cos\big(\mathrm{ang}(\mathbb{R}^+ \cdot X_1, \mathbb{R}^+ \cdot X_2)\big).$$

For half-lines H_1, H_2, *and* H_3 *we have*

$$\begin{aligned} \mathrm{ang}(H_1, H_2) &= -\mathrm{ang}(H_2, H_1), \\ \mathrm{ang}(H_1, H_3) &= \mathrm{ang}(H_1, H_2) + \mathrm{ang}(H_2, H_3), \end{aligned}$$

and the same for lines.

Any transformation $X \mapsto R(\theta)\cdot X$ of E is called a rotation by an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. □

For the action of compact groups we derive a simple, but very useful fact which gives the first impression of the emergence of invariant objects for the action of compact groups.

Proposition 1.11. *If a compact group G acts on a topological space X and x is a fixed point , that is, satisfies $Gx = \{x\}$, then x has a basis of G-invariant neighborhoods.*

More specifically, if U is any neighborhood of x, then the set

$$V = \bigcap_{g\in G} gU$$

is a G-invariant neighborhood of x contained in U.

Proof. Since all functions $y \mapsto hy: X \to X$, $h \in G$, are bijective and $hG = G$ we find $hV = h\bigcap_{g\in G} gU = \bigcap_{g\in G} hgU = \bigcap_{g\in G} gU = V$. Thus V is G-invariant. Clearly, $V \subseteq U$ since $\mathbf{1} \in G$ and $\mathbf{1}U = U$.

We now suppose that V is not a neighborhood of x in X and derive a contradiction; this will finish the proof. We may assume that U is open, for otherwise we can replace U by its interior. For any subset W of X we define

$$G_W = \{g \in G \mid gW \setminus U \neq \emptyset\}.$$

By our supposition that V is not a neighborhood of x, for any neighborhood W of x we compute $\emptyset \neq W \setminus V = W \setminus \bigcap_{g\in G} gU = \bigcup_{g\in G}(W \setminus gU)$, and so there is some $g \in G$ with $W \setminus gU \neq \emptyset$ and then $g^{-1}W \setminus U \neq \emptyset$. Thus $G_W \neq \emptyset$. Let $\mathcal{U}$ denote the neighborhood filter of x. Since $W \subseteq W'$ implies $G_W \subseteq G_{W'}$, the family $\{G_W \mid W \in \mathcal{U}\}$ is a filter basis on G. Hence by the compactness of G, there is an element $g \in \bigcap_{W\in\mathcal{U}} \overline{G_W}$. Then, for all neighborhoods N of $\mathbf{1}$ in G, we have $gN \cap G_W \neq \emptyset$, that is, $gNW \setminus U \neq \emptyset$. By the continuity of the action, given an arbitrary neighborhood W_0 of x we find N and W so that $NW \subseteq W_0$. Hence $gW_0 \setminus U \neq \emptyset$. Thus every neighborhood W_0 of x meets the set $X \setminus g^{-1}U$. This last set is closed as U and hence $g^{-1}U$ is open. Therefore $x \in X \setminus g^{-1}U$ and thus $gx \notin U$. But $x = gx$ since x is a fixed point. Thus $x \notin U$ and that is a contradiction since U is a neighborhood of x.

Second proof using nets. If V is not a neighborhood of x, then for each neighborhood W of x there is an element $g_W \in G$ such that $W \setminus g_W U \neq \emptyset$. Hence there is an $x_W \in W$ with $g_W^{-1}x_W \notin U$. Since G is compact, there is a subnet $g_{W(j)}$ converging to some g. On account of $x_W \in W$, the subnet $x_{W(j)}$ converges to x. By the continuity of the action, $g_{W(j)}^{-1}x_{W(j)}$ converges to $g^{-1}x$ which is x since x is a fixed point. We may assume that U is open. Then $g_{W(j)}^{-1}x_{W(j)} \notin U$ implies $x \notin U$, a contradiction.

Third proof using Wallace's Lemma. Let B be a compact subset of X and A a compact subset of G such that $A{\cdot}B \subseteq B$. Now let U be an open neighborhood of B. We shall show that $U' \stackrel{\text{def}}{=} \bigcap_{g\in A} g^{-1}U$ is a neighborhood of B which in the special case that $A = G$ is invariant. Specializing to $B = \{x\}$ again we get a proof of Proposition 1.11.

Define $f: G\times X \to X$ by $f(g,y) = g^{-1}{\cdot}x$. Then $f^{-1}(U)$ is an open neighborhood of $A^{-1}\times B \subseteq G\times X$. Then by *Wallace's Lemma* (see Proposition A4.29 of Appendix 4) there are open neighborhoods V of A in G and W of B in X such that $V^{-1}\times W \subseteq f^{-1}(U)$, and so $V{\cdot}W = f(V^{-1} \times W) \subseteq U$. Hence $W \subseteq g^{-1}U$ for all $g \in V$, that is, $W \subseteq \bigcap_{g\in V} g^{-1}U \subseteq \bigcap_{g\in A} g^{-1}{\cdot}U = U'$. Hence U' is a neighborhood of B. □

Corollary 1.12. *If G is a compact group and U any neighborhood of the identity, then*

$$V = \bigcap_{g\in G} gUg^{-1}$$

is a neighborhood of the identity which is contained in U and is invariant under all inner automorphisms.

Proof. The group G acts on G via $(g,x) \mapsto gxg^{-1}$, and $\mathbf{1}$ is a fixed point for this action. An application of Proposition 1.11 yields the result. □

A topological group is called a *SIN-group* or said to have *small invariant neighborhoods* if every neighborhood of the identity contains a neighborhood of the identity which is invariant under all inner automorphisms. Clearly every abelian topological group is a SIN-group. From Corollary 1.12, every compact group is also a SIN-group.

Corollary 1.13. *If G is a compact group acting on a topological vector space E in such a fashion that all maps $\pi(g) = (x \mapsto gx): E \to E$ are linear, then the family $\{\pi(g) \mid g \in G\}$ of continuous operators of E is equicontinuous, that is, given a neighborhood U of 0 there is a neighborhood V of 0 such that $\pi(g)(V) \subseteq U$ for all $g \in G$. In fact, V may be chosen to be $V = \bigcap_{g\in G} \pi(g)U$, in which case V is G-invariant.*

Proof. Since the action is linear, the origin 0 is a fixed point. Then we apply Proposition 1.11 to obtain the result. □

Products of Compact Groups

Using our elementary examples as raw material, we can construct a vast supply of compact groups.

Proposition 1.14. *If $\{G_j \mid j \in J\}$ is an arbitrary family of compact groups, then the product $G = \prod_{j\in J} G_j$ with the product topology is a compact group. Every closed subgroup H of G is a compact group.*

Proof. It is straightforward to observe that the product topology makes the cartesian product of any family of topological groups into a topological group. Since Tychonoff's Theorem [100] says that the product space of any family of compact spaces is compact, G is a compact group. By Remark 1.2, any closed subgroup H of G is a compact group. $\square$

As a simple example we see that any closed subgroup of any product $\prod_{j\in J} \mathrm{U}(n_j)$ or any product $\prod_{j\in J} \mathrm{O}(n_j)$ is a compact group. This is very elementary, but we shall see soon (namely, in Corollary 2.29 below) that *all compact groups are obtained in this fashion* up to isomorphy.

Applications to Abelian Groups

An important example arises out of the preceding proposition. For two sets X and Y the set of all functions $f\colon X \to Y$ will be denoted by Y^X.

Definition 1.15. If A is an abelian group (which we prefer to write additively) then the group

$$\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$$

of all morphisms of abelian groups into the underlying abelian group of the circle group (no continuity involved!) given the induced group structure and topology of the product group $\mathbb{T}^A$ (that is, pointwise operations and the topology of pointwise convergence) is called *the character group of* A and is written $\widehat{A}$. Its elements are called *characters* of A. $\square$

Proposition 1.16. *The character group $\widehat{A}$ of any abelian group A is a compact abelian group.*

Proof. By Proposition 1.14, the product $\mathbb{T}^A$ is a compact abelian group. For any pair $(a, b) \in A \times A$ the set $M(a, b) = \{\chi \in \mathbb{T}^A \mid \chi(a + b) = \chi(a) + \chi(b)\}$ is closed since $\chi \mapsto \chi(c)\colon \mathbb{T}^A \to \mathbb{T}$ is continuous by the definition of the product topology. But then $\widehat{A} = \bigcap_{(a,b)\in A\times A} M(a, b)$ is closed in $\mathbb{T}^A$ and therefore compact. $\square$

Let us look at a few examples: In order to recognize $\widehat{\mathbb{Z}}$ we note that the function $f \mapsto f(1)\colon \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \to \mathbb{T}$ is an algebraic isomorphism and is continuous by the definition of the topology of pointwise convergence. Since $\widehat{\mathbb{Z}}$ is compact and $\mathbb{T}$ Hausdorff, it is an isomorphism of compact groups. Hence

$$\widehat{\mathbb{Z}} \cong \mathbb{T}. \tag{1}$$

If $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n, then the function $z + n\mathbb{Z} \mapsto \frac{1}{n}z + \mathbb{Z}$ gives an injection $j\colon \mathbb{Z}(n) \to \mathbb{T}$ which induces an isomorphism $\mathrm{Hom}(\mathbb{Z}(n), j)\colon \mathrm{Hom}\big(\mathbb{Z}(n), \mathbb{Z}(n)\big) \to \mathrm{Hom}(\mathbb{Z}(n), \mathbb{T}) = \widehat{\mathbb{Z}(n)}$. Since the function $f \mapsto f(1 + n\mathbb{Z})\colon \mathrm{Hom}\big(\mathbb{Z}(n), \mathbb{Z}(n)\big) \to \mathbb{Z}(n)$ is an isomorphism, we have

$$\widehat{\mathbb{Z}(n)} \cong \mathbb{Z}(n). \tag{2}$$

If X is a set, and $\{A_x \mid x \in X\}$ a family of abelian groups, let us denote with $\bigoplus_{x\in X} A_x$ the direct sum of the A_x, that is, the subgroup of the cartesian product $\prod_{x\in X} A_x$ consisting of all elements $(a_x)_{x\in X}$ with $a_x = 0$ for all x outside some finite subset of X. A special case is $\mathbb{Z}^{(X)} = \bigoplus_{x\in X} A_x$ with $A_x = \mathbb{Z}$ for all $x \in X$. This is the *free abelian group on* X (cf. Appendix A1.6).

Proposition 1.17. *The function*

$$\Phi\colon \prod_{x\in X} \mathrm{Hom}(A_x, \mathbb{T}) \to \mathrm{Hom}(\bigoplus_{x\in X} A_x, \mathbb{T})$$

which associates with a family $(f_x)_{x\in X}$ *of morphisms* $f_x\colon A_x \to \mathbb{T}$ *the morphism*

$$(a_x)_{x\in X} \mapsto \sum_{x\in X} f_x(a_x)\colon \bigoplus_{x\in X} A_x \to \mathbb{T}$$

is an isomorphism of compact groups. Notably,

$$(\bigoplus_{x\in X} A_x)\widehat{\ } \cong \prod_{x\in X} \widehat{A_x}. \tag{3}$$

In particular

$$\mathbb{Z}^{(X)\widehat{\ }} \cong \widehat{\mathbb{Z}}^X \cong \mathbb{T}^X. \tag{4}$$

Proof. Abbreviate $\bigoplus_{x\in X} A_x$ by A. We notice that Φ is well defined, since the $f_x(a_x)$ vanish with only finitely many exceptions for $(a_x)_{x\in X}$. Clearly Φ is a morphism of abelian groups. Further $(f_x)_{x\in X} \in \ker\Phi$ if and only if $\sum_{x\in X} f_x(a_x) = 0$ for all $(a_x)_{x\in X} \in A$. Choosing for a given $y \in X$ the family (a_x) so that $a_x = 0$ for $x \neq y$ and $a_y = a$ we obtain $f_y(a) = 0$ for any $a \in A_y$. Thus $f_y = 0$ for all $y \in X$. Hence Φ is injective. If $f\colon A \to \mathbb{T}$ is a morphism, define $f_y\colon A_y \to \mathbb{T}$ by $f_y = f\circ\mathrm{copr}_y$ where $\mathrm{copr}_y\colon A_y \to A$ is the natural inclusion. Then $\Phi\bigl((f_x)_{x\in X}\bigr) = f$ follows readily. Thus Φ is surjective, too, and thus is an isomorphism of abelian groups. Next we show that Φ is continuous. By the definition of the topology on $\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$, it suffices to show that for each $(a_x)_{x\in X} \in A$, the function $(f_x)_{x\in X} \mapsto \Phi\bigl((f_x)_{x\in X}\bigr)\bigl((a_x)_{x\in X}\bigr) = \sum_{x\in X} f_x(a_x) : \prod_{x\in X} \mathrm{Hom}(A_x, \mathbb{T}) \to \mathbb{T}$ is continuous. Since only finitely many a_x are nonzero, this is the case if $(f_x)_{x\in X} \mapsto f_y(a_y)$ is continuous for each fixed y, and this holds if $f_y \mapsto f_y(a_y)\colon \mathrm{Hom}(A_y, \mathbb{T}) \to \mathbb{T}$ is continuous. However, by definition of the topology of pointwise convergence, this is indeed the case. Since the domain of Φ is compact by the theorem of Tychonoff and the range is Hausdorff, this suffices for Φ to be a homeomorphism.

The last assertion of the proposition is a special case. This remark concludes the proof of the proposition. □

The compact abelian groups $\mathbb{T}^X$ are called *torus groups*. The finite dimensional tori $\mathbb{T}^n$ are special cases.

We cite from the basic theory of abelian groups the fact that a finitely generated abelian group is a direct sum of cyclic groups (cf. Appendix A1.11). Thus (1), (2) and (3) imply the following remark:

Remark 1.18. If E is a finite abelian group, then $\widehat{E}$ is isomorphic to E (although not necessarily in any natural fashion!). If F is a finitely generated abelian group of rank n, that is, $F = E \oplus \mathbb{Z}^n$ with a finite abelian group E, then $\widehat{F} \cong \widehat{E} \times \mathbb{T}^n$.□

In particular, the character groups of finitely generated abelian groups are compact manifolds. (We shall not make any use of this fact right now. See for example [141].)

There are examples of compact abelian groups whose topological nature is quite different.

Example 1.19. Let $\{G_j \mid j \in J\}$ be any family of finite discrete nonsingleton groups. Then $G = \prod_{j\in J} G_j$ is a compact group. All connected components are singleton, and G is discrete if and only if J is finite. □

A topological space in which all connected components are singletons is called *totally disconnected*. Arbitrary products of totally disconnected spaces are totally disconnected, and all discrete spaces are totally disconnected. The standard Cantor middle third set C is a compact metric totally disconnected space. In fact it may be realized as the set of all real numbers r in the closed unit interval, whose expansion $r = \sum_{n=1}^{\infty} a_n 3^{-n}$ with respect to base 3 has all coefficients a_n in the set $\{0, 2\}$. Then the map $f\colon \{-1, 1\}^{\mathbb{N}} \to C$ given by $f\big((r_n)_{n\in\mathbb{N}}\big) = \sum_{n=1}^{\infty}(r_n + 1)3^{-n}$ is a homeomorphism. The set $\mathbb{S}^0 = \{-1, 1\}$ is a finite group, and thus, by Proposition 1.14, the domain of f is a compact group.

Hence the Cantor set can be given the structure of a compact abelian group. In this group, every element has order 2, so that in fact, algebraically, it is a vector space over the field GF(2) of 2 elements, and by (2) and (3) above, it is the character group of $\mathbb{Z}(2)^{(\mathbb{N})}$.

One can show that all compact metric totally disconnected spaces without isolated points are homeomorphic to C. In particular, all metric compact totally disconnected infinite groups are homeomorphic to C. (See [51], or e.g. Section 3.1, Theorem 4 of Fedorchuk, The Fundamentals of Dimension Theory in [14].)

Definition 1.20. Let X and Y be sets and $F \subseteq Y^X$ a set of functions from X to Y. We say that F *separates the points of* X if for any two different points x_1 and x_2 in X, there is an $f \in F$ such that $f(x_1) \neq f(x_2)$. □

If G and H are groups, then a set F of homomorphisms from G to H is easily seen to separate the points of G if and only if for each $g \neq 1$ in G there is an $f \in F$ with $f(g) \neq 1$.

For any abelian group A there is always a large supply of characters. In fact there are enough of them to separate the points. In order to see this we resort to some basic facts on abelian groups:

An abelian group A is called *divisible* if for each $a \in A$ and each natural number n there is an $x \in A$ such that $n{\cdot}x = a$. Examples of divisible groups are $\mathbb{Q}$ and $\mathbb{R}$. Every homomorphic image of a divisible group is divisible, whence $\mathbb{T}$ is divisible. The crucial property of divisible groups is that for every subgroup S of an abelian group A and a homomorphism $f\colon S \to I$ into a divisible group there is a homomorphic extension $F\colon A \to I$ of f. (See Appendix 1, A1.15.)

$$\begin{array}{ccc} I & \xrightarrow{\quad = \quad} & I \\ {\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle F} \\ S & \xrightarrow[\text{incl}]{\qquad} & A \end{array}$$

Lemma 1.21. *The characters of an abelian group A separate the points.*

Proof. Assume that $0 \neq a \in A$. We must find a morphism $\chi\colon A \to \mathbb{T}$ such that $\chi(a) \neq 0$. Let S be the cyclic subgroup $\mathbb{Z}{\cdot}a$ of A generated by a. If S is infinite, then S is free and for any nonzero element t in $\mathbb{T}$ (e.g. $t = \frac{1}{2} + \mathbb{Z}$) there is an $f\colon S \to \mathbb{T}$ with $f(a) = t \neq 0$. If S has order n, then S is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{T}$, and thus there is an injection $f\colon S \to \mathbb{T}$. If we let $\chi\colon A \to \mathbb{T}$ be an extension of f which exists by the divisibility of $\mathbb{T}$, then $\chi(a) = f(a) \neq 0$. □

Definitions 1.22. For a compact abelian group G a morphism of compact groups $\chi\colon G \to \mathbb{T}$ is called a *character of* G. The set $\mathrm{Hom}(G, \mathbb{T})$ of all characters is an abelian group under pointwise addition, called the *character group of* G and written $\widehat{G}$. Notice that we do not consider any topology on $\widehat{G}$. □

Now we can of course iterate the formation of character groups and oscillate between abelian groups and compact abelian groups. This deserves some inspection; the formalism is quite general and is familiar from the duality of finite-dimensional vector spaces.

Lemma 1.23. (i) *If A is an abelian group, then the function*

$$\eta_A\colon A \to \widehat{\widehat{A}}, \quad \eta_A(a)(\chi) = \chi(a)$$

is an injective morphism of abelian groups.

(ii) *If G is a compact abelian group, then the function*

$$\eta_G\colon G \to \widehat{\widehat{G}}, \quad \eta_G(g)(\chi) = \chi(g)$$

is a morphism of compact abelian groups.

Proof. (i) The morphism property follows readily from the definition of pointwise addition in $\widehat{A}$. An element g is in the kernel of η_A if $\chi(g) = 0$ for all characters. Since these separate the points by Lemma 1.21, we conclude $g = 0$. Hence η_A is injective.

(ii) Again it is immediate that η_G is a morphism of abelian groups. We must observe its continuity: The function $g \mapsto \chi(g): G \to \mathbb{T}$ is continuous for every character χ by the continuity of characters. Hence the function $g \mapsto \big(\chi(g)\big)_{\chi\in\widehat{G}}: G \to \mathbb{T}^{\widehat{G}}$ is continuous by the definition of the product topology. Since $\widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T}) \subseteq \mathbb{T}^{\widehat{G}}$ inherits its structure from the product, η_G is continuous. ☐

Exercise E1.7. For a discrete abelian group A and a compact abelian group G the members of $\widehat{\widehat{A}}$ and $\widehat{\widehat{G}}$ separate the points of $\widehat{A}$, respectively, $\widehat{G}$. Equivalently, the evaluation morphisms $\eta_{\widehat{A}}: \widehat{A} \to \widehat{\widehat{\widehat{A}}}$ and $\eta_{\widehat{G}}: \widehat{G} \to \widehat{\widehat{\widehat{G}}}$ are injective.
[Hint. Observe that already $\eta_A(A)$ separates the points of $\widehat{A}$.] ☐

Let us look at our basic examples: If A is a finite abelian group, then $\widehat{A}$ is isomorphic to A by Remark 1.18. Hence $\widehat{\widehat{A}}$ is isomorphic to A and $\eta_A: A \to \widehat{\widehat{A}}$ is injective by Lemma 1.23. Hence η_A is an isomorphism.

Every character $\chi: \mathbb{T} \to \mathbb{T}$ yields a morphism of topological groups $f: \mathbb{R} \to \mathbb{T}$ via $f(r) = \chi(r + \mathbb{Z})$. Let $q: \mathbb{R} \to \mathbb{T}$ be the quotient homomorphism. We set $V =]-\frac{1}{3}, \frac{1}{3}[\subseteq \mathbb{R}$ and $W = q(V)$. Then $q|V: V \to W$ is a homeomorphism. Assume that x and y are elements of W such that $x + y \in W$, too. Then $r = (q|V)^{-1}(x)$, $s = (q|V)^{-1}(y)$ and $t = (q|V)^{-1}(x + y)$ are elements of V such that $q(r + s - t) = q(r) + q(s) - q(t) = x + y - (x + y) = 0$ in $\mathbb{T}$. Hence $r + s - t \in \ker q = \mathbb{Z}$. But also $|r + s - t| \le |r| + |s| + |t| < 3\cdot\frac{1}{3} = 1$. Hence $r + s - t = 0$ and $(q|V)^{-1}(x) + (q|V)^{-1}(y) = r + s = t = (q|V)^{-1}(x + y)$. Now let U denote an open interval around 0 in $\mathbb{R}$ such that $f(U) \subseteq W$. If we set $\varphi = (q|V)^{-1} \circ f|U: U \to \mathbb{R}$ then for all $x, y, x + y \in U$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$. Under these circumstances φ extends uniquely to a morphism $F: \mathbb{R} \to \mathbb{R}$ of abelian groups (see Exercise E1.8 below). Now $q \circ F = f = \chi \circ q$ since F extends φ and U generates the abelian group $\mathbb{R}$. Then $\mathbb{Z} = \ker q \subseteq \ker(q \circ F)$, that is, $F(\mathbb{Z}) \subseteq \ker q = \mathbb{Z}$. Thus if we set $n = F(1)$, then $n \in \mathbb{Z}$. Since φ is continuous, then F is continuous at 0. As a morphism, F is continuous everywhere (see Exercise E1.2(iv)). As a morphism of abelian groups, F is quickly seen to be $\mathbb{Q}$-linear, and from its continuity it follows that it is $\mathbb{R}$-linear. Thus $F(t) = nt$ and $\chi(t + \mathbb{Z}) = nt + \mathbb{Z}$ follows. Thus the characters of $\mathbb{T}$ are exactly the endomorphisms $\mu_n = (g \mapsto ng)$ and $n \mapsto \mu_n: \mathbb{Z} \to \widehat{\mathbb{T}}$ is an isomorphism.

Exercise E1.8. Prove the following proposition:

The Extension Lemma. *Let U be an arbitrary interval in $\mathbb{R}$ containing 0 and assume that $\varphi: U \to G$ is a function into a group such that x, y, $x + y \in U$ implies*

$\varphi(x+y) = \varphi(x)\varphi(y)$. Then there is a morphism $F\colon \mathbb{R} \to G$ of groups extending φ. If U contains more than one point then F is unique.

[Hint. For $r \in \mathbb{R}$ and two integers m and n with r/m, $r/n \in U$ show $m{\cdot}\varphi(r/m) = n{\cdot}\varphi(r/n)$. Define $F(r)$ to be this unique element of G and show that F is a morphism.] □

We shall treat the Extension Lemma again in Lemma 5.8 below in a systematic fashion.

Now that we have determined $\widehat{\mathbb{T}}$ we look at $\eta_{\mathbb{Z}}$. We have $\eta_{\mathbb{Z}}(n)(\chi) = \chi(n) = n\chi(1) = \mu_n\bigl(\chi(1)\bigr)$ for any character χ of $\mathbb{Z}$. Since $\chi \mapsto \chi(1)\colon \widehat{\mathbb{Z}} \to \mathbb{T}$ is an isomorphism by (1) above and since every character of $\mathbb{T}$ is of the form μ_n, this shows that $\eta_{\mathbb{Z}}$ is an isomorphism.

Now we show that $\eta_{\mathbb{T}}$ is an isomorphism, too. We recall that $\widehat{\mathbb{T}}$ is infinite cyclic and is generated by the identity map $\varepsilon\colon \mathbb{T} \to \mathbb{T}$. In other words, any character $\chi\colon \mathbb{T} \to \mathbb{T}$ of $\widehat{\mathbb{T}}$ is of the form $\chi = n{\cdot}\varepsilon = \mu_n$. Now we observe $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon) = n{\cdot}\varepsilon(g) = n{\cdot}g$ for all $n \in \mathbb{Z}$. Taking $n = 1$ we note that the kernel of $\eta_{\mathbb{T}}$ is singleton and thus $\eta_{\mathbb{T}}$ is injective. In order to show surjectivity we assume that $\Omega\colon \widehat{\mathbb{T}} \to \mathbb{T}$ is a character of $\widehat{\mathbb{T}} \cong \mathbb{Z}$. Then $\Omega(\varepsilon)$ is an element $g \in \mathbb{T}$ and we see $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon) = n{\cdot}g = n{\cdot}\Omega(\varepsilon) = \Omega(n{\cdot}\varepsilon)$. Thus $\eta_{\mathbb{T}}(g) = \Omega$. This shows that $\eta_{\mathbb{T}}$ is surjective, too. Thus $\eta_{\mathbb{T}}$ is an isomorphism.

Remark 1.24. (i) Assume that A and B are abelian groups such that η_A and η_B are isomorphisms. Then $\eta_{A\oplus B}$ is an isomorphism.

(ii) If G and H are compact abelian groups and η_G and η_H are isomorphisms, then $\eta_{G\times H}$ is an isomorphism.

(iii) For any finitely generated abelian group A, the map $\eta_A\colon A \to \widehat{\widehat{A}}$ is an isomorphism.

(iv) If $G \cong \mathbb{T}^n \times E$ for a natural number n and a finite abelian group E then $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism.

(v) Every torus group $\mathbb{T}^n$ contains an element such that the subgroup generated by it is dense.

Proof. Exercise E1.9. □

Exercise E1.9. Prove Remarks 1.24(i)–(v).

[Hint. For (iii) and (iv) recall that the evaluation morphism is an isomorphism for cyclic groups, for $\mathbb{Z}$ and for $\mathbb{T}$. Also recall the Fundamental Theorem for Finitely Generated Abelian Groups (cf. Appendix A1.11).

For a proof of (v) set $T = \mathbb{T}^n$. Every quotient group of T modulo some closed subgroup is a compact group which is a quotient group of $\mathbb{R}^n$ and is, therefore, a torus by Appendix 1, Theorem 1.12(ii). Now let $x \in T$; then $T/\overline{\mathbb{Z}{\cdot}x}$ is a torus, and by (iv) above, its characters separate the points. Thus, $\mathbb{Z}{\cdot}x$ is dense in T iff all characters of T vanish on $\mathbb{Z}{\cdot}x$, i.e. on x, iff

$$(\forall\chi \in \widehat{T}) \quad [\chi(\mathbb{Z}{\cdot}x) = \{0\}]{\Rightarrow}[\chi = 0]$$

iff the map $\chi \mapsto \big(n \mapsto \chi(n{\cdot}x)\big){:}\,\widehat{T} \to \widehat{\mathbb{Z}}$ is injective iff the map $\chi \mapsto \chi(x){:}\,\widehat{T} \to \mathbb{T}$ is injective (via the natural isomorphism $\widehat{Z} \cong \mathbb{T}$). But since $\eta{:}\,T \to \widehat{\widehat{T}}$ is an isomorphism by (iv) above, any homomorphism $\alpha{:}\,\widehat{T} \to \mathbb{T}$ is an evaluation, i.e. there is a unique $x \in T$ such that for any $\chi \in \widehat{T}$ we have $\chi(x) = \alpha(\chi)$. Thus, in conclusion, we have an element $x \in T$ such that $\mathbb{Z}{\cdot}x$ is dense in T iff we have an injective morphism $\mathbb{Z}^n \cong \widehat{T} \to \mathbb{T}$. But the injective morphisms $\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z}$ abound (cf. Appendix A1.43).

Provide a direct proof of Remark 1.24(v) as follows: *Let $r_j \in \mathbb{R}$, $j = 1, \ldots, n$, be n real numbers such that $\{1, r_1, \ldots, r_n\}$ is a set of linearly independent elements of the $\mathbb{Q}$-vector space $\mathbb{R}$. Then the element $x + \mathbb{Z} \in \mathbb{R}^n/\mathbb{Z}^n$, $x = (r_1, \ldots, r_n)$ has the property that $\mathbb{Z}{\cdot}(x + \mathbb{Z})$ is dense.* See [34], Chap. 7, §1, n^o 3, Corollaire 2 de la Proposition 7.] □

Projective Limits

Definition 1.25. Let J be a directed set, that is, a set with a reflexive, transitive and antisymmetric relation $\leq$ such that every finite nonempty subset has an upper bound. A *projective system of topological groups over J* is a family of morphisms $\{f_{jk}{:}\, G_k \to G_j \mid (j,k) \in J \times J,\, j \leq k\}$, where G_j, $j \in J$ are topological groups, satisfying the following conditions:
 (i) $f_{jj} = \mathrm{id}_{G_j}$ for all $j \in J$
 (ii) $f_{jk} \circ f_{kl} = f_{jl}$ for all $j, k, l \in J$ with $j \leq k \leq l$. □

Lemma 1.26. (i) *For a projective system of topological groups, define the topological group P by $P = \prod_{j\in J} G_j$. Set*

$$G = \{(g_j)_{j\in J} \in P \mid (\forall j, k \in J)\, j \leq k \Rightarrow f_{jk}(g_k) = g_j\}.$$

Then G is a closed subgroup of P. If $\mathrm{inc}{:}\, G \to P$ denotes the inclusion and $\mathrm{pr}_j{:}\, P \to G_j$ the projection, then the function $f_j = \mathrm{pr}_j \circ \mathrm{inc}{:}\, G \to G_j$ is a morphism of topological groups for all $j \in J$, and for $j \leq k$ in J the relation $f_j = f_{jk} \circ f_k$ is satisfied.

(ii) *If all groups G_j in the projective system are compact, then P and G are compact groups.*

Proof. (i) Assume that $j \leq k$ in J. Define $G_{jk} = \{(g_l)_{l\in J} \in P \mid f_{jk}(g_k) = g_j\}$. Since f_{jk} is a morphism of groups, this set is a subgroup of P, and since f_{jk} is continuous, it is a closed subgroup. But $G = \bigcap_{(j,k)\in J\times J,\, j\leq k} G_{jk}$. Hence G is a closed subgroup. The remainder is straightforward.

(ii) If all G_j are compact, then P is compact by Tychonoff's Theorem, and thus G as a closed subgroup of P is compact, too. □

Definitions 1.27. If $\mathcal{P} = \{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\ j \leq k\}$ is a projective system of topological groups, then the group G of Lemma 1.26 is called its *projective limit* and is written $G = \lim \mathcal{P}$. As a rule it suffices to remind oneself of the entire projective system by recording the family of groups G_j involved in it; therefore the notation $G = \lim_{j \in J} G_j$ is also customary. The morphisms $f_j: G \to G_j$ are called *limit maps* and the morphisms $f_{jk}: G_k \to G_j$ are called *bonding maps*. □

Example 1.28. Assume that we have a sequence $\varphi_n: G_{n+1} \to G_n$, $n \in \mathbb{N}$ of morphisms of compact groups:

$$G_1 \stackrel{\varphi_1}{\leftarrow} G_2 \stackrel{\varphi_2}{\leftarrow} G_3 \stackrel{\varphi_3}{\leftarrow} G_4 \stackrel{\varphi_4}{\leftarrow} \cdots$$

Then we obtain a projective system of compact groups by defining $f_{jj} = \mathrm{id}_{G_j}$ and, for $j < k$ the morphisms

$$f_{jk} = \varphi_j \circ \varphi_{j+1} \circ \cdots \circ \varphi_{k-1}: G_k \to G_j.$$

Then $G = \lim_{n \in \mathbb{N}} G_n$ is simply given by $\{(g_n)_{n \in \mathbb{N}} \mid (\forall n \in \mathbb{N})\, \varphi_n(g_{n+1}) = g_n\}$.

(i) Choose a natural number p and set $G_n = \mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. Define $\varphi_n: \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ by $\varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}$:

$$\mathbb{Z}(p) \stackrel{\varphi_1}{\leftarrow} \mathbb{Z}(p^2) \stackrel{\varphi_2}{\leftarrow} \mathbb{Z}(p^3) \stackrel{\varphi_3}{\leftarrow} \mathbb{Z}(p^4) \stackrel{\varphi_4}{\leftarrow} \cdots$$

The projective limit of this system is called the *group* $\mathbb{Z}_p$ *of p-adic integers.*

(ii) Set $G_n = \mathbb{T}$ for all $n \in \mathbb{N}$ and define $\varphi_n(g) = p{\cdot}g$ for all $n \in \mathbb{N}$ and $g \in \mathbb{T}$. (It is customary, however, to write p in place of φ_n):

$$\mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \cdots$$

The projective limit of this system is called the *p-adic solenoid* $\mathbb{T}_p$. □

Let us discuss these examples in the following exercises:

Exercise E1.10. (i) Observe that the bonding maps $\varphi_1, \varphi_2, \ldots$ are morphisms of rings. Prove that $\mathbb{Z}_p$ is a compact ring with continuous multiplication so that all limit maps $f_n: \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ are morphisms of rings.

(ii) Define $\eta: \mathbb{Z} \to \mathbb{Z}_p$ by $\eta(z) = (z + p^n\mathbb{Z})_{n \in \mathbb{N}}$. Show that this is a well defined injective morphism of rings.

(iii) Prove the following statement: For an arbitrary element $g = (z_n + p^n\mathbb{Z})_{n \in \mathbb{N}} \in \mathbb{Z}_p$, the sequence $\big(\eta(z_n)\big)_{n \in \mathbb{N}}$ converges to g in $\mathbb{Z}_p$. Conclude that η has a dense image.

(iv) Show that $\mathbb{Z}_p$ is totally disconnected.

(v) Show that the limit map f_m has kernel $\{(z_n + p^n\mathbb{Z})_{n \in \mathbb{N}} \mid z_m \equiv 0 \pmod{p^m}\}$. Show that it is $p^m\mathbb{Z}_p = \overline{\eta(p^m\mathbb{Z})}$. Prove that the subgroups $p^m\mathbb{Z}_p$ are open and closed and form a basis for the filter of neighborhoods of 0.

(vi) Show that the limit of the system

$$(\frac{1}{p}\cdot\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^2}\cdot\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^3}\cdot\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^4}\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} \cdots$$

is a group $\mathbb{Z}'_p$ isomorphic to $\mathbb{Z}_p$.

(vii) Show that $\mathbb{Z}_p$ is torsion-free; that is, it has no elements of finite order). □

The following lemma will be useful in the next exercise and in many similar situations.

Lemma 1.29. *Assume that $f: G \to H$ is a morphism of topological groups. Then the following conditions are equivalent:*

(1) *The kernel N of f is discrete and f is open.*

(2) *There is an open neighborhood U of $\mathbf{1}$ such that $f|U: U \to f(U)$ is a homeomorphism onto an open identity neighborhood in H.*

Proof. (1)⇒(2) Since multiplication and inversion are continuous in G and since $\{\mathbf{1}\}$ is an open subset in N, we find an open neighborhood U of $\mathbf{1}$ in G so that $UU^{-1} \cap N = \{\mathbf{1}\}$. Now assume that $f(u) = f(v)$ with $u, v \in U$. Then $f(uv^{-1}) = f(u)f(v)^{-1} = \mathbf{1}$, whence $uv^{-1} \in UU^{-1} \cap N = \{\mathbf{1}\}$. Thus $u = v$ and $f|U: U \to f(U)$ is bijective. By hypothesis f and thus $f|U$ is both continuous and open.

(2)⇒(1) Since $f|U$ is injective, $f(u) = \mathbf{1} = f(\mathbf{1})$ implies $u = \mathbf{1}$. Thus $U \cap N = \{\mathbf{1}\}$. As U is an open set, this implies that $\{\mathbf{1}\}$ is open in N and thus N is discrete since translations on N are homeomorphisms. Let W be an open set in G and $w \in W$. Then $U \cap w^{-1}W$ is an open neighborhood of $\mathbf{1}$ in G. Thus, by (2), $f(U \cap w^{-1}W)$ is an open set W_w in H. Hence $f(w)W_w = f\big(w(U \cap w^{-1}W)\big) = f(wU \cap W) \subseteq f(W)$ is an open neighborhood of $f(w)$ in H contained in $f(W)$. Hence $f(W)$ is open. Thus f is an open map. □

Definition 1.30. A morphism of topological groups $f: G \to H$ is said to *implement a local isomorphism* if there is an open neighborhood U of $\mathbf{1}$ in G such that $f(U)$ is open in H and $f|U: U \to f(U)$ is a homeomorphism. □

This is precisely the situation of Lemma 1.29.

Exercise E1.11. (i) Let $f: \mathbb{T}_p \to \mathbb{T}$ be defined by $f\big((r_n + \mathbb{Z})_{n\in\mathbb{N}}\big) = pr_1 + \mathbb{Z}$ and show that $\mathbb{Z}'_p = \ker f$. In other words, there is an exact sequence:

$$0 \to \mathbb{Z}_p \xrightarrow{\alpha} \mathbb{T}_p \xrightarrow{f} \mathbb{T} \to 0.$$

(ii) Show that the morphisms in the sequence of abelian topological groups

$$0 \to \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}'_p \times \mathbb{R} \xrightarrow{\pi} \mathbb{T}_p \to 0$$

are well defined by

$$\varphi(z) = \big((p^{-n}z + \mathbb{Z})_{n\in\mathbb{N}}, -z\big), \qquad \pi\big((z_n + \mathbb{Z})_{n\in\mathbb{N}}, r\big) = (z_n + p^{-n}r + \mathbb{Z})_{n\in\mathbb{N}}.$$

Show that this sequence is exact. Conclude that π implements a local isomorphism between $\mathbb{Z}'_p \times \mathbb{R}$ and $\mathbb{T}_p$. Show that $\pi(\mathbb{Z}'_p \times \{0\}) = \ker f_1$.

In Theorem 8.22 of Chapter 8, we will give characterizations of finite dimensional compact abelian groups.

(iii) Visualize a compact subspace of $\mathbb{R}^3$ obtained as follows. Embed a solid torus (= doughnut) D_1 into euclidean space. Embed a second solid torus D_2 into D_1 by winding it around p times inside D_1. Embed a third solid torus D_3 into D_2 by winding it p times around inside D_2 (and hence p^2 times inside D_1). Continue recursively in this fashion and form the intersection D_∞ of the descending family $D_1 \supset D_2 \supset D_3 \supset \cdots$. Prove—at least visualize!—that D_∞ is homeomorphic to $\mathbb{T}_p$.

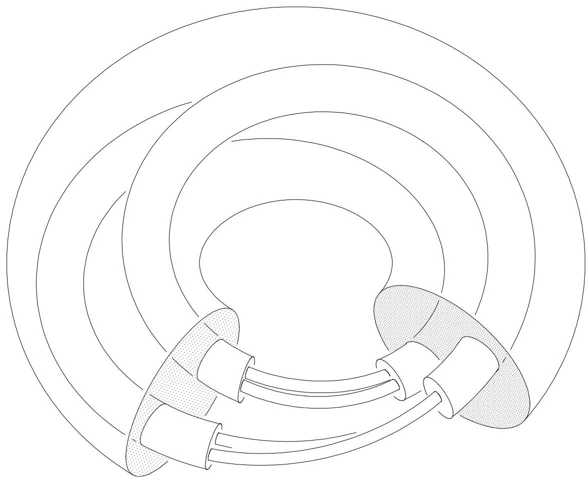


Figure 1.1: The dyadic solenoid

(iv) Show that $\mathbb{T}_p$ is connected but not arcwise connected. (The arc component of 0 in $\mathbb{T}_p$ is $\{(p^{-n}r + \mathbb{Z})_{n\in\mathbb{N}} \mid r \in \mathbb{R}\}$.)

Proposition 1.31. *Assume that* $G = \lim_{j\in J} G_j$ *for a projective system* $f_{jk}: G_k \to G_j$ *of compact groups,* $j \le k$ *in* J*, and denote with* $f_j: G \to G_j$ *the limit maps. Then the following statements are equivalent:*

(1) *All bonding maps* f_{jk} *are surjective.*
(2) *All limit maps* f_j *are surjective.*

Proof. (1)⇒(2) Fix $i \in J$. Let $h \in G_i$; we must find an element $g = (g_j)_{j\in J} \in G$ with $g_i = f_i(g) = h$. For all $k \in J$ with $i \le k$ we define $C_k \subseteq \prod_{j\in J} G_j$ by

$$\{(x_j)_{j\in J} \mid (\forall j \le k)\, x_j = f_{jk}(x_k) \text{ and } x_i = h\}.$$

Since f_{ik} is surjective, $C_k \ne \emptyset$. If $i \le k \le k'$ then we claim $C_{k'} \subseteq C_k$. Indeed $(x_j)_{j\in J} \in C_{k'}$ implies $f_{jk}(x_k) = f_{jk}f_{kk'}(x_{k'}) = f_{jk'}(x_{k'}) = x_j$ and $x_i = h$. Thus $(x_j)_{j\in J} \in C_k$ and the claim is established. Now $\{C_k \mid k \in J, i \le k\}$ is a filter basis of compact sets in $\prod_{j\in J} G_j$ and thus has nonempty intersection. Assume that $g = (g_m)_{m\in J}$ is in this intersection. Then, firstly, $g_i = h$. Secondly, let $j \le k$. Since J is directed, there is a k' with $i, k \le k'$. Then $(g_m)_{m\in J} \in C_{k'}$. Hence $g_j = f_{jk'}(g_{k'}) = f_{jk}f_{kk'}(g_{k'}) = f_{jk}(g_k)$ by the definition of $C_{k'}$. Hence $g \in \lim_{j\in J} G_j$. Thus g is one of the elements we looked for.

(2)⇒(1) Let $j \le k$. Then $f_j = f_{jk}f_k$. Thus the surjectivity of f_j implies that of f_{jk}. □

Definition 1.32. A projective system of topological groups in which all bonding maps and all limit maps are surjective is called a *strict projective system* and its limit is called a *strict projective limit*. □

Proposition 1.33. (i) *Let $G = \lim_{j\in J} G_j$ be a projective limit of compact groups. Let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then*

(a) *$\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$.*

(b) *$\mathcal{N}$ is a filter basis of compact normal subgroups converging to $\mathbf{1}$. (That is, given a neighborhood U of $\mathbf{1}$, there is an $N \in \mathcal{N}$ such that $N \subseteq U$.)*

(ii) *Conversely, assume that G is a compact group with a filter basis $\mathcal{N}$ of compact normal subgroups with $\bigcap\mathcal{N} = \{\mathbf{1}\}$. For $M \subseteq N$ in $\mathcal{N}$ let $f_{NM}: G/M \to G/N$ denote the natural morphism given by $f_{NM}(gM) = gN$. Then the f_{NM} constitute a strict projective system whose limit is isomorphic to G under the map $g \mapsto (gN)_{N\in\mathcal{N}}: G \to \lim_{N\in\mathcal{N}} G/N$. With this isomorphism, the limit maps are equivalent to the quotient maps $G \to G/N$.*

Proof. (i)(a) Let $V \in \mathcal{U}$. Then by the definition of the projective limit there is an identity neighborhood of $\prod_{j\in J} G_j$ of the form $W = \prod_{j\in J} W_j$ with $W_j \in \mathcal{U}_j$ for which there is a finite subset F of J such that $j \in J \setminus F$ implies $W_j = G_j$ such that $W \cap \lim_{j\in J} G_j \subseteq V$. Since J is directed, there is an upper bound $k \in J$ of F. There is a $U \in \mathcal{U}_k$ such that $f_{jk}(U) \subseteq W_j$ for all $j \in F$. Then $f_k^{-1}(U) \subseteq W \cap \lim_{j\in J} G_j \subseteq V$.

(i)(b) Evidently, each $\ker f_j$ is a compact normal subgroup. Since $i, j \le k$ implies $\ker f_k \subseteq \ker f_i \cap \ker f_j$ and J is directed, $\mathcal{N}$ is a filter basis. For each $j \in J$ we have $\ker f_j = f_j^{-1}(1) \subseteq f_j^{-1}(U)$ for any $U \in \mathcal{U}_j$. Since $f_j^{-1}(U)$ is a basic neighborhood of the identity by (a), we are done.

(ii) It is readily verified that the family of all morphisms $f_{NM}: G/M \to G/N$ for $M \subseteq N$ in $\mathcal{N}$ constitutes a strict projective system of compact groups. An element $(g_N N)_{N\in\mathcal{N}} \in \prod_{N\in\mathcal{N}} G/N$ with $g_N \in G$ is in its limit L if and only if for each pair $M \supseteq N$ in $\mathcal{N}$ we have $f_{MN}(g_N N) = g_M M$, that is, $g_M^{-1} g_N \in M$. Thus for each $g \in G$ certainly $(gN)_{N\in\mathcal{N}} \in L$. The kernel of the morphism $\varphi = (g \mapsto (gN)_{N\in\mathcal{N}}): G \to L$ is $\bigcap\mathcal{N} = \{\mathbf{1}\}$. Hence φ is injective. Assume $\gamma = (g_N N)_{N\in\mathcal{N}} \in L$. Then $\{g_N N \mid N \in \mathcal{N}\}$ is a filter basis of compact sets in G, for if $M \supseteq N$ then $g_M^{-1} g_N \in M$, and thus $g_N \in g_M M \cap g_N N$. Hence its intersection contains an element g and then $g \in g_N N$ is equivalent to $gN = g_N N$. Thus $\varphi(g) = \gamma$. We have shown that φ is also surjective and thus is an isomorphism of compact groups (see Remark 1.8). If $q_N\colon G \to G/N$ is the quotient map, and if $f_N\colon L \to G/N$ is the limit map defined by $f_N\big((g_N N)_{N\in\mathcal{N}}\big) = g_N N$, then clearly $q_N = f_N \circ \varphi$. The proof of the proposition is now complete. □

The significance of the preceding proposition is that we can think of a strict projective limit G as a compact group which is approximated by factor groups G/N modulo smaller and smaller normal subgroups N. This is not a bad image. The group G is decomposed into cosets gN whose size can be made as small as we wish using the normal subgroups in the filter basis $\mathcal{N}$.

Some special cases are of considerable theoretical interest.

Totally Disconnected Compact Groups

Theorem 1.34. *For a locally compact group G, the following statements are equivalent:*

(1) *The filter of neighborhoods of the identity has a basis of open subgroups.*

(2) *G is totally disconnected.*

If G is compact, these conditions are also equivalent to the following ones:

(3) *The filter of neighborhoods of the identity has a basis of open normal subgroups.*

(4) *G is a strict projective limit of finite groups.*

Proof. (1)⇒(2) Since every open subgroup is also closed, condition (1) implies that $\{1\}$ is the intersection of open and closed subsets. Therefore $\{1\}$ is the connected component of the identity and thus G is totally disconnected.

(2)⇒(1) Fix a compact identity neighborhood W. Since the component of 1 in the compact space W is singleton, 1 has a basis of open and closed neighborhoods in W. (Cf. Exercise E1.12 below.) Now let U be an open and closed neighborhood of $\{\mathbf{1}\}$ in the interior of W. Then U will be open compact in G. Now there is a compact neighborhood V of $\{\mathbf{1}\}$ such that $UV \subseteq U$, for if not, then the family $\{UV \setminus U \mid V \in \mathcal{U}\}$ (with the neighborhood filter $\mathcal{U}$ of $\mathbf{1}$) is a filter basis of compact sets, since U is compact open. A point g in its intersection is contained in the complement of U. On the other hand, $g \in U$, since the relation $g \in UV^{-1}$ implies $gV \cap U \neq \emptyset$ for any identity neighborhood V, whence $g \in \overline{U} = U$. Now choose a symmetric open neighborhood $V = V^{-1}$ with $UV \subseteq U$. Recursively, we find $UV^n \subseteq U$. But $H = \bigcup_{n\in\mathbb{N}} V^n$ is the subgroup generated by V in G. Hence $H \subseteq UH \subseteq U$. Thus any neighborhood U contains an open subgroup and (1) is proved.

Now we assume that G is compact.

(1)⇒(3) If H is an open subgroup of G, then

$$N = \bigcap_{g\in G} gHg^{-1}$$

is an open normal subgroup contained in H by Corollary 1.12. Thus (1) implies (3).

(3)⇒(4) Let $\mathcal{N}$ be a filter basis of the filter of identity neighborhoods consisting of open normal subgroups. An open subgroup N in any topological group is the complement of the union of all other cosets, each of which is open. Hence it is automatically closed. Thus G/N is a compact and discrete group, and hence it is finite. By Proposition 1.33, G is the strict projective limit of the factor groups G/N. This proves (4).

(4)⇒(1) If $G = \lim_{j \in J} G_j$ is a strict projective limit of finite groups then it is a subgroup of $\prod_{j \in J} G_j$. Products of totally disconnected spaces are totally disconnected, and subspaces of totally disconnected spaces are totally disconnected. □

The compact groups characterized by the equivalent conditions of Theorem 1.34 are also called *profinite groups*, in view of condition (4).

Exercise E1.12. (i) Prove that in a compact space the connectivity relation is the intersection of all equivalence relations with open compact equivalence classes.

(ii) Show that in any topological group, the identity component is a closed fully characteristic subgroup. (A subgroup of a topological group is called *characteristic*, if it is invariant under all (continuous and continuously invertible!) automorphisms. It is called *fully characteristic*, if it is invariant under all (continuous!) endomorphisms.)

(iii) Show that in every locally compact group G, the identity component G_0 is the intersection of the set of all open subgroups H such that H/G_0 is compact.

[Hint. For (iii): Consider the factor group G/G_0. Observe that it is totally disconnected and locally compact. Then utilize the equivalence of (1) and (2) in Theorem 1.34.] □

Exercise E1.13. Prove the following proposition.

A surjective homomorphic image of a totally disconnected compact group is totally disconnected.

[Hint. Let G be totally disconnected compact and $f\colon G \to H$ a surjective morphism of compact groups. Let $K \stackrel{\text{def}}{=} \ker f$ be its kernel. By Proposition 1.10(iv) we may assume that $H = G/K$ and that f is the quotient morphism. By 1.34, the identity of G has a neighborhood basis $\mathcal{N}(G)$ consisting of compact (normal) open subgroups N. Then the subgroup $KN = \bigcup_{k \in K} kN$ is open and $K = \bigcap_{N \in \mathcal{N}(G)} KN$ (since $K = \overline{K} = \bigcap_{U \in \mathcal{U}} KU$ with the filter $\mathcal{U}$ of identity neighborhoods. Thus the identity of G/K has a neighborhood basis of open subgroups, which are, therefore, closed. Thus G/K is totally disconnected.] □

Some Duality Theory

Let A be an arbitrary abelian group. Let $\mathcal{F}$ denote the family of all finitely generated subgroups. This family is directed, for if $F, E \in \mathcal{F}$ then $F + E \in \mathcal{F}$. Also, $A = \bigcup_{F\in\mathcal{F}} F$. If $E, F \in \mathcal{F}$ and $E \subseteq F$ then the inclusion $E \to F$ induces a morphism $f_{EF}: \widehat{F} \to \widehat{E}$ via $f_{EF}(\chi) = \chi|E$ for $\chi: F \to \mathbb{T}$. The family $\{f_{EF}: \widehat{F} \to \widehat{E} \mid E, F \in \mathcal{F}, E \subseteq F\}$ is a projective system of compact abelian groups. By the divisibility of $\mathbb{T}$, each character on $E \subseteq F$ extends to one on F and so this system is strict. The inclusion $F \to A$ induces a morphism $f_F: \widehat{A} \to \widehat{F}$ by $f_F(\chi) = \chi|F$ for each character $\chi: A \to \mathbb{T}$.

Proposition 1.35. *The map* $\chi \mapsto (\chi|F)_{F\in\mathcal{F}}: \widehat{A} \to \lim_{F\in\mathcal{F}} \widehat{F}$ *is an isomorphism of compact abelian groups.*

Proof. Define $\varphi: \mathrm{Hom}(A, \mathbb{T}) \to \lim_{F\in\mathcal{F}} \mathrm{Hom}(F, \mathbb{T})$ by $\varphi(\chi) = (\chi|F)_{F\in\mathcal{F}}$. This definition yields a morphism of compact groups. A character χ of A is in its kernel if and only if $\chi|F = 0$ for all $F \in \mathcal{F}$. But since $A = \bigcup_{F\in\mathcal{F}} F$ this is the case if and only if $\chi = 0$. Thus φ is injective. Now let $\gamma = (\chi_F)_{F\in\mathcal{F}} \in \lim_{F\in\mathcal{F}} \widehat{F}$. By the definition of the bonding maps, this means that for every pair of finitely generated subgroups $E \subseteq F$ in A we have $\chi_F|E = \chi_E$. Now we can unambiguously define a function $\chi: A \to \mathbb{T}$ as follows. We pick for each $a \in A$ an $F \in \mathcal{F}$ with $a \in F$ (for instance, $F = \mathbb{Z}{\cdot}a$). By the preceding, the element $\chi_F(a)$ in $\mathbb{T}$ does not depend on the choice of F. Hence we define a function $\chi: A \to \mathbb{T}$ by $\chi(a) = \chi_F(a)$. If $a, b \in A$, take $F = \mathbb{Z}{\cdot}a + \mathbb{Z}{\cdot}b$ and observe $\chi(a+b) = \chi_F(a+b) = \chi_F(a) + \chi_F(b) = \chi(a) + \chi(b)$. Thus $\chi \in \mathrm{Hom}(A, \mathbb{T})$ and $\chi|F = \chi_F$. Hence $\varphi(\chi) = \gamma$. Thus φ is bijective and hence an isomorphism of compact groups (see Remark 1.8). ☐

In short: *The character group $\widehat{A}$ of any abelian group A is the strict projective limit of the character groups $\widehat{F}$ of its finitely generated subgroups F.* We know that $\widehat{F}$ is a direct product of a finite group and a finite-dimensional torus group (see Remark 1.18). In particular, every character group of an abelian group is approximated by compact abelian groups on manifolds.

Assume that $G = \lim_{j\in J} G_j$ is a strict projective limit of compact abelian groups with limit maps $f_j: G \to G_j$. Every character $\chi: G_j \to \mathbb{T}$ gives a character $\chi \circ f_j: G \to \mathbb{T}$ of G. Since f_j is surjective, $\chi \mapsto \chi \circ f_j: \widehat{G_j} \to \widehat{G}$ is injective. Under this map, we identify $\widehat{G_j}$ with a subgroup of $\widehat{G}$.

Proposition 1.36. *If G is a strict projective limit* $\lim_{j\in J} G_j$ *then* $\widehat{G} = \bigcup_{j\in J} \widehat{G_j}$.

Proof. With our identification of $\widehat{G}_j$ as a subgroup of $\widehat{G}$, the right side is contained in the left one. Now assume that $\chi: G \to \mathbb{T}$ is a character of G. If we denote with V the image of $]-\frac{1}{3}, \frac{1}{3}[$ in $\mathbb{T}$, then $\{\mathbf{0}\}$ is the only subgroup of $\mathbb{T}$ which is contained in V. Now $U = \chi^{-1}(V)$ is an open neighborhood of 0 in G. Hence by Proposition 1.33(i) there is a $j \in J$ such that $\ker f_j \subseteq U$. Hence $\chi(\ker f_j)$ is a subgroup of

$\mathbb{T}$ contained in V and therefore is $\{\mathbf{0}\}$. Thus $\ker f_j \subseteq \ker \chi$ and there is a unique morphism $\chi_j : G_j \to \mathbb{T}$ such that $\chi = \chi_j \circ f_j$. With our convention, this means exactly $\chi \in \widehat{G_j}$. Thus $\widehat{G} \subseteq \bigcup_{j\in J} \widehat{G_j}$. □

The next theorem is one half of the famous Pontryagin Duality Theorem for compact abelian groups.

Theorem 1.37. *For any abelian group A the morphism $\eta_A : A \to \widehat{\widehat{A}}$ is an isomorphism.*

Proof. We know that $\widehat{A}$ is the strict projective limit $\lim_{F\in\mathcal{F}} \widehat{F}$ with the directed family $\mathcal{F}$ of finitely generated subgroups of A. (See Proposition 1.35.) The limit maps $f_F : \widehat{A} \to \widehat{F}$ are given by $f_F(\chi) = \chi|F$, and these surjective maps induce injective morphisms $\mathrm{Hom}(f_F, \mathbb{T}) : \mathrm{Hom}(\widehat{F}, \mathbb{T}) \to \mathrm{Hom}(\widehat{A}, \mathbb{T})$ with $\mathrm{Hom}(f_F, \mathbb{T})(\Sigma) = \Sigma \circ f_F$. By Proposition 1.36, $\mathrm{Hom}(\widehat{A}, \mathbb{T})$ is the union of the images of the injective morphisms $\mathrm{Hom}(f_F, \mathbb{T})$. Thus for any $\Omega \in \mathrm{Hom}(\widehat{A}, \mathbb{T})$ there is an $F \in \mathcal{F}$ such that Ω is in the image of $\mathrm{Hom}(f_F, \mathbb{T})$. Hence there is a $\Sigma \in \mathrm{Hom}(\widehat{F}, \mathbb{T})$ such that $\Omega = \mathrm{Hom}(f_F, \mathbb{T})(\Sigma) = \Sigma \circ f_F$. But $\eta_F : F \to \mathrm{Hom}(\widehat{F}, \mathbb{T})$ is an isomorphism by Remark 1.24(i). Hence there is an $a \in F$ such that $\Sigma = \eta_F(a)$. Thus $\Omega = \eta_F(a) \circ f_F$. Therefore, for any character $\chi : A \to \mathbb{T}$ of A we have $\Omega(\chi) = \eta_F(a)\big(f_F(\chi)\big) = \eta_F(a)(\chi|F) = (\chi|F)(a) = \chi(a) = \eta_A(a)(\chi)$. Thus η_A is surjective. The injectivity was established in Lemma 1.23. □

It is helpful to visualize our argument by diagram chasing:

$$\begin{array}{ccc}
F & \xrightarrow{\quad\eta_F\quad} & \mathrm{Hom}(\widehat{F}, \mathbb{T}) \\
{\scriptstyle \mathrm{inc}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\widehat{\mathrm{inc}}, \mathbb{T})} \\
A & \xrightarrow[\quad\eta_A\quad]{} & \mathrm{Hom}(\widehat{A}, \mathbb{T}).
\end{array}$$

The other half of the Pontryagin Duality Theorem claims that $\eta_G : G \to \widehat{\widehat{G}}$ is an isomorphism for any compact abelian group G, too. We cannot prove this at the present level of information. However, in practicing the concept of a projective limit we can take one additional step.

Let us, at least temporarily, use the parlance that a compact abelian group G is said *to have duality* if $\eta_G : G \to \widehat{\widehat{G}}$ is an isomorphism. We propose the following exercise whose proof we indicate rather completely since it is of independent interest.

Exercise E1.14. *If a compact abelian group G is the limit $\lim_{j\in J} G_j$ of a strict projective system of compact abelian groups G_j which have duality, then G has duality.*

Proof. After Lemma 1.23, we have to show that $\eta_G\colon G \to \widehat{\widehat{G}}$ is bijective. We attack the harder part first and show that η_G is surjective. Assume that $\Omega \in \widehat{\widehat{G}}$; that is, Ω is a morphism of abelian groups $\widehat{G} \to \mathbb{T}$. By Proposition 1.36, $\widehat{G} = \bigcup_{j\in J} \widehat{G_j}$. If we denote with Ω_j the restriction $\Omega|\widehat{G_j}$, then $\Omega_j\colon \widehat{G_j} \to \mathbb{T}$ is an element of $\widehat{\widehat{G_j}}$. Since G_j has duality by hypothesis, η_{G_j} is *surjective* and thus there is a $g_j \in G_j$ such that $\eta_{G_j}(g_j) = \Omega_j$. We claim that $g \stackrel{\rm def}{=} (g_j)_{j\in J} \in \prod_{j\in J} G_j$ is an element of $\lim_{j\in J} G_j = G$. For this purpose assume that $j \le k$ in J. We have a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\quad\eta_{G_k}\quad} & \widehat{\widehat{G_k}} \\ {\scriptstyle f_{jk}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f_{jk}}}} \\ G_j & \xrightarrow[\quad\eta_{G_j}\quad]{} & \widehat{\widehat{G_j}}. \end{array}$$

(We shall consider this claim in a separate exercise below.) We notice that

$$\widehat{\widehat{f_{jk}}}\colon \widehat{\widehat{G_k}} \to \widehat{\widehat{G_j}}$$

is the restriction map sending Ω_k to $\Omega_k|\widehat{G_j} = \Omega_j$. Thus

$$\eta_{G_j}\big(f_{jk}(g_k)\big) = \widehat{\widehat{f_{jk}}}\big(\eta_{G_k}(g_k)\big) = \widehat{\widehat{f_{jk}}}(\Omega_k) = \Omega_j = \eta_{G_j}(g_j).$$

But since G_j has duality, η_{G_j} is *injective*, and thus

$$f_{jk}(g_k) = g_j,$$

which establishes the claim $g \in \lim_{j\in J} G_j$. For each limit map $f_j\colon G \to G_j$, as before, we have a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\quad\eta_{G}\quad} & \widehat{\widehat{G}} \\ {\scriptstyle f_{j}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f_{j}}}} \\ G_j & \xrightarrow[\quad\eta_{G_j}\quad]{} & \widehat{\widehat{G_j}}. \end{array}$$

Thus $\widehat{\widehat{f_j}}\big(\eta_G(g)\big) = \eta_{G_j}\big(f_j(g)\big) = \eta_{G_j}(g_j) = \Omega_j$ for all $j \in J$. Now we observe that $\widehat{\widehat{f_j}}\colon \widehat{\widehat{G}} \to \widehat{\widehat{G_j}}$ is the restriction $\Sigma \to \Sigma|\widehat{G_j}$. Thus the restriction of the morphism $\eta_G(g)\colon \widehat{G} \to \mathbb{T}$ to each $\widehat{G_j}$ is Ω_j, and therefore this morphism is none other than the given map Ω. Hence $\eta_G(g) = \Omega$ and the claim of the surjectivity of η_G is proved.

As a second step we show that η_G is injective. We have observed before that this statement is equivalent to the assertion that the characters of G separate the points. Hence we assume that $0 \ne g \in G$. Set $\mathcal{N} = \{\ker f_j \mid j \in J\}$. From Proposition 1.33(i) we know that $\bigcap \mathcal{N} = \{0\}$. Hence there is a $j \in J$ such that $g \notin \ker f_j$, that is, $f_j(g) \ne 0$. Since the group G_j has duality, its characters separate its points. Hence there is a $\chi \in \widehat{G_j}$ such that $\chi\big(f_j(g)\big) \ne 0$. Hence $\chi \circ f_j \in \widehat{G}$ is a character of G which does not annihilate g. The assertion is now proved. $\square$

Exercise E1.15. Prove the following statements

(i) *Assume that* $f: A \to B$ *is a morphism of abelian groups. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \widehat{\widehat{A}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{\widehat{f}}} \\ B & \xrightarrow[\eta_B]{} & \widehat{\widehat{B}} \end{array}$$

is commutative.

(ii) *For a morphism* $f: G \to H$ *of compact abelian groups, the analogous statement is true.* □

In view of Remark 1.24, the Exercise E1.14 shows, in particular, that any strict projective limit of groups isomorphic to products of finite-dimensional torus groups and finite groups have duality. Let us observe that this applies to the examples in 1.28.

We set

$$\bigcup_{n\in\mathbb{N}} \frac{1}{p^n}\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z}$$

and notice that this set is subring of $\mathbb{Q}$ (cf. Appendix 1, Definition A1.30).

Example 1.38. (i) The group $\mathbb{Z}_p$ of all p-adic integers has duality and its character group is the group $\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$ of all elements in $\mathbb{T}$ of p-power order. This group is isomorphic to the subgroup of $\mathbb{S}^1$ of all p^n-th roots of unity for $n = 1, 2, \ldots$.

(ii) The p-adic solenoid $\mathbb{T}_p$ has duality and its character group is the group $\frac{1}{p^\infty}\mathbb{Z}$ of all rational numbers which can be represented as m/p^n, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

Proof. The groups $\mathbb{Z}(p^n)$ and $\mathbb{T}$ have duality by Remark 1.24. Hence in both examples Exercise E1.14 applies and shows that $\mathbb{Z}_p$ and $\mathbb{T}_p$ have duality. The identification of their respective character groups is in both cases a consequence of Proposition 1.36.

Example (i): The dual of the morphism $\varphi_n: \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ is the inclusion $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \to \frac{1}{p^{n+1}}\mathbb{Z}/\mathbb{Z}$. Hence Proposition 1.36 implies $\widehat{\mathbb{Z}_p} = \bigcup_{n\in\mathbb{N}} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty)$.

Example (ii): The dual of the morphism $\mu_p: \mathbb{T} \to \mathbb{T}$ is the morphism $\mu_p: \mathbb{Z} \to \mathbb{Z}$. This map is equivalent to the inclusion $\mathbb{Z} \to \frac{1}{p}\mathbb{Z}$. Hence by Proposition 1.36, we obtain $\widehat{\mathbb{T}_p} = \bigcup_{n\in\mathbb{N}} \frac{1}{p^n}\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z}$. □

Exercise E1.16. In 1.28(i) and E1.10 we introduced the compact ring $\mathbb{Z}_p$ of p-adic integers. Immediately prior to 1.38 we have also defined the ring $\frac{1}{p^\infty}\mathbb{Z}$. We now define a locally compact abelian group $\mathbb{Q}_p$ containing $\mathbb{Z}_p$ as an open, and $\frac{1}{p^\infty}\mathbb{Z}$ as a dense subgroup.

Define

$$\Phi_n: \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \to \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}, \quad \Phi_n(q + p^{n+1}\mathbb{Z}) = q + p^n\mathbb{Z}$$

and set

$$\mathbb{Q}_p = \lim\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p\mathbb{Z}} \xleftarrow{\Phi_1} \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^2\mathbb{Z}} \xleftarrow{\Phi_2} \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^3\mathbb{Z}} \xleftarrow{\Phi_3} \cdots\right).$$

Then we have a commutative diagram in which the rows are limit diagrams and the columns are exact

$$\begin{array}{ccccccc}
\cdots & 0 & \longleftarrow & 0 & \cdots & \longleftarrow & 0 \\
 & \downarrow & & \downarrow & & & \downarrow \\
\cdots & \frac{\mathbb{Z}}{p^n\mathbb{Z}} & \xleftarrow{\varphi_n} & \frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} & \cdots & \longleftarrow & \mathbb{Z}_p \\
 & \downarrow \text{incl} & & \downarrow \text{incl} & & & \downarrow \text{incl} \\
\cdots & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}} & \xleftarrow{\Phi_n} & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} & \cdots & \longleftarrow & \mathbb{Q}_p \\
 & \downarrow \text{quot} & & \downarrow \text{quot} & & & \downarrow \text{quot} \\
\cdots & \mathbb{Z}(p^\infty) & \xleftarrow{\text{id}} & \mathbb{Z}(p^\infty) & \cdots & \longleftarrow & \mathbb{Z}(p^\infty) \\
 & \downarrow & & \downarrow & & & \downarrow \\
\cdots & 0 & \longleftarrow & 0 & \cdots & \longleftarrow & 0.
\end{array}$$

Show that $\mathbb{Q}_p$ contains a copy of $\frac{1}{p^\infty}\mathbb{Z}$ densely. Show that $\mathbb{Q}_p$ is a $\frac{1}{p^\infty}\mathbb{Z}$-module such that the module operation $(q,r) \mapsto qr\colon \frac{1}{p^\infty}\mathbb{Z} \times \mathbb{Q}_p \to \mathbb{Q}_p$ extends the multiplication of $\frac{1}{p^\infty}\mathbb{Z}$. Show that

$$\mathbb{Q}_p = \mathbb{Z}_p \cup \frac{1}{p}\mathbb{Z}_p \cup \frac{1}{p^2}\mathbb{Z}_p \cup \cdots.$$

Thus $\mathbb{Q}_p$ is an ascending union of compact open subgroups, all isomorphic to $\mathbb{Z}_p$. Show that $\mathbb{Q}_p$ is torsion-free.

One can carry the investigation further by showing that $\mathbb{Q}_p$ is a field. (For more information see [147, 372].) □

The members of $\mathbb{Q}_p$ are called *p-adic rationals*.

The examples of abelian groups and compact abelian groups for which we have determined character groups are best summarized in a table, whose use should be self-explanatory. All groups which are listed have duality.

1.39. Tables of Basic Character Groups.

GROUP	$\mathbb{Z}$	$\mathbb{T}$	$\mathbb{Z}(n)$	$\mathbb{Z}_p$	$\mathbb{Z}(p^\infty)$	$\mathbb{T}_p$	$\frac{1}{p^\infty}\mathbb{Z}$
CHARACTER GROUP	$\mathbb{T}$	$\mathbb{Z}$	$\mathbb{Z}(n)$	$\mathbb{Z}(p^\infty)$	$\mathbb{Z}_p$	$\frac{1}{p^\infty}\mathbb{Z}$	$\mathbb{T}_p$

Table 1.1: Some elementary groups and their character groups.

GROUP	A	B	$A \oplus B$	E finite	$\mathbb{Z}^n \oplus E$	$\mathbb{T}^n \times E$	$\mathbb{R}$
CHARACTER GROUP	$\widehat{A}$	$\widehat{B}$	$\widehat{A} \times \widehat{B}$	E finite	$\mathbb{T}^n \times E$	$\mathbb{Z}^n \oplus E$	$\mathbb{R}$

Table 1.2: Elementary sums and products.

For $\mathbb{Z}$ and $\mathbb{T}$, see the comments after Proposition 1.16, Lemma 1.23.
For $\mathbb{Z}(n)$ and finite E, see the comments after Proposition 1.16, and Remark 1.18.
For $\mathbb{Z}^n \oplus E$ and $\mathbb{T}^n \times E$, see Remarks 1.18, 1.24.
For $\mathbb{Z}_p$ and $\mathbb{Z}(p^\infty)$, see Example 1.38(i).
For $\mathbb{T}_p$ and $\frac{1}{p^\infty}\mathbb{Z}$, see Example 1.38(ii).
The case $\mathbb{R}$ is a separate matter which is not subordinate to our discussions here, but which we consider in the following exercise. This will give an outlook to a theory of characters for locally compact abelian groups which would contain discrete and compact abelian groups as special cases. This exercise requires a certain familiarity with the topology of uniform convergence on compact sets in function spaces.

Exercise E1.17. Denote by $\widehat{\mathbb{R}}$ the group $\mathrm{Hom}(\mathbb{R},\mathbb{T})$ of all morphisms of abelian topological groups $f\colon\mathbb{R}\to\mathbb{T}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. Let $p\colon\mathbb{R}\to\mathbb{T}$ denote the quotient homomorphism. Use the Extension Lemma (E1.8) to show that $F\mapsto p\circ F : \mathrm{Hom}(\mathbb{R},\mathbb{R})\to\mathrm{Hom}(\mathbb{R},\mathbb{T})$ is, algebraically, an isomorphism. Show that it is a homeomorphism if domain and range are given the topology of uniform convergence on compact subsets of $\mathbb{R}$. Note that $F\mapsto F(1)\colon\mathrm{Hom}(\mathbb{R},\mathbb{R})\to\mathbb{R}$ is an isomorphism of topological groups. The group $\mathrm{Hom}(\mathbb{R},\mathbb{R})$ is the usual vector space dual of $\mathbb{R}$. Use the duality of finite-dimensional vector spaces to secure that $\eta_{\mathbb{R}}\colon\mathbb{R}\to\widehat{\widehat{\mathbb{R}}}$ is an isomorphism of topological groups. □

Postscript

The chapter is designed to provide a self-contained introduction to the most immediately accessible results of substance on compact groups. The background requirements for these are deliberately kept to a minimum. A barrier is Haar measure which we shall overcome in the next chapter. Chapter 1 should give an impression of the vastness of the class of compact groups.

The basic examples and building blocks have been introduced. These include the orthogonal and unitary groups, groups on the Cantor set—among them the group of p-adic integers—and the solenoids of van Dantzig. We have also identified the compact totally disconnected groups as the profinite ones.

It is consistent with the approach that we hasten to feature the character group of an arbitrary abelian group as a prime example of compact groups; we have to defer to Chapter 2 the insight that in this fashion we obtain all compact abelian groups. For the very formulation of that half of the duality theorem which we present here, we need, of course, the character group of a compact abelian group. On the basic level of this chapter we were able to see that every abelian group appears as the character group of some compact abelian group (namely, its character group). There is a more general context, which puts the duality between discrete and compact abelian groups in the light of a deeper understanding, namely, the context of locally compact abelian groups. We shall move into this context in Chapter 7.

References for this Chapter—Additional Reading

[14], [15], [33], [34], [51], [65], [66], [80], [87], [92], [100], [102], [113], [114], [115], [134], [135], [141], [147], [149], [152], [211], [230], [266], [277], [287], [295], [299], [309], [317], [331], [341], [349], [372].

Chapter 2

The Basic Representation Theory of Compact Groups

One of the most central results for the theory of compact groups is the Theorem of Peter and Weyl which says, among other things, that every compact group has sufficiently many finite dimensional unitary representations. We shall prove this result in the present chapter and elaborate on other ramifications in the next two chapters. One consequence of the Peter–Weyl Theorem is that every compact group is a strict projective limit of closed subgroups of unitary groups, but there are numerous other important consequences.

In Chapter 1 we proved that part of the Pontryagin Duality Theorem which says that the natural map of a (discrete) abelian group into the character group of its character group is an isomorphism of groups. In this chapter we prove the second part of the Pontryagin Duality Theorem which asserts that the canonical map of a compact abelian group into its second dual is an isomorphism of compact groups. A key ingredient in its proof is that compact abelian groups have sufficiently many characters to separate points, which is a special case of the result that sufficiently many finite dimensional unitary representations exist.

We give a definition of compact Lie groups and show that every compact group is a projective limit of compact Lie groups. This is of great significance in our later investigations.

Prerequisites. Our approach in this chapter uses integration of scalar valued continuous functions on compact spaces; we apply this to the integration of functions on a compact group with respect to Haar measure. We assume familiarity with some basic Hilbert space theory, including the rudiments of compact operators which we shall provide to the extent we need them, and we shall use some elementary Banach space theory. In an exercise with detailed directions, an overview of a proof of the existence and uniqueness of Haar measure on a compact group is given.

Some Basic Representation Theory for Compact Groups

We shall be concerned with linear actions of a compact group G on a topological vector space E. We recall for the record that a *topological vector space* is a vector space E over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ which is a topological group with respect to addition and for which scalar multiplication $(r, x) \mapsto r{\cdot}x\colon \mathbb{K}\times E \to E$ is continuous. We shall adopt the convention to write $\mathbb{K}$ for the field of real, respectively, complex numbers. A continuous linear self-map of E is called an *endomorphism of* E or a *continuous* *operator of* E. The vector space $\mathrm{Hom}(E,E) \subseteq E^E$ of these endomorphisms is a topological vector space relative to the structure induced from the topological vector space product E^E. The topology so obtained is the topology of pointwise convergence or, equivalently, the *strong operator topology* . One also writes $\mathcal{L}_p(E)$ for this topological vector space.

Recall that for a group G and a set E a function $(g,x) \mapsto gx\colon G \times E \to E$ is an *action* if $\mathbf{1}x = x$ and $(gh)x = g(hx)$ for all g, $h \in G$ and $x \in E$.

Definition 2.1. Let G be a topological group and E a topological vector space.

(i) We say that E is a G*-module* if there is an action $(g,x) \mapsto gx\colon G \times E \to E$ such that

(a) $x \mapsto gx\colon E \to E$ is a continuous vector space endomorphism of E for each $g \in G$,

(b) $g \mapsto gx\colon G \to E$ is continuous for each $x \in E$.

(ii) A *representation of* G *on* E is a continuous map $\pi\colon G \to \mathcal{L}_p(E)$ satisfying $\pi(\mathbf{1}) = \mathrm{id}_E$ and $\pi(gh) = \pi(g)\pi(h)$ for all $g,h \in G$. □

The following observation is immediate from the definitions:

Remark 2.2. (i) Let G be a topological group and E a G-module. Then the function $\pi\colon G \to \mathcal{L}_p(E)$ given by $\pi(g)(x) = gx$ is a representation of G on E.

(ii) If $\pi\colon G \to \mathcal{L}_p(E)$ is a representation then the function $(g,x) \mapsto gx \overset{\mathrm{def}}{=} \pi(g)(x)\colon G \times E \to E$ endows E with the structure of a G-module. □

After the preceding remark we are aware of the fact that, in reality, the concept of a G-module E is the same thing as that of a representation of G on E. There is a certain preference in algebraic circles toward the module aspect and a leaning towards the representation aspect among analysts. We shall freely move between the two concepts.

However, one also notices that in the spirit of topological algebra, for a G-module E one would expect a postulate demanding the continuity of the action $(g,x) \mapsto gx\colon G \times E \to E$. It is fortunate that, as shown in Theorem 2.3, for most situations this is in fact a consequence of the module concept as introduced in Definition 2.1. Let us first recall that a *Baire space* is a topological space in which every countable union of closed sets without interior points has no interior points. The Baire Category Theorem says that every locally compact space and every space whose topology can be defined through a complete metric is a Baire space. (See [34, 100, 147], or [230].)

Theorem 2.3. *Assume that E is a G-module for a topological group G and that $\pi\colon G \to \mathcal{L}_p(E)$ is the associated representation* (*see* 2.2(ii)). *If E is a Baire space, then for every compact subspace K of G the set $\pi(K) \subseteq \mathrm{Hom}(E,E)$ is equicontinuous at* 0, *that is, for any neighborhood V of* 0 *in E there is a neighborhood U of* 0 *such that $KU \subseteq V$.*

As a consequence, if G is locally compact, the function

$$(g,x) \mapsto gx: G \times E \to E$$

is continuous.

Proof. First step: Given V we find a closed 0-neighborhood W with $W - W \subseteq V$ and $[0,1]{\cdot}W \subseteq W$. Notice that also the interior, $\operatorname{int} W$, of W is star-shaped, that is, satisfies $[0,1]{\cdot} \operatorname{int} W = \operatorname{int} W$. Next we consider

$$C = \bigcap_{g\in K} g^{-1}W.$$

Since K is compact, Kx is compact for any $x \in E$ and thus, as $Kx \subseteq E = \bigcup_{n\in\mathbb{N}} n{\cdot} \operatorname{int} W$ and the $n{\cdot}W$ form an ascending family, we find an $n \in \mathbb{N}$ with $K \cdot x \subseteq n{\cdot}W$, that is, with $x \in \bigcap_{g\in K} n{\cdot}g^{-1}W$. Hence for each $x \in E$ there is a natural number n such that $x \in n{\cdot}C$. Therefore

$$E = \bigcup_{n\in\mathbb{N}} n{\cdot}C,$$

where all sets $n{\cdot}C$ are closed. But E is a Baire space, and so for some $n \in \mathbb{N}$, the set $n{\cdot}C$ has interior points, and since $x \mapsto n{\cdot}x$ is a homeomorphism of E, the set C itself has an interior point c. Now for each $g \in K$ we find $g(C - c) \subseteq W - W \subseteq V$. But $U = C - c$ is a neighborhood of 0, as $KU \subseteq V$, our first claim is proved.

Second step: For a proof of the continuity of the function $\alpha = \big((g,x) \mapsto gx\big)$: $G \times E \to E$, it suffices to show the continuity of α at the point $(\mathbf{1}, 0)$. To see this it suffices to note that for fixed $h \in G$ and fixed $y \in E$ the difference $\alpha(g,x) - \alpha(h,y) = gx - hy = h\big(h^{-1}g(x-y) + (h^{-1}gy - y)\big) = \pi(h)\big(\alpha(h^{-1}g, x-y) + (h^{-1}gy - y)\big)$ falls into any given neighborhood of 0 as soon as $h^{-1}g$ is close enough to $\mathbf{1}$ and the difference $x-y$ is close enough to zero, because α is continuous at $(\mathbf{1}, 0)$, because $\pi(h)$ is continuous by Definition 2.1(i)(a), and because $k \mapsto ky: G \to E$ is continuous by Definition 2.1(i)(b).

Third step: We now assume that G is locally compact and show that α is continuous at $(\mathbf{1}, 0)$. For this purpose it suffices to know that for a compact neighborhood K of $\mathbf{1}$ in G the set $\pi(K) \subseteq \operatorname{Hom}(E,E)$ is equicontinuous; for then any neighborhood V of 0 yields a neighborhood U of 0 in E with $\alpha(K \times U) = \pi(K)(U) \subseteq V$. This completes the proof of the second claim. □

According to the above theorem, if G is a compact group, and E is a G-module which is at the same time a Banach space, the compact group G acts on E in the sense of Definition 1.9; that is $(g,x) \mapsto gx : G \times E \to E$ is continuous.

Example 2.4. Let G be a compact group. Set $E = C(G, \mathbb{K})$; then E is a Banach space with respect to the sup-norm given by $\|f\| = \sup_{t\in G} |f(t)|$. We define ${}_gf = \pi(g)(f)$ by ${}_gf(t) = f(tg)$. Then $\pi: G \to \mathcal{L}_p(E)$ is a faithful (that is, injective) representation, and G acts on E.

Proof. We note $|f_1(tg_1) - f_2(tg_2)| \leq |f_1(tg_1) - f_1(tg_2)| + |f_1(tg_2) - f_2(tg_2)| \leq |f_1(tg_1) - f_1(tg_2)| + \|f_1 - f_2\|$. Since G is compact, f_1 is uniformly continuous. Hence the first summand is small if g_1 and g_2 are close. The second summand is small if f_1 and f_2 are close in E. This shows that $(g, f) \mapsto {}_g f\colon G \times E \to E$ is continuous. It is straightforward to verify that this is a linear action. Finally $\pi(g) = \mathrm{id}_E$ is tantamount to $f(tg) = f(t)$ for all $t \in G$ and all $f \in C(G, \mathbb{K})$. Since the continuous functions separate the points, taking $t = 1$ we conclude $g = 1$. □

In Example 2.4, under the special hypotheses present, we have verified the conclusion of Theorem 2.3 directly.

A G-module E for which π is injective (or faithful) is called a *faithful G-module.* Thus *every compact group G has at least one faithful Banach G-module.*

Corollary 2.5. *If E is a Banach G-module for a compact group G, then*

$$\sup\{\|\pi(g)\| \mid g \in G\} < \infty.$$

Proof. By Theorem 2.3, the set $\pi(G)$ is equicontinuous. Hence for the closed unit ball V around 0 there is closed ball $U = r{\cdot}V$ of radius $r > 0$ such that $GU \subseteq V$, equivalently, $GV \subseteq \frac{1}{r}V$. Hence $\|gx\| \leq 1/r$ for all $x \in V$, that is, $\|\pi(g)\| \leq 1/r$ for all $g \in G$. □

A small warning is in order: It is not true in general that $g \mapsto \pi(g)\colon G \to \mathrm{Hom}(E, E)^{-1} = \mathrm{Gl}(E)$ is continuous with respect to the operator norm. We therefore do not have the continuity of the function $g \mapsto \|\pi(g)\|\colon G \to \mathbb{R}$ available to us nor the quick proof of 2.5 which this information would entail.

The Haar Integral

For the moment, let G denote a compact Hausdorff space. An element μ of the topological dual E' of the Banach space $E = C(G, \mathbb{K})$ is a ($\mathbb{K}$-valued) *integral* or *measure*. (It is not uncommon in our context to use the words "integral" and "measure" synonymously; the eventual justification is, as is usual in the case of such an equivocation, a theorem; here it is the Riesz Representation Theorem of measure theory.) The number $\mu(f)$ is also written $\langle \mu,\, f\rangle$ or indeed $\int f\, d\mu = \int_G f(g)\, d\mu(g)$. It is not our task here to develop or review measure theory in full. What we need is the uniqueness and existence of one and only one particular measure on a compact group G which is familiar from the elementary theory of Fourier series as Lebesgue measure on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The formulation of the existence (and uniqueness theorem) will be easily understood.

Definition 2.6. Let G denote a compact group. A measure μ is called *invariant* if $\mu({}_g f) = \mu(f)$ for all $g \in G$ and $f \in E = C(G, \mathbb{K})$. It is called a *Haar measure* if it is invariant and *positive*, that is, satisfies $\mu(f) \geq 0$ for all $f \geq 0$. The measure μ

is called *normalized* if $\mu(1) = 1$ where 1 also indicates the constant function with value 1. □

Example 2.7. If $p: \mathbb{R} \to \mathbb{T}$ denotes the morphism given by $p(t) = t + \mathbb{Z}$ and $C_1(\mathbb{R}, \mathbb{K})$ denotes the Banach space of all continuous functions $f: \mathbb{R} \to \mathbb{K}$ with period 1, then $f \mapsto f \circ p: C(\mathbb{T}, \mathbb{K}) \to C_1(\mathbb{R}, \mathbb{K})$ is an isomorphism of Banach spaces. The measure γ on $\mathbb{T}$ defined by $\gamma(f) = \int_0^1 (f \circ p)(x)\, dx$ with the ordinary Riemann integral on $[0, 1]$ is a normalized Haar measure on $\mathbb{T}$. □

Exercise E2.1. Verify the assertion of Example 2.7. Give a normalized Haar measure on $\mathbb{S}^1$. For $n \in \mathbb{Z}$ define $e_n: \mathbb{T} \to \mathbb{C}$ by $e_n(t + \mathbb{Z}) = e^{2\pi int}$. Compute $\gamma(e_j \overline{e}_k)$ for $j, k \in \mathbb{Z}$. □

We now state the Existence and Uniqueness Theorem of Haar Measure. We shall present one of its numerous proofs in Appendix 5.

Theorem 2.8 (The Existence and Uniqueness Theorem of Haar measure). *For each compact group G there is one and only one normalized Haar measure.* □

The preceding theorem can be used to show that any Haar measure γ also satisfies the conditions specified in the following exercises.

Exercise E2.2. $\int_G f(gt)\, d\gamma(t) = \gamma(f)$ for all $g \in G$ and $f \in C(G, \mathbb{K})$. □

Exercise E2.3. $\int_G f(t^{-1})\, d\gamma(t) = \gamma(f)$ for all $f \in C(G, \mathbb{K})$. □

Definition 2.9. We shall use the notation $\gamma \in C(G, \mathbb{K})'$ for the unique normalized Haar measure, and we shall also write $\gamma(f) = \int_G f(g)\, dg$. □

Consequences of Haar Measure

Theorem 2.10 (Weyl's Trick). *Let G be a compact group and E a G-module which is, at the same time, a Hilbert space. Then there is a scalar product relative to which all operators $\pi(g)$ are unitary.*

Specifically, if $(\bullet \mid \bullet)$ is the given scalar product on E, then

$$\langle x \mid y \rangle = \int_G (gx \mid gy)\, dg \tag{1}$$

defines a scalar product such that

$$M^{-2}(x \mid x) \le \langle x \mid x \rangle \le M^2(x \mid x) \tag{2}$$

with

$$M = \sup\{\sqrt{(gx \mid gx)} \mid g \in G,\ (x \mid x) \leq 1\}, \tag{3}$$

and that

$$\langle gx \mid gy \rangle = \langle x \mid y \rangle \quad \textit{for all} \quad x, y \in E,\ g \in G. \tag{4}$$

Proof. For each x, $y \in E$ the integral on the right side of (1) is well-defined, is linear in x and conjugate linear in y. Since Haar measure is positive, the information $(gx \mid gx) \geq 0$ yields $\langle x \mid x \rangle \geq 0$. By Corollary 2.5, the positive number M in (3) is well-defined. Then $\langle x \mid x \rangle \leq \int_G M^2(x \mid x)\,dg = M^2(x \mid x)$ since γ is positive and normalized. Also, $(x \mid x) = (g^{-1}gx \mid g^{-1}gx) \leq M^2(gx \mid gx)$, whence $\langle x \mid x \rangle \geq \int_G M^{-2}(x \mid x)\,dg = M^{-2}(x \mid x)$. This proves (2) and thus also the fact that $\langle \bullet \mid \bullet \rangle$ is positive definite, that is, a scalar product. Finally, let $h \in G$; then $\langle hx \mid hy \rangle = \int_G (ghx \mid ghy)\,dg = \int_G (gx \mid gy)\,dg = \langle x \mid y \rangle$ by the invariance of γ. □

The idea of the construction is that for each $g \in G$ we obtain a scalar product $(x, y) \mapsto (gx \mid gy)$. The invariant scalar product is the "average" or "expectation" of this family with respect to the probability measure γ.

Definition 2.11. If G is a topological group, then a *Hilbert G-module* is a Hilbert space E and a G-module such that all operators $\pi(g)$ are unitary, that is, such that

$$(gx \mid gy) = (x \mid y) \quad \text{for all} \quad x, y \in E,\ g \in G. \qquad \square$$

Recall from Theorem 2.3, that in every Hilbert G-module of a locally compact group the action$(g, x) \mapsto gx\colon G \times E \to E$ is continuous. By Weyl's Trick 2.10, for compact G, it is never any true loss of generality to assume for a G-module on a Hilbert space that E is a Hilbert module. Every finite dimensional $\mathbb{K}$-vector space is a Hilbert space (in many ways). Thus, in particular, *every representation of a compact group on a finite dimensional $\mathbb{K}$-vector space may be assumed to be unitary.*

Hilbert modules are the crucial type of G-modules for compact groups G as we shall see presently. For the moment, let us observe, that every compact group G has at least one faithful Hilbert module.

Example 2.12. Let G be a compact group and $\mathcal{H}_0$ the vector space $C(G, \mathbb{K})$ equipped with the scalar product

$$(f_1 \mid f_2) = \gamma(f_1\overline{f_2}) = \int_G f_1(g)\overline{f_2(g)}\,dg.$$

Indeed the function $(f_1, f_2) \mapsto (f_1 \mid f_2)$ is linear in the first argument, conjugate linear in the second, and $(f \mid f) = \gamma(f\overline{f}) \geq 0$ since γ is positive. Also, if $f \neq 0$, then there is a $g \in G$ with $f(g) \neq 0$. Then the open set $U = \{t \in G \mid (f\overline{f})(t) > 0\}$ contains g, hence is nonempty. The relation $(f \mid f) = 0$ would therefore imply that U does not meet the support of γ, which is G—an impossibility. Hence the scalar product is positive definite and $\mathcal{H}_0$ is a pre-Hilbert space. Its completion is a Hilbert space $\mathcal{H}$, also called $\mathrm{L}^2(G,\mathbb{K})$.

The translation operators $\pi(g)$ given by $\pi(g)(f) = {}_gf$ are unitary since $(\pi(g)f \mid \pi(g)f) = \int_G f(tg)\overline{f(tg)}\,dt = \int_G f(t)\overline{f(t)}\,dt = (f \mid f)$ by invariance. Every unitary operator on a pre-Hilbert space $\mathcal{H}_0$ extends uniquely to a unitary operator on its completion $\mathcal{H}$, and we may denote this extension with the same symbol $\pi(g)$.

The space $\mathcal{L}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ is a C*-algebra (cf. Exercise E1.1) and $\mathcal{U}(\mathcal{H}) = \mathcal{U}\big(\mathcal{L}(\mathcal{H})\big)$ denotes its unitary group. Then $\pi: G \to \mathcal{U}(\mathcal{H})$ is a morphism of groups. We claim that it is continuous with respect to the strong operator topology, that is, $g \mapsto {}_gf: G \to \mathcal{H}$ is continuous for each $f \in \mathcal{H}$. Let $\varepsilon > 0$ and let $f_0 \in C(G,\mathbb{K})$ be such that $\|f - f_0\|_2 < \varepsilon$ where $\|f\|_2^2 = (f \mid f)$. Then $\|{}_gf - {}_hf\|_2 \leq \|{}_gf - {}_gf_0\|_2 + \|{}_gf_0 - {}_hf_0\|_2 + \|{}_hf_0 - {}_hf\|_2 = \|{}_gf_0 - {}_hf_0\|_2 + 2\|f - f_0\|_2 < \|{}_gf_0 - {}_hf_0\|_2 + 2\varepsilon$ in view of the fact that $\pi(g)$ is unitary. But $\|{}_gf_0 - {}_hf_0\|_2 \leq \|{}_gf_0 - {}_hf_0\|_\infty$ where $\|f_0\|_\infty$ is the sup-norm $\sup_{g\in G}|f_0(g)|$ for a continuous function f_0. By Example 2.4 the function $g \mapsto {}_gf$ is continuous with respect to the sup-norm; hence $\|{}_gf_0 - {}_hf_0\|_\infty$ can be made less than ε for g close enough to h. For these g and h we then have $\|{}_gf - {}_hf\|_2 < 3\varepsilon$. This shows the desired continuity. Since $\pi(g) = \mathrm{id}_\mathcal{H}$ implies $\pi(g)|\mathcal{H}_0 = \mathrm{id}_{\mathcal{H}_0}$ and this latter relation already implies $g = \mathbf{1}$ by Example 2.4, the representation π is injective. Thus $\mathrm{L}^2(G,\mathbb{K})$ *is a faithful Hilbert module.* It is called the regular G-module and the unitary representation $\pi: G \to \mathcal{U}\big(\mathrm{L}^2(G,\mathbb{K})\big)$ is called the *regular representation.* □

For the record we write:

Remark 2.13. Every compact group possesses faithful unitary representations and faithful Hilbert modules. □

The Main Theorem on Hilbert Modules for Compact Groups

We consider a Hilbert space $\mathcal{H}$. A *sesquilinear form* is a function $B: \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ that is linear in the first and conjugate linear in the second argument, and that is bounded in the sense that there is a constant M such that $|B(x,y)| \leq M\|x\|\cdot\|y\|$ for all x, $y \in \mathcal{H}$. If T is a bounded linear operator on $\mathcal{H}$, then $B(x,y) = (Tx \mid y)$ defines a sesquilinear form with $M = \|T\|$ in view of the Inequality of Cauchy and Schwarz saying that $|(x \mid y)| \leq \|x\|\cdot\|y\|$. (For our purposes we included continuity in the definition of sesquilinearity.)

Lemma 2.14. *If B is a sesquilinear form, then there exists a unique bounded operator T of $\mathcal{H}$ such that $\|T\| \leq M$ and that $B(x,y) = (Tx \mid y)$.*

Proof. Exercise. □

Exercise E2.4. Prove Lemma 2.14.

[Hint. Fix $x \in \mathcal{H}$. The function $y \mapsto B(x,y)$ is a bounded conjugate linear form on $\mathcal{H}$. Hence there is a unique element $Tx \in \mathcal{H}$ such that $B(x,y) = (Tx \mid y)$ by the elementary Riesz Representation Theorem for Hilbert spaces. The function $T = (x \mapsto Tx)\colon \mathcal{H} \to \mathcal{H}$ is linear. Use $|B(x,y)| \leq M\|x\|{\cdot}\|y\|$ to deduce $\|T\| \leq M$.]□

Lemma 2.15. *Let G denote a compact group and T a bounded operator on a Hilbert G-module E. Then there is a unique bounded operator $\widetilde{T}$ on E with $\|\widetilde{T}\| \leq \|T\|$ such that*

$$(5) \qquad (\widetilde{T}x \mid y) = \int_G (Tgx \mid gy)\, dg = \int_G (\pi(g)^{-1}T\pi(g)(x) \mid y)\, dg.$$

Proof. Since π is a unitary representation, $\pi(g)^* = \pi(g)^{-1}$ and so the last two integrals in (5) are equal. The prescription $B(x,y) = \int_G (Tgx \mid gy)\, dg$ defines a function B which is linear in x and conjugate linear in y. Because

$$|(Tgx \mid gy)| \leq \|Tgx\|{\cdot}\|gy\| = \|T\|{\cdot}\|gx\|{\cdot}\|gy\| = \|T\|{\cdot}\|x\|{\cdot}\|y\|$$

(as G acts unitarily on $\mathcal{H}$!) we obtain the estimate $|B(x,y)| \leq \int_G \|T\|{\cdot}\|x\|{\cdot}\|y\|\, dg = \|T\|{\cdot}\|x\|{\cdot}\|y\|$. Hence B is a sesquilinear form, and so by Lemma 2.14, there is a bounded operator $\widetilde{T}$ with $B(x,y) = (\widetilde{T}x \mid y)$ and $\|\widetilde{T}\| \leq \|T\|$. □

In any ring R, the *commutant* $\mathcal{C}(X)$ (or, in semigroup and group theory equivalently called the *centralizer* $Z(X,R)$) of a subset $X \subseteq R$ is the set of all elements $r \in R$ with $xr = rx$ for all $x \in X$. Using integration of no more than $\mathbb{K}$-valued functions, we have created the operator

$$\widetilde{T} = \int_G \pi(g)^{-1}T\pi(g)\, dg,$$

where the integral indicates an averaging over the conjugates $\pi(g)^{-1}T\pi(g)$ of T; we shall return to the integration of vector valued functions in Part 2 of Chapter 3, leading up to Proposition 3.30. It is clear that the averaging self-map $T \mapsto \widetilde{T}$ of $\mathrm{Hom}(\mathcal{H},\mathcal{H})$ is linear and bounded. Its significance is that its image is exactly the commutant $\mathcal{C}\big(\pi(G)\big)$ of $\pi(G)$ in $\mathrm{Hom}(\mathcal{H},\mathcal{H})$. Thus it is the set of all bounded operators S on $\mathcal{H}$ satisfying $S\pi(g) = \pi(g)S$. This is tantamount to saying that $S(gx) = g(Sx)$ for all $g \in G$ and $x \in \mathcal{H}$. Such operators are also called *G-module endomorphisms* or *intertwining operators*. In the present context the commutant is sometimes denoted also by $\mathrm{Hom}_G(\mathcal{H},\mathcal{H})$.

Lemma 2.16. *The following statements are equivalent for an operator S of $\mathcal{H}$:*

(1) $S \in \mathrm{Hom}_G(\mathcal{H},\mathcal{H})$.

(2) $S = \widetilde{S}$.

(3) *There is an operator T such that $S = \widetilde{T}$.*

Proof. (1)⇒(2) By definition, $(\widetilde{S}x \mid y) = \int_G (Sgx \mid gy)\, dg$. By (1) we know $Sgx = gSx$, and since $\mathcal{H}$ is a unitary G-module, $(Sgx \mid gy) = (gSx \mid gy) =$

$= (Sx \mid y)$. Since γ is normalized, we find $(\widetilde{S}x \mid y) = (Sx \mid y)$ for all x and y in $\mathcal{H}$. This means (2).

(2)⇒(3) Trivial.

(3)⇒(1) Let x and y be arbitrary in $\mathcal{H}$ and $h \in G$. Then

$$\begin{aligned}(Shx \mid y) = (\widetilde{T}hx \mid y) &= \int_G (Tghx \mid gy)\, dg = \int_G \big(Tghx \mid gh(h^{-1}y)\big)\, dg \\ &= \int_G (Tgx \mid gh^{-1}y)\, dg = (\widetilde{T}x \mid h^{-1}y) = (Sx \mid h^{-1}y) = (hSx \mid y)\end{aligned}$$

in view of the invariance of γ and the fact that $\pi(g)^{-1} = \pi(g)^*$. Hence $S\pi(h) = \pi(h)S$ for all $h \in G$ and thus (1) is proved. □

We see easily that $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$ is a closed C*-subalgebra of $\mathcal{L}(\mathcal{H})$.

An *orthogonal projection* of $\mathcal{H}$ is an idempotent operator P satisfying $P^* = P$, that is, $(Px \mid y) = (x \mid Py)$ for all $x, y \in \mathcal{H}$. The function $P \mapsto P(\mathcal{H})$ is a bijection from the set of all orthogonal projections of $\mathcal{H}$ to the set of all closed vector subspaces V of $\mathcal{H}$. Indeed every closed vector subspace V has a unique orthogonal complement $V^\perp$ and thus determines a unique orthogonal projection of $\mathcal{H}$ with image V and kernel $V^\perp$.

Definition 2.17. If G is a topological group and E a G-module, then a vector subspace V of E is called a *submodule* if $GV \subseteq V$. Equivalently, V is also called an *invariant subspace.* □

Lemma 2.18. *For a closed vector subspace V of a Hilbert G-module $\mathcal{H}$ and the orthogonal projection P with image V the following statements are equivalent:*

(1) *V is a G-submodule.*

(2) $P \in \mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$.

(3) *$V^\perp$ is a G-submodule.*

Proof. (1)⇒(2) Let $x \in \mathcal{H}$; then $x = Px + (1-P)x$ and thus

$$(*) \qquad gx = gPx + g(1-P)x$$

for all $g \in G$. But $Px \in V$ and thus $gPx \in V$ since V is invariant. Since the operator $\pi(g)$ is unitary, it preserves orthogonal complements, and thus $g(1-P)x \in V^\perp$. Then $(*)$ implies $gPx = P(gx)$ $\big($and $g(1-P)x = (1-P)(gx)\big)$.

(2)⇒(3) The kernel of a morphism of G-modules is readily seen to be invariant. Since $V^\perp = \ker P$ and P is a morphism of G-modules, clearly $V^\perp$ is invariant.

(3)⇒(1) Assume that $V^\perp$ is invariant. We have seen in the preceding two steps of the proof that the orthogonal complement $W^\perp$ of any invariant closed vector subspace W of $\mathcal{H}$ is invariant. Now we apply this to $W = V^\perp$. Hence $(V^\perp)^\perp$ is invariant. But $(V^\perp)^\perp = V$, and thus V is invariant. □

The problem of finding invariant subspaces of a Hilbert G-module therefore amounts to finding orthogonal projections in $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$.

Recall that an operator T on a Hilbert space $\mathcal{H}$ is called *positive* if it is selfadjoint or *hermitian* (i.e. satisfies $T = T^*$) and if $(Tx \mid x) \geq 0$ for all $x \in \mathcal{H}$.

Lemma 2.19. *If T is a hermitian (respectively, positive) operator on a Hilbert G-module $\mathcal{H}$, then so is $\widetilde{T}$.*

Proof. For $x, y \in \mathcal{H}$ we have

$$(\widetilde{T}x \mid y) = \int_G (gTg^{-1}x \mid y)\, dg = \int_G (Tg^{-1}x \mid g^{-1}y)\, dg.$$

If $T = T^*$, then $(Tg^{-1}x \mid g^{-1}y) = (T^*g^{-1}x \mid g^{-1}y) = (g^{-1}x \mid Tg^{-1}y) = \overline{(Tg^{-1}y \mid g^{-1}x)}$ and thus $(\widetilde{T}x \mid y) = \overline{(\widetilde{T}y \mid x)}$. Hence $\widetilde{T}$ is hermitian. If T is positive, then $\widetilde{T}$ is hermitian by what we just saw, and taking $y = x$ and observing $(Tg^{-1}x \mid g^{-1}x) \geq 0$ we find that $\widetilde{T}$ is positive, too. □

Next we turn to the important class of compact operators. Recall that an operator $T\colon V \to V$ on a Banach space is called *compact* if for every bounded subset B of V the image TB is precompact. Equivalently, this says that $\overline{TB}$ is compact, since V is complete.

Lemma 2.20. *If T is a compact operator on a Hilbert G-module $\mathcal{H}$, then $\widetilde{T}$ is also compact.*

Proof. Let B denote the closed unit ball of $\mathcal{H}$. We have to show that $\widetilde{T}B$ is precompact. Since all $\pi(g)$ are unitary, we have $gB = B$ for each $g \in G$. Hence $A \stackrel{\text{def}}{=} \overline{TGB}$ is compact since T is compact. Since the function $(g, x) \mapsto gx\colon G \times \mathcal{H} \to \mathcal{H}$ is continuous by Theorem 2.3, the set GA is compact. The closed convex hull K of GA is compact (see Exercise E2.5 below). Now let $y \in \mathcal{H}$ be such that $\operatorname{Re}(x \mid y) \leq 1$ for all $x \in K$. Then $x \in B$ implies $\operatorname{Re}(\widetilde{T}x \mid y) = \int_G \operatorname{Re}(gTg^{-1}x \mid y)\, dg \leq \int_G dg = 1$ since $gTg^{-1}x \in GTGB \subseteq GA \subseteq K$ for all $g \in G$. Hence $\widetilde{T}x$ is contained in every closed real half-space which contains K. From the Theorem of Hahn and Banach we know that a closed convex set is the intersection of all closed real half-spaces which contain it. Hence we conclude $\widetilde{T}x \in K$ and thus $\widetilde{T}B \subseteq K$. This shows that $\widetilde{T}$ is compact. □

It is instructive at this point to be aware of the information used in the preceding proof: the joint continuity of the action proved in 2.3, the precompactness of the convex hull of a precompact set in a Banach space (subsequent Exercise!), the Hahn–Banach Theorem, and of course the compactness of G.

Exercise E2.5. Show that in a Banach space V, the closed convex hull K of a precompact set P is compact.
[Hint. Since V is complete, it suffices to show that K is precompact. Thus let U be any open ball around 0. Since P is precompact, there is a finite set $F \subseteq P$ such that $P \subseteq F + U$. The convex hull S of F is compact (as the image of a

finite simplex under an affine map). Hence there is a finite set $F' \subseteq S$ such that $S \subseteq F' + U$. Now the convex hull of P is contained in the convex set $S + U$, hence in the set $F' + U + U$, and its closure is contained in $F' + U + U + U = F' + 3U$.]□

We can summarize our findings immediately in the following lemma.

Lemma 2.21. *On a nonzero Hilbert G-module $\mathcal{H}$ let x denote any nonzero vector and T the orthogonal projection of $\mathcal{H}$ onto $\mathbb{K}{\cdot}x$. Then $\widetilde{T}$ is a nonzero compact positive operator in* $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$.

Proof. This follows from the preceding lemmas in view of the fact that an orthogonal projection onto a one-dimensional subspace $\mathbb{K}{\cdot}x$ is a positive compact operator and that $(Tx \mid x) = \|x\|^2 > 0$, whence $(\widetilde{T}x \mid x) = \int_G (Tg^{-1}x \mid g^{-1}x)\,dg > 0$. □

Now we recall some elementary facts on compact positive operators. Notably, every compact positive nonzero operator T has a positive eigenvalue λ and the eigenspace $\mathcal{H}_\lambda$ is finite dimensional. (See also Dunford and Schwartz [94] or Rudin [307].)

Exercise E2.6. Let $\mathcal{H}$ be a Hilbert space and T a nonzero compact positive operator. Show that there is a largest positive eigenvalue λ and that $\mathcal{H}_\lambda$ is finite dimensional.

[Hint. Without loss of generality assume $\|T\| = 1$. Note $\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\} = \sup\{\mathrm{Re}(Tx \mid y) \mid \|x\|, \|y\| \leq 1\}$. Since T is positive, $0 \leq (T(x+y) \mid x+y) = (Tx \mid x) - 2\,\mathrm{Re}(Tx \mid y) + (Ty \mid y)$, whence $\mathrm{Re}(Tx \mid y) \leq \frac{1}{2}\big((Tx \mid x) + (Ty \mid y)\big) \leq \max\{(Tx \mid x), (Ty \mid y)\}$. It follows that $\|T\| = \sup\{(Tx \mid x) \mid \|x\| = 1\}$. Now there is a sequence $x_n \in \mathcal{H}$ with $1 - \frac{1}{n} < (Tx_n \mid x_n) \leq 1$ and $\|x_n\| = 1$. Since T is compact there is a subsequence $y_k = x_{n(k)}$ such that $z = \lim_{k\in\mathbb{N}} Ty_k$ exists with $\|z\| = 1$. Now $0 \leq \|Ty_n - y_n\|^2 = \|Ty_n\|^2 - 2{\cdot}(Ty_n \mid y_n) + \|y_n\|^2 \to 1 - 2 + 1 = 0$. Hence $z = \lim y_n$ and $Tz = z$.] □

We now have all the tools for the core theorem on the unitary representations of compact groups.

The Fundamental Theorem on Unitary Modules

Theorem 2.22. *Every nonzero Hilbert G-module for a compact group G contains a nonzero finite dimensional submodule.*

Proof. By Lemma 2.21 we find a nonzero compact positive operator $\widetilde{T}$ which is invariant by 2.16. But $\widetilde{T}$ has a finite dimensional nonzero eigenspace $\mathcal{H}_\lambda$ for an eigenvalue $\lambda > 0$ by Exercise E2.6. If $\widetilde{T}x = \lambda{\cdot}x$, then $\widetilde{T}gx = g\widetilde{T}x = g(\lambda{\cdot}x) = \lambda{\cdot}gx$. Thus $\mathcal{H}_\lambda$ is the desired submodule. □

Definition 2.23. A G-module E is called *simple* if it is nonzero and $\{0\}$ and E are the only invariant submodules. The corresponding representation of G is called *irreducible.* □

Corollary 2.24. *Every nonzero Hilbert G-module for a compact group G contains a simple nonzero G-module.*

Proof. By the Fundamental Theorem on Unitary Modules 2.22, we may assume that the given module $\mathcal{H}$ is finite dimensional. Every descending chain of nonzero submodules then is finite and thus has a smallest element. It follows that $\mathcal{H}$ has a nonzero minimal submodule which is necessarily simple. □

Corollary 2.25. *Every nonzero Hilbert G-module for a compact group G is a Hilbert space orthogonal sum of finite dimensional simple submodules.*

Proof. Let E be a Hilbert G-module and consider, by virtue of Corollary 2.24 and Zorn's Lemma, a maximal family $\mathcal{F} = \{E_j \mid j \in J\}$ of finite dimensional submodules such that $j \neq k$ in J implies $E_j \perp E_k$. Let H be the closed span of this family (that is, its orthogonal sum). Then H is a G-module. If $H \neq E$, then $H^\perp$ is a nonzero G-module by Lemma 2.18. Hence it contains a nonzero simple submodule K. Then $\mathcal{F} \cup \{K\}$ is an orthogonal family of finite dimensional simple submodules which properly enlarges the maximal family $\mathcal{F}$, and this is impossible. Thus $E = H$, and this proves the corollary. □

Definition 2.26. We say that a family $\{E_j \mid j \in J\}$ of G-modules, respectively, the family $\{\pi_j \mid j \in J\}$ of representations *separates the points of* G if for each $g \in G$ with $g \neq \mathbf{1}$ there is a $j \in J$ such that $\pi_j(g) \neq \mathrm{id}_{E_j}$, that is, there is an $x \in E_j$ such that $gx \neq x$. □

Corollary 2.27. *If G is a compact group, then the finite dimensional simple modules separate the points.*

Proof. By Example 2.12, there is a faithful Hilbert G-module E. By Corollary 2.25, the module E is an orthogonal direct sum $\bigoplus_{j\in J} E_j$ of simple finite dimensional submodules E_j. If $g \in G$ and $g \neq \mathbf{1}$, then there is an $x \in E$ such that $gx \neq x$. Writing x as an orthogonal sum $\sum_{j\in J} x_j$ with $x_j \in E_j$ we find at least one index $j \in J$ such that $gx_j \neq x_j$ and this is what we had to show. □

Corollary 2.28. *The orthogonal and the unitary representations $\pi\colon G \to \mathrm{O}(n)$, respectively, $\pi\colon G \to \mathrm{U}(n)$ separate the points of any compact group G.*

Proof. By Weyl's Trick 2.10, for a compact group G, every finite dimensional real representation is orthogonal and every complex finite dimensional representation is unitary for a suitable scalar product. The assertion therefore is a consequence of Corollary 2.27. □

In other words, given a compact group G, for each element $g \in G$ different from its identity e, and for each ground field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we find a natural number n_g and a representation $\pi_g : G \to O(n_g)$ in the case $\mathbb{K} = \mathbb{R}$, respectively, $\pi_g : G \to U(n_g)$ in the case $\mathbb{K} = \mathbb{C}$, such that $g \notin \ker \pi_g$. Since each of the groups $O(n_g)$ and $U(n_g)$ is compact by Example 1.6, the product $P \stackrel{\text{def}}{=} \prod_{e \neq g \in G} O(n_g)$ for $\mathbb{K} = \mathbb{R}$, respectively, $P \stackrel{\text{def}}{=} \prod_{e \neq g \in G} U(n_g)$ for $\mathbb{K} = \mathbb{C}$, is a compact group by Proposition 1.14. By the universal property of the product (see e.g. Definition A3.43(i)) we have a morphism $\pi : G \to P$, $\pi(x) = (\pi_g(x))_{g \neq e}$, which is injective by the selection of π_g. Then by Remark 1.8, π implements an isomorphism of compact groups $G \to \pi(G)$.

In the interest of the algebraic background which we shall pursue further in the next chapter we may take this argument a step further. Each orthogonal group $O(n_g)$ is contained in the real, respectively complex, algebra $M_{n_g}(\mathbb{K})$ of all $n \times n$ matrices over $\mathbb{K}$. Then the product $A \stackrel{\text{def}}{=} \prod_{e \neq g \in G} M_{n_g}(\mathbb{K})$ is an algebra over $\mathbb{K}$ with respect to componentwise addition, scalar multiplication, and multiplication, and a topological space with respect to the product topology. The algebraic operations are continuous with respect to the topology, and so A is what one calls a *topological algebra* (over $\mathbb{K}$). The subset of all multiplicatively invertible elements (or *units*) is denoted by A^{-1}. As inversion is calulated componentwise, it is also continuous, and so A^{-1} is indeed a topological group with respect to multiplication. It is clear that the product P of the orthogonal, respectively unitary groups, is a compact subgroup of A^{-1}, and thus the compact group G has an isomorphic copy in A^{-1}.

We summarize these conclusions in the following useful result:

Compact Groups as Subgroups of Generic Objects

Corollary 2.29. (i) Every compact group G is isomorphic to a closed subgroup of a product $\prod_{j \in J} \mathrm{O}(n_j)$ of orthogonal groups and of a product $\prod_{j \in J} \mathrm{U}(n_j)$ of unitary groups.

(ii) Every compact group is isomorphic to a compact multiplicative group of invertible elements of a real topological algebra $A = \prod_{j \in J} A_j$ for some family of finite dimensional real full matrix algebras $A_j = M_{n_j}(\mathbb{R})$. □

The first part of this result was announced immediately after Proposition 1.14. A finer version of Corollary 2.29 (i) will appear in Corollary 2.36. The second part of Proposition 2.29 will be put into a more systematic context in Part 3 of Chapter 3.

Lemma 2.30. *If E is an irreducible finite dimensional G-module, then* $\mathrm{Hom}_G(E, E)$ *is a division ring over* $\mathbb{K}$. *If* $\mathbb{K} = \mathbb{C}$, *then* $\mathrm{Hom}_G(E, E) = \mathbb{C} \cdot \mathrm{id}_E$.

If G is abelian, then $\dim_{\mathbb{C}} E = 1$, and there is a morphism $\chi : G \to \mathbb{C}^\times$ such that $gx = \chi(g) \cdot x$ for all $x \in E$.

Proof. Let $\varphi : E \to E$ be G-equivariant, i.e. satisfy $\varphi(g \cdot x) = g \cdot \varphi(x)$. Then $\ker \varphi$ and $\mathrm{im}\, \varphi$ are submodules of E. If $\varphi \neq 0$, then $\ker \varphi = \{0\}$ and $\mathrm{im}\, \varphi = E$ follow, and so φ is bijective, that is, has an inverse. Thus $\mathrm{Hom}_G(E,E)$ is a division ring over $\mathbb{K}$.

Assume now that $\mathbb{K}=\mathbb{C}$. If $0\neq\varphi\in\mathrm{Hom}_G(E,E)$, then φ has a nonzero eigenvalue λ. Then $\varphi-\lambda\cdot\mathrm{id}_E$ is an element of $\mathrm{Hom}_G(E,E)$ with a nonzero kernel, hence must be zero by the preceding. Thus $\varphi=\lambda\cdot\mathrm{id}_E$.

Now assume that G is abelian, that $\mathbb{K}=\mathbb{C}$, and that $\pi\colon G\to\mathrm{Hom}(E,E)$ is the associated representation. Commutativity of G implies $\pi(G)\subseteq\mathrm{Hom}_G(E,E)=\mathbb{C}\cdot\mathrm{id}_E$. Hence for each $g\in G$ there is a $\chi(g)\in\mathbb{C}$ such that $\pi(g)=\chi(g)\cdot\mathrm{id}_E$. We see immediately, that χ is a morphism $G\to\mathbb{C}^\times$. Moreover, every vector subspace of E is a submodule. Hence the simplicity of E implies $\dim E=1$. □

We remark that from a purely algebraic point of view, the finite dimensionality was not needed. If, however, G is a topological group and E a topological vector space, then the bijectivity of a continuous endomorphism does not necessarily imply its invertibility. The Inverse Mapping Theorem for bounded operators on Banach spaces (see [94]) allows this conclusion still for Banach spaces (at least). Also, if E is a G-module in the sense of Definition 2.1, then the morphism φ constructed in the abelian case is continuous. If G is, in addition, compact then $\chi(G)$ is a compact subgroup of $\mathbb{C}^\times$ and is, therefore, contained in $\mathbb{S}^1$. Hence χ is a character in the sense of Definition 1.22 (up to the isomorphy of $\mathbb{S}^1$ with $\mathbb{T}$).

Corollary 2.31. *The characters of a compact abelian group separate the points.*

Proof. This is an immediate consequence of Corollary 2.27 and Lemma 2.30. □

This allows us to prove the second half of the Pontryagin Duality Theorem:

Theorem 2.32. *For any compact abelian group G the morphism $\eta_G\colon G\to\widehat{\widehat{G}}$ is an isomorphism.*

Proof. By Corollary 2.31, $\eta_G\colon G\to\widehat{\widehat{G}}$ is injective, hence an isomorphism onto its image $\Gamma\subseteq\widehat{\widehat{G}}$ (cf. Remark 1.8). We claim that $\Gamma=\widehat{\widehat{G}}$; a proof of this claim will finish the proof. By Corollary 2.31 once again, the claim is proved if every character of $\widehat{\widehat{G}}/\Gamma$ is zero, that is, if every character of $\widehat{\widehat{G}}$ which vanishes on Γ is zero. By Theorem 1.37 we may identify $\widehat{G}$ with the character group of $\widehat{\widehat{G}}$ under the evaluation isomorphism. Thus a character f of $\widehat{\widehat{G}}$ vanishing on Γ is given by an element $\chi\in\widehat{G}$ such that $f(\Omega)=\Omega(\chi)$. But we have $0=f\big(\eta_G(g)\big)=\eta_G(g)(\chi)$ for all $g\in G$ since f annihilates Γ. By the definition of η_G we then note $\chi(g)=\eta_G(g)(\chi)=0$ for all $g\in G$, that is, $\chi=0$ and thus $f=0$. □

Theorems 1.37 and 2.32 constitute the object portion of the *Pontryagin Duality Theorem for discrete and compact abelian groups.* Up to natural isomorphism it sets up a bijection between the class of discrete and that of compact abelian groups.

It shall reveal its true power when it is complemented by the morphism part. This will set up a similar bijection between morphisms as we shall see in Chapter 7. However, this belongs to the domain of generalities and does, in fact, not require more work in depth. The nontrivial portion of the duality is accomplished.

The following consequence of the duality theorem turns out to be very useful.

Corollary 2.33. (i) *Let G be a compact abelian group and A a subgroup of the character group $\widehat{G}$. The following two conditions are equivalent:*
(1) *A separates the points of G.*
(2) *$A = \widehat{G}$.*

(ii) (The Extension Theorem for Characters) *If H is a closed subgroup of G, then every character of H extends to a character of G.*

Proof. (i) Corollary 2.31 says that (2) implies (1), and so we have to prove that (1) implies (2). Since the characters of the discrete group $\widehat{G}/A$ separate the points by Lemma 1.21, in order to prove (2) it suffices to show that every character of $\widehat{G}$ vanishing on A must be zero. Thus let Ω be a character of $\widehat{G}$ vanishing on A. By Theorem 2.32, there is a $g \in G$ with $\eta_G(g) = \Omega$. Thus $\chi \in A$ implies $0 = \Omega(\chi) = \eta_G(g)(\chi) = \chi(g)$. From (1) we now conclude $g = 0$. Hence $\Omega = \eta_G(g) = 0$.

(ii) The collection of all restrictions $\chi|H$ of characters of G to H separates the points of H since the characters of G separate the points of G by Corollary 2.31. Then (i) above shows that the function $\chi \mapsto \chi|H\colon \widehat{G} \to \widehat{H}$ is surjective, and this proves the assertion. □

Corollary 2.34. *For every compact abelian group G there is a filter basis $\mathcal{N}$ of compact subgroups such that G is the strict projective limit $\lim_{N\in\mathcal{N}} G/N$ of factor groups each of which is a character group of a finitely generated abelian group.*

Proof. Let $A = \widehat{G}$ denote the character group of G and $\mathcal{F}$ the family of finitely generated subgroups. If $F \in \mathcal{F}$, let $N_F = F^\perp$ denote the annihilator $\{g \in G \mid \chi(g) = 0 \text{ for all } \chi \in F\}$. Since $F \subseteq F'$ in $\mathcal{F}$ implies $N_{F'} \subseteq N_F$, the family $\mathcal{N} = \{N_F \mid F \in \mathcal{F}\}$ is a filter basis of closed subgroups. An element g is in $\bigcap \mathcal{N}$ if and only if it is in the annihilator of every finitely generated subgroup of A, hence if and only if it is annihilated by all of A, since A is the union of all of its finitely generated subgroups. Thus $g = 0$ by Corollary 2.31. By Proposition 1.33(ii), therefore, G is the strict projective limit $G = \lim_{F\in\mathcal{F}} G/N_F$.

Now we claim that the character group of G/N_F may be identified with F. This will finish the proof of the corollary. If $q_F\colon G \to G/N_F$ denotes the quotient map, then the function $\varphi \mapsto \varphi \circ q_F\colon (G/N_F)\widehat{\ } \to \widehat{G}$ is injective as q_F is surjective. Its image is precisely the group $F^{\perp\perp}$ of all characters vanishing on N_F. Since every character $\chi \in F$ vanishes on N_F, we have $F \subseteq F^{\perp\perp}$. We shall now show equality and thereby prove the claim. But when $F^{\perp\perp}$ is identified with the character group of G/N_F then the subgroup F separates the points of G/N_F since the only coset

$g + N_F \in G/N_F$ annihilated by all of F is N_F by the definition of N_F. Now Corollary 2.33 shows $F = F^{\perp\perp}$. □

Exercise E2.7. Assume that G is a compact abelian group whose character group is of the form $A \oplus B$ with two subgroups A and B. Show that the morphism $g \mapsto (g+N_A, g+N_B): G \to G/N_A \times G/N_B$ is an isomorphism and that $(G/N_A)\widehat{\ } \cong \widehat{N_B} \cong A$ and $(G/N_B)\widehat{\ } \cong \widehat{N_A} \cong B$. □

If G is a compact abelian group whose character group is finitely generated and thus is of the form $\widehat{G} = F \oplus E$ with $F \cong \mathbb{Z}^n$ and a finite group E then G is isomorphic to $\mathbb{T}^n \times E$ by a repeated application of Exercise E2.7 above in view of $\widehat{\mathbb{T}} \cong \mathbb{Z}$ and $\widehat{\mathbb{Z}(n)} \cong \mathbb{Z}(n)$, and in view of Remark 1.24(ii). Therefore, Corollary 2.34 yields the following remark:

Remark 2.35. Every compact abelian group is the strict projective limit of a projective system of groups G/N isomorphic to $\mathbb{T}^{n(N)} \times E_N$ with suitable numbers $n(N) = 0, 1, \ldots,$ and finite abelian groups E_N. □

More information will follow in 2.42 and 2.43 below.

But now we have further important conclusions from the Fundamental Theorem on Unitary Modules 2.22.

Corollary 2.36. *Every compact group is a strict projective limit of a projective system of groups each of which is isomorphic to a closed subgroup of an orthogonal* (*or a unitary*) *group.*

Proof. Assume that G is a compact group. We define $\mathcal{N}$ to be the set of all kernels of morphisms $f: G \to \mathrm{O}(n)$ for some n. All of these groups are compact normal subgroups, and $\bigcap \mathcal{N} = \{\mathbf{1}\}$ by Corollary 2.28. Now assume that $N_1, N_2 \in \mathcal{N}$. Then there are morphisms $f_j: G \to \mathrm{O}(n_j)$, $j = 1, 2$ and $N_j = \ker f_j$. Let $j: \mathrm{O}(n_1) \times \mathrm{O}(n_2) \to \mathrm{O}(n_1 + n_2)$ which in matrix form is given by

$$(T_1, T_2) \mapsto \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Define $f: G \to \mathrm{O}(n_1 + n_2)$ by $f(g) = j\big(f_1(g), f_2(g)\big)$. Then $\ker f = \ker f_1 \cap \ker f_2 = N_1 \cap N_2$. This shows that $N_1 \cap N_2 \in \mathcal{N}$. Hence $\mathcal{N}$ is a filter basis and thus G is the strict projective limit $\lim_{N \in \mathcal{N}} G/N$ by Proposition 1.33(ii). The same argument works with unitary instead of orthogonal groups. □

The preceding corollary says in effect that every compact group can be approximated with arbitrary accuracy by compact matrix groups (consisting of orthogonal matrices). Recall that this means that we find arbitrarily small compact normal subgroups N such that G/N is isomorphic to such a matrix group. This result is a sharper version of Corollary 2.29.

We now formulate an idea which is rather useful when dealing with topological groups.

Definition 2.37. Let G be a topological group. We say that G *has no small subgroups* (or is an *NSS-group*), respectively, *no small normal subgroups* if there is a neighborhood U of the identity such that for every subgroup, respectively, normal subgroup H of G the relation $H \subseteq U$ implies $H = \{\mathbf{1}\}$. □

According to this definition we shall say that a topological group *has small subgroups* if each of its identity neighborhoods contains a nonsingleton subgroup.

It is clear that $\mathbb{R}$ and $\mathbb{T}$ have no small subgroups. More generally, the additive groups of any Banach space and any topological groups locally isomorphic to it have no small subgroups. However, both the p-adic groups $\mathbb{Z}_p$ and the p-adic solenoids $\mathbb{T}_p$ (see Examples 1.28(i) and (ii)) have small subgroups. If G has no small subgroups and H is a subgroup of G then H has no small subgroups.

For a more general insight, let us return to Proposition 1.4 and the discussion which follows it.

Lemma 2.38. *If A is a Banach algebra and $G = A^{-1}$ is the group of units, then G has no small subgroups. As a consequence, every subgroup of G has no small subgroups.*

Proof. Let $B = \{a \in A \mid \|1-a\| < 1\}$ denote the unit ball around 1 in A. Then the function $\exp\colon A \to G$ given by $\exp x = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot x^n$ has a local inverse $\log\colon B \to A$ given by $\log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n}\cdot x^n$ for $\|x\| < 1$.

Now we select an open ball V of radius r around 0 in A such that $\exp 2V \subseteq B$. Then $U = \exp V$ is an open neighborhood of $\mathbf{1}$ in G. Let $\mathbf{1} \neq h \in U$. Then $a = \log h \in V$ and thus $0 < \|a\| < r$. If n is the largest natural number below $r/\|a\|$, then $\|n\cdot a\| = n\|a\| < r$ while $r \leq \|(n+1)\cdot a\| < \|n\cdot a\| + r < 2r$. Hence $(n+1)\cdot a \in 2V \setminus V$ and thus $h^{n+1} = \exp(n+1)\cdot a \in B \setminus U$ since exp maps $2V$ injectively into B. Therefore, if H is any subgroup of G contained in U, then $H = \{\mathbf{1}\}$ for every $h \in H$ different from $\mathbf{1}$ has a power outside U. The very last assertion is immediate. □

The following observation is practically trivial, but serves as a convenient reference.

Lemma 2.39. *Let $G = \lim_{j\in J} G_j$ be a strict projective limit of compact groups such that G has no small normal subgroups. Then there is an index $j \in J$ such that $G \cong G_j$.*

Proof. From Proposition 1.33(i) we know that the filter basis $\mathcal{N}$ of all kernels $\ker f_j$ of the limit maps $f_j\colon G \to G_j$ converges to $\mathbf{1}$. Assume now that U is an identity neighborhood in which $\{\mathbf{1}\}$ is the only normal subgroup. Then there is an index $j \in J$ such that $\ker f_j \subseteq U$. It follows that $\ker f_j = \{\mathbf{1}\}$. Hence f_j is injective. It is surjective anyhow since the projective limit is strict. Hence $f_j: G \to G_j$ is an isomorphism by Remark 1.8. □

Now we are ready for the following important consequence of our previous results.

Corollary 2.40. *For a compact group G, the following statements are equivalent:*

1. *G is isomorphic as a topological group to a* (*compact*) *group of orthogonal* (*or unitary*) *matrices.*
2. *G has a faithful finite dimensional orthogonal* (*or unitary*) *representation.*
3. *G has a faithful finite dimensional representation.*
4. *G is isomorphic as a topological group to a closed subgroup of the multiplicative group of some Banach algebra A.*
5. *There is a Banach algebra A and an injective morphism $j: G \to A^{-1}$ into the multiplicative group of A.*
6. *G has no small subgroups.*
7. *G has no small normal subgroups.*

Proof. (1)⇒(2) This is trivial.

(2)⇒(3) Again this is trivial

(3)⇒(4) Let $\pi: G \to \mathrm{Hom}(V, V)$ be an injective representation for a finite dimensional vector space V over $\mathbb{K}$. We take $A = \mathrm{Hom}(V, V)$ with the operator norm. Then (4) is an immediate consequence of Remark 1.8.

(4)⇔(5) Remark 1.8 again.

(4)⇒(6) Immediate from Lemma 2.38.

(6)⇒(7) This is trivial.

(7)⇒(1) By Proposition 1.33 and Corollary 2.36 there is a filter basis $\mathcal{N}$ of closed normal subgroups N of G converging to $\mathbf{1}$ such that $G = \lim_{N \in \mathcal{N}} G/N$ and such that G/N is isomorphic to a closed subgroup of an orthogonal group $\mathrm{O}(n_N)$ (respectively, unitary group $\mathrm{U}(m_N)$). By (7) and Lemma 2.39 there is an $N \in \mathcal{N}$ with $G \cong G/N$. Hence G is a closed subgroup of an orthogonal or unitary group in finitely many dimensions and (1) is proved. □

This allows us to make the following definition.

The Definition of a Compact Lie Group

Definition 2.41. A compact group G is called a *compact Lie group* if it satisfies one, and therefore all, of the equivalent conditions of Corollary 2.40. In particular, G is a Lie group if it has no small subgroups. □

We remark that this definition is consistent with all other definitions of a Lie group in the general domain of compact groups. Notice that all compact matrix groups are compact Lie groups.

Exercise E2.8. (i) Every finite group is a compact Lie group.

(ii) A totally disconnected compact group is a compact Lie group if and only if it is finite.

(iii) A finite direct product of compact Lie groups is a compact Lie group.

(iv) Every closed subgroup of a compact Lie group is a compact Lie group. □

Proposition 2.42. (i) *A compact abelian group is a compact Lie group if and only if it is isomorphic to* $\mathbb{T}^n \times E$ *for some natural number* n *and finite abelian group* E*, that is if and only if it is the character group of a finitely generated abelian group.*

(ii) *A compact connected abelian group is a compact Lie group if and only if it is isomorphic to* $\mathbb{T}^n$ *for some natural number* n*; i.e. is an* n*-torus.*

Proof. (i) No group $\mathbb{T}^n \times E$ has small subgroups. Hence all of these groups are Lie groups. Conversely, assume that G is a compact abelian Lie group. By Remark 2.35 and Lemma 2.39, G is isomorphic to $\mathbb{T}^n \times E$, where E is a finite group E. (ii) follows directly from (i). □

Exercise E2.9. If G is a closed subgroup of $\mathbb{T}^n$, then G is isomorphic to the direct product of a finite dimensional torus group and a finite group. □

Neither the p-adic groups $\mathbb{Z}_p$ nor the p-adic solenoids $\mathbb{T}_p$ are compact Lie groups because they have small subgroups. More trivially, an infinite product of nonsingleton compact groups is never a compact Lie group, because it, too, has small subgroups. We shall argue later that the underlying space of a compact Lie group always has to be a (real analytic) manifold (Corollary 5.37).

The converse question whether a topological group on a compact manifold is a Lie group has an affirmative answer, too, but that is a more delicate question. We shall be able to answer it when we have more information on the structure of compact groups.

Corollary 2.36 can now be reformulated in a smooth fashion:

Approximating a Compact Group by Compact Lie Groups

Corollary 2.43. *Every compact group is a strict projective limit of compact Lie groups.* □

Theorem 1.34 which we derived in an elementary fashion (that is, without the aid of integration on the group) is a forerunner of this theorem. The significance of Corollary 2.43 is that it reduces the theory of arbitrary compact groups in large measure to that of compact Lie groups. How this works we shall see when we learn more about compact Lie groups in Chapters 5 and 6.

Postscript

In this chapter we have leapt over the barrier of Haar measure in a somewhat unorthodox manner. The existence and uniqueness of Haar measure was outlined in Theorem 2.8 and its proof is given in Appendix 5 in a form that is due to James Wendel [373]. This proof was one of the earliest applications to harmonic analysis of the theory of compact topological semigroups which was in the process of being developed at the time Wendel's proof was published. It is an appropriate choice in the context of our presentation of compact groups; it provides information on compact groups beyond the existence of Haar measure as such, and it is compatible with the spirit of topological algebra. Wendel's proof, on the other hand, does not yield a proof of the existence of Haar measure on locally compact noncompact groups. The reader finds an updated contemporary presentation of this material in [177].

We have proceeded in this chapter to exploit Haar measure for the structure and representation theory as fully as possible at this point. We have proceeded to the Theorem of Peter and Weyl on a route which is somewhat different from that taken in other sources. We first aim for a proof of the existence of finite dimensional subrepresentations in any unitary representation of a compact group and develop everything from there. All proofs somehow rest on averaging operators; ours is more geometric and uses the averaging of compact operators rather than, say, Hilbert–Schmidt operators. Our proof of the existence of sufficiently many finite dimensional unitary representations of a compact group leading up to 2.22 is fairly direct and is taken from [160].

The definition of a *compact Lie group* in 2.41 is a bit unconventional, but it is appropriate to our approach. We shall show in Chapter 5 that it fits perfectly into the theory of linear Lie groups and of analytic groups. A significant outcome of the investigations of this chapter is the surprising result that every compact group is a strict projective limit of matrix groups. The next chapter is a generalisation of the classical theory of trigonometric functions and trigonometric polynomials expanding on the seminal work of Peter and Weyl.

References for this Chapter—Additional Reading

[34], [40], [38], [94], [100], [147], [149], [160], [177], [196], [211], [219], [229], [230], [263], [307], [317], [331], [373].

Chapter 3

The Ideas of Peter and Weyl, Tannaka, Hopf, and Hochschild

The emphasis in the first part of this chapter is on the algebra $C(G, \mathbb{K})$ of continuous functions on a compact group G. We know from Example 2.4 that $C(G, \mathbb{K})$ is a G-module. So the module aspect will be once more in the foreground. In particular, we shall find a dense submodule $R(G, \mathbb{K})$ whose structure we shall describe accurately. The prerequisites include a knowledge of the preceding part and such tools as the Approximation Theorem of Weierstraß and Stone.

In the second part of this chapter, however, we expand these ideas to a very wide class of G-modules on locally convex topological vector spaces which satisfy a certain (very weak) completeness condition. In order to appreciate this part of the chapter, the reader should have some familiarity with locally convex topological vector spaces. The reader may wish to skip this portion at the first reading and go directly to Part 1 of the next chapter; the price for skipping it is the loss of some of the beautiful generality inherent in the general theory of modules over a compact group, and in Part 2 of the next chapter we shall utilize Part 2 of this chapter.

Likewise, the reader primarily interested in the internal structure of compact groups may also skip at a first reading the third part of this chapter and return to it at a later time if the need arises. The following comments serve as an introduction to the remainder of the present Chapter 3.

In Chapters 1 and 2, specifically Theorem 2.32, we saw that the category of compact abelian groups is dual to the category of abelian groups. This is the Pontryagin Duality Theorem for compact abelian groups. The Tannaka Duality Theorem identifies a category which is dual to the category of compact groups. Part 3 will describe a conceptually new approach to the Tannaka Duality Theorem. For this purpose it systematically exploits the much more elementary duality between $\mathbb{K}$-vector spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and weakly complete $\mathbb{K}$-vector spaces. Here a weakly complete $\mathbb{K}$-vector space is any topological vector space that is isomorphic to a power $\mathbb{K}^J$ for a set J. Our approach leads to symmetric Hopf algebras in the symmetric monoidal categories of $\mathbb{R}$-vector spaces and of weakly complete $\mathbb{R}$-vector spaces. This allows us to identify organically a special category of weakly complete real symmetric Hopf algebras as completely equivalent to the category of compact groups. Then the dual category of weakly complete real symmetric Hopf algebras is the category of reduced Hopf algebras which now emerges as the dual category to that of compact groups in the Tannaka-Hochschild Duality Theorem.

In this process we learn that the weakly complete group algebra $\mathbb{R}[G]$ of a compact group also contains the Lie algebra of G which we shall get to know more explicitly in subsequent sections, and it contains, in addition, the compact monoid of probability measures on G. We shall return to measure theory later in the book, notably in Appendix 5.

In the process we formulate a theorem on the algebra structure of the real or complex group algebra $\mathbb{K}[G]$ and a dual result (Theorem 3.82 and Corollary 3.83) which may be viewed as refinements of the Peter–Weyl Theorems of the earlier parts of this chapter.

Prerequisites. As a tool we need vector valued integration of Radon measures on compact Hausdorff spaces, but we review it in this chapter. A certain familiarity with locally convex topological vector spaces is assumed.

The third part of this chapter requires familiarity with real and complex vector spaces and their duality; for this purpose and its ramifications we have expressly formulated Appendix 7 in order to secure self-contained access to this circle of basic linear algebra in this book. In the same vein, the methods of constructing group algebras in the category of weakly complete topological vector spaces, we use fundamental category theoretical methods such as the Adjoint Functor Existence Theorem for which we refer the reader to the compact course taught in Appendix 3, Theorem A3.28 through Theorem A3.60.

The concept of a symmetric Hopf algebra is present in many different corners of this book. The *systematic* background for this concept is that of a symmetric (or commutative) monoidal category as we expose it in Appendix 3 from Definition A3.62 onward to the end of the appendix. However, for the purposes of this chapter one needs from the subsection of Appendix 3 on "Commutative Monoidal Categories and its Monoid Objects" (beginning after Theorem A3.60) primarily Part 5: "Symmetric Hopf Algebras over $\mathbb{R}$ and $\mathbb{C}$" beginning with Definition A3.93, while the Parts 2, 3, and 4 which occasionally require technical efforts on Graded Commutative Hopf Algebras are not needed here.

However, the basic structure theory proper of the real or complex group algebra requires only the foundational material provided in the first 3 chapters of this book.

Part 1: The Classical Theorem of Peter and Weyl

Definition 3.1. Let G denote a topological group and assume that E is a G-module. (See Definition 2.1.) We say that an element $x \in E$ is *almost invariant* or *G-finite* if $\operatorname{span}(Gx)$, the linear span of its orbit Gx, is finite dimensional. The set of almost invariant vectors in E is denoted E_{fin}. □

Evidently, each orbit Gx is a G-invariant set and thus $\operatorname{span}(Gx)$ is a submodule. Therefore, *an element $x \in E$ is almost invariant if and only if there is a G-invariant finite dimensional vector subspace V of E with $x \in V$.*

Lemma 3.2. *Let E be a G-module. Then E_{fin} is a submodule. Moreover, if E is any algebra, that is, is equipped with a bilinear map $(x, y) \mapsto xy: E \times E \to E$ and if the actions $x \mapsto gx$ are algebra morphisms for all $g \in G$, then E_{fin} is a subalgebra.*

Proof. Clearly E_{fin} is closed with respect to scalar multiplication. Thus let $x, y \in E_{\text{fin}}$. We must show that $x + y$ and, where applicable, xy are elements of E_{fin}. Now

$$\text{(1)} \qquad \operatorname{span}\big(G(x+y)\big) \subseteq \operatorname{span}(Gx) + \operatorname{span}(Gy)$$

and

$$\text{(2)} \qquad \operatorname{span}(Gxy) \subseteq \operatorname{span}\big((Gx)(Gy)\big) \subseteq \operatorname{span}\big[\big(\operatorname{span}(Gx)\big)\big(\operatorname{span}(Gy)\big)\big],$$

provided that $g(xy) = (gx)(gy)$ for all $g \in G$ and $x, y \in E$. Condition (1) immediately implies that $\operatorname{span}\big(G(x+y)\big)$ is finite dimensional, whence $x + y \in E_{\text{fin}}$. Let us consider any two finite dimensional vector subspaces V and W in an arbitrary algebra E. Then $\operatorname{span}(VW)$ is finite dimensional. Indeed if $v_1, \ldots, v_m$ is a basis of V and $w_1, \ldots, w_n$ is a basis of W, then the finite family $\{v_j w_k \mid j = 1, \ldots, m, k = 1, \ldots, n\}$ spans $\operatorname{span}(VW)$. As a consequence, (2) implies $xy \in E_{\text{fin}}$. □

We specialize this immediately to the G-module $C(G, \mathbb{K})$ for a compact group G.

Definition 3.3. The set $C(G, \mathbb{K})_{\text{fin}}$ of almost invariant continuous functions on G is written $R(G, \mathbb{K})$ and its elements are called *representative functions.* □

Note that $R(G, \mathbb{K})$ is a subalgebra of $C(G, \mathbb{K})$, sometimes called the *representation ring of* G.

Thus a continuous function f on a compact group is a representative function if and only if the set of its translates ${}_g f$ spans a finite dimensional vector space. Clearly every constant function is a representative function. However, the door is open to a much larger supply of representative functions:

Proposition 3.4. *Let G be a compact group. Then the following statements are equivalent for a continuous function $f \in C(G, \mathbb{K})$:*

(1) *There is a finite dimensional G-module E and there are vectors $x \in E$ and $u \in E'$ (where E' is the dual of E, i.e. the vector space of all linear functionals on E), such that*
$$f(g) = \langle u, gx \rangle \quad \textit{for all} \quad g \in G.$$

(2) $f \in R(G, \mathbb{K})$.

Proof. (1)⇒(2): Assume (1), take a basis $x_1, \ldots, x_n$ is of E, and set $f_k(g) = \langle u, gx_k \rangle$. Now for any $h \in G$ we have $hx = \sum_{j=1}^n c_j \cdot x_j$ for suitable scalars $c_j \in \mathbb{K}$. Thus ${}_h f(g) = f(gh) = \langle u, ghx \rangle = \sum_{j=1}^n \langle u, g(c_j \cdot x_j) \rangle = \big(\sum_{j=1}^n c_j \cdot f_j\big)(g)$. So for any $h \in G$, the function ${}_h f$ is in the span of $\{f_1, \ldots, f_n\}$. Hence $f \in R(G, \mathbb{K})$.

(2)⇒(1): Let $E = \operatorname{span}({}_G f)$; then E is a submodule of $C(G, \mathbb{K})$. We define a functional $u \in E'$ as the restriction to E of the point measure $\delta_{\mathbf{1}} \in M(G, \mathbb{K})$

given by $\langle \delta_{\mathbf{1}}, \varphi\rangle = \varphi(1)$, that is, the evaluation of a continuous function at $\mathbf{1}$. Then $f(g) = {}_gf(\mathbf{1}) = \langle \delta_{\mathbf{1}}, {}_gf\rangle = \langle u, {}_gf\rangle$. Thus (1) holds. □

Let us observe what this means in terms of matrices.

Corollary 3.5. *For a compact group G, the representative functions are exactly the functions $G \to \mathbb{K}$ which appear as scalar multiples of the coefficient functions $g \mapsto a_{jk}(g)$ of all finite dimensional matrix representations*

$$g \mapsto \big(a_{jk}(g)\big)_{j,k=1,\dots,n}: G \to M_n(\mathbb{K})$$

of G.

Proof. Firstly, let f be a representative function. By Proposition 3.4, there is a finite dimensional representation $\pi: G \to \mathrm{Gl}(E)$ such that $f(g) = \langle u, \pi(g)(x)\rangle$ for vectors $x \in E$ and $u \in E'$.

If $\langle u, x\rangle \neq 0$ then we choose a basis $e_1 = x, e_2, \dots, e_n$ of E and a dual basis $u_1 = \langle u, x\rangle^{-1}{\cdot}u, u_2, \dots$ Then $\pi(g)$ has the matrix $a_{jk}(g) = \langle u_j, \pi(g)(e_k)\rangle$, $j, k = 1, \dots, n$ with respect to the basis $e_1, \dots, e_n$. In particular, we have $f(g) = \langle u, x\rangle^{-1}{\cdot}a_{11}(g)$.

If, however, $\langle u, x\rangle = 0$ and if $u \neq 0$, $x \neq 0$—which is the only case of interest—then we find a basis $e_1 = x, e_2, \dots, e_n$ of E and a dual basis $u_1, u_2 = u, \dots, u_n$. In this situation we observe $f(g) = a_{21}(g)$.

Thus in any case, f is a scalar multiple of a coefficient function of a matrix representation.

Secondly, assume that $f(g) = ca_{jk}(g)$ is a scalar multiple of a coefficient function of a matrix representation $P: G \to M_n(\mathbb{K})$. Then let $E = \mathbb{K}^n$ and let $\pi: G \to \mathrm{Gl}(E)$ be the representation given by P. If $e_1 = (1, 0, \dots, 0)$, and so on, is the standard basis of E and if we identify E with its own dual so that $\langle (a_1, \dots, a_n), (x_1, \dots, x_n)\rangle = a_1x_1 + \cdots + a_nx_n$, then $f(g) = ca_{jk}(g) = \langle c{\cdot}e_j, \pi(g)(e_k)\rangle$. Thus f is a representative function by Proposition 3.4. □

Exercise E3.1. Assume that $G = \mathbb{T}$. Describe the functions in $R(G, \mathbb{R})$ and $R(G, \mathbb{C})$. □

Let us observe that for each finite dimensional G-module E, the dual E' is also a G-module, called the *adjoint module*: Indeed if the representation associated with E is $\pi: G \to \mathrm{Gl}(E)$, then the adjoint operator $\pi(g)': E' \to E'$ given by $\pi(g)'(u) = u \circ \pi(g)$, that is, by $\langle \pi(g)'(u), x\rangle = \langle u, \pi(g)(x)\rangle$ defines a representation $\pi': G \to \mathrm{Gl}(E')$ given by $\pi'(g) = \pi(g^{-1})' = {\pi(g)'}^{-1}$.

This has an immediate consequence: Assume that for a function $f: G \to \mathbb{K}$ we set

$$\check{f}(g) = f(g^{-1}) \text{ for all } g \in G.$$

By Proposition 3.4, for a representative function f, we have $f(g) = \langle u, \pi(g)(x)\rangle = \langle \pi(g)'(u), x\rangle$ with a vector x from a finite dimensional G-module E and $u \in E'$. Let $\eta: E \to E''$ denote the natural isomorphism defined by $\eta(x)(u) = \langle u, x\rangle$. Then

$f(g) = \langle \pi(g)'(u), x\rangle = \langle \eta(x), \pi'(g^{-1})(u)\rangle$. Thus Proposition 3.4 again shows that $\check{f}$ is a representative function.

The assignment $f \mapsto \check{f}$ is an automorphism of algebras on $R(G, \mathbb{K})$ satisfying $\check{\check{f}} = f$, that is, it is an involution. The map $f \mapsto \overline{f}$ of $R(G, \mathbb{K})$ into itself (which is just the identity map in the case $\mathbb{K} = \mathbb{R}$) is an involutive conjugate linear automorphism of algebras. This involution is a G-module automorphism since ${}_g\overline{f} = \overline{({}_gf)}$. It therefore maps $R(G, \mathbb{K})$ into itself, that is $\overline{R(G, \mathbb{K})} = R(G, \mathbb{K})$.

There is a simple connection between the two involutions: If $f \in R(G, \mathbb{K})$, then by Proposition 3.4 there is a finite dimensional G-module E and vectors $x \in E$ and $u \in E'$ such that $f(g) = \langle u, gx\rangle$. By Weyl's Trick 2.10 we may assume that E is a finite dimensional Hilbert G-module. Let us consider the *conjugate vector space* $\widetilde{E}$ *of* E which has the same underlying addition as E and the scalar multiplication given by

$$c \bullet x = \overline{c}\cdot x \qquad \text{for all } c \in \mathbb{K}.$$

(In the case $\mathbb{K} = \mathbb{R}$ we have $\widetilde{E} = E$.) Now the function $\iota\colon \widetilde{E} \to E'$ given by $\langle \iota y, x\rangle = (x|y)$ is an isomorphism of G-modules since $\langle \iota(gy), x\rangle = \big(x|\pi(g)(y)\big) = \big(\pi(g)^*x|y\big) = \big(\pi(g)^{-1}x|y\big) = \langle \iota(y), g^{-1}x\rangle = \langle g\iota(y), x\rangle$ holds for all x, $y \in E$. □

If, for the moment, we set $f_{xy}(g) = \langle \iota(y), gx\rangle = (gx \mid y)$ then

$$\begin{aligned} f_{xy}(g) &= (\pi(g)x \mid y) = \big(x \mid \pi(g^{-1})y\big) \\ &= \overline{(\pi(g^{-1})y \mid x)} \\ &= \check{f}_{yx}(g). \end{aligned}$$

Let us denote by $\overline{\pi}$ the representation associated with $\widetilde{E}$. If $e_1, \ldots, e_n$ is an orthonormal basis of E, then the functions $a_{kj} = f_{e_je_k}$ are the matrix coefficients of π. By the definition of $\widetilde{E}$, the matrix coefficients of $\overline{\pi}(g)$ are $\overline{a_{jk}(g)}$.

For easy reference we summarize these observations:

Lemma 3.6. *Let E denote a finite dimensional G-module and E' its dual module.*

(i) *The dual module E' is isomorphic to the conjugate module $\widetilde{E}$ under $\iota\colon \widetilde{E} \to E'$ with $\langle \iota(y), x\rangle = (x|y)$.*

(ii) *The assignments $f \mapsto \overline{f}$, $\check{f}\colon C(G, \mathbb{K}) \to C(G, \mathbb{K})$ are involutions mapping $R(G, \mathbb{K})$ into itself. For $f_{xy}(g) = \langle \iota(y), gx\rangle$ we have*

$$(*) \qquad \overline{f_{xy}} = \check{f}_{yx}.$$

(iii) *If, for a suitable basis of E, the representation π has the matrix coefficients $g \mapsto a_{jk}(g)$, $j, k = 1, \ldots, n$, then $\overline{\pi}$, the representation associated with $\widetilde{E}$, has the matrix coefficients $g \mapsto \overline{a_{jk}(g)}$.* □

After Corollaries 2.27 and 2.28 we know that the finite dimensional representations of a compact group separate the points. As a consequence we know that there are plenty of representative functions. In fact, we immediately obtain a celebrated result which generalizes the fact that any continuous periodic function can be approximated uniformly by trigonometric polynomials. (See e.g. [231], p. 514.)

The Classical Theorem of Peter and Weyl

Theorem 3.7. *Let G be a compact group. Then $R(G,\mathbb{K})$ is dense in $C(G,\mathbb{K})$ and in $L^2(G,\mathbb{K})$.*

Proof. By Lemma 3.2 we know that $R(G,\mathbb{K})$ is a subalgebra of $C(G,\mathbb{K})$ containing the constant functions. By the preceding remarks we know that $R(G,\mathbb{K})$ is closed under conjugation. We shall show that it separates the points. For this purpose let $g_1 \neq g_2$ in G. Then by Corollary 2.28 there is a finite dimensional G-module E and a vector $x \in E$ such that $g_1x \neq g_2x$. Hence there must be a functional $u \in E'$ with $\langle u, g_1x\rangle \neq \langle u, g_2x\rangle$. Then by Corollary 3.5 we have found a representative function f with $f(g_1) \neq f(g_2)$. Thus $R(G,\mathbb{K})$ is a point separating subalgebra of $C(G,K)$ containing constants and being closed under conjugation. Then the Theorem of Stone and Weierstraß implies that $R(G,\mathbb{K})$ is dense in $C(G,\mathbb{K})$ with respect to the topology of uniform convergence. (Cf. [34], X.39, Proposition 7, or [331], p. 161.) Since uniform convergence certainly implies convergence in $L^2(G,\mathbb{K})$, and since $C(G,\mathbb{K})$ is dense in $L^2(G,\mathbb{K})$ by the definition of $L^2(G,\mathbb{K})$ (see Example 2.12), it follows that $R(G,\mathbb{K})$ is dense in $L^2(G,\mathbb{K})$ with respect to the topology of $L^2(G,\mathbb{K})$. □

Let us observe in the context of the Theorem of Peter and Weyl that $C(G,\mathbb{K})$ has also the structure of a G-module by translation on the left side of the arguments:

Remark 3.8. (i) The operation $(g,f) \mapsto {}^gf\colon G \times C(G,\mathbb{K}) \to C(G,\mathbb{K})$ given by ${}^gf(x) = f(g^{-1}x)$ makes $C(G,\mathbb{K})$ into a G-module.

(ii) For $g,h \in G$ and $f \in C(G,\mathbb{K})$ one has ${}_g({}^hf) = {}^h({}_gf)$. The action $\big((g,h),f\big) \mapsto \big(x \mapsto f(g^{-1}xh)\big)\colon (G\times G)\times C(G,\mathbb{K}) \to C(G,\mathbb{K})$ defines on $C(G,\mathbb{K})$ the structure of a $G\times G$-module.

(iii) If one restricts the action of $G\times G$ in (ii) to the diagonal of $G\times G$ one obtains $(g,f)\mapsto g{\cdot}f\colon G\times C(G,\mathbb{K}) \to C(G,\mathbb{K})$ given by $(g{\cdot}f)(x) = f(g^{-1}xg)$.

Proof. Exercise E3.2. □

Exercise E3.2. Verify the details of Remark 3.8. □

In order to distinguish the two actions let us call $(g,f) \mapsto {}_gf$ the *action on the right of the argument* and $(g,f)\mapsto {}^gf$ the *action on the left of the argument.*

Proposition 3.9. (i) *The involution $f \mapsto \check f\colon C(G,\mathbb{K}) \to C(G,\mathbb{K})$ is an isomorphism of G-modules if on the domain the action is on the right of the argument and on the range on the left—and vice versa.*

(ii) *The subset* $R(G,\mathbb{K})$ *of* $C(G,\mathbb{K})$ *is the set of almost invariant vectors with respect to both actions.*

Proof. (i) We observe $({}^g f)\check{}(x) = {}^g f(x^{-1}) = f(g^{-1}x^{-1}) = f\big((xg)^{-1}\big) = \check{f}(xg) = ({}_g\check{f})(x)$. Thus $\check{}$ exchanges the two actions.

(ii) Let $f \in R(G,\mathbb{K})$; we have to show that the span of all translates ${}^g f$ is finite dimensional. Now $({}^g f)\check{} = {}_g\check{f}$ by (i) and $\check{f} \in R(G,\mathbb{K})$ by $(*)$ in 3.6(ii). Thus $\operatorname{span}\{({}^g f)\check{} \mid g \in G\} = (\operatorname{span}\{{}_g f \mid g \in G\})\check{}$ is finite dimensional. Since $\check{}$ is a vector space automorphism, $\operatorname{span}\{{}^g f \mid g \in G\}$ is finite dimensional and the assertion is proved. □

By the very definition of $R(G,\mathbb{K})$ and by Proposition 3.9, every one of its elements is contained in a finite dimensional vector subspace which is invariant with respect to left translation, right translation, and conjugation of the argument. We wish to analyze the finite dimensional submodules of $R(G,\mathbb{K})$ more systematically. Proposition 3.4 suggests how such an investigation may be undertaken.

We need some elementary linear algebra which is absolutely indispensable in this context.

An Excursion into Linear Algebra

Let us consider an arbitrary finite dimensional vector space E. As usual, E' denotes the dual, and $E' \otimes E$ the tensor product of the two vector spaces over $\mathbb{K}$. We must know how the vector spaces $E' \otimes E$, $\operatorname{Hom}(E,E)$, and its dual $\operatorname{Hom}(E,E)'$ are related by natural isomorphisms.

Lemma 3.10. (i) *The vector space* $E' \otimes E$ *is isomorphic to* $\operatorname{Hom}(E,E)$ *under a morphism* $\theta_E\colon E' \otimes E \to \operatorname{Hom}(E,E)$ *given by* $\theta_E(v \otimes x)(y) = \langle v, y\rangle{\cdot}x$. *For* $x_1, x_2 \in E$ *and* $v_1, v_2 \in E'$ *we have*

$$\theta_E(v_2 \otimes x_2) \circ \theta_E(v_1 \otimes x_1) = (y \mapsto \langle v_2, x_1\rangle\langle v_1, y\rangle{\cdot}x_2). \tag{3}$$

(ii) *The function* $\tau_E\colon E' \otimes E \to \operatorname{Hom}(E,E)'$ *which assigns to an element* $v \otimes x$ *the functional of* $\operatorname{Hom}(E,E)$ *given by*

$$\langle \tau_E(v \otimes x), \varphi\rangle = \langle v, \varphi(x)\rangle$$

is an isomorphism of vector spaces and

$$\langle \tau_E(v_2 \otimes x_2), \theta_E(v_1 \otimes x_1)\rangle = \langle v_2, x_1\rangle\langle v_1, x_2\rangle. \tag{4}$$

(iii) *In particular,*

$$\begin{aligned}\langle \tau_E(v_2 \otimes x_2), \theta_E(v_1 \otimes x_1)\rangle{\cdot}x_3 &= \langle v_2, x_1\rangle\langle v_1, x_2\rangle{\cdot}x_3\\ &= \big(\theta_E(v_2 \otimes x_3) \circ \theta_E(v_1 \otimes x_1)\big)(x_2).\end{aligned}$$

Proof. Exercise E3.3. □

Exercise E3.3. Prove the statements of Lemma 3.10. □

The following exercise may be skipped in the context of representation theory; it illustrates, however, the semigroup structure of the set of all rank one endomorphisms of a vector space.

Exercise E3.4. Let Γ denote a group and Γ_0 the semigroup obtained from Γ by attaching a disjoint element 0 acting as zero, i.e. satisfying $\gamma 0 = 0\gamma = 0$. Let X and Y two arbitrary sets and $(y,x) \mapsto [y,x]: Y \times X \to \Gamma_0$ any function. Then $X \times \Gamma_0 \times Y$ becomes a semigroup Σ with respect to the multiplication $(x,\gamma,y)(x',\gamma',y') = (x,\gamma[y,x']\gamma',y')$. (Notice that the sequence of letters is the same on both sides of the equation!) The set $I = X \times \{0\} \times Y$ is an ideal and the quotient Σ/I obtained by collapsing the elements of I to one point is a semigroup which we call $[X,\Gamma_0,Y]$. Such semigroups are called *Rees matrix semigroups* (see e.g. [62]).

Show that the set $\{\theta_E(v \otimes x) \mid v \in E', \quad x \in E\}$ under composition of endomorphisms is a semigroup which is isomorphic to a Rees matrix semigroup. □

Definition 3.11. If $\mathbf{1}$ denotes the distinguished element id_E of $\mathrm{Hom}(E,E)$, then we use the abbreviation $\mathbf{T}$ for $\theta_E^{-1}(\mathbf{1})$. The functional $\tau_E(\mathbf{T}) = \tau_E\theta_E^{-1}(\mathbf{1})$ is called the *trace* and is written tr. □

Consider any basis $e_1,\dots,e_n$ of E and $u_1,\dots,u_n$ the dual basis characterized by $\langle u_j, e_k\rangle = \delta_{jk}$. Then for each $x \in E$ we note $\mathbf{1}(x) = x = \sum_{j=1}^n \langle u_j, x\rangle \cdot e_j = \theta_E(\sum_{j=1}^n u_j \otimes e_j)(x)$. This means $\theta_E(\sum_{j=1}^n u_j \otimes e_j) = \mathbf{1}$ and thus

(5) $$\mathbf{T} = \sum_{j=1}^n u_j \otimes e_j.$$

On the other hand, if $\varphi: E \to E$ is any endomorphism of E, then the coefficients of its matrix with respect to our basis are $a_{jk} = \langle u_j, \varphi(e_k)\rangle$. Thus

$$\begin{aligned}\operatorname{tr}\varphi &= \langle \tau_E(\sum_{j=1}^n u_j \otimes e_j), \varphi\rangle \\ &= \sum_{j=1}^n \langle u_j, \varphi(e_j)\rangle \\ &= a_{11} + a_{22} + \cdots + a_{nn}.\end{aligned}$$

If, in particular, $\varphi = \theta_E(v \otimes x)$, then

$$\begin{aligned}\langle \tau_E(\mathbf{T}), \varphi\rangle &= \sum_{j=1}^{n} \langle \tau_E(u_j \otimes e_j), \theta_E(v\otimes x)\rangle \\ &= \sum_{j=1}^{n}\langle u_j, x\rangle\langle v, e_j\rangle = \langle v, \sum_{j=1}^{n}\langle u_j, x\rangle{\cdot}e_j\rangle = \langle v, x\rangle.\end{aligned}$$

Thus *the trace is characterized by the fact that*

$$\operatorname{tr}\bigl(\theta_E(v\otimes x)\bigr) = \langle v, x\rangle. \tag{6}$$

Exercise E3.5. Show $\operatorname{tr}\varphi\psi = \operatorname{tr}\psi\varphi$. □

Lemma 3.12. *The isomorphism $\tau_E \circ \theta_E^{-1}\colon \operatorname{Hom}(E,E) \mapsto \operatorname{Hom}(E,E)'$ assigns to a morphism $\varphi\colon E \to E$ the functional $\psi \mapsto \operatorname{tr}\psi\varphi$.*

Proof. It suffices to verify the claim for $\psi = \theta_E(v_2 \otimes x_2)$ and $\varphi = \theta_E(v_1\otimes x_1)$. Then $\psi\bigl(\varphi(y)\bigr) = \langle v_2, x_1\rangle\langle v_1, y\rangle{\cdot}x_2 = \langle v_2, x_1\rangle{\cdot}\theta_E(v_1\otimes x_2)(y)$ by (3) in Lemma 3.10. Then we have $\operatorname{tr}\psi\varphi = \langle v_2, x_1\rangle\langle v_1, x_2\rangle$ by (6) above. But this means $\operatorname{tr}\psi\varphi = \langle\tau_E(v_2\otimes x_2), \theta_E(v_1\otimes x_1)\rangle = \langle \tau_E\theta_E^{-1}(\psi), \varphi\rangle$ by (4) in Lemma 3.10. This proves the assertion. □

Let us illustrate the situation in the diagram

$$\begin{array}{ccc} E'\otimes E & \xrightarrow{\quad\theta_E\quad} & \operatorname{Hom}(E,E) \\ {\scriptstyle\tau_E}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau_E\circ\theta_E^{-1}} \\ \operatorname{Hom}(E,E)' & \xrightarrow{\quad\mathrm{id}\quad} & \operatorname{Hom}(E,E)'. \end{array}$$

We keep in mind that, in this scheme of things, $\operatorname{Hom}(E,E)$ is a $\mathbb{K}$-algebra.

The G-Modules $E'\otimes E$, $\operatorname{Hom}(E,E)$ and $\operatorname{Hom}(E,E)'$

Equipped with these tools we return to G-modules and assume now, that E is a finite dimensional G-module. Then the dual E' is a G-module with respect to the *adjoint action* given by

$$\langle gv, x\rangle = \langle v, g^{-1}x\rangle, \tag{7}$$

that is, by $\pi_{E'}(g) = \pi_E(g^{-1})'$ if $\varphi'\colon E' \to E'$ is the adjoint map of an endomorphism $\varphi\colon E\to E$. As a consequence, $E'\otimes E$ is a module in at least three significant ways.

Firstly, it is a $G\times G$-module in such a fashion that $(g,h){\cdot}(v\otimes x) = gv\otimes hx$.

Secondly, if we consider E' as the trivial module characterized by $gv = v$ for all $g\in G$ and $v \in E'$, then $E'\otimes E$ is a G-module via $g(v\otimes x) = v\otimes gx$. This module is easily understood right away in terms of the given module E: For any $v \in E'$, the vector subspace $v \otimes E$ is a submodule of $E' \otimes E$ which is isomorphic to the G-module E. Thus if $u_1, \ldots, u_n$ is a basis of E' , then $E' \otimes E = \bigoplus_{j=1}^n u_j \otimes E$ is a direct sum of G-modules, and we have the observation

Remark 3.13. If the trivial action is considered on E', then the G-module $E' \otimes E$ is isomorphic to the G-module E^n with $n = \dim E$. □

Notice that this module action is derived from the first by restriction of the operators $(g, h) = (v \otimes x \mapsto gv \otimes hx)$ to the second component g.

Thirdly, however, we have on $E' \otimes E$ the G-module operation which is derived from the $G \times G$-modules structure by restriction of the action to the diagonal: it is given by $g{\cdot}(v \otimes x) = gv \otimes gx$.

It is clear that on $\mathrm{Hom}(E, E)$ and $\mathrm{Hom}(E, E)'$ there are unique module structures such that the isomorphisms θ_E and τ_E are isomorphisms of modules. Thus

firstly, $\mathrm{Hom}(E, E)$ is a $G \times G$-module in such a way that $\big((g, h)\varphi\big)(x) = h\varphi(g^{-1}x)$. In other words, $\pi_{\mathrm{Hom}(E,E)}(g, h)(\varphi) = \pi_E(h)\varphi\pi_E(g)^{-1}$. Indeed we now note $\big((g, h)\theta_E(v \otimes x)\big)(y) = \pi_E(h)\theta_E(v \otimes x)(g^{-1}y) = \langle v, g^{-1}y\rangle{\cdot}hx = \langle gv, y\rangle{\cdot}hx = \theta_E(gv \otimes hx)(y) = \theta_E\big((g, h)(v \otimes x)\big)(y)$. Thus θ_E is a module isomorphism with respect to the first module action.

Secondly, $\mathrm{Hom}(E, E)$ is a G-module so that $(g\varphi) = \pi_E(g) \circ \varphi$, and

thirdly, $\mathrm{Hom}(E, E)$ is a G-module such that $(g{\cdot}\varphi)(x) = g\varphi(g^{-1}x)$, that is, that $g{\cdot}\varphi = \pi(g)\varphi\pi(g)^{-1}$.

In the context of arbitrary Hilbert G-modules, the third action was crucial in Chapter 2. The subspace of fixed points of this module is exactly the space $\mathrm{Hom}_G(E, E)$ of endomorphisms commuting with all $\pi(g)$, that is, of all module endomorphisms with respect to the third action. By Remark 3.13 and the fact that θ_E is a module isomorphism with respect to the first, hence also with respect to the second action we know that in regard to the second action, $\mathrm{Hom}(E, E)$ is isomorphic to the G-module E^n.

On $\mathrm{Hom}(E, E)'$ we consider the $G \times G$-module structure given by

$$\langle (g, h)\omega, \varphi\rangle = \langle \omega, \pi(g)^{-1}\varphi\pi(h)\rangle. \tag{8}$$

If $\pi_{\mathrm{Hom}(E,E)'}$ is the representation of the dual $G \times G$-module of $\mathrm{Hom}(E, E)$ given for each $\omega \in \mathrm{Hom}(E, E)$ by

$$\begin{aligned}\langle (g, h)\omega, \varphi\rangle &= \langle \omega, (g, h)^{-1}\varphi\rangle \\ &= \langle \omega, \pi(h)^{-1}\varphi\pi(g)\rangle,\end{aligned}$$

and if we denote with $\kappa\colon G \times G \to G \times G$ the involutive automorphism given by $\kappa(g, h) = (h, g)$, then the module structure we have defined in (8) satisfies

$$(g, h)\omega = (\pi_{\mathrm{Hom}(E,E)'} \circ \kappa)(g, h)(\omega).$$

Exercise E3.6. Show that τ_E a module isomorphism. Conclude that τ_E is also an isomorphism for the adjoint actions on $\mathrm{Hom}(E,E)'$ of the second and third module structure on $\mathrm{Hom}(E,E)$. □

Exercise E3.7. Let E be a (finite dimensional) Hilbert space with respect to a scalar product. Let φ^* denote the adjoint operator of $\varphi \in \mathrm{Hom}(E,E)$. Show that $\mathrm{Hom}(E,E)$ is a Hilbert space over $\mathbb{K}$ with respect to the scalar product $\langle\varphi|\psi\rangle = \mathrm{tr}\,\varphi\psi^*$. □

The $G\times G$-module $E'\otimes E$ is simple whenever E is simple. This is, in fact, a consequence of a more general lemma:

Lemma 3.14. *If E is a simple G-module and F a simple H-module over $\mathbb{K}=\mathbb{C}$ for compact groups G and H, then the $G\times H$-module $E\otimes F$ characterized by $(g,h)(x\otimes y)=gx\otimes hy$ is simple.*

Proof. We may identify $\mathrm{Hom}(E\otimes F, E\otimes F)$ with $\mathrm{Hom}(E,E)\otimes\mathrm{Hom}(F,F)$ via $(\varphi\otimes\psi)(x\otimes y)=\varphi(x)\otimes\psi(y)$. The operator $T\mapsto\widetilde{T}$ of

$$\mathrm{Hom}(E,E)\otimes\mathrm{Hom}(F,F)$$

into itself according to Lemma 2.15 may be computed as follows

$$\begin{aligned}(\varphi\otimes\psi)^{\sim} &= \textstyle\int_{G\times H}\pi_E(g)\varphi\pi_E(g)^{-1}\otimes\pi_F(h)\psi\pi_F(h)^{-1}\,d(g,h)\\ &= \big(\textstyle\int_G\pi_E(g)\varphi\pi_E(g)^{-1}\,dg\big)\otimes\big(\textstyle\int_H\pi_F(h)\psi\pi_F(h)^{-1}\,dh\big)=\widetilde{\varphi}\otimes\widetilde{\psi}.\end{aligned}$$

If E is simple, then $\varphi\mapsto\widetilde{\varphi}$ is a projection onto $\mathrm{Hom}_G(E,E)$ by Lemma 2.16, and $\psi\mapsto\widetilde{\psi}$ is a projection onto $\mathrm{Hom}_H(F,F)$. If E and F are simple we have $\mathrm{Hom}_G(E,E)=\mathbb{C}\cdot\mathrm{id}_E$ and $\mathrm{Hom}_H(F,F)=\mathbb{C}\cdot\mathrm{id}_F$ by Lemma 2.30. Hence the self-map $T\mapsto\widetilde{T}$ of $\mathrm{Hom}(E\otimes F,E\otimes F)$ is a projection onto $\mathbb{C}\cdot\mathrm{id}_{E\otimes F}$. If V is an invariant subspace of $E\otimes F$, then the orthogonal projection of $E\otimes F$ onto V is in $\mathrm{Hom}_{G\times F}(E\otimes F,E\otimes F)$. It follows that $P=\mathrm{id}_{E\otimes F}$ or $P=0$, that is $V=E\otimes F$ or $V=\{0\}$. □

As a consequence of this lemma and the fact that θ_E and τ_E are $G\times G$-module isomorphisms, we have the following conclusion:

Lemma 3.15. *If $\mathbb{K}=\mathbb{C}$ and E is a simple G-module for a compact group, then* $\mathrm{Hom}(E,E)$ *and* $\mathrm{Hom}(E,E)'$ *are simple $G\times G$-modules.* □

The Fine Structure of $R(G,\mathbb{K})$

We continue to let E denote a finite dimensional G-module.

Now we can return to Proposition 3.4 and exploit the linear algebra which we have just prepared. The function $(u,x)\mapsto\big(g\mapsto\langle u,gx\rangle\big)\colon E'\times E\to C(G,\mathbb{K})$ takes its values in $R(G,\mathbb{K})$ and is bilinear. It therefore induces a linear map

$$\Phi_E\colon E'\otimes E\to R(G,\mathbb{K}). \tag{9}$$

It is characterized by the formula

$$\Phi_E(u\otimes x)(g)=\langle u,gx\rangle. \tag{10}$$

Definition 3.16. The image of $E'\otimes E$ under Φ_E in $R(G,\mathbb{K})$ is written $R_E(G,\mathbb{K})$. □

Lemma 3.17. *The set $R_E(G,\mathbb{K})$ is a finite dimensional vector subspace of $R(G,\mathbb{K})$ which is invariant under left and right translations of the arguments, and*

$$\Phi_E\colon E'\otimes E\to R_E(G,\mathbb{K})$$

is an equivariant surjective linear map.

Proof. It suffices to show the equivariance of Φ_E with respect to the $G\otimes G$-module structures. Now $\Phi_E\big((g,h)(v\otimes x)\big)(\gamma)=\Phi_E(gv\otimes hx)(\gamma)=\langle gv,\gamma hx\rangle=\langle v,g^{-1}\gamma hx\rangle=\Phi_E(v\otimes x)(g^{-1}\gamma h)=\big({}_h{}^g\Phi_E(v\otimes x)\big)(\gamma)$. This proves the lemma. □

Definition 3.18. We define

$$\sigma_E\colon \mathrm{Hom}(E,E)\to R_E(G,\mathbb{K})$$

and

$$\rho_E\colon \mathrm{Hom}(E,E)'\to R_E(G,\mathbb{K})$$

by $\sigma_E=\Phi_E\circ\theta_E^{-1}$ and $\rho_E=\Phi_E\circ\tau_E^{-1}$. □

The following observations are now readily verified. Equivariance here refers to the $G\times G$-module structure, from which the equivariance with respect to the restricted actions follows at once.

Remark 3.19. The function $\sigma_E=\Phi_E\circ\theta_E^{-1}\colon \mathrm{Hom}(E,E)\to R_E(G,\mathbb{K})$ is an equivariant surjective map which assigns to an endomorphism $\varphi\colon E\to E$ the function $g\mapsto \mathrm{tr}\,\big(\varphi\pi(g)\big)$.

The function $\rho_E=\Phi_E\circ\tau_E^{-1}\colon \mathrm{Hom}(E,E)'\to R_E(G,\mathbb{K})$ is an equivariant surjective linear map which associates with a functional ω on $\mathrm{Hom}(E,E)$ the function $(g\mapsto\langle\omega,\pi(g)\rangle)=\omega\circ\pi$. □

Exercise E3.8. Verify the details of Remark 3.19. □

We now have a network of $G \times G$-module maps which is summarized in the following diagram:

$$
\text{(11)} \qquad \begin{array}{ccc} E' \otimes E & \xrightarrow{\tau_E} & \operatorname{Hom}(E,E)' \\ {\scriptstyle \theta_E}\downarrow & & \downarrow{\scriptstyle \rho_E} \\ \operatorname{Hom}(E,E) & \xrightarrow[\sigma_E]{} & R_E(G,\mathbb{K}), \end{array} \qquad \Phi_E = \sigma_E \circ \theta_E = \rho_E \circ \tau_E.
$$

What we need next is information on the kernel of Φ_E or, equivalently, the kernel of ρ_E. Clearly, we have $\tau_E(\ker \Phi_E) = \ker \rho_E$.

Proposition 3.20. *For a finite dimensional G-module E we have the following conclusions:*

(i) *The kernel of ρ_E is described by*

$$
\begin{aligned} \ker \rho_E = \pi(G)^\perp &= \{\omega \in \operatorname{Hom}(E,E)' \mid (\forall g \in G)\, \langle \omega, \pi(g)\rangle = 0\} \\ &= \{\omega \mid \omega\big(\pi(G)\big) = \{0\}\}. \end{aligned}
$$

(ii) *If one denotes with* $\operatorname{res}\colon \operatorname{Hom}(E,E)' \to \operatorname{span}\big(\pi(G)\big)'$ *the restriction map given by* $\operatorname{res}(\omega) = \omega|\operatorname{span}\big(\pi(G)\big)$, *and by* $\operatorname{inc}\colon \pi(G)^\perp \to \operatorname{Hom}(E,E)'$ *the inclusion map, then the following sequence is an exact sequence of $G \times G$-module maps:*

$$
0 \to \pi(G)^\perp \overset{\text{inc}}{\to} \operatorname{Hom}(E,E)' \overset{\text{res}}{\to} \big(\operatorname{span} \pi(G)\big)' \to 0.
$$

(iii) *The $G \times G$-modules* $\operatorname{span}\pi(G)$ $\big($*in* $\operatorname{Hom}(E,E)\big)$ *and* $R_E(G,\mathbb{K})$ $\big($*in* $R(G,\mathbb{K})\big)$ *are isomorphic. In particular,*

$$
\dim R_E(G,\mathbb{K}) = (\dim E)^2 - \dim \pi(G)^\perp.
$$

Proof. (i) This is immediate from the definition of ρ_E (see Definition 3.18).

(ii) This is pure linear algebra: If X is a subset of a finite dimensional vector space V then the sequence

$$
0 \to X^\perp \overset{\text{inc}}{\to} V' \overset{\text{res}}{\to} (\operatorname{span} X)' \to 0
$$

is exact.

(iii) More linear algebra: The surjective maps $\rho_E\colon \operatorname{Hom}(E,E)' \to R_E(G,\mathbb{K})$ and res:$\operatorname{Hom}(E,E)' \to \big(\operatorname{span}\pi(G)\big)'$ have the same kernels by (i) and (ii) above. Hence their images are isomorphic. The assertion about the dimensions is elementary linear algebra. □

We are clearly motivated at this point to say something about the vector subspace $\mathbb{A} \overset{\text{def}}{=} \operatorname{span}\pi(G)$ of the algebra $\operatorname{Hom}(E,E)$. Firstly, apart from being a submodule with respect to the most general action we pursued, namely, the $G \times G$-action, $\mathbb{A}$ is clearly a subalgebra since $\pi(G)$ is multiplicatively closed. Let us denote with $\mathbb{F}$ the subalgebra $\operatorname{Hom}_G(E,E)$ of $\operatorname{Hom}(E,E)$. Notice that $\mathbb{F}$ is the commutant $\mathcal{C}(\mathbb{A})$ of $\mathbb{A}$ in $\operatorname{Hom}(E,E)$ (as introduced in the section preceding 2.16). The *bicommutant* $\mathcal{C}^2(\mathbb{A}) = \mathcal{C}(\mathbb{F})$ of $\mathbb{A}$ certainly contains $\mathbb{A}$. It is not an obvious matter to find that it actually agrees with $\mathbb{A}$. In fact, even if we momentarily assume that, firstly, $\mathbb{K} = \mathbb{C}$ and, secondly, that E is a simple module, so that after Lemma 2.30 we know $\mathbb{F} = \mathbb{C}\cdot\mathbf{1}$ and thus $\mathcal{C}^2(\mathbb{A}) = \mathcal{C}(\mathbf{1}) = \mathrm{Hom}(E, E)$, we have no immediate reason for the conclusion $\mathbb{A} = \operatorname{span}\pi(G) = \mathrm{Hom}(E, E)$. However Lemma 3.15 now allows us to conclude $\mathbb{A} = \mathrm{Hom}(E, E)$ because $\mathbb{A}$ is a $G\times G$-submodule and $\mathbf{1} \in \mathbb{A}$. Hence we have

Proposition 3.21. *If E is a simple G-module over $\mathbb{C}$ with a compact group G, then* $\operatorname{span}\pi(G) = \mathrm{Hom}(E, E)$. □

Exercise E3.9. (i) Show that for $\mathbb{K} = \mathbb{C}$, a compact group G, and a finite dimensional G-module E, one has $\mathcal{C}(\mathbb{F}) = \mathbb{A}$.

(ii) Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $E = \mathbb{R}^2$ be the G-module given by the representation $\pi: G \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\pi(t + \mathbb{Z}) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}.$$

(We identify an endomorphism of $\mathbb{R}^2$ with its matrix representation.) Set $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Show that $\operatorname{span}\pi(G) = \mathbb{R}\cdot\mathbf{1} + \mathbb{R}\cdot\mathbf{i} \neq \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$. □

Let us observe that there is an algebraic theory behind the preceding discussions which culminates in the so-called Density Theorem of Jacobson for semisimple modules. (See e.g. [35], §1, **n**^{o} 2, Theorème 1, p. 39, [218], p. 104, Theorem 11, or [219], p. 28 and p. 127.) This theorem would allow us to conclude that $\mathbb{A}$ agrees with its double commutant regardless of the ground field. We shall return to this point in Lemma 3.80, in whose proof we shall spell out the Jacobson Density Theorem.

If we now take Propositions 3.20 and 3.21 together, recalling Remark 3.19, we obtain the following conclusion.

Theorem 3.22. *Let G be a compact group and E a simple G-module, then the maps $\sigma_E: \mathrm{Hom}(E, E) \to R_E(G, \mathbb{C})$ and $\rho_E: \mathrm{Hom}(E, E)' \to R_E(G, \mathbb{C})$ are isomorphisms with respect to the following module actions:*

(i) *$G \times G$ acting so that (g, h) transforms $f \in R_E(G, \mathbb{C})$ into $\gamma \mapsto f(g^{-1}\gamma h)$ and $\varphi \in \mathrm{Hom}(E, E)$ into $\pi(h)\varphi\pi(g)^{-1}$ (while acting on* $\mathrm{Hom}(E, E,)'$ *by the action given by $\langle (g, h)\omega, \varphi\rangle = \langle \omega, \pi(g)^{-1}\varphi\pi(h)\rangle$),*

(ii) *G acting so that g transforms f into ${}_gf$ and φ into $\varphi\pi(g)^{-1}$,*

(iii) *G acting so that g transforms f into $\gamma \mapsto f(g^{-1}\gamma g)$ and φ into $\pi(g)\varphi\pi(g)^{-1}$ (and acting on* $\mathrm{Hom}(E, E)'$ *by the adjoint action).* □

Corollary 3.23. (i) *If E is a simple G-module for a compact group G, then the G-module $R_E(G, \mathbb{C})$ with respect to the action $(g, f) \mapsto {}_gf$ is isomorphic to the*

G-module E^n with $n = \dim E$. In particular,

$$\dim_{\mathbb{C}} R_E(G,\mathbb{C}) = (\dim_{\mathbb{C}} E)^2.$$

(ii) *The function $f \mapsto \check{f}$ maps the module $R_E(G,\mathbb{C})$ with respect to the action on the left of the argument isomorphically onto the module $R_{E'}(G,\mathbb{C})$ with respect to the action on the right of the argument. In particular, with respect to $(g,f) \mapsto {}^g f$, the module $R_E(G,\mathbb{C})$ is isomorphic to $(E')^n$.*

Proof. (i) By 3.13, the module $E' \otimes E$ is isomorphic to E^n when the action on E' is considered to be trivial. However, $\Phi_E\colon E' \otimes E \to R_E(G,\mathbb{C})$ is equivariant with respect to this action. By Theorem 3.22 and Remark 3.19, Φ_E is an isomorphism.

(ii) If $u \in E'$ and $x \in E$, and if we set $f(g) = \langle u, gx\rangle$, then $\check{f}(g) = \langle u, g^{-1}x\rangle = \langle gu, x\rangle$. Hence $f \mapsto \check{f}$ maps $R_E(G,\mathbb{C})$ into $R_{E'}(G,\mathbb{C})$, and since it is bijective and $\dim R_E(G,\mathbb{C}) = n^2 = \dim R_{E'}(G,\mathbb{C})$ it induces an isomorphism between these two vector spaces. Proposition 3.9(i) proves the asserted equivariance.

The remainder now follows from (i) above. □

As the next step we shall show that the vector space $R(G,\mathbb{K})$ is the direct sum of all finite dimensional subspaces $R_E(G,\mathbb{K})$ as E ranges through a set of representatives of the set of isomorphy classes of the class of all simple G-modules. Therefore, in the present discussion we fix a set $\mathcal{E}$ of simple G-modules which meets every isomorphy class of simple G-modules in precisely one element $E \in \mathcal{E}$.

Firstly, we generalize Corollary 3.23 to the case that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The following lemma will be a helpful tool.

Lemma 3.24. *Let E be a G-module which is a finite direct sum of simple submodules all of which are isomorphic to a simple module F. Then the following conclusions hold:*

(i) *If $f_1\colon F_1 \to E$ and $f_2\colon E \to F_2$ are equivariant morphisms and F_1 and F_2 are simple, then $F_1 \not\cong F$ implies $f_1 = 0$ and $F_2 \not\cong F$ implies $f_2 = 0$.*

(ii) *If $f_1\colon E_1 \to E$ is an injective equivariant morphism, and $f_2\colon E \to E_2$ is a surjective equivariant morphism, then E_1 and E_2 are modules all of whose simple summands are isomorphic to F.*

Proof. (i) We may write $E = F^n$. We then have n coprojections $\mathrm{copr}_j\colon F \to E$ mapping F isomorphically onto the j-th factor, and n projections $\mathrm{pr}_j\colon E \to F$. The n maps $\mathrm{pr}_j \circ f_1\colon F_1 \to F$ are necessarily zero if $F_1 \not\cong F$, and since the projections pr_j separate the points of E, we conclude $f_1 = 0$. Likewise, the n maps $f_2 \circ \mathrm{copr}_j\colon F \to F_2$ are all zero if $F_2 \not\cong F$. Since the sum of the images of the n coprojections copr_j is E, it follows that $f_2 = 0$.

(ii) By Corollary 2.25 we know (in view of Weyl's Trick 2.10!) that E_1 and E_2 are direct sums of simple modules. If F_1 is a simple submodule of E_1, then (i) shows at once that $F_1 \cong F$. If F_2 is a simple direct summand of E_2 and $p\colon E_2 \to F_2$ the orthogonal projection, then $p \circ f_2\colon E \to F_2$ is a surjective morphism. From (i) we conclude $F_2 \cong F$. This proves the lemma. □

Lemma 3.25. *If E is a simple G-module, then $R_E(G,\mathbb{K})$ is a direct sum of simple submodules each of which is isomorphic to E.*

Proof. By Lemma 3.17 the G-module $R_E(G,\mathbb{K})$ is a homomorphic image of the module $E'\otimes E$ which by Remark 3.13 is isomorphic to E^n with $n=\dim E$. The assertion follows from Lemma 3.24(ii). □

Proposition 3.26. (i) *The vector space $R(G,\mathbb{K})$ is the direct sum of the finite dimensional subspaces $R_E(G,\mathbb{K})$ where E ranges through the set $\mathcal{E}$ of representatives of the set of isomorphy classes of the class of all simple G-modules.*

(ii) *In the sense of $L^2(G,\mathbb{K})$, these summands $R_E(G,\mathbb{K})$, $E\in\mathcal{E}$, are orthogonal and $L^2(G,\mathbb{K})$ is their orthogonal direct sum.*

Proof. (i) For a simple G-module E, the finite dimensional vector space $R_E(G,\mathbb{K})$ is an invariant submodule of $L^2(G,\mathbb{K})$. Hence by Lemma 2.18, the orthogonal projection P_E of $L^2(G,\mathbb{K})$ onto $R_E(G,\mathbb{K})$ is equivariant. If now E_0 denotes any simple submodule of $L^2(G,\mathbb{K})$ which is not isomorphic to E, then $P_E(E_0)=\{0\}$ by Lemma 3.24. In view of Lemma 3.25, this means that $P_E\big(R_{E_0}(G,\mathbb{K})\big)=\{0\}$ for all simple E_0 which are not isomorphic to E. If, temporarily, we denote the span of all of these $R_{E_0}(G,\mathbb{K})$ with S_E, then $P_E(S_E)=\{0\}$.

Now we claim that $R(G,\mathbb{K})=R_E(G,\mathbb{K})+S_E$. Let $f\in R(G,\mathbb{K})$. Then by Proposition 3.4, there is a finite dimensional G-module V, a linear functional $u\in V'$ and a vector $x\in V$ such that $f(g)=\langle u,gx\rangle$. By Corollary 2.25 (in view of Weyl's Trick 2.10!) we have $V=E_1\oplus\cdots\oplus E_m$ with simple G-modules E_j and, accordingly, $V'=E_1'\oplus\cdots\oplus E_m'$. We write $x=x_1+\cdots+x_m$ with $x_j\in E_j$ and $u=u_1+\cdots+u_m$ with $u_j\in V_j'$ for $j=1,\dots,m$. Then $f(g)=\langle u,gx\rangle=\sum_{j=1}^m\langle u_j,gx_j\rangle$, and if we set $f_j(g)=\langle u_j,gx_j\rangle$, then $f=f_1+\cdots+f_m$ and $f_j\in R_{E_j}(G,\mathbb{K})$.

Hence $R(G,\mathbb{K})$ is the linear span of *all* the submodules $R_E(G,\mathbb{K})$ as E ranges through $\mathcal{E}$ and this certainly shows $R_E(G,\mathbb{K})+S_E=R(G,\mathbb{K})$. Since $R_E(G,\mathbb{K})$ and S_E are orthogonal direct summands and E was arbitrary in $\mathcal{E}$, then $R(G,\mathbb{K})$ is the direct vector space sum of the summands $R_E(G,\mathbb{K})$, $E\in\mathcal{E}$.

(ii) We have seen that the subspaces $R_E(G,\mathbb{K})$, $E\in\mathcal{E}$ form an orthogonal family in $L^2(G,\mathbb{K})$. Their sum is $R(G,\mathbb{K})$ and $R(G,\mathbb{K})$ is dense in $L^2(G,\mathbb{K})$ by the Peter–Weyl Theorem 3.7. Hence $L^2(G,\mathbb{K})$ is the orthogonal Hilbert space sum of the $R_E(G,\mathbb{K})$ as E ranges through $\mathcal{E}$. □

Corollary 3.27. *Let E and F denote two simple G-modules. Then the following two statements are equivalent:*

(1) $E\cong F$.
(2) $R_E(G,\mathbb{K})=R_F(G,\mathbb{K})$.

Proof. By Lemma 3.25 we have $R_E(G,\mathbb{K})\cong E^m$ and $R_F(G,\mathbb{K})\cong F^n$. Lemma 3.24 then shows that (2) implies (1). Conversely, suppose the negation of (2). Then Proposition 3.25 shows that $R_E(G,\mathbb{K})$ and $R_F(G,\mathbb{K})$ are orthogonal. In the notation of the proof of Proposition 3.26, this means that $R_F(G,\mathbb{K}) \subseteq \ker P_E = S_E$. The definition of S_E now shows that (1) fails. □

It is important that we understand the indexing of the direct sum representations of $R(G,\mathbb{K})$ and $L^2(G,\mathbb{K})$. The class of simple G-modules is a proper class (and not a set). However, isomorphy is a well-defined equivalence relation on this class, and the class of equivalence classes is a set which we shall denote with $\widehat{G}$. If E is a simple G-module, let $[E]$ denote its isomorphy class. If E_1 and E_2 are simple G-modules then $[E_1] = [E_2]$ if and only if $R_{E_1}(G,\mathbb{K}) = R_{E_2}(G,\mathbb{K})$ by Corollary 3.27. Thus if $\varepsilon \in \widehat{G}$, then we may write $R_\varepsilon(G,\mathbb{K}) = R_E(G,\mathbb{K})$ with any $E \in \varepsilon$, and the contents of Proposition 3.26 may be rewritten in the following more conclusive form:

We recall that for an orthogonal family $\{E_j \mid j \in J\}$ of closed vector subspaces of a Hilbert space we write $\sum_{j\in J} E_j$ for the algebraic direct sum and $\bigoplus_{j\in J} E_j$ for the orthogonal Hilbert space direct sum.

THE FINE STRUCTURE THEOREM FOR $R(G,\mathbb{K})$

Theorem 3.28.

(12) $$R(G,\mathbb{K}) = \sum_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K}) \qquad \textit{and} \qquad L^2(G,\mathbb{K}) = \bigoplus_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K})$$

are valid for any compact group, and

$$R_\varepsilon(G,\mathbb{K}) \cong E^m \qquad \textit{with} \quad E \in \varepsilon$$

and a suitable number $m \leq \dim_{\mathbb{K}} E$ with equality holding if $\mathbb{K} = \mathbb{C}$. □

In this fashion, the G-module $R(G,\mathbb{K})$ is a catalogue of all simple G-modules over $\mathbb{K}$. Likewise, $L^2(G,\mathbb{K})$ is a Hilbert space version of this catalogue. If $\mathbb{K} = \mathbb{C}$, we know that the *multiplicity* m with which the simple module E occurs in the catalogue is exactly the $\mathbb{K}$-dimension of E. After Corollary 3.83 in Part 3 of this chapter we shall know the exact size of the natural number m depending on ε for $\mathbb{K} = \mathbb{R}$ as well.

We have previously used the notation $\widehat{G}$ for the character group of a compact *abelian* (and also of a discrete abelian group). Now we have once more used the same notation $\widehat{G}$ to denote the set of all isomorphy classes of simple modules of an *arbitrary* compact group. However, there is no real conflict of notation: If G is a compact abelian group, then the elements of $\widehat{G}$ may be considered as the morphisms $G \to \mathbb{C}^\times$ into the multiplicative group of nonzero complex numbers. To be sure we did, in general, view $\widehat{G}$ as $\mathrm{Hom}(G,\mathbb{T}) \cong \mathrm{Hom}(G,\mathbb{S}^1)$, but every morphism $\chi\colon G \to \mathbb{C}^\times$ has a compact image as G is compact, and thus $\chi(G) \subseteq \mathbb{S}^1$, because $\mathbb{S}^1$ is the unique maximal compact subgroup of $\mathbb{C}^\times \cong \mathbb{R} \times \mathbb{T}$. Every morphism $\chi\colon G \to \mathbb{C}^\times$ is a representation $\pi_\chi\colon G \to \mathrm{Gl}(\mathbb{C})$ given by $\pi_\chi(g)(v) = \chi(g)v$. The associated module E on $\mathbb{C}$ is simple, and thus the function

$\chi \mapsto [E]$ from the set of characters of G to the set of isomorphy classes of simple modules of G is well defined. But conversely, every simple G-module E over $\mathbb{C}$ is one dimensional by Lemma 2.30 and defines a unique character $\chi: G \to \mathbb{C}^\times$ such that the module action is given by $g{\cdot}x = \chi(g)x$. An isomorphic module gives the same character χ (Exercise!) and the function $[E] \mapsto \chi$—therefore well-defined— inverts the function previously introduced. Thus we have a natural bijection between the set of all characters of G and the set of all equivalence classes of complex simple G-modules. It is true that we have considered on $\widehat{G}$ the structure of an abelian group whereas the set of equivalence classes of simple modules does not *a priori* carry such a structure.

Exercise E3.10. (i) Explain the details necessary to define the two functions $\chi \mapsto [E]$ and $[E] \mapsto \chi$ and verify explicitly that they are inverse functions of each other.

(ii) In what way can one endow the set of all isomorphy classes of irreducible G-modules for a compact abelian group with the structure of a group such that the two functions of (i) above become isomorphisms of groups?

(iii) Consider $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{K} = \mathbb{R}$; verify that $\widehat{G}$ can be identified with $\mathbb{N}_0$ in such a fashion that $0 \neq n \in \mathbb{Z}$ determines the irreducible representation π_n given by $\pi_n(t + \mathbb{Z}) = \begin{pmatrix} \cos 2\pi nt & \sin 2\pi nt \\ -\sin 2\pi nt & \cos 2\pi nt \end{pmatrix}$, and that $R_n(G, \mathbb{R}) = \operatorname{span}\{t \mapsto \cos 2\pi nt, \sin 2\pi nt \mid n \in \mathbb{N}_0\}$. Interpret Theorem 3.28 in the light of these observations. □

Part 2: The General Theory of G-Modules

The objective of this section is to complete a general structure theory of G-modules on rather arbitrary locally convex vector spaces. The Classical Theorem of Peter and Weyl 3.7 and the Fine Structure Theorem for $R(G, \mathbb{K})$ are the models for the general results for which we aim. As a principal tool from functional analysis we need a sufficiently general theory of integration for vector valued continuous functions on compact spaces.

One of the tools we need in Part 2 of this chapter is vector valued integration. If E is a G-module for a compact group G with Haar measure dg, then we want to form the so-called *averaging operator* $P: E \to E$ defined by $Px = \int_G gx\,dg = \int_G \pi_E(g)(x)dg$. This requires that we can integrate continuous functions $f: G \to E$ (here $f(g) = gx$). If $E = \mathbb{R}^n$ then $f(x) = \big(f_1(x), \ldots, f_n(x)\big)$, and the integration is reduced to scalar integration. Hence the averaging operator P is certainly well-defined for finite dimensional G-modules. The applications of the averaging operator to the character theory presented in the next chapter pertain almost exclusively to finite dimensional G-modules. Thus this material can be digested without going into vector valued integration on fairly arbitrary locally convex spaces which we outline by reducing the problem to the integration of scalar functions, somewhat in the spirit of integration of functions with values in $\mathbb{R}^n$.

Vector Valued Integration

We have managed quite well so far with the integration of continuous functions $f: G \to \mathbb{K}$. However, we have, in fact, repeatedly integrated continuous functions $f: G \to E$ with values in some topological vector space E and obtained, as an integral, an element of E, that is a vector. For instance, if E is a Hilbert space, then the function $x \mapsto \int_G (f(g) \mid x) dg: E \to \mathbb{K}$ is well defined through scalar integration and is easily seen to be a bounded $\mathbb{R}$-linear functional u (which happens to be conjugate linear over $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$). Hence, by the elementary Riesz Representation Theorem of Continuous Linear Functionals on Hilbert Space (cf. e.g. [331], p. 261, Theorem A) there is a unique element $I(f) \in E$ such that $u(x) = (I(f) \mid x)$. This vector $I(f)$ in the Hilbert space E is the integral of f and is written $\int_G f$ or $\int_G f(g)dg$. We used this procedure implicitly in Lemma 2.15 and the sections which followed it. In fact we even integrated operator valued functions such as $\varphi = (g \mapsto \pi(g)^{-1} T \pi(g))$. In this case we consider the topological vector space E of all bounded operators of a Hilbert space $\mathcal{H}$. What we did in Section 2 was to take a continuous linear functional $\omega: E \to \mathbb{K}$ given by $\omega(S) = (Sx \mid y)$ for fixed vectors $x, y \in \mathcal{H}$ and then to notice (with the aid of Lemma 2.14) that there is one and only one element $I(\varphi) \in E$ which satisfies the linear equation

$$\omega\big(I(\varphi)\big) = \int_G \omega\big(\varphi(g)\big) dg$$

for all continuous linear functionals ω of E. (It sufficed at that time to consider only a particular type of functional ω.)

We have seen that we have to deal with a variety of G-modules which are topological vector spaces, such as $C(G, \mathbb{K})$, or $L^2(G, \mathbb{K})$ or vector spaces of bounded operators. Evidently, we want to see a uniform theory handling vector valued integration so that, as first priority we can apply the averaging operator ("the expectation") to vector valued functions.

This requires a little functional analysis which we break down into a sequence of exercises. In particular, we have to understand vector valued integration which we shall survey. The caliber of the tools we need is that of the Theorem of Hahn and Banach in locally convex spaces ([40, 317]).

We consider a locally convex topological vector space E over $\mathbb{R}$. We denote by E' its topological dual, that is, the vector space of all continuous linear functionals. We fix a vector subspace $A \subseteq E'$ which separates the points of E. Certainly $A = E'$ is a valid choice and it is the one in which we are most interested. Another natural choice, however, arises if E is the topological dual F' of a topological vector space F (a *predual* of E); then we have a natural linear map $\eta: F \to E'$ given by $\eta(f)(e) = e(f)$ for $f \in F$, $e \in E = F'$; then $A \stackrel{\text{def}}{=} \eta(F)$ is a viable choice. We remember that there is an injection $\iota: E \to \mathbb{R}^A$ given by $\iota(x)(\omega) = \omega(x)$, due to the fact that the continuous linear functionals from A separate points. The weak topology (with respect to A) on E is that topology which makes ι an embedding, that is, a homeomorphism onto the image, when $\mathbb{R}^A$ is given the product topology.

Now we take a compact Hausdorff space G and a Radon measure μ ([37], Chap. III, §1, n^o 3, Définition 2) that is, a continuous linear functional of $C(G,\mathbb{R})$ for which we write $\langle\mu, f\rangle = \int f d\mu = \int f(g)d\mu(g)$. Let $F \in C(G, E)$ and note that we obtain a well defined element

$$p = (\langle\mu, \omega \circ F\rangle)_{\omega\in A} = \left(\textstyle\int \langle\omega, F(g)\rangle\, d\mu(g)\right)_{\omega\in A} \quad \text{in} \quad \mathbb{R}^A.$$

We hope that, under suitable circumstances, we can find an element $I(F) \in E$ such that $\iota\big(I(F)\big) = p$, in other words, that

$$\langle\omega, I(F)\rangle = \textstyle\int \langle\omega, F(g)\rangle\, d\mu(g) \quad \text{for all } \omega \in A. \tag{13}$$

For our purposes it will be sufficient to assume that μ is a probability measure, that is, $\langle\mu, f\rangle \geq 0$ for all nonnegative $f \in C(G,\mathbb{R})$ and $\langle\mu, 1\rangle = 1$. After all, we shall apply this theory to a compact group G and Haar measure. An open zero-set with respect to μ is any open set $U \subseteq G$ such that $\langle\mu, f\rangle = 0$ for any continuous function whose support is contained in U. The union of all open zero-sets is an open zero set, and its complement is called the *support* $\operatorname{supp}(\mu)$ of μ. For each $\omega \in A$ we set

$$m_\omega = \min\{\langle\omega, F(g)\rangle \mid g \in \operatorname{supp}(\mu)\}, \quad M_\omega = \max\{\langle\omega, F(g)\rangle \mid g \in \operatorname{supp}(\mu)\}.$$

Using positivity and $\int d\mu = 1$, we then compute the following estimate

$$m_\omega \leq \langle\mu, \omega \circ F\rangle = \textstyle\int \langle\omega, F(g)\rangle\, d\mu(g) \leq M_\omega. \tag{14}$$

In other words, p is contained in the compact plank

$$P = \prod_{\omega\in A} [m_\omega, M_\omega].$$

Note that $\langle\mu, \omega\circ F\rangle - m_\omega = \int_G(\langle\omega, F(g)\rangle - m_\omega)\, d\mu(g)$ is an integral with respect to a positive measure over a nonnegative function. If it vanishes, it follows that the integrand is zero over the support of μ which implies $M_\omega = m_\omega$. Likewise, $M_\omega - \langle\mu, \omega \circ F\rangle = 0$ implies $M_\omega = m_\omega$. Thus we can sharpen (14) as follows. For all $\omega \in A$,

$$m_\omega < M_\omega \quad \Longrightarrow \quad m_\omega < \langle\mu, \omega\circ F\rangle = \textstyle\int \langle\omega, F(g)\rangle\, d\mu(g) < M_\omega. \tag{15}$$

Let us denote the subspace of $\mathbb{R}^A$ consisting of all *linear* functions $L\colon A \to \mathbb{R}$ by A^*. (In fact this is the algebraic dual of A in the so-called weak *-topology, i.e. the topology of pointwise convergence.) The element $p = \big\{\omega \mapsto \langle\mu, \omega\circ F\rangle = \int\langle\omega, F(g)\rangle d\mu(g)\big\}$ is linear and is, therefore, in A^*.

We now want to observe that $\iota(E)$ is dense in A^*.

Exercise E3.11. Show $\overline{\iota(E)} = A^*$.

[Hint. (i) Show that A^* is closed in $\mathbb{R}^A$. (ii) Conclude that $\overline{\iota(E)} \subseteq A^*$. (iii) Consider an arbitrary finite dimensional subspace V of A and let res: $A^* \to V^*$ denote the restriction map given by $\mathrm{res}(L) = L|V$. Show that $\mathrm{res}\bigl(\iota(E)\bigr) = V^*$ by noting that $\mathrm{res}\bigl(\iota(E)\bigr)$ separates the points of V. (iv) Let M be any finite subset of A and consider the projection $\mathrm{pr}_M: \mathbb{R}^A \to \mathbb{R}^M$. Argue that $\mathrm{pr}_M(\iota E) = \mathrm{pr}_M(A^*)$ and conclude that this proves the assertion.] □

We set $K = \iota^{-1}(P)$. Then $\iota(K) = P \cap \iota(E)$ and we claim that $\iota(K)$ is dense in $P \cap A^*$.

Exercise E3.12. Show $\overline{\iota(K)} = P \cap A^*$.

[Hint. (i) The subset $\overline{\iota(K)}$ of P is closed and convex and is, therefore, a compact subset of $P \cap A^*$. (ii) By the Hahn–Banach Theorem, it therefore is the intersection of all closed half-spaces H of A^* which contain $\overline{\iota(K)}$. (iii) If A is any vector space over $\mathbb{R}$ and $A^* \subseteq \mathbb{R}^A$ its algebraic dual with the weak *-topology, i.e. the topology induced from $\mathbb{R}^A$, then the *continuous* linear functionals of A^* are the point evaluations $u \mapsto \langle u, a\rangle$. (iv) Each $\omega \in A$ defines a closed half-space H of A^* through $H = \{L \in A^* \mid L(\omega) \leq M\}$, and every closed half- space H of A^* is so obtained. (v) The set $\overline{\iota(K)}$ is the intersection of all $S_\nu = \{(r_\omega)_{\omega\in A} \in A^* \mid m_\nu \leq r_\nu \leq M_\nu\}$ as ν ranges through A. (vi) The intersection of all these S_ν, however, is $P \cap A^*$.] □

Now K is the set of all $x \in E$ with $m_\omega \leq \langle \omega, x\rangle \leq M_\omega$ and $\omega \in A$. Hence it is the intersection in E of all closed half-spaces containing $F(G)$ whose boundaries are parallel to hyperplanes which are kernels of functionals ω from A. Let us say that a subset of E is A-*convex* if it is the intersection of closed half-spaces defined by functionals from A. Such a set is automatically closed. If $A = E'$, then a subset is A-convex if it is closed and convex. The set K is the A-convex hull of $F(G)$ in E. What we have achieved so far is the conclusion

(16) $$p \in \overline{\iota(K)}, \quad \text{and } K \text{ is the } A\text{-convex hull of } F(G).$$

If $\iota(K)$ is closed in A^*, then we can indeed conclude the existence of a unique $I(F) \in E$ which satisfies (13). Since ι is an embedding as soon as E is equipped with the weak topology, the closedness of $\iota(K)$ in E'^* is guaranteed as soon as K is *weakly compact* (i.e. compact with respect to the smallest topology making all functions $x \mapsto \langle \omega, x\rangle: E \to \mathbb{R}$, $\omega \in E'$, continuous). Notice that this is certainly the case if K is compact in the given topology.

We recall that G is compact, whence $F(G)$ is compact. Hence K is precompact.

Exercise E3.13. In a locally convex vector space V, the closed convex hull K of a precompact set P is precompact.

[Hint: Let U_0 be an arbitrary closed neighborhood of 0 in V. It suffices to show that a finite union of translates of U_0 covers K. We find a closed convex neighborhood U of 0 such that $U + U \subseteq U_0$. Since P is precompact, there is a finite subset Q of

P such that $P \subseteq Q + U$. The convex hull

$$X = \Big\{ \sum_{x \in Q} r_x \cdot x : 0 \leq r_x, x \in Q, \text{ and } \sum_{x \in Q} r_x = 1 \Big\}$$

of Q is compact and $X + U$ is closed and convex, and $P \subseteq Q + U \subseteq X + U$, hence $X + U$ contains K. By the compactness of X there is a finite subset R of X such that $X \subseteq R + U$. Thus $K \subseteq X + U \subseteq R + U + U \subseteq R + U_0$.]

At this point we see very clearly which hypothesis on the space E will allow us to conclude what we want, namely, that all closed convex precompact sets are weakly compact. This is a very weak completeness condition.

However, in view of the fact that for the representation theory of a compact group G we keep the group G fixed, it is sensible to formulate a form of completeness which is adjusted to a particular given compact group G. We let $C(G, E)$ denote the topological vector space of all continuous functions $f: G \to E$ given the topology of uniform convergence.

Definitions 3.29. Let E be a locally convex topological vector space and A a point separating vector space $A \subseteq E'$ of continuous functionals.

(i) E will be called *feebly A-complete* if there is a point separating vector space $A \subseteq E'$ of continuous functionals such that every closed convex precompact set of E is compact for the weak topology (with respect to A). We say that E is *feebly complete* if E is feebly E'-complete.

(ii) Let G be a compact group. Then E is called *G-A-complete* if there is a continuous linear map

$$I: C(G, E) \to E$$

such that

$$(\forall \omega \in A)\, \langle \omega, I(f) \rangle = \int_G \langle \omega, f(g) \rangle \, dg,$$

where dg denotes Haar measure on G. We say that E is *G-complete* if it is G-E'-complete. □

If B is a subset of a feebly A-complete vector space , then we have called the intersection of all half spaces which are closed for the weak topology with respect to A and which contain B *the A-convex hull of B*.

If $A = E'$, then the E'-convex hull is the closed convex hull with respect to the given topology in view of the Hahn–Banach Theorem.

The first part of the following proposition summarizes our preceding discussion.

Proposition 3.30. (i) *Let E be a locally convex topological vector space and $A \subseteq E'$ a point separating vector space of continuous linear functionals. Assume that E is feebly A-complete and that G is a compact Hausdorff space. Assume further that μ is a Radon probability measure on G. Then for every continuous function $F: G \to E$ there is a unique element $\int_G F(g) d\mu(g) \in E$ such that for each* *linear functional* $\omega \in A$ *we have*

$$(*) \qquad \langle\omega, \int_G F(g)d\mu(g)\rangle = \int_G \langle\omega, F(g)\rangle\, d\mu(g).$$

If $\omega \in A$ *and* $\langle\omega, F\big(\mathrm{supp}(\mu)\big)\rangle = \{x\}$, *then* $\langle\omega, \int_G F(g)d\mu(g)\rangle = x$, *otherwise*

$$(**) \qquad \min\langle\omega, F\big(\mathrm{supp}(\mu)\big)\rangle < \langle\omega, \int_G F(g)d\mu(g)\rangle < \max\langle\omega, F\big(\mathrm{supp}(\mu)\big)\rangle.$$

The integral $\int_G F\,d\mu$ *is contained in the* A*-convex hull of* $F(G)$.

Moreover, if E *is feebly complete, then* $\int F d\mu$ *is contained in the closed convex hull of* $F(G)$ *in* E.

(ii) *If the vector space* $C(G, E)$ *is given the topology of uniform convergence, and if* E *is feebly complete, then* $F \mapsto \int F d\mu\colon C(G, E) \to E$ *is a continuous linear map.*

(iii) *If* G *is a compact group and* λ *is Haar measure on* G, *then the function* $I\colon C(G, E) \to E$ *given by* $I(F) = \int F\,d\lambda = \int_G F(g)\,dg$ *is linear and satisfies*

$$(***) \qquad \langle\omega, I(F)\rangle = \int_G \langle\omega, F(g)\rangle\, dg$$

for all $\omega \in A$. *If* E *is feebly complete, then* I *is continuous.*

(iv) *Every feebly complete locally convex vector space* E *is* G*-complete for any compact group.*

Proof. (i) is a summary of the preceding discussions.

(ii) Linearity is readily verified by $(*)$. Now let U be any closed convex identity neighborhood and $f \in C(G, E)$ a function with $f(G) \subseteq U$. Then, if E is feebly complete, the closed convex hull of $f(G)$ is contained in U. Hence $\int f d\mu \in U$ by the last assertion of (i) above.

(iii) is now a consequence of (i) and (ii), and (iv) follows from (iii) □

We recall from the theory of topological vector spaces that a subset B is called *bounded* if for every 0-neighborhood V there is a real number r with $B \subseteq rV$. A space is called *quasicomplete* if every bounded closed subset is complete. We notice the following chain of implications for a given locally convex topological vector space and, for the last implication, G-module E:

Hilbert space	⇒	Banach space	⇒	Fréchet space	⇒
complete	⇒	quasicomplete	⇒	feebly complete	⇒
G-complete.					

Table 3.1: Completeness conditions allowing integration over G-orbits.

Exercise E3.14. Verify the preceding implications. □

Exercise E3.15. Use Proposition 3.3(i) to develop the integration theory of continuous E-valued functions on a compact space G when E is a *complex vector space.* □

For two compact spaces G_1 and G_2 we may identify $C(G_1,\mathbb{K})\otimes C(G_2,\mathbb{K})$ with a dense subalgebra of $C(G_1\times G_2,\mathbb{K})$ closed under complex conjugation by writing $(f_1\otimes f_2)(g_1,g_2)=f_1(g_1)f_2(g_2)$. If for $j=1,2$ we have probability measures μ_j on compact spaces G_j, then the functional $\mu_1\otimes\mu_2\colon C(G_1,\mathbb{K})\otimes C(G_2,\mathbb{K})\to\mathbb{K}$ given by $\langle\mu_1\otimes\mu_2,f_1\otimes f_2\rangle=\langle\mu_1,f_1\rangle\langle\mu_2,f_2\rangle=\int f_1d\mu_1\int f_2d\mu_2$ extends to a unique probability measure on $C(G_1\times G_2,\mathbb{K})$ which we again denote by $\mu_1\otimes\mu_2$. It is called the product measure. (Sometimes, notably in set theoretical measure theory, one also writes $\mu_1\times\mu_2$.) The Fubini Theorem of elementary measure theory says

$$\begin{aligned}\int f(g_1,g_2)d(\mu_1\otimes\mu_2)(g_1,g_2)&=\int\left(\int f(g_1,g_2)d\mu_2(g_2)\right)d\mu_1(g_1)\\&=\int\left(\int f(g_1,g_2)d\mu_1(g_1)\right)d\mu_2(g_2).\end{aligned}$$

(Cf. [37], Chap. III, §4, n$^{\text{o}}$ 1, Théorème 2.)

Exercise E3.16. Prove the assertions in the preceding paragraph. □

For the application we have in mind we are exclusively interested in Haar measure λ on a compact group G.

Proposition 3.31. *Assume that E is a G-complete locally convex topological vector space, G a compact group and dg Haar measure on G. Then:*

(i) *If $f\in C(G,E)$ is a constant function with value x, then $\int f d\mu=x$.*

(ii) *If $T\colon E_1\to E_2$ is a continuous morphism between G-complete locally convex spaces, then for $f\in C(G,E_1)$ one has*

$$T\left(\int_G f(g)\,dg\right)=\int_G (T\circ f)(g)\,dg,$$

that is, integration commutes with linear operators.

(iii) (Fubini) *If G_1 and G_2 are two compact groups with Haar measures γ_1, respectively, γ_2 and $f\in C(G_1\times G_2,E)$ then*

$$\begin{aligned}\int f(g_1,g_2)d(\gamma_1\otimes\gamma_2)(g_1,g_2)&=\int\left(\int f(g_1,g_2)\,dg_2\right)dg_1\\&=\int\left(\int f(g_1,g_2)\,dg_1\right)dg_2.\end{aligned}\tag{17}$$

Proof. Since E is G-complete we have

$$(\dagger)\qquad \big(\forall\omega\in E',\ f\in C(G,E)\big)\quad \langle\omega,I(f)\rangle=\int_G\langle\omega,f(g)\rangle\,dg.$$

(i) By Proposition (†), $f(G)=\{x\}$ implies $\langle\omega,\int f d\lambda\rangle=\langle\omega,x\rangle$ for all $\omega\in E'$, whence the assertion.

(ii) T induces an adjoint map $T'\colon E_2'\to E_1'$ via $T'(\omega_2)=\omega_2\circ T$. Now we use (†) to compute

$$\begin{aligned}\langle\omega_2,\textstyle\int(T\circ f)d\mu\rangle&=\textstyle\int_G\langle\omega_2,T\big(f(g)\big)\rangle d\mu(g)=\int_G\langle T'(\omega_2),f(g)\rangle d\mu(g)\\&=\langle T'(\omega_2),\textstyle\int f d\mu\rangle=\langle\omega_2,T(\int f d\mu)\rangle.\end{aligned}$$

This proves the assertion.

(iii) Let $\omega \in E'$. Then by (†) we have $\langle\omega, \int f(g_1,g_2)d(\lambda_1\otimes\lambda_2)(g_1,g_2)\rangle = \int\langle\omega\circ f\rangle(g_1,g_2)d(\lambda_1\otimes\lambda_2)(g_1,g_2) = \int\big(\int(\omega\circ f)(g_1,g_2)d\lambda_2(g_2)\big)d\lambda_1(g_1)$ by the scalar Fubini Theorem. But by (†) again,

$$\int(\omega\circ f)(g_1,g_2)d\lambda_2(g_2) = \langle\omega, \int f(g_1,g_2)d\lambda_2(g_2)\rangle,$$

and applying this once more to integration with respect to λ_1, we find

$$\langle\omega, \int f(g_1,g_2)d(\lambda_1\otimes\lambda_2)(g_1,g_2)\rangle = \langle\omega, \int\left(\int f(g_1,g_2)d\lambda_1(g_1)\right)d\lambda_2(g_2)\rangle,$$

and this proves the first equation in (17). The second is proved analogously. □

The First Application: The Averaging Operator

The tools prepared in the preceding subsection allow us to deal in a very systematic way with the fixed points of a given G-module. We actually worked with this formalism in an ad hoc fashion in some crucial spots in Chapter 2.

We shall consider a compact group G with normalized Haar measure γ (see Definition 2.6ff.). We shall fix a vector subspace A of E' which separates points of E. In most cases $A = E'$. We shall consistently assume that E is G-A-complete. We recall right away that this is the case if $A = E'$ and E is feebly complete. We have a continuous linear map $I\colon C(G,E)\to E$ such that $\langle\omega, I(f)\rangle = \int_G\langle\omega, f(g)\rangle\,dg$ for all $\omega\in A$. We use the notation

$$\int_G f\,d\gamma = \int_G f(g)\,dg \stackrel{\text{def}}{=} I(f).$$

Definition 3.32. Assume that E is a G-A-complete locally convex G-module. We define

$$P = P_G : E\to E \qquad \text{by} \qquad Px = \int_G gx\,dg.$$

The function P is called the *averaging operator of the module E*. □

Let us first remark that P is well defined. Indeed the function $g\mapsto gx$ is in $C(G,E)$ by Definition 2.1.(i)(b). Since E is G-A-complete, $\int_G gxdg$ is well defined. The averaging operator is perhaps the single most important tool applying to linear actions of compact groups. In the next chapter we shall discuss generalizations of the averaging operator (see Definition 4.12).

We shall see presently that P is an idempotent operator. Therefore we record the following observation from linear algebra:

Lemma 3.33. *If $p\colon E\to E$ is a vector space endomorphism with $p^2 = p$ then we have the following conclusions:*

(i) $(\mathbf{1}-p)^2 = \mathbf{1}-p$.

(ii) $\ker p = \operatorname{im}(\mathbf{1}-p)$.

(iii) *The function $f\colon \operatorname{im}p\times\ker p\to E$ and $f'\colon E\to\operatorname{im}p\times\ker p$ given by $f(x,y) = x+y$ and $f'(z) = \big(P(z), z-P(z)\big)$ are inverse isomorphisms. If E is a* *topological vector space and* p *is continuous, then* f *and* f' *are isomorphisms of topological vector spaces.*

Proof. Exercise E3.17. □

Exercise E3.17. Prove Lemma 3.33. □

Definitions 3.34. (i) Let G operate on a set X (see Definition 1.9 and the discussion preceding it). We shall write

$$X_{\rm fix} = \big\{x \in X \mid Gx = \{x\}\big\}$$

for the set of fixed points.[1] □

(ii) Assume that G acts linearly on a vector space E. We write

$$E_{\rm eff} = \operatorname{span}\{gx - x \mid g \in G,\, x \in E\}$$

and call this space the *effective vector subspace*. If E is actually a G-module (see Definition 2.1) we set

$$E_{\rm Eff} = \overline{E_{\rm eff}}$$

and call this space *the effective submodule*. If A is a point separating vector subspace of E', we shall denote by

$\mathrm{cl}_A(X)$ the closure of a set X in E with respect to the coarsest topology making all functionals in A continuous. In particular we shall write $E^A_{\rm Eff} \stackrel{\rm def}{=} \mathrm{cl}_A(E_{\rm eff})$. We shall say that A is G-*invariant* if for every $\omega \in G$ and $g \in G$ also $\omega \circ \pi_E(g) \in A$. □

We note that $E_{\rm Eff}$ and $E^A_{\rm Eff}$ are one and the same thing as soon as $A = E'$. Otherwise $E^A_{\rm Eff}$ may be bigger.

For a vector space E on which G acts linearly we obviously have $E_{\rm fix} \subseteq E_{\rm fin}$ (see Definition 3.1).

Lemma 3.35. (i) *If* G *acts linearly on* E*, then* $G(E_{\rm eff}) \subseteq E_{\rm eff}$.

(ii) *If* E *is a* G*-module, then* $G(E_{\rm Eff}) \subseteq E_{\rm Eff}$.

(iii) *If* A *is a* G*-invariant and point separating vector subspace of* E'*, then* $G(E^A_{\rm Eff}) \subseteq E^A_{\rm Eff}$.

Proof. (i) Observe that $h(gx - x) = (hgx - x) - (hx - x)$ for all $g, h \in G$, $x \in E$. (ii) follows from (i) by the fact that each $\pi_E(g)\colon E \to E$ is continuous. (iii) follows from (i) again because each $\pi(g)\colon E \to E$ is continuous with respect to the weak topology induced by the functionals of A if these are permuted by the adjoint action of G. □

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1 We point out quickly that this notation deviates from other frequently used notation. The fixed point set $X_{\rm fix}$ is often denoted X^G; however, it appears that this notation is completely occupied by the set of all functions $G \to X$.

It follows that $E_{\rm Eff}$ is indeed a submodule; this justifies the choice of nomenclature in Definition 3.32. Now we proceed to the core definition in this subsection.

Theorem 3.36 (Splitting Fixed Points). *Let E denote a G-A-complete locally convex G module for a compact group G and $P\colon E \to E$ the averaging operator of E. Then*

(i) *Px is in the A-convex hull of the orbit Gx (that is, the closed convex hull if $A = E'$), and $Pgx = gPx = Px$ for all $g \in G$ and $x \in E$,*

(ii) *P is a linear projection, that is $P^2 = P$,*

(iii) $\operatorname{im} P = E_{\rm fix}$ *and* $E_{\rm eff} \subseteq \ker P \subseteq E^A_{\rm Eff}$*; in particular,* $\mathrm{cl}_A(\ker P) = E^A_{\rm Eff}$.

(iv) *The continuous linear bijection*

$$(x, y) \mapsto x + y : E_{\rm fix} \times \ker P \to E$$

is a G-module morphism with inverse

$$x \mapsto (Px, x - Px)\colon E \to E_{\rm fix} \times \ker P.$$

(v) *If E is a real vector space and the set $\omega(Gx)$ is not singleton for an $\omega \in A$, then*

$$\min\langle \omega, Gx\rangle < \langle \omega, Px\rangle < \max\langle \omega, Gx\rangle.$$

(vi) *If, in addition, E is feebly complete and the action $(g, x) \mapsto gx\colon G \times E \to E$ is continuous, then P is continuous,* $\ker P = E_{\rm Eff}$ *and* $E \cong E_{\rm fix} \times E_{\rm Eff}$ *via the map in* (iv) *above.*

Proof. (i) By Proposition 3.30, $Px = \int gxdg$ is in the A-convex hull of Gx. If $h \in G$, then $Phx = \int ghxdg = \int gxdg = Px$ by the right invariance of Haar measure. Similarly, $hPx = \pi(h)\int gxdg = \int \pi(h)gxdg$ by Proposition 3.31(ii). But $\int hgxdg = \int gxdg = Px$ by the left invariance of Haar measure.

(ii) $P^2x = \int g(Px)dg = \int Pxdg = Px$ in view of (i) above and Proposition 3.31(iii). The linearity of P follows from the linearity of the action of G and the linearity of the integral via Proposition 3.31(i).

(iii) If $x \in E_{\rm fix}$, then $Px = \int gxdg = \int xdg = x$ by Proposition 3.31(iii) again. Hence $x \in \operatorname{im} P$. If, on the other hand, $x = Py$, then $gx = gPy = Py = x$ by (i) above. Thus $PE = E_{\rm fix}$. Next observe $P(gx - x) = Pgx - Px = Px - Px = 0$ by (i) above. Hence $P(E_{\rm eff}) = \{0\}$ and thus $E_{\rm eff} \subseteq \ker P$. Next let $x \in \ker P$. We observe that $gx = x + (gx - x) \in x + E_{\rm eff}$ is contained in one and the same affine variety for all $g \in G$, i.e. $Gx \subseteq x + E_{\rm eff}$. It follows that the A-convex hull of Gx is contained in $\mathrm{cl}_A(x + E_{\rm eff}) = x + E^A_{\rm Eff}$. Hence $0 = Px \in x + E^A_{\rm Eff}$, whence $x \in E^A_{\rm Eff}$.

(iv) is a consequence of Lemma 3.32.

(v) The support of the Haar measure on a compact group G is G because of invariance. Hence $(**)$ in Proposition 3.30 yields the assertion.

(vi) We now assume that $A = E'$ and that $(g, x) \mapsto gx\colon G \times E \to E$ is continuous. Assume that U is a closed convex 0-neighborhood in E. Then by Corollary 1.13 we find a 0-neighborhood V with $GV \subseteq U$. If $x \in V$, then $Gx \subseteq V \subseteq U$. Then Px, which is in the closed convex hull of Gx by (i), is contained in U, that is $PV \subseteq U$. This shows that P is continuous.

In particular, $\ker P$ is closed, and now (iii) implies $\ker P = E_{\rm Eff}$ and the remainder follows from (iv). □

We recall that after Theorem 2.3, the action $(g, x) \mapsto g{\cdot}x$ is automatically continuous if E is a Baire space (for instance, if E is a Banach space); in this case the conclusion (vi) is instantaneously available. Every such module then splits neatly into an algebraic and topological direct sum of the fixed point module and the effective submodule, whereby the projection onto the former is implemented by the averaging operator P of E.

As noted before, the application of the averaging operator is one of the most effective tools in the representation theory of compact groups. Of course, in most situations, one can formulate what amounts to an application of the averaging operator in explicit terms without referring to its general background theory. But in the long run, this is unsatisfactory and a general theory is appropriate. Let us briefly review, where we have already seen the averaging operator at work:

Example 3.37. (i) (Weyl's Trick) Let E denote a G-module. We consider the vector space of bilinear forms $\mathrm{Bil}(E;\mathbb{K}) \subseteq \mathbb{K}^{E\times E}$, $B\colon E \times E \to \mathbb{K}$. Then G acts linearly on this vector space via $(g{\cdot}B)(x, y) = B(g^{-1}x, g^{-1}y)$. Then for each B the bilinear form PB is invariant and contained in the closed convex hull of $G{\cdot}B$.

(ii) (Equivariant Operators) Let E and F be two G-modules and $\mathrm{Hom}(E, F)$ the vector space of all continuous vector space morphisms $E \to F$ endowed with the topology of uniform convergence on bounded sets. For instance, if E and F are Banach spaces, this is the Banach space of all bounded operators $E \to F$ with the operator norm. Then $\mathrm{Hom}(E, F)$ is a G-module with the action $(g{\cdot}f)(x) = gf(g^{-1}x)$, that is, $g{\cdot}f = \pi_F(g)f\pi_E(g)^{-1}$. For every $f\colon E \to F$ the operator Pf is an equivariant map, or a morphism of G-modules. The averaging operator P of $\mathrm{Hom}(E, F)$ is a projection onto the submodule $\mathrm{Hom}_G(E, F) = \big(\mathrm{Hom}(E, F)\big)_{\rm fix}$ of all equivariant operators.

If E is a Hilbert space, then the space of all compact operators is a G-submodule of $\mathrm{Hom}(E, E)$, hence is respected by P. For the proof of the Fundamental Theorem 2.22 we took a rank one projection f and used that Pf was a nonzero compact equivariant self-map of E.

(iii) The modules $C(G, \mathbb{K})$ and $L^2(G, \mathbb{K})$.

Case (a): G acts by translation of the right of the argument. Then Pf is a constant function and the averaging operator maps both of these modules onto the submodule $\mathbb{K}{\cdot}\mathbf{1} = R_{[\mathbb{K}]}(G, \mathbb{K})$ where $[\mathbb{K}]$ denotes the class of modules isomorphic to the trivial one dimensional one and where $\mathbf{1}$ is the constant function on G with value 1.

Case (b): G acts by conjugation of the argument in $C(G, \mathbb{K})$ (and by L^2-extension of this action on $L^2(G, \mathbb{K})$). A function $f \in C(G, \mathbb{K})$ is called a *class function* if it is a function which is constant on conjugacy classes of G. Then Pf is a class function, that is, is constant on conjugacy classes. The averaging operator P maps both modules onto the submodule of all continuous class functions (respectively, the L^2 elements fixed under this action). We shall utilize this fact for a better understanding of characters in the next chapter. □

Compact Groups Acting on Convex Cones

The question whether $E_{\rm fix}$ is non-zero has no general answer. However, there is one instance which allows the conclusion $E_{\rm fix} \neq \{0\}$, namely, if a G-module contains an invariant convex cone. Experience confirms that indeed this is the only realistic instance in which the existence of non-zero fixed points can be secured. This should not come as a big surprise for the following reason. Assume that x is a non-zero fixed point, Then x has a closed convex invariant neighborhood B separated from 0 by a closed affine hyperplane (see Proposition 1.11). Then $W = \overline{\mathbb{R}^+{\cdot}B}$ is a closed convex invariant pointed cone having x in its interior.

If E is a topological vector space, a *cone* or *wedge* W is a nonempty closed convex and additively closed subset. In other words, $W + W \subseteq W$, $\mathbb{R}^+{\cdot}W \subseteq W$ and $\overline{W} = W$.

Exercise E3.18. (a) Show that the hyperquadrant $W = (\mathbb{R}^+)^n$ in $\mathbb{R}^n$ is a cone (recall $\mathbb{R}^+ = \{r \in \mathbb{R} \mid 0 \leq r\}$!).

(b) Show that the set $\{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 \leq x_n^2 \text{ and } 0 \leq x_n\}$ is a cone.

(c) Show that the set of all hermitian operators T of a Hilbert space $\mathcal{H}$ satisfying $(Tx \mid x) \geq 0$ is a cone. □

Theorem 3.38. *Let E be a G-complete real locally convex G-module with an invariant cone W which is not a vector space. Then $W \cap E_{\rm fix} \not\subseteq W \cap -W$. In particular $W \cap E_{\rm fix} \neq \{0\}$.*

Proof. Let P denote the averaging operator of E. Since W is not a vector space, there is an $x \in W$ with $-x \notin W$. The largest vector subspace of W is $W \cap -W$ which is closed since W is closed. In the factor space $E/(W \cap -W)$ we have a nonzero element $x+(W\cap -W)$ and hence there is a continuous functional assigning this point the value 1 by the Theorem of Hahn and Banach. When composed first with the quotient morphism $E \to E/(W \cap -W)$, this functional yields a continuous linear functional $\omega \in E'$ vanishing on $W\cap -W$ such that $\langle \omega, x\rangle = 1$ and $\langle \omega, W\rangle \subseteq [0, \infty[$. Since W is invariant, $Gx \subseteq W$. Hence Px, being contained in the closed convex hull of Gx by Theorem 3.36(i), is in W. If $\langle \omega, Gx\rangle = \{\langle \omega, x\rangle\}$, then $\langle \omega, Px\rangle = 1$. If $\langle \omega, Gx\rangle$ contains more than one element, then by Theorem 3.36(v) we have $0 < \min\langle \omega, Gx\rangle < \langle \omega, Px\rangle$. In either case $0 < \langle \omega, Px\rangle$. Since ω vanishes on $W \cap -W$ this means $Px \in W \setminus (W \cap -W)$. □

Corollary 3.39. (i) *If $P'\colon E' \to E'$ denotes the adjoint of P and $\omega \in E'$ then $P'\omega$ is a linear continuous functional fixed under the adjoint action. In particular, each* *affine variety* $(P'\omega)^{-1}(r)$ *is invariant as a whole. If* $\langle\omega, w\rangle \geq 0$ *for all* $w \in W$*, then* $\langle P'\omega, w\rangle \geq 0$ *for all* $w \in W$.

(ii) *Assume, in addition to the hypotheses of Theorem* 3.38*, that* $W \cap -W = \{0\}$*, i.e. that* W *is pointed, and that there exists a functional* $\omega \in E'$ *such that* $\langle\omega, w\rangle > 0$ *for all* $w \in W \setminus \{0\}$*. Then* $P'\omega$ *also satisfies* $\langle P'\omega, w\rangle > 0$ *for* $w \in W \setminus \{0\}$ *and* $B \overset{\text{def}}{=} (P'\omega)^{-1}(1)$ *is a* G*-invariant closed basis of* W*, that is,* $W = \mathbb{R}^{+}{\cdot}B$.

Proof. Exercise E3.19. □

Exercise E3.19. (i) Prove Corollary 3.39.

(ii) Adapt the proof of Theorem 3.38 to obtain a proof of the following generalisation of it: *If* E *is a* G*-*A*-complete* G*-module for a compact group* G *and* W *is an invariant cone which is closed for the weak topology with respect to* A*, then* W *contains a non- zero fixed point.* □

If E is the vector space of all compact operators on a G-Hilbert module $\mathcal{H}$, then the set W of all positive compact operators is a cone which is invariant under conjugation by all unitary operators. Hence by Theorem 3.38 and Example E3.18(ii) there is a compact equivariant operator. This is what we used in the proof of the Fundamental Theorem 2.22.

More Module Actions, Convolutions

Definition 3.40. If G is a compact group and E a G-complete locally convex G-module, for each $f \in C(G, \mathbb{K})$ and $x \in E$ we define $\widetilde{f_x} \in C(G, E)$ by $\widetilde{f_x}(g) = f(g){\cdot}gx$ and set

$$f * x = \int_G f(g){\cdot}gx\, dg = I(\widetilde{f_x}).$$ □

We note that the function $(f, x) \mapsto f * x\colon C(G; \mathbb{K}) \times E \to E$ is well defined and that $f * x$ is contained in the closed convex hull $C_f(x)$ of $\{f(g){\cdot}gx \mid g \in G\}$

If 1 denotes the constant function with value 1, then $1 * x = Px$ with the averaging operator P. The vector $f * x$ is a “weighted average”.

Lemma 3.41. (i) *The function* $(f, x) \mapsto f * x\colon C(G, \mathbb{C}) \times E \to E$ *is a bilinear map. The function* $f \mapsto f * x_0$ *is continuous.*

(ii) *If the action* $(g, x) \mapsto gx\colon G \times E \to E$ *is continuous, then* $(f, x) \mapsto f * x\colon C(G, \mathbb{K}) \times E \to E$ *is continuous.*

Proof. (i) Linearity in f follows from the linearity of $f \mapsto \widetilde{f_x}$ and the linearity of I, and linearity in x from the linearity of the G-action,yielding the linearity of $x \mapsto \widetilde{f_x}$, and the linearity of I.

Assume next that W is an arbitrary 0-neighborhood in E. Choose a closed convex 0-neighborhood U such that $U + U \subseteq W$. Let $f_0 \in C(G, \mathbb{K})$ and $x_0 \in E$ be given.

Since Gx_0 is compact for any x_0, then there is an $r > 0$ such that $r{\cdot}Gx_0 \subseteq U$. Hence $(f - f_0)(g){\cdot}gx_0 \in U$ if $\|f - f_0\| \leq r$ where $\|h\|$ denotes the sup-norm of h. Hence $(f - f_0) * x_0 \in U$ by Theorem 3.36(i).

(ii) Now set $M = \|f_0\| + 1$. According to Corollary 1.13 we find a zero neighborhood V with $GV \subseteq M^{-1}{\cdot}U$. Thus $x - x_0 \in V$ implies $G(x - x_0) \subseteq M^{-1}{\cdot}U$ and hence $f(g){\cdot}g(x-x_0) \in \|f\|M^{-1}{\cdot}U \subseteq U$ as soon as $\|f\| \leq M$. Hence $f*(x-x_0) \in U$ for these x and f by Theorem 3.36(i).

Finally, whenever f is so that $\|f - f_0\| \leq r$ and $\|f\| \leq \|f_0\| + 1$ and $x - x_0 \in V$, then $f * x - f_0 * x_0 = f * (x - x_0) + (f - f_0) * x_0 \in U + U \subseteq W$. This shows the continuity of $*$. ☐

Example 3.42. Let E be the G-module $C(G, \mathbb{K})$ with the operation $(g, f) \mapsto {}^g f$, ${}^g f(h) = f(g^{-1}h)$. Then for $f_1, f_2 \in C(G, \mathbb{K})$ we have

$$f_1 * f_2 = \int f_1(g)({}^g f_2)\, dg, \tag{18}$$

or, equivalently

$$(f_1 * f_2)(h) = \int f_1(g) f_2(g^{-1}h)\, dg = \int f_1(hg) f_2(g^{-1})\, dg. \tag{19}$$

Proof. (18) is just the application of Definition 3.40 to the special situation. Now let $h \in G$. Then $\delta_h \colon C(G, \mathbb{K}) \to \mathbb{K}$ given by $\langle \delta_h, f \rangle = f(h)$ is a probability measure, at any rate a continuous linear morphism of vector spaces. Hence Proposition 3.31(ii) applies and yields $\delta_h\big(\int f_1(g){}^g f_2\, dg\big) = \int \delta_h\big(f_1(g){}^g f_2\big)\, dg = \int f_1(g){}^g f_2(h)\, dg = \int f_1(g) f_2(g^{-1}h)\, dg$ on the one hand and $\delta_h(f_1 * f_2) = (f_1 * f_2)(h)$ on the other. Hence (18) implies (19). Conversely, two elements $\varphi_j \in C(G, \mathbb{K})$, $j = 1, 2$ agree if and only if $\delta_h(\varphi_1) = \varphi_1(h) = \varphi_2(h) = \delta_h(\varphi_2)$ for all $h \in G$. Hence (19) also implies (18). ☐

Lemma 3.43. *Let E be a G-complete G-module. For $f_j \in C(G, \mathbb{K})$, $j = 1, 2$ and $x \in E$, we have*

$$f_1 * (f_2 * x) = (f_1 * f_2) * x.$$

Proof. The left hand side of the equation is defined as $\int f_1(g){\cdot}g(f_2 * x)\, dg = \int f_1(g)\pi(g)\big(\int f_2(h){\cdot}hx\, dh\big)\, dg = \int f_1(g)\big(\int f_2(h){\cdot}\pi(g)hx\, dh\big)\, dg$ in view of Proposition 3.31(ii). The inner integral, by left invariance of Haar measure, equals $\int f_2(g^{-1}h)hx\, dh$. Thus

$$f_1 * (f_2 * x) = \int\big(\int f_1(g) f_2(g^{-1}h){\cdot}hx\, dh\big)\, dg. \tag{20}$$

On the other hand, the right hand side of the equation is given by $\int (f_1 * f_2)(h){\cdot}hx\, dh = \int\big(\int f_1(g) f_2(g^{-1}h)\, dg\big){\cdot}hx\, dh$ in view of (19) above. We consider the linear continuous map $T \colon \mathbb{K} \to E$ given by $T(r) = r{\cdot}hx$. Then for any $\varphi \in C(G, \mathbb{K})$ we have $(\int \varphi(g)\, dg){\cdot}hx = T(\int \varphi(g)\, dg) = \int (T \circ \varphi)(g)\, dg = \int \varphi(g){\cdot}hx\, dg$ by Proposition 3.31(ii). We therefore conclude

$$(f_1 * f_2) * x = \int\big(\int f_1(g) f_2(g^{-1}h){\cdot}hx\, dg\big)\, dh. \tag{21}$$

By the Fubini Theorem 3.31(iii), the right hand sides of (9) and (10) agree. Hence the lemma is proved. ☐

Lemma 3.44. *Let E be a G-module and a Banach space. According to Corollary* 2.5*, define $C = \sup\{\|\pi(g)\| \mid g \in G\}$. Then*

$$\|f * x\| \leq C\|f\|\|x\|.$$

Proof. Let $\omega \in E'$, $\|\omega\| = 1$. Then $|\langle\omega, f * x\rangle| = |\int\langle\omega, f(g)\cdot\pi(g)(x)\rangle\, dg| \leq \int |\langle\omega, f(g)\cdot\pi(g)(x)\rangle|\, dg \leq \|\omega\|\|f\|C\|x\| = C\|f\|\cdot\|x\|$. This proves the claim. ☐

Proposition 3.45. *Let G be a compact group and E a G-module and a Banach space. Then the following conclusions hold.*

(i) *If $g \in G$, $f \in C(G, \mathbb{K})$ and $x \in E$, then $g(f*x) = {}^g f*x$ and $f*(gx) = {}_{g^{-1}}f*x$ where ${}^g f(h) = f(g^{-1}h)$ and ${}_g f(h) = f(hg)$.*

(ii) *$(g\cdot f)(h) = f(g^{-1}hg)$ implies $(g\cdot f) * x = g\big(f * (g^{-1}x)\big)$.*

(iii) *If f is a class function (i.e. is constant on conjugacy classes of G), then $x \mapsto f * x\colon E \to E$ is an equivariant map for any G-module E.*

(iv) *The Banach space $C(G, \mathbb{K})$ is in fact a Banach algebra with respect to the multiplication $*$. Moreover, ${}^g(f_1 * f_2) = {}^g f_1 * f_2$ and ${}_{g^{-1}}f_1 * f_2 = f_1 * {}^g f_2$.*

Proof. (i) $g(f*x) = \pi(g)\int f(h)\cdot hx\, dh = \int f(h)\cdot\pi(g)hx\, dh = \int f(g^{-1}h)\cdot hx\, dh = \int {}^g f(h)\cdot hx\, dh = {}^g f*x$. Also $f*(gx) = \int f(h)\cdot hgx\, dh = \int f(hg^{-1}).hx\, dh = {}_{g^{-1}}f*x$.

(ii) $(g\cdot f) * x = {}_g f_{g^{-1}} * x = g\big(f * (g^{-1}x)\big)$.

(iii) $f \in C(G, \mathbb{K})$ is a class function if and only if $g\cdot f = f$. Then the assertion follows from (ii) above.

(iv) By Lemma 3.43, the multiplication $*$ is associative, and by Lemma 3.41, both distributive laws hold. With the notation of Lemma 3.44 we have $C = 1$, whence $\|f_1 * f_2\| \leq \|f_1\|\|f_2\|$.

For a proof of the remainder we apply the preceding statements to $E = C(G, \mathbb{K})$. ☐

Definition 3.46. The multiplication $*$ on $C(G, \mathbb{K})$ is called *convolution* and $C(G, \mathbb{K})$ endowed with this multiplication is called the *convolution algebra* $C(G, \mathbb{K})$. ☐

Lemma 3.47. *If F is a closed submodule of a G-complete G-module E, then $f * F \subseteq F$ for all $f \in C(G, \mathbb{K})$. If $x \in E_{\text{fix}}$, then $f * x = (\int f)\cdot x$.*

Proof. If $x \in F$, then $f(g)\cdot gx \in F$. Hence the closed convex hull of $\{f(g)\cdot gx \mid g \in G\}$ is in F, thus $f * x \in F$ by Proposition 3.30. The rest is straightforward. ☐

Proposition 3.48. *The vector space $R(G, \mathbb{K})$ is a dense ideal of the convolution algebra $C(G, \mathbb{K})$. In particular, $R(G, \mathbb{K})$ has itself a convolution algebra structure. For each $\varepsilon \in \widehat{G}$, the vector space $R_\varepsilon(G, \mathbb{K})$ is an ideal of $C(G, \mathbb{K})$, and $R(G, \mathbb{K}) =$*

$\sum_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K})$ *is an ideal direct sum decomposition of the convolution algebra* $R(G,\mathbb{K})$.

Proof. Let $\varepsilon \in \widehat{G}$. Then $R_\varepsilon(G,\mathbb{K})$ is a finite dimensional, hence closed submodule of $C(G,\mathbb{K})$. Hence $C(G,\mathbb{K}) * R_\varepsilon(G,\mathbb{K}) \subseteq R_\varepsilon(G,\mathbb{K})$ by Lemma 3.47. Thus $R_\varepsilon(G,\mathbb{K})$ is a left ideal. Considering $C(G,\mathbb{K})$ as a right module it follows in the same way that $R_\varepsilon(G,\mathbb{K})$ also is a right ideal, hence a two-sided ideal. Then $R(G,\mathbb{K}) = \sum_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K})$ is likewise an ideal and the remainder follows. □

Lemma 3.49. *If E is G-complete G-module, then for each 0-neighborhood U and each $x \in E$ there is a positive function $f \in C(G,\mathbb{K})$ such that $f * x - x \in U$. Thus the set $C(G,\mathbb{K}) * E$ is dense in E.*

Proof. We choose a closed convex 0-neighborhood V in E with $V \subseteq U$. Since $g \mapsto gx\colon G \to E$ is continuous by Definition 2.1((i)(b), there is an identity neighborhood W in G such that $Wx - x \in V$.

Now we choose a nonnegative function $f \in C(G,\mathbb{K})$ so that $\int f = 1$ and f vanishes outside W. If ω is a real continuous linear functional on E such that $\langle\omega, v\rangle \le 1$ for all $v \in V$, then $\langle\omega, f * x - x\rangle = \int f(g)\langle\omega, gx\rangle\, dg - \int f\cdot\langle\omega, x\rangle = \int f(g)\langle\omega, gx - x\rangle\, dg \le \int f(g)\cdot 1 = 1$ since $f(g) = 0$ for $g \notin W$, and $gx - x \in V$ for $g \in W$. But V is the intersection of all closed real half-spaces containing V, whence $f * x - x \in V \subseteq U$. □

Lemma 3.50. *For $f \in R(G,\mathbb{K})$ we have $f * E \subseteq E_{\rm fin}$ (see Definition 3.1).*

Proof. By Definition 3.3, the orbit ${}^G f$ spans a finite dimensional G-invariant vector subspace F of $C(G,\mathbb{K})$. By Lemma 3.41, $\varphi \mapsto \varphi * x\colon C(G,\mathbb{K}) \to E$ is linear. Hence $F * x$ is a finite dimensional vector subspace of E. Also by Proposition 3.45 we have $g(F * x) = {}^g F * x \subseteq F * x$ since F is invariant. Hence $f * x$ is contained in the invariant finite dimensional vector subspace $F * x$ and thus $f * x \in E_{\rm fin}$. □

We are now ready for the "big" version of the Theorem of Peter and Weyl 3.7. Recall the definition of a G-complete locally convex vector space from 3.29 and recall that every feebly complete G-module is G-complete by 3.30(iv).

The Big Peter and Weyl Theorem

Theorem 3.51. *Let E be a G-complete (Definition 3.29) locally convex G-module for a compact group G. Then $E = \overline{E_{\rm fin}}$: every element can be approximated by almost invariant elements.*

Moreover, $E_{\rm fin} = R(G,\mathbb{K}) * E$.

Proof. By Lemma 3.50 we know that $R(G,\mathbb{K}) * E \subseteq E_{\rm fin}$. By the Classical Peter and Weyl Theorem 3.7, $R(G,\mathbb{K})$ is dense in $C(G,\mathbb{K})$. By Lemma 3.41, the function $\varphi \mapsto \varphi * x$ is continuous, hence $R(G,\mathbb{K}) * x$ is dense in $C(G,\mathbb{K}) * x$ and thus

$R(G,\mathbb{K}) * E$ is dense in $C(G,\mathbb{K}) * E$. By Lemma 3.49, $C(G,\mathbb{K}) * E$ is dense in E. Hence $R(G,\mathbb{K}) * E$ is dense in E and thus E_{fin} is dense in E.

Finally we have noted $R(G,\mathbb{K}) * E \subseteq E_{\text{fin}}$. In order to argue the converse, take $x \in E_{\text{fin}}$. Then there is a finite dimensional submodule F of E containing x. By the first part of the theorem, $R(G,\mathbb{K}) * F$ is dense in F. But F is finite dimensional whence $R(G,\mathbb{K}) * F = F$. □

In the spirit of the Fine Structure Theorem for $R(G,\mathbb{K})$ 3.28 we shall explore the fine structure of E_{fin} for an arbitrary G-complete complex G-module. For this purpose we shall prepare in the next chapter the concept of characters which is fundamental for all of representation theory. This will require that we restrict our attention to the ground field $\mathbb{K} = \mathbb{C}$. In the meantime we draw some conclusions which are accessible now and hold for either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In Chapter 6 we shall need to return to the real representation theory of compact abelian groups.

Corollary 3.52. *Under the hypotheses of the Big Peter and Weyl Theorem* 3.51*,*

(i) *the G-module E_{fin} is the module direct sum of the submodules $R_\varepsilon(G,\mathbb{K}) * E$ as ε ranges through the set $\widehat{G}$ of equivalence classes of irreducible G-modules over $\mathbb{K}$, and*

(ii) *each element of $R_\varepsilon(G,\mathbb{K}) * E$ is contained in a finite direct sum of simple submodules each of which belongs to the class ε.*

Proof. (i) By Theorems 3.28 and 3.51 we know that $E_{\text{fin}} = R(G,\mathbb{K}) * E = \sum_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K}) * E$. We have to show that this sum is direct: Abbreviate $R(G,\mathbb{K})$ by R. Let $I = R_\eta(G,\mathbb{K})$ and $J = \sum_{\varepsilon\in\widehat{G},\varepsilon\neq\eta} R_\varepsilon(G,\mathbb{K})$; then $I * J \subseteq I \cap J = \{0\}$ by Proposition 3.48. Let F denote any finite dimensional submodule of the submodule $I * E \cap J * E$ which by 3.51 is contained in E_{fin}. Assume $x \in F$. Then $x = f_I * y = f_J * z$ with $f_I \in I$ and $f_J \in J$ with $y,\, z \in E$. Hence $I * x = I * f_J * z = \{0\}$ and $J * x = J * f_I * y = \{0\}$, whence $R * x = (I \oplus J) * x = \{0\}$. Thus $R * F = \{0\}$. Since $R * F$ is dense in F by 3.51 we have $F = \{0\}$ and thus $I * E \cap J * E = \{0\}$. Hence

$$E_{\text{fin}} = \bigoplus_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{K}) * E.$$

(ii) Let $x \in R_\varepsilon(G,\mathbb{K}) * E$. Then $x = f * y$ with $f \in R_\varepsilon(G,\mathbb{K})$ and $F \overset{\text{def}}{=} R_\varepsilon(G,\mathbb{K}) * y$ is a finite dimensional submodule. Now $R(G,\mathbb{K})$ is dense in $C(G,\mathbb{K})$ (3.7) and thus $C(G,\mathbb{K}) * x \subseteq \overline{R(G,\mathbb{K})} * f * y \subseteq R_\varepsilon(G,\mathbb{K}) * y = F$ since $R_\varepsilon(G,\mathbb{K})$ is an ideal in $R(G,\mathbb{K})$ and $R_\varepsilon(G,\mathbb{K}) * y$ is finite dimensional. But $x \in \overline{C(G,\mathbb{K}) * x} \subseteq F$ by Lemma 3.49. Now $\varphi \mapsto \varphi * y\colon R_\varepsilon(G,\mathbb{K}) \to F$ is a surjective G-module morphism by 3.45. By the Fine Structure Theorem 3.28 the module $R_\varepsilon(G,\mathbb{K})$ is a direct sum of simple G-modules all contained in ε. Hence by 3.24(ii) the module F is a direct sum of simple modules each contained in ε. □

The findings of this corollary will be refined in the next chapter in the case that $\mathbb{K} = \mathbb{C}$.

Corollary 3.53. *Assume that E is a Hilbert G-module over $\mathbb{K}$ (cf. Definition 2.11). Then each direct summand $R_\varepsilon(G,\mathbb{K}) * E$ is closed and E is a Hilbert space direct orthogonal sum of all of these summands. Each summand $R_\varepsilon(G,\mathbb{K}) * E$ is the Hilbert space direct sum $\bigoplus_{j\in J_\varepsilon} E_j$ of simple submodules E_j each of which belongs to the class ε.*

Proof. By 2.25, E is the orthogonal Hilbert space direct sum of simple submodules. Let $E_\varepsilon = \bigoplus_{j\in J_\varepsilon} E_j$ the Hilbert space direct sum within E of a maximal orthogonal family of submodules contained in ε; by Zorn's Lemma this exists. Then $E = \bigoplus_{\varepsilon\in\widehat{G}} E_\varepsilon$, because the orthogonal complement of this submodule, if nonzero would have to contain a nonzero simple submodule by 2.24 which is impossible by the construction of the E_ε. It follows from 3.52 that $R_\varepsilon(G,\mathbb{K}) * E \subseteq E_\varepsilon$. Conversely, if $x \in E_\varepsilon$, and $\varepsilon \neq \eta \in \widehat{G}$, then the module $R_\eta(G,\mathbb{K}) * x$ is the direct sum of simple modules out of η. Hence its orthogonal projection into any summand E_j of the orthogonal Hilbert space sum $E_\eta = \bigoplus_{j\in J_\eta} E_j$ is zero by 3.24(i). Hence $R_\eta(G,\mathbb{K}) * x = \{0\}$. Thus $\left(\sum_{\eta\neq\varepsilon} R(G,\mathbb{K})\right) * x = \{0\}$ and thus $x \in \overline{R(G,\mathbb{K}) * x} = R_\varepsilon(G,\mathbb{K}) * x \subseteq R_\varepsilon(G,\mathbb{K}) * E$. Thus $E_\varepsilon = R_\varepsilon(G,\mathbb{K}) * E$ and the corollary is proved. □

Complexification of Real Representations

There is general machinery which allows us to move comparatively freely between the representation (G-module) theory over the two ground fields $K = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. Trivially, we can consider every complex G-module as a real one by simply restricting the field of scalars. The somewhat more involved part is the ascending from real to complex modules. The background is pure linear algebra.

Let E denote a real vector space. The tensor product $E_\mathbb{C} \stackrel{\text{def}}{=} \mathbb{C} \otimes E$ is a complex vector space via the multiplication $c\cdot(d\otimes v) = cd \otimes v$. The function $v \mapsto 1 \otimes v$: $E \to E_\mathbb{C}$ implements an isomorphism of real vector spaces onto the real vector subspace $1 \otimes E$, and if we identify E with a real vector subspace of $E_\mathbb{C}$ via this map we may write $E_\mathbb{C} = E \oplus i\cdot E$. The function $\kappa: E_\mathbb{C} \to E_\mathbb{C}$ given by $\kappa(c\otimes v) = \overline{c} \otimes v$ is an involution (i.e. $\kappa^2 = \mathrm{id}_{E_\mathbb{C}}$) of real vector spaces satisfying $i\cdot\kappa(v) = -\kappa(i\cdot v)$, and the fixed point vector space of κ is E. If we set $P = \frac{1}{2}(\mathrm{id}_{E_\mathbb{C}} + \kappa)$, then P is a projection (i.e. $P^2 = P$) of real vector spaces satisfying $i\cdot P = \frac{1}{2}(i\cdot \mathrm{id}_{E_\mathbb{C}} + i\cdot\kappa) = \frac{1}{2}(i\cdot \mathrm{id}_{E_\mathbb{C}} - \kappa i\cdot \mathrm{id}_{E_\mathbb{C}}) = (1-P)i\cdot \mathrm{id}_{E_\mathbb{C}}$ such that $E = PE_\mathbb{C}$. Moreover, $P\kappa = P$. The following lemma is helpful.

Lemma A. *If V is a complex vector subspace of $E_\mathbb{C}$ with $\kappa V = V$, then $V = PV \oplus iPV$.*

Proof. We note $PV = \frac{1}{2}(\mathrm{id}_{E_\mathbb{C}} + \kappa)V \subseteq V + \kappa V = V$, and thus, since V is a complex vector subspace, $PV + iPV \subseteq V$. If $v \in V$ we write $v = x + iy$ with $x, y \in E$. Then $x = Pv \in PV$ and $y = P(-iv) \in PV$, whence $v = x + iy \in PV + iPV$. We observe that $PV \cap iPV \subseteq E \cap iE = \{0\}$. □

If E is a real Hilbert space, then $E_{\mathbb{C}}$ is a complex Hilbert space with respect to the scalar product given by $(c\otimes v \mid d\otimes w) = (c\overline{d})(v \mid w)$. The underlying real vector space of $E_{\mathbb{C}}$ is a real Hilbert space relative to the inner product $(v, w) \mapsto \operatorname{Re}(v \mid w)$. In particular, in this vector space, v and $i{\cdot}v$ are perpendicular since $\operatorname{Re}(v \mid i{\cdot}v) = 0$. The operator κ is orthogonal and P is an orthogonal projection with respect to this real Hilbert space structure on $E_{\mathbb{C}}$.

Now we apply this background information to G-modules and assume that E is a real G-module. Then $E_{\mathbb{C}}$ is a complex G-module relative to the operation $g(c \otimes v) = c \otimes gv$. If $x, y \in E$ then this is tantamount to saying that $g(x + i{\cdot}y) = gx + i{\cdot}gy$. If $\pi: G \to \operatorname{Gl}(E)$ denotes the representation associated with the real G-module E, we denote with $\pi_{\mathbb{C}}: G \to \operatorname{Gl}(E_{\mathbb{C}})$ the representation associated with the complex G-module $E_{\mathbb{C}}$. We note

$$(\forall g \in G) \quad \kappa\pi_{\mathbb{C}}(g) = \pi_{\mathbb{C}}(g)\kappa \text{ and } P\pi_{\mathbb{C}}(g) = \pi_{\mathbb{C}}(g)P.$$

If E is a real Hilbert G-module, then $E_{\mathbb{C}}$ is a complex Hilbert G- module.

Lemma B. *Let E be a real simple G-module and V a complex G-submodule of $E_{\mathbb{C}}$ which is invariant under κ. Then either $V = \{0\}$ or $V = E_{\mathbb{C}}$.*

Proof. From Lemma A we know that $V = PV \oplus iPV$. Since P is equivariant, PV is a submodule of the simple real G-module E. Thus if $PV = \{0\}$, then $V = PV + iPV = \{0\}$, and if $PV = E$, then $V = PV + iPV = E + iE = E_{\mathbb{C}}$. □

A *complex structure* on a real vector space is an endomorphism of real vector spaces $I \in \operatorname{Hom}(E, E)$ satisfying $I^2 = -\operatorname{id}_E$. Whenever such a complex structure exists, we have a scalar multiplication $*: \mathbb{C} \times E \to E$, $(a + ib)*v = a{\cdot}v + b{\cdot}(Iv)$, a claim whose verification is straightforward. The endomorphisms φ of the complex vector space so defined are exactly the real endomorphisms $\varphi \in \operatorname{Hom}(E, E)$ satisfying $\varphi I = I\varphi$.

Proposition 3.54. *For a real simple G-module E, the following conditions are equivalent:*

(1) *$E_{\mathbb{C}}$ fails to be a complex simple G-module.*
(2) *$E_{\mathbb{C}}$ contains a complex G-module F such that $E_{\mathbb{C}} = F \oplus \kappa F$ and $\kappa F \cong \widetilde{F}$, the conjugate complex module of F (see 3.6 and preceding discussion).*
(3) *The real G-module E is the underlying real vector space of a complex G-module; i.e. there is a complex structure $I \in \operatorname{Gl}(E)$ such that $I\pi(g) = \pi(g)I$ for all $g \in G$.*

If these conditions are satisfied, then the complex G-module (E, I) is isomorphic to the complex G-module F. Moreover, the real G-module E, the real G-module underlying F and the real G-module underlying κF are all isomorphic. In particular, F is a simple G-module both over $\mathbb{R}$ and $\mathbb{C}$.

If E is a real Hilbert module, then E, F and κF are isometrically isomorphic as real Hilbert G-modules. The real vector space automorphism I of E is orthogonal and satisfies $(v \mid Iv) = 0$ for all v. Let $I_{\mathbb{C}}$ denote the unique extension of I to *a complex involution of* $E_{\mathbb{C}}$. *Then we may take* F *to be the eigenspace of* $I_{\mathbb{C}}$ *for the eigenvalue* i, *and the function* $\iota \stackrel{\text{def}}{=} \frac{\sqrt{2}}{2}(\mathrm{id}_{E_{\mathbb{C}}} - i\cdot I_{\mathbb{C}})$ *maps the real Hilbert* G-*module* E *isometrically isomorphically onto* F.

Proof. (1)⇒(2). Assume $E_{\mathbb{C}}$ is not simple. Then there is a nonzero proper complex submodule $F \subseteq E_{\mathbb{C}}$, and κF is a complex G-module such that $i\cdot\kappa f = \kappa(-i)\cdot f = \kappa i \bullet f$ where $c \bullet f = \overline{c}\cdot v$ is the scalar multiplication of the conjugate vector space $\widetilde{F}$. Thus $\kappa\colon \widetilde{F} \to \kappa F$ is an isomorphism of complex vector spaces and G-modules. The vector subspaces $V_1 = F \cap \kappa F$ and $V_2 = F + \kappa F$ are complex G-submodules which are, in addition, invariant under κ. Hence Lemma B applies to both and shows that V_j is either $\{0\}$ or $E_{\mathbb{C}}$ for both $j = 1$ and $j = 2$. Suppose $V_1 = E_{\mathbb{C}}$; then $F = E_{\mathbb{C}}$ contradicting the assumption that F is proper. Hence $V_1 = \{0\}$. Secondly, V_2 contains F which was assumed to be nonzero, hence $V_2 = E_{\mathbb{C}}$. This shows that $E_{\mathbb{C}} = F \oplus \kappa F$.

(2)⇒(3). Assume (2) is true. We claim $F\cap E = \{0\}$. Suppose it is not true; then since $E \cap F$ is a real submodule of the simple G-module E, we have $E \subseteq F$ and then $F = E_{\mathbb{C}}$ since F is a complex vector subspace of $E_{\mathbb{C}}$. This is impossible since F is a *proper* subspace. This establishes the claim and also shows that $F \cap iE = i(F\cap E) = \{0\}$. Now set $\varphi \stackrel{\text{def}}{=} P|F\colon F \to E$. Then $\ker\varphi = F\cap \ker P = F\cap iE = \{0\}$, and $PF = P\kappa F = P(F + \kappa F) = PE_{\mathbb{C}} = E$. Thus φ is bijective. It is equivariant, and thus φ is an isomorphism or real G-modules. Defining $Ix = \varphi\big(i\varphi^{-1}(x)\big)$ we obtain the desired complex structure on E.

(3)⇒(1). Assume (3) is true. Since $I^2 = -\mathrm{id}_E$, the real vector space automorphism I of E is semisimple with spectrum $\{i, -i\}$. Now I extends uniquely to a complex vector space automorphism $I_{\mathbb{C}} = 1 \otimes I$ of $E_{\mathbb{C}}$ commuting with all $\pi_{\mathbb{C}}(g)$ and satisfying $I_{\mathbb{C}}^2 = -\mathrm{id}_{E_{\mathbb{C}}}$ and having the spectrum $\{i, -i\}$. Let F denote the eigenspace of $I_{\mathbb{C}}$ for the eigenvalue i. Then F is a nonzero proper $\pi_{\mathbb{C}}(G)$-invariant subspace.

In the process of the proof we have already established the isomorphy of the complex G-modules (E, I) and F, and the isomorphy of the real G-modules E, F and κF. Since κ is a real isometric automorphism of the real Hilbert space underlying $E_{\mathbb{C}}$, the real G-modules F and κF are isometrically isomorphic. We see that $I_{\mathbb{C}} \circ \iota = \frac{\sqrt{2}}{2}(I_{\mathbb{C}} + i\,\mathrm{id}_{E_{\mathbb{C}}}) = i\,\mathrm{id}_{E_{\mathbb{C}}} \circ \iota$. Since κ is a real isometry, for $v \in E$ we have $(v - iIv \mid v - iIv) = \|v - iIv\|^2 = \|\kappa(v - iIv)\|^2 = (v + iIv \mid v + iIv)$. Since the right, respectively, left hand sides are $\|v\|^2 \pm \mathrm{Re}(v \mid iIv) + \|iIv\|^2$ and since (v, Iv) is real it follows that $(v \mid Iv) = -\mathrm{Re}(-i)(v \mid Iv) = -\mathrm{Re}(v \mid iIv) = 0$. Because $I_{\mathbb{C}}^2 = -\mathrm{id}_{E_{\mathbb{C}}}$ this suffices for I to be orthogonal, and for ι to be isometric on E. □

If $\chi\colon G \to \mathbb{C}$ is a homomorphism with values in $\mathbb{S}^1$ we write

$$\chi(g) = \chi_1(g) + \chi_2(g)i, \quad \chi_1(g) = \mathrm{Re}\,(\chi(g))\,, \quad \chi_2(g) = \mathrm{Im}\,(\chi(g))\,.$$

As an example, if $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\chi(t + \mathbb{Z}) = e^{2\pi int}$ for some integer $n \in \mathbb{Z}$ (cf. E1.9). Then $\chi_1(t + \mathbb{Z}) = \cos 2\pi nt$ and $\chi_2(t + \mathbb{Z}) = \sin 2\pi nt$.

Corollary 3.55. *For a compact abelian group G the following conclusions hold:*

(i) *A simple real G-module is either one-dimensional or two-dimensional.*

(ii) *If E is a real simple two-dimensional G-module, then E has a complex structure I and there is a character $\chi \in \operatorname{Hom}(G, \mathbb{S}^1)$ such that $gv = \chi_1(g)\cdot v + \chi_2(g)\cdot(Iv) = \chi(g) * v$ for $v \in E$ (with the complex scalar multiplication $*$ given by $(a + ib)*v = a\cdot v + b\cdot(Iv)$).*

(iii) *If $\operatorname{Hom}\big(G, \mathbb{Z}(2)\big) = \{0\}$, then a nontrivial irreducible real G-module is two-dimensional. This applies, in particular, to all connected compact abelian groups, and thus to all torus groups.*

Proof. (i) By 2.30 a simple complex G-module is one-dimensional. Hence by Proposition 3.54, either $\dim_{\mathbb{C}} E_{\mathbb{C}} = 1$ or $\dim_{\mathbb{C}} F = 1$. In the first case, $\dim_{\mathbb{R}} E = 1$ and in the second case $\dim_{\mathbb{R}} E = \dim_{\mathbb{R}} F = 2$.

(ii) In Proposition 3.54 we saw that a two-dimensional simple real G-module has a complex structure I commuting with all $\pi(g)$ such that E with the complex scalar multiplication $*$ is isomorphic to the complex G-module F. By 2.30, there exists a character $\chi \in \operatorname{Hom}(G, \mathbb{S}^1)$ such that for $f \in F$ we have $gf = \chi(g)\cdot f$. Then the claim follows.

(iii) The maximal compact subgroup of the multiplicative group $\mathbb{R} \setminus \{0\}$ is $\{1, -1\}$. Thus if $\pi \colon G \to \operatorname{Gl}(1, \mathbb{R})$ is a representation, as G is compact we have to have $\pi(G) \subseteq \{1, -1\}$, giving an element of $\operatorname{Hom}\big(G, \mathbb{Z}(2)\big)$. If this set contains only one element, no nonconstant representation π is possible. If G is connected, then $\pi(G) \subseteq \{1, -1\}_0 = \{1\}$. □

Let A be an additively written abelian group. Then the multiplicative group $\mathbb{S}^0 = \{1, -1\} \subseteq \mathbb{Z}$ acts automorphically on A under the natural $\mathbb{Z}$-module action. Let $A/\mathbb{S}^0$ denote the orbit space. Then an orbit $\Gamma \in A/\mathbb{S}^0$ is either $\Gamma = \{0\}$, or $\Gamma = \{a\}$ for an involution a (i.e. $2\cdot a = 0$), or $\Gamma = \{a, -a\}$ for an element a which is not of order two. The fact that we have written A additively was just a matter of convenience; the concepts apply to any abelian group.

In the following proposition, for a compact abelian group G we let $\widehat{G} = \operatorname{Hom}(G, \mathbb{S}^1)$ denote the abelian group of multiplicative characters, and $\widehat{G}_{\mathbb{R}}$ the set of isomorphy classes of real simple G-modules. We note that a character $\chi \in \widehat{G}$ is an involution, i.e. satisfies $\chi^2 = 1$ if and only if $\chi(G) \subseteq \{1, -1\}$. In this situation, either χ is the constant character or $\chi(G) = \{1, -1\}$. We shall denote with $I \colon \mathbb{R}^2 \to \mathbb{R}^2$ the automorphism which has the matrix representation $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We abbreviate $\mathrm{id}_{\mathbb{R}^2}$ by $\mathbf{1}$.

Proposition 3.56. *Let G be a compact abelian group. Then there is a surjective function $\mathbf{re} \colon \widehat{G} \to \widehat{G}_{\mathbb{R}}$ such that $\mathbf{re}^{-1}\,\mathbf{re}(\chi) = \mathbb{S}^0\cdot\chi = \{\chi, \overline{\chi}\}$ and that the irreducible real linear representations are classified by the following three mutually exclusive cases:*

(i) *If $\chi = \mathbf{1}$ is the constant character, then $\mathbf{re}(\chi)$ is the class of the trivial G-module $\mathbb{R}$.*

(ii) *If* $\chi^2 = \mathbf{1}$ *and* $\chi(G) = \{1, -1\}$*, then* $\mathbf{re}(\chi)$ *is the class of modules isomorphic to the module* $\mathbb{R}$ *with the action* $\pi_{\mathbf{re}(\chi)}: G \to \mathrm{SO}(1) \subseteq \mathrm{Hom}(\mathbb{R}, \mathbb{R})$*,* $\pi_{\mathbf{re}(\chi)}(g)(r) = \chi(g)r$*.*

(iii) *If* χ *is not an involution then* $\mathbf{re}(\chi)$ *is the equivalence class of the* G*-module defined on* $\mathbb{R}^2$ *by* $\pi_{\mathbf{re}(\chi)}: T \to \mathrm{SO}(2) \subseteq \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$*,*

$$\pi_{\mathbf{re}(\chi)}(g) = \chi_1(g) \cdot \mathbf{1} + \chi_2(g) \cdot I.$$

Proof. From Corollary 3.55 we know that each nontrivial simple real G-module is either one- or two-dimensional. In the first case, $\pi(G) \subseteq \mathrm{O}(1)$ where $\mathrm{O}(1)$ contains the scalar multiplications by 1 and -1. If the module is trivial, we are in case (i), if not in case (ii).

Now assume that we have a two dimensional simple real G-module. Then by 3.55(ii) we may identify it with the underlying real vector space of $\mathbb{C}$ such that the module action is given by $gc = \chi(g)c$ for a character $\chi \in \mathrm{Hom}(G, \mathbb{S}^1)$. By Proposition 3.54(iii) these are exactly given by (iii) above.

Thus the function $\mathbf{re}$ is surjective. Proposition 3.54 shows that $\mathbf{re}^{-1}\,\mathbf{re}(\chi) = \{\chi, \overline{\chi}\}$. ☐

Proposition 3.57. *Let* E *be a real Hilbert* G*-module according to Definition* 2.11*. By the splitting of fixed points* (3.36) *we have a canonical orthogonal decomposition* $E = E_{\mathrm{fix}} \oplus E_{\mathrm{eff}}$*. Assume that* G *is a compact abelian group with* $\mathrm{Hom}\big(G, \mathbb{Z}(2)\big) = \{0\}$*. Then there is a unique orthogonal Hilbert space sum decomposition*

$$E_{\mathrm{eff}} = \bigoplus_{\varepsilon \in \widehat{G}_{\mathbb{R}} \setminus \{\mathbf{1}\}} E_\varepsilon.$$

There is a function $\varepsilon \mapsto \chi_\varepsilon : \widehat{G}_{\mathbb{R}} \to \widehat{G}$*, such that* $\mathbf{re}(\chi_\varepsilon) = \varepsilon$ *and there is a real orthogonal vector space automorphism* $I: E_{\mathrm{eff}} \to E_{\mathrm{eff}}$ *satisfying* $I^2 = -\mathrm{id}_{E_{\mathrm{eff}}}$ *such that for each* $\varepsilon \in \widehat{G}_{\mathbb{R}}$ *and each* $v \in E_{\mathbf{re}(\chi)}$ *we have*

$$gv = (\chi_\varepsilon)_1(g) \cdot v + (\chi_\varepsilon)_2(g) \cdot Iv.$$

Further, $(v \mid Iv) = 0$ *for all* $v \in E_{\mathrm{eff}}$*.*

If G *is connected then for each* $\varepsilon \in \widehat{G}_{\mathbb{R}}$ *there is an element* $g_\varepsilon \in G$ *such that* $I|E_\varepsilon = \pi_{E_{\mathrm{eff}}}(g_\varepsilon)$*.*

Proof. By Corollary 3.25 E_{eff} is the Hilbert space direct sum of submodules $R_\varepsilon(G, \mathbb{R}) * E$ where ε ranges through the set of equivalence classes of nontrivial simple modules and each $R_\varepsilon(G, \mathbb{R}) * E$ is an orthogonal Hilbert space direct sum $\bigoplus_{j \in J_\varepsilon} E_j$ of simple submodules E_j each of which belongs to the class ε. By Proposition 3.55 above, each element $\varepsilon \in \widehat{G}$ determines uniquely one pair of conjugate characters; let χ_ε be one of them. If $j \in J_\varepsilon$ then $E_j \in \varepsilon_D$ is endowed with an endomorphism I_j with $I_j^2 = -\mathrm{id}_{E_j}$ such that for $v \in E_j$ we have

$$\pi_{E_j}(g) = (\chi_\varepsilon)_1(g) \cdot \mathrm{id}_{E_j} + (\chi_\varepsilon)_2(g) \cdot I_j.$$

We write $E_\varepsilon \stackrel{\text{def}}{=} R_\varepsilon(G,\mathbb{R}) * E = \bigoplus_{j\in J_\varepsilon} E_j$ and set

$$I_\varepsilon \colon E_\varepsilon \to E_\varepsilon, \quad I_\varepsilon = \bigoplus_{j\in J_\varepsilon} I_j.$$

Finally we define

$$I \colon E_{\text{eff}} \to E_{\text{eff}}, \quad I = \bigoplus_{\varepsilon\in \widehat{G}_{\mathbb{R}}\setminus\{\mathbf{1}\}} I_\varepsilon.$$

Then $I^2 = -\operatorname{id}_{E_{\text{eff}}}$ and for $g \in G$, $v \in E_\chi$ we have

$$\pi_{E_{\text{eff}}}(g) = (\chi_\varepsilon)_1(g)\cdot \operatorname{id}_{E_{\text{eff}}} + (\chi_\varepsilon)_2(g)\cdot I.$$

From 3.5 we know that I is orthogonal and satisfies $(v \mid Iv) = 0$ for all $v \in E_{\text{eff}}$.

Finally assume that $G = G_0$. Then, for each nonconstant character ρ, the group $\rho(G)$ is a nonsingleton compact connected subgroup of the circle group $\mathbb{S}^1$. But $\mathbb{S}^1$ is the only such. (Its inverse image in $\mathbb{R}$ under $t \mapsto e^{2\pi it}$ is a nondiscrete closed subgroup, and hence is $\mathbb{R}$. Cf. e.g. Appendix A1.12.) Thus there is a $g_\varepsilon \in G$ such that $\chi_\varepsilon(g_\varepsilon) = i$. Then $(\chi_\varepsilon)_1(g_\varepsilon) = 0$ and $(\chi_\varepsilon)_2(g_\chi) = 1$. Consequently,

$$I|E_\varepsilon = \pi_{E_{\text{eff}}}(\mathfrak{g}_\varepsilon). \qquad \square$$

We should keep in mind that the existence of $\varepsilon \mapsto \chi_\varepsilon$ and, consequently of I is based on a choice. This cannot be helped, if indeed one wishes to realize E_{eff} as a complex G-module.

Exercise E3.20. Formulate and prove a generalisation of Proposition 3.57 without assuming that $\operatorname{Hom}\big(G, \mathbb{Z}(2)\big) \neq \{0\}$.

[Hint. E_{eff} decomposes into a direct sum $E_2 \oplus E_I$ where E_2 is the direct sum of all E_χ with $\chi(G) \subseteq \{1,-1\}$ and E_I is exactly as was E_{eff} in Proposition 3.57.] $\square$

Part 3: The Weakly Complete Group Algebra

In Part 2 we brought compact groups and locally convex topological vector spaces together. Now we focus on one particularly simple category of locally convex real or complex vector spaces which we call *weakly complete* vector spaces. Their basic aspects we treat in Appendix 7 in a self-contained way. In the context of weakly complete vector spaces, also weakly complete topological algebras arise in a natural fashion (see Definition A7.32).

In this section, as a tool, we shall use the language of category theory, for which we give a self-contained introduction in Appendix 3.

We begin by fixing some notation: Let $\mathcal{WA}$ be the category of weakly complete unital $\mathbb{K}$-algebras for the ground field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $\mathcal{TG}$ be the category of topological groups. For any unital algebra A, the subset A^{-1} of invertible elements, or *units*, forms a subgroup of $(A, \cdot)$. If A is a weakly complete unital algebra, then A^{-1} is a topological group (see Lemma A7.36). Then $(A \mapsto A^{-1}) : \mathcal{WA} \to \mathcal{TG}$

is a well defined functor. It is rather directly seen to preserve intersections and arbitrary products and hence, by Proposition A3.51, it is a continuous functor (see Definition A3.50), that is, it preserves *arbitrary* limits.

In order to conclude from this information that $(A \mapsto A^{-1})$ has in fact a left adjoint, we need to show that is satisfies the *Solution Set Condition* (see Definition A3.58).

Lemma 3.58. *The functor $A \mapsto A^{-1}$ from the category of weakly complete unital algebras to the category of topological groups satisfies the solution set condition.*

Proof. For a proof we claim that for any topological group G there is a *set* $S(G)$ of pairs (φ, A) with a continuous group morphism $\varphi{:}\, G \to A^{-1}$ for some object A of $\mathcal{WA}$ with the following property: For every pair (f, B), $f{:}\, G \to B^{-1}$ with a weakly complete unital algebra B, there is a pair (φ, A) in $S(G)$ and a $\mathcal{WA}$-embedding $e{:}\, A \to B$ such that

$$f = G \xrightarrow{\quad\varphi\quad} A^{-1} \xrightarrow{\quad e|A^{-1}\quad} B^{-1},$$

where $e|A^{-1}$ denotes the bijective restriction and corestriction of e.

Indeed if $f{:}\, G \to B^{-1}$ determines a unique smallest algebraic unital abstract subalgebra C of B generated by $f(G)$, then there is only a set of these "up to equivalence". Then on each of these there is only a set of algebra topologies and, a fortiori, only a set of them for which the corestriction is continuous; for each of these, there is at most a set of algebra completions up to isomorphism. So, up to equivalence there is only a set of pairs (φ, A), $\varphi{:}\, G \to A^{-1}$ such that the unital algebra generated by $\varphi(G)$ is dense in A. Any such set $S(G)$ will satisfy the claim. □

Now we are in a position to apply the Left Adjoint Functor Existence Theorem A3.60 to conclude that $(A \mapsto A^{-1}){:}\, \mathcal{WA} \to \mathcal{TG}$ has a left adjoint $\mathcal{TG} \to \mathcal{WA}$ which we shall denote $G \mapsto \mathbb{K}[G]$.

Definition 3.59. For each topological group G and each of the ground fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we shall call $\mathbb{K}[G]$ the *weakly complete unital topological group algebra* over $\mathbb{K}$, or also more succinctly, the *weakly complete group algebra* of G if the applicable ground field $\mathbb{K}$ is understood. □

We reformulate what we have observed:

The Weakly Complete Group Algebra Theorem

Theorem 3.60. *To each topological group G there is attached functorially a weakly complete group algebra $\mathbb{K}[G]$ with a natural morphism $\eta_G{:}\, G \to \mathbb{K}[G]^{-1}$ such that the following universal property holds:*
For each weakly complete unital $\mathbb{K}$-algebra A and each morphism of topological groups $f{:}\, G \to A^{-1}$ there exists a unique morphism of weakly complete unital

algebras $f'\colon \mathbb{K}[G] \to A$ restricting to a morphism $f''\colon \mathbb{K}[G]^{-1} \to A^{-1}$ of topological groups such that $f = f'' \circ \eta_G$.

Proof. This is now a consequence of the Adjoint Functor Existence Theorem A.3.60. □

We shall retain the diagram for this robust universal property of the group algebra in our mind:

$$\begin{array}{ccccc}
\text{top groups} & & & & \text{wc algebras}\\
\hline
G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} & & \mathbb{K}[G]\\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle f''} & & \big\downarrow{\scriptstyle \exists! f'}\\
A^{-1} & \xrightarrow[\text{id}]{} & A^{-1} & & A.
\end{array}$$

According to Definition A3.37, η is called the *front adjunction.*

Exercise E3.21. For any finite discrete group G the weakly complete group algebra $\mathbb{K}[G]$ is isomorphic to the finite dimensional elementary classical group algebra.

[Hint. Verify the universal property for the classical group algebra.]

We draw some immediate conclusions for the weakly complete group algebra of a topological group G.

Corollary 3.61. *Let G be any topological group. Then*

(i) *the subalgebra linearly spanned by $\eta_G(G)$ in $\mathbb{K}[G]$ is dense in $\mathbb{K}[G]$.*

(ii) *Let $\eta_G^{\mathbb{C}}\colon G \to \mathbb{C}[G]^{-1}$ be the front adjunction in the case of the complex ground field. Then the closed real linear span $\overline{\operatorname{span}_{\mathbb{R}}(\eta_G^{\mathbb{C}}(G))}$ inside the weakly complete real vector space underlying $\mathbb{C}[G]$ is in fact $\mathbb{R}[G]$ (up to natural isomorphism), and the morphism of topological groups $\eta_G^{\mathbb{R}}\colon G \to \mathbb{R}[G]^{-1}$, $\eta_G^{\mathbb{R}}(g) = \eta_G^{\mathbb{C}}(g)$ for all $g \in G$ is the front adjunction for $\mathbb{R}[G]$.*

(iii) *The real weakly complete group algebra $\mathbb{R}[G]$ is contained, up to natural isomorphism, in the complex weakly complete group algebra $\mathbb{C}[G]$.*

Proof. (i) Let $S = \overline{\operatorname{span}}\big(\eta_G(G)\big) \subseteq \mathbb{K}[G]$ be the closed subalgebra linearly spanned by $\eta_G(G)$. Let $f_S\colon G \to S^{-1}$ be a morphism of topological groups and $f\colon G \to \mathbb{K}[G]$ the coextension of f_S. Then by the universal property of $\mathbb{K}[G]$ there is a unique morphism $f'\colon \mathbb{K}[G] \to S$ of weakly complete unital algebras such that $f' \circ \eta_G = f$, implying that $(f'|S) \circ \eta_G^{\circ} = f_S$ with the corestriction $\eta_G^{\circ}\colon G \to S$ of η_G to S. Thus S has the universal property of $\mathbb{K}[G]$; then the uniqueness of $\mathbb{K}[G]$ implies $S = \mathbb{K}[G]$.

(ii) Set $S_{\mathbb{R}} = \overline{\operatorname{span}_G^{\mathbb{R}}(\eta_G(G))}$. Then $S_{\mathbb{R}}$ is a closed real unital subalgebra of the real weakly complete unital subalgebra underlying $\mathbb{C}[G]$. Moreover, $\eta_G^{\mathbb{R}}\colon G \to S_{\mathbb{R}}^{-1}$ is a well-defined morphism of topological groups as a corestriction of $\eta_G^{\mathbb{R}}$. We shall prove the claim by verifying that $\eta_G^{\mathbb{R}}: G \to S_{\mathbb{R}}^{-1}$ satisfies the required universal property of Theorem 3.60 for $\mathbb{K} = \mathbb{R}$ with $S_{\mathbb{R}}$ in place of $\mathbb{R}[G]$.

So let A be a weakly complete real unital algebra and $f: G \to A^{-1}$ a morphism of topological groups. Then $A_{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \otimes_{\mathbb{R}} A$, the complexification of A is a weakly complete complex unital algebra. Now $f_{\mathbb{C}}(g) \stackrel{\text{def}}{=} 1 \otimes f(g) \in 1 \otimes A^{-1} \subseteq A_{\mathbb{C}}^{-1}$ defines a morphism of topological groups $f_{\mathbb{C}}: G \to A_{\mathbb{C}}$. By the universal property of topological groups $\mathbb{C}[G]$ according to Theorem 3.60 there is a unique morphism $f'_{\mathbb{C}}: \mathbb{C}[G] \to A_{\mathbb{C}}$ $f_{\mathbb{C}}(g) = f'_{\mathbb{C}}\big(\eta_G^{\mathbb{C}}(g)\big)$. We claim that there is a morphism of weakly complete real unital algebras $f'_{\mathbb{R}}: S_{\mathbb{R}} \to A$ such that $f'_{\mathbb{C}}(s) = 1 \otimes f'_{\mathbb{R}}(s)$. Now $\eta_G^{\mathbb{C}}(g) = \eta_G^{\mathbb{R}}(g) \in S_{\mathbb{R}}$ by definition of $S_{\mathbb{R}}$, and likewise $\eta_G^{\mathbb{C}}(g)^{-1} = \eta_G^{\mathbb{C}}(g^{-1})$ exists in $S_{\mathbb{R}}$, whence $\eta_G^{\mathbb{R}}(G) \subseteq S_{\mathbb{R}}^{-1} \subseteq A^{-1}$. Further $f'_{\mathbb{C}}(\eta_G^{\mathbb{R}}(g)) = f'_{\mathbb{C}}(\eta_G^{\mathbb{C}}(g)) = f_C(g) = 1 \otimes f(g) \in 1 \otimes A$, whence $f'_{\mathbb{C}}(S_R) = f'_C\big(\overline{\text{span}^{\mathbb{R}}(\eta_G^{\mathbb{R}}(G))}\big) \subseteq \overline{f'_{\mathbb{C}}\big(\text{span}^{\mathbb{R}}\big(\eta_G^{\mathbb{R}}(G)\big)\big)} \subseteq \overline{1 \otimes f(G)} \subseteq 1 \otimes A$. Therefore $f'_{\mathbb{C}}|S_{\mathbb{R}}: S_{\mathbb{R}} \to 1 \otimes A$ is a morphism of weakly complete real unital algebras. Since $a \mapsto 1 \otimes a : A \to 1 \otimes A$ is an isomorphism of real weakly complete algebras, the existence of $f'_{\mathbb{R}}: S_{\mathbb{R}} \to A$ is shown as claimed. The uniqueness is clear from the uniqueness of $f'_{\mathbb{C}}: \mathbb{C}[G] \to A_{\mathbb{C}}$. Thus S_R has the required universal property and thus is naturally isomorphic to $\mathbb{R}[G]$.

(iii) Let $\mathbb{R}[G]$ be given. By Theorem 3.60 there is a complex weakly complete complex group algebra $\mathbb{C}[G]$ with the front adjunction $\eta_G: G \to \mathbb{C}[G]$. By Assertion (ii) above, $\mathbb{C}[G]$ contains a real weakly complete group algebra $\widetilde{\mathbb{R}}[G] = \overline{\text{span}_{\mathbb{R}}(\eta_G(G))}$ with the corestriction of η_G to $\widetilde{\mathbb{R}}[G]$ as its front adjunction. However, since the functor $G \mapsto \mathbb{K}[G] : \mathcal{TG} \to \mathcal{WA}$ like any left adjoint functor is uniquely determined up to natural isomorphism by its right adjoint $A \mapsto A^{-1} : \mathcal{WA} \to \mathcal{TG}$, we know that $\widetilde{\mathbb{R}}[G] \cong \mathbb{R}[G]$ and so the assertion follows. □

Convention. *Henceforth we shall always assume that for every topological group G, the weakly complete complex group algebra $\mathbb{C}[G]$ contains the weakly complete real group algebra $\mathbb{R}[G]$ in such a fashion that*

$$\eta_G(G) \subseteq \mathbb{R}[G] = \overline{\text{span}_{\mathbb{R}}(\eta_G(G))} \subseteq \mathbb{C}[G].$$

We recall from Corollary 2.29(ii) that it can be proved in any theory of compact groups at a very early stage that *every compact group has an isomorphic copy in the group of units of a weakly complete unital algebra.* As a consequence we have

Corollary 3.62. (The Group Algebra of a Compact Group) *If G is a compact group, then $\eta_G: G \to \mathbb{R}[G]^{-1}$ induces an isomorphism of topological groups onto its image.* □

By Corollary 3.61(iii), this remains true for the complex weakly complete group algebra as well.
In other words, *any compact group may be considered as a subgroup of the group of units of its weakly complete real or complex group algebras.*

Exercise E3.22. Let G be a topological group whose continuous finite dimensional representations (over $\mathbb{K}$) separate the points of G. Then $\eta_G: G \to \mathbb{K}[G]^{-1}$ is injective.

Before we move further into the subject let us pause for a comment and a warning:

The comment: The initial results show once again the power of the universal property provided by a left adjoint situation. We shall see more of this power shortly.

The warning: The close relationship being suggested between $\mathbb{R}[G]$ and $\mathbb{C}[G]$ in Corollary 3.61 is deceptive. Indeed, for any compact group G, we may and shall write $G \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G]$. We shall see later that even for compact abelian groups G, the fine structures of $\mathbb{R}[G]$ and $\mathbb{C}[G]$ are significantly different, even for the circle group $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

The Hopf Aspect of Weakly Complete Group Algebras

The category $(\mathcal{TG}, \times)$ of topological groups together with the cartesian product and the category $(\mathcal{WA}, \otimes_{\mathcal{W}})$ of weakly complete $\mathbb{K}$-algebras endowed with the tensor product of weakly complete $\mathbb{K}$-vector spaces are both symmetric monoidal categories (see Appendix 3, notably Definition A3.62ff.). We shall now establish the fact that $G \mapsto \mathbb{K}[G] : \mathcal{TG} \to \mathcal{WA}$ is a multiplicative functor (see Definition A3.66).

If A and B are weakly complete algebras, we have $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ which implies

$$A^{-1} \otimes B^{-1} \subseteq (A \otimes B)^{-1},$$

where we have used the natural inclusion function $j: A \times B \to A \otimes B$ and write $A^{-1} \otimes B^{-1}$ in place of $j(A^{-1} \times B^{-1})$.

Now let G and H be topological groups. Then

$$\eta_G(G) \otimes \eta_H(H) \subseteq \mathbb{K}[G]^{-1} \otimes \mathbb{K}[H]^{-1} \subseteq \big(\mathbb{K}[G] \otimes \mathbb{K}[H]\big)^{-1},$$

and so we have the morphism

$$G \times H \to \big(\mathbb{K}[G] \otimes \mathbb{K}[H]\big)^{-1},$$

$(g, h) \mapsto \eta_G(g) \otimes \eta_H(h)$ which, by the univeral property of $\mathbb{K}[-]$ gives rise to a unique morphism $\alpha: \mathbb{K}[G \times H] \to \mathbb{K}[G] \otimes \mathbb{K}[H]$ such that

$$(1) \qquad (\forall (g, h) \in G \times H)\, \alpha(\eta_{G\times H}(g, h)) = \eta_G(g) \otimes \eta_H(h).$$

On the other hand, the morphisms $j_G: G \to G \times H$, $j_G(g) = (g, 1_H)$ and $p_G: G \times H \to G$, $p_G(g, h) = g$ yield $p_G j_G = \mathrm{id}_G$. Therefore $\mathbb{K}[p_G]: \mathbb{K}[G \times H] \to \mathbb{K}[G]$ is an algebra retraction, and via $\mathbb{K}[j_G]$ we may identify $\mathbb{K}[G]$ with a subalgebra of $\mathbb{K}[G \times H]$; likewise $\mathbb{K}[H]$ is an algebra retract of the algebra $\mathbb{K}[G \times H]$. Since

$(g,1)(1,h)=(g,h)$ in $G\times H$, with the identifications of $\mathbb{K}[G],\mathbb{K}[H]\subseteq\mathbb{K}[G\times H]$ we have

$$(2)\qquad (\forall (g,h)\in G\times H)\,\eta_G(g)\eta_H(h)=\eta_{G\times H}(g,h)\in\mathbb{K}[G\times H].$$

The function

$$\mathbb{K}[G]\times\mathbb{K}[H]\to\mathbb{K}[G\times H],\quad (a,b)\mapsto ab$$

is a continuous bilinear map of weakly complete vector spaces; therefore the universal property of the tensor product in $\mathcal{W}$ yields a unique $\mathcal{W}$-morphism

$$\beta\colon\mathbb{K}[G]\otimes\mathbb{K}[H]\to\mathbb{K}[G\times H]$$

such that

$$(3)\qquad (\forall a\in\mathbb{K}[G],\, b\in\mathbb{K}[H])\,\beta(a\otimes b)=ab\in\mathbb{K}[G\times H].$$

Now, if for an arbitrary element $(g,h)\in G\times H$ we set $a=\eta_G(g)$ and $b=\eta_H(h)$, then we have

$$(4)\qquad \beta\big(\eta_G(g)\otimes\eta_H(h)\big)=a\otimes b=ab=\eta_G(g)\eta_H(h)=\eta_{G\times H}(g,h).$$

By Corollary 3.61, $\eta_G(G)$ generates $\mathbb{K}[G]$ as a weakly complete unital algebra and likewise $\eta_H(H)$ generates $\mathbb{K}[H]$ in this fashion, and the algebraic tensor product of $\mathbb{K}[G]$ and $\mathbb{K}[H]$ is dense in $\mathbb{K}[G]\otimes\mathbb{K}[H]$. Therefore, (4) implies $\beta\circ\alpha=\mathrm{id}_{\mathbb{K}[G\times H]}$. In other words, the diagram

$$\begin{array}{ccc}
\mathbb{K}[G\times H] & \xrightarrow{\mathrm{id}_{\mathbb{K}[G\times H]}} & \mathbb{K}[G\times H]\\
\alpha\big\downarrow & & \big\uparrow\beta\\
\mathbb{K}[G]\otimes\mathbb{K}[H] & \xrightarrow[\mathrm{id}_{\mathbb{K}[G]\otimes\mathbb{K}[H]}]{} & \mathbb{K}[G]\otimes\mathbb{K}[H]
\end{array}$$

commutes. Similarly, let us look at $\alpha\circ\beta:\mathbb{K}[G]\otimes\mathbb{K}[H]\to\mathbb{K}[G]\otimes\mathbb{K}[H]$: We recall (4) and (1) and verify

$$\alpha\big(\beta(\eta_G(g)\otimes\eta_H(h))\big)=\alpha(\eta_{G\times H}(g,h))=\eta_G(g)\otimes\eta_H(h)$$

By the same argument as above we conclude $\alpha\circ\beta=\mathrm{id}_{\mathbb{K}[G]\otimes\mathbb{K}[H]}$.

Taking everything together, we have proved the following important result:

Multiplicativity of the Group Algebra Functor $\mathbb{K}[-]$)

Theorem 3.63. *For two arbitrary topological groups G and H the natural morphisms of weakly complete unital algebras $\alpha\colon\mathbb{K}[G\times H]\to\mathbb{K}[G]\otimes_{\mathcal{W}}\mathbb{K}[H]$ and $\beta\colon\mathbb{K}[G]\otimes_{\mathcal{W}}\mathbb{K}[H]\to\mathbb{K}[G\times H]$ are inverse isomorphisms of each other.* □

The primary consequence of the Multiplicativity Theorem is that each weakly complete group algebra $\mathbb{K}[G]$ is in fact a symmetric Hopf algebra (see Appendix 3, Definition A3.65(iii)), that is, a group object (see Definition A3.64(ii)) in the symmetric monoidal category $(\mathcal{W},\otimes_{\mathcal{W}})$ of weakly complete vector spaces which we discuss at some length in Appendix 7 in order to have a self-contained reference in this book.

So let G be a topological group and $\delta_G\colon G \to G \times G$ the diagonal morphism $\delta_G(g) = (g, g)$. Together with the constant morphism $k_G\colon G \to \mathbf{E} = \{1\}$ we have a comonoid (δ_G, k_G) (according to the discussion preceding Definition A3.64). Since the group-algebra functor $\mathbb{K}[-]$ is multiplicative we have *morphisms of weakly complete unital algebras* $\mathbb{K}[\delta_G]\colon \mathbb{K}[G] \to \mathbb{K}[G \times G]$ and $\mathbb{K}[k_G]\colon \mathbb{K}[G] \to \mathbb{K}[\{1\}] = \mathbb{K}$. By Theorem 3.63 above we have an isomorphism $\alpha_G\colon \mathbb{K}[G \times G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$. Moreover, the natural inclusion $i_G\colon \mathbf{E} \to G$ provides us with a natural morphism $\iota_G\colon \mathbb{K} = \mathbb{K}[\mathbf{E}] \to \mathbb{K}[G]$ providing us with a natural endomorphism $\iota_G \circ \kappa_G : \mathbb{K}[G] \to \mathbb{K}[G]$ also called the *augmentation*.

This yields the following Lemma.

Lemma 3.64. *For any topological group G, the weakly complete group algebra $\mathbb{K}[G]$ is a Hopf algebra for the cocommutative and coassociative comultiplation*

$$\gamma_G\colon \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G], \quad \gamma_G = \alpha_G \circ \mathbb{K}[\delta_G]$$

the co-identity $\kappa_G\colon \mathbb{K}[G] \to \mathbb{K}$, $\kappa_G = \mathbb{K}[k_G]$ and an augmentation $\iota_G \circ \kappa_G : \mathbb{K}[G] \to \mathbb{K}[G]$.

Proof. By the preceding observations γ_G and κ_G are natural morphisms of weakly complete algebras which satisfy the required conditions for a comonoid since $\mathbb{K}[-]$ is a functor and since (G, δ_G, k_G) is a comonoid in the symmetric monoidal category $(\mathcal{TG}, \times)$ as is straightforwardly verified. □

In Appendix 3, we define in Definition A3.95 the concept of the set $\mathbb{G}(A)$ of grouplike elements in a weakly complete Hopf algebra A. So at this point, the following fairly immediate remark will be relevant:

Lemma 3.65. *For each topological group G we have $\eta_G(G) \subseteq \mathbb{G}\big(\mathbb{K}[G]\big)$.*

Proof. We have to observe that for all elements $x \in \mathbb{K}[G]$ with $x \in \eta_G(G)$ we have

$$\gamma_G(x) = x \otimes x \text{ and } \kappa_G(x) = 1.$$

Now we recall the definition

$$(*) \qquad \gamma_G = \big(\mathbb{K}[G] \xrightarrow{\mathbb{K}[\delta_G]} \mathbb{K}[G \times G] \xrightarrow{\alpha_G} \mathbb{K}[G] \otimes \mathbb{K}[G]\big).$$

If $a = \eta_G(g)$ for some $g \in G$, then $\gamma_G(a) = \alpha_G(a, a) = a \otimes a$ by $(*)$ and by (1) above. □

We now address the claim that the Hopf algebra $\mathbb{K}[G]$ is a symmetric one, that is, that it is a group object in the symmetric monoidal category $(\mathcal{W}, \otimes_{\mathcal{W}})$. For each topological group G the *opposite group* $G^{\rm op}$ is the underlying topological space of G together with the multiplication $(g, h) \mapsto g{*}h$ defined by $g{*}h = hg$. The groups G and $G^{\rm op}$ are isomorphic under the function $\mathrm{inv}_G\colon G \to G^{\rm op}$, $\mathrm{inv}_G(g) = g^{-1}$. Analogously, every topological algebra A gives rise to an opposite algebra $A^{\rm op}$ on the same underlying topological vector space but with the multiplication defined by $a*b = ba$, giving us

$$(A^{-1})^{\rm op} = (A^{\rm op})^{-1}$$

by definition, but not necessarily being isomorphic to A. Consequently,

$$((\mathbb{K}[G])^{-1})^{\rm op} = (\mathbb{K}[G]^{\rm op})^{-1}$$

and there are morphisms of topological groups $\eta_G\colon G \to \mathbb{K}[G]^{-1}$ and $\eta_{G^{\rm op}}\colon G^{\rm op} \to \mathbb{K}[G^{\rm op}]^{-1}$. So we have an isomorphism $\mathbb{K}[{\rm inv}_G]\colon \mathbb{K}[G] \to \mathbb{K}[G^{\rm op}]$ of weakly complete topological algebras and, accordingly, an involutive isomorphism of topological groups $\mathbb{K}[{\rm inv}_G]^{-1}\colon \mathbb{K}[G]^{-1} \to \mathbb{K}[G^{\rm op}]^{-1}$. This gives us a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} \\ {\scriptstyle {\rm inv}_G}\downarrow & & \downarrow{\scriptstyle \mathbb{K}[{\rm inv}_G]^{-1}} \\ G^{\rm op} & \xrightarrow[\eta_{G^{\rm op}}]{} & \mathbb{K}[G^{\rm op}]^{-1}. \end{array}$$

producing an isomorphism of weakly complete algebras $\mathbb{K}[G] \to \mathbb{K}[G^{\rm op}]$.

But we also have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} \\ {\scriptstyle {\rm inv}_G}\downarrow & & \downarrow{\scriptstyle {\rm inv}_{\mathbb{K}[G]^{-1}}} \\ G^{\rm op} & \xrightarrow[\eta_{G^{\rm op}}]{} & (\mathbb{K}[G]^{-1})^{\rm op} = (\mathbb{K}[G]^{\rm op})^{-1}. \end{array}$$

Let us abbreviate $f \stackrel{\rm def}{=} \big(\eta_{G^{\rm op}}\big) \circ {\rm inv}_G\colon G \to (\mathbb{K}[G]^{\rm op})^{-1}$. So by the adjunction formalism, there is a unique involutive isomorphism $f'\colon \mathbb{K}[G] \to \mathbb{K}[G]^{\rm op}$ of weakly complete algebras such that $f = f'|\mathbb{K}[G]^{-1} \circ \eta_G$.

We have a grounding functor $A \to |A|$ from the category $\mathcal{WA}$ of weakly complete algebras to the category $\mathcal{W}$ of weakly complete vector spaces, where $|A|$ is simply the weakly complete vector space underlying the weakly complete algebra A. With this convention we formulate the following definition:

Definition 3.66. For each topological group G there is a morphism of weakly complete vector spaces $\sigma_G \stackrel{\rm def}{=} |f'|\colon |\mathbb{K}[G]| \to |\mathbb{K}[G]^{\rm op}| = |\mathbb{K}[G]|$, called *symmetry* or *antipode.* □

So for any topological group G we have

$$(**) \qquad (\forall g \in G)\, \sigma_G(\eta_G(g)) = \eta_G(g^{-1}) = \eta_G(g)^{-1},$$

that is, on the subgroup $\eta_G(G)$ of $\mathbb{K}[G]^{-1}$ the morphism σ_G of weakly complete vector spaces agrees with the multiplicative inversion of $\mathbb{K}[G]$.

In the following main theorem, for any topological group G we write W_G for the underlying weakly complete vector space $|\mathbb{K}[G]|$ of the weakly complete group algebra $\mathbb{K}[G]$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We recall that $\mathbf{E}$ denotes the singleton group object. The weakly complete vector space morphism of the augmentation

$$\mathbb{K}[G] \xrightarrow{k_G} \mathbb{K}[\mathbf{E}] \xrightarrow{\mathbb{K}(\iota_G)} \mathbb{K}[G]$$

will be written $\omega_G\colon W_G \to W_G$. These are category theoretical formalities.

In Appendix 3 in Definition A3.101, a weakly complete Hopf algebra A is called *group-saturated* if the linear span of the subset $\mathbb{G}(A)$ of grouplike elements is dense in A.

The Weakly Complete Hopf Group Algebras

Theorem 3.67. *For any topological group G, the weakly complete group algebra $\mathbb{K}[G]$ is a weakly complete symmetric group-saturated Hopf algebra.*

Specifically, the following diagram involving natural morphisms of weakly complete vector spaces commutes:

$$\begin{array}{ccc} W_G \otimes W_G & \xrightarrow{\sigma_G \otimes \mathrm{id}_{W_G}} & W_G \otimes W_G \\ {\scriptstyle\gamma_G}\uparrow & & \downarrow{\scriptstyle\mu_G} \\ W_G & \xrightarrow[\omega_G]{} & W_G. \end{array}$$

Proof. Relation $(**)$ is equivalent to

$$(\forall x \in \eta_G(G))\, \sigma_G(x)x = x\sigma_G(x) = 1.$$

Using the bilinearity of multiplication $(x,y) \mapsto xy$ in $\mathbb{K}[G]$, defining $\mu_G\colon \mathbb{K}[G] \otimes \mathbb{K}[G] \to \mathbb{K}[G]$ by $\mu_G(x \otimes y) = xy$, and remembering from Lemma 3.65 that $\gamma_G(x) = x \otimes x$ for all $x = \eta_G(g)$ with some $g \in G$, once more, we obtain the equivalent relation

$$(***) \qquad (\forall x \in \eta_G(G))\, (\mu_G \circ (\sigma_G \otimes \mathrm{id}_{\mathbb{K}[G]}) \circ \gamma_G)(x) = 1.$$

By Corollary 3.61, the weakly complete algebra $\mathbb{K}[G]$ is the closed linear span of $\eta_G(G)$, and so equation $(***)$ holds in fact for all elements of $\mathbb{K}[G]$ which the theorem expresses in the form of a commutative diagram. According to Definition A3.65 this is exactly what we have to show for $\mathbb{K}[G]$ to be a weakly complete symmetric Hopf algebra. The fact that $\mathbb{K}[G]$ is group-saturated is a consequence of Lemma 3.65 and Corollary 3.61. □

If $\mathcal{WH}$ denotes the category of weakly complete symmetric Hopf algebras, then $A \mapsto \mathbb{G}(A)$ defines a functor from $\mathcal{WH}$ to the category of topological groups $\mathcal{TG}$. The functor $\mathbf{H} \stackrel{\mathrm{def}}{=} (G \mapsto \mathbb{K}[G]) : \mathcal{TG} \to \mathcal{WH}$ from the category of topological groups to the category of weakly complete symmetric Hopf algebras is known to us since Theorem 3.60 as the corestriction of the weakly complete group algebra functor $(G \mapsto \mathbb{K}[G]) : \mathcal{TG} \to \mathcal{WA}$ into the bigger category of all weakly complete unital algebras. In this context we shall now proceed to show a sharper functor adjunction theorem as follows:

The Weakly Complete Group Hopf Algebra Adjunction Theorem

Theorem 3.68. *The functor $\mathbf{H}\colon \mathcal{TG} \to \mathcal{WH}$ from the category of topological groups to the category of weakly complete symmetric Hopf algebras is left adjoint to the functor $(A \mapsto \mathbb{G}(A)) : \mathcal{WH} \to \mathcal{TG}$.*

Equivalently: for a topological group G there is natural morphism of topological groups $\eta_G\colon G \to \mathbb{G}(\mathbf{H}(G)) = \mathbb{G}(\mathbb{K}[G])$ such that for each morphism of topological groups $f\colon G \to \mathbb{G}(A)$ for a weakly complete Hopf algebra A there is a unique morphism of weakly complete symmetric Hopf algebras $f'\colon \mathbf{H}(G) \to A$ such that $f(g) = f'(\eta_G(g))$ for all $g \in G$:

$$\begin{array}{ccccc}
\mathcal{TG} & & & & \mathcal{WH}\\
\hline
G & \xrightarrow{\eta_G} & \mathbb{G}(\mathbf{H}(G)) & & \mathbf{H}(G) = \mathbb{K}[G]\\
{\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle \mathbb{G}(f')} & & \downarrow{\scriptstyle \exists! f'}\\
\mathbb{G}(A) & \xrightarrow[\mathrm{id}]{} & \mathbb{G}(A) & & A
\end{array}$$

Remark. Notice that the topological group morphism η_G in Theorem 3.68 is in general a corestriction of the morphism named η_G in Theorem 3.60.

Proof. Let A be a weakly complete symmetric Hopf algebra and $f\colon G \to \mathbb{G}(A)$ a continuous group morphism. Since A is, in particular, a weakly complete associative unital algebra and $\mathbb{G}(A) \subseteq A^{-1}$, by the Weakly Complete Group Algebra Theorem 3.60 there is a unique morphism $f'\colon \mathbb{K}[G] \to A$ of weakly complete algebras such that $f(g) = f'(\eta_G(g))$ for all $g \in G$. Since each $\eta_G(g)$ is grouplike by Lemma 3.65, we have $\eta_G(G) \subseteq \mathbb{G}(\mathbf{H}(G))$. We shall see below that the morphism f' of weakly complete algebras is indeed a morphism of weakly complete Hopf algebras and therefore maps grouplike element into grouplike elements. Hence f' maps $\mathbb{G}(\mathbf{H}(G))$ into $\mathbb{G}(A)$.

We now have to show that f' is a morphism of Hopf algebras, that is, f' respects
(a) comultiplication,
(b) coidentity, and
(c) symmetry.

For (a) we must show that the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{K}[G] & \xrightarrow{c_{\mathbb{K}[G]}} & \mathbb{K}[G] \otimes \mathbb{K}[G] & \cong & \mathbb{K}[G \times G]\\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f' \otimes f'} & & \\
A & \xrightarrow{c_A} & A \otimes A. & &
\end{array}$$

Since $\mathbb{K}[G]$ is generated as a topological algebra by $\eta_G(G)$ by Corollary 3.61, it suffices to track all elements $x = \eta_G(g) \in \mathbb{K}[G]$ for $g \in G$. Every such element is grouplike in $\mathbb{K}[G]$ by Lemma 3.65, and so $(f' \otimes f')\gamma_{\mathbb{K}[G]}(x) = (f' \otimes f')(x \otimes x) = f'(x) \otimes f'(x)$ in $A \otimes A$, while on the other hand $f'(x) = f(g) \in \mathbb{G}(A)$, whence $\gamma_A(f'(x)) = f'(x) \otimes f'(x)$ as well. This proves (a).

For (b) we must show that the following diagram commutes.

$$\begin{array}{ccccc}
\mathbb{K}[G] & \xrightarrow{k_{\mathbb{K}[G]}} & \mathbb{K} & \cong & \mathbb{K}[\{1_G\}]\\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{K}}} & & \\
A & \xrightarrow{k_A} & \mathbb{K}. & &
\end{array}$$

Again it suffices to check the elements $x = \eta_A(g)$. Since all grouplike elements are always mapped to 1, this is a trivial exercise.

Finally we consider (c), where again we follow all elements $x = \eta_A(g)$. On the one hand we have $f'(\sigma_{\mathbb{K}[G]}(x)) = f'(x^{-1}) = f'(x)^{-1}$ in A^{-1}. But again $f'(x)$ is grouplike, and thus $\sigma_A(f'(x)) = f'(x)^{-1}$, which takes care of case (c), and this completes the proof of the theorem. □

As with any adjoint pair of functors, there is an alternative way to express the adjunction in the preceding theorem: see e.g. Proposition A3.36:

Corollary 3.69. *For each weakly complete symmetric Hopf algebra A there is a natural morphism of symmetric Hopf algebras $\varepsilon_A: \mathbf{H}\big(\mathbb{G}(A)\big) \to A$ such that for any topological group G and any morphism of weakly complete symmetric Hopf algebras $\varphi: \mathbf{H}(G) \to A$ there is a unique continuous group morphism $\varphi': G \to \mathbb{G}(A)$ such that for each $x \in \mathbf{H}(G) = \mathbb{K}[G]$ one has $\varphi(x) = \varepsilon_A\big(\mathbb{K}[\varphi'](x)\big)$, where $\mathbb{K}[\varphi'] = \mathbf{H}(\varphi')$.* □

From Theorem 3.67 we recall that each weakly complete symmetric Hopf algebra $A = \mathbb{K}[G] = \mathbf{H}(G)$ is group-saturated, that is, $A = \mathbf{S}(A)$.

Corollary 3.70. *In the circumstances of* Corollary 3.69*,*

$$\operatorname{im}(\varepsilon_A) = \varepsilon_A\Big(\mathbf{H}\big(\mathbb{G}(A)\big)\Big) = \mathbf{S}(A).$$

Proof. Set $B = \mathbf{H}\big(\mathbb{G}(A)\big)$; then $\varepsilon_A: B \to A$ is a morphism of Hopf algebras, and so, in particular, a morphism of weakly complete vector spaces. Hence $\operatorname{im}(\varepsilon_A) = \varepsilon_A(B)$ is a closed Hopf subalgebra of A. Since ε_A is a morphism of Hopf algebras, $\varepsilon_A\big(\mathbb{G}(B)\big) \subseteq \mathbb{G}(A)$ and thus $\varepsilon_A\big(\mathbf{S}(B)\big) \subseteq \mathbf{S}(A)$. Since B is group-saturated, i.e., $B = \mathbf{S}(B)$ and so $\operatorname{im}(\varepsilon_A) = \varepsilon_A\big(\mathbf{S}(B)\big)$. Hence $\operatorname{im}(\varepsilon_A) \subseteq \mathbf{S}(A)$.

On the other hand, quite generally, by Proposition A3.38(2), we have $\mathbb{G}(A) \subseteq \operatorname{im}(\varepsilon_A)$, and since $\operatorname{im}(\varepsilon_A)$ is closed, we conclude $\mathbf{S}(A) \subseteq \operatorname{im}(\varepsilon_A)$ which completes the proof. □

In particular, we have the

Remark 3.71. ε_A is quotient homomorphism if and only if $A = \mathbf{S}(A)$). □

From the general theory of adjunctions (as e.g. in Appendix 3, Proposition A3.38), in view of the Weakly Complete Group Hopf Algebra Adjunction Theorem 3.68 we may draw some immediate conclusions.

Corollary 3.72. *For any weakly complete symmetric Hopf algebra A and any topological group G we have*

(1) $(\forall A)\left(\mathbb{G}(A) \xrightarrow{\eta_{\mathbb{G}(A)}} \mathbb{G}\big(\mathbb{K}[\mathbb{G}(A)]\big) \xrightarrow{\mathbb{G}(\varepsilon_A)} \mathbb{G}(A)\right) = \mathrm{id}_{\mathbb{G}(A)}$, *and*

(2) $\quad (\forall G)\left(\mathbb{K}[G] \xrightarrow{\mathbb{K}[\eta_G]} \mathbb{K}\big[\mathbb{G}(\mathbb{K}[G])\big] \xrightarrow{\varepsilon_{\mathbb{K}[G]}} \mathbb{K}[G]\right) = \mathrm{id}_{\mathbb{K}[G]}\,.$ □

This corollary suggests the specification of particular subcategories on the side of topological groups on the one hand and of weakly complete symmetric Hopf algebras on the other as follows:

Definition 3.73. (i) A topological group G such that $\eta_G\colon G \to \mathbb{G}(\mathbb{K}[G])$ is an isomorphism will be called $\mathbb{K}$-*linearizable*, if the natural morphism $\eta_G\colon G \to \mathbb{G}(\mathbb{K}[G])$ of topological groups is an isomorphism of topological groups. The full subcategory of $\mathcal{TG}$ consisting of all $\mathbb{K}$-linearizable topological groups will be written $\mathcal{TG}_{\mathbb{K}}$.

(ii) A weakly complete symmetric Hopf algebra A over $\mathbb{K}$ will be called *group-determined* if the natural morphism $\varepsilon_A\colon \mathbb{K}[\mathbb{G}(A)] \to A$ is bijective. The full subcategory of $\mathcal{WH}$ consisting of all group-determined weakly complete symmetric Hopf algebras over $\mathbb{K}$ will be written $\mathcal{WH}_{\mathbb{K}}$. □

All finite discrete topological groups are $\mathbb{K}$-linearizable, and one of our primary goals will be to establish that all compact groups are $\mathbb{R}$-linearizable. The group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ will be seen to fail to be $\mathbb{C}$-linearizable.

We apply the category theoretical result Proposition A3.40(ii) and observe that the functor $\mathbf{H} = (G \mapsto \mathbb{K}[G])$ from $\mathcal{TG}$ to $\mathcal{WH}$ maps $\mathcal{TG}_{\mathbb{K}}$ into $\mathcal{WH}_{\mathbb{K}}$, thereby inducing the functor $\mathbf{H}_{\mathbb{K}}\colon \mathcal{TG}_{\mathbb{K}} \to \mathcal{WH}_{\mathbb{K}}$. The functor $\mathbb{G}$ from $\mathcal{WH}$ to $\mathcal{TG}$ maps $\mathcal{WH}_{\mathbb{K}}$ into $\mathcal{TG}_{\mathbb{K}}$, thereby inducing the functor $\mathbb{G}_{\mathbb{K}}\colon \mathcal{WH}_{\mathbb{K}} \to \mathcal{TG}_{\mathbb{K}}$. For the concept of the equivalence of two categories, see Definition A3.39.

Theorem 3.74. *The categories $\mathcal{TG}_{\mathbb{K}}$ of $\mathbb{K}$-linearizable topological groups and the category $\mathcal{WH}_{\mathbb{K}}$ of group determined weakly complete symmetric Hopf algebras over $\mathbb{K}$ are equivalent under the pair of adjoint functors $\mathbf{H}_{\mathbb{K}}$ and $\mathbb{G}_{\mathbb{K}}$.* □

A group determined weakly complete symmetric Hopf algebra A is isomorphic to $\mathbb{K}[\mathbb{G}(A)]$. The comultiplication of a group is the diagonal morphism and is therefore automatically cocommutative. Therefore we note:

Remark 3.74a. Any group determined weakly complete symmetric Hopf algebra is cocommutative. □

The Dual of a Weakly Complete Group Hopf Algebra

From Theorem A3.94 we know that the topological dual $\mathbb{K}[G]'$ of a weakly complete group Hopf algebra $\mathbb{K}[G]$ is a symmetric Hopf algebra (of $\mathbb{K}$-vector spaces). In order to understand this duality clearly, we now define a morphism of weakly complete vector spaces:

Definition 3.75. Let G be an arbitrary topological group and denote by $\mathbb{K}[G]'$ the topological dual of the weakly complete symmetric $\mathbb{K}$-Hopf algebra $\mathbf{H}(G) = \mathbb{K}[G]$.

Define a morphism of vector spaces

$$F_G\colon \mathbb{K}[G]' \to R(G,\mathbb{K}), \qquad \text{by } F_G(\omega) = \omega \circ \eta_G$$

Exercise E3.23. (i) F_G is a natural transformation.

(ii) $R(G,\mathbb{K})$ is a $\mathbb{K}$-Hopf algebra.

(iii) F_G is a morphism of $\mathbb{K}$-Hopf algebras.

(iv) We recall the module actions $(g,f) \mapsto {}_gf, {}^gf : G \times R(G,\mathbb{K})$, ${}_gf(x) = f(xg)$, ${}^gf(x) = f(g^{-1}x)$, see 2.4 and 3.8. Similarly every associative algebra A, such as $\mathbb{K}[G]$ has the two A^{-1}-module actions $(g,a) \mapsto ga$ and $(g,a) \mapsto ag^{-1}$ which induce two module actions on A^*. Accordingly $\mathbb{K}[G]$ has two G-module actions on $\mathbb{K}[G]$ via $(g,a) \mapsto \eta_G(g)a$ and $(g,a) \mapsto a\eta_G(g)^{-1}$. These induce module actions on $\mathbb{K}[G]'$.

The function F_G respects both module actions.

[Hint. (i) is straightforward. (ii) The natural isomorphism $R(G\times H,\mathbb{K}) \cong R(G,\mathbb{K})\otimes R(H,K)$ implies that $R(G,\mathbb{K})$ is a symmetric Hopf algebra in a natural fashion. (iii) is to be made evident by diagam chasing, for instance for the associative multiplication $m\colon G \times G \to G$ by the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathbb{K}[G]' & \xrightarrow{F_G} & \mathbb{R}(G,\mathbb{K}) \\
{\scriptstyle \mathbb{K}[m]}\big\downarrow & & \big\downarrow{\scriptstyle R(m,\mathbb{K})} \\
\mathbb{K}[G\times G]' & & \mathbb{R}(G\times G,\mathbb{K}) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\
\mathbb{K}[G]' \otimes_{\mathcal{V}} \mathbb{K}[G]' & \xrightarrow[F_G \otimes_{\mathcal{V}} F_G]{} & R(G,\mathbb{K}) \otimes_{\mathcal{V}} R(G,\mathbb{K}).
\end{array}$$

The commuting of the other diagram involved is left as an exercise.

(iv) For $\omega \in \mathbb{K}[G]'$ and $g \in G$ we have $g{\cdot}\omega(a) = \omega(\eta_G(g)^{-1}a)$ and $g * \omega(a) = \omega(a\eta_G(g))$. Then we have e.g. $F_G(g.\omega)(h) = (g{\cdot}\omega)(h) = \omega(g^{-1}h) = F_G(\omega)(g^{-1}h) = (h{\cdot}F_G)(\omega)(h)$. The second action works analogously.] ☐

The Duals of Weakly Complete Group Hopf Algebras

Theorem 3.76. *For an arbitrary topological group G, the function $F_G\colon \mathbb{K}[G]' \to R(G,\mathbb{K})$ is a natural isomorphism of $\mathbb{K}$-Hopf algebras, and it respects the natural G-module actions.*

Proof. In view of Exercise E3.23(iii) we have to show the bijectivity of F_G. The injectivity of F_G is seen as follows: By Corollary 3.61(ii) we have $\operatorname{span}(\eta_G(G)) = \mathbb{K}[G]$ and so the relation $\{0\} = (\omega \circ \eta_G(G)) = \omega(\eta_G(G))$ implies $\omega = 0$.

For a proof of the surjectivity of F_G we use an argument that we also indicate in Appendix 3 following Corollary A3.107. So let $f \in R(G,\mathbb{K})$. By Proposition 3.34 there is a finite dimensional G-module V, an element ω in its dual V', and an element $v \in V$ such that $f(g) = \langle \omega, \pi(g)(v)\rangle$ with the representation π of G belonging to the G-module V. Then $\pi\colon G \to \operatorname{End}(V)$ where $\pi(G) \subseteq \operatorname{Aut}(V) = \operatorname{End}(V)^{-1}$. By the universal property of $\mathbb{K}[G]$, the morphism $\pi\colon G \to \operatorname{End}(V)$ provides an algebra morphism $\overline{\pi}\colon \mathbb{K}[G] \to \operatorname{End}(V)$ so that $\pi(g) = \overline{\pi}(\eta_G(g))$ for all

$g \in G$. Then indeed $L(a) = \langle \omega, \overline{\pi}(a)(v)\rangle$ defines a linear form $L \in \mathbb{K}[G]'$ such that $f(g) = L\big(\eta_G(g)\big)$. So the surjectivity of F_G is secured. □

With this theorem we connect the concept of weakly complete group Hopf algebras with the traditional concept of the algebra of representative functions of a topological group, which for compact groups G dominated the early chapters 1, 2 and 3 of this book. From the duality of the category $\mathcal{V}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and the category $\mathcal{W}_{\mathbb{K}}$ of weakly complete vector spaces (see Theorem A7.9) we see that Theorem 3.86 implies at once the following understanding of a weakly complete group algebra as a vector space dual:

Corollary 3.76a. *Any weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ may be viewed as the algebraic (G-module) dual $R(G,\mathbb{K})^*$ of $R(G,\mathbb{K})$.* □

Recall that a weakly complete symmetric Hopf algebra A is called *group-saturated* iff $\mathbf{S}(A) = A$, and recall also that we have a linear map $\tau_A\colon A' \to R(\mathbb{G}(A),\mathbb{K})$ defined by $\tau_A(\omega) = \omega|\mathbb{G}(A)$.

Proposition 3.77. *For a weakly complete symmetric group-saturated Hopf algebra A the function $\tau_A\colon A' \to R(\mathbb{G}(A),\mathbb{K})$ is an injective morphism of Hopf algebras. If A is also group-determined, then τ_A is an isomorphism of symmetric Hopf algebras.*

Proof. Since A is group-saturated, we know from Corollary A3.107 in Appendix 3 that τ_A is injective. We have a commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\ \mathrm{id}\ } & A' \\ {\scriptstyle \varepsilon'_A}\big\downarrow & & \big\downarrow{\scriptstyle \tau_A} \\ \mathbb{K}[\mathbb{G}(A)]' & \xrightarrow{\ F_{\mathbb{G}(A)}\ } & R(\mathbb{G}(A),\mathbb{K}). \end{array}$$

Since $\varepsilon_A\colon A \to \mathbb{K}[\mathbb{G}(A)]$ is a morphism of weakly complete symmetric Hopf algebras, its dual is a morphism of symmetric Hopf algebras. Since $F_{\mathbb{G}(A)}$ is an isomorphism of symmetric Hopf algebras, it follows that τ_A is a morphism of symmetric Hopf algebras.

If A is group determined, ε_A is an isomorphism, and then so is τ_A. □

Note that the last assertion of this proposition follows also from Theorem 3.76 directly. Also observe that τ_A respects the $\mathbb{G}(A)$-module actions.

Definition 3.78. A symmetric Hopf algebra R over $\mathbb{K}$ is called *weakly reduced* if its algebraic dual R^* is group-determined (see Definition 3.73(ii)). □

By Remark 3.74a we know that
any weakly reduced symmetric Hopf algebra is automatically commutative and is of the form $R(G,\mathbb{K})$ for some topological group G.

If it helps, we may reformulate the equivalence Theorem 3.74 into a duality theorem which, while not genuinely revealing new information, is a general backdrop of what has been generally accepted as "Tannaka Duality". However, it does not speak of *compact* groups yet.

Fact 3.79. *The category of* $\mathbb{K}$*-linearizable topological groups is dual to the category of weakly reduced symmetric Hopf algebras over* $\mathbb{K}$. □

A Principal Structure Theorem of $\mathbb{K}[G]$ for Compact G

In this subsection we finally assume that G is a *compact* group and assume, according to Corollary 3.62, that η_G is an inclusion map; that is,

for the remainder of this chapter we write

$$(\#) \qquad G \subseteq \mathbb{K}[G].$$

As in the Fine Structure Theorem for $R(G, \mathbb{K})$ 3.28, we let $\widehat{\mathbb{G}}_{\mathbb{K}}$ denote the set of equivalence classes of simple G-modules over $\mathbb{K}$. Since we do not consider the ground field $\mathbb{K}$ fixed on the case $\mathbb{K} = \mathbb{C}$ we choose the notation $\widehat{G}_{\mathbb{K}}$ in place of $\widehat{G}$ in order to indicate the dependence on $\mathbb{K}$.

For each element $\varepsilon \in \widehat{G}_{\mathbb{K}}$, we select a finite dimensional G-module $E_{\varepsilon,\mathbb{K}}$ from the class ε. If $\varepsilon \in \widehat{G}_K$, then the ring $\mathbb{L}_{\varepsilon,\mathbb{K}} = \mathrm{End}_G(E_{\varepsilon,\mathbb{K}})$ of all $\mathbb{K}$-linear endomorphisms of $E_{\varepsilon,\mathbb{K}}$ which commute with the G-action is, by Schur's Lemma 2.30, a finite dimensional division ring over $\mathbb{K}$. Hence

$$\begin{array}{lcll} \mathbb{L}_{\varepsilon,\mathbb{K}} & = & \mathbb{C} & \text{if} \quad \mathbb{K} = \mathbb{C}, \\ \mathbb{L}_{\varepsilon,\mathbb{K}} & \in & \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{array}$$

We view $E_{\varepsilon,\mathbb{K}}$ as a right module over $\mathbb{L}_{\varepsilon,\mathbb{K}}$. We denote the corresponding representation by $\rho_{\varepsilon,\mathbb{K}}: G \to \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}}) \subseteq \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$. Before we enter the presentation of the principal theorem on the weakly complete group algebra $\mathbb{K}[G]$ of a compact group we elaborate on some basic ideas of the Parts 1 and 2 of this chapter. The first lemma extends the details of Proposition 3.21 and the comments which precede Proposition 3.21 and follow Exercise E3.9.

Lemma 3.80. *Let E be a finite dimensional vector space over $\mathbb{K}$ and $\rho: G \longrightarrow \mathrm{End}_{\mathbb{K}}(E)$ the representation of the simple G-module E. Let A denote the $\mathbb{K}$-span of the set $\{\rho(g) \mid g \in G\}$. Then $A = \mathrm{End}_{\mathbb{L}}(E)$, where $\mathbb{L} = \mathrm{End}_A(E) = \mathrm{End}_G(E)$, $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. In particular, E is an $\mathbb{L}$-vector space such that* $\dim_{\mathbb{K}} E = \dim_{\mathbb{K}} \mathbb{L} \cdot \dim_{\mathbb{L}} E$*, and A is isomorphic to a full $m \times m$ matrix ring over $\mathbb{L}$ with $m = \dim_{\mathbb{L}} E$.*

Proof. First of all we note that A is a $\mathbb{K}$-algebra containing id_E. Hence every A-submodule of the additive group E is a G-invariant linear subspace, and vice versa. Therefore E is a simple A-module and so Jacobson's Density Theorem applies, which, for the sake of completeness, we cite here in its entirety (see e.g. [75]).

Jacobson's Density Theorem. *Let $M \neq 0$ be a vector space, let $A \subseteq \mathrm{End}(M)$ be a subring of* $\mathrm{End}(M)$*, and assume that M is simple as a left A-module. Put*

$\mathbb{L} = \mathrm{End}_A(M)$. *Then* $\mathbb{L}$ *is a division ring and* M *is a right* $\mathbb{L}$*-module in a natural way.*

Moreover, for every $2k$*-tuple* $(x_1, \ldots, x_k, y_1, \ldots, y_k) \in M^{2k}$*, such that the elements* $x_1, \ldots, x_k$ *are linearly independent, there exists an element* $a \in A$ *such that* $a(x_i) = y_i$ *holds for all* $i = 1, \ldots k$.

Now the division ring $\mathbb{L}$ is a finite dimensional $\mathbb{K}$-algebra over $\mathbb{K}$, and hence is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Moreover, $A = \mathrm{End}_{\mathbb{L}}(E)$. □

Specifically, let $x_1, \ldots, x_m$ be an $\mathbb{L}$-basis for E, and let $\varphi \in \mathrm{End}_{\mathbb{L}}(E)$ be arbitrary. Then there exists an element $a \in A$ such that $a(x_i) = \varphi(x_i)$ holds for all $i = 1, \ldots, m$. Therefore $\varphi = a$.

The following result now extends Lemma 3.14.

Lemma 3.81a. *Let* E *and* F *be finite dimensional vector spaces over* $\mathbb{K}$ *and assume that* $\rho\colon G \to \mathrm{End}_{\mathbb{K}}(E)$ *and* $\sigma\colon H \to \mathrm{End}_{\mathbb{K}}(F)$ *are irreducible representations of groups* G *and* H*. Assume also that* $\mathrm{End}_G(E) = \mathbb{L} = \mathrm{End}_H(F)$*. Then* $\mathrm{Hom}_{\mathbb{L}}(F, E)$ *is a simple* $G \times H$*-module over* $\mathbb{K}$*, where* $(g, h)(f) = \rho(g) \circ f \circ \sigma(h^{-1})$.

Proof. We define $A \subseteq \mathrm{End}_{\mathbb{K}}(E)$ and $B \subseteq \mathrm{End}_{\mathbb{K}}(F)$ as in Lemma 3.80. Then $A = \mathrm{End}_{\mathbb{L}}(E)$ and $B = \mathrm{End}_{\mathbb{L}}(F)$. The $\mathbb{K}$-vector space $\mathrm{Hom}(F, E)$ is in a natural way a left A-module and a right B-module. For every nonzero $f \in \mathrm{Hom}_{\mathbb{L}}(F, E)$ we have $AfB = \mathrm{Hom}_{\mathbb{L}}(F, E)$. Therefore $\mathrm{Hom}_{\mathbb{L}}(F, E)$ is simple as a left $G \times H$-module over $\mathbb{K}$. □

We are now ready to prove a principal structure theorem for the weakly complete group algebra $\mathbb{K}[G]$ of a compact group G for either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For each $\varepsilon \in \widehat{G}_{\mathbb{K}}$ we have the G-module $E_{\varepsilon,\mathbb{K}}$ and the corresponding irreducible representation $\rho_{\varepsilon,\mathbb{K}}\colon G \to \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$ into the group of units of the concrete *matrix ring* $M_\varepsilon \overset{\mathrm{def}}{=} \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}))$ over $\mathbb{L} = \mathrm{End}_G(E_{\varepsilon,\mathbb{K}})$ of $\mathbb{L}$-dimension $(\dim_{\mathbb{L}} E_{\varepsilon,\mathbb{K}})^2$. In the spirit of Proposition 3.20, we specify the following multiplicity lemma for M_ε whose proof was implicit in Lemma 3.80.

Lemma 3.81b. *The* one-sided *left* G*-module* $M_\varepsilon = \mathrm{Hom}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}, E_{\varepsilon,\mathbb{K}})$ *is a direct sum of* $\dim_{\mathbb{L}} E_{\varepsilon,\mathbb{K}} = \frac{\dim_{\mathbb{K}} E_{\varepsilon,\mathbb{K}}}{\dim_{\mathbb{R}} \mathbb{L}}$ *copies of the simple* G*-module* $E_{\varepsilon,\mathbb{K}}$. □

There is a unique function $\rho_G\colon G \to \prod_{\varepsilon \in \widehat{G}_{\mathbb{K}}} M_\varepsilon$, which is an injective group morphism into the multiplicative group of units of the product defined by the universal property of the product such that

$$\begin{array}{ccc} G & \xrightarrow{\ \rho_G\ } & \prod_{\varepsilon \in \widehat{G}_{\mathbb{K}}} M_\varepsilon \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_\chi} \\ G & \xrightarrow[\ \rho_{\chi,\mathbb{K}}\ ]{} & M_\chi \end{array}$$

commutes for all $\chi \in \widehat{G}_{\mathbb{K}}$.

The Algebra Structure of $\mathbb{K}[G]$

Theorem 3.82. *For any compact group G, the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is a direct product*

$$\mathbb{K}[G] = \prod_{\varepsilon \in \widehat{G}_{\mathbb{K}}} \mathbb{K}_\varepsilon[G]$$

of finite dimensional minimal two-sided ideals $\mathbb{K}_\varepsilon[G]$ such that for each $\varepsilon \in \widehat{G}_{\mathbb{K}}$ there is a $\mathbb{K}$-algebra isomorphism

$$\mathbb{K}_\varepsilon(G) \cong M_\varepsilon = \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}), \quad \mathbb{L} = \mathrm{End}_G(E_{\varepsilon,\mathbb{K}})$$

In particular, each of these two-sided ideals $\mathbb{K}_\varepsilon(G)$ is a two sided simple $G \times G$-module and as an algebra is isomorphic to a full matrix ring over $\mathbb{L}$.

Remark. The diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad \eta_G \quad} & \mathbb{K}[G] \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong} \\ G & \xrightarrow[\quad \rho_G \quad]{} & \prod_{\varepsilon\in\widehat{G}_{\mathbb{K}}} M_\varepsilon \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_\chi} \\ G & \xrightarrow[\quad \rho_{\chi,\mathbb{K}} \quad]{} & M_\chi \end{array}$$

commutes for all $\chi \in \widehat{G}_{\mathbb{K}}$.

Proof. By Theorem 3.76 and Theorem 3.28, the topological dual $\mathbb{K}[G]' \cong R(G,\mathbb{K})$ is the direct sum of the finite dimensional two-sided G-submodules $R_\varepsilon(G,\mathbb{K})$ as ε ranges through the set of isomorphy classes of simple G-modules over $\mathbb{K}$ in $\widehat{G}_{\mathbb{K}}$. The $G \times G$-module $R_\varepsilon(G,\mathbb{K})$ is defined as the image of the linear map

$$\varphi\colon E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} \longrightarrow R(G,\mathbb{K}),$$

where

$$\varphi(u \otimes v)(g) = \langle u, \rho_{\varepsilon,\mathbb{K}}(g)v\rangle.$$

If we put $\psi(f)(g) = \mathrm{tr}_{\mathbb{K}}(\rho_{\varepsilon,\mathbb{K}}(g)f)$, for $f \in \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$ and $g \in G$, then the diagram

$$\begin{array}{ccc} E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} & \xrightarrow{\quad \varphi \quad} & R_\varepsilon(G,\mathbb{K}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) & \xrightarrow{\quad \psi \quad} & R_\varepsilon(G,\mathbb{K}) \end{array}$$

commutes, where $s(u \otimes v) = [w \mapsto v\langle u, w\rangle]$. We recall that the group $G \times G$ acts on $R_\varepsilon(G,\mathbb{K})$ via $(a,b)(\lambda) = [g \mapsto \lambda(a^{-1}gb)]$. If we put $(a,b)(u \otimes v) = (u \circ \rho_{\varepsilon,\mathbb{K}}(a^{-1})) \otimes \rho_{\varepsilon,\mathbb{K}}(b)v$ and $(a,b)(f) = \rho_{\varepsilon,\mathbb{K}}(b) \circ f \circ \rho_{\varepsilon,\mathbb{K}}(a^{-1})$, then all maps in this diagram are $G \times G$-equivariant.

Assume that $\mathbb{K} = \mathbb{L}$. Then $\mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) = \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})$ is simple as a $G \times G$-module by Lemma 3.81a and thus ψ is an isomorphism.

Assume next that $\mathbb{K} \subsetneq \mathbb{L}$. Then $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ or $\mathbb{L} = \mathbb{H}$. By Weyl's Trick (see Theorem 2.10ff.) there exists a G-invariant positive definite $\mathbb{L}$-hermitian form $(\cdot|\cdot)$ on E, semi-linear in the first argument and linear in the second argument. This allows us to rewrite $R_\varepsilon(G,\mathbb{K})$ as the set of maps $g \mapsto \mathrm{Re}(u|gv)$, for $u, v \in E$. The G-invariance of $(\cdot|\cdot)$ yields that $\mathrm{Re}(au|gbv) = \mathrm{Re}(u|a^{-1}gbv)$. If we consider the algebra inclusion

$$j\colon \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}) \to \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}),$$

then $\mathrm{Re}(u|v) = \mathrm{tr}_{\mathbb{K}}[w \mapsto v(u|w)]$ holds for the trace map of $\mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$. It follows that the map $\psi \circ j$ in the diagram

$$\begin{array}{ccc}
E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} & \xrightarrow{\varphi} & R_\varepsilon(G,\mathbb{K}) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) & \xrightarrow{\psi} & R_\varepsilon(G,\mathbb{K}) \\
{\scriptstyle j}\uparrow & & \\
\mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}) & &
\end{array}$$

is surjective and $G \times G$-equivariant. Since $\mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})$ is a simple $G \times G$-module over $\mathbb{K}$ by Lemma 3.81a, the map $\psi \circ j$ is an isomorphism.

For the remaining part of the proof we apply standard duality theory (see e.g. A7.11). We put

$$R^\chi = \bigoplus_{\chi \neq \varepsilon \in \widehat{G}_{\mathbb{K}}} R_\varepsilon(G,\mathbb{K})$$

and define $\mathbb{K}_\chi[G]$ as the annihilator of R^χ. The annihilator mechanism supplies us with the diagram

$$\begin{array}{ccc}
\mathbb{K}[G] & \overset{\perp}{\longleftrightarrow} & \{0\} \\
| & & | \\
\mathbb{K}_\chi[G] & \overset{\perp}{\longleftrightarrow} & R^\chi \\
| & & | \\
\{0\} & \overset{\perp}{\longleftrightarrow} & R(G,\mathbb{K}).
\end{array}
\left.\vphantom{\begin{array}{c} a\\ a\\ a\end{array}}\right\} \cong R_\chi(G,\mathbb{K})$$

By the duality of $\mathcal{V}_{\mathbb{K}}$ and $\mathcal{W}_{\mathbb{K}}$ it follows that $\mathbb{K}[G] \cong \prod_{\varepsilon \in \widehat{G}_K} \mathbb{K}_\varepsilon[G]$ with

$$\mathbb{K}_\varepsilon[G] \cong R_\varepsilon(G,\mathbb{K})'.$$

Now, if any closed vector subspace J of $\mathbb{K}[G]$ satisfies $G{\cdot}J \subseteq J$ and $J{\cdot}G \subseteq J$, then we also have $\mathrm{span}(G){\cdot}J \subseteq J$ and $J{\cdot}\,\overline{\mathrm{span}(G)} \subseteq J$ (where we view G as a subset of $\mathbb{K}[G]$). Then Corollary 3.61 says that $\overline{\mathrm{span}(G)} = \mathbb{K}[G]$, and so $\mathbb{K}[G]{\cdot}J \subseteq J$ and $J{\cdot}\mathbb{K}[G] \subseteq J$. That is, J is a closed two-sided ideal of $\mathbb{K}[G]$. Therefore each $\mathbb{K}_\varepsilon[G]$ is a two-sided ideal in $\mathbb{K}[G]$.

It remains to clarify the multiplicative structure of the ideals $\mathbb{K}_\varepsilon[G]$. If we consider $\varepsilon \in \widehat{G}_{\mathbb{K}}$ and the representation $\rho_{\varepsilon,\mathbb{K}}$, then the map

$$G \xrightarrow{\rho_{\varepsilon,\mathbb{K}}} \mathrm{Gl}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}) = \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})^{-1} \xrightarrow{\mathrm{inc}} \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}),\ \mathbb{L} = \mathrm{End}_G(E_{\varepsilon,\mathbb{K}})$$

and the universal property of $\mathbb{K}[G]$ described in the Weakly Complete Group Algebra Theorem 3.60 provides a morphism of weakly complete algebras

$$\pi_\varepsilon: \mathbb{K}[G] \to \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})$$

extending $\rho_{\varepsilon,\mathbb{K}}$. We also have the product projection of weakly complete algebras $\mathrm{pr}_\varepsilon: \mathbb{K}[G] \to \mathbb{K}_\varepsilon[G]$. Both maps π_ε and pr_ε have the same kernel $\prod_{\varepsilon\neq\varepsilon'\in\widehat{G}_{\mathbb{K}}} \mathbb{K}_{\varepsilon'}[G]$. So there is an injective morphism

$$\alpha: \mathbb{K}_\varepsilon[G] \to \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})$$

such that $\pi_\varepsilon = \alpha \circ \mathrm{pr}_\varepsilon$. Since both algebras have the same dimension, α is an isomorphism of $\mathbb{K}$-algebras. □

Corollary 3.83. *There is an isomorphism of $G \times G$-modules*

$$R(G,\mathbb{K}) = \bigoplus_{\varepsilon\in\widehat{G}_{\mathbb{K}}} \mathrm{End}_{\mathbb{L}_\varepsilon}(E_{\varepsilon,\mathbb{K}}), \quad \mathbb{L}_\varepsilon = \mathrm{End}_G(E_{\varepsilon,\mathbb{K}}).$$ □

Thus the multiplicity m of $E_{\varepsilon,\mathbb{K}}$ as a G-module in $R(G,\mathbb{K})$ is

$$m = \dim_{\mathbb{L}_\varepsilon}(E_{\varepsilon,\mathbb{K}}) = \frac{\dim_{\mathbb{K}} E_{\varepsilon,\mathbb{K}}}{\dim_{\mathbb{K}} \mathbb{L}_\varepsilon}.$$

While this conclusion is well-known for $\mathbb{K} = \mathbb{C}$ (see e.g Theorems 3.22 and 3.28), a reference for the case $\mathbb{K} = \mathbb{R}$ is not easily found.

The algebra structure of the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is satisfactorily elucidated in Theorem 3.82, the comultiplication is not easily accessible due to the complications on the way to a representation theory of $G \times G$ in general, even if that of G is known as in Theorem 3.82. In the case of commutative compact groups G and the complex ground field $\mathbb{C}$ these complications go away, and so we shall clarify the situation in these circumstances in a subsequent subsection.

The crux of further information is the following argument. For each topological group G, our Theorem 3.76 provided an isomorphism of symmetric $\mathbb{K}$-Hopf algebras

$$F_G: \mathbb{K}[G]' \to R(G,\mathbb{K}).$$

So each member $f: G \to \mathbb{K}$ of $R(G,\mathbb{K})$ arises uniquely as a function $g \mapsto \langle \omega_f, \eta(g)\rangle$ for some linear form $\omega_f \in \mathbb{K}[G]'$. By the duality of $\mathcal{V}_{\mathbb{K}}$ and $\mathcal{W}_{\mathbb{K}}$ in Appendix 7 we may identify $\mathbb{K}[G]'^*$ and $\mathbb{K}[G]$ and obtain an isomorphism

$$F_G^*: R(G,\mathbb{K})^* \to \mathbb{K}[G].$$

Thus every linear form μ on $R(G,\mathbb{K})$ determines uniquely an element $F_G^*(\mu) \in \mathbb{K}[G]$ such that

$$\big\langle \mu, (g \mapsto \langle \omega_f, \eta_g\rangle)\big\rangle = \langle \omega_f, F_G^*(\mu)\rangle.$$

If μ happens to be of the special type that there is an element $g_\mu \in G$ such that $\langle \mu, f\rangle = f(g_\mu)$ for all $f \in R(G,\mathbb{K})$, that is, for all $f = (g \mapsto \langle \omega_f, \eta(g)\rangle)$, then we conclude that $\langle \omega, F_G^*(\mu)\rangle = \langle \omega, \eta(g_\mu)\rangle$ for all $\omega \in \mathbb{K}[G]$ and that implies $F_G^*(\mu) = \eta(g_\mu)$. As a summary of this argument, for each $g \in G$ we let $\mathrm{ev}_g\colon R(G,\mathbb{K}) \to \mathbb{K}$ be the point evaluation at g defined by $\mathrm{ev}_g(f) = f(g)$.

Lemma 3.84. *The isomorphism $F_G^*\colon R(G,\mathbb{K})^* \to \mathbb{K}[G]$ maps the point evaluation ev_g to $\eta_G(g) \in \mathbb{G}\big(\mathbb{K}[G]\big)$ for each $g \in G$, that is, $F_G^*(ev_g) = \eta_G(g)$.* □

If we write $\mathrm{ev}_g = \mathrm{ev}^G(g)$, then we may formulate this lemma as the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\eta_G\quad} & \mathbb{K}[G]^{-1} \\ {\scriptstyle \mathrm{ev}^G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inc}} \\ R(G,\mathbb{K})^* & \xrightarrow[F_G^*]{\qquad} & \mathbb{K}[G]. \end{array}$$

If A is a unital $\mathbb{K}$-algebra, then the set of morphisms $f\colon A \to \mathbb{K}$ of $\mathbb{K}$-algebras is called the *spectrum* of A, denoted $\mathrm{Spec}(A)$. If A^* denotes the algebraic dual of the underlying vector space of A, we have $\mathrm{Spec}(A) \subseteq A^*$, and the subset is closed in the topology of pointwise conergence on A^*, i.e., $\mathrm{Spec}(A)$ is a closed subset of the weakly complete vector space A^*. Occasionally, the elements of A^* are called the *characters* of A, a notation which we seek to avoid in the context of this book, since we us the word "character" in another context.

Now we recall from Proposition A3.99 in Appendix 3 that the isomorphism of weakly complete symmetric Hopf algebras F_G^* maps the set $\mathrm{Spec}(R(G,\mathbb{K})) \subseteq R(G,\mathbb{K})^*$ homeomorphically onto the set $\mathbb{G}(\mathbb{K}[G])$ of grouplike elements of $\mathbb{K}[G]$.

From Lemma 3.65 we recall $\eta_G(G) \subseteq \mathbb{G}(\mathbb{K}[G])$.

We summarize our arguments of the position of the sets $\eta(G) \subseteq \mathbb{G}\big(\mathbb{K}[G]\big) \subseteq \mathbb{K}[G]$ as follows:

Theorem 3.85. *For an arbitrary topological group G and its weakly complete group Hopf algebra $\mathbb{K}[G]$, the isomorphism $F_G^*\colon R(G,\mathbb{K})^* \to \mathbb{K}[G]$ of weakly complete Hopf algebras maps the set*

ev_G *of point evaluations* $\mathrm{ev}_g = (f \mapsto f(g)) : R(G,\mathbb{K}) \to \mathbb{K}$

bijectively onto the image $\eta(G)$ of G in $\mathbb{K}[G]$, and it maps the spectrum $\mathrm{Spec}(R(G,\mathbb{K}))$ bijectively onto the pro-Lie group $\mathbb{G}\big(\mathbb{K}[G]\big)$ of grouplike elements in $\mathbb{K}[G]$. In terms of a diagram:

$$\begin{array}{ccc} R(G,\mathbb{K})^* & \xrightarrow{\quad F_G^*\quad} & \mathbb{K}[G] \\ \Big| & & \Big| \\ \mathrm{Spec}\,\big(R(G,\mathbb{K})\big) & \xrightarrow{\quad\cong\quad} & \mathbb{G}\big(\mathbb{K}[G]\big) \\ \Big| & & \Big| \\ \mathrm{ev}^G(G) & \xrightarrow{\quad\cong\quad} & \eta_G(G). \end{array}$$ □

With the emergence of the spectrum we extended the scope of Part 3 of this Chapter from category theory to functional analysis.

The Spectrum of the $\mathbb{K}$-Algebra $R(G, \mathbb{K})$

At this point we address the claim, yet to be proved, that a compact group G is $\mathbb{R}$-linearizable. Recall that for compact G we write $G \subseteq \mathbb{G}\big(\mathbb{K}[G]\big) \subseteq \mathbb{K}[G]$. We consider $R(G,\mathbb{K})$ as an involutive unital $\mathbb{K}$-algebra, with complex conjugation as involution in the case of $\mathbb{K} = \mathbb{C}$ and as the identity as involution in the case of $\mathbb{K} = \mathbb{R}$. A *morphism* μ *of involutive algebras* is one preserving the involution, i.e., satisfying $\mu(f^*) = \mu(f)^*$. The set of morphisms of involutive algebras $\mu\colon R(G,\mathbb{K}) \to \mathbb{K}$ will be called $\mathrm{Spec}_*(R(G,\mathbb{K}))$. Note that it is a subset of $R(G,\mathbb{K})^*$.

Lemma 3.86. (The Point Evaluation Lemma) *For a compact group* G*, every element of* $\mathrm{Spec}_*(R(G,\mathbb{C}))$ *is a point evaluation.*

We present the proof through two lemmas in which we assume G to be compact and abbreviate the commutative involutive algebra $R(G,\mathbb{C})$ by A.

The first lemma explicitly uses the preservation of multiplication by elements of $\mathrm{Spec}(A)$. Its proof is based on the elementary theory of a finite dimensional Hilbert space over $\mathbb{K}$.

Lemma A. *For each* $f \in A$ *there is a nonnegative number* k_f *such that*

$$(\forall \mu \in \mathrm{Spec}_*(A))\, |\mu(f)| \le k_f.$$

Proof. For each $f \in R(G,\mathbb{C})$ there is a finite dimensional G-submodule E_f of $R(G,\mathbb{C})$ containing f and on which G acts unitarily via $(g\cdot f)(x) = f(xg)$. Then E_f is a finite dimensional Hilbert space with respect to the inner product $(f_1|f_2) = \int_{g\in G} f_1(g)\overline{f_2(g)}dm(g) \le \|f_1\|\cdot\|f_2\|$ with Haar measure m on G (see Appendix 5). Let $e_1, \dots, e_n$ be an orthonormal basis of E_f so that for some n-tuple of functions $F_m \in R(G,\mathbb{C})$ we have

$$(\forall x, g \in G), f(x\cdot g) = \sum_{m=1}^{n} F_m(g)e_m(x).$$

We observe $f(g) = \sum_{m=1}^{n} F_m(g)e_m(1)$ and

$$(\forall g \in G)\, (f|f) = (g\cdot f|g\cdot f) = \sum_{m=1}^{n} F_m(g)\overline{F_m(g)} = \Big(\sum_{m=1}^{n} F_m\overline{F}_m\Big)(g).$$

Since μ is a morphism of involutive complex algebras we conclude

$$(f|f) = \textstyle\sum_{m=1}^{n} \mu(F_m)\overline{\mu(F_m)},$$

and

$$|\mu(f)|^2 = \mu(f\overline{f}) = \mu\Big(\sum_{k,m=1}^{n} F_k\overline{F_m}e_k(1)e_m(1)\Big) = \sum_{k,m=1}^{n} \mu(F_k)\overline{\mu(F_m)}e_k(1)e_m(1).$$

Now let T_f be the hermitian operator on $\mathbb{C}^N$ with coefficient matrix

$$(e_k(1)e_m(1))_{k,m=1,\dots,n}$$

and $u \in \mathbb{C}^n$ the vector $(\mu(F_1), \dots, \mu(F_n))$. If we write

$$[(v_1,\dots,v_n)|(w_1,\dots,w_n)] = \textstyle\sum_{m=1}^n v_m w_m,$$

then we have

$$[(u|u)] = (f|f) \text{ and } [u|T_f u] = |\mu(f)|^2,$$

and therefore

$$|\mu(f)| \le k_f \stackrel{\text{def}}{=} \|T_f\|^{1/2}\cdot(f|f)^{1/2}. \qquad \square$$

As a consequence we derive

Lemma B. $\mathrm{Spec}_*(A)$ *is compact in the topology of pointwise convergence, and* $g \mapsto \mathrm{ev}_g\colon G \to \mathrm{Spec}_*(A)$, $ev_g(f) = f(g)$, *is a homeomorphic embedding.*

Proof. For $f \in A$ we let D_f be the compact complex disk of radius k_f. Then for all $\mu \in \mathrm{Spec}_*(A) \subseteq \mathbb{K}^A$ we have

$$\mu \in \prod_{f\in A} D_f \subseteq \mathbb{K}^A$$

by Lemma A. Since the set $\mathrm{Spec}_*(A)$ is closed in $\mathbb{K}^A$ in the topology of pointwise convergence, it is a closed subspace of the compact space $\prod_{f\in A} D_f$ and is, therefore compact. For each $g \in G$, clearly $\mathrm{ev}_g \in \mathrm{Spec}_*(A)$, and $g \mapsto \mathrm{ev}_g$ is clearly a continuous injection. Thus, by compactness of G, it is a homeomorphic embedding. $\square$

We can now complete the proof of the Point Evaluation Lemma 3.86, which asserts the surjectivity of $g \mapsto \mathrm{ev}_g : G \to X$, $X \stackrel{\text{def}}{=} \mathrm{Spec}_*(A)$. We claim that the dual surjection $C(X) \to C(G)$ is injective which will prove our claim. Let us suppose for the moment that $G \subseteq X$ (which we may) and that $G \ne X$. Then $C(X, \mathbb{K})$ would contain the dense unital involutive subalgebra A by the Stone Weierstraß Theorem which, when restricted to G, would yield an algebra $A|G$ isomorphic to A, which is impossible as G is a proper subspace of X.

In the case $\mathbb{K} = \mathbb{R}$, any morphism $\mu\colon R(G,\mathbb{R}) \to \mathbb{R}$ of real algebras, extends uniquely to a (complex) morphism

$$\widetilde{\mu}\colon \mathbb{C}\otimes_{\mathbb{R}} R(G,\mathbb{R}) \to \mathbb{C}, \text{ where } \mathbb{C}\otimes_{\mathbb{R}} R(G,\mathbb{R}) \cong R(G,\mathbb{C}),$$

such that $\overline{\widetilde{\mu}} = \widetilde{\overline{\mu}}$. Trivially, μ is a point evaluation of $R(G,\mathbb{R})$, if and only if $\widetilde{\mu}$ is a point evaluation of $R(G,\mathbb{C})$, and μ is continuous if and only if $\widetilde{\mu}$ is continuous. Hence the Point Evaluation Lemma 3.86 implies the analogous result over $\mathbb{R}$:

Corollary 3.87. *For a compact group G, the set* $\mathrm{Spec}(R(G,\mathbb{R}))$ *of algebra morphisms of* $R(G,\mathbb{R})$ *is precisely the set of point evaluations* $f \mapsto f(g)$, $g \in G$. $\square$

Recall that $R(G,\mathbb{R})^*$ and $\mathbb{R}[G]$ are natually isomorphic by the duality of $\mathcal{V}_{\mathbb{R}}$ and $\mathcal{W}_R$ (according to Appendix 7), since $\mathbb{R}[G]'$ and $R(G,\mathbb{R})$ are naturally isomorphic by Theorem 3.76.

After Corollary 3.62 we may identify a compact group G with its isomorphic image via η_G in $\mathbb{G}(\mathbb{R}[G])$, and now the preceding Corollary 3.87 and Theorem 3.68

allow us to conclude the following theorem, for which we recall that in Definition 3.73(i) we call a topological group $\mathbb{R}$-*linearizable* if $\eta_G: G \to \mathbb{G}(\mathbb{R}[G])$ is an isomorphism of topological groups. Also recall that a weakly complete symmetric real Hopf algebra A is *group-determined* if $\varepsilon_A: \mathbb{R}[\mathbb{G}(A)] \to A$ is an isomorphism.

R-Linearizability of Compact Groups

Theorem 3.88.
(i) *Every compact group is* $\mathbb{R}$-*linearizable.*
(ii) *The weakly complete symmetric real group-Hopf algebra* $\mathbb{R}[G]$ *of a compact group is group-determined.* □

It appears that a direct proof of the assertion that in $\mathbb{R}[G]$, for a compact group G, every grouplike element is a group element is nontrivial in general. If G happens to be finite then the proof, of course, is elementary linear algebra.

At this point, one piece of information seems to be open: Let A be a weakly complete symmetric real Hopf algebra A in which the group $\mathbb{G}(A)$ of grouplike elements is compact and algebraically and topologically generates A. Will the natural morphism $\varepsilon_A: \mathbf{H}\big(\mathbb{G}(A)\big) = \mathbb{R}[\mathbb{G}(A)] \to A$ be in fact an isomorphism? In other words:

If $\mathbb{G}(A)$ *is compact and* A *is group-saturated, is* A *group-determined?*

For the investigation of this question we need some elementary preparation. Assume that G is a compact group and $R(G,\mathbb{R}) \subseteq C(G,\mathbb{R})$ is the real symmetric Hopf algebra of all representative functions $f \in C(G,\mathbb{R})$ as usual (see Definition 3.3). We now let M be a subalgebra and a G-submodule of $R(G,\mathbb{R})$. Recall that $\operatorname{Spec} M$ denotes the set of all algebra morphisms $M \to \mathbb{R}$. From Definition 1.20 we recall that $M \subseteq C(G,\mathbb{R})$ is said to *separate points* if for two points $g_1 \neq g_2$ in G there is an $f \in M$ such that $f(g_1) \neq f(g_2)$. In other words, different points in G can be distinguished by different point evaluations of some function from M. The next lemma is just revisiting the Stone-Weierstraß Theorem: cf. Theorem 3.7 and its proof.

Lemma α**.** *If a unital subalgebra* M *of* $R(G,\mathbb{R})$ *separates points, then* M *is dense in* $C(G,\mathbb{R})$ *with respect to the sup norm and is dense in* $L^2(G,\mathbb{R})$ *with respect to the* L^2*-norm.*

Proof. Since M is a unital subalgebra of $\mathbb{R}(G,\mathbb{R})$, it contains the scalar multiples of the constant functions of value 1, that is, M contains all the constant functions. Moreover, the algebra $M \subseteq C(G,\mathbb{R})$ separates the points of G. Therefore the Stone-Weierstraß Theorem applies and shows that M is dense in $C(G,\mathbb{R})$ in the sup norm topology of $C(G,\mathbb{R})$. Since $L^2(G,\mathbb{R})$ is the L^2-norm completion of $C(G,\mathbb{R})$ and M is uniformly d ense in $C(G,R)$ it follows that M is dense in $L^2(G,\mathbb{R})$ in the L^2-norm. □

Lemma β**.** *If a G-submodule M of $R(G,E)$ is L^2-dense in $R(G,\mathbb{R})$ then it agrees with $R(G,\mathbb{R})$.*

Proof. Let $\widehat{G}_{\mathbb{R}}$ denote the set of isomorphy classes of irreducible real G-modules. By the Fine Structure Theorem of $R(G,\mathbb{R})$ (see Theorem 3.28)

$$R(G,R) = \sum_{\varepsilon\in\widehat{G}_{\mathbb{R}}} R(G,\mathbb{R})_\varepsilon,$$

where $\sum$ denotes the algebraic direct sum of (finite dimensional) vector subspaces and where $R(G,\mathbb{R})_\varepsilon$ is a finite direct sum of simple modules for each $\varepsilon \in \widehat{G}_{\mathbb{R}}$. (Cf. Theorem 3.82.) In particular, each $R(G,\mathbb{R})_\varepsilon$ is finite dimensional. Further $L^2(G,\mathbb{R}) = \bigoplus_{\varepsilon\in\widehat{G}} R(G,\mathbb{R})_\varepsilon$ where $\bigoplus$ denotes the Hilbert space direct sum.

The submodule M of $R(G,\mathbb{R})$ adjusts to the canonical decomposition of $R(G,\mathbb{R})$ since M_ε is necessarily a submodule of $R(G,\mathbb{R})_\varepsilon$. Hence $M = \sum_{\varepsilon\in\widehat{G}} M_\varepsilon$ and the L_2-closure of M in $L^2(G,\mathbb{R})$ is the Hilbert space sum $\bigoplus_{\varepsilon\in\widehat{G}} M_\varepsilon$. By hypothesis, this closure agrees with $L^2(G,\mathbb{R})$. This implies $M_\varepsilon = R_\varepsilon$ for all $\varepsilon \in \widehat{G}$. Hence $M = \sum_{\varepsilon\in\widehat{G}} M_\varepsilon = \sum_{\varepsilon\in\widehat{G}} R_\varepsilon = R(G,\mathbb{R})$. □

Now we are ready for another main result on compact groups in the present context of weakly complete group algebras.

Theorem 3.89. *Let A be a weakly complete real symmetric Hopf algebra satisfying the following two conditions:*

(i) *The subgroup $\mathbb{G}(A)$ of grouplike elements of A is compact,*

(ii) *$\mathbb{G}(A)$ generates A algebraically and topologically, that is, $\mathbf{S}(A) = A$.*

Then $\varepsilon_A\colon \mathbb{R}[\mathbb{G}(A)] = \mathbf{H}\big(\mathbb{G}(A)\big) \to A$ is a natural isomorphism.

Proof. We set $G = \mathbb{G}(A)$. By Remark 3.71, the morphism $\varepsilon_A\colon \mathbb{R}[G] \to A$ is a quotient homomorphism of weakly complete Hopf algebras. Now $\eta_G\colon G \to \mathbb{G}(\mathbb{R}[G])$ is an isomorphism by 3.88(i). Hence ε_A induces an isomorphism $\mathbb{G}(\varepsilon_A)\colon \mathbb{G}(\mathbb{R}[G]) \to G$ by Corollary 3.72. So $\mathbb{G}(\mathbb{R}[G])$ is identified with G if we consider G as included in $\mathbb{R}[G]$ as we agreed to to do in (#) at the beginning of this section on compact groups.

By the Duality between real Hopf algebras in $\mathcal{V}_{\mathbb{R}}$ and weakly complete real Hopf algebras in $\mathcal{W}_{\mathbb{R}}$, the dual morphism $\varepsilon'_A\colon A' \to \mathbb{R}[G]'$ is injective. Proposition 3.77 then gives us an inclusion $A' \subseteq R(G,\mathbb{R})$ of real Hopf algebras as well as of G-modules such that the natural map $\mathrm{Spec}(A') \to \mathrm{Spec}\,\big(R(G,\mathbb{R})\big)$ is the identity. Now applying Lemmas α and β above with $M = A'$ we conclude $A' = R(G,\mathbb{R})$ which, in turn, shows that ε_A is an isomorphism. □

The Tannaka-Hochschild Duality

For a concise formulation of the consequences let us use the following notation:

Definition 3.90. A real weakly complete symmetric Hopf algebra A will be called *compactlike* if its subgroup $\mathbb{G}(A)$ of grouplike elements is compact and $\mathbf{S}(A) = \overline{\operatorname{span}\big(\mathbb{G}(A)\big)} = A$. □

The Equivalence of Compact Groups and Compactlike Hopf algebras

Theorem 3.91. *The categories of compact groups and of weakly complete symmetric compactlike real Hopf algebras are equivalent.*

Proof. This now follows immediately from Theorems 3.88 and 3.89. □

Definition 3.92. A real symmetric Hopf algebra R over $\mathbb{K}$ is called *reduced* if its algebraic dual R^* is compactlike (see Definition 3.90). □

Since we chose our definitions in such a fashion that the categories of real symmetric Hopf algebras and of weakly complete real symmetric grouplike Hopf algebras are equivalent, we have as an immediate consequence of the Equivalence Theorem 3.91 the following celebrated result:

The Tannaka-Hochschild Duality Theorem

Theorem 3.93. *The category of compact groups is dual to the full category of real reduced Hopf algebras.* □

Whichever way one looks at various versions of this duality theorem, the concept of a **reduced** Hopf algebra is loaded with complexity. An application of this duality to a general structure theory of compact groups is therefore limited. This is quite in contrast with the Pontryagin–van Kampen Duality Theorem for compact abelian groups which we secured early on in Theorems 1.31 and 2.32. It is therefore not surprising, that in the case of compact abelian groups, our theory of weakly complete group algebras will be more attractive than the most general form which we have pursued up to this point.

Compact Abelian Groups

If G is a compact commutative group, some statements made for the general case become indeed simpler and more lucid. In the first part of this discussion we consider $\mathbb{K} = \mathbb{C}$. From Lemma 2.30 we know that there is a natural bijection from the character group $\widehat{G}$ according to Definition 1.15 onto the set $\widehat{G}_{\mathbb{C}}$ of isomorphy classes ε of irreducible G-modules E, all of which have complex dimension 1. This bijection associates with a character $\chi \in \widehat{G} = \operatorname{Hom}(G, \mathbb{T})$ the class of the module $E_\chi = \mathbb{C}$, $\chi{\cdot}c = e^{2\pi i\chi}c$. Accordingly, Theorem 3.28 (12) reads $R(G, \mathbb{C}) = \sum_{\chi\in\widehat{G}} \mathbb{C}{\cdot}f_\chi$, for a suitable basis f_χ, $\chi \in \widehat{G}$. In other words, as a G-module, $R(G, \mathbb{C}) \cong \mathbb{C}^{(\widehat{G})}$.

Accordingly, we expect $\mathbb{C}[G]$ to be uncomplicated. Our Theorem 3.82 makes this clear:

The complex algebra $\mathbb{C}[G]$ *may be naturally identified with the componentwise algebra* $\mathbb{C}^{\widehat{G}}$.

In the abelian case, our understanding of the comultiplication of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$ will be much more explicit than in the general situation of Theorem 3.82. Each character $\chi\colon G \to \mathbb{T}$ determines a morphism $f_\chi\colon G \to \mathbb{C}^{-1} = \mathbb{C}^\times$, $f_\chi(g) = e^{2\pi\chi(g)}$. By the universal property of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$, this value agrees with the χ-th projection of $\eta_G(g)$. Hence

$$(\forall g \in G, \chi \in \widehat{G})\, \eta_G(g)(\chi) = e^{2\pi i\langle \chi, g\rangle}.$$

Accordingly, if we write $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, then $\eta_G(g) \in \operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \cong \widehat{\widehat{G}} \cong G$. Then Corollary 3.61(ii) implies

$$\operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \subseteq \overline{\operatorname{span}_{\mathbb{R}}(\operatorname{Hom}(\widehat{G}, \mathbb{S}^1))} = \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^{\widehat{G}}.$$

Recall from Theorem 3.63 that we have an isomorphism $\alpha_G\colon \mathbb{C}[G \times G] \to \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G]$, and from Lemma 3.64 we recall the comultiplication $\gamma_G\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G]$ to be the composition

$$\mathbb{C}[G] \xrightarrow{\quad\mathbb{C}[\delta_G]\quad} \mathbb{C}[G \times G] \xrightarrow{\quad\alpha_G\quad} \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G].$$

Now, for a compact abelian group G, the diagonal morphism $\delta_G\colon G \to G \times G$ has the group operation of $\widehat{G}$ as its dual, namely:

$$\widehat{\delta_G}\colon \widehat{G} \times \widehat{G} \to \widehat{G}. \quad \widehat{\delta_G}(\chi_1, \chi_2) = \chi_1 + \chi_2,$$

as we write abelian group operations additively in general. If now we also write $\mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G] = \mathbb{C}^{\widehat{G} \times \widehat{G}}$ (identifying $\varphi \otimes \psi$ with $(\chi_1, \chi_2) \mapsto \varphi(\chi_1)\psi(\chi_2)$), then we have

$$\gamma_G = \mathbb{C}^{\widehat{\delta_G}}\colon \mathbb{C}^{\widehat{G}} \to \mathbb{C}^{\widehat{G} \times \widehat{G}}, \text{i. e., } (\forall \varphi \in \mathbb{C}^{\widehat{G}}), \gamma_G(\varphi)(\chi_1, \chi_2) = \varphi(\chi_1 + \chi_2).$$

This allows us to determine explicitly the elements of the group $\mathbb{G}(\mathbb{C}^{\widehat{G}})$ of all grouplike elements:

Indeed a nonzero element $\varphi \in \mathbb{C}^{\widehat{G}}$ is in $\mathbb{G}(\mathbb{C}^{\widehat{G}})$ iff

$$\gamma_G(\varphi) = \varphi \otimes \varphi \quad \text{in} \quad \mathbb{C}^{\widehat{G}} \otimes_{\mathcal{W}} \mathbb{C}^{\widehat{G}} = \mathbb{C}^{\widehat{G} \times \widehat{G}},$$

where $(\varphi \otimes \varphi)(\chi_1, \chi_2) = \varphi(\chi_1)\varphi(\chi_2)$. This is the case iff

$$(\forall \varphi_1, \varphi_2 \in \widehat{G})\, \varphi(\chi_1 + \chi_2) = \gamma_G(\varphi)(\chi_1, \chi_2) = (\varphi \otimes \varphi)(\chi_1, \chi_2) = \varphi(\chi_1)\varphi(\chi_2),$$

that is, iff φ is a morphism of groups from $\widehat{G}$ to $\mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \cdot)$.

Similarly, an element $\varphi \in \mathbb{C}^{\widehat{G}}$ is primitive iff

$$\varphi(\chi_1 + \chi_2) = \gamma_G(\varphi)(\chi_1, \chi_2)\big((\varphi \otimes 1) + 1 \otimes \varphi)\big)(\chi_1`, \chi_2) = \varphi(\chi_1) + \varphi(\chi_2)$$

iff $\varphi\colon \widehat{G} \to (\mathbb{C}, +)$ is a morphism of topological groups.

Let us summarize this discourse, recalling that $\mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[H]$ for compact groups G and H may be identified with $\mathbb{C}[G \times H]$:

The Weakly Complete Group Hopf Algebra $\mathbb{C}[G]$ for Compact Abelian G

Theorem 3.94. *Let G be a compact abelian group and $\widehat{G}$ its character group. Denote the weakly complete commutative symmetric group Hopf algebra $\mathbb{C}[G]$ by A. Then we have the following conclusions:*

(i) *A may be identified with $\mathbb{C}^{\widehat{G}}$ such that $\eta_G\colon G \to A^{-1}$ is defined by*

$$(\forall g \in G, \chi \in \widehat{G})\, \eta_G(g)(\chi) = e^{2\pi i\langle \chi, g\rangle} \in \mathbb{S}^1,$$

where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} \subseteq \mathbb{C}^\times$. The natural image of G in A^{-1} is

$$\eta_G(G) = \operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \cong \widehat{\widehat{G}} \cong G,$$

and

$$G \cong \operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^{\widehat{G}}.$$

(ii) *If the isomorphic weakly complete algebras $A \otimes_{\mathcal{W}} A$ and $\mathbb{C}^{\widehat{G}\times\widehat{G}}$ are identified, then the comultiplication $\gamma_G\colon A \to A \otimes_{\mathcal{W}} A$ of A is given by*

$$(\forall \varphi\colon \widehat{G} \to \mathbb{C},\ \chi_1, \chi_2 \in \widehat{G})\, \gamma_G(\varphi)(\chi_1, \chi_2) = \varphi(\chi_1 + \chi_2) \in \mathbb{C}.$$

(iii) *The group of grouplike elements of A is*

$$\mathbb{G}(A) = \operatorname{Hom}(\widehat{G}, \mathbb{C}^\times) \subseteq \mathbb{C}^{\widehat{G}}.$$

(iv) *The weakly complete Lie algebra of primitive elements of A is*

$$\mathbb{P}(A) = \operatorname{Hom}(\widehat{G}, \mathbb{C}) \subseteq \mathbb{C}^{\widehat{G}}.$$

□

We write $\mathbb{R}_+^\times$ for the multiplicative subgroup $\{z \in \mathbb{C} : 0 < z \in \mathbb{R} \subseteq \mathbb{C}\}$ of $\mathbb{C}^\times$. There is an elementary isomorphism of topological groups

$$(r, t + \mathbb{Z}) \mapsto e^r e^{2\pi i t} = e^{r+2\pi i t} : \mathbb{R} \times \mathbb{T} \to \mathbb{C}^\times.$$

For $A = \mathbb{C}^{\widehat{G}}$, from Statements (i) and (iii) in Theorem 3.94 we observe that $\mathbb{G}(A)$ is the direct product

$$\mathbb{G}(A) = \operatorname{Hom}(\widehat{G}, \mathbb{R}_+^\times)\cdot \operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \cong \mathfrak{L}(G) \times \eta_G(G),$$

where $\mathfrak{L}(G)$ denotes the weakly complete $\mathbb{R}$-vector space $\operatorname{Hom}(\widehat{G}, \mathbb{R})$. We observe that $\eta_G(G) \cong G$ is clearly the maximal compact subgroup of $\mathbb{G}(A)$. Recall that, according to Definition 3.73, the compact abelian group G is called $\mathbb{C}$-linearizable iff $\eta_G\colon G \to \mathbb{G}(\mathbb{C}[G]) = \mathbb{G}(A)$ is an isomorphism of topological groups. Here this is the case if and only if $\mathfrak{L}(G) = \{0\}$. Since $\widehat{\mathbb{T}} \cong \mathbb{Z}$ according to Table 1.39, and $\operatorname{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$, *the circle group $\mathbb{T}$ is not $\mathbb{C}$-linearizable*, while Theorem 3.88 showed that, like every compact group, it is $\mathbb{R}$-linearizable.

After Theorems 1.37 and 2.36 we know that the categories of abelian groups and the category of compact abelian groups are dual. So any abelian group $\mathcal{A}$ occurs in the form $\widehat{G}$ for some compact abelian group G. From the theory of abelian groups we know that $\operatorname{Hom}(\mathcal{A}, \mathbb{R}) = \{0\}$ if and only if $\mathcal{A}$ is a torsion group (cf. Remark A1.17, Propositions A1.33 and A1.39). So the question arises which compact abelian group G has a torsion group as character group $\widehat{G}$. We answer this question satisfactorily in Corollary 8.5 early in Chapter 8 on the structure of compact abelian groups. Indeed, the character group $\widehat{G}$ of a compact abelian group G is a torsion group if and only if G is totally disconnected; such groups were discussed in Theorem 1.34. Therefore we have

Corollary 3.95. *A compact abelian group G is $\mathbb{C}$-linearizable, that is, the equality $\eta_G(G) = \mathbb{G}(\mathbb{C}[G])$ holds, if and only if G is totally disconnected.* □

The exponential function of the algebra $A = \mathbb{C}^{\widehat{G}}$ is computed componentwise according to $\exp_A((c_\chi)_{\chi\in\widehat{G}}) = (e^{c_\chi})_{\chi\in\widehat{G}}$. It maps $\mathbb{P}(A) = \mathrm{Hom}(\widehat{G}, \mathbb{C})$ into $\mathbb{G}(A) = \mathrm{Hom}(\widehat{G}, \mathbb{C}^\times)$ as is established in Theorem A3.102 quite generally for all symmetric $\mathbb{K}$-Hopf algebras. If we define $\exp_G: \mathfrak{L}(G) \to G$ for a morphism $f: \widehat{G} \to \mathbb{R}$ of groups by $\exp_G(f)) = \exp_A(if)$, then we have the following set-up for $\exp_A: \mathbb{P}(A) \to \mathbb{G}(A)$:

Remark 3.96. For a compact abelian group G and the weakly complete complex symmetric Hopf algebra $A \stackrel{\text{def}}{=} \mathbb{C}[G]$ the following diagram is commutative:

$$\begin{array}{ccccc}
\mathbb{P}(A) & = & \mathrm{Hom}(\widehat{G}, \mathbb{R}) + \mathrm{Hom}(\widehat{G}, i\mathbb{R}) & \stackrel{\cong}{\to} & \mathfrak{L}(G) \times \mathfrak{L}(G) \\
{\scriptstyle \exp_A}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathfrak{L}(G)} \times \exp_G} \\
\mathbb{G}(A) & = & \mathrm{Hom}(\widehat{G}, \mathbb{R}_+^\times)\cdot \mathrm{Hom}(\widehat{G}, \mathbb{S}^1) & \stackrel{\cong}{\to} & \mathfrak{L}(G) \times G.
\end{array}$$

We understand $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$ rather explicitly, but $\mathbb{R}[G]$ only rather implicitly. However, in combination with our discussion in Corollary 3.55 and Proposition 3.56, Theorem 3.82 applies with $\mathbb{K} = \mathbb{R}$ in order to shed some light on its intrinsic structure. In particular, in the case of a compact abelian group G, we have $\mathbb{L}_{\varepsilon,\mathbb{R}} = \{\mathbb{R}, \mathbb{C}\}$.

We define the function $\sigma_G: \mathbb{C}[G] \to \mathbb{C}[G]$ as follows: For $\chi \in \widehat{G}$ we set $\check{\chi}(g) = \chi(-g) = -\chi(g)$ Then

$$(\forall \varphi \in \mathbb{C}^{\widehat{G}})\, \sigma(\varphi)(\chi) = \overline{\varphi}(\check{\chi}).$$

Exercise E3.24. For a compact group G, the function σ_G is an involution of weakly complete real algebras of $\mathbb{C}[G]$ whose precise fixed point algebra is $\mathbb{R}[G]$. Accordingly, $\mathbb{C}[G] = \mathbb{R}[G] \oplus i\mathbb{R}[G]$.

The Probability Semigroup of a Compact G inside $\mathbb{R}[G]$

We shall invoke measure theory in the form pioneered for arbitrary locally compact groups in [37]. For a *compact* group G it is less technical and is summarized in Appendix 5. In this form it adapts reasonably to the formalism of its real group algebra $\mathbb{R}[G]$. This discussion will help us to understand the power ot the group algebras $\mathbb{R}[G]$ for a compact group.

Indeed any compact Hausdorff topological group provides us with a real Banach algebra $C(G,\mathbb{R})$ endowed with the sup-norm. In the category of Banach spaces equipped with a suitable tensor product, $C(G,\mathbb{R})$ is a symmetric Hopf-algebra. Accordingly, its topological dual $C(G,\mathbb{R})'$ yields the Banach algebra and indeed Banach Hopf algebra $M(G,\mathbb{R})$ (see e.g. [163]). Its elements μ are the so called *Radon measures* on G. The general source books of this orientation of measure and probability theory are Bourbaki's book [37] and, for the foundations of harmonic analysis, the book of Hewitt and Ross [147]. For a measure theory in the context of compact groups see also Appendix 5 entitled “Measures on Compact Groups”.

So let W be a weakly complete real vector space. Then W may be identified with W'^* (see Appendix 7, Theorem A7.9). For $F \in C(G,W)$ and $\mu \in M(G,\mathbb{R})$ we obtain a unique element $\int_G F\,d\mu \in W$ such that we have

$$(*) \qquad (\forall \omega \in W') \quad \left\langle \omega, \int_G F\,d\mu \right\rangle = \int_{g\in G} \langle \omega, F(g)\rangle d\mu(g).$$

(See [37], Chap. III, §3, **n**° 1, Définition 1.) Let $\operatorname{supp}(\mu)$ denote the support of μ. (See [37], Chap. III, §2, **n**° 2, Définition 1.)

Lemma 3.97. *Let $T: W_1 \to W_2$ be a morphism of weakly complete vector spaces, G a compact Hausdorff space, and μ a measure on G. If $F \in C(G,W_1)$, then $T(\int_G F\,d\mu) = \int_G (T\circ F)d\mu$.*

Proof. See e.g. [37], Chap. III, §3, **n**° 2, Proposition 2. □

In [37] it is shown that the vector space $M(G) = M(G,\mathbb{R})$ is also a complete lattice with respect to a suitable natural partial order (see [37], Chap. III, §1, **n**° 6) so that each $\mu \in M(G)$ is uniquely of the form $\mu = \mu^+ - \mu^-$ for the two positive measures $\mu^+ = \mu \vee 0$ and $\mu^- = -\mu \vee 0$. One defines $|\mu| = \mu^+ + \mu^-$. If $M^+(G)$ denotes the cone of all positive measures, we have $M(G) = M^+(G) - M^+(G)$ ([37], Chap. III, §1, **n**° 5, Théorème 2). Moreover, $\|\mu\| = |\mu|(1) = \int d|\mu|$. A measure is called a *probability measure* if it is positive and $\mu(1) = 1$. We write $P(G)$ for the set of all probability measures on G and we note $M^+(G) = \mathbb{R}_+{\cdot}P(G)$ where $\mathbb{R}_+ = [0,\infty[\, \subseteq \mathbb{R}$. We denote by $M_p(G)$ the vector space $M(G,\mathbb{R})$ with the topology of pointwise convergence and recall that $P(G)$ has the structure of a compact submonoid of $M_p(G)^\times$; some aspects are discussed in Appendix 5 (see Lemma A5.8ff.). On $M^+(G)$ the topologies of $M_p(G)$ and the compact open topology of $M(G,\mathbb{R})$ agree ([37], Chap. III, §1, **n**° 10, Proposition 18), Also $M_p^+(G)$ is a locally compact convex pointed *cone* with the closed convex hull $P(G)$ of the set of point measures as *basis*. We also recall, that any positive linear form on $C(G,\mathbb{R})$ is in $M^+(G)$ (i.e., is continuous) (see [37], Chap. III, §1, **n**° 5, Théorème 1).

Now we allow this machinery and our structure theory of the weakly complete group Hopf algebra $\mathbb{R}[G]$ of for a compact group G to come together in order to further illuminate the structure of $\mathbb{R}[G]$.

So we let G be a compact group. By Corollary 3.62 we shall again assume that G is a compact subgroup of $\mathbb{R}[G]^{-1}$, and indeed that $G = \mathbb{G}(\mathbb{R}[G])$ is the subgroup of grouplike elements. The function $\mathrm{inc}_G\colon G \to \mathbb{R}[G]$ is the inclusion map. By Theorem 3.76 there is an isomorphism $F_G\colon \mathbb{R}[G]' \to R(G,\mathbb{R})$ where $F_G(\omega)(g) = \langle \omega, g\rangle$.

Therefore, in the spirit of relation $(*)$, in the present situation let $F = \mathrm{inc}_G \in C(G,\mathbb{R}[G])$

$$(\forall \omega \in \mathbb{R}[G]') \quad \left\langle \omega, \int_G F\,d\mu \right\rangle = \int_{g\in G} \langle \omega, g\rangle d\mu(g) = \int_{g\in G} F_G(\omega)(g)\,d\mu(g),$$

we are led to the following definition

Definition 3.98. Let G be a compact group. Then each $\mu \in M(G,\mathbb{R})$ gives rise to an element

$$\rho_G(\mu) \stackrel{\mathrm{def}}{=} \int_G \mathrm{inc}_G\ d\mu \in \mathbb{R}[G]$$

such that for all $\omega \in \mathbb{R}[G]'$ we have

$$(**) \qquad \langle \omega, \rho_G(\mu)\rangle = \int_{g\in G} \langle \omega, g\rangle\,d\mu(g) = \int_{g\in G} F_G(\omega)(g)\,d\mu(g) = \mu(F_G(\omega)).$$

Therefore we have a morphism of vector spaces

$$\rho_G\colon M(G,\mathbb{R}) \to \mathbb{R}[G].$$

We let $\tau_{R(G,\mathbb{R})}$ denote the weakest topology making the functions $\mu \mapsto \mu(f)$: $M(G,\mathbb{R}) \to \mathbb{R}$ continuous for all $f \in R(G,\mathbb{R})$, that is $(M(G,\mathbb{R}), \tau_{R(G,\mathbb{R})})$ is embedded into the weakly complete space $\mathbb{R}^{R(G,\mathbb{R})}$. □

On any compact subspace of $M_p(G)$ such as $P(G)$ the topology $\tau_{R(G,\mathbb{R})}$ agrees with the topology of $M_p(G)$, embedded into $\mathbb{R}^{C(G,\mathbb{R})}$.

Lemma 3.99. *The morphism ρ_G is injective and has dense image.*

Proof. We observe $\mu \in \ker \rho_G$ if for all $f \in R(G,\mathbb{R})$ we have $\int_{g\in G} f(g)\,d\mu(g) = 0$. Since μ is continuous on $C(G,\mathbb{R})$ in the norm topology and $R(G,\mathbb{R})$ is dense in $C(G,\mathbb{R})$ by the Theorem of Peter and Weyl (see e.g. Theorem 3.7), it follows that $\mu = 0$. So ρ_G is injective.

If $\mu = \delta_x$ is a measure with support $\{x\}$ for some $x \in G$, then $\rho_G(\mu) = \int_G \mathrm{inc}_G\,d\delta_x = x$. Thus $G \subseteq \rho_G(M(G))$. Since $\mathbb{R}[G]$ is the closed linear span of G by Corollary 3.61(i), it follows that ρ_G has a dense image. □

We note that in some sense ρ_G is dual to the inclusion morphism of vector spaces $R(G,\mathbb{R}) \to C(G,\mathbb{R})$.

Returning to (**) in Definition 3.98 for a compact group G we observe

Lemma 3.100. *The morphism*

$$\rho_G\colon (M(G,\mathbb{R}), \tau_{R(G,\mathbb{R})}) \to \mathbb{R}[G]$$

is a topological embedding. □

If μ is a probabililty measure, then the element $\rho_G(\mu) = \int_G \mathrm{inc}_G \, d\mu$ is contained in the compact closed convex hull $\overline{\mathrm{conv}}(G) \subseteq \mathbb{R}[G]$. Intuitively, $\int_G \mathrm{inc}_G \, d\mu = \int_{g\in G} g \, d\mu(g) \in \overline{\mathrm{conv}}(G)$ is the center of gravity of the "mass" distribution μ contained in $G \subseteq \mathbb{R}[G]$. In particular, if $\gamma \in M(G,\mathbb{R})$ denotes normalized Haar measure on G, then

$$\rho_G(\gamma) = \int_G \mathrm{inc}_G \, d\gamma = \int_{g\in G} g \, dg$$

is the center of gravity of G with respect to Haar measure.

We note that in the weakly complete vector space $\mathbb{R}[G]$ the closed convex hull

$$B(G) \stackrel{\mathrm{def}}{=} \overline{\mathrm{conv}}(G) \subseteq \mathbb{R}[G]$$

is compact. (See e.g. Exercise E3.13.)

Lemma 3.101. *The restriction $\rho_G|P(G)\colon P(G) \to B(G)$ is an affine homeomorphism.*

Proof. (i) Affinity is clear and injectivivity we know from Lemma 3.100.

(ii) Since $P(G)$ is compact in the weak topology and ρ_G is injective and continuous, $\rho_G|P(G)$ is a homeomorphism onto its image. But $G \subseteq \rho_G(P(G))$, and $B(G)$ is the closed convex hull of G in $\mathbb{R}[G]$. So it follows that $B(G) \subseteq \rho_G(P(G))$. □

If $\kappa_G\colon \mathbb{R}[G] \to \mathbb{R}$ is the coidentity of the Hopf algebra $\mathbb{R}[G]$ according to Lemma 3.64, then $\kappa_G(G) = \{1\}$ and so $\kappa_G(B(G)) = \{1\}$ as well. From $GG \subseteq G$ we deduce that $\mathrm{conv}(G)\,\mathrm{conv}(G) \subseteq \mathrm{conv}(G)$ and from there, by the continuity of the multiplication in $\mathbb{R}[G]$, and from $1 \in G \subseteq B(G)$, it follows that $B(G)$ is a compact submonoid of $\mathbb{R}[G]^\times$ contained in the submonoid $\kappa_G^{-1}(1)$.

Then the cone $\mathbb{R}_+[G] \stackrel{\mathrm{def}}{=} \mathbb{R}_+\cdot B(G)$, due to the compactness of $B(G)$, is a locally compact submonoid as well. The set

$$\kappa_G^{-1}(1) \cap \mathbb{R}_+[G] = \{x \in \mathbb{R}_+[G] : \kappa_G(x) = 1\} = B(G)$$

is a compact basis of the cone $\mathbb{R}_+[G]$.

Corollary 3.102. *The function $\rho_G|M^+(G) : M^+(G) \to \mathbb{R}_+[G]$ is an isomorphism of convex cones and $\rho_G(M(G)) = \mathbb{R}_+[G] - \mathbb{R}_+[G]$.*

Proof. Since $M^+(G) = \mathbb{R}_+\cdot P(G)$ and $\mathbb{R}_+[G] = \mathbb{R}_+\cdot B(G)$, Lemma 3.101 shows that $\rho_G|M^+(G)$ is an affine homeomorphism. Since $M(G) = M^+(G) - M^+(G)$, the corollary follows. □

Among other things this means that every element of $\mathbb{R}_+[G] - \mathbb{R}_+[G]$ is an integral $\int_G \mathrm{inc}_G\, d\mu$ in $\mathbb{R}[G]$ for some Radon measure $\mu \in M(G)$ on G.

In order to summarize our findings we firstly list the required conventions: Let G be a compact group viewed as a subgroup of the group $\mathbb{R}[G]^{-1}$ of units of the weakly complete group Hopf algebra $\mathbb{R}[G]$. Let $B(G) = \overline{\mathrm{conv}}(G)$ denote the closed convex hull of G in $\mathbb{R}[G]$ and define $\mathbb{R}_+[G] = \mathbb{R}_+\cdot B(G)$. Let $\kappa_G: \mathbb{R}[G] \to \mathbb{R}$ denote the co-identity and $I = \ker \kappa_G$ the *augmentation ideal*. We let $\mathrm{inc}_G: G \to \mathbb{R}[G]$ denote the inclusion map and consider $\rho_G: M(G,\mathbb{R}) \to \mathbb{R}[G]$ with $\rho_G(\mu) = \int_G \mathrm{inc}_G\, d\mu$.

The Probability Theorem of a Compact Group G inside $\mathbb{R}[G]$

Theorem 3.103. *For a compact group G the following conclusions hold:*

[a] $B(G) \supseteq G$ *is a compact submonoid of* $1 + I \subseteq (\mathbb{R}[G],\cdot)$ *with Haar measure* γ *of* G *as zero element.*

[b] $\mathbb{R}_+[G]$ *is a locally compact pointed cone with basis* $B(G)$*, and is a submonoid of* $(\mathbb{R}[G],\cdot)$.

[c] *The function* $\rho_G: (M(G), \tau_{R(G,\mathbb{R})}) \to \mathbb{R}[G]$ *is an injective morphism of topological vector spaces with dense image* $\mathbb{R}_+[G] - \mathbb{R}_+[G]$. *It induces a homeomorphism onto its image.*

[d] *The function* $\rho_G|M^+(G): M_p^+(G) \to \mathbb{R}_+[G]$ *is an affine homeomorphism from the locally compact convex cone of positive Radon measures on* G *onto* $\mathbb{R}_+[G] \supseteq B(G) \supseteq G$. □

Remark 3.104. The Haar measure γ is mapped by ρ_G onto the center of gravity $\int_G \mathrm{inc}_G\, d\gamma$ of G, $\gamma \in B(G) \subseteq 1 + I$. □

It should be noted that $\rho_G: M(G,\mathbb{R}) \to \mathbb{R}[G]$ is far from surjective if G is infinite: Indeed, if we identify $\mathbb{R}[G]$ with $R(G,\mathbb{R})^*$ according to Corollary 3.76a, then any element $u \in \mathbb{R}[G]$ representing a linear form on $R(G,\mathbb{R})$ which is discontinuous in the norm topology induced by $C(G,\mathbb{R})$ fails to be an element of $\rho_G(M(G))$.

In Appendix 7 we indicate in Definition A7.22 through Theorem A7.31 some features of a general theory of pro-Lie groups G, notably the focus on their Lie algebras $\mathfrak{L}(G)$ and exponential functions $\exp: \mathfrak{L}(G) \to G$: see Proposition A7.24ff. Using the terminology introduced there we know since Corollary 2.43, that every compact group is a pro-Lie group, and we shall return to this aspect of compact groups many times in this book. Here Theorem 3.103 shows that for a compact group G, the weakly complete real group algebra $\mathbb{R}[G]$ does not only contain G and its entire pro-Lie group theory encapsulated in the exponential function $\exp_G: \mathfrak{L}(G) \to G$ according to Theorem A3.102, but also its measure theory, notably, that of the monoid of probability measures $P(G) \cong B(G)$.

Recall the hyperplane ideal $I = \ker \kappa_G$ for the co-identity $\kappa_G: \mathbb{R}[G] \to \mathbb{R}$.

Corollary 3.105. *Let G be a compact group, $\mathbb{R}[G]$ its real symmetric group Hopf algebra, and $\gamma \in \mathbb{R}[G]$ its normalized Haar measure. Then $J \stackrel{\text{def}}{=} \mathbb{R}{\cdot}\gamma$ is a one-dimensional ideal, and $\mathbb{R}[G] = I \oplus J$ is the ideal direct sum of I and J. The vector subspace J is a minimal nonzero ideal and is in fact, in the terminology of* Theorem 3.82*, the minimal ideal $\mathbb{R}_{\varepsilon_0}[G]$ for the class ε_0 of trivial simple G-modules $\cong \mathbb{R}$ with $g{\cdot}r = r{\cdot}g = r$ for all $r \in \mathbb{R}$, $g \in G$. Consequently, $J \cong \mathbb{R}[G]/I \cong \mathbb{R}$ and $I \cong \mathbb{R}[G]/J$.*

Proof. In the multiplicative monoid $B(G) \subseteq \mathbb{R}[G]$, the idempotent element λ is a zero of the monoid $(I, \cdot)$, that is, $\lambda B(G) = B(G)\lambda = \{\lambda\}$. (See Corollary A5.12.) As a consequence, $JB(G) = B(G)J \subseteq J$. The vector space $\operatorname{span} B(G)$ contains $\operatorname{span} G$ which is dense in $\mathbb{R}[G]$ by Corollary 3.61(i). Hence $J\mathbb{R}[G] = \mathbb{R}[G]J \subseteq J$ and so J is a two-sided ideal. Since $\kappa_G(B(G)) = \{1\}$ by Theorem 3.103(a), we know $J \not\subseteq I$, and since I is a hyperplane, $\mathbb{R}[G] = I \oplus J$ follows. The reference to Theorem 3.82 is straightforward. □

We note that $\mathbb{R}[G]/J$ is a weakly complete topological algebra containing a copy of G and indeed of $P(G)$ with Haar measure in the copy of $P(G)$ being the zero of the algebra. It is neither a group algebra nor a Hopf algebra in general as the example of $G = \mathbb{Z}(3)$ shows.

While the group $\mathbb{G}(\mathbb{R}[G]) \cong G$ of grouplike elements of $\mathbb{R}[G]$ (and its closed convex hull $B(G)$) is contained in the affine hyperplane $1 + I$, in the light of Theorem A3.102 it is appropriate to observe that in the circumstances of Corollary 3.105, the Lie algebra of primitive elements $\mathbb{P}(\mathbb{R}[G]) \cong \mathfrak{L}(G)$ is contained in $I = \ker \kappa_G c$.

Indeed the ground field $\mathbb{R}$ is itself a Hopf algebra with the natural isomorphism $c_{\mathbb{R}}: \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ satisfying $c_{\mathbb{R}}(r) = r{\cdot}(1 \otimes 1) = r \otimes 1 = 1 \otimes r$. Now the coidentity κ of any coalgebra A is a morphism of coalgebras so that we have a commutative diagram for $A \stackrel{\text{def}}{=} \mathbb{R}[G]$:

$$(\dagger)\qquad \begin{array}{ccc} A & \xrightarrow{\quad c_A \quad} & A \otimes A \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa\otimes\kappa} \\ \mathbb{R} & \xrightarrow{\quad c_{\mathbb{R}} \quad} & \mathbb{R} \otimes \mathbb{R}. \end{array}$$

If $a \in A$ is primitive, then $c(a) = a \otimes 1 + 1 \otimes a$. The commutativity of (†) provides $\kappa(a) \otimes 1 = \alpha(\kappa(a))(\kappa \otimes \kappa)(c(a)) = (\kappa \otimes \kappa)(a \otimes 1 + 1 \otimes a) = \kappa(a) \otimes 1 + 1 \otimes \kappa(a)$, yielding $1 \otimes \kappa(a) = 0$, that is $\kappa(a) = 0$ which indeed means $a \in \ker \kappa = I$. We note that these matters are also compatible with Theorem A3.102 insofar as, trivially, $\exp(I) \subseteq 1 + I$.

While we have proved in this Part 3 the Tannaka-Hochschild duality theorem for compact groups, the reader who was focussing on just that theorem may have missed the much richer results that have been proved en route to that theorem. Therefore an exposition of what has been achieved is given in the Postscript to this chapter.

Postscript

Part 1 deals with the classical Theorem of Peter and Weyl while Part 2 is less conventional. We isolated the weakest feasible topological vector space condition on G-modules (G-completeness) which, on the representation space of a compact group, allows the averaging operator to function. In this book we introduced a rather weak condition referring to the topological vector space structure only which entails G-completeness, namely, feeble completeness. We exposed the averaging operator and its role in identifying fixed points. In the next chapter we generalize it and produce other projection operators which helps us to elucidate the full structure of very general types of G-modules. The Big Peter–Weyl Theorem not only says that the vector space of almost invariant vectors is dense in a feebly complete module, but also gives a canonical direct sum decomposition. We elaborate on this in the next chapter for the case of complex scalars. Through most of this chapter we proved all statements simultaneously for the real and the complex ground fields. In the last subsection, however, we explained, how real representations are analyzed in terms of complex ones. The special discussion of the representation theory of compact abelian groups in the last sections of the chapter are used in Chapter 6.

While in Chapters 1 and 2 we already proved the Pontryagin duality of compact abelian groups and discrete abelian groups, Chapter 7 of this book will be devoted to the Pontryagin Duality Theorem for locally compact abelian groups. That theorem tells us that if G is any locally compact abelian group, then the group $\widehat{G}$ of characters with the compact open topology is again a locally compact abelian group called the dual group or character group of G. Then if one forms $\widehat{\widehat{G}}$, the dual group of the dual group of G, one verifies that $\widehat{\widehat{G}}$ is isomorphic as a topological group to G. Two important observations follow: (i) in going from G to its dual group $\widehat{G}$, no information is lost since the original group G can be recreated from $\widehat{G}$ by taking its dual group; (ii) the structure of the dual group $\widehat{G}$ is rather simple, being a locally compact abelian group. Indeed, in the important case that G is a compact abelian group we saw already that the dual group $\widehat{G}$ is a discrete abelian group—a purely algebraic object and there is a wealth of known information about abelian groups.

The Tannaka Duality Theorem for compact groups was proved by the Japanese mathematician Tadao Tannaka in 1939 in [350]. The idea is to replace the characters in Pontryagin duality by finite-dimensional unitary representations. The task then is to put an algebraic structure on the representations such that it is possible to reconstruct the original compact group. Such was also the basis for Mark Grigorievich Krein's approach in 1949 in [232]. In his 1965 book [155], the German-born American mathematician Gerhard Paul Hochschild gave an approach to the Tannaka Duality Theorem which attached to every compact group a certain kind of Hopf algebra and showed how the compact group could be reconstructed from the Hopf algebra. However this approach of Hochschild has been largely overlooked as the exposition was, to say the least, not to everyone's taste.

Our approach in the third part of the chapter chooses a different avenue. In the classical period of (mostly finite) group theory it was a routine practice to generate from a finite group G a *group algebra* over a field $\mathbb{K}$: Consider the elements of G as basis elements of a $\mathbb{K}$-vector space, say, $\mathbb{K}[G]$, and extend the multiplication of G to one on $\mathbb{K}[G]$ by linear extension in the obvious fashion. (Cf. [209], p. 105.) This "linearisation process" served as a link from groups to their linear representations over $\mathbb{K}$. In this fashion any G-module, meaning every linear representation of G, became automatically a $\mathbb{K}[G]$-module and thus algebraists had a road from module theory to representation theory and obtained direct access to homological algebra.

In the case of topological groups, such a simple path from groups to algebras seemed not immediately available. However, the theory of operator algebras allowed the creation of "group algebras of topological groups" in terms of L^p-algebras, C^*-algebras or W^*-algebras, i.e. von Neumann algebras. The literature on such efforts is vast and the technical complications are considerable. An excellent example is the essay of M. Takesaki in [348].

In Part 3 of Chapter 3 we propose a procedure for creating group algebras of topological groups over $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ which, firstly, is still simple in the sense that is is close to basic $\mathbb{K}$-vector space theory, and, secondly, works perfectly for compact groups G.But what is basic $\mathbb{K}$-vector space theory? The first thing one learns in a course on linear algebra is that a vector space V has a dual $V^* \stackrel{\text{def}}{=} \operatorname{Hom}(V, \mathbb{K}) \subseteq \mathbb{K}^V$. One may not be taught at that time, that V^* inherits from $\mathbb{K}^V$ a topology called "the weak topology" which will make it isomorphic as a topological vector space to a topological vector space $\mathbb{K}^J$. But we call "weakly complete" any topological vector space that is isomorphic to a topological vector space $\mathbb{K}^J$. We have a good reason to introduce them in Chapter 7 on topological abelian groups in Definition 7.27, and for this purpose to collect all needed information in Appendix 7. It is that material that we mean by "basic $\mathbb{K}$-vector space theory". A *weakly complete unital $\mathbb{K}$-algebra*, accordingly, is a topological associative algebra with identity whose underlying vector space structure is that of a weakly complete $\mathbb{K}$-vector space. In Appendix 7 we show that, somewhat surprisingly, the group of its multiplicatively invertible elements, say U, is a pro-Lie group (see Appendix 7, Definition A7.22ff.) whose component factor group U/U_0 is always compact (indeed singleton if $\mathbb{K} = \mathbb{C}$!). (Cf. also [188] and [192].) In the present part we show that for every compact group G there is a functorially associated weakly complete unital algebra $\mathbb{K}[G]$ whose group of units contains G (more accurately: a naturally isomorphic copy of G) in such a way that every morphism of topological groups $f: G \to U$ into the group of units U of some weakly complete unital $\mathbb{K}$-algebra A extends uniquely to a morphism $f': \mathbb{K}[G] \to A$ of weakly complete $\mathbb{K}$-algebras. We call $\mathbb{K}[G]$ the *weakly complete group algebra* of G. If G is a finite group, then $\mathbb{K}[G]$ is precisely the "classical" group algebra. But there is much more than meets the eye. The category $\mathcal{W}$ of all weakly complete $\mathbb{K}$-vector spaces supports an appropriate tensor product $(W_1, W_2) \mapsto W_1 \otimes_{\mathcal{W}} W_2$ which has the known universal property encoding bilinearity. As is explained in detail in Appendix 3, this allows us to speak of weakly complete Hopf algebras. This is relevant since it turns out that every weakly complete group algebra $\mathbb{K}[G]$ is automatically a *weakly* *complete symmetric* $\mathbb{K}$*-Hopf algebra*. In Appendix 3 we learn, that symmetric Hopf algebras have *primitive* and *grouplike* elements. In the case of $\mathbb{R}[G]$ for a compact group G the grouplike elements are exactly the elements of $G \subseteq \mathbb{R}[G]$, and the set of primitive elements is naturally isomorphic to the Lie-algebra $\mathfrak{L}(G)$ of G (see Definition 9.44 and Proposition 9.45ff.), so that the Lie algebra $\mathfrak{L}(G)$ of any compact group G can be canonically identified with a subspace of $\mathbb{R}[G]$. Moreover, in Theorem A7.41, we learn that every weakly complete unital algebra A has a natural exponential function $\exp\colon A \to U$ from A to its group U of units. In the case of $A = \mathbb{R}[G]$ for a compact group G, it is this exponential function that induces the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ of the compact group G. The main result is an accurate description of those weakly complete symmetric real Hopf algebras $\mathbb{R}[G]$ for which the functor $G \mapsto \mathbb{R}[G]$ implements an equivalence of categories (see Definition A3.39).

One prominent feature of weakly complete $\mathbb{K}$-vector spaces and, notably, that of *real* weakly complete vector spaces, is their duality theory which we encounter in Chapter 7: See in particular Theorem 7.30. But we have in Appendix 7 a more comprehensive presentation: See Theorems A7.9 and A7.10. In the spirit of this duality one notices that the dual $\mathbb{K}[G]'$ of the weakly complete $\mathbb{K}$-group Hopf algebra $\mathbb{K}[G]$ of a compact group is a (purely algebraic) $\mathbb{K}$-Hopf algebra, namely, the Hopf algebra $R(G, \mathbb{K})$ which we have encountered as mere algebra under the name of algebra of representative functions earlier in this chapter and whose Hopf algebra structure was discussed for $\mathbb{K} = \mathbb{R}$ by Hochschild in [155] under the name of *reduced* Hopf algebra. Our approach elucidates what "reduced" means. So, as a corollary, our approach through weakly complete group algebras yields a very lucid form of the classical Tannaka Duality Theorem [350] for compact groups. For compact *abelian* groups G we illustrate and discuss very explicitly the isomorphism $\mathbb{C}[G] \cong \mathbb{C}^{\widehat{G}}$. The derivation of the *real* group Hopf algebra $\mathbb{R}[G]$ from this set-up is a bit more involved due to the occasionally intricate moving between real and complex representation theory; we encountered that in the subsection of Chapter 3 with the headline "Complexification of Real Representation" (following Corollary 3.53).

Finally we show that for a compact group G, the real symmetric group Hopf algebra $\mathbb{R}[G]$ supports indeed the entire probability measure theory of G which we touch upon in Appendix 5. Indeed the multiplicative topological semigroup of $\mathbb{R}[G]$ contains an isomorphic copy of the compact monoid $P(G)$ of probability measurs whose monoidal zero element is Haar measure γ. We see that $\mathbb{R}[G]$ contains not only the group G itself, but its entire pro-Lie theory in terms of the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$, and now we know that it also contains the potentially involved monoidal structure of its monoid $P(G)$ of probability measures.

References for this Chapter—Additional Reading

[34], [35], [37], [38], [40], [62], [77], [78], [110], [155], [218], [219], [231], [173], [243], [317], [331], [350].

Chapter 4

Characters

This chapter will continue the line of thought of Chapter 3. Chapter 3 was formulated without any reference to characters of (nonabelian) compact groups. As a consequence, almost all results could be phrased simultaneously for the real and the complex ground field. However, characters still constitute one of the most important set of tools of representation theory. Characters of an arbitrary compact group G, as we shall see are certain representative functions from $R(G, \mathbb{C})$. In this chapter, therefore, we restrict our attention to the complex ground field. In Chapters 1 and 2 we defined a character f of a compact (or discrete) abelian group G to be an element of $\mathrm{Hom}(G, \mathbb{T})$. The isomorphism $e: \mathbb{T} \to \mathbb{S}^1$ given by $e(r+\mathbb{Z}) = e^{2\pi i r}$ and the inclusion $\iota: \mathbb{S}^1 \to \mathbb{C}$ allows us to associate with f the function $\chi_f: G \to \mathbb{C}$ given by $\chi_f = \iota \circ e \circ f \in R(G, \mathbb{C})$. We noted at the end of Part 1 of the previous chapter that, conversely, every irreducible finite dimensional representation π of an abelian group gives a character f such that $\pi(g)(x) = \chi_f(g)\cdot x$. Under the identification of f and χ_f the characters of compact abelian groups are subsumed under that which we shall introduce here.

In Part 2 of this chapter we apply character theory in order to complete the theory of general complex G-modules over the general class of locally convex vector spaces which we began in Chapter 3. There we concluded the discourse with the Big Peter–Weyl Theorem 3.51 that generalized the classical Peter–Weyl Theorem 3.7. We expanded the Big Peter–Weyl Theorem and proved a first generalisation of the Fine Structure Theorem 3.28 for $R(G, \mathbb{K})$ in the form of Corollaries 3.52 and 3.53. In this chapter we will, for the complex ground field, arrive at the definitive generalisations of the Fine Structure Theorem 3.28 in the form of the Structure Theorem of G-Modules 4.22 and the Structure Theorem of Hilbert G-Modules 4.23.

Prerequisites. The prerequisites for this chapter are the same as those for Chapters 1 and 2. The second part demands the functional analytic background used in the second portion of the preceding chapter. We shall freely use nets where feasible, such as in the proof of Theorem 4.19.

Part 1: Characters of Finite Dimensional Representations

In the preceding remarks and those at the end of Part 1 of Chapter 3, we noted that the characters of an abelian group are linked with its irreducible representations. We shall now generalize this linkage to arbitrary groups. In the spirit of our present discourse, we carry this out for compact groups; the formalism as such does not depend on the compactness, but it does refer to finite dimensional representations, and compactness secures an abundance of these. As ground field, we shall now restrict our attention to $\mathbb{C}$. This does not surprise us if we recall the abelian theory. Thus we consider a compact group G and a finite dimensional G-module E. This gives us the finite dimensional function space $R_E(G,\mathbb{C})$ whose structure was completely elucidated in Part 1 of Chapter 3 (in contrast with the ground field $\mathbb{R}$ for which we had partial information). The key was contained in diagram (11) Proposition 3.20. Clearly, π shall denote the representation associated with the module E.

Lemma 4.1. *For a function $\chi \in R_E(G,\mathbb{C})$ the following conditions are equivalent:*

(i) $\chi = \sigma_E(\mathbf{1})$, *where again* $\mathbf{1} = \mathrm{id}_E$.

(ii) $\chi = \rho_E(\mathrm{tr})$.

(iii) *For any basis $e_1, \ldots, e_n$ of E and the corresponding dual basis $u_1, \ldots, u_n$ of E' the function χ equals $\sum_{j=1}^n \Phi_E(u_j \otimes e_j)$; that is, $\chi(g) = \sum_{j=1}^n \langle u_j, ge_j\rangle$.*

(iv) $\chi = \mathrm{tr} \circ \pi$*; that is,* $\chi(g) = \mathrm{tr}\big(\pi(g)\big)$.

These conditions imply

(v) *χ is constant on conjugacy classes of G and satisfies $\chi(\mathbf{1}) = \dim_{\mathbb{C}} E$.*

If E is simple then all these five conditions are equivalent.

Proof. By Remark 3.19 and in view of the remarks following Definition 3.11 of the trace, the first four conditions are equivalent. In order to prove that these conditions imply (v), we first observe that $\chi(\mathbf{1}) = \mathrm{tr}\big(\pi(\mathbf{1})\big) = \mathrm{tr}(\mathrm{id}_E) = \dim_{\mathbb{C}} E$. Further $\chi(g^{-1}xg) = \mathrm{tr}\,\pi(g)^{-1}\pi(x)\pi(g) = \mathrm{tr}\,\pi(x) = \chi(x)$ for all $x, g \in G$. This proves (v).

Finally assume that E is simple. Then σ_E is an isomorphism by Theorem 3.22. Hence (v) implies that $\sigma_E^{-1}(\chi)$ is a fixed element in $\mathrm{Hom}(E,E)$ under the action by conjugation. By Lemma 2.30 this element then must be a scalar multiple of id_E. Consequently, χ is a scalar multiple of the $\sigma_E(\mathrm{id}_E) = \mathrm{tr} \circ \pi$. Since by (v) the functions χ and $\mathrm{tr} \circ \pi$ assume the same value on $\mathbf{1}$, equality follows. This proves (iv). □

We have seen, notably, that for simple modules E, any function in $R_E(G,\mathbb{C})$ which is constant on conjugacy classes is a scalar multiple of $\chi = \mathrm{tr} \circ \pi$.

Definition 4.2. (i) If G is a compact group and E a finite dimensional G-module over $\mathbb{C}$, then the representative function $\chi_E = \mathrm{tr} \circ \pi_E \in R_E(G,\mathbb{C})$ is called the *character* of the module E and its associated representation π_E. If E is simple, then χ_E is also called a *simple* or an *irreducible character*.

(ii) We shall denote the linear span of all characters in $R(G,\mathbb{C})$ by $\mathrm{X}(G)$ and, accordingly, the linear span of all characters in $R_E(G,\mathbb{C})$ by $\mathrm{X}_E(G)$. The elements of $\mathrm{X}_E(G)$ are called *generalized characters* and sometimes, by abuse of language, also simply *characters of E*. □

Remark 4.3. Isomorphic modules have equal characters.

Proof. Exercise E4.1. □

Exercise E4.1. Prove Remark 4.3. □

This remark allows us to write for $\varepsilon \in \widehat{G}$, a class of isomorphic G-modules, $\chi_\varepsilon = \chi_E$ for $E \in \varepsilon$. □

Proposition 4.4. (i) *If $\varepsilon \in \widehat{G}$, then $\mathrm{X}_\varepsilon(G) = \mathbb{C}{\cdot}\chi_\varepsilon$.*

(ii) *$\mathrm{X}(G)$ is the direct sum $\sum_{\varepsilon\in\widehat{G}} \mathbb{C}.\chi_\varepsilon$ of one-dimensional L^2-orthogonal subspaces. Its closure in $L^2(G,\mathbb{C})$ is $\bigoplus_{\varepsilon\in\widehat{G}} \mathbb{C}.\chi_\varepsilon$.*

(iii) *If χ is a generalized character, then there is a unique family $(n_\varepsilon)_{\varepsilon\in\widehat{G}} \in \mathbb{C}^{\widehat{G}}$ of complex numbers vanishing with finitely many exceptions, such that*

$$\chi = \sum_{\varepsilon\in\widehat{G}} n_\varepsilon{\cdot}\chi_\varepsilon.$$

If χ is a character, then all n_ε are nonnegative integers.

(iv) *A representative function on G is constant on conjugacy classes of G if and only if it is a generalized character.*

Proof. (i) is just a summary of previous remarks and definitions.

(ii) In view of $\mathrm{X}_\varepsilon(G) = \mathrm{X}(G) \cap R_\varepsilon(G,\mathbb{C})$, the assertion is a consequence of Theorem 3.28.

(iii) By (ii), the family $\{\chi_\varepsilon \mid \varepsilon \in \widehat{G}\}$ is a basis of the vector space $\mathrm{X}(G)$, hence every generalized character is a finite linear combination of simple characters with unique coefficients. If $\chi = \chi_E$ for a finite dimensional module E, we write $E = E_1 \oplus \cdots \oplus E_n$ with simple modules E_j according to Corollary 2.25 and conclude from the definition of a character that $\chi = \mathrm{tr} \circ \pi_E = \sum_{j=1}^n \mathrm{tr} \circ \pi_{E_j} = \sum_{j=1}^n \chi_{[E_j]}$. Thus χ is a finite sum of simple characters and the assertion follows.

(iv) Assume that $f \in R(G,\mathbb{C})$ is constant on conjugacy classes. By Theorem 3.28 we may write $f = \sum_{\varepsilon\in\widehat{G}} f_\varepsilon$ with $f_\varepsilon \in R_\varepsilon(G,\mathbb{C})$. Then all f_ε are constant on conjugacy classes because fixed vectors under a module action are preserved under equivariant projections. Now Lemma 3.29 shows that f_ε is a scalar multiple of χ_ε. Thus f is a linear combination of simple characters and is, therefore, a generalized character. The converse is immediate from Lemma 3.29. □

In view of our main results in Theorem 3.28, this conclusion was immediate. The consequences are significant and far reaching even though they follow rather quickly.

Corollary 4.5. *Two finite dimensional G-modules for a compact group G are equivalent if and only if their characters agree.*

Proof. In Remark 4.3 we observed that the isomorphy of the modules implied equality of the characters. Conversely assume that E and F are two finite dimensional modules and that $\chi_E = \chi_F$. By Proposition 4.4(iii), $\chi_E = \sum_{\varepsilon\in\widehat{G}} n_\varepsilon(E)\cdot\chi_\varepsilon$ with unique nonnegative numbers $n_\varepsilon(E)$. The corresponding representation for χ_F holds with $n_\varepsilon(F) = n_\varepsilon(E)$. If we represent E and F as direct sums of simple modules, each simple module M occurs in E with multiplicity $n_{[M]}(E)$, but in F with multiplicity $n_{[M]}(F)$, and since these are equal, the isomorphy of E and F follows. □

We have remarked before that $R(G,\mathbb{C})$ is a "catalogue" for all simple G-modules over $\mathbb{C}$, each listed as often as its dimension indicates. Now $\mathrm{X}(G)$ is a more economical and still very canonical "list." But remember that this list does not contain any submodules; it is a $\mathbb{C}$-vector space with the characters as an L^2-orthogonal basis.

The following remarks require the knowledge of the averaging operator introduced and discussed in 3.32ff.

Proposition 4.6. *Let G be a compact group and denote with $C_{\mathrm{class}}(G,\mathbb{C})$ the space of all continuous complex class functions, that is functions constant on conjugacy classes of G. Let G act on $C(G,\mathbb{C})$ via $(g\cdot f)(x) = f(g^{-1}xg)$. Then the averaging operator P of this module is a continuous projection of $C(G,\mathbb{C})$ onto $C_{\mathrm{class}}(G,\mathbb{C})$. The submodule $R(G,\mathbb{C})$ is projected onto the space $\mathrm{X}(G)$ of generalized characters, and*

$$\overline{\mathrm{X}(G)} = C_{\mathrm{class}}(G,\mathbb{C}). \tag{1}$$

In other words, every continuous class function can be uniformly approximatedby generalized characters.

Further, every continuous function $f\colon G \to \mathbb{C}$ decomposes uniquely into a sum $f_0 + f_1$ with a class function f_0 and a function $f_1 \in C(G,\mathbb{C})_{\mathrm{eff}}$ with respect to the action by conjugation of the argument.

Proof. The action $(g,f) \mapsto g\cdot f\colon G \times C(G,\mathbb{C}) \to C(G,\mathbb{C})$ is continuous by Theorem 2.3. Hence P is continuous by Theorem 3.36(vi). From Theorem 3.36(iii) it retracts $C(G,\mathbb{C})$ onto its fixed point set, which is exactly $C_{\mathrm{class}}(G,\mathbb{C})$. Since $R(G,\mathbb{C})$ is dense in $C(G,\mathbb{C})$ by the Classical Peter and Weyl Theorem 3.7, we know that $PR(G,\mathbb{C})$ is dense in $PC(G,\mathbb{C}) = C_{\mathrm{class}}(G,\mathbb{C})$. Every submodule $R_\varepsilon(G,\mathbb{C})$ with $\varepsilon \in \widehat{G}$ is finite dimensional, hence closed, and thus is mapped by P into itself. Also, $PR_\varepsilon(G,\mathbb{C})$ is the fixed point space of $R_\varepsilon(G,\mathbb{C})$ and is, therefore, the space of class functions in this module. Hence $PR_\varepsilon(G,\mathbb{C}) = \mathbb{C}\cdot\chi_\varepsilon = \mathrm{X}_\varepsilon(G)$ by Lemma 4.1 and Proposition 4.4. From Theorem 3.28 it now follows that $PR(G,\mathbb{C}) = P\big(\sum_{\varepsilon\in\widehat{G}} R_\varepsilon(G,\mathbb{C})\big) = \sum_{\varepsilon\in\widehat{G}} \mathrm{X}_\varepsilon(G) = \mathrm{X}(G)$. Thus (6) follows. The remainder is a consequence of Theorem 3.36(vi) (because the action is continuous in view of Theorem 2.3!) □

We recall that $G \times G$ acts unitarily on $C(G, \mathbb{C})$ via $\big((g,h)\cdot f\big)(x) = f(g^{-1}xh)$ with respect to the scalar product $(f_1|f_2) = \int_G f_1(g)\overline{f_2(g)}\,dg$ (recall Example 2.12!). This action extends to $L^2(G, \mathbb{C})$. In particular, the action of Proposition 4.6 above extends unitarily.

Proposition 4.7. *The averaging operator of the G-module $L^2(G, \mathbb{C})$ with the action extending the action of G on $C(G, \mathbb{C})$ in* Proposition 4.6 *retracts $L^2(G, \mathbb{C}) = \bigoplus_{\varepsilon \in \widehat{G}} R_\varepsilon(G; \mathbb{C})$ onto the Hilbert subspace $\bigoplus_{\varepsilon \in \widehat{G}} \mathrm{X}_\varepsilon(G)$ in which $\mathrm{X}(G)$ is L^2-dense. Further, $L^2(G, \mathbb{C})$ is the orthogonal direct sum of $\overline{\mathrm{X}(G)}$ and $L^2(G, \mathbb{C})_{\mathrm{eff}}$ with respect to conjugation of the arguments.*

Proof. Exercise E4.2. □

Exercise E4.2. Prove Proposition 4.7. □

Exercise E4.3. Assume that a compact group G acts on a compact space X (see Definition 1.9). Then the orbit projection $x \mapsto Gx\colon X \to X/G$ is a continuous open map and X/G is a compact Hausdorff space. □

If we denote the orbit equivalence relation of conjugation on G by conj, then G/conj is a compact Hausdorff space, and $C(G/\mathrm{conj}, \mathbb{C})$ is canonically isomorphic to $C_{\mathrm{class}}(G, \mathbb{C})$.

Exercise E4.4. Let G be a finite group. Then a complex function on G is a class function if and only if it is a generalized character. The dimension of the space of class functions is the cardinality of G/conj, that is, the number of conjugacy classes and this is equal to the number of characters and equal to the number of elements in $\widehat{G}$. What is the dimension of $R(G, \mathbb{C})$? □

Lemma 4.8. *If E is a finite dimensional G-module and P its averaging operator, then $\operatorname{tr} P = \int \chi_E = \dim E_{\mathrm{fix}}$.*

Proof. $\operatorname{tr} P = \operatorname{tr} \int \pi(g)\,dg = \int \operatorname{tr}\big(\pi(g)\big)\,dg = \int \chi_E$. But P is a projection by the Splitting Fixed Points Theorem 3.36. Therefore, $\operatorname{tr} P = \dim \operatorname{im} P = \dim E_{\mathrm{fix}}$ by that same theorem. □

Proposition 4.9. (i) *If χ is a real linear combination of characters, then the relations $\check{\chi}(g) = \chi(g^{-1}) = \overline{\chi(g)}$ hold.*

(ii) *$\overline{\chi_E} = \chi_{E'}$ for every finite dimensional module E.*

(iii) *Assume that E and F are two finite dimensional modules, and consider on* $\mathrm{Hom}(E, F)$ *the action given by $g\cdot\varphi = \pi_F(g)\varphi\pi_E(g)^{-1}$. Then $\chi_{\mathrm{Hom}(E,F)} = \chi_F\overline{\chi_E}$. If* $\mathrm{Hom}_G(E, F)$ *denotes the space of all equivariant operators $E \to F$, then*

$$\dim \mathrm{Hom}_G(E, F) = (\chi_F|\chi_E) \quad \textit{in} \quad L^2(G, \mathbb{C}).$$

In particular, the simple characters form an orthonormal system in $L^2(G, \mathbb{C})$.

(iv) *If* χ_j, $j = 1, 2$ *are two characters, then* $(\chi_1 * \chi_2)(1) = (\chi_1|\chi_2)$.
(v) $\chi_\varepsilon * \chi_{\varepsilon'}(1) = \delta_{\varepsilon\varepsilon'}$ *for* $\varepsilon, \varepsilon' \in \widehat{G}$.

Proof. (i) Let π denote a representation on E with $\chi = \mathrm{tr} \circ \pi$. We may assume that π is unitary. Then $\pi(g^{-1}) = \pi(g)^*$ and $\chi(g^{-1}) = \mathrm{tr}\big(\pi(g^{-1})\big) = \mathrm{tr}\big(\pi(g)^*\big) = \overline{\mathrm{tr}\big(\pi(g)\big)}$. Thus $\chi(g^{-1}) = \overline{\chi(g)}$ for every character, and then also for every generalized real linear combination of characters.

(ii) By Lemma 3.6(iii), the matrix coefficients of the representation associated with $\widetilde{E}$ are the complex conjugates of those of the representation associated with E. By Lemma 3.6(i), the modules E' and $\widetilde{E}$ are isomorphic. The assertion is now clear from these facts.

(iii) By (ii) we have $\chi_{E' \otimes F} = \chi_F \overline{\chi_E}$. On the other hand, $E' \otimes F \cong \mathrm{Hom}(E, F)$ under the map θ_{EF} given by $\theta_{EF}(v \otimes x)(y) = \langle v, y \rangle \cdot x$, and this action is equivariant if on $\mathrm{Hom}(E, F)$ we consider the action stated in the proposition. (These assertions are simple generalisations of the set-up of Lemma 3.10 and the computation following Remark 3.13.) Thus $\chi_{\mathrm{Hom}(E,F)} = \chi_F \overline{\chi_E}$. Hence, in view of the preceding Lemma 4.8, we find $\dim \mathrm{Hom}_G(E, F) = \dim\big(\mathrm{Hom}(E, F)\big)_{\mathrm{fix}} = \int \chi_{\mathrm{Hom}(E,F)} = \int \chi_F \overline{\chi_E} = (\chi_F|\chi_E)$.

If both E and F are simple, then either they are nonisomorphic, in which case all members of $\mathrm{Hom}_G(E, F)$ are zero or else E and F are isomorphic, in which case $\mathrm{Hom}_G(E, F) \cong \mathrm{Hom}_G(E, E) = \mathbb{C} \cdot \mathrm{id}_E$, whence $(\chi_F|\chi_E) = \dim \mathrm{Hom}_G(E, F) = 1$. Thus the set of simple characters is an orthonormal system in $L^2(G, \mathbb{C})$.

(iv) $(\chi_1 * \chi_2)(\mathbf{1}) = \int \chi_1(g)\chi_2(g^{-1})\, dg = \int \chi_1(g)\overline{\chi_2(g)}\, dg = (\chi_1|\chi_2)$.

(v) This is now a consequence of the preceding statements. ☐

Theorem 4.10 (The Center Theorem). (i) *A function* $f \in C(G, \mathbb{C})$ *is in the center of the convolution algebra* $C(G, \mathbb{C})$ *if and only if it is a class function. In particular, a function* $f \in C(G, \mathbb{C})$ *is central if and only if it can be approximated by generalized characters; that is the center of* $C(G, \mathbb{C})$ *is* $\overline{\mathrm{X}(G)} = C_{\mathrm{class}}(G, \mathbb{C})$.

(ii) *The elements* $e_\varepsilon \stackrel{\mathrm{def}}{=} \chi_\varepsilon(\mathbf{1}) \cdot \chi_\varepsilon = \dim E \cdot \chi_E$ *(with* $[E] = \varepsilon$*) are central idempotents, and*

$$e_\varepsilon * f = f * e_\varepsilon = f \qquad \text{for all } f \in R_\varepsilon(G, \mathbb{C}). \tag{2}$$

Further, $e_\varepsilon * C(G, \mathbb{C}) = R_\varepsilon(G, \mathbb{C})$.

Proof. (i) For f, $f_1 \in C(G, \mathbb{C})$ we have $(f_1 * f)(h) = \int f_1(g) f(g^{-1}h)\, dg$. The transformation of variables $g \mapsto g' = g^{-1}h = (h^{-1}g)^{-1}$ on the compact group G does not change the integral; thus, noting $g = {g'}^{-1}h$ we have $\int f_1(g) f(hg^{-1})\, dg = \int f(g') f_1({g'}^{-1}h)\, dg' = (f * f_1)(h)$. Hence $(f_1 * f - f * f_1)(h) = \int f_1(g)\big(f(g^{-1}h) - f(hg^{-1})\big)\, dg = \big(f_1 * ({}_hf - {}^{h^{-1}}\!f)\big)(1)$. Now f is a class function if and only if ${}_hf - {}^{h^{-1}}\!f = 0$ for all $h \in G$. If this is satisfied, then $f_1 * f = f * f_1$ for all f_1 and f is central. Conversely, if f is central, then $0 = \big(f_1 * ({}_hf - {}^{h^{-1}}\!f)\big)(1) = ({}_hf - {}^{h^{-1}}\!f \mid (\overline{f_1})\check{}\,)$

for all $f_1 \in C(G,\mathbb{C})$ and $h \in G$. Since $C(G,\mathbb{C})$ is dense in $L^2(G,\mathbb{C})$ in the L^2-norm, it follows that ${}_hf - {}^{h^{-1}}\!f = 0$ for all $h \in G$, and thus f is a class function.

The remainder of Part (i) now follows from Proposition 4.6.

(ii) Let $\varepsilon \in \widehat{G}$. By Part (i) above χ_ε is central in $C(G,\mathbb{C})$, hence so is $\chi_\varepsilon * \chi_\varepsilon$, whence $\chi_\varepsilon * \chi_\varepsilon$ is central and hence is a generalized character by (i) again. By Proposition 3.48, $\chi_\varepsilon * \chi_\varepsilon \in R_\varepsilon(G,\mathbb{C})$, whence $\chi_\varepsilon * \chi_\varepsilon = c{\cdot}\chi_\varepsilon$ for a suitable $c \in \mathbb{C}$ by Proposition 4.4(i). We evaluate at $\mathbf{1}$ and obtain on the right side $c{\cdot}\chi_\varepsilon(\mathbf{1})$. On the left side, the preceding Proposition 4.9(iv) gives us $(\chi_\varepsilon * \chi_\varepsilon)(\mathbf{1}) = (\chi_\varepsilon|\chi_\varepsilon) = 1$. Hence $c = \chi_\varepsilon(\mathbf{1})^{-1}$. If we set $e_\varepsilon = \chi(1){\cdot}\chi_\varepsilon$ then $e_\varepsilon * e_\varepsilon = e_\varepsilon$, that is e_ε is a central idempotent and $\mathrm{X}_\varepsilon(G) = \mathbb{C}{\cdot}e_\varepsilon$. Since $R_\varepsilon(G,\mathbb{C})$ is an ideal of $C(G,\mathbb{C})$ by Proposition 3.48, we have $C(G,\mathbb{C}) * e_\varepsilon \subseteq R_\varepsilon(G,\mathbb{C})$. If $q\colon R_\varepsilon(G,\mathbb{C}) \to R_\varepsilon(G;\mathbb{C})$ is given by $q(f) = e_\varepsilon * f$, then $q^2 = q$ and q is equivariant with respect to all G-module structures on $R_\varepsilon(G,\mathbb{C})$ by Proposition 3.45(iii). Hence q is equivariant with respect to the actions of G on the left and the right of the argument. Thus, if we let $G \times G$ act on $R_\varepsilon(G,\mathbb{C})$ via $\big((g,h){\cdot}f\big)(x) = f(g^{-1}xh)$, then q is $G \times G$-equivariant. Let E denote a simple G-module with $[E] = \varepsilon$. Then $\mathrm{Hom}(E,E)$ is a simple $G \times G$-module by Lemma 3.15. In view of the equivariant isomorphism $\sigma_E\colon \mathrm{Hom}(E,E) \to R_\varepsilon(G,\mathbb{C})$ of Theorem 3.22 we know that $R_\varepsilon(G,\mathbb{C})$ is a simple $G \times G$-module. By Schur's Lemma 2.30, $q = c{\cdot}\,\mathrm{id}_{R_\varepsilon(G,\mathbb{C})}$ for some $c \in \mathbb{C}$. Since $q^2 = q$ we conclude $c = 0$ or $c = 1$, but since $q(e_\varepsilon) = e_\varepsilon$, we know that q cannot vanish. Hence $c = 1$ and q is the identity. Thus (2) follows. But then also $R_\varepsilon(G,\mathbb{C}) = e_\varepsilon * R_\varepsilon(G,\mathbb{C}) \subseteq e_\varepsilon * C(G,\mathbb{C})$. □

The central idempotents form an idempotent commutative subsemigroup under multiplication $*$. Such semigroups are also called *semilattices* (see e.g. [62, 196]).

One writes $e \le f$ if $e * f = e$. The minimal elements in this semilattice are exactly the e_ε, $\varepsilon \in \widehat{G}$. Thus the minimal elements in the semilattice of central idempotents classify the isomorphy classes of simple modules.

It is convenient for the following to have a name for the semilattice of all central idempotents of the convolution algebra $R(G,\mathbb{C})$.

Definition 4.11. The $*$-multiplicative semilattice of all central idempotents $e = e * e$ of $R(G,\mathbb{C})$ is denoted $ZI(G)$. □

Exercise E4.5. (i) Show that every central idempotent of $(R(G,\mathbb{C}),*)$ is a sum of a finite set of elements e_ε, $\varepsilon \in \widehat{G}$.

(ii) Show that with two central idempotents e, f in $R(G,\mathbb{C})$ also $e \vee f \stackrel{\mathrm{def}}{=} (e+f) - (e*f)$, and $e \oplus f \stackrel{\mathrm{def}}{=} (e \vee f) - (e*f) = (e+f) - 2(e*f)$ are central idempotents.

(iii) Show that $(ZI(G),\oplus,*)$ is a commutative ring in which every element e satisfies $e \oplus e = 0$ and is multiplicatively idempotent. Such a ring is also called a *Boolean ring*. An identity exists in $(ZI(G),\oplus,*)$ if and only if G is finite.

(iv) Show that $(ZI(G),\vee,*)$ is a distributive lattice. (On $ZI(G)$ define the partial order $\le$ by $e \le f$ iff $e*f = e$ and show that $e \vee f$ is the least upper bound

of $\{e, f\}$, that $e * (f \vee g) = (e * f) \vee (e * g)$ and that $e \vee (f * g) = (e \vee f) * (e \vee g)$.
[Hint. Establish the claim for an arbitrary Boolean ring.]

(v) Show that $e \mapsto \overline{e}: ZI(G) \to ZI(G)$ is an involutive automorphism of Boolean rings which sends $e_{[E]}$ with a simple module E to $e_{[E']}$.
[Hint. Use Proposition 4.9(iii).] □

Notice that by the Center Theorem 4.10, all elements e_ε belong to $ZI(G)$ as does the constant function with value 0.

Exercise E4.6. Let us denote, for any set X, by $\mathcal{F}(X)$ the Boolean ring of all finite subsets of $F \subseteq X$ under the addition

$$(F_1, F_2) \mapsto (F_1 \cup F_2) \setminus (F_1 \cap F_2)$$

and under the multiplication

$$(F_1, F_2) \mapsto (F_1 \cap F_2).$$

Show that the map which takes a finite subset $\{\varepsilon_1, \ldots, \varepsilon_n\} \subseteq \widehat{G}$ to the element $e_{\varepsilon_1} + \cdots + e_{\varepsilon_n} \in ZI(G)$ is an isomorphism of Boolean rings $\mathcal{F}(\widehat{G}) \to ZI(G)$.

Thus $ZI(G)$ *is isomorphic to the Boolean ring* $\mathcal{F}(\widehat{G})$. □

Part 2: The Structure Theorem of $E_{\rm fin}$

Armed with the arsenal of character theory we now attack the problem of describing G-modules in great generality. We consider a G-complete G-module for a compact group G and recall that every feebly complete G-module E is G-complete. We shall assume, in addition, that the action $(g, x) \mapsto gx: G \times E \to E$ is continuous. We have seen in the Splitting Fixed Points Theorem 3.36 that E has a continuous projection $P: E \to E$, the averaging operator whose image is the fixed point module $E_{\rm fix}$ and whose kernel is the closed effective submodule $E_{\rm Eff}$. Whenever one has a continuous projection $p: E \to E$, $p^2 = p$, then E splits algebraically and topologically into a direct sum of $\operatorname{im} p$ and $\ker p$. We now have the tools for a vast generalization of the averaging operator. For every element $e \in ZI(G)$ we can readily define a projection P_e as follows:

Definition 4.12. (i) For $\varepsilon \in \widehat{G}$ let $e_\varepsilon \in R(G, \mathbb{C})$ denote the central element $\chi_\varepsilon(\mathbf{1}) \cdot \chi_\varepsilon$.

(ii) For each $e \in ZI(G)$ we define $P_e: E \to E$ by

$$P_e(x) = \overline{e} * x \tag{3}$$

and set $E(e) = \operatorname{im} P_e$. Accordingly, we have

$$P_{e_\varepsilon}: E \to E, \quad P_{e_\varepsilon(x)} = \overline{e_\varepsilon} * x = \chi_\varepsilon(\mathbf{1}) \cdot \overline{\chi}_\varepsilon * x.$$

We write $E_\varepsilon = E(e_\varepsilon)$ for $\varepsilon \in \widehat{G}$. □

Notice that the averaging operator P is P_e for the constant function e on G with value 1. Indeed, e is the trace of the constant morphism $\pi\colon G \to \mathrm{Gl}(1, \mathbb{C})$ and is, therefore, the central idempotent associated with the one-dimensional simple module. By definition, $P_e(x) = \int_G \overline{1}\cdot gx\,dg = P(x)$.

Proposition 4.13. *Let E denote a G-complete G-module for a compact group G with jointly continuous action. Then the following conclusions hold:*

(i) *Each P_e is equivariant and $E(e)$ and $\ker P_e$ are submodules.*

(ii) *For e, $e' \in ZI(G)$ we have*

$$P_e P_{e'} = P_{e*e'} \qquad P_e + P_{e'} - P_e P_{e'} = P_{e\vee e'}.$$

In particular, $P_e^2 = P_e$ and $P_e P_{e'} = 0$ if $e * e' = 0$.

(iii) *For each $e \in ZI(G)$, the map*

$$(x, y) \mapsto x + y : E(e) \times \ker P_e \to E$$

is an isomorphism of G-modules whose inverse is given by

$$x \mapsto \big(P_e(x), (\mathbf{1} - P_e)(x)\big) : E \to E(e) \times \ker P_e.$$

(iv) *If $\varepsilon_1, \ldots, \varepsilon_n$ is any finite collection of different elements from $\widehat{G}$ and $e = e_{\varepsilon_1} + \cdots + e_{\varepsilon_n}$, then the map*

$$(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n : E_{\varepsilon_1} \times \cdots \times E_{\varepsilon_n} \to E(e)$$

is an isomorphism of G-modules with inverse

$$x \mapsto \big(P_{e_{\varepsilon_1}}(x), \ldots, P_{e_{\varepsilon_n}}(x)\big) : E(e) \to E_{\varepsilon_1} \times \cdots \times E_{\varepsilon_n}.$$

Proof. (i) It follows from Proposition 3.45(iii) that P_e is equivariant, and thus $E(e) = \operatorname{im} P_e$ and $\ker P_e$ are invariant.

(ii) $P_e\big(P_{e'}(x)\big) = \overline{e} * (\overline{e'} * x) = (\overline{e} * \overline{e'}) * x = P_{e*e'}(x)$ by Lemma 3.43. Also $(P_e + P_{e'} - P_e P_{e'})(x) = \overline{e} * x + \overline{e'} * x - \overline{e} * (\overline{e'} * x) = \overline{e + e' - (e * e')} * x = P_{e\vee e'}(x)$.

If $e' = e$, then $e' * e = e^2 = e$ since e is idempotent with respect to convolution. Thus $P_e^2 = P_e$. Also $P_0(x) = 0 * x = 0$ by Lemma 3.41(i).

(iii) follows from Lemma 3.36(iv), (vi) in conjunction with (i) above which contributes equivariance.

(iv) We define $\Phi\colon E(e) \to E_{\varepsilon_1} \times \cdots \times E_{\varepsilon_n}$ by $\Phi(x) = \big(P_{e_{\varepsilon_1}}(x), \ldots, P_{_{\varepsilon_n}}(x)\big)$ for $x \in E(e)$, and $\Psi\colon E_{\varepsilon_1} \times \cdots \times E_{\varepsilon_n} \to E(e)$ by $\Psi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Then $\Phi\Psi(x_1, \ldots, x_n) = \Phi(x_1 + \cdots + x_n) = \Phi(x_1) + \cdots + \Phi(x_n) = x_1 + \cdots + x_n$ since $P_{e_{\varepsilon_j}}(x_k) = P_{e_{\varepsilon_j}}\big(P_{e_{\varepsilon_k}}(x_k)\big) = \delta_{jk}\cdot x_k$. Since $e = e_{\varepsilon_1} + \cdots + e_{\varepsilon_n} = e_{\varepsilon_1} \oplus \cdots \oplus e_{\varepsilon_n}$, we have $\Psi\Phi = (P_{e_{\varepsilon_1}} + \cdots + P_{e_{\varepsilon_n}})|E(e) = P_e|E(e) = \mathrm{id}_{E(e)}$ by (ii) above. Hence Ψ and Φ are inverses of each other and (iv) is proved. □

Lemma 4.14. *Let F be a simple submodule of E and $\varepsilon = [F]$. Then $P_{e_\varepsilon}|F = \mathrm{id}\,F$. Moreover, $\varepsilon' \neq \varepsilon$ implies $P_{e_{\varepsilon'}}F = \{0\}$.*

Proof. Because of $e_{\varepsilon'} * e_\varepsilon = 0$ and $P_{e_{\varepsilon'}} \circ P_\varepsilon = P_{e_{\varepsilon'}} * e_\varepsilon$ according to Proposition 4.13(ii), the second assertion is a consequence of the first. Now let $x \in F$ and $u \in E'$. Then $f = (g \mapsto \langle u, gx\rangle)$ is in $R_{[F]}(G, \mathbb{C})$ by Definition 3.16 in view of the fact that $u|F \in F'$. Then $e_{[F]} * f = f$ by (2) in the Center Theorem 4.10. Hence

$$\begin{aligned}\langle u, x\rangle &= f(1) = (e_{[F]} * f)(1)\\ &= \textstyle\int_G e_{[F]}(g)\langle u, g^{-1}x\rangle\, dg\\ &= \langle u, \textstyle\int_G e_{[F]}(g)\cdot g^{-1}x\, dg\rangle\\ &= \langle u, \textstyle\int_G e_{[F]}(g^{-1})\cdot gx\, dg\rangle\\ &= \langle u, \overline{e_{[F]}} * x\rangle,\end{aligned}$$

as $\overline{e_{[F]}}(g) = e_{[F]}(g^{-1})$ by Proposition 4.9(i)). Thus $\langle u, x\rangle = \langle u, P_{e_{[F]}}(x)\rangle$ for all $u \in E'$, and this shows $x = P_{e_\varepsilon}(x)$. □

Note that this lemma justifies the use of $\overline{e}$ rather than that of e in the definition of P_e.

If $\{E_j \mid j \in J\}$ is a family of vector spaces, we shall denote the vector subspace of $\prod_{j\in J} E_j$ consisting of all J-tuples $(x_j)_{j\in J}$ whose components vanish with finitely many exceptions by $\sum_{j\in J} E_j$ and call it the (exterior) *direct sum* of the E_j.

Proposition 4.15. *In the circumstances of Proposition* 4.13*, we also have*

(i) $E_\varepsilon \subseteq E_{\rm fin}$ *for all* $\varepsilon \in \widehat{G}$.

(ii) *The function* $\iota \stackrel{\rm def}{=} (x \mapsto (P_{e_\varepsilon}(x))_{\varepsilon\in\widehat{G}}) : E \to \prod_{\varepsilon\in\widehat{G}} E_\varepsilon$ *induces an equivariant continuous bijection from* $E_{\rm fin}$ *onto the direct sum* $\sum_{\varepsilon\in\widehat{G}} E_\varepsilon$.

Proof. (i) By Lemma 3.49 it follows that $E_\varepsilon = P_{e_\varepsilon}(E) = \overline{e} * E \subseteq E_{\rm fin}$.

(ii) The function ι is equivariant by Proposition 4.13(i). Let $x \in E_{\rm fin}$ and $F = \operatorname{span} Gx$. Then F is a finite dimensional G-module. Thus F is a direct sum of simple submodules by Theorem 2.10 and Corollary 2.25. Hence $F = F_1 \oplus \cdots \oplus F_m$ such that there is a finite set $\{E_1, \ldots, E_m\}$ of pairwise nonisomorphic simple modules and an m-tuple of natural numbers $(n_1, \ldots, n_m)$ with $F_k \cong E_k^{n_k}$. If we set $e = e_{[E_1]} \oplus \cdots \oplus e_{[E_m]}$, we obtain that $P_e|F = {\rm id}_F$ by Proposition 4.13(iv) and Lemma 4.14. Now $x \in F$ and so $x = P_e(x)$. If $x \in \ker \iota$ then $0 = P_e(x) = x$ for all $e \in ZI(G)$ in view of Proposition 4.13(iv). Thus ι is injective. It remains to identify $\operatorname{im} \iota$. If $x \in E_{\rm fin}$ and F is as above, then $\iota(x) \in \sum_{\varepsilon\in\widehat{G}} E_\varepsilon$, since $\varepsilon \neq [E_k]$ for $k = 1, \ldots, m$ implies $P_{e_\varepsilon}(x) = P_{e_\varepsilon}P_e(x) = P_{e_\varepsilon * e}(x) = P_0(x) = 0$. Hence $\operatorname{im}\iota \subseteq \sum_{\varepsilon\in\widehat{G}} E_\varepsilon$. Conversely, let $y = (x_\varepsilon)_{\varepsilon\in\widehat{G}} \in \sum_{\varepsilon\in\widehat{G}} E_\varepsilon$. Since all of the x_ε vanish with the exception of a finite subcollection $\{x_{\varepsilon_1}, \ldots, x_{\varepsilon_n}\}$, the element $x = \sum_{k=1}^n x_{\varepsilon_k}$ satisfies

$$P_\varepsilon(x) = \begin{cases} x_k, & \text{if } \varepsilon = \varepsilon_k,\ k = 1, \ldots, n;\\ 0, & \text{otherwise.}\end{cases}$$

Hence $\iota(x) = (P_\varepsilon(x))_{\varepsilon\in\widehat{G}} = y$. The proof is complete. □

Now we need to understand the structure of E_ε.

Definition 4.16. A G-module E is called *cyclic* if there is an element $w \in E$ such that $E = \overline{\operatorname{span} Gw}$. Each element w with this property is called a *generator*. □

Clearly, every simple module is cyclic. The following example illustrates what might happen.

Example 4.17. Let F be a complex simple G-module for a compact group G. The nonzero submodules of F^2 which are different from $\{0\} \times F$, and F^2 are the graphs of equivariant maps $f\colon F \to F$. All of these are isomorphic to F. However, the simplicity of F implies that $f = c\cdot \operatorname{id}_F$ with $c \in \mathbb{C}$. Hence these submodules are all of the form $F_c \stackrel{\text{def}}{=} \{(x, c{\cdot}x) \mid x \in F\}$, $c \in \mathbb{C}$. If $\dim F > 1$ then we find a pair $\{x, y\}$ of linearly independent elements of F. Then (x, y) cannot be an element of any F_c, nor of $\{0\} \times F$. Hence the cyclic submodule generated by (x, y) is F^2. □

However, there is some control over cyclic modules contained in some power of a simple module, as the next observation shows.

Proposition 4.18. (i) *Assume that $x \in E$ satisfies $P_{e_\varepsilon} x = x \neq 0$, i.e. $x \in E_\varepsilon$, with $\varepsilon = [F]$. Then the cyclic G-submodule $V \subseteq E$ generated by x is isomorphic to F^k for some natural number $k \leq \dim F$.*

(ii) *Conversely, each module F^k with $k \leq \dim F$ is cyclic.*

In particular, every isotypic cyclic module V is finite dimensional and is, therefore, of the form $V = \operatorname{span} G.x$.

Proof. (i) We set $M = R_{F'}(G, \mathbb{C}) * x$. We know $e_{[F']} \in R_{F'}(G, \mathbb{C})$ from the Center Theorem 4.10. Hence $x = P_{e_\varepsilon} x = \overline{e_{[F]}} * x = e_{[F']} * x \in R_{F'}(G, \mathbb{C}) * x = M$. Now we consider the linear map $\alpha\colon R_{F'}(G, \mathbb{C}) \to M$, $\alpha(f) = f * x$. Note that $\alpha({}^g f) = {}^g f * x = g(f * x) = g\alpha(f)$ by Proposition 3.45. Thus α is equivariant if we consider on $R_{F'}(G, \mathbb{C})$ the module structure $(g, f) \mapsto {}^g f$, ${}^g f(h) = f(g^{-1}h)$. The Fine Structure Theorem of $R(G, \mathbb{C})$ 3.28 implies that $R_{F'}(G, \mathbb{C})$, when considered as a G-module under $(g, f) \mapsto {}^g f$, is the direct sum of n submodules isomorphic to $F'' = F$ with $n = \dim F$. Thus the G-module M is a homomorphic image of the G-Module F^n, hence is of the form F^m with $m \leq n$ by Lemma 3.24. Now let V be the G-submodule of M generated by x. Then by Lemma 3.24 again, V is a direct sum of k simple submodules isomorphic to F.

(ii) Since F^k is an equivariant homomorphic image of $F^{\dim F} \cong R_F(G, \mathbb{C})$ for the action $(g, f) \mapsto {}_g f$ (see 3.23), it suffices to observe that the latter module is cyclic. But $R_F(G, \mathbb{C}) \cong \operatorname{Hom}(E, E)$ with $(g, \varphi) \mapsto \pi_F(g)\varphi$ by 3.22(i). Now $\mathbf{1} = \operatorname{id}_F$ is a cyclic generator of $\operatorname{Hom}(F, F)$, since $G{\cdot}\mathbf{1} = \pi_F(G)$ and $\operatorname{span}\big(\pi_F(G)\big) = \operatorname{Hom}(F, F)$ by 3.21.

The last remark of the proposition follows from the fact that a finite dimensional vector subspace in a (Hausdorff) topological vector space is always closed. □

Next is some linear algebra of topological vector spaces! Let F denote a finite dimensional vector space with a basis $x_1, \ldots, x_n$ and E an arbitrary topological vector space, and consider on $\mathrm{Hom}(F, E)$ the topology of pointwise convergence. Then the function $\mathrm{Hom}(F, E) \to E^n$ given by $f \mapsto \big(f(x_1), \ldots, f(x_n)\big)$ is an isomorphism of topological vector spaces whose inverse assigns to $(y_1, \ldots, y_n) \in E^n$ the linear map $\sum_{k=1}^n r_k{\cdot}x_k \mapsto \sum_{k=1}^n r_k{\cdot}y_k$. This remark gives us a good visualisation of the topological vector space $\mathrm{Hom}(F, E)$.

Also, we recall the isomorphism $\theta\colon F' \otimes E \to \mathrm{Hom}(F, E)$ given by $\theta(u \otimes x)(y) = \langle u, y\rangle{\cdot}x$. This endows $F' \otimes E$ with a unique vector space topology such that θ is an isomorphism of topological vector spaces. If both F and E are G-modules then F' has the structure of the adjoint module and the isomorphism θ is equivariant if we consider on $F' \otimes E$ the action given by $g(u \otimes x) = gu \otimes gx$ and on $\mathrm{Hom}(F, E)$ the action given by $g{\cdot}f = \pi_E(g) \circ f \circ \pi_F(g)^{-1}$. If we denote $(F' \otimes E)_{\mathrm{fix}}$ by $F' \otimes_G E$ then θ maps $F' \otimes_G E$ isomorphically onto $\mathrm{Hom}_G(F, E)$.

In view of the duality of finite dimensional vector spaces (and G-modules), every finite dimensional vector space (or G-module) F is the dual of F'. Thus $F \otimes E$ has a unique vector space topology making it (equivariantly) isomorphic to $\mathrm{Hom}(F', E)$.

In this fashion, for a basis $x_1, \ldots, x_n$ of F, the function $(y_1, \ldots, y_n) \mapsto \sum_{k=1}^n x_k \otimes y_k\colon E^n \to F \otimes E$ is an isomorphism of topological vector spaces.

Exercise E4.7. Verify explicitly all claims in the preceding two paragraphs. □

The topologies on $\mathrm{Hom}(F, E)$, and on $F' \otimes E$ as well as on its vector subspaces such as $\mathrm{Hom}_G(F, E)$ and $F' \otimes_G E$ shall be called *the natural vector space topologies.* We shall now identify the structure of the submodules E_ε explicitly. In the following theorem and its proof we shall consider a simple G-module F and a G-module E. Then the G-modules $\mathrm{Hom}_G(F, E) \otimes F$ and $(F' \otimes_G E) \otimes F$ are well-defined with their natural vector space topologies and the tensor product actions obtained by letting G act trivially on $\mathrm{Hom}_G(F, E)$ and $F' \otimes_G E$, respectively. The following theorem gives, among other things, a complete structure theorem for the components E_ε.

Theorem 4.19. *Let F denote a simple module in $\varepsilon \in \widehat{G}$ and E a G-complete G-module such that $(g, x) \mapsto gx\colon G \times E \to E$ is continuous. Then the function $p\colon \mathrm{Hom}_G(F, E) \otimes F \to E$, uniquely determined by $p(f \otimes x) = f(x)$, is an isomorphism of topological vector spaces and G-modules onto its image $E_{[F]}$.*

Let $x_k \in F$, $k = 1, \ldots, \dim F$ be any basis and let $u_k \in F'$ denote the elements of the dual basis of F'. The function $q\colon E \to \mathrm{Hom}_G(F, E) \otimes F$ given by

$$q(x) = \dim F{\cdot} \sum_{k=1}^{\dim F} P_{\mathrm{Hom}(F,E)}\big(\theta(u_k \otimes x)\big) \otimes x_k$$

satisfies $pq = P_{e_\varepsilon}$ and $qp = \mathrm{id}_{\mathrm{Hom}_G(F,E)\otimes F}$. In particular, $q|E_\varepsilon$ inverts the corestriction of p onto its image.

Proof. The proof is a bit lengthy, and we proceed as follows: Firstly we note that p is well-defined and equivariant. Secondly, we prove that $\operatorname{im} p = E_\varepsilon$, thirdly we show that p is injective by showing that $\ker p = \{0\}$. As the fourth step, we establish the continuity of p, and as the fifth we show that q is the inverse of p. The observation that q is continuous will then complete the proof.

Step 1. We note that p is well-defined in view of the universal property of the tensor product since $(f,x) \mapsto f(x)\colon \operatorname{Hom}_G(F,E) \times F \to E$ is bilinear. Then equivariance of p is readily verified: $p(g(f \otimes x)) = p(f \otimes gx) = f(gx) = gf(x) = gp(f \otimes x)$ in view of the equivariance of f.

Step 2. If $f \in \operatorname{Hom}_G(F,E)$, then the corestriction $f\colon F \to f(F)$ of f maps F equivariantly into $f(F)$. By simplicity of F we know that $f(F) = \{0\}$ or $f(F) \cong F$. By Lemma 4.14 we note $P_{e_\varepsilon} f(x) = f(x)$ and thus $f(x) \in E_\varepsilon$. Conversely, let $x \in E_\varepsilon$. Then Lemma 4.18 shows that $x \in F_1 \oplus \cdots \oplus F_k \subseteq E$ with $F_j \cong F$. Then there exist elements $f_1, \ldots, f_k \in \operatorname{Hom}_G(F,E)$ such that f_j maps F isomorphically onto F_j and elements $x_j \in F_j$ such that $x = x_1 + \cdots + x_k$. Set $y_j = f_j^{-1}(x_j) \in F$. We now define the element $z \in \operatorname{Hom}_G(F,E) \otimes F$ by $z = f_1 \otimes y_1 + \cdots + f_k \otimes y_k$. Then $p(z) = f_1(y_1) + \cdots + f_k(y_k) = x_1 + \cdots + x_k = x$. Hence $\operatorname{im} p = E_\varepsilon$.

Step 3. We take a $z \in \operatorname{Hom}_G(F,E) \otimes F$ and assume $p(z) = 0$. There exist finitely many elements $f_1, \ldots, f_n \in \operatorname{Hom}_G(F,E)$ such that

$$z \in (f_1 \otimes F) \oplus \cdots \oplus (f_n \otimes F).$$

If we set $E_1 = \sum_{j=1}^n \operatorname{im} f_j$, and if we identify $\operatorname{Hom}(F,E_1)$ with a submodule of $\operatorname{Hom}(F,E)$ via the map induced by the inclusion $E_1 \to E$, then $z \in \operatorname{Hom}_G(F,E_1) \otimes F$. For our purpose it is, therefore, no loss of generality to assume that $\dim E$ is finite and that $P_{e_\varepsilon} = \mathrm{id}_E$. Now E is a finite direct sum of simple submodules, all of which have to be isomorphic to F by Lemma 4.18. Thus $E \cong F^m$ for some m. But $\operatorname{Hom}(F,F^m) \cong \operatorname{Hom}(F,F)^m$ and thus $\operatorname{Hom}_G(F,E) \cong \operatorname{Hom}_G(F,F)^m = \mathbb{K}^m$, that is $\dim \operatorname{Hom}_G(F,E) = m$. Hence $\dim\big(\operatorname{Hom}_G(F,E) \otimes F\big) = m \cdot \dim F = \dim F^m = \dim E$. Since p is surjective by Step 2, p is an isomorphism in our case and $p(z) = 0$ indeed implies $z = 0$.

Step 4. We show that $p\colon \operatorname{Hom}_G(F,E) \otimes F \to E_\varepsilon$ is continuous. Firstly, if $f \in \operatorname{Hom}_G(F,E)$ and $f \neq 0$, then $f(F) \cong F$ and hence $f(F) \subseteq E_\varepsilon$ in view of Lemma 4.14. Hence the inclusion $E_\varepsilon \to E$ induces an inclusion.

Now we recall that the simple module F is $\operatorname{span}(Gx)$ for any non-zero element $x \in F$. Hence there are elements $g_1 = \mathbf{1}, g_2, \ldots, g_n \in G$ such that $g_1 x, \ldots, g_n x$ is a basis of F. Then

$$(f_1, \ldots, f_n) \mapsto f_1 \otimes g_1 x + \cdots + f_n \otimes g_n x : \operatorname{Hom}_G(F,E_\varepsilon)^n \to \operatorname{Hom}_G(F,E_\varepsilon) \otimes F$$

is an isomorphism of vector spaces, and we have to observe that the map

$$\alpha = \big((f_1, \ldots, f_n) \mapsto f_1(g_1 x) + \cdots + f_n(g_n x)\big) : \operatorname{Hom}_G(F,E_\varepsilon)^n \to E_\varepsilon$$

is continuous. Notice $f_1(g_1 x) + \cdots + f_n(g_n x) = g_1 f_1(x) + \cdots + g_n f_n(x)$ since the f_j, $j = 1, \ldots, n$ are equivariant. Now a net $(f^{(j)})_{j \in J}$ of elements $f^{(j)} \in \operatorname{Hom}_G(F,E)$ converges to $f \in \operatorname{Hom}_G(F,E)$ if and only if $f^{(j)}(gx) = gf^{(j)}(x)$ converges to

$gf(x)$ if and only if $\lim_{j\in J} f^{(j)}(x) = f(x)$. Consequently, $\lim_{j\in J}(f_1^{(j)},\dots,f_n^{(j)}) = (f_1,\dots,f_n)$ in $\mathrm{Hom}_G(F,E)$ if and only if

$$\lim_{j\in J} f_k^{(j)}(x) = f_k(x) \quad \text{for} \quad k=1,\dots,n. \tag{4}$$

Clearly (4) implies

$$\lim_{j\in J}\alpha(f_1^{(j)},\dots,f_n^{(j)}) = g_1f_1(x)+\cdots+g_nf_n(x) = \alpha\big(\lim_{j\in J}(f_1^{(j)},\dots,f_n^{(j)})\big). \tag{5}$$

Thus α is continuous.

At this point we know that p a continuous equivariant algebraic isomorphism of vector spaces onto E_ε. It remains to show that its inverse is continuous. This requires the explicit identification of a left inverse q of p.

Step 5. Let $x_k \in F$ and $u_k \in F'$ the elements of dual bases and $q{:}\,E \to \mathrm{Hom}_G(F,E)\otimes F$ as in the statement of the theorem. Since $(g,x)\mapsto gx{:}\,G\times E\to E$ is continuous, then the action $G\times(F'\otimes E)\to F'\otimes E$ given by $(g,u\otimes x)\mapsto gu\otimes gx$ is continuous, too, whence $P_{F'\otimes E}{:}\,F'\otimes E\to F'\otimes E$ is continuous by Theorem 3.36(vi). For the remainder of the proof, we shall identify $F'\otimes E$ and $\mathrm{Hom}(F,E)$ under the isomorphism θ. Then $F'\otimes_G E = \mathrm{im}\,P$ with $P = P_{F'\otimes E}$ is identified with $\mathrm{Hom}_G(F,E)$. Now $q(x) = \dim F\cdot\sum_{k=1}^{\dim F} P(u_k\otimes x)\otimes x_k$. This map is continuous by the natural topologies on the tensor products and by the continuity of P. Since we know that p is bijective, in order to complete the proof, it now suffices to show that $pq(x) = x$ for all $x\in E$. Note that with our identification, $p{:}\,(F'\otimes_G E)\otimes F\to E$ is given by

$$p\big((v_1\otimes z_1+\cdots+v_m\otimes z_m)\otimes y\big) = \langle v_1,y\rangle{\cdot}z_1+\cdots+\langle v_m,y\rangle{\cdot}z_m.$$

Now let $x\in E$. First let us deduce from the definition of dual bases that

$$\begin{aligned}\sum_{k=1}^{\dim F}\langle gu_k,x_k\rangle &= \sum_{k=1}^{\dim F}\langle u_k,\pi_F(g^{-1})x_k\rangle\\ &= \mathrm{tr}\pi_F(g^{-1}) = \chi_F(g^{-1}) = \overline{\chi_\varepsilon}(g)\end{aligned}$$

in view of Proposition 4.9(i). Now we compute

$$\begin{aligned}pq(x) &= p\Big(\dim F\cdot\sum_{k=1}^{\dim F} P(u_k\otimes x)\otimes x_k\Big)\\ &= p\Big(\dim F\cdot\sum_{k=1}^{\dim F}\textstyle\int_G(gu_k\otimes gx)\,dg\otimes x_k\Big)\\ &= \dim F\cdot\textstyle\int_G\sum_{k=1}^{\dim F}\langle gu_k,x_k\rangle gx\,dg\\ &= \dim F\cdot\textstyle\int\overline{\chi_\varepsilon}(g){\cdot}gx\,dg = \overline{e_\varepsilon}*x = P_{e_\varepsilon}x.\end{aligned}$$

This shows $pq = P_{e_\varepsilon}$ and thus $pqp = P_{e_\varepsilon}p = p$ since $\mathrm{im}\,p = E_\varepsilon = P_{e_\varepsilon}(E)$ and $P_{e_\varepsilon}^2 = P_{e_\varepsilon}$. But p is injective after Step 3. Hence $qp = \mathrm{id}_{(F'\otimes_G E)\otimes F}$. The proof is complete. □

The G-modules E_ε are characterized by various properties, as we shall show now.

Proposition 4.20. *For a G-complete G-module E and a simple G-module F, the following statements are equivalent:*

(1) *Every simple submodule S of E is isomorphic to F.*
(2) *All equivariant homomorphisms $E \to S$ into a simple module are zero unless S is isomorphic to F.*
(3) *If $x \in E$ and $u \in E'$, then $g \mapsto \langle u, gx\rangle$ is in $R_F(G,\mathbb{C})$.*
(4) $P_{e_{[F]}} = \mathrm{id}_E$.
(5) *The map $\mathrm{Hom}_G(F,E) \otimes F \to E$ sending $f \otimes x$ to $f(x)$ is an isomorphism of G-modules.*

Proof. (1)⇒(2). Assume that S is simple and not isomorphic to F. Let $x \in E_{\mathrm{fin}}$ and $\varphi\colon E \to S$ an equivariant morphism. We claim that $\varphi(x) = 0$. If this is established, then $\varphi(E_{\mathrm{fin}}) = \{0\}$ and since, by the Big Peter and Weyl Theorem 3.51, E_{fin} is dense in E, we obtain $\varphi(E) = \{0\}$ as asserted. Now x is contained in a finite dimensional submodule M. But M is a direct sum of simple submodules M_j, every one of which is isomorphic to F by (1). Thus $f(M_j) = \{0\}$ since $M_j \cong F \ncong S$. Hence $f(x) \in f(M) = \{0\}$, as claimed.

(2)⇒(3). Fix an arbitrary functional $u \in E'$ and consider the map $\varphi\colon E \to R(G,\mathbb{C})$ given by $\varphi(x)(g) = \langle u, gx\rangle$. Then $\varphi(gx)(h) = \langle u, hgx\rangle = \big({}_g\varphi(x)\big)(h)$, whence φ is equivariant when $R(G,\mathbb{C})$ is considered as a G-module with the action $(g,f) \mapsto {}_gf$. For every simple module S, the submodule $R_S(G,\mathbb{C})$ is isomorphic to $S^{\dim S}$ by 3.28, the Fine Structure Theorem of $R(G,\mathbb{C})$. If p is any projection from $R_S(G,\mathbb{C})$ onto a direct summand S, then $p \circ \varphi = 0$ if $S \ncong F$. Since the p separate the points of $R_S(G,\mathbb{C})$, the map φ followed by the projection of $R(G,\mathbb{C})$ into $R_S(G,\mathbb{C})$ is zero. Hence $\operatorname{im}\varphi \subseteq R_F(G,\mathbb{C})$ by the Fine Structure Theorem 3.28. And this shows that $g \mapsto \langle u, gx\rangle$ is in $R_F(G,\mathbb{C})$.

(3)⇒(4). Let $u \in E'$ be arbitrary. Then $f = (g \mapsto \langle u, gx\rangle)$ is in $R_F(G,\mathbb{C})$ by (4). On the other hand,

$$\begin{aligned}\langle u, P_{e_{[F]}}x\rangle &= \textstyle\int_G \overline{e_{[F]}}(h)\langle u, hx\rangle\,dh\\ &= \textstyle\int_G e_{[F]}(h^{-1})f(h)\,dh = (e_{[F]} * f)(\mathbf{1}) = f(\mathbf{1})\\ &= \langle u, x\rangle\end{aligned}$$

by the Center Theorem 4.10. Since u was arbitrary, we conclude $P_{e_{[F]}}x = x$.

(4)⇒(5). This follows from Theorem 4.19.

(5)⇒(1). It suffices to show that a simple submodule of a G-module of the form $V \otimes F$ with a trivial G-module V is isomorphic to F. Let S denote a simple submodule of $V \otimes F$. Assume that $0 \neq x \in S$. Then $x = v_1 \otimes x_1 + \cdots + v_m \otimes x_m$, whence x is contained in the submodule $M = v_1 \otimes F + \cdots + v_m \otimes F$. Since $0 \neq x \in M \cap S$ and S is simple, $S \subseteq M$. By Lemma 4.14 we get $P_{e_{[F]}}|M = \mathrm{id}_M$. If we had $S \ncong F$, then, also by Lemma 4.14, we would have $S = P_{e_{[F]}}(S) = \{0\}$, a contradiction. Hence $S \cong F$, as asserted. □

Definition 4.21. (i) A G-module E satisfying the equivalent conditions of Proposition 4.20 is called *an isotypic G-module of type* $[F]$. If $\dim E$ is finite, then $\dim \mathrm{Hom}_G(F,E)$ is called *the multiplicity of E*.

(ii) The direct summands $E_\varepsilon = \mathrm{im}\, P_{e_\varepsilon}$ of E_{fin} according to Propositions 4.13 and 4.15 are called the *isotypic components of E*. □

We summarize our results in the following basic result on the general representation theory of compact groups:

THE STRUCTURE THEOREM OF G-MODULES

Theorem 4.22. *Let E denote a G-complete G-module with jointly continuous action $(g,x) \mapsto gx$. Then E_{fin} is algebraically the direct sum of the isotypic components E_ε, $\varepsilon \in \widehat{G}$ of E. Each one of these is the image of the continuous projection P_{e_ε} given by $P_{e_\varepsilon}x = \chi_\varepsilon(1)\cdot\overline{\chi_\varepsilon} * x$, and therefore is algebraically and topologically a direct summand of E. Each isotypic component $E_{[F]}$ is isomorphic to $\mathrm{Hom}_G(F,E) \otimes F$ as a topological vector space and G-module under the map given by $f \otimes x \mapsto f(x)$.* □

Together with the Big Peter and Weyl Theorem 3.51, this theorem completely describes the general representation theory of compact groups on topological vector spaces permitting integration; we have seen in the paragraphs surrounding Proposition 3.30 that a weak completeness condition is sufficient. While the Big Peter and Weyl Theorem generalizes the Classical Peter and Weyl Theorem 3.7, the Structure Theorem for G-Modules 4.22 above generalizes the Fine Structure Theorem 3.28. The dense vector space of almost invariant vectors is algebraically decomposed into a direct sum of topological vector spaces

(i) each of which is an algebraic and topological direct summand of E for which we have an explicit projection,

(ii) each of which is an isotypic G- module of precisely known structure $V \otimes F$, with a simple module F, and with a multiplicity counting topological vector space and trivial module V, whose structure depends in an explicit fashion of that of the given vector space E.

In this sense, everything we wish to know is essentially reduced to knowing the isotypic components and thus to (a) the structure of topological vector spaces, (b) the G-module structure of all simple modules. The word "essentially" indicates that we do not exactly know in which way E_{fin} is a direct sum in the topological sense (cf. Proposition 4.14), and exactly how E_{fin} is to be completed in order to obtain E. But all of these problems are already present in the module $E = C(G,\mathbb{C})$ and its submodule of representative functions $R(G,\mathbb{C})$, and indeed, more elementarily, we encounter the same deficiency in the basic theory of Fourier series, that is the theory of $G = \mathbb{R}/\mathbb{Z}$ where $R(\mathbb{R}/\mathbb{Z},\mathbb{C})$ is the space of trigonometric polynomials. One shows that every periodic continuous complex valued function on $\mathbb{R}$ can be uniformly approximated by trigonometric polynomials, but there is no Fourier series expansion for this topology. The appropriate topology for the approximation of a function by its Fourier series is L^2.

Indeed, our general results contained in Theorems 3.51 and 4.22 can be improved if E is a Hilbert space.

Theorem 4.23 (The Structure Theorem of Hilbert G-Modules). *Let E be a Hilbert G-module (cf. Definition 2.11). Then all operators P_e with $e \in ZI(G)$ are orthogonal projections, and E is a Hilbert space direct sum*

$$E = \bigoplus_{\varepsilon \in \widehat{G}} E_\varepsilon.$$

Each isotypic component is a Hilbert space direct sum

$$E_\varepsilon = \bigoplus_{j \in J(\varepsilon)} F_j, \qquad F_j \in \varepsilon \text{ for all } j \in J(\varepsilon).$$

Proof. Firstly, let $e \in ZI(G)$. We want to show that $P_e^* = P_e$, that is $(P_e x|y) = (x|P_e y)$. Observing Proposition 4.9(i), $\pi(g)^* = \pi(g^{-1})$, and $\int f = \int \check{f}$, we obtain

$$\begin{aligned}(P_e x|y) &= (\int_G e(g^{-1}){\cdot}\pi(g)x \mid y) = \int_G e(g^{-1})(\pi(g)x|y)\,dg \\ &= \int_G \overline{e}(g)(x \mid \pi(g^{-1}y)\,dg = \int_G (x \mid e(g){\cdot}\pi(g^{-1}y)\,dg \\ &= \int_G (x \mid e(g^{-1}){\cdot}\pi(g)y)\,dg = (x \mid P_e y),\end{aligned}$$

as asserted.

Thus Proposition 4.13(ii) implies that $E_{\varepsilon'} \subseteq \ker P_{e_\varepsilon}$ for $\varepsilon' \neq \varepsilon$, and thus the sum $E_{\text{fin}} = \sum_{\varepsilon \in \widehat{G}} E_\varepsilon$ is extended over mutually orthogonal closed subspaces. Its closure E, therefore, is the orthogonal Hilbert space sum $\bigoplus_{\varepsilon \in \widehat{G}} E_\varepsilon$. This proves the first assertion.

Now let E_ε denote the isotypic component of type ε. By Corollary 2.25, E_ε is the Hilbert space direct sum $\bigoplus_{j \in J(\varepsilon)} F_j$ of a family of simple submodules F_j. By Proposition 4.18, it follows that $F_j \in \varepsilon$. □

We note that the isotypic decomposition is unique and has canonically determined summands. On the other side, the sum decomposition

$$E_\varepsilon = \bigoplus_{j \in J(\varepsilon)} F_j, \qquad F_j \in \varepsilon$$

is not unique. In Corollary 2.25 one had to invoke the Axiom of Choice. The only isomorphy invariant for E_ε is the *multiplicity* $\operatorname{card} J(\varepsilon)$ which is, in essence, a Hilbert space dimension. In the general theory we have no such cardinal available. We therefore had to code multiplicity in terms of a locally convex topological vector space which is canonically attached to each isotypic component, namely the space $\operatorname{Hom}_G(F, E)$ of the Structure Theorem 4.22. When specialized to Hilbert modules, this vector space is a Hilbert space whose Hilbert space dimension is the multiplicity mentioned above.

A particular class of locally convex complete vector spaces is that of so-called weakly complete vector spaces. We shall discuss these vector spaces in greater detail in Appendix 7. Suffice it here to understand that a real vector space is called *weakly complete* iff it is algebraically and topologically isomorphic to a product $\mathbb{R}^I$ of a family of copies of the reals.

Exercise E4.8. Prove the following application of the Structure Theorem of G-Modules 4.22.

Let V be a locally convex weakly complete vector space isomorphic to $\mathbb{R}^I$ and an effective G-module for a compact group G. In particular, the associated representation $\pi\colon G \to \operatorname{Aut} V$ is injective. Assume that $V_{\rm fin} = V$. Then

(i) *V is a finite direct sum (and product) $V_1 \oplus \cdots \oplus V_k$ of isotypic components.*

(ii) *G is a compact Lie group.*

In particular, if G is profinite, then it is finite.

[Hint. (i) By Theorem 4.22, $V_{\rm fin}$ is a direct sum of its isotypic components V_ε, where $\varepsilon \in \widehat{G}$ is an equivalence class of irreducible representations of G. Each V_ε is a module retract of V under a canonical projection P_ε and is therefore complete, and thus as a closed vector subspace of a weakly complete vector space is weakly complete. The universal property of the product $W \stackrel{\rm def}{=} \prod_{\varepsilon\in\widehat{G}} V_\varepsilon$ gives us an equivariant G-module morphism $\varphi\colon V \to W$ of weakly complete G-modules such that $\mathrm{pr}_\varepsilon \circ \varphi = P_\varepsilon$. Since morphisms of weakly complete vector spaces have a closed image (see [188], Theorem A2.12(b)) and

$$\sum_{\varepsilon\in\widehat{G}} V_\varepsilon = \{(v_\varepsilon)_{\varepsilon\in\widehat{G}} \in W : v_\varepsilon = 0 \text{ for all but finitely many } \varepsilon \in \widehat{G}\} \subseteq W$$

is in the image of φ and is dense in W we know that φ is surjective. Now $V = V_{\rm fin}$ and $\varphi(V_{\rm fin}) \subseteq W_{\rm fin}$ whence $W_{\rm fin} = W$. It is readily verified that

$$W_\varepsilon = \{(v_\eta)_{\eta\in\widehat{G}} \in W : v_\eta = 0 \text{ for } \eta \neq \varepsilon\}.$$

Therefore $W_{\rm fin} = \sum_{\varepsilon\in\widehat{G}} V_\varepsilon$, and thus

$$\sum_{\varepsilon\in\widehat{G}} V_\varepsilon = W = \prod_{\varepsilon\in\widehat{G}} V_\varepsilon.$$

This equation, however, implies that the set $S \stackrel{\rm def}{=} \{\varepsilon \in \widehat{G} : V_\varepsilon \neq \{0\}\}$ is finite. List the members of $\{V_\varepsilon : \varepsilon \in S\}$ as $V_1, \ldots, V_k$. Then assertion (i) follows.

(ii) For a simple G-module F denote by $[F] \in \widehat{G}$ its equivalence class. Let F_j, $j = 1, \ldots, k$ be simple G-modules such that $S = \{[F_j] : j = 1, \ldots, k\}$, $V_j = V_{[F_j]}$. A simple G-module is finite-dimensional by Theorem 3.51, so the underlying vector space of each F_j is finite dimensional. By Weyl's Trick 2.10 the G-module structure of F_j is given by an orthogonal representation $\pi_j\colon G \to \mathrm{O}(F_j)$, $j = 1, \ldots, k$ defined by $\pi_j(g)(v) = g{\cdot}v$. By Theorem 4.22 again, $V_j \cong \operatorname{Hom}_G(F_j, V) \otimes F_j$, where $g \in G$, $f \in \operatorname{Hom}(F_j, V)$, and $v \in V$ imply $g{\cdot}(f \otimes v) = f \otimes g{\cdot}v$. Since V is an effective (or faithful) G-module, (i) implies $\bigcap_{j=1}^k \ker \pi_j = \{1\}$. Since $\mathrm{O}(F_j)$ is a compact Lie group, so $L \stackrel{\rm def}{=} \mathrm{O}(F_1) \times \cdots \times \mathrm{O}(F_k)$ is a compact Lie group as well. Thus the compact group G allows an injective representation $G \to L$ into a compact Lie group and is therefore a compact Lie group.

Since as profinite compact Lie group is finite, the assertion follows.] □

Cyclic Modules

The structure theorems yield for us a classification of cyclic modules.

Theorem 4.24. *Assume that E is a complex Hilbert G-module for a compact group G and that $E = \bigoplus_{\varepsilon\in\widehat{G}} E_\varepsilon$ is its isotypic decomposition. Then the following assertions are equivalent:*

(i) *E is cyclic.*

(ii) *There is a countable subset $\mathcal{C} \subseteq \widehat{G}$ such that $E = \bigoplus_{\varepsilon\in\mathcal{C}} E_\varepsilon$ and $E_\varepsilon \cong F^{m(\varepsilon)}$, $[F] = \varepsilon$ and $1 \le m(\varepsilon) \le \dim F$.*

(iii) *There is a countable subset $\mathcal{C} \subseteq \widehat{G}$ such that $E = \bigoplus_{\varepsilon\in\mathcal{C}} E_\varepsilon$ and E_ε is cyclic.*

Proof. (i) implies (ii) Assume that E is cyclic; i.e. $E = \overline{\operatorname{span}(G\cdot w)}$. The isotypic component E_ε is the image of E under the continuous equivariant projection P_{e_ε} according to the Structure Theorem 4.23. Hence it is cyclic and isotypic, and therefore, by 4.18, is of the form $E_\varepsilon = \operatorname{span} G\cdot w_\varepsilon \cong F^{m(\varepsilon)}$ with $[F] = \varepsilon$, $0 \le m(\varepsilon) \le \dim F$ and with $w_\varepsilon = P_{e_\varepsilon}(w)$. Now w is the Hilbert space orthogonal sum $w = \bigoplus_{\varepsilon\in\widehat{G}} w_\varepsilon$. Hence the family $(\|w_\varepsilon\|_2)_{\varepsilon\in\widehat{G}}$ is square summable (i.e. is in $\ell^2(\widehat{G})$), and thus the set $\mathcal{C} \stackrel{\text{def}}{=} \{\varepsilon \in \widehat{G} \mid w_\varepsilon \ne 0\}$ is countable. Then E has the form explained in (ii).

(ii)⇒(iii) is a trivial implication.

(iii)⇒(i) Assume $E_\varepsilon = \overline{\operatorname{span} Gw_\varepsilon}$ for all $\varepsilon \in \mathcal{C}$ in a countable subset $\mathcal{C} \subseteq \widehat{G}$ and that $E = \bigoplus_{\varepsilon\in\mathcal{C}} E_\varepsilon$. We may assume that each w_ε has norm 1. Let $\{\varepsilon_1, \varepsilon_2, \ldots\}$ be an enumeration of $\mathcal{C}$ and write w_n for w_{ε_n}. The orthogonal sum $w = \sum_{n=1}^\infty \frac{1}{n} w_n$ is a member of $E = \bigoplus_{n\in\mathbb{N}} E_{\varepsilon_n}$ since $(\frac{1}{n})_{n\in\mathbb{N}}$ is square summable. Define $F = \overline{\operatorname{span} G\cdot w}$. then $F_\varepsilon = P_{e_\varepsilon} F$. Since P_{e_η} is a continuous retraction, its application commutes with the formation of the closure. Hence by the equivariance of P_{e_η} we compute

$$F_\eta = \overline{P_{e_\eta} \operatorname{span} G\cdot w} = \overline{\operatorname{span} G\cdot P_{e_\eta} w} = \overline{\operatorname{span} G\cdot\big(\tfrac{1}{n}\cdot w_\eta\big)} = E_\eta.$$

Then from 4.23 it follows that $F = E$ which showsthat E is cyclic. □

From Proposition 4.18 we have a good idea what isotypic cyclic modules of type $[F]$ look like: their dimension is bounded by $(\dim F)^2$.

Certain portions of Theorem 4.24 work for any feebly complete G-module. The isotypic components of a cyclic module are always cyclic and are as specified in 4.18. If a family $(v_\varepsilon)_{\varepsilon\in\widehat{G}}$ consists of elements satisfying $\overline{\operatorname{span} G\cdot v_\varepsilon} = E_\varepsilon$ and is summable with a sum v, then $\overline{\operatorname{span} G\cdot v} = \bigoplus_{\varepsilon\in\widehat{G}} E_\varepsilon$.

Corollary 4.25. *Let G be a connected compact abelian group and E a Hilbert G-module over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ such that each isotypic component is simple. Assume that at most countable many isotopic components are nonzero. Then E is cyclic.*

Moreover, if E has only a finite number of nonzero isotypic components, then the set of generators is an open dense subset of E.

Proof. If $\mathbb{K} = \mathbb{C}$ then the assertion is an immediate consequence of the preceding Theorem 4.24. So assume now that $\mathbb{K} = \mathbb{R}$. The module $E = E_{\rm eff}$ is of the form $\bigoplus_{\varepsilon\in\widehat{G}_{\mathbb{R}}\setminus\{\mathbf{1}\}} E_\varepsilon$ where the isotypic component E_ε is simple by hypothesis. Since G is connected, it is two-dimensional (3.55(iii)). By Proposition 3.57 E has a complex G-module structure such that $E_\varepsilon = E_{\chi_\varepsilon}$ is a complex one-dimensional isotypic component determined by the character χ_ε according to 3.57. Hence by 4.24, E is a cyclic complex G-module generated by an element $w \in E$.

Let $F = \overline{\operatorname{span}_{\mathbb{R}} Gw}$ and $\varepsilon \in \widehat{G}_{\mathbb{R}} \setminus \{\mathbf{1}\}$. Abbreviate χ_ε by ρ. Recall the projection $P = P_{e_\rho}$ onto the isotypic component $E_\rho = E_\varepsilon$. Set $w_\varepsilon \overset{\rm def}{=} P(w) \in E_\varepsilon \cap F$. Since P is a continuous retraction we have $P(F) = \overline{P\operatorname{span}_R Gw} = \overline{\operatorname{span} GP(w)}$. If the module E_ε is $\{0\}$, then it is trivially cyclic. If not then $w_\varepsilon \neq 0$ since $E_\varepsilon = P(E)$ is a complex cyclic module with generator $P(w)$. Since the real G-module E_ε is simple by hypothesis, $E_\varepsilon = \operatorname{span}_{\mathbb{R}} Gw_\varepsilon = P(F)$. Thus $E_\varepsilon \subseteq F$ and therefore $F = E$. Hence E is cyclic.

Now assume that E has only a finite number of nonzero isotypic components $E_1, \dots, E_n$. Every nonzero element of E_j is a generator. Let P_j denote the equivariant projection onto E_j. Then $P_1^{-1}(0) \cup \cdots \cup P_n^{-1}(0)$ is the set of nongenerators, and it is clearly closed and nowhere dense. □

Postscript

The theory of characters is classical. However Part 2 of this chapter culminating in a fairly complete structure theory of G-modules such as is summarized in Theorem 4.22 is not. While the Hilbert space version 4.23 is an essential part of the common textbook literature of the subject, the developments leading up to 4.22 are not in the books and may, for all we know, be published here for the first time. They are taken from [169].

The description of the structure of the isotypic components of a feebly complete G-module E as described in 4.22 is as satisfactory as one might desire. It extends their characterisation from the case of finite dimensional modules verbatim. The proof in the general situation, however, required protracted arguments. Also, the fact that each isotypic component is a closed equivariant linear retract of E is gratifying.

On the other hand, the result that the subspace $E_{\rm fin}$ of almost invariant elements is algebraically the direct sum of the isotypic components is probably the best one can expect in terms of a global decomposition in view of the generality we have allowed for the locally convex vector space E. Theorem 4.23, the Structure

Theorem of Hilbert G-Modules, illustrates what one does obtain if E is as special as a Hilbert space.

The material on cyclic submodules in the last subsection will be used in Chapter 6.

References for this Chapter—Additional Reading

[62], [169], [196].

Chapter 5

Linear Lie Groups

We recall from Definition 2.41 that a compact group is called a *compact Lie group* if it has no small subgroups. Corollary 2.40 provided a number of equivalent conditions for a compact group to be a Lie group. For instance, every compact group of matrices is a compact Lie group, and every compact Lie group is isomorphic to a compact matrix group. We shall investigate the fundamentals of Lie group theory by paying particular attention to that aspect of Corollary 2.40 which says that for a compact group G the following two conditions are equivalent:

(1) *G is a compact Lie group.*

(2) *G is isomorphic as a topological group to a compact subgroup of the multiplicative group of some Banach algebra A.*

Typically, for the standard compact matrix groups such as $\mathrm{SO}(n)$ or $\mathrm{U}(n)$ we may take $A = M_n(\mathbb{R})$ or $A = M_n(\mathbb{C})$, the algebra of real or complex $n \times n$-matrices. Therefore, we shall concentrate on Lie groups which are closed subgroups of the multiplicative group of some Banach algebra.

This focus demands that we now learn some basic facts about Banach algebras which generalize familiar aspects of elementary real and complex analysis. Isolated aspects were anticipated in Proposition 1.4 and Lemma 2.38.

Prerequisites. We require an acquaintance with basic Banach algebra theory and elementary real and complex analysis, notably manipulation of power series. Often we shall apply the inverse function theorem and use the Picard–Lindelöf existence and uniqueness theorem for ordinary differential equations. We make reference to the spectral theory of elements in a Banach algebra, but for the reader interested in finite dimensional situations only, the theory of eigenvalues of endomorphisms of finite dimensional complex vector spaces suffices. In the last section (for 5.70) we shall use a partition of unity subordinate to an open cover in a paracompact space for which we shall give references.

Preliminaries

We let A be a Banach algebra with identity $\mathbf{1}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The set of invertible elements or *units* is again called A^{-1} (cf. Proposition 1.4). We recall some elementary facts from analysis.

A family $\{a_j \mid j \in J\}$ of elements of A is said to be *absolutely summable* if for each $\varepsilon > 0$ there is a finite set $F \subseteq J$ such that for every finite set $G \subseteq J$ with $G \cap F = \emptyset$ we have $\sum_{j\in G} \|a_j\| < \varepsilon$. We say that an element a is the *sum* of the family if $\{a_j \mid j \in J\}$ if for each $\varepsilon > 0$ there is a finite subset F of J such that for all finite subsets G of J with $F \subseteq G$ we have $\|a - \sum_{j\in G} a_j\| < \varepsilon$. Every absolutely summable family has a unique sum (see Exercise E5.1(i) below), and we write $a = \sum_{j\in J} a_j$. We shall deal always with countable index sets.

If we have a family $\{\alpha_n \in \mathbb{K} \mid n = 0, 1, 2, \ldots\}$ of scalars, and if, for some $x \in A$, the family $\{\alpha_n{\cdot}x^n \mid n = 0, 1, 2, \ldots\}$ (with $x^0 = \mathbf{1}$) is absolutely summable, then we say that the *power series with coefficients* α_n is absolutely convergent for x and has the sum $\sum_{n=0}^{\infty} \alpha_n x^n$. We also say simply that *the power series* $\sum_{n=0}^{\infty} \alpha_n{\cdot}x^n$ *is absolutely convergent.* The following exercise is a direct generalisation of the corresponding fact in elementary analysis:

Exercise E5.1. (i) Let E be a Banach space. Then for an absolutely summable family $\{a_j \mid j \in J\}$ the set $\{j \in J \mid a_j \neq 0\}$ is countable and the sum $a = \sum_{j\in J} a_j$ exists uniquely.

(ii) For $\rho = \left(\overline{\lim}_{n\in\mathbb{N}} \sqrt[n]{|\alpha_n|}\right)^{-1} \in [0, \infty]$, the power series with coefficients α_n converges absolutely for $\|x\| < \rho$.

(iii) The power series $\sum_{n=0}^{\infty} x^n$ is absolutely convergent for $\|x\| < 1$ and its sum is the inverse of $\mathbf{1} - x$ in A. In particular, the whole open unit ball around $\mathbf{1}$ belongs to the set A^{-1}.

(iv) The power series $\sum_{n=0}^{\infty} \frac{1}{n!}{\cdot}x^n$ is absolutely convergent for all $x \in A$.

(v) The power series $\sum_{n=1}^{\infty} \frac{1}{n}{\cdot}x^n$ is absolutely convergent for $\|x\| < 1$.

[Hint. (i) Firstly note that $J_n \stackrel{\text{def}}{=} \{j \in J : \|a_j\| \geq \frac{1}{n}\}$ is finite and then that $a_j \neq 0$ iff $j \in \bigcup_{n=1}^{\infty} J_n$. Secondly, define recursively a sequence $F_1 \subseteq F_2 \subseteq \cdots$ of finite subsets of J such that $G \cap F_n = \emptyset$ implies $\sum_{j\in G} |a_j| < \frac{1}{n}$. Show that $s_n = \sum_{j\in F_n} a_j$ yields a Cauchy sequence. Use completeness of E to get $a = \lim s_n$. Verify that a meets the requirements.] □

If E and F are Banach spaces and $A_n\colon E^n \to F$ is a continuous n-linear map, we shall write $A_n h^n$ instead of $A_n(h, \ldots, h)$, $h \in E$. If U and V are open subsets of E and F, respectively, then a function $f\colon U \to V$ is called *analytic* if for each $u \in U$ there is a positive number r and a family of nonnegative numbers $\{\alpha_n \mid n = 0, 1, \ldots\}$ with $\sum_{n=0}^{\infty} \alpha_n r^n$ convergent such that the following conditions are satisfied:

(i) $\|x - u\| < r$ implies $x \in U$,

(ii) there is a family $\{A_n \mid n = 1, 2, \ldots\}$ of continuous multilinear maps $A_n\colon E^n \to F$ and an element $A_0 \in E$ such that

$$\|A_0\| \leq \alpha_0 \text{ and } (\forall n \in \mathbb{N}) \sup_{\|h_1\|,\ldots,\|h_n\|\leq 1} \|A_n(h_1, \ldots, h_n)\| \leq \alpha_n, \text{ and}$$

(iii) $f(x) = \sum_{n=0}^{\infty} A_n(x - u)^n$ for all $\|x - u\| < r$.

We say that (iii) is a *power series expansion of* f *near* u.

Exercise E5.2. (i) Let $B_r(a)$ denote the open ball with radius r around a in a Banach algebra A. If $\sum_{n=0}^{\infty} \gamma_n{\cdot}x^n$ converges absolutely for $\|x\| < r$, then the function $f\colon B_r(0) \to A$ given by $f(x) = \sum_{n=0}^{\infty} \kappa_n{\cdot}x^n$ is analytic.

[Hint. Write $x = u + h$ and set

$$x_{Fj} = \begin{cases} h & \text{if } j \in F; \\ u & \text{if } j \notin F, \end{cases}$$

and

$$A_{kn}h^k = \sum_{\substack{F\subseteq\{1,2,\dots,n\}\\ |F|=k}} x_{F1}x_{F2}\cdots x_{Fn} \quad \text{for} \quad k \le n.$$

Compute $x^n = u^n + A_{1n}h + A_{2n}h^2 + \cdots + A_{n-1,n}h^{n-1} + h^n$. Define A_nh^n to be $\sum_{m=n}^{\infty}\kappa_m A_{nm}h^n$, show that this sum is majorized by $\sum_{m=n}^{\infty}\binom{m}{n}|\alpha_m|\|h\|^n\|u\|^{m-n}$ and that $\sum_{n=0}^{\infty}A_nh^n$ is the power series expansion of $f(u+h)$ around 0.]

(ii) The composition of analytic functions is analytic; that is if $g: V \to W$ and $f: U \to V$ are analytic, then $g\circ f: U \to W$ is analytic. In particular, if, in a Banach algebra A the function g has a power series expansion $\sum_{n=0}^{\infty}\alpha_n x^n$ around 0 and f has a power series expansion $\sum_{n=0}^{\infty}\beta_n(x-\alpha_0{\cdot}\mathbf{1})^n$ around $\alpha_0{\cdot}\mathbf{1}$ then $f\circ g$ has a power series expansion $f\big(g(x)\big) = \sum_{n=0}^{\infty}\gamma_n x^n$ at zero.

(iii) If two analytic functions $f, g: U \to V$ are given, then the set of points $u \in U$, at which they have coinciding power series expansions near u, is both open and closed in U. In particular, if U is connected and f and g agree on some nonempty open set, then they agree. □

The Exponential Function and the Logarithm

We have seen in Exercise E5.1(iv) that $\sum_{n=0}^{\infty}\frac{1}{n!}{\cdot}x^n$ is absolutely convergent for all x in a Banach algebra A. We shall call the sum $\exp x$. The following elementary exercise provides the first information on exp.

Exercise E5.3. If x and y commute in A, i.e. satisfy $xy = yx$, then

(i) $\frac{1}{n!}(x+y)^n = \sum_{p+q=n}\frac{1}{p!}\frac{1}{q!}{\cdot}x^py^q$ (*the binomial formula*), and

(ii) $\exp(x+y) = \exp x \exp y$ (*the functional equation of the exponential function*).

(iii) For every x the element $\exp x$ has inverse $\exp -x$.

[Hint. The elementary power series proof works.] □

Informally speaking, the binomial formula and the functional equation for the exponential function are equivalent things.

Definition 5.1. For a Banach algebra A with identity we define

(i) $\exp: A \to A^{-1}$ by $\exp x = \sum_{n=0}^{\infty}\frac{1}{n!}{\cdot}x^n$, and

(ii) $\log: B_1(\mathbf{1}) \to A$ by $\log(1+x) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}{\cdot}x^n$.

The function exp is called *the exponential function of* A and log the *logarithm.* □

By Exercise E5.1(iii) and (iv) the two functions are well-defined. By Exercise E5.2(ii) they are analytic.

The connected component of 0 of the open 0-neighborhood $\exp^{-1} B_1(\mathbf{1})$ is denoted N_0 and the set $\{(x,y) \in A \times A \mid \exp x \exp y \in B_1(\mathbf{1})\}$ is written D. It is instructive to contemplate briefly the real Banach algebra $A = \mathbb{C}$ and the homeomorphism $\exp_0\colon \{z = x+iy \in A : |y| < \pi\} \to \mathbb{C}\backslash]-\infty, 0], \quad \exp_0(z) = e^z$. We write $\log_0 = \exp_0^{-1}$. Then the domain N_0 is bounded by the curve $\{\log_0(1+e^{it}) : t \in]-\pi, \pi[\}$. The points $\log 2$, $\pm\frac{\pi i}{3}$ are on this curve and $\lim_{t\to\pm\pi} \log_0(1+e^{it}) = \pm\frac{\pi i}{2}$.

Lemma 5.2. $B_{\log 2}(0) \subseteq N_0$ *and* $\{(x,y) \in A \times A \mid \|x\| + \|y\| < \log 2\} \subseteq D$.

Proof. The second assertion implies the first via $y = 0$. Let $\|x\|+\|y\| < \log 2$. Then, since in a Banach algebra we have $\|x^n\| \leq \|x\|^n$ we compute $\|\exp x \exp y - \mathbf{1}\| = \|\sum_{0<p+q} \frac{1}{p!q!} x^p y^q\| \leq \sum_{0<p+q} \frac{1}{p!q!} \|x\|^p \|y\|^q = e^{\|x\|} e^{\|y\|} - 1 = e^{\|x\|+\|y\|} - 1 < e^{\log 2} - 1 = 2 - 1 = 1$. □

Proposition 5.3. (i) $\log(\exp x) = x$ *for all* $x \in N_0$.

(ii) $\exp(\log x) = x$ *for all* $x \in B_1(\mathbf{1})$.

(iii) $\exp|N_0\colon N_0 \to B_1(\mathbf{1})$ *is an analytic homeomorphism with analytic inverse* $\log\colon B_1(\mathbf{1}) \to N_0$.

Proof. Before we begin the proof we recall that Definition 5.1 applies, in particular, to $A = \mathbb{C}$, giving the classical exponential function $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ and $\log\colon B_1(1) \to \mathbb{C}$. The relations $\log e^z = z$ for z near 0 and $e^{\log(1+z)} = 1 + z$ for $|z| < 1$ are part of the classical theory. However, if one wants a proof directly from Definition 5.1, one should first argue that $\exp' z = \exp z$ and $\log' z = \frac{1}{z}$ (directly from the power series and the basic theorem on the differentiation of power series on the open disc of convergence). Now the derivative of $z \mapsto \log(\exp z)$ is $\frac{\exp z}{\exp z} = 1$ by the chain rule and agrees with the derivative of $z \mapsto z$. Both functions take the value 0 for $z = 0$ and therefore agree. Similarly, we note that the derivative of the function $\varphi\colon B_1(0) \to \mathbb{C}$, $\varphi(z) = \exp(\log(1+z))$ at z is $\varphi'(z) = \frac{\varphi(z)}{1+z}$ and $\varphi''(z) = 0$ is quickly verified via the quotient rule. Now $\frac{\varphi'(z)}{\varphi(z)}$ is the derivative of $\log \varphi(z)$ and $z \mapsto \frac{1}{1+z}$ is the derivative of $z \mapsto \log(1+z)$. Since $\varphi(0) = \varphi'(z) = 1$, $\varphi(z) = 1 + z$ follows for all small enough z and then for all z with $|z| < 1$. After this recalling of elementary complex function theory we can proceed with the proof.

(i) The function $\Psi\colon N_0 \to A$ given by $\Psi(x) = \log(\exp x)$ is analytic and has a power series expansion $\Psi(x) = \sum_{n=0}^{\infty} \gamma_n {\cdot} x^n$ at zero. For $z \in \mathbb{K}$ with $|z| < \log 2$ define $\psi(z) \in \mathbb{C}$ by $\psi(z){\cdot}\mathbf{1} = \Psi(z{\cdot}\mathbf{1}) = (\sum_{n=0}^{\infty} \gamma_n z^n){\cdot}\mathbf{1}$. Then $\psi(z) = \log(e^z) = z$ by our initial remarks, that is $\gamma_0 = \gamma_n = 0$ for $n > 1$. Thus the analytic functions Ψ and the inclusion map $N_0 \to A$ agree on a neighborhood of 0 and so agree on the connected domain N_0.

(ii) The function $\Phi\colon B_1(\mathbf{1}) \to A^{-1}$ given by $\Phi(x) = \exp(\log x)$ is analytic and has a power series expansion $\Phi(\mathbf{1} + h) = \exp\big(\log(\mathbf{1} + h)\big) = \sum_{n=0}^{\infty} \delta_n h^n$. Again define $\varphi(z)$ by $\varphi(1 + z){\cdot}\mathbf{1} = \Phi((1 + z){\cdot}\mathbf{1}) = e^{\log(1+z)}{\cdot}\mathbf{1}$ for $|z| < 1$ and notice $\varphi(z) = 1 + z$ by our initial remarks. This implies $\delta_0 = \delta_1 = 1$, $\delta_n = 0$ for $n > 1$.

Hence the analytic functions $h \mapsto \Phi(\mathbf{1}+h)$ and $h \mapsto \mathbf{1}+h$ from $B_1(0)$ into A agree on a neighborhood of 0 and hence agree on the ball $B_1(0)$.

(iii) By the definition of N_0 we have $\exp(N_0) \subseteq \exp\left(\exp^{-1} B_1(\mathbf{1})\right) \subseteq B_1(\mathbf{1})$, and by (ii) we have $\log B_1(\mathbf{1}) \subseteq \exp^{-1}\left(B_1(\mathbf{1})\right)$; since $B_1(\mathbf{1})$ and thus $\log B_1(\mathbf{1})$ are connected, $\log B_1(\mathbf{1}) \subseteq N_0$ follows from the definition of N_0. These facts together with (i) and (ii), however, prove assertion (iii). □

We note that we have used some facts on analytic functions from Exercise E5.2, namely, that two analytic functions which agree on a nonempty open subset of a connected common domain agree. If we do not wish to use this fact, our proof still shows in an elementary fashion that $\log(\exp x) = x$ and $\exp\log(\mathbf{1}+x) = \mathbf{1}+x$ for all x sufficiently close to 0.

Definition 5.4. We define the *Baker–Campbell–Hausdorff–Dynkin multiplication* or, *C–H-multiplication* for short, by

$$(x, y) \mapsto x * y = \log(\exp x \exp y) \colon D \to A. \qquad \square$$

This multiplication is analytic and is defined at least for all pairs (x, y) with $\|x\| + \|y\| < \log 2$, and thus is defined certainly on $B_{\frac{\log 2}{2}}(0) \times B_{\frac{\log 2}{2}}(0)$.

Proposition 5.5. *For $\|x\| + \|y\| < \log 2$, the element $x * y$ is the sum of an absolutely summable family which can be grouped as follows*

$$x * y = x + y + \frac{1}{2}\cdot[x, y] + H_3(x, y) + H_4(x, y) + \cdots,$$

where $H_n(x, y)$ denotes a homogeneous polynomial in the two (in general not commuting) variables x and y of homogeneous degree n and where $[x, y] = xy - yx$. Also

$$\|x * y\| \leq -\log\left(2 - e^{\|x\|+\|y\|}\right).$$

Proof. Assume $\|x\| + \|y\| < \log 2$. First notice that $(\exp x \exp y - \mathbf{1})^n$ is a well defined element of the open unit ball around 0 and is the sum of an absolutely summable family of elements

$$\frac{1}{\prod_{k=1}^{n} p_k! q_k!} x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}, \qquad 0 < p_1 + q_1, \ldots, p_n + q_n.$$

Set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. If we multiply each of these terms by $-(-1)^n/n$, let n range through $\mathbb{N}$, and the pairs $(p_1, q_1), \ldots, (p_n, q_n)$ through $\mathbb{N}_0 \times \mathbb{N}_0$ with the sole restriction $p_k + q_k > 0$ for all k, then we obtain an absolutely summable family whose sum is

$$x * y = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n}\cdot(\exp x \exp y - \mathbf{1})^n.$$

If r and s are real numbers with $r+s<\log 2$ such that $\|x\|\le r$ and $\|y\|\le s$, then a majorizing family of positive real numbers is

$$\frac{1}{n\prod_{k=1}^n p_k!q_k!}r^{\sum_{k=1}^n p_k}s^{\sum_{k=1}^n q_k}$$

with $n=1,2,\dots$ and $0<p_1+q_1,\dots,p_n+q_n$. Recall that $0\le r+s<\log 2$ entails $0<2-e^{r+s}<1$ and note that the sum of this family is $\sum_{n=1}^\infty \frac{1}{n}(e^re^s-1)^n = \sum_{n=1}^\infty \frac{1}{n}(e^{r+s}-1)^n = -\log\big(1-(e^{r+s}-1)\big) = -\log(2-e^{r+s})$.
We define

$$I_d \stackrel{\text{def}}{=} \big\{\big((p_1,q_1),\dots,(p_n,q_n)\big)\mid p_k+q_k>0,\ k=1,\dots,n;\ \ \textstyle\sum_{k=1}^n(p_k+q_k)=d\big\},$$

and set

$$H_d(x,y) = \sum_{\big((p_1,q_1),\dots,(p_n,q_n)\big)\in I_d} \frac{-(-1)^n}{n\prod_{k=1}^n p_k!q_k!}\cdot x^{p_1}y^{q_1}\cdots x^{p_n}y^{q_n}, \quad x,\,y\in A,$$

$$h_d(r,s) = \sum_{\big((p_1,q_1),\dots,(p_n,q_n)\big)\in I_d} \frac{1}{n\prod_{k=1}^n p_k!q_k!}\cdot r^{\sum_{k=1}^n p_k}s^{\sum_{k=1}^n q_k}, \quad r,\,s\in\mathbb{R}.$$

Then H_d and h_d are homogeneous of degree d, and

$$\|H_d(x,y)\|\le h_d(\|x\|,\|y\|),\quad x,\,y\in A.$$

For $d=1$ we have to sum over the index set $I_1=\{\big((1,0)\big),\ \big((0,1)\big)\}$, and for $d=2$ over the index set

$$I_2=\left\{\begin{matrix} \big((2,0)\big), & \big((1,0),\,(0,1)\big)\\ \big((0,2)\big), & \big((1,0),\,(1,0)\big)\\ \big((1,1)\big), & \big((0,1),\,(0,1)\big)\\ & \big((0,1),\,(1,0)\big)\end{matrix}\right\}.$$

A quick calculation gives $H_1(x,y)=x+y$ and $H_2(x,y)=\frac{1}{2}\cdot(xy-yx)$. Moreover, $\sum_{d=1}^\infty h_d(r,s)=-\log(2-e^{r+s})$. □

Exercise E5.4. Compute $H_d(x,y)$ for $d=3,4$, and $h_d(r,s)$ for $d=1,2$. □

Corollary 5.6. (i) *For all* $(x,y)\in D$ *we have*

$$\exp(x*y)=\exp x\exp y.$$

(ii) *If* B *is a zero neighborhood such that* $B*B$, $B*(B*B)$ *and* $(B*B)*B$ *are defined, then* $x,y,z\in B$ *implies* $x*(y*z)=(x*y)*z$.
(iii) *If* $(x,y)\in D$ *and* $[x,y]=0$ *then* $x*y=x+y$.
(iv) *If* $(x,-x)\in D$ *then* $x*(-x)=(-x)*x=0$.

Proof. (i) Since $(x,y)\in D$ we have $\|\exp x\exp y-\mathbf{1}\|<1$, hence $\exp(x*y) = \exp\big(\log\big(\mathbf{1}+(\exp x\exp y-\mathbf{1})\big)\big)=\exp x\exp y$ by Proposition 5.3(ii).

(ii) By (i) we obtain

$$\begin{aligned}\exp\big(x*(y*z)\big) &= \exp x(\exp y \exp z)\\ &= (\exp x \exp y)\exp z = \exp\big((x*y)*z\big).\end{aligned}$$

Since $(x, y*z) \in D$ we have $\|\exp\big(x*(y*z)\big)-\mathbf{1}\| = \|\exp x \exp(y*z)-\mathbf{1}\| < 1$, hence $B*(B*B) \subseteq N_0$. Therefore, $\log\big(\mathbf{1}+(\exp x\exp(y*z)-\mathbf{1})\big) = \log\big(\exp(x*(y*z))\big) = x*(y*z)$ by Proposition 5.3(i). Likewise we find $\log\big(\exp((x*y)*z)\big) = (x*y)*z$ for all $x, y, z \in B$. Hence (ii) follows.

(iii) If x and y commute, then $\exp(x+y) = \exp x \exp y$ by Exercise E5.3 and if $(x, y) \in D$, then $x*y = \log(\exp x \exp y) = \log\exp(x+y) = x+y$.

(iv) Since $[x, -x] = -[x, x] = 0$, by (iii) we have $x*(-x) = x+(-x) = 0$. Likewise, $(-x)*x = 0$. ☐

By the preceding results, there is a ball $B = B_r(0)$ around 0 which is a *local group* with respect to a partially defined multiplication $*\colon B\times B \to A$ in the sense of the properties listed in the preceding proposition, and the exponential function induces a local isomorphism $\exp|B : B \to \exp B$ into the multiplicative group A^{-1}. The local group operation $*$ is given by a power series, hence is analytic. Its definition therefore uses, from the given Banach algebra A, apart from the topology, the following information:

(i) the scalar multiplication,
(ii) the addition,
(iii) the multiplication.

Assuming that we are given $*$, can we recover (i), (ii), and (iii)?

Let us first observe, that the scalar multiplication is not too problematic. By Corollary 5.6(iii), for every $x \in A$, the function $t \mapsto t{\cdot}x\colon]-\varepsilon,\varepsilon[\to (B,*)$ is continuous and satisfies $(s{\cdot}x)*(t{\cdot}x) = (s+t){\cdot}x$ whenever $s, t, s+t \in]-\varepsilon,\varepsilon[$ and $t \mapsto \exp t{\cdot}x$ is a morphism of topological groups $\mathbb{R} \to A^{-1}$ in view of Exercise E5.3. This calls for a definition. First we note a piece of background information which, for topological abelian groups is discussed in great detail in Chapter 7. If G is a topological group and X a topological space, then the set $C(X, G)$ of continuous functions $f\colon X \to G$ is a group under pointwise operations. Let $\mathcal{U}$ denote the filter of all identity neighborhoods U of G and $\mathcal{K}$ the set of compact subsets K of X. For $K \in \mathcal{K}$ and $U \in \mathcal{U}$ we set $W(K, U) = \{f \in C(X, G) \mid f(K) \subseteq U\}$. Then $\{W(K, U) \mid (K, U) \in \mathcal{K}\times\mathcal{U}\}$ is a filterbasis of identity neighborhoods of a topology making $C(X, G)$ into a topological group. This topology, and each topology induced by it on a subset of $C(X, G)$ is called the topology of uniform convergence on compact sets. In this chapter this topology is used mostly in the more familiar case that G carries a metric compatible with its topology and that $X = \mathbb{R}$. Then a sequence f_n of continuous functions from $\mathbb{R}$ to G converges to a continuous function f if and only if for each number $r \in \mathbb{R}$ the sequence of function $f_n|[-r, r]$ converges uniformly to $f|[-r, r]$ in the sense of the metric of G.

Definition 5.7. (i) A *one parameter subgroup of a topological group* G is a homomorphism $X\colon \mathbb{R} \to G$ of topological groups, that is an element X of $\mathrm{Hom}(\mathbb{R}, G)$.

The topological space $\mathrm{Hom}(\mathbb{R}, G)$ which we obtain by endowing $\mathrm{Hom}(\mathbb{R}, G)$ with the topology of uniform convergence on compact subsets of $\mathbb{R}$ will be denoted by $\mathfrak{L}(G)$.

(ii) A *local one parameter subgroup of* G is a continuous function $f: I \to G$ with an interval $I \subseteq \mathbb{R}$ which is a neighborhood of 0 such that s, t, $s+t \in I$ implies $f(s+t) = f(s)f(t)$. □

One useful lemma which we discussed in an earlier exercise (Exercise E1.8) must be recorded at this point:

Lemma 5.8. *For every local one parameter subgroup $f: I \to G$ of a topological group there is a unique extension $X: \mathbb{R} \to G$ to a one parameter subgroup.*

Proof. Let $r \in \mathbb{R}$. There is at least one natural number n such that $\frac{r}{n} \in I$. Assume that $m, n \in \mathbb{N}$ satisfy $\frac{r}{m}, \frac{r}{n} \in I$. Then, since I is an interval, $\frac{kr}{mn} \in I$ for $k = 0, 1, \ldots, \max\{m, n\}$. We observe that $f(r/mn)^{mn} = \big(f(r/mn)^m\big)^n = \big(f(r/mn + \cdots + r/mn)\big)^n$ since $f(r/mn)^{k+1} = f(kr/mn)f(r/mn)$ for $k = 0, \ldots, \max\{m, n\} - 1$ as f is a local morphism. But $f(mr/mn) = f(r/n)$. Thus $f(r/mn)^{mn} = f(r/n)^n$. Exchanging the roles of m and n we also find $f(r/mn)^{mn} = f(r/m)^m$. Thus we can unambiguously define

(1) $$X(r) = f(r/n)^n \quad \text{for any natural number } n \text{ with } r/n \in I.$$

Next we observe that $X(r+s) = X(r)X(s)$ for all $r, s \in \mathbb{R}$; indeed let us find a natural number n such that r/n, s/n, and $(r+s)/n$ are all in I. Then $f(r+s) = f(r)f(s) = f(s)f(r)$. In particular, $f(r)$ and $f(s)$ are contained in some abelian subgroup of G and thus $X(r+s) = f(\frac{r+s}{n})^n = \big(f(\frac{r}{n})f(\frac{s}{n})\big)^n = f(\frac{r}{n})^n f(\frac{s}{n})^n = X(r)X(s)$.

Clearly, $X|I = f$ since for $r \in I$ we can apply definition (1) with $n = 1$. Thus X is a morphism of groups extending the continuous function f. In particular, X is continuous at 0. A morphism of groups between topological groups is continuous if and only if it is continuous at the identity (Exercise E1.2(iv)). Hence X is continuous.

Finally, let X and Y be two one parameter subgroups extending f. Then the *equalizer* $E = \{r \in \mathbb{R} \mid X(r) = Y(r)\}$ is a closed subgroup of $\mathbb{R}$. It contains the identity neighborhood I, hence is also open. As $\mathbb{R}$ is connected, $E = \mathbb{R}$ follows, and thus $X = Y$. The proof is complete. □

With this lemma we obtain at once:

Proposition 5.9 (Recovery of Scalar Multiplication). *Let A denote a Banach algebra. The function which associates with any $x \in A$ the one parameter subgroup $t \mapsto \exp t{\cdot}x: \mathbb{R} \to A^{-1}$ of A^{-1} is a homeomorphism $\alpha: A \to \mathfrak{L}(A^{-1})$.*

Proof. (i) α is injective: Assume that $\exp t{\cdot}x = \exp t{\cdot}y$ for all $t \in \mathbb{R}$. Choose $r > 0$ so small that $|t| < r$ implies $t{\cdot}x,\ t{\cdot}y \in N_0$. Then $t{\cdot}x = \log\exp t.x = \log\exp t{\cdot}y = t{\cdot}y$ for all of these t and thus $x = y$.

(ii) α is surjective: Let $X\colon\mathbb{R} \to A^{-1}$ denote a one parameter group. Choose $r > 0$ so small that $|t| < r$ implies $\|X(t) - \mathbf{1}\| < 1$. If we set $I = \{t \in \mathbb{R} \mid |t| < r\}$, then $f\colon I \to A$ given by $f(t) = \log X(t)$ is continuous and for $s,\ t,\ s+t \in I$ satisfies $f(s+t) = \log\big(X(s)X(t)\big) = \big(\log X(s)\big) * \big(\log X(t)\big) = f(s) * f(t)$. Now let A_X denote the closed subalgebra generated by $X(\mathbb{R})$ in A. Then A_X is a *commutative* subalgebra. By the definition of the logarithm through a power series, $f(t) \in A_X$. Also $(x,y) \in A_X \cap D$ implies $x * y = x + y$ by Corollary 5.6(iii). Thus $f(s) * f(t) = f(s) + f(t)$. Now $f\colon I \to (A,+)$ is a local one parameter subgroup of $(A,+)$. By Lemma 5.8, there is a unique extension to a one parameter subgroup $F\colon\mathbb{R} \to (A,+)$. For any integer n we have $n{\cdot}F(1) = F(n)$. If m is a natural number, then $m{\cdot}F(n/m) = F\big(m(n/m)\big) = F(n) = n{\cdot}F(1)$, whence $F(n/m) = (n/m){\cdot}F(1)$. Thus $F(t) = t{\cdot}F(1)$ for all rational, and then by continuity also for all real numbers. Set $x = F(1)$. Then $\alpha(x)\colon\mathbb{R} \to A^{-1}$ given by $\alpha(x)(t) = \exp t{\cdot}x = \exp F(t)$ is a one parameter group which for $|t| < r$ yields $\alpha(x)(t) = \exp f(t) = \exp\log X(t) = X(t)$. By the uniqueness part of Lemma 5.8 we conclude $X = \alpha(x)$. Thus the surjectivity of α is proved.

(iii) α is continuous: Let $\lim_n x_n = x$ in A. If $C \subseteq \mathbb{R}$ is compact, then $t{\cdot}x = \lim_n t{\cdot}x_n$ uniformly on C. In particular, there is a closed ball B around 0 containing $C{\cdot}x$ and $C{\cdot}x_n$ for all n. Since $\exp\colon A \to A^{-1} \subseteq A$ is uniformly continuous on B (Exercise E5.5), then $\alpha(x)(t) = \exp t{\cdot}x = \lim_n \exp t{\cdot}x_n = \lim_n \alpha(x_n)(t)$ uniformly on C.

(iv) α^{-1} is continuous: Assume that $X = \lim_n X_n$ in $\mathfrak{L}(A^{-1})$. Let $r > 0$ be such that $X(r), X_n(r) \in B_1(\mathbf{1})$ for all n. Then

$$r{\cdot}\alpha^{-1}(X) = \log X(r) = \lim_n \log X_n(r) = \lim_n r{\cdot}\alpha^{-1}(X_n),$$

and since $r \neq 0$, this implies $\alpha^{-1}(X) = \lim_n \alpha^{-1}(X_n)$. □

Exercise E5.5. Show that $\exp\colon A \to A$ is uniformly continuous on any bounded subset of A.

[Hint. The proof for the elementary exponential function on $\mathbb{C}$ works.] □

The preceding result 5.9 can be interpreted as saying among many other things that scalar multiplication can be recovered from the group A^{-1} and indeed from the local operation $*$ in A near 0. Next we try the recovery of addition.

Proposition 5.10 (Recovery of Addition). *Let A be a Banach algebra. Then for any $x, y \in A$ we have*

$$x + y = \lim_n n\left(\frac{1}{n}{\cdot}x * \frac{1}{n}{\cdot}y\right). \tag{2}$$

As a consequence,

$$\exp(x+y) = \lim_n \left(\exp \frac{1}{n}{\cdot}x \exp \frac{1}{n}{\cdot}y\right)^n. \tag{3}$$

Proof. First we choose a natural number p such that $\|x\|, \|y\| < \frac{p}{2}\log 2$. Then $p{\cdot}(\frac{1}{p}{\cdot}x * \frac{1}{p}{\cdot}y) = \sum_{k=1}^{\infty} p{\cdot}H_k(\frac{1}{p}{\cdot}x, \frac{1}{p}{\cdot}y)$ is absolutely summable with majorant

$$\sum_{k=1}^{\infty} \frac{1}{p^{k-1}} h_k(\|x\|, \|y\|).$$

Assume now that $\varepsilon > 0$ is given. Find q such that $\sum_{k=q+1}^{\infty} \frac{1}{p^{k-1}} h_k(\|x\|, \|y\|) < \varepsilon/2$. Choose $N \in \mathbb{N}$ such that $N \geq p$ and that $n > N$ implies

$$\left\| \sum_{k=2}^{q} \frac{1}{n^{k-1}} H_k(x,y) \right\| \leq \frac{1}{n} \sum_{k=2}^{q} \frac{1}{n^{k-2}} h_k(\|x\|, \|y\|) < \varepsilon/2.$$

Now let $n > N$. Then

$$\begin{aligned}
\left\| n{\cdot}(\frac{1}{n}{\cdot}x * \frac{1}{n}{\cdot}y) - (x+y) \right\| &= \left\| \sum_{k=2}^{\infty} n{\cdot}H_k(\frac{1}{n}{\cdot}x, \frac{1}{n}{\cdot}y) \right\| \\
&\leq \left\| \sum_{k=2}^{q} \frac{1}{n^{k-1}}{\cdot}H_k(x,y) \right\| + \left\| \sum_{k=q+1}^{\infty} n{\cdot}H_k(\frac{1}{n}{\cdot}x, \frac{1}{n}{\cdot}y) \right\| \\
&< \frac{\varepsilon}{2} + \sum_{k=q+1}^{\infty} \frac{1}{n^{k-1}}{\cdot}h_k(\|x\|, \|y\|) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=q+1}^{\infty} \frac{1}{p^{k-1}}{\cdot}h_k(\|x\|, \|y\|) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus (2) is proved. But then (3) is a consequence of (2) and Corollary 5.6(i). □

A reader may expect that we now proceed to recover the algebra multiplication from the locally defined $*$-operation in a similar way to that which we used for addition. Indeed for two elements $a, b \in B_1(\mathbf{1})$ we can write $ab = \exp\big((\log a) * (\log b)\big)$. But this does not extend to a global formula in a useful way. Instead, we can recover the bracket operation $[x,y] = xy - yx$. If we compute, for very small x and y, the $*$-product of

$$\frac{1}{n}{\cdot}x * \frac{1}{n}{\cdot}y = \frac{1}{n}{\cdot}(x+y) + \frac{1}{2n^2}{\cdot}[x,y] + \frac{1}{n^3}{\cdot}(\cdots)$$

and the negative of

$$\frac{1}{n}{\cdot}y * \frac{1}{n}{\cdot}x = \frac{1}{n}{\cdot}(x+y) + \frac{1}{2n^2}{\cdot}[y,x] + \frac{1}{n^3}{\cdot}(\cdots),$$

we obtain
$$\frac{1}{n}\cdot x * \frac{1}{n}\cdot y * \frac{-1}{n}\cdot x * \frac{-1}{n}\cdot y = \frac{1}{n^2}\cdot[x,y] + \frac{1}{n^3}\cdot(\cdots).$$
This suggests the following result. In a group G, we write $\operatorname{comm}(x,y) = xyx^{-1}y^{-1}$.

Proposition 5.11 (Recovery of the Bracket). *Let A be a Banach algebra. Then for any $x, y \in A$ we have*

$$(4)\qquad [x,y] = \lim_n n^2\left(\frac{1}{n}\cdot x * \frac{1}{n}\cdot y * \frac{-1}{n}\cdot x * \frac{-1}{n}\cdot y\right).$$

As a consequence,

$$(5)\qquad \exp[x,y] = \lim_n \operatorname{comm}\Big(\exp\frac{1}{n}\cdot x, \exp\frac{1}{n}\cdot y\Big)^{n^2}.$$

Proof. Exercise E5.6 □

Exercise E5.6. Prove Proposition 5.11.
[Hint. Use the proof of Proposition 5.10 as guide.] □

The bracket operation is a prime example of a multiplication in a non-associative algebra which we formally define as follows:

Definition 5.12. A *Lie algebra L* over a given field $\mathbb{K}$ is a vector space over $\mathbb{K}$ together with a bilinear multiplication
$$(x,y) \mapsto [x,y] : L \times L \to L$$
satisfying the following conditions:
(i) $(\forall x \in L)\quad [x,x] = 0$,
(ii) $(\forall x,\, y\, z \in L)\quad \big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] = 0$ (*Jacobi identity*). □

Since $0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x]$, condition (i) implies
(i′) $(\forall x,\, y \in L)\quad [x,y] = -[y,x]$.

If the characteristic of $\mathbb{K}$ is different from 2 then from (i′) we can also deduce (i) by letting $y = x$. Since we are interested here in the fields $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$, the axioms (i) and (i′) can be used interchangeably for these fields.

A function $f : L \to M$ between Lie algebras is a *morphism of Lie algebras* if it is linear and satisfies $f([x,y]) = [f(x), f(y)]$.

Example 5.13. (i) If A is any associative algebra over $\mathbb{K}$ then the bracket $[x,y] = xy - yx$ defines on A the structure of a Lie algebra $(A, [\cdot,\cdot])$. Every vector subspace L of A with $[L,L] \subseteq L$ is a Lie algebra with respect to the bracket operation. In particular, for any vector space E, the algebra $A = \operatorname{Hom}(E,E)$ becomes a Lie algebra with respect to the bracket. This Lie algebra is denoted $\mathfrak{gl}(E)$. For $E = \mathbb{K}^n$

we write $\mathfrak{gl}(n,\mathbb{K})$ instead; this algebra may be identified with the Lie algebra of all $n\times n$-matrices over $\mathbb{K}$. The vector subspace $\mathfrak{sl}(n,\mathbb{K})$ of all matrices with trace 0 is closed under the formation of brackets and is, therefore, a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{K})$.

(ii) If E is any vector space over $\mathbb{K}$ endowed with a bilinear multiplication $(x,y)\mapsto xy$—associative or not—then a vector space endomorphism $D\colon E\to E$ is called a *derivation* if $D(xy)=(Dx)y+x(Dy)$. The set $\mathrm{Der}(E)$ of all derivations is a Lie subalgebra of $(\mathrm{Hom}(E,E),[\cdot,\cdot])$. (In other words, the bracket of two derivations is a derivation.)

If L is a Lie algebra and $x\in L$, define $\mathrm{ad}(x)\colon L\to L$ by $\mathrm{ad}(x)(y)=[x,y]$, $y\in L$. Then $\mathrm{ad}(x)\in\mathrm{Der}(L)$. We say that $\mathrm{ad}(x)$ is *an inner derivation of* L.

Exercise E5.7. Verify the details of the preceding examples. □

The next step is to investigate the operation of inner automorphisms.

If, for a Banach algebra A, we denote with $\mathcal{A}$ the vector space $\mathrm{Hom}(A,A)$ of all continuous *linear* self-maps and consider on $\mathcal{A}$ the operator norm and composition as multiplication, then $\mathcal{A}$ is again a Banach algebra. In particular, $\mathcal{A}^{-1}$ is a topological group and we have an exponential function $\exp_{\mathcal{A}}\colon\mathcal{A}\to\mathcal{A}^{-1}$. In order to keep exponential functions cleanly separate, we shall frequently write $e^T=\mathrm{id}_A+T+\frac{1}{2!}\cdot T^2+\cdots$ for $T\in\mathcal{A}$. The group $\mathrm{Aut}\,A$ of all automorphisms of the Banach algebra A is a (closed) subgroup of $\mathcal{A}^{-1}$ and is, therefore, a topological group.

(i) For an element $x\in A$ we define linear operators $L(x)$, $R(x)\colon A\to A$ by $L(x)(y)=xy$ and $R(x)(y)=yx$. We shall later also write L_x and R_x in place of $L(x)$ and $R(x)$, respectively.

(ii) For each $g\in A^{-1}$ we denote the *inner automorphism* $x\mapsto gxg^{-1}$ of A by $\mathrm{Ad}(g)\colon A\to A$, that is $\mathrm{Ad}(g)(x)=gxg^{-1}$,

(iii) A *derivation of* A is a derivation $D\colon A\to A$ which is also continuous. The Lie algebra of all continuous derivations $\mathrm{Der}(A)$ is a (closed) Lie subalgebra of $(\mathcal{A},[\cdot,\cdot])$. For each $a\in A$ the map $\mathrm{ad}(a)\colon A\to A$, $\mathrm{ad}(a)(x)=[a,x]$ is a continuous derivation.

Lemma 5.14. (i) *$L(x),R(x)\in\mathcal{A}$ and $L\colon A\to\mathcal{A}$ is a contractive morphism of Banach algebras, while $R\colon A\to\mathcal{A}^{\mathrm{op}}$ is a contractive morphism of Banach algebras into the opposite of $\mathcal{A}$ with a multiplication given by $S\star T=TS$. Also, L and $g\mapsto R(g)^{-1}$ are morphisms of topological groups $A^{-1}\to\mathcal{A}^{-1}$. For all $x,y\in A$ we have $[L(x),R(y)]=0$.*

(ii) $\mathrm{Ad}(g)=L(g)R(g)^{-1}=R(g)^{-1}L(g)$ *for all* $g\in A^{-1}$, *and* $\mathrm{Ad}\colon A^{-1}\to\mathrm{Aut}(A)$ *is a morphism of topological groups.*

(iii) $\mathrm{ad}(x)=L(x)-R(x)$ *and* $\mathrm{ad}\colon(A,[\cdot,\cdot])\to\mathrm{Der}(A)$ *is a morphism of* (*topological*) Lie algebras.

Proof. (i) It is immediate that the distributive law $(x+y)z=xz+yz$ means $L(x+y)=L(x)+L(y)$ and that the associative law $(xy)z=x(yz)$ is expressed as $L(xy) = L(x)L(y)$. The algebra law $(t\cdot x)z = t\cdot(xz)$ for $t \in \mathbb{K}$ is equivalent to $L(t\cdot x) = t\cdot L(x)$. Also, $\|xy\| \leq \|x\|\|y\|$ translates into $\|L(x)\| \leq \|x\|$. Thus L is a contractive morphism of Banach algebras. The corresponding assertions on R are proved similarly. The restriction of L to A^{-1} is then a morphism of topological groups as is the function $g \mapsto R(g)^{-1}$.

Finally, $[L(x), R(y)](z) = L(x)R(y)(z) - R(y)L(x)(z) = x(zy) - (xz)y = 0$.

(ii) $\mathrm{Ad}(g)(z) = gzg^{-1} = L(g)R(g^{-1})(z)$. Hence $\mathrm{Ad}(g) = L(g)R(g)^{-1}$, and clearly $x \mapsto gxg^{-1}$ is a member of $\mathrm{Aut}(A) \subseteq \mathcal{A}^{-1}$, since multiplication and inversion in $\mathcal{A}$ are continuous, and L and $g \mapsto R(g)^{-1}$ are commuting morphisms of topological groups $A^{-1} \mapsto \mathcal{A}^{-1}$, it follows that $\mathrm{Ad}\colon A^{-1} \to \mathrm{Aut}(A)$ is a morphism of topological groups.

(iii) $\mathrm{ad}(x)(y) = xy - yx = L(x)(y) - R(x)(y)$, and the linearity and continuity of L and R shows that of ad. The Jacobi identity is equivalent to $\mathrm{ad}[x, y] = [\mathrm{ad}(x), \mathrm{ad}(y)]$. □

The automorphisms $\mathrm{Ad}(g)$ are called *inner automorphisms of* A and the derivations $\mathrm{ad}\, x$ are called *inner derivations.*

Lemma 5.15. *Assume that* $\sum_{j=0}^{\infty} a_j x^j$ *is a convergent power series in a Banach algebra.*

(i) *If* $\alpha\colon A \to A$ *is a continuous morphism of algebras, then* $\alpha\left(\sum_{j=0}^{\infty} a_j x^j\right) = \sum_{j=0}^{\infty} a_j \alpha(x)^j$.

(ii) $L(\sum_{j=0}^{\infty} a_j x^j) = \sum_{j=0}^{\infty} a_j L(x)^j$ *and* $R(\sum_{j=0}^{\infty} a_j x^j) = \sum_{j=0}^{\infty} a_j R(x)^j$.

(iii) $\alpha(\exp x) = \exp \alpha(x)$, $L(\exp x) = e^{L(x)}$ *and* $R(\exp x) = e^{R(x)}$.

Proof. (i) is straightforward from the fact that α is a continuous algebra morphism.

(ii) By 1.14(i) the function $L\colon A \to \mathcal{A}$ is a continuous morphism of algebras. Thus the assertion on L is a special case of (i). The proof for R is similar.

(iii) is a consequence of (i) and (ii). □

We shall presently apply this to $\exp x = \sum_{j=0}^{\infty} \frac{1}{n!}\cdot x^n$. A comment on our notation is in order. We will have to consider the exponential function on our primary Banach algebra A *and* on our secondary Banach algebra $\mathcal{A}$. The former we shall continue to denote by exp or $\exp_A$; the second we shall write in the form $T \mapsto e^T\colon \mathrm{Hom}(A, A) \to \mathrm{Gl}(A)$ (and rarely also in the form $\exp_{\mathcal{A}}$). Here the argument is always a bounded operator T. This notational distinction is quite helpful.

Proposition 5.16 (Inner Automorphisms and the Adjoint Representation). *Let* A *be a Banach algebra with identity.*

(i) $g(\exp y)g^{-1} = \exp \mathrm{Ad}(g)(y)$, $g \in A^{-1}$, $y \in A$.

(ii) *If* $x \in A$ *then* $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}(x)}$. *In other words,*

$$\mathrm{Ad} \circ \exp_A = \exp_{\mathcal{A}} \circ \mathrm{ad}\,.$$

Accordingly, the following diagram is commutative:

$$\begin{array}{ccccc} A & \xrightarrow{\ \mathrm{ad}\ } & \mathcal{A} & = & \mathrm{Hom}(A,A) \\ {\scriptstyle \exp_A}\Big\downarrow & & {\scriptstyle \exp_{\mathcal{A}}}\Big\downarrow & & \Big\downarrow{\scriptstyle T\mapsto e^T} \\ A^{-1} & \xrightarrow[\ \mathrm{Ad}\ ]{} & \mathcal{A} & = & \mathrm{Gl}(A). \end{array}$$

(iii) $\exp x \exp y \exp -x = \exp\big(e^{\mathrm{ad}(x)}y\big)$ *for all* $x, y \in A$.

(iv) $x * y * -x = e^{\mathrm{ad}\, x} y = y + [x,y] + \frac{1}{2}\cdot\big[x,[x,y]\big] + \cdots$ *for all sufficiently small elements* $x, y \in A$.

Proof. (i) If we apply Lemma 5.15(iii) with $\alpha = \mathrm{Ad}(g)$, we obtain $g(\exp y)g^{-1} = \mathrm{Ad}(g)(\exp y) = \exp \mathrm{Ad}(g)(y)$.

(ii) In view of Exercise E5.3, using Lemma 5.14(iii) we compute

$$e^{\mathrm{ad}(x)} = e^{L(x)-R(x)} = e^{L(x)}e^{-R(x)}.$$

By Lemma 5.15(iii) we know

$$e^{L(x)}e^{-R(x)} = L(\exp x)R(\exp x)^{-1}.$$

Lemma 5.14(ii) implies

$$L(\exp x)R(\exp x)^{-1} = \mathrm{Ad}(\exp x).$$

Taking all of this together we obtain (ii).

(iii) We apply (i) with $g = \exp x$ and use (ii) to deduce (iii).

(iv) Assume that x and y are so small that $e^{\mathrm{ad}\, x}y \in B_1(\mathbf{1})$ and that $x * y$ and $x * y * -x$ are defined and contained in N_0. Then $\exp(x * y * -x) \in B_1(\mathbf{1})$ and $x * y * -x = \log\exp(x * y * -x) = \log(\exp x \exp y \exp -x)$ by Corollary 5.6. Now (iii) above shows $\exp(x*y*-x) = \exp(e^{\mathrm{ad}\, x}y)$. Applying log to this equation gives $x * y * -x = e^{\mathrm{ad}\, y}x$. □

Notice that the analytic function $(x,y) \mapsto e^{\mathrm{ad}\, x}y : A \times A \to A$ is defined everywhere and is linear in y. It agrees with the analytic function $(x,y) \mapsto x*y*-x$ for all x and y sufficiently near 0, hence for all pairs (x,y) on any connected open neighborhood of $(0,0)$ on which it is defined.

Corollary 5.17. *Let V be a closed vector subspace of A and x, $y \in A$. Then the following three statements are equivalent:*

(i) $[x, V] \subseteq V$.

(ii) $e^{t\cdot \mathrm{ad}\, x}V = V$ *for all* $t \in \mathbb{R}$.

(iii) $\mathrm{Ad}(\exp t{\cdot}x)V = (\exp t{\cdot}x)V(\exp -t{\cdot}x) = V$ *for all* $t \in \mathbb{R}$.

Moreover, if $\|x\| < \frac{\log 2}{2}$, *then* $\mathrm{ad}\, x = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cdot(e^{\mathrm{ad}\, x} - \mathrm{id}_A)^n$ *and these conditions are also equivalent to the statement*

(iv) $e^{\mathrm{ad}\, x}V = V$.

Finally, the following proposition holds.

(v) *If* $[x,y] = 0$ *then* $e^{\mathrm{ad}\, x}y = y$, *and if* $\|x\| < \frac{\log 2}{2}$ *then both conditions are equivalent.*

Proof. The equivalence of (ii) and (iii) follows from Proposition 5.16(ii) and the implication (ii)⇒(iv) is trivial.

(i)⇒(ii) Condition (i) says that V is invariant under $(\operatorname{ad} x)$. Then $e^{t\cdot \operatorname{ad} x}V \subseteq V$ for all t and thus also $e^{-(t\cdot \operatorname{ad} x)}V = e^{-t\cdot \operatorname{ad} x}V \subseteq V$ which implies $V \subseteq e^{t\cdot \operatorname{ad} x}V$.

(ii)⇒(i) Let $v \in V$. Note that

$$[x,v] + t\cdot \sum_{n=0}^{\infty} \frac{t^n}{(n+2)!}(\operatorname{ad} x)^{n+2}v = \frac{1}{t}\cdot \left(e^{t\cdot \operatorname{ad} x}v - v\right) \in V$$

by (ii). Then $[x,v] = \lim_{0\neq t\to 0} \frac{1}{t}\cdot \left(e^{t\cdot \operatorname{ad} x}v - v\right) \in V$ since V is closed.

(iv)⇒(i) Assume that $\|x\| < \frac{\log 2}{2}$. Therefore, $\|(\operatorname{ad} x)(y)\| = \|[x,y]\| = \|xy - yx\| \leq 2\|x\|\|y\| < (\log 2)\|y\|$. Then $\|\operatorname{ad}(x)\| < \log 2$ in $\mathcal{A}$. Hence $\|e^{\operatorname{ad} x} - \operatorname{id}_A\| < 1$ by Lemma 5.2 and $\operatorname{ad}(x) = \log e^{\operatorname{ad} x} = \log\left(\operatorname{id}_A + (e^{\operatorname{ad} x} - \operatorname{id}_A)\right)$. Thus, if V is invariant under $e^{\operatorname{ad} x}$, it is invariant under $T = e^{\operatorname{ad} x} - \operatorname{id}_A$ and then under $\operatorname{ad} x = \log(\operatorname{id}_A + T) = T - \frac{1}{2}\cdot T^2 + \frac{1}{3}\cdot T^3 - \frac{1}{4}\cdot T^4 \pm \cdots$.

(v) The formula $e^{\operatorname{ad} x}y = y + \sum_{n=0}^{\infty} \frac{1}{n+1}(\operatorname{ad} x)^n[x,y]$ shows that $[x,y] = 0$ always implies $e^{\operatorname{ad} x}y = y$. If $\|x\| < \frac{\log 2}{2}$, then we saw $\operatorname{ad} x = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cdot(e^{\operatorname{ad} x} - \operatorname{id}_A)^n$. Thus $e^{\operatorname{ad} x}y = y$ implies $(e^{\operatorname{ad} x} - \operatorname{id}_A)y = 0$ and therefore $[x,y] = 0$. □

Differentiating the Exponential Function in a Banach Algebra

We need more information on the exponential function and the Campbell–Hausdorff formalism.

We define two analytic functions and begin with an entire one: define

$$(6) \qquad f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} z^n = \begin{cases} (1-e^{-z})z^{-1} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Observe

$$f(-z) = \frac{e^z - 1}{z} \quad \text{for} \quad z \neq 0.$$

Let us note

$$(*) \qquad f^{-1}(0) = 2\pi i\mathbb{Z} \setminus \{0\}.$$

Thus we get an analytic function

$$(6) \qquad g\colon \mathbb{C} \setminus f^{-1}(0) \to \mathbb{C}, \quad g(z) = \frac{1}{f(z)}.$$

The power series expansion of g on an open disc of radius 2π starts off by $g(z) = 1 + \frac{1}{2}z + \cdots$.

Lemma 5.18. *For two commuting elements u, v in a Banach algebra,*

$$(\exp u) f(u-v) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{p+q=n-1} u^p v^q. \tag{8}$$

Proof. Let u and v denote commuting elements in a Banach algebra with $u \neq v$. Then

$$(u^n - v^n)(u-v)^{-1} = \sum_{p=0}^{n-1} u^{n-p-1} v^p = \sum_{p+q=n-1} u^p v^q, \quad n = 1, \ldots,$$

whence

$$(\exp u) f(u-v) = (\exp u)\big(1 - \exp -(u-v)\big)(u-v)^{-1} = (\exp u - \exp v)(u-v)^{-1}$$

implies that (8) is valid. □

Now we consider a Banach algebra A with identity.

Proposition 5.19. *For $x \in A$ and $\|y\| \leq 1$ we have*

$$\exp(x+y) - \exp x = (\exp x) f(\operatorname{ad} x) y + \rho(x,y), \quad \|\rho(x,y)\| \leq \|y\|^2 e^{1+\|x\|}. \tag{9}$$

Proof. For each $x, y \in A$ and each natural number n, the binomial formula for noncommuting variables is as follows. Let $N = \{1, \ldots, n\}$, and for $J \subseteq N$ we set

$$u(J) = u_1(J) \cdots u_n(J), \quad u_k(J) = \begin{cases} y & \text{for } k \in J, \\ x & \text{for } k \in N \setminus J. \end{cases}$$

In particular $u(\varnothing) = x^n$ and $u(\{k\}) = x^{k-1} y x^{n-k} = \big(L_x^{k-1} R_x^{n-k}\big) y$, whence

$$\sum_{\substack{J \subseteq N \\ |J|=1}} u(J) = \sum_{p+q=n-1} \big(L_x^p R_x^q\big)(y),$$

where $|J|$ denotes the number of elements in J. Now

$$(x+y)^n = \sum_{J \subseteq N} u(J) = x^n + \sum_{\substack{J \subseteq N \\ |J|=1}} u(J) + \rho_n(x,y), \tag{10}$$

with a remainder $\rho_n(x,y) = \sum_{\substack{J \subseteq N \\ |J| \geq 2}} u(J)$ which, for $\|y\| \leq 1$, satisfies

$$\begin{aligned} \|\rho_n(x,y)\| &\leq \sum_{\substack{J \subseteq N \\ |J| \geq 2}} \|x\|^{n-|J|} \|y\|^{|J|} = \sum_{m=2}^{n} \binom{n}{m} \|y\|^m \|x\|^{n-m} \\ &\leq \|y\|^2 \sum_{m=0}^{n} \binom{n}{m} \|x\|^{n-m} = \|y\|^2 (1 + \|x\|)^n, \end{aligned}$$

whence

$$\left\| \frac{1}{n!} \cdot \rho_n(x,y) \right\| \leq \|y\|^2 \frac{(1+\|x\|)^n}{n!}.$$

Thus

$$\exp(x+y) = \exp x + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{p+q=n-1} L_x^p R_x^q \right) y + \rho(x,y), \tag{11}$$

with $\|\rho(x,y)\| \leq \|y\|^2 e^{(1+\|x\|)}$ as soon as $\|y\| \leq 1$.

The linear operators L_x and R_x commute, and $L_x - R_x = \operatorname{ad} x$. Accordingly, Lemma 5.18 (8) yields

$$e^{L_x} f(\operatorname{ad} x) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{p+q=n-1} L_x^p R_x^q \tag{12}$$

for all x, $y \in A$. As a consequence, for all $x \in A$ and all $\|y\| \leq 1$ we obtain

$$\begin{aligned} \exp(x+y) &= \exp x + e^{L_x} f(\operatorname{ad} x) y + \rho(x,y) \\ &= \exp x + L_{\exp x} f(\operatorname{ad} x) y + \rho(x,y) \\ &= \exp x + (\exp x) f(\operatorname{ad} x) y + \rho(x,y), \end{aligned}$$

which is (9). □

We draw several important conclusions. For the first we recall the concept of a derivative in the simplest context. If A_1 and A_2 are Banach spaces and $F\colon U \to A_2$ is a function on an open subset of A_1, the *derivative* of F at $u \in U$ is a linear operator at $F'(u)\colon A_1 \to A_2$ such that $F(u+h) - F(u) = F'(u)h + r(u,h)$ with a remainder term $r(u,v)$ satisfying $\lim_{0 \neq h \to 0} \frac{1}{\|h\|} \cdot r(u,h) = 0$.

Corollary 5.20 (The Derivative of the Exponential Function). *Let A be a Banach algebra with identity. The derivative $\exp'(a)\colon A \to A$ of the exponential function at $a \in A$ is computed as*

$$\exp'(a)(h) = (\exp a) f(\operatorname{ad} a)(h) = (\exp a) \frac{\operatorname{id}_A - e^{-\operatorname{ad} a}}{\operatorname{ad} a}(h). \tag{13}$$

Equivalently,

$$L_{\exp a}^{-1} \exp'(a) = L_{\exp -a} \exp'(a) = f(\operatorname{ad} a). \tag{14}$$

Proof. This follows at once from the definition of the derivative and from Proposition 5.19(9). □

We recall that $x * y = x + y + \frac{1}{2} \cdot [x,y] + \cdots$ is defined for all elements x and y in sufficiently small neighborhood B of 0 in a Banach algebra. In the following result we operate in such a B whose size we shall specify.

Theorem 5.21. *Assume that A is a Banach algebra with identity and B an open ball around 0 such that*

(a) *$(B * B) * B$ and $B * (B * B)$ is defined,*

(b) *$x, y \in B$ and $t \in [0,1]$ implies that $x * t{\cdot}y$ satisfies $\|x * t{\cdot}y\| < \pi$.*

Then the following statements hold:

(i) *For $x \in B$ there is a neighborhood V of 0 and there are real numbers C_1, C_2 such that for all $y \in V$*

$$(15)\qquad (-x) * (x + y) = f(\operatorname{ad} x)(y) + \mathrm{o}_1(x, y), \quad \|\mathrm{o}_1(x, y)\| \le \|y\|^2 C_1,$$

$$(16)\qquad x * y = x + g(\operatorname{ad} x)(y) + \mathrm{o}_2(x, y), \quad \|\mathrm{o}_2(x, y)\| \le \|y\|^2 C_2.$$

(ii) *Let $x, y \in B$ and assume that $\mathfrak{g}$ is the smallest closed Lie subalgebra of $(A, [\cdot,\cdot])$ containing x and y. Then $x * y - (x + y) \in \overline{[\mathfrak{g}, \mathfrak{g}]}$, and $x * y \in \mathfrak{g}$.*

(iii) *Assume that A_j, $j = 1, 2$ are Banach algebras and B a ball around 0 in A_1 satisfying* (a) *and* (b) *above. Let $\mathfrak{g}$ be a closed Lie subalgebra of $(A_1, [\cdot,\cdot])$. Assume that $T{:}\, \mathfrak{g} \to A_2$ is a continuous linear map such that $T[x, y] = [Tx, Ty]$. Then $x, y \in B_1 \cap \mathfrak{g}$ implies $T(x * y) = (Tx) * (Ty)$.*

(iv) *If B_0 is any connected open neighborhood such that $(x, y) \mapsto x{*}y{:}\, B_0 \times B_0 \to A$ is defined and analytic, if $\mathfrak{g}_0$ is any closed Lie subalgebra of $(A, [\cdot,\cdot])$, and if $\mathfrak{g}_0 \cap B_0$ is connected, then $(\mathfrak{g}_0 \cap B_0) * (\mathfrak{g}_0 \cap B_0) \subseteq \mathfrak{g}_0$. In particular, if B_0 is an open ball around 0, then $\mathfrak{g}_0 \cap B_0$ is automatically connected.*

Proof. (i) The functions F and G given by $F(y) = (-x){*}(x{+}y)$ and $G(y) = (-x){+}(x{*}y)$ are defined on suitable open neighborhoods U_1 and U_2 of 0, respectively, and are inverses of each other. Hence for the derivatives in 0 we have $G'(0) = F'(0)^{-1}$. Thus (16) follows from (15), and we have to prove (15). Assume that $-x, x{+}y \in B$ and $\|y\| \le 1$. From Proposition 5.19 we know

$$(17)\qquad \exp(x + y) = \exp x\big(1 + f(\operatorname{ad} x)y + \sigma(x, y)\big)$$

with $\sigma(x, y) = e^{-x}\rho(x, y)$ and thus $\|\sigma(x, y)\| \le \|y\|^2 e^{(1+2\|x\|)}$.

Then from Corollary 5.6(i) and (17) we know that $\exp\big((-x) * (x + y)\big) = (\exp -x)\exp(x + y) = 1 + f(\operatorname{ad} x)y + \sigma(x, y)$. For small enough y we may take logarithms on both sides and obtain $(-x) * (x + y) = \log\big(1 + f(\operatorname{ad} x)y + \sigma(x, y)\big) = f(\operatorname{ad} x)y + \tau(x, y)$ with $\|\tau(x, y)\| \le \|y\|^2 r(\|x\|, \|y\|)$ where r denotes a real analytic function of two variables. This completes the proof of (15).

Now let $x, y \in B$. By (a) we can define $\varphi(t) = x * t{\cdot}y \in A$ for $t \in [0, 1]$ and by (b) we know that $\|\varphi(t)\| < \pi$, whence $\|\operatorname{ad}\varphi(t)\| < 2\pi$. Hence (16) implies $\varphi(t + h) = x * (t + h){\cdot}y = x * (t{\cdot}y + h{\cdot}y) = x * (t{\cdot}y * h{\cdot}y) = (x * t{\cdot}y) * h{\cdot}y$ (by (a) and 5.6(ii)) $= \varphi(t) * h{\cdot}y = \varphi(t) + h{\cdot}g\big(\operatorname{ad}\varphi(t)\big)(y) + \mathrm{o}_2(x, h{\cdot}y)$, and thus $\varphi'(t) = g\big(\operatorname{ad}\varphi(t)\big)(y)$ with $\varphi(0) = x$.

(ii) Fix $x, y \in B$ and let $\mathfrak{g}$ be the smallest closed Lie subalgebra containing x and y; set $\mathfrak{v} \overset{\text{def}}{=} \overline{[\mathfrak{g}, \mathfrak{g}]}$. Let U be an open zero neighborhood such that $u \in U$ implies $\|u{+}x{+}t{\cdot}y\| < \pi$ for all $t \in [0, 1]$. Then $\Lambda_t{:}\, U \to A$, $\Lambda_t(u) = g\big(\operatorname{ad}(u{+}x{+}t{\cdot}y)\big)(y){-}y$ is a vector field (depending on t) such that $u \in U{\cap}\mathfrak{g}$ implies $\Lambda_t(u) \in \mathfrak{v}$. We consider the function $\alpha\colon [0,1] \to A$, $\alpha(t) = x * t{\cdot}y - (x + t{\cdot}y)$. Then $\alpha'(t) = \varphi'(t) - y = g\big(\operatorname{ad}\varphi(t)\big)(y) - y = \Lambda_t\big(\alpha(t)\big)$. The initial value problem

$$\omega'(t) = \Lambda_t\big(\omega(t)\big), \quad \omega(0) = 0$$

in the Banach space $\mathfrak{v}$ has a unique local solution $\omega\colon [0,\varepsilon[\to \mathfrak{g}\cap U$. The coextension $\Omega\colon [0,\varepsilon[\to A$, $\Omega(t) = \omega(t)$ is the unique solution of the initial value problem in A. We know such a solution in A to be α. Hence $\Omega = \alpha|[0,\varepsilon[$ by uniqueness. The analytic curve $t \mapsto \alpha(t) + \mathfrak{v}\colon [0,1] \to A/\mathfrak{v}$ is zero on a nonempty interval and thus, by analyticity, is zero. Hence $x*y-(x+y) = \alpha(1) \in \mathfrak{v}$. Finally $x*y \in x+y+\mathfrak{v} \subseteq \mathfrak{g}$ follows.

(iii) By (ii) and $[\mathfrak{g},\mathfrak{g}] \subseteq \overline{[\mathfrak{g},\mathfrak{g}]} \subseteq \mathfrak{g}$ we have $\varphi(t) = x * t{\cdot}y \in \mathfrak{g}$ for all $x, y \in B \cap \mathfrak{g}$ and $t \in [0,1]$. Then $y + \frac{1}{2}[\varphi(t), y] + \cdots = g\big(\operatorname{ad}\varphi(t)\big)(y) = \varphi'(t) \in \mathfrak{g}$, and since T respects brackets on $\mathfrak{g}$ by hypothesis, we have $T\Big(g\big(\operatorname{ad}\varphi(t)\big)(y)\Big) = g\big(\operatorname{ad}(T\varphi)(t)\big)(Ty)$. Therefore $\psi \overset{\text{def}}{=} T\varphi\colon [0,1] \to A_2$ is the unique solution of the initial value problem $\psi'(t) = g\big(\operatorname{ad}\psi(t)\big)(Ty)$, $\psi(0) = Tx$. Thus by (ii), applied to A_2, we find $T(x*y) = T\varphi(1) = \psi(1) = (Tx)*(Ty)$.

(iv) By (ii) we know $(\mathfrak{g}_0 \cap B) * (\mathfrak{g}_0 \cap B) \subseteq \mathfrak{g}_0$. The analytic function

$$\alpha\colon (\mathfrak{g}_0 \cap B_0) \times (\mathfrak{g}_0 \cap B_0) \to A/\mathfrak{g}_0, \quad \alpha(x,y) = x * y + \mathfrak{g}_0,$$

vanishes on the open subset $(\mathfrak{g}_0 \cap B_0 \cap B) \times (\mathfrak{g}_0 \cap B_0 \cap B)$ of the connected set open domain $(\mathfrak{g}_0 \cap B_0) \times (\mathfrak{g}_0 \cap B_0)$ and is, therefore, zero. This proves the claim. □

In 5.21(ii), a proof of the assertion $x * y \in \mathfrak{g}$ alone would have been simpler. Alternative proofs of 5.21(ii) require a more detailed analysis of the Campbell–Hausdorff formalism defining the local $*$-multiplication. Indeed, it is a fact that the homogeneous polynomial functions $(u,v) \mapsto H_n(u,v)$ in the Campbell–Hausdorff series (see 5.5) are Lie polynomials, i.e. their values are taken in the Lie subalgebra of $(A, [\cdot,\cdot])$ generated by u and v. We shall not go into this aspect here. At any rate, 5.21 shows once more, among other things, the significance of closed Lie subalgebras of $(A, [\cdot,\cdot])$ for the exponential function and the Campbell–Hausdorff multiplication.

We return briefly to the derivative of the exponential function. A differentiable function $F\colon U \to A$ from an open set U of a Banach space$A$ to A is called *regular* in u if $F'(u)$ is invertible. After we computed the derivative of $\exp\colon A \to A^{-1} \subseteq A$ for a Banach algebra, we are in a position to determine the points $a \in A$ in which exp is regular. For this purpose we need the concept of the spectrum, $\operatorname{Spec} a$, of an element a in a complex Banach algebra A with identity (cf. [39]):

$$\operatorname{Spec} a = \{\lambda \in \mathbb{C} \mid a - \lambda{\cdot}\mathbf{1} \notin A^{-1}\}.$$

If A is a real Banach algebra and $a \in A$ then $\operatorname{Spec} a$ is defined to be $\operatorname{Spec} 1 \otimes a$ for $1 \otimes a \in \mathbb{C} \otimes A$, the complexification of A.

The algebra of all continuous operators of a Banach space is a Banach algebra with respect to the operator norm, and thus the spectrum of an operator is well defined.

Lemma 5.22. *Let $a \in A$ for a Banach algebra A with identity. Then the following statements are equivalent:*

(i) $f(a) \notin A^{-1}$.
(i′) $0 \in \operatorname{Spec} f(a)$.
(ii) $\operatorname{Spec} a \cap (2\pi i\mathbb{Z} \setminus \{0\}) \neq \emptyset$.

Proof. (i) and (i′) are just reformulations of each other.

For a proof of the equivalence of (i′) and (ii) we invoke some spectral theory[1]. (A good reference is Bourbaki [39] at the end of the chapter. The part that is particularly relevant for our discourse in this chapter is Chapter I, §4, **n^o 8** and **n^o 9**, pp. 47ff. Unfortunately, these statements rest on a very general theorem whose proof is long and complicated, involving amongst other things holomorphic functions in several variables; its generality is not needed here.)

The Spectral Mapping Theorem (Bourbaki [39] Proposition 7, p. 47) says that $f(\operatorname{Spec} a) = \operatorname{Spec} f(a)$. Then $0 \in \operatorname{Spec} f(a) = f(\operatorname{Spec} a)$ iff $\operatorname{Spec} a \cap f^{-1}(0) \neq \emptyset$. In view of $(*)$ this implies the equivalence of (i′) and (ii).

Alternative proof: the set $F \overset{\text{def}}{=} (\operatorname{Spec} a) \cap (2\pi i\mathbb{Z} \setminus \{0\})$ is finite. Let U be an open neighborhood of $\operatorname{Spec} a$ in $\mathbb{C}$ such that $U \cap (2\pi i\mathbb{Z} \setminus \{0\}) = F$. The zeros of f are all of order one. Hence there is a holomorphic function on U vanishing nowhere such that $f(z) = (z - \lambda_1) \cdots (z - \lambda_k) g(z)$ for $z \in U$, $\{\lambda_1, \ldots, \lambda_k\} = F$ (with $k = 0$ and $f = g$ if $F = \emptyset$). Since $g(z) \neq 0$ for $z \in U$ the prescription $\frac{1}{g}(z) = g(z)^{-1}$ defines a holomorphic function on U. By the holomorphic functional calculus (see e.g. [39]) $g(a)$ and $\frac{1}{g}(a)$ are well-defined elements of A, and $g(a)\frac{1}{g}(a) = (g\frac{1}{g})(a) = \operatorname{const}_1(a) = \mathbf{1}$ (where const_c denotes the constant function with value c). Likewise $\frac{1}{g}(a)g(a) = \mathbf{1}$. Hence $g(a) \in A^{-1}$. Now $f(a) = (a - \lambda{\cdot}\mathbf{1}) \cdots (a - \lambda_k{\cdot}\mathbf{1}) g(a)$ is invertible if and only if $k = 0$, i.e. $F = \emptyset$. □

Exercise E5.8. Prove the Spectral Mapping Theorem in the case that $A = \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.

[Hint. Use the Jordan normal form of an endomorphism of $\mathbb{C}^n$.] □

The following corollary is now an immediate consequence.

Proposition 5.23. *Let $\mathfrak{g}$ be a closed Lie subalgebra of $(A, [\cdot,\cdot])$. Then the linear operator*

$$L_{\exp x}^{-1} \exp'(x)|\mathfrak{g} = f(\operatorname{ad}_{\mathfrak{g}} x) \colon \mathfrak{g} \to \mathfrak{g}, \quad \textit{where} \quad (\operatorname{ad}_{\mathfrak{g}} x)(y) = [x, y] \quad \textit{for} \quad x, y \in \mathfrak{g}$$

is invertible if and only if

$$\operatorname{Spec} \operatorname{ad}_{\mathfrak{g}} x \cap (2\pi i\mathbb{Z} \setminus \{0\}) = \emptyset. \qquad \square$$

1 The case that $A = \operatorname{Hom}(E, E)$ for a finite dimensional complex vector space is elementary and suffices for most of our purposes.

In particular, this allows us to say exactly in which points a of A the exponential function of A is regular. (In 5.41(iii) below we shall sharpen this information considerably.)

Local Groups for the Campbell–Hausdorff Multiplication

Let us consider in A an open ball $B = B_r(0)$ around 0 such that $r < \frac{\log 2}{2}$ such that $B * B$ is defined.

Definition 5.24. We shall say that a non-empty subset $\Gamma \subseteq B$ is *a local group with respect to B* if
(i) $(\Gamma * \Gamma) \cap B \subseteq \Gamma$,
(ii) $-\Gamma = \Gamma$.
We say that Γ is closed if it is closed in B. □

Notice that a local group contains 0, for if $a \in \Gamma$, then $-a \in \Gamma$ by (ii) and thus $0 = a * -a \in (\Gamma * \Gamma) \cap B \subseteq \Gamma$ by (i).

From 5.21(ii) we know that for every closed Lie subalgebra $\mathfrak{g}$ of $(A, [\cdot,\cdot])$ there are sufficiently small open balls B such that $\Gamma \stackrel{\text{def}}{=} \mathfrak{g} \cap B$ is a local group with respect to B.

Definition 5.25. If Γ is a subset of A, we let $\mathfrak{T}(\Gamma)$ denote the set of all elements $x \in A$ for which we find a sequence $x_n \in \Gamma$ converging to 0 and a sequence $m(n)$ of natural numbers such that $x = \lim_n m(n)\cdot x_n$. The elements of $\mathfrak{T}(\Gamma)$ are called *the subtangent vectors of* Γ *at* 0. If both x and $-x$ are subtangent vectors, then x is called a *tangent vector.* □

We observe right away $\mathfrak{T}(\overline{\Gamma} \cap B) = \mathfrak{T}(\Gamma)$. Indeed let $0 \neq y \in \mathfrak{T}(\Gamma \cap B)$. Then $y = \lim_n p(n)\cdot y_n$ with $y_n \in \overline{\Gamma}$. Pick for each $n \in \mathbb{N}$ an element $x_n \in \Gamma$ with $\|x_n - y_n\| < 1/p(n)^2$. Now $\lim_n p(n)\cdot(x_n - y_n) = 0$ and $y = \lim_n p(n)\cdot x_n \in \mathfrak{T}(\Gamma)$. Hence $\mathfrak{T}(\overline{\Gamma} \cap B) \subseteq \mathfrak{T}(\Gamma)$, and the reverse inclusion is obvious.

Also, if U is any open neighborhood of 0, then $\mathfrak{T}(U \cap \Gamma) = \mathfrak{T}(\Gamma)$.

Lemma 5.26. *For an element $x \in A$ and a closed local group Γ with respect to B, the following statements are equivalent:*
(i) $x \in \mathfrak{T}(\Gamma)$.
(ii) *There is a sequence $x_n \in \Gamma$ converging to 0 and a sequence $r(n)$ of natural numbers such that $x = \lim_n r(n)\cdot x_n$.*
(iii) $\mathbb{R}\cdot x \cap B \subseteq \Gamma$.

As a consequence,

$$B \cap \mathfrak{T}(\Gamma) = \Gamma \cap \mathfrak{T}(\Gamma). \tag{†}$$

Proof. (iii)⇒(i) is easy: let $x_n = \frac{1}{n}\cdot x$. Then $x = n\cdot x_n = \lim_n n\cdot x_n$ and $\lim_n x_n = 0$.

(i)⇒(ii) is trivial. (ii)⇒(iii) Assume $x = \lim t(n)\cdot x_n$ for a sequence $x_n \to 0$ with $x_n \in \Gamma$ and $t(n) \in \mathbb{N}$. Let r be an arbitrary real number r such that $r\cdot x \in B$. We must show $r\cdot x \in \Gamma$. Set $r(n) = rt(n)$. Then $r\cdot x = \lim r(n)\cdot x_n$. For a real number s let $[s] = \max\{z \in \mathbb{Z} \mid z \leq s\}$. Then $s = [s] + d(s)$ with $0 \leq d(s) = s - [s] < 1$. Then $0 \leq \|d\big(r(n)\big)\cdot x_n\| \leq d\big(r(n)\big)\|x_n\| \leq \|x_n\|$, whence $\lim d\big(r(n)\big)\cdot x_n = 0$. Thus $r\cdot x = \lim r(n)\cdot x_n = \lim [r(n)]\cdot x_n$. If $r\cdot x \in \Gamma$ then $(-r)\cdot x = -r\cdot x \in \Gamma$. It is therefore no loss of generality to assume that $r(n) \geq 0$. For all sufficiently large n, the elements $[r(n)]\cdot x_n$ are all in B. Now consider the assertion $k\cdot x_n \in \Gamma$ for $k = 1, 2, \ldots, [r(n)]$ for these n. For $k = 1$ this is true by hypothesis (i). If $k < [r(n)]$ and if we assume that $k\cdot x_n \in \Gamma$, then $(k+1)\cdot x_n = (k\cdot x_n) * x_n \in \Gamma$ since $(k+1)\cdot x_n = \frac{k+1}{[r(n)]}\cdot([r(n)]\cdot x_n) \in B$ by the convexity of B and since Corollary 5.6(iii) and condition 5.24(i) apply. By induction, it follows that $[r(n)]\cdot x_n \in \Gamma$ and thus, since Γ is closed in B we have $r\cdot x = \lim_n [r(n)]\cdot x_n \in \Gamma$. This completes the proof of the equivalence of (i), (ii), and (iii).

In order to see $B \cap \mathfrak{T}(\Gamma) = \Gamma \cap \mathfrak{T}(\Gamma)$ we first note that, trivially the right hand side is contained in the left hand side. If $x \in B \cap \mathfrak{T}(\Gamma)$, then $x \in \mathbb{R}\cdot x \cap B \subseteq B \cap \Gamma$ by the preceding. □

The following result provides a crucial insight.

Theorem 5.27. *Let A denote a Banach algebra with identity. Assume that B is an open ball around 0 whose radius is less than $\frac{\log 2}{2}$ and that Γ is a closed local group with respect to B. Then $\mathfrak{T}(\Gamma)$ is a closed Lie subalgebra of $(A, [\cdot\,,\cdot])$. If Γ is locally compact then there is an open ball B' around 0 in B such that*

$$B' \cap \Gamma = B' \cap \mathfrak{T}(\Gamma). \tag{‡}$$

Also, $\dim \mathfrak{T}(\Gamma)$ is finite.

Proof. (i) First we show that $\mathfrak{T}(\Gamma)$ is closed. Assume that $x \in \overline{\mathfrak{T}(\Gamma)}$, that is $x = \lim_n x_n$ with $x_n \in \mathfrak{T}(\Gamma)$. There is a real number $d > 0$ such that $r\cdot x_n \in B$ for all sufficiently large n and all $|r| \leq d$. Then by Lemma 5.26 we have $r\cdot x_n \in \Gamma$ for these n and for $|r| \leq d$. Since Γ is closed in B we deduce $r\cdot x \in \Gamma$ for $|r| \leq d$. By Lemma 5.26 this implies $x \in \mathfrak{T}(\Gamma)$.

From Lemma 5.26 the set $\mathfrak{T}(\Gamma)$ is closed under scalar multiplication. We shall now show that it is closed under addition and the bracket. For this purpose let $x, y \in \mathfrak{T}(\Gamma)$. Then $\frac{1}{n}\cdot x$, $\frac{1}{n}\cdot y \in \Gamma$ for all large n by Lemma 5.26. Thus also $\frac{1}{n}\cdot x * \frac{1}{n}\cdot y \in \Gamma$ for all large n by 5.24(i). Now by the Recovery of Addition 5.10, $x + y = \lim_n n(\frac{1}{n}\cdot x * \frac{1}{n}\cdot y) \in \mathfrak{T}(\Gamma)$. In a similar fashion, the Recovery of the Bracket 5.11 shows that $[x, y] \in \mathfrak{T}(\Gamma)$.

(ii) Now assume that Γ is locally compact. From Lemma 5.26 we know $\Gamma \cap \mathfrak{T}(\Gamma) = B \cap \mathfrak{T}(\Gamma)$. This set is a neighborhood of 0 in the Banach space $\mathfrak{T}(\Gamma)$ and is locally compact. Hence $\mathfrak{T}(\Gamma)$ is locally compact. But a Banach space is locally compact if and only if it is finite dimensional. Thus $\dim \mathfrak{T}(\Gamma) < \infty$. A finite dimensional vector subspace of a Banach space is complemented. Consequently, there is a vector space complement E such that $A = \mathfrak{T}(\Gamma) \oplus E$ topologically and algebraically. The derivative at 0 of the function $\mu\colon \big(B\cap\mathfrak{T}(\Gamma)\big)\oplus\big(B\cap E\big)\to A$, $\mu(x+y)=x*y=x+y+\frac{1}{2}\cdot[x,y]+H_3(x,y)+\cdots$ is the identity $\mu(0)'=\mathrm{id}_A\colon A=\mathfrak{T}(\gamma\oplus E)\to A$. Hence by the Theorem of the Local Inverse [237], μ is a local diffeomorphism and thus we find open balls C in $\mathfrak{T}(\Gamma)$ and D in E both centered at 0 in such a fashion that $x+y\mapsto x*y\colon C\oplus D\to C*D$ is a diffeomorphism of an open zero neighborhood of A onto an open zero neighborhood of A. Now suppose that there is no open ball B' around 0 with $B'\cap\mathfrak{T}(\Gamma)=B'\cap\Gamma$. Then there are sequences $x_n\in\mathfrak{T}(\Gamma)$ and $y_n\in E$ with $x_n*y_n\to 0$, $x_n*y_n\in\Gamma$ and $y_n\neq 0$. From 5.26(†) we know $B\cap\mathfrak{T}(\Gamma)\subseteq\Gamma$, and thus $x_n\in\Gamma$. Now $y_n=(-x_n)*x_n*y_n\in\Gamma*\Gamma\cap B\subseteq\Gamma$ for all large enough n by Definition 5.24. Let $r>0$ be the radius of a ball $B_r(0)$ such that $\Gamma\cap\overline{B_r(0)}$ is compact. Since $\lim_n\|[\frac{r}{2\|y_n\|}]\cdot y_n\|=\frac{r}{2}$, the sequence $[\frac{r}{2\|y_n\|}]\cdot y_n=y_n*\cdots*y_n$ is eventually in the compact set $\Gamma\cap\overline{B_r(0)}$. Hence it has a converging subsequence, and after renaming the sequence, if necessary, we may assume that $z=\lim_n[\frac{r}{2\|y_n\|}]\cdot y_n$ exists. By definition of $\mathfrak{T}(\Gamma)$ we have $z\in\mathfrak{T}(\Gamma)$. Also, $\|z\|=r/2>0$. On the other hand $y_n\in E$ implies $z\in E$. Thus $z\in\mathfrak{T}(\Gamma)\cap E=\{0\}$, a contradiction. □

This theorem has an important consequence:

Theorem 5.28. *Let A denote a Banach algebra with identity. Assume that B is an open ball around* 0 *whose radius is less than $\frac{\log 2}{2}$ and that Γ is a locally compact local group with respect to B. Then $\mathfrak{T}(\Gamma)\cap B$ is an open ball in the Banach space $\mathfrak{T}(\Gamma)$ and is open and closed in Γ. In particular, if Γ is also connected, then*

$$\Gamma=\mathfrak{T}(\Gamma)\cap B.$$

Proof. (a) Before we get into the technicalities of the proof, we note that *for each $x\in B$ and each neighborhood U of* 0 *in B such that $x*U\subseteq B$, the set $x*U$ is a neighborhood of x in B.* Indeed, let V be an open neighborhood of x in B such that $-x*V\subseteq U$, then $V=x*(-x*V)\subseteq x*U$.

(b) We claim that Γ is closed in B. Assume that $x=\lim_n x_n\in B$ with $x_n\in\Gamma$. Then we look for a closed ball C around 0 such that $C\cap\Gamma$ is compact; such a C exists because Γ is locally compact at 0. We may assume that we choose C so small that $x*C$ is defined and contained in B. By (a) above, $x*C$ is a neighborhood of x in B. Thus $x_n\in x*C$ for all large n and there is an N such that $-x_m*x_n\in C\subseteq B$ for $m,n\geq N$. Then $-x_N*x_n\in C\cap\Gamma$ implies $-x_N*x=\lim_n -x_N*x_n\in\Gamma$, since $C\cap\Gamma$ is compact. Thus $x=x_N*(-x_N*x)\in(\Gamma*\Gamma)\cap B\subseteq\Gamma$. Hence Γ is closed.

(c) By 5.27, $\mathfrak{T}(\Gamma)$ is a Banach subspace of A, and then B is an open ball around 0 in it. We claim that $B\cap\mathfrak{T}(\Gamma)$ is open and closed in Γ; thus if Γ is also connected, we shall conclude $\Gamma=\mathfrak{T}(\Gamma)\cap B$. Since obviously $B\cap\mathfrak{T}(\Gamma)$ is closed in Γ as $\mathfrak{T}(\Gamma)$ is closed, we must show that for every point $x\in B\cap\mathfrak{T}(\Gamma)$ the set $B\cap\mathfrak{T}(\Gamma)$ is a neighborhood of x in Γ. Since Γ is closed, Theorem 5.27 applies and yields an open ball B' around 0 such that $B'\cap\Gamma\subseteq\mathfrak{T}(\Gamma)$. Now we choose an open ball $B''\subseteq B'$ so small that the function $x*B''\subseteq B$. Then $x*B''$ is a neighborhood of x in B. We will now show that $(x*B'')\cap\Gamma$ is contained in $\mathfrak{T}(\Gamma)$. This will prove that $B \cap \mathfrak{T}(\Gamma)$ is a neighborhood of x in Γ and complete the proof. For a proof let $y \in (x * B'') \cap \Gamma$. Then

$$-x * y \in \big(-x * (x * B'')\big) \cap \big(-\mathfrak{T}(\Gamma) \cap B) * \Gamma \subseteq B'' \cap (\Gamma * \Gamma) \subseteq B' \cap \Gamma \subseteq \mathfrak{T}(\Gamma).$$

Now $\mathfrak{T}(\Gamma)$ is a closed Lie subalgebra of $(A, [\cdot, \cdot])$ by 5.27, and thus 5.21(iv) shows that $\mathfrak{T}(\Gamma) \cap B$ is a local group with respect to B. Then

$$y = x * (-x * y) \in (\mathfrak{T}(\Gamma) \cap B) * (\mathfrak{T}(\Gamma) \cap B) \subseteq \mathfrak{T}(\Gamma),$$

which is what we wanted to show. ☐

Subgroups of A^{-1} and Linear Lie Groups

Lemma 5.29. *Let G denote a subgroup of A^{-1} for a Banach algebra A. Let $B = B_r(0)$ be any open ball around 0 such that*

$$\exp B \subseteq B_1(\mathbf{1}).$$

Then $\Gamma = B \cap \log(B_1(\mathbf{1}) \cap G)$ is a local group with respect to B.

Proof. Firstly, $-\Gamma = \Gamma$ is clear, since $-B = B$ and $G^{-1} = G$. Secondly, assume that $x, y \in \Gamma$ and $x * y \in B$. Then $\exp x, \exp y \in B_1(\mathbf{1}) \cap G$. Also, $\exp(x * y) = \exp x \exp y \in G$ and $\exp(x * y) \in \exp B \subseteq B_1(\mathbf{1})$. Thus $x * y = \log \exp(x * y) \in B \cap \log(B_1(\mathbf{1}) \cap G) = \Gamma$. ☐

We set $\mathfrak{g} = \mathfrak{T}(\Gamma)$ and recall that $\mathfrak{g} = \mathfrak{T}(\overline{\Gamma} \cap U)$ for any open zero neighborhood U of A (see the comment following Definition 5.25). Thus we can write

$$\mathfrak{g} = \mathfrak{T}\big(\log(B_1(\mathbf{1}) \cap G)\big) = \mathfrak{T}\big(\log(B_1(\mathbf{1}) \cap \overline{G})\big). \tag{18}$$

Notice that after Theorem 5.27 the set $\mathfrak{g} = \mathfrak{T}(\Gamma)$ is a closed Lie algebra in $(A, [\cdot\,, \cdot])$.

Definition 5.30. Assume that G is any subgroup of A^{-1}. We define the completely normable Lie algebra $\mathfrak{g}$ by $\mathfrak{T}\big(\log(B_1(\mathbf{1}) \cap \overline{G})\big) \subseteq (A, [\cdot, \cdot])$ and call it *the Lie algebra of G in A.* ☐

Locally Compact Subgroups of Invertible Elements in a Banach Algebra

Theorem 5.31. *Let A denote a Banach algebra with identity and G a locally compact subgroup of A^{-1}. Then the Lie algebra $\mathfrak{g}$ of G in A is a closed finite dimensional Lie subalgebra of $(A, [\cdot\,, \cdot])$ and*

(i) *for every open ball B around 0 with $\exp B \subseteq B_1(\mathbf{1})$ the function*

$$\exp|(B \cap \mathfrak{g}) : B \cap \mathfrak{g} \to V \quad \textit{with} \quad V = \exp(B \cap \mathfrak{g})$$

is a diffeomorphism onto an identity neighborhood of G.

(ii) *For each such B there is an open ball C around* 0 *contained in B such that* $C \cap \exp^{-1} G = C \cap \mathfrak{g}$.

Proof. Let $\Gamma = B \cap \log(B_1(\mathbf{1}) \cap G)$. Then $\mathfrak{g} = \mathfrak{T}(\Gamma)$ is a closed finite dimensional Lie algebra by Theorem 5.27, and $\mathfrak{g} \cap B$ is open and closed in Γ by Theorem 5.28. Since exp maps B homeomorphically onto the open neighborhood $\exp B$ of $\mathbf{1}$ in A^{-1}, exp maps $B \cap \mathfrak{g}$ homeomorphically onto a neighborhood of $\mathbf{1}$ in G. This proves (i).

(ii) By Theorem 5.28, for each B as in (i), there exists an open ball C around 0 in B such that $C \cap \Gamma = C \cap \mathfrak{g}$. It follows that $\exp(C \cap \mathfrak{g}) \subseteq G$, whence $C \cap \mathfrak{g} \subseteq C \cap \exp^{-1} G$. Conversely, if $x \in C \cap \exp^{-1} G$, then $\exp x \in B_1(\mathbf{1}) \cap G$, whence $x = \log \exp x \in C \cap \Gamma = C \cap \mathfrak{g}$. This completes the proof of (ii). □

The Definition of Linear Lie Groups

Definition 5.32. A topological group G is called a *linear Lie group* if there is a Banach algebra A with identity and an isomorphism of topological groups from G onto a subgroup G_ℓ of the multiplicative group A^{-1} of A such that there is a closed Lie subalgebra $\mathfrak{g}$ of $(A, [\cdot\,,\cdot])$ with the property that exp maps some 0-neighborhood of $\mathfrak{g}$ homeomorphically onto a $\mathbf{1}$-neighborhood of G_ℓ.

Any group such as G_ℓ will be called *a Lie subgroup of* A^{-1}. □

The assumption that for some open ball B around 0 in A the exponential function exp maps $B \cap \mathfrak{g}$ homeomorphically onto an identity neighborhood in G entails in particular that $\Gamma = B \cap \mathfrak{g}$ is a local subgroup with respect to B.

The entire multiplicative group A^{-1} of any Banach algebra is a linear Lie group whose Lie algebra is $(A, [\cdot\,,\cdot])$. It is clear that we may assume for any linear Lie group that it is already embedded into the multiplicative group A^{-1} of some Banach algebra A if it is convenient. In this case G is simply a Lie subgroup of A^{-1}. If H is a Lie subgroup of A^{-1} and $H \subseteq G$, we shall also briefly say that H is a *Lie subgroup of* G. Recall that no subgroup of A^{-1} has small subgroups by Lemma 2.38. Hence, in particular, *linear Lie groups do not have small subgroups.*

Proposition 5.33. (i) *Every locally compact subgroup G of the multiplicative group of a Banach algebra is a linear Lie group with* $\mathfrak{g} = \mathfrak{T}(\Gamma)$, $\Gamma = B \cap \log(B_1(\mathbf{1}) \cap G)$.

(ii) *A locally compact subgroup H of an arbitrary linear Lie group G is a Lie subgroup. Moreover, G contains a subset C which is homeomorphic to a closed convex symmetric identity neighborhood in the Banach space $\mathfrak{g}/\mathfrak{h}$ such that* $(c, h) \mapsto ch \colon C \times H \to CH$ *is a homeomorphism onto a neighborhood of* 1 *in* G. (The Tubular Neighborhood Theorem for Subgroups)

(iii) *A closed subgroup H of a finite dimensional linear Lie group G is a Lie subgroup.*

(iv) *A compact Lie group is a linear Lie group.*

(v) *If A is a Banach algebra and G a Lie subgroup of A^{-1}, then G is closed in A^{-1}.*

Proof. (i) Use Theorem 5.31 and Definition 5.32 and note that the Lie algebra of G is $\mathfrak{g} = \mathfrak{T}(\Gamma)$!

(ii) The first assertion is immediate from (i). For a proof of the second, by Definition 5.32 we may assume that G is a closed subgroup of the group A^{-1} of invertible elements of a Banach algebra A, such that $\exp_A\colon A \to A^{-1}$ maps a closed Lie algebra $\mathfrak{g} = \mathfrak{T}(\Gamma)$ in A into G implementing a local homeomorphism from $B \cap \mathfrak{g}$ onto an identity neighborhood of G. The Lie algebra $\mathfrak{h} = \mathfrak{T}(B \cap \log(B_1(\mathbf{1}) \cap H)) \subseteq A$ is a direct summand of $\mathfrak{g}$ since $\dim \mathfrak{h} < \infty$. Thus $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{h}$ with a closed vector subspace $\mathfrak{v}$ of $\mathfrak{g}$. Let B_0 be a closed ball around 0 such that $B_0 * B_0$ is defined in A. We find open zero neighborhoods U and V of $\mathfrak{v}$ and $\mathfrak{h}$, respectively, contained in B_0 satisfying the following properties:

(a) $U * V \subseteq B_0$.

(b) By the Theorem of the Local Inverse, the function $u \oplus v \mapsto u * v\colon U \oplus V \to U * V$ is a homeomorphism onto a zero neighborhood of $\mathfrak{g}$.

(c) $U * V$ is mapped homeomorphically by $\exp|\mathfrak{g}$ onto an identity neighborhood of G, and

(d) $U * V$ is so small that $(U * V) \cap \exp^{-1}(H) = (U * V) \cap \mathfrak{h}$ (see 5.31(ii)).

Regarding (d): If $X \in U$ and $Y \in V$ are such that $X * Y \in \mathfrak{h}$, then $X = (X * Y) * (-Y) \in U \cap (\mathfrak{h} \cap B_0) * (\mathfrak{h} \cap B_0) \subseteq U \cap \mathfrak{h} = \{0\}$. Hence (d) implies

$$(U * V) \cap \exp^{-1}(H) = V \tag{$*$}$$

Now we let U_1 be a closed convex symmetric zero neighborhood of $\mathfrak{v}$ such that

$$U_1 \subseteq U \qquad \text{and} \qquad U_1 * U_1 \subseteq U * V \tag{$**$}$$

and set $C = \exp U_1$. We consider the continuous surjective function $m\colon C \times H \to CH$, $m(c, h) = ch$ and assume that $m(c_1, h_1) = m(c_2, h_2)$, i.e. $c_1 h_1 = c_2 h_2$ for $c_j \in C$ and $h_j \in H$, $j = 1, 2$. We write $c_j = \exp X_j$ for $X_j \in U_1$ for $j = 1, 2$. Then $\exp\big((-X_2) * X_1\big) = c_2^{-1} c_1 = h_2 h_1^{-1} \in H$, whence

$$-Y \stackrel{\text{def}}{=} (-X_2) * X_1 \in (-U_1 * U_1) \cap \exp^{-1} H \subseteq (U * V) \cap \exp^{-1} H = V$$

by $(**)$ and $(*)$. Hence $X_2 * 0 = X_2 = X_1 * Y$ and thus, by (b) above, $X_2 \oplus 0 = X_1 \oplus Y$, whence $Y = 0$ and $X_1 = X_2$, i.e. $c_1 = c_2$ and thus also $h_1 = h_2$. Therefore m is bijective. By (b) and (c) above and in view of $U_1 \subseteq U$ from $(**)$, we conclude that $m|(C \times V')$, $V' = \exp V$, implements a homeomorphism from the open neighborhood $C \times V'$ of $(1, 1)$ in $C \times H$ onto an open neighborhood CV' of 1 in CH; in particular, CH is a neighborhood of 1. The group H acts on the right on $C \times H = \bigcup_{h \in H} (C \times V')h$ by multiplication on the right factor and on $CH = \bigcup_{h \in H} (CV')h$ by multiplication on the right, and m is equivariant for this action. This allows us to conclude that m is a homeomorphism.

For (iii) note that a finite dimensional linear Lie group G, being locally homeomorphic to $\mathfrak{g}$ with $\mathfrak{g} \cong \mathbb{R}^n$ for some n as a vector space, is locally compact. Closed subspaces of locally compact spaces are locally compact. Thus (ii) applies.

In view of Definition 2.41, (iv) is a consequence of (iii), Corollary 2.40.
Proof of (v). By Definition 5.32 there is an open ball B around zero in A such that $\exp|B: B \to U \stackrel{\rm def}{=} \exp B$ is a homeomorphism onto an identity neighborhood U of A^{-1} and that $\exp|(B\cap\mathfrak{g}): (B\cap\mathfrak{g}) \to V \stackrel{\rm def}{=} \exp(B\cap\mathfrak{g})$ is a homeomorphism of an identity neighborhood of G. Since $\mathfrak{g}$ is closed in A by Definition 5.32, the set $B\cap\mathfrak{g}$ is closed in B and then the set V is closed in U since $\exp|B: B \to U$ is a homeomorphism. Therefore G is locally closed in A^{-1} (see Appendix 4, Definition A4.22. Hence G is closed in A^{-1} by A4.23. □

Note that in 5.33(ii) we have observed not only that a locally compact subgroup H of a linear Lie group G is itself a linear Lie group, but we have shown that it is very nicely embedded into G, namely, in a "tubular fashion." If G is finite dimensional, the set C may be taken to be an m-cell with $m = \dim G - \dim H$ so that H has a neighborhood in G homeomorphic to $H\times[-1,1]^m$. Local compactness of H was used in 5.33(ii) only in concluding that $\mathfrak{h}$ is algebraically and topologically a direct summand of $\mathfrak{g}$. The tubular neighborhood theorem holds if one postulates this as an assumption.

Exercise E5.9. Assume that A is a C^*-algebra (see Exercise E1.1 and Example 2.12). Then the group $U(A) = \{g\in A^{-1} \mid g^* = g^{-1}\}$ of unitary elements is a Lie subgroup of A^{-1} with Lie algebra $\mathfrak{u}(A) = \{a\in A \mid a^* = -a\}$.

Special case: $A = \mathbb{C}$, $U(A) = \mathbb{S}^1$, $\mathfrak{u}(A) = i\mathbb{R}$. Thus, as a consequence $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has the classical representation as a linear Lie group. We may say that $\mathfrak{L}(\mathbb{T}) = \mathbb{R}$ and $\exp_{\mathbb{T}}: \mathbb{R} \to \mathbb{T}$ is the quotient map $\exp_{\mathbb{T}}(r) = r + \mathbb{Z}$. □

Exercise E5.10. (i) We consider $\mathbb{K}^n$ and $A = \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n)$ and fix a scalar product $(x|y) = \sum_{j=1}^n x_j\overline{y_j}$. Then A is in fact a C^*-algebra with an involution defined by $(\varphi(x)|y) = (x|\varphi^*(y))$. Then

$$\mathrm{Gl}(n,\mathbb{K}) = A^{-1},\quad \mathrm{Sl}(n,\mathbb{K}),\quad \mathrm{O}(n),\quad \mathrm{SO}(n),\quad \mathrm{U}(n),\quad \mathrm{SU}(n)$$

are Lie subgroups of $\mathrm{Gl}(n,\mathbb{K})$ with Lie algebras

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{K}) &= (\mathrm{Hom}(\mathbb{K}^n,\mathbb{K}^n),[\cdot,\cdot]), & \mathfrak{sl}(n,\mathbb{K}) &= \{\varphi\in\mathfrak{gl}(n,\mathbb{K}) \mid \mathrm{tr}\,\varphi = 0\},\\
\mathfrak{o}(n) &= \{\varphi\in\mathfrak{gl}(n,\mathbb{R}) \mid \varphi^* = -\varphi\}, & \mathfrak{so}(n) &= \mathfrak{o}(n)\cap\mathfrak{sl}(n,\mathbb{R}),\\
\mathfrak{u}(n) &= \{\varphi\in\mathfrak{gl}(n,\mathbb{C}) \mid \varphi^* = -\varphi\}, & \mathfrak{su}(n) &= \mathfrak{u}(n)\cap\mathfrak{sl}(n,\mathbb{C}).
\end{aligned}$$

Determine the dimensions of all of these Lie groups.

(ii) Prove the following propositions.

1) *Every discrete group is a linear Lie group.*
2) *The additive group of every Banach space is a linear Lie group.*

[Hint. Regarding (ii) 1). Recall that a family $(s_j)_{j\in J}$ in a topological abelian additively written group S is called *summable* if the net of all finite partial sums $s_F = \sum_{j\in F} s_j$ for finite subsets $F\subseteq J$ converges as F ranges through the set $\mathrm{Fin}(J)$ of finite subsets of J.

Let G be a discrete group. Set

$$A \stackrel{\text{def}}{=} \ell^1(G) = \{(x_g)_{g\in G} \mid x_g \in \mathbb{C},\ (|x_g|)_{g\in G} \text{ is summable in } \mathbb{R}\}$$

with $\|(x_g)_{g\in G}\|_1 = \sum_{g\in G} |x_g|$. This is a Banach algebra with respect to the multiplication

$$(x_g)_{g\in G} * (y_h)_{h\in G} = (\sum_{gh=k} x_g y_h)_{k\in G}.$$

Prove that this multiplication is well-defined and turns A into a Banach algebra with identity $\mathbf{1} = (e_g)_{g\in G}$ where $e_{\mathbf{1}} = 1$ and $e_g = 0$ for $g \neq \mathbf{1}$. Now let

$$\underline{g} \stackrel{\text{def}}{=} (\delta_{gh})_{h\in G} \quad \text{with} \quad \delta_{gh} = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases}$$

Then $\underline{gh} = \underline{g} * \underline{h}$ and $g \mapsto \underline{g} : G \to A^{-1}$ is an injective morphism of groups. For two different elements g and h in G show that $\|\underline{g} - \underline{h}\|_1 = 2$. Hence $\{\underline{g} \mid g \in G\}$ is a discrete hence closed subgroup of A^{-1}.

Regarding (ii) 2). Let E be a real Banach space. Consider the Banach space $E_1 \stackrel{\text{def}}{=} E \times \mathbb{R}$ with the sup-norm. Set $A = \mathcal{L}(E_1)$, the Banach algebra of all bounded operators of E_1 with the operator norm. For $x \in E$ let $T_x \in A$ be defined by $T_x(y,r)) = (y + r{\cdot}x, r)$. Set $G = \{T_x \mid x \in E\}$ and show that G is a closed subgroup of A^{-1}. Show that $x \mapsto T_x\colon E \to G$ is an isomorphism of topological groups. Define $t_x(y,r) = (r{\cdot}x, 0)$ and $\mathfrak{g} \stackrel{\text{def}}{=} \{t_x \in A \mid x \in E\}$. Show that $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}$ and that $\exp\colon A \to A^{-1}$ maps $\mathfrak{g}$ homeomorphically onto G.] □

Analytic Groups

This subsection may be skipped without impairing the understanding of the remainder of the book; however, it is an important link to a more general theory of Lie groups.

Corollary 5.34. *Let G denote a Lie subgroup of the multiplicative group of a Banach algebra. Let B be an open ball around 0 such that $B * B$ is defined and mapped into $B_1(\mathbf{1})$ under* exp *and $(B*B)\cap\mathfrak{g}$ is mapped homeomorphically onto an identity neighborhood of G. Set $V = \exp(B\cap\mathfrak{g})$. Then $\mathcal{U} = \{gV \mid g \in G\}$ is an open cover of G. For each $g \in G$, the function $\varphi_g\colon gV \to B\cap\mathfrak{g}$, $\varphi_g(x) = \log(g^{-1}x)$ is an analytic and analytically invertible homeomorphism with the following property:*

(A) *If $\emptyset \neq U = gV \cap hV$ and $U_g = \varphi_g(U)$, $U_h = \varphi_h(U)$, then $\varphi_h \circ (\varphi_g|U_g)^{-1}\colon U_g \to U_h$ is an analytic and analytically invertible homeomorphism f of open subsets of $\mathfrak{g}$.*

Proof. It is clear that $\mathcal{U}$ is an open cover of G since V is an open identity neighborhood by definition. For $x \in U_g$ we note $\varphi_g^{-1}(x) = g\exp x$. Thus we compute $f(x) = \varphi_h\big(g\exp x\big) = \log\big(h^{-1}g\exp x\big)$. Therefore $\exp f(x) = h^{-1}g\exp x$, whence $h^{-1}g = \exp f(x)\exp -x = \exp(f(x) * -x) \in \exp(B * B) \subseteq B_1(\mathbf{1})$. If we set

$u = \log(h^{-1}g)$, then $u = f(x) * -x$ or $f(x) = u * x$. Hence $f: U_g \to U_h$ is an analytic function between open sets of $\mathfrak{g}$. □

Assume that a Hausdorff topological space M has an open cover $\{U_j \mid j \in J\}$ such that there is a family $\varphi_j: U_j \to B_j$ of homeomorphisms onto open sets of some Banach space E satisfying

(A) if $U_j \cap U_k \neq \emptyset$ then the function $\varphi_k \circ \big(\varphi_j|(U_j \cap U_k)\big)^{-1}$ maps $\varphi_j(U_j \cap U_k)$ analytically onto $\varphi_k(U_j \cap U_k)$.

Then M is called *an analytic manifold modelled on* E. If $E = \mathbb{R}^n$ then M is called an analytic manifold of dimension n. Therefore, Corollary 5.34 can also be expressed as follows:

Corollary 5.35. *Every Lie subgroup G of the multiplicative group of some Banach algebra is an analytic manifold modelled on $\mathfrak{g}$. If G is locally compact then G is a manifold of dimension* $\dim \mathfrak{g}$. □

If $f: M \to M'$ is a function between analytic manifolds then f is called *analytic* if for any $m \in M$ and any *coordinate system* $\varphi_j: U_j \to B_j$ with $m \in U_j$ and any coordinate system $\varphi'_{j'}: U'_{j'} \to B'_{j'}$ with $f(m) \in U'_{j'}$ the function $x \mapsto \varphi'_{j'}\big(f(\varphi_j^{-1}(x))\big)$ from a sufficiently small neighborhood C of $\varphi_j(m)$ in B_j into $B'_{j'}$ is analytic.

Exercise E5.11. Let G be a linear Lie group. Assume that it is a Lie subgroup of A^{-1} for a Banach algebra A with the analytic structure defined in Corollaries 5.34 and 5.35. With the obvious analytic structure on $G \times G$, show that $(x, y) \mapsto xy: G \times G \to G$ and $x \mapsto x^{-1}: G \to G$ are analytic functions. □

Any topological group G with the structure of an analytic manifold such that multiplication and inversion are analytic functions is called an *analytic group.* Thus once more Corollary 5.34 and Exercise E5.11 can be rephrased to read:

Corollary 5.36. *Every linear Lie group is an analytic group.* □

As a consequence of this corollary and Corollary 2.40 we now have:

Corollary 5.37. *Every compact Lie group is a finite dimensional analytic group.* □

The Intrinsic Exponential Function of a Linear Lie Group

Theorem 5.31 has an important consequence for the classification of one parameter groups which extends to all linear Lie groups. Recall Definition 5.7 in which we denoted with $\mathfrak{L}(G)$ the space $\mathrm{Hom}(\mathbb{R}, G)$ of all one parameter groups with the topology of uniform convergence on compact sets. We emphasize the fact that $\mathfrak{L}(G)$ is, to begin with, a topological space canonically (and functorially) associated with

G. In this subsection we shall see that G endows $\mathfrak{L}(G)$ with a lot of additional structure. In order to proceed, the following definition is helpful:

Definition 5.38. A topological vector space is called *completely normable* if it admits a norm with respect to which it is a Banach space. A *topological Lie algebra* is a Lie algebra $\mathfrak{g}$ which is a topological vector space such that $(x, y) \mapsto [x, y]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous. It is called *completely normable* if its topology is defined by a Banach space norm. □

Definition 5.39. If G is an arbitrary topological group then we call the function

$$\exp_G: \mathfrak{L}(G) \to G \qquad \exp_G(X) = X(1) \tag{19}$$

the *exponential function of* G. □

If G is a linear Lie group we assume that $G \subseteq A^{-1}$ for some Banach algebra A and that $\mathfrak{g} = \mathfrak{T}\big(\log(B_1(\mathbf{1}) \cap G)\big)$ is the Lie algebra of G in A. We recall from 5.3 that $\log_A: B_1(\mathbf{1}) \to \log(B_1(\mathbf{1}))$ is a well defined homeomorphism and set

$$\begin{aligned} M_1 =& \{g \in B_1(\mathbf{1}) \mid \exp_A[0,1]\cdot(\log_A g) \subseteq B_1(\mathbf{1}) \cap G\}, \quad \text{and} \\ N_1 =& \log M_1 = \{X \in \mathfrak{g} \mid [0,1]\cdot X \subseteq \log_A(B_1(\mathbf{1}) \cap G)\}. \end{aligned}$$

Proposition 5.40. *Assume that* G *is a Lie subgroup of the group* A^{-1} *of units in a Banach algebra* A *and that* $\alpha: A \to \mathfrak{L}(A^{-1})$ *is the homeomorphism of Proposition* 5.9 *given by* $\alpha(x)(t) = \exp_A t\cdot x$. *Then* $\exp_A x = \alpha(x)(1) = \exp_G\big(\alpha(x)\big)$ *and we have the following conclusions:*

(i) α *induces a homeomorphism* $\alpha_G: \mathfrak{g} \to \mathfrak{L}(G)$.

(ii) $\exp_A |N_1: N_1 \to M_1$ *is an analytic homeomorphism with analytic inverse* $\log: M_1 \to N_1$ *of an open connected zero neighborhood* N_1 *in* $\mathfrak{g}$ *to an open identity neighborhood of* G.

(iii) *If* G *is locally compact then* $\mathfrak{g} \cap \exp_A^{-1} G$ *is a zero-neighborhood of* $\exp_A^{-1} G$.

(iv) *If* X, Y, $X * Y \in N_1$, *then there are identity neighborhoods* U_X *and* U_Y *such that* $(X * U_Y) \cup (U_X * Y) \subseteq N_1$, *and the set* $\{(X, Y) \in N_1 \times N_1 \mid X * Y \in N_1\}$ *is a neighborhood of* $(0, 0) \in N_1 \times N_1$.

Proof. (i) The function α_G is just the restriction of the function $A \to \mathfrak{L}(A^{-1})$ of the Recovery of Scalar Multiplication 5.9 and is, therefore, a homeomorphism onto its image which is contained in $\mathfrak{L}(G)$ by Definition 5.32. Conversely, if $X: \mathbb{R} \to G$ is a one parameter subgroup of G, then by Proposition 5.9, $X = \alpha(x)$ for some $x \in A$, that is $X(t) = \exp t\cdot x$ for all $t \in \mathbb{R}$. If $\varepsilon > 0$ is such that $|t| < \varepsilon$ implies $X(t) \in \exp(B \cap \mathfrak{g})$ where $\Gamma = B \cap \mathfrak{g}$ is a local group mapped homeomorphically onto an identity neighborhood of G inside $B_1(\mathbf{1})$, then $t\cdot x = \log X(t) \in \Gamma \subseteq \mathfrak{g}$. Then $x \in \mathfrak{g}$ and α_G is surjective. Hence $\alpha: \mathfrak{g} \to \mathfrak{L}(G)$ is a homeomorphism.

(ii) We notice that $N_1 = \{x \in A \mid \exp[0,1]\cdot x \subseteq B_1(\mathbf{1}) \cap G\}$ by the definitions of M_1 and N_1. In particular, N_1 is a connected subset of the open set $U \stackrel{\text{def}}{=} \exp_A^{-1} B_1(\mathbf{1})$ and therefore is in the identity component of 0 in U. It is then clear from 5.3, that $\exp_A|N_1\colon N_1 \to M_1$ and $\log|M_1\colon M_1 \to N_1$ invert each other. Furthermore, $\exp[0,1]{\cdot}x \subseteq G$ implies $\exp \mathbb{R}{\cdot}x \subseteq G$ since $[0,1]$ generates the additive group $\mathbb{R}$. By (i) this implies $x \in \mathfrak{g}$. Thus $N_1 \subseteq \mathfrak{g}$. We claim that N_1 is open in $\mathfrak{g}$. If not, there is a point $x \in N_1$ which fails to be an inner point of $\mathfrak{g} \cap U$. Then there is a sequence $(t_n, x_n) \in [0,1] \times (\mathfrak{g} \cap U)$ with $x = \lim_n x_n$ but with $t_n{\cdot}x_n \notin \mathfrak{g} \cap U$. As $[0,1]$ a subsequence of t_n converges, and upon renaming it and x_n accordingly we may as well assume that $t = \lim t_n$ exists. But then, since $\mathfrak{g} \cap U$ is open in $\mathfrak{g}$ we would have $t{\cdot}x \notin \mathfrak{g} \cap U$ which would contradict $[0,1]{\cdot}x \subseteq \mathfrak{g} \cap U$ which we know from $x \in N_1$. A completely analogous proof shows that M_1 is open in G. This finishes the proof of (ii).

(iii) is a consequence of 5.31(ii).

(iv) follows from the fact that N_1 is open by (ii) and that the multiplication $*$ given by $X * Y = \log(\exp X \exp Y)$ is continuous. □

Every element g of N_1 is connected with $\mathbf{1}$ by the unique *local one parameter semigroup* $\exp_A([0,1]{\cdot}\log g)$. One sometimes says that the identity neighborhood M_1 of G is *uniquely ruled by one parameter semigroups* or simply by *arcs* (which is perhaps a bit too terse).

The Exponential Function of a Linear Lie Group

Theorem 5.41. *Let G denote a linear Lie group.*

(i) *The space $\mathfrak{L}(G)$ is a completely normable topological real Lie algebra with respect to the following operations:*

(s) *Scalar multiplication: $(r{\cdot}X)(t) = X(tr)$ for $X \in \mathfrak{L}(G)$, $r, t \in \mathbb{R}$.*

(a) *Addition: $(X+Y)(t) = \lim_n \big(X(t/n)Y(t/n)\big)^n$ for X, $Y \in \mathfrak{L}(G)$, $t \in \mathbb{R}$.*

(b) *Lie bracket: $[X,Y](t) = \lim_n \operatorname{comm}\big(X(t/n)Y(t/n)\big)^{n^2}$ with $\operatorname{comm}(g,h) = ghg^{-1}h^{-1}$ for X, $Y \in \mathfrak{L}(G)$, $t \in \mathbb{R}$.*

Furthermore,

(e) *the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$, $\exp_G X = X(1)$ maps all sufficiently small 0-neighborhoods of $\mathfrak{L}(G)$ homeomorphically onto $\mathbf{1}$-neighborhoods of G, that is it induces a local homeomorphism at 0.*

(ii) *There is an open zero neighborhood N_1 of $\mathfrak{L}(G)$, and an open identity neighborhood of G such that $\exp_G|N_1\colon N_1 \to M_1$ is a homeomorphism such that M_1 is ruled by one parameter semigroups. Its inverse is denoted $\log\colon M_1 \to N_1$.*

(iii) *The image $\operatorname{im}\exp_G = \exp_G \mathfrak{L}(G)$ of the exponential function algebraically generates the identity component G_0 of G, in symbols $G_0 = \langle \exp_G \mathfrak{g}\rangle$, and G_0 is open in G.*

(iv) *G is discrete if and only if $\mathfrak{L}(G) = \{0\}$.*

(v) *If $X \in \mathfrak{L}(G)$ and $(\operatorname{Spec}\operatorname{ad} X) \cap (2\pi i\mathbb{Z} \setminus \{0\}) = \emptyset$ then there is an open neighborhood U of X in $\mathfrak{g}$ such that $\exp_G|U\colon U \mapsto \exp U$ is a homeomorphism onto an open neighborhood of $\exp_G X$ in G and $Y \mapsto \log\big(\exp_G X)^{-1}\exp_G(X+Y)\big)\colon U \to U' \subseteq \mathfrak{g}$ is an analytic and analytically invertible homeomorphism onto an open zero neighborhood of $\mathfrak{g}$.*

In particular, $\exp_G \mathfrak{g}$ *is a neighborhood of each* $g \in G$ *such that* $\exp_G^{-1}(g)$ *contains an* X *for which* $\operatorname{Spec}\operatorname{ad} X$ *does not contain a nonzero integral multiple of* $2\pi i$.

(vi) *If* H *is a locally compact subgroup of* G, *then* $\mathfrak{L}(H) \cap \exp_G^{-1} G$ *is a neighborhood of* 0 *in* $\exp_G^{-1} H$.

Proof. (i) Since G is a linear Lie group we may assume that G is a closed subgroup of A^{-1} for a Banach algebra A. If we transport the Lie algebra structure and the topology from the Lie algebra $\mathfrak{g} \subseteq (A, [\cdot\,,\cdot])$ of G in A to $\mathfrak{L}(G)$ via α, then $\mathfrak{L}(G)$ becomes a completely normed topological Lie algebra and (e) is automatically satisfied. Furthermore, Recovery of Scalar Multiplication 5.9 proves (s), Recovery of Addition 5.10 proves (a), and Recovery of the Bracket 5.11 proves (b).

(ii) The assertion follows from Proposition 5.40(ii).

(iii) By (ii) we have $M_1 \subseteq \operatorname{im}\exp_G$. Hence $\operatorname{im}\exp_G$ is a neighborhood of the identity in G. We recall that a connected topological group C (such as G_0 here) is algebraically generated by any neighborhood U of the identity (such as $\operatorname{im}\exp_G$ in our case); indeed the subgroup $\langle U\rangle = \bigcup_{n=1}^{\infty}(U \cup U^{-1})^n$ is open and hence closed in C, since its cosets form a partition of C. This proof shows, in particular, that G_0 is open.

(iv) If G is discrete, then $\mathfrak{L}(G) = \operatorname{Hom}(\mathbb{R}, G)$ is singleton. Conversely, if $\mathfrak{L}(G) = \{\mathbf{0}\}$, then $G_0 = \langle \exp_G\{0\}\rangle = \{\mathbf{1}\}$ by (iii) and $\{\mathbf{1}\}$ is open in G.

(v) For a proof of (v) we may and shall assume that $G \subseteq A^{-1}$ for a Banach algebra A and that the exponential function of G is the restriction $\exp_G\colon \mathfrak{g} \to G$ of the exponential function of A to $\mathfrak{g}$. We define $\varphi\colon A \to A$ by

$$\varphi(Y) = (\exp_G -X)\big(\exp_G(X+Y)\big) = L_{\exp_G X}^{-1} \exp_G\big(T_X(Y)\big)$$

(with $L_a x = ax$ and $T_X Y = X+Y$ in A). Since $T_X|\mathfrak{g}\colon \mathfrak{g} \to \mathfrak{g}$ and $L_{\exp_G X}|G\colon G \to G$ are homeomorphisms, our proof will be accomplished if we can show that φ maps an open neighborhood U_0 of 0 bijectively onto an open identity neighborhood of G. By the chain rule $\varphi'(0) = \big(L_{\exp_G X}^{-1}\big)'(\exp_G X) \circ \exp_G'(X) \circ T_X'(0)$. Since L_a is linear, $\big(L_{\exp_G X}^{-1}\big)'(\exp_G X) = L_{\exp_G X}^{-1}$, and $T_X(0) = \mathrm{id}_A$. Hence $\varphi'(0) = L_{\exp_G X}^{-1} \exp_G'(X) = f(\operatorname{ad} X)$ by Proposition 5.23, and by the assumption on $\operatorname{Spec}\operatorname{ad} X$, the operator $\varphi'(0)$ is invertible on A and $\mathfrak{g}$ by 5.23. Now let V be a 0-neighborhood in $\mathfrak{g}$ mapped homeomorphically onto an identity neighborhood in G (by Definition 5.32). Now we define $W = \varphi^{-1}(\exp_G V)$ and $\Phi\colon W \to V$ by $\Phi(Z) = \log\big(\varphi(Z)\big)$. Now $\Phi'(0) = \log'(\mathbf{1}) \circ \varphi'(0)$ by the chain rule. We note that $\log'(\mathbf{1}) = \exp_G'(0)^{-1} = \mathrm{id}_A$, Hence $\Phi'(0) = \varphi'(0)\colon \mathfrak{g} \to \mathfrak{g}$ is invertible. Thus by the Theorem of the Local Inverse, Φ maps an open neighborhood U_0 of 0 in W onto an open neighborhood of 0. Hence φ maps U_0 onto an open identity neighborhood in G. This is what we had to show.

(vi) By 5.33(ii), H is a Lie subgroup of G. As in the proof of (v) we may assume that $G \subseteq A^{-1}$ and that $\mathfrak{L}(G)$ is identified with $\mathfrak{g} \subseteq A$ in such a fashion that $\exp_G\colon \mathfrak{g} \to G$ is the restriction of the exponential function $\exp\colon A \to A^{-1}$ to $\mathfrak{g}$. Then $\mathfrak{L}(H)$ becomes identified with a finite dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (see 5.31). By

5.31 we find an open ball C around 0 in A such that $C \cap \exp_A^{-1} H = C \cap \mathfrak{h}$. Since $C \cap \exp_A^{-1} H$ is a neighborhood of 0 in $\exp_A^{-1} H$, the set $C \cap \mathfrak{h}$ is a neighborhood of 0 in $\exp_A^{-1} H$. As $C \cap \mathfrak{h} = C \cap \mathfrak{g} \cap \exp_A^{-1} H = C \cap \mathfrak{g} \cap \exp_G^{-1} H \subseteq C \cap \mathfrak{h}$ the set $C \cap \mathfrak{h}$ is a neighborhood of 0 in $\exp_G^{-1} H$. Claim (vi) follows. (See also 5.40(iii).) □

We recognize after Theorem 5.41 that the Lie algebra $\mathfrak{L}(G)$ of G as a completely normable Lie algebra is uniquely and canonically determined by G. For any embedding of G as a closed subgroup of the multiplicative group A^{-1} of some Banach algebra A, the Lie algebra of G in A is isomorphic to $\mathfrak{L}(G)$ *as a completely normable Lie algebra*. The norm of $\mathfrak{g}$ induced by that of A is not determined by G.

We now recognize the *functorial property* of the prescription $G \mapsto \mathfrak{L}(G)$ (cf. Appendix 3). In the last part of the following result we make a brief reference to universal covering groups which are discussed in Appendix 2.

Functoriality of the Lie Algebra of a Lie Group

Theorem 5.42. (i) (Global Version) *Let G and H be linear Lie groups. If $f: G \to H$ is a morphism of topological groups, then the prescription*

$$\mathfrak{L}(f)(X) = f \circ X \text{ for } X \in \mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$$

defines a unique morphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ of topological real Lie algebras such that

$$\exp_H \circ \mathfrak{L}(f) = f \circ \exp_G$$

that is such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

The relation $\mathfrak{L}(f_1) = \mathfrak{L}(f_2)$ for two morphisms $f_j: G \to H$, $j = 1, 2$ implies $f_1|G_0 = f_2|G_0$.

(ii) (Local Version) *Assume that for two linear Lie groups G and H there are identity neighborhoods U and V of G and H, respectively, and a continuous map $f: U \to V$ such that $f(xy) = f(x)f(y)$ whenever $x, y, xy \in U$. Then there is a continuous Lie algebra morphism $\mathfrak{L}(f): \mathfrak{g} \to \mathfrak{h}$ such that for suitably small* 0-*neighborhoods B and C of $\mathfrak{g}$ and $\mathfrak{h}$, respectively, the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\mathfrak{L}(f)|B} & C \\ {\scriptstyle \exp_G|B}\downarrow & & \downarrow{\scriptstyle \exp_H|C} \\ U & \xrightarrow[f]{} & V. \end{array}$$

(iii) (Lie's Fundamental Theorem) *For two linear Lie groups G and H, assume that $T: \mathfrak{g} \to \mathfrak{h}$ is a morphism of completely normable Lie algebras. Then there are* *open identity neighborhoods* U *and* V *of* G *and* H*, respectively, and a continuous map* $f\colon U \to V$ *such that* $f(xy) = f(x)f(y)$ *whenever* $x, y, xy \in U$ *and that, for appropriately chosen* 0*-neighborhoods* B *and* C *of* $\mathfrak{g}$ *and* $\mathfrak{h}$*, respectively, the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\;T|B\;} & C \\ {\scriptstyle \exp_G |B}\big\downarrow & & \big\downarrow{\scriptstyle \exp_H |C} \\ U & \xrightarrow[\;f\;]{} & V. \end{array}$$

(iv) (Local Characterisation of Lie Groups) *Two linear Lie groups* G *and* H *have identity neighborhoods* U *and* V *linked by a homeomorphism* $f\colon U \to V$ *such that* $x, y, xy \in U$ *implies* $f(xy) = f(x)f(y)$ *if and only if the universal covering groups of* G_0 *and* H_0 *are isomorphic if and only if* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are isomorphic.*

Proof. (i) From $\mathfrak{L}(f)(X) = f \circ X$ we have $\exp_H\big(\mathfrak{L}(f)(X)\big) = (f \circ X)(1) = f\big(X(1)\big) = f(\exp_G X)$. Also

$$\begin{aligned} \mathfrak{L}(f)(r{\cdot}X)(t) &= \big((f \circ r{\cdot}X)\big)(t) = f\big(r{\cdot}X(t)\big) \\ &= f\big(X(tr)\big) = \big(r{\cdot}(f \circ X)\big)(t) \\ &= \big(r{\cdot}\mathfrak{L}(f)(X)\big)(t) \end{aligned}$$

by Theorem 5.41(i)(s). Similarly,

$$\begin{aligned} \mathfrak{L}(f)(X+Y)(t) &= f\big(X+Y)(t) = f\big\{\lim_n \big(X(t/n)Y(t/n)\big)^n\big\} \\ &= \lim_n \big\{f\big(X(t/n)\big)f\big(Y(t/n)\big)\big\}^n = \big((f \circ X) + (f \circ Y)\big)(t) \\ &= \big(\mathfrak{L}(f)(X) + \mathfrak{L}(f)(Y)\big)(t) \end{aligned}$$

by Theorem 5.41(i)(a). In just the same way Theorem 5.41(i)(b) implies

$$\mathfrak{L}(f)[X,Y] = [\mathfrak{L}(f)(X), \mathfrak{L}(f)(Y)].$$

The continuity of $\mathfrak{L}(f)$ follows if we establish continuity at 0. However, if B is an open ball around 0 in $\mathfrak{L}(H)$ which is mapped homeomorphically onto an open identity neighborhood of H under $\exp_H$, then $U = (f \circ \exp_G)^{-1}\big(\exp_H(B)\big)$ is an open zero neighborhood of $\mathfrak{L}(G)$, and $\mathfrak{L}(f)(U) \subseteq B$.

Next we show the uniqueness. If $F\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is a Lie algebra morphism with $\exp_H \circ F = f \circ \exp_G$, then $F(X)(1) = f\big(X(1)\big)$. Since F preserves scalar multiplication, we deduce $F(X)(t) = \big(t{\cdot}F(X)\big)(1) = \big(F(t{\cdot}X)\big)(1) = f\big((t{\cdot}X)(1)\big) = f\big(X(t)\big) = \mathfrak{L}(f)(X)(t)$ for all $t \in \mathbb{R}$, whence $F = \mathfrak{L}(f)$. (Note that the uniqueness proof requires the preservation of scalar multiplication by F only.)

Finally assume that $\mathfrak{L}(f_1) = \mathfrak{L}(f_2)$ for two morphisms. If $\exp_G$ maps the zero neighborhood B of $\mathfrak{L}(G)$ homeomorphically onto the zero neighborhood U of G, then $g \in U$ implies

$$f_1(g) = \exp_H \mathfrak{L}(f_1)(\exp_G |B)^{-1}(g) = \exp_H \mathfrak{L}(f_2)(\exp_G |B)^{-1}(g) = f_2(g).$$

Thus the equalizer $E \stackrel{\text{def}}{=} \{g \in G \mid f_1(g) = f_2(g)\}$ is a closed subgroup of G containing U. Hence it is open and thus contains G_0 which establishes the claim $f_1|G_0 = f_2|G_0$.

(ii) Let $f{:}\,U \to V$ be given as specified in (ii). Let $X{:}\,\mathbb{R} \to G$ be a one parameter group. There is an $\varepsilon > 0$ such that $X(]-\varepsilon,\varepsilon[) \subseteq U$. The function $t \mapsto fX(t){:}\,]\varepsilon,\varepsilon[\to H$ extends to a unique one parameter group $\mathfrak{L}\big(f \circ \mathfrak{L}(f)\big)X{:}\,\mathbb{R} \to H$. (For an elementary proof see the Extension Lemma 5.8 (cf. also E1.8), for a more sophisticated one see Appendix A2.26.) This gives us a function $\mathfrak{L}(f){:}\,\mathfrak{L}(G) \to \mathfrak{L}(H)$ such that $\exp_H \circ T$ agrees on a sufficiently small 0-neighborhood B with $f \circ \exp_G|B$. It respects scalar multiplication because of $r{\cdot}(\mathfrak{L}(f)X)(t) = \mathfrak{L}(f)X(rt) = \mathfrak{L}(f)(r{\cdot}X)(t)$. Exactly as we showed in the first part of the proof that $\mathfrak{L}(f)$ respected addition and brackets using recovery of addition and brackets we see now that $\mathfrak{L}(f)$ respects addition and brackets. Thus $\mathfrak{L}(f)$ is a morphism of completely normable Lie algebras. From its construction it follows that $\exp_H \circ \mathfrak{L}(f)$ agrees on a sufficiently small 0-neighborhood B of $\mathfrak{g}$ with $f \circ \exp_G|B$.

(iii) Let $T{:}\,\mathfrak{g} \to \mathfrak{h}$ be a morphism of completely normable Lie algebras. We may assume $G \subseteq A_1^{-1}$, $\mathfrak{g} \subseteq A_1$, $H \subseteq A_2^{-1}$, and $\mathfrak{h} \subseteq A_2$ for suitable Banach algebras. Pick open balls B_j around 0 in A_j such that $B_j * B_j$ is defined. Note that $\exp_G|(B_1 \cap \mathfrak{g}){:}\,B_1 \cap \mathfrak{g} \to U$ and $\exp_H|(B_2 \cap \mathfrak{h}){:}\,B_2 \cap \mathfrak{h} \to V$ are homeomorphisms onto open identity neighborhoods U and V of G and H, respectively, that $X, Y \in B_j$ implies $\|X * t{\cdot}Y\| < \pi$ for all $t \in [0,1]$, and that $TB_1 \subseteq B_2$. Then 5.21(iii) implies for all $X, Y \in B_1 \cap \mathfrak{g}$ that $T(X * Y) = (TX) * (TY)$. We set $f{:}\,U \to V$, $f(g) = \exp_H(T \log g)$ (where $\log = (\exp_G|B_1 \cap \mathfrak{g})^{-1}$). This f satisfies the requirements of (iii).

(iv) By (ii) and (iii) above, the homeomorphism $f{:}\,U \to V$ exists as stated iff $\mathfrak{g} \cong \mathfrak{h}$. By Appendix A2.21 every connected Lie group possesses a universal covering group. By A2.28, the homeomorphism $f{:}\,U \to V$ exists if and only if the universal covering groups of G and H are isomorphic. □

We notice that part (iii) and, consequently, part (iv) are more sophisticated than the rest because it is comparatively hard to create, from a morphism of the Lie algebras (an object of pure linear algebra) a local group morphism.

Exercise E5.12. Prove the following proposition:

The only connected linear Lie groups (*up to isomorphism*) *having a Lie algebra isomorphic to* $\mathfrak{so}(3)$ *are* SO(3) *and* $\mathbb{S}^3 \cong \mathrm{SU}(2)$.

[Hint. By Appendix A2.30 the topological groups listed are the only ones which are locally isomorphic to SO(3). They are linear Lie groups. By Theorem 5.42(iv) they are the only ones (up to isomorphism) whose Lie algebra is isomorphic to $\mathfrak{so}(3)$.] □

One should be aware of a subtlety involving the idea of topological groups being locally isomorphic. Assume that G and H are topological groups. It is fair to call them locally isomorphic if there are identity neighborhoods U and V of G and H, respectively, and a homeomorphism $f{:}\,U \to V$ such that u, u', $uu' \in U$ implies

$f(uu') = f(u)f(u')$. It is not fair, however, to call such a homeomorphism a local isomorphism because it is not necessarily the case that $f^{-1}(vv') = f^{-1}(v)f^{-1}(v')$ for all v, $v' \in V$ with $vv' \in V$. Example: $G = \mathbb{R}$, $H = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $U = \left]-\frac{1}{2}, \frac{1}{2}\right[$, $V = \mathbb{T} \setminus \{\frac{1}{2} + \mathbb{Z}\}$, $f: U \to V$, $f(t) = t + \mathbb{Z}$, $v = v' = \frac{1}{3} + \mathbb{Z} \in V$, and $v + v' = -\frac{1}{3} + \mathbb{Z}$, whence $f^{-1}(v) + f^{-1}(v') = \frac{2}{3} \neq -1/3 = f^{-1}(v + v')$. However, if f is given, passage to smaller identity neighborhoods U' and V' and to an invertible restriction $f' = f|U': U' \to V'$ has the local homomorphism property together with f'^{-1}.

If we are interested in analytic structures then one remarkable consequence of Theorem 5.42 is that *a continuous group morphism between linear Lie groups is automatically analytic with respect to the analytic structures on the groups.*

Exercise E5.13. (i) Let G and H be linear Lie groups and $f: G \to H$ a morphism of topological groups. Then f is analytic with respect to the analytic structures of G and H introduced in 5.34 and 5.35.

(ii) Let G, H, and K be linear Lie groups and $f: G \to H$ and $g: H \to K$ morphisms of topological groups. Then $\mathfrak{L}(g \circ f) = \mathfrak{L}(g) \circ \mathfrak{L}(f)$; that is Lie algebra morphisms $\mathfrak{L}(f)$ compose correctly. □

By a very slight abuse of notation, we shall often denote the Lie algebra $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ of a linear Lie group by $\mathfrak{g}$.

The Adjoint Representation of a Linear Lie Group

In view of the fact that we consider associative algebras as well as Lie algebras which are not associative it is appropriate to consider nonassociative algebras in general. They provide a large supply of Lie groups arising in various contexts. So let E be a Banach space and let $\mathcal{A}$ denote the Banach algebra of all bounded linear operators of E. Now assume that E is endowed with a continuous bilinear multiplication $(x, y) \mapsto xy$. For the purposes of the next theorem we shall call E a *not necessarily associative completely normable algebra*. The group $\mathrm{Aut}(E)$ of *automorphisms of* E consists of all $\alpha \in \mathcal{A}^{-1}$ satisfying $\alpha(xy) = \alpha(x)\alpha(y)$. Obviously $\mathrm{Aut}(E)$ is closed in $\mathcal{A}^{-1}$ with respect to the topology of pointwise convergence, hence a fortiori with respect to the operator norm topology which has at least as many closed sets. The vector space $\mathrm{Der}(E)$ of all *continuous derivations* $D \in \mathcal{A}$ defined by $D(xy) = (Dx)y + x(Dy)$ is a closed Lie subalgebra of the Lie algebra $(\mathcal{A}, [\cdot, \cdot])$.

Theorem 5.43. *Assume that E is a not necessarily associative completely normable algebra and consider the exponential function* $\exp = (T \mapsto e^T) : \mathcal{A} \to \mathcal{A}^{-1}$ *of the Banach algebra $\mathcal{A}$. Let $\mathcal{B}_1(\mathbf{1})$ denote the open unit ball in $\mathcal{A}$ around the identity operator and* $\log: \mathcal{B}_1(\mathbf{1}) \to \mathcal{A}$ *the logarithm. Then*

(i) $e^{\mathrm{Der}(E)} \subseteq \mathrm{Aut}(E)$ *and*

(ii) $\log\big(\mathcal{B}_1(\mathbf{1}) \cap \mathrm{Aut}(E)\big) \subseteq \mathrm{Der}(E)$.

(iii) $\mathrm{Aut}(E)$ *is a Lie subgroup with Lie algebra* $\mathfrak{aut}(E) = \mathrm{Der}(E)$.

Proof. Assume for the moment that we had established (i) and (ii). Let $\mathcal{N}_0$ denote the component of 0 in $\exp^{-1}\mathcal{B}_1(\mathbf{1})$. From Proposition 5.3(iii) it follows that

$$\exp|\mathcal{N}_0\colon \mathcal{N}_0 \to \mathcal{B}_1(\mathbf{1}) \quad \text{and} \quad \log\colon \mathcal{B}_1(\mathbf{1}) \to \mathcal{N}_0$$

are analytic inverses of each other. Thus, by (i) and (ii), the functions

$$\exp|\big(\mathcal{N}_0 \cap \operatorname{Der}(E)\big)\colon \big(\mathcal{N}_0 \cap \operatorname{Der}(E)\big) \to \big(\mathcal{B}_1(\mathbf{1}) \cap \operatorname{Aut}(E)\big), \quad \text{and}$$
$$\log|\big(\mathcal{B}_1(\mathbf{1}) \cap \operatorname{Aut}(E)\big)\colon \big(\mathcal{B}_1(\mathbf{1}) \cap \operatorname{Aut}(E)\big) \to \big(\mathcal{N}_0 \cap \operatorname{Der}(E)\big)$$

are analytic inverses of each other. In view of Definition 5.32, we then will have proved (iii). Thus it remains to prove (i) and (ii).

For Banach spaces V and W let $\operatorname{Hom}(V, W)$ denote the Banach space of all bounded linear operators $V \to W$ with the operator norm and note in passing that $\operatorname{Hom}(E, E) = \mathcal{A}$. We consider the Banach space $\mathcal{E} = \operatorname{Hom}\big(E, \operatorname{Hom}(E, E)\big)$ (which is isomorphic to the space of continuous bilinear mappings $E \times E \to E$) and three functions $T \mapsto T_l, T \mapsto T_r, T \mapsto T_\circ : \operatorname{Hom}(E, E) \to \operatorname{Hom}(\mathcal{E}, \mathcal{E})$ given by

$$(T_l\varphi)(x)(y) = \varphi(Tx)(y), \quad (T_r\varphi)(x)(y) = \varphi(x)(Ty), \quad (T_\circ\varphi)(x)(y) = T\big(\varphi(x)(y)\big)$$

for $\varphi \in \mathcal{E}$, $x, y \in E$. Also we shall denote with m the element of $\mathcal{E}$ which is given by $m(x)(y) = xy$.

Claim (i) *All three functions are contractive Banach space operators, and the first two reverse products while the last one preserves products. In particular,* $(T_l)^n = (T^n)_l$, $(T_r)^n = (T^n)_r$, *and* $(T_\circ)^n = (T^n)_\circ$.

Claim (ii) *For all* $R, S, T \in \operatorname{Hom}(E, E)$, *the operators* $R_\circ$, S_l, *and* T_r *commute pairwise.*

Claim (iii) *An element* $D \in \operatorname{Hom}(E, E)$ *is a derivation of* E *if and only if*

$$(D_l + D_r)(m) = D_\circ(m).$$

Claim (iv) *An element* $T \in \operatorname{Hom}(E, E)$ *is an endomorphism of* E (*i.e. respects multiplication*) *if and only if*

$$(T_l T_r)(m) = T_\circ(m).$$

The proof of these claims is Exercise E5.14. Now assume that $D \in \operatorname{Der}(E)$. Then $(D_l + D_r)(m) = D_\circ(m)$ by Claim (iii). In view of Claim (ii), the operators $D_l + D_r$ and $D_\circ$ commute. If α and β are two commuting operators on a Banach space which for some element x_0 satisfy $\alpha(x_0) = \beta(x_0)$, then for $k = 2, 3, \ldots$, recursively, we find $\alpha^k(x_0) = \alpha\big(\alpha^{k-1}(x_0)\big) = \alpha\big(\beta^{k-1}(x_0)\big) = \beta^{k-1}\big(\alpha(x_0)\big) = \beta^{k-1}\beta(x_0) = \beta^k(x_0)$. Thus we have $(D_l + D_r)^n(m) = D_\circ^n(m)$, $n = 0, 1, \ldots$ and therefore $p(D_1 + D_r)(m) = p(D_\circ)(m)$ for each power series $p(\xi)$ converging at $D_l + D_r$ and $D_\circ$. In particular, $e^{D_\circ}(m) = \big(e^{D_l + D_r}\big)(m)$. Now $e^{D_l + D_r} = e^{D_l}e^{D_r}$ in $\operatorname{Hom}(\mathcal{E}, \mathcal{E})$ by Claim (ii) and Exercise E5.3. By Claim (i) we have

$$e^{D_\circ} = (e^D)_\circ, \quad e^{D_l} = (e^D)_l, \quad e^{D_r} = (e^D)_r.$$

Then $(e^D)_\circ(m) = e^{D_\circ}(m) = \big((e^D)_l(e^D)_r\big)(m)$ and hence $e^D \in \operatorname{Aut} E$ by Claim (iv). This completes the proof of Statement 5.43(i).

For a proof of Statement 5.43(ii) we consider a $T \in \mathcal{B}_1(\mathbf{1})$. By Claim (i), the relation $\|T - \mathbf{1}\| < 1$ implies $\|T_l - \mathbf{1}\| = \|(T - \mathbf{1})_l\| \le \|T - \mathbf{1}\| < 1$ and similarly $\|T_r - \mathbf{1}\| < 1$ and $\|T_\circ - \mathbf{1}\| < 1$. The classical functional equation of the logarithm (cf. Exercise E5.3) and Claim (i) imply $\log(T_l T_r) = \log T_l + \log T_r$. Again by Claim (i) we have

$$\log T_\circ = (\log T)_\circ, \quad \log T_l = (\log T)_l, \quad \log T_r = (\log T)_r.$$

Now let $T \in \mathcal{B}_1(\mathbf{1}) \cap \operatorname{Aut} E$. Then $T_l T_r(m) = T_r T_l(m) = T_\circ(m)$ by Claims (ii) and (iv). Then $\big((\log T)_l + (\log T)_r\big)(m) = (\log T)_\circ(m)$, and thus $\log T$ is a derivation by Claim (iii). This completes the proof of the theorem. □

Observe that the preceding proof works without restriction on the dimension of E and on the nature of the multiplication on E. Assertion 5.43(ii) is noteworthy in so far as the entire unit ball around $\mathbf{1}$ inside the linear Lie group $G \stackrel{\text{def}}{=} \operatorname{Aut} E$ is mapped into the Lie algebra $\mathfrak{g} \stackrel{\text{def}}{=} \operatorname{Der} E$. In other words, $\mathcal{N}_0 \cap \exp_G^{-1} G \subseteq \mathfrak{g}$. As a rule, one can only show that some, possibly small, identity neighborhood of G is mapped into $\mathfrak{g}$ by the logarithm. The seemingly technical proof of 5.43 would be more lucid if we were to us the tensor product of Banach spaces; then we would have an isomorphism $\alpha: \mathcal{E} \to \operatorname{Hom}(E \otimes E, E)$, $\alpha(\varphi)(x \otimes y) = \varphi(x)(y)$. However, the machinery for developing tensor products of Banach spaces is intricate, and our approach avoids it.

Exercise E5.14. (a) Prove Claims (i), (ii), (iii), and (iv) in the preceding proof of Theorem 5.43.

(b) Formulate a different proof of Theorem 5.43 for finite dimensional E. The proof that e^D is an automorphism for a derivation D is a simple power series calculation in view of the Leibniz rule for $D^n(xy)$. In order to show that $\exp|B: B \to \exp B$ maps $B \cap \operatorname{Der} E$ homeomorphically onto $(\exp B) \cap \operatorname{Aut} E$ for some small ball around 0 first observe that $\operatorname{Aut} E$ is closed in $\operatorname{Hom}(E, E)^{-1}$, hence is locally compact as E and thus $\operatorname{Hom}(E, E)$ is finite dimensional. Apply Theorem 5.31 to see that $\operatorname{Aut} E$ is a Lie group. Show that an element $D \in \operatorname{Hom}(E, E)$ is in $\mathfrak{aut}(E)$ if and only if $e^{t \cdot D} \in \operatorname{Aut}(E)$ for all $t \in \mathbb{R}$. Differentiate the curve $t \mapsto e^{t \cdot D}(xy) = (e^{t \cdot D}x)(e^{t \cdot D}y)$ at $t = 0$. □

We now exploit the results of Theorem 5.43 for linear Lie groups in general.

The Adjoint Representation Theorem

Theorem 5.44. *Let G denote a linear Lie group with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$. Then the following conclusions hold:*

(i) *There is a unique morphism $\operatorname{Ad}: G \to \operatorname{Aut} \mathfrak{g}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \operatorname{ad}\ } & \operatorname{Der} \mathfrak{g} \\ {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle D \mapsto e^D} \\ G & \xrightarrow[\ \operatorname{Ad}\ ]{} & \operatorname{Aut} \mathfrak{g}. \end{array}$$

Equivalently,

$$(20)\qquad \begin{aligned} \mathrm{Ad}\circ\exp_G &= \exp_{\mathrm{Aut}\,\mathfrak{g}}\circ\,\mathrm{ad}, \quad \textit{i.e.}\\ \mathrm{Ad}(\exp_G X) &= e^{\mathrm{ad}\,X} \quad \textit{for all } X\in\mathfrak{g}. \end{aligned}$$

For all $g\in G$ and $X,Y\in\mathfrak{g}$, the following equations hold:

$$(21)\qquad g(\exp Y)g^{-1} = \exp\mathrm{Ad}(g)(Y), \quad \textit{and}$$

$$(22)\qquad \exp X\exp Y\exp -X = \exp(e^{\mathrm{ad}\,X}Y).$$

(ii) *The kernel of the adjoint representation* Ad *is the centralizer $Z(G_0,G)$ of the identity component of G in G.*

Proof. (i) We may assume that there is a Banach algebra A with $G\subseteq A^{-1}$. Then the Lie subalgebra $\mathfrak{g}\subseteq(A,[\cdot,\cdot])$ as in (18) may be identified with the Lie algebra $\mathfrak{g}$ of G such that $\exp_G\colon\mathfrak{g}\to G$ becomes identified with the function $\exp_A|\mathfrak{g}\colon\mathfrak{g}\to G$ with $\exp_A x = e^x$ in A (cf. Proposition 5.40). Let $g\in G\subseteq A^{-1}$. We claim that then the inner automorphism $\mathrm{Ad}_A(g)\colon A\to A$, $\mathrm{Ad}_A(g)(a) = gag^{-1}$ maps $\mathfrak{g}$ into itself. Indeed, $y\in\mathfrak{g}$ means $\exp\mathbb{R}{\cdot}y\subseteq G$, and $g(\exp t{\cdot}y)g^{-1} = \exp(t{\cdot}gyg^{-1})$ by Proposition 5.16. Hence $t\mapsto\exp(t{\cdot}gyg^{-1})$ is a one parameter subgroup of G and this yields $gyg^{-1}\in\mathfrak{g}$ by 5.40(i), proving the claim. Now $\mathrm{Ad}(g)\colon\mathfrak{g}\to\mathfrak{g}$ given by $\mathrm{Ad}(g)(y) = gyg^{-1}$ is a well-defined automorphism of $\mathfrak{g}$. The claim that $\mathrm{Ad}\colon G\to\mathrm{Aut}(\mathfrak{g})$ is a morphism of topological groups follows at once from 5.14(ii) where it was shown that $\mathrm{Ad}_A\colon A\to\mathrm{Aut}(A)$ was a morphism of topological groups.

Theorem 5.43 implies that $\mathrm{Aut}(\mathfrak{g})$ is a linear Lie group with Lie algebra $\mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ and with exponential function $D\mapsto e^D$. The adjoint representation $\mathrm{ad}_A : A\to\mathrm{Der}(A)$ given by $\mathrm{ad}_A(x)(y) = [x,y]$ maps $\mathfrak{g}$ into $\mathrm{Der}\,\mathfrak{g}$ and thus gives us the adjoint representation $\mathrm{ad}\colon\mathfrak{g}\to\mathrm{Der}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. From 5.16(ii) we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{ad}\ } & \mathrm{Der}\,A\\ {\scriptstyle\exp_A}\big\downarrow & & \big\downarrow{\scriptstyle D\mapsto e^D}\\ A^{-1} & \xrightarrow[\ \mathrm{Ad}\ ]{} & \mathrm{Aut}(A). \end{array}$$

After the preceding, restricting the maps in this diagram yields the commutativity of the diagram in the theorem and formula (20). Formula (21) arises from Proposition 5.16(i) and (22) comes from Proposition 5.16(iii). Formula (20) determines $\mathrm{Ad}(g)$ uniquely on a sufficiently small neighborhood of 0 in $\mathfrak{g}$, since exp is a local homeomorphism; as a linear operator, $\mathrm{Ad}(g)$ is then determined uniquely.

(ii) By (21), $\mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ holds exactly when $g(\exp X)g^{-1} = \exp X$ for all $X\in\mathfrak{g}$, that is if and only if g is in the centralizer of $\exp\mathfrak{g}$. But $\exp\mathfrak{g}$ is an identity neighborhood of G_0, and thus generated G_0. The assertion follows. □

Exercise E5.15. Show that $\varphi\mapsto\mathfrak{L}(\varphi)\colon\mathrm{Aut}\,G\to\mathrm{Aut}\,\mathfrak{g}$ is a morphism of groups whose kernel is the group of all automorphisms of G agreeing on G_0 with id_{G_0}. In particular, this morphism is an injection of groups if G is connected, in which case it identifies $\operatorname{Aut} G$ with a subgroup of the linear Lie group $\operatorname{Aut}\mathfrak{g}$. Note that the adjoint representation $\operatorname{Ad}: G \to \operatorname{Aut}\mathfrak{g}$ is the composition of $g \mapsto (x \mapsto gxg^{-1}) : G \to \operatorname{Aut} G$ and $\varphi \mapsto \mathfrak{L}(\varphi)$. □

Note that the entire set-up of the adjoint representation of a linear Lie group is canonically associated with the group.

For the following result involving the adjoint action of G on $\mathfrak{g} = \mathfrak{L}(G)$ we recall that, for an endomorphism $T: E \to E$ of a vector space of dimension $n < \infty$ over any field we always have a canonical decomposition $E = E^0 \oplus E^+$ into T-invariant subspaces such that $T|E^0: E^0 \to E^0$ is nilpotent and $T|E^+: E^+ \to E^+$ is an automorphism. One calls E^0 the *Fitting null component* or the *nil-space* and E^+ the *Fitting one component* of T. (Indeed $E^+ = T^n E$ and $E^0 = \ker T^n$.) We shall apply this as follows: For a linear Lie group G let $g \in G$. Then $T = \operatorname{Ad}(g) - \mathrm{id}_{\mathfrak{g}}: \mathfrak{g} \to \mathfrak{g}$ is a vector space endomorphism of $\mathfrak{g}$. We shall write $\mathfrak{g}^+(g)$ for its Fitting one component and $\mathfrak{g}^0(g)$ for its Fitting null component. Notice that $\mathfrak{g}^+$ and $\mathfrak{g}^0$ are invariant under $\operatorname{Ad}(g) = T + \mathrm{id}_{\mathfrak{g}}$ and that $-\operatorname{Ad}(g^{-1})T = -\operatorname{Ad}(g)^{-1}T = \operatorname{Ad}(g)^{-1} - \mathrm{id}_{\mathfrak{g}}$. Hence $\mathfrak{g}^+$ and $\mathfrak{g}^0$ are also the Fitting one and Fitting null component of $\operatorname{Ad}(g)^{-1}$, respectively.

Proposition 5.45. *Let G be a finite dimensional linear Lie group and $\mathfrak{g} = \mathfrak{L}(G)$ its Lie algebra, and consider any $g \in G$. Then there are open neighborhoods U^+ of 0 in $\mathfrak{g}^+(g)$ and U^0 of 0 in $\mathfrak{g}^0(g)$ and open neighborhoods V_X of 0 in $\mathfrak{g}$ and W_g of g in G such that the functions*

$$(*) \qquad Y \oplus Z \mapsto \operatorname{Ad}(g)^{-1}(Y) * Z * (-Y) : U^+ \oplus U^0 \to V_X,$$

$$(**) \qquad Y \oplus Z \mapsto (\exp Y) g \exp Z (\exp -Y) : U^+ \oplus U^0 \to W_g,$$

are analytic homeomorphisms.

Proof. By definition of $\mathfrak{g}^+(g)$ and the remarks preceding the proposition, the linear map $D: \mathfrak{g} \to \mathfrak{g}$, $D(Y \oplus Z) = (\operatorname{Ad}(g)^{-1} - 1)Y \oplus Z$ is invertible. For sufficiently small Y and Z we set $\Psi(Y \oplus Z) = \operatorname{Ad}(g)^{-1}Y * Z * (-Y)$ and compute $\Psi(Y \oplus Z) = \operatorname{Ad}(g)Y + Z - Y + \mathrm{o}(Y, Z; X)$, where $\mathrm{o}(Y, Z; X)$ is an analytic function satisfying $\|Y \oplus Z\|^{-1} \cdot \|\mathrm{o}(Y, Z; X)\| \to 0$ as $Y \oplus Z \to 0$. It follows that $\Psi'(0) = D$. The Theorem of the Local Inverse shows that for sufficiently small choice of zero neighborhoods U^+ and U^0 of $\mathfrak{g}^+(g)$ and $\mathfrak{g}^0(g)$, respectively, and a suitable V_X, the function in $(*)$ is an analytic homeomorphism.

Now we turn to $(**)$ and assume G to be a closed subgroup of A^{-1} for some Banach algebra (cf. 5.32). Consider the function $\Phi: \mathfrak{g} = \mathfrak{g}^+(g) \oplus \mathfrak{g}^0(g) \to G$ defined by $\Phi(Y \oplus Z) = g^{-1} \exp Y g \exp Z (\exp -Y)$ (in $G \subseteq A$). Now we compute, for all sufficiently small Y and Z in $\mathfrak{g}$,

$$\begin{aligned} g^{-1} \exp Y g \exp Z \exp -Y &= \exp \operatorname{Ad}(g)^{-1} Y \exp Z (\exp -Y) \\ &= \exp\big(\operatorname{Ad}(g^{-1})Y\big) \exp Z (\exp -Y) \\ &= \exp\big(\operatorname{Ad}(g)^{-1}Y * Z * (-Y)\big) = \exp \Psi(Y \oplus Z). \end{aligned}$$

Let B be a zero neighborhood of $\mathfrak{g}$ such that $\exp|B: B \to \exp B$ is a homeomorphism onto an identity neighborhood and let C be a zero-neighborhood of $\mathfrak{g}$ such that $\Phi(C) \subseteq \exp B$. Then $\log \circ \Phi: C \to B$ is analytic and agrees on some neighborhood of 0 with Ψ. Since $\log: \exp B \to B$ is a homeomorphism, and $x \mapsto g^{-1}x: G \to G$ is a homeomorphism, the map $Y \oplus Z \mapsto \exp Y g \exp Z \exp -Y: \mathfrak{g} \to G$ implements a local homeomorphism at 0 which is what we had to show. □

Subalgebras, Ideals, Lie Subgroups, Normal Lie Subgroups

Definitions 5.46. A subset $\mathfrak{a}$, respectively, $\mathfrak{i}$ in a Lie algebra $\mathfrak{g}$ is called a *subalgebra*, respectively, an *ideal*, if it is a vector subspace and satisfies $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$, respectively, $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. □

Definition 5.47. A subgroup H of a linear Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is a *Lie subgroup* if there is a (closed) Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that there is an open ball B in $\mathfrak{g}$ around 0 such that $\exp_G$ maps $B \cap \mathfrak{h}$ homeomorphically onto an open identity neighborhood of H. □

From Theorem 5.31 and Proposition 5.33 following it we know that every locally compact subgroup of a linear Lie group is a Lie subgroup, and that, in particular, every closed subgroup of a finite dimensional Lie group is a Lie subgroup.

A Lie subgroup satisfies itself the conditions of Definition 5.32. Therefore, *a Lie subgroup H of a linear Lie group G is a linear Lie group in its own right, and its Lie algebra is $\mathfrak{h}$, and its exponential function is the restriction of the exponential function of G to $\mathfrak{h}$.*

Proposition 5.48. *Let G be a linear Lie group, contained in A^{-1} for some Banach algebra A and let H be a subgroup of G and let $\mathfrak{h} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \exp \mathbb{R}{\cdot}x \subseteq H\}$. Assume that the following hypotheses are satisfied:*

(i) *$\mathfrak{h}$ is a closed subspace of $\mathfrak{g}$.*
(ii) *There exists an identity neighborhood V in G contained in $B_1(1) \cap G$ such that* $\log(V \cap H) \subseteq \mathfrak{h}$.

Then H is a linear Lie subgroup of G and thus, in particular, a linear Lie group.

Proof. By Proposition 5.40(ii) and with the notation used there, $\exp_A$ maps N_1 homeomorphically onto M_1. We may assume that $V \subseteq M_1$. We set $U \stackrel{\text{def}}{=} \log V$. Then $U \cap \exp^{-1} H = \log(V \cap H)$ and $\exp_A$ maps this set homeomorphically onto the identity neighborhood $V \cap H$ of H. Since $\exp \mathfrak{h} \subseteq H$ we have $U \cap \mathfrak{h} \subseteq U \cap \exp^{-1} H = \log(V \cap H)$. But $\log(V \cap H) \subseteq \mathfrak{h}$ by condition(ii), and $\log V = U$ by definition. Thus $\log(V \cap H) \subseteq U \cap \mathfrak{h}$. Hence $U \cap \mathfrak{h} = \log(V \cap H) = U \cap \exp^{-1} H$ and $V \cap H = \exp_G(U \cap \mathfrak{h})$. Thus $\exp_G$ maps $U \cap \mathfrak{h}$ homeomorphically onto the identity neighborhood $V \cap H$ of H. In order to see that the requirements of Definition 5.47 are satisfied for H it is sufficient to know that $\mathfrak{h}$ is a closed Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is closed by (i), the set $U \cap \mathfrak{h}$ is closed in U. Since $\exp_G|U: U \to V$ is a homeomorphism, the set $V \cap H = \exp_G(U \cap \mathfrak{h})$ is closed in H. Therefore H is locally closed and thus closed by Appendix 4, A4.32.

By condition (i), $\overline{\mathfrak{h}} = \mathfrak{h}$. From its definition, $\mathfrak{h}$ is closed under real scalar multiplication. Let $X, Y \in \mathfrak{h}$. Then $\frac{t}{n}\cdot X, \frac{t}{n}\cdot Y \in \mathfrak{h}$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. By Recovery of Addition 5.10 we have $t\cdot(X+Y) = \lim_n n(\frac{t}{n}\cdot X * \frac{t}{n}\cdot Y)$. But $(\exp_G \frac{t}{n}\cdot X)(\exp_G \frac{t}{n}\cdot Y) \in HH = H$ for all $t \in \mathbb{R}$ and thus $\exp_G t\cdot(X+Y) = \lim_n \big((\exp_G \frac{t}{n}\cdot X)\exp_G \frac{t}{n}\cdot Y)\big)^n \in \overline{H} = H$. Hence $X+Y \in \mathfrak{h}$ by definition of $\mathfrak{h}$. The proof of $[X,Y] \in \mathfrak{h}$ is analogous, but here we use Recovery of the Bracket 5.11. Thus $\mathfrak{h}$ is a closed Lie subalgebra of $\mathfrak{g}$ which is what remained to be verified. □

It is not our task here to develop a full-fledged Lie theory for linear Lie groups although all the tools for such a venture are now in our hands. Typically, structure problems on the Lie group level are transformed into algebraic problems on the Lie algebra level. The following proposition is an example.

Proposition 5.49. *Let H be a connected Lie subgroup of a linear Lie group G. Then the following statements are equivalent:*

(1) *H is normal in G.*

(2) *$\mathfrak{h}$ is an $\mathrm{Ad}(G)$-invariant subspace of $\mathfrak{g}$.*

These statements imply the following two equivalent statements:

(3) $(\forall X \in \mathfrak{g})\ e^{\mathrm{ad}\,X}\mathfrak{h} \subseteq \mathfrak{h}$,

(4) *$\mathfrak{h}$ is an ideal of $\mathfrak{g}$,*

and if G is also connected, then all four conditions are equivalent.

Proof. The subgroup H is normal if and only if $gHg^{-1} = H$ for all $g \in G$. Since H is a connected Lie subgroup, then by 5.41(iii), H is algebraically generated by all $\exp X$ with $X \in \mathfrak{h}$. Therefore, H is normal if and only if $g(\exp X)g^{-1} \in H$ for all $g \in G$ and $X \in \mathfrak{h}$.

An $X \in \mathfrak{g}$ is in $\mathfrak{h}$ if and only if $\exp t\cdot X \in H$ for all $t \in \mathbb{R}$. By Theorem 5.44(21) we know $g(\exp t\cdot X)g^{-1} = \exp t\cdot \mathrm{Ad}(g)(X)$; thus $g(\exp t\cdot X)g^{-1} \in H$ for all $t \in \mathbb{R}$ and all $g \in G$ if and only if $\mathrm{Ad}(g)(X) \in \mathfrak{h}$. Thus the equivalence of (1) and (2) is established. (2)⇒(3) If $X \in \mathfrak{g}$ then $g \stackrel{\text{def}}{=} \exp X$ gives $\mathrm{Ad}(g) = e^{\mathrm{ad}\,X}$ by 5.44(20). Hence (2) implies (3).

(3)⇒(4) Take $Y \in \mathfrak{h}$ and $X \in \mathfrak{g}$. Then $e^{\mathrm{ad}\,X}(Y) \in \mathfrak{h}$ by (3). But $e^{t\cdot\,\mathrm{ad}\,X}Y = Y + t\cdot[X,Y] + \mathrm{o}(t)$ by Theorem 5.44 (20) with a remainder satisfying $t^{-1}\mathrm{o}(t) \to 0$ for $0 \neq t \to 0$. Hence $t^{-1}\cdot(e^{t\cdot\,\mathrm{ad}\,X}(Y) - Y) = [X,Y] + t^{-1}\mathrm{o}(t)$ is in $\mathfrak{h}$ for all non-zero $t \in \mathbb{R}$. Passing to the limit with $t \to 0$ yields $[X,Y] \in \mathfrak{h}$.

(4)⇒(2) If $\mathfrak{h}$ is an ideal, then $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ implies $e^{\mathrm{ad}\,X}Y = Y + [X,Y] + \frac{1}{2!}\cdot\big[X,[X,Y]\big] + \cdots \in \mathfrak{h}$. By Theorem 5.44(22) this implies

$$\exp X(\exp Y)\exp -X = \exp e^{\mathrm{ad}\,X}(Y) \in \exp \mathfrak{h} \subseteq H.$$

Since H is connected, $\exp \mathfrak{h}$ generates H. Thus we conclude that H is invariant under all inner automorphism implemented by elements $g = \exp X$ with $X \in \mathfrak{g}$. But now we assume that G is connected, too. Thus G is generated by all of these elements g and thus H is invariant under all inner automorphisms. □

Proposition 5.50. *Let G and H be linear Lie groups and $f: G \to H$ a morphism of topological groups.* (*See Theorem* 5.42). *Then the kernel,* $\ker f$*, is a normal Lie subgroup of G with Lie algebra* $\ker \mathfrak{L}(f)$ *for the morphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ of topological real Lie algebras as in Theorem* 5.42. *Thus*

$$\ker\big(\mathfrak{L}(f)\big) = \mathfrak{L}\big(\ker(f)\big)$$

and the following diagram is commutative

$$\begin{array}{ccc} \ker \mathfrak{L}(f) & \xrightarrow{\text{incl}} & \mathfrak{L}(G) \\ {\scriptstyle \exp_{\ker(f)}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_G} \\ \ker(f) & \xrightarrow[\text{incl}]{} & G. \end{array}$$

Proof. From Definition 5.39, which tells us that $\mathfrak{L}(\ker f) = \operatorname{Hom}(\mathbb{R}, \ker f)$, we have $X \in \mathfrak{L}(\ker f)$ iff $\mathfrak{L}(f)(X)(r) = (f \circ X)(r) = f\big(X(r)\big) = \mathbf{1}$ for all $r \in \mathbb{R}$ iff $\mathfrak{L}(f)(X) = 0$ iff $X \in \ker \mathfrak{L}(f)$. Thus $\ker\big(\mathfrak{L}(f)\big) = \mathfrak{L}\big(\ker(f)\big)$. For any morphism of topological groups f it is true that $\ker f$ is a closed normal subgroup. It remains to observe that $\ker f$ is a Lie subgroup.

We may assume $G \subseteq A^{-1}$ for a Banach algebra A and that $\exp_G = \exp_A |\mathfrak{g}$. We want to apply Proposition 5.48 and notice that condition 5.48(i) has been verified. We prove 5.48(ii), i.e. we have to exhibit an identity neighborhood V in $\ker f$ such that $\exp^{-1}(V \cap \ker f) \subseteq \mathfrak{L}(\ker f)$.

Let B be an open ball around 0 in $\mathfrak{g}$ contained in N_1 (see Proposition 5.40 and Proposition 5.48 above) such that $\mathfrak{L}(f)(B)$ is contained in a neighborhood of 0 in $\mathfrak{h}$ on which $\exp_H$ is injective. Set $V = \exp_G B$. Let $K_0 \stackrel{\text{def}}{=} \exp_G^{-1}(V \cap \ker f)$. By 5.40(ii), this is the set of all $X \in B$ such that $\exp_H \mathfrak{L}(f)X = f(\exp_G X) = \mathbf{1}$. Since $\exp_H$ is injective on $\mathfrak{L}(f)(B)$ it follows that $\mathfrak{L}(f)X = 0$, i.e. $X \in \ker \mathfrak{L}(f)$. Thus $K_0 \subseteq \ker \mathfrak{L}(f) = \mathfrak{L}(\ker f)$. This verifies 5.48(ii) and $\ker f$ is a linear Lie group by Proposition 5.48. ☐

Proposition 5.51. (i) *Let $\{G_j \mid j \in J\}$ be a family of linear Lie groups contained in a linear Lie group G and set $H \stackrel{\text{def}}{=} \bigcap_{j\in J} G_j$, $\mathfrak{h} \stackrel{\text{def}}{=} \bigcap_{j\in J} \mathfrak{L}(G_j)$. Then $\mathfrak{L}(H) = \mathfrak{h}$ and if J is finite, then H is a linear Lie group.*

(ii) *Let $\{G_j \mid j \in J\}$ be a family of linear Lie groups and set $H \stackrel{\text{def}}{=} \prod_{j\in J} G_j$, $\mathfrak{h} \stackrel{\text{def}}{=} \prod_{j\in J} G_j$. Then $\mathfrak{L}(H) \cong \mathfrak{h}$, and if J is finite, then H is a linear Lie group.*

(iii) *Let f_1, $f_2: G \to H$ be morphisms of topological groups between linear Lie groups. Then $E \stackrel{\text{def}}{=} \{g \in G \mid f_1(g) = f_2(g)\}$, the equalizer of f_1 and f_2* (*see Appendix* 3, A3.43(ii)), *is a Lie subgroup and*

$$\mathfrak{L}(E) = \{X \in \mathfrak{L}(G) \mid \mathfrak{L}(f_1)(X) = \mathfrak{L}(f_2)(X)\}.$$

(iv) *Let $f_j: G_j \to H$, $j = 1, 2$, be morphisms of topological groups between linear Lie groups. Set $P = \{(g_1, g_2) \in G_1 \times G_2 \mid f_1(g_1) = f_2(g_2)\}$, the pullback* (*see*

Appendix 3, A3.43(iii)). *Then* P *is a linear Lie group and* $\mathfrak{L}(P)$ *is the pullback of* $\mathfrak{L}(f_1)$ *and* $\mathfrak{L}(f_2)$.

(v) *Let* $f: G \to H$ *be a morphism of topological groups between linear Lie groups, and let* S *be a Lie subgroup of* H. *Then the inverse image* $f^{-1}(S)$ *is a Lie subgroup of* G *and* $\mathfrak{L}\big(f^{-1}(S)\big) = \mathfrak{L}(f)^{-1}\big(\mathfrak{L}(S)\big)$.

Proof. (i) A one parameter subgroup $X: \mathbb{R} \to G$ has its image in H iff $X(r) \in G_j$ for all $r \in \mathbb{R}$ and $j \in J$. Thus $X \in \mathfrak{L}(G)$ iff $X \in \mathfrak{L}(G_j)$ for all $j \in J$. Hence $\mathfrak{L}(H) = \bigcap_{j\in J} \mathfrak{L}(G_j) = \mathfrak{h}$. We choose an open ball B around 0 in $\mathfrak{g}$ which is mapped homeomorphically onto an identity neighborhood V of G by $\exp_G$. Since G_j is a Lie subgroup for each j we find open balls $B_j \subseteq B$ in $\mathfrak{g}$ such that $\exp_G$ implements a homeomorphism of the zero neighborhood $B_j \cap \mathfrak{L}(G_j)$ to the identity neighborhood $V_j \cap G_j$, $V_j = \exp_G B_j$. Set $B_H \stackrel{\text{def}}{=} \bigcap_{j\in J} B_j$ and $V_H \stackrel{\text{def}}{=} \bigcap_{j\in J} V_j$. Then $\exp_G$ implements a homeomorphism from B_H onto V_H. However, if J is finite, then B_H is an open zero neighborhood of $\mathfrak{g}$ and V_H is an open identity neighborhood of G, and $\exp_G$ maps $B_H \cap \mathfrak{h} = \bigcap_{j\in J} B_j \cap \mathfrak{L}(G_j)$ homeomorphically onto $\bigcap_{j\in J} V_j \cap H_j = V_H \cap H$. In view of Definition 5.47 this shows that H is a Lie subgroup of G with $\mathfrak{L}(H) = \mathfrak{h}$.

(ii) Let $G_j \subseteq A_j^{-1}$, $j \in J$ for Banach algebras A_j. Then $A = \prod_{j\in J} A_j$ is a Banach algebra with $\|(a_j)_{j\in J}\| = \max\{\|a_j\| : j \in J\}$. Assume that $\exp_{A_j}$ maps $U_j \cap \mathfrak{g}_j$ for a zero neighborhood U_j in A_j homeomorphically onto an identity neighborhood V_j of G_j. If we set $G = \prod_{j\in J} G_j$ and $\mathfrak{g} = \prod_{j\in J} \mathfrak{g}_j$, then $\exp_A = \prod_{j\in J} \exp_{A_j}$ maps $(\prod_{j\in J} U_j) \cap (\prod_{j\in J} \mathfrak{g}_j)$ onto $\prod_{j\in J} V_j$ securing enough information for Definition 5.32 to apply to prove the claim.

We note that the case (v) of the inverse image is a special case of the case (iv) of the pullback, since the inverse image $f^{-1}(S)$ is the pullback of the morphism $f: G \to H$ and the inclusion morphism $S \to H$. A direct proof of the cases (iii) of the equalizer and (iv) of the pullback are recommended as an exercise along the lines of the proofs of (i) and (ii). However, in Appendix 3, A3.45 it is recorded how one constructs equalizers by using products and intersections, and in A3.44 how one constructs pullbacks using products and equalizers. Since $\mathfrak{L}(\cdot)$ preserves intersections and products by (i) and (ii), it preserves equalizers and pullbacks (cf. A3.51; we use the version here in which the word "arbitrary" is replaced by "finite"). □

Exercise E5.16. Supply all details of the proofs of parts (iii), (iv) and (v) of Proposition 5.51. □

We have seen in Theorem 5.31 and its consequences that in a locally compact linear Lie group G, every closed subgroup H determines a Lie subalgebra $\mathfrak{h}$ such that H is a linear Lie group with $\mathfrak{L}(H) = \mathfrak{h}$ and $H_0 = \langle \exp \mathfrak{h} \rangle$.

The converse is not even true for compact Lie groups as the following example shows:

Exercise E5.17. Let T denote the two dimensional compact Lie group of all matrices

$$\begin{pmatrix} \cos 2\pi s & \sin 2\pi s & 0 & 0 \\ -\sin 2\pi s & \cos 2\pi s & 0 & 0 \\ 0 & 0 & \cos 2\pi t & \sin 2\pi t \\ 0 & 0 & -\sin 2\pi t & \cos 2\pi t \end{pmatrix}, \quad s,\, t \in \mathbb{R}.$$

Determine the Lie algebra $\mathfrak{t}$ in $\mathfrak{gl}(4,\mathbb{R})$ of T. Let a be a real number and denote by $\mathfrak{h}$ the set of all

$$\begin{pmatrix} 0 & r & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & 0 & 0 & ar \\ 0 & 0 & -ar & 0 \end{pmatrix}, \quad r \in \mathbb{R}.$$

Show that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{t}$ and that there is a connected linear Lie subgroup H of T having $\mathfrak{h}$ as it Lie algebra if and only if a is rational. Thus, for instance, $a = \sqrt{2}$ is a case where such a linear Lie subgroup does not exist.

Furthermore, for $A = M_4(\mathbb{R})$ the Banach algebra of all 4×4 real matrices with the operator norm and for the unit ball $B_1(\mathbf{1})$ in A around $\mathbf{1}$, for any natural number n by choosing a suitable rational number a, the number of connected components of $B_1(\mathbf{1}) \cap H$ exceeds n. In particular, $\log(B_1(\mathbf{1}) \cap H)$ fails to be contained in $\mathfrak{h}$. ☐

This example indicates that the next best thing to finding the missing linear Lie group is the subgroup $H \stackrel{\text{def}}{=} \exp \mathfrak{h}$. In the case of the example, H is the image $\exp \mathfrak{h}$ of a one parameter group which happens to be dense (cf. Remark 1.24(v) for related density arguments). Thus one might consider, for a given closed subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of a linear Lie group G the subgroup $H = \langle \exp \mathfrak{h} \rangle$. But then, due to the algebraic generation process over which we have little control, it is not at all clear that we can recover from H the Lie algebra $\mathfrak{h}$ via $\mathfrak{L}(H) = \operatorname{Hom}(\mathbb{R}, H)$. That this is nevertheless the case, is not an entirely trivial matter which we discuss now.

In the following theorem and its proof we choose $N_1 \subseteq \mathfrak{g}$ and $M_1 \subseteq G$ as we did for 5.40(ii) in the paragraph preceding 5.40. Conclusion (iv) is of topological interest and refers to a topological construction described in Appendix 2A. For an understanding of the Lie theoretical implications of the theorem alone it may be skipped.

Analytic Subgroups and the Recovery of Subalgebras

Theorem 5.52. *Assume that G is a linear Lie group and that $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g} = \mathfrak{L}(G)$. Set $H = \langle \exp \mathfrak{h} \rangle$. Then*

(i) *there exists a unique arcwise connected locally arcwise connected group topology $\mathcal{O}$ on H containing the topology on H induced from that of G such that $\exp(N_1 \cap \mathfrak{h}) \in \mathcal{O}$ and that $\exp|(N_1 \cap \mathfrak{h}) \colon N_1 \cap \mathfrak{h} \to \exp(N_1 \cap \mathfrak{h})$ is a homeomorphism onto an open identity neighborhood of $(H, \mathcal{O})$. The function $\exp_G|\mathfrak{h} \colon \mathfrak{h} \to (H, \mathcal{O})$ is continuous.*

(ii) *Assume that* $\mathfrak{h}$*, as a completely normable vector space, is separable, i.e. has a countable dense subset. This is always the case if* $\mathfrak{h}$ *is finite dimensional. Then* $(H, \mathcal{O})$ *is separable.*

(iii) (Recovery of subalgebras) *If* $\mathfrak{h}$ *is separable, then* $\mathfrak{L}(H) = \mathfrak{h}$.

(iv) *The topology* $\mathcal{O}$ *on* H *is the arc component topology obtained from the topology induced by that of* G *on* H (*see* Lemma A4.1ff.).

Proof. We set $K \stackrel{\text{def}}{=} \exp(N_1 \cap \mathfrak{h})$ and since $\mathfrak{h} = \bigcup_{n\in\mathbb{N}} n\cdot(N_1 \cap \mathfrak{h})$ we also have $\exp\mathfrak{h} \subseteq \bigcup_{n\in\mathbb{N}}(\exp N_1 \cap \mathfrak{h})^n \subseteq \langle K\rangle \subseteq H$. Thus H is generated by K. For easy reference we cite here from the Appendix A2 the basic theorem on generating subgroups of topological groups A2.25.

Let K *be a symmetric subset* ($K = K^{-1}$) *of a group* G *containing* $\mathbf{1}$. *Assume that* K *is a connected topological space such that*

(i) x, y, $xy \in K$, *with* $xy \in V$ *for an open subset* V *of* K *imply the existence of open neighborhoods* U_x *and* U_y *of* x *and* y *such that* $xU_y \cup U_x y \subseteq V$,

(ii) $\{(x, y) \in K \times K \mid x, y, xy \in K\}$ *is a neighborhood of* $(\mathbf{1}, \mathbf{1})$ *in* $K \times K$, *and multiplication is continuous at* $(\mathbf{1}, \mathbf{1})$.

(iii) *inversion is continuous at* $\mathbf{1}$.

Then there is a unique topology $\mathcal{O}$ *on the subgroup* $H = \langle K\rangle$ *generated by* K *which induces on* K *the given topology and makes* H *a topological group such that* K *is an open identity neighborhood of* H.

We recall from 5.40(iv) that the hypotheses of this lemma are satisfied in the present circumstances and that it yields the desired topology $\mathcal{O}$. The topological group $(H, \mathcal{O})$ is generated by the arcwise connected subset K containing the identity. Therefore it is arcwise connected. It is locally arcwise connected since it contains an open subset K homeomorphic to the nonempty open subset $N_1 \cap \mathfrak{h}$ of a Banach space. In order to show continuity of $\exp_G|\mathfrak{h}\colon \mathfrak{h} \to (H, \mathcal{O})$ we recall that this function restricts to a continuous function $\exp_G|(N_1 \cap \mathfrak{h})\colon N_1 \cap \mathfrak{h} \to (H, \mathcal{O})$. If $0 \neq X \in \mathfrak{h}$ consider an open ball C in $\mathfrak{h}$ containing X and a natural number n such that $\frac{1}{n}\cdot C \subseteq N_1$. Let $\mu\colon C \to N_1$ be defined by $\mu(Y) = \frac{1}{n}\cdot Y$ and $p\colon (H, \mathcal{O}) \to (H, \mathcal{O})$ by $p(h) = h^n$. Since $(H, \mathcal{O})$ is a topological group, the function p is continuous. Then

$$e \stackrel{\text{def}}{=} p \circ (\exp|(N_1 \cap \mathfrak{h})) \circ \mu\colon C \to (H, \mathcal{O})$$

is continuous and because of $\exp Y = (\exp\frac{1}{n}\cdot Y)^n$ agrees with $\exp_G|C$. Thus $\exp_G|\mathfrak{h}\colon \mathfrak{h} \to (H, \mathcal{O})$ is continuous at X, and then everywhere on $\mathfrak{h}$ since X was arbitrary.

(ii) If $\mathfrak{h}$ is separable, then the continuous image $\exp\mathfrak{h}$ of $\mathfrak{h}$ in $(H, \mathcal{O})$ is separable, and since H is generated algebraically by this set, $(H, \mathcal{O})$ is separable.

(iii) Trivially, $\mathfrak{h} \subseteq \{X \in \mathfrak{g} \mid \exp\mathbb{R}\cdot X \subseteq H\} = \mathfrak{L}(H)$. We claim equality holds. Suppose that equality does not hold. Find a sufficiently small open ball around 0 in $\mathfrak{g}$ such that $B * B * B$ is defined and exp is injective on $B * B$. Then there is an $X \in B \setminus \mathfrak{h}$ with $[0, 1]\cdot X \subseteq \exp^{-1} H$. Set $B_0 \stackrel{\text{def}}{=} \mathfrak{h} \cap B$. Assume for the moment that $0 \leq r \leq s \leq 1$ and $(r\cdot X * B_0) \cap (s\cdot X * B_0) \neq \emptyset$. Then we find elements P, $Q \in B_0$ such that $r\cdot X * P = s\cdot X * Q$, i.e. $(s-r)\cdot X = (-r\cdot X)*(s\cdot X) = P*(-Q) \in B_0*B_0 \subseteq \mathfrak{h}$

(Theorem 5.21(iv)). But $X \notin \mathfrak{h}$ now implies $s - r = 0$. Thus the sets $r{\cdot}X * B_0$ form a disjoint family of subsets of $N_1 \cap \exp^{-1} H$. Since $\exp B_0 \in \mathcal{O}$ the family $\{\exp r\,X \exp B_0 \mid r \in [0,1]\}$ is an uncountable family of disjoint open subsets of $(H, \mathcal{O})$. Then certainly $(H, \mathcal{O})$ cannot be separable.

(iv) is a consequence of Appendix 4, A4.5. This completes the proof. □

In the more general theory of Lie groups the subgroups of a Lie group G of the form $\langle \exp \mathfrak{h} \rangle$ for a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are frequently called *analytic subgroups.* With their finer topology $\mathcal{O}$ they may be endowed with an analytic structure exactly as we did in 5.34ff. The recovery of subalgebras is somewhat subtle, although the topological apparatus used is, within the framework of general topology, elementary. The subtlety becomes apparent if one realizes that some hypothesis such as the separability of $\mathfrak{h}$ is needed. Therefore, the following example is an important illustration of what might happen on the level of topological groups. Let $E = \ell^1(\mathbb{R}_d)$ the Banach space of all real tuples $(x_r)_{r \in \mathbb{R}}$ such that the family $(|x_r|)_{r \in \mathbb{R}}$ is summable and let this sum designate the norm. The tuples $e_r = (\delta_{rs})_{s \in \mathbb{R}}$, $\delta_{rr} = 1$ and $\delta_{rs} = 0$ otherwise generate a free discrete subgroup. In the product Banach space $\mathfrak{g} = E \times \mathbb{R}$ the subgroup D generated by the family $\{(e_r, -r) \mid r \in \mathbb{R}\}$ is discrete and free. Set $G = \frac{E \times \mathbb{R}}{D}$ and write $\exp\colon \mathfrak{g} \to G$, $\exp(x, r) = (x, r) + D$. Then exp is a continuous open homomorphism of topological groups inducing a local isomorphism. The proper closed vector subspace $\mathfrak{h} \stackrel{\text{def}}{=} E \times \{0\}$, however, is mapped *surjectively* onto G. *The one parameter subgroup* $r \mapsto \exp(0, r)\colon \mathbb{R} \to G$ *does not lift to* $\mathfrak{h}$, i.e. it is not of the form $r \mapsto \exp r{\cdot}X$ for any $X \in \mathfrak{h}$. The completely normable vector space $\mathfrak{h}$ indeed fails to be separable. Recall from E5.10(ii) 2) that $E \times \mathbb{R}$ is a linear Lie group which may be identified with its own Lie algebra. Thus G is locally isomorphic to a linear Lie group.

Proposition 5.52 in fact contains more information than we pause to develop here. Indeed, the local faithfulness of $\exp|\mathfrak{h}\colon \mathfrak{h} \to (H, \mathcal{O})$ allows us to introduce an analytic structure on H whose underlying topology is $\mathcal{O}$.

Normalizers, Centralizers, Centers

The concepts governing the theory of Lie algebras are generally modelled after corresponding group theoretical concepts. We just saw that ideals correspond to normal subgroups. Another example follows. For a group G we shall denote by $\iota\colon G \to G$ the inversion given by $\iota(g) = g^{-1}$ and by $\kappa_h\colon G \to G$ the conjugation $\kappa_h(g) = ghg^{-1}$ of h by g. Notice that for a subset H of G one has $gHg^{-1} = H$ if and only if $g \in \bigcap_{h \in H} \kappa_h^{-1}(H) \cap \iota^{-1}\kappa_h^{-1}(H)$.

Definition 5.53. (i) If H is a subgroup of a group G then the largest subgroup

$$N(H, G) = \{g \in G \mid gHg^{-1} = H\} = \bigcap_{h \in H} \kappa_h^{-1}(H) \cap \iota^{-1}\kappa_h^{-1}(H)$$

of G containing H in which H is normal is called *the normalizer of H in G.*

If M is a subset of G then the set

$$Z(M,G) = \{g \in G \mid (\forall m \in M)\ gmg^{-1} = m\}$$

of elements commuting with each element of M is called *the centralizer of M in G*. The centralizer $Z(G,G)$ of all of G is called the *center* $Z(G)$ of G; it is the set of elements commuting with all elements of the group.

(ii) If $\mathfrak{h}$ is a subalgebra of a Lie algebra $\mathfrak{g}$ then the largest subalgebra $\mathfrak{n}(\mathfrak{h},\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,\mathfrak{h}] \subseteq \mathfrak{h}\}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ in which $\mathfrak{h}$ is an ideal is called the *idealizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$ or, alternatively, *the normalizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$.

If $\mathfrak{m}$ is a subset of $\mathfrak{g}$ then the set $\mathfrak{z}(\mathfrak{m},\mathfrak{g}) = \{X \in \mathfrak{g} \mid (\forall Y \in \mathfrak{m})\ [X,Y] = 0\}$ of elements commuting with each element of $\mathfrak{m}$ is called *the centralizer of* $\mathfrak{m}$ *in* $\mathfrak{g}$.

The centralizer $\mathfrak{z}(\mathfrak{g},\mathfrak{g})$ is called the *center* of $\mathfrak{g}$ written $\mathfrak{z}(\mathfrak{g})$. A Lie algebra $\mathfrak{g}$ is called *commutative* or *abelian* if all brackets vanish, i.e. $[\mathfrak{g},\mathfrak{g}] = \{0\}$, equivalently, $\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$. □

From the second variant of the definition of a normalizer one observes at once that in a topological group G the normalizer $N(H,G)$ of a closed subgroup is closed. Likewise the centralizer of any subset of G is closed. Similarly, the normalizer $\mathfrak{n}(\mathfrak{h},\mathfrak{g})$ of a closed subalgebra $\mathfrak{h}$ is closed, and the centralizer $\mathfrak{z}(\mathfrak{m},\mathfrak{g})$ of any subset $\mathfrak{m}$ is closed. It is, however, remarkable, that in the context of a linear Lie group G, the normalizer $N(H,G)$ of certain relevant subgroups H may turn out to be closed even though the subgroup H itself may fail to be closed.

Proposition 5.54. (i) *Assume that $\mathfrak{h}$ is a closed separable subalgebra of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of a linear Lie group G and set $H = \langle \exp \mathfrak{h}\rangle$. Then the normalizer $N(H,G)$ is a linear Lie subgroup of G such that*

$$(23)\qquad \mathfrak{L}\big(N(H,G)\big) = \mathfrak{n}\big(\mathfrak{h},\mathfrak{g}\big) \quad \textit{and} \quad N(H,G)_0 = \langle \exp \mathfrak{n}(\mathfrak{h},\mathfrak{g})\rangle.$$

In particular, the normalizer of H is closed.

(ii) *Let M be a subset of a linear Lie group G, then the centralizer $Z(M,G)$ is a linear Lie subgroup of G and*

$$\mathfrak{L}\big(Z(M,G)\big) = \{X \in \mathfrak{g} \mid (\forall m \in M)\ \mathrm{Ad}(m)(X) = X\}.$$

(iii) *If $H = \langle \exp \mathfrak{h}\rangle$ (as in* (i) *above), then*

$$(24)\qquad \mathfrak{L}\big(Z(H,G)\big) = \mathfrak{z}(\mathfrak{h},\mathfrak{g}) \quad \textit{and} \quad Z(H,G)_0 = \langle \exp \mathfrak{z}\big(\mathfrak{h},\mathfrak{g}\big)\rangle.$$

(iv) *The center Z of a connected linear Lie group G is a Lie subgroup of G, and its Lie algebra $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

(v) *A connected linear Lie group G is abelian if and only if $\mathfrak{g}$ is abelian.*

(vi) *Let E be a Banach space, F a closed vector subspace, and $\pi\colon G \to \mathrm{Gl}(E)$ be a morphism of topological groups. Write $N(F,G) \overset{\mathrm{def}}{=} \{g \in G \mid \pi(g)(F) \subseteq F\}$, $\mathfrak{n}(F,\mathfrak{g}) = \{X \in \mathfrak{g} \mid \mathfrak{L}(\pi)(X)(F) \subseteq F\}$; further write $Z(F,G) \overset{\mathrm{def}}{=} \{g \in G \mid (\forall v \in F)\, \pi(g)(v) = v\}$, $\mathfrak{z}(F,\mathfrak{g}) = \{X \in \mathfrak{g} \mid (\forall v \in F)\, \mathfrak{L}(\pi)(X)(v) = 0\}$. Then $N(F,G)$*

and $Z(F,G)$ *are closed Lie subgroups satisfying*

$$\mathfrak{L}\big(N(F,G)\big)=\mathfrak{n}(F,\mathfrak{g}) \quad \textit{and} \quad \mathfrak{L}\big(Z(F,G)\big)=\mathfrak{z}(F,\mathfrak{g}).$$

Proof. (i) If $gHg^{-1}=H$, the inner automorphism I_g (given by $I_g(x)=gxg^{-1}$) permutes the one parameter subgroups of G contained in H; i.e. $\mathrm{Ad}(g)\mathfrak{L}(H)=\mathfrak{L}(H)$. By the recovery of subalgebras 5.52(iii) we know $\mathfrak{h}=\mathfrak{L}(H)$. Hence $\mathrm{Ad}(g)(\mathfrak{h})=\mathfrak{h}$. Conversely, if $\mathrm{Ad}(g)(\mathfrak{h})=\mathfrak{h}$, then $gHg^{-1}=g\langle\exp\mathfrak{h}\rangle g^{-1}=\langle g(\exp\mathfrak{h})g^{-1}\rangle=\langle\exp\mathrm{Ad}(g)\mathfrak{h}\rangle=\langle\exp\mathfrak{h}\rangle=H$ in view of Theorem 5.44. Therefore,

$$(*)\qquad N(H,G)=\{g\in G\mid \mathrm{Ad}(g)\mathfrak{h}=\mathfrak{h}\}.$$

Now let B be an open ball around 0 in $\mathfrak{g}$ that is mapped homeomorphically by $\exp_G$ onto an identity neighborhood V of G and that $X\in B$ implies that $\|e^{\mathrm{ad}\,X}-\mathrm{id}_{\mathfrak{g}}\|<1$ and $\mathrm{ad}\,X=\log e^{\mathrm{ad}\,X}$. Then $X\in B$ is in $\mathfrak{n}(\mathfrak{h},\mathfrak{g})$ iff $[X,\mathfrak{h}]\subseteq\mathfrak{h}$ iff $e^{\mathrm{ad}\,X}\mathfrak{h}=\mathfrak{h}$ (cf. also 5.17 and 5.49). By 5.44(20) we have $\mathrm{Ad}(\exp X)=e^{\mathrm{ad}\,X}$; hence $e^{\mathrm{ad}\,X}\mathfrak{h}=\mathfrak{h}$ is equivalent to $\mathrm{Ad}(\exp X)\mathfrak{h}=\mathfrak{h}$, and by $(*)$ this means exactly $\exp X\in N(H,G)$. Thus $X\in B\cap\mathfrak{n}(\mathfrak{h},\mathfrak{g})$ iff $\exp X\in V\cap N(H,G)$. According to Definition 5.47, this implies that $N(H,G)$ is a Lie subgroup of G whose Lie algebra is $\mathfrak{n}(\mathfrak{h},\mathfrak{g})$. This establishes the first half of (23). The second half of (23) now follows from 5.41(iii) since $N(H,G)$ is a linear Lie group.

(ii) A one parameter subgroup $X\in\mathfrak{L}(G)$ is in $\mathfrak{L}\big(Z(M,G)\big)$ iff for each $m\in M$ we have $\exp_G t\cdot\mathrm{Ad}(m)(X)=I_m(\exp_G t\cdot X)=\exp_G t\cdot X$ for all $t\in\mathbb{R}$. This is the case iff $\mathrm{Ad}(m)(X)=X$. Thus, denoting by Fix_α the fixed point set of an endomorphism α of $\mathfrak{g}$, we get

$$(**)\qquad \mathfrak{L}\big(Z(M,G)\big)=\{X\in\mathfrak{g}\mid(\forall m\in M)\ \mathrm{Ad}(m)X=X\}=\bigcap_{m\in M}\mathrm{Fix}_{\mathrm{Ad}(m)}.$$

Since the fixed point set of a continuous endomorphism of $\mathfrak{g}$ is a closed Lie subalgebra, the set $\mathfrak{c}\overset{\mathrm{def}}{=}\mathfrak{L}\big(Z(M,G)\big)$ is a closed Lie subalgebra of $\mathfrak{g}$. The fixed point set of the automorphism $\mathrm{Ad}(m)$ of the Lie algebra $\mathfrak{g}$ is a closed Lie subalgebra. Hence $\mathfrak{c}=\bigcap_{m\in M}\mathrm{Fix}_{\mathrm{Ad}(m)}$ is a closed Lie subalgebra of $\mathfrak{g}$.

Since G is a linear Lie group we may assume $G\subseteq A^{-1}$ for a suitable Banach algebra A and we may consider $\mathfrak{g}$ to be a closed Lie subalgebra of $(A,[\cdot,\cdot])$ in such a way that $\exp_G\colon\mathfrak{g}\to G$ is the restriction of the exponential function of A. The function $\mathrm{Ad}(m)\colon G\to G$ extends to a function $\mathrm{Ad}_A(m)\colon A\to A$ given by $\mathrm{Ad}_A(m)(a)=mam^{-1}$ (cf. Lemma 5.14 and the discussion preceding it). Then each $\mathrm{Ad}_A(m)$ is an automorphism of Banach algebras and is, consequently, also an automorphism of the completely normable Lie algebra $(A,[\cdot,\cdot])$. Then by Lemma 5.15(iii), the exponential function maps the common fixed point set F_M of all $Ad_A(m)$, $m\in M$ into itself, and by 5.15(i), the logarithm $\log\colon B_1(\mathbf{1})\to A$ maps $F_M\cap B_1(\mathbf{1})$ into F_M. We note $F_M\cap\mathfrak{g}=\mathfrak{c}$ and $F_M\cap G=Z(M,G)$. Now let B be an open ball around 0 in A which under the exponential function is mapped homeomorphically onto an open identity neighborhood V of A^{-1} contained in $B_1(\mathbf{1})$ in such a way that $\exp(B\cap\mathfrak{g})$ is the identity neighborhood $V\cap G$ of G. Then the restriction of the exponential function implements a homeomorphism

$B\cap\mathfrak{L}\big(Z(M,G)\big)\to V\cap Z(M,G)$ onto an identity neighborhood of $Z(M,G)$. This shows that $Z(M,G)$ is a Lie subgroup with Lie algebra $\mathfrak{c}=\mathfrak{L}\big(Z(M,G)\big)$.

(iii) For a proof of (24), note that $X\in\mathfrak{L}\big(Z(H,G)\big)$ iff $\mathrm{Ad}(h)X=X$ for all $h\in H$ iff $e^{\mathrm{ad}\,Y}X=\mathrm{Ad}(\exp Y)X=X$ for all $Y\in\mathfrak{h}$. Then 5.17(iv) implies $[\mathfrak{h},X]=\{0\}$ (since we may assume $\mathfrak{g}\subseteq A$ for some Banach algebra A). Thus (24) is proved. Since $Z(H,G)$ is a linear Lie group, (26) follows from 5.41.(iii).

We prove (iv) straightforwardly from (ii) and (iii) by taking $M=H=G$, and (v) is an instant consequence of (iv).

(vi) One observes at once that $N(F,G)$ and $Z(F,G)$ are subgroups of G and that $\mathfrak{n}(F,\mathfrak{g})$ and $\mathfrak{z}(F,\mathfrak{g})$ are Lie subalgebras of $\mathfrak{g}$. The set $S_v\overset{\mathrm{def}}{=}\{g\in G\mid \pi(g)(v)\subseteq F\}$ is closed since $g\mapsto\pi(g)(v)\colon G\to E$ is a continuous function and F is closed in E. Since $N(F,G)=\bigcap_{v\in F}S_v$, the subgroup $N(F,G)$ of G is closed. Similarly, $Z(F,G)$, $\mathfrak{n}(F,E)$, and $z(F,\mathfrak{g})$ are closed in G and $\mathfrak{g}$, respectively.

Assume $X\in\mathfrak{L}\big(N(F,G)\big)\cap B$. Then $e^{t\cdot\mathfrak{L}(\pi)(X)}(v)=\pi(\exp t{\cdot}X)(v)\in F$ for all $v\in F$. The derivative of the function $t\mapsto e^{t\cdot\mathfrak{L}(\pi)(X)}(v)\colon\mathbb{R}\to F$ at 0 is $\mathfrak{L}(\pi)(X)(v)$. Hence $\mathfrak{L}(\pi)(X)(v)\in F$ for all $v\in V$ and thus $X\in\mathfrak{n}(F,G)$. Conversely, assume $X\in\mathfrak{n}(F,G)$. Then $\mathfrak{L}(\pi)(X)(v)\in F$ for all $v\in V$ and, recursively, $\big(\mathfrak{L}(\pi)(X)\big)^n(v)\in F$ for all $v\in V$ and all $n=1,2,\ldots$. Thus $e^{t\cdot\mathfrak{L}(\pi)(X)}(v)=v+t{\cdot}\mathfrak{L}(\pi)(X)(v)+\frac{t^2}{2!}{\cdot}\big(\mathfrak{L}(\pi)(X)\big)^2(v)+\cdots\in F$, whence $X\in\mathfrak{L}\big(N(F,G)\big)$. The same arguments show that $X\in\mathfrak{L}\big(Z(F,G)\big)$ iff $X\in\mathfrak{z}(F,\mathfrak{g})$. Thus

$$\mathfrak{L}\big(N(F,G)\big)=\mathfrak{n}(F,\mathfrak{g})\quad\text{and}\quad\mathfrak{L}\big(Z(F,G)\big)=\mathfrak{z}(F,\mathfrak{g}).$$

We assume that $G\subseteq A^{-1}$ for a Banach algebra A and let $0<r\le1$ be such that $\pi\big(G\cap B_r(\mathbf{1})\big)\subseteq B_1(\mathrm{id}_E)\subseteq\mathrm{Gl}(E)$. We note that $U\overset{\mathrm{def}}{=}\log B_r(\mathbf{1})$ is mapped homeomorphically onto $B_r(\mathbf{1})$ under $\exp\colon A\to A^{-1}$. Consider $g\in N(F,G)\cap B_r(\mathbf{1})$. Then $\|\mathrm{id}_E-\pi(g)\|<1$ and, recursively, from $\pi(g)(F)\subseteq F$, we get $(\mathrm{id}_E-\pi(g))^n(F)\subseteq F$, $n=1,2\ldots$. Hence

$$(*)\qquad \log\pi(g)(F)=\left(\sum_{n=1}^{\infty}\frac{1}{n}{\cdot}\big(\mathrm{id}_E-\pi(g)\big)^n\right)(F)\subseteq F.$$

If $X\in\mathfrak{g}\cap U$ then $\pi(\exp X)=e^{\mathfrak{L}(\pi)(X)}\in G\cap B_1(\mathrm{id}_E)$ and thus $\log\pi(\exp X)=\mathfrak{L}(\pi)(X)$, or equivalently, $\log\pi(g)=\mathfrak{L}(\pi)(\log g)$ for $g\in G\cap B_r(\mathbf{1})$. Thus $(*)$ and the definition of $\mathfrak{n}(F,g)$ yield $\log g\in\mathfrak{n}(F,G)$. Thus the exponential function maps $\mathfrak{n}(F,G)\cap U$ homeomorphically onto $N(F,G)\cap B_r(\mathbf{1})$. Hence $N(F,G)$ is a Lie subgroup of G by Definition 5.47 and is therefore a linear Lie group. The case of $Z(F,G)$ is similar. □

Part (vi) of the previous proposition applies, in particular, to the adjoint representation $\mathrm{Ad}\colon G\to\mathrm{Aut}(\mathfrak{g})\subseteq\mathrm{Gl}(\mathfrak{g})$. If one takes $F=\mathfrak{h}$, a subalgebra of $\mathfrak{g}$, then $\mathfrak{n}(F,G)=\mathfrak{n}(\mathfrak{h},\mathfrak{g})$ in accordance with the notation introduced prior to 5.54. This makes the connection of Part (vi) with the earlier parts of Proposition 5.54 evident.

Exercise E5.18. (i) Prove the following classification of abelian linear Lie groups.

The finite dimensional abelian linear Lie groups are exactly the topological groups which are isomorphic to $\mathbb{R}^m \times \mathbb{T}^n \times D$ *for suitable nonnegative integers* m *and* n *and some arbitrary discrete abelian group* D.

(ii) Prove the following remark which contributes to an understanding of the role of analytic subgroups in abelian linear Lie groups (cf. 5.52 above)

Let G *be a abelian linear Lie group. Let* Δ *be the kernel of* $\exp\colon \mathfrak{g} \to G$. *If* $\mathfrak{h}$ *is a vector subspace of* $\mathfrak{g}$ *and if* $\mathfrak{h}$ *has a basis of elements* $D_1, \ldots, D_r$ *with* $D_j \in \Delta$, *then* $H \stackrel{\text{def}}{=} \exp \mathfrak{h}$ *is a closed Lie subgroup of* G *with* $\mathfrak{L}(H) = \mathfrak{h}$.

(iii) Formulate a structure theorem for abelian linear Lie groups without restriction on the dimension.

[Hint. (i) If G is an abelian linear Lie group, then 5.54(v) shows that $\mathfrak{g} \cong \mathbb{R}^p$ with zero brackets for some nonnegative integer p. Then $\exp\colon \mathbb{R}^p \to G$ is a homomorphism, which is continuous, open with discrete kernel (by 5.41(ii)), and with image G_0 (by 5.41(iii)). Hence G_0 is a quotient of $\mathbb{R}^p$ modulo a discrete subgroup. These are classified in Appendix A1.12. This shows that G_0 is an open subgroup of G isomorphic to a divisible group $\mathbb{R}^m \times \mathbb{T}^n$. From Appendix A1.36, there is a subgroup D of G such that $G = G_0 \oplus D$ (when G is written additively). Since G_0 is open, D is discrete and G is isomorphic to the topological group $G_0 \times D$. Conversely, every discrete group is a linear Lie group by E5.10(ii). Hence $\mathbb{R}^m \times \mathbb{T}^n \times D$ is a linear Lie group by 5.51(ii).

(ii) Apply Appendix A1.12(iii).

(iii) Note that $\exp\colon \mathfrak{g} \to G$ is always an open homomorphism of topological groups with discrete kernel defined on the additive group of a Banach space.] $\square$

For the formulation of the following result we recall from linear algebra that an endomorphism $T\colon E \to E$ of a vector space is called *semisimple* if each T-invariant vector subspace has a T-invariant complement, and *nilpotent* if $T^n = 0$ for some n. It is an easy exercise to verify that a semisimple nilpotent endomorphism is zero. (Let n be the smallest nonzero integer such that $T^n = 0$. Assume $n > 0$. Write $E = E_1 \oplus \ker T^{n-1}$ with $TE_1 \subseteq E_1$, and let $x \in E_1$. Since $T^n x = 0$ we have $Tx \in E_1 \cap \ker T^{n-1}$ and thus $E_1 = \{0\}$. Hence $E = \ker T^{n-1}$ contradicting the choice of n.) Thus for a semisimple endomorphism the nil-space E^0 of T (see remarks preceding 5.45) is in fact the kernel, $\ker T$, of T.

Proposition 5.55. *Let* G *be a finite dimensional linear Lie group and* $\mathfrak{g} = \mathfrak{L}(G)$ *its Lie algebra. Let* $g \in G$ *be any element such that* $\operatorname{Ad} g$ *is semisimple. Then* $\bigcup_{x\in G} xZ(g,G)_0x^{-1}$ *is a neighborhood of* g *in* G.

Proof. We shall apply Proposition 5.45. What is more special here than in Proposition 5.45 is that $\operatorname{Ad} g$ is semisimple and that, as a consequence, the Fitting null component $\mathfrak{g}^0(g)$ of the semisimple vector space endomorphism $\operatorname{Ad}(g) - 1$ of $\mathfrak{g}$ is simply its kernel and therefore agrees with the set $\{X \in \mathfrak{g} \mid \operatorname{Ad}(g)(X) = X\}$, and by 5.54(ii) this set equals $\mathfrak{L}\big(Z(g,G)\big)$, i.e. the Lie algebra of the centralizer of g in G.

From Proposition 5.45 we find open neighborhoods U^+ of 0 in the Fitting one-component $\mathfrak{g}^+(g)$ of $\mathrm{Ad}(g)-1$ and U^0 of 0 in $\mathfrak{L}\big(Z(g,G)\big)$, and an open neighborhood W_g of g in G such that the function $Y \oplus Z \mapsto (\exp Y)g\exp Z(\exp -Y) : U^+ \oplus U^0 \to W_g$ is a homeomorphism. Since $g\exp Z \in Z(g,G)$, the assertion follows. □

The information provided by the proof is a bit sharper than that contained in the conclusion of the proposition. The proposition itself will be crucial in the proof of the important Maximal Torus Theorem 6.30 in the chapter on compact Lie groups.

The Commutator Subgroup

The center of a Lie group was, in principle, easy to handle. The commutator subgroup is much harder to treat.

Definition 5.56. (i) If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{a}$ and $\mathfrak{b}$ are subsets, then $[\mathfrak{a},\mathfrak{b}]$ denotes the *linear span* of all $[a,b]$ with $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. In particular, $[\mathfrak{g},\mathfrak{g}]$ is an ideal called the *commutator algebra of* $\mathfrak{g}$, also written $\mathfrak{g}'$.

(ii) If G is a group and A and B are subsets, then $\mathrm{comm}(A,B)$ denotes the *subgroup generated* by all $\mathrm{comm}(a,b) = aba^{-1}b^{-1}$. In particular, $\mathrm{comm}(G,G)$ is a normal subgroup called the *commutator group of* G, or *commutator subgroup of* G, also written G'. □

If G is a topological group we shall often refer to G' as *the algebraic commutator group of* G; it is a serious issue in topological group theory that G' is not closed in general. One will, therefore, often consider the closure $\overline{G'}$ and call this closed normal subgroup, in a grammatically somewhat imprecise fashion, the *closed commutator group.*

Exercise E5.19. Verify that the commutator algebra and the center of a Lie algebra are ideals preserved under all derivations.

Show that every morphism $\mathfrak{g} \to \mathfrak{c}$ into a commutative Lie algebra factors through the quotient homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}'$. Show that every morphism $G \to C$ of groups into a commutative group factors through the quotient morphism $G \to G/G'$. Every morphism $G \to C$ of Hausdorff topological groups factors through the quotient morphism $G \mapsto G/\overline{G'}$. □

Lemma 5.57. *Let A denote a Banach algebra with identity and $\mathfrak{g}$ a closed Lie subalgebra. Assume that B is an open ball around* 0 *such that $x*y$ and* $\mathrm{comm}_*(x,y) = x*y*(-x)*(-y)$ *are defined for all x, $y \in B$ and that $x*y$ has norm less than π for such x, y. Then*

$$\mathrm{comm}_*(x,y) \in \overline{[\mathfrak{g},\mathfrak{g}]} \quad \textit{for all} \quad x,\, y \in \mathfrak{g} \cap B. \tag{25}$$

Proof. By Proposition 5.16, we have $x * y * -x * -y = e^{\operatorname{ad} x} y * -y$, and from Theorem 5.21(ii) we know $e^{\operatorname{ad} x} y * -y - (e^{\operatorname{ad} x} y - y) \in \overline{[\mathfrak{g},\mathfrak{g}]}$. But $e^{\operatorname{ad} x} y - y = [x,y] + \frac{1}{2!}\cdot[x,[x,y]] + \cdots \in \overline{[\mathfrak{g},\mathfrak{g}]}$. These pieces of information, taken together yield $\operatorname{comm}(x,y) \in \overline{[\mathfrak{g},\mathfrak{g}]}$. □

For the following discussion recall from 5.25 how the set of subtangent vectors $\mathfrak{T}(\Gamma)$ of a set Γ in a Banach space was defined.

Theorem 5.58. *Let $\mathfrak{g}$ be a closed Lie subalgebra with respect to the bracket in a Banach algebra A and let B denote an open ball around 0 such that $B*B*B*B$ is defined and contained in a ball of radius π. Let Γ denote the smallest closed local subgroup with respect to B containing all commutators $x * y * -x * -y$ defined in B with x, $y \in \mathfrak{g} \cap B$. Then $\mathfrak{T}(\Gamma) = \overline{[\mathfrak{g},\mathfrak{g}]}$ and $\Gamma = B \cap \overline{[\mathfrak{g},\mathfrak{g}]}$.*

Proof. If x, y, and $[x,y]$ are in B, then $[x,y] = \lim_n n^2(\frac{1}{n}\cdot x * \frac{1}{n}\cdot y * \frac{-1}{n}\cdot x * \frac{-1}{n}\cdot y) \in B\cap\Gamma$ by the Recovery of the Bracket 5.11 and the definition of Γ. Hence $B\cap\overline{[\mathfrak{g},\mathfrak{g}]} \subseteq \mathfrak{T}(\Gamma)$ in view of Lemma 5.26. In particular, this implies $B \cap \overline{[\mathfrak{g},\mathfrak{g}]} \subseteq B \cap \mathfrak{T}(\Gamma) \subseteq B \cap \Gamma$. But Lemma 5.57(25) implies $x * y * -x * -y \in \overline{[\mathfrak{g},\mathfrak{g}]}$ for all $x, y \in B$ with $x * y * -x * -y \in B$. Hence $\Gamma \subseteq \overline{[\mathfrak{g},\mathfrak{g}]}$. □

Proposition 5.59. *If under the conditions of Theorem 5.58, the dimension of $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is finite, then the following statements hold:*

(i) *There are elements $X_j, Y_j \in \mathfrak{g}$, $j = 1,\ldots,n = \dim\mathfrak{g}'$ such that for each n-tuple $(r_1,\ldots,r_n)$ of real numbers with $0 < |r_j| \le 1$ there is an $\varepsilon > 0$ such that the function $\varphi\colon]-\varepsilon,\varepsilon[^n \to B \cap \mathfrak{g}$ given by*

$$\begin{aligned}\varphi(s_1,\ldots,s_n)&\\ &= (r_1\cdot X_1 * s_1\cdot Y_1 * -r_1\cdot X_1 * -s_1\cdot Y_1) * \cdots * (r_n\cdot X_n * s_n\cdot Y_n * -r_n\cdot X_n * -s_n\cdot Y_n)\\ &= \operatorname{comm}_*(r_1\cdot X_1, s_1\cdot Y_1) * \cdots * \operatorname{comm}_*(r_n\cdot X_n, s_n\cdot Y_n)\end{aligned}$$

is a homeomorphism onto some 0-neighborhood of $\mathfrak{g}'$.

(ii) *Each element of a whole zero neighborhood of $\mathfrak{g}'$ is a $*$-product of at most n $*$-commutators. In particular, there is an open ball B' around 0 in $\mathfrak{g}'$ such that $B' \cap \mathfrak{g}'$ is the smallest local subgroup with respect to B' containing all $\operatorname{comm}_*(X,Y) \in B'$, X, $Y \in B'$.*

Proof. (ii) is an immediate consequence of (i) and thus we have to prove (i). Assuming $\dim\mathfrak{g}' = n$ we find pairs $(X'_j, Y_j) \in \mathfrak{g} \times \mathfrak{g}$, $j = 1,\ldots,n$ such that $\{[X'_j, Y_j] \mid j = 1,\ldots,n\}$ is a basis of $\mathfrak{g}'$. Then for each n-tuple of nonzero real numbers r_j the vectors $[r_j\cdot X'_j, Y_j] = r_j\cdot[X'_j, Y_j]$, $j = 1,\ldots,n$, form a basis. Accordingly, there is a $\delta > 0$ such that for all $0 < t < \delta$ the elements $\frac{1}{t}\cdot(e^{t\cdot \operatorname{ad} r_j\cdot X'_j} Y_j - Y_j) = [r_j\cdot X'_j, Y_j] + \mathrm{o}(t)$ will be linearly independent. We select a $t \in]0,\delta[$ and an $\varepsilon_0 > 0$

so that for all $|s_j| < \varepsilon_0$ and all $|r_j| \le 1$, the element

$$\begin{aligned}
&\varphi(s_1,\dots,s_n)\\
&\stackrel{\text{def}}{=} (t{\cdot}r_1{\cdot}X'_1 * s_1{\cdot}Y_1 * -t{\cdot}r_1{\cdot}X'_1 * -s_1{\cdot}Y_1) * \cdots * (t{\cdot}r_n{\cdot}X'_n * s_n{\cdot}Y_n * -t{\cdot}r_n{\cdot}X'_n * -s_n{\cdot}Y_n)\\
&= (s_1{\cdot}e^{\operatorname{ad} r_1\cdot(t\cdot X'_1)}Y_1 * -s_1{\cdot}Y_1) * \cdots * (s_n{\cdot}e^{\operatorname{ad} r_n\cdot(t\cdot X'_n)}Y_n * -s_n{\cdot}Y_n)
\end{aligned}$$

is defined and is contained in an open ball B around 0 satisfying the conditions specified in Theorem 5.58. Then by Theorem 5.58 and the hypothesis $\dim \mathfrak{g}' < \infty$, the function φ takes its values in $\overline{[\mathfrak{g},\mathfrak{g}]} = [\mathfrak{g},\mathfrak{g}]$. We set $X_j = t{\cdot}X'_j$ and record that

$$(*) \qquad \{e^{r_j\cdot X_j}Y_j - Y_j \mid j = 1,\dots,n\} \text{ is a basis of } \mathfrak{g}'.$$

We compute the derivative of $\varphi\colon]-\varepsilon_0,\varepsilon_0[^n \to \mathfrak{g}'$ at $(0,\dots,0)$. For each $j = 1,\dots,n$, the derivative at 0 of the curve $s \mapsto s{\cdot}e^{\operatorname{ad} r_j\cdot X_j}Y_j * (-s{\cdot}Y_j) = s{\cdot}(e^{r_j\cdot X_j}Y_j - Y_j) + \mathrm{o}(s)$, $\frac{1}{s}{\cdot}\mathrm{o}(s) \to 0$ for $s \to 0$, is $e^{r_j\cdot X_j}Y_j - Y_j$. If U is a sufficiently small zero neighborhood of $\mathfrak{g}'$, then the function $(Z_1,\dots,Z_n) \mapsto Z_1 * \cdots * Z_n : U^n \to \mathfrak{g}'$ has the derivative $(Z_1,\dots,Z_n) \mapsto Z_1 + \cdots + Z_n : (\mathfrak{g}')^n \to \mathfrak{g}'$. By the chain rule, the derivative $\varphi'(0,\dots,0)$ now is the linear map

$$(h_1,\dots,h_n) \mapsto \sum_{j=1}^{n} h_j{\cdot}(e^{r_j\cdot X_j}Y_j - Y_j)$$

from $\mathbb{R}^n$ to $\mathfrak{g}'$. By $(*)$, this linear map has rank n and thus is a linear isomorphism of $\mathbb{R}^n \to \mathfrak{g}'$. Then the Theorem of the Local Inverse (see e.g. [237]) shows that a whole neighborhood $]-\varepsilon,\varepsilon[^n$ of 0 in $]-\varepsilon_0,\varepsilon_0[^n$ is mapped homeomorphically onto a 0 neighborhood of $[\mathfrak{g},\mathfrak{g}]$ and this is what we wanted to prove. □

For the purposes of proving (ii) it would have sufficed to take $r_1 = \cdots = r_n = 1$. But we shall use the sharper form of (i) in the proof of van der Waerden's Continuity Theorem 5.64 below. These results show the intimate relation between commutators on the local group level and the brackets on the Lie algebra level. In order to draw conclusions on the global structure of the commutator group let us recall a few group theoretical concepts. The commutator $ghg^{-1}h^{-1}$ in a group G will again be written $\operatorname{comm}(g,h)$. The commutator group G' of G is the group generated by all commutators. It is a characteristic, hence normal subgroup.

Exercise E5.20. Prove the identity

$$\begin{aligned}
\operatorname{comm}(g,mn) &= \operatorname{comm}(g,m)m\big(\operatorname{comm}(g,n)\big)m^{-1}\\
&= \operatorname{comm}(g,m)\operatorname{comm}(g,n)\operatorname{comm}\big(\operatorname{comm}(g,n)^{-1},m\big).
\end{aligned}$$

Use this identity to prove the following result: *Let $\Omega = \Omega^{-1}$ denote a subset of G generating G. Then G' is the smallest normal subgroup containing all commutators* $\operatorname{comm}(a,b)$ *with $a,b \in \Omega$ and the smallest subgroup H containing all commutators* $\operatorname{comm}(a,b)$ *with $a,b \in \Omega$ and being closed under the formation of commutators* $\operatorname{comm}(a,x)$ *with elements $a \in \Omega$ and $x \in H$. If Ω is invariant under* *inner automorphisms, then G' is generated by all commutators* $\mathrm{comm}(a,b)$ *with* $a,b \in \Omega$. □

The Commutator Subgroup Theorem

Theorem 5.60. *Let G denote a connected linear Lie group such that $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is finite dimensional. Then G' is the subgroup algebraically generated by* $\exp[\mathfrak{g},\mathfrak{g}]$*; i.e.*

$$G' = \langle \exp \mathfrak{g}' \rangle. \tag{26}$$

Furthermore,

$$\mathfrak{L}(G') = \mathfrak{g}'. \tag{27}$$

Proof. The Recovery of Subalgebras 5.52(iii) conclusion (26) implies (27); thus we shall have to prove (26).

Firstly, we show that $G' \subseteq \langle \exp \mathfrak{g}' \rangle$. Let B denote a ball around 0 in $\mathfrak{g}$ such that $B * B * B * B$ is defined and contained in a ball of radius π (with respect to a norm induced from a Banach algebra containing G and $\mathfrak{g}$). Since $\dim \mathfrak{g}' < \infty$ the Subalgebra $\mathfrak{g}'$ is closed and for $X, Y \in B$ we have $\mathrm{comm}(\exp X, \exp Y) = \exp \mathrm{comm}_*(X,Y) \in \exp \overline{\mathfrak{g}'} = \exp \mathfrak{g}'$ by 5.57(25). Thus the group generated by all $\mathrm{comm}(g,h)$ with $g,h \in \exp B$ is contained $\langle \exp \mathfrak{g}' \rangle$. The group $\langle \exp \mathfrak{g}' \rangle$ is invariant under inner automorphisms since $g(\exp \mathfrak{g}')g^{-1} = \exp \mathrm{Ad}(g)[\mathfrak{g},\mathfrak{g}] \subseteq \exp[\mathfrak{g},\mathfrak{g}]$. But according to Exercise E5.20, the commutator group G' is the smallest normal subgroup containing all commutators $\mathrm{comm}(\exp X, \exp Y)$ with $X, Y \in B$. Hence $G' \subseteq \langle \exp \mathfrak{g}' \rangle$. Secondly, we show that $\langle \exp \mathfrak{g}' \rangle \subseteq G'$. By 5.59(ii) there is an open ball $B' \subseteq B$ around 0 in $\mathfrak{g}'$ such that $B' \cap \mathfrak{g}'$ is the local subgroup with respect to B' generated by all $\mathrm{comm}_*(X,Y)$ in B' with X, $Y \in B'$. But $B' \cap \exp^{-1} G'$ is a local subgroup with respect to B' containing the commutators $\mathrm{comm}_*(X,Y) \in B'$, X, $Y \in B'$, since $\exp \mathrm{comm}_*(X,Y) = \mathrm{comm}(\exp X, \exp Y) \in G'$. Therefore $B' \cap \mathfrak{g}' \subseteq B' \cap \exp^{-1} G'$ and thus $\exp(B' \cap \mathfrak{g}') \subseteq G'$. Hence $\langle \exp \mathfrak{g}' \rangle = \langle \exp(B' \cap \mathfrak{g}') \rangle \subseteq G'$. Thus (26) is proved. □

Example 5.61. Let G denote the linear Lie group of all matrices

$$\begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^u \end{pmatrix}, \quad u, x, y, z \in \mathbb{R}.$$

Its commutator group G' consists of all matrices

$$\begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R},$$

and its center Z is the direct product of G' and the group of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^u \end{pmatrix}, \quad u \in \mathbb{R}.$$

The subgroup D consisting of the matrices

$$\begin{pmatrix} 1 & 0 & m & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-m\sqrt{2}+n} \end{pmatrix}, \quad m, n \in \mathbb{Z}$$

is discrete and central. The factor group $\Gamma = G/D$ (being analytic by Exercise E5.22 below, but not a linear Lie group by arguments given in the discussion of Example 5.76 below) has a nonclosed commutator group Γ' which is dense in the center $Z(\Gamma) = Z(G)/D \cong \mathbb{T}^2$. □

Remember that, in the following proposition, a Lie subalgebra of the Lie algebra of a linear Lie group is automatically closed and separable if it is finite dimensional.

Proposition 5.62. *Assume that a subgroup of the form $H = \langle \exp \mathfrak{h} \rangle$ with $\mathfrak{h}$ a closed separable Lie subalgebra of $\mathfrak{g}$ is dense in a linear Lie group G. Then it is normal and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$; in fact $\mathfrak{g}' \subseteq \mathfrak{h}$. Moreover, if $\mathfrak{g}'$ is finite dimensional, then $G' \subseteq H$.*

Proof. Since $N(H, G)$ is closed by Proposition 5.53 and H is dense in G we conclude that H is normal, and it then follows from (24) above that $\mathfrak{n}(\mathfrak{h}, \mathfrak{g}) = \mathfrak{g}$, i.e. that $\mathfrak{h}$ is an ideal. Thus $(\operatorname{ad} X)(\mathfrak{g}) \subseteq \mathfrak{h}$ for all $X \in \mathfrak{h}$. Then $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ implies $\operatorname{Ad}(\exp X)(Y) - Y = e^{\operatorname{ad} X}(Y) - Y = [X, Y] + \frac{1}{2!} \cdot \big[X, [X, Y]\big] + \cdots \in \mathfrak{h}$. The relation $\alpha_1\alpha_2 - 1 = (\alpha_1 - 1)(\alpha_2 - 1) + (\alpha_1 - 1) + (\alpha_2 - 1)$ in the ring of vector space endomorphisms of $\mathfrak{g}$ shows that X_1, $X_2 \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ implies $\operatorname{Ad}(\exp X_1 \exp X_2)(Y) - Y \in \mathfrak{h}$. By induction we conclude $\operatorname{Ad}(H)(Y) - Y = \operatorname{Ad}(\langle \exp \mathfrak{h} \rangle)(Y) - Y \subseteq \mathfrak{h}$. Since H is dense in G we have

$$(\forall g \in G)(\forall Y \in \mathfrak{g}) \quad \operatorname{Ad}(g)(Y) - Y \in \mathfrak{h}.$$

Now let X, $Y \in \mathfrak{g}$. Then the curve $t \mapsto e^{t \cdot \operatorname{ad} X} Y - Y \colon \mathbb{R} \to \mathfrak{h}$ has all of its tangent vectors in $\mathfrak{h}$, and thus $[X, Y] = \frac{d}{dt}\big|_{t=0} e^{t \cdot \operatorname{ad} X} Y \in \mathfrak{h}$. Hence $\mathfrak{g}' \in \mathfrak{h}$ and therefore $G' = \langle \exp \mathfrak{g}' \rangle \subseteq \langle \exp \mathfrak{h} \rangle = H$ if Theorem 5.60 applies. □

Forced Continuity of Morphisms between Lie Groups

By 5.59 in an n-dimensional Lie group G, whose Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{g} = \mathfrak{g}'$, there is a neighborhood in which every element is a product of n commutators. This property forces algebraic morphisms from G into any compact topological group to be continuous as we shall show now.

An independent (and elementary) observation on compact groups is needed.

Lemma 5.63. *Let H be a compact group and U an identity neighborhood. Consider a fixed natural number n. Then there is an identity neighborhood W of H such that for $x_1, \ldots, x_n \in W$,*

$$\mathrm{comm}(\{x_1\} \times H) \cdots \mathrm{comm}(\{x_n\} \times H) \subseteq U.$$

Proof. Since U is an identity neighborhood of H and we find an identity neighborhood V such that $V^n = V \cdots V \subseteq U$. For any $h \in H$ there is a neighborhood U_h of h and an identity neighborhood W_h such that $\mathrm{comm}(W_h, U_h) \subseteq V$. By the compactness of H there are finitely many $h(p)$, $p = 1, \ldots, P$ in H such that $H = U_{h(1)} \cup \cdots \cup U_{h(P)}$. Set $W \stackrel{\mathrm{def}}{=} W_{h(1)} \cap \cdots \cap W_{h(P)}$. Then $x \in W$ and $h \in H$ implies $\mathrm{comm}(x, h) \in V$. Hence W satisfies the requirement. □

Van der Waerden's Continuity Theorem

Theorem 5.64. *Assume that $f\colon G \to H$ is a group homomorphism where H is a compact group and G is an n-dimensional linear Lie group such that $\mathfrak{L}(G)' = \mathfrak{L}(G)$; i.e. $G_0 = (G_0)'$. Then f is continuous.*

Proof. Let U_H be an identity neighborhood in H. We must show that $f^{-1}(U_H)$ is an identity neighborhood of G. Using the preceding Lemma 5.63 we find an identity neighborhood W_H of H such that

$$(*) \qquad (\forall h_1, \ldots, h_n \in W_H) \quad \mathrm{comm}(\{h_1\} \times H) \cdots \mathrm{comm}(\{h_n\} \times H) \subseteq U_H.$$

Thus by $(*)$ it suffices to show that the set

$$\bigcup \{\mathrm{comm}(\{g_1\} \times G) \cdots \mathrm{comm}(\{g_n\} \times G) : f(g_1), \ldots, f(g_n) \in W_H\}$$

is an identity neighborhood of G.

Now by 5.59(i) we find vectors $X_j \in \mathfrak{g}$, $j = 1, \ldots, n$ such that for each n-tuple $(r_1, \ldots, r_n)$ of nonzero real numbers with $|r_j| \leq 1$ the set of all

$$\mathrm{comm}(\{\exp r_1 \cdot X_1\} \times G) \cdots \mathrm{comm}(\{\exp r_n \cdot X_n\} \times G)$$

is certainly an identity neighborhood. The proof of the theorem now boils down to finding, for each $j = 1, \ldots, n$, a real number r_j with $0 < |r_j| \leq 1$ such that $f(\exp r_j \cdot X_j) \in W_H$. Define a group homomorphism $\tau\colon \mathbb{R} \to H$ by $\tau(r) = f(\exp r \cdot X_j)$. If τ is constant, set $r_j = \frac{1}{2}$. In that case $f(\exp r_j \cdot X_j) = 1 \in W_H$. Now assume that τ is nondegenerate. Then the subset $\tau(]0, 1])$ of the compact space H is infinite and therefore has an accumulation point h. Let V be an identity neighborhood of H such that $VV^{-1} \subseteq W_H$. Find two real numbers s and t such that $0 < s < t \leq 1$, that $\tau(s) \neq \tau(t)$, and that $\tau(s), \tau(t) \in Vh$. We set $r_j = t - s$. Then $0 < r_j \leq 1$ and $f(\exp r_j \cdot X_j) = \tau(r_j) = \tau(t)\tau(s)^{-1} \in Vhh^{-1}V^{-1} = VV^{-1} \subseteq W_H$. Thus r_j satisfies our requirements. □

Corollary 5.65. *Let G be a connected linear Lie group such that $G' = G$. Then every homomorphism from G to* $\mathrm{O}(n)$ *or* $\mathrm{U}(n)$ *is automatically a continuous representation.*

Proof. By 5.60 we have $\mathfrak{L}(G)' = \mathfrak{L}(G') = \mathfrak{L}(G)$. The assertion then follows from 5.64. □

This applies in particular to groups like $G = \mathrm{SO}(n)$, $G = \mathrm{SU}(n)$ (cf. e.g., 6.8 below). In the next chapter we shall in fact observe, that the property $G = G'$ is a very prevalent phenomenon among connected compact Lie groups.

Corollary 5.66. *A compact connected Lie group G satisfying $G' = G$ supports only one compact group topology.*

Proof. By 5.60 we have $\mathfrak{g}' = \mathfrak{g}$. If $\mathcal{O}$ is a compact group topology on G, then $\mathrm{id}_G: G \to (G, \mathcal{O})$ is continuous by 5.64. Then, as G is compact, id_G is a homeomorphism. □

It should be emphasized that this statement means that the cardinality of the set of all compact group topologies on the underlying group G is one. This is to be contrasted with a different statement, applying for instance to the case of a torus group $\mathbb{T}^n$ where there are uncountably many different compact Lie group topologies, yet they are all isomorphic. The following exercise is to illustrate this point.

Exercise E5.21. (i) Show that $\mathbb{T}^n$ has discontinuous automorphisms and thus different compact Lie group topologies. Show that all compact Lie groups on the underlying group of $\mathbb{T}^n$ are isomorphic compact Lie groups.

(ii) Show that there are countably infinitely many isomorphy classes of connected Lie group topologies on the underlying group of $\mathbb{T}^n$.

[Hint. (i) Observe from the Appendix 1 that the underlying group of $\mathbb{T}^n$ is isomorphic to $\mathbb{Q}^{(2^{\aleph_0})} \oplus \bigoplus_{p\,\mathrm{prime}} \mathbb{Z}(p^\infty)^n$ (cf. A1.34). Exhibit discontinuous automorphisms from this information. Note that the dimension n is an algebraic invariant and that $\mathbb{T}^n$ is divisible. In 2.42 we have classified all compact Lie groups. Use these pieces of information for a proof of the uniqueness up to topological isomorphism.

(ii) Remark that $\mathbb{T}^n$ is algebraically isomorphic to $\mathbb{T}^n \times \mathbb{R}^m$ for $m,\, n = 0, 1, 2 \ldots.$ By E5.18, each connected linear abelian Lie group is isomorphic to one of these.] □

In fact there are compact group topologies on the underlying group of $\mathbb{T}^n$ which are not Lie group topologies. But this is the subject of a later chapter on compact abelian groups.

Quotients of Linear Lie Groups

We have introduced linear Lie groups; it is interesting to observe that the class of linear Lie groups is not closed under the formation of quotients even if these are relatively well behaved. A simple and theoretically interesting example is that of the so-called *Heisenberg algebra* $\mathfrak{h}$ and the corresponding *Heisenberg group* H. (The name arises from basic quantum mechanics; the quantum mechanical model of a free particle on a real line describes the position of the particle by a hermitian (unbounded) operator Q on separable Hilbert space (here being $L^2(\mathbb{R})$), its momentum by a hermitian (unbounded) operator P, and these do not commute; in particular, if I denotes the identity operator, they satisfy $[P,Q] = i{\cdot}I$ when physical quantities are appropriately normalized. If we set $p = -i{\cdot}P$, $q = i{\cdot}Q$, and $i{\cdot}I = \iota$, then all of these are skew hermitian operators satisfying $[p,q] = \iota$ and spanning a Lie algebra called the *Heisenberg algebra*. It generates a group of bounded unitary operators called the *Heisenberg group*. The names are generally extended to all objects which are isomorphic to these.)

Example 5.67. (i) Let T denote the Banach algebra of all upper triangular matrices

$$\langle t; x, y; z\rangle \stackrel{\text{def}}{=} \begin{pmatrix} t & x & z \\ 0 & t & y \\ 0 & 0 & t \end{pmatrix}, \qquad t, x, y, z \in \mathbb{R}.$$

Then $T^{-1} = \{\langle t; x, y; z\rangle \mid t \neq 0\}$. The *Heisenberg group* H is the subgroup of all elements $\langle 1; x, y; z\rangle \in T^{-1}$, and the *Heisenberg algebra* $\mathfrak{h}$ is the Lie subalgebra of $(T, [\cdot\,,\cdot])$ consisting of all $\langle 0; x, y; z\rangle \in \mathbb{T}$. Thus H is a linear Lie group. Its center Z consists of all $\langle 1; 0, 0; z\rangle$. It contains an infinite cyclic subgroup $F = \{\langle 1; 0, 0; n\rangle \mid n \in \mathbb{Z}\}$. Then F is a discrete subgroup, hence a closed subgroup, and it is central, hence normal. The quotient map $q\colon H \to H/F$ onto the quotient group $G = H/F$ induces a local isomorphism. (It is not hard to show that, in such circumstances, the factor group $G = H/F$ is an analytic group: Exercise E5.22.)

The center of G is the circle group Z/F. We claim that G is not a linear Lie group. Observe that this requires a proof that G cannot be embedded into the multiplicative group of *any* Banach algebra. In the context of compact groups, however, it is reassuring that we shall be able to show (in Theorem 6.7 below) that quotients of compact Lie groups are compact Lie groups. Here, in fact, we shall prove:

Lemma A. *If $f\colon H \to A^{-1}$ is a morphism of topological groups from the Heisenberg group H into the group of units of a Banach algebra A such that $f(F) = \{\mathbf{1}\}$, then $f(Z) = \{\mathbf{1}\}$.*

Proof. Suppose by way of contradiction that there is a morphism $f\colon H \to A^{-1}$ into a subgroup of the group of units of a Banach algebra with $f(F)$ singleton but $f(Z)$ not singleton. We shall derive a contradiction, and for this purpose we may and shall assume that A is the closed linear span of $f(H)$. Then $S = f(Z/F)$ is a central circle group.

If A is not complex, we consider the complexification $\mathbb{C}\otimes A$ and obtain a complex representation $h \mapsto \mathbf{1}\otimes f(h)\colon H \to \mathbb{C}\otimes A$. Thus, without losing generality, we assume A to be complex. The circle group S makes A into an S-module under multiplication. Consider the isotypic decomposition of $A_{\text{fin}} = \sum_{\chi\in\widehat{S}} A_\chi$ (see Theorem 4.39), where A_χ is the set of all $a \in A$ with $sa = \chi(s)\cdot a$. If $a \in A_\chi$ and $b \in A$, then $s(ab) = \chi(s)\cdot ab$, and thus A_χ is a right ideal. Since S is central, we also get $s(ba) = b(sa) = \chi(s)\cdot ba$, and so A_χ is likewise a left ideal. It follows that $A_\chi A_\rho \subseteq A_\chi \cap A_\rho$ and this intersection is zero if $\chi \neq \rho$. Thus A_χ annihilates all A_ρ with $\rho \neq \chi$ and thus annihilates the kernel of the canonical projection $P = P_{e_\chi}\colon A \to A$ onto A_χ since the sum of these A_ρ is dense in $\ker P = (\mathrm{id}_A - P)A$ by the Big Peter and Weyl Theorem 3.51 and since $A = PA \oplus (\mathrm{id}_a - P)A$ algebraically and topologically (see 4.13(iii)). Thus P is a ring morphism. Not all isotypic components can vanish. Each non-zero one of them is an algebra with identity, and $Pf\colon H \to PA$ is a morphism and $Pf(Z) = P(S)$ is still a nontrivial circle group. The improvement is that now $Pf(z) = Pf(z)P(\mathbf{1}) = \chi(z)\cdot P(\mathbf{1})$. Since S is nonsingleton, we must have at least one isotypic component other than A_{fix}. We shall henceforth restrict our attention to such a component and assume that $f(z) = \chi(z)\cdot\mathbf{1}$.

We define $P, Q, R \in \mathfrak{L}(H)$ by

$$P(t) = \langle 1; t, 0; 0\rangle, \quad Q(t) = \langle 1; 0, t; 0\rangle, \quad R(t) = \langle 1; 0, 0; t\rangle.$$

In the algebra T we have $P(t) = e^{t\cdot\langle 0;1,0;0\rangle}$, $Q(t) = e^{t\cdot\langle 0;0,1;0\rangle}$, and $Z(t) = e^{t\cdot\langle 0;0,0;1\rangle}$.

In T we notice

$$[\langle 0; 1, 0; 0\rangle, \langle 0; 0, 1; 0\rangle] = \langle 0; 0, 0; 1\rangle.$$

By Proposition 5.40, this implies $[P, Q] = R$ in $\mathfrak{L}(H)$. Then

$$[\mathfrak{L}(f)(P), \mathfrak{L}(f)(Q)] = \mathfrak{L}(f)(R)$$

by Theorem 5.42. By Proposition 5.9, we find unique elements p', q, $r \in A$ such that $\mathfrak{L}(f)(P)(t) = (f\circ P)(t) = \exp t\cdot p'$, $\mathfrak{L}(f)(Q)(t) = (f\circ Q)(t) = \exp t\cdot q$, and $\mathfrak{L}(f)(R)(t) = (f\circ R)(t) = \exp t\cdot r = e^{2\pi it}\cdot\mathbf{1}$. In particular, $r = 2\pi i\cdot\mathbf{1}$. Thus, in view of Proposition 5.40, we note $[p', q] = 2\pi i\cdot\mathbf{1}$. We set $p = (2\pi i)^{-1}\cdot p'$, and therefore have elements p, q in the Banach algebra A such that $[p, q] = \mathbf{1}$. We claim that this is impossible and prove this claim in a subsequent lemma. It will follow that the analytic group $G = H/F$ which is locally isomorphic to the linear Lie group H is not itself a linear Lie group. □

It is instructive to pursue the digression into the Heisenberg Lie algebra just a little further. Part (b) of the subsequent lemma will finish the proof of Lemma A above.

Lemma B. (a) *Let F denote a field. Then there is an associative algebra $A(p, q)$ over F such that for any associative algebra A' over F containing elements p', q'*

with $p'q'-q'p'=1$ *there is a unique morphism of associative algebras* $\varphi: A(p,q)\to A'$ *with* $\varphi(p)=p'$ *and* $\varphi(q)=q'$. *In particular, there is an automorphism of* $A(p,q)$ *with* $p\mapsto q$ *and* $q\mapsto -p$. *Moreover, the subalgebras* $A(p)$ *and* $A(q)$ *generated by* 1 *and* p, *respectively,* q *are isomorphic to the polynomial algebra* $\mathbb{K}[\zeta]$ *in one variable* ζ. *In* $A(p,q)$ *we have*

$$[p,q^n]=n{\cdot}q^{n-1}\quad \textit{for}\quad n=1,2\ldots. \tag{28}$$

Equation (28) *holds in any associative algebra for elements* p *and* q *satisfying* $[p,q]=1$. *In any such situation,* $q^n\neq 0$ *for all* $n\in\mathbb{N}$.

(b) *Let* A *denote an arbitrary Banach algebra with identity. Then* A *cannot contain elements* p,q *with* $pq-qp=\mathbf{1}$.

Proof. (a) We consider the free associative algebra $\mathbb{K}[\xi,\eta]$ in two noncommuting variables and let I denote the ideal generated by the element $\xi\eta-\eta\xi-1$. Then $A(p,q)=\mathbb{K}[\xi,\eta]/I$ has the universal property with $p=\xi+I$ and $q=\eta+I$.

In particular, the prescription $p\mapsto q$ and $q\mapsto -p$ extends to a unique automorphism whose inverse is the unique extension of $p\mapsto -q$ and $q\mapsto p$.

Now we consider the polynomial algebra $P=\mathbb{K}[\zeta]$ in one variable and the associative algebra $A'=\mathrm{Hom}(P,P)$ of all vector space endomorphisms of P. We define $p',q'\in A'$ by $p'f(\zeta)=f'(\zeta)$ with the derivative f' of the polynomial f, and $q'f(\zeta)=\zeta f(\zeta)$. Then $p'q'f(\zeta)=p'\big(\zeta f(\zeta)\big)=f(\zeta)+\zeta f'(\zeta)=f(\zeta)+q'p'f(\zeta)$. Thus $p'q'-q'p'=\mathrm{id}_{A'}$. Hence there is a unique morphism $\varphi: A(p,q)\to A'$ with $\varphi(p)=p'$ and $\varphi(q)=q'$. Since the algebra generated in A' by q' is clearly isomorphic to $\mathbb{K}[\zeta]$, it follows that $A(q)\cong\mathbb{K}[\zeta]$. The automorphism with $q\mapsto p$, $p\mapsto -q$ takes $A(q)$ to $A(p)$.

The function $D: A(p,q)\to A(p,q)$ given by $D=\mathrm{ad}\,p$ is a derivation with $Dq=1$. Hence $Dq^n=n{\cdot}q^{n-1}$ as one observes readily by induction. This shows (28). The universal property of $A(p,q)$ entails that this identity is true in any algebra as soon as $[p,q]=1$. In such a situation we certainly have $q\neq 0$, and then (28) shows inductively that $q^n\neq 0$ for $n\in\mathbb{N}$.

(b) Now suppose that A is a Banach algebra containing elements p and q with $[p,q]=1$. Then (28) implies

$$n\|q^{n-1}\|\le \|[p,q^n]\|\le 2\|p\|{\cdot}\|q^n\|\le 2\|p\|{\cdot}\|q\|{\cdot}\|q^{n-1}\|,$$

that is

$$(n-2\|p\|{\cdot}\|q\|)\|q^{n-1}\|\le 0$$

for all $n\in\mathbb{N}$. As soon as $n>2\|p\|{\cdot}\|q\|$, this implies $q^{n-1}=0$ a contradiction to (a) above. □

The proof of Part (b) in the preceding Lemma A is due to H. Wielandt [374].

The passing to a quotient group of a topological group modulo a discrete normal subgroup occurs frequently. It is important to keep in mind that this process produces groups which are locally isomorphic. Let us formulate this in a proposition:

Proposition 5.68. *Let G be a topological group and D a discrete normal subgroup. Let U be an open identity neighborhood of G such that $UU^{-1} \cap D = \{\mathbf{1}\}$. Then the quotient homomorphism $p: G \to G/D$, $p(g) = gD$ maps U homeomorphically onto $p(U) = UD/D$.*

Proof. Quotient homomorphisms are continuous and open. Thus $p|U: U \to p(U)$ is continuous and open. It remains to show that this function is injective. So consider u, $v \in U$ such that $p(u) = p(v)$. Then $p(uv^{-1}) = \mathbf{1}$ and so $uv^{-1} \in D \cap UU^{-1} = \{\mathbf{1}\}$, whence $u = v$. □

One says briefly that *quotient maps modulo discrete normal subgroups induce local isomorphisms.* In the following exercise one may elaborate this formalism a bit further:

Exercise E5.22. Assume that G is an analytic group and that D is a discrete normal subgroup. Show that G/D is an analytic group in a unique fashion such that the quotient homomorphism $G \to G/D$ is analytic. □

This result shows that quotient group $G = H/F$ which we produced in Example 5.67 is in fact an analytic group.

Exercise E5.23. Show that every associative algebra A generated by two elements p, q with $[p, q] = 1$ is a free $\mathbb{K}[\xi]$-module for the module operation $f(\xi).a = f(\operatorname{ad} p)(a)$ with a basis $1, q, q^2, \ldots$. Also, A has an ascending filtration A^n, $n = 0, 1, \ldots$, that is an ascending family of vector spaces satisfying $A^m A^n \subseteq A^{m+n}$ such that the graded algebra GA associated with A (see e.g., [36], Chap. III, §2, no. 3, p. 166) is commutative and is the polynomial algebra generated by two elements $\overline{p}$ and $\overline{q}$.

[Hint. We consider the polynomial algebra $A[t]$ with one commuting variable t. If in A we have $\sum_{j=0}^{n} a_n q^n = 0$ with $a_n \neq 0$, then an application of $\alpha_t = e^{t \cdot \operatorname{ad} p}$ yields $\sum_{j=0}^{n} a_n (q + t \cdot \mathbf{1})^n = 0$, that is $t^n a_n \cdot \mathbf{1} + R$ where R is a polynomial in t of degree $\leq n - 1$. It follows that $a_n = 0$, a contradiction. Then the subalgebra generated by $\mathbf{1}$ and q is isomorphic to $\mathbb{K}[\xi]$, the polynomial algebra in ξ under $\xi \mapsto q$. By the same token, applying $\beta_t = e^{-t \cdot \operatorname{ad} q}$ we obtain that the algebra generated by $\mathbf{1}$ and p is isomorphic to $\mathbb{K}[\xi]$.

Now we consider A as a left $\mathbb{K}[\xi]$-module under $f(\xi) \cdot a = f(\operatorname{ad} p)(a)$ and $\sum_{j=0}^{n} a_n . q^n = 0$ with $a_n \in \mathbb{K}(\xi)$ and $a_n \neq 0$. The preceding procedure implies $a_n = 0$. It follows that A is a free $\mathbb{K}[\xi]$ module with basis $1, q, q^2, \ldots$. The morphism $\varphi: \mathbb{K}[\xi, \eta] \to A$ of the free associative algebra in two generators given by $\varphi(\xi) = p$ and $\varphi(\eta) = q$ maps $\mathbb{K}[\xi]$ and $\mathbb{K}[\eta]$ isomorphically. We let $F^n[\xi, \eta]$ denote the vector subspace of all polynomials in ξ and η in degree $\leq n$. Then the family $A^n = \varphi(F^n[\xi, \eta])$, $n = 0, 1, \ldots$ defines on A an ascending filtration, that is a family of vector subspaces with $A^m A^n \subseteq A^{m+n}$. The graded algebra $GA = \bigoplus_{n=0}^{\infty} A^n / A^{n-1}$ (where $A^{-1} = \{0\}$) with the multiplication defined unambiguously by $(x + A^{m-1})(y + A^{n-1}) = xy + A^{m+n-1}$ is generated by $\overline{p} = p + A^0$ and

$\overline{q} = q + A^0$, and $\overline{pq} = (p+A^0)(q+A^0) = pq + A^1 = qp + A^1 = (q+A^0)(p+A^0) = \overline{qp}$. Hence GA is a graded commutative algebra generated by $\overline{p}$ and $\overline{q}$. Use the first part to prove that $GA(\overline{q})$ is a free $GA(\overline{p})$-module.] □

The Topological Splitting Theorem for Normal Vector Subgroups

A *vector subgroup* V of a topological group G is a closed subgroup which is isomorphic, in the category of topological groups, to the additive group of a real topological vector space.

Lemma 5.69. *If G is a linear Lie group and V a closed subgroup then the following conditions are equivalent.*

(i) *V is a vector subgroup.*

(ii) *V is a Lie subgroup with Lie algebra $\mathfrak{L}(V) = \mathfrak{v}$ such that* $\exp|\mathfrak{v}\colon \mathfrak{v} \to V$ *is an isomorphism of abelian topological groups.*

In particular, if these conditions are satisfied, then V is isomorphic to the additive subgroup of some Banach space.

Proof. Exercise E5.24. □

Exercise E5.24. Prove Lemma 5.69.

[Hint. For a proof of (i)⇒(ii) let $\varphi\colon E \to V$ be an isomorphism of abelian topological groups from the additive group of a real topological vector space E onto V. Recall $\mathfrak{L}(V) = \mathrm{Hom}(\mathbb{R}, V)$ and define a function $\alpha\colon E \to \mathfrak{L}(V)$ by $\alpha(x)(r) = \varphi(r{\cdot}x)$. Prove that α is an isomorphism of abelian topological groups. Conclude that $\exp_V\colon \mathfrak{L}(V) \to V$ is an isomorphism of topological groups with inverse $\alpha \circ \varphi^{-1}$. By the functoriality of $\mathfrak{L}$, the inclusion $j\colon V \to G$ gives an embedding $\mathfrak{L}(j)\colon \mathfrak{L}(V) \to \mathfrak{L}(G) = \mathfrak{g}$. Then the vector subspace $\mathfrak{v} = \mathfrak{L}(G)$ of $\mathfrak{g}$ is closed. It is therefore isomorphic to the underlying topological vector space of a Banach space, and $\exp_G|\mathfrak{v}\colon \mathfrak{v} \to V$ is an isomorphism of abelian topological groups. Conclude that V is a Lie subgroup.] □

By the Tubular Neighborhood Theorem for Subgroups 5.33(ii) for every linear Lie group G and a closed Lie subgroup N there is a closed subset C containing 1 such that

$$(n, c) \mapsto nc\colon N \times C \to NC \tag{29}$$

is a homeomorphism onto neighborhood of N of G. For a closed subgroup N of an arbitrary topological group G we shall say that N *has a tubular neighborhood* if such a C exists. If G is a linear Lie group, by 5.33(ii) we can choose C homeomorphic to a convex closed symmetric subset of $\mathfrak{g}/\mathfrak{n}$.

In the proof of the following theorem we shall use the concept of paracompactness and that of a partition of unity. For these topics see e.g. [34], Chap. 9, §4, n^o 3, 4, or [101], p. 299ff.

Theorem 5.70 (The Topological Splitting Theorem for Vector Subgroups). *Let N be a vector subgroup of a topological group G such that the following conditions are satisfied:*

(a) *N has a tubular neighborhood in G.*

(b) *The quotient space $G/N = \{Ng \mid g \in G\}$ is paracompact.*

Then there is a continuous map $\sigma: G/N \to G$ such that

$$(n, \xi) \mapsto n\sigma(\xi): N \times G/N \to G$$

is a homeomorphism satisfying $N\sigma(\xi) = \xi$. In particular, with $C \stackrel{\text{def}}{=} \sigma(G/N)$, the function

$$(n, c) \to nc: N \times C \to NC = G$$

is a homeomorphism, and G is homeomorphic to $N \times G/N$.

Proof. Let $C \subseteq G$ be such that NC is a tubular neighborhood of N in G according to hypothesis (a). The interior of NCg/N in G/N in an open neighborhood V'_g of $Ng \in G/N$. We note that $\{V_g \mid g \in G\}$ is an open cover of G/N. Since G/N is paracompact by hypothesis (b), there is a partition of unity subordinate to this cover, i.e. a family of continuous functions $f_j: G/N \to [0, 1]$ and a locally finite open cover $\{V_j \mid j \in J\}$ of G/N such that for each $j \in J$ there is a $g \in G$ such that $\operatorname{supp} f_j \subseteq V_j \subseteq V'_g$, and that $\sum_{j \in J} f_j = 1$ (where all but a finite number of summands $f_j(\xi)$ are nonzero for each $\xi \in G/N$).

Let $p: G \to G/N$ denote the quotient map and set $U_j = p^{-1}(V_j)$, $p_j: U_j \to V_j$, $p_j = p|U_j$. Let $j \in J$. Fix a $g \in G$ such that $V_j \subseteq V'_g$. Since $\rho_g: G \to G$, $\rho_g(x) = xg$ is a homeomorphism, the function

$$(n, c) \mapsto nc: N \times C \to NCg \tag{30}$$

is a homeomorphism onto a neighborhood of Ng. The continuous function $\tau_j: V_j \to U_j$, $\tau_j(Ncg) = cg$ satisfies $p_j\tau_j = \mathrm{id}_{V_j}$. For $i, j \in J$ and $\xi \in V_i \cap V_j$ we have $N\tau_i(\xi) = N\tau_j(\xi)$ and thus $\tau_i(\xi)\tau_j(\xi)^{-1} \in N$. We therefore can define

$$\kappa_{ij}: G/N \to \mathfrak{n}, \quad \exp \kappa_{ij}(\xi) = \begin{cases} \tau_i(\xi)\tau_j(\xi)^{-1} & \text{if } \xi \in V_i \cap V_k, \\ 1 & \text{otherwise.} \end{cases} \tag{31}$$

It is clear that $\kappa_{ji} = -\kappa_{ij}$. If $\xi \in V_i \cap V_j \cap V_k$, then $\exp\big(\kappa_{ij}(\xi) + \kappa_{jk}(\xi)\big) = \exp \kappa_{ij}(\xi) \exp \kappa_{jk}(\xi) = \tau_i(\xi)\tau_j(\xi)^{-1}\tau_j(\xi)\tau_k(\xi)^{-1} = \tau_i(\xi)\tau_k(\xi)^{-1} = \exp(\kappa_{ik}(\xi)$, that is

$$(\forall \xi \in V_i \cap V_j \cap V_k) \quad \kappa_{ij}(\xi) + \kappa_{jk}(\xi) = \kappa_{ik}(\xi). \tag{32}$$

Now we define for each $j \in J$ a function

$$\varphi_j: V_i \to \mathfrak{n}, \quad \varphi_j(\xi) = \sum_{i \in J} f_i(\xi) \cdot \kappa_{ij}(\xi). \tag{33}$$

Because $\operatorname{supp} f_i \subseteq V_i$, the function $\xi \mapsto f_i(\xi)\cdot\kappa_{ij}(\xi): U_j \to \mathfrak{n}$ is continuous for all $i \in J$, and thus φ_j is continuous. Now let $\xi \in U_j \cap U_k$. If $\xi \notin U_i$, then $f_i(\xi) = 0$. If $\xi \in U_i$, then (32) implies $f_i(\xi)\cdot\kappa_{ij} = f_i(\xi)\cdot\big(\kappa_{ik}(\xi) + \kappa_{kj}(\xi)\big)$. Hence

$$\begin{aligned} \varphi_j(\xi) &= \sum_{i\in J} f_i(\xi)\cdot\kappa_{ij}(\xi) \\ &= \sum_{i\in J} f_i(\xi)\cdot\kappa_{ik}(\xi) + \sum_{i\in J} f_i(\xi)\cdot\kappa_{kj}(\xi) \\ &= \varphi_k(\xi) + \kappa_{kj}(\xi). \end{aligned} \tag{34}$$

For each $j \in J$ we set

$$\sigma_j: V_j \to U_j, \quad \sigma_j(\xi) = \exp\varphi_j(\xi)\tau_j(\xi). \tag{35}$$

Then

$$p\big(\sigma_j(\xi)\big) = p\big(\tau_j(\xi)\big) = \xi \quad \text{for} \quad \xi \in V_j. \tag{36}$$

Now let $\xi \in U_j \cap U_k$. Then (34) and (31) imply

$$\begin{aligned} \sigma_j(\xi) &= \exp\big(\varphi_j(\xi)\big)\tau_j(\xi) \\ &= \exp\big(\varphi_k(\xi) + \kappa_{kj}(\xi)\big)\tau_j(\xi) \\ &= \exp\big(\varphi_k(\xi)\big)\big(\tau_k(\xi)\tau_j(\xi)^{-1}\big)\tau_j(\xi) \\ &= \exp\big(\varphi_k(\xi)\big)\tau_k(\xi) = \sigma_k(\xi). \end{aligned}$$

Hence we can define unambiguously

$$\sigma: G/N \to G, \quad \sigma(\xi) = \sigma_j(\xi) \quad \text{if} \quad \xi \in U_j. \tag{37}$$

Then σ is continuous since all σ_j are continuous, and $p \circ \sigma = \mathrm{id}_{G/N}$. The function

$$(n, \xi) \mapsto n\sigma(\xi): N \times G/N \to G$$

has the inverse $g \mapsto \big(g\sigma\big(p(g)\big), p(g)\big)$ and thus is the desired homeomorphism. □

Theorem 5.71 (The Vector Subgroup Splitting Theorem). *Let N be a normal vector subgroup of a topological group G such that the following three conditions are satisfied:*

(i) *G/N is compact.*
(ii) *N has a tubular neighborhood.*
(iii) *There is a feebly complete* (*see* 3.29) *real topological vector space $\mathfrak{n}$ and an isomorphism of topological groups $\exp_N: \mathfrak{n} \to N$.*

Then G contains a compact subgroup K such that the function $(n, k) \mapsto nk: N \times K \to G$ is a homeomorphism.

If G is a linear Lie group, then (ii), (iii) *are automatically satisfied.*

Proof. If G is a linear Lie group, then 5.33(ii) secures (ii); by Lemma 5.69, the vector subgroup N is a Lie subgroup and then $\mathfrak{n} = \mathfrak{L}(N)$ is a completely normable and thus certainly a feebly complete topological vector space (cf. chain of implications preceding 3.30). Hence (iii) will be satisfied, too. Let $\sigma: G/N \to G$ be a cross section according to Theorem 5.70 which we can apply by (i) and (ii). If $g \in G$ then $\sigma(Nhg)$ and $\sigma(Nh)g$ are in the same coset Ngh. Hence there is an $n \in N$ such that $\sigma(Nh)g = n\sigma(Nhg)$. If $\xi = Nh$ we write $Nhg = \xi{\cdot}g$. Then $\sigma(\xi{\cdot}g)^{-1}\sigma(\xi)g = \sigma(\xi{\cdot}g)^{-1}n\sigma(\xi{\cdot}g) \in N$. Define $\delta: G/N \times G \to N$ by

$$\delta(\xi, g) = \sigma(\xi{\cdot}g)^{-1}\sigma(\xi)g.$$

Then δ has the following properties:

(a) δ is continuous,

(b) $\delta(\xi, n) = n$ for all $(\xi, n) \in G/N \times N$.

(c) $\delta(\xi, gh) = \delta(\xi{\cdot}g, h)(h^{-1}\delta(\xi, g)h)$ for all $\xi \in G/N$, g, $h \in G$.

Indeed, (a) is clear, and for a proof of (b) let $\xi = Nh$, then $\xi{\cdot}n = Nhn = hNn = hN = Nh$ since N is normal. The assertion then is a consequence of the definition of δ. For (c) compute that, on the one hand, $\sigma(\xi{\cdot}gh)\delta(\xi, gh) = \sigma(\xi)gh$, and that $\sigma(\xi)gh = \sigma(\xi{\cdot}g)\delta(\xi, g)h = \sigma(\xi{\cdot}g)h(h^{-1}\delta(\xi, g)h) = \sigma(\xi.gh)\delta(\xi{\cdot}g, h)(h^{-1}\delta(\xi, g)h)$ on the other.

Now we let $d\xi$ denote Haar measure on G/N and define

$$\psi: G \to N, \quad \psi(g) = \exp_N \int_G \exp_N^{-1} \delta(\xi, g)\, d\xi.$$

The integral exists by (iii) and 3.30. Then by (b), $n \in N$ implies

$$\psi(n) = \exp_N \int_G \exp_N^{-1} \delta(\xi, n)\, d\xi = \exp_N \int_G \exp_N^{-1} n\, d\xi = n$$

because Haar measure is normalized. In particular, $\psi\big(\psi(g)\big) = \psi(g)$. From (c) we deduce

$$\begin{aligned} \exp_N^{-1} \psi(gh) &= \int_G \exp_N^{-1} \delta(\xi, gh)\, d\xi \\ &= \int_G \exp_N^{-1} \delta(\xi{\cdot}g, h)\, d\xi + \int_G \mathrm{Ad}(h)^{-1} \exp_N^{-1} \delta(\xi, g)\, d\xi \\ &= \exp_N^{-1} \psi(h) + \mathrm{Ad}(h)^{-1} \exp_N^{-1} \psi(g) \end{aligned}$$

since $\xi{\cdot}g = \xi(Ng)$ and Haar measure is invariant. Thus

$$(\forall g,\, h \in G) \quad \psi(gh) = \psi(h)\big(h^{-1}\psi(g)h\big).$$

From taking $g = h = 1$ it follows that $\psi(1) = 1$. Then $h = g^{-1}$ yields $\psi(g^{-1}) = g\psi(g)^{-1}g^{-1}$.

We observe, that $K = \psi^{-1}(1)$ is a closed subgroup. Moreover, $\psi\big(\psi(g)^{-1}g\big) = \psi(g)g^{-1}\psi\big(\psi(g)^{-1}\big)g = 1$, whence $\psi(g)^{-1}g \in K$. The function

$$g \mapsto (\psi(g), \psi(g)^{-1}g): G \to N \times K \text{ inverts the function}$$
$$(n, k) \mapsto nk: N \times K \to G$$

which therefore is a homeomorphism. □

We notice that the construction of the function ψ which provided us with the conclusion of the theorem was obtained via the averaging operator again. We systematically dealt with it in Chapter 3. The averaging concept is one of the most powerful and versatile tools in the theory of compact groups. Another example which is interesting in our context is the following exercise:

Exercise E5.25. Prove the following proposition.

Assume that the topological group G has a normal vector subgroup N satisfying 5.71(i), (iii) *and assume further that for two compact subgroups K_1 and K_2 the map $(n,k) \mapsto nk : N \times K_j \to G$ is a homeomorphism for $j = 1, 2$. Then K_1 and K_2 are conjugate.*

[Hint. The assumption provides isomorphisms $\pi_j\colon G/N \to K_j$ and then $\xi \mapsto \pi_2(\xi)^{-1}\pi_1(\xi)\colon G/N \to G$ has its image in N and thus defines a continuous function $\varphi\colon G/N \to \mathfrak{n}$ such that $\exp_N \varphi(\xi) = \pi_2(\xi)^{-1}\pi_1(\xi)$. Compute

$$(*) \qquad \varphi(\xi\eta) = \varphi(\eta) + \mathrm{Ad}\,\big(\pi_1(\eta)^{-1}\big)\varphi(\xi).$$

Now define $X \in \mathfrak{n}$ by $X = \int_{G/N} \varphi(\xi)\, d\xi$ and integrate with respect to Haar measure on G/N on both sides of $(*)$ with respect to ξ. Get $X = \varphi(\eta) + \mathrm{Ad}\,\big(\pi_1(\eta)^{-1}\big)(X)$. Set $n = \exp X$ and verify $\pi_2(\eta) = n\pi_1(\eta)n^{-1}$ for all $\eta \in G/N$.] ☐

We have seen that in the circumstances of the theorem we have a homeomorphism

$$\mu\colon N \times K \to G, \quad \mu(n,k) = nk.$$

It is not in general a morphism of groups. Of course we can transport the group structure of G to $N \times K$ and ask which group structure on $N \times K$ would make μ an isomorphism. Let us denote this group multiplication by

$$\big((n,k),(n',k')\big) \mapsto (n,k)*(n',k')\colon (N \times K) \times (N \times k) \to (N \times K).$$

Then

$$\mu\big((n,k)*(n',k')\big) = \mu(n,k)\mu(n',k') = nkn'k' = n(kn'k^{-1})kk' = \mu\big(n(kn'k^{-1}), kk'\big).$$

Thus the multiplication

$$(n,k)*(n',k') = (nI_k(n')), kk'\big), \quad I_g(x) = gxg^{-1}$$

makes the product space $N \times K$ a topological group such that μ becomes an isomorphism.

Group theory has long since prepared for this contingency with the concept of a semidirect product, which we recall in the following.

Let N and H be topological groups and assume that H *acts on N automorphically*; i.e. there is a morphism of groups $\alpha\colon H \to N$ such that

$$(h,n) \mapsto h{\cdot}n \stackrel{\text{def}}{=} \alpha(h)(n)\colon H \times N \to N$$

is continuous.

Exercise E5.26. Show that in these circumstances *the product space* $N \times H$ *is a topological group with respect to the multiplication* $(n,h)(n',h') = (n(h{\cdot}n'), hh')$, *identity* $(1_N, 1_H)$, *and inversion* $(n,h)^{-1} = \big(h^{-1}{\cdot}(n^{-1}), h^{-1}\big)$. □

Definition 5.72. The topological group constructed in E5.26 is called the *semidirect product of* N *with* H and is written $N \rtimes_\alpha H$. □

Exercise E5.27. Prove the following assertion.

Let G *be a topological group,* N *a normal subgroup, and* H *a subgroup such that* $G = NH$ *and* $N \cap H = \{1\}$. *Then* H *acts automorphically on* N *via inner automorphisms:*

$$(h,n) \mapsto h{\cdot}n = hnh^{-1} = I_h(n),\ \ I_g(x) = gxg^{-1},$$

and

$$\mu\colon N \rtimes_I H \to G, \quad \mu(n,h) = nh$$

is a bijective morphism of topological groups. If G *is locally compact and if* N *and* H *are closed and* σ*-compact, i.e. are countable unions of compact subsets, then* μ *is an isomorphism of topological groups. The latter conclusion is certainly true if* G *is a compact group and* N *and* H *are closed.*

[Hint. The verification of the first assertion is straightforward. The last one follows from the Open Mapping Theorem for Locally Compact Groups, Appendix 1, EA1.21.] □

Notably in the case of compact groups, the idea of a semidirect product will have many interesting applications as we shall begin to see in Chapter 6 (Lemma 6.37ff., notably Theorem 6.41).

Notice that for *any* normal subgroup N of a topological group G and any subgroup H the semidirect product $N \rtimes_I H$ and the morphism of topological groups

$$\mu\colon N \rtimes_I H \to G, \quad \mu(n,h) = nh$$

is well defined. If N is a normal subgroup of a topological group G such that there is a subgroup H such that $\mu\colon N \rtimes_I H \to G$ is an isomorphism of topological groups we shall say that N *is a semidirect factor*. Now Theorem 5.71 can be rephrased in the following fashion (with an addition originating from Exercise E5.25).

Corollary 5.73. *Any normal vector subgroup* N *of a linear Lie group* G *such that* G/N *is compact is a semidirect factor* (*and all cofactors are conjugate*). □

We observe that this is nontrivial even in the case that N is central and therefore N is a direct factor. Indeed even if G is abelian and additively written, knowing from Appendix 1, A1.36 that N as a divisible subgroup is a direct summand does not say at all that N is a direct summand in the sense of abelian topological groups, i.e. that there is a subgroup H such that $(n,h) \mapsto n + h\colon N \oplus H \to G$ is an isomorphism of topological groups.

Of course there is more general group theory in the function ψ which we produced in the proof of Theorem 5.71. Let us look at it in an exercise!

Exercise E5.28. Prove the following assertions.

(i) *Let G be a topological group and $\varphi: G \to G$ an endomorphism of topological groups. If $\varphi^2 = \varphi$, then*

$$\mu: \ker\varphi \rtimes_I \operatorname{im}\varphi \to G$$

is an isomorphism of topological groups, i.e. $\ker\varphi$ *is a direct summand.*

(ii) *Let G be a topological group and $\psi: G \to G$ a crossed endomorphism, i.e. assume that is satisfies*

$$(\forall g\, h \in G) \quad \psi(gh) = \psi(h)\big(h^{-1}\psi(g)h\big) = \psi(h)I_h\big(\psi(g)\big).$$

Write $\ker\psi = \psi^{-1}(1)$. *Then*

$$\mu: \operatorname{im}\psi \rtimes_I \ker\psi \to G$$

is an isomorphism of topological groups, i.e. $\operatorname{im}\psi$ *is a semidirect factor.* □

Corollary 5.73 has more applications than meet the eye. Let us state and prove a lemma and a proposition of independent interest.

Lemma 5.74. *Assume that $q: G \to H$ is a quotient map from a locally compact group G onto a compact space $H = \{gS \mid g \in G\}$ for some closed subgroup S. Then there is a compact symmetric identity neighborhood C of G such that $q(C) = H$.*

Proof. Let U be a compact identity neighborhood in G. If V is the interior of U, since q is open, $q(gV)$ is an open subset of H for each $g \in G$. As H is compact and q is surjective, there are finitely many elements $g_1, \ldots, g_n \in G$ such that $H = q(g_1U) \cup \cdots \cup q(g_nU)$. The set $C_1 \stackrel{\text{def}}{=} g_1U \cup \cdots \cup g_nU$ is a compact identity neighborhood of G such that $q(C_1) = H$ and $C \stackrel{\text{def}}{=} C_1 \cup C_1^{-1}$ is a symmetric compact identity neighborhood with $q(C) = H$. □

We shall say that a topological group G is *compactly generated* if G has a compact subspace C such that $G = \langle C \rangle$. By Corollary A4.26
every connected locally compact group is compactly generated. Indeed, every connected topological group is generated by each identity neighborhood.

Proposition 5.75. *Let G be a compactly generated locally compact group and H a closed subgroup such that the factor space $G/H = \{gH \mid g \in G\}$ is compact, then the following conclusions hold.*

(i) *There is a compact symmetric identity neighborhood C of G and a finitely generated subgroup F of H such that $G = FC$.*

(ii) *H is compactly generated.*

Proof. (i) As G is compactly generated, in view of Lemma 5.74, we find a compact symmetric identity neighborhood C of G generating G such that $CH = G$. Let U

denote the interior of C. Since C is compact, so is $C^2 = CC$ and there is a finite subset E of H such that $C^2 \subseteq EU \subseteq EC$. Let F denote the subgroup generated by E. Note $F \subseteq H$. Then

$$C^2 \subseteq FC. \tag{1}$$

We choose the abbreviation $C^n = \underbrace{C \cdots C}_{n \text{ factors}}$ and claim

$$C^{n+1} \subseteq FC. \tag{n}$$

For $n = 1$ this is already established. Assume (n) and note that this implies $C^{n+2} \subseteq FCC \subseteq FFC = FC$ in view of (1). This shows $(n+1)$. By induction, the claim is proved. We obtain

$$G = \bigcup_{n \in \mathbb{N}} C^n \subseteq FC \quad \text{and} \quad F \subseteq H.$$

(ii) If $h \in H$, then $h = fc$ with $f \in F$, $c \in C$, whence $c = hf^{-1} \in H \cap C$. Thus H is generated by $(C \cap H) \cup E$. This completes the proof. □

Exercise E5.29. Prove the "Closed Graph Theorem for Compact Range Spaces".

A function $f\colon X \to Y$ from a Hausdorff space X into a compact Hausdorff space Y is continuous if and only if the graph $\{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

[Hint. Assume f is continuous. Then the graph is the inverse image of the diagonal in $Y \times Y$ under the continuous map $f \times \mathrm{id}_Y$

Assume the graph graph_f is closed. Let $x \in X$ and let $\mathcal{U}$ be the filter of neighborhoods of x. Then $\mathcal{F} \overset{\text{def}}{=} \{\overline{f(U)} \mid U \in \mathcal{U}\}$ is a filter basis in the compact space Y. Let y be any point in its intersection. Let U be a neighborhood of x in X and V a neighborhood of y in Y. Then $V \cap f(U) \neq \emptyset$, i.e. $(U \times V) \cap \mathrm{graph}_f \neq \emptyset$. Since graph_f is closed, $(x, y) \in \mathrm{graph}_f$, i.e. $y = f(x)$. Thus $\bigcap_{U \in \mathcal{U}} \overline{f(U)} = \{f(x)\}$. Apply compactness of Y once more to conclude that for each neighborhood V of $f(x)$ there is a $U \in \mathcal{U}$ such that $f(U) \subseteq V$.] □

The Finite Discrete Center Theorem

Theorem 5.76. *Let G be a connected Hausdorff topological group and Z a discrete* (*hence closed*) *central subgroup such that the following conditions are satisfied:*

(a) *G/Z is compact.*

(b) *$(G/Z)'$, the algebraic commutator subgroup of G/Z, is dense in G/Z.*

Then Z is finite.

Proof. Since Z is discrete, G and G/Z are locally isomorphic; in particular, the group G is locally compact by (a). Since it is connected, it is also compactly generated. From Proposition 5.75(ii) Z is compactly generated. Thus, being discrete, Z is a finitely generated abelian group. By the Fundamental Theorem of Finitely

Generated Abelian Groups, Appendix 1, A1.11, the group Z is of the form is a direct product EZ_1 of a finite abelian subgroup and a free abelian group $Z_1 \cong \mathbb{Z}^n$, $n = 0, 1, \ldots$ We must show $n = 0$.

In order to simplify notation it is no loss of generality to assume, for the purposes of the proof, that $Z \cong \mathbb{Z}^n$. Then there is an algebraic and topological embedding $j: Z \to E$ into the additive group of a euclidean vector space $E = \mathbb{R}^n$ and $E/j(Z) \cong \mathbb{T}^n$. In particular, $E/j(Z)$ is compact.

The direct product $G \times E$ has a discrete subgroup $\Delta = \{(z, -j(z)) \mid z \in Z\}$. The group $\Gamma \stackrel{\rm def}{=} \frac{G\times E}{\Delta}$ contains a subgroup $G^* \stackrel{\rm def}{=} \frac{G\times j(Z)}{\Delta}$ isomorphic to G under $g \mapsto (\{g\} \times j(Z)))\Delta$ and a central subgroup $N \stackrel{\rm def}{=} \frac{Z\times E}{\Delta}$ isomorphic to E under $v \mapsto (Z \times \{v\})\Delta$. Let U be a closed symmetric identity neighborhood of G such that $U^2 \cap Z = \{1\}$ and define $C = \frac{(U\times\{1\})\Delta}{\Delta}$. Then

$$\nu: N \times C \to NC, \quad \nu\big((z, v)\Delta, (u, 0)\Delta\big) = \big(uz, v\big)\Delta,$$

is surjective, continuous and open. Whenever $(u_1 z_1, v_1)\Delta = \nu\big((z_1, v_1)\Delta, (u_1, 0)\Delta\big) = \big((z_2, v_2)\Delta, (u_2, 0)\Delta\big) = \big(u_2 z_2, v_2\big)\Delta$, then $(u_2^{-1}u_1 z_2^{-1} z_1, v_1 - v_2) \in \Delta$, i.e. firstly, $u_2^{-1}u_1 z_2^{-1} z_1 \in Z$, hence $u_2^{-1}u_1 \in U^2 \cap Z = \{1\}$ and thus $u_1 = u_2$, and, secondly $v_1 - v_2 = -j(z_2^{-1}z_1)$, i.e. $v_1 - v_2 = j(z_1) - j(z_2)$, and thus $v_1 - j(z_1) = v_2 - j(z_2)$, i.e. $(z_1, v_1)\Delta = (z_2, v_2)\Delta$. Hence ν is a homeomorphism and N has a tubular neighborhood in Γ.

Further note that $\Gamma/N \cong \frac{G\times E}{Z\times E} = NG^*/N \cong G^*/(N \cap G^*) \cong G/Z$. By hypothesis (a) the factor group G/Z is compact. Hence Γ/N is compact, and thus, by Corollary 5.73, N is a semidirect factor in Γ. Thus Γ contains a compact subgroup $K = S/\Delta \cong \Gamma/N \cong G/Z$, $\Delta \subseteq S \subseteq G \times E$ such that $\Gamma = NK$, semidirectly, i.e. $\Gamma \cong N \rtimes_I K$ and $\Gamma/(N\cap G^*) \cong \frac{N}{N\cap G^*} \rtimes_\iota K$ where $\iota: K \to \mathrm{Aut}\,\big(N/(N\cap G^*)\big)$ is given by $\iota(k)\big(n(N \cap G^*)\big) = knk^{-1}\big(n(N \cap G^*)\big)$. Note that $\frac{N}{N\cap G^*} \cong \frac{Z\times E}{Z\times j(Z)} \cong \frac{E}{j(Z)}$. The image of $G^*/(N \cap G^*)$ in $\frac{N}{N\cap G^*} \rtimes_\iota K$ is the graph of a morphism $\gamma: K \to \frac{N}{N\cap G^*}$. Since $K \cong E/j(Z)$ is compact, the graph of γ compact. Therefore, γ continuous by the Closed Graph Theorem for Compact Range Spaces (E5.29 above). Correspondingly, this gives us a continuous morphism $\gamma': G/Z \to E/j(Z)$. Since the range is abelian, the commutator group $(G/Z)'$ is contained in the kernel $\ker \gamma'$. Since $(G/Z)'$ is dense by hypothesis (b) the morphism γ' is constant. Hence $\gamma: K \to \frac{N}{N\cap G^*}$ is constant. This means that the image of $\frac{G^*}{N\cap G^*}$ in $\frac{N}{N\cap G^*} \rtimes_\iota K$ is $\{1\} \times K$. Hence $G^* = (N \cap G^*)K$. Since G^* is connected, $N \cap G^* \cong Z$ discrete, and $(n, k) \mapsto nk: (N \cap G^*) \times K \to (N \cap G^*)N$ is a homeomorphism, we get $G^* = K$, and thus $Z \cong N \cap G^* = \{1\}$. □

This gives us at once the following result.

Theorem 5.77. *Let G be a connected compact Lie group with dense commutator group. Then the fundamental group $\pi_1(G)$ is finite* (*See Appendix* 2, A2.17ff.)*, and the universal covering group $\widetilde{G}$ of G is compact.*

Proof. Let $\widetilde{p}: \widetilde{G} \to G$ denote the universal covering morphism (A2.21). Then $Z \stackrel{\text{def}}{=} \ker \widetilde{p}$ is isomorphic to $\pi_1(G)$. Theorem 5.76 applies to $\widetilde{G}$ and Z with $G \cong \widetilde{G}/Z$ and yields that Z is finite. As a consequence $\widetilde{G}$ is compact. (See Exercise E5.29 below.) □

Exercise E5.30. Prove the following.

Let G be a topological group and N a compact normal subgroup. If G/N is compact, then G is compact.

[Hint. Consider an open cover $\mathcal{U}$ of G. For each $g \in G$ find finitely many $U_1, \ldots, U_n$ covering Ng. Show that there is an open subset W_g of G such that $W_g N = W_g$ and $W_g \subseteq U_1 \cup \cdots \cup U_n$. Use compactness of G/N to cover G/N with finitely many W_{g_j}/N, $j = 1, \ldots, n$.] □

Exercise E5.31. Show that the 3-dimensional linear Lie group $\mathrm{Sl}(2, \mathbb{R})$ has a fundamental group isomorphic to $\mathbb{Z}$.

[Hint. Verify that $\mathrm{Sl}(2, \mathbb{R})$ is homeomorphic to the product space of $\mathrm{SO}(2) \cong \mathbb{S}^1$ and the space of all matrices $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, $a > 0$, $b \in \mathbb{R}$ which is homeomorphic to $\mathbb{R}^2$.] □

Theorem 5.78 (The Supplement Theorem). *Let G and H be connected linear Lie groups and $q: G \to H$ a quotient morphism of topological groups with kernel N. Assume that $\mathfrak{g}$ contains a subalgebra $\mathfrak{a}$ such that $\mathfrak{g}$ is algebraically and topologically the vector space direct sum $\mathfrak{n} \oplus \mathfrak{a}$.*

(i) *Let S be a simply connected* (*see Appendix* 2) *topological group and $f: S \to H$ a morphism of topological groups. Then there exists a unique morphism $f_{\mathfrak{a}}: S \to G$ of topological groups such that $f = q \circ f_{\mathfrak{a}}$ and that $f(S) = \langle \exp_G \mathfrak{a} \rangle$.*

$$\begin{array}{ccc} S & \xrightarrow{\quad f_{\mathfrak{a}} \quad} & G \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ H & \xrightarrow[\quad \mathrm{id} \quad]{} & H \end{array}$$

(ii) *Assume that H is compact and that H' is dense in H. Then $A \stackrel{\text{def}}{=} \langle \exp_G \mathfrak{a} \rangle$ is a compact Lie subgroup of G and $q|A: A \to H$ is a covering morphism with kernel $N \cap A$.*

(iii) *Under the assumptions of* (ii) *define $\iota: A \to \mathrm{Aut}(N)$ by $\iota(a)(n) = ana^{-1}$. Then the semidirect product $N \rtimes_\iota A$ is defined and the morphism $\mu: N \rtimes_\iota A \to G$, $\mu(n, A) = na$ is surjective, open, and has the discrete kernel*

$$\{(d^{-1}, d) \mid d \in N \cap A\} \cong N \cap A.$$

In particular, $G = NA \cong \frac{N \rtimes_\iota A}{D}$.

Proof. (i) The morphism $\mathfrak{L}(\mathfrak{q}): \mathfrak{g} \to \mathfrak{h}$ has kernel $\mathfrak{n} = \mathfrak{L}(N)$ by 5.50. Since the function q is a quotient morphism, it is open. Since $\exp_G: \mathfrak{g} \to G$ and $\exp_H: \mathfrak{h} \to H$

are local homeomorphisms at zero (5.41(i)(e)) the morphism $\mathfrak{L}(\mathfrak{q})$ of completely normable Lie algebras is open at 0 and is, therefore open and thus a quotient morphism. Since $\mathfrak{a}$ is algebraically and topologically a direct summand, $\mathfrak{L}(q)|\mathfrak{a}$: $\mathfrak{a} \to \mathfrak{h}$ is an isomorphism of completely normable Lie algebras.

Let B be an open convex symmetric neighborhood of 0 in $\mathfrak{g}$ such that the Campbell–Hausdorff multiplication $*: B \times B \to \mathfrak{g}$ is defined and that $\exp_G |B$: $B \to W$ is a homeomorphism onto an identity neighborhood of G. Accordingly, let C be an open convex symmetric neighborhood of 0 in $\mathfrak{h}$ such that firstly, $*$ is defined on $C \times C$, secondly, $\exp_H |C$ implements a homeomorphism of C onto an open identity neighborhood V of H, and thirdly, $\big(\mathfrak{L}(q)|\mathfrak{a}\big)^{-1}(C) \subseteq B$. Now let U be any identity neighborhood of S such that $f(U) \subseteq V$. Define $\varphi: U \to G$ by

$$\varphi(x) = \exp_G(\mathfrak{L}(q)|\mathfrak{a})^{-1}(\exp_H |C)^{-1} f(x), \quad x \in U.$$

If x, y, $xy \in U$ we set

$$\begin{aligned} X &= (\exp_H |C)^{-1} f(x), \quad Y = (\exp_H |C)^{-1} f(y), \text{ and} \\ X' &= (\mathfrak{L}(q)|\mathfrak{a})^{-1}(X), \quad Y' = (\mathfrak{L}(q)|\mathfrak{a})^{-1}(Y). \end{aligned}$$

Then $\exp_H(X * Y) = f(x)f(y) = f(xy)$, $\exp_G(X' * Y') = \varphi(x)\varphi(y)$. From the commuting of the diagrams

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathfrak{L}(q)} & \mathfrak{h} \\ {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[q]{} & H \end{array} \quad \text{restricting to} \quad \begin{array}{ccc} B & \xrightarrow{\mathfrak{L}(q)|B} & C \\ {\scriptstyle \exp_G |B}\big\downarrow & & \big\downarrow{\scriptstyle \exp_H |C} \\ W & \xrightarrow[q|W]{} & V \end{array}$$

we conclude $\mathfrak{L}(q)(X' * Y') = X * Y$ and thus $(\mathfrak{L}(q)|\mathfrak{a})^{-1}(X * Y) = X' * Y'$. Hence $\varphi(xy) = \exp_G(X' * Y')$. Thus $\varphi(xy) = \varphi(x)\varphi(y)$. Now the Extension Theorem A2.26, A2.27 applies and yields a unique extension $f_\mathfrak{a}: S \to G$ such that for $x \in U$ we have $q\big(f_\mathfrak{a}(x)\big) = q\big(\exp(\mathfrak{L}(q)|\mathfrak{a})^{-1}(\exp_H |C)^{-1} f(x) = \exp_H\big((\exp_H |C)^{-1} f(x)\big)$ $= f(x)$. Since the identity neighborhood U generates the connected group S, we conclude $q \circ f_\mathfrak{a} = f$.

Furthermore, $C' \stackrel{\text{def}}{=} (\mathfrak{L}(q)|\mathfrak{a})^{-1}(C)$ is a zero neighborhood in $\mathfrak{a}$ and $f_\mathfrak{a}(U) = \varphi(U) = \exp_G C'$. Therefore $f_\mathfrak{a}(S) = \langle \exp_H(C') \rangle = \langle \exp \mathfrak{a} \rangle$ (since $\exp \mathfrak{a} \subseteq \langle \exp C' \rangle$). This completes the proof of (i).

(ii) We apply (i) to the universal covering morphism $f: S \to H$ (see Appendix 2). Under the assumptions of (ii), the universal covering group S is compact by 5.77. Then $A = \langle \exp_G \mathfrak{a} \rangle = f_\mathfrak{a}(S)$ is compact as a continuous image of S. As a compact subgroup of a linear Lie group, A is a linear Lie group by 5.33. Since f implements a local isomorphism by 5.68, the function $\varphi = (\exp_G |B) \circ (\mathfrak{L}(q)|B)^{-1} \circ (\exp_H |C) \circ f|U: U \to A$ is a local homeomorphism at 1. Hence the corestriction $f_\mathfrak{a}: S \to A$ which extends φ is a local homeomorphism at 1. Then $f = (q|A) \circ f_\mathfrak{a}$ shows that $q|A: A \to H$ is a local homeomorphism and therefore has a discrete kernel. Thus it is a covering morphism by A2.3(iii). Since $N = \ker q$ we have $\ker q|A = N \cap A$.

(iii) The function $(a, n) \mapsto \iota(a)(n) = ana^{-1}: A \times N \to N$ is continuous. Thus the semidirect product is defined. It is straightforward to see that μ is a morphism which has the asserted kernel. It remains to observe that μ is surjective and open. Since H is connected, openness suffices. Now $\gamma: B_{\mathfrak{n}} \times B_{\mathfrak{a}} \to G$, $\gamma(X, Y) = X * Y$ for sufficiently small zero neighborhoods of $\mathfrak{n}$ and $\mathfrak{a}$, respectively, has derivative $\gamma'(0,0)$ given by $\gamma'(0,0)(X,Y) = X + Y$, and $g = \mathfrak{n} \oplus \mathfrak{a}$ is a vector space direct sum, algebraically and topologically. Hence $\gamma'(0,0)$ is an isomorphism of completely normable vector spaces and by the Theorem of the Local Inverse, γ is a local homeomorphism at $(0,0)$. For $(X,Y) \in B_{\mathfrak{n}} \times B_{\mathfrak{a}}$ we note $\mu(\exp_N X, \exp_A Y) = \exp_G(X * Y)$ and recall $\exp_N = \exp_G |\mathfrak{n}$, $\exp_A = \exp_G |\mathfrak{a}$. Since the exponential functions are local homeomorphisms at zero it follows that μ is a local homeomorphism at $(1,1)$ and since a morphism is open if it is open at the identity, this suffices. □

We note that in the case that $\mathfrak{g}$ is finite dimensional, for a subalgebra $\mathfrak{a}$ of $\mathbb{G}$ to satisfy the hypotheses of Theorem 5.78 it suffices that $\mathfrak{n} \cap \mathfrak{a} = \{0\}$ and $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}$. The extension $f_{\mathfrak{a}}$ of φ could have been obtained by invoking the Lifting Homomorphisms Theorem II of Appendix 2, A2.33 instead of directly calling on A2.26.

We shall show later that in the circumstances of 5.78(ii) we even have $A = \exp \mathfrak{a}$ (see 6.30(26)).

Postscript

There are many texts on the theory of Lie groups and Lie algebras; the most encyclopedic is the work of Bourbaki, extending over almost two decades; its untimely termination came before the travail was really complete.

However, Bourbaki is not a text for students; it builds on much of what Bourbaki has accumulated in his other volumes and aspires to the greatest possible generality.

We choose a different approach, and indeed an approach which is different from other texts. (The closest is a textbook of 1991 in German by J. Hilgert and K.-H. Neeb [153]; but these authors abandoned this approach in their later book of 2012 in English [154] in favor of a more comprehensive approach.) The approach here has been advocated by K. H. Hofmann for some time [164, 165, 166].

The present approach has two principal characteristics. Firstly, it emphasizes *linear* Lie Groups (cf. also [111]). This is perfectly sufficient for the theory of compact groups, since all compact Lie groups are linear. Secondly, much in line with the philosophy in the sources [164, 165, 166], it focuses on the exponential function rather than the analytic structure (which in passing is introduced, too). This works extremely well in the context of linear Lie groups based on the exponential function of Banach algebras with identity.

It is perhaps noteworthy that we get by in this chapter without squeezing the theory of the Campbell–Hausdorff series down to the point at which it yields that all homogeneous summands $H_n(x, y)$ are Lie polynomials which requires considerable additional machinery, either by a more systematic build-up or by invoking technical trickery.

As long as possible we retain a level of generality which requires only in isolated points (exemplified by certain results involving information on endomorphisms of finite dimensional vector spaces such as 5.45 or by the Commutator Subgroup Theorem 5.62) that the Lie groups treated are finite dimensional. All examples, which interest us, are.

In order to attach a Lie algebra $\mathfrak{L}(G)$ directly to a topological group G we consider it as the space $\mathrm{Hom}(\mathbb{R}, \cdot)$ of one parameter subgroups. This secures automatically relevant functorial properties. If G is a linear Lie group we endow this space with an addition and a Lie bracket.

A test for any presentation of Lie theory is the problem of associating with a given subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{L}(G)$ of a group G a subgroup of G, because often there is no closed subgroup H of G having $\mathfrak{h}$ as its Lie algebra $\mathfrak{L}(H)$. Our approach achieves this association without needing the machinery of immersed manifolds or the integration of distributions on manifolds. We take for H the subgroup $\langle \exp \mathfrak{h} \rangle$ generated by the image of $\mathfrak{h}$ under the exponential function and endow it with enough additional structure in an elementary fashion to allow us to recover $\mathfrak{h}$ from H.

We have seen that a quotient group of a linear Lie group (even modulo a discrete normal subgroup) is not necessarily a linear Lie group. We illustrated that in some detail by the three dimensional Heisenberg group, a quotient of which yields the classical example of a Lie group without faithful linear representation (5.67). Our discussion incorporates a famous elementary but highly elegant proof due to Wielandt that the operators P and Q in the quantum mechanics of one free particle in one dimension cannot be bounded [374]. The fact that the class of linear Lie groups is not closed under the formation of quotient groups does not concern us in the context of compact groups. We shall show in the next chapter that the class of compact Lie groups is closed under the formation of quotients (cf. 6.7 below).

The section on the Topological Splitting of Vector Subgroups (5.69ff.) produces a global topological cross section for a vector subgroup 5.70 (which is not easily accessible in the literature). The main application is the Vector Subgroup Splitting Theorem 5.71. This is to be found in Bourbaki [38], p. 74 and elsewhere. Our proof differs from Bourbaki's because Bourbaki gets by without a topological cross section and pays with extra technical complications. Our proof makes the Averaging Operator more explicit and follows a general scheme proposed in [195]. The access from this circle of ideas to the finiteness of the fundamental group of a connected compact Lie group with dense commutator group 5.77 is pioneered by Bourbaki [38], p. 78.

It is consistent with the general direction of this book that many results we present on the structure of linear Lie groups involve compact groups without always being directly theorems on compact groups. Representatives are van der Waerden's Continuity Theorem 5.64 and its corollaries, the Vector Subgroup Splitting Theorem 5.71, the Finite Discrete Theorem 5.76, and the Supplement Theorem 5.78. Corollary 5.66 persists for compact connected groups which are not necessarily Lie groups [344].

This book is about compact groups, and all compact Lie groups are linear Lie groups. By restricting our attention to *linear* Lie groups we have been able to proceed with minimum background, namely, about Banach algebras. It would require some extra effort to discuss Lie groups in general. But it is not a giant step in the spirit of our presentation as outlined below.

A *real Lie group* is a topological group G for which there is a completely normable Lie algebra $\mathfrak{g}$, an open ball B around zero in $\mathfrak{g}$ such that $B * B$ is defined, and a homeomorphism $e\colon B \to V$ onto an open identity neighborhood of G such that $e(X * Y) = e(X)e(Y)$ for $X, Y \in B$. Each topological group which is locally isomorphic to a linear Lie group clearly is a Lie group. On each Lie group there is a unique exponential function $\exp\colon \mathfrak{g} \to G$ extending e and classifying the one parameter subgroups as the functions $t \mapsto \exp t{\cdot}X$, $X \in \mathfrak{g}$. The ideas expressed in Corollary 5.34 through Exercise E5.11 and the subsequent comment then verify that every such group is in fact analytic. Conversely, one can show that every analytic finite dimensional group is locally isomorphic to a linear Lie group. In contrast with the situation in linear Lie groups, the class of Lie groups is closed under the formation of quotients. We shall prove this for compact Lie groups in the next chapter, and the proof is similar for the general Lie group case.

References for this Chapter—Additional Reading

[4], [28], [34], [38], [39], [36], [41], [43], [44], [58], [87], [88], [94], [101], [111], [134], [141], [153], [154], [155], [164], [165], [166], [195], [229], [237], [242], [263], [282], [299], [307], [308], [309], [331], [344], [353], [354], [360], [366], [374].

Chapter 6

Compact Lie Groups

This chapter is devoted to an exposition of the structure of compact Lie groups, which were defined in 2.41. We apply our knowledge of linear Lie groups, which were defined in 5.32, to the special case of compact Lie groups. From 2.40 and 2.41 we recall that every compact Lie group is a compact subgroup of the multiplicative group of some Banach algebra. Proposition 5.33 tells us that every compact Lie group is a linear Lie group. Thus the entire machinery of linear Lie groups prepared in Chapter 5 is available for the investigation of the structure of compact Lie groups.

In Chapter 5 we saw that the algebraic commutator subgroup of a Lie group G, obtained as a quotient of a connected linear Lie group modulo a discrete central subgroup, is not necessarily closed. This obstruction is shown to vanish for compact Lie groups G, where the commutator subgroup is closed (irrespective of the connectedness of G). Indeed it is remarkable that every element of the commutator subgroup of a connected compact Lie group is a commutator. Among the structural results we establish is the classic which says that a connected compact Lie group is almost the direct product of the identity component of its center and its commutator subgroup and the less classical result that the commutator subgroup of a connected compact Lie group is a semidirect factor. The second uses basic facts on maximal tori of a compact Lie group and their Lie algebras which we develop fully. This is at the heart of a study of the root space decomposition of the Lie algebra of a compact Lie group and of the automorphism group of a compact Lie group. We also use them to show that every connected compact Lie group contains a dense subgroup generated by 2 elements. Our techniques for the root space decomposition of the Lie algebra of a compact Lie group largely avoid complexification but rather use complex structures derived from the adjoint representation. The preliminaries were derived in Chapter 3. Maximal tori also play a significant role in our discussion of the cohomological structure of a connected compact Lie group. Clearly, one of the lead motives in the study of connected compact Lie groups is that of the maximal torus subgroups.

Prerequisites. In its main body this chapter demands no prerequisites beyond those in the previous chapters. However, in an exercise near the end of the chapter we refer to some literature on the Hausdorff–Banach–Tarski Paradox for the verification that two rotations of euclidean three space which we specifically cite generate a free subgroup of SO(3), and in another exercise we make reference to measure theory on manifolds. In the last section on the cohomology of compact Lie groups basic cohomology theory is required; this is inherent even in the formulation of the main results. At the end of that section the information from algebraic topology needed is more sophisticated.

Compact Lie Algebras

We must begin this section by introducing a terrible misnomer. Unfortunately it has become current in the subject area.

Definition 6.1. A *compact Lie algebra* is a real Lie algebra which is isomorphic to the Lie algebra of a compact Lie group. □

Of course, a compact Lie algebra is not a compact topological space. As the Lie algebra of a linear Lie group it has a natural topology, but unless it is singleton (in which case the Lie group whose Lie algebra it is must be finite), this topology is never a compact one.

Proposition 6.2. *The Lie algebra* $\mathfrak{g}$ *of a compact Lie group* G *is finite dimensional and supports a scalar product* $(\bullet \mid \bullet)$ *such that*

$$(1) \qquad (\mathrm{Ad}(g)X \mid \mathrm{Ad}(g)Y) = (X \mid Y) \quad \textit{for all} \quad X, Y \in \mathfrak{g},\ g \in G,$$

$$(2) \qquad ([X,Y] \mid Z) = (X \mid [Y,Z]) \quad \textit{for all} \quad X, Y, Z \in \mathfrak{g}.$$

Proof. Since $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, by Theorem 5.41(ii), the topological vector space $\mathfrak{g}$ is locally homeomorphic to G. Hence it is locally compact and thus finite dimensional. In particular, it is a finite dimensional real Hilbert space. By Theorem 5.44, $\mathfrak{g}$ is a G-module with respect to the adjoint representation, that is with a module action given by $gX = \mathrm{Ad}(g)(X)$. By Weyl's Trick 2.10, $\mathfrak{g}$ is an orthogonal G-module, that is there is a scalar product $(\bullet \mid \bullet)$ such that (1) holds.

Now let X, Y, and Z be arbitrary members of $\mathfrak{g}$. Consider the analytic function $f: \mathbb{R} \to \mathbb{R}$ given by $f(t) = \big(\mathrm{Ad}(\exp t{\cdot}Y)(X) \mid \mathrm{Ad}(\exp t{\cdot}Y)(Z)\big)$. Note $f(t) = (e^{t\cdot \mathrm{ad}\, Y}X \mid e^{t\cdot \mathrm{ad}\, Y}Z)$ by Theorem 5.44. In view of (1) this function is constant. Hence its derivative vanishes: $0 = \big(e^{t\cdot \mathrm{ad}\, Y}\,\mathrm{ad}(Y)(X) \mid e^{t\cdot \mathrm{ad}\, Y}Z\big) + \big(e^{t\cdot \mathrm{ad}\, Y}X \mid e^{t\cdot \mathrm{ad}\, Y}\,\mathrm{ad}(Y)(Z)\big)$. Setting $t = 0$ we obtain $0 = ([Y,X] \mid Z) + (X \mid [Y,Z])$. This proves (2). □

Exercise E6.1. Let G denote a linear Lie group with Lie algebra $\mathfrak{g}$, and let $F: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote a symmetric bilinear form. Then the condition

$$(1') \qquad F(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = F(X,Y) \quad \text{for all} \quad X, Y \in \mathfrak{g},\ g \in G$$

implies condition

$$(2') \qquad F([X,Y], Z) = F(X, [Y,Z]) \quad \text{for all} \quad X, Y, Z \in \mathfrak{g}.$$

If G is connected, then both conditions are equivalent.

The form F defined by $F(X,Y) = \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$ satisfies $(2')$.

[Hint. The proof of $(1')\Rightarrow(2')$ is a straightforward generalization of the proof of $(1)\Rightarrow(2)$ in 6.2. For the converse consider X, Y, $Z \in \mathfrak{g}$ and define $f: \mathbb{R} \to \mathbb{R}$ by $f(t) = F(\operatorname{Ad}(\exp t{\cdot}Y)X, \operatorname{Ad}(\exp t{\cdot}Y)Z)$. Verify with the aid of $(2')$ that $f'(t) = 0$ for all $t \in \mathbb{R}$. Conclude that $f(t) = F(X,Y)$ for all t. Now note that $(1')$ holds for all $g \in \exp(\mathfrak{g})$. Invoke 5.41(iii) to prove the assertion.

The last assertion is verified straightforwardly.] □

Bilinear maps satisfying $(2')$ are called *invariant*. An example familiar from elementary linear algebra is the ordinary euclidean scalar product on $\mathbb{R}^3$ which is invariant with respect to the "vector product" $(x,y) \mapsto x \times y$ which endows $\mathbb{R}^3$ with a Lie algebra structure isomorphic to that of so(3).

If $\mathcal{H}$ is a Hilbert space with a scalar product $(\cdot \mid \cdot)$ then a continuous linear operator $T: \mathcal{H} \to \mathcal{H}$ is called *skew symmetric* if $\mathbb{K} = \mathbb{R}$, respectively, *skew hermitian* if $\mathbb{K} = \mathbb{C}$, if $(Tx \mid y) = -(x \mid Ty)$ for all x, $y \in \mathcal{H}$; i.e. if $T^* = -T$. If $T^* = -T$ and E is a T-invariant closed subspace, then $E^\perp$ is also T-invariant: Indeed, $x \in E^\perp$ iff $(x \mid y) = 0$ for all $y \in E$ and thus $(Tx \mid y) = -(x \mid Ty) = 0$ for all $y \in E$ if $TE \subseteq E$. Thus T is semisimple (in the sense of the remarks made preceding E5.18).

Definition 6.3. (i) A *Hilbert Lie algebra* is a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ with a continuous Lie bracket $(x,y) \mapsto [x,y]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a positive definite inner product $(\bullet \mid \bullet)$ such that $\mathfrak{g}$ is a Hilbert space with respect to it and that (2) is satisfied. Equivalently, a Lie algebra $\mathfrak{g}$ which is also a Hilbert space is a Hilbert Lie-algebra if $(\operatorname{ad} X)^* = -\operatorname{ad} X$ for all of its elements X. Since the ground field is real this says that every $\operatorname{ad} X$ is skew symmetric.

(ii) An automorphism of a Hilbert Lie algebra $\mathfrak{g}$ is an automorphism of the Lie algebra and an isometry relative to the inner product. Thus the group of automorphisms of $\mathfrak{g}$ is $\operatorname{Aut} \mathfrak{g} \cap \mathrm{U}(\mathfrak{g})$ (cf. 1.6(ii)). Which we shall write $\operatorname{Aut} \mathfrak{g} \cap \mathrm{O}(\mathfrak{g})$.

(iii) A Lie algebra will be called *semisimple* if it has no nondegenerate abelian ideals. □

In any finite dimensional Lie algebra $\mathfrak{g}$, the Lie bracket is continuous. We also note that for a Hilbert Lie algebra every $\operatorname{ad} X$, being skew symmetric is semisimple, and every one of its automorphisms, being orthogonal, is semisimple, too.

We can rephrase Proposition 6.2 as saying that *every compact Lie algebra is a finite dimensional real Hilbert Lie algebra.* We shall see shortly, that every finite dimensional real Hilbert Lie algebra is a compact Lie algebra (see 6.6 below).

A Lie algebra $\mathfrak{g}$ is called *simple* if its only ideals are $\{0\}$ and $\mathfrak{g}$, and if $\dim \mathfrak{g} > 1$. As announced in Definition 5.56 we shall abbreviate the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ by $\mathfrak{g}'$.

Theorem 6.4. *Assume that $\mathfrak{g}$ is a Hilbert Lie algebra and $\mathfrak{z}$ its center. Then the following conclusions hold:*

(i) *If* $\mathfrak{i} \trianglelefteq \mathfrak{g}$, *then the orthogonal complement* $\mathfrak{i}^\perp$ *is a closed ideal such that* $[\mathfrak{i}, \mathfrak{i}^\perp] = \{0\}$, *and if* $\mathfrak{i}$ *is closed (which is automatically the case if* $\dim \mathfrak{i} < \infty$*), then* $\mathfrak{g}$ *is the orthogonal direct sum* $\mathfrak{i} \oplus \mathfrak{i}^\perp$.

(ii) *If* $\mathfrak{i}, \mathfrak{j} \trianglelefteq \mathfrak{g}$, $\overline{[\mathfrak{j}, \mathfrak{j}]} = \mathfrak{j}$, *and* $[\mathfrak{i}, \mathfrak{j}] = \{0\}$ *(e.g. if* $\mathfrak{i} \cap \mathfrak{j} = \{0\}$*), then* $(\mathfrak{i} \mid \mathfrak{j}) = \{0\}$; *that is if two ideals annihilate each other and one has a dense commutator algebra, they are orthogonal.*

(iii) *If* $\mathfrak{k} \trianglelefteq \mathfrak{i} \trianglelefteq \mathfrak{g}$ *and* $\mathfrak{i}$ *or* $\mathfrak{k}$ *is closed (which is always satisfied in the finite dimensional case) then* $\mathfrak{k} \trianglelefteq \mathfrak{g}$.

(iv) *If* $\mathfrak{i} \trianglelefteq \mathfrak{g}$ *is abelian, then* $\mathfrak{i} \subseteq \mathfrak{z}$. *In particular, a one-dimensional vector subspace is an ideal if and only if it is contained in* $\mathfrak{z}$. *Also,* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{z} = \{0\}$, *i.e. if it is centerfree.*

(v) *The orthogonal complement* $[\mathfrak{g}, \mathfrak{g}]^\perp$ *of the commutator algebra is the center* $\mathfrak{z}$. *In particular,* $\mathfrak{g}$ *is the orthogonal direct sum of* $\mathfrak{z}$ *and the closure of the commutator algebra* $\overline{[\mathfrak{g}, \mathfrak{g}]}$. *If* $\mathfrak{g}$ *is finite dimensional, then* $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$. *The algebra* $\mathfrak{g}$ *is semisimple if and only if it has a dense commutator algebra.*

(vi) *If* $\mathfrak{g}$ *is finite dimensional, then* $\mathfrak{g}$ *is the orthogonal direct sum of the center* $\mathfrak{z}$ *and a unique family* $\mathcal{S}$ *of simple ideals* $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ *and* $\mathfrak{g}' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$.

(vii) *If* $\mathfrak{g}$ *is finite dimensional and* $\mathfrak{i} \trianglelefteq \mathfrak{g}$ *then* $\mathfrak{i}$ *is an orthogonal direct sum of a vector subspace of* $\mathfrak{z}$ *and a unique subfamily* $\mathfrak{s}_{m_1}, \ldots, \mathfrak{s}_{m_k}$ *of* $\mathcal{S}$.

(viii) *If* $\mathfrak{g}$ *is finite dimensional, then* $\mathfrak{g}'' = \mathfrak{g}'$.

(ix) *If* $\mathfrak{g}$ *is finite dimensional and semisimple, there is an invariant inner product* $\langle \cdot \mid \cdot \rangle$ *given by*

$$(\forall X, Y \in \mathfrak{g}) \quad \langle X \mid Y \rangle = -\operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y).$$

On an arbitrary finite dimensional Hilbert Lie algebra $\mathfrak{g}$ *one finds invariant inner products* $(\cdot \mid \cdot)$ *by prescribing an arbitrary inner product* $(\cdot \mid \cdot)_\mathfrak{z}$ *on* $\mathfrak{z}(\mathfrak{g})$ *and defining* $(X \mid Y) = (X_\mathfrak{z} \mid Y_\mathfrak{z})_\mathfrak{z} - \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$, *where* $X \mapsto X_\mathfrak{z}$ *denotes the projection of* $\mathfrak{g}$ *onto* $\mathfrak{z}(\mathfrak{g})$ *with* $\mathfrak{g}'$ *as kernel.*

(x) *If* $\mathfrak{g}$ *is finite dimensional and semisimple, then* $\mathfrak{g}$ *has an invariant inner product such that all Lie algebra automorphisms are Hilbert Lie algebra automorphisms, i.e. are automatically unitary (respectively, orthogonal) relative to this inner product. Such an inner product is given by* $(X \mid Y) = -\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$.

Proof. (i) Let $y \in \mathfrak{g}$ and $x \in \mathfrak{i}^\perp$. Then for any $z \in \mathfrak{i}$ we have $[y, z] \in \mathfrak{i}$ and thus $([x, y] \mid z) = (x \mid [y, z]) = 0$. Hence $[x, y] \in \mathfrak{i}^\perp$, and $\mathfrak{i}^\perp$ is an ideal. As a consequence, $[\mathfrak{i}, \mathfrak{i}^\perp] \subseteq \mathfrak{i} \cap \mathfrak{i}^\perp = \{0\}$.

(ii) If $\mathfrak{i}, \mathfrak{j} \trianglelefteq \mathfrak{g}$ then $\mathfrak{i} \cap \mathfrak{j} = \{0\}$ implies $[\mathfrak{i}, \mathfrak{j}] \subseteq \mathfrak{i} \cap \mathfrak{j} = \{0\}$. If $[\mathfrak{i}, \mathfrak{j}] = \{0\}$ while $[\mathfrak{j}, \mathfrak{j}]$ is dense in $\mathfrak{j}$, then $\{0\} = (\{0\} \mid \mathfrak{j}) = ([\mathfrak{i}, \mathfrak{j}] \mid \mathfrak{j}) = (\mathfrak{i} \mid [\mathfrak{j}, \mathfrak{j}])$ is dense in $(\mathfrak{i} \mid \mathfrak{j})$. Thus $\mathfrak{i}$ and $\mathfrak{j}$ are orthogonal.

(iii) If $\mathfrak{k} \trianglelefteq \mathfrak{i}$, then $[\mathfrak{k}, \mathfrak{i}^\perp] \subseteq [\mathfrak{i}, \mathfrak{i}^\perp] = \{0\}$, and thus $[\mathfrak{k}, \mathfrak{i} \oplus \mathfrak{i}^\perp] \subseteq \mathfrak{k} + \{0\} = \mathfrak{k}$. If $\mathfrak{i}$ is closed, then $\mathfrak{i} + \mathfrak{i}^\perp = \mathfrak{g}$ by (i), and therefore $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. If $\mathfrak{k}$ is closed then the density of $\mathfrak{i} + \mathfrak{i}^\perp$ in $\mathfrak{g}$ and the continuity of the Lie bracket allows us to conclude $\mathfrak{k} \trianglelefteq \mathfrak{g}$.

(iv) If $[\mathfrak{i}, \mathfrak{i}] = \{0\}$ then $[\mathfrak{i}, \mathfrak{g}] = [\mathfrak{i}, \bar{\mathfrak{i}} \oplus \mathfrak{i}^\perp] = [\mathfrak{i}, \bar{\mathfrak{i}}] + [\mathfrak{i}, \mathfrak{i}^\perp] \subseteq \overline{[\mathfrak{i}, \mathfrak{i}]} = \{0\}$. If $\mathfrak{i}$ is a one-dimensional vector subspace, then $\mathfrak{i} = \mathbb{K}{\cdot}x$ and $[\mathfrak{i}, \mathfrak{i}] = \mathbb{K}{\cdot}[x, x] = \{0\}$, so the preceding applies. By 6.1(iii) $\mathfrak{g}$ is semisimple iff it has no nondegenerate ideals and thus, in the present case, iff $\mathfrak{z} = \{0\}$.

(v) If $x \in [\mathfrak{g}, \mathfrak{g}]^\perp$, then $(x \mid [y,z]) = 0$ for all $y, z \in \mathfrak{g}$. But then $([x,y] \mid z) = 0$ for all $y, z \in \mathfrak{g}$ and this means $[x,y] = 0$ for all $y \in \mathfrak{g}$ which is saying $x \in \mathfrak{z}$. Tracing this argument backwards and using the continuity of the Lie bracket we also see that any central element is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$. The remainder is immediate.

(vi) Assume that $\mathfrak{g}$ is finite dimensional. We can invoke (i) and (iii) to write $\mathfrak{g}$ as an orthogonal direct sum $\mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_p$ of ideals such that $\mathfrak{i}_j$ does not have any ideals other than $\{0\}$ and $\mathfrak{i}_j$. If $\mathfrak{i}_j$ is one-dimensional then $\mathfrak{i}_j \subseteq \mathfrak{z}$ by (iv). If $\dim \mathfrak{i}_j > 1$, then i_j is simple, since it cannot contain any nontrivial ideal by (iii). Hence $\mathfrak{i}_j = [\mathfrak{i}_j, \mathfrak{i}_j]$ by (v). In particular, $\mathfrak{i}_j \subseteq [\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{c}$ denotes the sum of all the one-dimensional ideals among the $\mathfrak{i}_j$ and $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_N$ the orthogonal sum of all simple ideals among them, then $\mathfrak{c} \subseteq \mathfrak{z}$ and $\mathfrak{s} \subseteq [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{s} \subseteq \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. It follows that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c} = \mathfrak{z}$. The uniqueness of the set $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_n\}$ of simple ideals follows from (vii) below.

(vii) Let $\mathfrak{i}$ denote any ideal of $\mathfrak{g}$. Then $\mathfrak{i}$ is the orthogonal direct sum of its center $\mathfrak{z}(\mathfrak{i})$ and its commutator algebra $[\mathfrak{i}, \mathfrak{i}]$ by (v). Then (iii) and (iv) imply $\mathfrak{z}(\mathfrak{i}) = \mathfrak{z} \cap \mathfrak{i}$, and $[\mathfrak{i}, \mathfrak{i}] \subseteq [\mathfrak{g}, \mathfrak{g}]$. Since the orthogonal projection of $\mathfrak{g}$ onto any ideal is a morphism of Lie algebras by (i), the projection of $\mathfrak{i}$ into any simple ideal $\mathfrak{s}$ is an ideal, hence is zero or all of $\mathfrak{s}$. Assume that $\mathfrak{i}$ projects onto $\mathfrak{s}$. Now $\mathfrak{s} \cap \mathfrak{i}$ is an ideal of $\mathfrak{s}$. If it were $\{0\}$ then $\mathfrak{s} \perp \mathfrak{i}$ by (ii) as $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$. But then $\mathfrak{i}$ could not map onto $\mathfrak{s}$ under the orthogonal projection. Hence $\mathfrak{s} \cap \mathfrak{i} = \mathfrak{s}$, that is $\mathfrak{s} \subseteq \mathfrak{i}$. Thus $\mathfrak{i}$ contains all simple ideals onto which it projects orthogonally. In particular, the orthogonal projection $\mathfrak{p}_j$ of $\mathfrak{i}$ into $\mathfrak{s}_j$ is $\mathfrak{s}_j \cap \mathfrak{i}$. Thus $\big(\mathfrak{i} \cap \mathfrak{z}(\mathfrak{g})\big) \oplus (\mathfrak{i} \cap \mathfrak{s}_1) \cdots \oplus (\mathfrak{i} \cap \mathfrak{s}_n) \subseteq \mathfrak{i} = \mathfrak{z}(\mathfrak{i}) \oplus [\mathfrak{i}, \mathfrak{i}] \subseteq \big(\mathfrak{i} \cap \mathfrak{z}(\mathfrak{g})\big) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n$. The relations $\mathfrak{i} \cap \mathfrak{s}_k = \mathfrak{p}_k$ now imply equality throughout, and the assertion is proved.

(viii) By (vi) we have $\mathfrak{g}' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ with simple ideals $\mathfrak{s}_j$ which satisfy $\mathfrak{s}_j' = \mathfrak{s}_j$ by simplicity. Since $[\mathfrak{s}_j, \mathfrak{s}_k] = \{0\}$ for $j \neq k$ we have $\mathfrak{g}'' = \mathfrak{s}_1' \oplus \cdots \oplus \mathfrak{s}_n' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n = \mathfrak{g}'$.

(ix) Notice first, that on any finite dimensional Lie algebra $\mathfrak{g}$, the function $(X, Y) \mapsto \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is bilinear and invariant; indeed $\operatorname{tr} \operatorname{ad}[X,Y] \operatorname{ad} Z = \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y \operatorname{ad} Z - \operatorname{tr} \operatorname{ad} Y \operatorname{ad} X \operatorname{ad} Z = \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y \operatorname{ad} Z - \operatorname{tr} \operatorname{ad} X \operatorname{ad} Z \operatorname{ad} Y = \operatorname{tr} \operatorname{ad} X \operatorname{ad}[Y,Z]$ as $\operatorname{tr} \varphi\psi = \operatorname{tr} \psi\varphi$. By (vi) above, $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ with simple ideals $\mathfrak{s}_j$. Therefore, if on each $\mathfrak{s}_j$ the function $(X, Y) \mapsto -\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$ is an inner product, the assertion follows. Thus without losing generality we assume that $\mathfrak{g}$ is simple and show that $\operatorname{tr} \operatorname{ad} X \operatorname{ad} X < 0$ for all nonzero $X \in \mathfrak{g}$. Since $\mathfrak{g}$ has an invariant inner product $(\cdot \mid \cdot)$, the vector space endomorphism $\operatorname{ad} X$ satisfies $(\operatorname{ad} X)^* = -\operatorname{ad} X$ with the adjoint operator with respect to $(\cdot \mid \cdot)$. Hence its eigenvalues $\lambda_1, \ldots, \lambda_n$ (not all necessarily distinct) are purely imaginary. Then $\operatorname{tr} \operatorname{ad} X \operatorname{ad} X = \sum_{j=1}^{n} \lambda_j^2 < 0$ if at least one of the λ_j is nonzero; since $\operatorname{ad} X$ is semisimple, this holds if $\operatorname{ad} X \neq 0$. Now $\operatorname{ad} X = 0$ means that $[X, \mathfrak{g}] = \{0\}$, i.e. $X \in \mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and so this is tantamount to $X = 0$. Therefore $\operatorname{tr} \operatorname{ad} X \operatorname{ad} X < 0$ iff $X \neq 0$, which proves the claim.

If $\mathfrak{g}$ is an arbitrary finite dimensional Hilbert Lie algebra, then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$ by (v) above. If $X \in \mathfrak{g}$ and $X \mapsto X_{\mathfrak{g}'}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{g}'$ with $\mathfrak{z}(\mathfrak{g})$ as kernel, then $\operatorname{ad} X = \operatorname{ad} X_{\mathfrak{g}'}$, and thus $\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y = \operatorname{tr} \operatorname{ad} X_{\mathfrak{g}'} \operatorname{ad} Y_{\mathfrak{g}'}$. Furthermore, $\mathfrak{g}'' = \mathfrak{g}'$ by (viii), whence $\mathfrak{g}'$ is semisimple. On an abelian Lie algebra, every bilinear form is trivially invariant. Thus the remainder of (ix) follows from (ix).

(x) By (ix) we may assume that $(X \mid Y) = -\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$. Let α be an endomorphism of the Lie algebra $\mathfrak{g}$. Then $\alpha[X,Y] = [\alpha X, \alpha Y]$, i.e. $\alpha \circ \operatorname{ad} X = \operatorname{ad}(\alpha X) \circ \alpha$. If α is invertible, we conclude $\operatorname{tr} \operatorname{ad}(\alpha X) \operatorname{ad}(\alpha Y) = \operatorname{tr}(\alpha(\operatorname{ad} X)\alpha^{-1}\alpha(\operatorname{ad} Y)\alpha^{-1} = \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$. This proves the claim. □

We now have (up to the classification of simple real finite dimensional Hilbert Lie algebras) a complete structure theory for finite dimensional real Hilbert Lie algebras and turn towards a proof of the fact that every such Lie algebra is the Lie algebra of a compact Lie group. Once this is shown, we know that a finite dimensional real Lie algebra is a compact Lie algebra if and only if it is a Hilbert Lie algebra.

The bilinear form $(X,Y) \mapsto \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ on a finite dimensional Lie algebra is called the *Cartan–Killing form*.

In the following proposition we use a piece of technical notation which we specify now. If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ for vector spaces and T is a vector space endomorphism of $\mathfrak{a}$ we shall denote by $T_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}$ the endomorphism given by $T_{\mathfrak{g}}(a \oplus b) = Ta$.

Proposition 6.5. *Assume that $\mathfrak{g}$ is a real Hilbert Lie algebra. Then*

(i) *its automorphism group* $\mathbb{G} \stackrel{\text{def}}{=} \operatorname{Aut} \mathfrak{g} \cap \mathrm{O}(\mathfrak{g})$ *is a linear Lie group with Lie algebra* $\mathfrak{L}(\mathbb{G}) = \operatorname{Der} \mathfrak{g} \cap \mathfrak{o}(\mathfrak{g})$, *the completely normable Lie algebra of all skew-symmetric derivations of* $\mathfrak{g}$. *If* $\dim \mathfrak{g} < \infty$. *Then* $\mathbb{G}$ *and* $\mathbb{G}_0$ *are compact Lie groups.*

(ii) *Each automorphism in* $\mathbb{G}$ *respects* $\mathfrak{z}$ *and* $\overline{[\mathfrak{g},\mathfrak{g}]}$ *and each derivation respects* $\mathfrak{z}(\mathfrak{g})$ *as well as each closed ideal* $\mathfrak{i}$ *satisfying* $\overline{[\mathfrak{i},\mathfrak{i}]} = \mathfrak{i}$, *in particular* $\overline{[\mathfrak{g},\mathfrak{g}]}$. *Any derivation satisfies*

$$(3) \qquad (\forall X \in \mathfrak{g}) \quad [D, \operatorname{ad} X] = \operatorname{ad}(DX)$$

and if D is skew symmetric, then $D\mathfrak{v} \subseteq \mathfrak{v}$ *implies* $D\mathfrak{v}^{\perp} \subseteq \mathfrak{v}^{\perp}$.

(iii) *Each of* $\big(\mathfrak{o}(\mathfrak{z})\big)_{\mathfrak{g}}$ *and* $\overline{\operatorname{ad} \mathfrak{g}}$ *are closed ideals of* $\mathfrak{L}(\mathbb{G})$, *and their intersection is trivial. If* $\dim \mathfrak{g} < \infty$, *then*

$$(4) \qquad \mathfrak{L}(\mathbb{G}) = \big(\mathfrak{o}(\mathfrak{z})\big)_{\mathfrak{g}} \oplus \operatorname{ad} \mathfrak{g}.$$

Now assume that $\dim \mathfrak{g}$ *is finite and thus that* $\mathbb{G}$ *and* $\mathbb{G}_0$ *are compact Lie groups by* (i). *Then the following conclusions hold.*

(iv) *If* $\mathfrak{g}$ *is semisimple, i.e. if* $\mathfrak{z} = \{0\}$, *then* $\mathfrak{L}(\mathbb{G}) = \mathfrak{L}(\mathbb{G}_0) = \operatorname{ad} \mathfrak{g} \cong \mathfrak{g}$.

(v) *Any derivation of a semisimple finite dimensional Hilbert Lie algebra is inner.*

Proof. (i) In E5.10 we noted that $\mathrm{O}(\mathfrak{g})$ is a linear Lie group with $\mathfrak{L}\big(\mathrm{O}(\mathfrak{g})\big) = \mathfrak{o}(\mathfrak{g})$.If $\dim \mathfrak{g} < \infty$ then $\mathrm{O}(\mathfrak{g})$ is a compact Lie group. In Theorem 5.43 we saw that $\operatorname{Aut} \mathfrak{g}$ is a linear Lie group with Lie algebra $\mathfrak{L}(\operatorname{Aut} \mathfrak{g}) = \operatorname{Der} \mathfrak{g}$. Now Proposition 5.51(i) proves $\mathfrak{L}(\mathbb{G}) = \mathfrak{L}(\mathbb{G}_0) = \operatorname{Der} \mathfrak{g} \cap \mathfrak{o}(\mathfrak{g})$. If $\dim \mathfrak{g} < \infty$, the group $\mathbb{G}$, as a closed subgroup of the compact Lie group $\mathrm{O}(\mathfrak{g})$, is a compact Lie group and so is its closed subgroup $\mathbb{G}_0$.

(ii) The ideals $\mathfrak{z}$ and $[\mathfrak{g},\mathfrak{g}]$ are characteristic, i.e. are preserved by any (algebraic) automorphism by their very definition; thus they and the closure of $[\mathfrak{g},\mathfrak{g}]$ are preserved by any continuous automorphism.

Now let D be any continuous derivation. If $X \in \mathfrak{z}$ and $Y \in \mathfrak{g}$, then $[DX,Y] = -[X,DY] + D[X,Y] = -0 + D0 = 0$. Thus $D\mathfrak{z} \subseteq \mathfrak{z}$. Now assume that $\mathfrak{i} \trianglelefteq \mathfrak{g}$ and $\overline{[\mathfrak{i},\mathfrak{i}]} = \mathfrak{i}$. Then $D\mathfrak{i} = D\overline{[\mathfrak{i},\mathfrak{i}]} \subseteq \overline{D[\mathfrak{i},\mathfrak{i}]} \subseteq \overline{[D\mathfrak{i},\mathfrak{i}]} + \overline{[\mathfrak{i},D\mathfrak{i}]} \subseteq \mathfrak{i}+\mathfrak{i} = \mathfrak{i}$.

We compute $[D,\operatorname{ad}X](Y) = D[X,Y] - [X,DY] = [DX,Y] + [X,DY] - [X,DY] = [DX,Y] = \big(\operatorname{ad}(DX)\big)(Y)$. If $X \in \mathfrak{v}^\perp$ and D is skew symmetric, then $(DX \mid \mathfrak{v}) = -(X \mid D\mathfrak{v}) \subseteq -(X \mid \mathfrak{v}) = \{0\}$.

(iii) Relation (3) shows that $\operatorname{ad}\mathfrak{g}$ and thus $\overline{\operatorname{ad}\mathfrak{g}}$ is an ideal of $\operatorname{Der}\mathfrak{g}$ and hence of $\mathfrak{L}(\mathbb{G})$ (by (i)). If D and D' are derivations and D annihilates $\overline{\mathfrak{g}'}$, then $[D,D']\overline{\mathfrak{g}'} = D(D'\overline{\mathfrak{g}'}) + D'(D\overline{\mathfrak{g}'}) = \{0\}$ since $D'\overline{\mathfrak{g}'} \subseteq \overline{\mathfrak{g}'}$ by (ii). We have $\mathfrak{g} = \mathfrak{z} \oplus \overline{\mathfrak{g}'}$ by 6.4(v) and so $\mathfrak{o}(\mathfrak{z})_\mathfrak{g}$ is the set of all elements in $\mathfrak{L}(\mathbb{G})$ annihilating $\overline{\mathfrak{g}'}$. Thus we see that $\mathfrak{o}(\mathfrak{z})_\mathfrak{g}$ is an ideal of $\mathfrak{L}(\mathbb{G})$.

We write $X = X_1 \oplus X_2$ with $X_1 \in \mathfrak{z}$ and $X_2 \in \overline{[\mathfrak{g},\mathfrak{g}]}$. Then $\operatorname{ad}X = \operatorname{ad}X_1 + \operatorname{ad}X_2 = \operatorname{ad}X_2$, whence $\operatorname{ad}\mathfrak{g} = \operatorname{ad}\overline{\mathfrak{g}'}$. Since $(\operatorname{ad}\mathfrak{g})(\mathfrak{z}) = \{0\}$ we have $\overline{\operatorname{ad}\mathfrak{g}} \subseteq \operatorname{Der}(\overline{\mathfrak{g}'})_\mathfrak{g}$ whence $\mathfrak{o}(\mathfrak{g})_\mathfrak{g} \cap \overline{\operatorname{ad}\mathfrak{g}} = \{0\}$. Hence $\mathfrak{o}(\mathfrak{g}) \oplus \overline{\operatorname{ad}\mathfrak{g}}$ is a direct sum of closed ideals of $\mathfrak{L}(\mathbb{G})$.

Now assume that $\mathfrak{g}$ is finite dimensional. Then $\mathrm{O}(\mathfrak{g})$ is a compact Lie group and so is $\mathbb{G}$ as a closed subgroup. Hence $\mathfrak{L}(\mathbb{G})$ is a finite dimensional Hilbert Lie algebra by 6.2. We have seen $\operatorname{ad}\mathfrak{g} = \operatorname{ad}[\mathfrak{g},\mathfrak{g}] = [\operatorname{ad}\mathfrak{g},\operatorname{ad}\mathfrak{g}] \subseteq \mathfrak{L}(\mathbb{G})'$. Let $D \in (\operatorname{ad}\mathfrak{g})^\perp$. Then $[D,\operatorname{ad}X] = 0$ by 6.4(i) and $\operatorname{ad}(DX) = [D,\operatorname{ad}X] = 0$ by (3). Hence $D\mathfrak{g} \subseteq \mathfrak{z}$ and thus $D\mathfrak{g}' = \{0\}$ since D respects the decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ by (ii). Thus $D \in \mathfrak{o}(\mathfrak{z})_\mathfrak{g}$. In other words, $(\operatorname{ad}\mathfrak{g}')^\perp \subseteq \mathfrak{o}(\mathfrak{z})$. But $[\mathfrak{o}(\mathfrak{z}),\operatorname{ad}\mathfrak{g}] = \{0\}$ by what we saw in the first part of this section of the proof. Further, $\operatorname{ad}\mathfrak{g} = [\operatorname{ad}\mathfrak{g},\operatorname{ad}\mathfrak{g}]$. Hence $\mathfrak{o}(\mathfrak{z}) \subseteq (\operatorname{ad}\mathfrak{g})^\perp$ by 6.4(ii). Therefore, $\mathfrak{o}(\mathfrak{z})_\mathfrak{g} = (\operatorname{ad}\mathfrak{g})^\perp$ and thus $\mathfrak{L}(\mathbb{G}) = \mathfrak{o}(\mathfrak{z})_\mathfrak{g} \oplus \operatorname{ad}\mathfrak{g}$, as asserted.

(iv) Assume that $\mathfrak{g} = \mathfrak{g}'$. Then $\mathfrak{z} = \ker\operatorname{ad} = \{0\}$ and thus $\mathfrak{g} \cong \operatorname{ad}\mathfrak{g} = \mathfrak{L}(\mathbb{G})$ by (4).

(v) Assume $\mathfrak{g} = \mathfrak{g}'$ again. By (4), any skew symmetric derivation is in $\operatorname{Der}(\mathfrak{g}) \cap \mathfrak{o}(\mathfrak{g}) = \mathfrak{L}(\mathbb{G}) = \operatorname{ad}\mathfrak{g}$. It remains to show that every derivation is skew symmetric. By Theorem 6.4(ix) we may assume that the inner product on $\mathfrak{g}'$ is the negative of the Cartan– Killing form $(X,Y) \mapsto -\operatorname{tr}\operatorname{ad}X\operatorname{ad}Y$. Let D be a derivation of $\mathfrak{g}$. Then $D[X,Z] = [DX,Z] + [X,DZ]$. Thus $\big(\operatorname{ad}(DX)\big)(Z) = [DX,Z] = (D \circ \operatorname{ad}X)(Z) - \big((\operatorname{ad}X) \circ D\big)(Z)$. Therefore $\operatorname{tr}\operatorname{ad}(DX)\operatorname{ad}Y = \operatorname{tr}D\operatorname{ad}X\operatorname{ad}Y - \operatorname{tr}(\operatorname{ad}X)D\operatorname{ad}Y = \operatorname{ad}X\operatorname{ad}Y\operatorname{tr}D - \operatorname{tr}(\operatorname{ad}X)D\operatorname{ad}Y = -\operatorname{tr}\operatorname{ad}X\operatorname{ad}(DY)$. This shows that D is skew symmetric and finishes the proof. ☐

A compact group G is called a *simple connected compact Lie group* if it is a connected compact Lie group such that every closed proper normal subgroup is discrete. Some authors call such groups *quasisimple*; they make this distinction since in abstract group theory a group is called *simple* if it has no nonsingleton proper normal subgroups. If N is a closed normal subgroup of a connected linear

Lie group G, then $\mathfrak{L}(N) \trianglelefteq \mathfrak{L}(G)$ is an ideal by 5.49. The relation $\mathfrak{L}(N) = \{0\}$ is equivalent to $N_0 = \langle \exp \mathfrak{L}(N) \rangle = \{1\}$ and thus to the discreteness of N. Hence *a connected compact Lie group G is simple if and only if its Lie algebra $\mathfrak{L}(G)$ is simple.* This remark accounts for a terminology that has been widely used in Lie theory even though it is at variance with terminology used in abstract group theory. In Theorem 9.90 below we shall offer further clarification of this issue.

Now we are ready for the

Characterisation of Compact Lie Algebras

Theorem 6.6. *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then the following conditions are equivalent:*

(i) $\mathfrak{g}$ *is a compact Lie algebra; i.e. there is a compact Lie group G with $\mathfrak{g} \cong \mathfrak{L}(G)$.*
(ii) $\mathfrak{g}$ *is a Hilbert Lie algebra with respect to a suitable inner product.*
(iii) *There is a connected linear Lie group $G \cong \mathbb{R}^m \times G_1 \times \cdots \times G_n$, for connected simple compact Lie groups G_k, $k = 1, \ldots, n$, such that $\mathfrak{g} \cong \mathfrak{L}(G)$.*
(iv) *There is a simply connected linear Lie group $G \cong \mathbb{R}^m \times S_1 \times \cdots \times S_n$, for simply connected simple compact Lie groups S_k, such that $\mathfrak{g} \cong \mathfrak{L}(G)$. The Lie group G is unique up to isomorphism.*

Proof. The implication (i)⇒(ii) was established in 6.2. The implication (iii)⇒(i) is clear because $\mathbb{T}^m \times G_1 \times \cdots \times G_n$ is a compact Lie group locally isomorphic to $\mathbb{R}^m \times G_1 \times \cdots \times G_n$ and therefore having a Lie algebra isomorphic to $\mathfrak{g}$.

We show that (ii) implies (iii) and thus complete the proof of the equivalence of (i), (ii) and (iii). By 6.4(v,vii), $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with a unique family of simple ideals $\mathfrak{g}_k$ of $\mathfrak{g}$. Then the additive group of $\mathfrak{z}$ is isomorphic to $\mathbb{R}^m$ for some m and $\mathbb{R}^m$ is a linear Lie group (see E5.10(ii) 2). By 6.5(iv) for each $k = 1, \ldots, n$ there is a connected compact Lie group G_k with $\mathfrak{L}(G_k) = \mathfrak{g}_k$. By remarks preceding this theorem, the simplicity of $\mathfrak{L}(G_k) \cong \mathfrak{g}_k$ implies that G_k is a simple connected compact Lie group. Now by 5.51(ii) the product $\mathbb{R}^m \times G_1 \times \cdots \times G_n$ is a linear Lie group whose Lie algebra is (isomorphic to) $\mathfrak{L}(\mathbb{R}^m) \times \mathfrak{L}(G_1) \times \cdots \times \mathfrak{L}(G_n) \cong \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$.

The proof of (iv)⇒(i) is again trivial. We show (iii)⇒(iv): By Appendix 2, A2.21 and by Theorem 5.77, the group G_k has a compact universal covering group S_k. Since S_k and G_k are locally isomorphic and G_k has no small subgroups, the group S_k has no small subgroups and thus is a compact Lie group by our introductory discussion and definition. The covering morphism $S_k \to G_k$ induces an isomorphism $\mathfrak{L}(S_k) \to \mathfrak{L}(G_k) = \mathfrak{g}_k$. Then by 5.51(ii) the product $\mathbb{R}^m \times S_1 \times \cdots \times S_n$ is a linear Lie group whose Lie algebra is (isomorphic to) $\mathfrak{L}(\mathbb{R}^m) \times \mathfrak{L}(S_1) \times \cdots \times \mathfrak{L}(S_n) \cong \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$. The uniqueness of G is a consequence of the fact that two locally isomorphic simply connected topological groups are isomorphic (see Appendix 2, A2.29). □

Exercise E6.2. Prove the following proposition.

(a) *Let G be a compact Lie group. Then the adjoint representation* $\mathrm{Ad}\colon G \to \mathbb{G} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{O}(\mathfrak{g})$ *is the composition $j \circ f$ of a surjective morphism $f\colon G \to \mathrm{Ad}(G)$ and an inclusion morphism $j\colon \mathrm{Ad}(G) \to \mathbb{G}$ of compact Lie groups. The kernel of f is $Z(G_0, G)$, the centralizer of the identity component in G. Accordingly,* $\mathrm{ad} = \mathfrak{L}(\mathrm{Ad}) = \mathfrak{L}(j) \circ \mathfrak{L}(f)$, *where $\mathfrak{L}(f)\colon \mathfrak{g} \to \mathrm{ad}\,\mathfrak{g} = \mathfrak{L}\big(\mathrm{Ad}(G)\big)$ is the corestriction of* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{L}(\mathbb{G}) = \mathrm{Der}\,\mathfrak{g} \cap \mathfrak{o}(\mathfrak{g})$ *to its image and has the center $\mathfrak{z}(\mathfrak{g})$ as its kernel, and where $\mathfrak{L}(j)\colon \mathrm{ad}\,\mathfrak{g} \to \mathfrak{L}(\mathbb{G})$ is the inclusion.*

If G is semisimple, then the kernel of f is discrete.

(b) *For each element x in a compact Lie algebra $\mathfrak{g}$ the following relation holds*

$$\mathfrak{z}(x, \mathfrak{g})^{\perp} = [x, \mathfrak{g}].$$

[Hint. (a) Consider 5.44.

(b) Let $\mathfrak{g}$ be a finite-dimensional vector space and κ a nondegenerate bilinear form. For an endomorphism φ of $\mathfrak{g}$ let φ^* denote the adjoint defined by

$$(\forall x, y \in \mathfrak{g})\ \kappa(\varphi^*(x), y) = \kappa(x, \varphi(y)).$$

Then $y \in \ker\varphi$ iff $(\forall x \in \mathfrak{g})\ \kappa(x, \varphi(y)) = 0$ iff $(\forall x \in \mathfrak{g})\ \kappa(\varphi^*(x), y) = 0$ iff $y \in \mathrm{im}(\varphi^*)^{\perp}$. Equivalently,

$$(*) \qquad (\ker\varphi^*)^{\perp} = \mathrm{im}\,\varphi.$$

Recall that a nondegenerate bilinear form κ on a finite-dimensional Lie algebra is called *invariant* if $(\mathrm{ad}\,x)^* = -\,\mathrm{ad}\,x$ for all $x \in \mathfrak{g}$. In that case, for $\varphi = \mathrm{ad}\,x$, relation $(*)$ reads

$$(\forall x \in \mathfrak{g})\, \mathfrak{z}(x, \mathfrak{g})^{\perp} = [x, \mathfrak{g}].$$

Theorem 6.6 now implies assertion (b).] □

As a first consequence we shall show, in order to fulfill a promise given in the previous chapter in the context of Example 5.67, that quotients of compact Lie groups are compact Lie groups.

Theorem 6.7. (i) *A quotient of a compact Lie group is a compact Lie group. A continuous homomorphic image of a compact Lie group is a compact Lie group.*

(ii) *Conversely, if G is a compact group with a closed normal subgroup N such that both N and G/N are Lie groups, then G is a Lie group.*

Proof. (i) By 1.10(iv), a continuous homomorphic image of a compact group is (isomorphic to) a quotient group. Thus the second assertion follows from the first. Now let G be a compact Lie group and N a closed normal subgroup. We shall show that G/N has no small subgroups, and by 2.40 this will establish the claim.

Reduction 1. The identity component N_0 of N is characteristic in N, hence normal in G. The morphism $gN_0 \mapsto gN : G/N_0 \to G/N$ maps a basic open identity neighborhood UN_0/N_0 (where U is an identity neighborhood in G onto an identity neighborhood UN/N and is therefore open. The induced bijective morphism $(G/N_0)/(N/N_0) \to G/N$ is, therefore, an isomorphism of topological groups. The closed subgroup N of G is a compact Lie group (since, like G, it does not have any small subgroups), and thus N_0 is open in N and so N/N_0 is discrete in G/N_0. Hence G/N and G/N_0 are locally isomorphic by Lemma 5.68. It therefore suffices to show that G/N_0 has no small subgroups. Therefore we shall henceforth assume that N is connected.

Reduction 2. We have $N = N_0 \subseteq G_0$, and G_0 is open in G by 5.41(iii). Therefore G_0/N is open in G/N. It suffices to show that G_0/N has no small subgroups. We shall therefore assume that G is connected, too.

Proof of the nonexistence of small subgroups in G/N: By Theorem 6.4(i) we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$. In the Lie algebra $\mathfrak{g}$ let B be a ball around 0 (with respect to some Banach algebra containing $\mathfrak{g}$ and G) such that $B * B$ is defined and that B is mapped homeomorphically onto an identity neighborhood of G by exp. Furthermore, by the Fundamental Theorem 5.31(ii) we may assume that B is chosen so small that

$$\exp^{-1} N \cap B = \mathfrak{n} \cap B. \tag{5}$$

The analytic function

$$\alpha\colon (B \cap \mathfrak{n}) \oplus (B \cap \mathfrak{n}^\perp) \to B * B, \quad \alpha(X+Y) \stackrel{\text{def}}{=} X * Y = X + Y + \frac{1}{2}\cdot[X,Y] + \cdots,$$

has the derivative $\mathrm{id}_{\mathfrak{g}}$ at 0 and the Theorem of the Local Inverse applies, securing the existence of open balls B_1 and B_2 around 0 in $B \cap \mathfrak{n}$ and $B \cap \mathfrak{n}^\perp$, respectively such that α maps $B_1 \oplus B_2$ homeomorphically onto an open neighborhood $B_1 * B_2 \subseteq B$ of 0 in $\mathfrak{g}$. Then $B_1 * B_2$ is a neighborhood of 0 in G by the claim. As a consequence, for any ball B_0 around 0 in B_2 the set $(\exp B_0)N = N(\exp B_0)$ contains $(\exp B_0)(\exp B_2) = \exp(B_0 * B_2)$ and thus is an identity neighborhood in G. Take B_0 so small that $B_0 * B_0 * B_0 * B_0 \subseteq B_2$. Then $V \stackrel{\text{def}}{=} (\exp B_0)N/N$ and $W \stackrel{\text{def}}{=} \big(\exp(B_0 * B_0)\big)N/N$ are identity neighborhoods in G/N. Assume $cN = dN$ with $c = \exp X$, $d = \exp Y$ with X, $Y \in B_0 * B_0$. Then $Z \stackrel{\text{def}}{=} (-Y) * X \in [(-B_0) * (-B_0)] * [B_0 * B_0] \subseteq B_2$, and $d^{-1}c = (\exp -Y)(\exp X) = \exp Z \in (\exp B_2) \cap N$, i.e. $Z \in B_2 \cap \exp^{-1} N \subseteq \mathfrak{n}^\perp \cap (B \cap \exp^{-1} N) \subseteq \mathfrak{n}^\perp \cap \mathfrak{n} = \{0\}$ by (5) above. Thus $Z = 0$ and therefore $c = d$. Thus

$$\mu \stackrel{\text{def}}{=} (X \mapsto (\exp X)N)\colon B_0 * B_0 \to W \subseteq G/N \quad \text{is a bijection.} \tag{6}$$

Now let $N \neq \gamma \in V$. Then there is a unique $c \exp X \in C$, $0 \neq X \in B_0$ such that $\gamma = cN$. There is a smallest natural number n such that $(n-1)\cdot X \in B_0$ but $n\cdot X = \big((n-1)\cdot X\big) * X \in B_0 * B_0 \setminus B_0$. Suppose $\gamma^n \in V$, then $\gamma^n = c^n N = (\exp n\cdot X)N = \mu(n\cdot X) \in W$. From (6) we would have to conclude that $n\cdot X \in B_0$ which is not the case. Thus $\gamma^n \notin V$. This means that V does not contain any subgroups other than the singleton one, and thus our proof of (i) is completed.

(ii) We must show that G has no small subgroups. First let V be an open identity neighborhood of G such that $VN = V$ and that V/N is an identity neighborhood of G/N which contains no nontrivial subgroup. Next let $U \subseteq G$ be an identity neighborhood of G such that $U \cap N$ does not contain a nontrivial subgroup of N. Then $U \cap V$ is indeed an identity neighborhood of G not containing any nontrivial subgroup of G. □

Exercise E6.3. Analyze the proof of Theorem 6.7 and distill a proof of the following statement: *Let G be a linear Lie group with a closed normal subgroup N such that there is a closed Lie subalgebra $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ algebraically and topologically, then G/N has no small subgroups.* In the finite dimensional case, it suffices to know that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ algebraically. □

In order to pursue further applications of Theorem 6.4 to compact Lie groups we shall require a blend of the Lie theoretical methods just introduced with the representation theory discussed in Chapter 2. The relevant tools in Chapter 2 contained simple complex Hilbert G-modules E. Equivalently, we shall consider irreducible representations $\pi: G \to \mathrm{U}$ into the group $\mathrm{U} = \{\varphi \in \mathrm{Gl}(E) \mid \varphi\varphi^* = \varphi^*\varphi = \mathbf{1}\}$ of isometries of a finite dimensional Hilbert space E (with $(\varphi x \mid y) = (x \mid \varphi^* y)$. Cf. Exercise E5.10.). If $\mathbb{K} = \mathbb{R}$ then U is the group $\mathrm{O}(E)$ of orthogonal transformations, and if $\mathbb{K} = \mathbb{C}$, then U is the group $\mathrm{U}(E)$ of unitary transformations of E. The group U is a compact Lie group with Lie algebra $\mathfrak{u}$, the Lie algebra of skew hermitian endomorphisms φ of the Hilbert space E, that is those endomorphisms which satisfy $\varphi^* = -\varphi$.

Recall that for any finite dimensional $\mathbb{K}$-vector space E, the Lie algebra of all endomorphisms of E is denoted by $\mathfrak{gl}(E)$. The map $(\varphi, \psi) \mapsto \operatorname{tr} \varphi\psi^*: \mathfrak{gl}(E) \times \mathfrak{gl}(E) \to \mathbb{C}$ is a real bilinear form, conjugate linear in the second argument if $\mathbb{K} = \mathbb{C}$. If $g \in \mathrm{Gl}(E)$, and if $\mathrm{Ad}(g)(\varphi) = g{\cdot}\varphi = g\varphi g^{-1}$ is the action by inner automorphisms, then for $g \in \mathrm{U}$ we have $g^{-1} = g^*$ and thus $(g{\cdot}\varphi \mid g{\cdot}\psi) = \operatorname{tr} g\varphi g^*(g\psi g^*)^* = \operatorname{tr} g(\varphi\psi^*)g^{-1} = \operatorname{tr} \varphi\psi = (\varphi \mid \psi)$. Hence this scalar product is invariant under the action of U. From E6.1 we note $([\varphi, \psi], \rho) = (\varphi, [\psi, \rho])$ for all φ, ψ, $\rho \in \mathfrak{u} \overset{\text{def}}{=} \{\varphi \in \mathfrak{gl}(E) \mid \varphi^* = -\varphi\}$. Moreover, $\operatorname{tr} \varphi\varphi^* \geq 0$, and if φ is diagonalisable, then $\operatorname{tr} \varphi\varphi^* = 0$ implies $\varphi = 0$. In particular, on the subalgebra $\mathfrak{u}$ the definition $(\varphi \mid \psi) = \operatorname{tr} \varphi\psi^*$ yields a scalar product making $\mathfrak{u}$ into a Hilbert Lie algebra, providing us with an example for Theorem 6.4.

Exercise E6.4. In the case $\mathbb{K} = \mathbb{C}$, the center Z of U is $\mathbb{S}^1 \cdot \mathrm{id}_E$ and thus the center of $\mathfrak{u} = \mathfrak{u}(E)$ is $\mathfrak{z}(\mathfrak{u}) = i\mathbb{R}\cdot \mathrm{id}_E$. Recall that $\mathrm{SU}(E)$ (respectively, $\mathfrak{su}(E)$) denotes the Lie group of automorphisms $\varphi \in \mathrm{U}$ with $\det \varphi = 1$ (respectively, the Lie algebra of endomorphisms $\varphi \in \mathfrak{u}$ with $\operatorname{tr}(\varphi) = 0$). Then $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{su}(E)$. (Cf. E5.9),

In the case $\mathbb{K} = \mathbb{R}$ we have $\mathrm{U} = \mathrm{O}(E)$, $\mathfrak{u} = \mathfrak{o}(E)$, $Z = \{1, -1\}\cdot \mathrm{id}_E$ and $\mathfrak{z}(\mathfrak{u}) = \{0\}$, $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{u}$.

[Hint. $K = \mathbb{C}$ (the case of most interest here): The vector space E is a simple U-module (since U acts transitively on the unit sphere of E), and thus, by Lemma 2.30, the center Z of U is the set $\mathrm{U} \cap \mathbb{C}\cdot \mathrm{id}_E = \mathbb{S}^1 . \mathrm{id}_E$. (Cf. Example 1.2(ii).) From Proposition 5.54(iv) it follows that $\mathfrak{z} = \mathfrak{L}(Z) = i\mathbb{R}\cdot \mathrm{id}_E$.

$\mathbb{K} = \mathbb{R}$: The commutant $\mathcal{C} = \mathrm{Hom}_{\mathrm{O}(E)}(E, E)$ of $\mathrm{O}(E)$ is a division ring over $\mathbb{R}$. Take $0 \neq \varphi \in \mathcal{C}$ and let $0 \neq \lambda \in \mathbb{C}$ be an eigenvalue of φ. Then so is $\overline{\lambda}$ because $\mathbb{K} = \mathbb{R}$. Then $\varphi^2 - |\lambda|^2\cdot \mathrm{id}_E = (\varphi - \lambda\cdot \mathrm{id}_E)(\varphi + \lambda\cdot \mathrm{id}_E) \in \mathcal{C}$ has a nonzero kernel, hence vanishes as E is a simple $\mathrm{O}(E)$-module. Thus $\varphi^2 = |\lambda|^2\cdot \mathrm{id}_E$. Since φ is orthogonal, $|\lambda|^2 = 1$. Hence $\lambda = \pm 1$, and $\varphi \mp \lambda\cdot \mathrm{id}_E \in \mathcal{C}$ has nonzero kernel, hence is zero. It follows that $\mathfrak{z} = 0$.

Now $\mathfrak{u}$ is a Hilbert Lie algebra with respect to the scalar product given by $(\varphi \mid \psi) = \operatorname{tr} \varphi\psi^*$. Then $\mathfrak{u} = \mathfrak{z}(\mathfrak{u}) \oplus [\mathfrak{u}, \mathfrak{u}]$ as a direct orthogonal sum by Theorem 6.4. We claim that $i\mathbb{R}\cdot \mathrm{id}_E$ is orthogonal to $\mathfrak{su}(E)$, for if $\varphi \in \mathfrak{su}(E)$, then $\operatorname{tr}\varphi = 0$, and

$$(it\cdot \mathrm{id}_E \mid \varphi) = \operatorname{tr}\big((it\cdot \mathrm{id}_E)\varphi^*\big) = \operatorname{tr}(\varphi(-it)\cdot \mathrm{id}_E)^* = it \operatorname{tr}\varphi = 0.$$

Thus $\mathfrak{su}(E) \subseteq \mathfrak{z}(\mathfrak{u})^\perp = [\mathfrak{u}, \mathfrak{u}]$, and since $\mathfrak{su}(E)$ is a real hyperplane in $\mathfrak{u}$, equality follows.] □

Lemma 6.8. (i) *Let Z denote the center of $\mathrm{U}(E)$ for a finite dimensional complex Hilbert space E. The function $(z, s) \mapsto zs\colon Z \times \mathrm{SU}(E) \to \mathrm{U}(E)$ is a surjective morphism μ whose kernel is isomorphic to $Z \cap \mathrm{SU}(E)$.*

(ii) *$Z = \mathbb{S}^1\cdot \mathrm{id}_E$, and*

$$Z \cap \mathrm{SU}(E) = \{e^{2\pi ik/n}\cdot \mathrm{id}_E \mid k = 0, \ldots, n-1\} \cong \mathbb{Z}(n), \quad n = \dim E.$$

The algebraic commutator group U' of U is $\mathrm{SU}(E)$.

Proof. (i) First observe, that for $\varphi \in \mathrm{U}(E)$ we have $|\det\varphi|^2 = \det\varphi\,\overline{\det\varphi} = \det\varphi\,\det\varphi^* = \det\varphi\varphi^* = \det\mathbf{1} = 1$, that is $\det\varphi = e^{it}$ for some $t \in \mathbb{R}$. Hence $\det e^{-it/n}\cdot\varphi = 1$, i.e. $e^{-it/n}\cdot\varphi \in \mathrm{SU}(E)$ and $\varphi = (e^{it/n}\cdot \mathrm{id}_E)(e^{-it/n}\cdot\varphi)$, where $e^{it/n}\cdot \mathrm{id}_E \in Z$. Thus μ is surjective. Clearly, μ is a morphism of topological groups, and $(z, s) \in \ker\mu$ iff $z = s^{-1}$. This element is obviously in $Z \cap \mathrm{SU}(E)$. We note that $z \mapsto (z^{-1}, z)\colon Z \cap \mathrm{SU}(E) \to \ker\mu$ is an isomorphism.

(ii) The first assertion of (ii) was observed in E6.2. For the second, we note that $\det e^{2\pi it}\cdot \mathrm{id}_E = 1$ iff $t = \frac{k}{n}$.

Finally, consider $g, h \in \mathrm{U}$, then $\det \mathrm{comm}(g, h) = 1$ and thus $\mathrm{U}' \subseteq \mathrm{SU}(E)$. Now we recall from Exercise E1.2 that $\mathbb{S}^3 \cong \mathrm{SU}(2)$. In the quaternionic unit sphere we quickly calculate $ie^{tj}i^{-1}e^{-tj} = e^{tij(-i)}e^{-tj} = e^{-2tj}$. Any unit quaternion in $\mathbb{R}\cdot i + \mathbb{R}\cdot j + \mathbb{R}\cdot k$ can be conjugated into j, since conjugation by an element of $\mathbb{S}^3$ acts as rotation on this copy of $\mathbb{R}^3$ and all rotations are so obtained. Hence every unit quaternion is conjugate to one of the form e^{-2tj} and hence is a commutator by the preceding. If $\dim E \geq 2$ then an element g in $\mathrm{SU}(E)$ is diagonalisable, i.e. is of the form

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{pmatrix}, \quad |\lambda_m| = 1,\ m = 1, \ldots, n, \quad \lambda_1 \cdots \lambda_n = 1.$$

If we write

$$g_k = \mathrm{diag}(1, \ldots, \lambda_k, \ldots, 1, \lambda_k^{-1}), \quad k = 1, \ldots, n-1,$$

then $g = g_1 \cdots g_{n-1}$, and each g_k is a member of a subgroup isomorphic to $\mathrm{SU}(2) \cong \mathbb{S}^3$ and thus is a commutator by the preceding remarks. Hence g is a member of the commutator subgroup of $\mathrm{SU}(n)$ (and then of $\mathrm{U}(n)$). □

Exercise E6.5. (i) Establish the conclusions of Lemma 6.8 via alternative routes.
(ii) Prove $\mathrm{Gl}(E)' = \mathrm{Sl}(E)$.

[Hint. An example of an alternative route is the Commutator Subgroup Theorem 5.60 in conjunction with Exercise E6.4 above.] □

Now we proceed to utilize this explicit information (which we could have established much earlier) on arbitrary compact groups. This turns out to be very important in the sequel.

Proposition 6.9. *In any compact Lie group G, the intersection of the center $Z(G)$ and the closure $\overline{G'}$ of the algebraic commutator group G' is finite.*

Proof. Let G be a connected compact Lie group. By Corollary 2.40 there is a faithful unitary representation $\pi: G \to \mathrm{U}(E)$ on a finite dimensional complex Hilbert space E. By Corollary 2.25, E is the orthogonal direct sum of simple G-modules $E_1 \oplus \cdots \oplus E_n$, and there are representations $\pi_k: G \to \mathrm{U}(E_k)$ such that $\pi(g) = \pi_1(g) \oplus \cdots \oplus \pi_n(g)$. For each $j \in \{1, \ldots, n\}$ and each $z \in Z = Z(G)$, we find $\pi_j(z) = \chi_j(z)\cdot \mathrm{id}_{E_j} \in Z\big(\mathrm{U}(E_j)\big)$ by Lemma 2.30 with a suitable character $\chi_j \in \widehat{Z}$ of Z. Furthermore, since any morphism maps commutators into commutators, we find $\pi_j(G') \subseteq \mathrm{U}(E_j)' = \mathrm{SU}(E_j)$ in view of Lemma 6.8 above. But $\mathrm{SU}(E_j)$ is closed, and thus $\overline{G'}$ is also mapped into $\mathrm{SU}(E_j)$. Recall from 2.6(ii) or 6.8(ii) above that $\mathbb{S}^1\cdot \mathrm{id}_{E_j} \cap \mathrm{SU}(E_j) = \{c\cdot \mathrm{id}_{E_j} \mid c^{\dim E_j} = 1\}$. It follows that the intersection of the group $Z\big(\mathrm{U}(E_1)\big) \oplus \cdots \oplus Z\big(\mathrm{U}(E_n)\big)$ with $\mathrm{SU}(E_1) \oplus \cdots \oplus \mathrm{SU}(E_n)$ is finite (and, of course, discrete). Since $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ is faithful, this completes the proof. □

The Commutator Subgroup of a Compact Lie Group

Proposition 6.9 motivates us to look more carefully at the commutator group of a compact Lie group. Recall from Proposition 2.42 that a compact connected abelian Lie group is isomorphic to $\mathbb{T}^m$ for some natural number m. For a natural number n, the function $t \mapsto n{\cdot}t : \mathbb{T}^n \to \mathbb{T}^n$ is an endomorphism the kernel of which is $\mathbb{T}^m[n] = \{t \in \mathbb{T}^m : n{\cdot}t = 0\} \cong \mathbb{Z}(n)^m$, a characteristic subgroup of $\mathbb{T}^m$. Before we get to the next result, a decisive (but not conclusive!) theorem on the commutator group of a compact Lie group, we establish a primarily group theoretical result:

Theorem 6.10. (Structure of Compact Lie Groups with Abelian Identity Component). *Let G be a compact Lie group such that G_0 is abelian. Then the following conclusions hold.*

(i) *There is a finite subgroup E of G such that $G = G_0E = EG_0$ and $G_0 \cap E = G_0[n]$, $n = |G/G_0|$, is normal in G.*

(ii) *The group $\mathrm{comm}(G, G_0)$ is a closed connected subgroup of G_0. In particular, it is a torus.*

(iii) G_0 *is the product of* $\mathrm{comm}(G, G_0)$ *and* $Z_0(G)$*, the identity component of the center of* G*, and the intersection of the two torus groups is finite. The order of every element in this intersection divides* $|G/G_0|$.

(iv) $G' = \mathrm{comm}(G, G_0)E'$. *In particular,* G' *is closed, and* $\mathrm{comm}(G, G_0) = (G')_0$.

(v) *The Lie algebra* $\mathfrak{g}$ *of* G *is an* E*-module under the adjoint action. If* E *contains an element* h *such that* $\mathrm{Ad}(h)$ *leaves no nonzero vector of* $\mathfrak{g}_{\mathrm{eff}}$ *fixed, then every element of comm*(G, G_0) *is a commutator.*

Proof. We denote the finite quotient group G/G_0 by F and select a function $\sigma: F \to G$ such that $\sigma(gG_0) \in gG_0$ for all $g \in G$. The group G operates on F via $g\cdot(hG_0) = ghG_0$. Since the element $g\sigma(\xi)$ is contained in $g\cdot\xi$, we have $g\sigma(\xi) \in \sigma(g\cdot\xi)G_0$. Thus there is a unique element $x \in G_0$ such that $g\sigma(\xi) = \sigma(g\cdot\xi)x$. Then the element $\gamma(g, \xi) \stackrel{\mathrm{def}}{=} g\sigma(\xi)\sigma(g\cdot\xi)^{-1} = \sigma(g\cdot\xi)x\sigma(g\cdot\xi)^{-1}$ is in G_0 since G_0 is normal, and we have

$$g\sigma(\xi) = \gamma(g, \xi)\sigma(g\cdot\xi) \quad \text{for } g \in G,\ \xi \in F. \tag{7}$$

For g, $h \in G$ and $\xi \in F$ we compute $\gamma(gh, \xi)\sigma(gh\cdot\xi) = gh\sigma(\xi) = g\gamma(h, \xi)\sigma(h\cdot\xi) = \big(g\gamma(h, \xi)g^{-1}\big)g\sigma(h\cdot\xi) = \big(g\gamma(h, \xi)g^{-1}\big)\gamma(g, h\cdot\xi)\sigma(gh\cdot\xi)$. If we write $I_g(x) = gxg^{-1}$ for $g \in G$ and $x \in G_0$ we obtain

$$\gamma(gh, \xi) = I_g\big(\gamma(h, \xi)\big)\gamma(g, h\cdot\xi) \quad \text{for } g,\ h \in G,\ \xi \in F. \tag{8}$$

Since G_0 is abelian and F is finite we can define a continuous function $\varphi: G \to G_0$ by

$$\varphi(g) = \prod_{\xi \in F} \gamma(g, \xi) \quad \text{for } g \in G. \tag{9}$$

Now (8) implies that φ satisfies the following functional equation:

$$\varphi(gh) = I_g\big(\varphi(h)\big)\cdot\varphi(g) \quad \text{for } g,\ h \in G. \tag{10}$$

If $x \in G_0$, and $\xi = gG_0$ then $x\cdot\xi = xgG_0 = xG_0g = G_0g = gG_0 = \xi$ whence $x\sigma(\xi) = \gamma(x, \xi)\sigma(x\cdot\xi)$ implies $\gamma(x, \xi) = x$ and therefore

$$\varphi(x) = x^n \quad \text{for } x \in G_0, \quad n = |G/G_0|. \tag{11}$$

In particular, $\varphi(\mathbf{1}) = \mathbf{1}$ and thus (10) implies

$$\varphi(g^{-1}) = I_g\big(\varphi(g)\big)^{-1} \quad \text{for } g \in G. \tag{12}$$

From (10) and (12) we conclude that $E \stackrel{\mathrm{def}}{=} \varphi^{-1}(\mathbf{1})$ is a subgroup. Now G_0 is isomorphic to $\mathbb{T}^m$ for some m by 2.42. By (11) we know that $G_0[n] = E \cap G_0$, and since $G_0 \cong \mathbb{T}^m$ we have $E \cap G_0 \cong \mathbb{Z}(n)^m$. The group G_0 is divisible. Hence $\varphi(G_0) = G_0$ and $\varphi: G \to G_0$ is surjective. Let $g \in G$. Then, since G_0 is divisible, there is an $x \in G_0$ such that $x^n = \varphi(g)$. We set $e = x^{-1}g$ and use (10) and (12) to perform, recalling the commutativity of G_0 the following computation: $\varphi(e) = I_{x^{-1}}\varphi(g)\varphi(x^{-1}) = \varphi(g)\varphi(x)^{-1} = \varphi(g)x^{-n} = \mathbf{1}$. Hence $e \in E$ and $g =$

$ex \in EG_0 = G_0E$. Finally let $x \in E \cap G_0$. Then $\mathbf{1} = \varphi(x) = x^n$. The function $\mu_n: G_0 \to G_0$, $\mu(x) = x^n$ is an endomorphism. From $G_0 \cong \mathbb{T}^m$ we conclude that $\ker \mu \cong \mathbb{Z}(n)^m$. Hence $E \cap G_0 \subseteq \ker \mu$ is finite. Since $E/(E \cap G_0) \cong EG_0/G_0 = F$ we deduce that E is finite. The normalizer of $E \cap G_0$ contains E as well as G_0. Hence the normalizer contains $EG_0 = G$ and thus $E \cap G_0$ is normal. (This also follows from the fact that $G_0[n]$ is a characteristic subgroup of G_0.)

(ii) The restriction of the adjoint representation $\mathrm{Ad}: G \to \mathrm{Aut}\,\mathfrak{g}$ of Theorem 5.44 to E yields a representation $\pi: E \to \mathrm{Aut}(\mathfrak{g})$. If $g \in G$ and $X \in \mathfrak{g}$ then $\mathrm{comm}(g, \exp X) = g(\exp X)g^{-1}(\exp X)^{-1} = \exp\big(\mathrm{Ad}(g)X - X\big)$. Since

$$G_0 \cong \mathbb{T}^m$$

by 2.42, we have $G_0 = \exp \mathfrak{g}$ and $\mathrm{comm}(G, G_0) = \langle \exp\big(\mathrm{Ad}(g)X - X\big) \mid g \in G, X \in \mathfrak{g}\rangle = \exp \mathrm{span}\{\mathrm{Ad}(g)X - X \mid g \in G,\ X \in \mathfrak{g}\}$ as $\exp : \mathfrak{g} \to G_0$ is a morphism because of the commutativity of G_0. By (i), every $g \in G$ is of the form $g = eg_0$ with $e \in E$ and $g_0 \in G_0$. Also $\mathrm{Ad}(g_0)X = X$ for $X \in \mathfrak{g}$, whence $\mathrm{Ad}(g)X - X = \mathrm{Ad}(e)X - X = \pi(e)X - X$. We have $\mathrm{span}\{\pi(e)X - X \mid e \in E,\ X \in \mathfrak{g}\} = \mathfrak{g}_{\mathrm{eff}}$ for the E-module $\mathfrak{g}$ in the sense of Definition 4.4. Thus

$$\mathrm{comm}(G, G_0) = \exp \mathfrak{g}_{\mathrm{eff}}. \tag{13}$$

An element is in the torus $Z_0(G)$ if and only if it is of the form $\exp X$ such that $\exp t{\cdot}X \in Z(G)$ for all $t \in \mathbb{R}$ which means that for every $g \in G$ we have $\exp t{\cdot}X = g(\exp t{\cdot}X)g^{-1} = \exp t{\cdot}\,\mathrm{Ad}(g)X$ for all $g \in G$, $t \in \mathbb{R}$. This is equivalent to $\mathrm{Ad}(g)X = X$ for all $g \in G$, and in our present situation this translates into $X \in \mathfrak{g}_{\mathrm{fix}}$ for the fixed point E-module $\mathfrak{g}$ via π. Thus

$$Z_0(G) = \exp \mathfrak{g}_{\mathrm{fix}}.$$

We let $D \overset{\mathrm{def}}{=} \ker \exp_G$, then $G_0 = \exp \mathfrak{g} \cong \mathfrak{g}/D$, and thus the compactness of G_0 implies

$$\mathfrak{g} = \mathrm{span}\, D. \tag{14}$$

Note that $\mathrm{Ad}(G)$ leaves D invariant. While it is clear that $Z_0(G)$ is closed in G_0 it is not a priori clear that $\mathrm{comm}(G, G_0)$ is closed. We have to prove this next. Theorem 4.4 told us how to compute $\mathfrak{g}_{\mathrm{fix}}$ and $\mathfrak{g}_{\mathrm{eff}}$ and that $\mathfrak{g} = \mathfrak{g}_{\mathrm{eff}} \oplus \mathfrak{g}_{\mathrm{fix}}$. We have to consider the averaging operator $P_E: \mathfrak{g} \to \mathfrak{g}$. Recall $n = |E|$. Then

$$P_E(X) = \frac{1}{n}\sum_{e \in E} e{\cdot}X. \tag{15}$$

We set $Q_E \overset{\mathrm{def}}{=} \mathbf{1} - P_E$ and note $Q_E^2 = Q_E$, $\ker P_E = \mathrm{im}\, Q_E$. From 4.4(iii) we know that

$$\mathfrak{g}_{\mathrm{eff}} = Q_E(\mathfrak{g}), \quad \mathfrak{g}_{\mathrm{fix}} = P_E(\mathfrak{g}). \tag{16}$$

Now (15) and (16) implies

$$D_1 \overset{\mathrm{def}}{=} n{\cdot}Q_E(D) \subseteq \mathfrak{g}_{\mathrm{eff}} \cap D, \quad D_2 \overset{\mathrm{def}}{=} n{\cdot}P_E(D) \subseteq \mathfrak{g}_{\mathrm{fix}} \cap D. \tag{17}$$

Also, (16) and (14) yields

$$\begin{aligned}\mathfrak{g}_{\text{eff}} &= \operatorname{im} Q_E = n{\cdot}Q_E(\mathfrak{g}) = n{\cdot}Q_E(\operatorname{span} D) = \operatorname{span}\big((n{\cdot}Q_E)(D)\big) = \operatorname{span} D_1,\\ \mathfrak{g}_{\text{fix}} &= \operatorname{im} P_E = n{\cdot}P_E(\mathfrak{g}) = n{\cdot}P_E(\operatorname{span} D) = \operatorname{span}\big((n{\cdot}P_E)(D)\big) = \operatorname{span} D_2.\end{aligned}\tag{18}$$

Thus $\mathfrak{g}_{\text{eff}}/D_1$ is compact. Hence the quotient $\mathfrak{g}_{\text{eff}}/(\mathfrak{g}_{\text{eff}} \cap D)$ is compact, too. Applying the bijective morphism

$$X + (\mathfrak{g}_{\text{eff}} \cap D) \mapsto X + D : \mathfrak{g}_{\text{eff}}/(\mathfrak{g}_{\text{eff}} \cap D) \to (\mathfrak{g}_{\text{eff}} + D)/D$$

we get that $\operatorname{comm}(G, G_0) = \exp \mathfrak{g}_{\text{eff}} \cong (\mathfrak{g}_{\text{eff}} + D)/\mathfrak{g}_{\text{eff}}$ is compact, too. (Cf. also Appendix A1.12(iii).)

(iii) We can utilize D_2 similarly. In both cases, the relevant conclusion is that

$$\operatorname{rank}(\mathfrak{g}_{\text{eff}} \cap D) = \dim_{\mathbb{R}} \mathfrak{g}_{\text{eff}}, \quad \text{and} \quad \operatorname{rank}(\mathfrak{g}_{\text{fix}} \cap D) = \dim_{\mathbb{R}} \mathfrak{g}_{\text{fix}}.$$

It follows from $\mathfrak{g} = \mathfrak{g}_{\text{eff}} \oplus \mathfrak{g}_{\text{fix}}$ that $D' \stackrel{\text{def}}{=} (\mathfrak{g}_{\text{eff}} \cap D) \oplus (\mathfrak{g}_{\text{fix}} \cap D) \subseteq D$ has rank $\dim_R \mathfrak{g}$, whence D/D' is finite. Since $D_1 \oplus D_2 \subseteq D'$ from (17), we conclude that $n{\cdot}D \subseteq D_1 \oplus D_2 \subseteq D'$. We record $\operatorname{comm}(G, G_0) \times Z_0(G) = (\exp \mathfrak{g}_{\text{eff}}) \times (\exp \mathfrak{g}_{\text{fix}}) \cong \frac{\mathfrak{g}_{\text{eff}}}{(\mathfrak{g}_{\text{eff}} \cap D)} \times \frac{\mathfrak{g}_{\text{fix}}}{\mathfrak{g}_{\text{fix}} \cap D} \cong \mathfrak{g}/D'$. The homomorphism $\mathfrak{g}/D' \to \operatorname{comm}(G, G_0) \times Z_0(G) \stackrel{\mu}{\to} G_0$, $\mu(c, z) = cz$ is given by $X + D \mapsto \exp_G X$ and thus factors through the quotient map $\mathfrak{g}/D' \to \mathfrak{g}/D$. Set $\Delta \stackrel{\text{def}}{=} \operatorname{comm}(G, G_0) \cap Z_0(G)$; then there is a commutative diagram of exact rows

$$\begin{array}{ccccccccc}\{0\} & \to & D/D' & \xrightarrow{\text{incl}} & \mathfrak{g}/D' & \xrightarrow{\text{quot}} & \mathfrak{g}/D & \to & \{0\}\\ & & \cong\downarrow & & \cong\downarrow & & \downarrow\cong & & \\ \{1\} & \to & \Delta & \xrightarrow[\delta]{} & \operatorname{comm}(G, G_0) \times Z_0(G) & \xrightarrow[\mu]{} & G_0 & \to & \{1\},\end{array}$$

$\delta(z) = (z, z^{-1})$. Since $n{\cdot}D \subseteq D'$ we conclude that $d \in \operatorname{comm}(G, G_0) \cap Z_0(G)$ implies $d^n = 1$.

(iv) Clearly, the group $\operatorname{comm}(G, G_0)E'$ is contained in G'. Set $G/\operatorname{comm}(G, G_0) \stackrel{\text{def}}{=} \Gamma$. Then $G_0/\operatorname{comm}(G, G_0)$ is open and connected in Γ and thus is the identity component Γ_0. It follows that $\operatorname{comm}(\Gamma, \Gamma_0) = \{\mathbf{1}\}$. Hence Γ_0 is central. Let $\Phi = \operatorname{comm}(G, G_0)E/\operatorname{comm}(G, G_0)$. Then $\Phi' = \frac{\operatorname{comm}(G,G_0)E'}{\operatorname{comm}(G,G_0)}$, and $\Gamma = \Gamma_0\Phi$. Therefore $\Gamma' = \Phi'$ and $G' = \operatorname{comm}(G, G_0)E'$. By (ii) above, $\operatorname{comm}(G, G_0)$ is closed, hence compact, and E' is finite, hence $G' = \operatorname{comm}(G, G_0)E'$ is compact and thus closed. Since G_0 is connected, the set $\{ghg^{-1}h^{-1} \mid g \in G,\, h \in G_0\} = \bigcup_{g \in G}\{I_g(h)h^{-1} \mid h \in G_0\}$ is connected, and thus the subgroup $\operatorname{comm}(G, G_0)$ generated by this set is connected. Hence $\operatorname{comm}(G, G_0) \subseteq (G')_0$. As $G' = \operatorname{comm}(G, G_0)E'$ we note that $(G')_0/\operatorname{comm}(G, G_0)$ is a connected subgroup of $G'/\operatorname{comm}(G, G_0) \cong E'/\big(E' \cap \operatorname{comm}(G, G_0)\big)$, a finite group. Hence it is singleton and $(G')_0 = \operatorname{comm}(G, G_0)$ follows.

(v) In the proof of (ii) in (13) we noted that $\operatorname{comm}(G, G_0) = \exp \mathfrak{g}_{\text{eff}}$ where $\mathfrak{g}_{\text{eff}} = \operatorname{span}\{\pi(e)X - X | e \in E, X \in \mathfrak{g}\}$. Now if $h \in E$ is such that $\pi(h)$ fixes no nonzero vector $X \in \mathfrak{g}_{\text{eff}}$, then $\pi(h)|\mathfrak{g}_{\text{eff}} - \operatorname{id}_{\mathfrak{g}_{\text{eff}}} : \mathfrak{g}_{\text{eff}} \to \mathfrak{g}_{\text{eff}}$ is bijective and so for each $Y \in \mathfrak{g}_{\rm eff}$ there is an $X \in \mathfrak{g}_{\rm eff}$ such that $\exp Y = \exp(\mathrm{Ad}(h)X - X) = \mathrm{comm}(h, \exp X)$. □

The proof of (i) yields an exact sequence with $n = |G/G_0|$, $m \dim G_0$:

$$1 \to \mathbb{Z}(n)^m \to E \to G/G_0 \to 1, \quad \text{and}| \quad E| = n^{m+1}.$$

We remark that the existence of the group E is interesting in itself, and we shall later generalize this fact to all compact Lie groups. For a proof of (ii) and the closedness of the commutator group part (i) can be avoided by noting that G_0 is in the kernel of the adjoint representation and that, as a consequence, the adjoint representation induces a representation $\pi_0: G/G_0 \to \mathrm{Gl}(\mathfrak{g})$. The proof of (ii) can be carried out almost verbatim with G/G_0 replacing E and π_0 replacing π.

The Closedness of the Commutator Subgroup

Theorem 6.11. *The algebraic commutator subgroup of a compact Lie group is closed.*

Proof. Let G be a compact Lie group. We shall show first that $(G_0)'$ is closed. Set $H = \overline{(G_0)'}$, and define Z to be the center of G_0. Then $Z \cap H$ is finite by Lemma 6.9 and therefore

$$\mathfrak{z} \cap \mathfrak{h} = \{0\}. \tag{20}$$

By Proposition 5.59 we have $(G_0)' = \langle \exp \mathfrak{g}' \rangle \subseteq H$ and thus

$$\mathfrak{g}' \subseteq \mathfrak{h}. \tag{21}$$

Now Theorem 6.4(v) yields

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'. \tag{22}$$

Now (20), (21) and (22) imply $\mathfrak{g}' = \mathfrak{h}$, whence $(G_0)' = \langle \exp \mathfrak{g}' \rangle = \langle \exp \mathfrak{h} \rangle$. But H is a linear Lie group by Proposition 5.33, whence $H = \langle \exp \mathfrak{h} \rangle$ by 5.41(iii). Hence $(G_0)' = H$ which we claimed.

Now $(G_0)'$ is a characteristic subgroup of G_0 and hence is normal on G. The group $\Gamma \stackrel{\rm def}{=} G/(G_0)'$ is a compact Lie group by Theorem 6.7. The abelian subgroup $G_0/(G_0)'$ is open and connected in Γ and thus is the identity component Γ_0 which is isomorphic to a torus $\mathbb{T}^m$ by 2.42. Clearly $(G_0)' \subseteq G'$, so G' is the full inverse image under the quotient homomorphism $G \to \Gamma$ of the commutator subgroup Γ'. It therefore suffices to observe that Theorem 6.10(iii) applies to Γ and shows that Γ' is closed in Γ. □

In the preceding proof we used Theorem 6.7 saying that quotients of compact Lie groups are compact Lie groups. In this particular instance there is an alternative route: From $G_0 = Z_0(G)(G_0)'$ we get $G_0/(G_0)' \cong Z_0(G)/(Z_0(G) \cap (G_0)'$ and this quotient is the quotient of a torus modulo a discrete subgroup which is again a torus. This argument, too, allows us to conclude that Γ is a compact Lie group.

Corollary 6.12. (i) *If G is a compact Lie group and $\mathfrak{h}$ is a subalgebra of $\mathfrak{L}(G)$ such that $\mathfrak{h}' = \mathfrak{h}$, then $H \stackrel{\text{def}}{=} \langle \exp \mathfrak{h} \rangle$ is closed and $\mathfrak{L}(H) = \mathfrak{h}$. Moreover, $H' = H$.*

(ii) *A connected compact Lie group G satisfies $G'' = G'$.*

Proof. (i) We may assume that $G = \overline{H}$. Then by 5.62 $\mathfrak{g}' \subseteq \mathfrak{h} \trianglelefteq \mathfrak{g}$. Then 6.4(viii) implies $\mathfrak{g}' = \mathfrak{g}'' \subseteq \mathfrak{h}' \subseteq \mathfrak{g}'$, whence $\mathfrak{h} = \mathfrak{h}' = \mathfrak{g}'$. By Theorem 6.11 G' is closed and since H is connected, $G = \overline{H}$ is connected, too. Hence G' is connected and thus $G' = \langle \exp \mathfrak{g}' \rangle$ by 5.60. Thus $H = \langle \exp \mathfrak{h} \rangle = \langle \exp \mathfrak{g}' \rangle = G'$ is closed. The relation $\mathfrak{L}(H) = \mathfrak{h}$ follows from the Recovery of Subalgebras 5.52(iii). Finally, $H' = \exp \mathfrak{h}'$ by 5.60 and H' is closed by 6.11. Then $H' = \exp \mathfrak{h}' = \exp \mathfrak{h} = H$.

(ii) By 6.4(viii) we have $\mathfrak{g}'' = \mathfrak{g}'$. Then we can apply (i) with $\mathfrak{h} = \mathfrak{g}'$ and $G' = H$. □

Exercise E5.19 shows that 6.12 fails if $\mathfrak{h} \neq \mathfrak{h}'$.

The issue of the closedness of the commutator subgroup of an arbitrary compact group is by no means settled to our satisfaction at this point of our discourse. We should bear in mind that, in general, the commutator group of a compact group need not be closed as is discussed in the following exercise. First define the *commutator degree* of a group P to be the smallest natural number n such that every element c of the commutator subgroup P' can be expressed in the form

$$c = \text{comm}(p_1, q_1) \cdots \text{comm}(p_m, q_m), \text{ with } p_j, q_j \in P, \quad 1 \le j \le m \le n.$$

Exercise E6.6. (i) *Fix a prime number $p \neq 2$. Let V_n denote an n-dimensional vector space over $\text{GF}(p)$ and define a finite p-group $P_n = V_n \oplus \bigwedge^2 V_n$ with the multiplication $(x, v)(y, w) = \big(x + y, v + w + 2^{-1}{\cdot}(x \wedge y)\big)$. Then the commutator degree of P_n is $\ge (n-1)/4$. Set $G = \prod_{n \in \mathbb{N}} P_n$. Then $G' \neq \overline{G'}$.*

(ii) *Let n be a natural number ≥ 2 and let $E = \{1, -1\}^n$ for the multiplicative group $\{1, -1\}$ of integers. We identify E with its character group $\widehat{E}$ by setting, for $\chi = (\alpha_1, \ldots, \alpha_n)$, and $e = (a_1, \ldots, a_n)$ in E,*

$$\langle \chi, e \rangle = \langle (\alpha_1, \ldots, \alpha_n), (a_1, \ldots, a_n) \rangle = \prod_{m=1}^{n} \alpha_m a_n \in \{1, -1\} \in \mathbb{S}^1.$$

Let $T \stackrel{\text{def}}{=} \mathbb{T}^{\widehat{E} \setminus \{1\}}$ and let E act on T as follows:

$$t{\cdot}(x_\chi)_{\chi \in \widehat{E} \setminus \{1\}} = (\langle \chi, t \rangle {\cdot} x_\chi)_{\chi \in \widehat{E} \setminus \{1\}} \in T.$$

Let $\varphi\colon E \to \text{Aut}\, T$ be the associated morphism and define $P_n = T \rtimes_\varphi E$. Then P_n is a metabelian Lie group and the commutator degree of P_n is $\ge 2^{n-1}$.

Set $G = \prod_{n=2}^{\infty} P_n$. Then G is a metabelian compact group in which $G' = \text{comm}(G, G_0)$ is not closed.

(iii) Dan Segal [322] proposes a more systematic approach to (ii) above: *Let E be any finite abelian group and $\mathbb{Z}[E]$ the discrete integral group ring of E. Then the additive group of $\mathbb{Z}[E]$ is free abelian on n generators, and so the ground ring* *extension* $A \stackrel{\text{def}}{=} \mathbb{Z}[E] \otimes_{\mathbb{Z}} \mathbb{T}$ *of the* $\mathbb{Z}$*-module* $\mathbb{T}$*, as abelian group, is isomorphic to* $\mathbb{T}^n$ *and thus carries the structure of an* n*-dimensional torus. The group* $E \subseteq \mathbb{Z}(E)$ *acts on the (commutative) ring* $\mathbb{Z}[E]$ *by multiplication and thus on* A *as a group of automorphisms. Hence* $P = A \rtimes E$ *is a compact metabelian group. If* d *is the cardinality of a minimal generating subset of* E*, then* SEGAL *shows that the commutator degree of* P *is* d*.*

Let E_n*,* $n \in \mathbb{N}$*, be any sequence of finite abelian groups such that the sequence of cardinalities* d_n *of a minimal generating set of* E_n *is unbounded. Set* $P_n = (\mathbb{Z}(E_n) \otimes_{\mathbb{Z}} \mathbb{T}) \rtimes E_n$ *and form* $G = \prod_{n\in\mathbb{N}} P_n$*. Then* G *is a metabelian compact group whose commutator subgroup is not closed.*

[Hint. (i) Show $\mathrm{comm}\big((x,v),(y,w)\big) = (0, x\wedge y)$ and $P_n' = \{0\} \oplus \bigwedge^2 V_n$. The set $S_n = \{x \wedge y \mid x, y \in V_n\}$ is closed under scalar multiplication and contains at most p^{2n} elements while $P_n' \cong \bigwedge^2 V_n$ contains $p^{\binom{n}{2}}$ elements. Now $\bigwedge^2 V_n = \underbrace{S_n + \cdots + S_n}_{k \text{ times}}$, whence $\frac{n(n-1)}{2} \le k{\cdot}2n$ and thus $k \ge (n-1)/4$. Hence there are elements in the commutator group of P_n which are products of no fewer than $(n-1)/4$ commutators. Use this to exhibit an element of $\overline{G'}$ which is not a finite product of commutators.

(ii) Let $\pi\colon E \to \mathrm{Gl}(\mathfrak{t})$ the representation of E associated with φ on the Lie algebra $\mathfrak{t} = \mathfrak{L}(T)$. Then the matrix of $\pi(e) - \mathrm{id}_{\mathfrak{t}}$ is diagonal with the diagonal entries d_χ, $\chi \in \widehat{E} \setminus \{1\}$, where

$$d_\chi = \begin{cases} 0 & \text{for } \langle \chi, e\rangle = 1, \\ -2 & \text{for } \langle \chi, e\rangle = -1. \end{cases}$$

Now let

$$N(e) = \mathrm{card}\{\chi \in \widehat{E} \setminus \{1\} : \langle \chi, e\rangle = 1\}.$$

Then $\dim \mathrm{im}(\pi(e) - \mathrm{id}_{\mathfrak{t}}) = 2^n - 1 - N(e)$. The annihilator of $e \ne 1$ in the GF(2)-vector space E is a hyperplane and thus has dimension $n-1$ and therefore contains 2^{n-1} elements. Hence $N(e) = 2^{n-1} - 1$. Therefore

$$\dim \mathrm{im}(\pi(e) - \mathrm{id}_{\mathfrak{t}}) = 2^n - 1 - N(e) = 2^n - 1 - (2^{n-1} - 1) = 2^{n-1}.$$

As a consequence, the sum of M subspaces of the form $\mathrm{im}(\pi(e) - \mathrm{id}_{\mathfrak{t}})$ has a dimension $\le M{\cdot}2^{n-1}$. In order that this sum is equal to $\mathfrak{t}$, we have to have $2^n - 1 \le M{\cdot}2^{n-1}$, that is,

$$M \ge \frac{2^n - 1}{2^{n-1}} = 2^{n-1} - \frac{1}{2^{n-1}}.$$

and so $M \ge 2^{n-1}$ since M is a natural number and $1/2^{n-1} \le 1/2 < 1$.

Recalling the proof of part (ii) of 6.10 we conclude that this means that every element of G_n' is the product of no fewer than 2^{n-1} commutators.

The rest of the claim follows as in (i) above.

Since (i) and (ii) suffice for the main objective of the Exercise, we may leave the details of (iii) to the reader.] □

Lemma 6.13. *A totally disconnected normal subgroup of a connected topological group is central.*

Proof. Exercise E6.7. □

Exercise E6.7. Prove Lemma 6.13.
[Hint. Cf. A4.27] □

If $\mathfrak{n}$ is an ideal of a compact Lie algebra $\mathfrak{g}$, then Theorem 6.4 shows that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$ with $\mathfrak{n}^\perp \cong \mathfrak{g}/\mathfrak{n}$. Now $\mathfrak{g}/\mathfrak{n}$ is centerfree, i.e. semisimple, if and only if $\mathfrak{z}$, the center of $\mathfrak{g}$ is contained in $\mathfrak{n}$. If $\mathfrak{g} = \mathfrak{L}(G)$ for a compact Lie group and N is a closed normal subgroup, then $\mathfrak{n} = \mathfrak{L}(N)$ is an ideal of $\mathfrak{g}$ which contains the center iff an only if G/N is equal to its commutator group.

Proposition 6.14. *Let G denote a connected compact Lie group. Then the following conclusions hold:*

(i) *Assume that $\mathfrak{n} \trianglelefteq \mathfrak{g}$ and that $\mathfrak{z} \subseteq \mathfrak{n}$. Then there are closed normal subgroups $N \trianglelefteq G$ and $S \trianglelefteq G$ such that $\mathfrak{L}(N) = \mathfrak{n}$ and $\mathfrak{L}(G) = \mathfrak{L}(N) \oplus \mathfrak{L}(S)$ and $G = NS$.*

(ii) *The function $(n, s) \mapsto ns: N \times S \to G$ is a homomorphism with a discrete kernel D isomorphic to $N \cap S$. In particular, $G \cong \frac{N \times S}{D}$.*

Proof. (i) We define $\mathfrak{s} = \mathfrak{n}^\perp$. Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{s}$ and since $\mathfrak{z} \subseteq \mathfrak{n}$, we have $\mathfrak{s}' = \mathfrak{s}$ from Theorem 6.4(vii). By 6.4(i), the two ideals $\mathfrak{n}$ and $\mathfrak{s}$ are orthogonal. After 6.12 the group $S \stackrel{\text{def}}{=} \langle \exp \mathfrak{s} \rangle$ is closed, connected and normal in G. Let $\pi: G \to \mathrm{O}(\mathfrak{s})$ denote the representation given by $\pi(g)(X) = \mathrm{Ad}(g)(X)$, $X \in \mathfrak{s}$. From Theorem 5.44 we deduce that the corresponding morphism of Lie algebras $\mathfrak{L}(\pi): \mathfrak{g} \to \mathfrak{o}(\mathfrak{s})$ is given by $\mathfrak{L}(\pi)(Y)(X) = \mathrm{ad}(Y)(X) = [Y, X]$. The kernel of $\mathfrak{L}(\pi)$ is the set of all $Y \in \mathfrak{g}$ with $[Y, X] = 0$ for all $X \in \mathfrak{s}$, i.e. $\ker \mathfrak{L}(\pi) = \mathfrak{z}(\mathfrak{s}, \mathfrak{g}) = \mathfrak{n}$ in view of $\mathfrak{s}' = \mathfrak{s}$.

Set $N \stackrel{\text{def}}{=} (\ker \pi)_0$. Then N is a closed connected normal subgroup of G and is, therefore, a Lie subgroup (cf. 5.33(iv)). By 5.51 we know $\mathfrak{L}(N) = \ker \mathfrak{L}(\pi) = \mathfrak{n}$.

We have $\mathfrak{L}(S) = \mathfrak{s}$ by the Recovery of Subalgebras 5.52(iii). From 5.51 we get $\mathfrak{L}(N \cap S) = \mathfrak{L}(N) \cap \mathfrak{L}(S) = \mathfrak{n} \cap \mathfrak{s} = \{0\}$. Thus the compact Lie group $N \cap S$ is discrete by 5.41(iv). Since the discrete group $N \cap S$ is normal, it is central in G by 6.13. As NS is a closed subgroup we find $\mathfrak{L}(N) + \mathfrak{L}(S) = \mathfrak{g}$, whence $G \subseteq NS$.

(ii) If $n \in N$ and $s \in S$, then $\mathrm{comm}(n, s) = nsn^{-1}s^{-1} \in N \cap S$. Since S is connected, it follows that $\mathrm{comm}(n, S) \subseteq (N \cap S)_0 = \{\mathbf{1}\}$. Thus $\mathrm{comm}(N, S) \subseteq \{\mathbf{1}\}$ and therefore $(n, s) \mapsto ns: N \times S \to G$ is a well-defined morphism of compact Lie groups with image NS. An element (n, s) is in its kernel D iff $n = s^{-1} \in N \cap S$. Thus $n \mapsto (n, n^{-1}): N \cap S \to D$ is an isomorphism and the remainder of (ii) follows. □

The Structure Theorem for Compact Lie Groups

The Structure Theorem for compact Lie groups

Theorem 6.15. *Let G be a compact Lie group, $\mathfrak{L}(G)$ its Lie algebra, $Z(G)$ its center, $Z_0(G)$ the identity component of the center, and G' the commutator subgroup of G. Let $\mathfrak{g} = \mathfrak{g}_{\rm fix} \oplus \mathfrak{g}_{\rm eff}$ denote the standard decomposition of the Lie algebra of G as the adjoint G-module (cf. Theorem 3.36(vi)). Then the following conclusions hold:*

(i) $\mathfrak{g}_{\rm fix} = \mathfrak{L}\big(Z(G)\big) \subseteq \mathfrak{z}(\mathfrak{g})$ *and* $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}_{\rm eff} = \mathfrak{L}(G')$.

(ii) $\mathfrak{g} = \mathfrak{L}\big(Z(G)\big) \oplus \mathfrak{L}(G')$.

(iii) $G_0 = Z_0(G)(G')_0 \subseteq Z(G)G'$ *and* $Z(G)\cap G'$ *is finite. In other words,* G_0 *is isomorphic to the identity component of the group*

$$\frac{Z(G)\times G'}{\Delta}, \qquad \Delta \cong Z(G)\cap G'.$$

(iv) *The subgroup $Z_0(G)$ is a torus, $G_0 = Z_0(G)(G')_0$, and $Z_0(G)\cap(G')_0$ is finite. In other words, G_0 is isomorphic to the group*

$$\frac{Z_0(G)\times (G')_0}{\Omega}, \qquad \Omega \cong Z_0(G)\cap (G')_0.$$

Proof. (i) Firstly, let us abbreviate $\mathrm{Ad}(g)(X)$ by gX for $X\in\mathfrak{g}$ and note that X generates a one-parameter subgroup of $Z(G)$ if and only if $\exp t{\cdot}X = g\exp t{\cdot}X g^{-1}$ $=\exp t{\cdot}gX$ for all $\mathfrak{t}\in\mathbb{R}$. This is equivalent to $gX = X$ and thus $\mathfrak{g}_{\rm fix} = \mathfrak{L}\big(Z(G)\big)$. By 5.54(iv) we have $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{L}(G_0) = \mathfrak{L}\big(Z(G_0)\big)$. Since $Z(G)\cap G_0 \subseteq Z(G_0)$ we have $\mathfrak{L}\big(Z(G)\big)\subseteq \mathfrak{L}\big(Z(G_0)\big) = \mathfrak{z}(\mathfrak{g})$. Secondly, $(G')_0\subseteq G'$ whence $\mathfrak{g}' = \mathfrak{L}\big((G')_0\big)\subseteq \mathfrak{L}(G')$ in view of 5.60. Next let us show $\mathfrak{g}_{\rm eff}\subseteq\mathfrak{L}(G')$. We note that for all $X\in B\cap g^{-1}B$ for an open ball B around 0 in $\mathfrak{g}$ with $B*B$ defined we have $t{\cdot}(gX-X) = \lim n(\frac{t}{n}{\cdot}gX * \frac{-t}{n}{\cdot}X)$ for all $|t|\le 1$ by the Recovery of Addition 5.10. Hence Theorem 6.11 stating the closedness of the commutator group of G yields

$$\begin{aligned}\exp t{\cdot}(gX-X) &= \lim\big((\exp \tfrac{t}{n}{\cdot}gX)(\exp\tfrac{-t}{n}{\cdot}X)\big)^n\\ &= \lim\big(g(\exp\tfrac{t}{n}{\cdot}X)g^{-1}(\exp\tfrac{t}{n}{\cdot}X)^{-1}\big)^n \in \overline{G'} = G'\end{aligned}$$

for all $|t|\le 1$. We conclude that $gX - X\in\mathfrak{L}(G')$ for all $X\in B\cap g^{-1}B$ and thus for all $X\in\mathfrak{g}$ since $\mathfrak{g}$ is spanned by $B\cap g^{-1}B$. But $g\in G$ was arbitrary. It follows that $\mathfrak{g}_{\rm eff}\subseteq\mathfrak{L}(G')$. Now, on the one hand, $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{g}_{\rm fix} = \mathfrak{L}\big(Z(G)\big)$ and $\mathfrak{g}_{\rm eff}$ by Theorem 4.8, and $\mathfrak{g}_{\rm eff}\subseteq\mathfrak{L}(G')$ on the other. But $\mathfrak{L}\big(Z(G)\big)\cap\mathfrak{L}(G') = \mathfrak{L}\big(Z(G)\cap G'\big) = \{0\}$ by Lemma 6.9 and Theorem 6.11. This implies $\mathfrak{g}_{\rm eff} = \mathfrak{L}(G')$ as asserted.

(ii) This follows at once from (i) and Theorem 3.36.

(iii) Note $G_0 = \langle\exp\mathfrak{g}\rangle$. Further, $\mathfrak{g} = \mathfrak{L}\big(Z(G)\big)\oplus\mathfrak{L}(G')$, and $\mathfrak{L}\big(Z(G)\big)\subseteq\mathfrak{z}(\mathfrak{g})$. Thus $\exp\mathfrak{g}\subseteq\langle\exp\mathfrak{L}\big(Z(G)\big)\rangle\langle\exp\mathfrak{L}(G')\rangle = Z_0(G)(G')_0\subseteq Z(G)G'$. By Lemma 6.9 and Theorem 6.11 the intersection of $Z(G)$ and G' is finite. Now we consider the surjective morphism $\mu\colon Z(G)\times G'\to Z(G)G'$ of compact groups given by

$\mu(z,g) = zg$. An element $(z,g) \in Z(G) \times G'$ is in $\ker\mu$ if and only if $zg = \mathbf{1}$, i.e. $z = g^{-1}$. In other words, $\ker\mu = \{(z^{-1}, z) \mid z \in Z(G) \cap G'\}$. The morphism $\delta\colon Z(G) \cap G' \to Z(G) \times G'$ given by $\delta(z) = (z^{-1}, z)$ is therefore an isomorphism onto $\ker\mu$. The assertion now follows from the Canonical Decomposition Theorem of morphisms.

(iv) is proved in the same manner as (iii); recall from 2.42(ii) that a connected compact abelian Lie group is a torus. The small adjustments required can be safely left to the reader. □

The inequalities in Part (i) of the preceding theorem may indeed be proper as is illustrated by the continuous dihedral group $G = \mathbb{T} \rtimes \{1, -1\}$.

A frequently used corollary of Theorem 6.15 is the structure theorem for connected compact Lie groups:

The First Structure Theorem for Connected Compact Lie Groups

Corollary 6.16. *Let G denote a connected compact Lie group, Z_0 the identity component of its center, and G' the algebraic commutator subgroup of G. Then G' is a closed Lie subgroup and $G = Z_0G'$ and $Z_0 \cap G'$ is finite. More specifically, there exists an exact sequence*

$$\{1\} \to Z_0 \cap G' \stackrel{\delta}{\to} Z_0 \times G' \stackrel{\mu}{\to} G \to \{1\},$$

where $\delta(z) = (z^{-1}, z)$ and $\mu(z,g) = zg$. In other words, G is the factor group of the direct product $Z_0 \times G'$ modulo a finite central subgroup Δ isomorphic to $Z_0 \cap G'$:

$$G \cong \frac{Z_0 \times G'}{\Delta}, \qquad \Delta \cong Z_0 \cap G',$$

where Z_0 is a torus.

Proof. This is an immediate consequence of Theorem 6.15. □

Theorem 6.4 has not yet been completely exploited insofar as it gives us precise information on the structure of the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ which is an orthogonal direct sum of simple ideals the set of which is uniquely determined.

Definition 6.17. (i) A connected compact Lie group G is called *simple* if every proper normal subgroup is discrete (hence central by 6.13).

(ii) It is called *semisimple* if $\{\mathbf{1}\}$ is the only connected central proper subgroup. □

Let us repeat that the definition of a simple connected compact Lie group deviates somewhat from the use of the term "simple group" in the algebraic theory of groups where simplicity means the absence of nontrivial normal subgroups; in our context simplicity means the absence of nontrivial connected compact normal subgroups. Therefore, as we pointed out before, simple connected compact Lie groups are occasionally called *quasisimple* in order to distinguish them from abstract simple groups.

The Commutator Subgroup of a Connected Compact Lie Group

Theorem 6.18. (i) *If G is a connected compact Lie group, then G' is a semisimple connected compact Lie group and there are simple normal connected compact Lie subgroups $S_1, \ldots, S_n$ of G such that the morphism of compact groups*

$$\mu\colon S_1 \times \cdots \times S_n \to G', \quad \mu(s_1, \ldots, s_n) = s_1 \cdots s_n \tag{23}$$

is surjective and has a finite central kernel isomorphic to a discrete central subgroup Δ of $S_1 \times \cdots \times S_n$. In particular,

$$G' \cong \frac{S_1 \times \cdots \times S_n}{\Delta}. \tag{24}$$

The set $\{S_1, \ldots, S_n\}$ is uniquely determined by these properties.

(ii) *The quotient morphism $\mu\colon S_1 \times \cdots \times S_n \to G'$ induces an isomorphism $\mathfrak{L}(\mu)\colon \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n \to \mathfrak{g}$.*

(iii) *For each connected compact normal subgroup N of G' there is a unique set $\{m_1, \ldots, m_k\}$, $1 \le m_1 < \cdots < m_k \le n$ such that*

$$N = S_{m_1} \cdots S_{m_k} = \mu(H_1 \times \cdots \times H_n), \quad H_j = \begin{cases} S_{m_p} & \textit{if } j = m_p \textit{ for } p \in \{1, \ldots, k\}, \\ \{1\} & \textit{otherwise.} \end{cases}$$

Proof. Note that By 5.49 and 6.4(iii), a connected compact subgroup N of G' is normal in G if and only if it is normal in G'.

(i) We know $\mathfrak{L}(G') = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ with a uniquely determined set of simple ideals $\mathfrak{s}_j$. (Theorem 6.4 and Theorem 6.11.) If A is a connected central subgroup of G' then so is $\overline{A}$, and $\mathfrak{L}(\overline{A})$ is a central ideal of $\mathfrak{g}'$ (cf. 5.54(iv)). But $\{0\}$ is the only such by what we just observed. Hence $\overline{A}$ is singleton (cf. 5.41(iii)) and thus A is singleton. Therefore G', being closed by 6.11, is a semisimple connected compact Lie group.

From Proposition 6.14 we deduce the existence of a closed connected normal subgroup N with $\mathfrak{L}(N) = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{n-1}$ and of a closed connected normal subgroup S_n with $\mathfrak{L}(S_n) = \mathfrak{s}_n$.

Proceeding by induction we find closed connected normal Lie subgroups S_j with $\mathfrak{L}(S_j) = \mathfrak{s}_j$ for $j = 1, \ldots, n-1$. The map μ in (23) is a well-defined morphism of compact groups. The Lie algebra of its image is $\mathfrak{s}_1 + \cdots + \mathfrak{s}_n = [\mathfrak{g}, \mathfrak{g}]$ and this image, therefore, agrees with G' on account of $\mathfrak{L}(G') = [\mathfrak{g}, \mathfrak{g}]$. Its kernel $\ker \mu$ is a Lie subgroup. Since $\mathfrak{L}(\mu)\colon \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_n \to [\mathfrak{g}, \mathfrak{g}]$ is given by $\mathfrak{L}(\mu)(X_1, \ldots, X_n) = X_1 + \cdots + X_n$, we know that $\mathfrak{L}(\mu)$ is an isomorphism and thus has zero kernel.

From Proposition 5.41(iii) it follows that Δ is discrete. From Proposition 6.14 we know that it must be central. Now conclusion (24) is a consequence, and the uniqueness of the set $\{S_1, \ldots, S_n\}$ of the groups S_j follows from the uniqueness of the set $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_n\}$ of their Lie algebras.

(ii) This is a reformulation of the fact that $\mathfrak{L}(\mu)\colon \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_n \to \mathfrak{s}_1 \oplus \cdots \oplus s_n$ is an isomorphism.

(iii) We assume that N is a closed connected normal subgroup of G'. Then $N = \langle \exp \mathfrak{L}(N) \rangle$ and $\mathfrak{L}(N)$ is an ideal in $\mathfrak{g}'$ by 5.49. Then $\mathfrak{L}(N) = \mathfrak{s}_{m_1} \oplus \cdots \oplus \mathfrak{s}_{m_k}$

for certain $1 \leq m_1 < \cdots < m_k \leq n$ by 6.4(vii). We conclude that $N = S_{m_1} \cdots S_{m_k}$. Consider the Lie group $G_k = \{1\} \times \cdots \times S_k \times \cdots \times \{1\} \cong S_k$. Then $\mu(G_k) = S_k$ and $N = \mu(G_{m_1} \cdots G_{m_k}) = \mu(H_1 \times \cdots \times H_n)$. □

Theorem 6.19. *Every connected compact Lie group is the quotient of the direct product of a torus and a finite set of simple connected compact Lie groups modulo a finite central subgroup.*

Proof. Exercise E6.8. □

Exercise E6.8. Prove Theorem 6.19. □

Maximal Tori

As we shall see in this subsection, the maximal connected compact abelian subgroups of compact Lie groups are crucial building blocks. We start off with a few very simple observations:

Lemma 6.20. (i) *Every connected abelian group $A \subseteq G$ of a compact group G (in particular, every subgroup of the form $\exp \mathbb{R}{\cdot}X$, $X \in \mathfrak{L}(G)$ for a compact Lie group) is contained in a maximal connected abelian subgroup T which is compact.*

(ii) *Every abelian Lie subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ (in particular every one-dimensional subalgebra $\mathbb{R}{\cdot}X$, $X \in \mathfrak{g}$), is contained in a maximal abelian subalgebra $\mathfrak{t}$.*

(iii) *If T is a maximal connected compact abelian subgroup of a compact Lie group G, then $\mathfrak{L}(T)$ is a maximal abelian subalgebra of $\mathfrak{g}$.*

(iv) *If $\mathfrak{t}$ is a maximal abelian subalgebra of the Lie algebra $\mathfrak{g}$ of a compact Lie group G, then $\exp \mathfrak{t}$ is a maximal connected compact abelian subgroup of G.*

Proof. (i) The set of all connected abelian subgroups of a topological group G containing a connected abelian subgroup A is inductive: The union of a tower is a connected abelian subgroup containing A. Hence by Zorn's Lemma there are maximal ones, say T. Then $\overline{T}$ is connected and abelian, so by maximality $T = \overline{T}$. If G is compact, this means the maximal connected abelian subgroups are compact.

(ii) The proof follows again with the aid of Zorn's Lemma.

(iii) Assume that T is a maximal connected compact subgroup of G and that $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ with $\mathfrak{L}(T) \subseteq \mathfrak{a}$. We must show that $\mathfrak{a} \subseteq \mathfrak{L}(T)$. Now $\overline{\exp \mathfrak{a}}$ is a connected compact abelian subgroup of G containing $\exp \mathfrak{L}(T)$ and thus T by 5.41(iii). From the maximality of T we get $T = \overline{\exp \mathfrak{a}}$. Hence $\mathfrak{a} \subseteq \mathfrak{L}(\overline{\exp \mathfrak{a}}) = \mathfrak{L}(T)$.

(iv) Assume that $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and that A is a connected closed subgroup of G containing $\exp \mathfrak{t}$. We claim $A = \exp \mathfrak{t}$ which will show that $\exp \mathfrak{t}$ is a maximal connected abelian subgroup (and is closed). Now $\overline{\exp \mathfrak{t}}$ is an abelian Lie subgroup (see 5.33(iii)), and $\mathfrak{t} \subseteq \mathfrak{L}(\overline{\exp \mathfrak{t}}) \subseteq \mathfrak{L}(A)$. By the maximality of $\mathfrak{t}$ we have $\mathfrak{L}(A) = \mathfrak{t}$ and thus $\exp \mathfrak{L}(A) = \exp \mathfrak{t}$. But A is a compact Lie group (see 5.33(iii)) and is abelian, whence $A = \exp \mathfrak{L}(A)$ by 5.41(iii). Thus $A = \exp \mathfrak{t}$ as asserted. □

For finite dimensional Lie algebras $\mathfrak{g}$ the Axiom of Choice is not required for a proof of (ii): Every ascending chain of abelian subalgebras containing $\mathfrak{a}$ will terminate after a finite number of members since $\dim \mathfrak{g} < \infty$. The last member will be maximal abelian and contain $\mathfrak{a}$. (iii) and (iv) do not require the Axiom of Choice.

Lemma 6.21. (i) *A maximal connected abelian subgroup of a compact Lie group is a torus.*

(ii) *Let $\mathfrak{g}$ be a compact Lie algebra and $\mathfrak{t}$ a maximal abelian subalgebra. Then $\mathfrak{t}$ is its own normalizer and its own centralizer.*

(iii) *In a compact Lie group G a maximal connected abelian subgroup T is open in its normalizer $N(T,G)$ and thus has finite index in it. The centralizer $Z(T,G_0)$ of T in the identity component is T itself.*

(iv) *If T is a maximal connected abelian subgroup of a compact Lie group G, then there is a finite group E such that $N(T,G) = TE$ and $T \cap E \trianglelefteq N(T,G)$. Moreover, $T \cap E = T[w]$, $w = |N(T,G)/T|$, and $E/(E \cap T) \cong N(T,G)/T$.*

Proof. (i) is immediate from 2.42.

For a proof of (ii) write $\mathfrak{n}(\mathfrak{t},\mathfrak{g}) = \mathfrak{t} \oplus \mathfrak{s}$ with $\mathfrak{s} = \mathfrak{t}^\perp \cap \mathfrak{n}(\mathfrak{t},\mathfrak{g})$. Then $[\mathfrak{t},\mathfrak{s}] = \{0\}$ by Theorem 6.4(i). Assume now that $\mathfrak{t} \neq \mathfrak{n}(\mathfrak{t},\mathfrak{g})$ and pick any nonzero $X \in \mathfrak{s}$. Then $[\mathfrak{t}, X] = \{0\}$, and thus $\mathfrak{t} \oplus \mathbb{R}{\cdot}X$ is abelian in contradiction with the maximality of $\mathfrak{t}$. Since $\mathfrak{t} \subseteq \mathfrak{z}(\mathfrak{t},\mathfrak{g}) \subseteq \mathfrak{n}(\mathfrak{t},\mathfrak{g}) = \mathfrak{t}$ we also have $\mathfrak{t} = \mathfrak{z}(\mathfrak{t},\mathfrak{g})$.

(iii) By 5.54(i)(24) we have $\mathfrak{L}\big(N(T,G)\big) = \mathfrak{n}(\mathfrak{L}(T),\mathfrak{g})$. By 6.20(iii) $\mathfrak{L}(T)$ is a maximal abelian subalgebra. Then $\mathfrak{n}(\mathfrak{L}(T),\mathfrak{g}) = \mathfrak{L}(T)$ by (ii) above. Thus $N(T,G)_0 = T$, and since $N(T,G)$ is a Lie group, T is open in $N(T,G)$. Thus $N(T,G)/T$ is discrete and compact, hence finite.

(iv) By (iii) we have $N(T,G)_0 = T$. The assertion then follows from Theorem 6.10(i). □

Exercise E6.9. (a) In view of the remarks following 6.20, formulate a proof of 6.20(i) for a compact Lie group G not using the Axiom of Choice.

(b) Verify the details of the construction described in the following discussion.

(i) One constructs examples for the possible $N(T,G)$ as follows. Let F be a finite group with an abelian normal subgroup A; then F/A acts on A via $(fA){\cdot}a = faf^{-1}$. Assume that we have an injective morphism $a \mapsto a_+ \colon A \to T$ into a torus T (additively written) such that F/A acts on T so that $(fA) \bullet a_+ = \big((fA){\cdot}a\big)_+$. Form the semidirect product $T \rtimes F$ with multiplication $(v,f)(w,g) = (v + (fA) \bullet w, fg)$. The subset $N = \{(-a_+, a) \mid a \in A\}$ is a subgroup isomorphic to A. Since $(0,f)(v,g)(0,f)^{-1} = ((fA) \bullet v, (fA){\cdot}g)$ we have $(0,f)(a_+,a)(0,f)^{-1} = ((fA){\cdot}a)_+, (fA){\cdot}a)$. Thus N is normal and we can form $G = \frac{T \rtimes F}{N}$. Then $G_0 = \frac{T \times A}{N} \cong T$ and with $W = \frac{A_+ \times F}{N} \cong F$ we have $G = G_0 W = W G_0$ and $G_0 \cap W =$

$\frac{A_+ \times A}{N} \cong A$. This does not and cannot assert, that G_0 fails to split as a semidirect factor, but there are certainly examples where it does not. Take a prime p and let F be the Heisenberg group of all matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathrm{GF}(p).$$

Let $T = \mathbb{T}$ and take A the center of F and $A_+ = \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ with trivial actions everywhere. The resulting group does not split and the intersection of G_0 and W cannot be removed, no matter how W is chosen.

(ii) Consider the following special case: Take $F = P_n$ of Exercise E6.6 and let A be the center and commutator subgroup $\bigwedge^2 V_n \cong \mathbb{Z}(p)^{\binom{n}{2}}$. Let $T = \mathbb{T}^{\binom{n}{2}}$, let the action of F/A be trivial. Then $G_0 \cap W \cong \bigwedge^2 V_n$ is the commutator group G'; it is contained in G_0, there are elements in it which are products of no fewer than $\frac{n-1}{4}$ commutators (see E6.6). □

This example shows that there are compact Lie groups G such that not every element of $G' \cap G_0$ is itself a commutator. This is to be seen in contrast with the Commutator Theorem for Connected Compact Lie Groups 6.55 below.

Definition 6.22. (i) If G is a compact group, let $\mathcal{T}(G)$ denote the set of all maximal connected abelian subgroups of G and if $\mathfrak{g}$ is a Lie algebra we write $\mathfrak{T}(\mathfrak{g})$ for the maximal abelian Lie subalgebras of $\mathfrak{g}$.

(ii) For a compact Lie group G, a member of $\mathcal{T}(G)$ is called a *maximal torus*. For a compact Lie algebra $\mathfrak{g}$ a member of $\mathfrak{T}(\mathfrak{g})$ is called a *Cartan subalgebra*.

(iii) For a maximal torus T of a compact Lie group G, the finite factor group $N(T,G)/T$ is called the *Weyl group* of G with respect to T and is written $\mathcal{W}(T,G)$. □

One can define the concept of a Cartan subalgebra for any finite dimensional Lie algebra whatsoever as follows. A subalgebra of a finite dimensional Lie algebra is called a *Cartan subalgebra* if it is nilpotent and agrees with its own normalizer. In view of 6.21(ii), for compact Lie algebras this concept agrees with the one introduced in 6.22(ii).

Notice from 6.21(iv) that the group $N(T,G)$ contains a finite group E such that $\mathcal{W}(T,G) \cong E/(E \cap T)$.

Proposition 6.23. *For a compact Lie group G, the functions $T \mapsto \mathfrak{L}(T) : \mathcal{T}(G) \to \mathfrak{T}(\mathfrak{g})$ and $\mathfrak{t} \mapsto \exp \mathfrak{t} : \mathfrak{T}(\mathfrak{g}) \to \mathcal{T}(G)$ are well-defined bijections which are inverses of each other.*

Proof. This is immediate from Lemma 6.20. □

Lemma 6.24. *Every Cartan subalgebra $\mathfrak{t}$ of a compact Lie algebra $\mathfrak{g}$ contains an element Y such that $\mathfrak{t} = \mathfrak{z}(Y, \mathfrak{g})$.*

Proof. By 6.6 there is a connected compact Lie group G such that $\mathfrak{g} = \mathfrak{L}(G)$. Let $\mathfrak{t}$ be a Cartan subalgebra. Then $T \stackrel{\text{def}}{=} \exp \mathfrak{t}$ is a maximal torus by 6.20(iii). By 1.24(v), the torus T contains an element t such that the group $\langle t \rangle$ generated by it is dense in T. Since $\exp\colon \mathfrak{t} \to T$ is surjective we find a Y such that $t = \exp Y$. Now $Z(t, G) = Z(\langle t \rangle, G) = Z(T, G)$ and $\mathfrak{t} \subseteq \mathfrak{L}\big(Z(T, G)\big) \subseteq \mathfrak{L}\big(N(T, G)\big) = \mathfrak{n}(\mathfrak{t}, \mathfrak{g}) = \mathfrak{t}$ by 5.54(i)(24) and 6.21(ii). Since $t \in \exp \mathbb{R}{\cdot}Y \subseteq T$ we also have $Z(\exp \mathbb{R}{\cdot}Y, G) = Z(T, G)$ and thus $\mathfrak{t} = \mathfrak{L}\big(Z(\exp \mathbb{R}{\cdot}Y, G)\big) = \mathfrak{z}(Y, \mathfrak{g})$ by 5.54(ii)(25). □

Lemma 6.25 (Hunt's Lemma [208]). *Let G be a compact Lie group, and X and Y two arbitrary elements of its Lie algebra $\mathfrak{g}$. Then there is an element $g \in G$ such that* $[\mathrm{Ad}(g)X, Y] = 0$.

Proof. We consider on $\mathfrak{g}$ an invariant scalar product according to 6.2 and note that the continuous real valued function $g \mapsto (\mathrm{Ad}(g)(X) \mid Y) : G \to \mathbb{R}$ on the compact space G attains a minimum in, say $g \in G$. We take an arbitrary $Z \in \mathfrak{g}$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(t) = \big((e^{t\cdot \mathrm{ad}\, Z})\, \mathrm{Ad}(g)(X) \mid Y\big) = (\mathrm{Ad}\,\big((\exp_G t{\cdot}Z)g\big)(X) \mid Y)$. Then f attains its minimal value for $t = 0$. Hence

$$0 = f'(0) = \big((\mathrm{ad}\, Z)\, \mathrm{Ad}(g)(X) \mid Y\big) = \big([\mathrm{Ad}(g)(X), Y] \mid Z\big)$$

by 6.2(2). Since Z was arbitrary and the scalar product is nondegenerate, the relation $[\mathrm{Ad}(g)(X), Y] = 0$ follows. □

Lemma 6.26. *Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra. For any $X \in \mathfrak{g}$ there is a $g \in G$ such that* $\mathrm{Ad}(g)X \in \mathfrak{t}$.

Proof. By Lemma 6.24 there is a $Y \in \mathfrak{t}$ such that $\mathfrak{t} = \mathfrak{z}(Y, \mathfrak{g})$. By Lemma 6.25 we find a $g \in G$ such that $[\mathrm{Ad}(g)(X), Y] = 0$, i.e. that $\mathrm{Ad}(g)(X) \in \mathfrak{z}(Y, \mathfrak{g}) = \mathfrak{t}$. □

The Transitivity Theorem

Theorem 6.27. *A compact Lie group G operates transitively on $\mathcal{T}(G)$ and $\mathfrak{T}(\mathfrak{g})$ such that the functions of Proposition* 6.23 *are equivariant. Further, for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{g})$ we have*

$$\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{t}. \tag{25}$$

Proof. Clearly G acts on the left on $\mathcal{T}(G)$ via $g \bullet T = gTg^{-1}$ and on $\mathfrak{T}(\mathfrak{g})$ via $g \bullet \mathfrak{t} = \mathrm{Ad}(g)(\mathfrak{t})$. Because $g(\exp \mathfrak{t})g^{-1} = \exp \mathrm{Ad}(g)(\mathfrak{t})$ and $\mathfrak{L}(gTg^{-1}) = \mathrm{Ad}(g)L(T)$ by 5.44 the two functions of Proposition 6.23 are equivariant.

Let $\mathfrak{t}_1$, $\mathfrak{t}_2 \in \mathfrak{T}(\mathfrak{g})$. By Lemma 6.21(ii) there are elements Y_j such that $\mathfrak{t}_j = \mathfrak{z}(Y_j, \mathfrak{g})$ for $j = 1, 2$. By 6.23 there is a $g \in G$ such that $\mathrm{Ad}(g)Y_1 \in \mathfrak{z}(Y_2, \mathfrak{g}) = \mathfrak{t}_2$. Then $\mathfrak{t}_2 \subseteq \mathfrak{z}(\mathrm{Ad}(g)Y_1, \mathfrak{g}) = \mathrm{Ad}(g)\mathfrak{z}(Y_1, \mathfrak{g}) = \mathrm{Ad}(g)\mathfrak{t}_1$. Because of the maximality of $\mathfrak{t}_2$ we have $\mathfrak{t}_2 = g \bullet \mathfrak{t}_1$. Thus the operation of G on $\mathfrak{T}(\mathfrak{g})$ is transitive. Since as G-sets, $\mathcal{T}(G)$ and $\mathfrak{T}(\mathfrak{g})$ are isomorphic by what we saw in the first part of the proof, the action of G on $\mathcal{T}(G)$ is transitive, too.

The last assertion of the theorem is a direct consequence of Lemma 6.26. □

Definition 6.28. By the commutativity of the torus T, the function $(g,t) \mapsto gtg^{-1}: G \times T \to G$ is constant on the sets $gT \times \{t\}$, $(g,t) \in G \times T$ and thus factors through $G/T \times T$, $G/T = \{gT \mid g \in G\}$.

We shall denote the continuous function which associates with (gT, t) the unique element gtg^{-1} by $\omega: G/T \times T \to G$. □

For compact groups, the map $\omega: G/T \times T \to G$ is almost as important as is the exponential function $\exp_G: \mathfrak{g} \to G$.

Lemma 6.29. *Let G be a compact Lie group with a maximal torus T and consider the following continuous functions:*

(i) $\exp: \mathfrak{g} \to G$, *and*

(ii) $\omega: G/T \times T \to G$.

Then both have the set $\exp_G \mathfrak{g} = \operatorname{im} \exp_G$ *as image.*

In particular, $\operatorname{im} \exp_G$ *is a compact subset of G.*

Proof. We compute $\omega(G/T \times T) = \bigcup_{g \in G} gTg^{-1} = \bigcup_{g \in G} \exp_G \operatorname{Ad}(g)\mathfrak{t} = \exp_G \bigcup_{g \in G} \operatorname{Ad}(g)\mathfrak{t} = \exp_G \mathfrak{g}$ by (25) in the Transitivity Theorem 6.27. □

The central theorem in this area now is the following

The Maximal Torus Theorem

Theorem 6.30. *Let G be a compact Lie group with Lie algebra $\mathfrak{L}(G) = \mathfrak{g}$. Then*

$$G_0 = \exp \mathfrak{g}. \tag{26}$$

Let T be any maximal torus of G. Then

$$G_0 = \bigcup_{g \in G} gTg^{-1}. \tag{27}$$

Proof. By Lemma 6.29 the conditions (26) and (27) are equivalent and are also equivalent to the following condition

$$\exp \mathfrak{g} \text{ is open in } G \tag{$*$}$$

which in turn is equivalent to

$$(\forall X \in \mathfrak{g}) \quad \exp \mathfrak{g} = \bigcup_{g \in G} gTg^{-1} \text{ is a neighborhood of } \exp X \text{ in } G. \tag{$**$}$$

We shall prove this by induction on the dimension of $\mathfrak{g}$ proceeding as follows. We suppose that the assertion is false and that G is a counterexample with minimal $\dim \mathfrak{g}$, and we shall derive a contradiction. ("Principle of the smallest criminal")

Firstly, $\dim \mathfrak{g} > 1$ since $\dim \mathfrak{g} = 1$ would mean that G_0 is a circle group, whence G could not be a counterexample to (26). Obviously there is no loss in generality to assume that G is connected.

As G is a counterexample to ($**$), there is an $X \in \mathfrak{g}$ such that $x \stackrel{\text{def}}{=} \exp X$ is a boundary point of $\exp \mathfrak{g}$ in G, since $\exp \mathfrak{g}$ is closed in G by Lemma 6.26. By 6.27(25), we may assume that $X \in \mathfrak{t} \stackrel{\text{def}}{=} \mathfrak{L}(T)$ and thus $x \in T$. Now $S \stackrel{\text{def}}{=} Z(x, G)_0$ is a connected linear Lie subgroup of G and $T \subseteq S$. If $\dim \mathfrak{L}(S) < \dim \mathfrak{L}(G)$, then the compact Lie group S is not a counterexample to the theorem. Hence $S = \exp \mathfrak{z}(X, \mathfrak{g}) = \bigcup_{s \in S} sTs^{-1}$. The automorphism $\operatorname{Ad} x$ of the Hilbert Lie algebra $\mathfrak{g}$ is orthogonal by 6.2(1) and hence is semisimple (see also the remarks following 6.3). Then Proposition 5.55 shows that $\bigcup_{g \in G} gSg^{-1}$ is a neighborhood of x in G. Thus

$$\bigcup_{g \in G} gTg^{-1} = \bigcup_{g \in G} g\Big(\bigcup_{s \in S} sTs^{-1}\Big)g^{-1} = \bigcup_{g \in G} gSg^{-1}$$

is a neighborhood of x in G. This contradicts our assumption that x is a boundary point of $\exp \mathfrak{g}$. Hence $\dim \mathfrak{L}(S) = \dim \mathfrak{L}(G)$ and thus $\mathfrak{L}(S) = \mathfrak{L}(G)$ and so $S = G_0$, that is, $x \in Z(G)$. Let $Y \in \mathfrak{L}(G)$. Then by Theorem 6.27(25) again, there is an $h \in G$ such that $Y \in \operatorname{Ad}(h)\mathfrak{t}$ and $\exp Y \in hTh^{-1}$. As $x \in Z(G)$ we have $x = hxh^{-1} \in hTh^{-1}$ and thus $x \exp Y \in hTh^{-1} \subseteq \exp \mathfrak{g}$. Since $\exp \mathfrak{g}$ is a neighborhood of 1, the translate $x \exp \mathfrak{g}$ is a neighborhood of x, but this again contradicts our assumption that x is a boundary point of $\exp \mathfrak{g}$. □

This theorem has several direct and important consequences.

Corollary 6.31. (i) *If H is a closed subgroup of a compact Lie group G, then $H_0 = \exp \mathfrak{L}(H)$.*

(ii) *If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ with $\mathfrak{h}' = \mathfrak{h}$, then $\exp \mathfrak{h}$ is a closed Lie subgroup with $\mathfrak{L}(\exp \mathfrak{h}) = \mathfrak{h}$.*

(iii) *In particular, for a connected compact group G, $G' = \exp \mathfrak{g}'$.*

Proof. (i) H_0 is a connected compact Lie group with Lie algebra $\mathfrak{L}(H)$, and so the first assertion is obvious from 6.30.

(ii) If $\mathfrak{h} = \mathfrak{h}' \leq \mathfrak{g}$, then $H \stackrel{\text{def}}{=} \langle \exp \mathfrak{h} \rangle$ is a closed subgroup with $\mathfrak{L}(H) = \mathfrak{h}$ by 6.12. The claim then follows from (i).

(iii) We have $\mathfrak{g}'' = \mathfrak{g}'$ by 6.4(viii). Thus (ii) applies to $\mathfrak{g}'$ and shows that $\exp \mathfrak{g}'$ is a closed subgroup. But then $\exp \mathfrak{g}' = \langle \exp \mathfrak{g}' \rangle = G'$ by 5.60. □

For connected compact Lie groups, this in fact provides a second proof for the closedness of the commutator groups (cf. the Closedness of the Commutator Subgroup 6.11).

Corollary 6.32. *Let G be a compact Lie group.*

(i) *For each $g \in G_0$ the following statements hold.*

(28a) $$\mathcal{T}\big(Z(g, G)\big) = \{T \in \mathcal{T}(G) \mid g \in T\},$$

(28b) $$Z(g,G)_0 = \bigcup\{T \mid T \in \mathcal{T}(G) \text{ and } g \in T\}.$$

(ii) *The center of a connected compact Lie group is contained in every maximal torus. More precisely,*

(29) $$Z(G) = \bigcap\{T \mid T \in \mathcal{T}(G)\}.$$

(I) *For each $X \in \mathfrak{g}$ we have*

(28aL) $$\mathfrak{T}(\mathfrak{z}(X,\mathfrak{g})) = \{\mathfrak{t} \in \mathfrak{T}(\mathfrak{g}) \mid X \in \mathfrak{t}\},$$

(28bL) $$\mathfrak{z}(X,\mathfrak{g}) = \bigcup\{\mathfrak{t} \mid \mathfrak{t} \in \mathfrak{T}(\mathfrak{g}) \text{ and } X \in \mathfrak{t}\}.$$

(II) *The center of a compact Lie algebra is contained in every Cartan subalgebra. More precisely,*

(29L) $$\mathfrak{z}(\mathfrak{g}) = \bigcap\{\mathfrak{t} \mid \mathfrak{t} \in \mathfrak{T}(\mathfrak{g})\}.$$

(iii) *For each $g \in G_0$ the following statement holds.*

(28c) $$\mathfrak{L}\big(Z(g,G)\big) = \bigcup_{X\in\mathfrak{g},\, \exp X = g} \mathfrak{z}(X,\mathfrak{g}).$$

Proof. (i) First we show (28a). If $g \in T \in \mathcal{T}(G)$, then $T \in Z(g,G)$, and T is maximal connected abelian in G hence in $Z(g,G)$. Thus the right hand side of (28a) is contained in the left hand side. Conversely, let T be a maximal torus of $Z(g,G)$. Let A be a maximal torus of G containing g; such an A exists by the Maximal Torus Theorem 6.30. Then $A \subseteq Z(g,G)$ and A, being maximal connected abelian in G is maximal connected abelian in $Z(g,G)$. Then by the Transitivity Theorem 6.27 applied to $Z(g,G)$, there is a $z \in Z(g,G)$ such that $T = zAz^{-1}$, whence $T \in \mathcal{T}(G)$; moreover, $g = zgz^{-1} \in T$. Hence T is a member of the right hand side of (28a). Thus (28a) is proved.

For a proof of (28b), let $T \in \mathcal{T}(G)$ with $g \in T$. Then $T \in \mathcal{T}\big(Z(g,G)\big)$ and then

$$Z(g,G)_0 = \bigcup_{z\in Z(g,G)_0} zTz^{-1} = \bigcup \mathcal{T}(Z(g,G)_0).$$

Now (28a) implies (28b).

(ii) Assume that G is connected. Let $z \in Z(G)$. This is equivalent to $Z(z,G) = G$. Then by (28a) every maximal torus of G contains z. It remains to show that the right hand side of (29) is contained in the center. Thus let $z \in \bigcap \mathcal{T}(G)$ and let $g \in G$ be arbitrary. According to 6.30 there is a $T \in \mathcal{T}(G)$ containing g. By hypothesis, $z \in T$. As T is abelian, g and z commute. Thus $z \in Z(G)$.

The proofs of (I) and (II) rest on the Transitivity Theorem 6.27 in place of the Maximal Torus Theorem 6.30 and follow otherwise exactly the same lines as the proofs of (i) and (ii).

(iii) From (28b) we deduce $\mathfrak{L}\big(Z(g,G)\big) = \bigcup\{\mathfrak{t} \mid \mathfrak{t} \in \mathfrak{T}(\mathfrak{g}) \text{ and } g \in \exp\mathfrak{t}\}$. Now a Cartan subalgebra $\mathfrak{t}$ satisfies $g \in \exp\mathfrak{t}$ if and only there is an $X \in \mathfrak{t}$ such that $g = \exp X$ if and only if there is an $X \in \mathfrak{g}$ such that $\mathfrak{t} \in \mathfrak{T}\big(\mathfrak{z}(X,\mathfrak{g})\big)$ such that

$g = \exp X$. Since $\mathfrak{z}(X,\mathfrak{g}) = \bigcup\{t \in \mathfrak{T}(\mathfrak{z}(X,\mathfrak{g}))\}$ by the Transitivity Theorem 6.27 applied to $\mathfrak{z}(X,\mathfrak{g})$, equation (28c) follows. □

Some comments are in order. Statements (I) and (II) are the Lie algebra analogs of the Lie group Statements (i) and (ii), respectively, and their proof is easier because the proof of 1.27 is easier than the proof of 1.30. However, (I) and (II) are not the “infinitesimal versions” of (i) and (ii). While it is true that $g \in G_0$ is always of the form $g = \exp X$ for some $X \in \mathfrak{g}$ by 1.30, and $\mathfrak{z}(X,\mathfrak{g}) = \mathfrak{L}\big(Z(\exp \mathbb{R}{\cdot}X, G)\big) \subseteq \mathfrak{L}\big(Z(g,G)\big)$ by 5.53(25), in general $\mathfrak{z}(X,\mathfrak{g}) \neq \mathfrak{L}\big(Z(X,G)\big)$. E.g. if $G = \mathbb{S}^3$, the group of unit quaternions, then $g = -\mathbf{1}$ is in the center, whence $Z(g,G) = G$. But in $\mathfrak{g} = \mathbb{R}.i + \mathbb{R}{\cdot}j + \mathbb{R}{\cdot}k$, the element $X = i$ (or any element in the unit ball of $\mathfrak{g}$) satisfies $\exp \pi{\cdot}X = e^{\pi i} = -1$ and $\mathfrak{z}(X,\mathfrak{g}) = \mathbb{R}.X \neq \mathfrak{g}$.

In Exercise E6.10(iv) below we shall see that the centralizer of an element in a connected compact Lie group need not be connected; this remark is pertinent to (28b) above.

Corollary 6.33. *Let G be a* connected *compact Lie group. Then the following assertions hold.*

(i) *If S is a torus subgroup of G then $Z(S,G) = \bigcup\{T \in \mathcal{T}(G) \mid S \subseteq T\}$. In particular, the centralizer of a torus subgroup of G is connected.*

(ii) *Each maximal torus T of G is its own centralizer $Z(T,G)$.*

(iii) *Each maximal torus of G is a maximal abelian subgroup.*

(iv) *There are connected compact groups containing abelian subgroups which are not contained in any torus.*

Proof. (i) Clearly the right hand side is contained in the left one. Conversely, let $g \in Z(S,G)$. Then $S \subseteq Z(g,G)$. Since S is connected, $S \subseteq Z(g,G)_0$. By 6.30 we find a maximal torus T_g of G containing g. Then also $T_g \subseteq Z(g,G)_0$. Now S is contained in a maximal torus T of $Z(g,G)$. By the Transitivity Theorem 6.27, the groups T and T_g are conjugate in $Z(g,G)$. Hence $T \in \mathcal{T}(G)$ and $S \subseteq T$.

(ii) By (i), $Z(T,G)$ is the union of all maximal tori containing T; but T is the only one of these, whence $Z(T,G) = T$.

(iii) Let $T \in \mathcal{T}(G)$ and $T \subseteq A$ for an abelian subgroup A of G. Then $A \subseteq Z(T,G)$; but $Z(T,G) = T$ by (ii). Hence $A = T$.

(iv) For an example that maximal abelian groups may not be connected take $G = \mathrm{SO}(3)$ and A the subgroup of all diagonal matrices in $\mathrm{SO}(3)$. Then $A \cong \mathbb{Z}(2)^2$ is a maximal abelian subgroup. (Exercise E6.10.) □

Exercise E6.10. Verify the details of the example $G = \mathrm{SO}(3)$ with A being the set of all

$$\mathrm{diag}(x_1,x_2,x_3) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \in \mathrm{SO}(3), \quad x_j \in \{1,-1\}, \quad x_1x_2x_3 = 1.$$

Show:

(i) $A \cong \mathbb{Z}(2)^2$.

(ii) The group T of all

$$\begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

is a maximal torus.

(iii) Two different maximal tori intersect in $\{\mathbf{1}\}$.

(iv) $N(T,G) = T \cup \operatorname{diag}(1,-1,-1)T = Z(\operatorname{diag}(-1,-1,1), G)$.

(v) A is maximal abelian. □

We note that the maximal torus T of $G = \mathrm{SO}(3)$ contains an element $g \stackrel{\text{def}}{=} \operatorname{diag}(-1,-1,1)$ whose centralizer is $Z(g,G) = N(T,G)$, a group which is not connected.

Corollary 6.34. *If S and T are maximal tori of a compact Lie group G, then the Weyl groups $\mathcal{W}(S,G)$ and $\mathcal{W}(T,G)$ are isomorphic.*

Proof. By the Maximal Torus Theorem 6.30 there is a $g \in G$ such that the inner automorphism $I_g\colon G \to G$, $I_g(x) = gxg^{-1}$ maps S onto T. It then maps $N(S,G)$ onto $N(T,G)$ and thus induces an isomorphism $\mathcal{J}_g\colon \mathcal{W}(S,G) \to \mathcal{W}(T,G)$, $\mathcal{I}_g(nS) = I_g(nS) = I_g(n)T$. □

In view of this fact one is also justified to speak in many circumstances of *the* Weyl group of G. For each $n \in N(T,G)$ the vector space automorphism $\operatorname{Ad}(n)|\mathfrak{t}$ of $\mathfrak{t}$ is orthogonal with respect to any invariant inner product on $\mathfrak{t}$ (see 6.2). Thus $n \mapsto \operatorname{Ad}(n)|\mathfrak{t}\colon N(T,G) \to \mathrm{O}(\mathfrak{t})$ is a well-defined morphism, whose kernel is the centralizer $Z(T,G)$; indeed if $\operatorname{Ad}(n)|\mathfrak{t} = \mathrm{id}_{\mathfrak{t}}$ then $n(\exp X)n^{-1} = \exp \operatorname{Ad}(n)X = \exp X$ for all $X \in \mathfrak{t}$. Since $T = \exp \mathfrak{t}$, this means that the element n is in the centralizer of T in G.

We shall presently use a very simple remark on group actions.

Lemma (Frattini Argument). *Let a group Γ act on a set M and assume that a subgroup Ω acts transitively. Then $\Gamma = \Omega\Gamma_x = \Gamma_x\Omega$, where $\Gamma_x = \{g \in \Gamma \mid g{\cdot}x = x\}$ is the isotropy subgroup at x.*

Proof. For each $g \in \Gamma$, since $\Omega{\cdot}x = M$, there are elements h, $h' \in \Omega$ such that $g{\cdot}x = h{\cdot}x$ and $g^{-1}{\cdot}x = h'{\cdot}x$. Then $h^{-1}g$, $gh' \in \Gamma_x$, whence $g \in \Omega\Gamma_x$ and $g \in \Gamma_x\Omega$. □

The first part of the following corollary illustrates this procedure.

Corollary 6.35. *Assume that G is a compact Lie group G and G_1 a normal subgroup. Let $T \in \mathcal{T}(G_1)$. Then $G = G_1N(T,G)$, and, in particular, the following conclusions hold:*

$$G = G_0N(T,G). \tag{30}$$

(31)
$$\begin{aligned} G/G_0 &\cong N(T,G)/(N(T,G)\cap G_0) = N(T,G)/N(T,G_0) \\ &\cong \mathcal{W}(T,G)/\mathcal{W}(T,G_0). \end{aligned}$$

Moreover, the representation

$$\rho\colon \mathcal{W}(T,G) \to \mathrm{O}(\mathfrak{t}), \quad \rho(nT) = \mathrm{Ad}(n)|\mathfrak{t},$$

maps $\mathcal{W}(T,G_0)$ *faithfully into* $\mathrm{O}(\mathfrak{t})$.

Proof. Since G_1 is normal, every inner automorphism leaves G_1 invariant. We apply the Frattini Argument with $\Gamma = G$ acting on $M = \mathcal{T}(G)_1$ under inner automorphisms with $\Omega = G_1$ operating transitively by the Transitivity Theorem 6.27. For $T \in \mathcal{T}(G)$, the isotropy group at T is $\Gamma_T = N(T,G)$. Thus $G = G_1N(T,G)$ by the preceding Lemma. We can apply this in particular to the normal subgroup G_0 and get (30).

Assertion (31) follows from the definitions and the standard isomorphy theorems for groups.

Lastly, $\ker\rho = Z(T,G)/T$ by the remarks preceding the corollary. However, $\mathcal{W}(T,G_0) = N(T,G_0)/T$. Accordingly, $\ker\big(\rho|\mathcal{W}(T,G_0)\big) = Z(T,G_0)/T = T/T = \{1\}$. □

It follows that we may view $\mathcal{W}(T,G)/\ker\rho$ as a subgroup of $\mathrm{O}(\mathfrak{t})$, in particular, $\mathcal{W}(T,G_0)$ may be considered as a subgroup of $\mathrm{O}(\mathfrak{t})$.

The present information also yields a quick proof of a preliminary version of Lee's Theorem on supplements for the identity component [238]. The final version will be shown below in 6.74.

Theorem 6.36. *Every compact Lie group contains a finite group E such that $G = G_0E = EG_0$. If T is a maximal torus, E may be chosen so that $E \subseteq N(T,G)$ and $E\cap G_0 = E\cap N(T,G_0)$. The order of each element in $E\cap T$ divides the order of the Weyl group of G.*

Proof. Let $T \in \mathcal{T}(G)$. By 6.21(iv) there is a finite group E such that $N(T,G) = TE = ET$. Since $T \subseteq G_0$, the assertion follows at once from 6.35. □

We conclude this section with another application of the Frattini Argument. This exercise requires knowledge of the Sylow Theorems for finite groups, notably this information ([209], p. 33ff.): *If G is a finite group of order p^nm for a prime p not dividing m then G contains subgroups of order p^n, the* Sylow subgroups *of G, and all of them are conjugate.*

Exercise E6.11. Prove the following fact:

Let G_1 be a finite normal subgroup of a group G and P a Sylow subgroup of G_1. Then $G = G_1N(P,G)$.

[Hint. Apply the Frattini Argument with G acting under inner automorphisms on the set of Sylow subgroups of G_1 conjugate to P.] □

The Second Structure Theorem for Connected Compact Lie Groups

We have seen previously in 6.16 that a connected compact Lie group G is almost a direct product of a connected central torus group $Z_0(G)$ and a semisimple Lie group. The example of $\mathrm{U}(n)$ shows that one cannot do better. However, one can do better in one regard, if one renounces the requirement that the product decomposition be direct. There is a semidirect decomposition available which is topologically clean (while algebraically being a bit less comfortable). We discuss this as our next topic.

The first step is a lemma on the decomposition of groups. To begin with, we secure some terminology. If N is a closed normal subgroup of a compact group G then every closed subgroup A of G with $G = NA$ and $N \cap A = \{\mathbf{1}\}$ is called a *semidirect cofactor of* N. If N has a semidirect cofactor A, then according to a general group theoretical formalism, $N \times A$ is a compact topological group relative to the multiplication $(m,a)(n,b) = (m(I_a n), ab)$ with $I_a n = ana^{-1}$, called $N \rtimes_I A$, and $(m,a) \mapsto ma\colon N \rtimes_I A \to G$ is an isomorphism of compact groups. Any cofactor of N is isomorphic to G/N. If A is a semidirect cofactor of N, then any conjugate gAg^{-1} is a cofactor, too. For a subgroup A of G let us denote with $\mathrm{cls}(A)$ the set $\{gAg^{-1} \mid g \in G\}$ of conjugate subgroups. We let $\mathcal{C}(N)$ denote the set of cofactors of N in G and $\mathcal{C}_{\mathrm{conj}}(N)$ the set of conjugacy classes; so $\mathrm{cls}\colon \mathcal{C}(N) \to \mathcal{C}_{\mathrm{conj}}(N)$ is the orbit map of the action of G under inner automorphisms.

We say that a function $f\colon H \to N$ between topological groups is a 1-*cocycle* if there is a continuous action $(h,n) \mapsto h{\cdot}n\colon H \times N \to N$ (see Definition 1.9) such that every $n \mapsto h{\cdot}n$ is an automorphism of n and such that the following functional equation is satisfied:

$$f(h_1h_2) = \big(h_1{\cdot}f(h_2)\big)f(h_1).$$

An action of the type we need here is also called an *automorphic action*. If the action is constant and N is abelian, the 1-cocycle is a homomorphism. We have encountered cocycles in the proof of Theorem 6.10. Let $Z^1(H,N)$ denote the set of all cocycles $f\colon H \to N$. If N and H are subgroups of G and H is in the normalizer of N, acting on N under inner automorphisms, let $j\colon N \cap H \to N$ be the inclusion morphism and let $Z^1_{N\cap H}(H,N)$ denote the set of cocycles $f\colon H \to N$ with $f|(N \cap H) = j$. Using this notation we formulate the group theoretical background of our discussion in the following proposition.

Proposition 6.37. (i) *Assume that G is a compact group, N a compact normal subgroup, and H a compact subgroup such that $G = NH$. The group H acts on N automorphically via $h{\cdot}n = hnh^{-1}$. The following conditions are equivalent:*

(1) *There is a compact subgroup A such that $G = NA$ and $N \cap A = \{\mathbf{1}\}$, that is G is a semidirect product of N and A.*

(2) *The inclusion function $j\colon N \cap H \to N$ extends to a 1-cocycle $f\colon H \to N$ with respect to the action of H on N.*

(ii) *The function*

$$\Phi\colon Z^1_{N\cap H}(H,N)\to \mathcal{C}(N),\quad \Phi(f)=\{f(z)^{-1}z \mid z\in H\},$$

is a bijection.

(iii) *The kernel of the morphism* $\mu\colon N\rtimes_I H\to G$ *is* $\ker\mu=\{(h^{-1},h)\mid h\in N\cap H\}$, *the function* $\kappa\colon N\cap H\to\ker\mu$, $\kappa(h)=(h^{-1},h)$ *is an isomorphism satisfying* $\kappa(xhx^{-1})=(1,x)\kappa(h)(1,x)^{-1}$ *for all* $x\in H$. *The group* $N\cap H$ *is normal in* H.

(iv) *Assume that* $\mathbf{N}$ *and* $\mathbf{H}$ *are compact groups and that* $\iota\colon\mathbf{H}\to\operatorname{Aut}\mathbf{N}$ *is a morphism of groups defining an action* $h{\cdot}n=\iota(h)(n)$ *such that* $(h,n)\mapsto h{\cdot}n\colon\mathbf{H}\times\mathbf{N}\to\mathbf{N}$ *is continuous. If* $\mathbf{D}\trianglelefteq\mathbf{H}$ *a normal subgroup of* $\mathbf{H}$, *and if* $f\colon\mathbf{D}\to\mathbf{N}$ *is a cocycle satisfying* $f(h{\cdot}d)=h{\cdot}f(d)$ *for* $h\in\mathbf{H}$, $d\in\mathbf{D}$. *Then* $\mathbf{K}=\{(f(d)^{-1},d)\mid d\in\mathbf{D}\}$ *is a normal subgroup of* $\mathbf{H}\rtimes_\iota\mathbf{N}$ *such that* $\Gamma\stackrel{\text{def}}{=}(\mathbf{N}\rtimes_\iota\mathbf{H})/\mathbf{D}$ *is the product of the normal subgroup* $(\mathbf{N}\rtimes_\iota\{\mathbf{1}\})\mathbf{D}/\mathbf{D}\cong\mathbf{N}$ *and the subgroup* $(\{\mathbf{1}\}\rtimes_\iota\mathbf{H})/\mathbf{D}\cong\mathbf{H}$, *whose intersection is* $(\mathbf{D}\rtimes_\iota\mathbf{D})/\mathbf{D}\cong\mathbf{D}$.

Proof. (i) First we show (1)⇒(2). Every $g\in G$ decomposes uniquely and continuously into a product na with $n\in N$ and $a\in A$. In particular, each $h\in H$ defines a unique element $f(h)\in N$ and a $\varphi(h)\in A$ such that $h=f(h)\varphi(h)$. If $h\in N\cap H$, then $\varphi(h)=\mathbf{1}$ and thus $f(h)=h$. Also, $f(h_1h_2)\varphi(h_1h_2)=h_1h_2=h_1f(h_2)\varphi(h_2)=\big(h_1{\cdot}f(h_2)\big)h_1\varphi(h_2)=\big(h_1{\cdot}f(h_2)\big)f(h_1)\varphi(h_1)\varphi(h_2)$. Then f is the desired cocycle.

(2)⇒(1) Assume that f is a 1-cocycle with respect to the action of H on N. We define $p\colon H\to G$ by $p(h)=f(h)^{-1}h$. Then p is continuous. We note $p(h_1h_2)=\{\big(h_1{\cdot}f(h_2)\big)f(h_1)\}^{-1}h_1h_2=f(h_1)^{-1}h_1f(h_2)^{-1}h_1^{-1}h_1h_2=p(h_1)p(h_2)$. Thus p is an endomorphism. We set $A=p(H)$. Clearly, $Np(h)=Nh$, whence $NA=NH=G$. Also $g\in N\cap A$ means the existence of an $h\in H$ with $f(h)^{-1}h=p(h)=g\in N$. Then $h=f(h)g\in N$ and thus $f(h)=h$ by assumption. This implies $g=1$. The proof of (i) is complete.

(ii) Let $f\in Z^1_{N\cap H}(H,N)$. We saw that the set $A=\{f(z)^{-1}z\mid z\in H\}$ is a cofactor of N and that every cofactor arises in this way. Hence the function Φ is well-defined and is surjective. In order to see its injectivity, let $f_j\in Z^1_{N\cap H}(H,N)$, $j=1,2$ be two cocycles such that $\Phi(f_1)=\Phi(f_2)$. Write $A=\Phi(f_1)$. Define the morphisms $\varphi_j\colon H\to A$ by $h=f_j(h)\varphi_j(h)$. Since the product $G=NA$ is semidirect, we have a projection $p\colon G\to A$ such that every element $g\in G$ is uniquely written as $g=np(g)$ with $n\in N$. Accordingly, $\varphi_1(h)=p(h)=\varphi_2(h)$ for $h\in H$. It follows that $f_1=f_2$. This completes the proof.

(iii) It is straightforward to verify that μ is a morphism. (Cf. Definition 5.72 and Exercise E5.72.) An element $(n,h)\in N\rtimes_I H$ is in $\ker\mu$ iff $1=\mu(n,h)=nh$ iff $n=h^{-1}\in N\cap H$. The function κ is the inverse of the restriction and corestriction $\ker\mu\to N\cap H$ of the projection $p\colon N\rtimes_I H\to H$ and thus is an isomorphism of groups. The relation $\kappa(xhx^{-1})=(1,x)\kappa(h)(1,x)^{-1}$ for all $x\in H$ is straightforward. The group $N\cap H$ is normal in H since it is the image of the normal subgroup $\ker\mu$ of $N\rtimes_I H$ under the surjective morphism p.

(iv) All assertions made are based on straightforward calculations. □

The preceding proposition exhibits the role of cocycles in group theory; Parts (iii) and (iv) illustrate the circumstances, under which groups G arise which are the product of a normal subgroup N and a subgroup H. In the light of these observations one should review the construction of Exercise E6.9(b).

If $G = NH$ and H centralizes N, then $f\colon H \to N$ is a cocycle iff $h \mapsto f(h)^{-1}$ is a morphism; so the issue is whether the inclusion morphism of $N \cap H \to N$ extends to a morphism $H \to N$; this is an important special case as we shall see presently. Note that $N \cap H$ is central in H. If N and G/N are abelian, then any morphism $f\colon H \to N$ annihilates the commutator group and $H' \subseteq N$; if f induces the identity on $N \cap H$, then $H' = \{1\}$. In other words, if H is a nilpotent group of class 2 and H' can be identified with a nontrivial subgroup of N then $N \times H/\{(d^{-1}, d) \mid d \in H'\}$ is a nilpotent group of class 2 in which N cannot be a direct factor; this at once gives examples of compact Lie groups G in which the identity component is a torus which is not a direct factor (cf. Exercise E6.9(b)(ii)).

The simplest case in which 6.37(i) applies is that of a compact abelian group.

Lemma 6.38. *Let T and H be closed abelian subgroups of a compact group such that T is a torus group and T and H commute elementwise. Then there exists a closed subgroup $K \subseteq TH$ such that $(t,k) \mapsto tk\colon T \times K \to TH$ is an isomorphism of compact abelian groups. In particular, TK is a direct product and equals TH.*

Proof. By the preceding Lemma 6.37 it suffices to extend the inclusion morphism $T \cap H \to T$ to a homomorphism $H \to T$. Since $T \cong \mathbb{T}^M$ for some set M this is accomplished by the Extension Theorem for Characters 2.33(ii). □

This conclusion carries much further as the following proposition shows:

Proposition 6.39. *Assume that G is a compact group, H a closed abelian subgroup and N a normal subgroup of G such that $G = NH$ and that $H \cap N$ is contained in a torus subgroup T of N which commutes elementwise with H. Then there is a closed abelian subgroup A such that G is the semidirect product $AN = NA$ of the normal subgroup N with the not necessarily normal subgroup $A \cong G/N$.*

Proof. The set TH is a compact subgroup of G to which Lemma 6.38 applies. Hence there is a closed abelian subgroup A such that $A \cap T = \{\mathbf{1}\}$. Note $NA = NTA = NTH = NH = G$. In order to complete the proof we have to show that $N \cap A = \{1\}$. Now $a \in N \cap A$ implies $a \in N \cap TA = N \cap TH$. Hence $a = th \in A$ with $t \in T \subseteq N$ and $h \in H$. Thus $h = t^{-1}a \in H \cap N \subseteq T$ by hypothesis. Hence $a = th \in T \cap A = \{\mathbf{1}\}$. This is what we had to show. □

The main applications in the context of compact Lie groups leads us to the Second Structure Theorem for Connected Compact Lie Groups. Let G be a compact Lie group, take an abelian subgroup Z satisfying $Z(G)_0 \subseteq Z \subseteq Z((G')_0, G)$, and set $G_1 \stackrel{\text{def}}{=} (G')_0 Z$. The commutator subgroup G' is closed by 6.11. By Corollary 6.32(ii) to the Maximal Torus Theorem, the subgroup $D = Z \cap (G')_0$ is contained in the center of $(G')_0$. Then D is contained in some maximal torus T of $(G')_0$ which we fix. We recall from 6.15(iii) that $G_0 \subseteq (G')_0 Z(G)_0 \subseteq G_1$, i.e. G_1 is substantial from our point of view.

For $D = N \cap H$, $H, N \subseteq G$ we define $\mathrm{Hom}_D(H,N)$ to be the set of all morphisms $f\colon H \to N$ which agree on D with the inclusion $j\colon D \to N$. If it happens that H and N commute elementwise and the action of H on N is by inner automorphisms, and if H happens to be commutative, then for $f \in Z^1(H,N)$ we have $f(xy) = f(yx) = (yf(x)y^{-1})f(y) = f(x)f(y)$. In particular, in these circumstances, $Z^1_D(H,N) = \mathrm{Hom}_D(H,N)$.

Corollary 6.40. *Let G be a compact Lie group with the subgroup $G_1 \supseteq G_0$ defined as in the preceding paragraphs, and let $\mathcal{C}\big((G')_0\big)$ denote the set of cofactors of $(G')_0$ in G_1. Then the following conclusions hold:*

(i) *$(G')_0$ has a semidirect cofactor in G_1, i.e. $G_1 = (G')_0 A$, $(G')_0 \cap A = \{1\}$.*

(ii) *The function*

$$\Phi\colon \mathrm{Hom}_D(Z,(G')_0) \to \mathcal{C}(G'), \quad \Phi(f) = \{f(z)^{-1}z \mid z \in Z\},$$

is a bijection.

Proof. (i) We apply Proposition 6.39 with $H = Z$ and $N = (G')_0$. Then $H \cap N = D$ is central in $(G')_0$ and by 6.32(ii), there is a torus subgroup T of $(G')_0$ containing D. The group $T \subseteq (G')_0$ clearly commutes elementwise with $Z \subseteq Z((G')_0, G)$. Hence Proposition 6.39 proves the assertion.

(ii) By a remark preceding this corollary, $Z^1_D(H,N) = \mathrm{Hom}_D(H,N)$. Then the assertion directly follows from 6.37. □

Examples arise in the situation we described in 6.10, notably in 6.10(iii), 6.10(iv). But now we consider a *connected* compact Lie group G. Then G' is connected and $Z(G',G) = Z(G)$ is the center. We let Z_0 denote the identity component $Z_0(G)$ of the center of G and set $D = Z_0 \cap G'$.

The Second Structure Theorem for Connected Compact Lie Groups

Theorem 6.41. (i) *For a connected compact Lie group G the commutator subgroup G' has a semidirect cofactor isomorphic to G/G'.*

(ii) *Let $\mathcal{C}(G')$ denote the set of cofactors of G'. Then*

$$\Phi\colon \mathrm{Hom}_D(Z_0(G),G') \to \mathcal{C}(G'), \quad \Phi(f) = \{f(z)^{-1}z \mid z \in Z_0(G)\},$$

is a bijection.

(iii) *Let T be a maximal torus of G' and $i\colon T \to G'$ the inclusion morphism and $\mathrm{Hom}_D(Z_0(G),i)\colon \mathrm{Hom}_D(Z_0(G),T) \to \mathrm{Hom}_D(Z_0(G),G')$ the induced injection. Then the composition*

$$\mathrm{Hom}_D(Z_0(G),T) \xrightarrow{\mathrm{Hom}_D(Z_0(G),i)} \mathrm{Hom}_D(Z_0(G),G') \overset{\Phi}{\to} \mathcal{C}(G') \overset{\mathrm{cls}}{\to} \mathcal{C}_{\mathrm{conj}}(G')$$

is surjective. Thus every cofactor for G' is conjugate to one that is generated by an $f \in \mathrm{Hom}_D(Z_0(G), T)$.

Proof. (i) We note that $G = Z_0 G'$ by the First Structure Theorem 6.16. Hence Part (i) is an immediate consequence of 6.40.

(ii) Let $f \in \mathrm{Hom}_D(Z_0, G')$. Since Z_0 is central, f is in particular a 1-cocycle. Hence by Lemma 6.37 the set $A = \{f(z)^{-1}z \mid z \in Z_0\}$ is a cofactor of G' and thus the function Φ is well-defined. By Lemma 6.37, Φ is surjective. In order to see injectivity, let $f_j \in \mathrm{Hom}_D(Z_0, G')$, $j = 1, 2$ be two morphisms such that $\Phi(f_1) = \Phi(f_2)$. Write $A = \Phi(f_1)$. Define the morphisms $\varphi_j \colon Z_0 \to A$ by $z = f_j(z)\varphi_j(z)$. Since the product $G = G'A$ is semidirect, we have a projection $p \colon G \to A$ such that every element $g \in G$ is uniquely written as $g = g'p(g)$ with $g \in G'$. Accordingly, $\varphi_1(z) = p(z) = \varphi_2(z)$ for $z \in Z_0$. It follows that $f_1 = f_2$. This completes the proof.

(iii) Let $A \in \mathcal{C}_{\mathrm{conj}}(G')$. By (ii) there is an $f \in \mathrm{Hom}_D(Z_0(G), G')$ such that $\Phi(f) = A$. The image $f\bigl(Z_0(G)\bigr)$ is a connected abelian subgroup of G' and therefore, by the Transitivity Theorem 6.27, there is a $g \in G'$ such that $gf\bigl(Z_0(G)\bigr)g^{-1} \subseteq T$. Then the function $g{\cdot}f$ given by $(g{\cdot}f)(z) = gf(z)g^{-1}$ belongs to $\mathrm{Hom}_D(Z_0, T)$ and $\Phi(g{\cdot}f) = \{(gf(z)^{-1}g^{-1}z \mid z \in Z_0(G)\} = g\{f(z)^{-1}z \mid z \in Z_0(G)\}g^{-1} = gAg^{-1} \in \mathrm{cls}(A)$ in view of the centrality of $Z_0(G)$. □

If $f \in \mathrm{Hom}_D(Z_0(G), T)$ and $g \in Z\bigl(f\bigl(Z_0(G)\bigr), G'\bigr)$, then the morphism $g{\cdot}f$ is contained in $\mathrm{Hom}_D(Z_0(G), T)$ and the cofactors generated by f and $g{\cdot}f$ are conjugate. The set $\mathrm{Hom}_D(Z_0(G), T)$ is rich by the injectivity of T in the category of compact abelian groups (see Appendix 1, A1.35 and compare the proof of 6.38 above). Notice that even if $D = Z_0(G) \cap G' = Z_0(G) \cap T = \{1\}$, i.e. if G is a direct product of G' and $Z_0(G)$ and if $Z_0(G) \neq \{1\}$ there are numerous nonnormal cofactors for G'.

It is a constructive exercise to contemplate the difference between the First Structure Theorem 6.16 for connected compact Lie groups and the second one in 6.41. The decomposition given in Theorem 6.16 is algebraically clean and canonical, but is defective topologically, since the direct decomposition is available only in the covering group $Z_0 \times G'$ of G. The decomposition $G = G'A \cong G' \rtimes_I A$ of 6.41 is topologically clean. Algebraically it is satisfactory, but the product is only semidirect and A is not unique, not even up to conjugation in general. However, we have some control over the possible complements through 6.41(ii).

Compact Abelian Lie Groups and their Linear Actions

We have discussed compact abelian groups and their duality in Chapters 1 and 2, and their general representation theory was discussed at the end of Chapter 3. Now that the aspects of Lie groups are added we have to review some of the facts accumulated earlier.

Let T denote a torus and $\exp = \exp_T: \mathfrak{t} \to T$ its exponential function. It is a homomorphism of abelian topological groups. Let $\mathfrak{k} = \mathfrak{K}(T)$ be its kernel. Thus we have an exact sequence of abelian groups

$$0 \longrightarrow \mathfrak{k} \xrightarrow{\text{incl}} \mathfrak{t} \xrightarrow{\exp_T} T \longrightarrow 0.$$

In the context of general representation theory, it was natural to choose the complex ground field, but in the context of Lie groups, the ground field of reals is appropriate. For an abelian topological group, a character is in the first place a continuous homomorphism from it into the circle group. The circle group G has three different guises:

(i) $\mathbb{R}/\mathbb{Z}$, preferred from the view point of the structure theory of abelian groups and duality theory (cf. Chapters 1, 2 above and 7 and Appendix 1),

(ii) $\mathbb{S}^1$, the multiplicative group of complex numbers of norm 1, preferred in representation theory and harmonic analysis (cf. Chapters 2, 3, and 4), and

(iii) SO(2), the group of matrices

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R}$$

preferred in the theory of real representations and in the context of (real) Lie groups.

The Lie algebras $\mathfrak{g}$ coming along with these variants are the following.

(i) Take $\mathbb{R}$ with $\exp: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ being the quotient map;

(ii) set $A = \mathbb{C}$, $\mathbb{S}^1 \subseteq A^{-1} = \mathbb{C} \setminus \{0\}$, and $\mathfrak{g} = i\mathbb{R}$ with $\exp ir = e^{ir}$; and

(iii) set $A = M(2, \mathbb{R})$, the algebra of real 2×2-matrices, $\mathrm{SO}(2) \subseteq A^{-1} = \mathrm{Gl}(2, \mathbb{R})$, $\mathfrak{g} = \mathbb{R}\cdot I = \mathfrak{so}(2)$, $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\exp r\cdot I = e^{r\cdot I}$.

For our present discussion we opt for alternative (iii). However, as the Lie algebra of the circle group SO(2) we shall choose $\mathbb{R}$ with the exponential function

$$\exp = \exp_{\mathrm{SO}(2)}: \mathbb{R} \to \mathrm{SO}(2), \quad \exp r = e^{r\cdot I} = \mathbf{1} + r\cdot I - \frac{r^2}{2!}\mathbf{1} - + \cdots = \cos r \cdot \mathbf{1} + \sin r \cdot I.$$

For each character $\chi: T \to \mathrm{SO}(2)$ from the functoriality of $\mathfrak{L}$ (5.42) we have a commutative diagram

$$(\&) \qquad \begin{array}{ccc} \mathfrak{k} & \xrightarrow{\mathfrak{L}(\chi)|\mathfrak{k}} & 2\pi\mathbb{Z} \\ {\scriptstyle\text{incl}}\downarrow & & \downarrow{\scriptstyle\text{incl}} \\ \mathfrak{t} & \xrightarrow{\mathfrak{L}(\chi)} & \mathbb{R} \\ {\scriptstyle\exp_T}\downarrow & & \downarrow{\scriptstyle\exp_{\mathrm{SO}(2)}} \\ T & \xrightarrow[\chi]{} & \mathrm{SO}(2). \end{array}$$

Let $\mathfrak{t}^* \stackrel{\text{def}}{=} \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ denote the vector space dual of $\mathfrak{t}$. We define $\mathfrak{k}_* \stackrel{\text{def}}{=} \{\omega \in \mathfrak{t}^* \mid \omega(\mathfrak{k}) \subseteq 2\pi\mathbb{Z}\}$. If for $\omega \in \mathfrak{k}_*$ we let $\omega' \stackrel{\text{def}}{=} \omega|\mathfrak{k}: \mathfrak{k} \to 2\pi\mathbb{Z}$ denote the restriction and corestriction, then

$$\omega \mapsto \omega': \mathfrak{k}_* \to \mathrm{Hom}(\mathfrak{k}, 2\pi\mathbb{Z})$$

is an isomorphism of abelian groups.

Lemma 6.42. *The functions*

$$\widehat{T} \xrightarrow{\chi\mapsto\mathfrak{L}(\chi)} \mathfrak{k}_* \xrightarrow{\omega\mapsto\omega'} \mathrm{Hom}(\mathfrak{k}, 2\pi\mathbb{Z})$$

are isomorphisms of abelian groups.

Proof. The injectivity of $\chi \mapsto \mathfrak{L}(\chi)$ follows from 5.42. If $\omega \in \mathfrak{k}_*$, then a character $\chi: T \to \mathrm{SO}(2)$ is well-defined by $\chi(\exp_T X) = \exp_{\mathrm{SO}(2)} \omega(X)$ such that $\mathfrak{L}(\chi) = \omega$. Thus the first of the two maps is an isomorphism. We have observed above that the second one is an isomorphism, too. □

By this lemma we have several manifestations of the character group $\widehat{T}$. If some inner product on $\mathfrak{t}$ is arbitrarily given and fixed, another manifestation of it exists in $\mathfrak{t}$. Recall that a character χ of T right now is a morphism $\chi: T \to \mathrm{SO}(2)$. We let $\mathbf{1}$ be the identity of $\mathrm{SO}(2)$ and $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Lemma 6.43. *Assume that on the Lie algebra $\mathfrak{t}$ of a torus group T we are given an inner product $(\cdot \mid \cdot)$ making $\mathfrak{t}$ into a real Hilbert space. Then*

(i) *there exist, firstly, a discrete subgroup $\Delta \subseteq \mathfrak{t}$ and, secondly, an isomorphism $\alpha: \widehat{T} \to \Delta$, both depending on $(\cdot \mid \cdot)$ such that*

$$\begin{aligned}(\forall \chi \in \widehat{T},\, X \in \mathfrak{t}) \quad (\alpha(\chi) \mid X) &= \mathfrak{L}(\chi)(X), \ \textit{and} \\ \chi(\exp X) &= \exp_{\mathrm{SO}(2)} \mathfrak{L}(\chi)(X) \\ = e^{(\alpha(\chi)\mid X)I} &= \cos(\alpha(\chi) \mid X)\cdot\mathbf{1} + \sin(\alpha(\chi) \mid X)\cdot I.\end{aligned}$$

(ii) $\Delta = \{X \in \mathfrak{t} \mid (X \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}\}$.

(iii) *The equality $\Delta = \mathfrak{k}$ holds if and only if*

$$(\forall D_1 \in \mathfrak{t},\, D_2 \in \mathfrak{k}) \quad \big((D_1 \mid D_2) \in 2\pi\mathbb{Z}\big) \Leftrightarrow (D_1 \in \mathfrak{k}).$$

(iv) *Let $e_1, \ldots, e_n$ be a set of free generators of $\mathfrak{k}$ and define an inner product $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{t}$ by $\langle \sum_{j=1}^n x_j{\cdot}e_j \mid \sum_{j=1}^n y_j{\cdot}e_j \rangle = 2\pi \sum_{j=1}^n x_j y_j$ Then, with respect to this inner product, $\Delta = \mathfrak{k}$.*

Proof. (i) Given the inner product $(\cdot \mid \cdot)$, the function $\varphi: \mathfrak{t} \to \mathfrak{t}^*$ given uniquely by $\varphi(X)(Y) = \langle X \mid Y \rangle$ is an isomorphism of vector spaces. We set $\Delta \overset{\mathrm{def}}{=} \varphi^{-1}(\mathfrak{k}_*)$ and define $\alpha: \widehat{T} \to \Delta$ by $\alpha(\chi) = \varphi^{-1}\big(\mathfrak{L}(\chi)\big)$ i.e. by the commutative diagram

$$\begin{array}{ccccc} \widehat{T} & \xrightarrow{\alpha} & \Delta & \xrightarrow{\mathrm{incl}} & \mathfrak{t} \\ {\scriptstyle \chi\mapsto\mathfrak{L}(\chi)}\downarrow & & {\scriptstyle \varphi|\mathfrak{k}_*}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathfrak{k}_* & \xrightarrow[\mathrm{id}_{\mathfrak{k}}]{} & \mathfrak{k}_* & \xrightarrow[\mathrm{incl}]{} & \mathfrak{t}^*. \end{array}$$

Then α is clearly an isomorphism and $(\alpha(\chi) \mid X) = (\varphi^{-1}(\mathfrak{L}(\chi)) \mid X) = \mathfrak{L}(\chi)(X)$. Then the equations expressing $\chi(\exp_T X)$ follow from the definition of $\mathfrak{L}(\chi)$ in the commuting diagram (&) preceding 6.42.

(ii) For $\chi \in \widehat{T}$ we have $(\alpha(\chi) \mid \mathfrak{k}) = \mathfrak{L}(\chi)(\mathfrak{k}) \subseteq 2\pi\mathbb{Z}$ by (&). Conversely, assume that $(X \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$, then $Y \mapsto e^{(X|Y)\cdot I}: \mathfrak{t} \to \mathrm{SO}(2)$ vanishes on $\mathfrak{k}$ and thus induces a character $\chi: T \to \mathrm{SO}(2)$ such that $\chi(\exp_T Y) = e^{(X|Y)\cdot I}$. Then $X = \alpha(\chi)$.

(iii) By (ii), the inclusion $\Delta \subseteq \mathfrak{k}$ holds iff every $D \in \mathfrak{t}$ such that $(D \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$ is contained in $\mathfrak{k}$.

The relation $\mathfrak{k} \subseteq \Delta$ means that $(\mathfrak{k} \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$ by (ii).

(iv) By (iii) $X = \sum_{j=1}^{n} x_j \cdot e_j \in \Delta$ iff for all $j = 1, \dots, n$, we have $2\pi x_j = \langle X \mid e_j \rangle \in 2\pi\mathbb{Z}$ iff for all j we have $x_j \in \mathbb{Z}$ iff $X \in \mathfrak{k}$. ☐

We shall call Δ *the lattice associated with the inner product* $(\cdot \mid \cdot)$

With respect to a basis $e_1, \dots, e_r$ of Δ and therefore of $\mathfrak{t}$ let $a = a_1 \cdot e_1 + \cdots + a_r \cdot e_r$ and assume that the elements $a_1, \dots, a_r$ of the rational vector space $\mathbb{R}$ are linearly independent over $\mathbb{Q}$. Then $a^{\perp} = \{x_1 \cdot e_1 + \cdots + x_r \cdot e_r \mid a_1 x_1 + \cdots + a_r x_r = 0\}$ does not contain any element of Δ except 0. If we set $\Delta^{+} \stackrel{\text{def}}{=} \{D \in \Delta : (D \mid a) > 0\}$, then Δ^{+} is a subsemigroup of Δ such that $\Delta = \Delta^{+} \cup \{0\} \cup -\Delta^{+}$. It is incidental to our present purposes that the relation $D \prec D'$ iff $D' - D \in \Delta^{+}$ is a total order on Δ. But we note that each orbit of the action of $\mathbb{S}^0 = \{1, -1\} \subseteq \mathbb{R}$ on Δ by multiplication contains exactly one element of Δ^{+} while each orbit on $\Delta \setminus \{0\}$ has two elements. By these remarks and Lemma 6.43 we classify all real simple T-modules according to 3.56 in terms of the elements of Δ^{+}. Indeed we can deal with arbitrary real representations of T as follows. Let E be a real Hilbert T-module according to Definition 2.11. By the splitting of fixed points (3.36) there exists a canonical orthogonal decomposition $E = E_{\text{fix}} \oplus E_{\text{eff}}$. Let $(\cdot \mid \cdot)$ be an inner product on $\mathfrak{t}$ and let Δ the associated lattice in $\mathfrak{t}$. We apply Proposition 3.57 and choose the cross section $\chi \mapsto \chi_{\varepsilon}$ for $\widehat{T} \to \widehat{T}_{\mathbb{R}}$ judiciously. With the aid of Lemma 6.43 above we can do this in such a fashion that $(\alpha(\chi_{\varepsilon}) \mid a) > 0$ for all isomorphy classes $\varepsilon \in \widehat{T}_{\mathbb{R}}$ of real simple nontrivial T-modules. Thus $\varepsilon \mapsto \alpha(\chi_{\varepsilon}) : \widehat{T}_{\mathbb{R}} \to \Delta^{+}$ is a bijection. We may therefore use Δ^{+} as an index set for the equivalence classes of nontrivial real irreducible representations of T and write E_D for E_{ε} iff $D = \alpha(\chi_{\varepsilon})$. Of course, there is considerable arbitrariness in the choice of the indexing of the set $\widehat{T}_{\mathbb{R}}$ by Δ^{+}, but this choice is practical. Now there is a unique orthogonal Hilbert space sum decomposition

$$E_{\text{eff}} = \bigoplus_{D \in \Delta^{+}} E_D$$

into isotypic components.

The set $R^{+} \stackrel{\text{def}}{=} \{D \in \Delta^{+} \mid E_D \neq \{0\}\}$ is called *a set of positive weights* and $R = R^{+} \cup -R^{+}$ *the set of real weights.* Note that this set is canonically attached to the T-module E_{eff} and to the invariant inner product $(\cdot \mid \cdot)$ on $\mathfrak{t}$. We shall refer to the decomposition

$$E = E_{\text{fix}} \oplus \bigoplus_{D \in R^{+}} E_D$$

as the *weight decomposition of* E of the T-module E (associated with a choice of positive weights).

Proposition 6.44. *For a real Hilbert T-module E there is a real orthogonal vector space automorphism $I\colon E_{\text{eff}} \to E_{\text{eff}}$ satisfying $I^2 = -\,\mathrm{id}_{E_{\text{eff}}}$ such that for each $X \in \mathfrak{t}$ and $v \in E_D$, $D \in R^+$,*

$$(\exp X)v = \cos(X \mid D)\cdot v + \sin(X \mid D)\cdot Iv,$$

and that $(v \mid Iv) = 0$ for all $v \in E_{\text{eff}}$. Moreover,

$$I|E_D = \pi_E\big(\exp(\frac{\pi}{2\|D\|^2}\cdot D)\big)|E_D$$

for some $D \in \mathfrak{t}$ with the representation π_E of T on E. In particular, $I\colon E_{\text{eff}} \to E_{\text{eff}}$ is T-equivariant.

Proof. We recall that for each character χ of T we have $\chi_1(\exp X) = \cos(\alpha(\chi) \mid X)$ and $\chi_2(\exp X) = \sin(\alpha(\chi) \mid X)$. Then the proposition follows directly from Proposition 3.57. If $v = \sum_{D\in\Delta^+} v_D \in E_{\text{eff}}$ with $v_D \in E_D$ and $g = \exp X$, $X \in \mathfrak{t}$, then

$$g(Iv) = \sum_{D\in\Delta^+} \exp(X + \frac{\pi}{2\|D\|^2}\cdot D)v_D = I(gv). \qquad \square$$

Action of a Maximal Torus on the Lie Algebra

We apply this to the following situation. Let G denote a compact Lie group and T a maximal torus. By 6.2 and 6.3, $\mathfrak{g}$ is a Hilbert Lie algebra with respect to some invariant inner product $(\,\cdot \mid \cdot\,)$ which we fix. Let $\Delta \subseteq \mathfrak{t}$ be the lattice associated with the restriction of this inner product to $\mathfrak{t}$. Then $\mathrm{Ad}\,|T\colon T \to \mathrm{Aut}(\mathfrak{g})$ is a representation of T on the Hilbert Lie algebra $\mathfrak{g}$. By 6.21(ii) we know that

$$\mathfrak{g}_{\text{fix}} = \{Y \in \mathfrak{g} \mid (\forall X \in \mathfrak{t})e^{\mathrm{ad}\,X}Y = \mathrm{Ad}(X)Y = Y\} = \mathfrak{z}(\mathfrak{t}, \mathfrak{g}) = \mathfrak{t}.$$

The set $R \subseteq \Delta \subseteq \mathfrak{t}$ of weights (respectively, a set $R^+ \subseteq R$ of positive weights) of this representation is called the *the set of real roots* (respectively, a set of *positive roots* of $\mathfrak{g}$) with respect to the Cartan algebra $\mathfrak{t}$ (and a choice of an invariant inner product $(\cdot \mid \cdot)$ on $\mathfrak{g}$). This is a fair description since each Cartan algebra $\mathfrak{t}$ of $\mathfrak{g}$ determines a unique maximal torus $T = \exp\mathfrak{t}$ and each maximal torus T determines a unique Cartan subalgebra. It is clear that $R^+ = \emptyset$ if and only if $\mathfrak{g}_{\text{eff}} = \{0\}$ iff $\mathfrak{g} = \mathfrak{t}$ iff $\mathfrak{g}$ is abelian. We now simply apply Proposition 6.44 with $E = \mathfrak{g}$ and $\pi_E = \mathrm{Ad}$ and obtain what is called the *real root space decomposition* of $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{D\in R^+} \mathfrak{g}^D,$$

where for $D \in R^+$ the $\mathrm{Ad}(T)$-invariant subspace $\mathfrak{g}^D$ is the isotypic component of E for the irreducible representation associated with the root D and where

$$\mathfrak{g}_{\mathrm{eff}} = \bigoplus_{D\in R^+} \mathfrak{g}^D.$$

Proposition 6.45. *For each compact Lie group G and a maximal torus T, a fixed invariant inner product $(\cdot \mid \cdot)$ on $\mathfrak{g}$ and a set R^+ of positive roots there is an orthogonal vector space automorphism I of $\bigoplus_{D\in R^+}\mathfrak{g}^D$ with $I^2 = -\mathrm{id}$, commuting elementwise with the action of $\mathrm{Ad}(T) = e^{\mathrm{ad}\,\mathfrak{t}}$ on this vector subspace such that $I|\mathfrak{g}^D = e^{\mathrm{ad}\left(\frac{\pi}{2\|D\|^2}\cdot D\right)}\Big|\mathfrak{g}^D$ where I satisfies the following relation for all $Y \in \mathfrak{g}^D$:*

$$(\#) \qquad e^{\mathrm{ad}\,X}Y = \cos(X \mid D)\cdot Y + \sin(X \mid D)\cdot IY.$$

(i) *For $Y \in \bigoplus_{D\in R^+}\mathfrak{g}^D$,*

$$\text{(A)} \qquad Y \in \mathfrak{g}^D \Leftrightarrow (\forall X \in \mathfrak{t}) \quad [X, Y] = (X \mid D)\cdot IY.$$

(ii) *The following relation holds*

$$\text{(B)} \qquad (\forall Y \in \mathfrak{g}^D) \quad [Y, IY] = \|Y\|^2\cdot D,$$

and $\mathbb{R}.D + \mathbb{R}\cdot Y + \mathbb{R}\cdot IY$ is a subalgebra and $R \subseteq \mathfrak{g}'$.

(iii) *$\mathfrak{t} \oplus \mathbb{R}\cdot Y \oplus \mathbb{R}\cdot IY$ is a subalgebra with center $D^{\perp_{\mathfrak{t}}} = \{X \in \mathfrak{t} \mid (X \mid D) = 0\}$.*

(iv) *Assume that G is connected and center free, i.e. that G may be identified with $e^{\mathrm{ad}\,\mathfrak{g}}$. Then $\mathfrak{g}$ is semisimple, i.e. $\mathfrak{z}(\mathfrak{g}) = \{0\}$, $\mathfrak{g}' = \mathfrak{g}$, and $\mathfrak{t} = \mathrm{span}_{\mathbb{R}} R^+$. Further assume, using 6.4(x), that $(X \mid Y) = -\frac{1}{4\pi}\,\mathrm{tr}\,\mathrm{ad}\,X\,\mathrm{ad}\,Y$. Let Δ be the lattice associated to the restriction of this inner product to $\mathfrak{t}$ according to 6.43, and set $\mathfrak{k} = \ker\exp_T$. Then $\mathfrak{k} \subseteq \Delta = \mathrm{span}_{\mathbb{Z}} R^+$, and $\Delta/\mathfrak{k}$ is finite. The relation between Δ and $\mathfrak{k}$ is as follows:*

$$X \in \mathfrak{k} \Leftrightarrow (X \mid \Delta) \subseteq 2\pi\mathbb{Z},$$
$$X \in \Delta \Leftrightarrow (\mathfrak{k} \mid X) \subseteq 2\pi\mathbb{Z}.$$

Proof. (i) The first assertion follows directly from 6.44. As to (A), set $\varphi(t) = e^{t\cdot\,\mathrm{ad}\,X}Y$; then $[X, Y] = \frac{d}{dt}\big|_{t=0}\varphi(t)$. Thus if $Y \in \mathfrak{g}^D$ then $\varphi(t) = \cos t(X \mid D)\cdot Y + \sin t(X \mid D)\cdot IY$ by (#) and thus

$$[X, Y] = \varphi'(0) = (X \mid D)\cdot IY.$$

Conversely, assume $[X, Y] = (X \mid D)\cdot IY$ for all $X \in \mathfrak{t}$. Since $I\colon \mathfrak{g}_{\mathrm{eff}} \to \mathfrak{g}_{\mathrm{eff}}$ is T-equivariant by 6.44, we have $\big((\mathrm{ad}\,X)\circ I\big)|\mathfrak{g}_{\mathrm{eff}} = \big(I\circ(\mathrm{ad}\,X)\big)|\mathfrak{g}_{\mathrm{eff}}$. We claim that $(\mathrm{ad}\,X)^nY = \big((X \mid D)\cdot I\big)^nY$ for $n \in \mathbb{N}$, $Y \in \mathfrak{g}_{\mathrm{eff}}$. By assumption this is true for $n = 1$. Assume that the equation holds for n. Then

$$(\mathrm{ad}\,X)^{n+1}Y = (\mathrm{ad}\,X)\big((X \mid D)\cdot I\big)^nY = \big((X \mid D)\cdot I\big)^n[X, Y]Y = \big((X \mid D)\cdot I\big)^{n+1}Y.$$

This proves the claim by induction. Thus $e^{\operatorname{ad} X}Y = e^{(X|D)\cdot I}Y$, and thus $Y \in \mathfrak{g}^D$ by definition of $\mathfrak{g}^D$ as the isotypic component for the irreducible representation of T corresponding to D.

(ii) We consider $Y_1 \in \mathfrak{g}$, $Y_2 \in \mathfrak{g}^D$ and try to obtain information on $[Y_1, Y_2]$. For this purpose we first compute the inner product $(X \mid [Y_1, Y_2])$ with an element $X \in \mathfrak{t}$ and then the bracket $[X, [Y_1, Y_2]]$. Using (A) above, we compute

$$(\forall X \in \mathfrak{t}) \quad ([Y_1, Y_2] \mid X) = (Y_1 \mid [Y_2, X]) = -(X \mid D)(Y_1 \mid IY_2) = -\big((Y_1 \mid IY_2)\cdot D \mid X\big).$$

Thus $([Y_1, Y_2] + (Y_1 \mid IY_2)\cdot D \mid \mathfrak{t}) = \{0\}$; i.e.

$$(*) \qquad [Y_1, Y_2] + (Y_1 \mid IY_2)\cdot D \in \mathfrak{t}^\perp = \bigoplus_{C \in R^+} \mathfrak{g}^C.$$

Now we also take $Y_1 \in \mathfrak{g}^D$ and use (A) again to calculate

$$(**) \qquad (\forall X \in \mathfrak{t}) \quad \big[X, [Y_1, Y_2]\big] = \big[[X, Y_1], Y_2\big] + \big[Y_1, [X, Y_2]\big] = (X \mid D)\cdot([IY_1, Y_2] + [Y_1, IY_2]).$$

A special case is of interest here: $Y_2 = IY_1$; then $[IY_1, Y_2] = 0 = [Y_1, IY_2]$ whence $\big[X, [Y, IY]\big] = 0$ for all $X \in \mathfrak{t}$, $Y \in g^D$. Then $[Y, IY] \in \mathfrak{z}(\mathfrak{t}, \mathfrak{g})$. Since $\mathfrak{z}(\mathfrak{t}, \mathfrak{g}) = \mathfrak{t}$ by 6.21(ii), we have $[Y, IY] \in \mathfrak{t}$. Then $(*)$ above implies $[Y, IY] = -(Y|I^2Y)\cdot D = \|Y\|^2\cdot D$. Thus (B) is proved. Then (A) and (B) together imply that D, Y, IY span a subalgebra. This completes the proof of (ii). Statement (iii) is an immediate consequence of (i) and (ii).

(iv) Since G is centerfree, $\operatorname{Ad}(T)$ acts faithfully on $\mathfrak{g}_{\text{eff}}$ and so for $X \in \mathfrak{t}$ we have $X \in \mathfrak{k}$ iff $\exp X = 1$ iff $e^{\operatorname{ad} X} = \operatorname{id}_{\mathfrak{g}}$ iff $e^{\operatorname{ad} X}|\mathfrak{g}^D = e^{(X|D)\cdot I}|\mathfrak{g}^D = \operatorname{id}_{\mathfrak{g}^D}$ for all $D \in R^+$ iff $(X \mid D) \in 2\pi\mathbb{Z}$ for all $D \in R^+$. This means that the characters $\chi_D \in \widehat{T}$, $\chi_D(\exp X) = e^{(X|D)\cdot I}$ for $D \in \mathbb{R}^+$ separate points. Thus, when Δ is identified with the character group of T according to 6.43, the subgroup $\operatorname{span}_{\mathbb{Z}} R^+$ separates points. Hence by 2.33(i) the equality $\Delta = \operatorname{span}_{\mathbb{Z}} R^+$ follows.

With this information we now know that $X \in \mathfrak{k}$ iff $(\Delta \mid X) \subseteq 2\pi\mathbb{Z}$. The analogous characterisation of Δ was already given in 6.43(ii). The containment $\mathfrak{k} \subseteq \Delta$ will follow below.

We let $P_D\colon \mathfrak{g} \to \mathfrak{g}^D$ denote the orthogonal projection and write $I_D = I \circ P_D$ with I as in 6.44. The linear form $\omega_D \in \mathfrak{t}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ given by $[X, Y] = \omega_D(X)\cdot IY$ for $Y \in \mathfrak{g}^D$ satisfies $\omega_D(X) = (D \mid X)$ by 6.45(A) but is independent of $(\cdot \mid \cdot)$. Note that

$$(*) \qquad \omega_D(\mathfrak{k}) \subseteq 2\pi\mathbb{Z}.$$

Then

$$(\forall X \in \mathfrak{t}) \quad \operatorname{ad} X = \sum_{D \in R^+} \omega_D(X)\cdot I_D,$$

$$(\forall X,\ X' \in \mathfrak{t}) \quad \operatorname{ad} X \operatorname{ad} X' = -\sum_{D \in R^+} \omega_D(X)\omega_D(X')\cdot P_D.$$

Since $\dim \mathfrak{g}^D = 2$ by (ii) above we have $\operatorname{tr} P_D = 2$ and thus

$$(\forall X,\, X' \in \mathfrak{t})\quad (X \mid X') = -\frac{1}{4\pi}\operatorname{tr}\operatorname{ad} X \operatorname{ad} X' = \frac{1}{2\pi}\sum_{D\in R^+}\omega_D(X)\omega_D(X').$$

Then $(*)$ implies $(\mathfrak{k} \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$. Since $\Delta = \{X \in \mathfrak{t} \mid (X \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}\}$ by what we saw above, $\mathfrak{k} \subseteq \Delta$.

Since $\operatorname{span}_R \Delta = \operatorname{span}_{\mathbb{R}} R^+ = \mathfrak{t}$ by (vi) above, the relation $\mathfrak{k} \subseteq \Delta$ implies that $\Delta/\mathfrak{k}$ is finite. □

The proof of the very last part provided additional information, notably the following. If $\omega_D(X) = (D \mid X)$, then $\omega_D \in \mathfrak{t}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ does not depend on the choice of $(\cdot \mid \cdot)$ and for $X,\, X' \in \mathfrak{t}$ we have

$$\text{(C)}\qquad (X \mid X') = \frac{1}{2\pi}\sum_{D\in R^+}\omega_D(X)\omega_D(X')$$

In particular,

$$(\forall D_1,\, D_2 \in R^+)\quad (D_1 \mid D_2) = \frac{1}{2\pi}\sum_{D\in R^+}\omega_D(D_1)\omega_D(D_2).$$

A different proof of some of the results of 6.45(iv) may be outlined as follows. Let $\mathfrak{g}$ be semisimple and set $(X \mid Y) = -\frac{1}{4\pi}\operatorname{tr}\operatorname{ad} X \operatorname{ad} Y$ (cf. 6.4(ix)). Recall

$$\mathfrak{k} \stackrel{\text{def}}{=} \ker(\operatorname{Ad}\circ\exp_G)|\mathfrak{t} = \{D \in \mathfrak{t} \mid e^{\operatorname{ad} D} = \operatorname{id}_{\mathfrak{g}}\}.$$

Then we claim $(\mathfrak{k} \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$ and $\mathfrak{k} \subseteq \Delta$. Indeed recall that for $D \in \mathfrak{t}$ we have $D \in \mathfrak{k}$ iff $e^{\operatorname{ad} D} = \operatorname{id}_{\mathfrak{g}}$. Since $\operatorname{ad} D$ is semisimple, verify that for $D \in \mathfrak{t}$ this implies $\operatorname{Spec}\operatorname{ad} D \subseteq 2\pi i\mathbb{Z}$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C},1}\oplus\cdots\oplus\mathfrak{g}_{\mathbb{C},n}$ be the joint eigenspace decomposition of the semisimple abelian family $\operatorname{ad}_{\mathbb{C}}\mathfrak{t}$ of vector space endomorphisms of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Let $D_1,\, D_2,\, X \in \mathfrak{t}$. Then $\operatorname{ad} D_j(X) = \lambda_{j1}{\cdot}X_1 + \cdots + \lambda_{j2}{\cdot}X_n$ for $j = 1, 2$ and $\operatorname{tr}_{\mathbb{C}}\operatorname{ad} D_1 \operatorname{ad} D_2 = \sum_{m=1}^n \lambda_{1m}\lambda_{2m}$. If $\lambda_{jm} \in 2\pi i\mathbb{Z}$ for all $j = 1, 2,\ m = 1, \ldots,$ then since $(\cdot \mid \cdot)$ is defined in terms of a trace of the reals, $(D_1 \mid D_2) \in 2\pi\mathbb{Z}$. By Lemma 6.43(ii), an element of $\mathfrak{t}$ is in Δ if $(X \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$. We know $(\mathfrak{k} \mid \mathfrak{k}) \subseteq 2\pi\mathbb{Z}$; hence $\mathfrak{k} \subseteq \Delta$. This proves the claim.

In order to identify the Lie algebra which emerged in 6.45(ii) we let $\mathbb{H}$ denote the skew field of quaternions; then $\mathfrak{s}^3 \stackrel{\text{def}}{=} \mathbb{R}{\cdot}i + \mathbb{R}{\cdot}j + \mathbb{R}{\cdot}k$ is a Lie subalgebra of $(\mathbb{H}, [\cdot,\cdot])$. We let $\mathbb{Z}(3) = \mathbb{Z}/3\mathbb{Z}$ as usual denote the group of integers $0, 1, 2$ modulo 3.

We retain the hypotheses and the notation of Proposition 6.45.

Proposition 6.46. (I) *Let $D \in R^+$ be such that $\mathfrak{g}^D \neq \{0\}$ and pick a nonzero $Y \in \mathfrak{g}^D$. Set $\mathfrak{g}_D \stackrel{\text{def}}{=} \mathbb{R}{\cdot}D + \mathbb{R}{\cdot}Y + \mathbb{R}{\cdot}IY$. We find a basis E_ν, $\nu \in \mathbb{Z}(3)$ such that the following relations hold.*

$$(\dagger)\qquad E_0 = \frac{\mathbf{1}}{\|D\|^2}{\cdot}D,\quad E_1 = \frac{1}{\|D\|{\cdot}\|Y\|}{\cdot}Y,\quad \textit{and}\quad E_2 = \frac{1}{\|D\|{\cdot}\|Y\|}{\cdot}IY,$$

$$(\ddagger)\qquad (\forall \nu \in \mathbb{Z}(3)) \quad [E_\nu, E_{\nu+1}] = E_{\nu+2}.$$

Define linear maps $\varphi\colon \mathfrak{g}_D \to \mathfrak{so}(3)$, $\psi\colon \mathfrak{g}_D \to \mathfrak{su}(2)$ *and* $\rho\colon \mathfrak{g}_D \to \mathfrak{s}^3$ *by*

$$\varphi(E_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \varphi(E_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \varphi(E_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\psi(E_0) = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \psi(E_1) = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \psi(E_2) = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \textit{and}$$

$$\rho(E_0) = \tfrac{1}{2}i, \qquad \rho(E_1) = \tfrac{1}{2}j, \qquad \rho(E_2) = \tfrac{1}{2}k.$$

Then φ, ψ *and* ρ *secure the following isomorphisms:*

$$\mathfrak{g}_D \cong \mathfrak{so}(3) \cong \mathfrak{su}(2) \cong \mathfrak{s}^3.$$

Let $(\mathfrak{g}_D)_\mathbb{C} = \mathbb{C} \otimes \mathfrak{g}_D$ *denote the complexification of* $\mathfrak{g}_D$ *and identify* $1 \otimes \mathfrak{g}_D$ *and* $\mathfrak{g}_D$. *Set*

$$(*)\qquad H = -\frac{2i}{\|D\|^2}\cdot D, \quad P_\pm = \frac{1}{\|D\|\cdot\|Y\|}\cdot(-i\cdot Y \mp IY).$$

Then

$$(**)\qquad [H, P_+] = 2P_+, \quad [H, P_-] = -2P_-, \quad \textit{and} \quad [P_+, P_-] = H.$$

Moreover, the linear map $\Psi\colon (\mathfrak{g}_D)_\mathbb{C} \to \mathfrak{sl}(2,\mathbb{C})$ *given by*

$$\Psi(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Psi(P_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \textit{and} \quad \Psi(P_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is an isomorphism.

(II) *Assume that* $D' \in R^+$ *and* $0 \neq Y' \in \mathfrak{g}^{D'}$ *are elements such that* $\mathbb{R}\cdot D = \mathbb{R}\cdot D'$. *Then the adjoint representation of* T *induces isomorphic* T*-module structures on* $\mathbb{R}\cdot Y + \mathbb{R}\cdot IY$ *and* $\mathbb{R}\cdot Y' + \mathbb{R}\cdot IY'$.

(III) *For all* $D \in R$, *with* $E_0 = \frac{1}{\|D\|^2}\cdot D$, *the following statement holds*

$$(\forall D' \in R) \quad 2\frac{(D \mid D')}{(D \mid D)} = 2(E_0 \mid D') \in \mathbb{Z}.$$

(IV) *Set* $G_D = \exp_G \mathfrak{g}_D$. *Then* G_D *is a closed subgroup of* G *with the following properties.*

(i) $\mathfrak{L}(G_D) = \mathfrak{g}_D$,

(ii) $G_D \cong \mathrm{SO}(3)$ *or* $G_D \cong \mathbb{S}^3 \cong \mathrm{SU}(2)$,

(iii) $\mathrm{Ad}(G_D)|\mathfrak{g}_D \cong \mathrm{SO}(3)$.

(iv) *There is a surjective homomorphism* $\Phi\colon G_D \to \mathrm{SO}(3)$ *with a discrete central kernel such that* $\mathfrak{L}(\Phi) = \varphi\colon \mathfrak{g}_D \to \mathfrak{so}(3)$.

(v) $\exp_G(\mathfrak{t} \oplus \mathbb{R}\cdot E_1 \oplus \mathbb{R}\cdot E_2)$ *is the closed subgroup* TG_D *in which* T *is a maximal torus and* $\exp_G \mathfrak{t}_D$, $\mathfrak{t}_D \stackrel{\text{def}}{=} D^{\perp_\mathfrak{t}}$, *is the identity component of its center.*

Proof. (I) We pick any nonzero element $Y \in \mathfrak{g}^D$. Then $\mathfrak{g}_D = \operatorname{span}\{D, Y, IY\}$ is a subalgebra by Proposition 6.45(ii). Let us consider positive numbers λ_ν, $\nu \in \mathbb{Z}(3)$ and set $E_0 = \lambda_0 \cdot D$, $E_2 = \lambda_1 \cdot Y$, and $E_2 = \lambda_2 \cdot IY$. Then Condition (A) yields

$$\begin{aligned}[E_0, E_1] &= \quad \lambda_0\lambda_1 \cdot [D, Y] = \|D\|^2 \lambda_0 \lambda_1 \lambda_2^{-1} \cdot E_2, \text{ and} \\ [E_2, E_0] &= -\lambda_0\lambda_2 \cdot [D, IY] = \|D\|^2 \lambda_0 \lambda_2 \lambda_1^{-1} \cdot E_1,\end{aligned}$$

while Condition (B) gives

$$\begin{aligned}[E_1, E_2] &= \lambda_1\lambda_2 \cdot [Y, IY] = \lambda_1\lambda_2 \|Y\|^2 \cdot D = \lambda_1\lambda_2\|Y\|^2\lambda_0^{-1} \cdot E_0 \\ &= \lambda_0^{-1}\lambda_1\lambda_2\|Y\|^2 \cdot E_0.\end{aligned}$$

Now condition (‡) holds iff we can solve the equations

$$\begin{aligned}\|D\|^2\lambda_0\lambda_1 &= \lambda_2, \\ \|Y\|^2\lambda_1\lambda_2 &= \lambda_0, \\ \|D\|^2\lambda_2\lambda_0 &= \lambda_1.\end{aligned}$$

with positive numbers λ_ν. The first and the third equation are equivalent to the first equation plus $\lambda_1/\lambda_2 = \lambda_2/\lambda_1$, i.e. $\lambda_1^2 = \lambda_2^2$, which holds for nonnegative λ_j iff $\lambda_1 = \lambda_2$. The first equation then yields $\lambda_0 = \|D\|^{-2}$, and the second one gives $\lambda_1^2 = \frac{\lambda_0}{\|Y\|^2} = (\|D\|^2\|Y\|^2)^{-1}$, and these values for $\lambda_\nu > 0$ solve the equations uniquely. Thus

$$\lambda_0 = \frac{1}{\|D\|^2} \quad \text{and} \quad \lambda_1 = \lambda_2 = \frac{1}{\|D\| \cdot \|Y\|},$$

are the unique positive numbers yielding (‡), and with these numbers we obtain (†). We verify quickly that φ, ψ, and ρ are the required Lie algebra isomorphisms.

The verification of the claims regarding H and $P_\pm$ is elementary; it is helped by considering the isomorphism $\psi\colon \mathfrak{g}_D \to \mathfrak{su}(2)$ and following the computations in terms of 2×2 complex matrices inside $\mathfrak{sl}(2\mathbb{C})$.

(II) We set

$$(\dagger') \qquad E_0' = \frac{\mathbf{1}}{\|D'\|} \cdot D', \quad E_1' = \frac{1}{\|D'\| \cdot \|Y'\|} \cdot Y', \quad \text{and} \quad E_2' = \frac{1}{\|D'\| \cdot \|Y'\|} \cdot IY'.$$

Then (‡′) holds, i.e. (‡) with E_ν' replacing E_ν. Further, E_0 and E_0' are unit vectors in the one dimensional subspace $\mathbb{R} \cdot D = \mathbb{R} \cdot D'$ of $\mathfrak{t}$ and thus $E_0' = E_0$ or $E_0' = -E_0$. In the latter case we set $E_0'' = -E_0' = E_0$, $E_1'' = E_2'$, and $E_2'' = E_1''$. Then the E_ν'' satisfy (‡″). Thus we may and will assume now that $E_0' = E_0$ holds. Now we define a function $\alpha\colon \mathfrak{t} \oplus \mathbb{R} \cdot Y \oplus \mathbb{R} \cdot IY \to \mathfrak{t} \oplus \mathbb{R} \cdot Y' \oplus \mathbb{R} \cdot IY'$ by $\alpha|\mathfrak{t} = \mathrm{id}_{\mathfrak{t}}$ and $\alpha(E_1) = E_1'$, $\alpha(E_2) = E_2'$. Then α is a vector space isomorphism which is a Lie algebra isomorphism because of (‡) and (‡′) and the fact that $D^{\perp_\mathfrak{t}} = (D')^{\perp_\mathfrak{t}}$ and $[D^{\perp_\mathfrak{t}}, \mathbb{R} \cdot Y + \mathbb{R} \cdot IY] = [D^{\perp_\mathfrak{t}}, \mathbb{R} \cdot Y' + \mathbb{R} \cdot IY'] = \{0\}$. For $X \in D^{\perp_\mathfrak{t}}$ we have $[X, E_\nu] = [X, E_\nu'] = 0$, whence $\alpha \circ \operatorname{ad} X = \operatorname{ad} X \circ \alpha$, and $\alpha[E_0, E_\nu] = \alpha E_{-\nu} = E_{-\nu}' = [E_0', E_\nu'] = [E_0, \alpha E_\nu]$ by (‡) and (‡′), whence $\alpha \circ \operatorname{ad} E_0 = \operatorname{ad} E_0 \circ \alpha$. Thus $\alpha \circ \operatorname{ad} X = \operatorname{ad} X \circ \alpha$ for all $X \in \mathfrak{t}$. It follows that $\alpha \circ \operatorname{Ad}(\exp X) = \alpha \circ e^{\operatorname{ad} X} = e^{\operatorname{ad} X} \circ \alpha = \operatorname{Ad}(\exp X) \circ \alpha$. Thus α is a module isomorphism and induces, in particular, a T-module isomorphism from $\mathbb{R}{\cdot}Y + \mathbb{R}{\cdot}IY = \mathbb{R}{\cdot}E_1 + \mathbb{R}{\cdot}E_2$ onto $\mathbb{R}{\cdot}Y' + \mathbb{R}{\cdot}IY' = \mathbb{R}{\cdot}E_1' + \mathbb{R}{\cdot}E_2'$.

(III) Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i{\cdot}\mathfrak{g}$ be the complexification of $\mathfrak{g}$ to which we extend the inner product to a symmetric bilinear nondegenerate form in the obvious way. Then $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra and a complex T-module via $\operatorname{Ad}(g)_\mathbb{C}(c\otimes X) = c\otimes \operatorname{Ad}(g)(X)$, $X \in \mathfrak{g}$. We make reference to the information contained in 3.54–3.57 and to 6.45. For each $D \in \Delta$ let $\chi_D\colon T \to \mathbb{S}^1 \subseteq \mathbb{C}^\times$ be the character given uniquely by

$$(\forall X \in \mathfrak{t}) \quad \chi_D(\exp T) = e^{(X|D)i}.$$

Then $\chi_{-D} = \overline{\chi_D}$. For a character $\chi \in \operatorname{Hom}(T, \mathbb{S}^1)$ of T we let $(\mathfrak{g}_\mathbb{C})_\chi$ denote the isotypic component of all $Z \in \mathfrak{g}_\mathbb{C}$ satisfying $\operatorname{Ad}(t)_\mathbb{C}(Z) = \chi(t){\cdot}Z$. For $D \in \Delta$ we shall write $\mathfrak{g}_C^D \stackrel{\text{def}}{=} (\mathfrak{g}_\mathbb{C})_{\chi_D}$. Thus for $Z \in \mathfrak{g}_\mathbb{C}$ we have

$$\begin{aligned} Z \in \mathfrak{g}_\mathbb{C}^D &\Leftrightarrow \big((\forall X \in \mathfrak{t}) \quad \operatorname{Ad}(\exp X)_\mathbb{C}(Z) = e^{(X|D)i}{\cdot}Z, \quad \text{and} \\ &\Leftrightarrow \big((\forall X \in \mathfrak{t}_\mathbb{C}) \quad [X, D] = (X \mid D)i{\cdot}Z. \end{aligned} \tag{1}$$

We observe

$$(\forall D \in R^+) \quad \mathfrak{g}^D \oplus i{\cdot}\mathfrak{g}^D = \mathfrak{g}_\mathbb{C}^D \oplus \mathfrak{g}_\mathbb{C}^{-D},$$

and the $\mathfrak{g}_\mathbb{C}^D$ are the isotypic components of $(\mathfrak{g}_\mathbb{C})_{\text{eff}}$ while $\mathfrak{t} \oplus i{\cdot}\mathfrak{t}$ is the isotypic component $(\mathfrak{g}_\mathbb{C})_{\text{fix}}$. Since $\dim_\mathbb{R} \mathfrak{g}^D = 2$ we have $\dim_\mathbb{C} \mathfrak{g}_\mathbb{C}^{\pm D} = 1$.

If $Z \in \mathfrak{g}_\mathbb{C}^{D'}$ for $D' \in R$, then the definition of H and (1) imply

$$[H, Z] = (H \mid D')i{\cdot}Z = -2\frac{(D \mid D')}{(D \mid D)}{\cdot}Z. \tag{2}$$

Thus the root spaces $\mathfrak{g}_\mathbb{C}^{D'}$ happen to be the (one-dimensional) eigenspaces of $\operatorname{ad} H$ on $\mathfrak{g}_\mathbb{C}$. The claim will therefore be proved if we note that all eigenvalues of $\operatorname{ad} H$ are integral. This information follows from the elementary representation theory of $\mathfrak{sl}(2, \mathbb{C})$ (see Exercise E6.12(i) below.)

(IV) Now we turn to the group generated by $\mathfrak{g}_D$. Since $\mathfrak{g}_D \cong \mathfrak{so}(3)$ and $\mathfrak{so}(3)' = \mathfrak{so}(3)$, Corollary 6.31(ii) applies to show that $G_D \stackrel{\text{def}}{=} \exp \mathfrak{g}_D$ is a closed subgroup with $\mathfrak{L}(G_D) = \mathfrak{g}_D$. By E5.12, the compact Lie groups with a Lie algebra isomorphic to $\mathfrak{so}(3)$ are isomorphic to $\mathrm{SO}(3)$ or $\mathbb{S}^3 \cong \mathrm{SU}(2)$. This establishes the existence of G_D and the validity of (i) and (ii). The subgroup $\operatorname{Ad}(G_D)|\mathfrak{g}_D$ of the orthogonal group $\mathrm{O}(\mathfrak{g}_D)$ on $\mathfrak{g}_D \cong \mathbb{R}^3$ is connected, hence is contained in $\mathrm{O}(\mathfrak{g}_D)_0 = \mathrm{SO}(\mathfrak{g}_D)$. The Lie algebra $\operatorname{ad} \mathfrak{g}_D|\mathfrak{g}_D = \mathfrak{L}(\operatorname{Ad}(G_D)|\mathfrak{g}_D)$ is isomorphic to $\mathfrak{g}_D \cong \mathfrak{so}(\mathfrak{g}_D)$. Hence $\mathfrak{L}(\operatorname{Ad}(G_D)|\mathfrak{g}_D) = \mathfrak{so}(\mathfrak{g}_D)$ and $\operatorname{Ad}(G_D)|\mathfrak{g}_D = \mathrm{SO}(\mathfrak{g}_D) \cong \mathrm{SO}(3)$. Thus (iii) is proved. For a proof of (iv) we define a vector space automorphism $L\colon \mathfrak{g}_D \to \mathbb{R}^3$ by $L(E_j) = e_j$, $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$, and $\Phi\colon G_D \to \mathrm{SO}(3) \subseteq \operatorname{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ by $\Phi(g) = L\circ(\operatorname{Ad}(g)|\mathfrak{g}_D)\circ L^{-1}$. Then $\Phi(\exp t{\cdot}E_j)(v) = Le^{t\cdot \operatorname{ad} E_j}L^{-1}v$ and differentiation at $t = 0$ yields $\mathfrak{L}(\Phi)(E_j)(e_k) = L([E_j, E_k]) = \varphi(E_j)(e_k)$ by the definition of φ. Hence $\mathfrak{L}(\Phi) = \varphi$. Thus $\mathfrak{L}(\varphi)$ is an isomorphism and hence Φ implements a local isomorphism by 5.42. This proves (iv). The proof of (v) is a consequence of 6.45(iii). □

Exercise E6.12. Prove the following assertions:
(i) *Set* $L = \mathfrak{sl}(2,\mathbb{C})$ *and define*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 & 1, \\ 0 & 0 \end{pmatrix}, \quad P_= \begin{pmatrix} 0 & 0, \\ 1 & 0 \end{pmatrix}.$$

Let V *denote a finite dimensional complex vector space, and consider the Lie algebra* $\mathfrak{gl}(V)$ *of endomorphisms with the bracket* $[\alpha,\beta] = \alpha\beta - \beta\alpha$. *Let* $\pi\colon L \to \mathfrak{gl}(V)$ *be a representation, i.e. a morphism of* $\mathbb{K}$*-Lie algebras). For* $\lambda \in \mathbb{K}$ *define* $V_\lambda = \{v \in V \mid \pi(H)(v) = \lambda{\cdot}v\}$ *and assume that we know* $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$. *Then*

$$(\pm) \qquad (\forall \lambda \in \mathbb{C},) \quad \pi(P_\pm)(V_\lambda) \subseteq V_{\lambda\pm 2}.$$

For $\lambda \in \mathbb{C}$ *with* $V_\lambda \neq \{0\}$ *let*

$$\mu_\lambda = \max\big\{m \in \mathbb{Z} \mid V_{\lambda+2m} \neq \{0\}\big\}.$$

For $0 \neq v \in V_{\lambda+2\mu}$, *set* $v_p = \frac{1}{p!}{\cdot}\pi(P_-)^p(v)$ *for* $p = 0, 1, \ldots$. *Then*

$$(H) \qquad \pi(H)(v_p) = (\lambda + 2\mu - 2p){\cdot}v_p,$$

$$(+) \qquad \pi(P_+)(v_p) = (\lambda + 2\mu - p + 1){\cdot}v_{p-1},$$

$$(-) \qquad \pi(P_-)(v_p) = (p+1){\cdot}v_{p+1}.$$

Let $\nu = \max\{n \in \mathbb{Z} \mid v_{m+1} = 0, v_m \neq 0\}$. *Then* $\lambda = \nu - 2\mu$. *In particular, all* $\lambda \in \mathbb{C}$ *for which* $V_\lambda \neq \{0\}$ *are integers.*

(ii) *Let* G *be a compact Lie group,* T *a maximal torus of* G, *and* $R \subseteq \mathfrak{t}$ *a set of real roots with respect to some invariant inner product* $(\cdot \mid \cdot)$ *on* $\mathfrak{g}$. *Fix a set* R^+ *of positive roots and let* B^+ *be the basis attached to* R^+. *Let* $D_j \in R$, $j = 1, 2$. *Then*

$$[\mathfrak{g}_{\mathbb{C}}^{D_1}, \mathfrak{g}_{\mathbb{C}}^{D_2}] = \{0\} \quad \textit{or} \quad [\mathfrak{g}_{\mathbb{C}}^{D_1}, \mathfrak{g}_{\mathbb{C}}^{D_2}] = \mathfrak{g}_{\mathbb{C}}^{D_1+D_2}.$$

[Hint. (i) For a proof of $(\pm)$, argue

$$\pi(H)\pi(P_\pm)(v) = \pi([H,P_\pm])(v) + \pi(P_\pm)\pi(H)(v) = \pm 2\pi(P_\pm)(v) + \lambda{\cdot}\pi(P_\pm)v.$$

Recursively, $(\pm)$ proves (H) and $(-)$ is a consequence of the definition of v_p. Assertion $(+)$ is proved by induction. Using the definition of v_p, equations (H) and $(-)$, set $v_{-1} = 0$ and calculate

$$\begin{aligned} p\pi(P_+)(v_p) &= \pi(P_+)\pi(P_-)(v_{p-1}) = \pi([P_+,P_-])(v_{p-1}) + \pi(P_-)\pi(P_+)(v_{p-1}) \\ &= \big(\lambda + 2\mu - 2(p-1)\big){\cdot}v_{p-1} + (\lambda + 2\mu - p + 2)\pi(P_-)(v_{p-2}) \\ &= (\lambda + 2\mu - 2p - 2){\cdot}v_{p-1} + (p-1)(\lambda + 2\mu - p + 2){\cdot}v_{p-1} \\ &= p(\lambda + 2\mu - p + 1){\cdot}v_{p-1}. \end{aligned}$$

By the definition of ν note that $0 = \pi(P_-)(v_{\nu+1}) = (\lambda + 2\mu - \nu){\cdot}v_\nu$, whence $\lambda + 2\mu - \nu = 0$.

(ii) Let $D_j \in R$, $j = 1, 2$ and $Z_j \in \mathfrak{g}_{\mathbb{C}}^{D_j}$; pick any $X \in \mathfrak{t}$. Then

$$\begin{aligned} \mathrm{Ad}(\exp X)_{\mathbb{C}}[Y_1, Y_2] &= [\mathrm{Ad}(\exp X)_{\mathbb{C}}(Y_1), \mathrm{Ad}(\exp X)_{\mathbb{C}}(Y_2)] \\ &= [e^{(X|D_1)i}{\cdot}Y_1, e^{(X|D_2)i}{\cdot}Y_2] = e^{(X|D_1+D_2)i}{\cdot}[Y_1, Y_2]. \end{aligned}$$

Therefore, $[\mathfrak{g}_{\mathbb{C}}^{D_1}, \mathfrak{g}_{\mathbb{C}}^{D_2}] \subseteq \mathfrak{g}_{\mathbb{C}}^{D_1+D_2}$. We have

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{D_1+D_2} = \begin{cases} 1 & \text{if } D_1 + D_2 \in R, \\ 0 & \text{otherwise.} \end{cases}$$

This implies (ii).] □

For more information on the representations of $\mathfrak{sl}(2,\mathbb{C})$, consult any source on complex Lie algebras (e.g. [207]).

We remain in the circumstances, and continue the notation, of the preceding discussions. For a $D \in R^+$ and any $0 \neq Y \in \mathfrak{g}^D$ we set

$$E_0 = \frac{1}{\|D\|}\cdot D, \quad E_1 = \frac{1}{\|D\|\cdot\|Y\|}\cdot Y, \quad \text{and} \quad E_2 = \frac{1}{\|D\|\cdot\|Y\|}\cdot IY$$

as in 6.46. In the following proposition we use the techniques just developed in order to attach yet another element of structure to D.

Proposition 6.47. *For $D \in R^+$ set $n_D \overset{\text{def}}{=} \exp \pi E_1$. Then $n_D \in N(T, G_0) \cap G_D$ and thus*

$$\sigma_D \overset{\text{def}}{=} \mathrm{Ad}(n_D)|\mathfrak{t} = e^{\pi \,\mathrm{ad}\, E_1}|\mathfrak{t}$$

is well-defined and

$$(\sigma) \qquad \sigma_D(X) = X - \frac{2(X \mid D)}{\|D\|^2}\cdot D.$$

In other words, σ_D is the unique reflection of $\mathfrak{t}$ leaving the hyperplane $\mathfrak{t}_D$ of $\mathfrak{t}$ elementwise fixed and satisfies $\sigma_D(D) = -D$.

Finally, if $G_D \cong \mathbb{S}^3$ then G_D contains a unique element $z_D \in Z(G_D)$ of order 2 corresponding to $-1 \in \mathbb{S}^3$, and

$$n_D^2 = \exp \frac{2\pi}{\|D\|^2}\cdot D = \begin{cases} 1 & \text{if } G_D \cong \mathrm{SO}(3), \\ z_D & \text{if } G_D \cong \mathrm{SU}(2). \end{cases}$$

In particular, $n_D^2 \in T$ and $n_D^4 = 1$.

Proof. By definition, $n_D \in G_D \subseteq G_0$. We refer to explicit information we have on SO(3) (see E6.10). We know from 6.46(iv) that there is a surjective homomorphism $\Phi: G_D = \exp \mathfrak{g}_D \to \mathrm{SO}(3)$ with discrete kernel such that $\Phi(\exp t\cdot E_j) = e^{t\varphi(E_j)}$, $j = 1, 2, 3$, where $\varphi = \mathfrak{L}(\Phi) =: \mathfrak{L}(G_D) = \mathfrak{g}_D \to \mathfrak{so}(3)$ maps E_ν as follows:

$$\varphi(E_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \varphi(E_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \varphi(E_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$\Phi(\exp t{\cdot}E_0) = \exp t{\cdot}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&\cos t&-\sin t\\0&\sin t&\cos t\end{pmatrix}, \text{ and}$$

$$\Phi(\exp \pi{\cdot}E_1) = \exp \pi\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \text{ whence}$$

$$\begin{aligned}\Phi(\exp t{\cdot}e^{\pi\,\mathrm{ad}\,E_1}E_0) &= \Phi((\exp \pi E_1)(\exp t{\cdot}E_0)(\exp \pi E_1)^{-1})\\ &= \begin{pmatrix}1&0&0\\0&\cos t&\sin t\\0&-\sin t&\cos t\end{pmatrix}\\ &= \Phi(\exp -t{\cdot}E_0).\end{aligned}$$

Thus the function $t\mapsto \exp(t{\cdot}e^{\pi\,\mathrm{ad}\,E_1}E_0)\exp t{\cdot}E_0 : \mathbb{R}\to G_D$ maps $\mathbb{R}$ continuously into the center of G_D which has one or two elements (cf. 6.46(iv)) and thus is constant. Since for $t=0$ the value $1\in G_D$ results, we conclude $\exp(t{\cdot}e^{\pi\,\mathrm{ad}\,E_1}E_0) = \exp t{\cdot}E_0$ for all $t\in\mathbb{R}$ and thus

$$\sigma_D(E_0) = \mathrm{Ad}(n_D)(E_0) = e^{\pi\,\mathrm{ad}\,E_1}(E_0) = -E_0.$$

Hence $\sigma_D(D) = -D$. As $[\mathfrak{t}_D,\mathfrak{g}_D]=\{0\}$ by 6.47(iii), $E_0\in\mathfrak{g}_D$, and $\sigma_D = e^{\pi\cdot\,\mathrm{ad}\,E_0}|\mathfrak{t}$, we conclude $\sigma_D|\mathfrak{t}_D = \mathrm{id}_{\mathfrak{t}_D}$. It follows that σ_D is the orthogonal reflection of $\mathfrak{t}$ at the hyperplane $D^{\perp_\mathfrak{t}}$ whose explicit elementary formula is given in (σ). Since $\mathrm{Ad}(n_D)(\mathfrak{t}) = \mathfrak{t}$ we have $n_DTn_D^{-1} = n_D(\exp\mathfrak{t})n_D^{-1} = \exp\mathrm{Ad}(n_D)(\mathfrak{t}) = \exp\mathfrak{t} = T$. Hence $n_D\in N(T,G_0)$. Also $\Phi(n_D^2) = \mathrm{Ad}(n_D^2)|\mathfrak{t} = \sigma_D^2 = \mathrm{id}_\mathfrak{t}$. Thus $n_D^2\in Z(T,G_0) = T$. Hence $n_D^2\in T\cap G_D = \exp\mathbb{R}{\cdot}D$. Also $n_D^2\in\ker\Phi = Z(G_D)$ by 6.46(iv). By the Transitivity Theorem 6.27 applied to $\mathfrak{g}_D\cong\mathfrak{so}(3)$ (where $\mathrm{Ad}\big(\mathrm{SO}(3)\big)$ acts on $\mathfrak{so}(3)$ as the rotation group), there is a $g\in G_D$ such that $\mathrm{Ad}(g)(E_1) = E_0$. Then, since n_D^2 is central in G_D we have $n_D^2 = gn_D^2g^{-1} = \exp 2\pi\,\mathrm{Ad}(g)(E_1) = \exp 2\pi E_0 = \exp\frac{2\pi}{\|D\|^2}{\cdot}D$ by the definition of E_0. If $G_D\cong\mathrm{SO}(3)$, then $n_D^2 = \exp 2\pi E_0 = \mathbf{1}$, if $G_D\cong\mathrm{SU}(2)\cong\mathbb{S}^3$, then n_D^2 in $\mathbb{S}^3$ corresponds to the element -1. Thus n_D has order 4 in this case. □

All groups G_D are connected and thus contained in G_0. Their global structure is readily elucidated.

Corollary 6.48. *Let G be a compact Lie group, T a maximal torus and $\mathfrak{g} = \mathfrak{t}\oplus\bigoplus_{D\in R^+}\mathfrak{g}^D$ the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then for each $D\in R^+$ let $\mathfrak{g}_D$ and $\mathfrak{t}_D$ be defined as in Proposition* 6.47*. Then there are connected compact subgroups $G_D \overset{\mathrm{def}}{=} \exp_G\mathfrak{g}_D$ (as in Proposition* 6.46*), and $T_D = \exp_G\mathfrak{t}_D$ such that $\mathfrak{t}_D \overset{\mathrm{def}}{=} \mathfrak{L}(T_D) = D^{\perp_\mathfrak{t}}$ and that $TG_D = T_DG_D$ is isomorphic to $(T_D\times G_D)/F$ where F is a finite central subgroup of the product isomorphic to $T_D\cap G_D\subseteq Z(G_D)$.*

Proof. For each $D \in R^+$, the results of Proposition 6.46 apply to $\mathfrak{t}+\mathfrak{g}_D = \mathfrak{t}_D \oplus \mathfrak{g}_D$ with $\mathfrak{t}_D = \mathfrak{z}(\mathfrak{t}+\mathfrak{g}_D)$ and $\mathfrak{g}_D = \operatorname{span}\{E_0, E_1, E_2\}$. Then $\mathfrak{g}_D = (\mathfrak{t}+\mathfrak{g}_D)'$. We proved in 6.46 that G_D is a closed subgroup such that TG_D is a closed subgroup with $\mathfrak{L}(TG_D) = \mathfrak{t}+\mathfrak{g}_D$ and that $\exp_G \mathfrak{t}_D = T_D$ is the identity component of the center of TG_D and $G_D = (TG_D)'$. The remainder follows from the First Structure Theorem for Connected Compact Lie Groups 6.16. □

From the action of T on $\mathfrak{g}$ we derived the information that G, when noncommutative, contains many copies of SO(3) or SU(2) and that $\mathfrak{g}$ contains many copies of $\mathfrak{so}(3)$. We further elucidate some of the structure of $\mathfrak{g}$ by refining the information we have and continue the notation introduced in this subsection.

Theorem 6.49. *Let G be a compact Lie group. Assume that its Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is nonabelian, and fix an invariant inner product $(\cdot \mid \cdot)$ on $\mathfrak{g}$. Then for any $D \in R^+$ the following statements hold.*

(i) *For all $X \in R^+$ there is at most one $D \in R^+ \cap \mathbb{R}\cdot X$.*

(ii) *If $D \in R^+$, then $\dim \mathfrak{g}^D = 2$, and $\mathfrak{g}_D \overset{\text{def}}{=} \mathbb{R}\cdot D \oplus \mathfrak{g}^D \cong \mathfrak{so}(3)$.*

(iii) *$\mathfrak{t} \oplus \mathfrak{g}^D$ is a subalgebra with center $\mathfrak{t}_D \overset{\text{def}}{=} D^{\perp_\mathfrak{t}} = \{X \in \mathfrak{t} \mid (X \mid D) = 0\} = \mathfrak{z}(\mathfrak{g}^D, \mathfrak{t})$ and thus is isomorphic to $\mathbb{R}^{r-1} \oplus \mathfrak{so}(3)$, $r = \dim \mathfrak{t}$.*

(iv) *$\mathfrak{t}_D = \mathfrak{z}(Y, \mathfrak{t})$ for each nonzero $Y \in \mathfrak{g}^D$.*

(v) *$\mathfrak{z}(\mathfrak{g}) = (R^+)^{\perp_\mathfrak{t}} = \bigcap_{D \in R^+} \mathfrak{t}_D$.*

(vi) *$\mathfrak{t} \cap \mathfrak{g}' = \operatorname{span} R^+$ and $\mathfrak{g}' = \operatorname{span} R^+ \oplus \mathfrak{g}_{\text{eff}}$.*

(vii) *If $r = \dim \mathfrak{t} = 1$, i.e. if the maximal torus subgroups of G are circles, then G is isomorphic to $\mathbb{S}^1$, SO(3) or SU(2).*

Proof. (i) Let $D, D' \in R^+$, $D' = r\cdot D$. We take $0 \neq Y \in \mathfrak{g}^D$ and $0 \neq Y' \in \mathfrak{g}^{D'}$. Then by 6.46(II) the simple T-modules $\mathbb{R}\cdot Y \oplus \mathbb{R}\cdot IY$ and $\mathbb{R}\cdot Y' \oplus \mathbb{R}\cdot IY$ are isomorphic. Since the T-modules $\mathfrak{g}^D$ and $\mathfrak{g}^{D'}$ are isotypic components, they agree, and this implies $D' = D$ (cf. 6.44 and 6.45).

(ii) Let $D \in R^+$. We claim that $\mathfrak{t} \oplus \mathfrak{g}^D$ is a subalgebra. For a proof, since $[\mathfrak{t}, \mathfrak{g}^D] \subseteq \mathfrak{g}^D$, we must show that

$$(***) \qquad [\mathfrak{g}^D, \mathfrak{g}^D] \subseteq \mathfrak{t} + \mathfrak{g}^D.$$

To prove this, we consider, for $C \in R^+$, the orthogonal projection P_C of $\mathfrak{g}$ onto $\mathfrak{g}^C$. Assume that C is $\mathbb{R}$-linearly independent from D and let $Y_j \in \mathfrak{g}^D$, $j = 1, 2$. Since $\operatorname{ad} X$ and I commute with P_C we have

$$(\forall X \in \mathfrak{t}) \quad P_C\big[X, [Y_1, Y_2]\big] = \big[X, P_C[Y_1, Y_2]\big] = (X \mid C)\cdot IP_C[Y_1, Y_2].$$

From this and from equation $(**)$ in the proof of 6.45, in view of $I^2 = -\operatorname{id}_\mathfrak{g}$, we conclude

$$(\forall X \in \mathfrak{t}) \quad (X \mid C)\cdot P_C[Y_1, Y_2] = -(X \mid D)\cdot IP_C([IY_1, Y_2] + [Y_1, IY_2]).$$

There are two cases possible:
Case 1. Both sides are zero for all X. Considering the left hand side and taking $X = C$ we get $(C \mid C) = \|C\|^2 \neq 0$; thus $P_C[Y_1, Y_2] = 0$.

Case 2. The vectors $v_1 = P_C[Y_1, Y_2]$ and $v_2 = IP_C([IY_1, Y_2]+[Y_1, IY_2])$ are nonzero but linearly dependent, say, $v_1 = r{\cdot}v_2$ for a nonzero $r \in \mathbb{R}$. Then $(\forall X \in \mathfrak{t})\,(X \mid C) = (X \mid D)r = (X \mid r{\cdot}D)$ whence $C = r{\cdot}D$, contrary to our supposition that D and C are linearly independent. Thus Case 1 is the only possible one. This completes the proof of (∗∗∗). In view of the action of $\mathfrak{t}$ on $\mathfrak{g}^D$ by 6.45(A) we know that the Hilbert Lie algebra $\mathfrak{t}\oplus\mathfrak{g}_D$ has center (a) $\mathfrak{z}(\mathfrak{t}\oplus\mathfrak{g}^D) = D^{\perp_\mathfrak{t}}$. Also by 6.45(A,B) we have (b) $\mathfrak{g}_D \stackrel{\text{def}}{=} \mathbb{R}{\cdot}\mathfrak{g}^D \subseteq (\mathfrak{t}\oplus\mathfrak{g}^D)'$. By 6.4(v), the algebra $\mathfrak{t}\oplus\mathfrak{g}^D$ is the ideal direct sum $\mathfrak{z}(\mathfrak{t}\oplus\mathfrak{g}^D)\oplus(\mathfrak{t}\oplus\mathfrak{g}^D)'$. Because of (a) and (b) we get $\mathfrak{g}_D = (\mathfrak{t}\oplus\mathfrak{g}^D)'$. In particular, $\mathfrak{g}_D$ is a semisimple subalgebra with Cartan subalgebra $\mathbb{R}{\cdot}D$. The set $\mathfrak{T}(\mathfrak{g})_D$ of Cartan subalgebras is the set of one dimensional vector subspaces, and by the Transitivity Theorem 6.27, the group $\mathrm{Ad}(G_D)$ operates transitively on this set. Consequently, for a nonzero element $Y \in \mathfrak{g}_D$, the vector space $\mathbb{R}{\cdot}Y$ is a Cartan subalgebra of $\mathfrak{g}_D$ and thus

$$(Z) \qquad (\forall 0 \neq Y \in \mathfrak{g}_D) \quad \mathfrak{z}(Y, \mathfrak{g}_D) = \mathbb{R}{\cdot}Y.$$

In order to complete the proof of (ii), in view of 6.46 it now suffices to show that $\dim\mathfrak{g}^D = 2$. Suppose it is not true. Then there exists a nonzero $Z \in \{Y, IY\}^\perp \cap \mathfrak{g}^D$. Since $I|\mathfrak{g}^D$ is an orthogonal linear map, $IZ \in \{Y, IY\}^\perp \cap \mathfrak{g}^D$ and thus $(Y \mid IZ) = 0$. We apply (∗) (in the proof of 6.45) with $Y_1 = Y$ and $Y_2 = Z$, and get $[Y, Z] \in \mathfrak{t}^\perp = \mathfrak{g}^D$. Then $[D, [Y, Z]] = \|D\|^2{\cdot}I[Y, Z]$ by 6.45(A). But $I|g^D = \mathrm{Ad}\left(\exp(\frac{\pi}{2\|D\|^2}{\cdot}D)\right)|\mathfrak{g}^D$ by 6.45. Since $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$ for all $g \in G$, the vector space involution I of $\mathfrak{g}^D$ agrees on $\mathfrak{g}^D$ with an automorphism of $\mathfrak{g}$. Hence we have $I[Y, Z] = [IY, IZ]$, whence $[Y, Z] = -I^2[Y, Z] = -[I^2Y, I^2Z] = -[Y, Z]$, and thus $[Y, Z] = 0$. Hence $Z \in \mathfrak{z}(Y, \mathfrak{g}_D) = \mathbb{R}{\cdot}Y$, contradicting the assumption that $Y \perp Y$. This contradiction shows that $\dim\mathfrak{g}^D = 2$ as asserted.

(iii) From the proof of (ii) we know that $\mathfrak{t}\oplus\mathfrak{g}^D = D^{\perp_\mathfrak{t}}\oplus\mathfrak{g}_D$ with $D^{\perp_\mathfrak{t}}$ as center and that $\mathfrak{g}_D = \mathbb{R}{\cdot}D \oplus \mathbb{R}{\cdot}Y \oplus \mathbb{R}{\cdot}IY$ for any nonzero $Y \in \mathfrak{g}^D$. But then $\mathfrak{g}_D \cong \mathfrak{so}(3)$ by 9.46.

(iv) We know that $\mathfrak{t}_D \subseteq \mathfrak{z}(Y, \mathfrak{t})$. Conversely, if $X \in \mathfrak{z}(Y, \mathfrak{t})$, then

$$\big[X, [X, IY]\big] = (X \mid D){\cdot}[X, I^2Y] = -(X \mid D){\cdot}[X, Y] = 0.$$

The kernel of the morphism $\mathrm{ad}\,X|\mathbb{R}{\cdot}Y \oplus \mathbb{R}{\cdot}IY$ is zero. Since $[X, IY]$ is a scalar multiple of Y it follows that $[X, IY] = 0$. Thus $X \in \mathfrak{z}(\mathfrak{g}_D, \mathfrak{t}) = \mathfrak{t}_D$.

(v) Clearly $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}(Y, \mathfrak{t})$ for all $Y \in \mathfrak{g}^D$, $D \in R^+$. Conversely, let $X \in \mathfrak{t}\cap(R^+)^\perp$. Then $0 = [X, Y]$ for all $Y \in \mathfrak{g}^D$, $D \in R^+$ by (iii). Thus $\{0\} = [X, \mathfrak{g}_{\text{eff}}]$. Trivially $\{0\} = [X, \mathfrak{t}] = [X, \mathfrak{g}_{\text{fix}}]$. Thus $\{0\} = [X, \mathfrak{g}_{\text{fix}} \oplus \mathfrak{g}_{\text{eff}}] = [X, \mathfrak{g}]$. Hence $X \in \mathfrak{z}(\mathfrak{g})$.

(vi) Quite generally, we have $\mathrm{span}\,R^+ = (R^+)^{\perp\perp} = (\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\text{eff}})^\perp$ by (v). But $(\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\text{eff}})^\perp = \mathfrak{z}(\mathfrak{g})^\perp \cap \mathfrak{g}_{\text{eff}}^\perp = \mathfrak{g}' \cap \mathfrak{t}$ by 6.4.

(vii) Assume $\dim\mathfrak{t} = 1$. If $\mathfrak{t} = \mathfrak{g}$, then $G \cong \mathbb{S}^1$. Assume that $\mathfrak{t} \neq \mathfrak{g}$. Then $R \neq \emptyset$. By (i) above and in view of $\dim\mathfrak{t} = 1$ we have $R = \{D, -D\}$ for a nonzero D, and $R^+ = \{D\}$. Then $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}^D$ by 6.45. By (ii) this implies $\mathfrak{g} = \mathfrak{g}_D \cong \mathfrak{so}(3)$. Hence $G \cong \mathrm{SO}(3)$ or $G \cong \mathrm{SU}(2) \cong \mathbb{S}^3$ by E5.12. □

The Weyl Group Revisited

We recall the notation of Proposition 6.47 and define $\mathcal{W}_\sigma(T,G)$ to be the subgroup of the Weyl group $\mathcal{W}(T,G)$ generated by all elements $n_DT \in N(T,G_0)/T$. In the end we will show that $\mathcal{W}_\sigma(T,G) = \mathcal{W}(T,G)$ (see 6.52 below), but the proof will require a considerable amount of additional work.

We let $\mathcal{W} \subseteq \mathrm{O}(\mathfrak{t})$ be the faithful (cf. 6.35) image of $\mathcal{W}_\sigma(T,G)$ under the adjoint representation on $\mathfrak{t}$, i.e. the group generated by all orthogonal reflections at hyperplanes $\mathfrak{h}_D = D^\perp$, $D \in R$. Since the Weyl group is finite, $\mathcal{W}_\sigma(T,G)$ and thus the subgroup $\mathcal{W}$ of $\mathrm{O}(\mathfrak{t})$ is finite.

With the notation just introduced, we may quickly draw the following conclusions from Proposition 6.47 above.

Corollary 6.50. *In the circumstances of Proposition* 6.49 *and with the preceding conventions*

(i) $\mathfrak{z}(\mathfrak{g}) = R^{\perp_\mathfrak{t}} = \mathfrak{t}_{\mathrm{fix}}$, *the fixed point module of* $\mathfrak{t}$ *as a* $\mathcal{W}(T,G)$*-module, and*

(ii) $\mathfrak{t} \cap \mathfrak{z}(\mathfrak{g})^\perp = \mathfrak{t} \cap \mathfrak{g}' = \operatorname{span} R$.

(iii) $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \operatorname{span} R$, *and this direct sum representation is a* $\mathcal{W}(T,G)$*-module decomposition.*

Proof. (i) From 6.49(v) we know that $\mathfrak{z}(\mathfrak{g}) = R^{\perp_\mathfrak{t}}$. The set $R^{\perp_\mathfrak{t}}$ is the intersection of all hyperplanes $\mathfrak{t}_D = D^\perp \cap \mathfrak{t}$, $D \in R$. Hence it is the precise set of vectors fixed by all σ_D, $D \in R$ and thus equals the precise set of fixed points of $\mathcal{W}$. On the other hand $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{t}_{\mathrm{fix}}$ by the definition of the action of $\mathcal{W}(T,G)$ via the adjoint representation of $N(T,G)$.

(ii) By Theorem 6.4(v) $\mathfrak{z}(\mathfrak{g})^\perp = \mathfrak{g}'$. Hence $\mathfrak{z}(\mathfrak{g})^{\perp_\mathfrak{t}} = \mathfrak{g}' \cap \mathfrak{t}$. By the duality theory of finite dimensional Hilbert spaces we have $\operatorname{span} R = R^{\perp_\mathfrak{t}\perp_\mathfrak{t}} = \mathfrak{z}(\mathfrak{g})^{\perp_\mathfrak{t}}$ by (i) above.

(iii) By the general annihilator mechanism of Hilbert spaces applied to $\mathfrak{t}$ we have $\mathfrak{t} = R^{\perp_\mathfrak{t}} \oplus R^{\perp_\mathfrak{t}\perp_\mathfrak{t}} = \mathfrak{z}(\mathfrak{g}) \oplus \operatorname{span} R$. Since $\mathfrak{z}(G)$ is a $\mathcal{W}(T,G)$-submodule of $\mathfrak{t}$ and the Weyl group acts orthogonally, $\operatorname{span} R = \mathfrak{z}(\mathfrak{g})^\perp$ is a submodule, too. ☐

The finiteness of $\mathcal{W}$ places strong restrictions on the geometry of R. It is not surprising that we have to look more closely at the geometric structure which is introduced on $\mathfrak{t}$ by the set of finite hyperplanes $\mathfrak{t}_D$, $D \in R^+$ and the group $\mathcal{W}$ generated by the reflections in these hyperplanes. This allows us to detour into the area of basic euclidean geometry for a while; we will not return to compact Lie groups before Theorem 6.52.

Therefore, in the following paragraphs we consider a finite dimensional vector space $\mathfrak{t}$ with a positive definite bilinear symmetric form $(\cdot \mid \cdot)$ and a finite system of vectors $R^+ \subseteq \mathfrak{t}$ such that

(W0) $(\forall D \in R^+)\quad R^+ \cap \mathbb{R}{\cdot}D = \{D\}$.

(W1) $(\exists a \in \mathfrak{t})(\forall D \in R^+)\quad (a \mid D) > 0$.

(W2) The group $\mathcal{W}_\sigma$ generated by the orthogonal reflections σ_D with the fixed point set $D^\perp$, $D \in R^+$ is finite.

If these conditions are satisfied we shall call R^+ a *positive set of vectors generating the reflection group* $\mathcal{W}_\sigma$ or a *positive generating set* for short if the group is understood.

Notice that on this level of generality, the length of the vectors D in R^+ plays a subordinate role: the relevant ingredients are the hyperplanes $\mathfrak{t}_D \stackrel{\text{def}}{=} D^\perp$ in $\mathfrak{t}$. We also observe right away that for a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of a compact Lie group any system of positive roots R^+ as defined in comments preceding Proposition 6.45 is a positive set of vectors generating a reflection group $\mathcal{W}$ after 6.49 and the subsequent remarks. Later we shall even see that $\mathcal{W}$ leaves the set $R^+ \cup -R^+$ invariant (cf. 6.59(v)ff.). As soon as this is the case, then even in the general geometric setting, the vectors of R^+ themselves and their lengths enter the picture significantly.

The union S of the finitely many hyperplanes $\mathfrak{t}_D$, $D \in R^+$ is a closed subset of $\mathfrak{t}$ whose complement is an open subset. Each connected component is the intersection of finitely many open half-spaces and is called a *Weyl chamber*. For a better understanding of the Weyl chambers as geometric objects, let us briefly review some facts concerning convex cones. A *convex cone* C in a real vector space E is a subset such that $C + C \subseteq C$ and $]0, \infty[\, \cdot C \subseteq C$. A convex cone is indeed convex. Each open or closed half-space bounded by a hyperplane through the origin is a convex cone. Intersections of convex cones are convex cones. Hence each Weyl chamber is an open convex cone.

The set $C \cap -C$ is either empty or the largest vector subspace contained in C. In this case it is called the *edge* of C. A convex cone C is called *pointed* if its closure has a zero edge, i.e. if $\overline{C} \cap -\overline{C} = \{0\}$.

A *face* F of a convex cone C is a subset which is a convex cone in some unique hyperplane $\mathfrak{t}_D$ for $D \in R^+$. We call the interior of an $r-1$ dimensional F with respect to the hyperplane $\mathfrak{t}_D$ it generates the *algebraic interior of* F and denote it by $\operatorname{algint} F$; note that the points are not inner points of C with respect to $\mathfrak{t}$. Since $\operatorname{algint} F$ is dense in F, the union of the algebraic interiors of the $r-1$ dimensional maximal faces is dense in $\partial\overline{C}$.

We set $\mathfrak{z} = (R^+)^\perp$ and recall from 6.49(v, vi) that $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ if $\mathfrak{t}$ is the Cartan subalgebra of a compact Lie algebra $\mathfrak{g}$ and that $\operatorname{span} R^+ = \mathfrak{t} \cap \mathfrak{g}'$.

In the present discussion we need elementary facts on finite orthogonal groups on the euclidean plane, i.e. on cyclic rotation groups and dihedral groups generated by reflections.

Exercise E6.13. *The Geometric Dihedral Groups.* Recall the concepts of (oriented) angles between half-lines and lines introduced in Exercise E1.6(iv). Prove the facts claimed in the following discussion.

If u is any unit vector in the euclidean plane E and $L = u^\perp$ is the line perpendicular to it, then the element $\sigma_u \in \mathrm{O}(2)$, $\sigma_u(X) = X - 2(X \mid u){\cdot}u$ depends on L only and is the *reflection about* L whose precise fixed point set is L. We recall that $(X \mid u) = \|X\| \cos\big(\operatorname{ang}(\mathbb{R}^+{\cdot}X.\mathbb{R}^+{\cdot}u)\big)$. Recall that an involution in a group is an element of order two. Every involution $\sigma \in \mathrm{O}(2)$ is a reflection about its line of fixed points.

Let $\mathcal{W} \subseteq \mathrm{O}(2)$ be a finite group of orthogonal automorphisms of $\mathbb{R}^2$. The following conclusions hold:

(i) If $\mathcal{W} \subseteq \mathrm{SO}(2)$ then $\mathcal{W}$ is cyclic of order n and consists of rotations by the angles $\{\frac{2\pi m}{n} + 2\pi Z \mid m = 0, 1, \ldots, n-1\}$.

(ii) If $\mathcal{W} \not\subseteq \mathrm{SO}(2)$, then $\mathcal{W}_* \stackrel{\text{def}}{=} \mathcal{W} \cap \mathrm{SO}(2)$ is a proper normal subgroup of index 2 in $\mathcal{W}$ which is cyclic of order n, and for any $\sigma \in \mathcal{W} \setminus \mathrm{SO}(2)$, $\sigma^2 = \mathrm{id}_E$ and the product $\mathcal{W} = \mathcal{W}_*\{\mathrm{id}_E, \sigma\}$ is semidirect. The involution σ is an orthogonal reflection about a line $L = E_{\mathrm{fix}}(\{\mathrm{id}_E, \sigma\}$.

For $n = 3, 4, \ldots$ the group $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$ has exactly one involutive automorphism $g \mapsto -g$ and there is a semidirect product

$$D(n) \stackrel{\text{def}}{=} \mathbb{Z}(n) \rtimes \{\mathrm{id}_{\mathbb{Z}(n)}, -\mathrm{id}_{\mathbb{Z}(n)}\}.$$

There exists an isomorphism $\varphi\colon D(n) \to \mathcal{W}$. The group $D(n)$ (and any group isomorphic to it) is called a *dihedral group* of order $2n$.

(iii) Let L_1 and L_2 be two different lines such that $\mathrm{ang}(L_1, L_2) = \frac{\pi}{n} + \pi\mathbb{Z}$, and let $\sigma^{(j)}$ be the reflection about L_j. Then $\sigma^{(2)}\sigma^{(1)}$ is a rotation by the angle $\frac{2\pi}{n} + 2\pi\mathbb{Z}$. The group generated by $\sigma^{(j)}$, $j = 1, 2$, is a dihedral group whose normal subgroup of index two is a rotation group of order n. *Notice that the angle between the lines is half of the angle of the rotation which generates $\mathcal{W}_*$.*

(iv) Assume that (H_1, H_2) is a pair of half lines and that $\sigma^{(j)}$ is the reflection about $L_j \stackrel{\text{def}}{=} \mathbb{R}{\cdot}H_j$, $j = 1, 2$. Let C be the interior of the closed convex hull of $H_1 \cup H_2$. If none of the lines $\mathcal{W}{\cdot}L_j$, $L_j = \mathbb{R}{\cdot}H_j$ meets C, then $\mathcal{W} = \langle \sigma^{(1)}, \sigma^{(2)} \rangle$ is a dihedral group of order $2n$ for some natural number n and $\mathrm{ang}(L_1, L_2) = \frac{\pi}{n} + \pi\mathbb{Z}$.

(v) With the notation of (iv), let D_j, $j = 1, 2$ be two vectors in E such that the oriented angles $\mathrm{ang}(\mathbb{R}^+{\cdot}D_1, H_1)$ and $\mathrm{ang}(H_2, \mathbb{R}^+{\cdot}D_2)$ between half lines are $\frac{\pi}{2}$ (modulo 2π) for $j = 1, 2$. Let R denote the finite set of vectors $\mathcal{W}_*(D_1) \mathbin{\dot\cup} \mathcal{W}_*(D_2)$. Then $\mathcal{W}$ leaves each of the disjoint orbits $\mathcal{W}_*(D_j)$ and, in particular R, invariant. Let $D \in R$, further $H \stackrel{\text{def}}{=} D^\perp$ the line perpendicular to D, and $\sigma_D \in \mathcal{W}$ the reflection about H. Then the following conditions are equivalent:

(1) $n = 2, 3, 4, 6$ where the order of $\mathcal{W}$ is $2n$.
(2) $\mathrm{ang}(H_1, H_2) = \frac{\pi}{n}$ (modulo π), $n = 2, 3, 4, 6$.
(3) For any pair $(D, D') \in R \times R$, the (oriented) angle $\mathrm{ang}(\mathbb{R}^+{\cdot}D, \mathbb{R}^+{\cdot}D')$ is a multiple of $\frac{2\pi}{n} + 2\pi\mathbb{Z}$.
(4) For any pair $(D, D') \in R \times R$, the number $\frac{2(D|D')}{(D|D)} = 2(\frac{1}{\|D\|^2}{\cdot}D, D')$ is an integer.
(5) $R \subseteq \mathrm{span}_{\mathbb{Z}}\{D_1, D_2\}$.

[Hint for (v). By the preceding sections, (1), (2), and (3) are equivalent. Since $\frac{2(D|D')}{(D|D)} = 2\|D'\| \cos\big(\mathrm{ang}(\mathbb{R}^+{\cdot}D, \mathbb{R}^+, D')\big)$, (3) is seen to be equivalent to (4). Assume (4). Then for $j, k = 1, 2$ we have $\sigma_j D_k \in \Delta \stackrel{\text{def}}{=} \mathrm{span}_{\mathbb{Z}}\{D_1, D_2\}$. Thus $\sigma_j(\Delta) \subseteq \Delta$. Hence Δ is invariant under $\mathcal{W}$. Hence $R = \mathcal{W}\{D_1, D_2\} \subseteq \Delta$. Thus (4) implies (5). Assume (5) and let $(D, D') \in R \times R$. Then applying a transformation of $\mathcal{W}$ we may assume $D = D_j$, $j = 1$ or $j = 2$, say $D = D_1$. Then $\frac{2(D_1|D')}{(D_1|D_1)}D_1 = \sigma_{D_1}(D') - D' \in \Delta$. Then $\frac{2(D_1|D')}{(D_1|D_1)}D_1 = m_1{\cdot}D_1 + m_2{\cdot}D_2$ for $m_j \in \mathbb{Z}$. Since D_1 and D_2 are linearly independent, $n_2 = 0$ and $m_1 = \frac{2(D_1|D')}{(D_1|D_1)}D_1$. The case $D = D_2$ is analogous. Hence (4) follows.] □

We define the subset B^+ of R^+ by

(BAS) $$B^+ = B^+(R^+) \stackrel{\text{def}}{=} \{D \in R^+ \mid D^\perp \cap \overline{C} \text{ is a face of } \overline{C}\}.$$

The importance of this set will appear presently.

Proposition 6.51. *Let $\mathfrak{t}$ be a finite dimensional vector space with a positive definite bilinear symmetric form $(\cdot \mid \cdot)$ and a $R^+ \subseteq \mathfrak{t}$ a positive set of vectors generating the finite reflection group $\mathcal{W}$. Let $\mathfrak{z} \stackrel{\text{def}}{=} (R^+)^\perp$. Then the following conclusions hold.*

(i) *Each Weyl chamber C is of the form $\mathfrak{z} \oplus (C \cap \operatorname{span} R^+)$. Likewise $\overline{C} = \mathfrak{z} \oplus (\overline{C} \cap \operatorname{span} R^+)$.*

(ii) *Each $\overline{C} \cap \operatorname{span} R^+$ is pointed.*

(iii) *Let C be a Weyl chamber and $\mathcal{W}^C$ the subgroup of $\mathcal{W}$ generated by $\{\sigma_D \mid D \in B^+\}$. Then $\mathcal{W}^C$ acts transitively on the set of Weyl chambers. In particular, $\mathcal{W}$ acts transitively on the Weyl chambers.*

(iv) *The set*
$$C(R^+) = \{X \in \mathfrak{t} \mid (\forall D \in R^+) \quad (D \mid X) < 0\}$$
is a Weyl chamber. If C is a Weyl chamber, then
$$R^+(C) = \{D \in R^+ \cup -R^+ \mid (\forall X \in C) \quad (D \mid X) < 0\}$$
is a positive set of vectors generating the reflection group $\mathcal{W}$.

(v) *If $D_1, D_2 \in B^+$, then the subgroup $\mathcal{V} = \mathcal{V}(\sigma_{D_1}, \sigma_{D_2}) \stackrel{\text{def}}{=} \langle \sigma_{D_1}, \sigma_{D_2} \rangle$ of $\mathcal{W}^C$ is dihedral of order $2n$ where $\operatorname{ang}(\mathbb{R}^+{\cdot}D_1, \mathbb{R}^+{\cdot}D_2) = \pi - \frac{\pi}{n}$ modulo $2\pi\mathbb{Z}$. In particular, $(D_1 \mid D_2) \le 0$. Further,*
$$\mathbb{R}{\cdot}(R^+ \cap \operatorname{span}_{\mathbb{R}}\{D_1, D_2\}) = \mathbb{R}{\cdot}\big(\mathcal{V}(D_1) \cup \mathcal{V}(D_2)\big).$$

(vi) *B^+ is a basis of $\operatorname{span}_{\mathbb{R}} R^+$ and $R^+ \subseteq \sum_{D \in B^+} \mathbb{R}^+{\cdot}D$.*

(vii) *There is a bijective function $j \mapsto D_j \colon \{1, \ldots, r\} \to B^+$ such that the fixed point set of $\sigma_{D_1} \cdots \sigma_{D_r}$ is $\mathfrak{z}$.*

(viii) *For every $D \in R^+$ there is an element $\delta \in \mathcal{W}^C$, and a real number $r > 0$ such that $r{\cdot}\delta(D) \in B^+$.*

Assume now for the remainder of the proposition that $\mathcal{W}^C(R^+) \subseteq R^+ \cup -R^+$. Then

(ix) *for every $D \in R^+$ then the orbit $\mathcal{W}^C(D)$ meets B^+,*

(x) *if D_1, $D_2 \in B^+$, then $(R^+ \cup -R^+) \cap \operatorname{span}_{\mathbb{R}}\{D_1, D_2\} = \mathcal{V}(\pm D_1) \cup \mathcal{V}(\pm D_2)\big)$, and*

(xi) *the following statements are equivalent:*

(1) *For any pair* $(D_1, D_2) \in B^+ \times B^+$,

$$\frac{2(D_1 \mid D_2)}{(D_1 \mid D_1)} = 2(\frac{1}{\|D_1\|^2}\cdot D_1, D_2) \in \mathbb{Z}.$$

(2) *For any pair* $(D_1, D_2) \in B^+ \times B^+$, *with* $\mathcal{V} = \mathcal{V}(\sigma_{D_1}, \sigma_{D_2})$,

$$\mathcal{V}(D_1) \cup \mathcal{V}(D_2) \subseteq \operatorname{span}_{\mathbb{Z}}\{D_1, D_2\}.$$

(2′) *For any pair* $(D_1, D_2) \in B^+ \times B^+$,

$$(R^+ \cup R^+) \cap \operatorname{span}_{\mathbb{R}}\{D_1, D_2\} \subseteq \operatorname{span}_{\mathbb{Z}}\{D_1, D_2\}.$$

(3) $R^+ \cup -R^+ \subseteq \operatorname{span}_{\mathbb{Z}} B^+$.
(4) $\operatorname{span}_{\mathbb{Z}}(B^+) = \operatorname{span}_{\mathbb{Z}}(R^+)$.
(5) $R^+ \subseteq \sum_{D\in B^+} \mathbb{Z}^+\cdot D$, *where* $\mathbb{Z}^+$ *denotes the set of nonnegative integers.*

Proof. (i) The closure $\overline{C}$ is the intersection of finitely many closed half-spaces bounded by hyperplanes $\mathfrak{h}_D$ for certain $D \in R$. Since $\mathfrak{z} = (R^+)^\perp$ is contained in each $\mathfrak{t}_R$ then $\mathfrak{z} \subseteq \overline{C}$. Since translations with elements $X \in \mathfrak{z} \subseteq \overline{C} \cap -\overline{C}$ leave the interior $\operatorname{int}\overline{C} = C$ invariant, we have $C = \mathfrak{z} + C$. Since $\mathfrak{z}^\perp = (R^+)^{\perp\perp} = \operatorname{span}_{\mathbb{R}} R^+$, we have $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \operatorname{span} R$. Hence every $X \in C$ is uniquely of the form $X_1 \oplus X_2$ with $X_1 \in \mathfrak{z}(\mathfrak{g})$ and $X_2 \in \operatorname{span}_{\mathbb{R}} R^+$. Hence $X_2 = -X_1 + X \in \mathfrak{z} + C = C$ and thus $X_2 \in C \cap \operatorname{span}_{\mathbb{R}}^+ R$. The same argument applies to the closure of C.

(ii) We project everything orthogonally along $\mathfrak{z}$ into $z^\perp = \operatorname{span}_{\mathbb{R}} R^+$ and show the claim for $\operatorname{span}_{\mathbb{R}} R^+$ in place of $\mathfrak{t}$. In other words we assume that $\mathfrak{z} = \{0\}$ and show that each Weyl chamber is pointed. Let $R_C \subseteq R^*$ denote the set of those elements D of R^* for which $\mathfrak{t}_D$ is generated by an $(r-1)$-dimensional face of $\overline{C}$. Then $E \overset{\text{def}}{=} \bigcap_{D\in R_C} \mathfrak{t}_D$ is the edge of $\overline{C}$. If $D \in R_C$, then $E \subseteq \mathfrak{t}_D$. But σ_D leaves $\mathfrak{t}_D$, hence E elementwise fixed. Thus $\sigma_D C$ has the same edge E. Thus all Weyl chambers adjacent to C have the same edge. The same holds for Weyl chambers adjacent to these. Proceeding recursively we find that all Weyl chambers have the same edge E. Thus $E = \bigcap_{D\in R} \mathfrak{t}_D = (R^*)^\perp = \mathfrak{z}$ by definition of $\mathfrak{z}$. But we assumed $\mathfrak{z} = \{0\}$ and thus $E = \{0\}$.

(iii) We prove the claim in several steps. If $\dim \mathfrak{t} = 1$, then there are two Weyl chambers which are permuted by W; thus the assertion is true in this case. We therefore assume now that $\dim \mathfrak{t} > 1$.

Let C be a Weyl chamber and $\mathcal{C} \overset{\text{def}}{=} \{\omega C \mid \omega \in \mathcal{W}^C\}$ its orbit under $\mathcal{W}^C$. If $C' \in \mathcal{C}$ and $D' \in R^+$ is such that $\mathfrak{t}_{D'} \cap \overline{C'}$ is a maximal face of C', then we claim $\sigma_{D'} \in \mathcal{W}^C$. Indeed let $\gamma \in \mathcal{W}^C$ be such that $C' = \gamma(C)$; then $D^* \overset{\text{def}}{=} \gamma^{-1}(D')$ is such that $\mathfrak{t}_{D^*} \cap \overline{C}$ is a maximal face of $\overline{C}$. Hence there is a $D \in B^+$ with $\mathfrak{t}_D = \mathfrak{t}_{D^*} = \gamma^{-1}\mathfrak{t}_{D'}$. By definition of $\mathcal{W}^C$ we have $\sigma_D \in \mathcal{W}^C$. Therefore, $\sigma_{D'} = \delta\sigma_D\delta^{-1} \in \mathcal{W}^C$; this proves the claim.

Now let $U = \bigcup \mathcal{C}$ be the union of the $\mathcal{W}^C$-orbit of C. We have to show that it contains all Weyl chambers. Suppose that this fails. Then there is a Weyl chamber C' with $U \cap C' = \emptyset$. We pick points $u \in U$ and $v \in C'$. Denote the straight line segment connecting u and v by $[u, v]$. Let S denote the finite union of hyperplanes $\mathfrak{t}_D$, $D \in R^+$. Then S meets $[u, v]$ in a finite set since u, $v \notin S$. Each of the points

$d \in [u, v] \cap S$ is contained in an $r-1$ dimensional bounding face $F \subseteq \mathfrak{t}_D$ of some Weyl chamber. Since algint F is open dense in F with respect to $\mathfrak{t}_D$, by displacing u and v slightly we may assume that

(a) *each $d \in [u, v] \cap S$ is contained in the algebraic interior* algint F *of some $(r-1)$-dimensional face F of a Weyl chamber with $d \in F$.*

The complement $[u, v] \setminus S$ is a finite union of intervals each belonging to some Weyl chamber, some pertaining to U, others being disjoint from U. There is no loss in generality if we assume that

(b) *$[u, v] \cap S = \{d\}$, further $[u, d[\subseteq \alpha C$, $\alpha \in \mathcal{W}^C$, and $]d, v] \subseteq C'$.*

From (a) and (b) above, d is contained in algint F of a bounding face of the Weyl chamber αC for an $\alpha \in \mathcal{W}^C$. Then there is a unique $D \in R^+$ such that $\mathfrak{t}_D$ contains algint F. But then $\sigma_D(\alpha C) \cap C' \neq \emptyset$, and $\sigma_D \in \mathcal{W}^C$ by the claim we proved above. Since the collection of Weyl chambers is a partition of $\mathfrak{t} \setminus S$ we have $C' = \sigma_D \omega C$. Thus $C' \in \mathcal{C}$. This is a contradiction which finishes the proof of Claim (iii).

(iv) For $D \in R^+$ let H_D be the open half-space $\{X \in \mathfrak{t} \mid (D \mid X) > 0\}$. Then $C(R^+) \stackrel{\text{def}}{=} \bigcap_{D \in \mathbb{R}^+} H_D$ is an open convex cone. If $D \in R$, then $D \in R^+$ or $-D \in R^+$, and the hyperplane $\mathfrak{t}_D = \{X \in \mathfrak{t} \mid (D \mid X) = 0\}$ is the bounding hyperplane of one of the half-spaces $H_{D'}$, $D' \in \mathbb{R}^+$ and thus fails to meet C. By (W1) there is an $a \in \mathfrak{t}$ such that $R^+ = \{D \in R \mid (D \mid a) > 0\}$. Then $a \in C(R^+)$ and thus $C(R^+) \neq \emptyset$. Then $C(R^+)$ is the required Weyl chamber.

If C is a Weyl chamber then by (iii) there is a $\gamma \in \mathcal{W}$ with $C = \gamma\big(C(R^+)\big)$. The group $\mathcal{W}$ permutes the hyperplanes $D^\perp$, $D \in \mathbb{R}^+$ and, accordingly, the set $\mathcal{H}$ of half-lines $\pm\mathbb{R}^+{\cdot}D$ perpendicular to these. A vector $0 \neq D \in \bigcup \mathcal{H}$ satisfies $(D \mid a) > 0$ iff $\big(\gamma(D) \mid \gamma(a)\big) > 0$. Thus $D \in R^+ \cup -R^+$ is in $R^+(C)$ iff $\big(D \mid \gamma(a)\big) > 0$. It follows that this set of vectors is a positive set of vectors generating the reflection group $\mathcal{W}$.

By (i) and (ii) it is no loss of generality for the proof of the remaining assertions to assume that $\mathfrak{z} = \{0\}$, that $\operatorname{span}_{\mathbb{R}} \mathbb{R}^+ = \mathfrak{t}$, and that C is pointed. We will do that.

(v) $\mathcal{V}$ is generated by the reflection σ_{D_1} and the rotation $\gamma \stackrel{\text{def}}{=} \sigma_{D_2}\sigma_{D_1}$. Let n be the order of γ. The fixed point set $\mathcal{V}_{\text{fix}}$ is $D_1^{\perp_\mathfrak{t}} \cap D_2^{\perp_\mathfrak{t}}$ and the effective set is the two dimensional space $\mathfrak{t}_2 \stackrel{\text{def}}{=} \mathcal{V}_{\text{eff}} = \operatorname{span}_{\mathbb{R}}\{D_1, D_2\}$. Then γ induces in the plane $\mathfrak{t}_2$ a rotation by the angle $\frac{2\pi}{n}$. Consider the orthogonal projection $p \colon \mathfrak{t} \to \mathfrak{t}_2$. The maximal faces $F_i \stackrel{\text{def}}{=} D_i^{\perp_\mathfrak{t}} \cap \overline{C}$ of $\overline{C}$, $i = 1, 2$ yield closed half-lines $L_i = p(F_i)$ which bound the "quadrant" $C_2 \stackrel{\text{def}}{=} \overline{p(C)}$. The hyperplanes $\nu(D_i^\perp)$, $i = 1, 2$, $\nu \in \mathcal{V}$ do not meet C. Hence the lines $\mu(\mathbb{R}{\cdot}L_i)$, $i = 1, 2$, $\mu \in \mathcal{V}|\mathfrak{t}_2$ do not meet the open quadrant $\operatorname{int} C_2$. The group $\mathcal{V}|\mathfrak{t}_2$ is generated by the reflections $\sigma^{(i)} = \sigma_{D_i}|\mathfrak{t}_2$ about the lines $\mathbb{R}{\cdot}L_i$. It follows from the elementary theory of finite subgroups of the orthogonal group $\mathrm{O}(2)$ (see Exercise E6.13 above) of the plane that the oriented angle between the lines L_1 and L_2 is $\frac{\pi}{n}$ (modulo π) where $n > 1$ (see Exercise E6.13 above), and that $\mathcal{V}|\mathfrak{t}_2$ is a dihedral group of order $2n$. Then the oriented angle between the half-lines $\mathbb{R}^+{\cdot}D_1$ and $\mathbb{R}^+{\cdot}D_2$ then is $2\pi - \pi - \frac{\pi}{n} = \pi - \frac{1}{n}{\cdot}\pi$ (modulo 2π). Since $n > 1$ this angle is in the interval $\left[\frac{\pi}{2}, \pi\right[$ and thus $(D_1 \mid D_2) \leq 0$.

An element of $D \in \mathbb{R}^+$ is in $\operatorname{span}_{\mathbb{R}}\{D_1, D_2\}$ iff $D_1^\perp \cap D_2^\perp \subseteq D^\perp$. Since $D^\perp$ does not meet any of the Weyl chambers $\gamma(C)$, $\gamma \in \mathcal{V}$, it follows that $D^\perp$ agrees with $\gamma_1(D_1^\perp)$ for some $\gamma_1 \in \mathcal{V}$ or with some $\gamma_2(D_2^\perp)$ for some $\gamma_2 \in \mathcal{V}$. Equivalently, $D \in \mathcal{V}(\mathbb{R}\cdot D_1) \cup \mathcal{V}(\mathbb{R}\cdot D_2)$. Conversely, $\gamma_j \cdot D_j \in \mathbb{R}\cdot(R^+ \cap \operatorname{span}_{\mathbb{R}}\{D_1, D_2\})$ as $\mathcal{V}$ leaves $\mathbb{R}\cdot R^+$ invariant.

(vi) Since $B^+ \subseteq R^+$ we have $\operatorname{span}_{\mathbb{R}} B^+ \subseteq \operatorname{span}_{\mathbb{R}} R^+$. The intersection of the maximal faces of a closed convex cone is its edge; indeed the edge is the intersection of all faces and every face is the intersection of maximal ones. The edge of $\overline{C}$ is $\{0\}$ by our assumption. Hence $\{0\} = (B^+)^{\perp_{\mathfrak{t}}} \supseteq (R^+)^{\perp_{\mathfrak{t}}} = \{0\}$. Hence $\operatorname{span}_{\mathbb{R}} B^+ = \operatorname{span}_{\mathbb{R}} R^+$ and we have to show that B^+ is a linearly independent set. Thus assume that $r_D \in \mathbb{R}$, $D \in B^+$ are such that $\sum_{D\in B^+} r_D\cdot D = 0$. We have to show that $r_D = 0$ for all $D \in B^+$. Then, in view of $(D_1 \mid D_2) \le 0$, for different D_i from B^+ we have

$$\begin{aligned} 0 &= \Big\|\sum\nolimits_{D\in B^+} r_D\cdot D\Big\|^2 = \Big(\sum\nolimits_{D\in B^+} r_D\cdot D\Big| \sum\nolimits_{D\in B^+} r_D\cdot D\Big) \\ &= \sum\nolimits_{D\in B^+} r_D^2(D \mid D) + \sum\nolimits_{D_1,D_2\in B^+, D_1\neq D_2} r_{D_1} r_{D_2}(D_1 \mid D_2) \\ &\ge \sum\nolimits_{D\in B^+} |r_D|^2(D \mid D) + \sum\nolimits_{D_1,D_2\in B^+, D_1\neq D_2} |r_{D_1}|\cdot|r_{D_2}|(D_1 \mid D_2) \\ &= \Big(\sum\nolimits_{D\in B^+} |r_D|\cdot D\Big| \sum\nolimits_{D\in B^+} |r_D|\cdot D\Big) = \Big\|\sum\nolimits_{D\in B^+} |r_D|\cdot D\Big\|^2 \ge 0. \end{aligned}$$

Thus $\sum_{D\in B^+} |r_D|\cdot D = 0$. Now

$$0 = (a \mid \sum_{D\in B^+} |r_D|\cdot D) = \sum_{D\in B^+} |r_D|\cdot(a \mid D) \quad \text{and } (a \mid D) > 0$$

by (W1). This implies $|r_D| = 0$ for all $D \in B^+$ and this shows that B^+ is a linearly independent set.

Since a $D \in R^+$ is in B^+ iff $D^\perp$ is a maximal face of C, we have $X \in C$ iff $(\forall D \in B^+)\,(X \mid D) > 0$. The closed convex hull $\sum_{D\in B^+} \mathbb{R}^+\cdot D$ of $\bigcup_{D\in B^+} \mathbb{R}^+\cdot D$ is therefore the set of all $Y \in \mathfrak{t}$ such that $(\forall X \in C)\,(Y \mid C) \ge 0$. By (iv) we know $X \in C$ iff $(\forall D \in R^+)\,(X \mid D) > 0$. Thus the closed convex hull of $\bigcup_{D\in R^+} \mathbb{R}^+\cdot D$ is the same set and therefore agrees with $\sum_{D\in B^+} \mathbb{R}^+\cdot D$. In particular this set contains R^+. This finishes the proof of (vi).

(vii) By (i) we may assume $\mathfrak{z} = \{0\}$ and then must show that $j \mapsto D_j$ may be chosen such that $\sigma_{D_1} \cdots \sigma_{D_r}$ has no nonzero fixed points.

The proof passes through a bit of elementary graph theory. We consider a finite graph Γ whose vertices are the elements of B^+ and whose edges consist of those two element sets $\{D, D'\}$ with $(D \mid D') < 0$. This is tantamount to saying that σ_D and $\sigma_{D'}$ don't commute. We prove a few facts about this graph; the notation we use is intuitive enough so as to be understood.

(a) Γ has no closed paths: If $(D_1, \dots, D_p)$, $p > 2$, is a closed path then set $E_j = \|D_j\|^{-1}\cdot D_j$ and $E_{p+1} = E_1$. If $X = E_1 + \cdots + E_p$, then we obtain $0 \le (X|X) = p + 2\sum_{i<j}(E_i \mid E_j)$. But $(E_i \mid E_j) \le -\cos\frac{\pi}{n_{ij}} < -\frac{1}{2}$ for all $i \neq j$ (with integers $n_{ij} > 2$) by the proof of (v) above and the definition of the graph. All this adds up to the contradiction $0 \le p + p(p-1)\cdot\frac{-1}{2} = \frac{(2-p)p}{2}$.

(b) Γ has a terminal vertex: find a maximal chain in Γ. Both of its ends must be terminal vertices.

(c) Assume $\operatorname{card} B^+ \ge 2$. Then $B^+ = B_1 \dot\cup B_2$, $B_1 \ne \emptyset \ne B_2$ such that if $D \ne D'$ in B^+ both belong to B_i for $i = 1$ or $i = 2$, then D and D' are not linked: we prove this by induction on the number r of vertices. For $r = 1, 2$ everything is clear. Assume the claim holds for $r-1$, $r > 2$. Let D be a terminal vertex. By the induction hypothesis write $\Gamma \setminus \{D\} = B_1' \dot\cup B_2'$ with the properties stated. Since D is terminal it is linked with at most one other vertex; if it is, we may assume that this vertex belongs to B_1'. We set $B_1 = B_1'$ and $B_2 = B_2' \cup \{D\}$. The sets B_i, $i = 1, 2$ are as required.

Now, we choose $j \mapsto D_j\colon \{1, \ldots, r\} \to B^+$ such that $\{D_1, \ldots, D_q\} = B_1$ and $\{D_{q+1}, \ldots, D_r\} = B_2$. Let $\mathfrak{t}(i) = \operatorname{span}_{\mathbb{R}} B_i$ for $i = 1, 2$ and set $\gamma_1 = \sigma_{D_1} \cdots \sigma_{D_q}$ and $\gamma_2 = \sigma_{D_{q+1}} \cdots \sigma_{D_r}$. Then the $D \in B_1$ are orthogonal and γ_1 is an orthogonal transformation such that $\gamma_1|\mathfrak{t}(1) = -\operatorname{id}_{\mathfrak{t}(1)}$ and $\gamma_1|\mathfrak{t}(1)^\perp = \operatorname{id}_{\mathfrak{t}(1)^\perp}$. Analogous conclusions hold for γ_2 and $\mathfrak{t}(2)$. Assume that $X \in \mathfrak{t}$ satisfies $\gamma_1\gamma_2(X) = X$. Then $\gamma_2(X) = \gamma_1^{-1}(X) = \gamma_1(X)$. Let $Y = X - \gamma_1(X) = X - \gamma_2(X)$. Then $\gamma_i(Y) = -Y$ for $i = 1, 2$. If $Y = Y' + Y''$ with $Y' \in \mathfrak{t}(1)$ and $Y'' \in \mathfrak{t}(1)^\perp$, then $-Y_1 - Y_2 = -Y = \gamma_1(Y) = -Y_1 + Y_2$. Thus $Y_2 = 0$ and $Y \in \mathfrak{t}(1)$. Similarly, $Y \in \mathfrak{t}(2)$. Hence $Y \in \mathfrak{t}(1) \cap \mathfrak{t}(2) = \{0\}$. Thus $X = \gamma_1(X) = \gamma_2(X)$. Then X is in the fixed point set $\mathfrak{t}(i)^\perp$ of γ_i for both $i = 1$ and $i = 2$. Hence $X \in \mathfrak{t}(1)^\perp \cap \mathfrak{t}(2)^\perp = \big(\mathfrak{t}(1) + \mathfrak{t}(2)\big)^\perp = \{0\}$. This proves the claim.

(viii) Let $D \in R^+$. Then there is a Weyl chamber C such that $\mathfrak{t}_D \cap \overline{C}$ is a face of $\overline{C}$. By (iii) above, there is a $\gamma \in \mathcal{W}^C$ such that $C = \gamma^{-1}\big(C(R^+)\big)$. Then the hyperplane $\gamma(t_D) = \mathfrak{t}_{\gamma(D)}$ intersects $\overline{C}$ in a face. Hence there is a $D' \in B^+$ such that $\mathfrak{t}_{\gamma(D)} = \mathfrak{t}_{D'}$. Thus $\mathbb{R}\cdot\gamma(D) = \mathbb{R}\cdot D'$. Hence there is a nonzero $t \in \mathbb{R}$ with $t\cdot\gamma(D) = D'$. If $t > 0$ we are done with $r = t$ and $\delta = \gamma$. If not, we note $t\cdot\gamma(D) = |t|\cdot\sigma_{D'}\gamma(D)$; now we set $r = |t|$, $\delta = \sigma_{D'}\gamma$.

(ix) Retain the notation of the proof of (viii) above. (a) If $\mathcal{W}(R^+) \subseteq R^+ \cup -R^+$, then $\delta(D) \in \mathbb{R}^+\cdot D' \cap (R^+ \cup -R^+) = \mathbb{R}\cdot D \cap R^+ = \{D'\}$ by (W0), whence $\delta(D) = D'$.

(x) is a consequence of the last part of (v) and (W0).

(xi) We first note that the equivalence of (1) and (2) follows from (v) and basic facts about dihedral groups (cf. Exercise E6.13(v)). By (b) above $\mathcal{V}(\pm D_1) \cup \mathcal{V}(\pm D_2) = (R^+ \cup R^+) \cap \operatorname{span}_{\mathbb{R}}\{D_1, D_2\}$; this establishes the equivalence of (2) and (2′).

Trivially, (2′) and (3) are equivalent and (4) implies (3). We show (2) implies (4). Let $\langle B^+ \rangle$ denote the subgroup $\operatorname{span}_{\mathbb{Z}} B^+$ generated by B^+ in $\Delta \overset{\text{def}}{=} \langle R^+ \rangle = \operatorname{span}_{\mathbb{Z}} R^+$. By (2), for each $D \in B^+$, the reflection σ_D maps B^+ into $\langle B^+ \rangle$. Hence $\mathcal{W}^C(\langle B^+ \rangle) \subseteq \langle B^+ \rangle$. By the first part of (ix) above, $R^+ \subseteq \mathcal{W}^C(B^+)$. Hence $\Delta \subseteq \langle B^+ \rangle \subseteq \Delta$. Trivially, (5) implies (4). Assume (4). We know that $R^+ \subseteq \sum_{D \in B^+} \mathbb{R}^+\cdot D$ from (vi). Also, by (4), $R^+ \subseteq \sum_{D \in B^+} \mathbb{Z}\cdot D$. Hence $R^+ \subseteq \sum_{D \in B^+} \mathbb{Z}^+\cdot D$. □

The set B^+ will be called the *basis attached to the set* R^+. We note from 6.51(xi) that the finer geometric properties of the sets R^+ and B^+ are determined by the behavior of *two* basis vectors from B^+. If R^+ is the system of positive roots for a compact Lie group, then the equivalent conditions of 6.51(xi) are satisfied by 6.46(III).

Let now $\mathfrak{t}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of a compact connected Lie group G, and let $\mathcal{W}(T,G) = N(T,G)/T$ be the Weyl group of G with respect to T. Define

$$\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g}) = \{\alpha \in \mathrm{Aut}\,\mathfrak{g} \cap \mathrm{O}(\mathfrak{g}) \mid \alpha(\mathfrak{t}) = \mathfrak{t}\}.$$

Then $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ is, in particular, a closed subgroup of $\mathrm{O}(\mathfrak{g})$. We recall that by 6.4(x) $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{O}(\mathfrak{g})$ if $\mathfrak{g}$ is semisimple. Notice that an adjoint morphism $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ with $g \in G$ is in $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ if and only if $g \in N(T,G)$. Thus $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})|\mathfrak{t}$ contains $\mathrm{Ad}\big(N(T,G)\big)|\mathfrak{t}$ and by Corollary 6.35, the subgroup $\mathrm{Ad}(N(T,G)|\mathfrak{t}$ of $\mathrm{O}(\mathfrak{t})$ is naturally isomorphic to the Weyl group $\mathcal{W}(T,G)$. Let R^+ be a set of positive roots with respect to $\mathfrak{t}$ and $B^+ \subseteq R^+$ the unique basis of $\mathfrak{t}$ defined by (BAS) for R^+ discussed in Proposition 6.51, and let, for each basis root $D \in B^+$, an element $n_D \in N(t,G)$ be selected according to Proposition 6.47. These notions prepare us for the following theorem:

The Role of the Weyl Group for Compact Connected Lie Groups

Theorem 6.52. *Let $\mathfrak{t}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of a compact connected Lie group G, and $\mathcal{W}(T,G) = N(T,G)/T$ the Weyl group of G with respect to T. Then the following conclusions hold.*

(i) *$\mathcal{W}(T,G)$ is generated by the set $\{n_D T \mid D \in B^+\}$ of involutions and acts* simply *transitively on the set of Weyl chambers.*

(ii) *There is at least one element $n \in N(T,G)$ such that the fixed point set of* $\mathrm{Ad}(n)|\mathfrak{t}$ *is* $\mathfrak{z}(\mathfrak{g})$.

(iii) $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ *acts on $R = R^+ \cup -R^+$ and permutes the root spaces $\mathfrak{g}^D$ according to*

$$\gamma(\mathfrak{g}^D) = \mathfrak{g}^{\gamma D}, \text{ and}$$
$$(\forall X \in \mathfrak{t},\, Y \in \mathfrak{g}^D) \quad [X, \gamma Y] = (X \mid \gamma Y){\cdot}I(\gamma Y),$$
$$[\gamma Y, I(\gamma Y)] = (\gamma Y \mid \gamma Y){\cdot}\gamma D = (Y \mid Y){\cdot}\gamma D.$$

In particular, the Weyl group $\mathcal{W}(T,G)$ acts on $R = R^+ \cup -R^+$.

(iv) $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ *acts on the set of subalgebras $\mathfrak{g}_D = \mathbb{R}{\cdot}D \oplus \mathfrak{g}^D \cong \mathfrak{so}(3)$ such that $\gamma(\mathfrak{g}_D) = \mathfrak{g}_{\gamma D}$.*

(v) *If $B^+ = B^+(R^+)$ is the basis attached to the set R^+ of positive roots then the $\mathcal{W}(T,G)$ orbit of every $D \in R^+$ meets B^+.*

(vi) *Every element in R^+ is an* integral *nonnegative linear combination of elements in B^+.*

Proof. (i) Let C be the Weyl chamber $C(R^+)$. We now let $\mathcal{W}^C$ be the group of orthogonal transformations of $\mathfrak{t}$ generated by the reflections σ_D in the hyperplanes $\mathfrak{t}_D = D^{\perp_{\mathfrak{t}}}$, $D \in B^+$ for a basis attached to a positive root system R^+. We set

$\mathcal{W}_C = \{\mathrm{Ad}(n)|\mathfrak{t} \in \mathrm{O}(\mathfrak{t}) \mid n \in N(T,G),\ \mathrm{Ad}(n)(C) = C\}$. We shall show that $\mathcal{W}_C$ is singleton. This will in fact complete the proof as follows. Applying 6.51(iii) we see that the group $\mathcal{W}^C$ acts transitively on the set $\mathcal{C}$ of all Weyl chambers; then by the Frattini Argument (preceding Lemma 6.35), $\{\mathrm{Ad}(n)|\mathfrak{t} \mid n \in N(T,G)\} = \mathcal{W}^C\mathcal{W}_C$. Thus $\mathcal{W}_C = \{\mathrm{id}_\mathfrak{t}\}$ will show at the same time that the action of the Weyl group $\mathcal{W}(T,G)$ on $\mathfrak{t}$ is by the group generated by the set of reflections σ_D, $D \in B^+$ and that this action is simply transitive. Since the linear action of $\mathcal{W}(T,G)$ on $\mathfrak{t}$ is faithful, and since $\sigma_D = \mathrm{Ad}(n_D)|\mathfrak{t}$, the Weyl group is generated by the set of involutions $\{n_DT \mid D \in B^+\}$.

In order to prove $\mathcal{W}_C = \{\mathrm{id}_\mathfrak{t}\}$, let $\mathfrak{t}(C)$ denote the fixed point vector space in $\mathfrak{t}$ of the linear group $\mathcal{W}_C$ on $\mathfrak{t}$ which leaves the closed cone $\overline{C}$ invariant. By Theorem 3.38 on the linear action of compact groups on cones applied to the group $\mathcal{W}_C$, we conclude that $\mathfrak{t}(C)$ meets the interior C of $\overline{C}$. Let $X \in C \cap \mathfrak{t}(C)$. and set $T(C) = \exp \mathfrak{t}(C)$. Let $N_C = \{n \in N(T,G_0) \mid \mathrm{Ad}(n)C = C\}$, then $\mathcal{W}_C = \mathrm{Ad}(N_C)|\mathfrak{t}$. Then $N_C \subseteq Z(T(C),G_0)$. Now $G_1 \overset{\mathrm{def}}{=} Z(T(C),G_0)$ is a closed subgroup of G_0 containing T, and thus $\mathfrak{g}_1 = \mathfrak{L}(G_1)$ is, in particular, a T-submodule of $\mathfrak{g}$ containing T. As a consequence, we have a subset $R_1^+ \subseteq R^+$ such that

$$\mathfrak{g}_1 = \mathfrak{t} \oplus \bigoplus_{D\in R_1^+} \mathfrak{g}^D \subseteq \mathfrak{t} \oplus \bigoplus_{D\in R^+} \mathfrak{g}^D.$$

We claim that $R_1^+ = \emptyset$, i.e. $\mathfrak{g}_1 = \mathfrak{t}$. Suppose the contrary holds. Then we would have a $D \in R_1^+$ and then a nonzero $Y \in \mathfrak{g}^D$. Accordingly, $[X,Y] = (X \mid D)\cdot IY$ for $X \in \mathfrak{t}$ by 6.45(ii)(A). Now $Y \in \mathfrak{g}_1 = \mathfrak{L}\big(Z(T(C),G_0)\big) = \mathfrak{z}(\mathfrak{t}(C),\mathfrak{g})$, and thus $X \in \mathfrak{t}(C)$ implies $[X,Y] = 0$. Therefore $(X \mid D) = 0$ and thus $X \in \mathfrak{t}_D = D^{\perp_\mathfrak{t}}$. However, $X \in C$ and $C \cap \mathfrak{t}_D = \emptyset$ by the definition of a Weyl chamber. This contradiction proves the claim $R_1^+ = \emptyset$. Now $\mathfrak{L}\big(Z(T(C),G_0)\big) = \mathfrak{t}$ and since $Z(T(C),G_0)$ is connected by Corollary 6.33(i) we conclude $Z(T(C),G_0)_0 = T$ and thus $N_C \subseteq T$. This shows that $\mathcal{W}_C = \mathrm{Ad}(N_C)|\mathfrak{t} = \{\mathrm{id}_\mathfrak{t}\}$ and thus concludes the proof of (i).

(ii) We apply 6.51(vii) with $\mathcal{W} = \{\mathrm{Ad}(n)|\mathfrak{t} \mid n \in N(T,G)\}$. Since $\mathfrak{z}(\mathfrak{g}) = \mathfrak{t} \cap (R^+)^\perp$ by 6.50(i), assertion (ii) follows at once.

(iii) and (iv) Let $\gamma \in \mathrm{Aut}_\mathfrak{t}\,\mathfrak{g}$ and recall $\gamma\,\mathrm{ad}\,Z = \mathrm{ad}(\gamma Z)\gamma$. Recall from 6.45 that $I|\mathfrak{g}^D$ is $e^{\frac{\pi}{2(D|D)}\,\mathrm{ad}\,D}\Big|\mathfrak{g}^D$. Hence, since γ is orthogonal by the definition of $\mathrm{Aut}_\mathfrak{t}(\mathfrak{g})$, for $Y_D \in \mathfrak{g}^D$, we have

$$\begin{aligned}\gamma(IY_D) &= \gamma e^{\frac{\pi}{2(D|D)}\,\mathrm{ad}\,D}\gamma^{-1}\gamma(Y_D)\\ &= e^{\frac{\pi}{2(D|D)}\,\mathrm{ad}\,\gamma D}\gamma(Y_D) = e^{\frac{\pi}{2(\gamma D|\gamma D)}\,\mathrm{ad}\,\gamma D}\gamma(Y_D)\\ &= I\gamma(Y_D).\end{aligned}$$

For $X \in \mathfrak{t}$ set $X' = \gamma^{-1}X$. Then

$$[X,\gamma Y] = \gamma[X',Y] = (X' \mid D)\cdot\gamma I_D Y = (\gamma X' \mid \gamma D)\cdot I_{\gamma D}(\gamma Y) = (X \mid \gamma D)\cdot I_{\gamma D}(\gamma Y).$$

Similarly, $[\gamma Y, I(\gamma Y)] = \gamma[Y,IY] = (Y \mid Y)\cdot\gamma D = (\gamma Y \mid \gamma Y)\cdot\gamma D$. Thus γ permutes $R^+ \cup -R^+$ and the root spaces; specifically, $\gamma(\mathfrak{g}^D) = \mathfrak{g}^{\gamma(D)}$.

All inner automorphisms $\mathrm{Ad}(n)$ with $n \in N(T,G)$ belong to $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ hence the preceding applies with $\gamma = \mathrm{Ad}(n)$.

(iv) This follows from (iii) above and 6.51(ix)

(v) By (iii) above, 6.51(xi) applies. From 6.46(III) we know that 6.51(xi)(1) is satisfied. Hence 6.51(xi)(3) holds and this is our assertion. □

Since the Exercise E6.13 on Geometric Dihedral Groups applies to the situation of Theorem 6.52 we observe in passing that the oriented angle $\mathrm{ang}(\mathbb{R}^+\cdot D, \mathbb{R}^+\cdot D')$ between any two roots $D,\ D' \in R$ is a multiple of $\frac{2\pi}{n}$ modulo 2π where $n = 2, 3, 4, 6$.

In the wake of Theorem 6.52 let $\mathcal{W}^*(T,G) \stackrel{\text{def}}{=} \langle n_D \mid D \in B^+\rangle \subseteq N(T,G)$ be the subgroup generated by the elements $n_D \in N(T,G)$ of order 2 or 4, selected, one for each basis root $D \in B^+$, as in Proposition 6.47. Then Part (i) of Theorem 6.52 implies

$$N(T,G) = T\mathcal{W}^*(T,G) \text{ and } \mathcal{W}(T,G) \cong \frac{\mathcal{W}^*(T,G)}{\mathcal{W}^*(T,G)\cap T}.$$

Without further information, however, it is not clear whether the countable group $\mathcal{W}^* = \mathcal{W}^*(T,G)$ in fact is finite. As we observed in the paragraph preceding Proposition 6.23, our Theorem 6.10(i) implies the existence of a finite subgroup E of $N(T,G)$ such that

$$N(T,G) = TE \text{ and } \mathcal{W}(T,G) \cong \frac{E}{T[n]},\ n = |\mathcal{W}(T,G)|,$$

where $T \cap E = T[n]$ with the understanding that

$$T[n] = \{t \in T : t^n = 1\}.$$

Exactly how E and $\mathcal{W}^*$ are related is also not clear on the basis of the information we provided here.

However, let us assume for the moment, that additional information is available on so-called *Coxeter groups* , that is, groups W generated by a set S of elements $s \in W$ such that $s^2 = 1$ and that W is free subject only to the conditions $s^2 = 1$ for $s \in S$ and $(ss')^{m(s,s')} = 1$ for all pairs $(s,s') \in S \times S$ for which ss' has finite order $m(s,s')$. (See Bourbaki, [42], IV, §1, n° 3, Definition 3.) The elements $n_D T$ and $n_{D'}T$ satisfy such relations by Proposition 6.51(v). Based on information of Coxeter groups, in [43], Exercice 12 for IX, §4, Bourbaki provides the following information (and more):

Bourbaki's Lemma on $\mathcal{W}^*$. *If G is a simply connected compact (hence semi-simple) Lie group, then*

$$\mathcal{W}^*(T,G) \cap T = T[2].$$

The smallest example illustrating this situation is the group $\mathbb{S}^3$ of unit quaternions with $T = \mathbb{S}^1 = \exp^{i\mathbb{R}}$ and $\mathcal{W}^*(T,G) = \{\pm 1, \pm i, \pm j, \pm k\}$ the 8 element "quaternion group".

Notice that $T[2]$ is what in Definition A1.20 we call the 2-socle of T. We have $\mathrm{rank}_2(T[2]) = \dim T$. If the center Z of the simply connected compact group G has

2-rank $\mathrm{rank}_2(Z) < \dim T = \mathrm{rank}(G)$, then for the semisimple Lie group $\underline{G} \stackrel{\mathrm{def}}{=} G/Z$ the group $N(\underline{T}, \underline{G})$ *is not* the semidirect product of $\underline{T}$ and $\mathcal{W}^*(\underline{T}, \underline{G})$. There are only rare exceptions to this general case, such as $\underline{G} = \mathrm{SO}(3)$. (Cf. Exercise E6.10.) Thus, in general, a maximal torus T of a nonabelian compact connected Lie group G does not split as a semidirect normal factor in its normalizer $N(T, G)$.

The Commutator Subgroup of Connected Compact Lie Groups

Recall the set R of real roots of of $\mathfrak{g}$ with respect to the Cartan algebra $\mathfrak{t}$.

Lemma 6.53. *There is an element* $n \in N(T, G)$ *such that* $\mathrm{Ad}(n)|\mathfrak{t} - \mathrm{id}_{\mathfrak{t}}$ *has kernel* $\mathfrak{z}(\mathfrak{g})$ *and* $\mathrm{Ad}(n)|\mathfrak{t} - \mathrm{id}_{\mathfrak{t}} \,|\, \mathrm{span}\, R$ *is a vector space automorphism of* $\mathrm{span}\, R$.

Proof. By Theorem 6.52(ii) there in an $n \in N(T, G)$ such that $\mathrm{Ad}(n)|\mathfrak{t}$ has the precise fixed point set $\mathfrak{z}(\mathfrak{g})$. Obviously, this n satisfies the requirements since $\mathrm{Ad}(n)|\mathfrak{t}$ is an orthogonal transformation and $\mathfrak{t} \cap \mathfrak{z}(\mathfrak{g})^\perp = \mathrm{span}_{\mathbb{R}} R^+$ by 6.50(ii). □

Recall that for two subgroups A and B of a group G we write

$$\mathrm{comm}(A, B) \stackrel{\mathrm{def}}{=} \langle aba^{-1}b^{-1} \mid a \in A,\, b \in B \rangle.$$

For the following proposition also recall from Theorem 6.30(26) that for a compact connected Lie group G we have $G = \exp \mathfrak{g}$ and that, therefore, there is an $X \in \mathfrak{g}$ such that $n = \exp X$ for the element $n \in N(T, G)$ in Lemma 6.52.

Proposition 6.54. *Let* T *be a maximal torus of a connected semisimple compact Lie group. Then*

(i) $T = \{ntn^{-1}t^{-1} \mid n \in N(T, G),\, t \in T\} = \mathrm{comm}\big(N(T, G), T\big)$. *In particular, every element of* T *is a commutator in the normalizer* $N(T, G)$.

(ii) *There is a* $Z \in \mathfrak{g}$ *such that* $\mathfrak{t} \subseteq [Z, \mathfrak{g}]$. *In particular, every element of* $\mathfrak{t}$ *is a bracket in* $\mathfrak{g}$.

(iii) *Let* $\mathfrak{t}'$ *be any Cartan subalgebra of* $\mathfrak{g}$ *containing* Z, *then* $\mathfrak{t}$ *and* $\mathfrak{t}'$ *are orthogonal to each other.*

Proof. By Lemma 6.53 there is an element $n \in N(T, G)$ such that $\mathrm{Ad}(n)|\mathfrak{t} - \mathrm{id}_{\mathfrak{t}}$ is a vector space automorphism of $\mathfrak{t}$.

(i) If $g \in T$, then there is an $X \in \mathfrak{t}$ with $g = \exp X$ and after the preceding Lemma 6.53, there is a $Y \in \mathfrak{t}$ and an $n \in N(T, G)$ such that $X = \mathrm{Ad}(n)Y - Y$. Then $g = \exp X = \exp \mathrm{Ad}(n)Y \exp -Y = n(\exp Y)n^{-1}(\exp Y)^{-1} = \mathrm{comm}(n, \exp Y)$. Hence every element of the maximal torus is a commutator $\mathrm{comm}(n, t)$ with $t \in T$, $n \in N(T, G)$. (ii) Let $Z \in \mathfrak{g}$ be such that $n = \exp n$. In view of Theorem 5.44(i)(20) we can write the linear endomorphism $L \stackrel{\mathrm{def}}{=} \mathrm{Ad}(n) - \mathrm{id}_{\mathfrak{g}}$ of the vector space $\mathfrak{g}$ as

$$L = e^{\mathrm{ad}\, Z} - \mathrm{id}_{\mathfrak{g}}\,. \tag{1}$$

Equation (1) may be expressed equivalently as

$$(2)\qquad L = \operatorname{ad} Z \circ F, \text{ where } F = (\mathrm{id}_{\mathfrak{g}} + \sum_{m=2}^{\infty} \frac{1}{m!} (\operatorname{ad} Z)^{m-1}),$$

and from (2) and Lemma 6.53 we obtain $\mathfrak{t} = L(\mathfrak{t}) = [Z, F(\mathfrak{t})] \subseteq [Z, \mathfrak{g}]$.

(iii) By E6.2(b) we have $[Z, \mathfrak{g}] = \mathfrak{z}(Z, \mathfrak{g})^{\perp}$. From Corollary 6.32(28bL) we record that

$$\mathfrak{z}(Z, \mathfrak{g}) = \bigcup \{\mathfrak{t}' | \mathfrak{t}' \in \mathfrak{T}(\mathfrak{g}) \text{ and } Z \in \mathfrak{t}'\}.$$

Accordingly,

$$\mathfrak{z}(Z, \mathfrak{g})^{\perp} = \bigcap \{\mathfrak{t}'^{\perp} | \mathfrak{t}' \in \mathfrak{T}(\mathfrak{g}) \text{ and } Z \in t'\}.$$

Thus by (ii) above $\mathfrak{t} \subseteq [Z, \mathfrak{g}] = \mathfrak{z}(Z, \mathfrak{g})^{\perp} \subseteq \mathfrak{t}'^{\perp}$ for each Cartan subalgebra $\mathfrak{t}'$ containing Z. □

Among other things we see that the orthogonality of Cartan subalgebras is not a rare ocurrence in a compact Lie algebra.

A remarkable consequence of these facts is the following theorem.

Gotô's Commutator Theorem for Connected Compact Lie Groups

Theorem 6.55 ([127]). *Let G be a compact Lie group. Then every element of $(G_0)' = \exp \mathfrak{g}'$ is a commutator, and every element of $\mathfrak{g}'$ is a bracket.*

Proof. We may assume that $G = G_0$. Then $G' = \exp \mathfrak{g}'$ by 6.31(iii) and this is a compact group. It suffices to prove the theorem for G'. Thus we assume for the remainder of the proof that G is connected and semisimple; i.e. $G' = G = G_0$. Let T be a maximal torus. Then by 6.54, for each $s \in T$ there is an $n \in N(T, G)$ and a $t \in T$ such that $s = \operatorname{comm}(n, t)$. By the Maximal Torus Theorem 6.30(27), every element of G is conjugate to an element in the maximal torus T. Hence every element of G is a commutator.

From Theorem 6.27 we know $\mathfrak{g} = \bigcup_{g \in G} \operatorname{Ad}(g)\mathfrak{t}$ and so $\mathfrak{g} \subseteq \bigcup_{g \in G} [\operatorname{Ad}(g)Z, \mathfrak{g}]$, that is, every element of $\mathfrak{g}$ is a bracket. □

According to this theorem, for a connected compact Lie group G, the function $(g, h) \mapsto \operatorname{comm}(g, h): G \times G \to G$ has the image G'. This gives a new proof of the closedness of the commutator group of a *connected* compact Lie group (see the first part of the proof of 6.11). The information here, however, is much sharper than that of 6.11 in the connected case. On the other hand, the methods and the information involved in 6.55 cannot lead to a proof of the closedness of the commutator group of a (not necessarily connected) compact Lie group. Indeed in 6.10 and its proof we saw that it requires effort to get the closedness of the commutator group even in the case that G_0 is abelian (and thus, in particular, has no nontrivial commutators at all).

Corollary 6.56. *Every element in a connected compact semisimple Lie group is a commutator.* □

On the Automorphism Group of a Compact Lie Group

Let $\mathrm{Aut}(G)$ as usual denote the group of automorphisms of the compact group G and $\mathrm{Inn}(G)$ the compact normal subgroup of all inner automorphisms I_g, $I_g(x) = gxg^{-1}$. We have considered automorphism groups of linear Lie groups only in passing and merely as abstract groups (e.g. in Theorem 5.42(iv) and in Exercise E5.15 following Theorem 5.44). On the other hand, we have studied the automorphism group of a Lie algebra as a linear Lie group in detail (see 5.43), but gave the special case of the automorphism group of a compact Lie algebra only cursory attention (see 6.5). Now we expand in two directions: firstly, we provide more detailed information on the structure of $\mathrm{Aut}\,\mathfrak{g}$ for a compact Lie algebra $\mathfrak{g}$, and, secondly, we consider the automorphism group $\mathrm{Aut}(G)$ of a compact group as a topological group in its own right. For this purpose we must endow $\mathrm{Aut}(G)$ with a group topology; this requires a certain effort in its own right. We shall then systematically extend the subject foreshadowed in Exercise E5.15 and look more closely at the relation of $\mathrm{Aut}\,G$ and $\mathrm{Aut}\,\mathfrak{g}$ if G is a compact Lie group. We elucidate the structure of $\mathrm{Aut}(G)$ by identifying its identity component as a compact Lie group isomorphic to $G_0/\big(G_0 \cap Z(G)\big)$. If G is connected, it is isomorphic to $G/Z(G) \cong G'/Z(G')$ and is therefore semisimple.

We begin the first step by considering a compact Lie algebra $\mathfrak{g}$ and focusing on its automorphism group.

Lemma 6.57. *Let $\mathfrak{g}$ be a compact Lie algebra and write $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$ and $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with isotypic components $\mathfrak{g}_j$, each being a finite orthogonal ideal direct sum of simple modules each isomorphic to a simple module $\mathfrak{s}_j$, when $j \neq j'$ implies $\mathfrak{s}_j \not\cong s_{j'}$. Then the functions*

$$\eta\colon \mathrm{Aut}\,\mathfrak{g} \to \mathrm{Aut}\,\mathfrak{z}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{g}'), \quad \eta(\alpha) = (\alpha|\mathfrak{z}(\mathfrak{g}), \alpha|\mathfrak{g}'),$$
$$\zeta\colon \mathrm{Aut}\,\mathfrak{g}' \to \mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k), \quad \zeta(\alpha) = (\alpha|\mathfrak{g}_1, \ldots, \alpha|\mathfrak{g}_k),$$

are isomorphisms. Thus

$$\mathrm{Aut}\,\mathfrak{g} \cong \mathrm{Gl}\left(\mathfrak{z}(\mathfrak{g})\right) \times \mathrm{Aut}\,\mathfrak{g}' \cong \mathrm{Gl}\left(\mathfrak{z}(\mathfrak{g})\right) \times \mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k).$$

Proof. From Theorem 6.4 we know that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$ and $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is an orthogonal ideal direct sum where the ideals $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}_j$, $j = 1, \ldots, k$ are fully characteristic (in the sense that even every endomorphism of $\mathfrak{g}$ maps each of them into itself). Hence each $\alpha \in \mathrm{Aut}\,\mathfrak{g}$ induces an automorphism on each of these ideals. Thus $\alpha \mapsto (\alpha|\mathfrak{z}(\mathfrak{g}), \alpha|\mathfrak{g}_1, \ldots, \alpha|\mathfrak{g}_k)\colon \mathrm{Aut}\,\mathfrak{g} \to \mathrm{Aut}\,\mathfrak{z}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)$ is well-defined and has as inverse morphism the assignment $(\alpha_{z(\mathfrak{g})}, \alpha_{\mathfrak{g}_1}, \ldots, \alpha_{\mathfrak{g}_k}) \mapsto \alpha$, where $\alpha(X_{\mathfrak{z}(\mathfrak{g})} + X_{\mathfrak{g}_1} + \cdots + X_{\mathfrak{g}_k}) = \alpha_{\mathfrak{z}(\mathfrak{g})}(X_{\mathfrak{z}(\mathfrak{g})}) + \alpha_{\mathfrak{g}_1}(X_{\mathfrak{g}_1}) + \cdots + \alpha_{\mathfrak{g}_k}(X_{\mathfrak{g}_k})$. □

Before we go into the next lemma, let us recall some general group theoretical concepts. Let A be a group and let S_n denote the group of permutations (bijections) of $\{1,\dots,n\}$. The elements of A^n we consider as functions $\alpha\colon\{1,\dots,n\}\to A$ with componentwise multiplication $\alpha\beta(j)=\alpha(j)\beta(j)$. Then S_n operates on A^n by $\sigma{\cdot}\alpha=\alpha\circ\sigma^{-1}$. The function $s\colon S_n\to\mathrm{Aut}(A^n)$, $s(\sigma)(\alpha)=\sigma{\cdot}\alpha$ is a morphism, and the semidirect product $A^n\rtimes_s S_n$ is called the *wreath product* of A and S_n. The multiplication in the wreath product is given by $(\alpha,\sigma)(\beta,\tau)=(\alpha(\sigma{\cdot}\beta),\sigma\tau)=(\alpha(\beta\circ\sigma^{-1}),\sigma\tau)$. If A is the automorphism group of an object $\mathbf{g}$ (in some set-based category), then we consider the elements of $\mathbf{g}^n$ as functions $g\colon\{1,\dots,n\}\to\mathbf{g}$.

The group $\mathrm{Aut}(\mathbf{g}^n)$ contains a subgroup N isomorphic to $\mathrm{Aut}(\mathbf{g})^n$ containing all elements $\widetilde{\alpha}$ so that $\widetilde{\alpha}(g)(j)=\alpha(j)\big(g(j)\big)$ for $\alpha\in\mathrm{Aut}(\mathbf{g})^n$. It also contains a subgroup H isomorphic to S_n consisting of elements $\widetilde{\sigma}$ such that such that $\widetilde{\sigma}(g)=g\circ\sigma^{-1}$ for each $\sigma\in S_n$. Now $\big(\widetilde{\sigma}\circ\widetilde{\alpha}\circ\widetilde{\sigma}^{-1}\big)(g)(j)=\widetilde{\sigma}(h)(j)=h\big(\sigma^{-1}(j)\big)$ with $h=\big(\widetilde{\alpha}\circ\widetilde{\sigma}^{-1}\big)(g)=\widetilde{\alpha}\big(\widetilde{\sigma}^{-1}(g)\big)=\widetilde{\alpha}(g\circ\sigma)$, i.e. $h(k)=\alpha(k)\big(g(\sigma(k))\big)$. Then $h\big(\sigma^{-1}(j)\big)=\alpha\big(\sigma^{-1}(j)\big)\big(g(j)\big)=(\alpha\circ\sigma^{-1})^{\sim}(g)(j)=(\sigma{\cdot}\alpha)^{\sim}(g)(j)$. This shows that H is in the normalizer of N. An automorphism of $\mathrm{Aut}(\mathbf{g}^n)$ is of the form $\widetilde{\alpha}=\widetilde{\sigma}$ iff $\alpha(j)\big(g(j)\big)=g\big(\sigma^{-1}(j)\big)$ for all g and all j; appropriately specializing g we see that only the identity automorphism satisfies this condition. Therefore NH is a semidirect product and thus the morphism $(\widetilde{\alpha},\widetilde{\sigma})\mapsto\widetilde{\alpha}\circ\widetilde{\sigma}:N\rtimes_I H\to N\circ H$ is an isomorphism. We have seen, however, that $I_{\widetilde{\sigma}}(\widetilde{\alpha})=\widetilde{\sigma}\circ\widetilde{\alpha}\circ\widetilde{\sigma}^{-1}=s(\sigma)(\alpha)^{\sim}$, and thus $(\alpha,\sigma)\mapsto(\widetilde{\alpha},\widetilde{\sigma})\colon\mathrm{Aut}(\mathbf{g})^n\rtimes_s S_n\to N\rtimes_I H$ is an isomorphism. Hence the function $\Phi\colon\mathrm{Aut}(\mathbf{g})^n\rtimes_s S_n\to\mathrm{Aut}(\mathbf{g}^n)$, $\Phi(\alpha,\sigma)=\widetilde{\alpha}\circ\widetilde{\sigma}$ is an injective morphism.

Lemma 6.58. *Assume that $\mathfrak{g}=\mathfrak{s}^n$ for a simple compact Lie algebra $\mathfrak{s}$. Then the function $\Phi\colon\mathrm{Aut}(\mathfrak{s})^n\rtimes_s S_n\to\mathrm{Aut}(\mathfrak{g})$, is an isomorphism; i.e. $\mathrm{Aut}(\mathfrak{g})$ is a wreath product of $\mathrm{Aut}(\mathfrak{s})$ and S_n.*

Proof. After the preceding remarks we have to show that Φ is surjective. Thus let $\alpha\in\mathrm{Aut}\,\mathfrak{g}$. Let $\mathrm{copr}_j\colon\mathfrak{s}\to\mathfrak{g}$ (where $\mathrm{copr}_j(s)=(s_1,\dots,s_n)$, $s_j=s$ and $s_k=1$ otherwise) and $\mathrm{pr}_j\colon\mathfrak{g}\to\mathfrak{s}$ be the j-th coprojection, respectively, projection and set $\alpha_{jk}\overset{\mathrm{def}}{=}\mathrm{pr}_j\circ\alpha\,\mathrm{copr}_k$. By the simplicity of $\mathfrak{s}$, the morphism $\alpha_{jk}\colon\mathfrak{s}\to\mathfrak{s}$ is either the constant morphism 0 or is an automorphism. We claim that there is a bijection $\sigma\in S_n$ such that

$$(*)\qquad \alpha_{jk}\begin{cases}\in\mathrm{Aut}\,\mathfrak{s} & \text{if } j=\sigma(k),\\ =0 & \text{otherwise.}\end{cases}$$

Indeed, for each $k\in\{1,\dots,n\}$ the isomorphic copy $\mathrm{copr}_k(\mathfrak{s})$ of $\mathfrak{s}$ is an ideal. By 6.4(vi), the automorphism α permutes the set $\{\mathrm{copr}_j(\mathfrak{s})\mid j=1,\dots,n\}$ of ideals. Hence there is a unique $\sigma(k)\in\{1,\dots,n\}$ such that $\mathrm{copr}_{\sigma(k)}(\mathfrak{s})=(\alpha\circ\mathrm{copr}_k)(\mathfrak{s})$. According to 6.4 we have a unique decomposition $\mathfrak{g}=\mathrm{copr}_{\sigma(k)}\oplus\mathfrak{n}_k$ with a semisimple ideal $\mathfrak{n}_j=\mathrm{copr}_{\sigma(n)}(\mathfrak{s})^{\perp}$. The projection pr_j maps the summand $\mathrm{copr}_j(\mathfrak{s})$ isomorphically onto $\mathfrak{s}$ and maps all summands $\mathrm{copr}_{j'}(\mathfrak{s})$ trivially for $j'\neq j$. Thus $(*)$ follows. Now define $\beta\in(\mathrm{Aut}\,\mathfrak{s})^n$, $\beta\colon\{1,\dots,n\}\to\mathrm{Aut}\,\mathfrak{s}$ by $\beta(k)=\alpha_{j,\sigma^{-1}(j)}$. We write the elements of $\mathfrak{g}=\mathfrak{s}^n$ as functions $X\colon\{1,\dots,n\}\to\mathfrak{s}$ and compute $\alpha(X)(j) = \mathrm{pr}_j\big(\alpha(X)\big) = \alpha_{j\sigma^{-1}(j)}\big(X(\sigma^{-1}(j))\big) = \beta(j)\big(X(\sigma^{-1}(j))\big) = \beta(j)\big(\widetilde{\sigma}(X)(j)\big) = \widetilde{\beta}\big(\widetilde{\sigma}(X)\big)(j) = (\widetilde{\beta}\circ\widetilde{\sigma})(X)(j)$, and thus $\alpha = \widetilde{\beta}\circ\widetilde{\sigma} = \Phi(\beta,\sigma)$. □

If $\mathfrak{g}$ is a compact Lie algebra, then $\mathrm{ad}\colon \mathfrak{g}\to \mathrm{Der}\,\mathfrak{g}$ is a representation with kernel $\mathfrak{z}(g)$. In view of 6.4, ad maps $\mathfrak{g}'$ isomorphically onto $\mathrm{ad}\,\mathfrak{g}$. Hence $(\mathrm{ad}\,\mathfrak{g})' = \mathrm{ad}\,\mathfrak{g}$. If an invariant inner product of $\mathfrak{g}$ is chosen, then $\mathrm{ad}\,\mathfrak{g}$ consists of skew symmetric automorphisms with respect to this inner product and thus $\mathrm{ad}\,\mathfrak{g}\subseteq \mathfrak{o}(\mathfrak{g})$. Now $\mathfrak{o}(\mathfrak{g})$ is the Lie algebra of the compact Lie group $\mathrm{O}(\mathfrak{g})$. Thus by Corollary 6.31(ii), the subspace $e^{\mathrm{ad}\,\mathfrak{g}}\subseteq \mathrm{O}(\mathfrak{g})$ is a compact subgroup with $\mathfrak{L}(e^{\mathrm{ad}\,\mathfrak{g}}) = \mathrm{ad}\,\mathfrak{g}\cong \mathfrak{g}'$. Let us write

$$\mathrm{Inn}\,\mathfrak{g} \stackrel{\mathrm{def}}{=} e^{\mathrm{ad}\,\mathfrak{g}}$$

and call this compact Lie group the group of *inner automorphisms* of $\mathfrak{g}$.

If G is a connected compact Lie group and $\mathfrak{g}$ its Lie algebra, then $\mathrm{Ad}\,G\subseteq \mathrm{Aut}\,\mathfrak{g}$ is exactly the connected compact subgroup whose Lie algebra is $\mathrm{ad}\,\mathfrak{g}$ and $\mathrm{Ad}\,G\cong G/Z(G)\cong \mathrm{Inn}(G)$ (cf. the Adjoint Representation Theorem 5.44), and thus $\mathrm{Inn}(G)\cong \mathrm{Ad}\,G = e^{\mathrm{ad}\,\mathfrak{g}} = \mathrm{Inn}\,\mathfrak{g}$. This justifies the notation. However, if $G = \mathbb{T}\rtimes\{\mathrm{id}_{\mathbb{T}}, -\mathrm{id}_{\mathbb{T}}\}$ (the group of orthogonal transformations of the euclidean plane), then $Z(G) = \{(\mathbb{Z},\mathrm{id}_{\mathbb{T}}), (\frac{1}{2}+\mathbb{Z},\mathrm{id}_{\mathbb{T}})\}$ has order 2 and $\mathrm{Inn}(G)\cong G/Z(G)\cong G$ while $\mathrm{Inn}\,\mathfrak{g} = \{\mathrm{id}_{\mathfrak{g}}\}$.

The factor group $\mathrm{Aut}\,\mathfrak{g}/\,\mathrm{Inn}\,\mathfrak{g}$ is, not entirely appropriately, called the group of *outer automorphisms*, at any rate we shall write

$$\mathrm{Out}\,\mathfrak{g} \stackrel{\mathrm{def}}{=} \mathrm{Aut}\,\mathfrak{g}/\,\mathrm{Inn}\,\mathfrak{g}.$$

We note $\mathrm{ad}\,\mathfrak{g} = \mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}' = \{\mathrm{ad}_{\mathfrak{g}}\,X \mid X\in\mathfrak{g}'\}$, $\mathrm{ad}_{\mathfrak{g}}\,X\colon \mathfrak{g}\to\mathfrak{g}$, and $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}'\cong \mathrm{ad}\,\mathfrak{g}' = \{\mathrm{ad}_{\mathfrak{g}'}\,X \mid X\in\mathfrak{g}'\}$, $\mathrm{ad}_{\mathfrak{g}'}\,X\colon\mathfrak{g}'\to\mathfrak{g}'$. Thus $\mathrm{Inn}\,\mathfrak{g} = e^{\mathrm{ad}\,\mathfrak{g}} = e^{\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}'}\cong e^{\mathrm{ad}_{\mathfrak{g}'}\,\mathfrak{g}'} = \mathrm{Inn}(\mathfrak{g}')\subseteq \mathrm{Aut}(\mathfrak{g}')$, and the isomorphism η of 6.57 maps $\mathrm{Inn}\,\mathfrak{g}$ onto $\{\mathrm{id}_{\mathfrak{z}(\mathfrak{g})}\times \mathrm{Inn}(\mathfrak{g}')$.

Lemmas 6.57 and 6.58 boil the determination of $\mathrm{Aut}\,\mathfrak{g}$ for a compact Lie algebra $\mathfrak{g}$ down to knowing $\mathrm{Aut}\,\mathfrak{g}$ for a simple compact Lie algebra $\mathfrak{g}$. Let us assume now that $\mathfrak{g}$ is a simple compact Lie algebra given the inner product $(X,Y)\mapsto (X\mid Y) = -\,\mathrm{tr}\,\mathrm{ad}\,X\,\mathrm{ad}\,Y$. (See 6.4(ix).) From 6.4(x) we know that $\mathrm{Aut}\,\mathfrak{g}\subseteq \mathrm{O}(\mathfrak{g})$ and that it is, therefore, a compact Lie group. We continue the notation of 6.5 and note that $\mathfrak{L}(\mathrm{Aut}\,\mathfrak{g}) = \mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{g}$ in this case by 6.5, whence $\mathrm{Inn}\,\mathfrak{g} = e^{\mathrm{ad}\,\mathfrak{g}} = (\mathrm{Aut}\,\mathfrak{g})_0$; therefore we record

$$\mathbb{G} \stackrel{\mathrm{def}}{=} \mathrm{Aut}\,\mathfrak{g}\subseteq \mathrm{O}(\mathfrak{g}) \quad\text{and}\quad \mathrm{Out}(\mathfrak{g}) = \mathbb{G}/\mathbb{G}_0.$$

Since $\mathbb{G}$ is a compact Lie group, the group $\mathrm{Out}(\mathfrak{g})$ is finite; we might refer to it as the *outer automorphism group*. If $\mathfrak{g} = \mathfrak{so}(3)\cong (\mathbb{R}^3,\times)$ with the vector product on euclidean 3-space, then $\mathbb{G} = \mathrm{Aut}\,\mathfrak{g}\cong \mathrm{Aut}(\mathbb{R}^3,\times) = \mathrm{SO}(3)$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and set $\mathrm{Aut}_{\mathfrak{t}}\,\mathfrak{g} = \{\alpha\in\mathbb{G}\mid \alpha(\mathfrak{t}) = \mathfrak{t}\}$. Fix a Weyl chamber $C\subseteq\mathfrak{t}$; then $\overline{C}$ is a pointed convex cone since $\mathfrak{g}$ is simple and thus $\mathfrak{z}(\mathfrak{g}) = \overline{C}\cap -\overline{C} = \{0\}$. Set $\mathrm{Aut}_C\,\mathfrak{g} = \{\alpha\in \mathrm{Aut}_{\mathfrak{t}}\,\mathfrak{g}\mid \alpha(C) = C\}$ and $\mathbf{T} = e^{\mathrm{ad}\,\mathfrak{t}}\subseteq\mathbb{G}$; then $\mathfrak{L}(\mathbf{T}) = \mathfrak{t}$, and $\mathbf{T}$ is a maximal torus of $\mathbb{G}$. Let

$$\mathfrak{k} \stackrel{\mathrm{def}}{=} \ker\exp_{\mathbf{T}} = \{X\in\mathfrak{t}\mid \exp_{\mathbf{T}}X = \exp_{\mathbb{G}}X = e^{\mathrm{ad}\,X} = \mathrm{id}_{\mathfrak{g}}\}.$$

Then $\mathfrak{k}$ is a point lattice in $\mathfrak{t}$ isomorphic to $\mathbb{Z}^r$, $r = \dim \mathfrak{t}$. Let us denote by $\operatorname{Aut}\mathfrak{k}$ the set of those linear automorphisms of $\mathfrak{t}$ inducing an automorphism of $\mathfrak{k}$. Then $\operatorname{Aut}_{\mathfrak{t}}\mathfrak{g}|\mathfrak{t} \subseteq \mathrm{O}(\mathfrak{t}) \cap \operatorname{Aut}\mathfrak{k}$; thus $\gamma \mapsto \gamma|\mathfrak{t}\colon \operatorname{Aut}_{\mathfrak{t}}\mathfrak{g} \to \operatorname{Aut}_{\mathfrak{t}}\mathfrak{g}|\mathfrak{t}$ is a surjective morphism onto a finite group whose kernel is $N(\mathbf{T}, \mathbb{G})$.

We now recall (from 6.45ff) the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{D \in R^+} \mathfrak{g}^D.$$

For $X \in \mathfrak{t}$ and $Y \in \mathfrak{g}^D$ we have $[X, Y] = (X \mid D)\cdot I_D Y$, whence $e^{\operatorname{ad} X} Y = e^{(X|D) I_D} Y$, and $[Y, I_D Y] = (Y \mid Y)\cdot D$, where $I_D = e^{\frac{\pi}{2(D|D)}\cdot \operatorname{ad} D}\Big|\mathfrak{g}^D$. In particular,

$$X \in \mathfrak{k} \Leftrightarrow (\forall D \in R^+) \quad (X \mid D) \in 4\mathbb{Z}.$$

After 6.51(iv) we may and will assume that the Weyl chamber C was picked in such a fashion that $(X \mid D) > 0$ for all $X \in C$ and all $D \in R^+$. If this choice is made, then a vector $D \in R^+ \cup -R^+$ is in R^+ if and only if $(X \mid D) > 0$ for all $X \in C$.

Proposition 6.59. *Assume that* $\mathfrak{g}$ *is a compact semisimple Lie algebra.*

(i) $\mathfrak{L}(\mathbb{G}) = \operatorname{Der}\mathfrak{g} \subseteq \mathfrak{o}(\mathfrak{g})$, *and* $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g} \cong \mathfrak{g}$.

(ii) *The Lie algebra of* $\mathbb{G}$ *may be identified with* $\mathfrak{g}$ *in such a way that the exponential function of* $\mathbb{G}$ *may be written* $\exp_{\mathbb{G}}\colon \mathfrak{g} \to \mathbb{G}$, $\exp_{\mathbb{G}} X = e^{\operatorname{ad} X}$, *and* $\mathbb{G}_0 = \exp_{\mathbb{G}} \mathfrak{g} = e^{\operatorname{ad}\mathfrak{g}} = \operatorname{Ad}(\mathbb{G})$.

(iii) *With the identifications of* (ii), *the adjoint representation agrees with the natural action of* $\mathbb{G} = \operatorname{Aut}\mathfrak{g}$ *on* $\mathfrak{g}$, *i.e. for* $\gamma \in \mathbb{G}$ *and* $X \in \mathfrak{g}$, $\operatorname{Ad}(\gamma)(X) = \gamma(X)$, *and if* $\gamma = e^{\operatorname{ad} Y}$, *then* $\gamma(X) = e^{\operatorname{ad} Y} X$.

(iv) $\mathbb{G} = \mathbb{G}_0 \cdot \operatorname{Aut}_C(\mathfrak{g})$ *and* $\mathbb{G}_0 \cap \operatorname{Aut}_C(\mathfrak{g}) = \mathbf{T}$. *In particular,* $\mathbf{T} = \big(\operatorname{Aut}_C(\mathfrak{g})\big)_0$ *and* $\operatorname{Out}(\mathfrak{g}) \cong \operatorname{Aut}_C(\mathfrak{g})/\mathbf{T}$.

(v) *The group* $\operatorname{Aut}_C(\mathfrak{g})$, *when acting on* $\mathfrak{t}$, *leaves* R^+ *and* B^+ *invariant.*

Proof. (i) is a direct consequence of preceding remarks and 6.5.

(ii) By 5.43, the exponential function $E_{\mathbb{G}}\colon \operatorname{Der}(\mathfrak{g}) \to \mathbb{G}$ is given by $E_{\mathbb{G}}(X) = \mathrm{id}_{\mathfrak{g}} + X + \frac{1}{2!}\cdot X^2 + \cdots$, $X \in \operatorname{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$. The map $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{Der}\mathfrak{g}$ is an isomorphism by 6.5. We may therefore consider $\mathfrak{g}$ as the Lie algebra of $\mathbb{G}$ and the exponential function to be the composition

$$\mathfrak{g} \overset{\operatorname{ad}}{\to} \operatorname{Der}(\mathfrak{g}) \overset{E}{\to} \mathbb{G}, \quad \exp_{\mathbb{G}} = E_{\mathbb{G}} \circ \operatorname{ad}, \quad \exp_{\mathbb{G}} X = e^{\operatorname{ad} X}.$$

(iii) Let $\gamma \in \mathbb{G}$ and $X \in \mathfrak{g}$. Then $\gamma[X, Y] = [\gamma X, \gamma Y]$ whence $\gamma \circ \operatorname{ad} X = \operatorname{ad}(\gamma X) \circ \gamma$. Therefore $\exp_{\mathbb{G}} \operatorname{Ad}(\gamma)(X) = \gamma(\exp_{\mathbb{G}} X)\gamma^{-1} = \gamma e^{\operatorname{ad} X}\gamma^{-1} = e^{\gamma \circ \operatorname{ad} X \circ \gamma^{-1}} = e^{\operatorname{ad}(\gamma X)} = \exp_{\mathbb{G}}(\gamma X)$; since this holds for all $t\cdot X$ in place of X with $t \in \mathbb{R}$, we conclude $\operatorname{Ad}(\gamma)(X) = \gamma(X)$ and thus $\operatorname{Ad}(\gamma) = \gamma$. The remainder now follows from 5.44.

(iv) The group $\mathbb{G}_0$ acts transitively on the set $\mathfrak{T}(\mathfrak{g})$ of Cartan algebras under its natural action by (iii) above and the Transitivity Theorem 6.27. The stability group of this action by definition is $\operatorname{Aut}_{\mathfrak{t}}(\mathfrak{g})$. Hence the Frattini Argument (preceding 6.35) shows $\mathbb{G} = \mathbb{G}_0 \cdot \mathrm{Aut}_{\mathfrak{t}}\,\mathfrak{g}$. The group $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g})$ acts on the set $\mathcal{C}$ of Weyl chambers with stability group $\mathrm{Aut}_C(\mathfrak{t})$, and the subgroup $\mathrm{Ad}\big(N(\mathbf{T},\mathbb{G}_0)\big)$ acts simply transitively by 6.52. So by the Frattini Argument once more, $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{g}) = \mathrm{Ad}\big(N(\mathbf{T},\mathbb{G}_0)\big)\cdot \mathrm{Aut}_C(\mathfrak{t}) \subseteq \mathbb{G}_0\cdot \mathrm{Aut}_C(t)$. Hence $\mathbb{G} = \mathbb{G}_0\cdot \mathrm{Aut}_C(\mathfrak{t})$ follows. One notes that $\gamma \in \mathbb{G}_0 \cap \mathrm{Aut}_C\,\mathfrak{g}$ means that $\gamma \in N(\mathbf{T},\mathbb{G}_0)$ and that the morphism $\mathrm{Ad}(\gamma)|\mathfrak{t}$ induced by γ on $\mathfrak{t}$ (cf. (iii) above!) fixes C as a whole. Since the action of $\mathrm{Ad}\big(N(\mathbf{T},\mathbb{G}_0)\big)$ on $\mathcal{C}$ is *simply* transitive by 6.52 we have $\gamma = e^{\mathrm{ad}\,X}$ with some $X \in \mathfrak{t}$, i.e. $\gamma \in \mathbf{T}$. Clearly $\big(\mathrm{Aut}_C(\mathfrak{g})\big)_0 \subseteq \mathbb{G}_0 \cap \mathrm{Aut}_C(\mathfrak{g}) = \mathbf{T} \subseteq \big(\mathrm{Aut}_C(\mathfrak{g})\big)_0$, so equality holds. Now $\mathrm{Out}(\mathfrak{g}) = \mathbb{G}/\mathbb{G}_0 = \mathbb{G}_0\,\mathrm{Aut}_C\,\mathfrak{g}/\mathbb{G}_0 \cong \mathrm{Aut}_C\,\mathfrak{g}/(\mathbb{G}_0 \cap \mathrm{Aut}_C\,\mathfrak{g}) = \mathrm{Aut}_C\,\mathfrak{g}/\mathbf{T}$.

(v) Now we prove that $\gamma(C) = C$ implies $\gamma(R^+) = R^+$. We have

$$(\forall X \in C,\, D \in R^+) \quad (X \mid D) > 0.$$

Then $\gamma \in \mathrm{Aut}_C\,\mathfrak{g}$ and $X \in C$ implies $X' \stackrel{\mathrm{def}}{=} \gamma^{-1}X \in C$ and thus

$$(\forall D \in R^+) \quad (X \mid \gamma D) = (\gamma X' \mid \gamma D) = (X' \mid D) > 0.$$

Hence

$$(\forall X \in C,\, D \in R^+) \quad (X \mid \gamma D) > 0.$$

Since for $D \in R^+ \cup -R^+$ we have $D \in R^+$ iff $(X \mid D) > 0$ for all $X \in C$, we conclude that $\gamma R^+ = R^+$. The elements $D' \in R^+$ are unique nonnegative real linear combinations $D' = \sum_{D\in B^+} r_D\cdot D$; the relation $D' \in B^+$ means that exactly one of the r_D is nonzero (and 1). Thus the invariance of R^+ under $\mathrm{Aut}_C(\mathfrak{g})$ implies the invariance of B^+ under this group. □

We know from 6.10(i) that we get a finite subgroup $\mathbb{E} \subseteq \mathrm{Aut}_C(\mathfrak{t})$ such that $\mathrm{Aut}_C(\mathfrak{g}) = \mathbf{T}\mathbb{E}$ and $\gamma \in \mathbb{E} \cap \mathbf{T}$ implies $\gamma^{|\,\mathrm{Out}\,\mathfrak{g}|} = \mathrm{id}$. Since $\mathbf{T} \subseteq \mathbb{G}_0$, we then have $\mathbb{G} = \mathbb{G}_0\,\mathrm{Aut}_C\,\mathfrak{g} = \mathbb{G}_0\mathbb{E}$.

However, in the present situation we can do better and choose the group $\mathbb{E} \subseteq \mathbb{G}$ so that $\mathbb{E} \cap \mathbb{G}_0 = \{1\}$. This requires a little bit of preparation because we have to pick the supplementary subgroup $\mathbb{E}$ judiciously. For this purpose we fix a semisimple compact Lie algebra $\mathfrak{g}$ with the canonical inner product given by $(X \mid Y) = -\operatorname{tr}\operatorname{ad}X\operatorname{ad}Y$ (see 6.4.(ix)) and abbreviate $\mathrm{Aut}_C\,\mathfrak{g}$ by $\mathbb{A}$. The group $\mathbb{A}$ acts orthogonally on $\mathfrak{g}$ (cf. 6.4(x)). Moreover, $\mathbb{A}$ leaves both R^+ and B^+ invariant by 6.59(v).

Set $r = \dim\mathfrak{t}$ and write

$$(\mathbb{S}^1)^r \stackrel{\mathrm{def}}{=} \left\{ \sum_{D\in B^+} Y_D \mid (\forall D \in B^+) \quad (Y_D \mid Y_D) = 1 \right\}.$$

Since $\{Y \in \mathfrak{g}^D \mid (Y \mid Y) = 1\} \subseteq \mathfrak{g}^D$ is a one-sphere by 6.49(ii), the subspace $(\mathbb{S}^1)^r$ of the $2r$-dimensional vector space $\bigoplus_{D\in B^+}\mathfrak{g}^D$ is an r-torus.

If $Y = \sum_{D\in B^+} Y_D \in (\mathbb{S}^1)^r$ and $\alpha \in \mathbb{A}$, then for each $D \in B^+$ we have $\alpha D \in B^+$ and $Y'_{\alpha D} \stackrel{\mathrm{def}}{=} \alpha Y_D \in \mathfrak{g}^{\alpha D}$ by 6.59(v). Also $(Y'_{\alpha D} \mid Y'_{\alpha Y}) = (\alpha Y_D \mid \alpha Y_D) = (Y_D \mid$

$Y_D) = 1$. Thus

$$\alpha Y = \sum_{D\in B^+} \alpha(Y_D) = \sum_{D\in B^+} Y'_{\alpha D} = \sum_{\alpha D\in B^+} Y'_{\alpha D} = \sum_{D\in B^+} Y'_D \in (\mathbb{S}^1)^r.$$

Hence $\mathbb{A}$ acts on $(\mathbb{S}^1)^r$.

The r-dimensional torus group $\mathbf{T} = e^{\operatorname{ad}\mathfrak{g}} \subseteq \mathbb{A}$ is the identity component $\mathbf{T}$ of $\mathbb{A}$ by 6.59(iv). By 6.45 and the fact that $\mathfrak{g}$ may be viewed as the Lie algebra of $\mathbb{G}$ (see 6.59(ii)) it acts on $(\mathbb{S}^1)^r$ via

$$e^{\operatorname{ad}X}(\sum_{D\in B^+} Y_D) = \sum_{D\in B^+} \cos(X \mid D)\cdot Y_D + \sin(X|D)\cdot IY_D = \sum_{D\in B^+} e^{(X|D)\cdot I} Y_D.$$

Since B^+ is a basis of $\mathfrak{t}$ by 6.51(iv), the function $X \mapsto \big((X \mid D)\big)_{D\in B^+}: \mathfrak{t} \to \mathbb{R}^r$ is an isomorphism of vector spaces. Hence the function $\mathfrak{t} \to \{z \in \mathbb{C} : |z| = 1\}^r$ mapping X to $(e^{(X|D)i})_{D\in B^+}$ is surjective and thus

$$(*) \quad (\forall Y = \sum_{D\in B^+} Y_D \in (\mathbb{S}^1)^r) \quad X \mapsto \sum_{D\in B^+} e^{(X|D)\cdot I} Y_D : \mathfrak{t} \to (\mathbb{S}^1)^r \text{ is surjective.}$$

Let $X \in \mathfrak{t}$. The relation $e^{\operatorname{ad}X}\cdot Y = Y$ for all $Y \in (\mathbb{S})^r$ is equivalent to

$$(**) \qquad (\forall D \in B^+) \quad (X \mid D) \in 2\pi i\mathbb{Z}.$$

By 6.52(vii), every $D' \in R^+$ is an integral linear combination $\sum_{D\in B^+} n_D\cdot D$. Hence $(**)$ implies $(X \mid D') = \sum_{D\in B^+} n_D(X \mid D) \in 2\pi i\mathbb{Z}$. Thus $e^{(X|D')\cdot I}(Y_{D'}) = Y_{D'}$ for all $Y_{D'} \in \mathfrak{g}^{D'}$. In view of $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{D\in R^+} \mathfrak{g}^D$ this implies that $e^{\operatorname{ad}X} = \mathrm{id}_\mathfrak{g} \in \mathbb{A}$. We conclude that for all $Y \in (\mathbb{S}^1)^r$ the function

$$(\dagger) \qquad \alpha \mapsto \alpha(Y): \mathbf{T} \to (\mathbb{S}^1)^r \text{ is a homeomorphism,}$$

being surjective by $(*)$ and injective by what we just saw. In particular, $\mathbf{T}$ *operates transitively on* $(\mathbb{S}^1)^r \subseteq \sum_{D\in B^+} \mathfrak{g}^D$.

Lemma 6.60 (K.-H. Neeb). *Let $\mathfrak{g}$ be a compact semisimple Lie algebra and continue the notation just introduced. Fix an element $Y \in (\mathbb{S}^1)^r$ and let $\mathbb{A}_Y$ be the isotropy group at Y of the group $\mathbb{A} = \operatorname{Aut}_C \mathfrak{g}$ acting on $(\mathbb{S}^1)^r$. Then the following statements hold.*

(i) $\operatorname{Aut}_C \mathfrak{g} = \mathbf{T}\mathbb{A}_Y$ *and* $\mathbf{T} \cap \mathbb{A}_Y = \{\mathrm{id}_\mathfrak{g}\}$, *i.e.* $\operatorname{Aut}_C \mathfrak{g}$ *is a semidirect product of the normal torus* $\mathbf{T}$ *and the subgroup* $\mathbb{A}_Y \cong \operatorname{Out}\mathfrak{g}$;

(ii) $\operatorname{Aut}\mathfrak{g} = (\operatorname{Aut}\mathfrak{g})_0\cdot \operatorname{Aut}_C \mathfrak{g}$ *and* $(\operatorname{Aut}\mathfrak{g})_0 \cap \operatorname{Aut}_C \mathfrak{g} = \{\mathrm{id}_\mathfrak{g}\}$, *i.e.* $\operatorname{Aut}\mathfrak{g}$ *is a semidirect product of its identity component with the group* $\mathbb{A}_Y$ *isomorphic to* $\operatorname{Out}\mathfrak{g}$.

Proof. (ii) will be a consequence of (i) in view of 6.59(iv).

We prove (i). We apply the Frattini Argument (see Lemma preceding 6.35) on the group $\mathbb{A}$ acting on $(\mathbb{S}^1)^r$ and the normal subgroup $\mathbf{T} = \mathbb{A}_0$ which acts transitively on $(\mathbb{S}^1)^r$ by $(\dagger)$ above. Then $\mathbb{A} = \mathbf{T}\mathbb{A}_Y$. Now assume that $\alpha \in \mathbf{T} \cap \mathbb{A}_Y$.

Since $\alpha \in \mathbf{T}$ we have $\alpha = e^{\operatorname{ad} X}$ for some $X \in \mathfrak{t}$. Since $\alpha \in \mathbb{A}_Y$ we have $\alpha Y = Y$, and that implies $\alpha = \mathrm{id}_{\mathfrak{g}}$ by (†) above. ☐

We can now summarize the salient points of our discussion of $\operatorname{Aut}\mathfrak{g}$ in the following theorem.

The Automorphism Group of a Compact Lie Algebra

Theorem 6.61. *Let $\mathfrak{g}$ be a compact Lie algebra. Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$,*
$\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ *with* $\mathfrak{g}_j \cong \mathfrak{s}_j^{n_j}$, $\mathfrak{s}_j$ *simple,* $\mathfrak{s}_j \not\cong \mathfrak{s}_{j'}$ *for* $j \neq j'$, $n_j \in \mathbb{N}$.
Accordingly

(i) $\operatorname{Aut}\mathfrak{g} \cong \operatorname{Aut}\mathfrak{z}(\mathfrak{g}) \times \operatorname{Aut}(\mathfrak{g}') = \operatorname{Gl}\big(\mathfrak{z}(\mathfrak{g})\big) \times \operatorname{Aut}\mathfrak{g}'$,

(ii) $\operatorname{Gl}\big(\mathfrak{z}(\mathfrak{g})\big) = \operatorname{Sl}\big(\mathfrak{z}(\mathfrak{g})\big) \rtimes \big(\mathbb{R} \times \mathbb{Z}(2)\big)$,

(iii) $\operatorname{Aut}\mathfrak{g}' \cong \operatorname{Aut}\mathfrak{g}_1 \times \cdots \times \operatorname{Aut}\mathfrak{g}_k$,

(iv) $\operatorname{Aut}\mathfrak{g}_j \cong (\operatorname{Aut}\mathfrak{s}_j)^{n_j} \rtimes S_{n_j}, \quad j = 1, \ldots, k$,

(v) $\operatorname{Aut}\mathfrak{s}_j \cong e^{\operatorname{ad}\mathfrak{s}_j} \rtimes \operatorname{Out}(\mathfrak{s}_j), \quad j = 1, \ldots, k$.

In particular, there is a finite subgroup $F \subseteq \operatorname{Aut}\mathfrak{g}$ *such that*

(vi)
$$\begin{aligned}\operatorname{Aut}\mathfrak{g} &\cong (\operatorname{Aut}\mathfrak{g})_0 \rtimes F \cong \big(\operatorname{Sl}\big(\mathfrak{z}(\mathfrak{g})\big) \rtimes \mathbb{Z}(2)\big) \times (\operatorname{Inn}(\mathfrak{g}') \rtimes \operatorname{Out}(\mathfrak{g}'))\\ &= \operatorname{Inn}(\mathfrak{g}') \rtimes \big(\operatorname{Gl}(\mathfrak{z}(\mathfrak{g})) \times \operatorname{Out}(\mathfrak{g}')\big)\\ &= \operatorname{Inn}(\mathfrak{g}) \rtimes \operatorname{Out}(\mathfrak{g}).\end{aligned}$$

The factor $\operatorname{Gl}\big(\mathfrak{z}(\mathfrak{g})\big)$ *is the only noncompact one if* $\mathfrak{g}$ *is not semisimple. For a given Cartan subalgebra* $\mathfrak{t}$ *of* $\mathfrak{g}$ *a complementary semidirect factor for* $(\operatorname{Aut}\mathfrak{g})_0$ *may be chosen so that all of its elements* α *satisfy* $\alpha(\mathfrak{t}) = \mathfrak{t}$.

Proof. We collect the content of the preceding lemmas and note that only the assertion on the semidirect splitting in (v) and (vi) has yet to be proved. We note that a Cartan subalgebra $\mathfrak{t}$ is obtained as $\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_k$ with a Cartan subalgebra t_j of $\mathfrak{g}_j$, where $\mathfrak{g}_j \cong \mathfrak{s}_j^{n_j}$ in such a way that $t_j \cong \mathfrak{t}_{j1} \times \cdots \times \mathfrak{t}_{jn_j}$ with Cartan subalgebras $\mathfrak{t}_{jp}$ of $\mathfrak{s}_j$. Further we note that $\operatorname{Gl}\big(\mathfrak{z}(\mathfrak{g})\big)$ is a semidirect product of a $\operatorname{Sl}(\mathfrak{z}(\mathfrak{g})\big)$ and the multiplicative group of nonzero real numbers which is isomorphic to $\mathbb{R} \times \mathbb{Z}(2)$. We now observe that it suffices to prove the assertion for the case that $\mathfrak{g} = \mathfrak{s}^n$. By 6.59 we have $\operatorname{Aut}\mathfrak{s}^n \cong A^n \rtimes S_n$ where $A = \operatorname{Aut}\mathfrak{s}$. By 6.60 we have $A = NE$ with a normal subgroup $N \cong \big(\operatorname{Aut}\mathfrak{g})_0 \cong e^{\operatorname{ad}\mathfrak{s}}$ and where E is a semidirect complement isomorphic to $\operatorname{Out}\mathfrak{s}$ which we may choose to leave a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{s}$ invariant. Now $N_0 = N^n$ is characteristic, hence normal in $A^n \rtimes S_n$. The subgroup $E \stackrel{\text{def}}{=} E^n$ of A^n is invariant under the action of S_n which operates by permuting the factors of A^n. Hence $F \stackrel{\text{def}}{=} E \rtimes S_n$ is a finite subgroup meeting N_0 trivially and satisfying $A^n \rtimes S_n = N_0 F$ and this product is semidirect as asserted in (∗). However, F so far respects the Cartan subalgebra $\mathfrak{t}^n$. Now let $\mathfrak{t}^* \stackrel{\text{def}}{=} \mathfrak{t}_1 \times \cdots \times \mathfrak{t}_n$ be an arbitrary Cartan subalgebra of $\mathfrak{s}^n$. Then by the Transitivity Theorem 6.27, there are elements $X_j \in \mathfrak{s}_j$ such that $e^{\operatorname{ad} X_j}\mathfrak{t}_j = \mathfrak{t}$.

Let $\gamma \in \operatorname{Aut} \mathfrak{s}^n$ be defined by $\gamma = e^{\operatorname{ad}(X_1,\ldots,X_n)}$ and let $\varphi \in F$. Then

$$(\gamma^{-1} \circ \varphi \circ \gamma)(\mathfrak{t}_1 \times \cdots \times \mathfrak{t}_n) = (\gamma^{-1} \circ \varphi)(\mathfrak{t} \times \cdots \times \mathfrak{t}) = \gamma^{-1}(\mathfrak{t} \times \cdots \times \mathfrak{t}) = \mathfrak{t}_1 \times \cdots \times \mathfrak{t}_n.$$

Hence $\gamma^{-1}F\gamma$ is a semidirect complement for N_0 respecting $\mathfrak{t}^*$. This establishes (v) and (vi).

If $\mathfrak{g}$ is not semisimple, then $\mathfrak{z}(\mathfrak{g}) \cong \mathbb{R}^m$ with $m > 0$ and $\operatorname{Gl}\big(\mathfrak{z}(\mathfrak{g})\big) = \operatorname{Gl}(m, \mathbb{R})$, and this group is not compact. Since all $(\operatorname{Aut} \mathfrak{s}_j)_0 = e^{\operatorname{ad} \mathfrak{g}_j}$ are connected compact Lie groups, all other factors are finite extensions of compact Lie groups and are therefore compact. □

Remark 6.61a. Looking at the very explicit description of the automorphism group $\operatorname{Aut}(\mathfrak{g})$ of a compact Lie algebra in Theorem 6.61 we notice that the one item that was not detailed was the group $\operatorname{Out}(\mathfrak{s})$ of outer automorphisms of a simple compact Lie group. The information on these belongs to the subject of classification of simple compact Lie algebras via their Dynkin diagrams; we do not deal with this issue here because there are numerous sources one can consult on this subject [4], [42], [43], [111], [158], [282], [296], [332], [353], [354]. The isomorphy classes of simple compact Lie algebras are in bijective correspondence with the isomorphy classes of complex simple Lie algebras ([43], p. 16ff.) and these are in bijective correspondence to isomorphy classes of root systems, and these are in bijective correspondence to Dynkin diagrams [42]. An enormous amount of cataloging and tabulating is present in this area. The group $\operatorname{Out}(\mathfrak{g})$ is isomorphic to the isomorphy group of the Dynkin diagram corresponding to $\mathfrak{g}$. These groups are either trivial, or of order two, and in one case isomorphic to the six element group S_3. Suffice it to say at this point that the entries $\operatorname{Out}(\mathfrak{s}_i)$ are quite small in general. □

The second major step in this section deals with the automorphism group of compact Lie groups. We first have to discuss function space topologies. (More of this will follow in Chapter 7 for abelian topological groups.) For a topological space K let $C(K, G)$ denote the set of all continuous functions $f\colon K \to G$. We define a topology on $C(K, G)$ as follows: Let $\mathcal{U}$ be the set of all identity neighborhoods of G. For $f_0 \in C(K, G)$ we set $W(U; f_0) = \{f \in C(K, G) \mid (\forall k \in K)\, f(k) = Uf_0(k)\}$. Let $\mathcal{O}$ be the set of all subsets $V \subseteq C(K, G)$ such that for every $f_0 \in V$ there is a $U \in \mathcal{U}$ such that $W(U; f_0) \subseteq V$.

Exercise E6.14. Prove the following statement:

$\mathcal{O}$ is a Hausdorff topology on $C(K, G)$ such that each $f \in C(K, G)$ has a neighborhood basis $\{W(U; f) \mid U \in \mathcal{U}\}$.

[Hint. In order to prove the last statement, for a given $U_0 \in \mathcal{U}$ pick a $U \in \mathcal{U}$ such that $U^2 \subseteq U$. Let $f \in W(U; f_0)$; then $f' \in W(U; f)$ implies $f'(k) \in Uf(k) \in UUf_0(k) \subseteq U_0 f_0(k)$ for all $k \in K$, whence $W(U; f) \subseteq W(U_0; f_0)$.] □

The topology we constructed is called the *topology of uniform convergence* on $C(K, G)$. Now let $K = G$.

Lemma 6.62. *For a compact group, the composition*

$$(f_1, f_2) \mapsto f_1 \circ f_2 : C(G,G) \times C(G,G) \to C(G,G)$$

is continuous with respect to the topology of uniform convergence and thus makes $C(G,G)$ into a topological semigroup.

Proof. Let $U_0 \in \mathcal{U}$ be the set of identity neighborhoods of G and let $f_j \in C(G,G)$, $j = 1,2$. Pick $U_1 \in \mathcal{U}$ such that $U_1^2 \subseteq U_0$ and select $U \in \mathcal{U}$ such that $U \subseteq U_1$ and $f_1(Ux) \subseteq U_1 f(x)$ for all $x \in G$. (This choice is possible since G is compact: f_1 is *uniformly continuous*.) Consider $f_j' \in W(U; f_j)$, $j = 1,2$, and let $g \in G$. Then $(f_1' \circ f_2')(g) = f_1'\big(f_2'(g)\big) \in f_1'\big(Uf_2(g)\big)$. Now $f_1'(h) \in Uf_1(h)$ for all $h = uf_2(g)$, $u \in U$ since $f_1' \in W(U; f_1)$. Thus $(f_1' \circ f_2')(g) \in Uf_1\big(Uf_2(g)\big) \subseteq U_1U_1f_1\big(f_2(g)\big) \subseteq U_0(f_1 \circ f_2)(g)$, whence $f_1' \circ f_2' \in W(U_0; f_1 \circ f_2)$. ☐

For a compact group we endow $\mathrm{Aut}(G)$ with the topology $\mathcal{O}_1$ induced by the topology of uniform convergence of $C(G,G)$. Consider the opposite group $\mathrm{Aut}(G)^{\mathrm{op}}$, i.e. the group defined on $\mathrm{Aut}(G)$ with multiplication $\alpha * \beta = \beta \circ \alpha$. Then $J : \mathrm{Aut}(G) \to \mathrm{Aut}(G)^{\mathrm{op}}$, $J(g) = g^{-1}$, is an isomorphism of groups. Multiplication of $\mathrm{Aut}(G)^{\mathrm{op}}$ is continuous with respect to $\mathcal{O}_1$. Let $\mathcal{O}_2$ be the unique topology on $\mathrm{Aut}(G)$ making J a homeomorphism. Now let $\mathcal{O}^\vee = \mathcal{O}_1 \vee \mathcal{O}_2$ be the common refinement of the two topologies. Then $\mathrm{Aut}\, G$ has a continuous multiplication and a continuous inversion $\alpha \mapsto \alpha^{-1}$ with respect to $\mathcal{O}^\vee$ and therefore is a topological group. We shall consider $\mathrm{Aut}\, G$ as a topological group equipped with the topology $\mathcal{O}^\vee$.

Now let G be a compact Lie group. The Lie algebra functor $\mathfrak{L}$ gives a morphism $\mathfrak{L}_G : \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$, $\mathfrak{L}_G(\alpha) = \mathfrak{L}(\alpha)$ which is uniquely determined by the validity of the equation $\alpha(\exp_G X) = \exp_G \mathfrak{L}_G(\alpha)(X)$ for all $X \in \mathfrak{g}$. Recall that $\mathrm{Aut}\,\mathfrak{g}$ is a linear Lie group (cf. 5.43) which inherits its topology from the (finite dimensional) Banach algebra $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ of all linear self-maps of $\mathfrak{g}$. In the following theorem we shall consider two technical hypotheses on G:

- (#) There is a system of coset representatives $\{1, g_1, \dots, g_m\}$ such that $G = G_0 \cup G_0 g_1 \cup \cdots \cup G_0 g_m$ and that for arbitrarily small identity neighborhoods U_1 of G the set $\mathfrak{L}_G\{\alpha \in \mathrm{Aut}(G) \mid \alpha(g_i) \in U_1 g_i,\ i = 1, \dots, m\}$ is open in $\mathrm{im}\,\mathfrak{L}_G$.
- (##) If $\pi : \widetilde{G} \to G$ is the universal covering then $\ker \pi$ is a characteristic subgroup of $Z(G)$.

Condition (#) controls the behavior of the morphism $\mathfrak{L}_G$ off the identity component. If G is connected, then it holds trivially. Condition (##) secures that an automorphism of $\widetilde{G}$ pushes down to an automorphism of G. It is automatically satisfied if G is simply connected, or if G is centerfree, or if $Z(\widetilde{G})$ is cyclic. Here, as we did in the case of Lie algebras, we call a group *centerfree* if its center is singleton.

By 6.6, the simply connected covering group of a connected compact Lie group G is isomorphic to $\widetilde{G} = \mathbb{R}^p \times S_1 \times \cdots \times S_q$ for simply connected simple compact Lie groups S_i, $i = 1, \dots, q$. If $\ker q \subseteq \mathbb{R}^p \times Z(S_1) \times \cdots \times Z(S_q) = Z(\widetilde{G})$ is characteristic in $Z(\widetilde{G})$, then necessarily $p = 0$ and thus $\widetilde{G}$ and G are semisimple. We may identify $\mathfrak{L}(\widetilde{G})$ with $\mathfrak{g}$ so that

$$\begin{array}{ccc}
\mathfrak{g} = \mathfrak{L}(\widetilde{G}) & \xrightarrow{\ \mathrm{id}_{\mathfrak{g}}\ } & \mathfrak{g} \\
{\scriptstyle \exp_{\widetilde{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_G} \\
\widetilde{G} & \xrightarrow[\pi]{\qquad} & G
\end{array}$$

commutes where π is the covering morphism. The lifting $\alpha \mapsto \widetilde{\alpha}\colon \operatorname{Aut} G \to \operatorname{Aut} \widetilde{G}$ is injective and $\mathfrak{L}_G(\alpha) = \mathfrak{L}_{\widetilde{G}}(\widetilde{\alpha})$. However, every automorphism $\varphi\colon \widetilde{G} \to \widetilde{G}$ maps the finite center $Z(\widetilde{G})$ bijectively. Hence if $\ker \pi$ is characteristic in $Z(\widetilde{G})$, then φ preserves $\ker \pi$ and thus induces an automorphism of G. Consequently (##) implies that the lifting $\alpha \mapsto \widetilde{\alpha}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(\widetilde{G})$ is bijective.

The Topological Automorphism Group of a Compact Lie Group

Theorem 6.63. *Assume that G is a compact Lie group G and consider the homomorphism of groups $\mathfrak{L}_G\colon \operatorname{Aut}(G) \to \operatorname{Aut} \mathfrak{g}$; then the following conclusions hold.*

(i) $\ker \mathfrak{L}_G = \{\alpha \in \operatorname{Aut}(G) \mid \alpha|G_0 = \mathrm{id}_{G_0}\}$ *and there is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Aut}(G) & \xrightarrow{\ \mathfrak{L}_G\ } & \operatorname{Aut} \mathfrak{g} \\
{\scriptstyle \mathrm{quot}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{incl}} \\
\operatorname{Aut}(G)/\ker \mathfrak{L}_G & \xrightarrow[\mathfrak{L}'_G]{\qquad} & \operatorname{im} \mathfrak{L}_G,
\end{array}$$

where $\mathfrak{L}'_G$ is the isomorphism of groups given by $\mathfrak{L}'_G(\alpha \ker \mathfrak{L}_G) = \mathfrak{L}_G(\alpha)$. If G is connected, $\mathfrak{L}_G$ is injective.

(ii) *The map $\mathfrak{L}_G$ is a morphism of topological groups.*

(iii) $\operatorname{im} \mathfrak{L}_G$ *is closed in* $\operatorname{Aut} \mathfrak{g}$. *In particular,* $\operatorname{im} \mathfrak{L}_G$ *is a linear Lie group.*

(iv) *If (#) holds, then $\mathfrak{L}_G$ is open onto its image, and the corestriction $\mathfrak{L}'_G$:* $\operatorname{Aut}(G) \to \operatorname{im}(\mathfrak{L}_G)$ *of $\mathfrak{L}_G$ is an isomorphism of topological groups. In particular, if G is connected, $\mathfrak{L}_G$ is an isomorphism onto its image and* $\operatorname{Aut}(G)$ *is a linear Lie group. If G is simply connected, then $\mathfrak{L}_G$ is an isomorphism of topological groups.*

(v) *If G is connected and (##) holds, then G is semisimple and the morphism* $\mathfrak{L}_G\colon \operatorname{Aut} G \to \operatorname{Aut} \mathfrak{g}$ *is an isomorphism of compact Lie groups.*

(vi) *If G is a semisimple connected compact Lie group, then the adjoint group* $\operatorname{Ad}(G) \cong G/Z(G)$, *the simply connected covering group $\widetilde{G}$, and the Lie algebra $\mathfrak{g}$ have isomorphic automorphism groups containing a closed subgroup isomorphic to the automorphism group of G, and all of these groups are compact Lie groups.*

Proof. We need some general preparation before we go into proving (i)–(iv). If U is an identity neighborhood of G, then the relation $\alpha \in W(U; \mathrm{id}_G)$ is equivalent to $\exp_G \mathfrak{L}_G(\alpha)(X) = \alpha(\exp_G X) \in U \exp_G X$ by the definition of $W(U; \mathrm{id}_G)$ and since $\exp_G: \mathfrak{g} \to G$ is surjective by the Maximal Torus Theorem 6.30. Thus

$$(*)\qquad \begin{aligned} W(U; \mathrm{id}_G) &= \{\alpha \in \mathrm{Aut}(G) \mid (\forall X \in \mathfrak{g})\ \exp_G \mathfrak{L}_G(\alpha)(X) \in U \exp_G X\} \\ &= \bigcap_{X \in \mathfrak{g}} \{\alpha \in \mathrm{Aut}(G) \mid \exp_G \mathfrak{L}_G(\alpha)(X) \in U \exp_G X\}. \end{aligned}$$

We fix once and for all an open ball B around 0 in $\mathfrak{g}$ such that $B * B$ is defined and contained in a ball B' around 0 for which $\exp|B': B' \to V$ is a homeomorphism onto an identity neighborhood of G so that B' is the 0-component of $\exp_G^{-1}(V)$ (see 5.41). If C is any ball around 0 contained in B, then $U \stackrel{\text{def}}{=} \exp_G C$ is an identity neighborhood of G and for $X \in B$ we have $U \exp_G X = \exp_G C \exp_G X = \exp_G(C * X)$. Thus $(*)$ entails

$$(\dagger)\qquad W(U; \mathrm{id}_G) \subseteq \bigcap_{X \in B} \{\alpha \in \mathrm{Aut}(G) \mid \exp_G \mathfrak{L}_G(\alpha)(X) \in \exp_G(C * X)\}.$$

If for all $X \in B$, $\exp_G \mathfrak{L}_G(\alpha)(X) \in \exp_G(C * X)$, then $\exp_G \mathfrak{L}_G(\alpha)(B) \subseteq \exp_G(C * X) \subseteq \exp_G B'$, and so $\mathfrak{L}_G(\alpha)(B)$ is in the 0-component of $\exp_G^{-1} \exp_G B'$ which is B'. Thus $\mathfrak{L}(\alpha)(B) \subseteq B'$, and then upon applying $(\exp_G |B')^{-1}$ to $\exp_G \mathfrak{L}(\alpha)(X) \in \exp_G(C * X)$, we get $\mathfrak{L}_G(\alpha)(X) \in C * X$. Conversely, this last relation in turn implies $\exp_G L_G(\alpha) \subseteq \exp_G(C * X) = U \exp_G X$. Hence

Step 1. *If $U = \exp_G C$ for any open ball around 0 in $\mathfrak{g}$ contained in B, then*

$$(\ddagger)\qquad W(U; \mathrm{id}_G) \subseteq \bigcap_{X \in B} \{\alpha \in \mathrm{Aut}(G) \mid \mathfrak{L}_G(\alpha)(X) \in C * X\}.$$

Next let U_1 be an identity neighborhood such that $U_1^2 \subseteq U = \exp_G C$ and that U_1 is invariant under all inner automorphisms (according to 1.12). Let m be such that $(\exp_B G)^m = G_0$ and choose an invariant identity neighborhood U' of G which satisfies $(U')^m \subseteq U_1$. Pick a ball C' around 0 so that $\exp C' \subseteq U'$. Now assume $(\forall X \in B)\, \mathfrak{L}_G(\alpha)(X) \in C' * X$ and let $g \in G_0$. Then there are elements $X_j \in B$, $j = 1, \ldots, m$ such that $g = \exp_G X_1 \cdots \exp_G X_m$. Now

$$\begin{aligned} \alpha(g) &= \exp_G \mathfrak{L}(\alpha)(X_1) \cdots \exp_G \mathfrak{L}(\alpha)(X_m) \subseteq \exp_C(C' * X_1) \cdots \exp_G(C' * X_m) \\ &= \exp_G C' \exp_G X_1 \cdots \exp_G C' \exp_G X_m \subseteq U' \exp_G X_1 \cdots U' \exp_G X_m \\ &= (U')^m \exp_G X_1 \cdots \exp_G X_m \subseteq U_1 g. \end{aligned}$$

Write $G = G_0 \cup G_0 g_1 \cup \ldots \cup G_0 g_k$ for a system of representatives of the cosets of G_0. If $\alpha(g_i) \in U_1 g_i$ for $i = 1, \ldots, k$ and $g \in G$, then we represent g uniquely as $g = g_0$ or $g = g_0 g_i$ with $g_0 \in G_0$ and $i \in \{1, \ldots, m\}$ and note $\alpha(g) = \alpha(g_0)\alpha(g_i) \subseteq U_1\alpha(g_0)U_1\alpha(g_i) = U_1U_1\alpha(g_0)\alpha(g_i) \subseteq U\alpha(g)$, using the invariance of U_1. Therefore, $\alpha \in W(U; \mathrm{id}_G)$. Thus

Step 2. *If $\{1, g_1, \ldots, g_m\}$ is a system of coset representatives for G/G_0 then for $U = \exp_G C$ with any open ball C around 0 in $\mathfrak{g}$ contained in B there is an open ball C' around 0 and an identity neighborhood U_1 in G such that*

$$(\dagger\dagger)\qquad \bigcap_{\substack{X \in B \\ i=1,\ldots,m}} \{\alpha \in \mathrm{Aut}(G) \mid \mathfrak{L}_G(\alpha)(X) \in C' * X \text{ and } \alpha(g_i) \in U_1 g_i\} \subseteq W(U; \mathrm{id}_G).$$

Now we shall prove assertions (i)–(iv).

Conclusion (i) is immediate: $\mathfrak{L}_G(\alpha) = \mathrm{id}_{\mathfrak{g}}$ iff $\alpha|G_0$. The remainder is simply the Canonical Decomposition of a Morphism.

Now we prove Conclusion (ii). We must show that $\mathfrak{L}_G$ is continuous at id_G. For this purpose let D be an open ball around 0. Then we let C be a ball around 0 contained in B so small that $(C * X) - X \subseteq D$ for all $X \in B$ (which is possible by the compactness of $\overline{B}$). Now we define the identity neighborhood U of G as $\exp_G C$. Then by Step 1 and (†), the relation $\alpha \in W(U; \mathrm{id}_G)$ implies $\mathfrak{L}_G(\alpha)(X) \in C * X \subseteq D + X$ for all $X \in B$. Hence $(\mathfrak{L}_G(\alpha) - \mathrm{id}_G)(X) \in D$ for all $X \in \overline{B}$. This shows that $\mathfrak{L}_G: (\mathrm{Aut}(G), \mathcal{O}_1) \to \mathrm{Aut}\,\mathfrak{g}$ is continuous. Then, a fortiori, $L_G: \mathrm{Aut}(G) \to \mathrm{Aut}\,\mathfrak{g}$ is continuous with respect to $\mathcal{O}^\vee$. Thus we see that $\mathfrak{L}_G$ is a morphism of topological groups.

Proof of (iii). Let $\varphi \in \mathrm{Aut}\,\mathfrak{g}$. Then by 5.42(iii) there are open identity neighborhoods U and V of G, respectively, and a continuous map $\Phi: U_1 \to U_2$ such that $\Phi(xy) = \Phi(x)\Phi(y)$ whenever x, y, $xy \in U_1$ and that, for appropriately chosen 0 neighborhoods B_1 and B_2 of $\mathfrak{g}$, the following diagram commutes:

$$\begin{array}{ccc} B_1 & \xrightarrow{\varphi|B_1} & B_2 \\ {\scriptstyle \exp_G|B_1}\big\downarrow & & \big\downarrow{\scriptstyle \exp_G|B_2} \\ U_1 & \xrightarrow[\Phi]{} & U_2. \end{array}$$

Now assume that $\varphi \in \overline{\mathrm{im}\,\mathfrak{L}_G}$. Then there is a sequence $\alpha_n \in \mathrm{Aut}(G)$ such that $\varphi = \lim_n \mathfrak{L}_G(\alpha_n)$ in the linear Lie group $\mathrm{Aut}\,\mathfrak{g}$, and this convergence is uniform on each ball around zero in $\mathfrak{g}$. The function $\exp_G \circ \varphi: \mathfrak{g} \to G$ satisfies $\exp_G \varphi(X) = \lim_n \exp_G \mathfrak{L}_G(\alpha_n)(X) = \lim_n \alpha_n(\exp_G X)$. For every $g \in G_0$, by the Maximal Torus Theorem 6.30, we have some $X_g \in \mathfrak{g}$ with $g = \exp_G X_g$ and the pointwise limit $\alpha(g) \overset{\mathrm{def}}{=} \lim_n \alpha_n(g)$ exists for all g, giving a function $\alpha: G_0 \to G_0$ such that $\alpha(g) = \exp_G \varphi(X_g)$ (irrespective of the selection of X_g). Since α is the pointwise limit of a sequence of automorphisms $\alpha_n|G_0$ of G, it is a group homomorphism. If $X \in B_1$ and $g = \exp_G X \in U_1$, then $\Phi(g) = \Phi(\exp_G X) = \exp_G \varphi(X) = \alpha(g)$. Thus α agrees with the continuous function Φ on U_1. Thus α is continuous at 1 and therefore is a morphisms of topological groups. The relation $\alpha \circ \exp_G = \exp_G \circ \varphi$ shows that $\varphi = \mathfrak{L}(\alpha)$.

The same arguments show that a morphism $\alpha': G \to G$ of topological groups is defined by $\alpha'(g) = \lim_n \alpha_n^{-1}(g)$, and that $\varphi^{-1} = \mathfrak{L}(\alpha')$. Then $\mathfrak{L}(\alpha \circ \alpha') = \mathfrak{L}(\alpha) \circ \mathfrak{L}(\alpha') = \varphi \circ \varphi^{-1} = \mathrm{id}_{\mathfrak{g}}$ since $\mathfrak{L}$ is a functor. Thus $\alpha \circ \alpha'$ agrees with id_G on some identity neighborhood of G and consequently on all of G_0. The same holds for $\alpha' \circ \alpha$. Thus $\alpha' = \alpha^{-1}$ and $\alpha \in \mathrm{Aut}(G_0)$. We shall now show that α is in fact the restriction to G_0 of an automorphism α^* of G. Since G is a compact space, G^G is compact in the product topology. Hence the sequence $(\alpha_n)_{n\in\mathbb{N}}$ in G^G has a convergent subnet $(\alpha_{n(j)})_{j\in J}$ in G^G. Let $\alpha^* \in G^G$ be its limit. Then α^* is a homomorphism of groups and $\alpha^*|G_0 = \alpha$. Since G_0 is open in G and α is continuous and open, α^* is continuous and open. Let us write $G = G_0 g_1 \cup \ldots \cup G_0 g_k$ with a system of representatives of the cosets of G_0. Then $\lim_{j\in J} G_0 \alpha_{n(j)}(g_m)$ exists for each $m = 1, \ldots, k$ in the finite set G/G_0 and therefore is eventually constant. Hence there is a $j_0 \in J$ such that $G_0\alpha(g_m) = G_0\alpha_{n(j)}(g_m)$ for all $m = 1, \ldots, k$ and all $j > j_0$. Take any $j > j_0$; then $G = G_0\alpha_{n(j)}(g_1) \cup \cdots \cup G_0\alpha_{n(j)}(g_k)$ since $\alpha_{n(j)}$ is an automorphism of G. Hence α^* is surjective and thus induces an automorphism of G/G_0 because this factor group is finite. Since $\alpha^*|G_0 = \alpha$ we have $\ker \alpha^* \cap G_0 = \{1\}$, and thus $\ker \alpha = \{1\}$ since α induces a bijection of G/G_0. Hence, finally, $\alpha^* \in \mathrm{Aut}(G)$ and $\mathfrak{L}_G(\alpha^*) = \mathfrak{L}(\alpha) = \varphi$. Thus $\varphi \in \mathrm{im}\,\mathfrak{L}_G$ and therefore $\mathrm{im}\,\mathfrak{L}_G$ is a closed subgroup of the linear Lie group $\mathrm{Aut}\,\mathfrak{g}$ and consequently is a linear Lie group (see 5.53(iii)).

(iv) Next we show that $\mathfrak{L}_G$ is open onto its image. Let C be any ball around 0 in G contained in B and set $U \overset{\mathrm{def}}{=} \exp_G C$. We have to show that $\mathfrak{L}_G\big(W(U;\mathrm{id}_G)\big)$ is an identity neighborhood of $\mathrm{im}\,\mathfrak{L}_G$. Now by Step 2 there is an open ball C' around 0 in $\mathfrak{g}$ and an identity neighborhood U_1 in G such that (††) holds. Let D' be an open ball around 0 so that $D' + X \subseteq C' * X$ for all $X \in \overline{B}$ (which exists by the compactness of $\overline{B}$). Now set $W' \overset{\mathrm{def}}{=} \{\varphi \in \mathrm{Aut}\,\mathfrak{g} \mid (\varphi - \mathrm{id}_{\mathfrak{g}})(B) \subseteq D'\}$; then W' is an identity neighborhood of $\mathrm{Aut}\,\mathfrak{g}$. Set $W_* \overset{\mathrm{def}}{=} \{\alpha \in \mathrm{Aut}(G) \mid \alpha(g_i) \in U_1 g_i,\ i = 1, \ldots, m\}$. For any $\varphi \in W' \cap \mathfrak{L}_G(W_*)$ there is an $\alpha \in \mathrm{Aut}(G)$ such that $\varphi = \mathfrak{L}_G(\alpha)$, that for all $X \in B$ we have $\mathfrak{L}_G(\alpha)(X) = \varphi(X) \in D' + X \subseteq C' * X$, and that $\alpha(g_i) \in U_1 g_i$ for $i = 1, \ldots, n$. Then (††) implies that $\alpha \in W(U;\mathrm{id}_G)$. This proves that $W'' \overset{\mathrm{def}}{=} W' \cap \mathfrak{L}_G(W_*) \subseteq \mathfrak{L}_G\big(W(U;\mathrm{id}_G)\big)$ and thus $W'' \cap W''^{-1} \subseteq \mathfrak{L}_G\big(W(U;\mathrm{id}_G) \cap W(U;\mathrm{id}_G)^{-1}\big)$. Now if Condition (#) holds, this establishes the claim that $\mathfrak{L}_G$ is open onto its image.

If G is connected, then (#) is satisfied and thus in view of (i), the morphism $\mathfrak{L}_G$ implements an isomorphism of topological groups $\mathrm{Aut}(G) \to \mathrm{im}(\mathfrak{L}_G)$. By Theorem 5.33(iv), every compact Lie group is a linear Lie group and thus Theorem 5.42 applies and shows that every automorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ gives rise to a local isomorphism $f\colon U_1 \to U_2$ between identity neighborhoods U_1 and U_2 of G such that there are zero neighborhoods B_1 and B_2 of $\mathfrak{g}$ satisfying $f(B_1) = B_2$ and that $\exp_G\colon B_j \to U_j$ are homeomorphisms for $j = 1, 2$ and the following diagram is commutative:

$$\begin{array}{ccc} B_1 & \xrightarrow{\ \varphi|B_1\ } & B_2 \\ {\scriptstyle \exp_G|B_1}\big\downarrow & & \big\downarrow{\scriptstyle \exp_G|B_2} \\ U_1 & \xrightarrow[\ f\ ]{} & U_2. \end{array}$$

Now assume that G is simply connected. Then by Corollary A2.26 there is a morphism $F_1\colon G \to G$ such that $F|U_1 = f$. Likewise there is a $F_2\colon G \to G$ such that $F|U_2 = f^{-1}$. Then $F_2 \circ F_1|U_1 = \mathrm{id}_{U_1}$. Since G is connected and thus generated by U_1 by Proposition 4.25(iii), we conclude that F_2 is the inverse of F_1. Then F_1 is an isomorphism and the diagram above shows that $\mathfrak{L}_G(F_1) = \mathfrak{L}(G)(F_1) = \varphi$. Therefore $\mathrm{im}\,\mathfrak{L}_G = \mathrm{Aut}\,\mathfrak{g}$ and thus $\mathfrak{L}_G$ is an isomorphism.

(v) If G is simply connected , then $\mathfrak{L}_G$ is an isomorphism of topological groups by (iv) above. If G is semisimple and connected, then $\widetilde{G}$ is compact by 5.77. Hence $\mathfrak{L}_{\widetilde{G}}\colon \mathrm{Aut}(\widetilde{G}) \to \mathrm{Aut}\,\mathfrak{g}$ is an isomorphism by what we just remarked. By the observations preceding the theorem, Condition (#) implies that the lifting $\alpha \mapsto$

$\widetilde{\alpha} : \mathrm{Aut}(G) \to \mathrm{Aut}(\widetilde{G})$ is bijective. Thus $\mathfrak{L}_G\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is bijective and hence an isomorphism of topological groups by (ii) and (iv). From Theorem 6.4(x) we know that the closed subgroup $\mathrm{Aut}\,\mathfrak{g}$ of $\mathrm{Gl}(\mathfrak{g})$ is contained in the orthogonal group $\mathrm{O}(\mathfrak{g})$ with respect to some inner product on $\mathfrak{g}$, namely, the one given by $(X \mid Y) = -\operatorname{tr}\operatorname{ad} X \operatorname{ad} Y$. Since $\mathrm{O}(\mathfrak{g})$ is compact, $\mathrm{Aut}\,\mathfrak{g}$ is compact.

(vi) If G is a semisimple compact Lie group, by what we just saw, $\mathrm{Aut}(\widetilde{G})$, $\mathrm{Aut}\big(G/Z(G)\big)$, and $\mathrm{Aut}\,\mathfrak{g}$ are isomorphic compact Lie groups, and by (i) through (iv), $\mathrm{Aut}(G)$ is isomorphic to a closed subgroup of $\mathrm{Aut}(\mathfrak{g})$ and therefore is a compact Lie group. □

We reiterate what the preceding theorem achieves: $\mathrm{Aut}\,\mathfrak{g}$ is a subgroup of the group of units of a finite dimensional Banach algebra and is a matrix group, inheriting its topology and its linear structure from $\mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \cong M_n(\mathbb{R})$, $n = \dim \mathfrak{g}$. Chapter 5 said a lot about linear Lie groups of the form $\mathrm{Aut}\,\mathfrak{g}$ in general and Theorem 6.61 gives us detailed information about this group if $\mathfrak{g}$ is a compact Lie algebra. But $\mathrm{Aut}(G)$ is a function space with a function space topology. The preceding theorem links the two and establishes the fact that, at least for connected compact Lie groups G, the topological group $\mathrm{Aut}(G)$ can be regarded as a closed subgroup of $\mathrm{Aut}\,\mathfrak{g}$. One should not think, however, that the linear Lie group $\mathrm{Aut}(G)$ is compact in general as we shall observe in the following exercise.

Exercise E6.15. Prove the following statements:

(i) *If* $G = \mathbb{T}^p$, *then* $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) = \mathbb{R}^p$ *and* $\mathrm{Aut}(G) \cong \mathrm{Aut}(\widehat{G}) \cong \mathrm{Aut}(\mathbb{Z}^p) = \mathrm{Gl}(p,\mathbb{Z})$ *with the discrete topology.*

(ii) *If* G *is connected and semisimple,* $\mathrm{Aut}(G)$ *is compact; e.g. if* $G = \mathrm{SO}(3)$, *then* $\widetilde{G} \cong \mathrm{SU}(2)$ *and* $\mathfrak{g} \cong (\mathbb{R}^3, \times)$ *where* $\times$ *is the vector product. Then* $\mathrm{Aut}(G) \cong \mathrm{Aut}(\widetilde{G}) \cong \mathrm{Aut}\,\mathfrak{g} = \mathrm{SO}(3)$.

[Hint. (i) Use information from Chapters 1 and 2, on the duality of compact and discrete abelian groups to verify that $f \mapsto \widehat{f}\colon \mathrm{Hom}(G,G) \to \mathrm{Hom}(\widehat{G},\widehat{G})$, $\widehat{f}(\chi) = \chi \circ f$, is a bijective morphism of abelian groups mapping $\mathrm{Aut}(G)$ isomorphically onto $\mathrm{Aut}\,\widehat{G}$. The topology of uniform convergence of $\mathrm{Hom}(G,G)$ is discrete, since G has an identity neighborhood which contains no subgroup except the singleton one. (In a systematic way, we shall investigate such matters in Chapter 7 where we deal with the duality of locally compact abelian groups.) The discrete group $\mathrm{Gl}(p,\mathbb{Z})$ is compact for $p < 2$ only.

(ii) Apply 6.63(vi).] □

Condition (#) is rather technical and may be difficult to verify in special instances. At the moment, the best we have in terms of sufficient conditions is that G is connected. We do need more information on the structure of $\mathrm{Aut}(G)$.

For a compact group let $\iota\colon G \to \mathrm{Inn}(G)$ be the morphism given by $\iota(g) = I_g$. Its kernel is the center $Z(G)$ of G and thus it induces an isomorphism of compact groups $G/Z(G) \to \mathrm{Inn}(G)$. Note that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ is represented in the form $\mathrm{Ad} = \mathfrak{L}_G \circ \iota$.

The connected open subgroup $G_0Z(G)/Z(G)$ of $G/Z(G)$ is the identity component $\big(G/Z(G)\big)_0$. Hence $\iota(G_0) = \mathrm{Inn}(G)_0$. We write

$$\mathrm{Inn}_0(G) = \iota(G_0) = \mathrm{Inn}(G)_0.$$

Lemma 6.64. *The restriction morphism* $\rho_G\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G_0)$, $\rho(\alpha)\alpha|G_0$ *and the induced mapping morphism* $\sigma_G\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/G_0)$, $\sigma_0(\alpha)(gG_0) = \alpha(g)G_0$ *are continuous and the following conclusions hold:*

(i) $\ker \rho_G = \{\alpha \in \mathrm{Aut}(G) \mid \alpha|G_0 = \mathrm{id}_{G_0}\}$, *and*
$\ker \sigma_G = \{\alpha \in \mathrm{Aut}(G) \mid (\forall g \in G)\, \alpha(g) \in gG_0\}$.

(ii) $\mathrm{Inn}_0(G) \subseteq \mathrm{Aut}(G)_0 \subseteq \ker \sigma_G$.

(iii) ρ_G *induces an exact sequence*

$$\{1\} \to Z(G_0)/\big(Z(G_0) \cap Z(G)\big) \to \mathrm{Inn}_0(G) \xrightarrow{\rho_G|\,\mathrm{Inn}_0(G)} \mathrm{Inn}(G_0) \to \{1\}.$$

Proof. The continuity of ρ_G and σ_G follows directly from the definition of the topology of uniform convergence and the topology defined on the automorphism groups. Conclusion (i) is straightforward from the definition of ρ_G and σ_G. For a proof of the exactness of (ii) note that due to the continuity of σ_G we have $\sigma_G(\mathrm{Aut}(G)_0) \subseteq \mathrm{Aut}(G/G_0)_0 = \{G_0\}$. Thus $\mathrm{Aut}(G)_0 \subseteq \ker \sigma_G$. Next we prove (iii). Let $\alpha \in \ker \rho_G|\,\mathrm{Inn}_0(G)$. Then there is a $g \in G_0$ such that $I_g|G_0 = \mathrm{id}_{G_0}$. This means $g \in Z(G_0)$. Thus $\ker(\rho_G|\,\mathrm{Inn}_0(G)) = \iota\big(Z(G_0)\big) \cong Z(G_0)Z(G)/Z(G) \cong Z(G_0)/\big(Z(G_0) \cap Z(G)\big)$. On the other hand, let $\beta \in \mathrm{Inn}(G_0)$. Then there is a $g \in G_0$ such that $\beta(x) = gxg^{-1}$ for all $x \in G_0$. Set $\alpha = I_g = \iota(g) \in \mathrm{Aut}(G)$. Then $\alpha \in \mathrm{Inn}_0(G)$ by the definition of $\mathrm{Inn}_0(G)$ and $\rho_G(\alpha) = \alpha|G_0 = \beta$. ☐

The next theorem will actually show that $\mathrm{Aut}(G)_0 = \mathrm{Inn}_0(G)$ and, accordingly, $\mathrm{Aut}(G_0)_0 = \mathrm{Inn}(G_0)$.

Remarkably, in its proof we shall (again) be confronted with the need to argue at one point along the lines of homological algebra. We shall therefore first recall some concepts and establish a lemma. If a group Γ acts on an additively written abelian topological group A via $(\xi, a) \mapsto \xi{\cdot}a : \Gamma \times A \to A$ such that $a \mapsto \xi{\cdot}a$ is an automorphism of the topological group A we shall say that A is a Γ-*module.* A function $\Phi\colon \Gamma \to A$ is called a 1-*cocycle* if it is continuous and satisfies

(‡) $(\forall \xi, \eta \in \Gamma) \quad \Phi(\xi\eta) = \Phi(\xi) + \xi{\cdot}\Phi(\eta)$

(cf. definition preceding 6.37). It is called a 1-*coboundary* if

(‡‡) $(\exists a \in A)(\forall \xi \in \Gamma) \quad \Phi(\xi) = \xi{\cdot}a - a.$

A quick calculation shows that every 1-coboundary is a 1-cocycle. Let $Z^1 = Z^1(\Gamma, A)$ be the additive group under pointwise addition of all 1-cocycles and $B^1 = B^1(\Gamma, A)$ the subgroup of all 1-coboundary. We say that the quotient group $H^1 = H^1(\Gamma, A)$ is *the first cohomology group of the* Γ-*module* A.

Lemma 6.65. *Let A be an additively written compact abelian Lie group, which is a Γ-module for a finite group Γ. Then its first cohomology group $H^1(\Gamma, A)$ is finite.*

Proof. For each $\Phi \in Z^1$ we define an element $z_\Phi \in A$ by

$$-z_\Phi \stackrel{\text{def}}{=} \sum_{\xi\in\Gamma} \Phi(\xi).$$

Forming in the equation (‡) on both sides the sum over all η we get for each $\xi \in \Gamma$ the equation $-z_\Phi = N{\cdot}\Phi(\xi) - \xi{\cdot}z_\Phi$, $N = \operatorname{card}\Gamma$; i.e.

$$(\forall \xi \in \Gamma) \quad N{\cdot}\Phi(\xi) = -z_\Phi + \xi{\cdot}z_\Phi = \xi{\cdot}z_\Phi - z_\Phi.$$

Thus $N{\cdot}Z^1 \subseteq B^1$; hence $N{\cdot}H^1 = \{0\}$, i.e. the order of every element in H^1 divides N; we will briefly say that H^1 *has exponent dividing* N.

The finite product A^Γ is a compact abelian Lie group; the subgroup Z^1 is a closed subgroup in view of the definition of cocycles in (‡) above. Hence Z^1 is a compact abelian Lie group. Every compact abelian Lie group is of the form $\mathbb{T}^r \oplus E$ with a finite abelian group E (see 2.42(i)). Let $\operatorname{div}(Z^1)$ be the largest divisible subgroup of Z^1 (cf. Appendix 1, A1.31). Since $\operatorname{div}(\mathbb{T}^r \oplus E) = \mathbb{T}^r$, it follows that $Z^1/\operatorname{div}(Z^1)$ is finite. Since H^1 has exponent N we conclude $\operatorname{div}(H^1) = \{1\}$. The surjective homomorphism $Z^1 \to H^1$ therefore annihilates $\operatorname{div}(Z^1)$ and thus induces a surjective morphism from the finite abelian group $Z^1/\operatorname{div}(Z^1)$ onto H^1. Hence H^1 is finite as asserted. □

Iwasawa's Automorphism Group Theorem for Lie Groups

Theorem 6.66. *Let G be a compact Lie group. Then*

(i) $\operatorname{Aut}(G)_0 = \operatorname{Inn}_0(G)$ (Iwasawa [217]).

(ii) $\operatorname{Inn}_0(G)$ *is a compact Lie group isomorphic to* $G_0/(Z(G)\cap G_0)$. *If* $Z_0(G_0) \subseteq Z(G)$, *which is trivially the case if G is connected, then* $\operatorname{Inn}_0(G)$ *is isomorphic to* $(G_0)'/\big(Z(G)\cap(G_0)'\big)$ *and is, therefore, a semisimple connected compact Lie group. If G is connected,* $\operatorname{Inn}_0(G) \cong G'/(Z(G)\cap G')$.

Proof. Let us prove the simple observation (ii) first. By definition of ι and $\operatorname{Inn}_0(G)$, the kernel of ι is $Z(G)$, and the group $\operatorname{Inn}_0(G)$ is the image $\iota(G_0)$ of the compact Lie group G_0 under the morphism ι; it is, therefore, a compact Lie group by 6.7. Now $G_0 = Z_0(G_0)(G_0)'$ by 6.16. Thus if we assume that $Z_0(G_0) \subseteq Z(G) = \ker\iota$ then $\operatorname{Inn}_0(G) = \iota(G_0) = \iota\big((G_0)'\big)(G_0)'Z(G)/Z(G) \cong (G_0)'/\big((G_0)'\cap Z(G)\big)$, as asserted. Since this group agrees with its commutator group, by 6.16, it is semisimple. If G is connected, then $G = G_0$ and the notation simplifies.

Now we shall prove (i) in several steps.

Step 1. Assume that G is connected and semisimple. By 6.5(iv), $\mathfrak{L}(\operatorname{Aut}\mathfrak{g}) = \operatorname{ad}\mathfrak{g}$, whence

$$\begin{aligned}(\operatorname{Aut}\mathfrak{g})_0 &= \exp_{\operatorname{Aut}\mathfrak{g}}\mathfrak{L}(\operatorname{Aut}\mathfrak{g}) \quad \text{since } \operatorname{Aut}\mathfrak{g} \text{ is compact by 6.58(vi)}\\ &= e^{\operatorname{ad}\mathfrak{g}} = \operatorname{Ad}(\exp_G \mathfrak{g}') = \operatorname{Ad}(G) \quad \text{(by 5.44).}\end{aligned}$$

Thus, if G is connected and semisimple,

$$(*) \qquad (\operatorname{Aut}\mathfrak{g})_0 = \operatorname{Ad}(G).$$

Further $\operatorname{Inn}(G) \subseteq \operatorname{Aut}(G)_0$ whence $\mathfrak{L}_G\big(\operatorname{Inn}(G)\big) \subseteq \mathfrak{L}_G(\big(\operatorname{Aut}(G)\big)_0) \subseteq (\operatorname{Aut}\mathfrak{g})_0 = \operatorname{Ad}_g(G)$ by $(*)$. Since $\operatorname{Ad}_{\mathfrak{g}}(G) = \mathfrak{L}_G\big(\iota(G)\big) = \mathfrak{L}_G\big(\operatorname{Inn}(G)\big)$, we have $\mathfrak{L}_G\big(\operatorname{Inn}(G)\big) = \mathfrak{L}_G(\operatorname{Aut}(G)_0)$, and since $\mathfrak{L}_G$ is an embedding if G is connected, we get

$$(**) \qquad \operatorname{Inn}(G) = \operatorname{Aut}(G)_0 \quad \text{and} \quad \mathfrak{L}_G\big(\operatorname{Inn}(G)\big) = \big(\operatorname{Aut}\mathfrak{g}\big)_0.$$

Step 2. Assume that G is connected. Since $\mathfrak{g}$ is a finite dimensional real Hilbert Lie algebra we know from Lemma 6.57 that $\eta\colon \operatorname{Aut}\mathfrak{g} \to \operatorname{Aut}\mathfrak{z}(\mathfrak{g}) \times \operatorname{Aut}\mathfrak{g}'$, $\eta(\alpha) = (\alpha|\mathfrak{z}(\mathfrak{g}), \alpha|\mathfrak{g}')$ is an isomorphism whose inverse is given by $\eta^{-1}(\alpha,\beta)(X_{\mathfrak{z}} + X_{\mathfrak{g}'}) = \alpha(X_{\mathfrak{z}}) + \beta(X'_{\mathfrak{g}})$. Hence $\operatorname{Aut}\mathfrak{g} \cong \operatorname{Gl}(\mathfrak{z}) \times \operatorname{Aut}\mathfrak{g}'$. We note that η embeds $\mathfrak{L}_G\big(\operatorname{Aut}(G)\big)$ into

$$\mathfrak{L}_G\big(\operatorname{Aut}(G)\big)|\mathfrak{z}(\mathfrak{g}) \times \mathfrak{L}_G\big(\operatorname{Aut}(G)\big)|\mathfrak{g}'.$$

If we have a characteristic connected compact subgroup H of G such as $H = Z_0(G)$ or $H = G'$, then by 6.4 we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Aut}(G) & \xrightarrow{\alpha\mapsto\alpha|H} & \operatorname{Aut}(H) \\ {\scriptstyle \mathfrak{L}_G}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{L}_H} \\ \operatorname{Aut}(\mathfrak{g}) & \xrightarrow[\alpha\mapsto\alpha|\mathfrak{g}]{} & \operatorname{Aut}(\mathfrak{h}). \end{array}$$

In particular, $\mathfrak{L}_G\big(\operatorname{Aut}(G)\big)|\mathfrak{h} = \mathfrak{L}_H\big(\operatorname{Aut}(G)|H\big) \subseteq \operatorname{Aut}(\mathfrak{h})$. Hence η gives an embedding

$$\begin{aligned}\mathfrak{L}_G\big(\operatorname{Aut}(G)\big) &\to \mathfrak{L}_{Z_0(G)}\big(\operatorname{Aut}(G)|Z_0(G)\big) \times \mathfrak{L}_{G'}\big(\operatorname{Aut}(G)|G'\big)\\ &\subseteq \mathfrak{L}_{Z_0(G)}\big(\operatorname{Aut}(Z_0(G))\big) \times \mathfrak{L}_{G'}\big(\operatorname{Aut}(G')\big).\end{aligned}$$

Now $\operatorname{Aut}\big(Z_0(G)\big) \cong \operatorname{Gl}(p,\mathbb{Z})$ with $p = \dim\mathfrak{z}$; this is a discrete group. By Step 1 we have $\operatorname{Aut}(G')_0 = \operatorname{Inn}(G')$. Further

$$\mathfrak{L}_G\big(\operatorname{Inn}(G)\big)|\mathfrak{z}(\mathfrak{g}) = \{\operatorname{id}_{\mathfrak{z}(\mathfrak{g})}\} \quad \text{and} \quad \mathfrak{L}_G\big(\operatorname{Inn}(G)\big)|\mathfrak{g}' = \mathfrak{L}_{G'}\big(\operatorname{Inn}(G')\big).$$

Thus

$$\begin{aligned}\eta\big(\mathfrak{L}_G(\mathrm{Inn}(G))\big) &= \{\mathrm{id}_{\mathfrak{z}(\mathfrak{g})}\} \times \mathfrak{L}_{G'}\big(\mathrm{Inn}(G')\big)\\ &= \{\mathrm{id}_{\mathfrak{z}(\mathfrak{g})}\} \times (\mathrm{Aut}\,\mathfrak{g}')_0 \quad \text{by } (**) \text{ above}\\ &\supseteq \{\mathrm{id}_{\mathfrak{z}(\mathfrak{g})}\} \times \mathfrak{L}_{G'}(\mathrm{Aut}(G')_0)|\mathfrak{g}'\\ &= \mathfrak{L}_{Z_0(G)}\big(\mathrm{Aut}(G)_0|Z_0(G)\big) \times \mathfrak{L}_{G'}(\mathrm{Aut}(G')_0)|\mathfrak{g}'\\ &\qquad \text{(since Aut}\,\big(Z_0(G)\big) \text{ is discrete)}\\ &= \mathfrak{L}_G\big(\mathrm{Aut}(G)\big)|\mathfrak{z}(\mathfrak{g}) \times \mathfrak{L}_{G'}(\mathrm{Aut}(G')_0)|\mathfrak{g}' \supseteq \eta\big(\mathfrak{L}_G(\mathrm{Aut}(G)_0)\big).\end{aligned}$$

Since $\eta \circ \mathfrak{L}_G$ is an embedding, $\mathrm{Inn}(G) = \mathrm{Aut}(G)_0$ follows.

Step 3. In the terminology of Lemma 6.64 we write

$$\begin{aligned}\mathrm{Aut}^*(G) &= \ker \rho_G \cap \ker \sigma_G\\ &= \{\alpha \in \mathrm{Aut}(G) \mid (\forall g \in G_0)\, \alpha(g) = g \text{ and } (\forall g \in G)\, \alpha(g) \in gG_0\}.\end{aligned}$$

From Lemma 6.64(iii) and Step 2 above we have exact sequences

$$\{1\} \to \mathrm{Aut}^*(G) \cap \mathrm{Aut}(G)_0 \xrightarrow{\mathrm{incl}} \mathrm{Aut}(G)_0 \xrightarrow{\rho_G|\,\mathrm{Aut}(G)_0} \mathrm{Inn}(G_0) \to \{1\},$$

$$\{1\} \to Z(G_0) \cap Z(G) \xrightarrow{\mathrm{incl}} \mathrm{Inn}_0(G) \xrightarrow{\rho_G|\,\mathrm{Inn}_0(G)} \mathrm{Inn}(G_0) \to \{1\}.$$

It follows that $\mathrm{Aut}(G)_0 = \mathrm{Inn}_0(G)\cdot(\mathrm{Aut}^*(G) \cap \mathrm{Aut}(G)_0\big)$. The left side is connected, thus in order to show (i) it suffices to show that $\mathrm{Inn}_0(G)$ is the identity component of the right hand side. This is the case iff $\mathrm{Aut}^*(G) \cap \mathrm{Inn}_0(G)$ is open in $\mathrm{Aut}^*(G) \cap \mathrm{Aut}(G)_0$, i.e. iff $\big(\mathrm{Aut}^*(G) \cap \mathrm{Aut}(G)_0\big)/\big(\mathrm{Aut}^*(G) \cap \mathrm{Inn}_0(G)\big)$ is discrete. Therefore, in order to prove (i) it now suffices to show that $\mathrm{Aut}^*(G)/\big(\mathrm{Aut}^*(G) \cap \mathrm{Inn}_0(G)\big)$ is finite, hence discrete. This we will show in the remainder of the proof.

Step 4. From here on it is convenient to abbreviate the finite group G/G_0 by Γ and the abelian group $Z(G_0)$ by A. For any $\alpha \in \mathrm{Aut}^*(G)$ and $g' \in gG_0 = G_0g$ write $g' = gg_0$ for some $g_0 \in G_0$. Then $\alpha(g') = \alpha(g)\alpha(g_0) = \alpha(g)g_0$ and the element $\varphi(g) \overset{\mathrm{def}}{=} g\alpha(g)^{-1}$ is contained in G_0. Thus $\varphi(g') = gg_0g_0^{-1}\alpha(g)^{-1} = g\alpha(g)^{-1} = \varphi(g)$. Hence we get a function $\Phi_\alpha\colon \Gamma \to G_0$ such that, writing $\pi\colon G \to \Gamma$ for the quotient morphism, we have $\varphi = \Phi_\alpha \circ \pi$ and $\Phi_\alpha\big(\pi(g)\big) = g\alpha(g)^{-1}$, i.e. $g = \Phi_\alpha\big(\pi(g)\big)\alpha(g)$. Then $\Phi_\alpha\big(\pi(gh)\big)\alpha(gh) = gh = \Phi_\alpha\big(\pi(g)\big)\alpha(g)\Phi_\alpha\big(\pi(h)\big)\alpha(h)$ whence

$$(\dagger) \qquad (\forall g,\, h \in G) \quad \Phi_\alpha\big(\pi(gh)\big) = \Phi_\alpha\big(\pi(g)\big)\big(\alpha(g)\Phi_\alpha\big(\pi(h)\big)\alpha(g)^{-1}\big).$$

In particular, for all $\xi \in \Gamma$ and $x \in G_0$, we compute

$$\begin{aligned}\alpha(g)x\Phi_\alpha(\xi)x^{-1}\alpha(g)^{-1} &= \alpha(gx)\Phi_\alpha(\xi)\alpha(gx)^{-1}\\ &= \Phi_\alpha\big(\pi(gx)\big)^{-1}\Phi_\alpha(\pi(g)\xi) = \Phi_\alpha\big(\pi(g)\big)^{-1}\Phi_\alpha(\pi(g)\xi)\\ &= \alpha(g)\Phi_\alpha(\xi)\alpha(g)^{-1}\end{aligned}$$

whence

$$(\forall \xi \in \Gamma,\, x \in G_0)\, x\Phi_\alpha(\xi) = \Phi_\alpha(\xi)x.$$

Thus $\operatorname{im}\Phi_\alpha \subseteq Z(G_0) = A$, and since $\alpha(g) = gg_0$ for some $g_0 \in G_0$ we have $\alpha(g)\Phi_\alpha(\xi)\alpha(g)^{-1} = g\Phi_\alpha(\xi)g^{-1}$, and this element depends (for fixed ξ) on gG_0 only. Thus for $z \in Z(G_0)$ and $\xi = gG_0$ we set $I_\xi(z) = gzg^{-1}$, unambiguously. Then $\Phi = \Phi_\alpha$ satisfies the following condition

$$(\ddagger) \qquad (\forall \xi, \eta \in \Gamma)\, \Phi(\xi\eta) = \Phi(\xi) I_\xi\big(\Phi(\eta)\big),$$

i.e. Φ_α is a 1-cocycle. If $\alpha \in \operatorname{Aut}^*(G) \cap \operatorname{Inn}_0(G)$, then $\alpha = I_z$ for some $z \in G_0$, and since $I_z|G_0 = \mathrm{id}_{G_0}$ iff $z \in A$, we have $\Phi_\alpha(\xi) = g\alpha(g)^{-1} = gzg^{-1}z^{-1}$. Under these circumstances

$$(\ddagger\ddagger) \qquad (\exists z \in A)(\forall \xi \in \Gamma) \quad \Phi_\alpha(\xi) = I_\xi(z)z^{-1},$$

i.e. Φ_α is a 1-coboundary. Since $g = \Phi_\alpha\big(\pi(g)\big)\alpha(g)$ for all $\alpha \in \operatorname{Aut}^*(G)$, the function

$$\alpha \mapsto \Phi_\alpha\colon \operatorname{Aut}^*(G) \to Z^1\big(\Gamma, A\big)$$

is a bijection mapping $\operatorname{Aut}^*(G) \cap \operatorname{Inn}_0(G)$ to $B^1\big(\Gamma, A\big)$. Thus we have an isomorphism

$$\operatorname{Aut}^*(G)/(\operatorname{Aut}^*(G) \cap \operatorname{Inn}_0(G)\big) \cong \frac{Z^1\big(\Gamma, A\big)}{B^1\big(\Gamma, A\big)} = H^1(\Gamma, A\big).$$

Therefore, we have to argue that $H^1\big(\Gamma, A\big)$ is finite. But that is the content of Lemma 6.65. The application of this lemma completes the proof the theorem. $\square$

Note that in the group $G = \mathrm{O}(2)$, the semidirect product of the circle group $\mathrm{SO}(2)$ with the two element group $\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$, we have $Z_0(G_0) = G_0 = \mathrm{SO}(2)$, but $Z(G) = \left\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Thus $Z_0(G_0) \not\subseteq Z(G)$ and indeed $\operatorname{Aut}(G)_0 = \operatorname{Inn}_0(G)$ is a circle group.

It may serve a useful purpose to depict some of the groups we discussed in a diagram. Recall $\rho_G\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G_0)$, $\rho_G(\alpha) = \alpha|G_0$ and $\sigma_G\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/G_0)$, $\sigma(\alpha)(gG_0) = \alpha(g)G_0$. We make the following definitions.

$$\begin{aligned}
\operatorname{Aut}^{\#}(G) &\stackrel{\text{def}}{=} \ker \rho_G = \{\alpha \in \operatorname{Aut}(G) \mid \alpha|G_0 = \mathrm{id}_{G_0}\},\\
\operatorname{Aut}^*(G) &\stackrel{\text{def}}{=} \ker \rho_G \cap \ker \sigma_G = \{\alpha \in \operatorname{Aut}^{\#}(G) \mid (\forall g \in G)\, \alpha(g) \in gG_0\},\\
G^1 &\stackrel{\text{def}}{=} \ker \rho_G/(\ker \rho_G \cap \ker \sigma_G) \xrightarrow{\text{bij}} \sigma_G(\ker \rho) \subseteq \operatorname{Aut}(G/G_0) \text{ finite},\\
H^1 &= H^1\big(G/G_0, Z(G_0)\big) \text{ finite.}
\end{aligned}$$

6.67. Diagram. There is a commutative diagram:

$$
\begin{array}{ccccccc}
 & & & & \mathrm{Aut}(\widetilde{G_0}) & \xrightarrow[\cong]{\mathfrak{L}_{\widetilde{G_0}}} & \mathrm{Aut}\,\mathfrak{g} \\
 & & & & \widetilde{\bullet}\big\uparrow & & \big\uparrow\mathrm{incl} \\
 & & & & \mathrm{Aut}(G_0) & \xrightarrow[\cong]{\mathfrak{L}_{G_0}} & \mathrm{im}\,\mathfrak{L}_{G_0} \\
 & & & & \mathrm{incl}\big\uparrow & & \big\uparrow\mathrm{incl} \\
\mathrm{Aut}^{\#}(G) & \xrightarrow{\mathrm{incl}} & \mathrm{Aut}(G) & \xrightarrow{\rho_G} & \mathrm{Aut}(G)|G_0 & \xrightarrow[\cong]{\mathfrak{L}_G|\rho_G(\mathrm{Aut}\,G)} & \mathrm{im}\,\mathfrak{L}_G \\
G^1\Big\{\big\uparrow & & \mathrm{incl}\big\uparrow & & \mathrm{incl}\big\uparrow & & \big\uparrow\mathrm{incl} \\
\mathrm{Aut}^*(G) & & \mathrm{Inn}(G) & \xrightarrow[\rho_G|\,\mathrm{Inn}(G)]{} & \mathrm{Inn}(G)|G_0 & \xrightarrow[\cong]{\mathfrak{L}_{G_0}|\rho_G(\mathrm{Inn}\,G)} & \mathrm{Ad}(G) \\
H^1\Big\{\big\uparrow & & \big\uparrow\mathrm{incl} & & \mathrm{incl}\big\uparrow & & \big\uparrow\mathrm{incl} \\
\mathrm{Aut}^*(G)\cap\mathrm{Inn}_0(G) & \xrightarrow[\mathrm{incl}]{} & \begin{matrix}\mathrm{Inn}_0(G)=\\ \mathrm{Aut}(G)_0\end{matrix} & \xrightarrow[\rho_G|\,\mathrm{Inn}_0(G)]{} & \mathrm{Inn}(G_0) & \xrightarrow[\cong]{\mathfrak{L}_{G_0}|\,\mathrm{Inn}(G_0)} & \begin{matrix}\mathrm{Ad}(G_0)=\\ e^{\mathrm{ad}\,\mathfrak{g}}=e^{\mathrm{ad}\,\mathfrak{g}'}\end{matrix}
\end{array}
$$

From 6.61 we recall $\mathrm{Aut}\,\mathfrak{g} \cong e^{\mathrm{ad}\,\mathfrak{g}'} \rtimes \big(\mathrm{Gl}\big(\mathfrak{z}(\mathfrak{g})\big) \times \mathrm{Out}\,\mathfrak{g}\big)$. Let us write $\mathrm{Aut}\,\mathfrak{g} = e^{\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}'}D$ with a semidirect cofactor $\cong \mathrm{Gl}\big(\mathfrak{z}(\mathfrak{g})\big) \times \mathrm{Out}\,\mathfrak{g}$.

Lemma 6.68. *Let P be a semidirect product of a normal subgroup N and a subgroup H and let $f\colon \Gamma \to P$ be a group homomorphism. Assume that M is a normal subgroup of Γ such that $N = f(M)$. Set $S = f^{-1}(H)$. Then $G = MS$ and $M \cap S \subseteq \ker f \subseteq f^{-1}(N) \cap S$.*

Proof. Let $g \in G$. Then there are $n \in N$ and an $h \in H$ such that $f(g) = nh$. Since $N = f(M)$ there is an $m \in M$ such that $n = f(m)$. Set $s = m^{-1}g$. Then $f(s) = f(m^{-1}g) = n^{-1}nh = h \in H$, i.e. $s \in S$. Hence $G = MS$. Now let $d \in M \cap S$. Then $f(d) \in f(M) \cap f(S) \subseteq N \cap H = \{1\}$ whence $d \in \ker f$. Conversely, $d \in \ker f$ implies $d \in f^{-1}(\{1\}) = f^{-1}(N \cap H) = f^{-1}(N) \cap f^{-1}(H)$. ☐

Corollary 6.69. (i) *For a compact Lie group G, the compact Lie subgroup $\mathrm{Aut}(G)_0 = \mathrm{Inn}_0(G)$ of $\mathrm{Aut}(G)$ is open and is isomorphic to $G_0/\big(G_0 \cap Z(G)\big)$.*

(ii) *The subgroup $\mathrm{Aut}(G)_0 \cdot \mathrm{Aut}^{\#}(G)$ is a compact open normal Lie subgroup of $\mathrm{Aut}(G)$ and*

$$\mathrm{Aut}(G)/\big(\mathrm{Aut}(G)_0 \cdot \mathrm{Aut}^{\#}(G)\big) \cong (\mathrm{Aut}(G)|G_0)/\,\mathrm{Inn}(G_0) \cong \mathrm{im}\,\mathfrak{L}_G/\,\mathrm{Ad}(G_0).$$

(iii) *The group $\mathrm{Aut}(G)$ contains a closed subgroup S such that $\mathrm{Aut}(G) = \mathrm{Inn}_0(G)S$ and $\mathrm{Inn}_0(G) \cap S \subseteq \mathrm{Aut}^{\#}(G) \subseteq \big(\mathrm{Inn}_0(G)\,\mathrm{Aut}^{\#}(G)\big) \cap S$.*

Proof. We first prove (i) and (ii). From Theorem 6.63 we know that $\mathrm{Aut}(G_0)$ is a linear Lie group, and from Theorem 6.66 that $\mathrm{Inn}(G_0)$ is its identity component, which is therefore open in $\mathrm{Aut}(G_0)$. This implies that $\rho_G^{-1}\big(\mathrm{Inn}(G_0)\big)$ is open in $\mathrm{Aut}(G)$. In order to show that $\mathrm{Inn}_0(G)$ is open in $\mathrm{Aut}(G)$ it therefore suffices to show that it is open in $\rho_G^{-1}\big(\mathrm{Inn}(G_0)\big)$. Since $\rho_G\big(\mathrm{Inn}_0(G)\big) = \mathrm{Inn}(G_0)$, we have

$$\rho_G^{-1}\big(\mathrm{Inn}(G_0)\big) = \mathrm{Inn}_0(G)\cdot \ker\rho_G = \mathrm{Inn}_0(G)\cdot \mathrm{Aut}^{\#}(G).$$

Now

$$\begin{aligned}\mathrm{Inn}_0(G)\cdot \mathrm{Aut}^{\#}(G)/\mathrm{Inn}_0(G) &= \mathrm{Aut}^{\#}(G)/\big(\mathrm{Aut}^{\#}(G)\cap \mathrm{Inn}_0(G)\big)\\ &= \mathrm{Aut}^{\#}(G)/\big(\mathrm{Aut}^{*}(G)\cap \mathrm{Inn}_0(G)\big).\end{aligned}$$

As is seen from Diagram 6.67, this group is an extension of the finite group H^1 by the finite group G^1 and therefore is finite. It follows that $\rho_G^{-1}\big(\mathrm{Inn}(G_0)\big)$ has $\mathrm{Inn}_0(G)$ as a compact closed normal subgroup of finite index and is, therefore a compact group in which $\mathrm{Inn}_0(G)$ is open. Moreover, $\mathrm{Aut}(G)/\rho^{-1}\big(\mathrm{Inn}(G_0)\big) \cong \rho_G\big(\mathrm{Aut}(G)\big)/\mathrm{Inn}(G_0)$.

Finally we prove (iii). We apply Lemma 6.68 with $\Gamma = \mathrm{Aut}(G)$, $P = \mathrm{Aut}(\widetilde{G_0})$, $N = \mathrm{Inn}(G_0)$, and $f = \rho_G$. Indeed since $\mathrm{Aut}\,\mathfrak{g} = \mathrm{Inn}\,\mathfrak{g} \rtimes \mathrm{Out}\,\mathfrak{g}$ by Theorem 6.61(iii), in view of Diagram 6.67 above we also have $P = \mathrm{Aut}(\widetilde{G_0}) = \mathrm{Inn}(G_0)\rtimes H$ with a suitable subgroup H of P. We let $S = \rho_G^{-1}\,\mathrm{Inn}(G_0)$. The final assertion of the corollary now follows from Lemma 6.68 in view of the fact that $\rho_G\big(\mathrm{Inn}_0(G)\big) = \mathrm{Inn}(G_0)$ implies $\rho_G^{-1}\big(\mathrm{Inn}_0(G)\big) = \mathrm{Inn}_0(G)\ker\rho_G$. □

Note that $\mathrm{Aut}(G)$ is a Lie group since $\mathrm{Aut}(G)_0$ is a Lie group (see Postscript to Chapter 5).

Definition 6.70. Let G be a compact group and T a maximal torus. Define $\mathrm{Aut}_T(G)$ to be the set of all $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(T) = T$. Let C be a Weyl chamber of $\mathfrak{t} = \mathfrak{L}(T)$. Set $\mathrm{Aut}_C(G) = \{\alpha \in \mathrm{Aut}_T(G) \mid \mathfrak{L}(\alpha)(C) = C\}$. □

Notice that

$$\mathrm{Aut}_T(G)\cap \mathrm{Inn}_0(G) = \{I_g \mid g\in G_0,\ I_g(T)T = T\} = \iota\big(N(T,G)\big)$$

and that $\mathrm{Aut}^{\#}(G) \subseteq \mathrm{Aut}_T(G)$ for all T.

Proposition 6.71. *For a compact Lie group G and a maximal torus T, the following statements hold.*

(i) $\mathrm{Aut}(G) = \mathrm{Inn}_0(G)\cdot \mathrm{Aut}_T(G)$ *and* $\mathrm{Inn}_0(G)\cap \mathrm{Aut}_T(G) = \mathrm{Inn}_T(G) = \iota\big(N(T,G_0)\big) \cong N(T,G_0)Z(G)/Z(G) \cong N(T,G_0)/\big(N(T,G_0)\cap Z(G)\big)$.

(ii) $\mathrm{Aut}_T(G) = \iota\big(N(T,G_0)\big)\cdot \mathrm{Aut}_C(G)$ *and* $\iota\big(N(T,G)\big)\cap \mathrm{Aut}_C(G) = \iota(T)$.

(iii) $\mathrm{Aut}(G) = \mathrm{Inn}_0(G)\cdot \mathrm{Aut}_C(G)$ *and* $\mathrm{Inn}_0(G)\cap \mathrm{Aut}_C(G) = \iota(T)$.

(iv) $\iota(T)$ *is an open subgroup of* $\mathrm{Aut}_C(G)$ *and* $\mathrm{Aut}_C(G)_0 = \iota(T)$.

Proof. In this proof, we use the Frattini Argument (preceding Corollary 6.35) several times.

(i) The group $\mathrm{Aut}(G)$ acts on the set $\mathcal{T}(G)$ of maximal tori of G. The subgroup $\mathrm{Inn}_0(G)$ acts transitively by the Transitivity Theorem 6.27 (applied to the identity component G_0). The isotropy group of $\mathrm{Aut}(G)$ at T is $\mathrm{Aut}_T(G)$. Thus the Frattini

Argument shows $\mathrm{Aut}(G) = \mathrm{Inn}_0(G)\cdot \mathrm{Aut}_T(G)$. An automorphism α of G is in $\mathrm{Inn}_0(G) \cap \mathrm{Aut}_T(G)$ iff there is a $g \in G_0$ such that $\alpha = I_g$ and $T = \alpha(T) = gTg^{-1}$ iff for some $g \in N(T, G_0)$ we have $\alpha = \iota(g)$. Since $\ker \iota = Z(G)$ the remainder follows from the canonical decomposition of the morphism $\iota|N(T,G_0)\colon N(T,G_0) \to \iota\big(N(T,G_0)\big)$.

(ii) The group $\mathrm{Aut}_T(G)$ acts on $\mathfrak{t}$ via $(\alpha, X) \mapsto \mathfrak{L}(\alpha)(X) : \mathrm{Aut}_T(G) \times \mathfrak{t} \to \mathfrak{t}$. Accordingly, it acts on the set $\mathcal{C}$ of Weyl chambers via $(\alpha, C) \mapsto \mathfrak{L}\big(\alpha(C)\big) : \mathrm{Aut}_T(G) \times \mathcal{C} \to \mathcal{C}$. By 6.52, the subgroup $\iota\big(N(T,G_0)\big)$ acts transitively on $\mathcal{C}$. The Frattini Argument now shows $\mathrm{Aut}_T(G) = \iota\big(N(T,G_0)\big)\cdot \mathrm{Aut}_C(G)$. Now $\alpha \in \mathrm{Aut}(G)$ is in $\iota\big(N(T,G_0)\big) \cap \mathrm{Aut}_C(G)$ if there is a $g \in N(T,G_0)$ such that $C = \alpha(C) = \mathfrak{L}(\alpha)(C) = \mathrm{Ad}(g)(C)$. Since $\mathcal{W}(T,G)$ acts simply transitively on $\mathcal{C}$ by 6.52, we conclude $g \in T$. Then $\mathrm{Ad}(g)|\mathfrak{t} = \mathrm{id}_{\mathfrak{t}}$.

(iii) By (i) and (ii) we have

$$\begin{aligned}\mathrm{Aut}(G) &= \mathrm{Inn}_0(G)\cdot \mathrm{Aut}_T(G)\\ &= \mathrm{Inn}_0(G)\cdot\iota\big(N(T,G_0)\big)\cdot \mathrm{Aut}_C(G) = \mathrm{Inn}_0(G)\cdot \mathrm{Aut}_C(G).\end{aligned}$$

Finally, $\alpha \in \mathrm{Aut}(G)$ is contained in $\mathrm{Inn}_0(G) \cap \mathrm{Aut}_C(G)$ iff $\alpha = I_g$ for some $g \in N(T,G_0)$ and $C = \mathfrak{L}(\alpha)(C) = \mathrm{Ad}(g)(C)$, and this holds iff $\alpha = I_g$ with $g \in T$ by 6.52.

(iv) The group $\iota(T) \subseteq \mathrm{Aut}(G)$ contains exactly the inner automorphisms $\iota(t) = I_t$, $t \in T$, and these morphisms fix T elementwise. Since $G_0 \subseteq Z(T,G)$ we have $\iota(T) = \mathrm{Inn}_0(G) \cap \mathrm{Aut}_C(G) \subseteq \mathrm{Aut}_C(G)$. From (iii) we obtain $\mathrm{Aut}_C(G)/\iota(T) = \mathrm{Aut}(G)/\big(\mathrm{Inn}_0(G) \cap \mathrm{Aut}_C(G)\big) \cong \mathrm{Aut}(G)/\mathrm{Inn}_0(G)$, and from Iwasawa's Theorem 6.66 we know that this group is discrete. Hence $\iota(T) \subseteq \mathrm{Aut}_C(G)_0 \subseteq \iota(T)$. ☐

Corollary 6.72. *The automorphism group* $\mathrm{Aut}(G)$ *of a compact Lie group* G *is a product of the identity component* $\mathrm{Aut}(G)_0 = \mathrm{Inn}_0(G)$ *and a subgroup* $\mathrm{Aut}_C(G) \subseteq \mathrm{Aut}_T(G)$ *containing* $\iota(T)$ *as an open connected subgroup.* ☐

If $G = \mathbb{T}^2$, then $G = T$ and $\mathrm{Aut}(G) = \mathrm{Aut}_T(G) = \mathrm{Aut}_C(G) \cong \mathrm{Gl}(2,\mathbb{Z})$. If $G = \mathrm{SO}(3)$ then $\mathrm{Aut}(G) \cong \mathrm{Aut}\,\mathfrak{g} \cong \mathrm{Aut}(\mathbb{R}^3, \times)$, where $\times$ denotes the vector product on $\mathbb{R}^3$. We may take T to be the group of rotations with the z-axis as axis. We consider $\mathfrak{g}$ as $(\mathbb{R}^3, \times)$ with $\mathfrak{t} = \mathbb{R}\cdot(0,0,1)$ and then identify $\mathrm{Aut}(G)$ with $\mathrm{SO}(3)$. Thus $\iota(T)$ gets identified with the T. Then $\mathrm{Aut}_T(G) = T\cdot D$ where D is the group of diagonal matrices $\mathrm{diag}(\pm1,\pm1,\pm1)$ of determinant 1. We may take $C = \mathbb{R}^+\cdot(0,0,1) = \{0\} \times \{0\} \times [0,\infty[$. Then $\mathrm{Aut}_C(G) = T\cdot D_1$ where $D_1 = \{\mathrm{diag}(1,1,1), \mathrm{diag}(-1,-1,1)\}$.

Corollary 6.72 applies, in particular, to the case that G is connected. Then the situation becomes even clearer. Diagram 6.67 above reduces to a much more compact one as follows:

$$\begin{array}{ccc}
\operatorname{Aut}(\widetilde{G}) & \xrightarrow[\cong]{\mathfrak{L}_{\widetilde{G}}} & \operatorname{Aut}\mathfrak{g} \\
\widetilde{\bullet}\big\uparrow & & \big\uparrow\text{incl} \\
\operatorname{Aut}(G) & \xrightarrow[\cong]{\mathfrak{L}_G} & \operatorname{im}\mathfrak{L}_G \\
\text{incl}\big\uparrow & & \big\uparrow\text{incl} \\
\operatorname{Inn}(G) & \xrightarrow[\cong]{\mathfrak{L}_G|\operatorname{Inn}(G)} & \operatorname{Ad}(G)= e^{\operatorname{ad}\mathfrak{g}}
\end{array}.$$

For an arbitrary compact Lie group G we write

$$\operatorname{Out}(G) \overset{\text{def}}{=} \operatorname{Aut}(G)/\operatorname{Inn}(G).$$

Structure of the automorphism group of a compact Lie group

Theorem 6.73. *Let G be a connected compact Lie group. Then $\operatorname{Inn}(G) = \operatorname{Inn}_0(G)$ is a compact open subgroup of $\operatorname{Aut}(G)$, and for a given maximal torus T there is a discrete subgroup D of $\operatorname{Aut} G$ contained in $N(T, \operatorname{Aut}(G)) \overset{\text{def}}{=} \{\alpha \in \operatorname{Aut}(G) \mid \alpha(T) = T\}$ such that*

$$\operatorname{Aut}(G) = \operatorname{Inn}(G){\cdot}D, \quad \operatorname{Inn}(G) \cap D = \{1\}.$$

Thus

$$\operatorname{Aut}(G) \cong \operatorname{Inn}(G) \rtimes \operatorname{Out}(G) \cong e^{\operatorname{ad}\mathfrak{g}'} \rtimes \frac{\operatorname{Aut}(G)}{e^{\operatorname{ad}\mathfrak{g}'}}.$$

Proof. We apply Lemma 6.68 with $\Gamma = \operatorname{Aut}(G)$, $P = \operatorname{Aut}\mathfrak{g}$, $N = \operatorname{Inn}\mathfrak{g} = \operatorname{Ad}(G)$, H a semidirect complement according to 6.61(viii), which we may choose in $N(\mathfrak{t}, \operatorname{Aut}\mathfrak{g}) = \{\alpha \in \operatorname{Aut}\mathfrak{g} \mid \alpha(\mathfrak{t}) = \mathfrak{t}\}$. We further take $f = \mathfrak{L}_G\colon \Gamma \to P$. Set $D = f^{-1}(H)$. We claim $f^{-1}\big(N(\mathfrak{t}, \operatorname{Aut}\mathfrak{g})\big) = N(\iota(T), \operatorname{Aut} G)$: indeed $\alpha \in \operatorname{Aut}(G)$ is in the left hand side iff $\mathfrak{L}_G(\alpha)(\mathfrak{t}) = \mathfrak{t}$ iff $\alpha(T) = T$. Since G is connected, the morphism $f = \mathfrak{L}_G$ is injective, and thus the theorem follows from 6.68 in view of the fact that $\operatorname{Ad}(G)$ is open in $\operatorname{Aut}\mathfrak{g}$ and thus $\operatorname{Inn}(G)$ is open in $\operatorname{Aut}(G)$. □

From Diagram 6.67 we see that for not necessarily connected groups G, the morphism f is the composition of $\rho_G\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G)|G_0$ and an injective map. Thus f is injective if and only if $\rho_G = (\alpha \mapsto \alpha|G_0)$ is injective. If this holds, the conclusions of the theorem persist.

This structure theorem now allows us to sharpen the preliminary version of Lee's Theorem 6.36.

Dong Hoon Lee's Supplement Theorem for Lie groups

Theorem 6.74. *Let G be a topological group such that G_0 is a compact Lie group and let T be an arbitrary maximal torus of G_0. Then*

(I) *$N(T,G) \subseteq G$ contains a closed subgroup S such that $G = (G_0)'S$ and $(G_0)' \cap S \subseteq Z\big((G_0)'\big)$.*

(II) *If G is a compact Lie group, or if G_0 is semisimple, then $N(T,G) \subseteq G$ contains a finite subgroup E such that $G = G_0E$ and that $G_0 \cap E \subseteq Z(G_0)$. The finite subgroup $G_0 \cap E$ of G is normal in G.*

Proof. (I) We consider the homomorphism $f\colon G \to \mathrm{Aut}(G_0)$, $f(g)(x) = gxg^{-1}$, which maps $(G_0)'$ onto the identity component $\mathrm{Inn}(G_0) = \mathrm{Aut}(G_0)_0$. By 6.73 this open subgroup is a semidirect factor of $\mathrm{Aut}(G_0)$. Let H be a semidirect complement contained in $N\big(T, \mathrm{Aut}(G_0)\big)$. Then by Lemma 6.68 the closed subgroup $S \stackrel{\mathrm{def}}{=} f^{-1}(H)$ satisfies $G = (G_0)'S$ and $(G_0)' \cap S \subseteq \ker f = Z(G_0, G)$. Moreover, $s \in S$ implies $f(s) \in N\big(T, \mathrm{Aut}(G)\big)$, i.e. $sTs^{-1} = T$ and thus $s \in N(T,G)$. Since $(G_0)' \cap Z(G_0, G) = Z\big((G_0)'\big)$ we have $(G_0)' \cap S \subseteq Z\big((G_0)'\big)$. This proves (I).

For a proof of (II) we note that $\mathrm{Inn}(G_0) = f(G_0) \subseteq \mathrm{Aut}(G_0)_0$ and apply Lemma 6.68 to $f\colon G \to \mathrm{Aut}(G_0)$ again, this time with $M = G_0$ instead of $M = (G_0)'$. This yields $G_0 \cap S \subseteq \ker f = Z(G_0, G)$, i.e. $S_0 \subseteq G_0 \cap S \subseteq Z(G_0)$. Since $Z(G_0) \subseteq \ker f \subseteq S$ we get $S_0 = Z(G_0)_0$. If G_0 is semisimple, then (I) and (II) have equivalent conclusions. If G is a compact Lie group, then S is a Lie group and Theorem 6.10 applies to S and yields a finite subgroup $E \subseteq S \subseteq N(T,G)$ such that $S = S_0E$ and the order of each element of $E \cap Z(G_0)_0$ divides $|S/Z(G_0)_0|$. Now $G = G_0S = G_0Z(G_0)_0E = G_0E$ and $G_0 \cap E \subseteq G_0 \cap S = Z(G_0)$.

The centralizer and thus the normalizer of $G_0 \cap E$ contains G_0, and the normalizer also contains E. Hence $G_0 \cap E$ is normal in $G_0E = G$. ☐

The proof yielded also that $x \in Z(G_0)_0 \cap E$ implies $x^{|S/Z(G_0)_0|} = 1$. The group S in 6.74 is a Lie group with a torus as identity component S_0; such groups were discussed in Theorem 6.10.

We shall call the subgroup E of Theorem 6.74 as *Lee supplement* and also observe that the normality of the finite group $G_0 \cap E$ in G implies its normality in G_0; it is therefore necessarily central in G_0 by 6.13.

Sandwich Theorem for Compact Lie Groups

Corollary 6.75. *For any compact Lie group G, there is a finite subgroup E of G providing a "sandwich situation"*

$$G_0 \rtimes_\iota E \xrightarrow{\mu} G \to G/(G_0 \cap E) \cong \frac{G_0}{G_0 \cap E} \rtimes \frac{E}{G_0 \cap E}$$

with $\iota\colon E \to \mathrm{Aut}(G_0)$ defined by $\iota(e)(g_0) = eg_0e^{-1}$, $\mu\colon G_0 \rtimes_\iota E \to G$ by $\mu(g_0, e) = g_0e$, and where the morphism $G \to G/(G_0 \cap E)$ is the quotient map. Both morphisms are surjective and have kernels isomorphic to $G_0 \cap E$.

Proof. We let T be an arbitrary maximal torus of G and let E be as in Lee's Supplement Theorem 6.74(II). The quotient group $G/(G_0 \cap E)$ then is a semidirect product of $G/(G_0 \cap E)$ and $E/(G_0 \cap E)$. The semidirect product $G_0 \rtimes_\iota E$ and the morphism μ are well-defined and surjective; the kernel of μ consists of all (e^{-1}, e) with $e \in G_0 \cap E$. Thus $e \mapsto (e^{-1}, e) : G_0 \cap E \to \ker \mu$ therefore is an isomorphism.□

In the sense of these previous results, each compact Lie group comes close to splitting over its component.

In Exercise E6.9(b) we saw that these results cannot be improved in the sense that one might be able to eliminate the intersection $G_0 \cap E$. A ready-made example is the subgroup $\mathbb{S}^1 \cdot \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbb{S}^3$.

Let us emphasize from Proposition 6.37 what, in Lee's Supplement Theorem 6.74, is the precise obstruction to the splitting of G_0. Recall that for a subgroup E of G and $D = G_0 \cap E$ we let $Z^1_D(E, G_0)$ denote the set of all cocycles $f\colon E \to G_0$ which agree on D with the inclusion map $D \to G_0$.

Remark 6.75a. Let G be a compact Lie group and E a finite subgroup such that $G = G_0 E$ and $D \overset{\text{def}}{=} G_0 \cap E \subseteq Z(G_0)$ as in Theorem 6.74. Then $G \cong G_0 \rtimes G/G_0$, i.e. the set $\mathcal{C}(G_0)$ of semidirect cofactors of G_0 is not empty if and only if $Z^1_D(E, G_0) \neq \emptyset$. The function

$$\Phi\colon Z^1_D(E, G_0) \to \mathcal{C}(G_0), \quad \Phi(f) = \{f(e)^{-1} e \mid e \in E\},$$

is a bijection.

Proof. This follows at once from Proposition 6.37. □

Covering Groups of Disconnected Compact Lie Groups

Assume that N is a connected compact Lie group and H a topological group for which there is a morphism of topological groups $f\colon H \to \operatorname{Aut} N$. Let $\widetilde{N}$ be the universal covering group and $p_N\colon \widetilde{N} \to N$ is the universal covering morphism. We consider the morphisms $\mathfrak{L}_N\colon \operatorname{Aut}(N) \to \operatorname{Aut}\mathfrak{L}(N)$ and $\mathfrak{L}_{\tilde{N}}\colon \operatorname{Aut}(\widetilde{N}) \to \operatorname{Aut}\mathfrak{L}(\widetilde{N})$ of Theorem 6.63. The morphism $\mathfrak{L}(p_N)\colon \mathfrak{L}(\widetilde{N}) \to \mathfrak{L}(N)$ is an isomorphism of Lie algebras, which induces an isomorphism $\iota_N\colon \operatorname{Aut}\mathfrak{L}(\widetilde{N}) \to \mathfrak{L}(N)$, $\iota_N(\alpha) = \mathfrak{L}(p_N)^{-1} \circ \alpha \circ \mathfrak{L}(p_M)$, and $\mathfrak{L}_{\tilde{N}}\colon \operatorname{Aut}(\widetilde{N}) \to \operatorname{Aut}\mathfrak{L}(\widetilde{N})$ is an isomorphism by Theorem 6.63(iv).

We now recall that we have an injective morphism

$$\eta_N = \mathfrak{L}_{\tilde{N}}^{-1} \circ \iota_N^{-1} \circ \mathfrak{L}_N : \operatorname{Aut} N \to \operatorname{Aut} \widetilde{N},$$

$\eta_N(\beta) = \widetilde{\beta}$, filling in correctly in the commuting diagram

$$\begin{array}{ccc} \operatorname{Aut}\widetilde{N} & \xrightarrow{\quad\mathfrak{L}_{\tilde{N}}\quad} & \operatorname{Aut}\mathfrak{L}(\widetilde{N}) \\ {\scriptstyle\eta_N}\big\uparrow & & \big\downarrow{\scriptstyle\iota_N} \\ \operatorname{Aut} N & \xrightarrow{\quad\mathfrak{L}_N\quad} & \operatorname{Aut}\mathfrak{L}(N). \end{array}$$

We define $F \overset{\text{def}}{=} \eta_N \circ f\colon H \to \operatorname{Aut} \widetilde{N}$; in other words, for $h \in H$ we have $F(h) = \widetilde{f(h)}$ and there is a commuting diagram

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\quad F(h) \quad} & \widetilde{N} \\ {\scriptstyle p_N}\big\downarrow & & \big\downarrow{\scriptstyle p_N} \\ N & \xrightarrow[\quad f(h) \quad]{} & N. \end{array}$$

Accordingly, for each $\widetilde{n} \in \widetilde{N}$ we obtain

$$(*) \qquad p_N\big(F(h)(\widetilde{n})\big) = f(h)\big(p_N(\widetilde{n})\big).$$

We shall call F the *lifting* of f and construct two semidirect products:

$$\widetilde{P} \overset{\text{def}}{=} \widetilde{N} \rtimes_F H,$$
$$P \overset{\text{def}}{=} N \rtimes_f H$$

and define $\psi_f\colon \widetilde{P} \to P$ by $\psi_f\big((\widetilde{n}, h)\big) = (p_N(\widetilde{n}), h)$. Then formula $(*)$ permits us to compute
$\psi_f\big((\widetilde{n}_1, h_1)(\widetilde{n}_2, h_2)\big) = \psi_f\big((\widetilde{n}_1 F(h_1)(\widetilde{n}_2), h_1 h_1)\big) = (p_N\big(\widetilde{n}_1 F(h_1)(\widetilde{n}_2)\big), h_1 h_2) =$
$(p_N(\widetilde{n}_1) f(h)\big(p_N(\widetilde{n}_2)\big), h_1 h_2) = (p_N(\widetilde{n}_1), h_1)(p_N(\widetilde{n}_2), h_2) =$
$\psi_f\big((\widetilde{n}_1, h_1)\big)\psi_f\big((\widetilde{n}_2, h_2)\big)$. That is, ψ_f is a morphism of topological groups with the discrete kernel $\ker p_N \times \{1\} \cong \ker p_N \cong \pi_1(N)$.

For a compact Lie group G let us call a finite subgroup E as described in Dong Hoon Lee's Supplement Theorem 6.74 and in the Sandwich Theorem 6.75 a *Lee supplement* (for G_0 in G). We recall that there is an exact sequence

$$1 \to G_0 \cap E \longrightarrow E \longrightarrow G/G_0 \to 1,$$

and $G_0 \cap E$ is central in G_0.

We fix the following notation: Let $f\colon E \to \operatorname{Aut}(G_0)$ be the morphism defined by $f(h)(g) = hgh^{-1}$. Let $F\colon E \to \operatorname{Aut}(\widetilde{G_0})$ be the lifting of f and $\mu\colon G_0 \rtimes_f E \to G$ the morphism given by $\mu(g_0, e) = g_0 e$. Finally we set $\widetilde{G} \overset{\text{def}}{=} \widetilde{G_0} \rtimes_F E$ and define $\Phi\colon \widetilde{G} \to G$ to be the morphism defined by $\Phi = \mu \circ \psi_f$, $\Phi(\widetilde{g_0}, e) = p_{G_0}(\widetilde{g_0})e$.

Corollary 6.76. *Let G be a compact Lie group and E a Lee supplement. Then*

(i) *$\widetilde{G}_0$ is the universal covering group of G_0 and Φ is a surjective morphism inducing on the identity components the universal covering morphism.*

(ii) *Its kernel $\ker \Phi$ is a discrete normal finitely generated abelian subgroup of $\widetilde{G}$ for which there is an exact sequence*

$$(1) \qquad 1 \to \pi_1(G_0) \longrightarrow K \longrightarrow G_0 \cap E \to 1.$$

(iii) *$\widetilde{G}/\widetilde{G}_0 \cong E$ and there is an exact sequence*

$$(2) \qquad 1 \to G_0 \cap E \longrightarrow \widetilde{G}/\widetilde{G}_0 \longrightarrow G/G_0 \to 1.$$

Proof. (i) It is clear from the construction that $\widetilde{G}_0 = \widetilde{G_0} \times \{1\}$ and that Φ implements the universal covering on the components, and induces a local isomorphism.

The finite group $G_0 \cap E$ is contained in the center of G_0 and therefore is abelian. We have $\ker\mu = \{(g_0, e) \in G_0 \times E : e = g_0^{-1}\} \cong G_0 \cap E = \mathbb{Z}(G_0) \cap E$, and $\ker\mu \subseteq (Z(G_0) \cap E)^2$. Therefore $\ker\Phi = \{(\widetilde{g_0}, e) : e = p_N(\widetilde{g_0})^{-1}\} \subseteq Z(\widetilde{G_0}) \cap E$. In particular, K is abelian. The morphism $\psi_f\colon \widetilde{G} \to G_0 \rtimes_f E$ maps maps K onto $\ker\mu \cong G_0 \cap E$, and K contains $\ker\psi_f = \ker p_{G_0} \times \{1\} \cong \pi_1(G_0)$. Thus $K/\ker\psi_f \cong G_0 \cap E$. This amounts to the existence of the exact sequence (1).

(ii) The group $\ker p_{G_0} \cong \pi_1(G_0)$ is central in $\widetilde{G_0}$ and thus is finitely generated abelian. Thus K is a finitely generated abelian group.

(iii) This is immediate from the construction and the comments preceding the Corollary. □

We remark that according to the Sandwich Theorem 6.75, the covering construction of semidirect products also applies to the group $G/(G_0 \cap E)$, yielding for this group a "covering group" of the form $\widetilde{G_0} \rtimes G/G_0$. This is the kind of covering group which one would naïvely want for G but cannot get in general because of the obstruction $G_0 \cap E$.

Let L be a connected compact Lie group and F a finite group with a central subgroup C. Assume that there is an injective morphism $\kappa\colon C \to Z(L)$. Set $D = \{(\kappa(c)^{-1}, c) : c \in C\} \subseteq L \times F$. Set $G = (L \times E)/D$. Then $G_0 = (L \times C)/D \cong L$, $E = (\kappa(C \times F)/D \cong F$. Then $G = G_0 E$ and $G_0 \cap E = (\kappa(C) \times C)/D \cong C$. Here $G_0 \cap E \subseteq Z(G)$ and K is abelian. Example: Let $p \neq 2$ be a prime, $L = \mathbb{R}/p\mathbb{Z}$, $F = (\mathbb{Z}/p\mathbb{Z})^2 \times \mathbb{Z}/p\mathbb{Z}$ such that $(v, \xi)(w, \eta) = \big(v + w, \xi + \eta + 1/2\det(v, w)\big)$, and $C = \{0\} \times \mathbb{Z}/p\mathbb{Z}$; we let $\kappa(0, n + p\mathbb{Z}) = n + p\mathbb{Z}$. Then $G' = G_0 \cap E$ and thus G_0 does not split. We have $\widetilde{G} = \mathbb{R} \times F$, but there is no group H with $H_0 \cong \mathbb{R}$ and $H/H_0 \cong G/G_0$ with a surjective morphism $H \to G$ inducing a universal covering $H_0 \to G_0$ and an isomorphism $H/H_0 \to G/G_0$.

Let X be a finite set and S the group of permutations on X acting on the right. Then S acts on the groups $(\mathbb{R}/4\mathbb{Z})^X$ and $(\mathbb{Z}/4\mathbb{Z})^X$ on the left as a group of automorphisms. Let $F \stackrel{\text{def}}{=} (\mathbb{Z}/4\mathbb{Z})^X \rtimes S$ and let F act on $(\mathbb{R}/4\mathbb{Z})^X$ as a group of automorphisms via S. In $\Gamma \stackrel{\text{def}}{=} (\mathbb{R}/4\mathbb{Z})^X \rtimes ((\mathbb{R}/4\mathbb{Z})^X \rtimes S)$ let D be the normal subgroup of all $\big(-(2n_x + 4\mathbb{Z})_{x\in X}, ((2n_x + 4\mathbb{Z})_{x\in X}, 1)\big)$, $n_x \in \mathbb{Z}$. Then let $G = \Gamma/D$; then $G/G_0 \cong (\mathbb{Z}/2\mathbb{Z})^X \rtimes S$, $E \cong (\mathbb{Z}/4\mathbb{Z})^X \rtimes S$ and $G_0 \cap E \cong (\mathbb{Z}/2\mathbb{Z})^X$. The group $\widetilde{G} \stackrel{\text{def}}{=} \mathbb{R}^X \rtimes ((\mathbb{R}/4\mathbb{Z})^C \rtimes S)$ is mapped onto G via Φ and $\ker\Phi$ is not central.

Auerbach's Generation Theorem

The information accumulated on the action of a maximal torus on the Lie algebra of a compact group allows us to gather structural information on the generation of a Lie group.

Lemma 6.77. *Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_{\text{eff}}$, $\mathfrak{g}_{\text{eff}} = \bigoplus_{D\in R^+} \mathfrak{g}^D$ the root decomposition of the Lie algebra of a compact Lie group G with respect to a maximal torus T. Then $\mathfrak{g}_{\text{eff}}$ is a cyclic T-module* (*cf.* 4.16) *and the set of generators is a dense open set in $\mathfrak{g}_{\text{eff}}$.*

Proof. By 6.45, the isotypic components $\mathfrak{g}^D$ of $\mathfrak{g}_{\text{eff}}$ are simple real T-modules since their dimension is two and $I|\mathfrak{g}^D = \text{Ad}(\exp \frac{\pi}{2\|D\|^2}\cdot D)$. Hence 4.25 applies and proves the assertion. □

Proposition 6.78. *Assume that $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, and that T is a maximal torus. It acts on $\mathfrak{g}$ under the adjoint action. Let R^+ denote a set of positive roots with respect to $\mathfrak{t}$ and P the T-averaging operator which is the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{t}$ (cf.* 3.32*). For D, $D' \in R^+$ set $\mathfrak{t}_D = \mathfrak{t} \cap D^\perp$ and $\mathfrak{t}_{DD'} = \mathfrak{t} \cap (D - D')^\perp$. Assume that the pair $(X, Y) \in \mathfrak{t} \times \mathfrak{g}$ satisfies the following conditions:*

(i) $X \in \mathfrak{t} \setminus (\bigcup_{D \in R^+} \mathfrak{t}_D \cup \bigcup_{D,\, D' \in R^+} \mathfrak{t}_{DD'})$, *and*

(ii) $(\mathbf{1} - P)(Y)$ *is a generator of the cyclic T-module $\mathfrak{g}_{\text{eff}}$.*

Then the Lie subalgebra $\langle X,\, Y\rangle$ generated in $\mathfrak{g}$ by X and Y is $\mathbb{R}\cdot X + \mathbb{R}\cdot PY + \mathfrak{g}'$. In particular, if $\mathfrak{g}$ is semisimple, i.e. $\mathfrak{g}' = \mathfrak{g}$, then $\langle X,\, Y\rangle = \mathfrak{g}$.

The set of all $(X, Y) \in \mathfrak{t} \times \mathfrak{g}$ satisfying (i) *and* (ii) *above is open and dense in $\mathfrak{t} \times \mathfrak{g}$.*

The set

$$\mathbf{C}_T \stackrel{\text{def}}{=} \{Y \in \mathfrak{g} \mid \text{Spec}\,\text{ad}\,Y \cap (2\pi i\mathbb{Z} \setminus \{0\}) = \emptyset \text{ and } Y \text{ satisfies Condition (ii)}\}$$

is open dense in $\mathfrak{g}$ and $D_T(G) \stackrel{\text{def}}{=} \exp \mathbf{C}_T$ is open dense in G.

Proof. The assertion is true if $\mathfrak{g}$ is abelian. We therefore assume that $\mathfrak{g}' \neq \{0\}$. We abbreviate $\langle X,\, Y\rangle$ by $\mathfrak{h}$. Then $\mathbb{R}\cdot X \subseteq \mathfrak{h}$, and $T_X \stackrel{\text{def}}{=} \overline{\exp \mathbb{R}\cdot X} \subseteq T$ is contained in the closed subgroup $\{g \in G \mid \text{Ad}(g)\mathfrak{h} \subseteq \mathfrak{h}\}$. Thus T_X acts on $\mathfrak{h}$ under the adjoint action of G. By 6.45(ii)(∗) we have $\text{Ad}(\exp r\cdot X)Y_D = e^{r\cdot \text{ad}\, X} Y_D = \cos r(X \mid D)\cdot Y_D + \sin r(X \mid D)\cdot IY_D$. Since $(X \mid D) \neq 0$ for all roots D by (i), then $\mathfrak{g}^D$ is a cyclic T_X-module. Each $\mathfrak{g}^D$ is a simple T_X-module and for $D \neq D'$ in R^+ the T_X-modules $\mathfrak{g}^D$ and $\mathfrak{g}^{D'}$ are not isomorphic since $(X \mid D) \neq (X \mid D')$ by (i). Thus $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{D \in R^+} \mathfrak{g}^D$ is the isotypic decomposition of the T_X-module $\mathfrak{g}$. Accordingly, the isotypic decomposition of the T_X-submodule $\mathfrak{h}$ is

$$\mathfrak{h} = (\mathfrak{t} \cap \mathfrak{h}) \oplus \bigoplus_{D \in R^+} (\mathfrak{g}^D \cap \mathfrak{h}). \tag{$*$}$$

We write Y in the form $Y = Y_0 + \sum_{D \in R^+} Y_D$ with $Y_0 \in \mathfrak{t}$ and a $Y_D \in \mathfrak{g}^D$ for all positive roots D. By (∗) we have Y_0 and Y_D in $\mathfrak{h}$ for all $D \in R^+$. Since $(\mathbf{1} - P)(Y)$ is a generator of $\mathfrak{g}_{\text{eff}}$ all elements Y_D are nonzero. Then by 4.25, the T_X-module $\mathfrak{g}_{\text{eff}} = \bigoplus_{D \in R^+} \mathfrak{g}^D$ is cyclic generated by $\sum_{D \in R^+} Y_D = (\mathbf{1} - P)(Y)$. But $\mathfrak{h}$ is a T_X-module containing Y. Hence $\mathbb{R}\cdot Y_0 \oplus \mathfrak{g}_{\text{eff}} \subseteq \mathfrak{h}$. Any Lie algebra containing Y_D and IY_D contains $\mathfrak{g}_D$ and thus D (see 6.46 and 6.49). Hence $\mathfrak{h}$ contains $\text{span}\, R^+$ and thus contains $\text{span}\, R^+ \oplus \mathfrak{g}_{\text{eff}} = \mathfrak{g}'$ (see 6.49(vi)). As a consequence, $\mathbb{R}\cdot X + \mathbb{R}\cdot PY + \mathfrak{g}' \subseteq \mathfrak{h}$. For the converse inclusion we may now note that $\mathfrak{h}/\mathfrak{g}' = \langle X, Y\rangle/\mathfrak{g}' \subseteq (\mathbb{R}\cdot X + \mathbb{R}\cdot Y + \mathfrak{g}')/\mathfrak{g}'$ whence $\mathfrak{h} = \mathbb{R}\cdot X + \mathbb{R}\cdot Y + \mathfrak{g}'$ Since $Y = PY + (1 - P)Y$ with $(1 - P)Y \in \mathfrak{g}_{\text{eff}} \subseteq \mathfrak{g}'$, the relation $\mathfrak{h} = \mathbb{R}\cdot X + \mathbb{R}\cdot PY + \mathfrak{g}'$ follows.

If $\mathfrak{g} = \mathfrak{g}'$, then X, $PY \in \mathfrak{g}'$, and we conclude $\mathfrak{h} = \mathfrak{g}$.

The set of all $X \in \mathfrak{t}$ satisfying (i) is open dense in $\mathfrak{t}$, and since Y satisfies (ii) iff $Y_D \neq 0$ for all $D \in R^+$, the set of all $Y \in \mathfrak{g}$ satisfying (ii) is open and dense in $\mathfrak{g}$.

Finally, the set of all $Y \in \mathfrak{g}$ such that $\operatorname{Spec}\operatorname{ad} Y \cap (2\pi i\mathbb{Z} \setminus \{0\}) = \emptyset$ is open and dense in $\mathfrak{g}$, and by 5.41(v) for any such Y there is an open neighborhood of Y mapped homeomorphically onto an open neighborhood of $\exp_G Y$ under $\exp_G$. Thus $\mathbf{C}_T$ is a still open and dense in $\mathfrak{g}$ and since $\exp_G$ is open at any of its points, its image $D_T(G)$ under $\exp_G$ is open and dense in G. ☐

Remark 6.79. Assume the hypotheses of 6.78 and the additional one that $\mathfrak{g}$ is semisimple. Then

(a) the set $\Gamma(\mathfrak{g}) \stackrel{\text{def}}{=} \{(X, Y) \in \mathfrak{g} \times \mathfrak{g} \mid \langle X, Y\rangle = \mathfrak{g}\}$ is a dense open subset of $\mathfrak{g} \times \mathfrak{g}$, and

(b) for each Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ the set $\Gamma(\mathfrak{g}) \cap (\mathfrak{t} \times \mathfrak{g})$ is open dense in $\mathfrak{t} \times \mathfrak{g}$.

Proof. First we abbreviate $\Gamma(\mathfrak{g})$ by Γ and fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and claim that Γ is open in $\mathfrak{g}\times\mathfrak{g}$ (irrespective of any assumption on the Lie algebra $\mathfrak{g}$) and that $\Gamma \cap (\mathfrak{t} \times \mathfrak{g})$ is open in $\mathfrak{t} \times \mathfrak{g}$. By 6.78, there exists at least one $(X, Y) \in \Gamma \cap (\mathfrak{t} \times \mathfrak{g})$. Consider one of these pairs (X, Y) and set $n = \dim \mathfrak{g}$. Then there are n Lie monomials $p_j(U, V)$, $j = 1, \ldots, n$, such as U, V, $[U, V]$, $\big[U, [U, V]\big]$, $\big[V, [V, U]\big], \ldots,$ such that $p_1(X, Y), \ldots, p_n(X, Y)$ are linearly independent. This means exactly that $p_1(X, Y) \wedge \cdots \wedge p_n(X, Y) \in \bigwedge^n \mathfrak{g} \cong \mathbb{R}$ is nonzero. (In coordinates with respect to some basis of $\mathfrak{g}$ this means that the determinant of the matrix formed by the components of the vectors $p_j(X, Y)$ is nonzero.) Since the function $\delta\colon \mathfrak{g}\times\mathfrak{g} \to \bigwedge^n \mathfrak{g}$, $\delta(U, V) = p_1(U, V) \wedge \cdots \wedge p_n(U, V)$ is continuous, the set $\delta^{-1}(\bigwedge^n \mathfrak{g} \setminus \{0\})$ is an open subset of $\mathfrak{g} \times \mathfrak{g}$ containing (X, Y). This proves the claim. (In fact the argument shows that the set is open in the Zariski topology and therefore is dense; but we shall give a separate argument for the density.)

Secondly we claim that Γ is dense. By 6.78, the set $\Gamma \cap (\mathfrak{t} \times \mathfrak{g})$ is dense in $\mathfrak{t} \times \mathfrak{g}$. This completes the proof of (b). The group $\operatorname{Ad}(G)$ acts on $\mathfrak{g}\times\mathfrak{g}$ via $\operatorname{Ad}(g)\cdot(U, V) = (\operatorname{Ad}(g)(U), \operatorname{Ad}(g)(V))$ and the set Γ is invariant under this action. Now $\operatorname{Ad}(g)\cdot\big(\Gamma\cap(\mathfrak{t}\times\mathfrak{g})\big) = \Gamma\cap\big(\operatorname{Ad}(g)(\mathfrak{t}\times\mathfrak{g})\big)$ is dense in $\operatorname{Ad}(g)\mathfrak{t}\times\mathfrak{g}$. Thus the set $\bigcup_{g\in G} \operatorname{Ad}(g)\cdot\big(\Gamma\cap(\mathfrak{t}\times\mathfrak{g})\big)$, which is contained in Γ, is dense in $\bigcup_{g\in G}(\operatorname{Ad}(g)(\mathfrak{t})\times\mathfrak{g}) = \Big(\bigcup_{g\in G} \operatorname{Ad}(g)(\mathfrak{t})\Big)\times\mathfrak{g}$. But by the Transitivity Theorem 6.27, this last set is $\mathfrak{g} \times \mathfrak{g}$. Hence Γ is dense in $\mathfrak{g} \times \mathfrak{g}$ as asserted in (a). ☐

Thus two randomly chosen elements X and Y of a semisimple compact Lie algebra have a very good chance of generating it. Remark 6.79 makes this statement precise.

We say that a subset $M \subseteq G$ is *a generating set of* G if $G = \overline{\langle M\rangle}$. The following lemma generalizes 1.24.

Lemma 6.80. *If $G = \mathbb{R}^n/\mathbb{Z}^n$ is a torus, then the set $\mathbf{g}(G)$ of generators of G is a dense set whose complement is the union of the countable family $\mathcal{S}(G)$ of all closed proper subgroups and has Haar measure* 0.

Proof. Exercise E6.16. □

Exercise E6.16. Prove Lemma 6.80.

[Hint. It is clear that g fails to be a generator iff $\overline{\langle g\rangle} \in \mathcal{S}(G)$. Thus it remains to show that $\mathcal{S}(G)$ is countable. The pullback along the quotient morphism $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ provides a bijection from this set to the set $\mathcal{S}_{\mathbb{Z}}(\mathbb{R}^n)$ of all closed subgroups of $\mathbb{R}^n$ containing $\mathbb{Z}^n$. The closed subgroups of $\mathbb{R}^n$ are classified in Appendix 1, Theorem A1.12. Indeed, if S is a closed subgroup of $\mathbb{R}^n$, then there is a basis $e_1, \ldots, e_n$ and there are natural numbers p, q such that

$$S = \mathbb{R}{\cdot}e_1 \oplus \cdots \oplus \mathbb{R}{\cdot}e_p \oplus \mathbb{Z}{\cdot}e_{p+1} \oplus \cdots \oplus \mathbb{Z}{\cdot}e_{p+q}.$$

We note $p + q = n$. Set

$$S_0 = \mathbb{R}{\cdot}e_1 \oplus \cdots \oplus \mathbb{R}{\cdot}e_p.$$

If $\mathbb{R}^n$ is identified with $\mathfrak{L}(G)$, then S_0 gets identified with $L(S/\mathbb{Z}^n)$, and thus the identity component of $S/\mathbb{Z}^n$ in G, being $(S_0 + \mathbb{Z}^n)/\mathbb{Z}^n \cong S_0/(S_0 \cap \mathbb{Z}^n)$, is a p-dimensional torus. This is the case iff the $e_1, \ldots, e_p$ may be chosen in $\mathbb{Z}^n$. There are countably many possibilities to do this. Once S_0 is fixed, $\mathbb{R}^n/(S_0 + \mathbb{Z}^n)$ is an $n-p$-torus T, and the image of S in T is discrete. In order to see that a torus can contain only a countable set of discrete subgroups, let us go back to S and assume now that $p = 0$. Then $S = \mathbb{Z}{\cdot}e_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}e_n$ and $\mathbb{Z}^n \subseteq S$. Then the Elementary Divisor Theorem A1.10 shows that there are natural numbers $m_1|\cdots|m_n$ such that $m_1{\cdot}e_1, \ldots, m_n{\cdot}e_n$ is a basis of $\mathbb{Z}^n$. Thus $S \subseteq \mathbb{Q}^n$. Thus there are only countably many possibilities for S in this case, too.

So the set of generators is the complement of a meager set in the sense of Baire category theory. Moreover, it is of Haar measure 0 since each proper closed subgroup of G is of lower dimension than that of G and therefore has Haar measure 0. The proof of this last statement in 6.80 pertaining to the Haar measure requires enough Haar measure theory to conclude that each proper subtorus of a torus has Haar measure 0. In order to see this in an elementary fashion one shows first that every subtorus is a direct factor. (We shall have very appropriate ways of seeing that in Chapter 8 below, see e.g. 8.78.) Then the issue is that $\mathbb{T}^p \times \{0\}^q$ has measure zero in $\mathbb{T}^{p+q}$. Since Haar measure on $\mathbb{T}^n$ agrees with Lebesgue measure, we may invoke Lebesgue measure theory for this conclusion.] □

Lemma 6.81. *Let G be a connected compact Lie group. Then there is a dense subset $D(G)$ of G with the following properties.*

(i) *$G \setminus D(G)$ is a countable union of nowhere dense compact subsets.*

(ii) *$D(G)$ is invariant under all inner automorphisms.*

(iii) *For every maximal torus T the set $\mathbf{g}(T)$ of generators of T is precisely $D(G) \cap T$.*

(iv) *For every $g \in D(G)$ the subgroup $\overline{\langle g\rangle}$ is a maximal torus.*

(v) *If S and T are maximal tori and $S \cap T \cap D(G) \neq \emptyset$, then $T = S$. Thus the sets $T \cap D(G)$, $T \in \mathfrak{T}(G)$ partition $D(G)$.*

(vi) *If $X \in \mathfrak{g}$ satisfies $\exp_G X \in T \cap D(G)$ for some $T \in \mathfrak{T}(G)$, then $X \in \mathfrak{t}$, and X satisfies Condition* 6.78(i).

Proof. We fix a maximal torus T of G. By Lemma 6.80 above, $\mathbf{g}(T) = T \setminus C$, $C = \bigcup \mathcal{S}(T)$, where $\mathcal{S}(T)$ is the *countable* set of proper subgroups of T. We note that any automorphism of T permutes the set $\mathcal{S}(T)$ and thus leaves the nowhere dense meager set C invariant. Define $D(G) = G \setminus C(G)$ where $C(G) \stackrel{\rm def}{=} \bigcup_{g\in G} gCg^{-1} = \bigcup_{H\in\mathcal{S}(T)} K_H$, where $K_H = \bigcup_{g\in G} gHg^{-1}$ is a compact subset invariant under inner automorphisms. We claim that K_H is nowhere dense.

Indeed, if k is an inner point of K_H, then by the Transitivity Theorem 6.27 we may assume that k is an inner point of $T \cap K_H$ in T, i.e. that there is an open set V of T with $k \in V \subseteq T \cap H_K$. However, $T \cap K_H = \bigcup_{g\in G}(T \cap gHg^{-1}) \subseteq \bigcup_{H'\in\mathcal{S}(T)} H' = C$, whence $V \subseteq C$. But C is nowhere dense in T, and this contradiction proves the claim. Thus $D(G)$ is the complement of the meager set $C(G)$ and thus is dense in G. Obviously, C(G) and $D(G)$ are invariant under inner automorphisms. If S and T are maximal tori then either $S = T$ or $S \cap T \in \mathcal{S}(T)$ whence $S \cap T \subseteq C \subseteq C(G)$. Hence

$$C(G) \cap T = C \quad \text{and} \quad D(G) \cap T = \mathbf{g}(T).$$

Thus Statements (i), (ii), and (iii) are proved.

Now let $g \in D(G)$. Then by the Maximal Torus Theorem 6.30 we find a maximal torus S containing g. Then $g \in S \cap D(G) = \mathbf{g}(S)$ by (iii) and thus $\overline{\langle g\rangle} = S$. This proves (iv), and (v) is an immediate consequence.

Finally we prove (vi). Let $\exp_G X \in T \cap D(G)$. The connected abelian subgroup $\exp \mathbb{R}{\cdot}X$ is contained in some maximal connected abelian subgroup $S \in \mathfrak{T}(G)$ and then $X \in \mathfrak{s} \stackrel{\rm def}{=} \mathfrak{L}(S)$. But now $\exp_X \in S \cap T \cap D(G)$. Then (v) implies $S = T$ and thus $X \in \mathfrak{t}$.

Using the notation of 6.78 we note that $\exp \mathfrak{t}_D$ and $\exp \mathfrak{t}_{D,D'}$ are closed subgroups of $T = \exp \mathfrak{t}$ as kernels and equalizers of representations of T. Hence $\exp_G(\bigcup_{D\in R^+} \mathfrak{t}_D \cup \bigcup_{D,\,D'\in R^+} \mathfrak{t}_{DD'}) \subseteq C$. But $\exp_G X \notin C$. Thus we have $X \notin \exp^{-1} C \supseteq (\bigcup_{D\in R^+} \mathfrak{t}_D \cup \bigcup_{D,\,D'\in R^+} \mathfrak{t}_{DD'})$. This is 6.78(i). ☐

Auerbach's Generation Theorem

Theorem 6.82. *Let G be a connected compact Lie group. Then the set $\Gamma(G) \stackrel{\rm def}{=} \{(g,h) \in G \times G \mid \overline{\langle g,h\rangle} = G\}$ is dense in $G \times G$.*

For every maximal torus T, the set $\Gamma(G)$ contains the set $(D(G)\cap T) \times \big(D(G) \cap D_T(G)\big)$ where $D(G)$ is as in 6.81 *and $D_T(G)$ is the dense open subset of G introduced in* 6.78. *In particular, the complement of $(T \times G) \setminus \Gamma(G)$ is contained in the countable union of closed nowhere dense subsets of $T \times G$.*

Proof. Let T be a maximal torus of G. By 6.81 the set $D(G)$ is dense and its complement is meager, and the set $D(G) \cap T$ is dense in T with a meager complement. By 6.78 the set D_T is open dense in G. Thus $\overline{\Gamma(G)}$ contains $T \times G$. Since T was arbitrary, by the Maximal Torus Theorem 6.30 it follows that $\overline{\Gamma(G)}$ contains $G\times G$. It therefore suffices now to show that $(g,h)\in \big(D(G)\cap T\big)\times\big(D(G)\cap D_T(G)\big)$ implies that $H\stackrel{\rm def}{=}\overline{\langle g,h\rangle}=G$. By 6.30 and the definition of $D_T(G)$ we find elements $(X,Y)\in\mathfrak{t}\times\mathbf{C}_T$ such that $\exp_G X=g$ and $\exp_G Y=h$. Since $\exp_G X=g\in\mathbf{g}(T)$, by 6.81(vi), the element X belongs to $\mathfrak{t}$ and satisfies Condition (i) of 6.78, and since $Y\in\mathbf{C}_T$, the element Y satisfies Condition (ii) of 6.78. Thus Proposition 6.78 shows that $\mathfrak{g}'\subseteq\overline{\langle X,Y\rangle}$. Since $g\in D(G)\cap T$, by 6.81(iv, v) we conclude that $T=\overline{\langle g\rangle}\subseteq H$, whence $\mathfrak{t}\subseteq\mathfrak{h}$. Thus $\mathfrak{g}=\mathfrak{t}+\mathfrak{g}'\subseteq\overline{\mathfrak{h}}=\mathfrak{h}$. Therefore $G\subseteq H$ which shows $G=H$. This completes the proof of the theorem. □

The theorem gives us some information on the location of the pairs (g,h) which have to generate a dense subgroup. It gives us no information on the algebraic structure of these groups. There is a method in the spirit of universal algebra which, together with certain information pertaining to compact Lie groups, will tell us that there are plenty of pairs among these which generate free subgroups.

The elements of the free group $F(\xi,\eta)$ in two generators are 1, and monomials $\mathbf{w}(\xi,\eta)$ of the form $\xi^{n_1}\eta^{n_2}\cdots\xi^{n_j}$, $\xi^{n_1}\eta^{n_2}\cdots\eta^{n_j}$, $\eta^{n_1}\xi^{n_2}\cdots\xi^{n_j}$, or $\eta^{n_1}\xi^{n_2}\cdots\xi^{n_j}$, with $j=1,2,\ldots,n_j\in\mathbb{Z}\backslash\{0\}$, the multiplication being implemented by associative juxtaposition and the obvious rules such as $1\gamma=\gamma 1=\gamma$ for all $\gamma\in F(\xi,\eta)$, $\xi^0=\eta^0=1$, and $\xi^m\xi^n=\xi^{m+n}$, $\eta^m\eta^n=\eta^{m+n}$. The monomials $\mathbf{w}(\xi,\eta)$ are sometimes called *words*; the identity 1 is the *empty word*. A homomorphism of groups $\varphi\colon F(\xi,\eta)\to G$ is uniquely determined by the elements $g=\varphi(\xi)$ and $h=\varphi(\eta)$. If $\mathbf{w}(\xi,\eta)\in F(\xi,\eta)$ is a word, then we write $w(g,h)=\varphi\big(\mathbf{w}(\xi,\eta)\big)$; e.g. if $\mathbf{w}(\xi,\eta)=\xi\eta$, then $w(g,h)=gh$. The relations $\mathbf{w}(\xi,\eta)\in\ker\varphi$ and $w(g,h)=1$ are equivalent. Thus each word $\mathbf{w}(\xi,\eta)$ yields a function $w\colon G\times G\to G$. The subgroup $\langle g,h\rangle$ of G is free in two generators iff for every nonempty word $\mathbf{w}(\xi,\eta)$ we have $w(g,h)\neq 1$. We shall write

$$\text{(W)}\qquad U\big(\mathbf{w}(\xi,\eta)\big)\stackrel{\rm def}{=}w^{-1}(G\setminus\{1\})=\{(g,h)\in G\times G\mid w(g,h)\neq 1\}.$$

Lemma 6.83. *Let G be a connected linear Lie group. For each nonempty word $\mathbf{w}(\xi,\eta)\in F(\xi,\eta)$, the subset $U\big(\mathbf{w}(\xi,\eta)\big)$ is either empty or open dense.*

Proof. The continuity of $w\colon G\times G\to G$ and the fact that in a Hausdorff space G, the set $G\setminus\{1\}$ is open show that the set $U\big(\mathbf{w}(\xi,\eta)\big)$ is open. We assume that it is nonempty and prove its density by showing that for every nonempty open subset W of $G\times G$, the set $w(W)$ fails to be $\{1\}$. Suppose the contrary and assume that the closed set $w^{-1}(1)=\{(g,h)\in G\times G\mid w(g,h)=1\}$ has nonempty interior. By 5.36 every linear Lie group is an analytic group. Hence $w\colon G\times G\to G$ is an analytic function between analytic manifolds (cf. paragraph following 5.34). The constant function $e\colon G\times G\to G$ with value 1 is trivially analytic. The two analytic functions w and e agree on some nonempty open subset of the connected analytic manifold G; therefore they agree. But $w=e$ means $U\big(\mathbf{w}(\xi,\eta)\big)=\emptyset$, contrary to our assumption. This contradiction proves the lemma. □

We should make some remarks about the status of the prerequisites we made in the proof of the last lemma. In Chapter 5 we stated that we would not utilize the analyticity of a linear Lie group which we proved in 5.36; but we used it now. Secondly, we used the fact that two analytic functions on a connected analytic manifold agree if they agree on some nonempty open subset. In E5.5(iii) we proved this for analytic functions between open subsets of Banach spaces. But the definition of an analytic manifold (given in the paragraph preceding 5.35) permits readily a proof of the more general fact we used here.

Recall (e.g. from the discussion in Chapter 2 preceding Theorem 2.3) that a *Baire space* X is a topological space in which for every meager set M (i.e. a countable union of nowhere dense closed sets) the interior of $\overline{M}$ is empty (cf. References cited there). Every locally compact Hausdorff space and every completely metrizable space is a Baire space; we shall use here only the fact that a locally compact Hausdorff space is a Baire space.

Lemma 6.84. *For a finite dimensional connected linear Lie group G, the following statements are equivalent:*

(1) *G contains a free group on two generators.*

(2) *There is a dense subset D of $G \times G$ such that $(g, h) \in D$ implies that the subgroup $\langle g, h\rangle$ is free and that $(G \times G) \setminus D$ is meager.*

Proof. Trivially (2)⇒(1). Assume (1). Then there is a free subgroup $\langle g, h\rangle$ of two generators. Then $w(g, h) \neq 1$ for all $\mathbf{w}(\xi, \eta) \in F(\xi, \eta) \setminus \{1\}$. Thus none of the countably many open sets $U\big(\mathbf{w}(\xi, \eta)\big)$ is empty. Hence by Lemma 6.83 all of these sets are dense. Hence

$$D \stackrel{\text{def}}{=} \bigcap\nolimits_{1 \neq \mathbf{w}(\xi,\eta) \in F(\xi,\eta)} U\big(\mathbf{w}(\xi, \eta)\big)$$

is dense in $G \times G$ since $G \times G$ is a Baire space. □

The preceding lemmas remain intact for Lie groups that are not necessarily linear.

Lemma 6.85. *The groups* SO(3) *and* SU(2) *both contain a free group on two generators.*

Proof. If SO(3) contains a free group $\langle g, h\rangle$ and $p\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering morphism, then $\langle \widetilde{g}, \widetilde{h}\rangle$ is free in SU(2) for any $\widetilde{g}$ and $\widetilde{h}$ with $p(\widetilde{g}) = g$ and $p(\widetilde{h}) = h$. It therefore suffices to exhibit a nonabelian free group of rotations of the euclidean space $\mathbb{R}^3$. This is a classical task which we outline in Exercise E6.17. □

Exercise E6.17. Show that the following rotations by an angle of arccos $\frac{1}{3}$ generate a free group in SO(3):

$$\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ -\frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{8}}{3} \\ 0 & -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{pmatrix}.$$

(For details cf. [368].) □

The context of the preceding exercise is that of the so-called Hausdorff–Banach–Tarski Paradox for which [368] is a comprehensive reference. Hausdorff shows in [137], p. 469ff that SO(3) contains a rotation a of order 2 and a rotation b of order 3 which generate a subgroup isomorphic with the coproduct $\mathbb{Z}(2) * \mathbb{Z}(3)$ in the category of groups (i.e. a product in the opposite category: see Appendix A3.43); such a coproduct is called a *free product*. The elements $g = ab$ and $h = ab^2$ then generate a free group. The free product $\mathbb{Z}(2) * \mathbb{Z}(3)$, incidentally, is isomorphic to the group $\mathrm{PSl}(2, \mathbb{Z})$ of all fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$ of the Riemann Sphere with $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$ and a, b, c, $d \in \mathbb{Z}$ (see e.g. [339], p. 187, 188). In particular this implies that SO(3) contains an algebraic copy of $\mathrm{PSl}(2, \mathbb{Z})$.

Proposition 6.86. *Let G be a nonabelian connected compact Lie group. Then the set $\Phi(G)$ of all pairs $(g, h) \in G \times G$ such that $\langle g, h \rangle$ is free is dense in $G \times G$ and its complement is meager.*

Proof. By 6.48, the group G contains a copy of SO(3) or SU(2). Then by Lemma 6.85 it contains a nonabelian free group of two generators. Then Lemma 6.84 completes the proof. □

The following exercise demands some extra prerequisites from advanced real analysis.

Exercise E6.18. Prove the following complement of Proposition 6.86:

The complement of $\Phi(G)$ has Haar measure zero in $G \times G$.

[Hint. Let $(g, h) \in \Phi(G)$; find open neighborhoods U and U' of 0 in $\mathfrak{g}$ such that $(X, Y) \mapsto (g \exp_G X, h \exp_G Y)$ is a homeomorphism $\alpha: U \times U \to V$ onto an open neighborhood V of (g, h) in $G \times G$, that $X \mapsto gh(\exp_G X)$ is a homeomorphism of $\beta: U' \to W$ onto an open neighborhood W of gh in G, and that

$$\mu: G \times G \to G,$$

$\mu(x, y) = xy$, maps V into W. Pick $\dim \mathfrak{g}$ linearly independent functionals $\omega_j: \mathfrak{g} \to G$. Then each $(\omega_j | U_1) \circ \beta^{-1} \circ \mu \circ \alpha: U \times \mathbb{R}$ is analytic. (Cf. 5.34ff.).

Show that, in view of this argument, it suffices to know that (i) the inverse image of any singleton under an analytic function $f: O \to \mathbb{R}$ of a connected open subset O of $\mathbb{R}^m$ is either O or a Lebesgue zero set, and (ii) the exponential function of a compact Lie group G maps any bounded Lebesgue measure zero set of $\mathfrak{g}$ to a Haar measure zero set of G.

For a proof of (i) note that for $m = 1$ each $f^{-1}(r)$ without interior points is discrete, and that we have

$$\lambda_n\bigl(f^{-1}(r)\bigr) = \int_{-\infty}^{\infty} \lambda_{n-1}(\{(x_1, \dots, x_n - 1) \in \mathbb{R}^{n-1} \mid f(x_1, \dots, x_n) = r\})\, d\lambda.$$

Use induction. For a proof of (ii) use the fact that the normalized Haar integral on a compact Lie group is integration with respect to a unique invariant volume form of integral 1.] ☐

The following theorem finally generalizes information given in Auerbach's Generation Theorem 6.82.

THE GENERATION THEOREM REVISITED

Corollary 6.87. *Let G be a nonabelian connected compact Lie group and let*

$$\Omega(G) = \{(g, h) \in G \times G \mid \langle g, h\rangle \text{ is nonabelian free and dense in } G\}.$$

Then $\Omega(G)$ is dense in $G \times G$.

For every maximal torus T, the set $(T \times G) \cap \Omega(G)$ is dense with a meager complement in $T \times G$.

Proof. As in the proof of 6.82, it suffices to prove the second part. If $\langle g, h\rangle$ is one of them, then there is an $x \in G$ such that $xgx^{-1} \in T$ and $\langle xgx^{-1}, xhx^{-1}\rangle = x\langle g, h\rangle x^{-1}$ is free. Thus $(T \times G) \cap U\bigl(\mathbf{w}(\xi, \eta)\bigr)$ is open dense in $T \times G$. With the notation and the information in the proof of 6.82 and with

$$D = \bigcap\nolimits_{1 \neq \mathbf{w}(\xi,\eta) \in F(\xi,\eta)} U\bigl(\mathbf{w}(\xi, \eta)\bigr)$$

we get from the Baire Category Theorem that

$$B \overset{\text{def}}{=} \Bigl(\bigl(D(G) \cap T\bigr) \times \bigl(D(G) \cap D_T(G)\bigr)\Bigr) \cap D$$

is dense in $T \times G$ and has a meager set as complement. By Theorem 6.82 and Lemma 6.84 however, $B \subseteq \Omega$. This completes the proof of the corollary. ☐

The Topology of Connected Compact Lie Groups

We shall explore the rudiments of a cohomology theory of connected compact Lie groups enough so that we shall be able to determine when a compact Lie group is a sphere. Indeed we consider a connected compact Lie group G, fix a field K of characteristic 0 such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, and denote by $H(G)$ the graded cohomology group $H^*(G, K) = H^0(G, K) \oplus \cdots \oplus H^n(G, K)$ with $n = \dim G$. Since the underlying space is a compact manifold, the Alexander–Čech–Spanier–Wallace cohomology agrees with singular cohomology, and in the case of $K = \mathbb{R}$

other options are available from the analytic structure of G as a manifold. However, for an understanding of the basic features of the cohomology of a compact group we do not have to go into the mechanics of cohomology theory; we need to record that it provides a contravariant functor

$$H\colon \mathbb{CTOP} \to \mathbb{AB}_K^*$$

from the category of compact spaces and continuous functions to the category of graded K vector spaces such that $H(M)$ is a finite dimensional vector space for a compact manifold M, that we have a natural isomorphism between the graded vector spaces $H(X \times Y)$ and $H(X) \otimes^* H(Y)$; this is the content of the Künneth Theorem (cf. [338]), and that

$$H^m(\mathbb{S}^n, K) \cong \begin{cases} K & \text{if } m = 0 \text{ or } m = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

We have introduced in Appendix 3 the concept of a *connected graded commutative Hopf algebra* (Definition A3.65 and the paragraph preceding A3.69). A very important and typical example is the exterior algebra $\bigwedge V$ over a graded vector space $V = V^1 \oplus V^2 \oplus \cdots \oplus V^N$ with the homogeneous components V^j. The exterior algebra has a canonical comultiplication $\bigwedge V \to (\bigwedge V) \otimes^* (\bigwedge V) \cong \bigwedge(V \oplus V)$ induced by the diagonal morphism $V \to V \oplus V$ of graded vector spaces. For the details concerning graded Hopf algebras we refer to the last section of Appendix 3. We explain there the importance, for any graded Hopf algebra A, of the graded vector subspace $P(A)$ of A of all primitive elements $x \in A$ which are characterized by the equation $c_A(x) = x \otimes 1 + 1 \otimes x$ with the comultiplication c_A of A. Finally, in the Appendix we proved purely algebraic characterisation theorems for graded commutative Hopf algebras which allowed us to establish, using only very basic properties of cohomology, in Theorem A3.90 the Hopf–Samelson Theorem which describes the cohomology Hopf algebra of a connected compact monoid G under the hypothesis that $\dim_K H(G) < \infty$. This applies at once to the case of a compact Lie group and gives us, via specialization, the following result.

The Hopf–Samelson Theorem for Connected Compact Lie Groups

Theorem 6.88. *Let G be a connected compact Lie group G. Let $H(G) = K \oplus H^1(G,K) \oplus \cdots \oplus H^n(G,K)$ denote the cohomology Hopf algebra of G and let $P\big(H(G)\big)$ denote the graded vector subspace of primitive elements of the Hopf algebra $H(G)$. Then the following conclusions hold:*

(i) *There is a natural isomorphism of graded Hopf algebras*

$$H(G) \cong \bigwedge P\big(H(G)\big).$$

(ii) *The graded vector space $P\big(H(G)\big)$ of primitive elements of $H(G)$ has only odd dimensional nonvanishing homogeneous components and determines*

$$H(G) \cong \bigwedge P\big(H(G)\big)^1 \otimes \bigwedge P\big(H(G)\big)^3 \otimes \cdots \otimes \bigwedge P\big(H(G)\big)^{2N-1}$$

uniquely and functorially.

(iii) *Define* $d_{2j-1} \stackrel{\text{def}}{=} \dim P\big(H(G)\big)^{2j-1}$, $j = 1, 2, \ldots, N$ *and define* S *to be the product of spheres*

$$S \stackrel{\text{def}}{=} (\mathbb{S}^1)^{d_1} \times (\mathbb{S}^3)^{d_3} \times \cdots \times (\mathbb{S}^{2N-1})^{d_{2N-1}}.$$

Then the graded commutative K*-algebras* $H(G)$ *and* $H(S)$ *are isomorphic.*

(iv) $\dim G = d_1 + 3d_3 + \cdots + (2N-1)d_{2N-1}$. □

The Hopf–Samelson Theorem associates with a compact Lie group contravariantly functorially the graded vector space $P\big(H(G)\big)$ from which the entire cohomology can be completely reconstituted.

Let G be an n-dimensional connected compact Lie group. For any natural number m define the power function $\pi_m\colon G \to G$ by $\pi_m(x) = x^m$. On a K-vector space V let $\sigma_m = \sigma_m^V$ denote the scalar multiplication given by $\sigma_m(v) = m{\cdot}v$ for $v \in V$.

Example 6.89. Let $T \cong \mathbb{T}^r$ be an r-dimensional torus group, $r \in \mathbb{N}_0$, then the following statements hold:

(i) All homogeneous primitive elements of $H(T)$ have degree 1 and

$$P\big(H(T)\big)^m \cong \begin{cases} K \otimes \widehat{T} & \text{if } m = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

(ii) $H(T) \cong \bigwedge(K \otimes \widehat{T})$.

(iii) The endomorphism induced on $H^r(T)$ by $x \mapsto x^m$ is $H^r(\pi_m^T) = \sigma_{m^r}^{H^r(T)}$, i.e. if $\xi \in H^r(T)$ then $H^r(\pi_m^T)(\xi) = m^r{\cdot}\xi$.

Proof. By Theorem 6.88 we have $H(T) \cong \bigwedge P\big(H(T)\big)$ naturally. Thus (i) implies (ii). In order to determine $\dim_K P\big(H(T)\big)^m$ we note that $H(\mathbb{S}^1) = K \oplus H^1(\mathbb{S})$ with $H^1(\mathbb{S}) \cong K$. Then $H\big((\mathbb{S}^1)^r\big) \cong \bigotimes^r H(\mathbb{S}^1) \cong \bigwedge K^r$. Hence $\dim_K P\big(H(T)\big) = r$, i.e. on the full subcategory $\mathcal{F}$ of finitely generated free abelian groups, the assignment $F \mapsto P\big(H(\widehat{F})\big)$ is a functor into $\mathbb{AB}_K^*$, the category of graded K-vector spaces, with

$$\dim_K P\big(H(\widehat{F})\big)^m = \begin{cases} \operatorname{rank} F & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus this function is naturally isomorphic to

$$F \mapsto \bigoplus_{m \in \mathbb{N}_0} V^m, \quad V^m = \begin{cases} K \otimes F & \text{if } m = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Accordingly, $P\big(H(T)\big)^1 \cong P\big(H(T)\big) \cong K \otimes \widehat{T}$ and $P\big(H(T)\big)^m = \{0\}$ for $m > 1$.

(iii) By passing to the dual we have $\widehat{\pi_m^T} = \pi_m^{\widehat{T}}$. Since $\widehat{T}$ may be identified with $\mathbb{Z}$, the morphism $\pi_m^{\mathbb{Z}}$ is simply multiplication with the integer m. Now $\bigwedge^r(K \otimes \pi_m^{\mathbb{Z}^r})$ is multiplication with the determinant of the scalar multiplication by m on the r-dimensional vector space $K \otimes \mathbb{Z}^r$ and therefore is scalar multiplication with m^r. □

Lemma 6.90. *Assume that G is an n-dimensional connected compact Lie group. The K-linear function $H(\pi_m): H(G) \to H(G)$ maps $P\big(H(G)\big)$ into itself, and*

(i) $$H(\pi_m)|P\big(H(G)\big) = \sigma_m^{P\left(H(G)\right)},$$

(ii) $$H^n(\pi_m) = \sigma_{m^q}^{H^n(G)} \quad \text{with } q = d_1 + d_3 + \cdots + d_{2N-1},$$

(iii) $$\dim_K P\big(H(G)\big) = \log_m e, \quad \text{where } \sigma_e^{H^n(G)} = H^n(\pi_m).$$

Proof. (i) Let $\Delta_m: G \to G^m$ denote the diagonal map, $\Delta_m(x) = (x, \ldots, x)$, and $\nabla_m: G^m \to G$ the multiplication, $\nabla_m(x_1, \ldots, x_m) = x_1 \cdots x_m$. Then $\pi_m = \nabla_m \circ \Delta_m$. Now

$$\mu_m = \Big(\bigotimes^m H(G) \xrightarrow{\cong} H(G^m) \xrightarrow{H(\Delta_n)} H(G)\Big)$$

is the algebra multiplication given by $\mu_m(h_1 \otimes \cdots \otimes h_m) = h_1 \cdots h_m$ by the definition of the algebra multiplication in $H(G)$ (cf. Appendix 3, A3.67(ii), A3.70). The algebra morphism

$$c_m = \Big(H(G) \xrightarrow{H(\nabla_m)} H(G^m) \xrightarrow{\cong} \bigotimes^m H(G)\Big)$$

by the definition of the comultiplication and its associativity is recursively given by $c_1 = c$, $c_{k+1} = (c_k \otimes \mathrm{id}_{H(G)}) \circ c: H(G) \to \Big(\bigotimes^k H(G)\Big) \otimes H(G) \cong \bigotimes^{k+1} H(G)$.

Now let $x \in P\big(H(G)\big)$. We claim that

$$c_m(x) = x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes 1 \otimes x.$$

Indeed, this is readily verified by induction. Now $\mu_m(x \otimes 1 \otimes 1 \otimes \cdots \otimes 1) = x{\cdot}1{\cdot}1 \cdots 1 = x$, etc. Hence

$$H(\pi_m)(x) = H(\Delta_m)H(\nabla_m)(x) = \mu_m\big(c_m(x)\big) = m{\cdot}x = \sigma_m^{P(H(G))}(x).$$

In particular, $H(\pi_m)$ maps $P\big(H(G)\big)$ into itself.

(ii) If V is a q-dimensional vector space, then $\dim_K \bigwedge^q V = \binom{q}{q} = 1$, and if $\varphi: V \to V$ is a morphism, then $\big((\bigwedge^q)(\varphi)\big)(v) = \det \varphi{\cdot}v$. We apply this to $V = P\big(H(G)\big)^{2j-1}$ and $\varphi = H(\pi_m)|V$. Thus $\bigwedge(\varphi)(v) = \det \varphi{\cdot}v = m^{d_{2j-1}}$. Therefore

$$\bigotimes_{j=1}^N \bigwedge^{d_{2j-1}} H(\pi_m)|P\big(H(G)\big)^{2j-1} = \sigma_{m^{d_1} \cdot m^{d_3} \cdots m^{d_{2N-1}}} = \sigma_{m^q}.$$

(iii) follows from the preceding results. □

Recall that the *rank* of a connected compact Lie group G is the dimension of a maximal torus. In order to make further progress we consider again the map

$$\omega: \frac{G}{T} \times T \to G, \quad \omega(gT, t) = gtg^{-1}$$

of Lemma 6.29 which we used heavily in the proof of the Maximal Torus Theorem; the surjectivity of this map is equivalent to the statement that every element of

G lies in a conjugate of T. Some of the new information we will now uncover is of independent interest; here we will use it mainly to derive information on the degree of the power map.

We shall shortly show that the function ω induces a surjective linear map in top cohomology:

Lemma 6.91. *For an n-dimensional connected compact Lie group G,*

$$H^n(\omega)\colon H^n(G) \to H^n(\tfrac{G}{T} \times T) \quad \textit{is an isomorphism.}$$

However before we prove 6.91 after 6.98 below, we shall draw conclusions which show the significance of this lemma.

Lemma 6.92. *If r is the rank of a connected compact Lie group G of dimension n, then for any $m \in \mathbb{N}$ we have $H^n(\pi_m) = \sigma_{m^r}^{H^n(G)}$ for cohomology over a field K of characteristic* 0.

Proof. Let $\omega\colon \frac{G}{T} \times T \to G$ be as above, given by $\omega(gT, t) = gtg^{-1}$. Set $n = \dim G$. For $g \in G$ and $t \in T$ and for any natural number m we have $g\big(\pi_m(t)\big)g^{-1} = gt^m g^{-1} = (gtg^{-1})^m = \pi_m(gtg^{-1})$. In other words, $\omega\big(gT, \pi_m(t)\big) = \pi_m\big(\omega(gT,t)\big)$; in other words, the following diagram is commutative:

$$(*) \qquad \begin{array}{ccc} \frac{G}{T} \times T & \xrightarrow{\omega} & G \\ {\scriptstyle \mathrm{id}_{\frac{G}{T}} \times \pi_m^T}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_m^G} \\ \frac{G}{T} \times T & \xrightarrow[\omega]{} & G. \end{array}$$

This entails a commutative diagram of graded commutative K-algebras

$$(**) \qquad \begin{array}{ccccc} H(G) & \xrightarrow{H(\omega)} & H(\frac{G}{T} \times G) & \xleftarrow{\cong} & H(\frac{G}{T}) \otimes^* H(T) \\ {\scriptstyle H(\pi_m^G)}\Big\downarrow & & {\scriptstyle H\left(\frac{G}{T} \times \pi_m^T\right)}\Big\downarrow & & {\scriptstyle \mathrm{id}_{H\left(\frac{G}{T}\right)} \otimes^* H(\pi_m^T)}\Big\downarrow \\ H(G) & \xrightarrow[H(\omega)]{} & H(\frac{G}{T} \times T) & \xleftarrow[\cong]{} & H(\frac{G}{T}) \otimes^* H(T). \end{array}$$

Now we claim that

(1) $\qquad H^k(T) = \{0\} \quad \text{for} \quad k = r+1, r+2, \ldots,$

(2) $\qquad H^k(\frac{G}{T}) = \{0\} \quad \text{for} \quad k = n-r+1, n-r+2, \ldots.$

Indeed T is an r-dimensional compact manifold, and G/T is an $n-r$-dimensional compact manifold by the Tubular Neighborhood Theorem 5.33. Then (1) and (2) follow from [96], p. 314, Theorem 6.8(i). (For homology see also [89], p. 260, Proposition 3.3., [49], p. 344, Theorem 7.8).

Given (1) and (2) we see that the homogeneous component of degree n in $H(\frac{G}{T}) \otimes^* H(T)$ is $H^{n-r}(\frac{G}{T}, K) \otimes H^r(T, K)$. From diagram $(**)$ we then derive the following commuting diagram by singling out the homogeneous components of degree $n = \dim G$:

$$
(***)\qquad
\begin{array}{ccccc}
H^n(G) & \xrightarrow{H^n(\omega)} & H^n(\frac{G}{T}\times T) & \xleftarrow{\cong} & H^{n-r}(\frac{G}{T},K)\otimes H^r(T,K)\\
{\scriptstyle H^n(\pi_m^G)}\big\downarrow & & {\scriptstyle H^n(\mathrm{id}_{\frac{G}{T}}\times\pi_m^T)}\big\downarrow & & {\scriptstyle \mathrm{id}_{H^{n-r}(\frac{G}{T},K)}\otimes H^r(\pi_m^T)}\big\downarrow\\
H^n(G) & \xrightarrow[H^n(\omega)]{} & H^n(\frac{G}{T}\times T) & \xleftarrow[\cong]{} & H^{n-r}(\frac{G}{T},K)\otimes H^r(T,K).
\end{array}
$$

From Example 6.89(iii) we have $H^r(\pi_m^T) = \sigma_{m^r}^{H^r(T)}$. Then diagram $(***)$ proves our assertion in view of (1), (2), and Lemma 6.91. □

Hopf's Rank Theorem

Theorem 6.93. *The rank of G agrees with $\dim_K P\big(H(G,K)\big)$ for any field K of characteristic 0.*

There is a K-vector space isomorphism $K\otimes\widehat{T}\cong P\big(H(G)\big)$ for any maximal torus T of G.

Proof. This follows from Lemmas 6.90 and 6.92. □

In particular this means that the rank of G is a topological invariant: two homeomorphic connected compact Lie groups have the same rank. In fact, a better statement is still true. Two spaces X and Y are called *homotopy equivalent* if there are continuous maps $f\colon X\to Y$ and $g\colon Y\to X$ such that fg is homotopic to id_Y and gf is homotopic to id_X. (Cf. Appendix 3, Example A3.3.7, Exercise EA3.1ff.) The cohomology functor has the property that for two homotopic maps φ and φ' we have $H(\varphi)=H(\varphi')$. Hence two homotopy equivalent connected compact Lie groups have the same rank.

Hopf's Theorem on the Cohomology of Lie Groups

Corollary 6.94. *The cohomology of a connected compact Lie group G over a field of characteristic 0 is that of a product of r odd dimensional spheres, where $r=\operatorname{rank}G$.*

Proof. This follows at once from Theorem 6.88 and Theorem 6.93. □

In particular, no space which is homotopy equivalent to a product of spheres one of which is even dimensional can carry a Lie group structure.

With very little extra effort we can now prove a theorem which has been known since the early history of global Lie group theory when it was observed as a consequence of the classification of simple compact Lie groups and the structure theorem for Lie groups. The proof based on the results of Hopf and Samelson elucidates, however, the deeper reasons why the theorem holds.

The Sphere Theorem for Connected Compact Lie Groups

Theorem 6.95. *If G is a connected compact Lie group on a sphere then $G \cong \mathbb{S}^1 \cong \mathrm{SO}(2)$ or $G \cong \mathbb{S}^3 \cong \mathrm{SU}(2)$.*

Proof. From 6.88 and the hypothesis that G is homeomorphic to a sphere we conclude $q = \dim_K P\big(H(G)\big) = 1$. By 6.93 this implies $\operatorname{rank} G = \dim T = q = 1$. Then 6.49(vii) implies $G \cong \mathbb{S}^1$ or $G \cong \mathrm{SO}(3)$ or $G \cong \mathbb{S}^3 \cong \mathrm{SU}(2)$. The space $\mathrm{SO}(3)$ is not simply connected (having $\mathbb{S}^3$ as universal covering) and so cannot be a sphere. Hence the assertion is proved. □

We finally must prove Lemma 6.91; the remainder of this section is devoted to this proof.

The Weyl group $\mathcal{W} \stackrel{\text{def}}{=} N(T,G)/T$ operates on $\frac{G}{T} \times T$ via $nT{\cdot}(gT, s) = (gn^{-1}T, nsn^{-1})$. This action is *free*, i.e. all isotropy groups are trivial: Indeed $nT{\cdot}(gT,s) = (gT,s)$, iff $gn^{-1}T = gT$ and $nsn^{-1} = s$ iff $n \in T$ iff $nT = T$, the identity of $\mathcal{W}$. We set $M \stackrel{\text{def}}{=} \left(\frac{G}{T} \times T\right)/\mathcal{W}$ and note that the orbit map $q\colon \frac{G}{T} \times T \to M$ is a covering map (see Appendix 2, Definition A2.1, Examples A2.3(iii)).

We observe that $\omega\big(nT{\cdot}(gT,s)\big) = \omega(gn^{-1}T, nsn^{-1}) = gn^{-1}(nsn^{-1})ng^{-1} = \omega(gT,s)$. Hence we have a unique continuous map $\Omega\colon M \to G$ such that the following diagram commutes.

(†)
$$\begin{array}{ccc} \frac{G}{T} \times T & \xrightarrow{\ \omega\ } & G \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_G} \\ M & \xrightarrow[\ \Omega\ ]{} & G. \end{array}$$

We note that $\mathcal{W}{\cdot}(gT,s) \subseteq \omega^{-1}\omega(gT,s)$.

The map Ω is far from being a covering; since $\omega^{-1}(1) = \frac{G}{T} \times \{1\}$, the set $\Omega^{-1}(1) = q\big(\omega^{-1}(1)\big) = (\frac{G}{T} \times \{1\})/\mathcal{W} \cong G/N(T,G)$ is a submanifold (which, in the case of $G = \mathrm{SO}(3)$ is the real projective plane) while in other points Ω induces a very strong form of local homeomorphism, as the following lemma will show.

Lemma 6.96. *Let $t \in T$ be a generator, i.e. $T = \overline{\langle t \rangle}$ and set $m \stackrel{\text{def}}{=} \mathcal{W}{\cdot}(T,t) \in M$. There is an open neighborhood U of m in M which satisfies*
- (a) *$\Omega^{-1}\Omega(U) = U$, and*
- (b) *$\Omega|U\colon U \to W$, $W = \Omega(U)$, is a homeomorphism onto an open neighborhood of t in G.*

Proof. We proceed in steps.

Step 1. $\omega^{-1}\big(\omega(T,t)\big) = \mathcal{W}{\cdot}(T,t) = m$.

Proof of Step 1. The relation $\omega(gT,s) = \omega(T,t)$ is equivalent to $t = gsg^{-1} \in gTg^{-1}$. This implies $T = \overline{\langle t \rangle} \subseteq gTg^{-1}$ which, by the maximality of T, is equivalent to $T = gTg^{-1}$, i.e. to $g \in N(T,G)$. Then $(gT,s) = (gT, g^{-1}tg) = g^{-1}T{\cdot}(T,t)$. Thus the left side is contained in the right side; the converse inclusion is always true by the preceding remarks.

Step 2. The function ω maps a suitable open neighborhood of (T, t) homeomorphically onto an open neighborhood of t.

Proof of Step 2. We have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^+$ with an orthogonal complement $\mathfrak{t}^+$ of $\mathfrak{t}$ in $\mathfrak{g}$.

By the Tubular Neighborhood Theorem for Subgroups 5.33(ii) and its proof there is an open neighborhood V of 0 in $\mathfrak{t}^+$ such that

$$\tau: V \times T \to (\exp V)T, \quad \tau(X, t) = (\exp X)t$$

is a homeomorphism onto an open neighborhood of T in G.

Since $T = \overline{\langle t \rangle}$, the centralizer of t in G agrees with the centralizer of T in G, and the latter is T by 6.21(iii). Then 5.45 and 5.55 show that there is an open neighborhood U^+ of 0 contained in $\mathfrak{t}^+$ with $U^+ \subseteq V$, an open neighborhood U^0 of 0 in $\mathfrak{t}$, and an open neighborhood W_t of t in G such that the function

$$\Psi: U^+ \oplus U^0 \to W_t, \quad \Psi(Y \oplus Z) = (\exp Y)t \exp Z(\exp -Y)$$

is a homeomorphism.

Let $U^* = (\exp U^+)T/T \subseteq \frac{G}{T}$ and $U_t^* = t \exp_T(U^0)$. Then $p: U^+ \to U^*$, $p(Y) = (\exp Y)T$ and $e: U^0 \to tU_0^*$, $e(Z) = t \exp Z$ are homeomorphisms. Then the function

$$\Psi \circ (p^{-1} \times e^{-1}): U^* \times U_t^* \to W_t$$

is given as follows: Let $gT \in U^*$, say $gT = p(Y)$, and $s \in U_t^*$, say $s = e(Z) = t \exp Z$; then $\Psi\big(p^{-1}(gT), e^{-1}(s)\big) = \Psi(Y \oplus Z) = (\exp Y)t(\exp Z)(\exp -Y) = gsg^{-1} = \omega(gT, s)$. Hence

$$\omega | U^* \times U_0^*: U^* \times U_t^* \to W_t$$

is the required homeomorphism.

Step 3. There is an open neighborhood U of m in M which is mapped homeomorphically under Ω.

Proof of Step 3. Since U^* is homeomorphic to U^+ and U_0^* to U^0, the neighborhood $U^* \times U_0^*$ of (T, t) has arbitrarily small open neighborhoods. Let U_1 be one of them which is so small that $\big(nT, (gT, s)\big) \mapsto nT{\cdot}(gT, s) : \mathcal{W} \times U_1 \to \mathcal{W}{\cdot}U_1$ is a homeomorphism. (Cf. A2.3(iii).) Then q maps U_1 homeomorphically onto the open neighborhood $U = q(U_1)$ of $m \in M$ and then Ω maps U homeomorphically to $W \stackrel{\text{def}}{=} \omega(U_1)$

Step 4. The neighborhood U of m in M may be chosen so small, that $\Omega^{-1}(W) = U$.

Proof of Step 4. By Step 1, the set $\Omega^{-1}(t)$ is the singleton $\{m\}$. The open neighborhood U is mapped homeomorphically under Ω onto W. Hence $w \in W$ implies $\Omega^{-1}(w) \cap U$ is singleton. Suppose the claim of Step 4 is false. Then for every V from the neighborhood filter $\mathcal{U}$ of m in M there is a pair $(m_V, m'_V) \in V \times (M \setminus U)$ such that $\Omega(m_V) = \Omega(m'_V)$. Then $m = \lim_{V \in \mathcal{U}} m_V$. Since $M \setminus U$ is compact, the net $(m_V)_{V \in \mathcal{U}}$ has a subnet converging to $m' \in M \setminus U$; then $\Omega(m') = \Omega(m)$. This contradicts $\Omega^{-1}\Omega(m) = \{m\}$ and thus proves the claim of Step 4 and thereby the lemma. □

We extract information from this lemma which is of independent interest.

Proposition 6.97. *Let G be an n-dimensional connected compact Lie group, T a maximal torus and $t \in T$ an element with $T = \overline{\langle t \rangle}$. Let the Weyl group $\mathcal{W}$ act on $\frac{G}{T} \times T$ via $nT{\cdot}(gT, t) = (gn^{-1}, ntn^{-1})$. Then*

(i) *t has a neighborhood D homeomorphic to a closed n-cell such that $C_t \stackrel{\text{def}}{=} \omega^{-1}(D) \subseteq \frac{G}{T} \times T$ has the following properties:*
   (a) *C_t is invariant under the action of $\mathcal{W}$.*
   (b) *The component C of (T, t) in C_t is mapped homeomorphically onto D by $\omega|C{:}\, C \to D$.*
   (c) *The function $(\gamma, c) \mapsto \gamma(c) : \mathcal{W} \times C \to C_t$ is an equivariant homeomorphism, where $\mathcal{W}$ acts on $\mathcal{W} \times C_t$ via $\gamma{\cdot}(\gamma', c) = (\gamma\gamma', c)$.*
   (d) *$(\forall(\gamma, c) \in \mathcal{W} \times C)\, \omega(\gamma{\cdot}c) = \omega(c)$.*

(ii) *$\omega^{-1}(1) = \frac{G}{T} \times \{1\} \cong G/T$.*

Proof. (i) Let $U \subseteq M$ be as in Lemma 6.96. Since $p{:}\, \frac{G}{T} \times T \to M$ is a covering map we may assume that U is so small that it is a component of $p^{-1}(U)$. Let C^* be a closed n-cell neighborhood of $\mathcal{W}{\cdot}(T, t)$ contained in U and set $D \stackrel{\text{def}}{=} \Omega(C^*)$. Now $C_t = p^{-1}(C^*)$ and the assertions now follow from the fact that p is a covering with $\mathcal{W}$ as Poincaré group (cf. A2.17) and from Lemma 6.96 and the remarks preceding it.

(ii) is straightforward and was observed above. □

With additional input from basic algebraic topology we can do the following exercise.

Exercise E6.19. Prove the following proposition.

For a connected compact Lie group G with a maximal torus T, the quotient space G/T is a simply connected orientable manifold.

[Hint. By 6.31(ii), $Z(G) \subseteq T$. Since $\big(G/Z(G)\big)/((T/Z(G)\big) \cong G/T$ we may assume that G is semisimple centerfree. Then let $\widetilde{G}$ the universal covering group of G; then $G \cong \widetilde{G}/Z(\widetilde{G})$. Now $G/T \cong \big(\widetilde{G}/Z(G)\big)/\big(T^*/Z(\widetilde{G})\big) \cong \widetilde{G}/T^*$ where T^* is a maximal torus of $\widetilde{G}$ mapping onto T under the universal covering. Hence we may assume that G is semisimple and simply connected. In particular, $\pi_1(G) = \{1\}$ by EA2.6. The coset map $q{:}\, G \to G/T$ is a fibration by the Tubular Neighborhood Theorem 5.33(ii) (see [338], p. 66, p. 96, Theorem 13). The exact homotopy sequence of q (see [338], p. 377, Theorem 10) yields an exact sequence

$$\pi_1(G) \xrightarrow{\pi_1(q)} \pi_1(G/T) \to \pi_0(T),$$

in which the ends are singleton, $\pi_0(T)$ being the group of pathcomponents of T. Hence $\pi_1(G/T)$ is singleton. Thus G/T is simply connected. A simply connected compact manifold is orientable (see e.g. [89], p. 255, 2.12).] □

In the following lemma we must use more cohomology theory in the area of compact manifolds than we have used so far, or in the appendix. However, most of the information we use is on the level of properties shared by *all* cohomology theories. We therefore will refer to the book by Eilenberg and Steenrod [96].

If (X, Y) is a pair of compact spaces with $Y \subseteq X$, then (X, Y) is called a *relative n-cell* if $X \setminus Y$ is homeomorphic to $\mathbb{R}^n$.

Lemma 6.98. (i) *If (M, N) is a pair of compact spaces and a relative n-cell and M is a manifold with $H^n(M, K) \neq \{0\}$, then the inclusion $j: (M, \emptyset) \to (M, N)$ induces an isomorphism $H^n(j): H^n(M, N; K) \to H^n(M, K)$.*

(ii) *Let (M_i, N_i), $i = 1, 2$ denote pairs of compact spaces, $N_i \subseteq M_i$ such that (M_1, N_1) is a relative n-cell, that $H^n(M_i, K) \neq \{0\}$, and that a continuous function $f: M_1 \to M_2$ maps N_1 into N_2 and maps $M_1 \setminus N_1$ homeomorphically onto $M_2 \setminus N_2$. Then $H^n(f, K): H^n(M_2, K) \to H^n(M_1, K)$ is an isomorphism.*

Proof. (i) By [96], p. 314, Theorem 6.8(ii), $H^n(j)$ is surjective. Since K is a field both $H^n(M, N; K)$ and $H^n(M, K) \neq \{0\}$ are one-dimensional K-vector spaces, $H^n(j)$ is an isomorphism.

(ii) Consider the commutative diagram

$$\begin{array}{ccc} H^n(M_2, N_2) & \xrightarrow{j_{M_2}} & H^n(M_2) \\ {\scriptstyle H^n(f)}\downarrow & & \downarrow{\scriptstyle H^n(f)} \\ H^n(M_1, N_1) & \xrightarrow[j_{M_1}]{} & H^n(M_1). \end{array}$$

The horizontal maps are isomorphisms by (i) above, the left vertical map is an isomorphism by [96], p. 266, Theorem 5.4. Hence the right vertical map is an isomorphism as asserted. □

Finally, we finish the missing proof of Lemma 6.91. The manifolds $\frac{G}{T} \times T$ and G are orientable. Hence $H^n(G, K)$ and $H^n(\frac{G}{T} \times T)$ are both nonzero ([96], p. 314, 6.8; [49], p. 347, 7.14). Proposition 6.97 allows us to apply 6.98(ii) with $M_1 = \frac{G}{T} \times T$, $N_1 = M_1 \setminus \operatorname{int} C$, $M_2 = G$, and $N_2 = G \setminus \operatorname{int} C$. Then by Lemma 6.97(ii), $H^n(\omega, K): H^n(G, K) \to H^n(\frac{G}{T} \times T, K)$ is an isomorphism. Thus Lemma 6.91 is proved.

Exercise E6.20. Prove the following statement.

A continuous function $f: M \to G$ between two connected compact manifolds is surjective if there is a point $m \in M$ such that the following two conditions are satisfied:

(i) $f^{-1}\big(f(m)\big) = \{m\}$.

(ii) *There are open neighborhoods U of m and V of $f(m)$ such that $f|U: U \to V$ is a homeomorphism.*

[Hint. Assume the contrary, pick an open n-cell W in $G \setminus f(M)$ and consider $j: G\big(= (G, \emptyset)\big) \to (G, G \setminus W)$ and $k: G \setminus W\big(= (G \setminus W, \emptyset)\big) \to G\big(= (G, \emptyset)\big)$. Then $H(k) \circ H(j) = H(jk) = 0$; observe

$$\begin{array}{ccccc}
H^n(G, G\setminus W) & \xrightarrow{H^n(j)} & H^n(G) & \xrightarrow{H^n(k)} & H^n(G\setminus W) \\
{\scriptstyle H^n(f')}\downarrow & & {\scriptstyle H^n(f)}\downarrow & & \downarrow{\scriptstyle H^n(f'')} \\
H^n(M) & \xrightarrow[\mathrm{id}]{} & H^n(M) & \xrightarrow[\mathrm{id}]{} & H^n(M)
\end{array}$$

and derive a contradiction.] □

Exercise E6.20 provides us with a proof of the surjectivity of the function $\Omega\colon M \to G$ and thus an alternative proof of the surjectivity of the function $\omega\colon \frac{G}{T} \times T \to G$ which was at the root of the Maximal Torus Theorem 6.30. However, a considerable body of information on algebraic topology enters this present approach.

Postscript

The existence of an invariant scalar product on the Lie algebra of a compact Lie group causes us to call such Lie algebras *Hilbert Lie algebras*. Their structure is very quickly derived and is at the heart of the first basic structure theorems. We prove at an early stage that the class of compact Lie groups is closed under the formation of quotients, and under the passage to algebraic commutator subgroups; this latter fact, despite it being a clean group theoretical statement, appears not to be well known or widely used. The fact that the quotient of a compact Lie group, i.e. a compact group without small subgroups is of the same type is not trivial. In fact it is shown in [265] that a quotient group of a topological group with no small subgroups can have small subgroups.

The approach to the maximal torus theory is the one presented by Bourbaki [43]; it avoids the use of such tools as cohomology or degree theory or Stokes' Theorem all of which have been used at this point by one author or another. The Lie algebra version (which we call the Transitivity Theorem) is fairly direct, and on the group level, Bourbaki's proof has the advantage, in our opinion, of being intimately group theoretical while not being devoid of geometric arguments. We prepared for this proof in Chapter 5 by providing Proposition 5.55 which used the Open Mapping Theorem and was proved on the level of our discussion of linear Lie groups in Chapter 5.

Inspired by Scheerer's Theorem [319] that the commutator subgroup of a connected compact Lie group is a topological factor we present Theorem 6.41 which says that it is indeed a semidirect factor—which was proved in [167]. Applications of this Theorem 6.41 appear e.g. in [179].

There is a considerable literature on the structure of compact semisimple Lie algebras $\mathfrak{g}$ and their root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{D\in R} \mathfrak{g}^D$ with respect to the Lie algebra $\mathfrak{t}$ of a maximal torus. We are not exhaustive in this respect. While usually the route leads through the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ we stay in the real domain and use the complex structure already present on $\mathfrak{g}_{\mathrm{eff}} = \bigoplus_{D\in R} \mathfrak{g}^D$ which we have seen in Chapter 3 (cf. 3.54ff.). Thus our approach is not the usual one.

We deal with the Weyl group in Theorem 6.52 in an unconventional fashion. In general we do not delve into matters involving the classification of complex simple Lie algebras or, what amounts to the same, simple compact real Lie algebras; there are many sources accessible for this material. The geometry of root systems and Weyl groups is excellently treated in one of the Bourbaki volumes [42]; what makes this highly recommended reading is the fact that, in contrast with the other volumes of the series on "Groupes et algèbres de Lie" this one is completely self contained and largely accessible on a comparatively elementary level; this commentary certainly does not apply to the chapter on compact Lie groups [43]. With our Theorem 6.52 and the developments leading up to it we have shown that what we called the real root system R of a compact Lie group with respect to a maximal torus is a *root system* in the technical sense of the literature on the subject [42]. Thus the reader who wishes to apply the information on the complete classification of root systems to the classification of compact Lie groups is poised to do so at that point.

The structure theory of the automorphism groups of compact Lie algebras and compact Lie groups is presented rather fully—with the caveat mentioned above that we do not go into the classification of the automorphism groups of simple complex or simple compact Lie algebras. The theory of the automorphism group $\mathrm{Aut}(G)$ of a general compact Lie group G, notably the identification in Theorem 6.67 of the identity component $\mathrm{Aut}(G)_0$ as the group of inner automorphisms $x \mapsto gxg^{-1}: G \to G$ with $g \in G_0$, is due to Iwasawa [217]; the overall presentation here is a bit different.

Lee's Theorem 6.74 [238] that a compact Lie group G contains a finite group E such that $G = G_0E$ and that $G_0 \cap E$ is central in the component G_0 deserves to be better known. In its full power it requires the information that for a semisimple real compact Lie algebra $\mathfrak{g}$, the group $\mathrm{Aut}\,\mathfrak{g}$ of automorphisms splits over its identity component $e^{\mathrm{ad}\,\mathfrak{g}}$ (6.60); the proof we give here follows a suggestion by Karl-Hermann Neeb. Lee's Theorem will be generalized in Chapter 9 to arbitrary compact groups. The fact that for a compact group G the groups G and $G_0 \times G/G_0$ are homeomorphic (trivial in the case of Lie groups but not trivial in the general case) will be discussed in Chapter 10.

The classical sources [16, 235] show that every connected compact Lie group can be topological generated by two elements. Indeed Auerbach's Generation Theorem 6.71 says that the set of ordered pairs (g, h) such that the subgroup $\langle g, h\rangle$ is dense in the connected compact Lie group G is dense in $G \times G$. One might ask whether the subgroup generated by $\{g, h\}$ is a free group. The existence of an abundance of free subgroups in a Lie group belongs to universal topological algebra, as one notices in [273] and [106]. The special question for compact Lie groups requires classical information belonging to the vicinity of the Hausdorff–Banach–Tarski paradox because it needs the existence of one free subgroup of the rotation group of euclidean 3-space. A good source book for this topic is [368]. In this chapter it was proved that if S is the set of ordered pairs (g, h) satisfying the two conditions: (i) $\langle g, h\rangle$ is dense in the connected compact Lie group G, and (ii) the subgroup generated by $\{g, h\}$ is a free group, then S is dense in $G \times G$. Indeed more precise statements were given in 6.88.

There is a vast literature on the topology of compact Lie groups [29, 314]. We opted for presenting enough cohomology to prove Hopf's Theorem on the cohomology of compact Lie groups and to derive from it the classification Theorem 6.95 of compact Lie groups on spheres which says that the only spheres which admit a Lie group structure are $\mathbb{S}^1$ and $\mathbb{S}^3$—this theorem is presented primarily as an example of how information on cohomology works. Naturally this demands more background knowledge from the reader; but we have tried to keep at least the ideas rather self-contained by presenting the necessary tools and functorial arguments for the understanding of Hopf algebras in Appendix 3. In the proof of Hopf's Rank Theorem 6.95 we have employed an additional share of basic cohomology. An alternative tool is the rank of a differentiable map , which we have not used here.

In Chapter 9 we shall generalize most of the structural results accumulated here to compact groups, but the generalisation will not always be straightforward—in contrast with what is sometimes believed. For a general structure theory of compact groups it will be necessary to address the special case of compact abelian groups in considerable detail, and this we do in the next two chapters.

References for this Chapter—Additional Reading

[4], [16], [29], [42], [43], [46], [48], [49], [55], [58], [89], [96], [101], [106], [111], [137], [155], [209], [158], [167], [169], [170], [179], [181], [196], [201], [205], [207], [217], [230], [235], [238], [245], [247], [259], [265], [273], [282], [295], [296], [312], [314], [313], [319], [338], [339], [352], [353], [354], [368].

Chapter 7

Duality for Abelian Topological Groups

By the end of Chapter 2 we had the full power of the Pontryagin Duality Theorem for compact abelian groups and for discrete abelian groups. Locally compact abelian groups are much closer to compact abelian groups than is apparent at first sight. We deal in a very low key fashion with the locally compact abelian groups $\mathbb{R}^n$ in Appendix 1. These locally compact groups are the basis of most of analysis. In this chapter we prove the Pontryagin Duality Theorem for locally compact abelian groups and derive a structure theory of locally compact abelian groups in terms of their main ingredients, namely, vector groups $\mathbb{R}^n$, compact abelian groups, and discrete abelian groups. This then reduces a more detailed structure theory for locally compact abelian groups to a structure theory of compact abelian groups which is produced in the next chapter.

One of the main tools we again use is the exponential function as in Chapters 5 and 6. We will apply this tool now to abelian topological groups where it will be immediately clear that the Lie algebra $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$, i.e. the space of all one-parameter subgroups of G according to 5.7, is a topological vector space. But even for compact abelian groups G, the functor $\mathfrak{L}$ takes us outside of the category of locally compact abelian groups. This motivates us to derive the apparatus of duality in sufficient generality. In the category $\mathbb{TAB}$ of all abelian topological groups the fact, known to us for compact or discrete abelian groups, that G is canonically isomorphic to its double dual $\widehat{\widehat{G}}$ fails in general. Yet for certain categories of topological vector spaces relevant to the exponential map of compact groups, important features of duality theory apply. Thus in this chapter we work in categories of abelian topological groups which properly contain the category of locally compact abelian groups. In the next chapter we apply it to compact abelian groups.

The exponential function plays a significant part in the literature on Lie groups. In the preceding Chapters 5 and 6 we saw how effective it is in the context of linear Lie groups and compact Lie groups. We show now that it is an equally powerful tool for the analysis of the structure of locally compact abelian groups in this chapter. It will remain a *Leitmotiv* in the subsequent ones.

Prerequisites. We will use some topological vector space theory; for this purpose we have collected the required background information on *weakly complete* topological vector spaces in Appendix 7. In this fashion we are keeping matters self-contained. As we progress into more advanced chapters, the demands on the reader's maturity and willingness to accept abstract reasoning increases. In this chapter, functorial thinking and category theoretical concepts become overt, and it may be time to consult Appendix 3 which was expressly written to accommodate a framework for reasoning such as duality theory in its full bloom.

The Compact Open Topology and Hom-Groups

In Chapter 1, 1.15, we introduced the character group of a discrete abelian group A as $\widehat{A} = \mathrm{Hom}(A, \mathbb{T})$ and gave it the topology of pointwise convergence. In 1.22 we considered a compact abelian group G and introduced its character group $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ giving it the discrete topology. In Chapter 5 we started a discussion of the exponential function and introduced in 5.7, for a topological group G, the space $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ and gave it the topology of uniform convergence on compact subsets of $\mathbb{R}$. We now more systematically consider $\mathrm{Hom}(G, H)$ for abelian topological groups G and H and overview the possible topologies which are reasonable for these groups.

For the record we formulate the basic characterisation of group topologies on abelian groups.

Exercise E7.1. Prove the following facts.

(i) *If G is an abelian Hausdorff topological group, then the filter of identity neighborhoods $\mathcal{U}$ of G satisfies the following conditions:*

(1) *Every U in $\mathcal{U}$ contains a V such that $V - V \subseteq U$.*

(2) $\bigcap \mathcal{U} = \{0\}$.

(3) *A subset W of G is open if and only if for every $g \in W$ there is a $U \in \mathcal{U}$ such that $g + U \subseteq W$.*

(ii) *If G is an abelian group and $\mathcal{U}$ a filter of subsets satisfying* (1) *and* (2), *then the set*

$$\mathcal{O}(G) = \{W \subseteq G \mid (\forall w \in W)(\exists U \in \mathcal{U}) \quad w + U \subseteq W\}$$

is a Hausdorff topology on G relative to which G is an abelian Hausdorff topological group with $\mathcal{U}$ as the neighborhood filter of 0. □

If necessary, to avoid confusion, we shall write $\mathcal{U}(G)$ for the filter of identity neighborhoods of G.

In this chapter all topological groups considered are assumed to be Hausdorff unless the contrary is stated explicitly.

In the following proposition we shall consider a set G endowed with a family of subsets $\mathcal{C}$ satisfying the following conditions:

(a) $\mathcal{C}$ is directed; i.e. for C_1, C_2 in $\mathcal{C}$ there is a $C_3 \in \mathcal{C}$ with $C_1 \cup C_2 \subseteq C_3$.

(b) $\bigcup \mathcal{C} = G$.

Let H denote an abelian topological group. We let $\mathcal{U}$ denote the filter of its identity neighborhoods. For $C \subseteq G$ and $U \subseteq H$, set

$$W(C, U) = \{f \in H^G \mid f(C) \subseteq U\}.$$

Proposition 7.1. *Given these data,*

(i) *the relations $C \subseteq D$ and $U \subseteq V$ imply $W(D,U) \subseteq W(C,V)$.*

(ii) *For C_1, $C_2 \in \mathcal{C}$, $U \in \mathcal{U}$,*

$$W(C_1 \cup C_2, U) = W(C_1, U) \cap W(C_2, U).$$

(iii) *For $C \in \mathcal{C}$ and U, $V \in \mathcal{U}$*

$$W(C, U \cap V) = W(C,U) \cap W(C,V).$$

(iv) *$W(C,V) - W(C,V) \subseteq W(C, V - V)$.*

The set $\mathcal{U}_\mathcal{C}$ of all subsets of H^G containing some set $W(C,U)$, $C \in \mathcal{C}$, $U \in \mathcal{U}$ is a filter defining on H^G the structure of an abelian Hausdorff topological group with a topology $\mathcal{O}_\mathcal{C}$ which is finer than or equal to the topology of pointwise convergence.

Proof. The properties (i), (ii), (iii) and (iv) are straightforward from the definitions. On account of $W(C_1 \cup C_2, U_1 \cap U_2) \subseteq W(C_1, U_1) \cap W(C_2, U_2)$ and hypothesis (a), the set $\mathcal{U}_\mathcal{C}$ is a filter. From (iv) one readily deduces that it satisfies condition E7.1(1). If $f \in \bigcap \mathcal{U}_\mathcal{C}$ then $f(C) \subseteq U$ for all $C \in \mathcal{C}$ and $U \in \mathcal{U}$. Hence $f(C) \subseteq \bigcap \mathcal{U} = \{0\}$ by E7.1(2). Now hypothesis (b) implies $f = 0$. Hence $\mathcal{U}_\mathcal{C}$ satisfies E7.1(2). It follows from E7.1(ii) that H^G is an abelian topological group with respect to a topology $\mathcal{O}_\mathcal{C}$ for which the $W(C,U)$ form a basis of the identity neighborhoods.

If $\mathcal{C}_{\text{fin}}$ is the collection of all finite subsets of G, then this collection satisfies conditions (a) and (b), and the group topology $\mathcal{O}_{\mathcal{C}_{\text{fin}}}$ defined according to our construction is exactly the topology of pointwise convergence, i.e. the product topology of H^G. If F is any finite subset of G, then by hypothesis (b) there is a finite family $C_1, \ldots, C_n \in \mathcal{C}$ such that $F \subseteq C_1 \cup \cdots \cup C_n$ and by hypothesis (a) there is a $C \in \mathcal{C}$ with $C_1 \cup \cdots \cup C_n \subseteq C$, and thus $F \subseteq C$. Hence $W(C,U) \subseteq W(F,U)$ and thus $\mathcal{U}_{\mathcal{C}_{\text{fin}}} \subseteq \mathcal{U}_\mathcal{C}$. Hence $\mathcal{O}_{\mathcal{C}_{\text{fin}}} \subseteq \mathcal{O}_\mathcal{C}$. □

The topology $\mathcal{O}_\mathcal{C}$ is called *the topology of uniform convergence on the sets of* $\mathcal{C}$. If G is a topological space and $\mathcal{C}$ is the set of all compact subsets of G, then it is called *the topology of uniform convergence on compact sets* or *the compact open topology.*

The construction which associates with $(G, \mathcal{C})$ and an abelian topological group H the abelian topological group H^G has certain functorial properties which are often used.

Proposition 7.2. *Let G and H be sets equipped with families $\mathcal{C}$, respectively, $\mathcal{D}$ of subsets satisfying* (a) *and* (b) *preceding Proposition* 7.1. *Assume that S and T are abelian topological groups and $f: S \to T$ is a morphism of abelian topological groups; assume further that $g: G \to H$ is a function such that for each $C \in \mathcal{C}$ there is $D \in \mathcal{D}$ with $g(C) \subseteq D$. Then*

(i) *the function $f^G: S^G \to T^G$ given by $f^G(\varphi) = f \circ \varphi$ is a morphism of abelian topological groups with respect to the topologies of uniform convergence on the sets of the family $\mathcal{C}$, and*

(ii) *the function* $S^g: S^H \to S^G$ *given by* $S^g(\psi) = \psi \circ g$ *is a morphism of topological groups with respect to the topologies of uniform convergence on the sets of the families* $\mathcal{D}$ *and* $\mathcal{C}$*, respectively.*

Proof. (i) Clearly f^G is a morphism of abelian groups. Assume that $C \in \mathcal{C}$ and $V \in \mathcal{U}(T)$. Since f is continuous at 0 we find a $U \in \mathcal{U}(S)$ with $f(U) \subseteq V$. Then $\varphi(C) \subseteq U$ implies $f\bigl(\varphi(C)\bigr) \subseteq f(U) \subseteq V$, whence $f^G\bigl(W(C,U)\bigr) \subseteq W(C,V)$. Thus f^G is continuous at 0 and, being a morphism of abelian groups, is then a morphism of abelian topological groups.

(ii) Again one observes readily that S^g is a morphism of abelian groups. If $C \in \mathcal{C}$ and $U \in \mathcal{U}(\mathcal{G})$ we find $D \in \mathcal{D}$ with $g(C) \subseteq D$, whence $\psi(D) \subseteq U$ implies $\psi\bigl(g(C)\bigr) \subseteq \psi(D) \subseteq U$. Thus $S^g\bigl(W(D,U)\bigr) \subseteq W(C,U)$. Hence the morphism of abelian groups S^g is continuous at 0 and thus a morphism of abelian topological groups. □

If G and H are abelian topological groups we shall denote with $\mathrm{Hom}(G,H)$ the abelian group of all *continuous* homomorphisms from G to H with the group structure inherited from H^G. *We shall always equip* $\mathrm{Hom}(G,H)$ *with the topology of uniform convergence on the compact sets of* G unless something is explicitly stated to the contrary.

Proposition 7.3. *For any abelian topological group* G *and any morphism* $f: S \to T$ *of abelian topological groups there are morphisms of abelian topological groups*

$$\mathrm{Hom}(G,f): \mathrm{Hom}(G,S) \to \mathrm{Hom}(G,T), \quad \mathrm{Hom}(G,f)(\varphi) = f \circ \varphi,$$
$$\mathrm{Hom}(f,G): \mathrm{Hom}(T,G) \to \mathrm{Hom}(S,G), \quad \mathrm{Hom}(f,G)(\varphi) = \varphi \circ f.$$

Proof. This is immediate from Proposition 7.2. □

The following definition is crucial for duality theory. We shall no longer hesitate to use the concept of a category rigorously. For all material used on categories and functors we refer to Appendix 3.

Definition 7.4. (i) If G is an abelian topological group, then the abelian topological group

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$$

is called *the character group* or *the dual group* of G. The elements of the character group are called *characters.*

(ii) If $f: G \to H$ is a morphism of abelian topological groups, then the morphism of abelian topological groups

$$\widehat{f}: \widehat{H} \to \widehat{G}, \quad \widehat{f} = \mathrm{Hom}(f, \mathbb{T})$$

is called the *adjoint morphism of* f.

(iii) For a compact subset C of G and an identity neighborhood U of $\mathbb{T}$ we shall write $V_G(C,U) \overset{\mathrm{def}}{=} W(C,U) \cap \widehat{G}$ or simply $V(C,U)$ if no confusion is likely. □

We first hasten to show that Definition 7.4(i) extends Definitions 1.15 and 1.22 correctly. We shall also prove that the character group of the additive group of a real topological vector space is its topological dual (with the compact open topology) thereby linking character theory with vector space duality.

Proposition 7.5. (i) *The character group of a discrete abelian group is compact, and the character group of a compact abelian group is discrete.*

(ii) *Assume that* E_1 *and* E_2 *are* $\mathbb{R}$*-vector spaces such that the underlying additive groups are topological groups and that all functions* $r \mapsto r{\cdot}v : \mathbb{R} \to E_j$, $v \in E_j$, $j = 1, 2$ *are continuous. Then every morphism* $f: E_1 \to E_2$ *of abelian topological groups is linear.*

(iii) *Let* E *be a real topological vector space and* $E' = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ *the space of all continuous linear forms* $E \to \mathbb{R}$ *endowed with the compact open topology. Then* $E' = \mathrm{Hom}(E, \mathbb{R})$ (*in the sense of topological* Hom*-groups*)*, and if* $q: \mathbb{R} \to \mathbb{T}$ *is the quotient morphism, then* $\mathrm{Hom}(E, q): E' = \mathrm{Hom}(E, \mathbb{R}) \to \mathrm{Hom}(E, \mathbb{T}) = \widehat{E}$ *is an isomorphism of topological vector spaces.*

Proof. (i) If G is discrete, then the topology of compact convergence is the topology of pointwise convergence and $\widehat{G} = \mathrm{Hom}(G, \mathbb{T}) \subseteq \mathbb{T}^G$ has the topology inherited from the product topology of $\mathbb{T}^G$ which is compact, by the Theorem of Tychonoff. But since G is discrete, $\mathrm{Hom}(G, \mathbb{T})$ is the group of all algebraic group morphisms $G \to \mathbb{T}$ and is, therefore, closed in $\mathbb{T}^G$. Hence $\widehat{G}$ is a compact abelian group.

Now let G be compact and U the neighborhood $\left(\left]-\frac{1}{3}, \frac{1}{3}\right[+ \mathbb{Z}\right)/\mathbb{Z}$ of zero in $\mathbb{T}$. Then $\{0\}$ is the only subgroup of $\mathbb{T}$ contained in U. Then any morphism $f: G \to \mathbb{T}$ contained in $W(G, U)$ satisfies $f(G) \subseteq U$ for a subgroup $f(G)$ of $\mathbb{T}$ and must, therefore, vanish. Hence $\{0\} = W(G, U)$ is an identity neighborhood. Hence $\widehat{G}$ is discrete.

(ii) Let $f: E_1 \to E_2$ be additive. If $m \in \mathbb{N}$, $n \in \mathbb{Z}$, then $m{\cdot}f(\frac{n}{m}{\cdot}v) = f(n{\cdot}v) = n{\cdot}f(v)$, whence $f(\frac{n}{m}{\cdot}v) = \frac{m}{n}{\cdot}f(v)$. Thus f is $\mathbb{Q}$-linear, i.e. $r{\cdot}f(v) = f(r{\cdot}v)$ for $r \in \mathbb{Q}$. By the continuity of all $r \mapsto r{\cdot}v$ and the continuity of f we get the desired $\mathbb{R}$-linearity.

(iii) Each continuous linear form $E \to \mathbb{R}$ is trivially a member of $\mathrm{Hom}(E, \mathbb{R})$. Conversely, every member f of $\mathrm{Hom}(E, \mathbb{R})$ is $\mathbb{R}$-linear by (iii). It follows that $E' = \mathrm{Hom}(E, \mathbb{R})$. Now $\mathrm{Hom}(E, q): E' \to \widehat{E}$ is a morphism of topological groups by 7.3.

The additive topological group of E as that of a real topological vector space is simply connected (see A2.6, A2.9, A2.10(i)). Hence every character $\chi: E \to \mathbb{T}$ has a unique lifting $\widetilde{\chi}: E \to \mathbb{R}$ such that

$$\mathrm{Hom}(E, q)(\widetilde{\chi}) = q \circ \widetilde{\chi} = \chi$$

(see Appendix 2, A2.32). Thus $\chi \mapsto \widetilde{\chi}: \widehat{E} \to E'$ is an inverse of $\mathrm{Hom}(E, q)$. It remains to be verified that it is continuous. Let C be a compact subset of E and $U = \left]-\varepsilon, \varepsilon\right[\subseteq \mathbb{R}$ with $0 < \varepsilon \leq \frac{1}{4}$. Now $\mathbb{D}{\cdot}C$ is compact connected and contains C. Consider $\chi \in V_E\big(\mathbb{D}{\cdot}C, q(U)\big)$. Then $\widetilde{\chi}(\mathbb{D}{\cdot}C)$ is a connected subset of $q^{-1}(q(U))$

containing 0. The component of 0 in $q^{-1}\big(q(U)\big) = U + \mathbb{Z}$ is U. Hence $\widetilde{\chi}(\mathbb{D}{\cdot}C) \subseteq U$. Thus $\widetilde{\chi} \in V_E(\mathbb{D}{\cdot}C, U) \subseteq V_{\mathbb{R}}(C, U)$. This proves the continuity of $\chi \mapsto \widetilde{\chi}: \widehat{E} \to E'$. This completes the proof. □

Instead of A2.32 we could have used the Extending Local Homomorphisms Theorem A2.26. But the full power of either A2.26 or A2.32 from Appendix 2 is not absolutely needed in the proof of 7.5(iii). The content of the following exercise, whose proof can be patterned after that of 5.8 will serve as a more elementary replacement.

Exercise E7.2. Verify the following statement.

Let E denote a real vector space and K a subset which absorbs E (i.e. satisfies $\bigcup_{n\in\mathbb{N}} nK = E$) and is balanced (i.e. satisfies $[-1,1]{\cdot}K = K$). If $f: K \to G$ is a function into a group such that $r, s, r+s \in K$ implies $f(r+s) = f(r) + f(s)$ then there is a unique extension $F: E \to G$ to a morphism of groups with $F|K = f$. □

Local Compactness and Duality of Abelian Topological Groups

With each abelian topological group we now associate a natural homomorphism $\eta = \eta_G: G \to \widehat{\widehat{G}}$. Indeed if we set $\eta(g)(\chi) = \chi(g)$ for $g \in G$ and $\chi \in \widehat{G}$, then clearly $\eta(g): \widehat{G} \to \mathbb{T}$ is a morphism of abelian groups for each g; it is continuous with respect to the topology of pointwise convergence on $\widehat{G}$. Since the topology of $\widehat{G}$ (namely, the topology of uniform convergence on compact sets) is finer than or equal to the topology of pointwise convergence, $\eta(g) = \big(\chi \mapsto \chi(g)\big)$ is continuous and thus is a character. Hence $\eta: G \to \widehat{\widehat{G}}$ is a well-defined function and is also readily seen to be a morphism of abelian groups. The function $(\chi, g) \mapsto \chi(g)$ is continuous in each variable separately. However, we now must determine conditions under which the two functions relevant for duality are continuous:

(†) the evaluation function $(\chi, g) \mapsto \chi(g): \widehat{G} \times G \to \mathbb{T}$,

(‡) the evaluation morphism $\eta_G: G \to \widehat{\widehat{G}}$, $\eta_G(g)(\chi) = \chi(g)$.

A set C in a topological group G is called *precompact* if for each identity neighborhood U there is a finite subset $F \subseteq G$ such that $C \subseteq FU$. Every compact subset is clearly precompact. If G is the additive group $\mathbb{Q}$ of rational numbers with the order topology (i.e. the topology induced by the natural topology of $\mathbb{R}$), then $[-1,1]$ is a precompact identity neighborhood of G which is not compact. Clearly every subset $K \subseteq G$ whose closure is compact is precompact; the preceding example shows that the converse fails. If K is a precompact subset of G and $f: G \to H$ is a morphism of abelian topological groups, then $f(K)$ is precompact in H.

Proposition 7.6. *Let G and T be abelian topological groups and $K \subseteq \operatorname{Hom}(G, T)$. Then each of the following conditions implies the next:*

(1) *K is compact.*
(2) *K is precompact.*
(3) *For each compact subset C of G and each $U \in \mathcal{U}(T)$ there is an $M \in \mathcal{U}(G)$ such that $K(C \cap M) \subseteq U$.*
(4) *The topology of K agrees with the topology induced from the product topology T^G, that is the topology of pointwise convergence.*

Consider the following conditions:

(i) *$\underline{K(x)}$ is compact for all $x \in G$.*
(ii) *$\overline{K(x)}$ is compact for all $x \in G$.*
(iii) *K is closed in T^G in the product topology, i.e. the topology of pointwise convergence.*

Then (1)⇒(i)⇒(ii) *and* (1)⇒(iii). *Conversely, conditions* (4), (ii), *and* (iii) *together imply* (1).

Finally, if T is compact, then (3) *implies* (2)

Proof. As we remarked above, it is clear that (1) implies (2).

(2)⇒(3) We shall abbreviate

$$V_G(C, U) = W_G(C, U) \cap \mathrm{Hom}(G, T).$$

Assume $U \in \mathcal{U}(T)$ and let C be an arbitrary compact subset of G. Pick $U' \in \mathcal{U}(T)$ such that $U' + U' \subseteq U$. Since K is precompact there is a finite set $F \subseteq K$ such that $K \subseteq F + V_G(C, U')$. Since each $\kappa \in F$ is continuous, we find an $M_\kappa \in \mathcal{U}(G)$ such that $\kappa(M_\kappa) \subseteq U'$. Set $M = \bigcap_{\kappa \in F} M_\kappa$. Then $K(M \cap C) \subseteq \big(F + V_G(C, U')\big)(M \cap C) \subseteq F(M) + V_G(C, U')(C) \subseteq U' + U' \subseteq U$.

(3)⇒(4) Let $f \in K$. Since the compact open topology on K is finer than or equal to the topology of pointwise convergence, we have to show the reverse containment. Thus for each $f \in K$ and for each basic neighborhood $V_G(C, U)$, C compact in G, $U \in \mathcal{U}(T)$ we must find a finite set $F \subseteq K$ and a zero neighborhood $N \in \mathcal{U}(T)$ such that $(f + V_G(F, N)) \cap K \subseteq f + V_G(C, U)$. Select $N \in \mathcal{U}(T)$ in such a fashion that $N + N + N \subseteq U$. By (3) and by the continuity of f, for each $c \in C$ we find an $M_c \in \mathcal{U}(G)$ such that $K((C - c) \cap M_c) \subseteq N$ and $f(M_c) \subseteq -N$. Since C is compact, there is a finite set $F \subseteq C$ such that $C \subseteq \bigcup_{x \in F} x + M_x$. Let $\kappa \in \big((f + V_G(F, N)\big) \cap K$, say $\kappa = f + \varphi$ with $\varphi(F) \subseteq N$. We must show $\varphi \in V_G(C, U)$. If $c \in C$, then there is an $x \in F$ such that $c = x + m$ with $m \in M_x$. Then $\kappa(m) = \kappa(c - x) \in N$ by the definition of M_x. Now $\varphi(c) = -f(c) + \kappa(c) = -f(c) + \kappa(m) + \kappa(x) = f(x - c) + \kappa(m) + \varphi(x) \in N + N + N \subseteq U$. Thus $\varphi \in V_G(C, U)$, and this is what we had to show.

If K is compact, since evaluation is continuous, $K(x)$ is compact for each $x \in G$. Then trivially $\overline{K(x)}$ is compact for all $x \in G$. Also, since the compact open topology is finer than the topology of pointwise convergence, K is compact and hence closed in T^G in the latter topology.

Now we assume that $\overline{K(x)}$ is compact for each $x \in G$ and that K is closed in T^G. Then $P \stackrel{\text{def}}{=} \prod_{x \in G} \overline{K(x)}$ is compact by the Theorem of Tychonoff, and the closed subset $K \subseteq P$ is compact in the topology of pointwise convergence. Now (4) clearly implies (1).

If T is compact, then (ii) is automatic and each subset $K \subseteq T^G$ is precompact. Since (3) implies (4), we conclude that K is precompact in the topology induced from the compact open topology. □

This result is a version of Ascoli's Theorem (see e.g. [34], Chap 10., §2, n^o 4 and n^o 5, notably Théorème 2).

As a consequence of this proposition, a hom-group of the form $\mathrm{Hom}(G,T)$ can be shown to be locally compact if and only if there exists a closed identity neighborhood U of T, and a compact subset C of G such that $V_G(C,U)$ is closed in T^G and satisfies condition (3) of Proposition 7.6. The following key theorem shows that for $T = \mathbb{T}$ one can verify this condition.

Since compactness is built into the definition of the compact open topology and therefore into the definition of a character group it does not come as a complete surprise, that local compactness is connected with these questions. In fact a certain bigger class of topological spaces plays a role. We recall that a space X is said to be a k-*space* if a set $U \subseteq X$ is open if and only if for each compact subset $K \subseteq X$ the set $U \cap K$ is open in K. A function $f\colon X \to Y$ from a Hausdorff k-space to a topological space is continuous if and only if $f|K\colon K \to Y$ is continuous for each compact subset K of X. All locally compact Hausdorff spaces and all first countable spaces are k-spaces (because if every neighborhood filter has a countable basis, then the open sets are determined by sequences and the underlying subspace of a convergent sequence plus its limit is compact).

Duality and Local Compactness

Theorem 7.7. *Let G be an abelian topological group. We set $U_1 \stackrel{\mathrm{def}}{=} ([-\frac{1}{4}, \frac{1}{4}] + \mathbb{Z})/\mathbb{Z} \in \mathcal{U}(\mathbb{T})$ and consider the evaluation morphism $\eta_G\colon G \to \widehat{\widehat{G}}$, $\eta_G(g)(\chi) = \chi(g)$, and the evaluation function $\mathrm{ev}_G\colon \widehat{G} \times G \to \mathbb{T}$, $\mathrm{ev}_G(\chi, g) = \chi(g)$. Now the following statements hold.*

(i) *If N is any neighborhood of 0 in G, then the subspace $V_G(N, U_1) \stackrel{\mathrm{def}}{=} W(N, U_1) \cap \widehat{G}$ is compact. If N is compact, then this set is a 0-neighborhood of $\widehat{G}$.*

(ii) *If G is locally compact then the character group $\widehat{G}$ is locally compact.*

(iii) *If the underlying space of G is a k-space, then $\eta_G\colon G \to \widehat{\widehat{G}}$ is continuous.*

(iv) *For every compact subspace K of G, the restriction*

$$\mathrm{ev}_G |(\widehat{G} \times K)\colon \widehat{G} \times K \to \mathbb{T}$$

is continuous. If G is locally compact, then η_G and ev_G are continuous.

(v) *If $\mathrm{ev}_G\colon \widehat{G} \times G \to \mathbb{T}$ is continuous, then $\widehat{G}$ and $\widehat{\widehat{G}}$ are locally compact.*

(vi) *Assume that $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups. Then the following conditions are equivalent:*

(a) *G is locally compact.*

(b) *$\mathrm{ev}_G\colon \widehat{G} \times G \to \mathbb{T}$ is continuous.*

Proof. (i) We set

$$U_n = \{t \in \mathbb{T} \mid (\forall k = 1, \dots, n)\, k{\cdot}t \in U_1\}.$$

If $\frac{1}{4} > \varepsilon > 0$ and $n > 1/4\varepsilon > 1$ then we claim $U_n \subseteq (]-\varepsilon, \varepsilon[+\mathbb{Z})/\mathbb{Z}$. Proof of this claim: If $t = x + \mathbb{Z} \in U_n$, then we may assume $0 \le x \le \frac{1}{4}$ and we have

$$(*) \qquad \{x, 2{\cdot}x, 3{\cdot}x, \dots, n{\cdot}x\} \subseteq \left[-\frac{1}{4}, \frac{1}{4}\right] + \mathbb{Z}.$$

Set $m = \max\{k \mid x, 2{\cdot}x, \dots, k{\cdot}x \le \frac{1}{4}\}$. Then $\frac{1}{4} < (m+1){\cdot}x$. We assert that $n \le m$. Suppose, on the contrary, that $m < n$. Then $m + 1 \le n$ and thus $(m+1){\cdot}x \in [-\frac{1}{4}, \frac{1}{4}] + \mathbb{Z}$ by $(*)$. Hence $\frac{3}{4} = 1 - \frac{1}{4} \le (m+1){\cdot}x$. Hence $\frac{1}{4} \ge x = (m+1){\cdot}x - m{\cdot}x \ge \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$, a contradiction. Hence $n \le m$, as asserted, and thus $n{\cdot}x \le \frac{1}{4}$, whence $0 \le x \le \frac{1}{4n} < \varepsilon$ and, therefore, $t = x + \mathbb{Z} \in (]-\varepsilon, \varepsilon[+\mathbb{Z})/\mathbb{Z}$, and this proves the claim. Now we know that $\{U_n \mid n \in \mathbb{N}\}$ is a basis for $\mathcal{U}(\mathbb{T})$.

Lemma A. Assume that N is a zero neighborhood of G. Then for each $n \in \mathbb{N}$ there is an identity neighborhood V_n of G such that

$$(**) \qquad V_G(N, U_1) \stackrel{\text{def}}{=} \widehat{G} \cap W(N, U_1) \subseteq \bigcap_{n \in \mathbb{N}} W(V_n, U_n).$$

Proof of Lemma A. For $n \in \mathbb{N}$ we find a neighborhood V_n of 0 in G such that $\underbrace{V_n + \cdots + V_n}_{n\,\text{times}} \subseteq N$. If now $\chi \in \widehat{G} \cap W(N, U_1)$ then for all $k = 1, 2, \dots, n$ we have

$$k{\cdot}\chi(V_n) = \chi(k{\cdot}V_n) \subseteq \chi(\underbrace{V_n + \cdots + V_n}_{n\,\text{times}}) \subseteq \chi(N) \subseteq U_1,$$

whence $\chi(V_n) \subseteq U_n$ by the definition of U_n which proves $(**)$.

Lemma A gives us two pieces of information. Firstly, we claim that $V_G(N, U_1)$ is closed in $\mathbb{T}^G$. Let G_d denote the discrete group underlying G. A morphism of groups between topological groups is continuous if it is continuous at the identity element. Since all $f \in \bigcap_{n \in \mathbb{N}} W(V_n, U_n)$ are continuous at 0, we have $\mathrm{Hom}(G_d, \mathbb{T}) \cap \bigcap_{n \in \mathbb{N}} W(V_n, U_n) \subseteq \widehat{G}$. If $\mathrm{cl}\,\widehat{G}$ denotes the closure of $\widehat{G}$ in $\mathbb{T}^G$ in the topology of pointwise convergence, then we note from $(**)$ that $\mathrm{cl}\,\widehat{G} \cap W(N, U_1) \subseteq \mathrm{Hom}(G_d, \mathbb{T}) \cap \bigcap_{n \in \mathbb{N}} W(V_n, U_n) \subseteq \widehat{G}$. (In fact, equality holds.) Since U_1 is closed in $\mathbb{T}$, the set $W(N, U_1)$ is closed in the topology of pointwise convergence in $\mathbb{T}^G$. Therefore the set $\widehat{G} \cap W(N, U_1) = \mathrm{cl}\,\widehat{G} \cap W(N, U_1)$ is closed in $\mathbb{T}^G$ in the product topology as asserted.

Secondly, we claim that condition (3) of Proposition 7.6 is satisfied for $K = V_G(N, U_1)$. Let C be a compact subset of G and $U \in \mathcal{U}(\mathbb{T})$. Find an $n \in \mathbb{N}$ such that $U_n \subseteq U$. Set $M = V_n \subseteq N$. By Lemma A we have $V_G(N, U_1) \subseteq V_G(V_n, U_n)$. Then

$$K(C \cap M) \subseteq V_G(N, U_1)(V_n) \subseteq V_G(V_n, U_n)(V_n) \subseteq U_n \subseteq U,$$

which proves 7.6(3). Hence, by 7.6, the set $V_G(N, U_1)$ is compact.

Finally, if N is compact, then by the definition of the compact open topology, the compact set $V_G(N, U_1)$ is a 0-neighborhood of $\widehat{G}$. This completes the proof of (i).

(ii) By Proposition 7.1 and Definition 7.4, the topology of $\widehat{G}$ is Hausdorff. Hence it suffices to show that every point has one compact neighborhood. Since $\widehat{G}$ is homogeneous as an abelian topological group it suffices to show that one of the basic 0-neighborhoods $W(C, U) \cap \widehat{G}$ is compact. This is what we did in (i).

(iii) It is worthwhile to be pedestrian at this point. Let us therefore go through several steps.

Step (iii)1. *For each zero-neighborhood V of $\widehat{\widehat{G}}$ and each compact subspace C of G containing 0 we have a zero neighborhood M of G such that $\eta_G(M \cap C) \subseteq V$.*

Proof. Let $V_{\widehat{G}}(K, U)$ be a basic zero neighborhood of $\widehat{\widehat{G}}$ contained in V with a compact set K in $\widehat{G}$ and with $U \in \mathcal{U}(\mathbb{T})$. By Proposition 7.6, for each compact subset C of G, we find a zero neighborhood M in G such that $\eta_G(M \cap C)(K) = K(M \cap C) \subseteq U$, i.e. such that $\eta_G(M \cap C)(K) \subseteq W(K, U) \cap \widehat{\widehat{G}} = V_{\widehat{G}}(K, U) \subseteq V$. This finishes Step 1.

Next consider a function $f: X \to Y$ between Hausdorff spaces. Let $x_0 \in X$ and let $\mathcal{K}$ denote the set of all compact subspaces of X containing x_0, $\mathcal{U}$ the set of all open neighborhoods of x_0, and $\mathcal{V}$ the set of all open neighborhoods of $f(x_0)$. Next consider the following five conditions:

(A) f is continuous at x_0.

(B) $(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}) \quad f(U) \subseteq V$.

(C) $(\forall V \in \mathcal{V})(\forall K \in \mathcal{K})(\exists U_K \in \mathcal{U}) \quad f(U_K \cap K) \subseteq V$.

(D) $(\forall K \in \mathcal{K})$ $f|K: K \to Y$ is continuous.

(E) $(\forall V \in \mathcal{V})(\forall K \in \mathcal{K}) \quad f^{-1}(V) \cap K$ is a neighborhood of x_0 in K.

Then clearly (A)⇔(B)⇒(C)⇔(D) ⇔(E), and if x_0 has a compact neighborhood in X, then all are equivalent.

Step (iii)2. *Assume that X is a Hausdorff k-space and Y a Hausdorff space. Assume that $f: X \to Y$ satisfies* (C) *at all points x_0 of X. Then f is continuous.*

Proof. Let V be open in Y. Then the hypothesis on f and (E) imply that $f^{-1}(V) \cap K$ is open in K for all compact subsets K of X. Since X is a k-space, $f^{-1}(V)$ is open. Step 2 is proved.

Step (iii)3. *Let G and H be topological groups and assume that the underlying space of G is a k-space. Let $f: G \to H$ be an algebraic homomorphism which satisfies* (C) *at one point. Then f is continuous.*

Proof. If f satisfies (C) at one point, it satisfies (C) at all points since left translations are continuous. Hence f is continuous by Step 2. This completes the proof of Step 3.

Step (iii)4. Proof of Assertion (iii). By Step 1, the algebraic homomorphism η_G satisfies (C) at 0. Hence it is continuous by Step 3.

(iv) Again we go through several steps.

Step (iv)1. *Let X be an arbitrary subset of G. The evaluation function*

$$\mathrm{ev}_G\colon \widehat{G} \times G \to \mathbb{T}, \quad \mathrm{ev}_G(\chi, g) = \chi(g)$$

has a continuous restriction

$$\mathrm{ev}_G |(\widehat{G} \times X)\colon \widehat{G} \times X \to \mathbb{T}$$

if and only if

(∗) *for each $g_0 \in X$, and $U \in \mathcal{U}(\mathbb{T})$ there is a compact subset $C \subseteq G$ and zero neighborhoods $M \in \mathcal{U}(G)$, $N \in \mathcal{U}(\mathbb{T})$ such that $V_G(C, N)\big(M \cap (X - g_0)\big) \subseteq U$.*

Proof. The evaluation function ev_G is separately continuous and bilinear (i.e. it is a morphism of abelian topological groups in each argument if the other argument is fixed). The difference

$$\chi(g) - \chi_0(g_0) = (\chi - \chi_0)(g - g_0) + \chi_0(g - g_0) + (\chi - \chi_0)(g_0),$$

is small if all summands on the right side are small; the second summand can be made small by taking g close to g_0 by the continuity of χ_0, the last one can be made small by taking χ close enough to χ_0 by the continuity of $\eta_G(g_0)$. Thus $\mathrm{ev}_G |\widehat{G} \times X$ is continuous at $(\chi_0, g_0) \in \widehat{G} \times X$ if and only if $\mathrm{ev}_G |\big(\widehat{G} \times (X - g_0)\big)$ is continuous at $(0, 0)$ and this is the case if and only if (∗) holds. This proves Step 1.

The following is straightforward:

Step (iv)2. *The following statements are equivalent for $N, U \subseteq \mathbb{T}$, $C, M \subseteq G$:*

(a) $$V_G(C, N)\big(M \cap (X - g_0)\big) \subseteq U.$$

(b) $$\big((\forall \chi \in \mathrm{Hom}(G, \mathbb{T})\big) \quad (\chi(C) \subseteq N) \Rightarrow (\chi\big(M \cap (X - g_0)\big) \subseteq U.$$

(c) $$V_G(C, N) \subseteq W(M \cap (X - g_0), U).$$

(d) $$V_G(C, N) \subseteq V_G(M \cap (X - g_0), U).$$

Step (iv)3. *Let K an arbitrary compact subset of an abelian topological group G. Then*

$$\mathrm{ev}_G |(\widehat{G} \times K)\colon \widehat{G} \times K \to \mathbb{T}$$

is continuous.

Proof. Let $g_0 \in K$ and $U \in \mathcal{U}(\mathbb{T})$. Set $C = K - g_0$, $N = U$ and $M = G$. Then Step 2(d) holds trivially and Condition (∗) of Step 1 is satisfied. This completes the proof of Step 3 and thus the first part of Assertion (iv).

Step (iv)4. *If G is locally compact then the functions η_G and ev_G are continuous.*

Proof. Regarding η_G: every locally compact space is a k-space. Hence (iii) above proves the claim.

Regarding ev_G: every point $(\chi, g) \in \widehat{G} \times G$ has a neighborhood of the form $\widehat{G} \times K$ with a compact neighborhood K of g. Then Step 3 shows that ev_G is continuous at (χ, g).

(v) Assume that $\mathrm{ev}_G: \widehat{G} \times G \to \mathbb{T}$ is continuous. Then there is a zero neighborhood U of $\widehat{G}$ and a zero neighborhood N of G such that $\mathrm{ev}_G(U \times N) \subseteq U_1$. This implies that $U \subseteq V_G(N, U_1)$. Thus $V_G(N, U_1)$ is a zero neighborhood of $\widehat{G}$, which by (i) is compact in $\widehat{G}$. Thus $\widehat{G}$ is locally compact. It then follows from (ii) that $\widehat{\widehat{G}}$ is locally compact, too.

(vi) If G is locally compact, then ev_G is continuous by (iv). Assume now that $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism algebraically and topologically and that ev_G is continuous. Then $\widehat{\widehat{G}}$ is locally compact by (v). Then $G \cong \widehat{\widehat{G}}$ implies that G is locally compact. □

We have seen in particular, that the evaluation morphism η_G is a morphism of abelian topological groups whenever the abelian topological group G is locally compact or satisfies the first axiom of countability. For Proposition (iii) of Theorem 7.7 see Banaszczyk [18], p. 133, 14.4. The equivalence of 7.7(v) was shown by Martin-Peinador [253] only recently. The quintessence of this section is that we should expect a duality theory for abelian topological groups to be perfect in all aspects at best within the full category of locally compact abelian groups. We shall confirm this in the course of this chapter. Nevertheless we shall see other categories of abelian topological groups for which duality is of interest even though their objects in general fail to be locally compact.

It is reasonable in view of our predominant interest in locally compact abelian groups that we wish to endow $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ once and for all with one particular topology which we choose to be the compact open one. If we were interested in a very general theory of duality we would be well advised to allow a greater flexibility in choosing, on an abelian topological group A a family $\mathcal{C}$, and thus a group topology $\mathcal{O}_{\mathcal{C}}|\mathbb{TAB}(A, \mathbb{T})$ on the character group $\mathbb{TAB}(A, \mathbb{T}) \subseteq \mathbb{T}^A$, according to Proposition 7.1. For instance, on the abelian topological group $A = \widehat{G}$ of all continuous characters of an abelian topological group G, it is quite reasonable to consider the set $\mathcal{C}_{\mathrm{e}}$ of all *equicontinuous* subsets, i.e. subsets $K \subseteq \mathbb{TAB}(G, \mathbb{T})$ satisfying

$$\big(U \in \mathcal{U}(\mathbb{T})\big)\big(\exists M \in \mathcal{U}(G)\big)\, K(M) \subseteq U.$$

This family satisfies conditions (a) and (b) preceding Proposition 7.1. The virtue of the topology $\mathcal{O}_{\mathcal{C}_e}$ on the double dual $\mathbb{TAB}(\widehat{G}, \mathbb{T})$, called the *topology of uniform convergence on equicontinuous sets*, has the advantage that the evaluation morphism $\eta_G: G \to (\mathbb{TAB}(\widehat{G}, \mathbb{T}), \mathcal{O}_{\mathcal{C}_e})$ is continuous. In view of the last conclusion of 7.6 every member of $\mathcal{C}_e$ is precompact. Therefore, if *every closed precompact subset of G is compact*, then $\mathcal{O}_{\mathcal{C}_e}$ is contained in the compact open topology. In so far as η_G

should be continuous, the topology of uniform convergence on equicontinuous sets is the correct one on a double dual; the fortunate fact for locally compact groups G, established in 7.7, is that the topologies of uniform convergence on compact sets and that on equicontinuous sets agree.

Basic Functorial Aspects of Duality

Definitions 7.8. The category of all abelian topological groups and morphisms of abelian topological groups will be denoted by $\mathbb{TAB}$ and the full subcategory (cf. Appendix 3, EA3.10(i)) of all locally compact abelian groups is denoted by $\mathbb{LCA}$. The full subcategory of compact abelian groups is written $\mathbb{CAB}$ and the category of (discrete) abelian groups $\mathbb{AB}$. The category of $\mathbb{K}$-vector spaces and linear maps will be denoted by $\mathbb{AB}_{\mathbb{K}}$ and the category of topological $\mathbb{K}$-vector spaces and continuous linear maps by $\mathbb{TAB}_{\mathbb{K}}$. We let $\mathbb{TAB}_\eta$, respectively, $\mathbb{ABD}$ denote the full subcategory of $\mathbb{TAB}$ of all abelian topological groups G such that $\eta_G\colon G \to \widehat{\widehat{G}}$ is a morphism of abelian topological groups, respectively, is an isomorphism of abelian topological groups. We say that an abelian topological group is *reflexive* or *has duality* if it is contained in $\mathbb{ABD}$. We shall call G *semireflexive* if $\eta_G\colon G \to \widehat{\widehat{G}}$ is bijective. □

An abelian topological group therefore is semireflexive if and only if every character of $\widehat{G}$ is an evaluation.

In Appendix 4, A4.21 shows there are semireflexive abelian topological groups which are not reflexive.

We know from 1.37 and 2.32 that $\mathbb{AB}$ and $\mathbb{CAB}$ are contained in $\mathbb{ABD}$. We will show later that $\mathbb{LCA} \subseteq \mathbb{ABD}$ (Theorem 7.63 below). Theorems that show that certain abelian topological groups G belong to $\mathbb{ABD}$ are called *duality theorems.*

Here is a straightforward exercise:

Exercise E7.3. Show that $\widehat{\eta_G} \circ \eta_{\widehat{G}} = \mathrm{id}_{\widehat{G}}$. □

Next we record quickly that the character group is a functor (cf. Appendix 3, A3.17).

Proposition 7.9. (i) *If $f\colon A \to B$ is a morphism of abelian topological groups with dense image, then $\widehat{f}\colon \widehat{B} \to \widehat{A}$ is injective.*

(ii) *Assume that η_G and $\eta_{\widehat{G}}$ are continuous. If η_G or $\eta_{\widehat{G}}$ has a dense image, then $\widehat{G}$ is reflexive.*

(iii) *The assignment $\widehat{\cdot}\ : \mathbb{TAB} \to \mathbb{TAB}^{\mathrm{op}}$ is a functor, i.e. $\widehat{\cdot}\ : \mathbb{TAB} \to \mathbb{TAB}$ is a contravariant functor, mapping $\mathbb{ABD}$ into itself, and exchanging $\mathbb{CAB}$ and $\mathbb{AB}$.*

Proof. (i) Let $\chi \in \widehat{B}$. Then $\widehat{f}(\chi) = 0$ means that for all $a \in A$ we have $\chi\big(f(a)\big) = \widehat{f}(\chi)(a) = 0$. Since $B = \overline{f(A)}$ we conclude $\chi = 0$.

(ii) The following diagram is commutative. (Exercise E7.3.)

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\ \mathrm{id}\ } & \widehat{G} \\ {\scriptstyle \eta_{\widehat{G}}}\Big\downarrow & & \Big\uparrow{\scriptstyle \widehat{\eta_G}} \\ \widehat{\widehat{\widehat{G}}} & \xrightarrow[\ \mathrm{id}\ ]{} & \widehat{\widehat{\widehat{G}}} \end{array}$$

If η_G has a dense image, by (i), $\widehat{\eta_G}$ is injective. Because $\widehat{\eta_G} \circ \eta_{\widehat{G}} = \mathrm{id}_{\widehat{G}}$ the morphism $\widehat{\eta_G}$ is an injective retraction. Hence it is an isomorphism. (See Appendix 3, A3.13(iii).) Hence the coretraction $\eta_{\widehat{G}}$ is an isomorphism, too.

If $\eta_{\widehat{G}}$ has dense image, then $\mathrm{id}_{\widehat{\widehat{\widehat{G}}}}\eta_{\widehat{G}} = \eta_{\widehat{G}} = \eta_{\widehat{G}}\widehat{\eta_G}\eta_{\widehat{G}}$ and this relation implies that the continuous functions $\mathrm{id}_{\widehat{G}}$ and $\eta_{\widehat{G}}\widehat{\eta_G}$ agree on a dense set of their domain and thus are equal, since all spaces considered are assumed to be Hausdorff. Thus $\widehat{\eta_G} = \eta_{\widehat{G}}^{-1}$.

(iii) In order to show that $\widehat{\cdot}$ is a functor, we have to show that $\widehat{\mathrm{id}_G} = \mathrm{id}_{\widehat{G}}$ of any abelian topological group G, and that for morphisms $G \xrightarrow{f} H \xrightarrow{g} K$ we have $\widehat{gf} = \widehat{f}\widehat{g}$. But this is straightforward.

It has been observed in 7.5(i) and (ii) that the functor $\widehat{\cdot}$ exchanges $\mathbb{CAB}$ and $\mathbb{AB}$. We must show that it preserves $\mathbb{ABD}$. So take G in $\mathbb{ABD}$. Then $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism. Then $\eta_{\widehat{G}}$ is an isomorphism by E7.3.

Thus $\widehat{G}$ is in $\mathbb{ABD}$. □

Proposition 7.10. (i) *Let G, H, and T be abelian topological groups. Then $\sigma\colon \mathrm{Hom}(G \times H, T) \to \mathrm{Hom}(G, T) \times \mathrm{Hom}(H, T)$, $\sigma(\Phi) = (\Phi_1, \Phi_2)$, $\Phi_1(g) = \Phi(g, 0)$, $\Phi_2(h) = \Phi(0, h)$, and $\rho\colon \mathrm{Hom}(G, T) \times \mathrm{Hom}(H, T) \to \mathrm{Hom}(G \times H, T)$, $\rho(\varphi, \psi)(g, h) = \varphi(g) + \psi(h)$ are inverse morphisms of abelian topological groups.*

(ii) *The function $\alpha_{G,H}\colon \widehat{G\times H} \to \widehat{G} \times \widehat{H}$, $\alpha(\chi) = (\chi_1, \chi_2)$, $\chi_1(g) = \chi(g, 0)$, $\chi_2(h) = \chi(0, h)$, is an isomorphism of abelian topological groups whose inverse is given by $\alpha_{G,H}^{-1}(\chi_1, \chi_2)(g, h) = \big(\chi_1(g), \chi_2(h)\big)$.*

(iii) *There is an isomorphism $\alpha_{\widehat{G},\widehat{H}} \circ \widehat{\alpha_{G,H}}^{\,-1}\colon \widehat{\widehat{G\times H}} \to \widehat{\widehat{G}} \times \widehat{\widehat{H}}$ of abelian topological groups. Define $\eta'_{G,H}\colon G \times H \to \widehat{\widehat{G}\times\widehat{H}}$ by $\eta'_{G,H}(g, h)(\chi_1, \chi_2) = \chi_1(g) + \chi_2(h)$. Then the following diagram is commutative:*

$$\begin{array}{ccccc} G \times H & \xleftarrow{\ \mathrm{id}_{G\times H}\ } & G \times H & \xrightarrow{\ \mathrm{id}_{G\times H}\ } & G \times H \\ {\scriptstyle \eta_{G\times H}}\Big\downarrow & & {\scriptstyle \eta'_{G,H}}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta_G \times \eta_H} \\ \widehat{\widehat{G\times H}} & \xleftarrow[\ \widehat{\alpha_{G,H}}\ ]{} & \widehat{\widehat{G}\times\widehat{H}} & \xrightarrow[\ \alpha_{\widehat{G},\widehat{H}}\ ]{} & \widehat{\widehat{G}} \times \widehat{\widehat{H}}. \end{array}$$

(iv) $\mathbb{ABD}$ *is closed under the formation of finite products; that is finite products of reflexive abelian topological groups are reflexive.*

Proof. (i) The verification of the assertions is a straightforward exercise. (E7.4 below). (ii) is an immediate consequence of (i). Also the verification of (iii) is straightforward. Now (iii) directly implies (iv). □

Exercise E7.4. Fill in the details of the proof of 7.10. □

The functor $\widehat{\cdot}$ has functorial and universal properties as we will record now.

Proposition 7.11. (i) *Assume that $f: A \to B$ is a morphism of abelian topological groups. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \widehat{\widehat{A}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{\widehat{f}}} \\ B & \xrightarrow[\eta_B]{} & \widehat{\widehat{B}} \end{array}$$

is commutative.

In particular, $\eta: \mathrm{id}_{\mathbb{TAB}_\eta} \to \widehat{\widehat{\cdot}}$ is a natural transformation from the inclusion functor $\mathbb{TAB}_\eta \to \mathbb{TAB}$ to the self-functor $\widehat{\widehat{\cdot}}$ of $\mathbb{TAB}_\eta$ (see Appendix 3, A3.31).

(ii) *For each pair of abelian topological groups G and H and for each morphism $f: G \to \widehat{H}$ there is a unique morphism of abelian groups $f': H \to \widehat{G}$ such that $\widehat{f'} \circ \eta_G = f$. In fact, for $h \in H$ and $g \in G$ we have $f'(h)(g) = f(g)(h)$. If $\eta_H: H \to \widehat{\widehat{H}}$ is continuous, then f' is continuous. If the underlying space of H is a k-space, in particular if it is locally compact, then η_H is continuous.*

$$\begin{array}{ccccc} \mathbb{TAB}_\eta & & & & \mathbb{TAB}_\eta \\ \hline \\ G & \xrightarrow{\eta_G} & \widehat{\widehat{G}} & & \widehat{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f'}} & & \uparrow{\scriptstyle f'} \\ \widehat{H} & \xrightarrow[\mathrm{id}_{\widehat{H}}]{} & \widehat{H} & & H \end{array}$$

(iii) *For abelian topological groups G and H in the category $\mathbb{TAB}_\eta$, the function $f \mapsto f': \mathrm{Hom}(G, \widehat{H}) \to \mathrm{Hom}(H, \widehat{G})$ is an isomorphism of abelian groups whose inverse is given by exchanging the roles of G and H. If G and H are locally compact, then $f \mapsto f'$ is continuous.*

(iv) *The functor $\widehat{\cdot} : \mathbb{LCA}^{\mathrm{op}} \to \mathbb{LCA}$ is adjoint to $\widehat{\cdot} : \mathbb{LCA} \to \mathbb{LCA}^{\mathrm{op}}$. In particular, the contravariant functor $\widehat{\cdot} : \mathbb{LCA} \to \mathbb{LCA}$ transforms colimits into limits.*

Proof. (i) The first claim is a straightforward exercise from the definitions. (E7.10.)

(ii) We attempt to define a function $f': H \to \widehat{G}$ by $f'(h)(g) = f(g)(h)$. This attempt is successful as soon as we know that for each $h \in H$ the function $f'(h) = (g \mapsto f(g)(h): G \to \mathbb{T}$ is continuous. However, $\chi \mapsto \chi(h): \widehat{H} \to \mathbb{T}$ is continuous by the definition of the compact open topology; then the continuity of f and the continuity of $\chi \mapsto \chi(h)$ guarantees the continuity of $f'(h)$. It is straightforward that f' is a morphism of abelian groups.

Now we compute $\big(\widehat{f}\eta(h)\big)(g) = \widehat{f}\big(\eta(h)\big)(g) = \eta(h)\big(f(g)\big) = f(g)(h) = f'(h)(g)$ whence $f' = \widehat{f} \circ \eta_H$. Thus if η_H is continuous, i.e. if H is in $\mathbb{TAB}_\eta$, then f' is continuous, i.e. $f' \in \mathrm{Hom}(H, \widehat{G})$. By 7.7(iii), if H is a k-space then this condition is satisfied. This holds, in particular, if H is locally compact.

The symmetry in the relation of f and f' suggests the analogous computation $\big(\widehat{f'}\eta(g)\big)(h) = \widehat{f'}\big(\eta(g)\big)(h) = \eta(g)\big(f'(h)\big) = f'(h)(g) = f(g)(h)$. Hence $\widehat{f'} \circ \eta_G = f$. If $f''\colon H \to \widehat{G}$ satisfies $\widehat{f''} \circ \eta_G = f$, then the preceding computation shows $f(g)(h) = \eta(g)\big(f''(h)\big) = \cdots = f''(g)(h)$, and so f' is indeed uniquely determined by the universal property.

(iii) It is clear that $f \mapsto f'$ is a morphism of abelian groups and that it has an inverse morphism obtained by exchanging the roles of G and H. It therefore remains to observe its continuity—since the continuity of its inverse then follows by the same argument. Let $V\big(K_H, V(C_G, U)\big)$ denote a basic 0-neighborhood of $\mathrm{Hom}(H, \widehat{G})$. Then we must find a 0-neighborhood $V\big(K_G, V(C_H, U')\big)$ such that $f \in V\big(K_G, V(C_H, U')\big)$, i.e. $f(K_G)(C_H) \subseteq U'$ implies $f' \in V\big(K_H, V(C_G, U')\big)$, i.e. $f(C_G)(K_H) = f'(K_H)(C_G) \subseteq U$. It clearly suffices to take $K_G = C_G$, $C_H = K_H$, and $U' = U$.

(iv) By (iii) the function $f \mapsto f'\colon \mathbb{LCA}(G, \widehat{H}) \to \mathbb{LCA}^{\mathrm{op}}(\widehat{G}, H)$ is a natural bijection. The assertion on the adjunction then follows from Appendix 3, A3.35. Now left adjoints preserve limits (see Appendix 3, A3.52). Since $\widehat{\cdot}$ is contravariant, the asserted transformation rule follows. □

Exercise E7.10. Prove 7.11(i). (Cf. Exercise E1.15). □

The Annihilator Mechanism

Definition 7.12. (i) If H is a subset of an abelian topological group G, then

$$(*) \qquad H^\perp = \{\, \chi \in \widehat{G} \mid (\forall g \in H)\, \langle \chi, g\rangle = 0 \,\}$$

is called the *annihilator of H in $\widehat{G}$*. If A is a subset of $\widehat{G}$, we also write

$$(**) \qquad A^\perp = \{\, g \in G \mid (\forall \chi \in A)\, \langle \chi, g\rangle = 0 \,\}$$

and $A^\perp$ is called *the annihilator of A in G*.

(ii) We shall say that *a pair* (G, H) consisting of a topological group G and a subgroup H *has enough compact sets* if for each compact subset K of G/H there is a compact subset C of G such that $(CH)/H \supseteq K$. □

Note that, in Definition 7.12(i), $H^\perp$ (or $A^\perp$) have a unique meaning only when it is known in which group it is being considered as a subgroup. There is potential confusion in using the symbol $(\,\cdot\,)^\perp$ as in both $(*)$ and $(**)$. But the context will make clear which is meant. Therefore we have to specify where the annihilator is taken in order to avoid having to write an additional argument.

Annihilators are closed subgroups. Also it is clear that $H \mapsto H^\perp$ reverses containment. The annihilator is a very effective tool. Its full power becomes apparent for locally compact abelian topological groups, but certain aspects are more general and come in handy for proofs of duality theorems.

We also record that the notation $(\cdot)^\perp$ has already been used in the context of orthogonal complements in a Hilbert space. No confusion should result.

Exercise E7.11. (i) Prove the following proposition.

Let $f: X \to Y$ be a continuous open surjective function between locally compact spaces. Then for each compact set K in Y there is a compact set C such that $f(C) \supseteq K$.

[Hint. By the surjectivity of f (and the Axiom of Choice) there is a function $\sigma: K \to X$ with $f\big(\sigma(k)\big) = k$ for $k \in K$. By the local compactness of X, for each $k \in K$ there is a compact neighborhood U_k of $\sigma(k)$ in X. The image under f of the interior of U_k is an open neighborhood of k in Y since f is open. The compactness of K allows us to find $k_1, \ldots, k_n \in K$ such that $K \subseteq V_1 \cup \cdots \cup V_n$. Set $C = U_1 \cup \cdots \cup U_n$; use the continuity of f.]

(ii) Verify that each of the following conditions is sufficient to ensure that a pair (G, H) as in 7.12(ii) has enough compact sets.

(a) G is locally compact.

(b) H is compact.

(c) There is a continuous function $\sigma: G/H \to G$ such that $\sigma(\xi) \in \xi$ for all $\xi \in G/H$. □

Assume that $H \subseteq G$ is a subgroup of the abelian topological group G, and let $H^\perp$ be its annihilator in $\widehat{G}$. Let $q: G \to G/H$ denote the quotient morphism. Then the morphism $\mathrm{Hom}(q, \mathbb{T}): \widehat{G/H} = \mathrm{Hom}(G/H, \mathbb{T}) \to \mathrm{Hom}(G, \mathbb{T}) = \widehat{G}$ has $H^\perp$ as its image and thus defines a morphism $\lambda_{G,H}: \widehat{G/H} \to H^\perp$ of abelian topological groups.

Lemma 7.13. *For a subgroup H of an abelian Hausdorff topological group G,*

(i) *the annihilator $H^\perp$ of H in $\widehat{G}$ is a closed subgroup, and $\overline{H}^\perp = H^\perp$.*

(ii) *The morphism of abelian topological groups $\lambda_{G,H}: \widehat{G/H} \to H^\perp$ is bijective. The following formulae apply.*

$$\lambda_{G,H}(\chi)(g) = \chi(g + H),$$
$$\big(\lambda_{G,H}\big)^{-1}(\chi)(g + H) = \chi(g).$$

(iii) *If the pair (G, H) has enough compact sets, $\lambda_{G,H}$ is an isomorphism of topological groups.*

(iv) *Let $j\colon H \to \overline{H}$ be the inclusion morphism. Then the restriction morphism $\widehat{j}\colon \widehat{\overline{H}} \to \widehat{H}$, $\widehat{j}(\chi) = \chi|H$ is bijective.*

Proof. (i) We have

$$H^{\perp} = \{\chi \in \mathrm{Hom}(G, \mathbb{T}) \subseteq \mathbb{T}^{\mathbb{G}} \mid (\forall h \in H)\, \chi(h) = 0\},$$

and thus $H^{\perp}$ is closed in $\mathrm{Hom}(G, \mathbb{T})$ with respect to the topology of pointwise convergence which is contained in the compact open topology. Hence $H^{\perp}$ is closed in $\widehat{G}$. Since $\mathbb{T}$ is Hausdorff, for each $\chi \in \widehat{H}$ the kernel $\chi^{-1}(0)$ is closed. Thus if $\chi(H) = \{0\}$ then also $\chi(\overline{H}) = \{0\}$. Thus $H^{\perp} \subseteq \overline{H}^{\perp}$. The reverse inclusion is trivial.

(ii) The function $\alpha\colon H^{\perp} \to \widehat{G/H}$ given by $\alpha(\chi)(g+H) = \chi(g)$ for each character $\chi \in H^{\perp}$ of G vanishing on H is a well-defined algebraic homomorphism, which is an inverse of $\lambda_{G,H}$. Thus $\lambda_{G,H}$ is a bijective morphism of topological groups.

(iii) Assume now that (G, H) has enough compact sets. We claim that α is a character; i.e. that it is continuous. Let an identity neighborhood $V(C, U)$ in $\widehat{G/H}$ be given, where C is compact in G/H and U is a zero neighborhood of $\mathbb{T}$. Pick a compact set C' in G such that $(C' + H)/H \supseteq C$. If $\chi \in H^{\perp}$ satisfies $\chi(C') \subseteq U$, then $\alpha(\chi)(C) \subseteq \alpha(\chi)\big((C' + H)/H\big) = \chi(C') \subseteq U$. Thus $\alpha\big(V(C', U)\big) \subseteq V(C, U)$, which proves the claim.

(iv) Clearly, since G is a Hausdorff space, $\widehat{j}$ is injective. We must show that every character χ of H extends to a character of $\overline{H}$. Now χ is uniformly continuous since for each identity neighborhood V in $\mathbb{T}$) there is an identity neighborhood U in G with $\chi(U) \subseteq V$ and thus $\chi(g+V) \subseteq \chi(g)+U$. Let $h \in \overline{H}$. If $\mathcal{U}$ is the filter of zero neighborhoods of $\overline{H}$ then $\{\overline{\chi\big((h+U) \cap H\big)} \mid U \in \mathcal{U}\}$ is a filter basis of closed subsets of the compact space $\mathbb{T}$. Hence it has a nonempty intersection $\Phi(h) \subseteq \mathbb{T}$. Let V be a closed zero neighborhood of $\mathbb{T}$; find a zero neighborhood U of $\overline{H}$ such that $\chi\big((U-U) \cap H\big) \subseteq V$. If $h_1,\ h_2 \in (h+U) \cap H$ then $\chi(h_2 - h_1) \subseteq V$. We conclude that $\Phi(h_2) - \Phi(h_1) \subseteq V$ for all V. Hence $\Phi(h) = \{\chi'(h)\}$ for a unique element $\chi'(h)$ agreeing with $\chi(h)$ if $h \in H$. It is then readily checked that $\chi'\colon \overline{H} \to \mathbb{T}$ is a continuous extension of χ. □

The proof of (iv) could have been abbreviated by using the apparatus of Cauchy filters and completeness.

Note that even though G is a Hausdorff topological group, the factor group G/H is Hausdorff if and only if H is closed. Thus, in contrast with our convention that all topological groups are assumed to be Hausdorff, the factor groups G/H occurring in the present discussion fail to be Hausdorff if H fails to be closed. Since $\mathbb{T}$ is Hausdorff, every character of G vanishing on H also vanishes on $\overline{H}$. Hence the natural morphism $q\colon G/H \to G/\overline{H}$, $q(gH) = g\overline{H}$ induces a bijection $\widehat{q}\colon \widehat{G/\overline{H}} \to \widehat{G/H}$. This is implicit in 7.13(i), but may be useful to realize the full implication of what 7.13 says.

The situation is best illustrated in the following combination of Hasse diagrams:

$$
\begin{array}{ccc}
G & \{0\} & \\
| & | & \left.\right\} \cong \widehat{G/H} \cong \widehat{G/\overline{H}} \\
\overline{H} & \overline{H}^{\perp} & \\
| & | & \left.\right\} \text{are equal} \\
H & H^{\perp} & \\
| & | & \\
\{0\} & \widehat{G}. &
\end{array}
$$

It will be very helpful throughout in the context of annihilator arguments to have the Hasse diagrams above in mind.

It is a useful exercise to note that for any subset A of $\widehat{G}$ the annihilator $A^{\perp} = \bigcap_{\chi \in A} \ker \chi$ is always a closed subgroup of G. Also, if f is in the closure in $\mathbb{T}^G$ of A (in the topology of pointwise convergence), then $f(A^{\perp}) = \{0\}$. In particular, if $\chi \in \overline{A}$ in $\widehat{G}$, then $\chi(A^{\perp}) = \{0\}$. Thus $A^{\perp} \subseteq \overline{A}^{\perp}$ whence $A^{\perp} = \overline{A}^{\perp}$.

Definition 7.14. If G is a semireflexive abelian topological group, for $g \in G$ and $\chi \in \widehat{G}$ we write $\langle g, \chi \rangle = \langle \chi, g \rangle = \chi(g) = \eta_G(g)(\chi)$, thereby emphasizing that G and $\widehat{G}$ play largely interchangeable roles, and thereby identifying G and $\widehat{\widehat{G}}$ via η_G as groups. □

We should keep the following in mind. If G is a semireflexive abelian topological group and if we consider G as the character group of $\widehat{G}$ by evaluation, even if we assume η_G to be continuous, the topology we are given on G may be properly finer than the natural topology on $\widehat{\widehat{G}}$, i.e. the compact open topology when G is regarded as the set of characters on $\widehat{G}$.

Lemma 7.15. *Assume that G is a semireflexive abelian topological group, and that G and $\widehat{\widehat{G}}$ are identified as abelian groups. Let H be an arbitrary subset of G.*

(i) *If $H \subseteq G$, then $H^{\perp\perp\perp} = H^{\perp}$.*

(ii) *$\overline{\langle H \rangle} \subseteq H^{\perp\perp}$, and if $H^{\perp}$ (via $\lambda_{G,\overline{\langle H \rangle}}$) separates the points of $G/\overline{\langle H \rangle}$, equality holds.*

Proof. (i) For all $\chi \in H^{\perp}$ and $h \in H$ we have $\langle \chi, h \rangle = 0$. Hence $H \subseteq H^{\perp\perp}$. Since $\{\cdot\}^{\perp}$ is containment reversing, $(H^{\perp\perp})^{\perp} \subseteq H^{\perp}$. Applying the same reasoning to $H^{\perp}$ in place of H we get $H^{\perp} \subseteq (H^{\perp})^{\perp\perp}$.

(ii) Since $H^{\perp\perp}$ is a closed subgroup of G and contains H, it contains $\overline{\langle H \rangle}$. By 7.13, $H^{\perp}$ may be identified with the character group of $G/\overline{\langle H \rangle}$ in such a way that $\langle \chi, g + \overline{\langle H \rangle} \rangle = \langle \chi, g \rangle = \chi(g)$ for $g \in G$ and $\chi \in H^{\perp}$. Let $g \in H^{\perp\perp}$. Then

$\langle \chi, g + \overline{\langle H \rangle} \rangle = \chi(g) = 0$ for all $\chi \in H^\perp$. If the characters of $G/\overline{\langle H \rangle}$ separate the points, this implies $g + \overline{\langle H \rangle} = 0$; i.e. $g \in \overline{\langle H \rangle}$. □

Proposition 7.16 (The Separation Theorem). *Let G be a semireflexive abelian topological group and let A be a subgroup of the character group $\widehat{G}$. Consider the following conditions:*

(1) *A is dense in $\widehat{G}$.*

(2) *$A^\perp = \{0\}$ in G.*

(3) *$A^{\perp\perp} = \widehat{G}$.*

(4) *A separates the points of G.*

Then (1)⇒(2)⇔(3)⇔(4) *and if the characters of $\widehat{G}/\overline{A}$ separate points* (*e.g. if the characters on* all *factor groups of G modulo closed subgroups separate points*), *all four conditions are equivalent.*

Proof. We continue to identify G and $\widehat{\widehat{G}}$ as groups.

(1)⇒(2) If $g \in G$ then the set $\{\chi \in \widehat{G} \mid \langle \chi, g \rangle = 0\}$ is a closed subgroup S of $\widehat{G}$. If $\langle \alpha, g \rangle = 0$ for all $\alpha \in A$, then $A \subseteq S$ and by (1) we conclude $S = \widehat{G}$. Thus $g = 0$ since G is semireflexive.

(2)⇔ (3) In view of 7.15(i), this is trivial since $\{0_G\}^\perp = \widehat{G}$.

(2)⇔(4) is clear from the definitions.

(2)⇒(1) $g \in A^\perp$ implies $0 = \langle g, \chi \rangle = \chi(g)$ for all $\chi \in A$. By (2) this implies $g = 0$. Thus $A^\perp = \{0\}$. This means that every character of $\widehat{G}$ vanishing on A is zero. Thus all characters on $\widehat{G}/\overline{A}$ vanish. But if the characters of this group separate the points, then it is singleton and thus $\widehat{G} = \overline{A}$. □

The relation between density in the dual and separating points is reminiscent of the Stone–Weierstraß Approximation Theorem (see e.g. [34]).

We continue to let H be a subgroup of the abelian topological group G. Let $j: H \to G$ denote the inclusion morphism. Then the adjoint $\widehat{j}: \widehat{G} \to \widehat{H}$ of j given by $\widehat{j}(\chi) = \chi|H$ has precisely the kernel $H^\perp$. It thus induces an injective morphism of topological groups

$$\kappa_{G,H}: \widehat{G}/H^\perp \to \widehat{H}, \quad \kappa_{G,H}(\chi + H^\perp) = \chi|H.$$

Let $\eta_G^H: H \to (H^\perp)^\perp$ denote the morphism induced by $\eta_G: G \to \widehat{\widehat{G}}$, where $(H^\perp)^\perp$ is the annihilator of $H^\perp \subseteq \widehat{G}$ in $\widehat{\widehat{G}}$. We will abbreviate $\widehat{G}/H^\perp$ by Γ, and we will use the following hypotheses as we progress with the material:

(a) The pair (G, H) has enough compact sets (cf. Definition 7.12(ii)).

(b) G is semireflexive.

(B) G is reflexive.

(c) The characters of G/H separate points.

(c*) $\eta_G^H\colon H \to (H^\perp)^\perp$ is bijective.
(d) Γ is reflexive.
(e) H is reflexive.
(f) H is in $\mathbb{TAB}_\eta$, i.e. $\eta_H\colon H \to \widehat{\widehat{H}}$ is continuous.

Notice right away that each of the hypotheses (c) and (c*) implies that H is closed.

For abelian topological groups A and B in the category $\mathbb{TAB}_\eta$ we recall from 7.11(ii), (iii) the isomorphism

$$f \mapsto f'\colon \mathrm{Hom}(A, \widehat{B}) \to \mathrm{Hom}(B, \widehat{A}), \quad f'(b)(a) = f(a)(b),\ a \in A,\ b \in B.$$

We shall have two main applications of the tools provided in the following lemma. One of these will be to a duality between certain classes of topological vector spaces and to locally compact abelian groups. In the case of the latter which is our main concern, all of the hypotheses (a)–(f) are automatically satisfied. In the former case, however, special effort will still be required to verify the required hypotheses.

The Annihilator Mechanism

Lemma 7.17. *Let H be a closed subgroup of an abelian topological group G and $\Gamma = \widehat{G}/H^\perp$. Define morphisms of abelian topological groups*

$$\lambda_{G,H}\colon \widehat{G/H} \to H^\perp, \quad \lambda_{G,H}(\chi)(g) = \chi(g+H),$$

$$\kappa_{G,H}\colon \widehat{G}/H^\perp \to \widehat{H}, \quad \kappa_{G,H}(\chi + H^\perp) = \chi|H,$$

and (*cf.* 7.11(ii))

$$\kappa'_{G,H}\colon H \to \widehat{\Gamma}, \quad \kappa'_{G,H}(h)(\chi + H^\perp) = \chi(h).$$

The definition of λ also yields a morphism

$$\lambda_{\widehat{G},H^\perp}\colon \widehat{\Gamma} \to (H^\perp)^\perp \subseteq \widehat{\widehat{G}}, \quad \lambda_{\widehat{G},H^\perp}(\Omega)(\chi) = \Omega(\chi + H^\perp) \text{ for } \Omega \in \widehat{\Gamma},\ \chi \in \widehat{G}.$$

Let $\eta_G^H\colon H \to (H^\perp)^\perp$ denote the morphism induced by $\eta_G\colon G \to \widehat{\widehat{G}}$, where $(H^\perp)^\perp$ is the annihilator of $H^\perp \subseteq \widehat{G}$ in $\widehat{\widehat{G}}$.

Then the following conclusions hold.

(i) *The morphism of abelian topological groups $\lambda_{G,H}$ is bijective. If hypothesis* (a) *holds it is an isomorphism.*

(ii) $\kappa_{G,H}$ *maps Γ bijectively and continuously onto $\widehat{G}|H \subseteq \widehat{H}$ and*

$$(*) \qquad \lambda_{\widehat{G},H^\perp} \circ \kappa'_{G,H} = \eta_G^H.$$

$$\begin{array}{ccc} G & \{0\} & \\ | & | & \left.\right\} \overset{\leftarrow}{\cong} \widehat{G/H} \\ H & H^{\perp} & \\ | & | & \left.\right\} \Gamma \overset{\rightarrow}{\cong} \widehat{G}|H \\ \{0\} & \widehat{G}. & \end{array}$$

(The symbols $\overset{\leftarrow}{\cong}$ *and* $\overset{\rightarrow}{\cong}$ *represent the continuous algebraic isomorphisms* $\lambda_{G,H}$*, respectively,* $\widehat{G}/H^{\perp} \to \widehat{G}|H$*.)*

(iii) *If hypothesis* (b) *holds, and if, for a subset* M *of* G*, the characters separate the points of* $G/\overline{\langle M\rangle}$*, then* $\overline{\langle M\rangle} = M^{\perp\perp}$*, where* $M^{\perp\perp}$ *denotes the annihilator of* $M^{\perp} \subseteq \widehat{G}$ *in* G*. In particular,* (b) *and* (c) *imply* (c*).

(iv) *If hypotheses* (b), (c*), *and* (f) *hold, then the morphism of abelian group* $\kappa'_{G,H}: H \to \widehat{\Gamma}$ *is continuous and bijective.*

(v) *If hypotheses* (B), (c*), *and* (f) *hold, then* $(\eta_G^H)^{-1}\lambda_{\widehat{G},H^{\perp}}: \widehat{\Gamma} \to H$ *is an isomorphism.*

(vi) *If hypotheses* (B), (c), (d), *and* (f) *hold, then* $\kappa_{G,H}: \Gamma \to \widehat{H}$ *is an isomorphism. If hypotheses* (B), (c), (e), *and* (f) *hold, then* $(\kappa_{G,H})\widehat{\ }: \widehat{\widehat{H}} \to \widehat{\Gamma}$ *is an isomorphism.*

Proof. (i) is Lemma 7.13.

(ii) The image of $\kappa_{G,H}$ is $\widehat{G}|H$, the subgroup of all those characters of H which are restrictions of characters of G. The surjectivity of $\kappa_{G,H}$ thus is equivalent to the statement that every character of H extends to a character to G which is a rather strong property.

In order to prove $(*)$ let $h \in H$ and $\chi \in \widehat{G}$. Then we compute, abbreviating $\lambda_{\widehat{G},H^{\perp}}$ by λ and $\kappa'_{G,H}$ by κ' and using the definitions of these maps, that

$$(\lambda \circ \kappa')(h)(\chi) = \lambda\big(\kappa'(h)\big)(\chi) = \kappa'(h)(\chi + H^{\perp}) = \chi(h) = \eta_G^H(h)(\chi),$$

and this proves $(*)$.

(iii) is Lemma 7.15(ii).

(iv) By (c*) the morphism $\eta_G^H: H \to (H^{\perp})^{\perp}$ is bijective. By (i) above, the morphism $\lambda_{\widehat{G},H^{\perp}}: (\widehat{G}/H^{\perp})\widehat{\ } \to (H^{\perp})^{\perp}$ is a bijective morphism of abelian topological groups. By 7.11(ii) hypothesis (f) implies that $\kappa'_{G,H}$ is continuous, and $(*)$ shows that $\kappa'_{G,H}$ is bijective.

(v) If $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism of abelian topological groups by (B), then it restricts and corestricts to an isomorphism $\eta_G^H: H \to (H^{\perp})^{\perp}$ of abelian topological groups; i.e. (c*) holds. By (B) and (c*) conclusion (iv) applies and shows that $\kappa'_{G,H}$ is bijective. Then $(*)$ and (iv) prove that $\kappa'_{G,H}$ and $(\eta_G^H)^{-1}\lambda_{\widehat{G},H^{\perp}}$ are inverse isomorphisms of each other.

(vi) From 7.11(ii) we get a commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\eta_\Gamma} & \widehat{\widehat{\Gamma}} & & \widehat{\Gamma} \\ {\scriptstyle \kappa_{G,H}}\downarrow & & \downarrow{\scriptstyle (\kappa'_{G,H})\widehat{\ }} & & \uparrow{\scriptstyle \kappa'_{G,H}} \\ \widehat{H} & \xrightarrow[\mathrm{id}_{\widehat{H}}]{} & \widehat{H}, & & H. \end{array}$$

We have just established that under the assumption of (B), (c), and (f) the morphism $\kappa'_{G,H}$ is an isomorphism which yields that $(\kappa'_{G,H})\widehat{\ }$ is an isomorphism. From (e) we get that η_Γ is an isomorphism. This suffices for $\kappa_{G,H}$ to be an isomorphism.

If (B), (c), (e), and (f) hold we consider the commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{\eta_H} & \widehat{\widehat{H}} & & \widehat{H} \\ {\scriptstyle \kappa'_{G,H}}\downarrow & & \downarrow{\scriptstyle (\kappa_{G,H})\widehat{\ }} & & \uparrow{\scriptstyle \kappa_{G,H}} \\ \widehat{\Gamma} & \xrightarrow[\mathrm{id}_{\widehat{\Gamma}}]{} & \widehat{\Gamma} & & \Gamma \end{array}$$

From (d) we get that $\eta_{\widehat{H}}$ is an isomorphism. By (v) above, $\kappa'_{G,H}$ is an isomorphism. It then follows that $(\kappa_{G,H})\widehat{\ }$ is an isomorphism. □

Let us summarize for a closed subgroup H of G the conclusions in the following diagram:

$$\begin{array}{ccccccc} & & G & & \{0\} & & \\ \widehat{H^\perp} & \cong & \left\{ \Big| \right. & & \left. \Big| \right\} & \overset{\leftarrow}{\cong} & \widehat{G/H} \\ & & H & & H^\perp & & \\ (\widehat{G}/H^\perp)\widehat{\ } & \cong & \left\{ \Big| \right. & & \left. \Big| \right\} & \cong & \widehat{H} \\ & & \{0\} & & \widehat{G}. & & \end{array}$$

Hypotheses which suffice for the establishing of various isomorphism are as follows, where $\overset{\rightarrow}{\cong}$ means the existence of a bijective morphism of abelian topological groups:

for	$\widehat{G/H} \overset{\rightarrow}{\cong} H^\perp$	without extra hypothesis,
for	$\widehat{G/H} \cong H^\perp$	hypothesis (a),
for	$H \overset{\rightarrow}{\cong} (\widehat{G}/H^\perp)\widehat{\ }$	hypotheses (b), (c*), and (f),
for	$H \overset{\rightarrow}{\cong} (\widehat{G}/H^\perp)\widehat{\ }$	hypotheses (b), (c), and (f),
for	$H \cong (\widehat{G}/H^\perp)\widehat{\ }$	hypotheses (B), (c*), and (f),
for	$H \cong (\widehat{G}/H^\perp)\widehat{\ }$	hypotheses (B), (c), and (f),
for	$\widehat{G}/H^\perp \cong \widehat{H}$	hypotheses (B), (c), (d), and (f),
for	$\widehat{H^\perp} \cong G/H$	hypothesis (a) and G/H is reflexive.

We observe that in parts (i) and (ii) we need not assume that H is closed. If H is not closed, then G/H is not Hausdorff. But $\mathbb{T}$ is Hausdorff, and so every continuous character vanishing on H vanishes on $\overline{H}$. The quotient map $G/H \to G/\overline{H}$ induces a bijection on the topologies of the two spaces and thus yields an isomorphism $\widehat{G/\overline{H}} \to \widehat{G/H}$.

Hypothesis (b) allows us to think of G as the character group of $\widehat{G}$ at least group theoretically (while the given topology of the group G, when considered as a function space on $\widehat{G}$, may be finer than the topology of uniform convergence on equicontinuous sets of $\widehat{G}$ and thus, in most instances, finer that the compact-open topology). Hypothesis (c) is used only to secure that $H = H^{\perp\perp}$ (i.e. (c*)) under these circumstances. Hypothesis (B) that G *"is"* the character group of $\widehat{G}$ is, so to speak, the general hypothesis under which we work. If G is reflexive and $H^{\perp\perp} = H$, then H *"is"* the character group of $\widehat{G}/H^{\perp}$. The hardest thing to get is that $\widehat{G}/H^{\perp}$ *"is"* the character group of H; it appears to require strong information on $\widehat{G}/H^{\perp}$ itself. But this is a powerful conclusion because it entails, among other things that every character of H extends to one on G and this, of itself, asks a lot.

Banasczcyk [18] calls an abelian topological group G *strongly reflexive* if every closed subgroup and every Hausdorff quotient group of G and of $\widehat{G}$ is reflexive. Such a group will automatically satisfy all hypotheses of 7.17 except (a). 17.1.2 of [18] points out that all countable products of locally compact abelian groups are strongly reflexive. On the other hand, every infinite dimensional Banach space is an example of a reflexive but not strongly reflexive abelian topological group. We will see in due course that all locally compact abelian groups are strongly reflexive in this sense (and satisfy property (a)).

Corollary 7.18. *Assume that G is an abelian topological group and that $H_1 \subseteq H_2 \subseteq G$ are closed subgroups such that the bijective morphism*

$$(\lambda) \qquad \lambda_{G,H_1}\colon (G/H_1)\widehat{\ } \to H_1^{\perp}$$

is an isomorphism. (Cf. 7.17(i).*) Then $\left(\lambda_{G,H_1}\right)^{-1}\colon H_1^{\perp} \to (G/H_1)\widehat{\ }$ and the injective morphism*

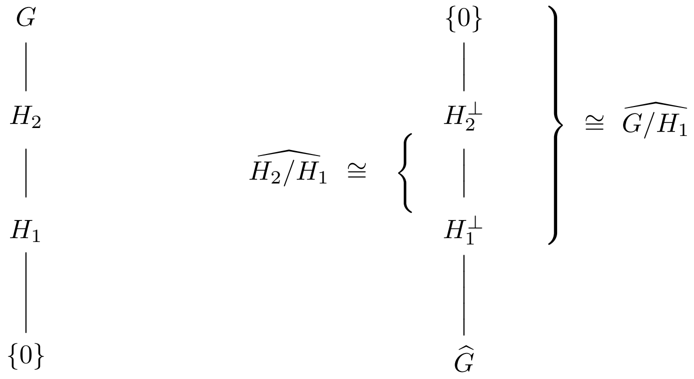
$$(\kappa) \qquad \kappa_{G/H_1,H_2/H_1}\colon (G/H_1)\widehat{\ }/(H_2/H_1)^{\perp} \to (H_2/H_1)\widehat{\ }$$

implement an injective morphism of abelian topological groups

$$(\kappa^*) \qquad \kappa^*\colon H_1^{\perp}/H_2^{\perp} \to (H_2/H_1)\widehat{\ }$$

with image $(G/H_1)\widehat{\ }|\,H_2/H_1$.

Assume that G/H_1 and $H_1^{\perp}/H_2^{\perp}$ are reflexive, the characters of G/H_2 separate points, and H_2/H_1 is in $\mathbb{TAB}_\eta$. Then κ^* is an isomorphism of abelian topological groups.



Proof. By condition (λ) we may in a natural fashion identify $(G/H_1)\hat{}$ with $H_1^\perp$. The morphism κ^* is an isomorphism if $\kappa_{G/H_1,H_2/H_1}$ is an isomorphism. The claim then follows from the Annihilator Mechanism 7.17(vi). The Hasse diagrams help to visualize the situation. □

The results in the Annihilator Mechanism Lemma 7.17 can also be expressed in terms of exact sequences:

Corollary 7.19. *Assume that*

$$0 \to H \xrightarrow{j} G \xrightarrow{q} Q \to 0$$

is an exact sequence of abelian topological groups where j is an embedding morphism and q is a quotient morphism. Assume that G, Q, and $\widehat{G}/(\ker q)^\perp$ are reflexive. Then the dual sequence

$$0 \to \widehat{Q} \xrightarrow{\widehat{q}} \widehat{G} \xrightarrow{\widehat{j}} \widehat{H} \to 0$$

is exact. Further $\widehat{H} \cong \widehat{G}/H^\perp$, and there is a bijective morphism of abelian topological groups $\widehat{Q} \to H^\perp$; and if the pair (G,H) has enough compact sets, then it is an isomorphism of topological groups. □

One may rephrase this by saying *the contravariant duality functor $G \mapsto \widehat{G}$ is exact*, under the given hypotheses.

Theorem 7.20 (The Extension Theorem for Characters). *Let H be a closed subgroup of an abelian topological group G such that the following hypotheses are satisfied:*

(I) *G is reflexive.*
(II) *η_H is continuous; i.e. H belongs to $\mathbb{TAB}_\eta$.*
(III) *The characters of G/H separate points.*
(IV) *$\widehat{G}/H^\perp$ is reflexive.*

Then the following conclusions hold.

(i) *Every character, i.e. continuous morphism* $H \to \mathbb{T}$*, extends to a character of* G*.*

(ii) *The restriction homomorphism* $\chi \mapsto \chi|H\colon \widehat{G} \to \widehat{H}$*; i.e. the adjoint of the inclusion* $H \to G$*, is surjective.*

Proof. Clearly, (i) and (ii) are equivalent statements. But conclusion (ii) is a consequence of the fact that $\kappa_{G,H}$ is a bijective morphism by the Annihilator Mechanism 7.17(vi). □

The terminology introduced in the following definition will facilitate communication in subsequent discussions.

Definition 7.21. We shall say that an abelian topological group G *has sufficient duality* if it is reflexive and if for all closed subgroups H of G the characters of G/H separate the points and $\widehat{G}/H^\perp$ is reflexive, too. □

For an abelian topological group G, let $\mathrm{Lat}(G)$ denote the lattice of closed subgroups of G. Since the intersection of any family of closed subgroups is a closed subgroup, every subset of $\mathrm{Lat}(G)$ has a greatest lower bound; i.e. $\mathrm{Lat}(G)$ is a complete lattice (see e.g. [122], p. 1f). As in every complete lattice, the least upper bound of a family $\{H_j \mid j \in J\}$ of closed subgroups is the greatest lower bound of the set of its upper bounds, i.e. the intersection of all closed subgroups containing all of the H_j. Since this group contains $\sum_{j\in J} H_j = \left\langle \bigcup_{j\in J} H_j \right\rangle$, this least upper bound can also be described as $\overline{\sum_{j\in J} H_j}$.

Corollary 7.22. *Let* G *be an abelian topological group with sufficient duality. Then*

$$H \mapsto H^\perp\colon \mathrm{Lat}(G) \to \mathrm{Lat}(\widehat{G})$$

is an antitone (*i.e. order reversing*) *function which is involutive* (*i.e. satisfies* $H^{\perp\perp} = H$). *It maps* $\mathrm{Lat}(G)$ *isomorphically onto the sublattice of all subgroups* S *of* $\widehat{G}$ *satisfying* $S = S^{\perp\perp}$*. If* $\widehat{G}$ *also has sufficient duality, then it is a lattice anti-isomorphism.*

Proof. As observed in the beginning, $H \mapsto H^\perp$ reverses the order, and because $H = H^{\perp\perp}$ by the Annihilator Mechanism Lemma 7.17, and by 7.15, its image is exactly the set of all closed subgroups S satisfying $S = S^{\perp\perp}$. If $\widehat{G}$ also has sufficient duality, by 7.17 this is satisfied for all closed subgroups S. □

Proposition 7.23. *Assume that* G *is an abelian topological group with sufficient duality and* $\{H_j \mid j \in J\}$ *is a family of subgroups. Let* $D = \bigcap_{j\in J} \overline{H_j}$ *and* $H = \sum_{j\in J} H_j$*, then*

(i) $H^\perp = \bigcap_{j\in J} H_j^\perp$,

and if $\widehat{G}$ *also has sufficient duality, then*

(ii) $D^\perp = \overline{\sum_{j\in J} H_j^\perp}$.

Proof. (i) A character is in $H^\perp$ if and only if it vanishes on every H_j, i.e. if and only if it is contained in every $H_j^\perp$ if and only if it is in $\bigcap_{j\in J} H_j^\perp$.

(ii) Since D is the greatest lower bound in $\mathrm{Lat}(G)$ of the family $\{\overline{H}_j \mid j \in J\}$ by 7.21 we know that $D^\perp$ is the least upper bound of the family of all $\overline{H_j}^\perp = H_j^{\perp\perp\perp} = H_j^\perp$, $j \in J$. This least upper bound, however, is $\overline{\sum_{j\in J} H_j^\perp}$. □

Let us remark in conclusion of the section, that the formalism of duality for abelian topological groups based on character groups endowed with the compact open topology is rather general, but has, on a level of great generality some delicate points. Is the function $(\chi, g) \mapsto \chi(g): G \times \widehat{G} \to \mathbb{T}$ continuous? Is the evaluation map $\eta_G: G \to \widehat{\widehat{G}}$ continuous? Is it bijective? Is it an isomorphism of topological groups? To what extent does the annihilator mechanism for subgroups function? These questions have motivated many authors to write on this subject. We have collected what we need and what appears feasible to present within the bounds of our overall objective which is the structure of compact groups. The next section illustrates some of the issues raised; it is still preliminary to our core subject; but for an understanding of the exponential function for locally compact abelian groups we need an understanding of the duality theory for certain topological vector spaces [317].

Character Groups of Topological Vector Spaces

For topological vector spaces the study of vector space duals turned out to be eminently fruitful. We want to make the connection between character theory and vector space duality. A first step was already taken in 7.5(iii) where we saw that the character group $\widehat{E}$ of the additive group of a real topological vector space E may be identified with its topological dual E' given the topology of uniform convergence on compact sets. The material we have in this section is adequate for our purposes, but a comprehensive treatment can be found in Appendix 7.

Let us recall some basic introductory facts on topological vector spaces in an exercise. We shall in this section, as in earlier chapters, denote the real or complex ground field by $\mathbb{K}$. We set $\mathbb{D} = \{z \in \mathbb{K} : |z| \le 1\}$. A subset U in a $\mathbb{K}$-vector space is called *balanced* if $\mathbb{D}{\cdot}U = U$. It is called *absorbing* if

$$(\forall v \in E)(\exists r > 0)(\forall t \in \mathbb{K}) \quad (|t| > r) \Rightarrow (v \in t{\cdot}U).$$

A balanced set is absorbing if every vector is contained in a multiple of the set.

Recall that a *topological vector space over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ (or over any topological field $\mathbb{K}$, for that matter) is an abelian topological group E with a continuous vector space scalar multiplication $(t, v) \mapsto t{\cdot}v: \mathbb{K} \times E \to E$.

Exercise E7.12. Show that the filter of zero neighborhoods $\mathcal{U}$ in a topological vector space satisfies

(0) $\bigcap \mathcal{U} = \{0\}$.
(i) $(\forall U \in \mathcal{U})(\exists V \in \mathcal{U}) \quad V - V \subseteq U$.
(ii) $(\forall U \in \mathcal{U})(\exists V \in \mathcal{U}) \quad \mathbb{D}\cdot V \subseteq U$.
(iii) Every $U \in \mathcal{U}$ is absorbing.

Conversely show that, if a filter $\mathcal{U}$ satisfies (i), (ii), (iii), then the set $\mathcal{O}$ of all subsets U of E such that for $v \in U$ there is a $W \in \mathcal{U}$ with $v + W \subseteq U$ is a vector space topology whose filter of identity neighborhoods in $\mathcal{U}$. If it also satisfies (0), then $\mathcal{O}$ is a Hausdorff topology. (We shall always assume that.) □

We call an abelian topological group G *locally precompact* if there is a precompact identity neighborhood.

Proposition 7.24. (i) *On a one-dimensional* $\mathbb{K}$*-vector space* E*, outside the indiscrete topology* $\{\emptyset, E\}$ *there is only one vector space topology. For each* $0 \neq v \in E$ *the map* $r \mapsto r\cdot v : \mathbb{K} \to E$ *is an isomorphism of topological vector spaces.*

(ii) *A locally compact subgroup* H *of a Hausdorff topological group* G *is a closed subset.*

(iii) *A finite dimensional* $\mathbb{K}$*-vector space admits one and only one vector space topology. If* E *is a* $\mathbb{K}$*-vector space with* $\dim E = n$ *and* $E \to \mathbb{K}^n$ *is an isomorphism then it is an isomorphism of topological vector spaces where* $\mathbb{K}^n$ *has the product topology.*

(iv) *A locally precompact topological vector space over* $\mathbb{K}$ *is finite dimensional.*

Proof. Exercise E7.13. □

Exercise E7.13. Prove Proposition 7.24.

[Hint. See Appendix 7, Proposition 7.2.] □

For a vector space E over $\mathbb{K}$ we shall denote the set of all finite dimensional vector subspaces by $\mathrm{Fin}(E)$. For a topological vector space E over $\mathbb{K}$ we denote the set of cofinite dimensional *closed* vector subspaces (i.e. closed vector subspaces M with $\dim E/M < \infty$) by $\mathrm{Cofin}(E)$.

A topological vector space E over $\mathbb{K}$ is called *locally convex* if every zero neighborhood contains a convex one. Now let E be any real vector space and let $\mathcal{B}(E)$ denote the set of all balanced, absorbing and *convex* subsets of E. Let us observe that there are plenty of those, in fact enough to allow only 0 in their intersection. Let F be a basis of E and $\rho: F \to]0, \infty[$ any function. Then the set

$$U(F;\rho) = \{\sum_{e \in F} r_e \cdot e : |r_e| < \rho(e)\}$$

is balanced, absorbing and convex. We call it a *box neighborhood* with respect to F. The box neighborhoods with respect to a single basis already intersect in 0. Thus the filter of all supersets of sets from $\mathcal{B}(E)$ satisfies (0), (i), (ii), and (iii) of E7.12. If we set

$$\mathcal{O}(E) = \{W \subseteq E \mid (\forall w \in W)\big(\exists U \in \mathcal{B}(E)\big) \quad w + U \subseteq W\},$$

then $\mathcal{O}(E)$ is a locally convex vector space topology. From its definition it is immediate that it contains every other locally convex vector space topology. It is clearly an algebraic invariant in so far as it depends only on the vector space structure of E. A convex subset U of E belongs to $\mathcal{O}(E)$ if and only if for every $u \in U$ and every $x \in E$ the set $\{r \in \mathbb{R} \mid u + r{\cdot}x \in U\}$ is an open interval of $\mathbb{R}$ containing 0. It follows that a convex subset U of E belongs to $\mathcal{O}(E)$ if and only if for each finite dimensional vector subspace F and each $v \in E$ the intersection $F \cap (U - v)$ is open in F (in the unique vector space topology of F).

Let us record some of the basic properties of $\mathcal{O}(E)$.

Proposition 7.25. *Let E be an arbitrary vector space over $\mathbb{K}$.*

(i) *If E_1 and E_2 are vector spaces and $T: E_1 \to E_2$ is a linear map, and E_2 is a locally convex topological vector space then T is continuous for the topology $\mathcal{O}(E_1)$.*

In particular, every algebraic linear form $E \to \mathbb{K}$ is $\mathcal{O}(E)$-continuous; i.e. the algebraic dual $E^* = \mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$ is the underlying vector space of the topological dual $E' = \mathrm{Hom}(E, \mathbb{K})$ (which is considered to carry the compact open topology). If $\mathbb{K} = \mathbb{R}$ then $E' \cong \widehat{E}$.

(ii) *Every vector subspace of E is $\mathcal{O}(E)$-closed and is a direct summand algebraically and topologically. Moreover, the topology induced on each vector subspace is its finest locally convex topology.*

(iii) *Let F be a linearly independent subset of E. Then there is a zero neighborhood $U \in \mathcal{O}(E)$ such that $\{v + U \mid v \in F\}$ is a disjoint open cover of F. In particular, any linearly independent subset of $\big(E, \mathcal{O}(E)\big)$ is discrete.*

(iv) *If C is an $\mathcal{O}(E)$-precompact subset, then $\mathrm{span}_{\mathbb{K}}(C)$ is finite dimensional.*

Proof. See Appendix 7, Proposition 7.3. □

We shall call the topology $\mathcal{O}(E)$ the *finest locally convex topology on E*.

Our main interest will be with vector spaces dual to those we just discussed. Their topology was determined by the finite dimensional vector subspaces. Dually we may consider vector space topologies which are determined by the cofinite dimensional closed vector subspaces.

Let E be a topological vector space. Then for M, $N \in \mathrm{Cofin}(E)$ with $N \subseteq M$, there is a canonical quotient map $q_{MN}: E/N \to E/M$. Since $\mathrm{Cofin}(E)$ is a filter basis, there is an inverse system and, in the category of topological vector spaces, there is a projective limit $E^\star = \lim_{M\in\mathrm{Cofin}(E)} E/M$, the vector subspace of all $(v_M + M)_{M\in\mathrm{Cofin}(E)} \in \prod_{M\in\mathrm{Cofin}(E)} E/M$ such that $N \subseteq M$ implies $v_N - v_M \in M$. The function $\gamma_E: E \to E^\star$, $\gamma_E(v) = (v + M)_{M\in\mathrm{Cofin}(E)}$ is a morphism of topological vector spaces which is injective if and only if $\bigcap \mathrm{Cofin}(E) = \{0\}$.

Lemma 7.26. *For a topological $\mathbb{K}$-vector space E, the following statements are equivalent:*

(1) *There is a set J and an isomorphism of topological vector spaces $E \to \mathbb{K}^J$.*

(2) *There exists a* $\mathbb{K}$*-vector space* P *such that* $E = P^* = \mathrm{Hom}(P, \mathbb{K}) \subseteq \mathbb{K}^{\mathbb{P}}$ *with the topologgy of pointwise convergence on* P^**.*
(3) *The evaluation map* $\mathrm{ev}: E \to E'^*$, $\mathrm{ev}(v)(f) = f(x)$ *is an isomorphism of topological vector spaces.*
(4) *The function* $\gamma_E: E \to E^\star = E_{\mathrm{Cofin}(E)}$ *is an isomorphism of topological vector spaces.*
(5) E *is isomorphic to a closed vector subspace of* $\mathbb{K}^X$ *for some set* X*.*

Definition 7.27. The topology $\mathcal{O}^\star(E)$ on E making $\gamma_E: E \to E^\star$ a topological embedding is called the *weak topology* on E. We shall call a topological vector space E *weakly complete* if $\gamma_E: E \to E^\star$ is an isomorphism of topological vector spaces. □

The *weak topology* on a $\mathbb{K}$-vector space is the smallest topology making all linear functionals $f: E \to \mathbb{K}$ continuous. All finite dimensional vector spaces are weakly complete. On a weakly complete topological vector space, the continuous functionals separate points. In this book we do not talk about Cauchy filters on topological groups (let alone uniform spaces) and therefore we do not deal with the concept of *completeness* in the technical sense. Our definition in terms of the projective limit (which we do need in the context of compact groups and which we have at our disposal since Chapter 1) replaces the completeness in the weak topology as the name of a *"weakly complete vector space"* suggests.

Recall from 7.25(i) that the algebraic dual E^* of a $\mathbb{K}$-vector space E is at the same time the topological dual E' of all continuous linear functionals on E when E is endowed with the finest locally convex topology. On the basis of just linear algebra one always has the weak $*$-topology on E^*, i.e. the topology of pointwise convergence induced by the natural inclusion $E^* \to \mathbb{K}^E$. The first item in the following lemma will show that this topology agrees with the topology of uniform convergence on compact sets which is the topology we consider in order to have, in the case of $\mathbb{K} = \mathbb{R}$, the isomorphism $E' \cong \widehat{E}$ according to 7.5.

Lemma 7.28. *Let* E *be a* $\mathbb{K}$*-vector space endowed with its finest locally convex vector space topology. Then*

(i) *the compact open topology on* E' *is the weak***-topology, i.e. the topology of pointwise convergence.*

(ii) *Every continuous linear functional* $\Omega: E' \to \mathbb{K}$ *is of the form* $\omega \mapsto \omega(x) : E' \to \mathbb{K}$ *for a unique* $x \in E$*.*

(iii) $F \mapsto F^\perp: \mathrm{Fin}(E) \to \mathrm{Cofin}(E')$ *is an order reversing bijection.*

(iv) E' *is weakly complete.*

Proof. For the proof of (i), (ii), and (iii) see Appendix 7, Lemma A7.4. By (i) we have $E' = E^*$ with the topology of pointwise convergence on E^*. Thus E' satisfies 7.26(ii) and therefore is weakly complete by Definition 7.27. □

Let E be a locally convex topological vector space over $\mathbb{K}$ and E' its topological dual. If $\eta_E\colon E \to E''$, $\eta_E(x)(\omega) = \omega(x)$, denotes the evaluation morphism, then for each subset $H \subset E$ we set $H^\circ \stackrel{\text{def}}{=} \{\omega \in E' : |\omega(H)| \subseteq [0,1]\} = \bigcap_{h\in H} \eta_E(h)^{-1}B_1$ with $B_1 = \{r \in \mathbb{K} : |r| \leq 1\}$ and call this set the *polar* of H in E'. Similarly for a subset $\Omega \subseteq E'$ we define the polar of Ω in E to be $\Omega^\circ \stackrel{\text{def}}{=} \{x \in E : |\Omega(x)| \subseteq [0,1]\} = \bigcap_{\omega\in\Omega} \omega^{-1}(B_1)$. Again as in the case of annihilators of subsets of abelian topological groups one must specify where the polars are taken. Polars are always closed.

Lemma 7.29 (The Bipolar Lemma). *Let E be a locally convex vector space and U be a convex balanced subset of E. Let Ω be a convex balanced subset of E'. Then*
(i) *$U^{\circ\circ} = \overline{U}$, and*
(ii) *if E is semireflexive, $\Omega^{\circ\circ} = \overline{\Omega}$.*

Proof. See Appendix 7, Lemma A7.5. □

Now we focus onf the case $\mathbb{K} = \mathbb{R}$.

Theorem 7.30. (Duality of Real Vector Spaces) Let E be a real vector space and endow it with its finest locally convex vector space topology, and let V be a weakly complete real topological vector space. Then

(i) E is reflexive; i.e. $\eta_E\colon E \to \widehat{\widehat{E}}$ is an isomorphism of topological vector spaces. Thus E belongs to $\mathbb{ABD}$.

(ii) V is reflexive; i.e. $\eta_V\colon V \to \widehat{\widehat{V}}$ is an isomorphism of topological vector spaces, and thus V belongs to $\mathbb{ABD}$.

(iii) The contravariant functor $\widehat{\cdot}: \mathbb{ABD} \to \mathbb{ABD}$ exchanges the full subcategory of real vector spaces (given the finest locally convex topology) and the full subcategory of weakly complete vector spaces.

(iv) Every closed vector subspace V_1 of V is algebraically and topologically a direct summand; that is there is a closed vector subspace V_2 of V such that $(x, y) \mapsto x + y : V_1 \times V_2 \to V$ is an isomorphism of topological vector spaces. Every surjective morphism of weakly complete vector spaces $f\colon V \to W$ splits; i.e. there is a morphism $\sigma\colon W \to V$ such that $f \circ \sigma = \mathrm{id}_W$.

(v) For every closed vector subspace H of E, the relation $H^{\perp\perp} = H \cong (E'/H^\perp)\widehat{}$ holds and $E'/H^\perp$ is isomorphic to $\widehat{H}$. The map $F \mapsto F^\perp$ is an antiisomorphism of the complete lattice of vector subspaces of E onto the lattice of closed vector subspaces of E'.

Proof. For the proof see Appendix 7, Theorem A7.10 and Theorem A7.11. □

By the Theorem of Alaoglu, Banach and Bourbaki ([40, 317]), a subset of E' is compact if and only if it is closed and bounded (i.e. is absorbed by each zero neighborhood). We have used here Proposition 7.6 instead, which belongs to the same class of theorems pertaining to the Ascoli Theorem.

Notice that we have not specified the nature of the map $A \mapsto A^\perp$ from the complete lattice of all closed subgroups of E into the complete lattice of closed subgroups of E'

The real topological vector spaces of the form $V = E'$ and their additive groups $\widehat{E}$, i.e. the weakly complete real topological vector spaces are in bijective correspondence with the real vector spaces E. In this sense they are purely algebraic entities, and the cardinal $\dim E$ is their only isomorphy invariant. We shall use these real topological vector spaces in the context of the exponential function of locally compact abelian groups.

Real vector spaces and weakly complete vector spaces have a perfect annihilator mechanism for closed vector subgroups. But so far we are lacking information on closed subgroups. Considering its dual, at the very least we need information on the closed subgroups of a vector group in its finest locally convex vector space topology. The following result is a generalisation of the Fundamental Theorem on Closed subgroups of $\mathbb{R}^n$ (Appendix 1, A1.12). We recall that an abelian group is $\aleph_1$-free if and only if each countable subgroup is free. (See Appendix 1, A1.6.173).

Proposition 7.31. *Let E be a real vector space with its finest locally convex vector space topology and let H be a closed subgroup. Let H_0 denote the largest vector subspace of H, let E_1 be a vector space complement of H_0 in $\operatorname{span}_{\mathbb{R}} H$ and E_2 a vector space complement of $\operatorname{span}_{\mathbb{R}} H$ in E. Set $A \overset{\text{def}}{=} H \cap E_1$. Then*

(i) *E_1, E_2 are closed vector subspaces and A is closed subgroup such that*

$$E = H_0 \oplus E_1 \oplus E_2 \quad \textit{algebraically and topologically,}$$
$$H = H_0 \oplus A \quad \textit{algebraically and topologically.}$$

(ii) *A is an $\aleph_1$-free totally arcwise disconnected subgroup.*

In particular, every countable closed subgroup A of E is free.

For the remaining conclusions assume that H/H_0 is countable.

(iii) *The characters of E/H separate points.*

(vi) *The characters of H separate points.*

(v) *If $H^\perp \in \widehat{E}$ is the annihilator of H in $\widehat{E}$, then $H^{\perp\perp} = H$.*

(vi) *If $\eta_H \colon H \to \widehat{\widehat{H}}$ is continuous, then $H \cong (\widehat{E}/H^\perp)\widehat{\ }$.*

Proof. (i) The subgroup generated by the union of all vector subspaces of H is a vector subspace H_0, clearly the largest vector subspace of H. By 7.25(iii) we have $\operatorname{span}_{\mathbb{R}} H = H_0 \oplus H_1$ and $E = \operatorname{span}_{\mathbb{R}} H \oplus E_2$ with a closed vector subspaces E_1 and E_2, algebraically and topologically. Thus the morphism

$$(h_0, e_1, e_2) \mapsto h_0 + e_1 + e_2 \colon H_0 \times E_1 \times E_2 \to E$$

is an isomorphism of topological vector spaces. By the modular law, $H = H_0 \oplus A$ where $A = H \cap E_1$. Hence the isomorphism

$$(h_0, e_1) \mapsto h_0 + e_1 \colon H_0 \times A \to H$$

is an isomorphism of abelian topological groups.

We notice that $E/H \cong E_1/A \times E_2$. In order to determine the structure of A in E_1 and to prove that the characters of E/A separate points it suffices to show that the characters of E_1/A separate points. Hence we simplify notation by assuming henceforth that H has no vector subspaces and linearly spans E.

(ii) We have to show that $H = A$ is $\aleph_1$-free. Let A_1 be a finite rank subgroup of A. Then $F \stackrel{\text{def}}{=} \operatorname{span}_{\mathbb{R}} A_1$ is a finite dimensional vector subspace. Thus $A \cap F$ is a closed subgroup of F without any nontrivial vector subspaces. Hence it is discrete and free by Appendix 1, A1.12. Thus A is $\aleph_1$-free by Definition A1.63. By Appendix 1, A1.64, every countable subgroup of A is free. If C is any arc in A then, being compact, it lies in a finite dimensional vector subspace F (see 7.25(iv)). Since $A \cap F$ is discrete, C is singleton.

(iii) Assume that A countable, that $E = \operatorname{span}_{\mathbb{R}} A$, and that $x \in E \setminus A$. It suffices to show that the characters of E/A separate the points. By (ii) the subgroup A is free. With the aid of the Axiom of Choice we find a $\mathbb{Z}$-basis B of A. Then B is an $\mathbb{R}$-basis of E. Then there is a finite subset $B_x \subseteq B$ such that $x = \sum_{b\in F} r_b{\cdot}b$. We define $E_x = \operatorname{span}_{\mathbb{R}} B_x$ and $E^x = \operatorname{span}_{\mathbb{R}}(B \setminus B_x)$. Then $A_x = E_x \cap A$ and $A^x = E^x \cap A$ are closed subgroups. If $a \in A$. Then $a = \sum_{b\in B} s_b{\cdot}b$ with $s_b \in \mathbb{Z}$. Set $a_x = \sum_{b\in B_x} s_b{\cdot}b$ and $a^x = \sum_{b\in B^x} s_b{\cdot}b$. Then $a = a_x + a^x$ and, since $B \subseteq A$, we have $a_x \in A_x$ and $a^x \in A^x$. Thus the isomorphism of topological vector spaces $(e_x, e^x) \mapsto e_x + e^x \colon E_x \times E^x \to E$ induces an isomorphism of abelian topological groups

$$(a_x, a^x) \mapsto a_x + a^x \colon A_x \times A^x \to A.$$

Notice that $x \in E_x \setminus A_x$. If we find a character $\chi\colon E_x \to \mathbb{T}$ with $\chi(x) \neq 0$ we are done because χ extends to a character of E vanishing on E^x. But $E_x \cong \mathbb{R}^n$ and A_x is a closed subgroup. Hence $E_x/A_x \cong \mathbb{R}^{m_1} \times \mathbb{T}^{m_2}$ by Appendix 1, A1.12. But the characters of $\mathbb{R}^{m_1} \times \mathbb{T}^{m_2}$ separate points because characters of $\mathbb{R}$ and $\mathbb{T}$ separate points. This completes the proof.

(iv) Let $x \in X$ proceed as in the proof of (iii) and find $x \in A_x$. It suffices to observe that the characters of A_x separate points. But A_x is isomorphic to a closed subgroup of $\mathbb{R}^n$ and thus is isomorphic to $\mathbb{R}^{p_1} \times \mathbb{Z}^{p_2}$ by Appendix 1, A1.12.

(v) This is a consequence of (iii) and 7.15(ii).

(vi) This is a consequence of 7.17(v). □

Example 7.32. The group $A = \mathbb{Z}^{\mathbb{N}}$ of all functions $\mathbb{N} \to \mathbb{Z}$ is an $\aleph_1$-free subgroup of the vector space $E = \mathbb{R}^{\mathbb{N}}$ of all real valued function s in the finest locally convex topology. Now A is not a Whitehead group and thus is not free by Appendix 1, Example A1.65. Clearly, A does not contain any vector subspaces. Clearly $U \stackrel{\text{def}}{=} \left]-\frac{1}{2}, \frac{1}{2}\right[^{\mathbb{N}} \in \mathcal{O}(\mathbb{R}^{\mathbb{N}})$ and $U \cap \mathbb{Z}^{\mathbb{N}} = \{0\}$, whence $\mathbb{Z}^{\mathbb{N}}$ is discrete, and thus is in particular closed. The real span $\operatorname{span}_{\mathbb{R}} A$ is properly smaller than E. □

Later we will discover that the domain, $\mathfrak{L}(G)$, of the exponential function of a (locally) compact abelian group is a weakly complete vector space. (7.35(vii), 7.66.) Its kernel is a closed subgroup; therefore we are interested in clarifying the role of closed subgroups in weakly complete vector spaces. We use the Duality of Real Vector Spaces (7.30) in the process.

Theorem 7.33 (Subgroups of a Weakly Complete Real Vector Space). *Let V be a real weakly complete vector space and let S be a closed subgroup satisfying $S = S^{\perp\perp}$. Then*

(i) *there are unique closed vector subspaces S_0, the maximal vector subspace contained in S, and V_S, the closed span $\overline{\mathrm{span}_{\mathbb{R}} S}$ of S in V. Furthermore there are a generally nonunique closed vector subspaces $V_1 \supseteq S_0$ and V_2 such that*

$$\begin{aligned} V &= S_0 \oplus V_1 \oplus V_2 && \textit{algebraically and topologically,} \\ V_S &= S_0 \oplus V_1 && \textit{algebraically and topologically, and} \\ S &= S_0 \oplus V_1 \cap S && \textit{algebraically and topologically.} \end{aligned}$$

(ii) *The largest vector subspace contained in $V_1 \cap S$ is $\{0\}$, and the closed span $\overline{\mathrm{span}_{\mathbb{R}}(V_1 \cap S)}$ is V_1. Moreover, $(V_1 \cap S)^{\perp\perp} = V_1 \cap S$.*

(iii) *$\lambda_{V,S}\colon \widehat{V/S} \to S^\perp$ is a bijective morphism of abelian topological groups, and $\lambda_{\widehat{V},S^\perp}\colon (\widehat{V}/S^\perp)\widehat{\ } \to S$ is a bijective morphism of topological abelian groups.*

(iv) *Assume that $S_0 = \{0\}$ and that $V_S = V$. Then $S^\perp$ is $\aleph_1$-free.*

Proof. (i) By 7.30, every closed vector subspace of a weakly complete vector space is algebraically and topologically a direct summand. Hence $V = V_S \oplus V_2$ for some closed vector subspaces V_2. Likewise, $V_S = S_0 \oplus V_1$. Thus we have an isomorphism of vector spaces $(s_0, v_1, v_2) \mapsto s_0 + v_1 + v_2\colon S_0 \times V_1 \times V_2 \to V$. Since $S = S_0 + (V_1 \cap S)$ by the modular law we know that this isomorphism induces an isomorphism $(s_0, v_1) \mapsto s_0 + v_1\colon S_0 \times (V_1 \cap S) \to S$.

(ii) The assertions follow from (i) above and from the fact that passing to the character groups on direct products is implemented factorwise by Proposition 7.10.

(iii) Then $\widehat{V}$ is a vector space with its finest locally convex topology and $S^\perp$ is a closed subgroup of $\widehat{V}$ such that $S^{\perp\perp} = S$ by hypothesis. Then 7.17(i) applies and proves the claims on the morphisms λ.

(iv) Parts (i) and (ii) of the proposition have reduced the general case to the case assumed here. Since $\overline{\mathrm{span}_{\mathbb{R}} S} = V$ the largest vector subspace of $S^\perp$ is zero. Then Proposition 7.31 shows that $S^\perp$ is $\aleph_1$-free. □

Not all technical ramifications of Theorem 7.34 will be used later; however, the domain of the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ of a locally compact group (Theorem 7.66 below) is a vector space of the type of V in 7.33, and its kernel $\mathfrak{K}(G)$ is an $\aleph_1$-free totally arcwise disconnected closed subgroup of $\mathfrak{L}(G)$, a special instance of a subgroup of the type of the subgroup S of V in 7.33.

Before we conclude the excursion into vector groups, we point out how dual vector spaces arise in the context of hom-groups. The following proposition will be applied to the exponential function, and it largely motivates the preceding discussion.

Proposition 7.34. (i) *For each abelian topological group G there exists a real locally convex topological vector space $\mathbb{R}\mathbin{\overline{\otimes}}G$ and a unique morphism of abelian topological groups $\iota_G\colon G \to \mathbb{R}\mathbin{\overline{\otimes}}G$ such that for each morphism $f\colon G\to V$ into a real locally convex topological vector space there is a unique continuous linear map $f'\colon\mathbb{R}\mathbin{\overline{\otimes}}G\to V$ such that $f = f'\circ\iota_G$.*

$$
\begin{array}{ccccc}
\multicolumn{3}{c}{\mathbb{TAB}} & & \text{LocConvVect} \\
\hline
G & \xrightarrow{\eta_G} & \mathbb{R}\mathbin{\overline{\otimes}}G & & \mathbb{R}\mathbin{\overline{\otimes}}G \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f'} \\
V & \xrightarrow[\mathrm{id}_V]{} & V & & V
\end{array}
$$

(ii) $\operatorname{span}_{\mathbb{R}}\big(\iota_G(G)\big) = \mathbb{R}\mathbin{\overline{\otimes}}G$.

(iii) *Assume that G is a discrete abelian group, Then the topological vector space $\mathbb{R}\mathbin{\overline{\otimes}}G$ may be identified with the tensor product $\mathbb{R}\otimes G$ (as in Appendix* 1, A1.44) *equipped with the finest locally convex vector space topology, and ι_G with the function given by $\iota_G(g) = 1\otimes g$.*

Proof. Exercise E7.14. □

Exercise E7.14. Prove Proposition 7.34.

[Hint. (i) The existence of $\mathbb{R}\mathbin{\overline{\otimes}}G$ in general follows from the Adjoint Functor Existence Theorem in Appendix 3, A3.30. (The solution set condition mentioned in Appendix 3, A3.58 is readily verified here.)

(ii) Note that the corestriction $\iota_G\colon G\to\operatorname{span}_G\big(\iota_G(G)\big)$ has the universal property and apply the uniqueness assertion in the universal property.

(iii) Verify that $g\mapsto 1\otimes g\colon G\to\mathbb{R}\otimes G$ has the universal property if G is discrete and if the abelian group $\mathbb{R}\otimes G$ is made into a real vector space equipped with the scalar multiplication characterized by $r\cdot(s\otimes g) = rs\otimes g$ and is given the finest locally convex topology.] □

Alternative existence proofs for $\mathbb{R}\mathbin{\overline{\otimes}}G$ can be given (see [140]).

Motivated by the discrete situation, which is later of primary interest to us, we will write $\iota_G(g) = 1\mathbin{\overline{\otimes}}g$ in the general case.

Proposition 7.35. (i) *For any abelian topological group G the hom-group $V \stackrel{\text{def}}{=} \operatorname{Hom}(G,\mathbb{R})$ with the compact open topology is a real topological vector space with respect to the scalar multiplication given by $(r\cdot f)(g) = rf(g)$ for $r\in\mathbb{R}$, $g\in G$.*

(ii) *The morphism* $\operatorname{Hom}(\iota_G,\mathbb{R})\colon(\mathbb{R}\mathbin{\overline{\otimes}}G)' = \operatorname{Hom}(\mathbb{R}\mathbin{\overline{\otimes}}G,\mathbb{R})\to\operatorname{Hom}(G,\mathbb{R})$, $\operatorname{Hom}(\iota_G,\mathbb{R})(F) = F\circ\iota_G$, *is a continuous isomorphism of abelian groups. Its inverse is $f\mapsto f'\colon\operatorname{Hom}(G,\mathbb{R})\to(\mathbb{R}\mathbin{\overline{\otimes}}G)'$. This morphism is continuous if $(\mathbb{R}\mathbin{\overline{\otimes}}G)'$ is given the weak $*$-topology, i.e. the topology of pointwise convergence.*

(iii) *Assume that the following hypothesis is satisfied:*

(∗) *The topology of the locally convex vector space* $\mathbb{R}\,\overline{\otimes}\,G$ *agrees with the finest locally convex vector space topology.*

Then $V = \mathrm{Hom}(G,\mathbb{R})$ *is naturally isomorphic with the dual* $(\mathbb{R}\,\overline{\otimes}\,G)'$ *with the compact open topology and thus is naturally isomorphic to the character group* $(\mathbb{R}\,\overline{\otimes}\,G)\widehat{\ }$.

(iv) *If* (∗) *holds then* V *is a weakly complete vector space, and the dual of* V *is naturally isomorphic with* $E = \mathbb{R}\,\overline{\otimes}\,G$.

(v) *The group* $\mathrm{Hom}(G,\mathbb{Z})$ *may be identified with the subgroup* S *of all* $\omega \in V = \mathrm{Hom}(G,\mathbb{R})$ *with* $\omega(G) \subseteq \mathbb{Z}$.

a) *Under the hypothesis* (∗), $\mathrm{Hom}(G,\mathbb{Z})$ *is the annihilator in* $V \cong \widehat{E}$ *of* $A \stackrel{\mathrm{def}}{=} 1\,\overline{\otimes}\,G$. *Consequently,* $\overline{A} \subseteq S^{\perp}$.

b) S *satisfies* $S = S^{\perp\perp}$.

c) $\lambda_{V,S}\colon \widehat{V/S} \to S^{\perp}$ *is a bijective morphism of abelian topological groups.*

d) *If the largest vector subspace of* $\overline{A} = \overline{1\,\overline{\otimes}\,G}$ *is denoted by* $\overline{A}_0$ *then* $(\overline{A}_0)^{\perp} = \overline{\mathrm{span}_{\mathbb{R}} S}$*, is the smallest closed real vector subspace of* $\mathrm{Hom}(G,\mathbb{R})$ *containing* $S = \mathrm{Hom}(G,\mathbb{Z})$. *Moreover,* $\overline{A}_0$ *is also the largest vector subspace contained in* $S^{\perp}$.

e) *If* $\overline{A}/\overline{A}_0$ *is countable, then* $S^{\perp} = \overline{A} = \overline{1\,\overline{\otimes}\,G}$.

(vi) *Assume that* $G = E \oplus H$ *where* E *is the additive group of a vector space having the finest locally convex vector space topology and where* H *is an abelian topological group having an open compact subgroup* C. *Then* G *satisfies* (∗). *Accordingly,* $\mathrm{Hom}(G,\mathbb{R})$ *is a weakly complete vector space with dual* $\mathbb{R}\,\overline{\otimes}\,G$.

Proof. (i) Since $\mathrm{Hom}(G,\mathbb{R})$ is a vector subspace of the topological vector space $C(G,\mathbb{R})$ of continuous functions given the compact open topology, clearly V is a real topological vector space.

(ii) The fact that the morphisms of abelian groups $\mathrm{Hom}(\iota_G,\mathbb{R})$ and $f \mapsto f'$ are inverses of each other is equivalent to the universal property of $\mathbb{R}\,\overline{\otimes}\,G$ in 7.34. The continuity of $\mathrm{Hom}(\iota_G,\mathbb{R})$ with respect to the compact open topologies is straightforward with the continuity of ι_G; indeed let $W \stackrel{\mathrm{def}}{=} W(C,U_\varepsilon)$, $U_\varepsilon = \,]-\varepsilon,\varepsilon[$ be a basic zero neighborhood of V with a compact subset C of G. Then $K \stackrel{\mathrm{def}}{=} \iota_G(C)$ is compact in $\mathbb{R}\,\overline{\otimes}\,G$, and if $f \in W(K,U_\varepsilon)$ implies that $\big(\mathrm{Hom}(\iota_G,\mathbb{R})(f)\big)(C) = f\big(\iota_G(C)\big) = f(K) \in U_\varepsilon$, i.e. $\mathrm{Hom}(\iota_G,\mathbb{R})(f) \in W$. For a proof of the continuity of $f \mapsto f'$ with respect to the topology of pointwise convergence on $(\mathbb{R}\,\overline{\otimes}\,\mathbb{R})'$ let $g \in G$ be arbitrary. Then $f \mapsto f(g)\colon V \to \mathbb{R}$ is continuous since the compact open topology is finer than (or equal to) the topology of pointwise convergence on $V = \mathrm{Hom}(G,\mathbb{R})$. But $f(g) = f'\big(\iota_G(g)\big)$, and every element $x \in \mathbb{R}\,\overline{\otimes}\,G$ is a finite linear combination of elements $\iota_G(g)$ by 7.34(ii). Hence $f \mapsto f'(x)\colon V \to \mathbb{R}$ is continuous for each $x \in \mathbb{R}\,\overline{\otimes}\,G$. This proves the asserted continuity of $f \mapsto f'$.

(iii) If $E \stackrel{\mathrm{def}}{=} \mathbb{R}\,\overline{\otimes}\,G$ has the finest locally convex topology, then on $V = \mathrm{Hom}(\mathbb{R}\,\overline{\otimes}\,G,\mathbb{R}) = E'$ the compact open topology agrees with the topology of pointwise convergence by 7.28(i). But then $V \cong \widehat{E}$ by 7.5(iii).

(iv) follows from (iii), 7.28, and 7.30.

(v) The first assertion is straightforward. Proof of a). Assuming $(*)$ we have seen that the morphism $\mathrm{Hom}(\iota_G, \mathbb{R})$ identifies $\mathrm{Hom}(G, \mathbb{R})$ with $E' = \mathrm{Hom}(E, \mathbb{R})$, $E = \mathbb{R} \mathbin{\overline{\otimes}} G$. It identifies $S = \mathrm{Hom}(G, \mathbb{Z})$ with $\mathrm{Hom}(A, \mathbb{Z})$, where $A \stackrel{\mathrm{def}}{=} \iota(G) = 1 \mathbin{\overline{\otimes}} G$. Now we compute $A^\perp$. By definition, $A^\perp$ in the character group $\widehat{E}$ of E is the set of characters $\chi: E \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ vanishing on A. We have identified $\widehat{E} = \mathrm{Hom}(E, \mathbb{R}/\mathbb{Z})$ canonically with $E' = \mathrm{Hom}(E, \mathbb{R})$ via $\mathrm{Hom}(E, \exp_{\mathbb{R}}): \mathrm{Hom}(E, \mathbb{R}) \to \mathrm{Hom}(E, \mathbb{R}/\mathbb{Z})$. Under this identification, a linear functional $\omega: E \to \mathbb{R}$ gets identified with a character $\chi = \exp_{\mathbb{R}} \circ \omega$ which vanishes on A exactly if ω maps A into $\mathbb{Z}$. Hence the annihilator $A^\perp$ in $E' = \mathrm{Hom}(A, \mathbb{R})$ is exactly $\mathrm{Hom}(A, \mathbb{Z})$. Re-identifying $\mathrm{Hom}(A, \mathbb{R})$ with $\mathrm{Hom}(G, \mathbb{R})$ we see that the annihilator $A^\perp$ in $\mathrm{Hom}(G, \mathbb{R})$ is exactly $\mathrm{Hom}(G, \mathbb{Z})$. By 7.15(ii) we get $\overline{A} \subseteq S^\perp$. Proof of b). The relation $S = S^{\perp\perp}$ now follows from 7.15(i). Claim c) follows from 7.33(iii). Proof of d). This is a consequence of 7.30(v). Proof of e). Assume that $\overline{A}/\overline{A}_0$ is countable. Then 7.31(v) shows that $\overline{A} = A^{\perp\perp} = \mathrm{Hom}(G, \mathbb{Z})^\perp = S^\perp$ in $\mathbb{R} \mathbin{\overline{\otimes}} G$.

(vi) We have $\mathbb{R} \mathbin{\overline{\otimes}} G = (\mathbb{R} \mathbin{\overline{\otimes}} E) \oplus (\mathbb{R} \mathbin{\overline{\otimes}} H)$. Thus G satisfies $(*)$ if each of the groups E and H does. In the case of E this is trivial because $\mathbb{R} \mathbin{\overline{\otimes}} E$ may be identified with E. In the case of H, the open compact subgroup C is in the kernel of the morphism ι_G because its range is a real topological vector space. If we take the algebraic tensor product $\mathbb{R} \otimes (H/C)$ (in the sense of Appendix 1, A1.44) and equip it with the natural scalar multiplication uniquely determined by $r\cdot\big(s \otimes (h + C)\big) = rs \otimes (h + C)$, and give it the finest locally convex vector space topology, then $\iota_G: H \to \mathbb{R} \otimes (H/C)$, $\iota_G(h) = 1 \otimes (h + C)$ is readily verified to have the universal property. Since universal objects are unique (up to natural isomorphism) this shows $\mathbb{R} \mathbin{\overline{\otimes}} H \cong \mathbb{R} \otimes (H/C)$ and verifies $(*)$. ☐

The main application of 7.35 will be made in 7.66. There the abelian topological group G of 7.35 will be the (locally compact) character group of a given locally compact abelian group.

We notice that all discrete groups G, and all real vector spaces with the finest locally convex vector space topology satisfy hypothesis $(*)$.

The Exponential Function

The exponential function has served us well in our theory of linear Lie groups. However, it is also of considerable importance for compact and locally compact abelian groups in general. We therefore take up this theme time and again. In this section we emphasize the environment of abelian topological groups.

In Definition 5.7, for a topological group G, we introduced the space $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ of all one-parameter groups given the topology of uniform convergence on compact sets and equipped with a scalar multiplication given by $r\cdot X(t) = X(rt)$. In 5.39 we defined the exponential function of G by

$$\exp_G: \mathfrak{L}(G) \to G, \quad \exp_G(X) = X(1).$$

In the case of an *abelian topological group* G, the situation improves instantly.

Proposition 7.36. *If G is an abelian topological group, then $\mathfrak{L}(G)$ is a topological vector space with respect to pointwise addition and the topology of uniform convergence on compact sets.*

Proof. The set $C(M,G)$ of all continuous functions from a locally compact space M to a topological group G is a group under pointwise operation given by $(X+Y)(t) = X(t)+X(t)$ (where G is written additively). This group is topological with respect to the topology of uniform convergence on compact sets; the filter of identity neighborhoods for this topology is generated by the sets $W(C,U) = \{f \in C(M,G) \mid f(C) \subseteq U\}$, where C ranges through the compact sets of M and U through the identity neighborhoods of G. We have $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G) \subseteq C(\mathbb{R}, G)$, and since G is abelian, $\mathfrak{L}(G)$ is a subgroup. It is readily verified that the definition of scalar multiplication endows this additive group with a vector space scalar multiplication which is separately continuous in each variable. If $V(C,U)$ is a basic identity neighborhood of $\mathrm{Hom}(\mathbb{R}, G)$ with an open identity neighborhood U of G and $C = [-r,r]$, say, then the function $(r,X) \mapsto r{\cdot}X$ maps $]-1,1[\,\times V(C,U)$ into $V(C,U)$, for if $|r| < 1$ and $X(C) \subseteq U$, then $(r{\cdot}X)(C) = X(Cr) \subseteq X(C) \subseteq U$. Hence scalar multiplication is jointly continuous at $(0,0)$; this together with separate continuity of scalar multiplication implies that it is jointly continuous. □

In accordance with discussions in Chapter 5 we shall call the vector space $\mathfrak{L}(G)$ *the Lie algebra* of G even when there is no nonzero Lie bracket present.

Exercise E7.15. Prove the following remark.

Assume that G and H are abelian topological groups and that $f\colon G \to H$ is a morphism of topological groups. Then the morphism $\mathfrak{L}(f) = \mathrm{Hom}(\mathbb{R}, f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ given by $\mathfrak{L}(f)(X) = f \circ X$ introduced in 5.42 *is a morphism of topological vector spaces.* □

The difference between Theorem 5.42 and the following proposition is that there the groups were linear Lie groups whereas here they need not be, and that here the groups are commutative whereas there they were not necessarily abelian.

Proposition 7.37. (i) *If $f\colon G \to H$ is a morphism of abelian topological groups, then there is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\ {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

(ii) *If f induces a local isomorphism, then $\mathfrak{L}(f)$ is an isomorphism of topological vector spaces.*

Proof. (i) This is immediate from the definitions since

$$\exp_H \mathfrak{L}(f)(X) = \big(\mathfrak{L}(f)(X)\big)(1) = \big(f \circ X\big)(1) = f\big(X(1)\big) = f\big(\exp_G X\big).$$

(ii) We shall first show that $\mathfrak{L}(f)$ is bijective. Assume that $Y: \mathbb{R} \to H$ is given. We know that there are open identity neighborhoods U and V in G and H, respectively, such that $(f|U): U \to V$ is a homeomorphism. Now let I be an open symmetric interval around 0 in $\mathbb{R}$ such that $Y(I) \subseteq V$. Then $r \mapsto (f|U)^{-1}Y(r): I \to G$ is a local morphism which, by Lemma 5.8, extends *uniquely* to a morphism $X: \mathbb{R} \to G$ so that $f\big(X(r)\big) = Y(r)$ for all $r \in I$, and then, since I generates $\mathbb{R}$ for all $r \in \mathbb{R}$. Thus $\mathfrak{L}(f)$ is bijective. Now let U_1 denote an open zero neighborhood of G contained in U. Then since $f|U$ is a homeomorphism, $V_1 = f(U_1)$ is an open zero neighborhood of H. Let C denote any compact subset of $\mathbb{R}$. Then $V(C, U_1)$ is a basic open zero neighborhood of $\mathfrak{L}(G)$ and $V(C, V_1)$ is a basic open 0-neighborhood of $\mathfrak{L}(H)$. Also,

$$\mathfrak{L}(f)(V(C, U_1)\big) = V(C, V_1).$$

Thus $\mathfrak{L}(f)$ is continuous and open. □

We note in passing that 7.37(i) can be expressed in category theoretical language. Let $I: \mathbb{TAB}_{\mathbb{R}} \to \mathbb{TAB}$ be the forgetful functor (see Appendix 3, A3.19) of the category of real topological vector spaces to the category of abelian topological groups, $\mathfrak{L}: \mathbb{TAB} \to \mathbb{TAB}_{\mathbb{R}}$ the Lie algebra functor, and $\mathrm{id}_{\mathbb{TAB}}$ the identity functor from the category $\mathbb{TAB}$ to itself. Then $\exp: I \circ \mathfrak{L} \to \mathrm{id}_{\mathbb{TAB}}$ is a natural transformation.

In the following we call two morphisms $f_j: A_j \to B$, $j = 1, 2$ in a category *equivalent* if we are given a natural isomorphism $\alpha: A_1 \to A_2$ such that $f_1 = f_2 \circ \alpha$. (This language can be easily made technically compatible with the language used in category theory for natural transformations. Cf. Appendix 3, A3.31.)

Proposition 7.38. (i) *If $\{G_j \mid j \in J\}$ is a family of topological groups and $P = \prod_{j\in J} G_j$, then $\mathfrak{L}(P) \cong \prod_{j\in J} \mathfrak{L}(G_j)$ and $\exp_P: \mathfrak{L}(P) \to \prod_{j\in J} G_j$ is equivalent to $\prod_{j\in J} \exp_{G_j}: \prod_{j\in J} \mathfrak{L}(G_j) \to \prod_{j\in J} G_j$ in a natural way.*

(ii) *If $K = \ker f$ for a morphism $f: G \to H$ of topological groups then $\mathfrak{L}(K) \cong \ker \mathfrak{L}(f)$, and $\exp_{\ker f}: \mathfrak{L}(\ker f) \to \ker f$ is equivalent to the function $\exp_G | \ker \mathfrak{L}(f): \ker \mathfrak{L}(f) \to \ker f$ in a natural way.*

(iii) *If f_1, $f_2: G \to H$ are two morphisms of topological groups and $j: E \to G$ is their equalizer, i.e. the inclusion map of the subgroup $E = \{g \in G \mid f_1(g) = f_2(g)\}$, then $\mathfrak{L}(j): \mathfrak{L}(E) \to \mathfrak{L}(G)$ is the equalizer of $\mathfrak{L}(f_1)$, $\mathfrak{L}(f_2)$, and $\exp_E: \mathfrak{L}(E) \to E$ is equivalent to $\exp_G |\mathfrak{L}(E): \mathfrak{L}(E) \to E$ in a natural way.*

(iv) *If G is the projective limit $\lim G_j$ of a projective system $f_{jk}: G_k \to G_j$ of topological groups, then $\mathfrak{L}(G)$ is the projective limit $\lim \mathfrak{L}(G_j)$ of the projective system*

$$\mathfrak{L}(f_{jk}): \mathfrak{L}(G_k) \to \mathfrak{L}(G_j),$$

and $\exp_{\lim_{j\in J} G_j}\colon \mathfrak{L}(\lim_{j\in J} G_j) \to \lim_{j\in J} G_j$ *is equivalent to*

$$\lim_{j\in J} \exp_{G_j}\colon \lim_{j\in J} \mathfrak{L}(G_j) \to \lim_{j\in J} G_j$$

in a natural way.

(v) *Let* $\mathrm{incl}\colon G_0 \to G$ *denote the inclusion morphism of the identity component. Then* $\mathfrak{L}(\mathrm{incl})\colon \mathfrak{L}(G_0) \to \mathfrak{L}(G)$ *is an isomorphism. In other words,* $\mathfrak{L}(G_0) = \mathfrak{L}(G)$.

Proof. Exercise E7.16. □

Exercise E7.16. Prove Proposition 7.38.

[Hint. (i) Each morphism $X\colon \mathbb{R} \to \prod_{j\in J} G_j$ is uniquely of the form $X(r) = \big(X_j(r)\big)_{j\in J}$ with $X_j \in \mathfrak{L}(G_j)$. (ii) $f \circ X$ is in the kernel of $\mathfrak{L}(f)$ iff $X(\mathbb{R}) \subseteq \ker f$ iff X corestricts to a one-parameter group of $\ker G$. (iii) A one-parameter subgroup $X\colon \mathbb{R} \to G$ is in the equalizer E iff $\mathfrak{L}(f_1)(X)(r) = (f_1\big(X(r)\big) = f_2\big(X(r)\big) = \mathfrak{L}(f_2)(X)(r)$ for all $r \in \mathbb{R}$, i.e. iff $\mathfrak{L}(f_1)(X) = \mathfrak{L}(f_2)(X)$. (iv) Use (i) and (iii) to verify that $X\colon \mathbb{R} \to \prod_{j\in J} G_j$, $X(r) = \big(X_j(r)\big)_{j\in J}$ has its image in $\lim G_j$ iff $X_j = f_{jk} \circ X_k$ for $j \le k$ iff $X \in \lim \mathfrak{L}(G_j)$. (v) is straightforward.] □

We remark that we have formulated 7.38 in such a fashion that it remains valid when $\mathfrak{L} = \mathrm{Hom}(\mathbb{R}, \cdot)$ is considered as a functor from the category of arbitrary topological groups to the category of pointed spaces. In light of Appendix 3 we then observe that $\mathfrak{L}$ is a covariant Hom-functor from the category of topological groups into the category of pointed topological spaces, and that it is limit preserving because it preserves products and equalizers (see Appendix 3, A3.50ff.). The restriction of $\mathfrak{L}$ to the category $\mathbb{TAB}$ of abelian topological groups allows its range category to be the category $\mathbb{TAB}_{\mathbb{R}}$ of real topological vector spaces.

Example 7.39. (i) If G is a totally arcwise disconnected topological group (i.e. G has singleton arc components) then $\mathfrak{L}(G) = \{0\}$. This applies, in particular, to totally disconnected groups.

(ii) If G is an abelian linear Lie group (cf. E5.18) then $\mathfrak{L}(G)$ agrees with the Lie algebra of G considered in Chapter 5.

In particular, if $G = \mathbb{R}^m \times \mathbb{T}^n \times D$ with a discrete abelian group D according to E5.18, then $\mathfrak{L}(G) \cong \mathbb{R}^m \times \mathbb{R}^n \times \{0\}$.

(iii)

$$\mathfrak{L}\big(\lim(\mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \cdots)\big) = \lim(\mathbb{R} \xleftarrow{p} \mathbb{R} \xleftarrow{p} \cdots) = \mathbb{R}. \qquad \square$$

Exercise E7.17. Let G be the additive topological group underlying the Banach space $L^1([0,1],\lambda)$ for Lebesgue measure λ. Define $h_t(f)$ for $t \in [0,1]$ and an L^1-function by

$$h_t(f)(x) = \begin{cases} f(x) & \text{for } 0 \le x \le t, \\ 0 & \text{for } t < x \end{cases}$$

where f is an L^1-function on the unit interval. Show that this definition is compatible with passage $f \mapsto [f]$ to classes modulo null-functions and thus yields a bounded linear operator on G. The continuous function $(t,[f]) \mapsto [h_t(f)]$: $[0,1] \times G \to G$ is a homotopic contraction. Consider the closed subgroup H of all $[f]$ with L^1-functions f such that $f(t) \in \mathbb{Z}$ almost everywhere. Then $h_t(H) \subseteq H$. Hence H is a closed contractible, hence arcwise connected subgroup of G. Now G is a linear Lie group. We may identify $\mathfrak{L}(G)$ and G. Since H allows no one-parameter subgroups, $\mathfrak{L}(H) = \{0\}$. Thus, while G, as the additive group of a Banach space, is a linear Lie group by E5.10(ii)2), the closed arcwise connected subgroup H is not a linear Lie group. □

Exercise E7.18. (i) Let $q\colon G \to H$ be a surjective morphism between finite dimensional abelian linear Lie groups, inducing a local isomorphism such that $\ker q$ is finitely generated free. We may identify the Lie algebras of G and H and write $E \cong \mathbb{R}^n$ with $\exp_G\colon E \to G$ inducing a local isomorphism and $q \circ \exp_G = \exp_H\colon E \to H$. Then $F_H = \ker \exp_H$ is a finitely generated free subgroup of E and $F_G = \ker \exp_G$ is a pure subgroup of F_H and there is a subgroup F' of F_H such that $F_H = F' \oplus F_G$ and which is mapped isomorphically onto $G_0 \cap \ker q$ by $\exp_G$. Let $E' = \operatorname{span}_{\mathbb{R}} F'$, $E_G = \operatorname{span}_{\mathbb{R}} F_G$ and find a vector space $E_{\mathbb{R}}$ such that $E = E_{\mathbb{R}} \oplus E' \oplus E_G$. Then

$$G \cong E_R \oplus E' \oplus \frac{E_G}{F_G} \oplus D,$$

where D is a discrete group $D \cong G/G_0$ such that $\ker q$ is isomorphic to $F' \oplus D'v$ with a free subgroup of D, and where E_G/F_G is a torus group whose dimension equals $\operatorname{rank} F_G$. Also,

$$H \cong E_R \oplus \frac{E'}{F'} \oplus \frac{E_G}{F_G} \oplus \frac{D}{D'},$$

where $(E'/F') \oplus (E_G/F_G)$ is a torus group of dimension $\operatorname{rank}(G_0 \cap \ker q) + \operatorname{rank} F_G = \operatorname{rank} F_H$.

(ii) If H is compact, then $E_{\mathbb{R}} = 0$ and D/D' is finite. □

The exponential function of abelian groups can be reinterpreted in a useful fashion for reflexive groups. From 7.11, for each pair of abelian topological groups in $\mathbb{TAB}_\eta$, we have an isomorphism of abelian topological groups

$$\alpha_{G,H}\colon \operatorname{Hom}(G,\widehat{H}) \to \operatorname{Hom}(H,\widehat{G}), \quad \alpha_{G,H}(f)(h)(g) = f(g)(h),$$

given that G and H are locally compact. By 7.5(iii) there is an isomorphism

$$\rho\colon \mathbb{R} \to \widehat{\mathbb{R}}, \quad \rho(r)(s) = rs + \mathbb{Z} \in \mathbb{T}.$$

We let G be a reflexive group. Then $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism and we obtain an isomorphism $\varepsilon_G\colon \operatorname{Hom}(\widehat{G},\mathbb{R}) \to \mathfrak{L}(G)$ defined by the following commutative diagram

$$\begin{array}{ccccc}
\operatorname{Hom}(\widehat{G},\mathbb{R}) & \xrightarrow{\varepsilon_G} & \mathfrak{L}(G) & = & \operatorname{Hom}(\mathbb{R},G) \\
{\scriptstyle \operatorname{Hom}(G,\rho)}\big\downarrow & & {\scriptstyle \mathfrak{L}(\eta_G)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(\mathbb{R},\eta_G)} \\
\operatorname{Hom}(\widehat{G},\widehat{\mathbb{R}}) & \xrightarrow[\alpha_{\widehat{G},\mathbb{R}}]{} & \mathfrak{L}(\widehat{\widehat{G}}) & = & \operatorname{Hom}(\mathbb{R},\widehat{\widehat{G}}).
\end{array}$$

Now $\alpha_{\widehat{G},\mathbb{R}}^{-1} = \alpha_{\mathbb{R},\widehat{G}}$, and, setting $e_G = \varepsilon_G^{-1}$ we get

$$e_G = \mathrm{Hom}(\widehat{G},\rho)^{-1} \circ \alpha_{\mathbb{R},\widehat{G}} \circ \mathfrak{L}(\eta_G), \quad \text{and} \quad \mathrm{Hom}(\widehat{G},\rho)\circ e_G = \alpha_{\mathbb{R},\widehat{G}} \circ \mathfrak{L}(\eta_G),$$

in a diagram:

$$\begin{array}{ccc}
\mathfrak{L}(G) & \xrightarrow{e_G} & \mathrm{Hom}(\widehat{G},\mathbb{R}) \\
{\scriptstyle \mathfrak{L}(\eta_G)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(G,\rho)} \\
\mathrm{Hom}(\mathbb{R},\widehat{\widehat{G}}) & \xrightarrow[\alpha_{\mathbb{R},\widehat{G}}]{} & \mathrm{Hom}(\widehat{G},\widehat{\mathbb{R}}),
\end{array}$$

and in formulae

$$(*)\quad \big((\rho\circ e_G)(X)\big)(\chi)(r) = \chi\big(X(r)\big) = \chi(\exp r{\cdot}X) \quad \text{for } X\in\mathfrak{L}(G),\ \chi\in\widehat{G},\ r\in\mathbb{R}.$$

We may use $(*)$ as a definition on arbitrary abelian topological groups and formulate the following proposition.

Later we shall prove that all locally compact abelian groups are reflexive. (See 7.63.)

Proposition 7.40. *Assume that G is a locally compact abelian group. Then*

(i) *formulae $(*)$ defines a natural morphism of abelian topological groups $e_G\colon \mathfrak{L}(G) \to \mathrm{Hom}(\widehat{G},\mathbb{R})$ and then (by 7.5(ii)) a continuous linear map between topological vector spaces. If G is also reflexive, then e_G is an isomorphism of real topological vector spaces.*

(ii) *The quotient morphism $\exp_{\mathbb{T}}\colon \mathbb{R}\to\mathbb{T}$ induces a commutative diagram*

$$\begin{array}{ccc}
\mathfrak{L}(G) & \xrightarrow{e_G} & \mathrm{Hom}(\widehat{G},\mathbb{R}) \\
{\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\widehat{G},\exp_{\mathbb{T}})} \\
G & \xrightarrow[\eta_G]{} & \widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G},\mathbb{T}).
\end{array}$$

(iii) *For G reflexive, the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{\text{incl}} \mathbb{R} \xrightarrow{\exp_{\mathbb{T}}} \mathbb{T} \to 0$$

gives rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Hom}(\widehat{G},\mathbb{Z}) & \xrightarrow{\mathrm{Hom}(\widehat{G},\,\text{incl})} & \mathrm{Hom}(\widehat{G},\mathbb{R}) & \xrightarrow{\mathrm{Hom}(\widehat{G},\,\exp_T)} & \mathrm{Hom}(\widehat{G},\mathbb{T}) \\
\big\uparrow & & {\scriptstyle \alpha_{\mathbb{T},\widehat{G}}}\big\uparrow & & {\scriptstyle e_G}\big\uparrow & & \big\uparrow{\scriptstyle \eta_G} \\
0 & \to & \mathrm{Hom}(\mathbb{T},G) & \xrightarrow[\mathrm{Hom}(\exp_{\mathbb{T}},\,G)]{} & \mathfrak{L}(G) & \xrightarrow[\exp_G]{} & G.
\end{array}$$

Proof. Exercise E7.19. ☐

Exercise E7.19. Prove Proposition 7.40.

[Hint. (i) Show first that e_G is continuous and then verify for each morphism $f\colon G \to H$, the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{e_G} & \operatorname{Hom}(\widehat{G},\mathbb{R}) \\ {\scriptstyle \mathfrak{L}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(\widehat{f},\mathbb{R})} \\ \mathfrak{L}(H) & \xrightarrow{e_H} & \operatorname{Hom}(\widehat{H},\mathbb{R}). \end{array}$$

(ii) Proceed straightforwardly.

(iii) The vertical maps are all isomorphisms. The top row is exact because $\operatorname{Hom}(\widehat{G},\mathbb{Z})$ can be identified with a subgroup of $\operatorname{Hom}(\widehat{G},\mathbb{R})$ so as to be exactly the kernel of $\operatorname{Hom}(\widehat{G},\exp_{\mathbb{R}})$. Compare Appendix 1, A1.49 and A1.55, but notice that the appendix deals with discrete abelian groups.] □

Let $f\colon G \to H$ be a *surjective* morphism of topological groups. Is

$$\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$$

surjective? In other words: does the functor $\mathfrak{L}$ preserve surjectivity? The answer is obviously negative, as the example of the identity map from the additive group of real numbers $\mathbb{R}_d$ with the discrete topology to the group $\mathbb{R}$ shows. In general, this fails to be true even for quotient morphisms between connected abelian linear Lie groups in the absence of countability assumptions as we have already seen in a somewhat tricky example following the proof of Theorem 5.52 on the Recovery of Subalgebras. The problem is one of lifting one-parameter groups: If $Y\colon \mathbb{R} \to H$ is a one-parameter group of H, can we find a one-parameter group $X\colon \mathbb{R} \to G$ such that $Y = f \circ X$?

Thus an only slightly more special question is the following. Assume that $f\colon G \to \mathbb{R}$ is a quotient morphism of abelian topological groups: Is $\mathfrak{L}\big(f\big)\colon \mathfrak{L}(G) \to \mathfrak{L}(\mathbb{R}) = \mathbb{R}$ surjective, i.e. is $\mathfrak{L}(f)$ a nonzero linear functional?

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(\mathbb{R}) = \mathbb{R} \\ {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\mathbb{R}}=\mathrm{id}_{\mathbb{R}}} \\ G & \xrightarrow[f]{} & \mathbb{R}. \end{array}$$

Writing the exponential function of $\mathbb{R}$ as the identity, we would then find an $X \in \mathfrak{L}(G) \setminus \ker \mathfrak{L}(f)$ such that $\mathfrak{L}(G) = \mathbb{R}{\cdot}X \oplus \ker \mathfrak{L}(f)$. Upon proper scaling of X we would then have $f(\exp t{\cdot}X) = t$, and f would be a retraction with $t \mapsto \exp t{\cdot}X\colon \mathbb{R} \to G$ as the coretraction. In other words, if we set $N = \ker f$, then the surjectivity of $\mathfrak{L}(f)$ is tantamount to a decomposition $G = N \oplus E$ with $E = \exp \mathbb{R}{\cdot}X$ such that $(k,e) \mapsto k + e\colon N \times E \to G$ is an isomorphism of topological groups.

In the following lemma we need not assume commutativity. It is of independent interest.

Lemma 7.41. *Assume that G is a locally compact group such that every identity neighborhood contains a compact subgroup K such that G/K is a linear Lie group. Then the following conclusions hold.*

(i) *If $f\colon G \to \mathbb{R}$ is a quotient morphism, then there is a morphism $\sigma\colon \mathbb{R} \to G$ such that $f\sigma = \mathrm{id}_{\mathbb{R}}$, i.e. f splits.*

(ii) *If N is a closed normal subgroup of G and $X\colon \mathbb{R} \to G/N$ a one parameter subgroup, then there is a one parameter subgroup $\widetilde{X}\colon \mathbb{R} \to G$ such that $\widetilde{X}(r)N = X(r)$ for $r \in \mathbb{R}$, i.e. the one parameter groups of G/N lift to one parameter groups of G.*

(iii) *The quotient morphism $q\colon G \to G/N$ induces a surjective morphism $\mathfrak{L}(q)\colon \mathfrak{L}(G) \to \mathfrak{L}(G/N)$.*

Proof. We prove (i) is several steps.

(a) Assume first that G itself is a linear Lie group. We may write $\mathfrak{L}(\mathbb{R}) = \mathbb{R}$ and $\exp_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}\colon \mathfrak{L}(\mathbb{R}) \to \mathbb{R}$. Then f induces a morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(R)$ such that the following diagram is commutative (cf. 5.42):

$$\begin{array}{ccccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(\mathbb{R}) & = & \mathbb{R} \\ {\scriptstyle \exp_G}\downarrow & & {\scriptstyle \mathfrak{L}(\exp_{\mathbb{R}})}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{R}}} \\ G & \xrightarrow[f]{} & \mathbb{R} & = & \mathbb{R}. \end{array}$$

Since the morphism f is open it maps the open component G_0 onto an open subset of $\mathbb{R}$, and thus onto $\mathbb{R}$. Therefore $\mathfrak{L}(f)$ is surjective. Since every surjective linear map between finite dimensional vector spaces splits, there is a linear map $s\colon \mathfrak{L}(\mathbb{R}) = \mathbb{R} \to \mathfrak{L}(G)$ such that $\mathfrak{L}(f) \circ s = \mathrm{id}_{\mathbb{R}}$. Then $\sigma = \exp_G \circ s\colon \mathbb{R} \to G$ satisfies $f \circ \sigma = f \circ \exp_G \circ s = \mathfrak{L}(f) \circ s = \mathrm{id}_{\mathbb{R}}$.

(b) Let K be a compact subgroup of G_0 such that G_0/K is a linear Lie group. Then $f(K)$ is a compact subgroup of $\mathbb{R}$ and thus is $\{0\}$. Hence $K \subseteq \ker f$ and there are quotient morphisms $q_K\colon G \to G/K$ and $f_K\colon G/K \to \mathbb{R}$ such that $f_K \circ q_K = f$. By (a) there is a morphism $\sigma_K\colon \mathbb{R} \to G/K$ such that $f_K \circ \sigma_K = \mathrm{id}_{\mathbb{R}}$. Let $C_K = q_K^{-1}\big(\sigma_K(\mathbb{R})\big)$. Then $f|C_K\colon C_K \to \mathbb{R}$ satisfies $f|C_K = \big(f_K|(C_K/K)\big) \circ q_K|C_K$ where $\sigma_K = \big(f_K|(C_K/K)\big)^{-1}$. Thus $f_K|(C_K/K)$ is an isomorphism and $f|C_K$ is a quotient morphism. In particular, since K is compact, for every compact subset $M \subseteq \mathbb{R}$ the set $(f|C_K)^{-1}(M)$ is compact. This is exactly saying that $f|C_K\colon C_K \to \mathbb{R}$ is a proper morphism.

(c) Now let $\mathcal{C}$ denote the family of all closed subgroups C such that $f|C\colon C \to \mathbb{R}$ is a surjective proper morphism. By (b) and our hypothesis, $\mathcal{C} \neq \emptyset$. If $\{C_j \mid j \in J\}$ is a filter basis in $\mathcal{C}$ then for each compact subset $M \in \mathbb{R}$ we have $f^{-1}(M) \cap C_j = (f|C_j)^{-1}(M) \neq \emptyset$, and the family $\{f^{-1}(M) \cap C_j \mid j \in J\}$ is a filter basis of compact sets and thus has nonempty intersection. Thus $C = \bigcap_{j\in J} C_j$ is a closed subgroup satisfying $f(C) = \mathbb{R}$; moreover $(f|C)^{-1}(M) = f^{-1}(M) \cap C = \bigcap_{j\in J}(f|C_j)^{-1}(M)$ is compact. Hence $f|C\colon C \to \mathbb{R}$ is a surjective proper morphism. Thus $\mathcal{C}$ is inductive and we find a minimal element $E \in \mathcal{C}$. We claim that $E \cap \ker f = \{0\}$. In view of our hypothesis we find a filter basis $\mathcal{N}$ of compact subgroups of E converging to 0 such that E/K is a linear Lie group for all $K \in \mathcal{N}$. Now let $K \in \mathcal{N}$. We apply (b) with E in place of G and $f|E$ in place of f. We thus find a subgroup $C_K \subseteq E$, $C_K \in \mathcal{C}$, $C_K \cap \ker f = K$. By the minimality of E we conclude $C_K = E$. Thus $E \cap \ker f = K$ for all $K \in \mathcal{N}$. Thus the filter basis $\mathcal{N}$ is singleton and since it converges to 0, we have $E \cap \ker f = \{0\}$. Thus the surjective proper morphism $f|E: E \to \mathbb{R}$ is injective and thus is an isomorphism. Let $j: E \to G$ denote the inclusion. Then the morphism $\sigma \stackrel{\text{def}}{=} j \circ (f|E)^{-1}: \mathbb{R} \to G$ satisfies $f \circ \sigma = \mathrm{id}_{\mathbb{R}}$.

Now a proof of (ii) is a relatively easy exercise. Indeed let $q: G \to G/N$ denote the quotient map and consider a one parameter subgroup $X: \mathbb{R} \to G/N$. Form the pull back (cf. Appendix 3, A3.43(iii))

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle\xi}\Big\downarrow & & \Big\downarrow{\scriptstyle X} \\ G & \xrightarrow[q]{} & G/N, \end{array}$$

$P = \{(g, r) \in G \times \mathbb{R} \mid q(g) = X(r)\}$, $f(g, r) = r$, $\xi(g, r) = g$. We claim that f is open. Indeed a basic identity neighborhood of P is given by $P \cap (U \times V)$ where U is an identity neighborhood of G and V an identity neighborhood of $\mathbb{R}$. Then $q(U)$ is an identity neighborhood of G/N and $f\big(P \cap (U \times V)\big) = \{v \mid v \in V, X(v) \in q(U)\} = X^{-1}q(U) \cap V$. This is an identify neighborhood of $\mathbb{R}$ which proves the claim. Now by Part (i) of the lemma we find a cross section morphism $\sigma: \mathbb{R} \to P$ such that $f \circ \sigma = \mathrm{id}_{\mathbb{R}}$. Set $\widetilde{X} = \xi \circ \sigma$. Then $q \circ \widetilde{X} = q \circ \xi \circ \sigma = X \circ f \circ \sigma = X$.

Assertion (iii) is nothing but a reformulation of (ii) if view of the equations $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$, $\mathfrak{L}(G/N) = \mathrm{Hom}(\mathbb{R}, G/N)$, and $\mathfrak{L}(q) = \mathrm{Hom}(\mathbb{R}, q)$. □

We shall see a little later in 7.54 that each locally compact abelian group G has arbitrarily small compact subgroups K such that G/K is a linear Lie group. An immediate consequence is the following:

Lemma 7.42. *Assume that G is a locally compact abelian group such that every identity neighborhood contains a compact subgroup K such that G/K is a linear Lie group. Assume further that $f: G \to \mathbb{R}^n$ is a quotient morphism. Then $G = N \oplus E$ with a vector group $E \cong \mathbb{R}^n$ mapping isomorphically onto $\mathbb{R}^n$ by f.*

Proof. We identify $\mathfrak{L}(\mathbb{R}^n)$ with $\mathbb{R}^n$ under $\exp_{\mathbb{R}^n}$. By Lemma 7.41, the morphism of vector spaces $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathbb{R}^n$ is surjective. There is a morphism of topological vector spaces $\sigma: \mathbb{R}^n \to \mathfrak{L}(G)$ such that $\mathfrak{L}(f) \circ \sigma = \mathrm{id}_{\mathbb{R}^n}$. Then $E \stackrel{\text{def}}{=} \exp\big(\sigma(\mathbb{R}^n)\big)$ satisfies the requirement. □

Weil's Lemma and Compactly Generated Abelian Groups

The basic ingredients of a group are its cyclic subgroups. In the case of topological groups one wants to know the structure of the closures of cyclic subgroups; these are usually called *monothetic* subgroups. For locally compact groups one knows precisely what monothetic groups look like. An important first step is the following result whose proof is elementary but not trivial.

Proposition 7.43 (Weil's Lemma). *Let g be an arbitrary element in a locally compact topological group G and assume that the cyclic subgroup generated by g is dense in G. Then G is either compact or isomorphic to the discrete group $\mathbb{Z}$. Moreover, in the first case, $\{g, g^2, g^3, \ldots\}$ is dense in G.*

Proof. Assume that G is not isomorphic to $\mathbb{Z}$. We show that G is compact.

Step 1. We claim that $\mathbb{N}.g$ is dense in G. Let U be nonempty and open in G. Then since $\mathbb{Z}\cdot g$ is dense in G there is an integer n such that $n\cdot g \in U$. Let V denote a symmetric 0-neighborhood with $n\cdot g + V \subseteq U$. If we had some natural number N such that $\{\, n \in \mathbb{Z} \mid |n| > N \text{ and } n\cdot g \in V \,\} = \emptyset$ then $\mathbb{Z}\cdot g \cap V = (\mathbb{Z} \cap [-N, N])\cdot g$, and $\mathbb{Z}\cdot g$ would be discrete and isomorphic to $\mathbb{Z}$ contrary to our assumption. Hence for any $n \in \mathbb{N}$ there is some $m \in \mathbb{N}$ with $|n| < m$ and $m\cdot g \in V$. Then $m + n \in \mathbb{N}$ and $(m + n)\cdot g \in n\cdot g + V \subseteq U$. So $\mathbb{N}\cdot g$ is dense in G.

Step 2. Let C denote a compact symmetric 0-neighborhood. We show that there is a finite subset $E \subseteq \mathbb{N}$ with $G \subseteq E\cdot g + C$. This will finish the proof. If $x \in G$ and U is an open symmetric 0-neighborhood contained in C, then by Step 1 there is an $n \in \mathbb{N}$ such that $n\cdot g \in x + U$, whence $x \in n\cdot g - U = n\cdot g + U$. Thus $G \subseteq \mathbb{N}\cdot g + C$. Since C is compact there is a finite subset $E = \{1, \ldots, k\}$ of $\mathbb{N}$ such that $C \subseteq E\cdot g + U$. Now let $h \in G$ be arbitrary and set $m = \min\{\, n \in \mathbb{N} \mid n\cdot g \in h + U \,\}$. Then $m\cdot g - h \in V \subseteq C \subseteq E\cdot g + U$ and thus there is an $e \in E$ such that $m\cdot g - h \in e\cdot g + U$, i.e. $(m - e)\cdot g \in h + U$. Because of the minimality of m, the integer $m - e$ must be nonpositive. Hence $m \le e \le k$, and $h \in m\cdot g - V = m\cdot g + V \subseteq E + C$. ☐

Definition 7.44. For a topological group G we write

$$\operatorname{comp}(G) \stackrel{\text{def}}{=} \{g \in G : \overline{\langle g \rangle} \text{ is compact}\}$$

and call an element of this set a *compact element*. ☐

Corollary 7.45. *If G is a locally compact abelian group then the set* $\operatorname{comp}(G)$ *is a subgroup, and for each element g in the complement $G \setminus \operatorname{comp}(G)$, the subgroup $\langle g \rangle$ which it generates is discrete and isomorphic to $\mathbb{Z}$. Moreover, $\langle g \rangle \cap \operatorname{comp}(G) = \{0\}$.*

Proof. If $g, h \in \operatorname{comp}(G)$ then $C \stackrel{\text{def}}{=} \overline{\langle g \rangle} + \overline{\langle h \rangle}$ is compact as a continuous image of the compact group $\overline{\langle g \rangle} \times \overline{\langle h \rangle}$ under the continuous homomorphism $(x, y) \mapsto x + y$. Now $g - h \in C$ and thus $\overline{\langle g - h \rangle}$ is compact; i.e. $g - h \in \operatorname{comp}(G)$. So $\operatorname{comp}(G)$ is a subgroup of G. The assertions on $\langle g \rangle$ follow from 7.43. Finally, if

$n{\cdot}g \in \text{comp}(G) \setminus \{0\}$, then $\overline{\langle n{\cdot}g\rangle}$ is compact on the one hand and discrete infinite on the other which is impossible. □

If G is a discrete abelian group, then $\text{comp}(G) = \text{tor}(G)$ (cf. Appendix 1, A1.16).

Proposition 7.46. (i) *If G is a locally compact group and N a compact normal subgroup, then* $\text{comp}(G)$ *contains N and* $\text{comp}(G)/N = \text{comp}(G/N)$.

(ii) *For two locally compact groups G and H,*

$$\text{comp}(G \times H) = \text{comp}(G) \times \text{comp}(H).$$

Proof. Exercise E7.20. □

Exercise E7.20. Prove Proposition 7.46. □

Definition 7.47. A topological group is said to be *compactly generated* if there is a compact subset K such that

$$G = \bigcup_{n=1}^{\infty} (K \cup K^{-1})^n. \qquad \square$$

In 1.28(i) and E1.11 we introduced the compact ring $\mathbb{Z}_p$ of p-adic integers. In E1.16 we have also defined the ring $\frac{1}{p^\infty}\mathbb{Z}$ and the locally compact ring $\mathbb{Q}_p$ containing $\mathbb{Z}_p$ as an open and $\frac{1}{p^\infty}\mathbb{Z}$ as a dense subring.

Remark 7.48. (i) A discrete group is compactly generated if and only if it is finitely generated.

(ii) Every compact group is compactly generated.

(iii) Assume that G is a locally compact group such that G/G_0 is compact. Then G is compactly generated.

(iv) Every connected locally compact group is compactly generated.

(v) The identity component of every locally compact group is the intersection of open compactly generated subgroups.

(vi) The additive group $\mathbb{Q}_p$ of p-adic rationals is not compactly generated.

Proof. (i) and (ii) are trivial.

(iii) By Lemma 5.73, there is a symmetric compact identity neighborhood C in G such that $C + G_0 = G$. The identity neighborhood $C \cap G_0$ of G_0 generates G_0 by Corollary A4.26. Hence C generates G.

(iv) follows from (iii).

(v) By E1.13(iii), in every locally compact group G, the identity component is the intersection of open subgroups H (of course containing G_0) such that H/G_0 is compact. By (iii) above, such groups H are compactly generated.

(vi) If a group is the union of its compact open subgroups and is not compact, then it is not compactly generated, for any compact subset is contained in some open compact subgroup. □

The preceding remark shows, in particular, that every locally compact abelian group G has an open compactly generated subgroup H. Accordingly the quotient G/H is a discrete abelian group and thus a purely algebraic object. In this sense, the compactly generated locally compact abelian groups, modulo extension theory, carry the bulk of the structure theory of locally compact abelian groups.

With a compactly generated locally compact abelian group G we can associate a set of free subgroups of G as follows. Let $\mathcal{F}(G)$ be the set of all finitely generated free subgroups F of G satisfying the following two conditions:

(a) There is a compact subset C of G such that $G = F + C$.

(b) $F \cap \operatorname{comp}(G) = \{0\}$.

Lemma 7.49. *Let G be a compactly generated locally compact abelian group. Then there is a compact subset C of G and a finitely generated subgroup F_1 such that $G = F_1 + C$.*

Proof. This follows at once from 5.75(i) (with $H = G$). □

Lemma 7.50. *If G is a compactly generated locally compact abelian group, then $\mathcal{F}(G) \neq \emptyset$.*

Proof. Lemma 7.49 provides us with a compact subset C of G and a finitely generated subgroup F_1 of G such that $G = F_1 + C$. By the Fundamental Theorem of Finitely Generated Abelian Groups A1.11, the group F_1 is a direct sum of a finitely generated free group and a finite group M. Then $M + C$ is compact and we may replace C by $C + M$ and assume that F_1 is free. If $g \in F_1$ and there is a natural number n such that $n{\cdot}g \in P \stackrel{\text{def}}{=} F_1 \cap \operatorname{comp}(G)$, then $\overline{\langle g\rangle}$ is compact since $A \stackrel{\text{def}}{=} \overline{\langle n{\cdot}g\rangle}$ is compact and the compact group $A \cup (g + A) \cup \cdots \cup ((n-1){\cdot}g + A)$ contains $\langle g\rangle$ and is contained in $\overline{\langle g\rangle}$. It follows that $g \in P$ and thus P is a pure subgroup of F_1. Then $F_1 = F \oplus P$ for some free subgroup F by Appendix 1, A1.24(ii). Now P is finitely generated, say, $P = \langle p_1, \ldots, p_k\rangle$. Since $p_j \in \operatorname{comp}(G)$ we have $\overline{P} \subseteq \overline{\langle p_1\rangle} + \cdots + \overline{\langle p_k\rangle}$, and thus $\overline{P}$ is compact. Hence $G = F + \overline{P} + C$. Since $\overline{P} + C$ is compact, this shows that $F \in \mathcal{F}(G)$. □

Now we can define for any compactly generated locally compact abelian group G the following canonically associated concepts:

$$m(G) = \min\{\operatorname{rank} F \mid F \in \mathcal{F}(G)\},$$
$$\mathcal{M}(G) = \{F \in \mathcal{F}(G) \mid \operatorname{rank} F = m(G)\}.$$

Let us look at some simple examples to understand better what has been said.

Examples 7.51. (i) If $G = \mathbb{Z}^n \oplus K$ for a compact abelian group K, then $m(G) = n$.

(ii) Assume $G = \mathbb{R}$ then $\mathbb{R} = \mathbb{Z} + [0,1]$ and $m(G) = 1$. For every $\varepsilon > 0$, $G = (\mathbb{Z} \oplus \sqrt{2}\mathbb{Z}) + [0,\varepsilon]$, and $\mathbb{Z} \oplus \sqrt{2}\mathbb{Z}$ is free of rank 2.

Proof. Exercise E7.21. □

Exercise E7.21. Prove 7.51.

[Hint. (i) Obviously, $m(G) \leq n$. If $G = F + C$ with C compact we may assume that $K \subseteq C$. Now $G/K \cong \mathbb{Z}^n$ is a discrete group in which $(F+K)/K + C/K$ with a finite C/K and $(F+K)/K$ is finitely generated. As a subgroup of a free group the latter group is free. Argue that its rank has to be at least n. Complete the proof.

(ii) Since $\mathbb{R}$ is not compact, $m(G) > 0$. Because $\mathbb{R} = \mathbb{Z}+[0,1]$ we have $m(G) \leq 1$. For the second part note that $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is dense in $\mathbb{R}$.] □

In particular, the example of $\mathbb{R}$ shows that finitely generated free subgroups need not be discrete.

Lemma 7.52. *Let G be a compactly generated locally compact abelian group. Then each member of $\mathcal{M}(G)$ is discrete.*

Proof. We prove the claim by induction with respect to $m(G)$. For this purpose suppose that G is a counterexample with minimal $m(G)$ and we shall derive a contradiction. If $m(G) = 0$, then F in $\mathcal{M}(G)$ is singleton and therefore discrete. But G is a counterexample to the claim; therefore $m(G) > 0$. Assume that G is written $F + C$ with $F \in \mathcal{M}(G)$ and some compact subset C of G and that F is not discrete. We must derive a contradiction. (We note that $\overline{F}$ is a counterexample, since $F \in \mathcal{M}(\overline{F})$.) Let us write $F = F_1 + F_2$ with a rank one pure subgroup F_1 and a pure subgroup F_2 of rank $m(G) - 1$. Set $H = \overline{F}_2$. We claim that $F_1 \cap H = \{0\}$. If not, then for a generator f of F_1 and some natural number $n > 0$ we would have $n{\cdot}f \in H$. Then $G = H + \{f, 2{\cdot}f, \ldots, (n-1){\cdot}f\} = F_2 + C + \{f, 2{\cdot}f, \ldots, (n-1){\cdot}f\}$, whence $m(G) \leq \operatorname{rank}(F_2) = m(G) - 1$, which is a contradiction.

Now F_1 is mapped faithfully (and densely) into G/H. Hence, by Weil's Lemma 7.43, we have two cases: Case A: G/H is compact, or Case B: G/H is isomorphic to $\mathbb{Z}$. Assume Case A. Then H is compactly generated and $G = H + K$ for some compact subset K of G by E7.11 or [38], Chap. VII, §3, n^o 2, Lemme 3. Assume Case B. Then $G/H \cong \mathbb{Z}$. In particular, H is open in G and $G = F_1 \oplus H$ algebraically and topologically. But then $H \cong G/F_1$ is compactly generated.

Thus H is compactly generated in either case. But $m(H) \leq \operatorname{rank} F_2 = m(G) - 1$. Hence H is not a counterexample. If $H = F' + C'$ with $F' \in \mathcal{F}(H)$ and $\operatorname{rank} F' = m(H)$, then F' is discrete. In Case A, $G = H + K = F' + C' + K$, whence $m(G) \leq \operatorname{rank} F' = m(H) \leq m(G) - 1$ which is a contradiction. In Case B, $G = (F_1 \oplus F') + C'$ with a discrete free group $F_1 \oplus F'$ of rank $m(G)$. Hence $m(H) = \operatorname{rank} F' = m(G) - 1 = \operatorname{rank} F_2$. Therefore $F_2 \in \mathcal{M}(H)$ and thus F_2 is discrete, whence $H = \overline{F_2} = F_2$. Hence $F = F_1 \oplus F_2$ algebraically and topologically, whence F is discrete in contradiction to our assumption. □

Reducing Locally Compact Abelian Groups to Compact Abelian Groups

Theorem 7.53 (The Reduction Theorem). *Let G be a compactly generated locally compact abelian group. Then there is a discrete, hence closed, subgroup $F \cong \mathbb{Z}^n$, n a nonnegative integer, such that G/F is compact.*

Proof. Let $F \in \mathcal{M}(G)$. Then F is a free group of rank $m(G)$ which is discrete by Lemma 7.52. Hence F is closed. By the definition of $\mathcal{M}(G)$ we have $G = F + C$ for a compact C. Therefore G/F is compact. □

Corollary 7.54. *Let G be a locally compact abelian group. Then for each identity neighborhood U there is a compact subgroup N contained in U such that $G/N \cong \mathbb{R}^m \times \mathbb{T}^n \times D$, m and n nonnegative integers, for a discrete abelian group D. In particular, G/N is a linear Lie group.*

Proof. Let G_1 be a compactly generated open subgroup of G according to 7.48(v). Find a discrete finitely generated free group $F \in \mathcal{M}(G_1)$ such that G_1/F is compact by Theorem 7.53. Choose a symmetric compact identity neighborhood V in G_1 so small that $V + V + V \subseteq U \cap \big(G \setminus (F \setminus \{0\})\big)$. The identity neighborhood $V' \stackrel{\text{def}}{=} (V + F)/F$ of this compact group contains a compact subgroup, say M/F, such that $K \stackrel{\text{def}}{=} G_1/M \cong (G_1/F)/(M/F)$ is a compact Lie group. Then $M \subseteq V+F$. If $x, y \in F$, then the relation $z \in V+x \cap V+y$ implies $z = u+x = u'+y$ for suitable elements u, $u' \in V$, whence $y - x = -u' - u \in (V + V) \cap F = \{0\}$. Hence $x = y$. Thus M is contained in the disjoint union of the sets $V + x$, $x \in F$. Set $N = M \cap V$. We claim that $N = M \cap (V + V)$. Indeed let $m \in M \cap (V + V)$. Then either $m \in N$ or $m \in M \setminus N$. But $(V + V) \cap (M \setminus N) = \emptyset$. Since $m \in V + V$, the relation $m \in N$ follows. In particular $N + N \subseteq M \cap (V + V) = N$. Hence N is a compact subgroup. Furthermore let $m \in M$. Then $m = u + x$ with $u \in V$ and $x \in F$. Then $u = m-x \in M \cap (V+V) = N$. Hence $M = N+F$ and since $N \cap F = \emptyset$, this sum is algebraically direct, and then also topologically because N is compact and F is discrete. Now $K = G_1/M \cong G_1/(N \oplus F) \cong (G_1/N)/(N \oplus F)/N$. Set $L \stackrel{\text{def}}{=} G_1/N$ and let $q\colon L \to K$ be the quotient morphism whose kernel $(N \oplus F)/N$ is a discrete free group of rank $m(G_1)$. In particular, q induces a local isomorphism, and $K \cong \mathbb{T}^p \oplus E$ with a finite group E by 2.42. We claim that the identity component L_0 is a quotient of $\mathbb{R}^p$ modulo a discrete subgroup. One may use E7.2 to prove this claim. An argument using covering groups proceeds as follows.

The underlying space of L is a topological manifold (cf. Appendix A2.15 and the preceding paragraph) and so is the open identity component L_0. The image $q(L_0)$ is open and connected in K, and thus agrees with $K_0 \cong \mathbb{T}^p$. The universal covering $\widetilde{p}\colon \widetilde{L_0} \to L_0$ (see Appendix A2.21), when followed by the covering $(q|L_0)\colon L_0 \to K_0 \cong \mathbb{T}^p$ is a universal covering of $\mathbb{T}^p$. Hence $\widetilde{L_0} = \mathbb{R}^p$ up to isomorphism (Appendix A2.22(i) together with the uniqueness statement in A2.28). This establishes the claim that L_0 is a quotient group of $\mathbb{R}^p$ modulo a discrete subgroup.

Then L_0 is isomorphic to $\mathbb{R}^m \times \mathbb{T}^n$ (see Appendix A1.12(ii)). Now L_0 is a divisible subgroup of G/N. Hence G/N contains a subgroup D such that $G/N = L_0 \oplus D$ algebraically (see Appendix A1.36). Since L_0 is open, G/N and $L_0 \times D$ are isomorphic as topological groups. Thus $G/N \cong \mathbb{R}^m \times \mathbb{T}^n \times D$. By E5.18 all of these groups are linear Lie groups. □

Corollary 7.55. *In a locally compact abelian group G the subgroup* $\mathrm{comp}(G)$ *is closed.*

Proof. By 7.54 we find a compact subgroup N such that $G/N \cong \mathbb{R}^m \times \mathbb{T}^n \times D$ for a discrete group D. By E7.20(ii) we have

$$\mathrm{comp}(G/N) \cong \mathrm{comp}(\mathbb{R}^m) \times \mathrm{comp}(\mathbb{T}^n) \times \mathrm{comp}(D) = \{0\} \times \mathbb{T}^n \times \mathrm{tor}\, D$$

and thus $\mathrm{comp}(G/N)$ is closed. By E7.20(i) we know that $\mathrm{comp}(G)$ contains N. Hence $\mathrm{comp}(G)$ is the full inverse image of the closed subset $\mathrm{comp}(G)/N = \mathrm{comp}(G/N)$ of G/N in G. The closedness of $\mathrm{comp}(G)$ follows. □

Exercise E7.22. Consider the linear Lie group G of 3×3-matrices

$$M(x, y, t) \stackrel{\text{def}}{=} \begin{pmatrix} \cos t & -\sin t & x \\ \sin t & \cos t & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,\, y,\, t \in \mathbb{R}.$$

Show (i) G' is the set of all $M(x, y, 0)$, $x, , y \in \mathbb{R}$ and is isomorphic to $\mathbb{R}^2$. Accordingly, $G/G' \cong \mathbb{S}^1 \cong \mathbb{T}$.

(ii) G is homeomorphic to $\mathbb{R}^2 \times \mathbb{T}$.

(iii) $\mathrm{comp}(G) = (G \setminus G') \cup \{1\}$. In particular, $G = \overline{\mathrm{comp}(G)}$. □

The group in Exercise E7.22 is the group of rigid motions of the euclidean plane. Such a motion is either a translation or a rotation around a suitable center. The group is metabelian; i.e. its commutator group is abelian. But the set of compact elements is dense. Corollary 7.55 is therefore a strictly abelian phenomenon.

Corollary 7.56. *A locally compact abelian group without nonsingleton compact subgroups is isomorphic to $\mathbb{R}^n \times D$, for some nonnegative integer n and a discrete torsion-free subgroup D.* □

We are now able to prove the principal structure theorem for locally compact abelian groups. For a full exploitation of the information it provides, recall the fully characteristic closed subgroup $\mathrm{comp}(G)$ of Definition 7.44 and Corollary 7.55 and let us define the subgroup G_1 of G by $G_0 + \mathrm{comp}(G)$.

A Major Structure Theorem

The Vector Group Splitting Theorem

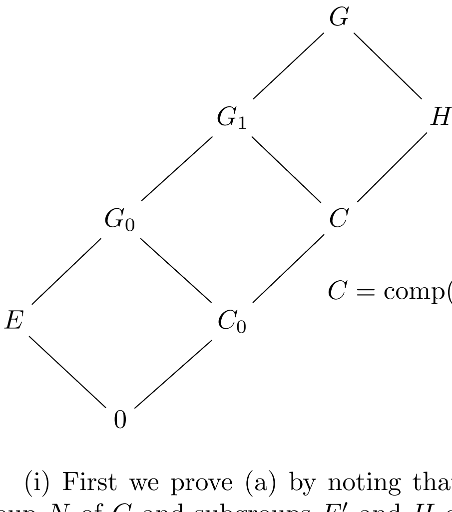
Theorem 7.57. (i) *Every locally compact abelian group G is algebraically and topologically of the form $G = E \oplus H$ for a subgroup $E \cong \mathbb{R}^n$ and a locally compact abelian subgroup H which has the following properties*

(a) *H contains a compact subgroup which is open in H.*
(b) *Every compact subgroup of G is contained in H.*
(c) *H contains* $\mathrm{comp}(G)$.
(d) $H_0 = (\mathrm{comp}(G))_0 = \mathrm{comp}(G_0)$ *is the unique maximal compact connected subgroup of G.*
(e) *The subgroup G_1 is an open, hence closed, fully charactersistic subgroup which is isomorphic to $E \oplus \mathrm{comp}(G)$.*
(f) *G/G_1 is a discrete torsion-free group and G_1 is the smallest open subgroup with this property.*

(ii) *Every compactly generated locally compact abelian group is isomorphic to a direct product $\mathbb{R}^m \times K \times \mathbb{Z}^n$ for a compact abelian group K and nonnegative integers m and n.*

(iii) *Every connected locally compact abelian group is a direct sum, algebraically and topologically, of a finite dimensional vector group and a unique maximal compact subgroup.*

(iv) The lattice of closed subgroups of G contains the following key lattice diagram of closed subgroups:



$$C = \mathrm{comp}(G), \quad G_1 = G_0 + C = E \oplus C.$$

Proof. (i) First we prove (a) by noting that Corollary 7.54 gives us a compact subgroup N of G and subgroups E' and H of G containing N such that $G/N = E'/N \oplus H/N$, that $E'/N \cong \mathbb{R}^n$, and that H/N has a compact open subgroup K/N. Then K is a compact open subgroup of H. Now $G/H \cong (G/N)/(H/N) \cong E'/N \cong \mathbb{R}^n$. By 7.54, G has arbitrarily small compact subgroups M such that

G/M is a linear Lie group. Now 7.42 applies and shows that $G = E \oplus H$ with $E \cong \mathbb{R}^n$ and H containing an open and compact subgroup.

For a proof of (b) let C be a compact subgroup of G and $p: G \to E$ the projection onto E with kernel H. Then $p(C)$ is a compact subgroup of $E \cong \mathbb{R}^n$. Since $\mathbb{R}$, and thus $\mathbb{R}^n$, has no nonsingleton compact subgroups we have $p(C) = \{0\}$ and thus $C \in \ker p = H$.

Condition (c) follows immediately from (b) and the definition of comp(G) in 7.44.

For a proof of (d) we recall from (a) that H has compact open subgroups implying that H_0 is compact. Thus $H_0 \subseteq \mathrm{comp}(G)$, and since $(\mathrm{comp}(G))_0 \subseteq H_0$ the equality $H_0 = (\mathrm{comp}(G))_0$ follows. Since $G = E \oplus H$ is the product of $E \cong \mathbb{R}^n$ and H we have $G_0 = E \oplus H_0$, whence $H_0 = \mathrm{comp}(G_0)$.

In order to prove (e) we let K be a compact and open subgroup of H according to (a). Then $E \oplus K$ on the one hand is open in $E \oplus H = G$ and on the other is contained in $G_0 + \mathrm{comp}(G) = G_1$, and so G_1 is open in G. Since G_0 and comp(G) are fully characterstic subgroups of G, so is $G_1 = G_0 + \mathrm{comp}(G)$. The sum $E + \mathrm{comp}(G)$ is direct algebraically and topologically by (c) since $G = E \oplus H$. Clearly, $E \oplus \mathrm{comp}(G) \subseteq G_0 + \mathrm{comp}(G) = G_1$ and

$$G_1 = G_0 + \mathrm{comp}(G) = (E + (\mathrm{comp}(G))_0) + \mathrm{comp}(G) = E + \mathrm{comp}(G)$$

in view of (d). Thus $G_1 = E \oplus \mathrm{comp}(G)$.

For a proof of (f), since G_1 is open in G by (e) and

$$\frac{G}{G_1} = \frac{E \oplus H}{E \oplus \mathrm{comp}(G)} \cong \frac{H}{\mathrm{comp}(G)},$$

we know that G/G_1 is discrete and torsion-free by 7.45. If U is an open subgroup of G, then $E \subseteq U$ and so $U = E \oplus (H \cap U)$ by the modular law. If G/U is torsion-free, then $H/(H \cap U)$ is torsion-free and so $\cong (G) \subseteq U$. Hence $G_1 = E + \cong (G) \subseteq U$. Thus G_1 is indeed minimal with the properties stated in (e).

(ii) By (i) we may assume that G contains an open compact subgroup C. Now G/C is a discrete compactly generated, hence finitely generated group. Thus $G/C \cong E \times \mathbb{Z}^n$ with a finite group E and a nonnegative integer n by the Fundamental Theorem of Finitely Generated Abelian Groups (Appendix 1, A1.11). The full inverse image K of E in G is a maximal compact subgroup such that $G/K \cong \mathbb{Z}^n$. Then, algebraically, $G = K \oplus F$ with $F \cong \mathbb{Z}^n$ by Appendix 1, A1.15. Since K is open, then map $(k, f) \mapsto k + f: K \times F \to G$ is an isomorphism of abelian topological groups.

(iii) This follows from 7.48(iv) and (ii) above, and

(iv) is the summary in a diagram of the information accumulated in (i), (ii) and (iii). □

The direct summand E is not uniquely determined in general. But $\dim E$ is uniquely determined because it is the dimension of the identity component of $G/\mathrm{comp}(G) \cong E \times D$ with a discrete group D. We call this number the *vector rank* and write $\mathrm{vrank}(G) \stackrel{\mathrm{def}}{=} \dim E$. The summand H in Theorem 7.57 is not uniquely determined either. However, the subgroups G_0, $\mathrm{comp}(G)$, $\mathrm{comp}(G_0) = (\mathrm{comp}(G))_0$ and $G_1 = G_0 + \mathrm{comp}(G) = E \oplus \mathrm{comp}(G)$ are canonically determined.

A further splitting of the direct summand H is not to be expected; e.g. $H = \mathbb{Q}_p$ does not split a compact open subgroup K, because each of them is of the form $p^n\mathbb{Z}_p$ for some $n \in \mathbb{Z}$, and $H/K \cong \mathbb{Z}(p^\infty)$ (cf. E1.16) which is a torsion group, while $\mathbb{Q}_p$ is torsion-free.

Corollary 7.58. (i) *If G is a locally compact abelian group, then there is a unique maximal compact connected subgroup $\big(\mathrm{comp}(G)\big)_0 = \mathrm{comp}(G_0)$.*

(ii) *Let C be a compact abelian group and $G = \mathbb{R}^n \times C$. Then $K \stackrel{\text{def}}{=} \{0\}\times C$ is the unique largest compact subgroup, and for each morphism $f\colon \mathbb{R}^n \to C$ the subgroup $\mathrm{graph}(f) \stackrel{\text{def}}{=} \{\big(v, f(v)\big) \mid v \in \mathbb{R}^n\}$ is isomorphic to $\mathbb{R}^n$ and $G = \mathrm{graph}(f) \oplus K$ algebraically and topologically.*

The map $f \mapsto \mathrm{graph}(f)$ from $\mathrm{Hom}(\mathbb{R}^n, C)$ to the set of complements of K in G is surjective.

(iii) *For a locally compact abelian group, the following statements are equivalent:*

(a) *G is a linear Lie group.*

(b) *There are nonnegative integers m and n and a discrete abelian group D such that $G \cong \mathbb{R}^m \times \mathbb{T}^n \times D$.*

(c) *G has no small subgroups.*

Proof. (i) Applying Theorem 7.57 to $\mathrm{comp}(G)$ we find that $\mathrm{comp}(G)$ has an open compact subgroup C. Thus the identity component $\big(\mathrm{comp}(G)\big)_0$, being contained in C is clearly compact, and is the unique largest compact connected subgroup contained in G. The two containments making up the relation $\big(\mathrm{comp}(G)\big)_0 = \mathrm{comp}(G_0)$ are easily seen to hold.

(ii) Let $\mathrm{pr}_1\colon G \to \mathbb{R}^n$ and $\mathrm{pr}_2\colon G \to C$ denote the projections. The following functions are inverse morphisms of each other:

$$v \mapsto \big(v, f(v)\big)\colon \mathbb{R}^n \to \mathrm{graph}(f)$$
$$\mathrm{pr}_1 \,|\, \mathrm{graph}(f)\colon \mathrm{graph}(f) \to \mathbb{R}^n,$$

as are the following ones:

$$\big(\big(v, f(v)\big), (0, c)\big) \mapsto (v, f(v) + c)\colon \mathrm{graph}(f) \times K \to G$$
$$g \mapsto \big(\big(\mathrm{pr}_1(g), f\big(\mathrm{pr}_1(g)\big)\big), \big(0, -f\big(\mathrm{pr}_1(g)\big) + \mathrm{pr}_2(g)\big)\big)\colon G \to \mathrm{graph}(f) \times K.$$

This shows $G \cong \mathrm{graph}(f) \oplus K$.

If $G = E \oplus K$, then $p \stackrel{\text{def}}{=} (\mathrm{pr}_1 \,|E)\colon E \to \mathbb{R}^n$ is an isomorphism. Set $f = \mathrm{pr}_2 \circ p^{-1}\colon \mathbb{R}^n \to C$. Then $E = \mathrm{graph}(f)$.

(iii) For the equivalence of (a) and (b) see E5.18(i); the implication (b)⇒(c) is clear. We show (c) implies (b). If G has no small subgroups, by 7.57 there is an open subgroup which is isomorphic to $\mathbb{R}^m \times C$ with a compact group without small subgroups. Then C is a compact (hence linear) Lie group by 2.41. Therefore the identity component of G is of the form $\mathbb{R}^m \times \mathbb{T}^n$ (cf. E5.18(i) again) and divisible and open in G. It is therefore a direct summand, algebraically (see A1.36) and topologically (since it is open). This proves (b). $\square$

Corollary 7.59. *Let G be a locally compact abelian group. Then $\mathfrak{L}(G) = V \oplus \mathfrak{L}_{\mathrm{comp}}(G)$ where*

$$V \cong \mathbb{R}^{\mathrm{vrank}(G)} \quad \textit{and} \quad \mathfrak{L}_{\mathrm{comp}}(G) = \mathfrak{L}(H) = \mathfrak{L}(\mathrm{comp}(G)) = \mathfrak{L}(\mathrm{comp}(G)_0).$$

Proof. Since $\mathfrak{L}$ preserves products by 7.38(i), from Theorem 7.57 we derive $\mathfrak{L}(G) = \mathfrak{L}(E) \oplus \mathfrak{L}(H)$. Clearly $V \stackrel{\mathrm{def}}{=} \mathfrak{L}(E)$ is a vrank(G)-dimensional real vector space, and $\mathfrak{L}(H) = \mathfrak{L}(H_0) = \mathfrak{L}(\mathrm{comp}(G)_0)$. $\square$

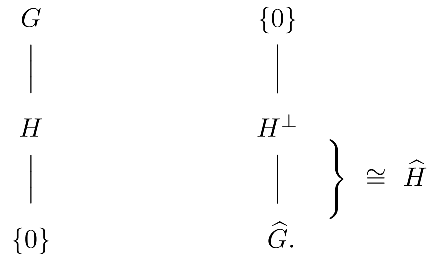
Note that $\mathfrak{L}_{\mathrm{comp}}(G)$ is a uniquely determined cofinite dimensional vector subspace of $\mathfrak{L}(G)$; the complement is not uniquely determined. We have

$$\dim \mathfrak{L}(G)/\mathfrak{L}_{\mathrm{comp}}(G) = \mathrm{vrank}(G).$$

The Duality Theorem

In this section we shall prove that every locally compact abelian group is reflexive (see Definition 7.8). We shall utilize the Annihilator Mechanism 7.17.

Lemma 7.62. *If H is a compact open subgroup of G. Then $H^\perp$ is a compact open subgroup of $\widehat{G}$ and $\kappa_{G,H}\colon \widehat{G}/H^\perp \to \widehat{H}$ is an isomorphism of topological groups.*



Proof. Firstly, since H is open, G/H is discrete, and thus its character group is compact by 7.5(i). Now $H^\perp \cong \widehat{G/H}$ by 7.17(i). Hence $H^\perp$ is compact. Furthermore, let U_1 be the zero neighborhood $\left]-\frac{1}{4}, \frac{1}{4}\right[+ \mathbb{Z}$ in $\mathbb{T}$. Then U_1 does not contain any nontrivial subgroups and thus the zero neighborhood $V_{\widehat{G}}(H, U_1)$ of $\widehat{G}$ is exactly $H^\perp$. Thus $H^\perp$ is open.

If $\chi\colon H \to \mathbb{T}$ is a character on H then by the injectivity of $\mathbb{T}$ (cf. A1.34, A1.35) it extends to an algebraic character $\widetilde{\chi}\colon G \to \mathbb{T}$. But since H is open, the extension $\widetilde{\chi}$ is continuous and thus is a character. Clearly $\kappa_{G,H}(\widetilde{\chi}) = \widetilde{\chi}|H = \chi$. Thus $\kappa_{G,H}$ is bijective. Since the domain $\widehat{G}/H^\perp$ and the range $\widehat{H}$ are both discrete, $\kappa_{G,H}$ is trivially an isomorphism of abelian topological groups. $\square$

We have all the ingredients to prove the duality theorem. Certain aspects of the proof were anticipated in the proof of 2.32.

The Pontryagin–van Kampen Duality Theorem

Theorem 7.63. *Every locally compact abelian group is reflexive; that is for every locally compact abelian group G the evaluation morphism $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.*

Proof. By Proposition 7.10(iv) a direct product $G \times H$ of abelian topological groups is reflexive if each of the factors is reflexive. (The converse holds, too.) Since $\mathbb{R}$ is reflexive, the group $\mathbb{R}^n$ is reflexive. Thus from the Vector Group Splitting Theorem 7.57 we obtain that all locally compact abelian groups are reflexive if we can show that a locally compact group G with an open compact subgroup H is reflexive. Thus we assume, for the remainder of the proof, that G has a compact open subgroup H. From 7.62 we know that $H^\perp$ is a compact open subgroup of $\widehat{G}$.

Firstly, we shall show that the characters of G separate points, which will prove the injectivity of η_G. By 1.21, the characters of the discrete abelian group G/H separate the points; hence for every $g \notin H$, by 7.62 there is a character χ of G with $\chi(g) \neq 0$.

Now any character χ from the subgroup $H^\perp \subseteq \widehat{G}$ of characters vanishing on H induces a character χ' on the discrete group G/H via $\chi'(g+H) = \chi(g)$, and if χ' is a character of G/H we get a character $\chi \in \widehat{G}$ via $\chi(g) = \chi'(g+H)$. Thus $\chi \mapsto \chi': H^\perp \to \widehat{G/H}$ is an isomorphism. By 2.31, the characters of the compact group H separate the points. Hence by 7.62, for every $h \in H$ there is a character χ of G such that $\chi(g) \neq 0$. Hence the characters of G separate points and η_G is injective.

Secondly, we claim that $\eta_G(H) = H^{\perp\perp}$ in $\widehat{\widehat{G}}$. Because $\eta_G(h)(\chi) = \chi(h) = 0$ for all $\chi \in H^\perp$ and $h \in H$ the left side is contained in the right one. On the other hand, an $\Omega \in \widehat{\widehat{G}}$ is in $H^{\perp\perp}$ iff it vanishes on $H^\perp$ and thus induces a character of $\widehat{G}/H^\perp$. Every such corresponds to a character of $\widehat{H}$ when $\widehat{G}/H^\perp$ is identified with $\widehat{H}$ via $\kappa_{G,H}$ in 7.62. But every character of $\widehat{H}$ is an evaluation $\eta_H(h)$ by 2.32. Thus $\Omega = \eta_G(h_\Omega)$ for some $h_\Omega \in H$. This establishes the claim.

Thirdly, we claim that η_G is surjective. Let $\Gamma = \eta_G(G) \subseteq \widehat{\widehat{G}}$. By 7.62 applied to $\widehat{G}$ we know that $H^{\perp\perp}$ is compact open in $\widehat{\widehat{G}}$. Thus by what we just finished showing, $\eta_G(H) = H^{\perp\perp}$ is open in $\widehat{\widehat{G}}$ and therefore Γ is open and closed in $\widehat{\widehat{G}}$. We want to show that $\Gamma = \widehat{\widehat{G}}$; this means $\widehat{\widehat{G}}/\Gamma = \{0\}$. For this we show that every character f on $\widehat{\widehat{G}}$ vanishing on Γ is zero. But since f vanishes on $H^{\perp\perp}$, by what we have seen in the proof of the second claim above, there is a $\chi \in H^\perp$ such that $f(\Omega) = \Omega(\chi)$. As f annihilates Γ we note $\chi(g) = \eta_G(g)(\chi) = f\big(\eta_G(g)\big) = 0$ for all $g \in G$. Hence $\chi = 0$ and thus $f = 0$.

We have therefore also shown that η_G is surjective. Since η_G maps the open compact subgroup H isomorphically onto the open closed subgroup $\eta_G(H) =$

$H^{\perp\perp}$ of $\widehat{\widehat{G}}$, the morphism η_G is also open and thus is an isomorphism of abelian topological groups. □

With the duality theorem available, the Annihilator Mechanism now functions for all locally compact abelian groups and all closed subgroups without additional hypothesis. Just for the record we repeat information presented in 7.17, 7.18, 7.19, 7.20, 7.22, 7.23.

The Annihilator Mechanism

Theorem 7.64. *Assume that G is a locally compact abelian group and H is a subgroup. Define morphisms of abelian topological groups*

$$\lambda_{G,H}\colon \widehat{G/H} \to H^\perp, \quad \lambda_{G,H}(\chi)(g) = \chi(g+H),$$

and

$$\kappa_{G,H}\colon \widehat{G}/H^\perp \to \widehat{H}, \quad \kappa_{G,H}(\chi + H^\perp) = \chi|H.$$

Then the following conclusions hold:

(i) $\lambda_{G,H}$ *and* $\kappa_{G,H}$ *are isomorphisms of abelian topological groups.*

$$\begin{array}{ccccccc}
 & & G & \qquad & \{0\} & & \\
\widehat{H^\perp} & \cong \left\{ \right. & | & & | & \left. \right\} \cong & \widehat{G/H} \\
 & & \overline{H} & & H^\perp & & \\
\widehat{\widehat{G}/H^\perp} & \cong \left\{ \right. & | & & | & \left. \right\} \cong & \widehat{H} \\
 & & \{0\} & & \widehat{G}. & &
\end{array}$$

(ii) *Moreover, assume that $H_1 \subseteq H_2 \subseteq G$ are closed subgroups. Then*

$$\kappa_{G/H_1,H_2/H_1}\colon \widehat{G/H_1}/(H_2/H_1)^\perp \to \widehat{H_2/H_1}$$

implements via the isomorphism $\lambda_{G,H_1}\colon \widehat{G/H_1} \to H_1^\perp$ an isomorphism of abelian topological groups $\kappa'\colon H_1^\perp/H_2^\perp \to \widehat{G/H_1}|H_2/H_1 = (H_2/H_1)\widehat{\ }$.

$$\begin{array}{ccc}
G & & \{0\} \\
| & & | \\
H_2 & & H_2^{\perp} \\
| & (H_2/H_1)\widehat{\ } \cong \left\{ \right. & | \\
H_1 & & H_1^{\perp} \\
| & & | \\
\{0\} & & \widehat{G}
\end{array}
\qquad \left.\begin{array}{c} \{0\} \\ | \\ H_2^{\perp} \\ | \\ H_1^{\perp} \end{array}\right\} \cong \widehat{G/H_1}$$

(iii) *Assume that S is a subset of G. Then $S^{\perp\perp}$ is the smallest closed subgroup $\overline{\langle S\rangle}$ containing S.*

(iv) *Assume that H is a closed subgroup of G. Then $H^{\perp\perp} = H$.*

(v) *The function $H \mapsto H^{\perp}$ maps the lattice of closed subgroups of G antiisomorphically onto the lattice of closed subgroups of $\widehat{G}$.*

(vi) *Assume that*

$$0 \to H \stackrel{j}{\to} G \stackrel{q}{\to} Q \to 0$$

is an exact sequence of locally compact abelian groups where j is an embedding morphism and q is a quotient morphism. Then the dual sequence

$$0 \to \widehat{Q} \stackrel{\widehat{q}}{\to} \widehat{G} \stackrel{\widehat{j}}{\to} \widehat{H} \to 0$$

is exact, where $\widehat{q}$ is an embedding and $\widehat{j}$ is a quotient morphism. Further $\widehat{Q} \cong H^{\perp}$, and $\widehat{H} \cong \widehat{G}/H^{\perp}$.

(vii) *Assume that $\{H_j \mid j \in J\}$ is a family of subgroups of G. Let $D = \bigcap_{j\in J} \overline{H_j}$ and $H = \sum_{j\in J} H_j$, then*

(a) $H^{\perp} = \bigcap_{j\in J} H_j^{\perp}$,

(b) $D^{\perp} = \overline{\sum_{j\in J} H_j^{\perp}}$.

(viii) *Every character of a subgroup H of G extends to a character of G.* □

Exercise E7.23. Determine all Hausdorff quotient groups of $\mathbb{T}^n$ for $n \in \mathbb{N}$. (Cf. Exercise E2.9.) □

The following is a sharpening of 7.64(vi).

Proposition 7.65. *Let $f: G_1 \to G_2$ be a morphism of locally compact abelian groups and denote by $\widehat{f}: \widehat{G}_2 \to \widehat{G}_1$ its adjoint. Then*

(i) $(\operatorname{im} f)^{\perp} = \ker \widehat{f}$.

(ii) $(\ker \widehat{f})^{\perp} = \overline{\operatorname{im} f}$.

In particular, f has a dense image if and only if $\widehat{f}$ is injective, and f is injective if and only if $\widehat{f}$ has dense image.

Proof. The relation $\chi \in (\operatorname{im} f)^\perp$ means $\chi\big(f(g)\big) = 0$ for all $g \in G$ which is equivalent to $\big(\widehat{f}(\chi)\big)(g) = 0$ for all $g \in G$ and this is tantamount to $\widehat{f}(\chi) = 0$ which says $\chi \in \ker \widehat{f}$.

(ii) By Theorem 7.64(iii) we know $\overline{\operatorname{im} f} = (\operatorname{im} f)^{\perp\perp}$. Hence (ii) follows from (i) by applying $(\cdot)^\perp$.

The final conclusions are consequences in view of the fact that the roles of f and $\widehat{f}$ may be exchanged by duality. □

Before we formulate and prove the major general results around the exponential function $\exp\colon \mathfrak{L}(G) \to G$ of a locally compact abelian group, we recall some of the background facts which were discussed earlier in the chapter, notably in 7.35 and 7.40. Let $q\colon \mathbb{R} \to \mathbb{T}$ denote the quotient map given by $q(r) = r + \mathbb{Z}$, and let $\nu\colon \mathbb{R} \to \mathfrak{L}(\mathbb{T})$ be the natural isomorphism given by $\nu(r)(s) = rs + \mathbb{Z}$. We have $\mathfrak{L}(G) = \operatorname{Hom}(\mathbb{R}, G)$, and $\exp X = X(1)$. The adjoint of the exponential function is $\varepsilon_G \stackrel{\text{def}}{=} \widehat{\exp_G}\colon \widehat{G} \to \widehat{\mathfrak{L}}(G) \stackrel{\text{def}}{=} \operatorname{Hom}(\mathfrak{L}(G), \mathbb{T})$ given by $\varepsilon_G(\chi)(X) = \chi(\exp_G X) = \exp_{\mathbb{T}}\big(\mathfrak{L}(\chi)(X)\big)$.

$$\begin{array}{ccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\chi)} & \mathfrak{L}(\mathbb{T}) & \xleftarrow{\ \nu\ } & \mathbb{R} \\
{\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\mathbb{T}}} & & \big\downarrow{\scriptstyle q} \\
G & \xrightarrow[\chi]{} & \mathbb{T} & \xleftarrow{\ \mathrm{id}_{\mathbb{T}}\ } & \mathbb{T}.
\end{array}$$

We also recall that the morphism q induces an isomorphism

$$q^* = \operatorname{Hom}(\mathfrak{L}(G), q)\colon \mathfrak{L}'(G) \stackrel{\text{def}}{=} \operatorname{Hom}(\mathfrak{L}(G), \mathbb{R}) \to \operatorname{Hom}(\mathfrak{L}(G), \mathbb{T}) = \widehat{\mathfrak{L}}(G),$$

$q^*(\omega) = q \circ \omega$ for $\omega \in \mathfrak{L}'(G)$ whose inverse is given by $(q^*)^{-1}(\gamma) = \nu^{-1}\mathfrak{L}(\gamma)$:

$$\begin{array}{ccccc}
\mathfrak{L}\big(\mathfrak{L}(G)\big) & \xrightarrow{\mathfrak{L}(\gamma)} & \mathfrak{L}(\mathbb{T}) & \xleftarrow{\ \nu\ } & \mathbb{R} \\
{\scriptstyle \exp_{\mathfrak{L}(G)}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\mathbb{T}}} & & \big\downarrow{\scriptstyle q} \\
\mathfrak{L}(G) & \xrightarrow{\ \gamma\ } & \mathbb{T} & \xleftarrow{\ \mathrm{id}_{\mathbb{T}}\ } & \mathbb{T},
\end{array}$$

where $\exp_{\mathfrak{L}(G)}\colon \mathfrak{L}\big(\mathfrak{L}(G)\big) \to \mathfrak{L}(G)$ is an isomorphism of topological vector spaces.

Thus we also have an isomorphism $\iota_G\colon \mathfrak{L}(G)\widehat{\ } \to \mathfrak{L}'(G) = \operatorname{Hom}(\mathfrak{L}(G), \mathbb{R})$ given by $\iota_G(\chi)(X) = \nu^{-1}\mathfrak{L}(\chi)(X)$.

Let us calculate the annihilator $\varepsilon_G(\widehat{G})^\perp$ of the subgroup $\varepsilon_G(\widehat{G})$ of $\widehat{\mathfrak{L}}(G)$ in $\mathfrak{L}(G)$. Indeed by definition an element $X\colon \mathbb{R} \to G$ in $\mathfrak{L}(G)$ is in this annihilator iff $0 = \varepsilon_G(\chi)(X) = \chi(\exp_G X)$ for all $\chi \in \widehat{G}$. Since the characters of a locally compact abelian group separate the points, this is equivalent to $\exp_G X = 0$, i.e. to $X \in \ker \exp_G$. Thus

$$\varepsilon_G(\widehat{G})^\perp = \mathfrak{K}(G) \stackrel{\text{def}}{=} \ker \exp_G$$

for any locally compact abelian group G.

Recall that for any locally compact abelian group G the characteristic group $\mathrm{comp}(G)_0$ is the largest compact connected subgroup and that by 7.59 we have $\mathfrak{L}_{\mathrm{comp}}(G) = \mathfrak{L}(\mathrm{comp}(G)_0)$.

The Exponential Function for Locally Compact Abelian Groups

Theorem 7.66. *Assume that G is a locally compact abelian group, say, $G = E \oplus H$, where $E \cong \mathbb{R}^n$ and $\mathrm{comp}(G)$ is open in H according to 7.57.*

(i) *$\mathfrak{L}(G) = \mathfrak{L}(E) \oplus \mathfrak{L}_{\mathrm{comp}}(G) \cong \mathbb{R}^n \oplus \mathfrak{L}(\mathrm{comp}(G)_0)$ (see Corollary 7.59) is a weakly complete topological vector space (see Definition 7.32) with dual*

$$\mathfrak{L}'(G) \cong \mathbb{R} \mathbin{\overline{\otimes}} \widehat{G} \cong \mathbb{R}^n \oplus (\mathbb{R} \otimes (\mathrm{comp}(G)_0)\widehat{\ }).$$

(ii) *The kernel $\mathfrak{K}(G)$ of $\exp_G\colon \mathfrak{L}(G) \to G$ consists of all one-parameter subgroups $X\colon \mathbb{R} \to G$ such that $X(1) = 0$, equivalently $X(\mathbb{Z}) = \{0\}$. It satisfies $\mathfrak{K}(G) \cong \mathrm{Hom}(\widehat{G}, \mathbb{Z})$ and is a closed subgroup of the weakly complete topological vector space $\mathfrak{L}_{\mathrm{comp}}(G)$ satisfying $\mathfrak{K}(G)^{\perp\perp} = \mathfrak{K}(G)$. The morphism of abelian topological groups $\lambda_{\mathfrak{L}(\mathrm{comp}(G)_0),\mathfrak{K}(G)}\colon \mathfrak{L}\big(\mathrm{comp}(G)_0\big)/\mathfrak{K}(G) \to \mathfrak{K}(G)^\perp$ (in $\mathbb{R}\otimes(\mathrm{comp}(G)_0)\widehat{\ }$) is bijective. The topological group $\mathfrak{K}(G)$ is totally disconnected.*

Give $\mathbb{R} \otimes \big(\mathrm{comp}(G)_0\big)\widehat{\ }$ the finest locally convex vector space topology; then this topological vector space is the dual of $\mathfrak{L}_{\mathrm{comp}}(G)$. Denote the subgroup $1 \otimes (\mathrm{comp}(G)_0)\widehat{\ }$ by A. Let $\overline{A}_0$ denote the largest vector subspace of $\overline{A}$. Then $\overline{A} = \overline{A}_0 \oplus \overline{A}_1$ algebraically and topologically with an $\aleph_1$- free closed subgroup A_1. The annihilator $(A_0)^\perp$ is $\overline{\mathrm{span}_{\mathbb{R}}\big(\mathrm{Hom}(\widehat{G}, \mathbb{Z})\big)} \cong \overline{\mathrm{span}_{\mathbb{R}}\big(\mathbb{K}(G)\big)}$ in $\mathrm{Hom}(\widehat{G}, \mathbb{R}) \cong \mathfrak{L}(G)$, and $\overline{A}_0$ is also the largest vector subspace of $\mathfrak{K}(G)^\perp \supseteq \overline{A}$. If the group $\overline{A}/\overline{A}_0 \cong \overline{A}_1$ is countable then it is free and $\mathfrak{K}(G)^\perp = \overline{A}$.

Alternatively, if $\varepsilon_G\colon \widehat{G} \to \widehat{\mathfrak{L}}(G) \cong \mathfrak{L}'(G)$ is the adjoint of the exponential function then $(\mathrm{im}\,\varepsilon_G)^\perp = \mathfrak{K}(G)$, and $\mathfrak{K}(G)^\perp = \overline{\mathrm{im}\,\varepsilon_G}$.

We consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{p} \mathbb{T} = \mathbb{R}/\mathbb{Z} \to 0$. Then there is a commutative diagram whose rows are exact and whose vertical maps are isomorphisms:

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}(\widehat{G}, \mathbb{Z}) & \xrightarrow{\mathrm{Hom}(\widehat{G},j)} & \mathrm{Hom}(\widehat{G}, \mathbb{R}) & \xrightarrow{\mathrm{Hom}(\widehat{G},p)} & \mathrm{Hom}(\widehat{G}, \mathbb{T}) \\
& \cong\uparrow & & e_G\uparrow & & \uparrow\eta_G \\
0 \to & \mathfrak{K}(G) & \xrightarrow{\mathrm{incl}} & \mathfrak{L}(G) & \xrightarrow{\exp_G} & G \\
& \rho\downarrow & & \mathrm{id}_{\mathfrak{L}(G)}\downarrow & & \downarrow\sigma \\
0 \to & \mathrm{Hom}(\mathbb{T}, G) & \xrightarrow[\mathrm{Hom}(p,G)]{} & \mathrm{Hom}(\mathbb{R}, G) & \xrightarrow[\mathrm{Hom}(j,G)]{} & \mathrm{Hom}(\mathbb{Z}, G),
\end{array}$$

where $\sigma(g)(n) = n{\cdot}g$ and where $\rho(X)\colon \mathbb{T} \to G$ is the morphism induced by $X\colon \mathbb{R} \to G$ with $X(\mathbb{Z}) = \{0\}$.

There is an exact sequence

$$0 \to \mathfrak{K}(G) \xrightarrow{\mathrm{incl}} \mathfrak{L}(G) \xrightarrow{\exp_G} \exp_G \mathfrak{L}(G) \to 0$$

(*where* $\exp_G$ *is identified with its corestriction to its image*)*. Equivalently,* $\exp_G$ *induces a bijective morphism of abelian topological groups* $\mathfrak{L}(G)/\mathfrak{K}(G) \to \exp_G \mathfrak{L}(G)$.

(iii) *If* $f\colon G \to H$ *is a quotient morphism of locally compact abelian groups, then* $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ *is surjective. If* $K = \ker f$*, then the following commuting diagram has exact rows*

$$(*)\qquad \begin{array}{ccccccccc} 0 \to & \mathfrak{L}(K) & \xrightarrow{\mathfrak{L}(\text{incl})} & \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) & \to 0 \\ & {\scriptstyle\exp_K}\big\downarrow & & {\scriptstyle\exp_G}\big\downarrow & & \big\downarrow{\scriptstyle\exp_H} & \\ 0 \to & K & \xrightarrow{\text{incl}} & G & \xrightarrow{f} & H & \to 0. \end{array}$$

The top row is split exact; i.e. there is a continuous linear map $\sigma\colon \mathfrak{L}(H) \to \mathfrak{L}(G)$ *such that* $\mathfrak{L}(f) \circ \sigma = \mathrm{id}_{\mathfrak{L}(H)}$*. In particular,* $\mathfrak{L}(f)$ *is open.*

(iv) *For each identity neighborhood* U *of* G *there is a compact subgroup* K *of* G *and a finite dimensional vector subspace* $\mathfrak{F}$ *of* $\mathfrak{L}(G)$ *such that*

(a) *the morphism* $\psi\colon K \times \mathfrak{F} \to G$, $\psi(k, X) = k + \exp_G X$ *is open and has a discrete kernel, thus implementing a local isomorphism, and*

(b) *there is an open identity neighborhood* V *in* $\mathfrak{F}$ *such that* $K + \exp V \subseteq U$ *and* $\psi|(K \times V)\colon K \times V \to K + \exp V$ *is a homeomorphism.*

Proof. By the Vector Group Splitting Theorem 7.57, we may indeed write $G = E \oplus H$ in the form explained in the first sentence of the theorem. Its character group $\widehat{G}$ is locally compact by 7.7(ii), and $\widehat{G} = H^\perp \oplus E^\perp \cong \widehat{E} \times \widehat{H}$ by the Annihilator Mechanism 7.64. Since H has an open compact subgroup K, H/K is discrete and by the annihilator mechanism, $K^\perp \cong \widehat{H/K}$ is compact, and since $\widehat{G}/K^\perp \cong \widehat{K}$ is discrete, $K^\perp$ is open.

(i) By the Duality Theorem 7.63 and Proposition 7.40, the natural morphism $e_G\colon \mathfrak{L}(G) \to \mathrm{Hom}(\widehat{G}, \mathbb{R})$ is an isomorphism of topological groups. We shall apply Theorem 7.35 to the character group $\widehat{G}$. By the preceding remarks the hypotheses of Proposition 7.35(vii) are then satisfied. Hence by 7.35(vii), $\mathrm{Hom}(\widehat{G}, \mathbb{R})$ and thus $\mathfrak{L}(G)$ are weakly completely vector spaces whose dual is $\mathbb{R} \mathbin{\overline{\otimes}} \widehat{G}$ (see 7.35(iii)). Since $\mathfrak{L}(G) = \mathfrak{L}(G_0)$ and $G_0 = E \oplus \mathrm{comp}(G)_0$ with a compact connected group $\mathrm{comp}(G)_0 = \mathrm{comp}(G_0)$ (see 7.58). The dual of $\mathfrak{L}(G)$ (see 7.30) thus satisfies $\mathfrak{L}(G)' \cong E' \times \mathfrak{L}\big(\mathrm{comp}(G)_0\big)'$. Now $\mathfrak{L}\big(\mathrm{comp}(G)_0\big)' \cong \mathbb{R} \mathbin{\overline{\otimes}} \big(\mathrm{comp}(G)\big)\widehat{\ } = \mathbb{R} \otimes \big(\mathrm{comp}(G)\big)\widehat{\ }$ since $\big(\mathrm{comp}(G)\big)\widehat{\ }$ is discrete and thus 7.34(iii) applies.

(ii) We apply Proposition 7.35 with the discrete group $\mathrm{comp}(G)_0\widehat{\ }$ in place of G. Then $\mathfrak{K}(G)$, naturally identifiable with $\mathrm{Hom}(\widehat{G}, \mathbb{Z})$, is the annihilator $A^\perp$ of A in $\mathfrak{L}(\mathrm{comp}(G)_0)$ by 7.35(v)a). From 7.33 we get the continuous bijective morphism $\lambda_{\mathfrak{L}(\mathrm{comp}(G)_0), \mathfrak{K}(G)}\colon \mathfrak{L}\big(\mathrm{comp}(G)_0\big)/\mathfrak{K}(G) \to \mathfrak{K}(G)^\perp$ as asserted. The topological group $\mathrm{Hom}(\widehat{G}, \mathbb{Z})$ is a subgroup of $\mathbb{Z}^{\widehat{G}}$ and thus is totally disconnected. Hence $\mathfrak{K}(G)$ is totally disconnected.

By 7.31(i), the group $\overline{A}$ is algebraically and topologically a direct product of $\overline{A}_0$ and an $\aleph_1$-free group $\overline{A}_1$. By 7.35(v)d) the annihilator $(A_0)^\perp$ is

$$\overline{\mathrm{span}_{\mathbb{R}}\left(\mathrm{Hom}(\widehat{G}, \mathbb{Z})\right)} \cong \overline{\mathrm{span}_{\mathbb{R}}\left(\mathbb{K}(G)\right)} \quad \text{in} \quad \mathrm{Hom}(\widehat{G}, \mathbb{R}) \cong \mathfrak{L}(G),$$

and $\overline{A}_0$ is the largest vector subspace of $\mathfrak{K}(G)^\perp$. If $\overline{A}/\overline{A}_0 \cong \overline{A}_1$ is countable, it is free, and then $\mathfrak{K}(G)^\perp$ in $\mathbb{R} \mathbin{\overline{\otimes}} \widehat{G}$ is $\overline{A}$ by 7.35(v)e).

The commutativity of the diagram derived from the morphisms $j: \mathbb{Z} \to \mathbb{R}$ and $p: \mathbb{R} \to \mathbb{T}$ is straightforward from the definitions. The exactness of the top and bottom row are standard homological algebra (cf. Appendix 1, A1.55). Since the vertical maps are isomorphisms, the exactness of the middle row follows. The assertion concerning the exact sequence involving $\exp_G$ is a simple application of the canonical decomposition theorem for homomorphisms.

(iii) By 7.54, G has arbitrarily small compact subgroups K such that G/K is a linear Lie group. Then the surjectivity of $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ follows directly from 7.41(iii).

Finally, by 7.30(iv) the existence of σ follows. Thus $\mathfrak{L}(f)$ is a continuous linear retraction. Retractions are quotient maps; indeed if $V \subseteq \mathfrak{L}(H)$ and $U \stackrel{\text{def}}{=} \mathfrak{L}(f)^{-1}(V)$ is open in $\mathfrak{L}(G)$, then $V = \sigma^{-1}U$ is open. But quotient morphisms of topological groups are always open.

Proof of (iv). Let U be an open identity neighborhood of G. By 7.54 there is a compact subgroup K contained in U and a quotient homomorphism $q: G \to H = \mathbb{R}^m \times \mathbb{T}^n \times A$ with a discrete abelian group A and $\ker q = K$. Let $j: K \to G$ denote the inclusion morphism. We note $\mathfrak{L}(H) \cong \mathbb{R}^m \times \mathbb{R}^n$ and apply (iii) above. In particular, we have the diagram $(*)$ and the cross section morphism $\sigma: \mathfrak{L}(H) \to \mathfrak{L}(G)$ with $\mathfrak{L}(q)\sigma = \mathrm{id}_{\mathfrak{L}(H)}$. We define $\varphi: K \times \mathfrak{L}(H) \to G$, $\varphi(k, X) = k + \exp_G \sigma(X)$. Then $q\big(\varphi(k,X)\big) = q\big(\exp_G \sigma(X)\big) = \exp_H \big(\mathfrak{L}(q)\sigma(X)\big) = \exp_H X$. Since $\exp_H \mathfrak{L}(H) = H_0$ is open in H in the present situation, and since $\operatorname{im}\varphi$ contains $K = \ker q$, we know that $\operatorname{im}\varphi = q^{-1}(H_0)$ is an open subgroup G_1 of G. The corestriction $\varphi_1: K \times \mathfrak{L}(H) \to G_1$ is a quotient morphism since $K \times \mathfrak{L}(H) \cong K \times \mathbb{R}^{m+n}$ is a countable union of compact subsets (see Appendix 1, EA1.21). We note that $(k, X) \in \ker\varphi$ iff $k = \exp_G -\sigma(X)$. Thus $\ker\varphi = \{(\exp_G -\sigma(X), X) \mid X \in \mathfrak{L}(H), \exp_G \sigma(X) \in K\}$. Thus the projection $\mathrm{pr}_2: K \times \mathfrak{L}(H) \to \mathfrak{L}(H)$ maps $\ker\varphi$ isomorphically onto the closed subgroup $D \stackrel{\text{def}}{=} \sigma^{-1} \exp_G^{-1} K$ of $\mathfrak{L}(H)$. Assume that $\mathbb{R}{\cdot}X \subseteq D$, then $\exp_H \mathbb{R}{\cdot}X = \exp_H \mathfrak{L}(q)\sigma\mathbb{R}{\cdot}X = q(\exp_G \sigma\mathbb{R}{\cdot}X) \subseteq q(K) = \{0\}$. This shows $X = 0$. Hence D does not contain any nondegenerate vector subspaces and is therefore discrete (see Appendix 1, A1.12). Thus $\ker\varphi$ is discrete and φ implements a local isomorphism. Since $K \subseteq U$ is compact, there is an open identity neighborhood V in $\mathfrak{L}(H)$ such that $\varphi|(K \times V)$ maps $K \times V$ homeomorphically onto an open identity neighborhood of G_1 and hence of G, and thus $K + \exp V \subseteq U$. □

The fact that any group of the form $\exp_G W$ with a closed vector subspace W of $\mathfrak{L}(G)$ should be compact, as is the case e.g. when $W = \mathfrak{L}(G) = \mathbb{R}^n$ and $\mathfrak{K}(G) \cong \mathbb{Z}^n$, is a relatively rare occurrence. Later we shall verify that $G_\ell \stackrel{\text{def}}{=} \exp_G \overline{\operatorname{span}_{\mathbb{R}} \mathfrak{K}(G)}$ is locally connected (8.35).

If G is the group of real numbers with the discrete topology and $H = \mathbb{R}$, and if $f: G \to H$ is the identity function, then f is a surjective morphism of locally compact abelian groups, but $\mathfrak{L}(G) = \{0\}$ and $\mathfrak{L}(H) \cong \mathbb{R}$. Thus $\mathfrak{L}(f) = 0$ and this morphism is not surjective. Thus $\mathfrak{L}$ does not preserve surjectivity.

Conclusion (iv) tells us much about the local structure of G; there are arbitrarily small open identity neighborhoods homeomorphic to $K \times V$ where K is a compact subgroup of G and V is homeomorphic to $\mathbb{R}^n$ for some n.

The Identity Component

Recall from Theorem 1.34 and the subsequent Exercise E1.13(iii) that in a locally compact group G, the identity component G_0 is the intersection of all open subgroups.

From 7.55 we also recall that in a locally compact abelian group G the subgroup $\operatorname{comp}(G)$ of all compact elements is closed.

The role of G_0 and $\operatorname{comp}(G)$ is elucidated by the Vector Group Splitting Theorem 7.57ff. even before we finally had the full Duality Theorem 7.63. Given a locally compact abelian group G, we naturally want to apply the Vector Space Splitting Theorem to $\widehat{G}$ and obtain, in particular, the key lattice diagram of 7.57(iv) of $\widehat{G}$ involving $\widehat{G}_0$ and $\operatorname{comp}(\widehat{G})$ on the one hand, and we are equally eager to apply the Annihilator Mechanism 7.64 to the closed subgroups of the group G in its key lattice diagram of 7.57(iv). We should obtain two new sets of information for G from both processes.

However, the following theorem will show us that both processes yield the same information on the two subgroup lattices of of G, respectively, $\widehat{G}$:

Theorem 7.67. *For a locally compact abelian group G we have*

(1) $$\big(\operatorname{comp}(G)\big)^\perp = \widehat{G}_0 \quad \textit{and} \quad (G_0)^\perp = \operatorname{comp}(\widehat{G}), \quad \textit{and}$$

(2) $$(G_1)^\perp = \operatorname{comp}((\widehat{G})_0) \quad \textit{and} \quad \operatorname{comp}(G_0)^\perp = (\widehat{G})_1.$$

Proof. Since G_0 is closed we have $G_0^{\perp\perp} = G_0$ by the Annihilator Mechanism Theorem 7.64. Hence, by duality, the second assertion follows from the first which we prove now.

Let $\mathcal{K}$ denote the set of all compact subgroups of G. Then $\operatorname{comp}(G) = \sum_{K\in\mathcal{K}} K$. Thus $\big(\operatorname{comp}(G)\big)^\perp = \bigcap_{K\in\mathcal{K}} K^\perp$ by Proposition 7.23(i). Since K is compact, its character group, $\widehat{K}$, which is isomorphic to $\widehat{G}/K^\perp$ by the Annihilator Mechanism Theorem 7.64, is discrete. Hence $K^\perp$ is an open subgroup of $\widehat{G}$. Thus $\big(\operatorname{comp}(G)\big)^\perp$ being an intersection of open subgroups contains $\widehat{G}_0$. Conversely, if U is an open subgroup of $\widehat{G}$, then the character group of the discrete group $\widehat{G}/U$ is compact and is isomorphic to $U^\perp$ again by Theorem 7.64. Hence $U^\perp \in \mathcal{K}$. But U, being open is also closed, and thus $U^{\perp\perp} = U$ by 7.17(iii). Hence $\big(\operatorname{comp}(G)\big)^\perp$ is the intersection of all open subgroups and thus agrees with $\widehat{G}_0$ by E1.13(iii) (following Theorem 1.34). Thus the first part of (1) is proved.

In order to prove the first part of assertion (2) we recall that G_1, by Theorem 7.57(i)(f), is the unique smallest open subgroup U such that G/U is torsion-free. Hence by the Annihilator Mechanism 7.64 and duality, $(G_1)^\perp$ is the unique largest compact subgroup $C = U^\perp$ of $\widehat{G}$ such that $C = U^\perp$, being isomorphic to the character group of G/U, is compact connected. That is, $(G_1)^\perp$ is the unique largest compact connected subgroup of $\widehat{G}$, and that is $\mathrm{comp}(G_0)$. □

The Annihilator Mechanism "reflects" the center of the key lattice diagram of 7.57(iv) consisting of the characteristic subgroups G_0, $\mathrm{comp}(G)$, $G_0 \cap \mathrm{comp}(G) = \mathrm{comp}(G)_0$, and $G_1 = G_0 + \mathrm{comp}(G)$. It also "reflects" the noncharacteristic subgroups $E = E_G$ and $H = H_G$: every possible H_G is the annihilator $(E_{\widehat{G}})^\perp$ of some maximal vector subgroup $E_{\widehat{G}}$ of $\widehat{G}$, yielding

$$G = E_G \oplus (E_{\widehat{G}})^\perp.$$

There are several direct consequences of the preceding Theorem 7.67:

Corollary 7.68. *For a locally compact abelian group G and its character group $\widehat{G}$ the following statements are equivalent:*

(1) *G has no nontrivial compact subgroups.*

(2) *$\widehat{G}$ is connected.*

Also, the following statements are equivalent:

(i) *For every element $g \in G$ the cyclic subgroup $\mathbb{Z}{\cdot}g$ is not isomorphic to $\mathbb{Z}$ as a topological group.*

(ii) *$\widehat{G}$ is totally disconnected.*

Proof. Condition (1) means $\mathrm{comp}(G) = \{0\}$, hence $\widehat{G} = \widehat{G}_0$ by Theorem 7.67. This is (2).

Next we recall Weil's Lemma 7.43 and note that, consequently, (i) means $\mathrm{comp}(G) = G$, and thus $\widehat{G}_0 = \{0\}$ by Theorem 7.67, which is (ii). □

Corollary 7.69. *For a compact abelian group G,*

(i) *the annihilator $(G_0)^\perp$ of the identity component in the discrete group $\widehat{G}$ is* $\mathrm{tor}\,\widehat{G}$*, the torsion subgroup, and*

(ii) $\widehat{G_0} = \widehat{G}/\mathrm{tor}(\widehat{G})$*.*

(iii) *A closed subgroup D of G is totally disconnected if and only if $\widehat{G}/D^\perp$ is a torsion group if and only if for each $\chi \in \widehat{G}$ there is a natural number n such that $n{\cdot}\chi \in D^\perp$.*

Proof. (i) We apply Theorem 7.67 by noticing that in a discrete abelian group A we have $\mathrm{comp}(A) = \mathrm{tor}\,A$.

(ii) This follows from (i) by the Annihilator Mechanism 7.64(i).

(iii) By the Annihilator Mechanism Theorem 7.64(i) we have $\widehat{D} \cong \widehat{G}/D^\perp$. Thus by (i) above $D_0 = \{0\}$ iff $\widehat{G}/D^\perp$ is a torsion group. □

Corollary 7.70. *A compact abelian group G is connected if and only if its character group is torsion-free, and it is totally disconnected if and only if its character group is a torsion group.* □

Theorem 7.71. *For a locally compact abelian group G the identity component G_0 agrees with $\overline{\exp_G \mathfrak{L}(G)}$. In particular, if G_a is the arc component of 0 in G, then $G_0 = \overline{G_a}$, and G is totally disconnected if and only if G is totally arcwise disconnected.*

Proof. As a real topological vector space, $\mathfrak{L}(G) = \mathfrak{L}(G_0)$ is arcwise connected, hence connected. Thus $\exp \mathfrak{L}(G) \subseteq G_a \subseteq G_0$ and therefore $H \stackrel{\text{def}}{=} \overline{\exp \mathfrak{L}(G)}$ is a closed connected subgroup, whence $H \subseteq G_0$. Suppose now that contrary to the claim, $H \neq G_0$. Then G_0/H is a nondegenerate locally compact connected abelian group. By Corollary 7.54, there is a closed subgroup N of G_0 containing H such that G_0/N is a nondegenerate connected linear Lie group and thus is isomorphic to $\mathbb{R}^m \times \mathbb{T}^n$ with $m + n > 0$. The quotient morphism $f\colon G_0 \to G_0/N$ gives a surjective morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G_0) \to \mathfrak{L}(G_0/N) \cong \mathbb{R}^{m+n}$. If $X \in \mathfrak{L}(G_0)$, then $X(\mathbb{R}) = \exp \mathbb{R}{\cdot}X \subseteq H \subseteq N$, whence $f\big(X(\mathbb{R})\big) = \{0\}$ and thus $\mathfrak{L}(f)(X) = f \circ X = 0$. Hence $\{0\} = \operatorname{im} \mathfrak{L}(f) \cong \mathbb{R}^{m+n}$. Therefore $m + n = 0$, a contradiction. This shows $H = G_0$ which is what we had to prove. □

In the next chapter in Theorem 8.30 we shall show that $\exp_G \mathfrak{L}(G)$ is indeed equal to G_a, the arc component of the identity.

Corollary 7.72. *A locally compact abelian group is not totally disconnected if and only if it has a nontrivial one-parameter subgroup.* □

Corollary 7.73. *If $f\colon G \to H$ is a quotient morphism of locally compact abelian groups, then $\overline{f(G_0)} = H_0$. Hence if G is totally disconnected then H is totally disconnected.*

If G is a compact abelian group then $f(G_0) = H_0$.

Proof. From 7.66(iii) we know that $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective. Then $\overline{f(G_0)} = \overline{f(\overline{\exp \mathfrak{L}(G)})} = \overline{f\big(\exp \mathfrak{L}(G)\big)} = \overline{\exp \mathfrak{L}(f)\big(\mathfrak{L}(G)\big)} = \overline{\exp_H \mathfrak{L}(H)} = H_0$ by 7.71. Further if G is totally disconnected, then H_0 is singleton by the preceding and the claim follows.

If G is compact, the G_0, being closed, is compact. Hence the image $f(G_0)$ is compact and therefore closed. From the preceding we get $f(G_0) = H_0$. □

The example of $\mathrm{id}_{\mathbb{T}}\colon \mathbb{T}_d \to \mathbb{T}$ with the discrete circle group $\mathbb{T}_d$ shows that 7.73 fails for surjective morphisms of locally compact groups, even if H is compact.

We have seen in Chapter I, Exercise E1.11, that there are quotient homomorphisms $f\colon G \to H$ of locally compact abelian groups for which $f(G_0) \neq H_0$, even when H is compact and f has a discrete kernel. and we shall return to this point in Chapter 8 (see Example 8.31).

The Weight of Locally Compact Abelian Groups

For the *weight* $w(X)$ of a topological space X refer to Appendix A4, A4.7ff. We now want to estimate the weight of a hom-group $\mathrm{Hom}(A,B)$. For this purpose it is sensible to give an alternative definition of the compact open topology. This may be of independent interest as a complement to the developments at the beginning of the chapter beginning with 7.1.

Proposition 7.74. *Let A and B be abelian topological groups, $\mathcal{O}(B)$ a basis for the topology on B, and $\mathcal{K}(A)$ be a set of compact subsets of A satisfying the following condition: $(*)$ If C is compact in A then for each $c \in C$ and each neighborhood U of c there is a $C' \in \mathcal{K}(A)$ such that $C' \subseteq U$ and $C \cap C'$ is a neighborhood of c in C.*

For $C \in \mathcal{K}(A)$ and $U \in \mathcal{O}(B)$ set $W(C,U) = \{f \in \mathrm{Hom}(A,B) \mid f(C) \subseteq U\}$. Then

$$\{W(C,U) \mid (C,U) \in \mathcal{K}(A) \times \mathcal{O}(B)\}$$

is a subbasis for the compact open topology of $\mathrm{Hom}(A,B)$.

Proof. Let $\mathcal{U}$ denote the open identity neighborhoods of B. By the definition following 7.1, the compact open topology has a basis of sets of the form $f+W(C,U)$ with $f \in \mathrm{Hom}(A,B)$, C compact in A and $U \in \mathcal{U}$.

Claim (a) If $f \in W(C,U)$ for $(C,U) \in \mathcal{K}(A) \times \mathcal{O}(B)$, then there is a $V \in \mathcal{U}$ such that

$$f + W(C,V) \subseteq W(C,U).$$

Claim (b) For $f \in \mathrm{Hom}(A,B)$, and C compact in A and $U \in \mathcal{U}$ there are finitely many elements $(C_j,U_j) \in \mathcal{K}(A) \times \mathcal{O}(B)$, $j = 1,\dots,n$ such that

$$f \in W(C_1,U_1) \cap \cdots \cap W(C_n,U_n) \subseteq f + W(C,U).$$

A proof of these claims will establish the proposition.

Proof of Claim (a). Since $f \in W(C,U)$, the compact space $f(C) \subseteq U$. We assert that there is an open identity neighborhood V in B such that $f(C) + V \subseteq U$. Indeed for each $c \in C$ there is a $V(c) \in \mathcal{U}$ such that $f(c) + V(c) + V(c) \subseteq U$. By the compactness of C we find $c_1,\dots,c_N \in C$ such that

$$f(C) \subseteq \big(f(c_1) + V(c_1)\big) \cup \cdots \cup \big(f(c_N) + V(c_N)\big).$$

Set $V = V(c_1) \cap \cdots \cap V(c_N) \in \mathcal{U}$. Let $c \in C$. Then there is a $j \in \{1,\dots,N\}$ such that $f(c) \in f(c_j) + V(c_j)$; hence $f(c) + V \subseteq f(c_j) + V(c_j) + V(c_j) \subseteq U$, which proves the assertion. Now let $\varphi \in W(C,V)$. Then $(f+\varphi)(C) \subseteq f(C) + \varphi(C) \subseteq f(C) + V \subseteq U$. Hence $f + W(C,V) \subseteq W(C,U)$ as claimed in (a).

Proof of Claim (b). Select $V \in \mathcal{U}$ so that $V - V \subseteq U$. For each $c \in C$, there is an open set $V(c) \in \mathcal{O}(B)$ such that $f(c) \in V(c) \subseteq f(c) + V$, and by continuity of f and hypothesis $(*)$ there is a $C(c) \in \mathcal{K}(A)$ such that $C(c) \cap C$ is a neighborhood of c in C and that $f\big(C(c)\big) \subseteq V(c)$. By the compactness of C there are elements $c_1,\dots,c_n \in C$, such that with $C_j = C(c_j)$ we have $C \subseteq C_1 \cup \cdots \cup C_n$. Set

$V_j = V(c_j)$. Then $f \in W(C_j, V_j)$ for $j = 1, \ldots, n$. Now let $\varphi \in W(C_j, V_j)$ for all $j = 1, \ldots, n$. We shall prove that $\varphi \in f + W(C, U)$. Now let $c \in C$. Then there is a j such that $c \in C_j$. Then $(\varphi - f)(c) \in \varphi(C_j) - f(C_j) \subseteq V_j - V_j \subseteq (f(c) + V) - (f(c) + V) \subseteq U$ which shows $\varphi - f \in W(C, U)$ as we had to prove.□

The simplest example of $\mathcal{K}(A)$ is the set of all compact subsets of A and the simplest example of $\mathcal{O}(B)$ is the topology of B.

Corollary 7.75. *Assume that A and B are abelian topological groups at least one of which is infinite. If A is locally compact, then*

$$w(\mathrm{Hom}(A, B)) \leq \max\{w(A), w(B)\}.$$

Proof. Let $\mathcal{O}(A)$ be a basis for the topology of A of cardinality $w(A)$ and $\mathcal{O}(B)$ a basis for the topology of B of cardinality $w(B)$. Let $\mathcal{K}(A)$ be the set of all sets of the form $\overline{V}$ such that $V \in \mathcal{O}(A)$ and $\overline{V}$ is compact; since A is locally compact, $\mathcal{K}(A)$ satisfies condition $(*)$ of 7.74. By 7.74, the set of all $W(C, U)$, as (C, U) ranges through $\mathcal{K}(A) \times \mathcal{O}(B)$ is a subbasis $\mathcal{B}$ for the compact open topology of $\mathrm{Hom}(A, B)$. Thus $\mathrm{card}\,\mathcal{B} \leq w(A)w(B) = \max\{w(A), w(B)\}$ since at least one of the two cardinals is infinite. By Lemma A4.8, it follows that $w\big(\mathrm{Hom}(A, B)\big) \leq \max\{w(A), w(B)\}$. □

For any group H and any subgroup K we write $(H : K) \stackrel{\mathrm{def}}{=} \mathrm{card}(H/K)$ and call this cardinal the *index* of K in H.

Theorem 7.76 (The Weight of Locally Compact Abelian Groups). (i) *Let G be a locally compact abelian group. Then $w(G) = w(\widehat{G})$.*

(ii) *If G is a compact abelian group and A its discrete character group, then $w(G) = \mathrm{card}\,A$.*

(iii) *If $G \cong \mathbb{R}^n \times H$ with a compact open subgroup K of H, according to the Vector Group Splitting Theorem* 7.57*, then*

$$w(G) = \begin{cases} \mathrm{card}\,G & \textit{if } G \textit{ is finite,} \\ \max\{\aleph_0, \mathrm{card}\,\widehat{K}, (H : K)\} & \textit{otherwise.} \end{cases}$$

Proof. (i) If G is finite, then $\widehat{G} \cong G$ by 1.18, and the assertion is clear. We now assume that G is infinite, whence $w(G) \geq \aleph_0$. Also $w(\mathbb{T}) = \aleph_0 \leq w(G)$. Similarly, $w(\mathbb{T}) \leq w(\widehat{G})$. As $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ and $\widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T})$, by Corollary 7.75 we have $w(\widehat{G}) \leq \max\{w(G), w(\mathbb{T})\} = w(G)$, and similarly $w(\widehat{\widehat{G}}) \leq w(\widehat{G})$. By Duality 7.63, $\widehat{\widehat{G}} \cong G$. Hence $w(G) = w(\widehat{G})$.

(ii) is a special case of (i).

(iii) If K is a compact open subgroup of H, then H is the disjoint sum of cosets $K + g$, each homeomorphic to K. Then $w(H) = \mathrm{card}(H/K)w(K) = (H : K)\,\mathrm{card}\,\widehat{K}$ by(ii). If G is finite, $w(G) = \mathrm{card}\,G$. If G is infinite, at least one of the groups $\mathbb{R}^n$, K, or H/K is infinite. The assertion then follows from the preceding calculation of $w(H)$ and $w(\mathbb{R}^n) = 0$ if $n = 0$ and $w(\mathbb{R}^n) = \aleph_0$ for $n > 0$ in view of $w(G) = \max\{w(\mathbb{R}^n), w(H)\}$ by A4.9. $\square$

Proposition 7.77. *Let G be an infinite compact abelian group and assume $\aleph_0 \leq \aleph < w(G)$. Then G has a closed subgroup M with $w(M) = \aleph$. .*

In particular, every compact abelian group has an infinite closed metric subgroup.

Proof. By the Annihilator Mechanism Theorem 7.64, for a subgroup H of G and its annihilator $H^\perp$ in $\widehat{G}$, one has $\widehat{H} \cong \widehat{G}/H^\perp$ and thus by Theorem 7.76, $w(H) = \operatorname{card}\widehat{H} = \operatorname{card}\widehat{G}/H^\perp$. Since $\operatorname{card}\widehat{G} = w(G)$, we have $\operatorname{card}\widehat{G} > \aleph$.

The Corollary is therefore equivalent to the result recorded in Exercise EA1.12 in Appendix 1. $\square$

Exercise E7.24. Prove that the conclusions of Proposition 7.77 remain intact for infinite *locally* compact abelian groups. $\square$

[Hint. Use the Vector Group Splitting Theorem 7.57, or check [144].]

Postscript

Our aim in this book is to expose the structure of compact groups; we have discussed compact Lie groups and have seen that compact abelian Lie groups are an important ingredient in structure theorems. We therefore need a general structure theory of compact abelian groups for which we paved the way as early as Chapters 1 and 2 by establishing the duality between compact abelian groups and discrete abelian groups.

The four primary results in this chapter are the Vector Group Splitting Theorem 7.57, the Pontryagin–van Kampen Duality Theorem 7.63, the Annihilator Mechanism 7.17 and 7.64, and the Exponential Function for Locally Compact Abelian Groups 7.66. The first one of these essentially reduces the study of the structure of locally compact abelian groups to the investigation of the structure of compact abelian groups, which is the topic of Chapter 8. The Pontryagin–van Kampen Duality Theorem for compact abelian groups shows that questions on the structure of compact abelian groups reduce to purely algebraic questions on their dual groups which are (discrete) abelian groups. But the full power of this reduction is possible only because of the Annihilator Mechanism which provides a precise containment reversing correspondence between the lattice of closed subgroups of a compact abelian group and the lattice of all subgroups of its character group. The last of the primary results deals with the exponential function and to a large part determines the flavor of this chapter. Our theorem on the relation between duality and local compactness (7.7) gives internal mathematical reasons why the category of locally compact abelian groups is the primary concern of duality theory based on the compact open topology on character groups.

In Chapters 5 and 6 we saw the effectiveness of the exponential function. Our strategy is to carry forth the tool of the exponential function into the study of abelian topological groups. Since the Lie algebra of many a compact abelian group is an infinite dimensional topological vector space which cannot be locally compact, we are forced to examine a class of abelian topological groups that is wide enough to include these vector spaces.

Our presentation therefore aims from the very beginning to encompass more than compact abelian groups, more than discrete abelian groups, and indeed more than the classical domain of locally compact abelian groups. It is then natural that we adopt, in this chapter, a category theoretical stance towards duality. Since the work of Kaplan in the forties, many authors have made excursions outside the realm of locally compact abelian groups; the theory of locally convex topological vector spaces and their duality theory added an incentive to do so. Since, in the end, the structure theory of compact groups is our goal, we chose to be general and selective at the same time. The generality has the added advantage, that it brings out the contours of certain arguments such as those involving the Annihilator Mechanism more sharply than a conventional treatment restricted to locally compact abelian groups would have.

Our treatment of the character theory of real topological vector spaces therefore does not go as far as some of the literature, but on the other hand provides another example of a full duality between two abelian categories, namely, the category of all real topological vector spaces endowed with their finest locally convex topology and the category of what we call weakly complete topological vector spaces. This duality is needed since the domain, $\mathfrak{L}(G)$, of the exponential function of any compact abelian group, G, is drawn from the pool of objects of this latter category. The theory of weakly complete vector spaces and their duality is treated in a more detailed fashion in Appendix A7. We have referred to that material for some of the proofs in this chapter.

The topological structure of the domain of the exponential function of a compact abelian group is more complex than that of the Lie algebra of a linear Lie group. But its algebraic structure is much simpler. In order to recognize that it is a vector space, no particular argument such as the recovery of addition (cf. 5.41) is needed, and the Lie bracket does not play any role. We recall that finite dimensional abelian linear Lie groups were classified in E5.18 (following the discussion of the center of a linear Lie group in 5.54) with the result that a finite dimensional abelian linear Lie group is a product of a copy of $\mathbb{R}^n$, a finite dimensional torus $\mathbb{T}^m$, and some discrete abelian group D. In the more general circumstance, we show in the Vector Group Splitting Theorem that every locally compact abelian group is a direct product of a group isomorphic to $\mathbb{R}^n$, and a group possessing an open compact subgroup. This specifically reduces the study of the exponential function of a locally compact abelian group largely to the investigation of the exponential function of a compact abelian group.

Important structural invariants of the vector space $\mathfrak{L}(G)$ are the kernel $\mathfrak{K}(G)$ of the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ and the closure $\overline{\operatorname{span}_{\mathbb{R}} \mathfrak{K}(G)}$ of its linear span. They are discussed in this chapter and the next one. Once the importance of this remark is accepted one realizes that the duality theory of our vector space categories will also have to deal with closed additive subgroups of the vector spaces in question, and not only with closed vector subspaces. This requires that the full power of the Annihilator Mechanism be available for the duality between weakly complete topological real vector spaces and real vector spaces. The establishing of this machinery necessitates some technical complications.

We utilize information on the exponential function in the Vector Group Splitting Theorem 7.57 which precedes and is used in the proof of the Pontryagin–van Kampen Duality Theorem for locally compact abelian groups. Other proofs of the Duality Theorem are in the literature, but since we are interested in the structure theory of compact and locally compact abelian groups this approach suits our goals. In particular, we shall use the Vector Group Splitting Theorem time and again.

The theorem on the Exponential Function for Locally Compact Abelian Groups 7.66 is a test case for basic techniques of general Lie theory as applied to non-Lie groups. It provides the link between the duality theory of locally compact abelian groups and the duality theory of weakly complete vector spaces. It is not entirely obvious that the identity component G_0 of a locally compact abelian group is the closure $\overline{\exp_G \mathfrak{L}(G)}$ of the image of the exponential function; i.e. that every point in G_0 can be approximated by points on one-parameter groups. We prove this in our discussion of the identity component (Theorems 7.67 and 7.71). Apart from the characterisation of the identity component via duality in 7.67, this is the link between connectivity and the exponential function in locally compact abelian groups. Theorem 7.66 on the exponential function for locally compact abelian groups is the first part of a sequence of three major results in Chapter 8, namely,

—Theorem 8.30 showing that the image of the exponential function is the arc component of the identity,

—Theorem 8.41 showing that, for a compact abelian group, G, the closure of the image of $\operatorname{span} \mathfrak{K}(G)$ under the exponential function is the locally connected component of the identity, i.e. the closed subgroup which is at the same time the smallest closed subgroup containing all torus subgroups and the largest closed locally connected subgroup, and

—Theorem 8.62 showing that the kernel of the exponential function is algebraically isomorphic to the first homotopy group $\pi_1(G)$ and describing all homotopy groups of the space underlying G.

Relatively little of the information contained in these theorems is to be found in textbooks.

Finally, one notices that locally compact abelian groups can be very large in some sense. Accordingly, various cardinal invariants are used to measure the size of a locally compact abelian group. Most of these are topological invariants of the underlying space. One of them is the *weight* underlying the space of a locally compact abelian group. It is a traditional result that the weight of the character group of a locally compact abelian group is that of the original group. This is contained in Theorem 7.76, whose proof necessitates that we go back to the definition of the compact open topology and present the definition of this topology in terms of spaces of continuous functions between spaces (rather than between groups). Weight is a coarse, but useful cardinal invariant as is exemplified in Proposition 7.77. We shall present other cardinal invariants (such as dimension, density, and generating degree) in subsequent chapters.

References for this Chapter—Additional Reading

[15], [18], [34], [40], [53], [83], [113], [114], [115], [122], [131], [140], [144], [147], [150], [211], [225], [226], [227], [228], [230], [241], [253], [266], [278], [279], [281], [290], [295], [303], [304], [306], [317], [362], [364], [371].

Chapter 8

Compact Abelian Groups

Compact abelian groups form the most important class of abelian topological groups, as is evidenced by the considerable literature on them and their applications. This chapter is devoted to describing their structure in detail.

As well as being important in its own right, an understanding of the structure of compact abelian groups is essential for the task of describing the structure of compact not necessarily abelian groups. In Chapter 6 we saw that compact connected Lie groups decompose in various ways into abelian and semisimple components. We shall see later, in Chapter 9, that such decompositions persist for compact non-Lie groups.

In Appendix 1 we have a presentation of basic features of the structure theory of discrete abelian groups. We exploit this here using the duality theory in Chapter 7 to establish the structure theory of compact abelian groups.

The material in this chapter is organized in five parts, specifically,

Part 1: Aspects of the Algebraic Structure,

Part 2: Aspects of the Point Set Topological Structure,

Part 3: Aspects of Algebraic Topology—Homotopy,

Part 4: Aspects of Homological Algebra,

Part 5: Aspects of Algebraic Topology—Cohomology

Part 6: Aspects of Set Theory—Arc Components and Borel sets.

Prerequisites. Beyond the prerequisites required in the previous chapters it is now desirable that the reader have had some acquaintance with homotopy and cohomology. At two points we encounter the axiomatics of set theory and such notions as independence of axioms from ZFC, at another point we meet commutative C*-algebras. We will have to refer to Borel subsets of a space and to the utilisation of Lusin spaces.

Our presentation of dimension theory of locally compact abelian groups is meant to give even the uninitiated reader a good grasp of it. But a complete understanding of dimension theory of topological groups would require more than a passing acquaintance with the various concepts of dimension of topological spaces. The section on the cohomology of the space underlying a compact abelian group requires, aside from some knowledge of Čech cohomology, a certain routine in functorial reasoning.

Part 1: Aspects of the Algebraic Structure

Divisibility, Torsion, Connectivity

Torsion and divisibility of abelian groups are expressed in terms of the endomorphisms

$$\mu_n = \{x \mapsto n{\cdot}x\}: G \to G$$

given by $n{\cdot}x = \underbrace{x + \cdots + x}_{n \text{ times}}$. We will analyze the question of torsion and divisibility in the context of compact abelian groups using the Annihilator Mechanism. First some notation is in order.

Definition 8.1. For an abelian topological group G we shall write

(1) $$nG = \operatorname{im}\mu_n = \{n{\cdot}x \mid x \in G\} \quad \text{and} \quad G[n] = \ker\mu_n = \{x \in G \mid n{\cdot}x = 0\},$$

(2) $$\mathrm{DIV}(G) = \bigcap_{n\in\mathbb{N}} \overline{nG}.$$

□

Furthermore we note from Appendix 1

(3) $$\mathrm{tor}(G) = \bigcup_{n\in\mathbb{N}} G[n],$$

(4) $$\mathrm{div}(G) = \bigcup\{H \mid H \text{ is a divisible subgroup}\},$$

(5) $$\mathrm{Div}(G) = \bigcap_{n\in\mathbb{N}} nG = \{g \in G \mid (\forall n \in \mathbb{N})(\exists x \in G)\ n{\cdot}x = g\}.$$

The *torsion subgroup*, $\mathrm{tor}(G)$, of all elements of finite order is the union of all the kernels, $G[n]$, of the endomorphisms μ_n. Notice that these form a directed family, because $G[m] + G[n] \subseteq G[mn]$.

We observe that for every $m \in \mathbb{N}$, trivially, $\bigcap_{n\in\mathbb{N}} nG \subseteq \bigcap_{n\in\mathbb{N}} mnG$. As $mnG \subseteq mG \cap nG$, the family $\{nG \mid n \in \mathbb{N}\}$ is a filter basis. We conclude that the reverse inclusion, holds, too, and thus

(6) $$(\forall m \in \mathbb{N}) \ \bigcap_{n\in\mathbb{N}} mnG = \bigcap_{n\in\mathbb{N}} nG = \mathrm{Div}(G).$$

Obviously we have

$$\overline{\mathrm{Div}(G)} \subseteq \mathrm{DIV}(G).$$

Proposition 8.2. (i) *If G is a compact abelian group or a discrete abelian group, then* $\mathrm{Div}\,G = \mathrm{DIV}\,G$.

(ii) *If G is a compact abelian group or a discrete torsion-free abelian group, then* $\mathrm{div}\,G = \mathrm{Div}\,G$.

Proof. (i) In both cases nG is closed for all $n \in \mathbb{N}$. Hence $\operatorname{DIV} G = \operatorname{Div} G$.

(ii) Assume first that G is discrete torsion-free. Then μ_m is injective and thus maps nG bijectively onto mnG. Hence $\mu_m\big(\operatorname{Div}(G)\big) = \bigcap_{n\in N} mnG$. It follows from (6) that $\operatorname{Div}(G)$ is divisible and thus is contained in $\operatorname{div}(G)$.

Secondly, assume that G is compact. If $\mathcal{F}$ is any filter basis of compact subsets of G and $f\colon G \to G$ any continuous self-map, then $f(\bigcap \mathcal{F}) = \bigcap_{F\in\mathcal{F}} f(F)$. (Indeed the left side is trivially contained in the right side, and if y is an element of the right side, then for each $F \in \mathcal{F}$ the set $X_F \stackrel{\text{def}}{=} f^{-1}(y) \cap F$ is nonempty compact. The set $\{X_F : F \in \mathcal{F}\}$ is a filter basis of compact sets and thus has an element $x \in \bigcap \mathcal{F}$ in its intersection. Then $f(x) = y$ and thus y is in the left side.) Applying this with $\mathcal{F} = \{nG \mid n \in \mathbb{N}\}$ and $f = \mu_m$, we again obtain $\mu_m(\bigcap_{n\in\mathbb{N}} nG) = \bigcap_{n\in\mathbb{N}} mnG$, whence $\operatorname{Div}(G)$ is divisible as in (i) above. Once more we conclude $\operatorname{Div}(G) \subseteq \operatorname{div}(G)$. □

Proposition 8.3. *In a locally compact abelian group G, the following conclusions hold:*

(i) $(nG)^\perp = \widehat{G}[n]$, $G[n] = (n\widehat{G})^\perp$, *and* $\overline{nG} = (\widehat{G}[n])^\perp$.

(ii) $\operatorname{DIV}(G) = (\operatorname{tor}\widehat{G})^\perp$, *and* $(\operatorname{DIV} G)^\perp = \overline{\operatorname{tor}\widehat{G}}$.

Proof. If $\mu_n^{(G)}\colon G \to G$ is multiplication with n, then we have $\langle \chi, \mu_n^{(G)}(g)\rangle = \langle \chi, n{\cdot}g\rangle = \langle n{\cdot}\chi, g\rangle = \langle \mu_n^{(\widehat{G})}(\chi), g\rangle$ and thus $\mu_n^{(\widehat{G})} = \big(\mu_n^{(G)}\big)\widehat{\ }$. Then (i) follows from Proposition 7.65. For a proof of (ii) we compute

$$\begin{aligned}\operatorname{DIV} G &= \bigcap_{n\in\mathbb{N}} \overline{nG} = \bigcap_{n\in\mathbb{N}} \big(\widehat{G}[n]\big)^\perp \\ &= \Big(\bigcup_{n\in\mathbb{N}} \widehat{G}[n]\Big)^\perp = (\operatorname{tor}\widehat{G})^\perp.\end{aligned}$$

The remainder then follows from 7.64(iii). □

We are now well prepared for a crucial structure theorem for compact abelian groups through which the property of connectivity is expressed in purely algebraic terms.

Divisibility and Connectivity in Compact Abelian Groups

Theorem 8.4. *Let G denote a compact abelian group, G_0 its identity component, and G_a its identity arc component. Then*

(7) $$G_0 = \operatorname{Div}(G) = \operatorname{div}(G) = (\operatorname{tor}\widehat{G})^\perp = \overline{\exp\mathfrak{L}(G)} = \overline{G_a} \quad\text{and}\quad G_0^\perp = \operatorname{tor}\widehat{G},$$

(8) $$\overline{\operatorname{tor} G}^\perp = (\operatorname{tor} G)^\perp = \operatorname{Div}\widehat{G}, \quad (\operatorname{Div}\widehat{G})^\perp = \overline{\operatorname{tor} G}.$$

Proof. (7) is a consequence of Corollary 7.69, Theorem 7.71, Propositions 8.2 and 8.3. □

This theorem says that there is a two-fold algebraic way to describe connectivity in a compact abelian group. The first way expresses the fact that the connected component of the identity is precisely the largest divisible subgroup, and the second says that its annihilator is the torsion subgroup of the dual. But there are also two topological ways to describe it. Namely, it is the largest topologically connected subset containing 0 and the closure of the arc component G_a of 0. This calls for a description of the arc component G_a and its "index" G_0/G_a in the connected component; we will discuss this later in 8.33.

Let us recall that for an abelian group A and a set X, the group $A^{(X)}$ is the direct sum of X copies of A, that is the subgroup of all $(a_x)_{x\in X} \in A^X$ with $a_x = 0$ for all but a finite number of the elements $x \in X$. Let $\mathcal{P}$ denote the set of all prime numbers.

Corollary 8.5. *For a compact abelian group G, the following conditions are all equivalent:*
(i) *G is connected.*
(ii) *G is divisible.*
(iii) *$\widehat{G}$ is torsion-free.*
(iv) *Each nonempty open subset of G contains a point on a one parameter subgroup.*
(v) *Each nonempty open subset of G contains a point on an arc emanating from 0.*

Also, the following statements are equivalent:
(a) *G is totally disconnected.*
(b) $\operatorname{Div} G = \{0\}$.
(c) *$\widehat{G}$ is a torsion group.*
(d) *G has no nondegenerate one parameter subgroups.*
(e) *G is totally arcwise disconnected.*

Finally, the following conditions are also equivalent:
(1) *G is torsion-free.*
(2) *$\widehat{G}$ is divisible.*
(3) *There is a family of sets $\{X_p \mid p \in \{0\}\cup\mathcal{P}\}$ such that $\widehat{G} \cong \mathbb{Q}^{(X_0)} \oplus \bigoplus_{p\in\mathcal{P}} \mathbb{Z}(p^\infty)^{(X_p)}$.*
(4) *There is a family of sets $\{X_p \mid p \in \{0\}\cup\mathcal{P}\}$ such that $G \cong (\widehat{\mathbb{Q}})^X \times \prod_{x\in\mathcal{P}} \mathbb{Z}_p^{X_p}$.*

Proof. The equivalences of the first and the second group are immediate consequences of Theorem 8.4. We prove the equivalences of the third group. The equivalence of conditions (3) and (4) is a consequence of Proposition 1.17 and the Duality Theorem 2.32 (or, more generally, 7.63). The equivalence of (2) and (3) is the Structure Theorem for Divisible Abelian Groups in Appendix 1, A1.42. Finally, (1) means $\operatorname{tor} G = \{0\}$, and thus $\widehat{G} = \{0\}^\perp = (\operatorname{tor} G)^\perp = \operatorname{Div} \widehat{G}$ by Propositions 8.2(i) and 8.3(ii) applied to $\widehat{G}$ in place of G and by duality. But $\operatorname{Div} A = A$ for an abelian group A is tantamount to the divisibility of A. Thus (1) and (2) are equivalent, too. □

Corollary 8.6. *A compact abelian group G has no nondegenerate torsion-free direct factors if and only if $\widehat{G}$ is reduced.*

Proof. The abelian group $\widehat{G}$ is reduced if and only if it has no nondegenerate divisible direct summand (see A1.42). By duality, this is equivalent to the absence of nondegenerate direct factors with divisible character group. By the equivalences of the last group in 8.5, this proves the claim. □

Definitions 8.7. (i) A compact abelian group is called a *compact p-group* if its character group is a p-group. A locally compact abelian group G is called a *p-group*, if it is a union of compact p-groups.

(ii) Let A be a torsion group. Then $A = \bigoplus_{p\in\mathcal{P}} A_p$ with the p-primary components (or p-Sylow subgroups) of A according to Appendix 1, A1.19. If $G = \widehat{A}$ we set $G_p = \left(\bigoplus_{p\neq q\in\mathcal{P}} A_q\right)^{\perp}$. We call G_p the *p-primary component* or the *p-Sylow subgroup* of G. □

The last part of the nomenclature needs justification which we give next.

Corollary 8.8. (i) *The p-primary component G_p of a compact totally disconnected group is a compact p-group whose character group is the p-Sylow subgroup $(\widehat{G})_p$ of the character group (up to natural isomorphism).*

(ii) *A totally disconnected compact abelian group G is the product $\prod_{p\in\mathcal{P}} G_p$ of its p-primary components.*

(iii) *A compact abelian group is totally disconnected if and only if it is a direct product of p-groups.*

Proof. Assertion (i) follows from the Annihilator Mechanism 7.64:

$$\begin{array}{ccccccc} & & G & & \{0\} & & \\ \big((G_p)^{\perp}\big)^{\wedge} & \cong & \Big\{ \quad \big| & & \big| \quad \Big\} & \cong & \widehat{G/G_p} \\ & & G_p & & \bigoplus_{p\neq q\in\mathcal{P}}(\widehat{G})_p & & \\ \big(\widehat{G}/(G_p)^{\perp}\big)^{\wedge} & \cong & \Big\{ \quad \big| & & \big| \quad \Big\} & \cong & \widehat{G_p} \\ & & \{0\} & & \widehat{G}. & & \end{array}$$

(ii) Since products are dual to direct sums according to Proposition 1.17, this is a consequence of the splitting of torsion groups into their primary components (Appendix 1, A1.19).

(iii) If the group G is totally disconnected, it is a product of p-groups by (ii), and if, conversely, it is a product of p-groups, then it is totally disconnected, since every p-group is totally disconnected by its very definition (8.7(i)) combined with the second group of equivalences in 8.5. □

Corollary 8.9. (i) *A compact connected abelian group is torsion-free if and only if it is isomorphic to* $(\widehat{\mathbb{Q}})^X$ *for some set* X*.*

(ii) *A compact abelian group* G *has a dense torsion group if and only if its character group has no divisible elements, i.e. if and only if* $\operatorname{Div}\widehat{G} = \{0\}$*. A compact* connected *abelian group has a dense torsion group if and only if its character group is a reduced torsion-free group.*

(iii) *A compact abelian group* G *is a torsion group if and only if it is a finite product of totally disconnected compact* p*-groups* G_p *each of which has bounded exponent* (*that is* $p^{n(p)}\cdot G_p = \{0\}$ *for some natural number* $n(p)$)*.*

Proof. (i) is a consequence of the third group of equivalent statements of Corollary 8.5.

(ii) By the Annihilator Mechanism 7.64(ii), the torsion group $\operatorname{tor} G$ of G is dense in G if and only if $G = (\operatorname{tor} G)^{\perp\perp}$ which in turn is equivalent to $(\operatorname{tor} G)^{\perp} = \{0\}$. By Propositions 8.2(i) and 8.3(ii) this is the case if and only if $\operatorname{Div}\widehat{G} = \{0\}$. This proves the first assertion. If G is also connected, then $\widehat{G}$ is torsion-free by Proposition 8.5. But in a torsion-free group A one has $\operatorname{Div} A = \operatorname{div} A$ by Proposition 8.2(ii). Hence it has no divisible elements if and only if it is reduced. The second assertion follows.

(iii) It is clear that a finite product of p-groups of bounded exponent is a torsion group. We have to prove the converse. From 8.1(3) we have $G = \bigcup_{n\in\mathbb{N}} G[n]$. By the Baire Category Theorem, there is an n such that $G[n]$ has interior points and thus is an open closed subgroup of finite index. It readily follows that G has finite exponent that is, $G \subseteq G[N]$ for some $N \in \mathbb{N}$. This implies the assertion. □

Exercise E8.1. Show that the additive groups $\mathbb{R}$ and $\mathbb{T}$ support uncountably many compact group topologies. Show that the compact abelian groups on $\mathbb{R}$ are all isomorphic as compact abelian groups. The groups on $\mathbb{T}$ fall into two isomorphy classes of compact abelian groups. (Compare Exercise E5.21.) □

Exercise E8.2. (i) Describe the structure of $\widehat{\mathbb{R}_d}$, where $\mathbb{R}_d$ is the additive group of real numbers endowed with the discrete topology.

(ii) Describe the structure of $\widehat{\mathbb{T}_d}$, where $\mathbb{T}_d$ is the additive group of real numbers modulo 1 endowed with the discrete topology. □

Exercise E8.3. Prove the following variation of Proposition 7.77:

Let G be an infinite compact abelian group and let $\aleph$ be an infinite cardinal satisfying $\aleph \le w(G_0) < w(G)$. Then there is a closed connected subgroup H with $w(H) = \aleph$. □

[Hint. Follow the proof of Proposition 7.77 and note that H is connected iff $\widehat{G}/H^{\perp}$ is torsion free by Corollary 8.5. Then Exercise E1.12 in Appendix 1 proves the assertion by duality.] □

Exercise E8.4. Show that an abelian p-group of bounded exponent is a direct sum of cyclic groups. Deduce the following result:

A compact abelian p-group is a direct product of cyclic groups of bounded order. $\square$

Exercise E8.5. Show that a nondegenerate free abelian group cannot support a compact group topology. Deduce that the only locally compact group topology on a free abelian group is the discrete one.

[Hint. Any connected subgroup is divisible hence trivial. Also, every subgroup of a free abelian group is a free abelian group (A1.9).] $\square$

We notice that dealing with a dense torsion group in a compact abelian group is easier than dealing with the hypothesis that G be a torsion group itself. From the above it is clear that the structure of compact abelian torsion groups is rather special. In some sense, the conditions of compactness and torsion are not very compatible. We recall from Corollary 8.9(ii) that G has a dense torsion group iff $\widehat{G}$ has no divisible elements, and if G is also connected, then this is the case iff $\widehat{G}$ is a reduced torsion free group.

Corollary 8.10. *In a compact abelian group G, the identity component G_0 is a direct factor if and only if the torsion subgroup of $\widehat{G}$ is a direct summand.*

Proof. Since $G_0^{\perp} = \operatorname{tor}\widehat{G}$ by Theorem 8.4, the claim follows from duality (see 1.17). $\square$

Example 8.11. Let ∇ denote the abelian group whose properties are described in Appendix 1, Theorem A1.32. Let G be its compact character group. Then G has the following properties:

(i) G has no nondegenerate torsion-free direct factor.
(ii) Set $T = \overline{\operatorname{tor} G}$. Then $T \cong \prod_{n=2}^{\infty} \mathbb{Z}(n)$ and G/T is a circle group.
(iii) $G_0 \cong \widehat{\mathbb{Q}}$ and G_0 is torsion-free divisible.
(iv) G_0 is not a direct factor.
(v) $G_0 \cap T \cong \prod_{p\,\text{prime}} \mathbb{Z}_p$, and $G = G_0 + T$.

Proof. (i) Since ∇ is reduced by A1.32(vi), the group G has no torsion-free direct summands by 8.6.

(ii) By 8.2(i), $\operatorname{DIV}\nabla = \operatorname{Div}\nabla$ and by 8.3(ii), $\overline{\operatorname{tor} G} = (\operatorname{DIV}\nabla)^{\perp}$. By A1.32(iii), (vi), $\operatorname{Div}\nabla \cong \mathbb{Z}$, and by A1.32(v) $\nabla/\operatorname{Div}\nabla \cong \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$. Duality then proves (ii).

(iii) By A1.32(ii), (iv), $\nabla/\operatorname{tor}\nabla \cong \mathbb{Q}$. Thus $G_0 = (\operatorname{tor}\nabla)^{\perp}$ is isomorphic to the character group of $\mathbb{Q}$ by the annihilator mechanism. Then it is torsion-free and divisible by 8.9 and 8.5 (first group of equivalences).

(iv) By A1.32(vii), $\operatorname{tor}\nabla$ is not a direct summand of ∇. The assertion (iv) then follows from 8.10.

(v) By the Annihilator Mechanism, by $G_0 = (\operatorname{tor}\nabla)^\perp$, and by $T = \operatorname{Div}(\nabla)^\perp$ the group $G_0 \cap T$ is the annihilator of $\operatorname{Div}(\nabla) \oplus \operatorname{tor}\nabla$ and is thus isomorphic to the character group of $\nabla/(\operatorname{Div}(\nabla) \oplus \operatorname{tor}\nabla) \cong \mathbb{Q}/\mathbb{Z} = \bigoplus_{p\,\text{prime}} \mathbb{Z}(p^\infty)$ by A1.32 (see the second of the diagrams illustrating some aspects of the subgroup lattice of ∇).

Similarly, the annihilator of G_0+T is $\operatorname{tor}\nabla \cap \operatorname{Div}(\nabla) = \{0\}$. By the annihilator mechanism, this shows $G = G_0 + T$. □

Compact Abelian Groups as Factor Groups

The concept of dimension of a topological space is notoriously delicate. We shall see that for compact abelian groups, and for compact groups in general, topological dimension is a very lucid idea. In the present section we shall provide a structure theorem which will allow us to discuss dimension in a very self-contained and convincing fashion in the next section.

We begin here with one concept in dimension theory which is not controversial.

Definition 8.12. A topological space X is said to be *zero-dimensional* or to satisfy $\dim X = 0$ if its topology has a basis consisting of open closed sets. □

Exercise E8.6. *Let X be a locally compact Hausdorff space. Then the following are equivalent.*

(i) $\dim X = 0$.

(ii) *X is totally disconnected.*

[Hint. The implication (i)⇒(ii) is true for any topological T_0-space. Indeed let C be a component of X, and suppose that $c_1 \neq c_2$ in C. Since X is a T_0-space there is an open set U in X containing one of c_1 and c_2 but not the other, say $c_1 \in U$ and $c_2 \notin U$. By (i) there is an open closed neighborhood V of c_1 contained in U. Then $c_2 \in X \setminus V$. Thus C is decomposed into a disjoint union $(C \cap V) \,\dot\cup\, (C \setminus V)$ of nonempty open subsets in contradiction with the connectivity of C.

For a proof of (ii)⇒(i) we first observe that it is sufficient to prove the claim for all compact spaces; for if $x \in X$, let U be an open set and C a compact set such that $x \in U \subseteq C$. If the implication (ii)⇒(i) holds for compact spaces, then there is an open closed neighborhood V of x in C which is contained in U. Then the set V, on the one hand, is open in U, hence in X, and on the other, being closed and contained in the compact Hausdorff space C, it is compact, and thus closed in the Hausdorff space X. Thus x has arbitrarily small open closed neighborhoods in X.

Thus we may and shall assume now that X is compact. In Exercise E1.12(i) we showed that, in a compact Hausdorff space, the connectivity relation is the intersection of all equivalence relations with open compact equivalence classes. Condition (ii) means that the binary relation of connectivity is equality. Let $x \in X$ and U an open neighborhood of x. Let $\mathcal{B}$ denote the set of all $R(x)$ where R ranges through the set $\mathcal{R}$ of all binary relations with open compact classes. We know $\bigcap_{R\in\cap R} R(x) = \{x\}$. We claim that there is an $R \in \mathcal{R}$ such that $R(x) \subseteq U$. Suppose not, then $\{R(x)\setminus U : R \in \mathcal{R}\}$ is a filter basis of compact sets; then there is a point y in its intersection. Now $y \in \bigcap R(x) = \{x\}$ on the one hand and $y \in X\setminus U$ on the other, a contradiction. We have shown that x has arbitrarily small open closed neighborhoods, which establishes (ii) as x was arbitrary.] □

This piece of information is sometimes called Vedenissoff's Theorem [363].

Our first step is to find compact subgroups of dimension 0 in a compact abelian group. We begin by considering a compact abelian group G and its character group $A = \widehat{G}$. A closed subgroup D of G is zero-dimensional iff it is totally disconnected iff $A/D^\perp = \widehat{G}/D^\perp$ is a torsion group by 7.69(ii). Thus we aim for finding subgroups F of an abelian group A such that A/F is a torsion group.

Recall that a subset $\mathcal{X} \subseteq A$ is *free* if the subgroup $\langle\mathcal{X}\rangle$ generated by $\mathcal{X}$ in A is free. In other words, $\sum_{x\in\mathcal{X}} n_x{\cdot}x = 0$ implies $n_x = 0$ for all $x \in \mathcal{X}$. The collection $\mathcal{F}$ of all free subsets of A containing a given free subset $\mathcal{X}_0$ (e.g. the empty set) is clearly inductive with respect to $\subseteq$, and thus, by Zorn's Lemma, there exists a maximal free subset $\mathcal{X}$ in A containing $\mathcal{X}_0$.

Lemma 8.13. *Let $F = \langle\mathcal{X}\rangle$ be the free abelian group generated by a free subset $\mathcal{X}$ of A. Then the following statements are equivalent:*

(i) *$\mathcal{X}$ is maximal in $\mathcal{F}$.*
(ii) *A/F is a torsion group.*
(iii) *The inclusion $j\colon F \to A$ induces an isomorphism $\mathbb{Q}\otimes j\colon \mathbb{Q}\otimes F \to \mathbb{Q}\otimes A$.*

If these conditions are satisfied, then $\operatorname{card}\mathcal{X} = \operatorname{rank} A$.

Proof. (i)⇒(ii) Let $a \in A$ and assume (i). We claim that there is an $n \neq 0$ with $n{\cdot}a \in F$. If not, then $\mathbb{Z}{\cdot}a$ is a free group with $\mathbb{Z}{\cdot}a \cap F = \{0\}$. But then $\mathbb{Z}{\cdot}a + F$ is a direct sum and is free, since $m{\cdot}a + \sum_{x\in\mathcal{X}} n_x{\cdot}x = 0$ implies $m{\cdot}a = -\sum_{x\in\mathcal{X}} n_x{\cdot}x \in \mathbb{Z}{\cdot}a \cap F = \{0\}$ and thus $m = 0$ and, consequently $n_x = 0$ for all $x \in \mathcal{X}$ as F is free. But this contradicts the maximality of $\mathcal{X}$.

(ii)⇒(iii) The exact sequence

$$0 \to F \xrightarrow{j} A \to A/F \to 0$$

remains exact when the functor $\mathbb{Q}\otimes\{\cdot\}$ is applied by Appendix 1, A1.45(v). But since A/F is a torsion group by (ii), we know that $\mathbb{Q}\otimes(A/F) = \{0\}$ (see A1.46(ii). Thus

$$0 \to \mathbb{Q}\otimes F \xrightarrow{\mathbb{Q}\otimes j} \mathbb{Q}\otimes A \to 0$$

is exact which is (iii).

(iii)⇒(i) If $\mathcal{X}'$ is a free subset properly containing $\mathcal{X}$ and $F' = \langle\mathcal{X}'\rangle$ then F'/F is a nonzero free group. By tensoring the exact sequence $0 \to F \to F' \to F'/F \to 0$ we obtain the exact sequence

$$0 \to \mathbb{Q}\otimes F \to \mathbb{Q}\otimes F' \to \mathbb{Q}\otimes(F'/F) \to 0$$

(by A1.45(v) again) in which $\mathbb{Q}\otimes(F'/F)$ is not zero. Hence $\mathbb{Q}\otimes F \to \mathbb{Q}\otimes F'$ is not surjective. Since the inclusion $F' \to A$ induces an injective morphism $\mathbb{Q}\otimes F' \to \mathbb{Q}\otimes A$ (in view of A1.45(v)), the injective morphism $\mathbb{Q}\otimes F \to \mathbb{Q}\otimes A$ cannot be surjective.

If these equivalent conditions are satisfied, then $\operatorname{card}\mathcal{X} = \dim_{\mathbb{Q}}\mathbb{Q}\otimes A$ and this last cardinal is the rank of A by Definition A1.59. □

For easy reference we formalize the widely known concept of a *torus* or *torus group*. For these groups we have an intuitively clear concept of dimension:

Definition 8.14. A topological group G is called a *torus* if there is a set $\mathcal{X} \neq \emptyset$ such that $G \cong \mathbb{T}^{\mathcal{X}}$. We shall write $\dim G = \operatorname{card}\mathcal{X}$. □

By duality, *G is a torus iff it is a nonsingleton compact abelian group whose character group is free.* The dimension of a torus is the rank of its character group. We call G a finite extension of a torus if G_0 is a torus and G/G_0 is finite. We write $\dim G = \dim G_0$.

If G_0 is a torus then it is divisible. Hence it is algebraically a direct summand (see Appendix 1, A1.36). If it is also an open subgroup, then it is an algebraic and topological direct summand because in a complementary subgroup C the singleton $\{0\} = G_0 \cap C$ is open, whence C is discrete. Hence a finite extension of a torus G is isomorphic as a compact abelian group to $G_0 \times G/G_0$. An abelian group A is the character group of a finite extension of a torus group if and only if it is the direct sum of a finite abelian group and a free abelian group. If B is a subgroup, then $\operatorname{tor} B = B \cap \operatorname{tor} A$ and B is an extension of $\operatorname{tor} B$ by $B/\operatorname{tor} B \cong (B + \operatorname{tor} A)/\operatorname{tor} A$ which is free as a subgroup of the free group $A/\operatorname{tor} A$ (see A1.9). Thus B is a direct sum of $\operatorname{tor} B$ and a free abelian group (see A1.15) and thus is the character group of a finite extension of a torus. By duality this says that every factor group of a finite extension of a torus is a finite extension of a torus.

The following proposition now secures the existence of compact zero-dimensional subgroups with nice factor groups.

Proposition 8.15. *Let G be a compact abelian group with closed subgroups $D_1 \subseteq G_1 \subseteq G$ such that $\dim D_1 = 0$ (e.g. $D_1 = \{0\}$) and G/G_1 is a finite extension of a torus (e.g. $G_1 = G$). Then there is a closed subgroup D with the following properties.*

(i) *$D_1 \subseteq D \subseteq G_1$.*

(ii) *$\dim D = 0$.*

(iii) *G/D is a finite extension of a torus.*

The set $\mathcal{D}[D_1, G_1]$ of subgroups satisfying (i), (ii), *and* (iii) *is a sup-semilattice; i.e. it is closed under the formation of finite sums. The set of subgroups D satisfying* (i) *and* (ii) *is a lattice, i.e. is also closed under finite intersections.*

Every $D \in \mathcal{D}[D_1, G_1]$ satisfies $\operatorname{rank}\widehat{G} = \operatorname{rank}(D^{\perp}) = \dim(G/D)$, *and $G/D \cong \mathbb{T}^{\operatorname{rank}\widehat{G}} \times E$ with a finite abelian group E.*

If $D \in \mathcal{D}[D_1, G_1]$, then there is a finite extension $\widetilde{D}$ of D such that
(iii′) *$G/\widetilde{D}$ is a torus.*

Proof. Let the closed subgroup G_2 be defined such that $G_1 \subseteq G_2 \subseteq G$ and that G_2/G_1 is the identity component $(G/G_1)_0$ of G/G_1. Then G/G_2 is finite and G_2/G_1 is a torus. If we construct D so that (i), (ii), and (iii) are satisfied with G_2 in place of G, then G/D will still be a finite extension of a torus. So we may simplify our notation by assuming that G/G_1 is a torus. Then the group $(G_1)^\perp \cong \widehat{G/G_1}$ (see 7.64(i)) is free. Hence it has a free generating set $F_0 \subseteq (G_1)^\perp \subseteq (D_1)^\perp$ (see 7.64(v)). Let F be a maximal free subset of $(D_1)^\perp$ containing F_0 and set $E = \langle F \rangle$. Since D_1 is totally disconnected, $\widehat{G}/(D_1)^\perp \cong \widehat{D_1}$ (see 7.64(i)) is a torsion group (see 7.69(ii)). By 8.13, $(D_1)^\perp/E$ is a torsion group. Hence $\widehat{G}/E$, being an extension of a torsion group by a torsion group is a torsion group. Hence F is a maximal free subset of $\widehat{G}$ by 8.13. We set $D \overset{\text{def}}{=} E^\perp$. Then (i) follows from 7.64(v). Since $\widehat{G}/E$ is a torsion group, $D \cong (\widehat{G}/E)\widehat{\ }$ (see 7.64(i)) is totally disconnected (see 7.69(ii)). This proves (ii). Since $E \cong \widehat{G/D}$ (see 7.64(i)) is free, G/D is a torus. Thus (iii) is established.

Now let C and D be two subgroups satisfying (i), (ii), (iii). Then obviously $D \cap C$ and $D + C$ satisfy (i). The group $C \times D$ is compact totally disconnected, and $m: C \times D \to C + D$, $m(c, d) = c + d$ is a surjective morphism of compact groups. Then $C + D$ is totally disconnected by 7.73. Hence $\dim(C + D) = 0$. Trivially, $C \cap D$ is totally disconnected and thus $C + D$ and $C \cap D$ satisfy (ii).

We have $(C \cap D)^\perp = C^\perp + D^\perp$ and $(C + D)^\perp = C^\perp \cap D^\perp$ by 7.64(v). Since each of the groups $C^\perp$ and $D^\perp$ is a free extension of a finite group, the group $(C+D)^\perp = C^\perp \cap D^\perp$, as a subgroup of $C^\perp$, is of the same type. Hence $G/(C+D)$ is a finite extension of a torus. Thus $C + D$ satisfies (iii).

Let $D \in \mathcal{D}[D_1, G_1]$. Then $\widehat{G}/D^\perp \cong \widehat{D}$ is a torsion group since D is totally disconnected and $D^\perp$ is a direct sum of a finite abelian group E and a free abelian group F since G/D is finite extension of a torus. Then 8.13 applies and shows that $\dim G/D = \dim(G/D)_0 = \operatorname{rank} F = \dim \mathbb{Q} \otimes F = \dim \mathbb{Q} \otimes \widehat{G} = \operatorname{rank} \widehat{G}$. As a consequence, $G/D \cong \mathbb{T}^{\operatorname{rank} \widehat{G}} \times E$.

Finally let $D \in \mathcal{D}[D_1, G_1]$. Then $G/D = E \oplus T$ with a finite group E and a torus T. Let $\widetilde{D}$ denote the full inverse image of E under the quotient morphism $G \to G/D$. Then $G/\widetilde{D} \cong T$ is a torus group. □

The preceding proposition yields the following

Corollary 8.15a. *Every compact abelian group G contains a compact totally disconnected subgroup Δ such that G/Δ is a torus whose dimension is* $\operatorname{rank} \widehat{G}$. □

Let B be any abelian group and $\pi: F \to B$ a surjective morphism of abelian groups from a free group (see A1.8). Then the group $A \overset{\text{def}}{=} F \times B$ is the sum of the two free groups $F_1 \overset{\text{def}}{=} F \times \{0\}$ and $F_2 \overset{\text{def}}{=} \{(x, \pi(x)) : x \in F\}$; it is free, however, only if B is free. If we set $G = \widehat{A}$, $D_j = F_j^\perp$ then $D_1 \cap D_2 = \{0\}$ and G/D_j is a torus for $j = 1, 2$. But if B is not free, then G is not a torus even though it is a subdirect product of the tori G/D_j, $j = 1, 2$; that is a closed subgroup of the torus $G/D_1 \times G/D_2$ projecting *onto* each of the factors.

Lemma 8.16. *Let A be an abelian group containing a free subgroup F such that A/F is finite. Then $A = \operatorname{tor} A \oplus \widetilde{F}$ with the finite torsion group of A and a free group $\widetilde{F}$ such that $A/(\widetilde{F} \cap F)$ is finite. In fact, $\widetilde{F}$ contains a subgroup F^* isomorphic to F and $|A/\widetilde{F}|$ divides $|A/F^*| = |A/F|$.*

Proof. The torsion group $\operatorname{tor} A$ of A meets the torsion-free group F trivially; hence it is mapped faithfully into the quotient group A/F which is finite. Hence it is finite. Set $A' = A/\operatorname{tor} A$. Then A' is torsion-free (see Appendix 1, A1.17(i)). The image $F' \stackrel{\text{def}}{=} (F + \operatorname{tor} A)/\operatorname{tor} A \cong F/(F \cap \operatorname{tor} A) \cong F$ is free and $A'/F' \cong A/(F + \operatorname{tor} A)$ is a homomorphic image of A/F and is therefore finite. Let $\{e_j \mid j \in J\}$ be a free generating set of F. Then the elements $e_j' = e_j + \operatorname{tor} A$ in A' form a free basis of F'. Let $a_1', \dots, a_n'$ be elements of A' such that $A' = \mathbb{Z}{\cdot}a_1' + \cdots + \mathbb{Z}{\cdot}a_n' + F'$. Since A'/F' is finite and A' is torsion-free, for each $i = 1, \dots, n$ there is an integer k_i and a finite subset $I_i \subseteq J$ such that $k_i{\cdot}a_i' = \sum_{j \in I_i} m_{ij}{\cdot}e_j'$ for suitable integers m_{ij}. The set $I \stackrel{\text{def}}{=} \bigcup_{i=1}^n I_i \subseteq J$ is finite. The subgroup $F_1' \stackrel{\text{def}}{=} \langle a_1', \dots, a_n'; e_j', j \in I\rangle$ is finitely generated and torsion-free, hence it is free by the Fundamental Theorem of Finitely Generated Abelian Groups (A1.11). We claim that the free group $F_2' \stackrel{\text{def}}{=} \langle e_j' \mid j \in J \setminus I\rangle$ meets F_1' trivially. Indeed, $\mathbb{Q} \otimes F_1' = \mathbb{Q} \otimes \langle e_j' \mid j \in I\rangle$ and $\mathbb{Q} \otimes F_2' = \mathbb{Q} \otimes \langle e_j' \mid j \in J \setminus I\rangle$, and these two vector spaces contain the isomorphic copies of F_1' and F_2' in $\mathbb{Q} \otimes F'$, respectively and intersect in zero, since $\{1 \otimes e_j' \mid j \in J\}$ is a basis of $\mathbb{Q} \otimes F'$. Thus $A' = F_1' \oplus F_2'$ is free. Let P be the full inverse image of F_1' in A and $F_2 = \langle e_j \mid j \in J \setminus I\rangle \subseteq F$. Since $\operatorname{tor} A \subseteq P$ is finite and $P/\operatorname{tor} A \cong F_1'$ is finitely generated, P itself is finitely generated. Also, $A = P \oplus F_2$ and $F = (F \cap P) \oplus F_2$. Moreover, $P/(F \cap P) \cong A/F$ is finite.

In the finitely generated subgroup P we can write $P = \operatorname{tor} P \oplus F_3$ with a finitely generated free group F_3. The projection of P onto the second summand maps $F \cap P$ isomorphically onto a subgroup $F_4 \subseteq F_3$. We set $\widetilde{F} \stackrel{\text{def}}{=} F_3 \oplus F_2$ and $F^* \stackrel{\text{def}}{=} F_4 \oplus F_2$. Then $F^* \cong (F \cap P) \oplus F_2 = F$, and $F_2 \subseteq F \cap F^*$. Also, $A/F^* \cong P/F_4$ is an extension of $(F + F^*)/F^* \cong F/(F \cap F^*) \cong (P \cap F)/(F \cap F_4) \cong F_4/(F \cap F_4)$ by $A/(F + F^*)$ while $A/F \cong P/(P \cap F)$ is an extension of $(F + F^*)/F \cong F^*/(F \cap F^*) \cong F_4/(F \cap F_4)$ by $A/(F + F^*)$. Thus $|A/F| = |A/F^*|$. ☐

Corollary 8.17. *Retain the hypotheses of Proposition* 8.15. *Assume that $D \in \mathcal{D}[D_1, G_1]$ and that D' is a compact open subgroup of D containing D_1. Then $D' \in \mathcal{D}[D_1, G_1]$. In particular, $G/D \cong \mathbb{T}^{\operatorname{rank} \widehat{G}} \times E$ and $G/D' \cong \mathbb{T}^{\operatorname{rank} \widehat{G}} \times E'$ for some finite groups E and E'.*

Proof. Conditions (i) and (ii) of 8.15 are clearly satisfied by D'. We must show that G/D' is a finite extension of a torus. Since $D^\perp$ is a finite extension of a free group, also $(D')^\perp$ is a finite extension of a free group by Lemma 8.16 and thus, by Lemma 8.16, a direct sum of a free group and a finite group. Thus also (iii) of 8.15 is satisfied. $\square$

The relevance of the preceding Corollary 8.17 becomes particularly clear in the case $D_1 = \{0\}$ and $G_1 = G$. Then we may rephrase 8.17 as in Corollary 8.18 below.

Recall that we say that a set of subgroups of a topological group contains arbitrarily small members if every identity neighborhood contains a member of the set.

Corollary 8.18. *In any compact abelian group G there are arbitrarily small* 0*-dimensional compact subgroups $D \in \mathcal{D}[\{0\}, G]$; for each such D, the factor group G/D is a finite extension of a torus.*

In a compact connected *abelian group G there are arbitrarily small* 0*-dimensional compact subgroups such that G/D is a torus.*

Proof. Let U be a neighborhood of 0 in G. Let $D_0 \in \mathcal{D}[\{0\}, G]$. Then D_0 contains a compact open subgroup D which is contained in U since D_0 is compact totally disconnected. (Cf. 1.34.) By 8.17 we have $D \in \mathcal{D}[\{0\}, G]$, and this is the first assertion. The second follows because the connectivity of G entails the connectivity of G/D. $\square$

Corollary 8.19. *Assume the hypotheses of Proposition* 8.15 *and consider a $D \in \mathcal{D}[D_1, G_1]$. Set $T = G/D$ and write $p\colon G \to T$ for the quotient morphism and let $q\colon T \to \mathbb{T}^{\mathcal{X}} \oplus E$ with $E \cong T/T_0$ denote any isomorphism whose existence is guaranteed by the fact that T is a finite extension of a torus. Then $\mathfrak{L}(p)\colon \mathfrak{L}(G) \to \mathfrak{L}(T)$ is an isomorphism and if $\mathfrak{L}(\mathbb{T}^{\mathcal{X}})$ is identified with $\mathbb{R}^{\mathcal{X}}$, then $\mathfrak{L}(q)\colon \mathfrak{L}(T) \to \mathbb{R}^{\mathcal{X}}$ is an isomorphism, whence $\mathfrak{L}(qp)\colon \mathfrak{L}(G) \to \mathbb{R}^{\mathcal{X}}$ is an isomorphism.*

Proof. By 7.66(iii), the morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(T)$ is surjective and open and has a kernel isomorphic to $\mathfrak{L}(D)$. Since D is totally disconnected, $\mathfrak{L}(D) = \{0\}$ by 7.72. Hence $\mathfrak{L}(f)$ is injective and is, therefore, an isomorphism of topological vector spaces. We may identify $\mathfrak{L}(\mathbb{T}) = \mathrm{Hom}(\mathbb{R}, \mathbb{T})$ (by E5.9) with $\mathbb{R}$ and $\mathfrak{L}(\mathbb{T}^{\mathcal{X}})$ with $\mathfrak{L}(\mathbb{T})^{\mathcal{X}}$ by 7.38(i). The remainder follows. $\square$

For the following result recall from 8.15 that every compact abelian group contains a 0-dimensional compact subgroup Δ such that G/Δ is a torus of dimension

$\operatorname{rank}\widehat{G}$.

$$\begin{array}{rcl}
 & \begin{array}{c} G \\ | \\ \Delta \\ | \\ \{0\} \end{array} \qquad \begin{array}{c} \{0\} \\ | \\ \Delta^\perp \\ | \\ \widehat{G}. \end{array} & \\
\end{array}$$

$$\begin{array}{lccl}
\mathbb{T}^{\operatorname{rank}\widehat{G}} \cong \left\{ \begin{array}{c} G \\ | \\ \Delta \end{array} \right. & & \left. \begin{array}{c} \{0\} \\ | \\ \Delta^\perp \end{array} \right\} \cong \mathbb{Z}^{(\operatorname{rank}\widehat{G})} \\
\text{0-dimensional} \left\{ \begin{array}{c} \Delta \\ | \\ \{0\} \end{array} \right. & & \left. \begin{array}{c} \Delta^\perp \\ | \\ \widehat{G}. \end{array} \right\} \cong \widehat{G}|\Delta, \text{ torsion}
\end{array}$$

The Resolution Theorem for Compact Abelian Groups

Theorem 8.20. *Assume that G is a compact abelian group and that Δ is any compact totally disconnected subgroup such that G/Δ is a torus. Let $p\colon G \to G/\Delta$ denote the quotient map. Then the homomorphism*

$$\varphi\colon \Delta \times \mathfrak{L}(G) \to G, \quad \varphi(d, X) = d \exp X$$

satisfies the following conditions:

(i) *φ is continuous, surjective and open, i.e. is a quotient morphism. There is a compact subset C of $\mathfrak{L}(G)$ such that $\varphi(\Delta \times C) = G$.*

(ii) *The kernel $K \stackrel{\text{def}}{=} \ker\varphi \subseteq \Delta \times \mathfrak{L}(G)$ is mapped algebraically and topologically isomorphically onto $D \stackrel{\text{def}}{=} \exp_G^{-1}(\Delta) = \ker(p \circ \exp_G)$ under the projection $\Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$. Further, D is a closed totally disconnected subgroup of $\mathfrak{L}(G)$ whose annihilator $D^\perp$ in $E(G) \stackrel{\text{def}}{=} \mathbb{R} \mathbin{\overline{\otimes}} \widehat{G} \cong \widehat{\mathfrak{L}(G)}$ (according to 7.66(ii)) is an $\aleph_1$-free closed subgroup of the locally convex topological vector space $E(G)$. In particular, D does not contain any nonzero vector spaces.*

(iii) *The identity component of $\Delta \times \mathfrak{L}(G)$ is $\{0\} \times \mathfrak{L}(G)$ and agrees with the arc component of 0. Moreover, $\varphi(\{0\} \times \mathfrak{L}(G)) = \exp\mathfrak{L}(G)$ is dense in the identity component G_0 of G.*
The intersection $K \cap \big(\{0\} \times \mathfrak{L}(G)\big)$ is of the form $\{0\} \times \mathfrak{K}(G)$ where $\mathfrak{K}(G) \stackrel{\text{def}}{=} \ker\exp_G$ and the morphism φ factors through the quotient morphism

$$\Delta \times \mathfrak{L}(G) \to \Delta \times (\mathfrak{L}(G)/\mathfrak{K}(G))$$

with a quotient morphism

$$\Phi\colon \Delta \times (\mathfrak{L}(G)/\mathfrak{K}(G)) \to G, \quad \Phi(d, X + \mathfrak{K}(G)) = d \exp X$$

(iv) *The character group of $\Delta \times \mathfrak{L}(G)$ may be identified with $\widehat{\Delta} \times E(G)$, $E(G) \stackrel{\text{def}}{=} \mathbb{R} \mathbin{\overline{\otimes}} \widehat{G}$, equipped with the finest locally convex topology (see 7.66(ii)), in such a fashion that*

$$\langle(\delta, r \otimes \chi), (d, X)\rangle = \delta(d) + \chi\big(X(r)\big), \quad \delta \in \widehat{\Delta},\ d \in \Delta,\ r \in \mathbb{R},\ \chi \in \widehat{G},\ X \in \mathfrak{L}(G).$$

With this identification,

$$\ker\varphi = \big\{(\delta, r \otimes \chi) \mid \big(\forall X \in \mathfrak{L}(G)\big)\ \delta\big(X(1)\big) = -\chi\big(X(r)\big),\ X(1) \in \Delta\big\}^\perp$$

and $\widehat{\varphi}\colon \widehat{G} \to \widehat{\Delta} \times E(G)$ *is given by*

$$\widehat{\varphi}(\chi) = (\chi|\Delta, 1 \otimes \chi).$$

$$\begin{array}{cccccc}
 & & \Delta \times \mathfrak{L}(G) & \{0\} & & \\
G \ \cong & \left\{\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right. & \Big| & \Big| & \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} & \cong \ \widehat{G} \\
 & & K \cong D & K^{\perp} & & \\
\begin{array}{c}\text{totally}\\ \text{disconnected}\end{array} & \left\{\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right. & \Big| & \Big| & & \\
 & & \{0\} & \widehat{\Delta} \oplus E(G). & &
\end{array}$$

The group $(\ker\varphi)^{\perp}$ *in* $\widehat{\Delta} \oplus E(G)$ *is algebraically isomorphic to the discrete group* $\widehat{G}$*, and there is a bijective morphism from* $\big(\widehat{\Delta} \oplus E(G)\big)/(\ker\varphi)^{\perp}$ *onto the character group of* D*.*

Proof. Clearly, $\varphi\colon \Delta \times \mathfrak{L}(G) \to G$ defined by $\varphi(d, X) = d\exp X = dX(1)$ is a well-defined continuous homomorphism. The torus $T \overset{\text{def}}{=} G/\Delta$ is isomorphic to $\mathbb{T}^{\operatorname{rank}\widehat{G}}$ by 8.15. We note that $\exp_T\colon \mathfrak{L}(T) \to T$ is equivalent to the function $q^{\operatorname{rank}\widehat{G}}\colon \mathbb{R}^{\operatorname{rank}\widehat{G}} \to \mathbb{T}^{\operatorname{rank}\widehat{G}}$ with the quotient morphism $q\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Hence $\exp_T$ is a quotient morphism, mapping some compact subset C of $\mathfrak{L}(T)$ onto T. Let $p\colon G \to T$ denote the quotient morphism. The morphism $\mathfrak{L}(p)\colon \mathfrak{L}(G) \to \mathfrak{L}(T)$ is an isomorphism of topological vector spaces by 8.19. As a consequence $p \circ \exp_G = \exp_T \circ \mathfrak{L}(p)$ is surjective and open, and $\varphi(\Delta \times C) = \Delta \exp_G(C) = p^{-1}\exp_T(C) = G$.

Proof of (i). (a) The surjectivity of φ: from $p(\exp_G \mathfrak{L}(G)) = T$ and $\Delta = \ker p$ we conclude $\Delta \exp \mathfrak{L}(G) = (\ker p)\exp \mathfrak{L}(G) = G$. Thus φ is surjective.

(b) The openness of φ: we have to show that a basic zero neighborhood V of $\Delta \times \mathfrak{L}(G)$ is mapped onto a zero neighborhood of G. Now $\Delta \times \mathfrak{L}(G)$ has a basis of zero neighborhoods of the form $\Omega \times U$ with Ω a compact open subgroup of Δ and U a zero neighborhood of $\mathfrak{L}(G)$. We write $\varphi_\Omega\colon \Omega \times \mathfrak{L}(G) \to G$, $\varphi_\Omega(d, X) = d\exp_G X$, and denote by $p_\Omega\colon G \to G/\Omega$ the quotient morphism. Then $p_\Omega \circ \exp_G = \exp_{G/\Omega} \circ \mathfrak{L}(p_\Omega)$ is open since $\mathfrak{L}(p_\Omega)$ is an isomorphism by 8.19, and since $\exp_{G/\Omega}$ is open, being equivalent to the map $(r_x)_{x\in\mathcal{X}} \mapsto \big((r_x + \mathbb{Z})_{x\in\mathcal{X}}, 0\big)\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{T}^{\mathcal{X}} \times E$. Thus the set $p_\Omega(\varphi(\{0\} \times U)) = p_\Omega(\exp_G U)$ is a zero neighborhood of G/Ω. Then $\varphi_\Omega(\Omega \times U) = \Omega\varphi_\Omega(\{0\} \times U) = p_\Omega^{-1}p_\Omega\big(\varphi_\Omega(\{0\} \times U)\big)$ is a zero neighborhood of G. But $\Omega\varphi_\Omega(\{0\} \times U) = \Omega\exp_G(U) = \varphi(\Omega \times U)$. This shows that the image of the basic zero neighborhood $\Omega \times U$ of $\Delta \times \mathfrak{L}(G)$ under φ is a zero neighborhood of G. Hence φ is open.

Proof of (ii). We have $\ker\varphi = \{(d, X) \in \Delta \times \mathfrak{L}(G) : d\exp X = 1\}$. The map $X \mapsto (\exp -X, X) : D \to \ker\varphi$ is therefore bijective. This morphism has inverse $(d, X) \mapsto X\colon \ker\varphi \to D$, the restriction of the projection $\mathrm{pr}_{\mathfrak{L}(G)}\colon \Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$. Hence it is an algebraic and topological isomorphism. It follows that $K \subseteq \Delta \times \mathfrak{K}(G)$,

$\mathfrak{K}(G) = \ker \exp_G$. By 7.67(ii), the group $\mathfrak{K}(G)$ is totally disconnected. Hence $K = \ker \varphi$ and consequently D are totally disconnected. Since $D = \ker(p \circ \exp_G)$ the assertions about D follow from 7.66(ii).

Proof of (iii). Since $\{0\}$ is the identity component of Δ, clearly $\{0\} \times \mathfrak{L}(G)$ is the identity component of $\Delta \times \mathfrak{L}(G)$. Since $\varphi\big(\{0\} \times \mathfrak{L}(G)\big) = \exp_G \mathfrak{L}(G)$ the closure of this set is G_0 by 7.71. The remainder of (iii) is straightforward from the definitions.

Proof of (iv). From 7.10 we know that the character group of $\Delta \times \mathfrak{L}(G)$ may be canonically identified with $\widehat{\Delta} \times \widehat{\mathfrak{L}(G)}$ under the evaluation map given by

$$\langle (\delta, \omega), (d, X) \rangle = \delta(d) + \omega(X).$$

The character group of $\mathfrak{L}(G)$ has been identified in 7.35(i) (and 7.66(ii)) as $\mathbb{R} \otimes \widehat{G}$. The isomorphism $\chi \mapsto \widetilde{\chi} : (\mathbb{R} \otimes \widehat{G})\widehat{\ } \to (\mathbb{R} \otimes \widehat{G})'$ of 7.5, the map $\iota : \widehat{G} \to \mathbb{R} \otimes \widehat{G}$, $\iota(g) = 1 \otimes g$, and the isomorphism $\mathrm{Hom}(\widehat{G}, \mathbb{R}) \to \mathrm{Hom}(\mathbb{R}, G) = \mathfrak{L}(G)$ of 7.11 (with the identification of $\mathbb{R}$ with its character group) give the isomorphism α

$$(\mathbb{R} \otimes \widehat{G})\widehat{\ } \to (\mathbb{R} \otimes \widehat{G})' = \mathrm{Hom}(\mathbb{R} \otimes \widehat{G}, \mathbb{R}) \to \mathrm{Hom}(\widehat{G}, \mathbb{R}) \to \mathrm{Hom}(\mathbb{R}, G) = \mathfrak{L}(G),$$

$\alpha(r \otimes \chi)(s) = \widetilde{\chi}(rs)$ with inverse $\alpha^{-1}(X)(r \otimes \chi) = \chi\big(X(r)\big)$. This gives the identification of $\big(\Delta \times \mathfrak{L}(G)\big)\widehat{\ }$ which we claimed.

The topological vector space $\mathfrak{L}(G)$ is weakly complete by 7.64(i) and thus is reflexive by 7.30. Hence $\Delta \times \mathfrak{L}(G)$ is reflexive by 7.10. Now the hypotheses (a), (B), and (c) of the Annihilator Mechanism 7.17 are satisfied for the group $\Delta \times \mathfrak{L}(G)$ and the subgroup K with $\big(\Delta \times \mathfrak{L}(G)\big)/K \cong G$; then 7.17(i) shows that there is an isomorphism $G \to (K^\perp)\widehat{\ }$, and 7.17(ii), (iii) show that there is a bijective continuous morphism $\big((\widehat{\Delta} \oplus E(G))/(\ker \varphi)^\perp\big)\widehat{\ } \to \mathfrak{K}(G)$. □

Part 2: Aspects of the Point Set Topological Structure

Topological Dimension of Compact Abelian Groups

In the previous section we dealt with dimension zero in the process of proving the Resolution Theorem 8.20. We shall now address the question of arbitrary topological dimension in the context of compact abelian groups.

Proposition 8.21. *Let G be a compact abelian group and U an arbitrary identity neighborhood. Then U contains a compact subset homeomorphic to*

$$[-1, 1]^{\mathrm{rank}\,\widehat{G}}.$$

Proof. By Corollary 8.18 we find a $\Delta \in \mathcal{D}[\{0\}, G]$ such that Δ is contained in the interior of U. Since Δ is compact, there is an identity neighborhood V_0 such that $V_0\Delta \subseteq U$. If we set $V \stackrel{\mathrm{def}}{=} V_0\Delta$, then V is an identity neighborhood satisfying

$V\Delta = V \subseteq U$. Set $p: G \to T \stackrel{\rm def}{=} G/\Delta$. Then $\mathfrak{L}(p): \mathfrak{L}(G) \to \mathfrak{L}(T)$ is an isomorphism by 8.19.

We recall from 8.15 that the subgroup $\Delta^\perp \cong \widehat{T}$ of $\widehat{G}$ is of the form $F \oplus E$ with a free group F and a finite group E and that F is generated by a maximal free set $(\varepsilon_j)_{j\in J}$ such that $\operatorname{card} J = \operatorname{rank} \widehat{G}$. Then $T_0 = E^\perp/\Delta \cong \widehat{F}$ is a torus isomorphic to $\mathbb{T}^J$. Define $\rho_0: \mathbb{R}^J \to \operatorname{Hom}(F, \mathbb{R})$ by $\rho_0((r_j)_{j\in J})(\sum_{j\in J} s_j\cdot\varepsilon_j) = \sum_{j\in J} \varepsilon_j(r_j s_j)$. Then ρ_0 is an isomorphism of topological groups, and since

$$\operatorname{Hom}(F, \mathbb{R}) \cong \operatorname{Hom}(\mathbb{R}, T_0) = \mathfrak{L}(T_0) = \mathfrak{L}(T)$$

by 7.11(iii) we have an isomorphism of topological groups $\rho: \mathbb{R}^J \to \mathfrak{L}(T)$. Moreover, if we write $F = \bigoplus_{j\in J} \mathbb{Z}\cdot\varepsilon_j$ and, accordingly, $T_0 = \prod_{j\in J} \widehat{\mathbb{Z}\cdot\varepsilon_j} \stackrel{\sigma}{\to} \mathbb{T}^J$, then $\exp' = \sigma \exp_{T_0} \rho: \mathbb{R}^J \to \mathbb{T}^J$ has the kernel $\mathbb{Z}^J$. It follows that $N \stackrel{\rm def}{=} \ker \exp_T \mathfrak{L}(p) = \ker(p \exp_G)$ maps isomorphically onto $\mathbb{Z}^J$ under $\rho^{-1}\mathfrak{L}(p)$. Now $p(V)$ is an identity neighborhood of $T \cong \mathbb{T}^J \times E$. Then we find an $\frac{1}{2} > r > 0$ so that $S_r = \mathfrak{L}(p)^{-1}\rho([-r, r]^J)$ satisfies $S_{2r} \cap N = \{0\}$ and $p(\exp_G S_r) = \exp_T \mathfrak{L}(p)(S_r) \subseteq p(V)$. Then $p \exp_G$ maps S_r homeomorphically into T and then a fortiori $\exp_G$ maps S_r homeomorphically into G. But $V = V\Delta = p^{-1}p(V) \subseteq U$. Hence $\exp_G S_r \subseteq U$, and $e = \exp_G |S_r: S_r \to U$ is a homeomorphism onto the image. Since S_r can be trivially rescaled to become homeomorphic to $[-1, 1]^J$, the proposition is proved.□

Note that we constructed a homeomorphism ε from $[-1, 1]^{\operatorname{rank} \widehat{G}} \subseteq \mathbb{R}^{\operatorname{rank} \widehat{G}}$ into U so that $\varepsilon(0) = 0$ and $\varepsilon(-x) = -\varepsilon(x)$.

The proof of the following theorem will twice use the fact that if a compact topological m-cell is contained in $\mathbb{R}^n$, then $m \leq n$. This is true by whatever topological dimension theory applies to compact subsets of $\mathbb{R}^n$, e.g. small inductive dimension. We shall call this fact, with a little license, “invariance of domain”. (See e.g. [103], p. 4, 1.1.2, and p. 73ff., 1.8.3.)

Characterisation of Finite Dimensional Compact Abelian Groups

Theorem 8.22. *The following conditions are equivalent for a compact abelian group G and a natural number n:*

(1) $\operatorname{rank} \widehat{G} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = n$.

(2) *There is an exact sequence*

$$0 \to \operatorname{tor}(\widehat{G}) \to \widehat{G} \to \mathbb{Q}^n \to E \to 0$$

with the torsion subgroup $\operatorname{tor}(\widehat{G})$ *and some torsion group E.*

(3) *There is an exact sequence*

$$0 \to \mathbb{Z}^n \to \widehat{G} \to E \to 0$$

for some torsion group E.

(4) *There is a compact zero-dimensional subgroup Z of $(\widehat{\mathbb{Q}})^n$ and an exact sequence*

$$0 \to Z \to (\widehat{\mathbb{Q}})^n \to G \to G/G_0 \to 0,$$

where G_0 is the identity component of G.

(5) *There is a compact zero-dimensional subgroup Δ of G and an exact sequence*

$$0 \to \Delta \to G \to \mathbb{T}^n \to 0.$$

(6) $\dim \mathfrak{L}(G) = n$.

(7) *There is a compact zero-dimensional subgroup Δ of G and quotient homomorphism $\varphi: \Delta \times \mathbb{R}^n \to G$ which has a discrete kernel. In particular, φ yields a local isomorphism of $\Delta \times \mathbb{R}^n$ and G and is a covering map in the sense of Appendix* 2, A2.1

(8) *The identity of G has a neighborhood basis each member of which is homeomorphic to $D \times [0,1]^n$ for some 0-dimensional compact space D.*

(9) *The identity of G has a basis of open neighborhoods each member of which is homeomorphic to $D \times \mathbb{R}^n$ for some 0-dimensional compact space D.*

(10) $\operatorname{rank} \widehat{G_0} = n$.

Proof. By Appendix A1, A1.59, for any abelian group we define $\operatorname{rank} A = \dim_{\mathbb{Q}} \mathbb{Q} \otimes A = \dim_{\mathbb{Q}} \mathbb{Q} \otimes (A/\operatorname{tor} A)$. For the implication (1)⇒(2) see Appendix 1, A1.45. If (2) holds, then there is an injection of $j: \widehat{G}/\operatorname{tor}\widehat{G} \to \mathbb{Q}^n$ which induces an injection $\mathbb{Q} \otimes (\widehat{G}/\operatorname{tor}\widehat{G}) \to \mathbb{Q} \otimes \mathbb{Q}^n \cong \mathbb{Q}^n$ whose cokernel is $\mathbb{Q} \otimes (\mathbb{Q}^n/\operatorname{im} j) \cong \mathbb{Q} \otimes E = \{0\}$ by A1.45(v). Hence $\mathbb{Q} \otimes \widehat{G} \cong \mathbb{Q} \otimes (\widehat{G}/\operatorname{tor}\widehat{G}) \cong \mathbb{Q}^n$. Thus $\operatorname{rank}\widehat{G} = n$. Therefore (1) and (2) are equivalent.

Next let $\mathcal{X}$ be a free set in $\widehat{G}$ and $F = \langle \mathcal{X} \rangle$. By 8.13 this set is a maximal free set iff $\widehat{G}/F$ is a torsion group iff the inclusion $j: F \to \widehat{G}$ induces an isomorphism $\mathbb{Q} \otimes j: \mathbb{Q} \otimes F \to \mathbb{Q} \otimes \widehat{G}$. If this is the case, $\operatorname{card} \mathcal{X} = \operatorname{rank} F = \dim_{\mathbb{Q}} \mathbb{Q} \otimes F = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = \operatorname{rank}\widehat{G}$. These remarks prove that (3) and (1) are equivalent statements.

By 7.69, $(G_0)^\perp = \operatorname{tor}\widehat{G}$ and thus by the Annihilator Mechanism 7.64(vi), Conditions (4) and (2) are equivalent, and, similarly, (5) is equivalent to (3).

Since G is compact and thus $\widehat{G}$ is discrete, in the terminology of Theorem 7.66 on the Exponential Function of Locally Compact Abelian Groups, the dual $\mathfrak{L}(G)'$ of $\mathfrak{L}(G)$ is $\mathbb{R} \mathbin{\overline{\otimes}} \widehat{G} = \mathbb{R} \otimes \widehat{G} = \mathbb{R} \otimes \mathbb{Q} \otimes \widehat{G}$. Hence $\dim_{\mathbb{R}} \mathfrak{L}(G)' = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = \operatorname{rank}\widehat{G}$.

As a consequence (1) and (6) are equivalent.

(6)⇒(7) By (6) we have $\mathfrak{L}(G) \cong \mathbb{R}^n$. By the Resolution Theorem for Compact Abelian Groups 8.20 we obtain the quotient morphism $\varphi: \Delta \times \mathbb{R}^n \to G$ as asserted, because $\ker \varphi \cong \exp^{-1}(\Delta)$ and this closed subgroup of $\mathfrak{L}(G) \cong \mathbb{R}^n$ does not contain vector subgroups, hence is discrete (see Appendix A1, A.1.12(i)).

The implications (7)⇒(8)⇒(9) are immediate.

(9)⇒(6) Let U be an identity neighborhood of G and $h: D \times \mathbb{R}^n \to U$ a homeomorphism for a compact 0-dimensional space D.

We recall from the last part of the proof of Proposition 8.21 that there is a subset S_r in $\mathfrak{L}(G)$ which is homeomorphic to $[-1,1]^{\operatorname{rank}\widehat{G}}$, such that the restriction $e = \exp_G |S_r: S_r \to U$ maps S_r homeomorphically into U. Then $h^{-1}e: S_r \to D \times \mathbb{R}^n$ is a homeomorphism onto the image. If $(d,c) = h^{-1}(1)$, then $h(\{d\} \times \mathbb{R}^n)$ is the connected component of 1 in U, and thus $h^{-1}e(S_r)$ is a homeomorphic copy of S_r

contained in the euclidean n-space $\{d\} \times \mathbb{R}^n$. But S_r is homeomorphic to $[-r, r]^J$ hence to $[-1, 1]^J$. Since $[-1, 1]^J$ contains $[-1, 1]^m$ for $m = 0, 1, 2, \ldots, \operatorname{card} J$, this entails $\operatorname{card} J \leq n$ by the invariance of domain.

Thus $n' \stackrel{\text{def}}{=} \dim \mathfrak{L}(G) = \operatorname{card} J \leq n$. Then by "(6) implies (9)," there are identity neighborhoods homeomorphic to $D' \times \mathbb{R}^{n'}$ for a totally disconnected compact space D'. Thus, by hypothesis (9), some compact n-cell must be contained in $\mathbb{R}^{n'}$, and this implies $n \leq n'$ by invariance of domain. Thus $\dim \mathfrak{L}(G) = n$ which we had to show.

(1)⇔(10) Since $\widehat{G_0} \cong \widehat{G}/\operatorname{tor}(\widehat{G})$ by 7.69(ii), by A1.45(iv) we have

$$\operatorname{rank} \widehat{G_0} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \left(\widehat{G}/\operatorname{tor}(\widehat{G})\right) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = \operatorname{rank} \widehat{G}. \qquad \square$$

Clearly, the equivalence of (1) and (10) allows us to write down equivalent conditions $(2'), \ldots, (9')$ which arise from $(2), \ldots, (9)$ by replacing G by G_0.

Definitions 8.23. Let G be a compact abelian group. Then we set $\dim G = \operatorname{rank} \widehat{G} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G}$ and call this cardinal the *dimension* of G. If $\dim G$ is finite, then G is called *finite dimensional* and otherwise *infinite dimensional.* □

Corollary 8.24. (i) *If for a compact abelian group G there is a natural number n such that the equivalent conditions of Theorem* 8.22 *are satisfied, then*

$$n = \dim G = \dim G_0.$$

(ii) *If no such number exists, then* $\dim G = w(G_0)$.

(iii) *Any finite dimensional compact connected abelian group satisfies the second axiom of countability* (*and is, therefore, metrizable*).

(iv) *If G is a compact abelian group and D a totally disconnected closed subgroup, then* $\dim G/D = \dim G$.

(v) *A compact abelian group contains a cube* $\mathbb{I}^{\dim G}$, $\mathbb{I} = [0, 1]$.

Proof. (i) is a restatement of Definition 8.23.

We prove (ii). In this case, $\operatorname{rank} \widehat{G} = \operatorname{rank}(\widehat{G}/\operatorname{tor} \widehat{G}) = \operatorname{rank} \widehat{G_0}$ is infinite. Then $\operatorname{rank} \widehat{G_0} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G_0} = \operatorname{card}(\mathbb{Q} \otimes \widehat{G_0}) = \operatorname{card} \widehat{G_0}$ since the cardinality of an infinite dimensional rational vector space is the cardinality of any basis and thus of any generating set. By Theorem 7.77, however, $\operatorname{card} \widehat{G_0} = w(G_0)$.

(iii) Since G is connected, $\widehat{G}$ is torsion-free by 7.70. Hence $\widehat{G}$ is isomorphic to a subgroup of the rational vector space $\mathbb{Q} \otimes \widehat{G}$ which is finite dimensional because $\dim G < \infty$. Thus $\widehat{G}$ is countable, and so $w(G) = 1$ or $w(G) = \aleph_0$. (A topological group is metrizable iff the filter of identity neighborhoods has a countable basis, see A4.16)

(iv) By the Annihilator Mechanism 7.64, $\widehat{G}/D^\perp$ is isomorphic to the character group of the totally disconnected compact group D and is, therefore, a torsion group. Hence $\mathbb{Q} \otimes (\widehat{G}/D^\perp) = \{0\}$ by A1.46. Then the inclusion map $j\colon D^\perp \to \widehat{G}$ induces an isomorphism $\mathbb{Q} \otimes j\colon \mathbb{Q} \otimes D^\perp \to \mathbb{Q} \otimes \widehat{G}$ by A1.45(v). By the Annihilator

Mechanism 7.64 again, $D^\perp \cong (G/D)\widehat{\ }$. Hence $\mathrm{rank}(G/D)\widehat{\ } = \dim_\mathbb{Q} \mathbb{Q} \otimes D^\perp = \dim_\mathbb{Q} \mathbb{Q} \otimes \widehat{G} = \mathrm{rank}\, \widehat{G}$ and thus $\dim G/D = \dim G$.

(v) By Proposition 8.21, G contains a cube $\mathbb{I}^{\mathrm{rank}\, \widehat{G}}$. By Definition 8.23, $\mathrm{rank}\, \widehat{G} = \dim G$. □

8.25. Scholium. *Assume that* DIM *is a function defined on the class $\mathcal{C}$ of all locally compact spaces with values in $\{0, 1, 2, \dots; \infty\}$ such that the following conditions are satisfied.*

(Da) *If $f: X \to Y$ is a covering map (see Appendix 2, A2.1) for $X, Y \in \mathcal{C}$ then $\mathrm{DIM}\, X = \mathrm{DIM}\, Y$. In particular, $\mathrm{DIM}\, X = \mathrm{DIM}\, Y$ if X and Y are homeomorphic.*

(Db) *If $X = \mathbb{R}^n$ or $X = [0,1]^n$ for $n \in \mathbb{N}$ then $\mathrm{DIM}\, X = n$.* (The Euclidean Fundamental Theorem.)

(Dc) *For every paracompact space $Y \in \mathcal{C}$ and each closed subspace X of Y the relation $\mathrm{DIM}\, X \leq \mathrm{DIM}\, Y$ holds.* (The Closed Subspace Theorem.)

(Dd) *Assume that X is the underlying space of a compact group whose topology has a basis of compact open sets, and assume that $Y = \mathbb{R}^n$. Then $\mathrm{DIM}(X \times Y) \leq n$.* (The Special Product Theorem.)

Then for any compact abelian group G,

$$\mathrm{DIM}(G) = \begin{cases} \dim G & \textit{if } \dim G \textit{ is finite,} \\ \infty & \textit{otherwise.} \end{cases}$$

and, more generally, for any locally compact abelian group G with largest compact connected characteristic subgroup $K = (\mathrm{comp}\, G)_0$ and vector rank n (see Theorem 7.57 and subsequent remarks)

$$\text{(DIM)} \qquad \mathrm{DIM}(G) = \begin{cases} n + \dim K & \textit{if } \dim K \textit{ is finite,} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. First assume that G is a compact abelian group. If $\dim G$ is infinite, then by 8.21, the space G contains a cube homeomorphic to $[0,1]^{\dim G}$ and thus a cube homeomorphic to $[0,1]^n$ for $n = 1, 2, \dots$. Hence $\mathrm{DIM}\, G = \infty$ by the Euclidean Fundamental Theorem (Db) and the Closed Subspace Theorem (Dc).

If, however, $\dim G$ is a finite number n, then the Characterisation Theorem 8.21 applies and by 8.21(7) there is a covering morphism $\varphi: \Delta \times \mathbb{R}^n \to G$ with a compact zero-dimensional group Δ and with a kernel isomorphic to $\mathbb{Z}^n$, $n = 0, 1, \dots$ Then $\mathrm{DIM}\, G = \mathrm{DIM}(\Delta \times \mathbb{R}^n) = n$ by (Da). Now $\{0\} \times \mathbb{R}^n$ is a closed subspace of the paracompact space $\Delta \times \mathbb{R}^n$, whence $\mathrm{DIM}(\{0\} \times \mathbb{R}^n) \leq \mathrm{DIM}(\Delta \times \mathbb{R}^n)$ by the Closed Subspace Theorem (Dc). By the Euclidean Fundamental Theorem (Db) we have $\mathrm{DIM}(\{0\} \times \mathbb{R}^n) = n$. Thus $n \leq \mathrm{DIM}(\Delta \times \mathbb{R}^n)$. From the Special Product Theorem (Dd) we get $\mathrm{DIM}(\Delta \times \mathbb{R}^n) \leq n$. Thus $\mathrm{DIM}\, G = \mathrm{DIM}(\Delta \times \mathbb{R}^n) = n$.

Now assume that G is a locally compact abelian group. Then by the Vector Group Splitting Theorem 7.57, there is an open subgroup $U = E \oplus C$ with a compact group C and $E \cong \mathbb{R}^n$. Let $\sigma: G/U \to G$ be a cross section for the quotient map with $\sigma(U) = 0$. Then the projection map $f: G = \bigcup_{\xi \in G/U} \big(U + \sigma(\xi)\big) \to U$, $f\big((u + \sigma(\xi)\big) = u$ is a covering map. By Condition (Da), $\mathrm{DIM}(G) = \mathrm{DIM}(U)$.

Thus we know $\mathrm{DIM}\, G = \mathrm{DIM}(\mathbb{R}^n \times C)$ and by (Da), applied to the covering map $f\colon \mathbb{R}^n \times C \to \mathbb{T}^n \times C$, $f\big((r_j)_{j=1,\dots,n}, c\big) = \big((r_j+\mathbb{Z})_{j=1,\dots,n}, c\big)$, we get $\mathrm{DIM}(\mathbb{R}^n \times C) = \mathrm{DIM}\big((\mathbb{R}/\mathbb{Z})^n \times C\big)$. Now $\dim\big((\mathbb{R}/\mathbb{Z})^n \times C\big) = \operatorname{rank}\big((\mathbb{R}/\mathbb{Z})^n \times C\big)\widehat{\ } = \operatorname{rank}(\mathbb{Z}^n \times \widehat{C}) = n + \operatorname{rank} \widehat{C} = n + \dim C = n + \dim C_0 = n + \dim K$ since $C_0 = K$. By the first part of the proof

$$\mathrm{DIM}\big(\mathbb{T}^n \times C\big) = \begin{cases} \dim(\mathbb{T}^n \times C) & \text{if } \dim(\mathbb{T}^n \times C) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{DIM}\, G = \begin{cases} n + \dim C & \text{if } \dim C < \infty \\ \infty & \text{otherwise.} \end{cases}$$ □

One expects that any "reasonable" dimension function should satisfy the conditions (Da)–(Dd) on the class of locally compact spaces. In the following we discuss what is on record on topological dimension theory. Its intricate nature of course prevents us here from being self-contained; we must refer to the literature for the details. We hope that in light of these circumstances the reader will appreciate the value of Theorem 8.22 and the Scholium 8.25 for an understanding of the concept of topological dimension for spaces underlying locally compact abelian groups.

There are several viable concepts of topological dimension. We mention the following:

(1) small inductive dimension $\operatorname{ind} X$ or Menger–Urysohn dimension ([103], p. 3),

(2) large inductive dimension $\operatorname{Ind} X$ or Brouwer–Čech dimension ([103], p. 52),

(3) Lebesgue covering dimension $\operatorname{cdim} X$ or Čech–Lebesgue dimension ([103], p. 54),

(4) cohomological dimension over a given coefficient group ([103], p. 95),

(5) sheaf theoretical dimension $\dim_L X$ over a ground ring L ([46], p. 74).

—We need not distinguish between (4) and (5); for locally compact spaces relative cohomology $H^*(X, A; G)$ for a closed subspace A of a locally compact space is naturally isomorphic to $H^*_c(X \setminus A; G)$ with cohomology with compact support. The definitions (see [103], p. 95, [46], p. 73) then show that the sheaf theoretical concept of dimension is the more modern generalisation of cohomological dimension, certainly on the class $\mathcal{C}$ of locally compact spaces. Also, on paracompact spaces, cohomological dimension over the ring of integers agrees with cdim. (See e.g. [275], p. 206, 36-15 and p. 210, 37-7.)

The definition of the various dimensions show directly that (Db) is trivially satisfied. The dimensions ind, cdim, $\dim_L$ are defined locally and thus satisfy (Da). (The sheaf theoretical concept is particularly suitable for this situation as is seen e.g. from [46], p. 143. For cdim see e.g. [286], p. 196, 3.4.) The large inductive dimension is not defined locally, but a local version loc Ind has been defined (see [286], p. 188ff.). It is shown that loc Ind and Ind agree at least on the class of so-called weakly paracompact totally normal spaces ([286], p. 197) which compact abelian group spaces fail to be as soon as their weight is uncountable.

—All these dimensions satisfy the Euclidean Fundamental Theorem (Dc) (see e.g. ([103], p. 73, p. 95, [46], p. 144.)

—The Closed Subspace Theorem (Dd) holds for $\dim_L X$ on the class of locally compact spaces X ([46], p. 74, 15.8) and for ind even on the class of regular spaces ([103], p. 4, 1.1.2 (where it holds even for all subspaces). It is satisfied by the covering dimension $\operatorname{cdim} X$ on the class of normal spaces X ([103], p. 209, 3.1.4), hence for all paracompact spaces. The large inductive dimension $\operatorname{Ind} X$ satisfies the Closed Subspace Theorem for normal spaces X ([103], p. 170, 2.2.1).

—The Special Product Theorem (De) is satisfied (in more general form)

- by sheaf dimension: ([46], p. 143, 7.3),
- by covering dimension: ([103], p. 236, 3.2.14),
- by small and large inductive dimension: by the subsequent lemma on dimensions.

Dimension Lemma. *Let G be a compact totally disconnected group and Y a closed subset of some euclidean space. Then* $\operatorname{ind}(G \times Y) \le \operatorname{Ind}(G \times Y) \le \operatorname{Ind} Y = \operatorname{ind} Y$.

Proof. For the first inequality, see e.g. [103], p. 52, 1.6.3, and for the last equality see [103], p. 53, 1.6.4. We prove the inequality by induction on n. There is no loss in generality in assuming Y compact. Let $\operatorname{Ind} Y = \operatorname{ind} Y \le n+1$. Let A be a closed subspace of $G \times Y$ and V a neighborhood of A in $G \times Y$. For each $a = (a', a'') \in A$ we find a compact open identity neighborhood N_a of G and an open neighborhood U_a of a'' in Y such that $N_a a' \times U_a \subseteq V$ and that $\operatorname{Ind} \partial U_a = \operatorname{ind} \partial U_a \le n$. Since A is compact, there is a finite sequence of points $a_1, \dots, a_n$ such that $A \subseteq \bigcup_{j=1}^n N_{a_j} a'_j \times U_{a_j}$. Since all N_a are compact open subgroups, $N \stackrel{\text{def}}{=} \bigcap_{j=1}^n N_{a_j}$ is a compact open subgroup and each N_{a_j} is a finite union of cosets modulo N. Hence we can write

$$A \subseteq (Ng_1 \times U'_1) \cup \cdots \cup (Ng_q \times U'_q) \subseteq V$$

for open subsets U'_j of Y with $\operatorname{ind} \partial U'_j \le n$ and a family of cosets Ng_j such that Ng_j and Ng_k are either disjoint or agree. Renumber the g_j so that $Ng_1, \dots, Ng_p$ is a maximal collection of disjoint cosets among these. Set $U_m = \bigcup\{U'_j : Ng_j = Ng_m\}$, $m = 1, \dots, p$. Then $\partial U_m \subseteq \bigcup\{\partial U'_j : Ng_j = Ng_m\}$ and

$$(*) \qquad A \subseteq W \stackrel{\text{def}}{=} (Ng_1 \times U_1) \,\dot\cup \cdots \dot\cup\, (Ng_p \times U_p) \subseteq V.$$

By the Subspace Theorem for ind (see e.g. [103] p. 4, 1.1.2) and the Sum Theorem for ind on separable metric spaces (see e.g. [103], p. 42, 1.5.3), we obtain $\operatorname{ind} \partial U_m = \operatorname{Ind} \partial U_m \le n$ for $m = 1, \dots, p$. Now

$$\partial W = (Ng_1 \times \partial U_1) \,\dot\cup \cdots \dot\cup\, (Ng_p \times \partial U_p) \subseteq V$$

since each Ng_j has empty boundary. By the induction hypothesis, $\operatorname{Ind}(Ng_j \times \partial U_j) \le n$. Now the (trivial) Sum Theorem for finitely many disjoint open subsets

sets shows that $\operatorname{Ind} \partial W \leq n$. Hence by definition (see e.g. [103], p. 52) we conclude $\operatorname{Ind} G \times Y \leq n+1$. ☐

We conclude that ind, loc Ind, cdim and $\dim_L$ satisfy all conditions (D0)–(De) for all ground rings L. By the Scholium 8.25 it thus follows that they all agree with the unique dimension DIM defined on the class of locally compact group spaces by (DIM).

Corollary 8.26. *On the class of spaces underlying locally compact abelian groups, small inductive dimension, local large inductive dimension, Lebesgue covering dimension, cohomological dimension, and sheaf theoretical dimension* (*for any ground ring*) *all agree and when finite, take the value* $\dim G = \operatorname{rank} \widehat{G} = \dim G_0$. *The dimension of a locally compact abelian group is a topological invariant; i.e. two homeomorphic locally compact abelian groups have the same dimension.* ☐

We shall prove later that for arbitrary compact groups, topological dimension behaves as well as it does for compact abelian groups, and indeed this remains true for locally compact groups in general (although we shall not address this degree of generality in this book).

With the approach we have chosen, it remains open whether large inductive dimension also agrees on locally compact abelian groups. This is the case as was shown by Pasynkov [284] using a projective limit argument. What we are lacking here is an argument which would show that Ind fulfils condition (Da) and that could be verified if a Sum Theorem for countable families of closed subsets were available on the spaces we consider.

There are other numerical functions on classes of spaces such as e.g. the Hausdorff dimension of metric spaces. This one is not a *topological* dimension function at all, and the issue of such “dimensions” is an entirely different matter.

The dimension function dim which we have introduced in Definition 8.23 on the class of all compact abelian groups takes its values in the class of all cardinal numbers, and every cardinal number $\aleph$ occurs as a dimension since $\dim \mathbb{T}^{\aleph} = \operatorname{rank} \mathbb{Z}^{(\aleph)} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \mathbb{Z}^{(\aleph)} = \dim_{\mathbb{Q}} \mathbb{Q}^{(\aleph)} = \aleph$.

Arc Connectivity

For a pointed space (X, x_0) let $C_0(\mathbb{I}, X)$ denote the set of arcs $\gamma: \mathbb{I} = [0,1] \to X$ with $\gamma(0) = x_0$. We shall call these arcs *pointed arcs*.

Definition 8.27. A continuous function $p: E \to B$ between pointed spaces is said to have *arc lifting* if for any $\gamma \in C_0(\mathbb{I}, B)$ there is a $\widetilde{\gamma} \in C_0(\mathbb{I}, E)$ such that $p \circ \widetilde{\gamma} = \gamma$, in other words if $C_0(\mathbb{I}, p): C_0(\mathbb{I}, E) \to C_0(\mathbb{I}, B)$ is surjective. ☐

The interval $\mathbb{I}$ is simply connected (see Appendix A2, A2.8(iii)). Hence in view of A2.6, every covering of pointed spaces has arc lifting.

Lemma 8.28. (i) *Assume that* $p_j\colon E_j \to B_j$, $j \in J$ *is a family of pointed spaces with arc lifting. Then the product* $p \stackrel{\text{def}}{=} \prod_{j\in J} p_j\colon \prod_{j\in J} E_j \to \prod_{j\in J} B_j$ *has arc lifting.*

(ii) *Every product of pointed covering maps has arc lifting.*

Proof. (i) Let $\gamma\colon \mathbb{I} \to \prod_{j\in J} B_j$ be a pointed arc. Then $\gamma_j \stackrel{\text{def}}{=} \mathrm{pr}_j \circ \gamma\colon \mathbb{I} \to B_j$ is a pointed arc which has a lifting $\widetilde{\gamma}_j\colon \mathbb{I} \to E_j$ by hypothesis. Then $\widetilde{\gamma}(r) = (\gamma_j(r))_{j\in J}$ defines a pointed lifting $\widetilde{\gamma} \in C_0(\mathbb{I}, \prod_{j\in J} E_j)$ of γ.

(ii) is an immediate consequence of (i) in view of the fact that every covering map has arc lifting. □

Example 8.29. The exponential function $\exp_T\colon \mathfrak{L}(T) \to T$ of every torus has arc lifting.

Proof. Each torus T is (isomorphic to) a product $\mathbb{T}^J$ of circles and $\exp_{\mathbb{T}}\colon \mathfrak{L}(\mathbb{T}) \to \mathbb{T}$ is equivalent to the covering $\mathbb{R} \to \mathbb{T}$. The exponential function of a product is the product of the exponential functions of the factors (see 7.38), so the claim follows from 8.28. □

For a topological space X one frequently denotes the set of all arc components by $\pi_0(X)$. For any topological group G, the arc component G_a of the identity is clearly a *fully characteristic subgroup*; i.e. it is mapped into itself by all (continuous) endomorphisms. In particular, it is a normal subgroup.

Hence $\pi_0(G) = G/G_a$ is a group and is, therefore, a more sensible concept than it is for spaces without additional structure. Since G_a, in contrast with the connected component of the identity, is not closed in general, the quotient topology on $\pi_0(G)$ is not Hausdorff. One therefore is not eager to consider this topology, and we shall regard $\pi_0(G)$ as a group (without topology).

The Arc Component of a Locally Compact Abelian Group

Theorem 8.30. (i) *The exponential function* $\exp_G\colon \mathfrak{L}(G) \to G$ *of a locally compact abelian group has arc lifting.*

(ii) *The arc component* G_a *of the identity in a locally compact abelian group* G *is* $\exp \mathfrak{L}(G)$. *In other words, the sequence* $\mathfrak{L}(G) \xrightarrow[\exp_G]{} G \xrightarrow[p]{} \pi_0(G) \to 0$, $p(g) = g + \exp_G \mathfrak{L}(G)$ *is exact.*

(iii) *If* G *is a compact abelian group, then there is a natural isomorphism of abelian groups* $\alpha_G\colon \pi_0(G) \to \mathrm{Ext}(\widehat{G}, \mathbb{Z})$ *and thus there is an exact sequence*

$$\text{(exp)} \qquad 0 \to \mathfrak{K}(G) \to \mathfrak{L}(G) \xrightarrow{\exp_G} G \xrightarrow{\alpha_G p} \mathrm{Ext}(\widehat{G}, \mathbb{Z}) \to 0$$

where $\mathfrak{K}(G) = \ker \exp_G$.

If G is a locally compact abelian group, such that G/G_0 is compact, then $\pi_0(G) \cong \operatorname{Ext}((\operatorname{comp} G)\widehat{\ }, \mathbb{Z})$.

(iv) *A compact abelian group G is arcwise connected if and only if $\widehat{G}$ is a Whitehead group, i.e. satisfies* $\operatorname{Ext}(\widehat{G}, \mathbb{Z}) = \{0\}$ (*Appendix* 1, *Definition* A1.63).

(v) *If $f: G \to H$ is a quotient morphism of locally compact abelian groups, then* $f(G_a) = H_a$.

Proof. (i) A product of two functions has arc lifting if each one has arc lifting. By the Vector Group Splitting Theorem 7.57, $G = E \oplus H$ with a vector group $E \cong \mathbb{R}^n$ and a group H possessing a compact open subgroup. By 7.38 the exponential function of G may be written $\exp_E \times \exp_H: \mathfrak{L}(E) \times \mathfrak{L}(H) \to E \times H$. Then $\exp_E$ is an isomorphism and thus has arc lifting. The morphism $\exp_H: \mathfrak{L}(H) \to H$ has arc lifting iff its corestriction $\mathfrak{L}(H) \to H_0$ has arc lifting since each pointed arc of H maps into H_0, as does $\exp_H$. But H_0 is compact by 7.57. It therefore suffices now to prove arc lifting for a compact abelian group G. We let $\Delta \in \mathcal{D}[\{0\}, G]$ (see 8.15 and 8.20). Then $T \stackrel{\text{def}}{=} G/\Delta$ is a finite extension of a torus T_0 and the quotient morphism $p: G \to T$ yields an isomorphism $\mathfrak{L}(p): \mathfrak{L}(G) \to \mathfrak{L}(T)$ by 8.19.

Now let $\gamma \in C_0(\mathbb{I}, G)$. Then $p \circ \gamma \in C_0(\mathbb{I}, T)$. Since T_0 has arc lifting by 8.29 and the connected image of $p \circ \gamma$ is in T_0 there is a $\gamma^* \in C_0\big(\mathbb{I}, \mathfrak{L}(T)\big)$ with $\exp_T \circ \gamma^* = p \circ \gamma$. Set $\widetilde{\gamma} = \mathfrak{L}(p)^{-1} \circ \gamma^* \in C_0\big(\mathbb{I}, \mathfrak{L}(G)\big)$. Then $p \circ \exp_G \circ \widetilde{\gamma} = \exp_T \circ \mathfrak{L}(p) \circ \widetilde{\gamma} = \exp_T \circ \gamma^* = p \circ \gamma$. The arc $\delta \in C_0(\mathbb{I}, G)$ defined by $\delta(r) = \gamma(r)^{-1} \exp_G\big(\widetilde{\gamma}(r)\big)$ satisfies $p\big(\delta(r)\big) = 0$ for all $r \in I$. Hence $\delta(\mathbb{I})$ is a connected subspace of the subgroup $\ker p = \Delta$ which is totally disconnected. Hence $\delta(\mathbb{I})$ is singleton whence δ is constant, and thus $\gamma = \exp_G \circ \widetilde{\gamma}$. Thus $\widetilde{\gamma}$ is the required lifting of γ across $\exp_G$. This proves (i).

(ii) Trivially $\exp_G \mathfrak{L}(G) \subseteq G_a$. Now let $g \in G_a$. Then there is a $\gamma \in C_0(\mathbb{I}, G)$ with $\gamma(1) = g$. Let $\widetilde{\gamma} \in C_0\big(\mathbb{I}, \mathfrak{L}(G)\big)$ according to (ii) above and set $X = \widetilde{\gamma}(1)$. Then $\exp_G X = (\exp_G \circ \widetilde{\gamma})(1) = \gamma(1) = g$. Hence $g \in \exp_G \mathfrak{L}(G)$.

(iii) By the Vector Group Splitting Theorem $G = E \oplus H$ as in (i) above. Now $G_a = E \oplus H_a$ since E is arcwise connected, whence $G/G_a \cong H/H_a$. If G/G_0 is compact, then $G = E \oplus \operatorname{comp}(G)$ and $\operatorname{comp}(G)$ is compact. Assertion (iii) will therefore be proved if we prove it for compact G which we shall assume henceforth. Let $j: \mathbb{Z} \to \mathbb{R}$ the inclusion and $q: \mathbb{R} \to \mathbb{T}$ denote the quotient homomorphism. From the homological algebra of abelian groups (see Appendix 1, A1.56), we know the exact sequence

$$\operatorname{Hom}(\widehat{G}, \mathbb{R}) \xrightarrow{\operatorname{Hom}(\widehat{G}, q)} \operatorname{Hom}(\widehat{G}, \mathbb{T}) \xrightarrow{\delta} \operatorname{Ext}(\widehat{G}, \mathbb{Z}) \to 0.$$

From 7.66 we recall the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(\widehat{G}, \mathbb{R}) & \xrightarrow{\operatorname{Hom}(\widehat{G}, q)} & \operatorname{Hom}(\widehat{G}, \mathbb{T}) \\
{\scriptstyle e_G}\big\uparrow & & \big\uparrow{\scriptstyle \eta_G} \\
\mathfrak{L}(G) & \xrightarrow[\exp_G]{} & G.
\end{array}$$

By (ii), the cokernel $G/\exp_G \mathfrak{L}(G)$ of $\exp_G$ is $\pi_0(G)$. Thus we get an isomorphism $\alpha_G\colon \pi_0(G) \to \operatorname{Ext}(\widehat{G}, \mathbb{Z})$ and a commuting diagram with exact rows

$$\begin{array}{ccccccc}
\mathfrak{L}(G) & \xrightarrow{\exp_G} & G & \longrightarrow & \pi_0(G) & \to 0 \\
{\scriptstyle e_G}\big\downarrow & & {\scriptstyle \eta_G}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_G} & \\
\operatorname{Hom}(\widehat{G}, \mathbb{R}) & \xrightarrow[\operatorname{Hom}(\widehat{G},q)]{} & \operatorname{Hom}(\widehat{G}, \mathbb{T}) & \xrightarrow[\delta]{} & \operatorname{Ext}(\widehat{G}, \mathbb{Z}) & \to 0.
\end{array}$$

(iv) is a direct consequence of (iii).

For a proof of (v) we take $h \in H_a$. Then by (ii) above there is a $Y \in \mathfrak{L}(H)$ such that $\exp_H Y = h$. Now we invoke Theorem 7.66(iii) in which we saw that for a quotient morphism f the morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective. Hence there is an $X \in \mathfrak{L}(G)$ such that $\mathfrak{L}(f)(X) = Y$. Set $g = \exp_G X$. Then $f(g) = f(\exp_X) = \exp_H \mathfrak{L}(f)(X) = \exp_H Y = h$. Hence $H_a \subseteq f(G_a)$. The reverse inclusion is trivial.□

Notice that in Part (iii) of the theorem the assumption that G/G_0 be compact is no real restriction of generality, because every locally compact abelian group has an open subgroup U isomorphic to $\mathbb{R}^n \times H$ with a compact group H by 7.57. Thus $\pi_0(G)$ is an extension of the group $\operatorname{Ext}(\widehat{H}, \mathbb{Z})$ with the discrete group G/U. Any discrete group D satisfies $\pi_0(D) = D$. The portion of a locally compact abelian group relevant for the arc component G_a is anyhow the identity component G_0, and G_0 is always a product of a vector group with a compact group.

Since products of arcwise connected spaces are arcwise connected, every group of the form $\mathbb{R}^m \times \mathbb{T}^n$ for arbitrary cardinals m and n is arcwise connected. The fact that tori are arcwise connected is recovered from 8.30(iv) in view of the fact that free abelian groups, being projective, are Whitehead groups. In the section on metric compact abelian groups below we shall see more about the converse: What arcwise connected compact abelian groups do we know?

Theorem 8.30 complements Theorem 7.66 on the exponential function of a locally compact abelian group and a compact abelian group. The exact sequence (exp) appeared in 7.66 for the first time but without a precise identification of the cokernel of $\exp_G$, i.e. without the two rightmost arrows.

In 7.73 we noted that in the circumstances of Theorem 8.30 $\overline{f(G_0)} = H_0$.

Example 8.31. Let H be the one dimensional group $\widehat{\mathbb{Q}}$. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives us an exact sequence $0 \to \Delta \to H \to \mathbb{T} \to 0$ where Δ is the annihilator of $\mathbb{Z}$ in $H = \widehat{\mathbb{Q}}$. Thus Δ is a compact totally disconnected subgroup of H such that H/Δ is a torus. We apply the Resolution Theorem 8.20 and look at 8.22(7); we see that we have a quotient and even covering morphism $\varphi\colon G \overset{\text{def}}{=} \Delta\times\mathbb{R} \to H$. We notice $G_0 = G_a = \{0\}\times\mathbb{R}$ and $H_0 = H$. Thus $f(G_0) \neq H_0$. □

The example shows that this phenomenon is quite prevalent. In fact, let H be a finite dimensional compact connected but not arcwise connected group. Then any morphism $\varphi\colon \Delta \times \mathfrak{L}(H) \to H$ such as arises in the Resolution Theorem gives a locally compact abelian group $G \stackrel{\text{def}}{=} \Delta \times \mathfrak{L}(H)$ (it is here that the finiteness of the dimension is used!) such that $\varphi(G_0) = \exp_G \mathfrak{L}(H) \neq H = H_0$ by 8.30 because H is not arcwise connected.

In the following corollary we are referring to the fact that with each space X there is (functorially) associated the space X^α on X with the canonical arcwise connected topology (see Appendix 4, A4.1 and A4.2).

Corollary 8.32. *Let G be a compact abelian group and Δ a totally disconnected compact subgroup such that G/Δ is a torus. As in the Resolution Theorem* 8.20, *we let*

$$\varphi\colon \Delta \times \mathfrak{L}(G) \to G, \quad \varphi(d, X) = d \exp X.$$

and

$$\Phi\colon \Delta \times (\mathfrak{L}(G)/\mathfrak{K}(G)) \to G, \quad \Phi(d, X + \mathfrak{K}(G)) = d \exp X.$$

Then the following conclusions hold:

(i) *φ and Φ have arc lifting, and if Δ_d is the discrete space on the set underlying Δ and $\mathfrak{L}_{\text{red}}(G) = (\mathfrak{L}(G)/\mathfrak{K}(G))^\alpha$, then*

$$\varphi^\alpha\colon \Delta_d \times \mathfrak{L}(G) \to G^\alpha, \quad \varphi^\alpha(d, X) = d \exp X$$

is a quotient map, and

$$\Phi^\alpha\colon \Delta_d \times \mathfrak{L}_{\text{red}}(G) \to G^\alpha, \quad \Phi^\alpha(d, X + \mathfrak{K}(G)) = d \exp X$$

is a covering map.

(ii) *Let*

$$\Pi = \{d \in \Delta : \big(\exists X \in \mathfrak{L}(G)\big)\, d \exp X = 0\}.$$

Then $G^\alpha \cong (\Delta_d/\Pi) \times \mathfrak{L}_{\text{red}}(G)$ with $\Delta_d/\Pi \cong \operatorname{Ext}(\widehat{G}, \mathbb{Z})$ and $(G_a)^\alpha = \mathfrak{L}_{\text{red}}(G)$.

Proof. (i) By 8.30(i), the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ has arc lifting. If $j\colon \mathfrak{L}(G) \to \Delta \times \mathfrak{L}(G)$ is the coretraction given by $j(X) = (0, X)$, then $\varphi \circ j = \exp_G$. If $\gamma \in C_0(\mathbb{I}, G)$ is a pointed arc in G, and if $\widetilde{\gamma} \in C_0\big(\mathbb{I}, \mathfrak{L}(G)\big)$ is a lifting of it across $\exp_G$, then $j \circ \widetilde{\gamma} \in C_0\big(\mathbb{I}, \Delta \times \mathfrak{L}(G)\big)$ is a lifting of it across φ. Since $\exp_G$ has arc lifting, it follows that $X + \mathfrak{K}(G) \mapsto \exp_G X\colon \mathfrak{L}(G)/\mathfrak{K}(G) \to G$ has arc lifting. As before it follows that Φ has arc lifting. By the Resolution Theorem 8.20 the function φ is a quotient morphism and thus Φ is a quotient morphism. Now A4.6 applies and shows that

$$\varphi^\alpha\colon \big(\Delta \times \mathfrak{L}(G)\big)^\alpha \to G^\alpha$$

and

$$\Phi^\alpha\colon \big(\Delta \times \mathfrak{L}(G)/\mathfrak{K}(G)\big)^\alpha \to G^\alpha$$

are quotient morphisms. We note $\big(\Delta \times \mathfrak{L}(G)\big)^\alpha = \Delta^\alpha \times \mathfrak{L}(G)^\alpha = \Delta_d \times \mathfrak{L}(G)$ by A2.35, similarly $\big(\Delta \times \mathfrak{L}(G)/\mathfrak{K}(G)\big)^\alpha = \Delta_d \times \big(\mathfrak{L}_{\text{red}}(G)\big)\big)^\alpha$. In order to see that Φ is a covering map, it now suffices to show that $\ker \Phi$ is discrete. However, $\ker \Phi \cap \big(\{0\} \times \mathfrak{L}_{\text{red}}(G)\big) = \{(0, \mathbf{0})\}$, and thus the projection onto the first factor maps $\ker \Phi$

injectively into the discrete group Δ_d. Hence $\ker\Phi$ itself is discrete. The proof is now complete.

(ii) If we set $\Pi = \mathrm{pr}_1(\ker\varphi)$ with the projection onto the first factor of $\Delta_d \times \mathfrak{L}(G)$, then we can write $\ker\Phi = \{(d, \alpha(d)) : d \in \Pi\}$ and thereby define a morphism $\mu: \Pi \to \mathfrak{L}_{\mathrm{red}}(G)$ of abelian groups into a divisible group, which extends to a morphism $\overline{\mu}: \Delta_d \to \mathfrak{L}_{\mathrm{red}}(G)$. Then $\Delta' \stackrel{\mathrm{def}}{=} \{(d, \overline{\mu}(d)) : d \in \Delta\}$ is a complement for $\{0\} \times \mathfrak{L}_{\mathrm{red}}(G)$ containing $\ker\Phi$. Then $G^\alpha \cong (\Delta_d \times \mathfrak{L}(G))/\mathfrak{K}(G) \cong (\Delta'/\ker\Phi) \times \mathfrak{L}_{\mathrm{red}}(G) \cong (\Delta_d/\Pi) \times \mathfrak{L}_{\mathrm{red}}(G)$. By Theorem 8.30(iii) we have $G/G_a \cong \mathrm{Ext}(\widehat{G}, \mathbb{Z})$ as abelian groups. Hence $\Delta_d/\Pi \cong \mathrm{Ext}(\widehat{G}, \mathbb{Z})$. □

If the compact abelian group G is finite dimensional, then the group $\mathfrak{L}_{\mathrm{red}}(G) = (\mathfrak{L}(G)/\mathfrak{K}(G))^\alpha$ is more easily understood. Indeed $\mathfrak{L}(G) \cong \mathbb{R}^n$ with $n = \dim G$ by Theorem 8.22(vi) and then $\mathfrak{L}(G)/\mathfrak{K}(G) \cong \mathbb{R}^p \times \mathbb{T}^q$ with $n = p + q$ (see Appendix A1, A1.12(ii)). Thus this space is locally arcwise connected and then agrees with $\mathfrak{L}_{\mathrm{red}}(G)$. Therefore we have

Corollary 8.33. *Let Δ be a totally disconnected compact subgroup of the finite dimensional compact abelian group G such that G/Δ is a torus. Let*

$$\Pi = \{d \in \Delta : (\exists X \in \mathfrak{L}(G))\, d \exp X = 0\}.$$

Then $G^\alpha \cong (\Delta_d/\Pi) \times \mathfrak{L}(G)/\mathfrak{K}(G)$ with $\Delta_d/\Pi \cong \mathrm{Ext}(\widehat{G}, \mathbb{Z})$ and $(G_a)^\alpha \cong \mathbb{R}^p \times \mathbb{T}^q$ and with $p + q = \dim G$. □

For more information on arc components see Part 6 of this Chapter under "Arc Components and Borel Sets". (Numbers 8.86–8.99.)

Local Connectivity

Recall that a topological space is *locally connected* if its topology has a basis of connected open sets. An equivalent formulation states that every point has arbitrarily small connected open neighborhoods. In a locally connected space, every connected component is open. In other words, a locally connected space is the disjoint union of open closed subsets each of which is connected and locally connected. The image of a locally connected space under an open continuous map is locally connected. Finite products of locally connected spaces are locally connected. Conversely, since the projections of a product onto its factors are open and continuous, the local connectivity of a product entails the local connectivity of the factors.

Proposition 8.34. *Let G be a locally compact abelian group. Then the following statements are equivalent.*

(1) *G is locally connected.*
(2) *The identity component G_0 is open and locally connected.*

(3) *G is isomorphic to a group $\mathbb{R}^n \times K \times D$ for $n \in \{0, 1, 2, \ldots\}$, K is a connected locally connected compact group, and D is a discrete group.*

Proof. (1)⇒(2) If G is locally connected, then the identity component is open and locally connected.

(2)⇒(3) By the Vector Group Splitting Theorem 7.57 we have $G_0 = E \oplus K$ with a subgroup $E \cong \mathbb{R}^n$ and the maximal compact connected subgroup $K = \big(\mathrm{comp}(G)\big)_0$ of G. Since G_0 is locally connected, K is locally connected. As a compact connected group K is divisible by Theorem 8.4. Since $\mathbb{R}^n$ is divisible, the subgroup G_0 is divisible, and thus there is a subgroup D such that G is a direct sum of G_0 and D by Appendix 1, A1.36. But as G_0 is open, the subset $\{0\} = D \cap G_0$ is open in D, i.e. D is discrete, and the idempotent endomorphism p of G with image D and kernel G_0 is continuous. Hence $(g, d) \mapsto g + d\colon G_0 \times D \to G$ has the continuous inverse $x \mapsto \big(x - p(x), p(x)\big)$. Thus $G = G_0 \oplus D$ algebraically and topologically. Thus $G = E \oplus K \oplus D$ and $E \cong \mathbb{R}^n$.

(3)⇒(1) Since the three factors $\mathbb{R}^n$, K and D are locally connected, so is their product. ☐

After this proposition it is clear that local connectivity of locally compact abelian groups will be completely understood if it is understood for compact connected groups.

The following lemma is technical but it isolates the essential features on which the proof of the subsequent theorem rests.

Lemma 8.35 (The Local Structure of Locally Compact Abelian Groups). *Assume that $\mathcal{P}$ is a property of topological spaces satisfying the following conditions.*

(a) *$\mathcal{P}$ is preserved under continuous maps.*

(b) *If an identity neighborhood U of a compact group H has property $\mathcal{P}$ then the underlying space of the subgroup $\langle U \rangle$ generated by U in H has property $\mathcal{P}$.*

(c) *If K is a compact space having property $\mathcal{P}$, then $K \times \mathbb{R}^n$ has property $\mathcal{P}$ for $n = 0, 1, \ldots$.*

Then for a locally compact abelian group G, the following statements are equivalent.

(1) *G has arbitrarily small open identity neighborhoods with property $\mathcal{P}$.*

(2) *For each identity neighborhood U there is a compact subgroup N of G with property $\mathcal{P}$ and a finite dimensional vector subspace $\mathfrak{F}$ of $\mathfrak{L}(G)$ such that for some connected identity neighborhood V in $\mathfrak{F}$ the function $(n, X) \mapsto n \exp X\colon N \times \mathfrak{F} \to G$ maps $N \times V$ homeomorphically onto an open identity neighborhood contained in U.*

(3) *There is a filter basis $\mathcal{N}$ of compact subgroups N possessing property $\mathcal{P}$ such that $\bigcap \mathcal{N} = \{1\}$ and for each $N \in \mathcal{N}$, the quotient G/N is a (finite dimensional) linear Lie group.*

Proof. (1)⇒(2) Assume (1) and let U be an identity neighborhood; we must find N and $\mathfrak{F}$ so that (2) holds. Using 7.66(iv) we choose a compact subgroup K and a finite dimensional vector subspace $\mathfrak{F}$ of $\mathfrak{L}(G)$, together with an open euclidean ball V in $\mathfrak{F}$ around 0 such that $W \stackrel{\mathrm{def}}{=} K \exp V \subseteq U$ and $(n, x) \mapsto n \exp X\colon K \times V \to W$

is a homeomorphism onto an open neighborhood of the identity in G. By (i), the zero neighborhood W contains an open neighborhood U' having property $\mathcal{P}$. The projection $\mathrm{pr}_1: W \to K$ onto the first factor in the product $K \exp V$ is continuous and open. Hence $\mathrm{pr}_1(U')$ is an identity neighborhood of K, which, because of hypothesis (a) has property $\mathcal{P}$. Then the subgroup $N \stackrel{\mathrm{def}}{=} \langle \mathrm{pr}_1(U') \rangle$ has property $\mathcal{P}$ by hypothesis (b). Thus $W' \stackrel{\mathrm{def}}{=} N \exp V$ is an open identity neighborhood in the product $W = K \exp V$ and thus in G. Since $W \subseteq U$ we also have $W' \subseteq U$ and this proves assertion (2).

(2)⇒(3) Let $\mathcal{N}$ denote the set of all compact subgroups occurring in (2). In the circumstances of (2), the morphism $\psi: N \times \mathfrak{F} \to G$, $\psi(n, X) = n \exp X$ has a discrete kernel D projecting onto a discrete subgroup of $\mathfrak{F}$ under the second projection (cf. proof of 7.66(iv)). Thus $(N \times \{0\})D/(N \times \{0\})$ is the discrete kernel of the morphism $\Psi: (N \times \mathfrak{F})/(N \times \{0\}) \to G/N$, $\Psi\big((n, X) + (N \times \{0\})\big) = \psi(n, X)$. Thus G/N is locally isomorphic to the linear Lie group $\mathfrak{F} \cong (N \times \{0\})/(N \times \{0\})$. and is, therefore, a linear Lie group by E5.18. Since $N \subseteq U$ where U was an arbitrary identity neighborhood of G we have $\bigcap \mathcal{N} = \{1\}$.

Finally assume that N_1 and N_2 are in $\mathcal{N}$. The morphism $g \mapsto (gN_1, gN_2)$: $G \to G/N_1 \times G/N_2$ has the kernel $N_1 \cap N_2$. Thus we have an injective morphism $G/(N_1 \cap N_2) \to G/N_1 \times G/N_2$. The linear Lie group $G/N_1 \times G/N_2$ has no small subgroups; hence the locally compact abelian $G/(N_1 \cap N_2)$ has no small subgroups. By 7.58(ii) it is therefore a linear Lie group. By 7.66(iv) we find a compact subgroup $K \subseteq N_1 \cap N_2$ and a finite vector subspace $\mathfrak{F}_K$ of $\mathfrak{L}(G)$ and an open morphism $\psi_K: G_K \stackrel{\mathrm{def}}{=} K \times \mathfrak{F}_K \to G$ with discrete kernel, implementing a local isomorphism. Since G satisfies condition (2), also the locally isomorphic group G_K satisfies condition (2). Moreover we may identify $\mathfrak{L}(G_K)$ and $\mathfrak{L}(G)$ via $\mathfrak{L}(\psi_K)$. Let U be an identity neighborhood. By Condition (2) applied to G_K we pick an identity neighborhood U_K of G_K in $\psi_K^{-1}(U)$ and a compact subgroup N_K of G_K with property $\mathcal{P}$, such that the conclusion of (2) holds with G_K is place of G and with a suitable finite dimensional vector subspace $\mathfrak{F}$ of $\mathfrak{L}(G_K) = \mathfrak{L}(G)$. Now $K \times \{0\}$ is the unique maximal compact subgroup of G_K by 7.58(ii). Hence N_K is necessarily of the form $N \times \{0\}$ with $N \subseteq K$. But then (2) is satisfied with the given U, N, $\mathfrak{F}$ and G and thus $N \in \mathcal{N}$. Since $N \subseteq N_1 \cap N_2$ we have shown that $\mathcal{N}$ is a filter basis.

(3)⇒(1) Let $N \in \mathcal{N}$. Then the quotient morphism $f: G \to H \stackrel{\mathrm{def}}{=} G/N$ has finite dimensional linear Lie group as image. Hence there is a morphism $\sigma: \mathfrak{L}(H) \to \mathfrak{L}(G)$ with $\mathfrak{L}(f) \circ \sigma = \mathrm{id}_{\mathfrak{L}(H)}$ by 7.66(iii). Now let W be an open convex symmetric 0 neighborhood of $\mathfrak{L}(H)$ which is mapped homeomorphically onto an open identity neighborhood V of H. Let $s: V \to G$ be defined by $s = \exp_G \circ \sigma \circ (\exp_H |W)^{-1}$. Let $h \in V$ and set $Y = (\exp_H |W)^{-1} h$ and $X = \sigma(Y)$ so that $\exp_H Y = h$ and $\mathfrak{L}(f)(X) = Y$. Then $f\big(s(h)\big) = f\big(\exp_G(\sigma(\exp_H |W)^{-1}(h)))\big) = f(\exp_G X) = \exp_H \mathfrak{L}(f)(X) = \exp_H = Y = h$. Set $U = Ns(V)$. We claim $U = f^{-1}(V)$. Indeed, firstly, $f(U) = f(N)f(s(V) = V$, giving $U \subseteq f^{-1}(V)$. Secondly, for $g \in f^{-1}(V)$ we set $n = g - s(f(g))$ and get $f(n) = f(g) - f(g) = 0$, i.e. $n \in N$; thus $g = ns\big(f(g)\big) \in Ns(V)$, and the claim is established. Thus U is an identity neighborhood of G, and the function $(n, v) \mapsto n + s(v) \colon N \times V \to U$ is a homeomorphism with inverse $u \mapsto (u - s\big(f(u)\big), f(u))$. By hypothesis (c), the space $N \times V$ and thus the homeomorphic space U has property $\mathcal{P}$ by (a). □

The properties $\mathcal{P}$ = "being connected" and $\mathcal{P}$ = "being arcwise connected" are the ones that interest us; they satisfy hypotheses (a), (b), (c) of Lemma 8.35. For an easy formulation of the following result, we shall call an abelian group A *super-$\aleph_1$-free* if it is torsion-free and for every finite subset F the pure subgroup $[F]$ generated by F (cf. A1.25) is free and $A/[F]$ is a Whitehead group (cf. A1.63(ii)).

Recall from 7.57 and 7.58 that a locally compact connected abelian group G is isomorphic to $\mathbb{R}^n \times \operatorname{comp} G$ for some n and the maximal compact subgroup $\operatorname{comp} G$.

Characterisation of Local Connectivity

Theorem 8.36. (i) *For a connected locally compact abelian group G, the following conditions are equivalent.*

(1) *G is locally connected, respectively, locally arcwise connected.*
(2) *For each identity neighborhood U there is a closed* connected*, respectively,* arcwise connected *subgroup N and a finite dimensional vector subspace $\mathfrak{F}$ of $\mathfrak{L}(G)$ such that for some connected identity neighborhood V in $\mathfrak{F}$ the function $(n, X) \mapsto n \exp X \colon N \times \mathfrak{F} \to G$ maps $N \times V$ homeomorphically onto an open identity neighborhood contained in U.*
(3) *G has arbitrarily small* connected*, respectively,* arcwise connected *compact subgroups N such that G/N is a finite dimensional linear Lie group.*
(4) $(\operatorname{comp} G)\widehat{\ }$ *is $\aleph_1$-free, respectively,* $(\operatorname{comp} G)\widehat{\ }$ *is super-$\aleph_1$-free.*
(5) *Every nondegenerate homomorphic image of* $\operatorname{comp} G$ *whose topology has a countable basis is a (possibly infinite dimensional) torus, respectively, every nondegenerate homomorphic image of* $\operatorname{comp} G$ *whose topology has a countable basis is a (possibly infinite dimensional) torus and the kernel of the homomorphism is arcwise connected.*
(6) *Every finite dimensional quotient group is a torus, respectively, every finite dimensional quotient group is a torus and the kernel of the quotient map is arcwise connected.*

(ii) *A connected locally arcwise connected topological group is arcwise connected. There is a compact connected locally connected abelian group which is not arcwise connected and is a fortiori not locally arcwise connected.*

(iii) *Every arcwise connected locally compact abelian group is locally connected.*

(iv) *For a compact connected abelian group G, the following statements are equivalent:*

(1) *There are arbitrarily small compact connected subgroups N for which there is a finite dimensional torus subgroup T_N of G such that $(n, t) \mapsto n + t : N \times T_N \to G$ is an isomorphism of topological groups.*
(2) *The character group $\widehat{G}$ is the directed union of finitely generated free* split *subgroups.*

(3) $\widehat{G}$ *is an* S-*group* (*see* Definition A1.63(iii)).

Proof. (i) The equivalence of (1), (2), and (3) in both cases is a consequence of Lemma 8.35.

Since $G \cong \mathbb{R}^n \times \operatorname{comp} G$ and $\mathbb{R}^n$ is locally arcwise connected, for the remainder of the proof it is no loss of generality to assume $G = \operatorname{comp} G$, i.e. to assume that G is a compact connected abelian group.

(3)⇔(4) In (3) we state that the compact connected group G has arbitrarily small connected compact subgroups N such that G/N is a compact connected Lie group, i.e. a torus by 2.42(ii). By the Annihilator Mechanism 7.64 this means that $\widehat{G}$ is the directed union of subgroups $N^\perp$ such that $N^\perp \cong (G/N)\widehat{\ }$ is finitely generated free and that $\widehat{G}/N^\perp \cong \widehat{N}$ is torsion-free by 8.5.

$$
\begin{array}{ccccccc}
 & & G & \qquad\qquad & \{0\} & & \\
\widehat{N^\perp} & \cong & \left\{\ \Big| \right. & & \left. \Big|\ \right\} & \cong & \widehat{G/N} \\
 & & N & & N^\perp & & \\
(\widehat{G}/N^\perp)\widehat{\ } & \cong & \left\{\ \Big| \right. & & \left. \Big|\ \right\} & \cong & \widehat{N} \\
 & & \{0\} & & \widehat{G}. & &
\end{array}
$$

Thus $N^\perp$ is pure in $\widehat{G}$ (see A1.24). Therefore (3) is equivalent to saying that $\widehat{G}$ is the directed union of pure finitely generated free subgroups. But this means exactly that $\widehat{G}$ is $\aleph_1$-free. (See A1.63.)

The case of local arcwise connectivity is treated in the same fashion except that here we know that the groups N are arcwise connected. By the Annihilator Mechanism and 8.30(iv) this is tantamount to saying that $\widehat{G}/N^\perp$ is a Whitehead group. This means precisely that $\widehat{G}$ is super-$\aleph_1$-free.

(4)⇔(5) By A1.64, the group $\widehat{G}$ is $\aleph_1$-free if and only if every countable subgroup of $\widehat{G}$ is free. By the Annihilator Mechanism 7.64, a subgroup A of $\widehat{G}$ is countable if and only if the quotient group $G/A^\perp \cong \widehat{A}$ is the character group of a countable discrete abelian group. By the Weight Theorem 7.76 we have $w(G/A^\perp) = w(A) = \operatorname{card} A \le \aleph_0$. Since $\widehat{G}$ is torsion-free, these cardinals are actually equal unless A is singleton. Since every homomorphic image of G is isomorphic to a quotient $G/A^\perp$ for some subgroup of A by the Annihilator Mechanism, the equivalence of (4) and (5) follows.

(4)⇔(6) We apply duality like in the proof of (4)⇔(5), denote the character group of $(\operatorname{comp} G)\widehat{\ }$ by A, and observe that (6) is equivalent to

$(\widehat{6})$ Every finite rank subgroup F of A is free, respectively, every finite rank subgroup F is free and A/F is a Whitehead group.

We note the following facts from Appendix 1:

Fact 1. By Definition A1.63 an abelian torsion free group A is $\aleph_1$-free iff every finite rank pure subgroup is free.

Fact 2. By A1.25, in a torsion free group A every finite rank subgroup F is contained in a pure subgroup $[F]$ of the same rank.

Fact 3. By A1.9, any subgroup of a free abelian group is free.

If every finite rank subgroup of A is free, then trivially, A is $\aleph_1$-free by Fact 1. Conversely, if A is $\aleph_1$-free and F is a finite rank subgroup, then $[F]$ is free by Facts 1 and 2. Then by Fact 3, F is free. Thus the first part of $(\widehat{6})$ is equivalent to the first part of (4) and the second part of $(\widehat{6})$ is equivalent to the second part of (4).

(ii) In any locally arcwise connected topological space, the arc components are open; hence, being equivalence classes, they are also closed. Thus any connected locally arcwise connected space is arcwise connected. Hence the first assertion of (ii) follows.

Let $G \stackrel{\text{def}}{=} \widehat{\mathbb{Z}^{\mathbb{N}}}$. Then G is a compact connected locally connected group which is not arcwise connected. Indeed, by A1.65 we know that $\widehat{G} \cong \mathbb{Z}^{\mathbb{N}}$ is $\aleph_1$-free, but that $\operatorname{Ext}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \neq \{0\}$. Let G_a denote the arc component of the identity. Then $\pi_0(G) = G/G_a \cong \operatorname{Ext}(\widehat{G}, \mathbb{Z}) \neq \{0\}$ by 8.30.

(iii) Again because $G \cong \mathbb{R}^n \times \operatorname{comp} G$ for a connected locally compact abelian group we may assume that G is a compact arcwise connected group. Now let H be a continuous homomorphic image of G and assume that H has a countable basis; then H is also arcwise connected and thus has a countable Whitehead group as character group by Theorem 8.30. Hence by A1.62 $\widehat{H}$ is free. Therefore H is a torus. Now by (5) above we conclude that G is locally connected.

(iv) See [193]. □

A locally compact abelian group is said to be *strongly locally connected* if its identity component is open and its unique maximal compact connected subgroup satisfies the equivalent conditions of Theorem 8.36(iv).

In particular, *a compact connected abelian group is strongly locally connected if and only if its character group is an* S*-group.*

If G is the character group of the discrete group $\mathbb{Z}^{\mathbb{N}}$, then according to Example A1.64, G is a strongly locally connected and connected but not arcwise connected compact abelian group. According to Proposition A1.66(ii), there is a compact connected, locally connected, but not strongly locally connected group H of weight $2^{\aleph_0}$ containing G such that H/G is a circle group. The group G has a metric torus group quotient which is not a homomorphic retract.

The property of being strongly locally connected has a remarkable characterisation in terms of the exponential function.

Characterisation of Strong Local Connectivity

Theorem 8.36bis. *For a compact connected abelian group G and its zero arc-component G_a, the following conditions are equivalent:*

(1) *G is strongly locally connected.*

(2) *The exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ is open onto its image.*

(3) *G_a is locally arcwise connected.*

(4) *$\widehat{G}$ is an* S*-group, that is, every finite rank pure subgroup of $\widehat{G}$ is free and is a direct summand.*

Proof. See [193]. □

As one by-product of this theorem and the example of $G = \widehat{\mathbb{Z}_d^N}$ above based on Example A1.65, we get the following example:

Example. The uncountable product $V \stackrel{\rm def}{=} \mathbb{R}^{\mathbb{R}}$ has a closed totally disconnected algebraically free subgroup K of countable rank such that the quotient V/K is incomplete and its completion G is a compact connected and strongly locally connected abelian group of continuum weight.

In passing we observe the following result one encounters in researching this question and finds to be of independent interest:

Proposition. *Let G be a compact connected group, F a maximal rank free subgroup of $\widehat{G}$, Δ the annihilator of F in G (so that $\widehat{\Delta} \cong \widehat{G}/F$). Then there is an exact sequence*

$$0 \to \pi_1(G) \to \mathbb{Z}^{\operatorname{rank} F} \to \Delta \to \pi_0(G) \to 0.$$

Proof. Apply the *Snake Lemma* [245] to

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{K}(G) & \xrightarrow{\mathfrak{K}(p)} & \mathbb{Z}^{\operatorname{rank} F} & & \\
 & & & & \downarrow{\scriptstyle\rm inc} & & \downarrow{\scriptstyle\rm inc} & & \\
 & & 0 & \longrightarrow & \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(p)} & \mathbb{R}^{\operatorname{rank} F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\exp_G} & & \downarrow{\scriptstyle\exp_{\mathbb{T}^{\operatorname{rank} F}}} & & \\
0 & \longrightarrow & \Delta & \xrightarrow[\rm inc]{} & G & \xrightarrow[p]{} & \mathbb{T}^{\operatorname{rank} F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \\
 & & \operatorname{Ext}(\frac{\widehat{G}}{F}, \mathbb{Z}) & \longrightarrow & \pi_0(G) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

□

A Hausdorff topological group is metric if and only if if and only if the filter of its identity neighborhoods has a countable basis (cf. A4.16). We shall address metric compact abelian groups in greater detail in the next section.

We recall that a torus is a product of circles. For a topological group G let $\mathcal{T}(G)$ denote the set of all torus subgroups of G.

Proposition 8.37. (i) *Every compact abelian group G contains a characteristic arcwise connected subgroup*

$$G_T = \left\langle \bigcup \mathcal{T}(G) \right\rangle = \bigcup \mathcal{T}(G).$$

(ii) *The union of all finite dimensional torus subgroups is* $\exp \operatorname{span}_{\mathbb{R}} \mathfrak{K}(G)$ *and is dense in* G_T.

Proof. (i) We claim that the collection $\mathcal{T}(G)$ of all torus subgroups of a compact abelian group G is directed, i.e. the product of two torus subgroups is a torus. Indeed the family $\mathcal{T}(\widehat{G})$ of all subgroups T of $\widehat{G}$ such that $\widehat{G}/T$ is free is a filter basis by the remarks preceding Appendix 1, Lemma A1.27. The claim is simply the dual of this fact.

As a consequence of the claim the union of all torus subgroups is a group, and it is obviously arcwise connected and mapped into itself by all continuous endomorphisms of G.

(ii) Let $g = \exp X$ with $X = \sum_{j=1}^{N} r_j{\cdot}X_j$, $r_j \in \mathbb{R}$, $X_j \in \mathfrak{K}(G) = \ker \exp_G$. The finite dimensional real vector space $V \overset{\text{def}}{=} \mathbb{R}{\cdot}X_1 + \cdots + \mathbb{R}{\cdot}X_N$ is the real span of the closed subgroup $D \overset{\text{def}}{=} V \cap \mathfrak{K}(G)$ which does not contain a vector subspace. Hence by the Theorem on Closed Subgroups of $\mathbb{R}^n$ in Appendix 1, A1.12 there is a basis $Y_1, \ldots, Y_n$ of V such that $V = \mathbb{R}{\cdot}Y_1 \oplus \cdots \oplus \mathbb{R}{\cdot}Y_n$ and $V \cap \mathfrak{K}(G) = \mathbb{Z}{\cdot}Y_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}Y_n$. Hence $e\colon \mathbb{R}^n \to \exp_G V$, $e(r_1, \ldots, r_n) = \exp_G(r_1{\cdot}Y_1 + \cdots + r_n{\cdot}Y_1)$ yields an exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{\text{incl}} \mathbb{R}^n \xrightarrow{e} \exp_G V \to 0,$$

whence $\exp_G V \cong \mathbb{R}^n/\mathbb{Z}^n$ is a finite dimensional torus containing g.

Conversely, let T be a finite dimensional torus in G. Then $\mathfrak{L}(T) \subseteq \mathfrak{L}(G)$ with an obvious identification of $\operatorname{Hom}(\mathbb{R}, T)$ with a vector subspace of $\operatorname{Hom}(\mathbb{R}, G)$. Since $T \cong \mathbb{R}^n/\mathbb{Z}^n$, we have $\mathfrak{L}(T) = \operatorname{span}_{\mathbb{R}} \mathfrak{K}(T) \subseteq \operatorname{span}_{\mathbb{R}} \mathfrak{K}(G)$. Hence $T = \exp_T \mathfrak{L}(T) \subseteq \exp_G \operatorname{span}_{\mathbb{R}} \mathfrak{K}(G)$.

Since for any set J, the torus $\mathbb{T}^J$ is the closure of the union of the family of all finite partial products it follows that the union of all finite dimensional tori is dense in G_T. □

Proposition 8.37 implies that the smallest closed subgroup

$$G_\ell \overset{\text{def}}{=} \overline{\bigcup \mathcal{T}(G)}$$

of G containing all torus subgroups of G is a fully characteristic closed connected subgroup of G. (Recall that a subgroup of a topological group is fully characteristic if it is mapped into itself by all continuous endomorphisms.)

As is pointed out in Appendix 1, in the paragraph preceding A1.27, each abelian group A contains a characteristic subgroup $K_\infty(A)$ which is the intersection of all subgroups K such that A/K is free.

Theorem 8.38 (Characteristic Locally Connected Subgroup). *In every abelian topological group G with compact identity component G_0, the fully characteristic subgroup G_ℓ is the unique smallest closed subgroup containing all torus subgroups, and it is locally connected. If G is locally compact, then the annihilator of G_ℓ in $\widehat{G}$ is $K_\infty(\widehat{G})$.*

There exists an example of a locally connected connected compact abelian group G such that G/G_ℓ is a circle.

Proof. Since G_T is arcwise connected, we have $G_T \subseteq G_a \subseteq G_0$. We assume that G_0 is compact. It is therefore no loss of generality for the proof to assume that G is compact so that duality theory becomes available. By Proposition A1.66, the discrete abelian group $\widehat{G}$ contains a characteristic pure subgroup $K_\infty(\widehat{G})$ such that $\widehat{G}/K_\infty(\widehat{G})$ is $\aleph_1$-free. The group $K_\infty(\widehat{G})$ is the intersection of the kernels of all morphisms $\widehat{G} \to \mathbb{Z}$. By duality (see 7.64(v)), the annihilator $K_\infty(\widehat{G})^\perp$ is exactly the smallest closed subgroup of G containing all torus subgroups and is therefore exactly G_ℓ.

Since $G_\ell \cong \big(\widehat{G}/K_\infty(\widehat{G})\big)\widehat{\ }$ by the Annihilator Mechanism 7.64, we know that the character group of G_ℓ is $\aleph_1$-free. Therefore, G_ℓ is locally connected by Theorem 8.35. By A1.66(ii) there exists an example A of an $\aleph_1$-free group A such that $K_\infty(A) \cong \mathbb{Z}$. Dually, this yields a compact, connected and locally connected example $G = \widehat{A}$ such that $G/G_\ell \cong K_\infty(A)\widehat{\ } \cong \mathbb{T}$. ☐

In the example constructed in 8.36(ii), we have $G_\ell = G$ and we see that G_ℓ need not be a torus. On the other hand, Theorem 8.38 says, in particular, that the closure G_ℓ of the union G_T of all torus subgroups of G is locally connected—a fact which is not at all obvious a priori.

Corollary 8.39. *Let G be a connected locally compact abelian group. Then G contains connected locally connected subgroups of the form $H = E \oplus \big(\mathrm{comp}(G)\big)_\ell$ for a vector subgroup E, where $\big(\mathrm{comp}(G)\big)_\ell$ contains all torus subgroups and maps onto $G/\mathrm{comp}(G)$.*

Proof. Exercise E8.7. ☐

Exercise E8.7. Prove 8.39.

[Hint. By the Vector Group Splitting Theorem 7.57 we have $G = E \oplus \mathrm{comp}(G)$ with the maximal compact subgroup $\mathrm{comp}(G)$ and some vector subgroup E. Then $E \oplus \big(\mathrm{comp}(G)\big)_\ell$ is locally connected, contains all torus subgroups, and maps onto $G/\mathrm{comp}(G)$.] ☐

Notice that in the locally compact but noncompact case, there may be no unique maximal locally connected connected closed subgroup. For example, let $G = \mathbb{R} \times \widehat{\mathbb{Q}}$ with the discrete group $\mathbb{Q}$ of rationals. Then $\mathrm{comp}(G) = \{0\} \times \widehat{\mathbb{Q}}$ and $\big(\mathrm{comp}(G)\big)_\ell$ is singleton. Let $X \in \mathfrak{L}(\widehat{\mathbb{Q}})$. Then $E = \{(r, \exp r{\cdot}X) \mid r \in \mathbb{R}\}$ and

$\mathbb{R}\times\{0\}$ are both maximal locally connected connected closed subgroups which are different if $X \neq 0$.

Despite this remark, for *locally compact* abelian groups we fix the following notation:

Definition 8.40. In a locally compact abelian group G in which G_0 is compact, G_ℓ is called the *locally connected component* (of the identity) of G. □

An attentive reader may have noticed that a group by the name of G_ℓ was introduced in the Exponential Function for Locally Compact Abelian Groups Theorem 7.66. We have to make sure at this time that both definitions agree. Theorem 8.41(ii) below will prove this.

We recall that $\mathfrak{K}(G)$ denotes the kernel $\ker\exp_G$ of the exponential function and that it is isomorphic to $\mathrm{Hom}(\widehat{G},\mathbb{Z})$. Further recall the adjoint $\varepsilon\colon \widehat{G}\to\widehat{\mathfrak{L}}(G)\cong\mathfrak{L}'(G)$ of the exponential map, and that $\mathfrak{L}'(G)\cong\mathbb{R}\otimes\widehat{G}$ with the finest locally convex topology (as was pointed out in remarks preceding 7.35). By 7.66, $\mathfrak{K}(G)$ is the annihilator of $1\otimes\widehat{G}\subseteq\mathbb{R}\otimes\widehat{G}$ and $\mathfrak{K}(G)^\perp = (1\otimes\widehat{G})^{\perp\perp}$.

The Locally Connected Component of a Compact Abelian Group

Theorem 8.41. *Let G be a compact abelian group. Then the locally connected component G_ℓ has the following properties.*

(i) $\mathfrak{L}(G_\ell)=\overline{\mathrm{span}_{\mathbb{R}}\,\mathfrak{K}(G)}$.

(ii) *The annihilator ${G_\ell}^\perp$ of G_ℓ in $\widehat{G}$ is*

$$K_\infty(\widehat{G})=\bigcap\{\ker f\mid f\in\mathrm{Hom}(\widehat{G},\mathbb{Z})\}=\bigcap\{\ker F\mid F\in\mathrm{Hom}(\widehat{G},\mathbb{R}),\ F(\widehat{G})\subseteq\mathbb{Z}\}.$$

Therefore $\widehat{G_\ell}\cong\widehat{G}/K_\infty(\widehat{G})$.

(iii) *Let* $\mathbb{R}\otimes\widehat{G}\cong\mathrm{Hom}(\mathfrak{L}(G),\mathbb{R})\cong\big(\mathfrak{L}(G)\big)\widehat{\ }$ *be the vector space dual* $\mathfrak{L}'(G)$ *of* $\mathfrak{L}(G)$ *(see* 7.66(i)*) and let* $(\mathfrak{K}(G)^\perp)_0$ *denote the largest vector subspace of the group* $\mathfrak{K}(G)^\perp\subseteq\mathbb{R}\otimes\widehat{G}$ *which is also the largest vector subspace of* $\overline{1\otimes\widehat{G}}$*. Then* $\mathfrak{K}(G)^\perp/(\mathfrak{K}(G)^\perp)_0$ *is the image of the character group of* $\mathfrak{L}(G_\ell)/\mathfrak{K}(G)$ *under a continuous bijective morphism, and is an* $\aleph_1$*-free group.*

The factor group $\overline{1\otimes\widehat{G}}/\big(\mathfrak{K}(G)^\perp\big)_0$ *contains densely an algebraically isomorphic copy of* $\widehat{G_\ell}$.

$$
\begin{array}{ccccccc}
 & & & & & & \{0\} \\
 & & & & & & | \\
\mathfrak{L}(G) & & \{0\} & & & & \operatorname{tor}\widehat{G} \\
| & & | & & & & | \\
\mathfrak{L}(G_\ell)=\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)} & & \big(\mathfrak{K}(G)^\perp\big)_0 & & & & K_\infty(\widehat{G}) \\
| & & | & \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \leftarrow & \widehat{G_\ell} & \cong \left\{\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right. & | \\
\mathfrak{K}(G) & & \overline{1\otimes\widehat{G}} & & & & \widehat{G} \\
| & & | & & & & \\
\{0\} & & \mathbb{R}\otimes\widehat{G} & & & &
\end{array}
$$

(iv) *The union of all finite dimensional torus subgroups is dense in G if and only if the group G_T is dense in G if and only if G is connected and* $\operatorname{span}\mathfrak{K}(G)$ *is dense in $\mathfrak{L}(G)$.*

Proof. (i) and (ii). Let us recall that for a compact abelian group G we have an isomorphism $X \mapsto \widehat{X}\colon \mathfrak{L}(G) = \operatorname{Hom}(\mathbb{R}, G) \to \operatorname{Hom}(\widehat{G}, \mathbb{R})$ where $\widehat{\mathbb{R}}$ is identified in the obvious fashion with $\mathbb{R}$, and that, under this isomorphism, $\mathfrak{K}(G) = \ker\exp$ is mapped to $\operatorname{Hom}(\widehat{G}, \mathbb{Z})$, identified in the obvious way with a subgroup of $\operatorname{Hom}(\widehat{G}, \mathbb{R})$ (see 7.66). Now we note that $\mathfrak{L}(G_\ell) = \operatorname{Hom}(\mathbb{R}, G_\ell) \cong \operatorname{Hom}(\widehat{G_\ell}, \mathbb{R}) \cong \operatorname{Hom}\big((\widehat{G}/K_\infty(\widehat{G})\big), \mathbb{R})$. In other words, if we identify $\mathfrak{L}(G_\ell)$ with a vector subspace of $\mathfrak{L}(G)$ we have

$$\mathfrak{L}(G_\ell) = \{X \in \mathfrak{L}(G) : \widehat{X}(K_\infty(\widehat{G})) = \{0\}\}.$$

Now

$$
\begin{aligned}
K_\infty(\widehat{G}) &= \bigcap\{\ker f \mid f \in \operatorname{Hom}(\widehat{G}, \mathbb{Z})\} \\
&= \{\chi \in \widehat{G} \mid \big(\forall \widehat{X} \in \operatorname{Hom}(\widehat{G}, \mathbb{R})\big)\,(\operatorname{im}\widehat{X} \subseteq \mathbb{Z}) \Rightarrow (\widehat{X}(\chi) = 0)\} \\
&= \{\chi \in \widehat{G} \mid \big(\forall X \in \mathfrak{K}(G)\big)\,(\widehat{X}(\chi) = 0)\}.
\end{aligned}
$$

Remembering that $\widehat{X}(\chi) \in \mathbb{R}$ is to be considered as an element of $\widehat{\mathbb{R}}$ via $\langle \widehat{X}(\chi), r\rangle = \widehat{X}(\chi)r + \mathbb{Z} \in \mathbb{T}$ for all $r \in \mathbb{R}$, we see that $\widehat{X}(\chi) = 0$ means that $\chi(\exp_G r{\cdot}X) = \chi\big(X(r)\big) = \langle \widehat{X}(\chi), r\rangle = 0$ for all $r \in \mathbb{R}$. We also recall the commutative diagram (involving the quotient morphism $q\colon \mathbb{R} \to \mathbb{T}$)

$$
(*)\qquad
\begin{array}{ccccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\chi)} & \mathfrak{L}(T) & \cong & \mathbb{R} \\
{\scriptstyle \exp_G}\big\downarrow & & {\scriptstyle \exp_{\mathbb{T}}}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
G & \xrightarrow[\chi]{} & \mathbb{T} & = & \mathbb{T}.
\end{array}
$$

This means that $\chi(\exp_G r{\cdot}X) = \exp_{\mathbb{T}}\big(r{\cdot}\mathfrak{L}(\chi)(X)\big)$. Now $r{\cdot}\mathfrak{L}(\chi)(X) \in \ker\exp_T$ for all $r \in \mathbb{R}$ holds iff $\mathfrak{L}(\chi)(X) = 0$. Thus

$$(\dagger)\qquad K_\infty(\widehat{G}) = \{\chi \in \widehat{G} \mid \big(\forall X \in \mathfrak{K}(G)\big)\,\mathfrak{L}(\chi)(X) = 0\}.$$

From $G_\ell = K_\infty(\widehat{G})^\perp = \bigcap_{\chi\in K_\infty(\widehat{G})}\ker\chi$ we conclude

$$(\ddagger)\qquad \mathfrak{L}(G_\ell) = \bigcap_{\chi\in K_\infty(\widehat{G})}\mathfrak{L}(\ker\chi) = \bigcap_{\chi\in K_\infty(\widehat{G})}\ker\mathfrak{L}(\chi)$$

since $\mathfrak{L}$ commutes with the formation of intersections and kernels by 7.38.

From (†) and (‡) we derive that

$$\mathfrak{L}(G_\ell) = \{X \in \mathfrak{L}(G) \mid \big(\forall\chi \in \widehat{G},\, Y \in \mathfrak{K}(G)\big)\quad \big(\mathfrak{L}(\chi)(Y) = 0\big)\Rightarrow\big(\mathfrak{L}(\chi)(X) = 0\big)\}.$$

The dual $\mathfrak{L}'(G) = \operatorname{Hom}(\mathfrak{L}(G),\mathbb{R})\big(\cong \mathbb{R}\otimes\widehat{G}\big)$ is the linear span of the set $\{\mathfrak{L}(\chi) \in \mathfrak{L}'(G) \mid \chi \in \widehat{G}\}\big(\cong 1\otimes\widehat{G}\big)$. Hence $\mathfrak{L}(G_\ell)$ consists of all $X \in \mathfrak{L}(G)$ such that for all $\omega \in \mathfrak{L}(G)'$ which are in the vector space annihilator $\mathfrak{K}(G)^\perp$ we have $\omega(X) = 0$. Thus $\mathfrak{L}(G_\ell) = \mathfrak{K}(G)^{\perp\perp}$ in the sense of annihilators of reflexive topological vector spaces. But $\mathfrak{K}(G)^{\perp\perp} = \overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}$ by the Annihilator Mechanism for subsets of locally convex topological vector spaces (cf. 7.35). Thus $\mathfrak{L}(G_\ell) = \overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}$ as asserted. In Theorem 7.66(ii) we noted that the character group of $\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}/\mathfrak{K}(G)$ is the $\aleph_1$-free group $\overline{1\otimes\widehat{G}}/\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}^{\perp}$. This concludes the proofs of the assertions concerning $\mathfrak{L}(G)$ and $\mathfrak{L}'(G)$

By the Annihilator Mechanism for Locally Compact Groups 7.64, since the annihilator of G_ℓ in $\widehat{G}$ is $K_\infty(\widehat{G})$ by 8.38, we have $\widehat{G_\ell} \cong \widehat{G}/K_\infty(\widehat{G})$.

(iii) By 7.33(iii) there is a bijective morphism from the character group of $\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}/\mathfrak{K}(G)$ onto the $\aleph_1$-free group $\frac{\mathfrak{K}(G)^\perp}{(\mathfrak{K}(G)^\perp)_0}$.

We shall now prove that $\mathbb{R}\otimes\widehat{G}/\overline{1\otimes\widehat{G}}$ contains a dense isomorphic copy of $\widehat{G_\ell}$. For this purpose we consider the map $\varepsilon\colon\widehat{G} \to \mathbb{R}\otimes\widehat{G}$ given by $\varepsilon(\chi) = 1\otimes\chi$. We recall that its kernel is $\operatorname{tor}\widehat{G}$. (Cf. Appendix 1. A1.45(iii).) It is no loss of generality to assume that G is connected, i.e. that $\operatorname{tor}\widehat{G} = \{0\}$. In that case, the image $\varepsilon(\widehat{G})$ is an algebraically isomorphic copy of $\widehat{G}$ in $\mathbb{R}\otimes\widehat{G}$. It is dense in $\overline{\varepsilon(\widehat{G})} \cong \overline{1\otimes\widehat{G}}$. So we induce a morphism $\varepsilon'\colon\widehat{G} \to \overline{1\otimes\widehat{G}}/V$ where $V = (\overline{1\otimes\widehat{G}})_0$ is the largest vector subspace contained in $\overline{1\otimes\widehat{G}} \subseteq \mathbb{R}\otimes\widehat{G}$ and where $\varepsilon'(\chi) = 1\otimes\chi + V$. We would like to show that ε' is injective. We note that V is divisible, and since a closed abelian divisible subgroup of the topological vector space $\mathbb{R}\otimes\widehat{G}$ is a real vector subspace, V is the unique largest divisible subgroup of $\overline{1\otimes\widehat{G}}$. Since $\overline{1\otimes\widehat{G}}/V$ is $\aleph_1$-free, on the one hand we have $K_\infty(\overline{1\otimes\widehat{G}}) \subseteq V$; since V is divisible, the reverse inclusion holds on the other. Hence $V = K_\infty(\overline{1\otimes\widehat{G}})$. We claim that $V\cap(1\otimes\widehat{G}) = 1\otimes K_\infty(\widehat{G}) = K_\infty(1\otimes\widehat{G})$. By the definition of $K_\infty(A)$ (as the intersection of all $\ker\varphi$, $\varphi \in \operatorname{Hom}(A,\mathbb{Z})$ clearly the right side is in the left one. Now let $1\otimes\chi \in (1\otimes\widehat{G})\setminus K_\infty(1\otimes\widehat{G})$. Then there is a morphism of abelian groups $\varphi\colon 1\otimes G \to \mathbb{Z}$ such that $\varphi(1\otimes\chi)\neq 0$. It extends in a unique fashion to an $\mathbb{R}$-linear map $\Phi\colon \mathbb{R}\otimes\widehat{G}\to\mathbb{R}$. This map is continuous by 7.25(i). Hence the subgroup $\Phi^{-1}(\mathbb{Z})$ is closed and thus, containing $1\otimes\widehat{G}$, it contains $\overline{1\otimes\widehat{G}}$. This means that $\varphi\colon 1\otimes\widehat{G}\to\mathbb{Z}$ extends continuously to a morphism $\overline{\varphi}\colon \overline{1\otimes\widehat{G}}\to\mathbb{Z}$. Now $\overline{\varphi}\in\mathrm{Hom}(\overline{1\otimes\widehat{G}},\mathbb{Z})$ and $\overline{\varphi}(1\otimes\chi)=\varphi(1\otimes\chi)\neq 0$. Hence $1\otimes\chi\neq K_\infty(\overline{1\otimes\widehat{G}})=V$. This proves the claim. But now $\varepsilon'\colon\widehat{G}\to\overline{1\otimes\widehat{G}}/V$ is injective as asserted.

(iv) By 8.37(ii), the union G_T of all tori in G is dense in G if and only if the union $\exp\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)$ of all finite dimensional tori in G is dense in G.

By definition, $G_\ell=\overline{G_T}$ and thus G_T is dense in G iff $G_\ell=G$. This implies $\mathfrak{L}(G_\ell)=\mathfrak{L}(G)$, and this condition implies $G_\ell=\overline{\exp_{G_\ell}\mathfrak{L}(G_\ell)}=\overline{\exp_G\mathfrak{L}(G)}=G_0$. If G is connected then this implies $G_\ell=G$. Thus G_T is dense in G iff $\mathfrak{L}(G_\ell)=\mathfrak{L}(G)$. By (i) above $\mathfrak{L}(G_\ell)=\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}$. This proves the assertion. ☐

Theorem 8.41 may be viewed as another complement to Theorem 7.66 on the exponential function. It clarifies rather explicitly the nature of the locally connected component G_ℓ in its relation to the exponential function and its kernel.

For further clarification of the locally connected component we now assume $G=G_\ell$ and $\mathfrak{L}(G)=\overline{\operatorname{span}_{\mathbb{R}}\mathfrak{K}(G)}$, and, correspondingly, $(\overline{1\otimes\widehat{G}})_0=\{0\}$.

Corollary 8.42. *The exponential function defines a canonical decomposition diagram*

$$\begin{array}{ccc}
\mathfrak{L}(G_\ell) & \xrightarrow{\exp_{G_\ell}} & G_\ell \\
{\scriptstyle\text{quot}}\big\downarrow & & \big\uparrow{\scriptstyle\text{incl}} \\
\mathfrak{L}(G_\ell)/\mathfrak{K}(G) & \xrightarrow[e]{} & \exp_{G_\ell}\mathfrak{L}(G_\ell).
\end{array}$$

Consider the following conditions.

(i) $\mathfrak{L}(G_\ell)/\mathfrak{K}(G)$ *is compact,*

(ii) G_ℓ *is arcwise connected, and* $\exp_{G_\ell}$ *is open,*

(iii) $1\otimes\widehat{G_\ell}$ *is discrete.*

Then (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii), *and if* $\mathfrak{L}(G_\ell)/\mathfrak{K}(G)$ *is reflexive, these conditions are all equivalent.*

Proof. (i)$\Leftrightarrow$(ii) If (i) is satisfied, then e is an isomorphism of abelian topological groups. Hence $\exp_G\mathfrak{L}(G_\ell)$ is compact on the one hand, but is also dense in G_ℓ by 7.71. Hence $\exp_G$ is surjective and open. By 8.32, the group G_ℓ is arcwise connected.

Conversely, if (ii) holds, then $\exp_G\mathfrak{L}(G_\ell)=G_\ell$ and $e\colon\mathfrak{L}(G_\ell)/\mathfrak{K}(G)\to G_\ell$ is an isomorphism of abelian topological groups. Hence (i) follows.

(i)$\Rightarrow$(iii) We apply the Annihilator Mechanism 7.17 with $\mathbb{R}\otimes\widehat{G}$ in place of G and $\overline{1\otimes\widehat{G_\ell}}$ in place of H. Then $\mathfrak{L}(G_\ell)$ and $\mathfrak{K}(G)$ take on the roles of $\widehat{G}$ and $H^\perp$ in 7.17, respectively, and since $\mathfrak{L}(G_\ell)/\mathfrak{K}(G)$ is compact, this group is reflexive. The characters of $(\mathbb{R}\otimes G_\ell)/\overline{1\otimes G_\ell}$ separate the points (see 7.33(ii) or directly in the proof of 7.33 the verification that the characters of E/H separate the points). Thus by 7.17(iv) there is an isomorphism of topological groups $(\mathfrak{L}(G_\ell)/\mathfrak{K}(G))\widehat{\ } \to 1 \otimes \widehat{G_\ell}$. Since the character group of a compact group is discrete, (iii) is established.

(iii)⇒(i) Now assume that $1 \otimes \widehat{G_\ell}$ is discrete. Then by 7.17(i) there is a bijective morphism $\big(\mathfrak{L}(G_\ell)/\mathfrak{K}(G)\big)\widehat{\ } \to 1 \otimes \widehat{G_\ell}$. It follows that the factor group $\mathfrak{L}(G_\ell)/\mathfrak{K}(G)$ has a discrete character group. If it is reflexive, it is compact. □

Compact Metric Abelian Groups

We know exactly when a topological group admits a left invariant metric which defines the topology (see Appendix 4, A4.16). We apply this in the compact situation.

Proposition 8.43. *For a compact group G, the following statements are equivalent.*

(i) *The topology of G has a countable basis.*
(ii) *The filter of identity neighborhoods has a countable basis.*
(iii) *The topology of G is defined by a biinvariant metric.*
(iv) *The topology of G is defined by a metric.*

Proof. A compact metric space always has a countable basis for its topology; thus (iv)⇒(i) trivially, (iii)⇒(iv) and (i)⇒(ii). The implication (ii)⇒(iii) is proved in Appendix 4, A4.19. □

Definition 8.44. A compact group satisfying the equivalent conditions of 8.43 will be called a *compact metric group.* □

Compact metric abelian groups have certain special properties which we discuss in this section.

Theorem 8.45 (Characterisation of Compact Metric Abelian Groups). *For a compact abelian group, the following statements are equivalent.*

(i) *G is a compact metric group.*
(ii) *$\widehat{G}$ is countable.*
(iii) *G is isomorphic to a subgroup of the torus $\mathbb{T}^{\aleph_0}$.*
(iv) *G is isomorphic to a quotient group of $(\widehat{\mathbb{Q}})^{\aleph_0} \times \prod_{p\,\mathrm{prime}}(\mathbb{Z}_p)^{\aleph_0}$.*

Proof. (i)⇔(ii) By 8.43, (i) is equivalent to $w(G) \le \aleph_0$. By the Weight of Locally Compact Abelian Groups Theorem 7.76 this in turn is equivalent to $\operatorname{card}\widehat{G} = w(\widehat{G}) \le \aleph_0$.

(ii)⇔(iii) Condition (iii) by duality is equivalent to $\widehat{G}$ is a homomorphic image of a free group $\mathbb{Z}^{(\aleph)}$, i.e. to the statement that $\widehat{G}$ is the homomorphic image of a countable free group. Using A1.8 we see that is clearly equivalent to (ii).

(Alternatively, use the fact that every compact abelian group is a subgroup of a torus and then apply Proposition A4.20(i).)

(ii)⇔(iv) Every abelian group A is a subgroup of a divisible abelian group D such that A is countable if and only if D is countable. (See Appendix 1, A1.33.) By the Structure Theorem for Divisible Abelian Groups A1.42(ii), a divisible group is countable if and only if it is a subgroup of the countable divisible group $\mathbb{Q}^{(\aleph_0)} \oplus \bigoplus_{p\,\mathrm{prime}} \mathbb{Z}(p^\infty)$. The asserted equivalence is thus established by duality. □

Connectivity in Compact Metric Abelian Groups

Theorem 8.46. (i) *Let G be a compact abelian group and assume that the locally connected component G_ℓ (see Definition 8.40 and Theorem 8.41) is metric. This is the case, in particular, if G is metric. Then G_ℓ is a torus and there is a closed subgroup H such that $G = G_\ell \oplus H$ algebraically and topologically.*

(ii) *The group H does not contain any nondegenerate torus groups, i.e. $H_\ell = \{0\}$.*

(iii) *The following statements are equivalent for a compact metric abelian group:*

(1) *$G = G_\ell$.*
(2) *G is a torus.*
(3) *G is arcwise connected.*
(4) *G is connected and locally connected.*
(5) *The linear span of the kernel $\mathfrak{K}(G)$ of the exponential function of G is dense in $\mathfrak{L}(G)$.*

Proof. (i), (ii) By definition, $G_\ell = K_\infty(\widehat{G})^\perp$. By hypothesis G_ℓ is metric, whence $\widehat{G}/K_\infty(\widehat{G}) \cong \widehat{G_\ell}$ is countable by the Annihilator Mechanism 7.64 and 8.45. Then Appendix 1, A1.66 applies and shows that $\widehat{G} = F \oplus K_\infty(\widehat{G})$ with a countable free summand F and $K_\infty(\widehat{G})$ has no nontrivial free quotient groups. Now let $H = F^\perp$. Then all assertions of (i) and (ii) follow by duality.

(iii) By (i) the equivalence of (1) and (2) is clear, and (2)⇒(3) is trivial. By 8.30(iv), Condition (3) means exactly that $\widehat{G}$ is a Whitehead group.

By Theorem A1.62 from Appendix 1 we know that a *countable* group is a Whitehead group if and only if it is free. Thus (3) is equivalent to saying that $\widehat{G}$ is free and that, by duality, is (2).

By Theorem 8.35 we know that (4) is equivalent to the statement that $\widehat{G}$ is $\aleph_1$-free. By Proposition A1.64 of the Appendix, an abelian group is $\aleph_1$-free if and only if every countable subgroup is free. Since $\widehat{G}$ is countable this shows that (4) is equivalent to the freeness of $\widehat{G}$ and thus to (2).

By 8.37, conditions (1) and (5) are equivalent. □

Let us say that a compact group is *torus free* iff $G_\ell = \{0\}$. Then we can formulate the following corollaries.

Corollary 8.47. (i) *A compact metric abelian group is algebraically and topologically the direct sum of a characteristic maximal torus and a torus free compact subgroup.*

(ii) *For a compact metric abelian group G, the following conditions are equivalent:*

(1) *G is torus free.*
(2) $\mathrm{Hom}(\widehat{G}, \mathbb{Z}) = \{0\}$.
(3) *The exponential function* $\exp_G : \mathfrak{L}(G) \to G$ *is injective.*
(4) $\mathfrak{K}(G) = \{0\}$.
(5) $K_\infty(\widehat{G}) = \widehat{G}$. □

In light of the beautiful results in 8.46 on connectivity in metric compact groups it is now high time to face the question in what measure the hypothesis of metrizability is necessary for their validity. A good deal of information is available quite generally in such theorems as 8.30, 8.35, 8.40. In combination with 8.46, the following proposition is a reasonable statement, and one might well ask, whether it is true.

Torus Proposition. *A compact arcwise connected abelian group is a torus.*

For metric groups this is true as 8.46 shows. Due to a theorem proved by Shelah [329] the quest for verifying the Torus Proposition which was rather intensive in the sixties and early seventies takes the following surprising turn. For the set theoretical axioms mentioned here we refer to Appendix 1.

The Undecidability of the Torus Proposition

Theorem 8.48. (i) *Assume that the axioms of ZFC, Zermelo–Fraenkel Set Theory with the Axiom of Choice, and the Diamond Principle $\diamond$ are valid. Then every compact arcwise connected abelian group is a torus.*

(ii) *Assume the axioms of ZFC, Martin's Axiom, and $\aleph_1 < 2^{\aleph_0}$. Then given any uncountable cardinal $\aleph$ there exists a compact arcwise connected abelian group G of weight $w(G) = \aleph$ which is not a torus.*

(iii) *If ZFC is consistent, then ZFC + Torus Proposition and ZFC + $\neg$Torus Proposition are consistent; i.e. the Torus Proposition is undecidable in ZFC.*

Proof. Let G be a compact connected abelian group. Then by 8.30, G is arcwise connected iff $\widehat{G}$ is a Whitehead group, i.e. satisfies $\mathrm{Ext}(\widehat{G}, \mathbb{Z}) = \{0\}$. On the other hand, G is a torus iff $\widehat{G}$ is free. Thus the Torus Proposition is equivalent to Proposition W of the Section on Whitehead's Problem in Appendix 1, further, by duality, (i) is equivalent to Theorem A1.67, and (ii) is equivalent to Theorem A1.69, while (iii) is equivalent to Theorem A1.70. □

Torus groups are the natural examples of arcwise connected locally connected groups. A quotient of a torus group is a torus group. Any group of dimension $\aleph_1$ which is a Whitehead group but not a torus group (8.48(ii)) has the property that all quotients of properly smaller dimension (namely, $\aleph_0$) are tori. Some remarkable facts arise if the dimension (or, for these large groups, the weight) is a singular cardinal, i.e. one whose cofinality is properly smaller (see Appendix 1, discussion preceding A1.67).

Exercise E8.8. Prove the following assertion.

If G is a compact abelian group whose weight is a singular cardinal, and if each quotient of properly smaller weight is a torus, then G is a torus.

[Hint. Dualize Shelah's Singular Compactness Theorem A1.82.] □

We conclude the section by remarking that a finite dimensional connected compact abelian group is always metric.

Theorem 8.49. *For a finite dimensional compact abelian group G the following conditions are equivalent:*
(i) *G is metric.*
(ii) *G/G_0 is metric.*
In particular, a finite dimensional compact connected abelian group is always metric.

Proof. Assume that $\dim G = n < \infty$. Then by 8.22

$$n = \operatorname{rank} \widehat{G} = \operatorname{rank} \widehat{G}/\operatorname{tor} \widehat{G} = \operatorname{rank} \widehat{G_0}.$$

Thus the torsion-free group $\widehat{G_0}$ is isomorphic to a subgroup of $\mathbb{Q}^n$ and is therefore countable. Thus G_0 is metric by 8.45.

(i)⇒(ii) If G is metric then it satisfies the first axiom of countability. Then G/G_0, as a continuous open image of G, satisfies the first axiom of countability. Hence G/G_0 is metric by A4.16.

(ii)⇒(i) If G/G_0 is metric, then $\operatorname{tor} \widehat{G} \cong (G/G_0)\widehat{\ }$ is countable by 8.45. If $\widehat{G}/\operatorname{tor} \widehat{G} \cong \widehat{G_0}$ is countable and $\operatorname{tor} \widehat{G}$ is countable, then $\widehat{G}$ is countable and thus G is metric. □

Part 3: Aspects of Algebraic Topology—Homotopy

Free Compact Abelian Groups

The study of free objects in the category of compact abelian groups turns out to be especially rich. As in other contexts, every object is a quotient of a free object, for example in the category of groups or in the category of abelian groups (see A1.8). In this case it will follow immediately from the definition that every compact abelian group is a quotient group of a free compact abelian group.

In the investigations of this section we need some function spaces. Let us denote by $C_0(X, G)$, for a topological space X with base point x_0 and an abelian topological group G, the group of all continuous functions $f\colon X \to G$ with $f(x_0) = 0$. This space is a topological group with respect to the pointwise group operations and the topology of uniform convergence. If $C_0(X,G)_0$ denotes, as usual, the identity component of $C_0(X,G)$ we set

$$[X,G] \overset{\text{def}}{=} \frac{C_0(X,G)}{C_0(X,G)_0}.$$

Exercise E8.9. Verify the following proposition.

(i) *If G is a linear Lie group, then $C_0(X,G)$ is a linear Lie group with*

$$\mathfrak{L}\big(C_0(X,G)\big) = C_0\big(X,\mathfrak{L}(G)\big), \text{ and}$$
$$\exp_{C_0(X,G)} = C_0(X,\exp_G)\colon C_0\big(X,\mathfrak{L}(G)\big) \to C_0(X,G).$$

(ii) *The morphism $C_0(X,q)\colon C_0(X,\mathbb{R}) \to C_0(X,\mathbb{T})$ may be identified with the exponential function of the linear Lie group $C_0(X,\mathbb{T})$.*

[Hint. For (i). Let A be a Banach algebra containing G as a closed subgroup of A^{-1} (cf. Chapter 5). Then $C(X,A)$ is a Banach algebra and its exponential function is given as follows. Let $f\colon X \to A$ be an element of $C(X,A)$, then $e^f(x) = e^{f(x)}$. If we identify $\mathfrak{L}(G)$ with a closed Lie subalgebra of A (with the Lie bracket) then each $f \in C_0(X,\mathfrak{L}(G))$ yields a function $e^{r\cdot f}(x) = e^{r\cdot f(x)} \in G$ for all r. Deduce that the exponential function is correctly defined and satisfies the requirements of 5.32. Part (ii) is straightforward.] □

Let G be a linear Lie group. Since the identity component $C_0(X,G)_0$ of the linear Lie group $C_0(X,G)$ is the arc component of the identity, two functions $f, f'\colon X \to \mathbb{T}$ in the same coset modulo $C_0(X,\mathbb{T})$ are connected by an arc. If X is a compact space, then this means precisely that they are homotopic. Thus $[X,\mathbb{T}] = C_0(X,\mathbb{T})/C_0(X,\mathbb{T})_0 = \pi_0\big(C_0(X,\mathbb{T})\big)$ is the group of all homotopy classes of maps $f\colon X \to \mathbb{T}$, also called *Bruschlinsky group* of X. Since in a linear Lie group the connected component of the identity is open, it follows that $[X,G]$ is a discrete abelian group in the quotient topology.

As usual we consider the standard exact sequence

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{q} \mathbb{T} \to 0.$$

It induces a sequence

$$(*) \qquad \{0\} \to C_0(X,\mathbb{Z}) \xrightarrow{C_0(X,j)} C_0(X,\mathbb{R}) \xrightarrow{C_0(X,q)} C_0(X,\mathbb{T}) \xrightarrow{\text{quot}} [X,\mathbb{T}] \to \{0\}.$$

Proposition 8.50. (i) *For any compact pointed topological space X, the sequence $(*)$ is exact.*

(ii) *$C_0(X,\mathbb{T})$ contains a discrete and torsion-free subgroup $B[X,\mathbb{T}] \cong [X,\mathbb{T}]$ such that*

$$C_0(X,\mathbb{T}) = C_0(X,\mathbb{T})_0 \oplus B[X,\mathbb{T}]$$

algebraically and topologically. The identity component $C_0(X,\mathbb{T})_0$ *is the image of the Banach space* $C_0(X,\mathbb{R})$ *under* $C_0(X,q)$ *and is isomorphic to* $\frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Z})}$.

(iii) *If* X *is a compact connected pointed space, then* $C_0(X,\mathbb{T})$ *is isomorphic to* $C_0(X,\mathbb{R})\times[X,\mathbb{T}]$, *and* $[X,\mathbb{T}]$ *is discrete and torsion-free.*

Proof. (i) Exactness at $C_0(X,\mathbb{Z})$ and $C_0(X,\mathbb{R})$ is immediate, but exactness at $C_0(X,\mathbb{T})$ needs comment. For a proof of exactness there we have to show that the image $\operatorname{im} C_0(X,p)$ is the path component $C_0(X,\mathbb{T})_a$ of 0. Since $C_0(X,\mathbb{R})$ is a topological $\mathbb{R}$-vector space, its continuous image $\operatorname{im} C_0(X,q)$ is arcwise connected hence is contained in $C_0(X,\mathbb{T})_a$. Conversely, an element $f\in C_0(X,\mathbb{T})_a$ is homotopic to 0. Let $\operatorname{cone}(X)$ denote the cone $\big([0,1]\times X\big)/\big(\{0\}\times X\big)$ (i.e. the compact arcwise connected space obtained from $[0,1]\times X$ upon collapsing the compact subspace $\{0\}\times X$ to a point), and let $i: X\to\operatorname{cone}(X)$ be defined by $i(x)=\operatorname{class}(1,x)$, where $\operatorname{class}(r,x)$ is the equivalence class of the element $(r,x)\in[0,1]\times X$ in the cone. Saying that f is homotopic to zero is tantamount to saying that there exists a continuous function $F:\operatorname{cone}(X)\to\mathbb{T}$ with $f=F\circ i$. Since $\operatorname{cone}(X)$ is contractible and hence simply connected, we have a lifting $\widetilde{F}:\operatorname{cone}(X)\to\mathbb{R}$ across $q:\mathbb{R}\to\mathbb{T}$ with $q\circ\widetilde{F}=F$ (see A2.9).

Then set $\varphi=\widetilde{F}\circ i: X\to\mathbb{R}$. Then $C_0(X,q)(\varphi)=q\circ\widetilde{F}\circ i=F\circ i=f$. Thus $f\in\operatorname{im} C_0(X,q)$. This proves the claim

$$C_0(X,\mathbb{T})_a=\operatorname{im} C_0(X,q).$$

(ii) If we identify $C_0(X,\mathbb{Z})$ in the obvious fashion with a subgroup of $C_0(X,\mathbb{R})$, then this subgroup is discrete, because the set of all functions $f: X\to\mathbb{R}$ in $C_0(X,\mathbb{R})$ with $\|f\|=\sup\{|f(x)| : x\in X\}\le\frac{1}{2}$ meets $C_0(X,\mathbb{Z})$ in $\{0\}$. By the exactness of $(*)$ we know that $\operatorname{im} C_0(X,q)\cong C_0(X,\mathbb{R})/C_0(X,\mathbb{Z})$. As a quotient of the $\mathbb{R}$-vector space $C_0(X,\mathbb{R})$, this group is divisible, and hence splits in $C_0(X,\mathbb{T})$ by Proposition A1.36. Since $C_0(X,\mathbb{T})_0$ is open, the splitting is also topological.

It remains to show that $C_0(X,\mathbb{T})/C_0(X,\mathbb{T})_0$ is torsion-free. Let $f: X\to\mathbb{T}$ be a continuous base point preserving map such that for some nonnegative integer n one has $n{\cdot}f\in C_0(X,\mathbb{T})_0$. By 8.50 there is an $F: X\to\mathbb{R}$ such that $n.f=qF$. Set $g=\frac{1}{n}.F$. Then $n{\cdot}(f-qg)=0$ and $f+C_0(X,\mathbb{T})_0=(f-qg)+C_0(X,\mathbb{T})_0$. We assume henceforth that $n{\cdot}f=0$ in $C_0(X,\mathbb{T})$. Thus $f\in C_0\left(X,\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right)$. Let $C(n)\overset{\text{def}}{=}\left(\frac{1}{n}\mathbb{Z}\right)/\mathbb{Z}$. For $z\in C(n)$ let $U_z=f^{-1}(z)$. Then X is the disjoint sum of the compact open sets U_z and $f=\sum_{z\in C(n)} f_z$ where

$$f_z(x)=\begin{cases} z & \text{if } x\in U_z,\\ 0 & \text{otherwise.}\end{cases}$$

Now we define $F_z: X\to\mathbb{R}$ for $z=m/n+\mathbb{Z}$, $m=0,\dots,n-1$ by

$$F_z(x)=\begin{cases} m/n & \text{if } x\in U_z,\\ 0 & \text{otherwise.}\end{cases}$$

Then $f_z=q\circ F_z$, and if $F=\sum_{z\in C(n)} F_z$, then $f=q\circ F$. Thus $f\in q\circ C_0(X,\mathbb{R})=C_0(X,\mathbb{T})_0$ (by the preceding results). Thus $f+C_0(X,\mathbb{T})_0=0$.

(iii) If X is also connected, then $C_0(X,\mathbb{Z}) = \{0\}$, and the assertion follows from (ii). □

In algebraic topology one proves that $[X,\mathbb{T}]$ is naturally isomorphic to the Čech cohomology group $H^1(X,\mathbb{Z})$. The space $K(\mathbb{Z},1)$ underlying $\mathbb{T}$ has the property that

$$\pi_n(K(\mathbb{Z},1),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=1, \text{ and} \\ \{0\} & \text{otherwise.} \end{cases}$$

It is a very simple *Eilenberg–MacLane space* (see e.g. [338], p. 424ff., p. 428, Theorem 10. For the fact that this isomorphism continues to be valid for Čech cohomology on paracompact spaces, see [206]).

Exercise E8.10. Show that

$$\pi_1(\mathbb{S}^1) = [\mathbb{S}^1,\mathbb{T}] \cong \mathbb{Z}.$$

The isomorphism may be implemented by the function $\varphi\colon \mathbb{Z} \to [K(\mathbb{Z},1),\mathbb{T}]$, $\varphi(n) = \mu_n + C_0(K(\mathbb{Z},1),\mathbb{T})_0$, $\mu_n\colon \mathbb{T} \to \mathbb{T}$, $\mu_n(t) = n{\cdot}t$.

[Hint. Each function $f \in C_0([0,1],\mathbb{T})$ $f(0) = 0$ has a unique lifting to a continuous function $\widetilde{f}\colon [0,1] \to \mathbb{R}$ since $[0,1]$ is simply connected. (See A2.7.) Let $p\colon [0,1] \to \mathbb{T}$ be given by $p(t) = t + \mathbb{Z}$. Then we have maps

$$f \mapsto \widetilde{f\circ p}\colon C_0(\mathbb{T},\mathbb{T}) \to C_0([0,1],\mathbb{R}), \text{ and}$$
$$f \mapsto (\widetilde{f\circ p})(1)\colon C_0(\mathbb{T},\mathbb{T}) \to \mathbb{Z}.$$

Show that they are morphisms of abelian topological groups and that the second one induces an isomorphism $[\mathbb{T},\mathbb{T}] \to \mathbb{Z}$ whose inverse is given by $\varphi\colon \mathbb{Z} \to [\mathbb{T},\mathbb{T}]$.

Alternatively, for each member $f \in C_0(\mathbb{S}^1,\mathbb{S}^1)$ find f_1 homotopic to f in $C_0(\mathbb{C}\setminus\{0\},\mathbb{C})$ and define $w\colon C_0(\mathbb{S}^1,\mathbb{S}^1) \to \mathbb{Z}$ by the *winding number*

$$w(f) = \frac{1}{2\pi i}\int_{\text{path f}} \frac{dz}{z} = \int_0^1 f'(e^{2\pi i t})e^{-2\pi i t}\,dt.$$

Show that both approaches yield the same result on piecewise differentiable closed curves. For a more detailed discussion see A2.6] □

Free abelian groups are treated in Appendix 1, A1.4ff. If we accept the philosophy that free objects should be defined in terms of their universal property, then the issue of free compact abelian groups is defined rather straightforwardly as follows:

Definition 8.51. Let X denote an arbitrary topological space with a base point x_0. We say that $F(X)$ is *a free compact abelian group over* X if it is a compact abelian group and there is a continuous base point preserving function $\varepsilon_X\colon X \to F(X)$ such that for every continuous function $f\colon X \to G$ into a compact abelian group with $f(x_0) = 0$, there is a unique morphism of compact abelian groups $f'\colon F(X) \to G$

such that $f = f' \circ \varepsilon_X$. We shall also say that $F(X)$ is *a free compact abelian group with respect to the universal map* $\varepsilon_X : X \to F(X)$.

Proposition 8.52. (i) *Assume that* $\varepsilon_X^{(j)} : X \to F^{(j)}(X)$, $j = 1, 2$ *are universal maps for free compact abelian groups over* X. *Then there is a unique isomorphism* $f : F^{(1)}(X) \to F^{(2)}(X)$ *such that* $\varepsilon^{(2)} = f \circ \varepsilon^{(1)}$.

(ii) *Let* $F(X)$ *be a free compact abelian group over* X *with respect to the universal map* $\varepsilon_X : X \to F(X)$. *Set* $X' \stackrel{\text{def}}{=} \varepsilon_X(X)$. *Then the subgroup* $\langle X' \rangle$ *is dense in* $F(X)$ *and is, as an abelian group, the free abelian group over the set* X'.

Proof. (i) By the universal property there is a morphism $f : F^{(1)}(X) \to F^{(2)}(X)$ such that $f\varepsilon_X^{(1)} = \varepsilon_X^{(2)}$. Similarly, there is a morphism $g : F^{(2)}(X) \to F^{(1)}(X)$ such that $g\varepsilon_X^{(2)} = \varepsilon_X^{(1)}$. Then $fg\varepsilon_X^{(2)} = \varepsilon_X^{(2)} = \mathrm{id}_{F^{(2)}(X)}\,\varepsilon_X^{(2)}$. The uniqueness in the universal property of $\varepsilon_X^{(2)}$ shows $fg = \mathrm{id}$. Exchanging roles yields $gf = \mathrm{id}$. Hence $g = f^{-1}$ and thus f is an isomorphism.

(ii) Set $G = \overline{\langle X' \rangle}$. Then the corestriction $\varepsilon' : X \to G$ is readily seen to have the universal property. Thus by (i) there is a unique isomorphism $f : G \to F(X)$ such that $f\varepsilon' = \varepsilon_X$. Thus $f|X' : X' \to X'$ is the identity. Hence f and the inclusion $G \to F(X)$ agree on G. Since f is an isomorphism, $G = F(X)$ follows.

Now we prove that $\langle X' \rangle$ is free over X'. For this purpose we invoke the following

Lemma. *Let* C *be a completely regular Hausdorff space with base point* c_0 *and* $(c_0, c_1, \ldots, c_n)$ *a sequence of different points* $c_j \in C$. *Let*

$$\mathbb{I}_j = \{(r_1, \ldots, r_n) \in [0,1]^n \mid r_k = 0 \text{ for } k \neq j\} \subseteq [0,1]^n.$$

Then there is a continuous function $f : C \to \bigsqcup_{j=1}^n \mathbb{I}_j$ *such that* $f(c_0) = (0, \ldots, 0)$ *and* $f(c_j) = (e_1, \ldots, e_n)$ *with* $e_j = 1$ *and* $e_k = 0$ *otherwise.*

The proof of this lemma is a simple exercise in point set topology based on the definition of complete regularity.

We apply this with $C = X'$. Let $t_1, \ldots, t_n \in \mathbb{T}$ be n elements such that $\langle t_1, \ldots, t_n \rangle$ is a free abelian group of rank n. Such groups exist (see Appendix 1, A1.43(ii)). Since $\mathbb{T}$ is arcwise connected, there is a continuous base point preserving function $\Phi : \bigsqcup_{j=1}^n \mathbb{I}_j \to \mathbb{T}$ such that $\Phi\big(f(c_j)\big) = t_j$. By the universal property of $F(X)$ the continuous function $\Phi \circ f \circ \varepsilon_X : X \to \mathbb{T}$ uniquely provides a morphism of compact abelian groups $\varphi : F(X) \to \mathbb{T}$ such that $\varphi \circ \varepsilon_X = \Phi \circ f \circ \varepsilon_X$. Since $X' = \varepsilon_X(X)$ we have $\varphi|X' = \Phi \circ f$. In particular, φ restricts to a homomorphism of $\langle c_1, \ldots, c_n \rangle \to \langle t_1, \ldots, t_n \rangle$ mapping c_j to t_j. This proves that $\{c_1, \ldots, c_n\}$ is a free subset of $F(X)$. Hence $\langle X' \rangle$ is free. □

The appropriate general background for the concept of a free compact abelian group is that of adjoint functors which we discuss in Appendix 3, A3.28, A3.29. The existence of free compact abelian groups can be secured by the general existence theorem for adjoints (see A3.60). But we opt for a direct construction in the present context.

Theorem 8.53 (Construction of Free Compact Abelian Groups). *Let X be any pointed topological space and let $C_0(X,\mathbb{T})_d$ denote the abelian group of all continuous base point preserving functions $X \to \mathbb{T}$ equipped with pointwise multiplication* and the discrete topology. *Set $F(X) = C_0(X,\mathbb{T})_d\widehat{\ }$ and define $\varepsilon_X: X \to F(X)$ by $\varepsilon_X(x)(f) = f(x)$ for all $f \in C_0(X,\mathbb{T})$. Then*

(i) *$F(X)$ is the free compact abelian group over X with respect to the universal map ε_X.*

(ii) *$F(X) \cong \big(C_0(X,\mathbb{R})_d/C_0(X,\mathbb{Z})\big)\widehat{\ } \times [X,\mathbb{T}]\widehat{\ }$.*
If X is connected, then

$$F(X) \cong \big(C_0(X,\mathbb{R})_d\big)\widehat{\ } \times [X,\mathbb{T}]\widehat{\ }.$$

(iii) *For each compact space X and each abelian group G, the function $f \mapsto f \circ \varepsilon_X: \mathrm{Hom}(F(X), G) \to C_0(X, G)$ is a bijective morphism of topological groups.*

Proof. (i) Let $f: X \to G$ be a continuous base point preserving map into a compact abelian group G. We define a function $\varphi: \widehat{G} \to C(X,\mathbb{T})$ by $\varphi(\chi)(x) = \langle \chi, f(x)\rangle$. This is a morphism of abelian groups. Let $f' = \widehat{\varphi}: F(X) \to G$ be its adjoint. Then $(f' \circ \varepsilon_X)(x) = f'\big(\varepsilon_X(x)\big) = \widehat{\varphi}\big(\varepsilon_X(x)\big)$, and thus for each $\chi \in \widehat{G}$ we have

$$\begin{aligned}\langle \chi, \widehat{\varphi}\big(\varepsilon_X(x)\big)\rangle &= \langle \varphi(\chi), \varepsilon_X(x)\rangle = \varepsilon_X(x)\big(\varphi(\chi)\big)\\ &= \varphi(\chi)(x) = \langle \chi, f(x)\rangle\end{aligned}$$

in view of the definition of $\varepsilon_X(x)$ and of $\varphi(\chi)$. Since the characters of G separate the points, we conclude $f'\big(\varepsilon_X(x)\big) = f(x)$, that is $f = f' \circ \varepsilon_X$. This secures the existence of the desired morphism f'. Its uniqueness is seen as follows. Assume that $f' \circ \varepsilon_X = f'' \circ \varepsilon_X$. We claim $f' = f''$. Set $f^* = f'' - f'$, then $0 = f^*\big((\varepsilon_X(x)\big)$ for all $x \in X$. But we claim that $F(X)$ is the smallest closed subgroup containing $\varepsilon_X(X)$, and if this is so, we conclude $f^* = 0$ and thus $f' = f''$ as asserted. Now an element $f \in C(X,\mathbb{T})$ is in the annihilator $A \stackrel{\text{def}}{=} \big(\varepsilon_X(X)\big)^\perp$ if and only if $0 = \langle \varepsilon_X(x), f\rangle = f(x)$ for all $x \in X$, and this means $f = 0$. Hence $A = \{0\}$ and thus $F(X) = A^\perp$ is indeed the smallest subgroup of $F(X)$ containing $\varepsilon_X(X)$ by 7.64(iii).

(ii) This is an immediate consequence of 8.50 and duality.

(iii) The universal property shows that $\alpha \stackrel{\text{def}}{=} (f \mapsto f \circ \varepsilon_X)$ is an isomorphism of abelian groups. It remains to show its continuity. Let V be an identity neighborhood of $C_0(X,G)$. Then there is an identity neighborhood U of G such that $W(X,U) \subseteq V$. Since X is compact, the subset $C \stackrel{\text{def}}{=} \varepsilon_X(X)$ of $F(X)$ is compact. Then $W(C,U)$ is an identity neighborhood of $\mathrm{Hom}(F(X),G)$ in the compact open topology, and $\alpha\big(W(C,U)\big) \subseteq W(X,U) \subseteq V$. This proves continuity of α. □

Homotopy of Compact Abelian Groups

For a topological group G we shall denote by $|G|$ the topological space underlying G; e.g. $|\mathbb{T}| = K(\mathbb{Z},1)$ is the one-sphere. In what follows we shall need information on $[|G|, \mathbb{T}]$ for compact abelian groups G. For a compact abelian group G let $\mathcal{N}$ denote the filter basis of compact subgroups such that G/N is a Lie group. Recall from 2.34 that $G = \lim_{N\in\mathcal{N}} G/N$.

Lemma 8.54. *For a compact abelian group G and $N \in \mathcal{N}$ let $C_0^N(|G|, \mathbb{T})$ denote the subgroup of all $f \in C_0(|G|, \mathbb{T})$ which are constant on cosets $g + N$. Then*

$$C_0(|G|, \mathbb{T}) = \overline{\bigcup_{N\in\mathcal{N}} C_0^N(|G|, \mathbb{T})}.$$

Proof. We consider the commutative C*-algebra $C(|G|, \mathbb{C})$ and let $C^N(|G|, \mathbb{C})$ denote the C*-subalgebra of all complex valued continuous functions on $|G|$ which are constant on cosets $g + N$. Let $A = \bigcup_{N\in\mathcal{N}} C^N(|G|, \mathbb{C})$. Then A is involutive and separates points. Hence the Theorem of Stone and Weierstraß (cf. [34], X.39, Proposition 7, or [331], p. 161.) implies $\overline{A} = C(|G|, \mathbb{C})$. Since the group $C(|G|, \mathbb{C})^{-1}$ of units is open,

$$A^{-1} = A \cap C(|G|, \mathbb{C})^{-1} \quad \text{is dense in} \quad C(|G|, \mathbb{C})^{-1}.$$

Now $A^{-1} = \bigcup_{N\in\mathcal{N}} C^N(|G|, \mathbb{C})^{-1}$. If X is a compact space, then $C(X, \mathbb{C})^{-1} = C(X, \mathbb{C} \setminus \{0\})$. Let $P =]0, \infty[$ denote the multiplicative group of positive real numbers, and for $0 \neq z \in \mathbb{C}$ set $\rho(z) = \frac{z}{|z|}$. Then the polar decomposition $z \mapsto \big(|z|, \rho(z)\big) : \mathbb{C}\setminus\{0\} = P \times \mathbb{S}^1$ is an isomorphism of multiplicative abelian topological groups whose inverse is $(r, c) \mapsto rc$. Accordingly. $C(X, \mathbb{C})^{-1} = C(X, P) \times C(X, \mathbb{S}^1)$ via $f(x) = |f(x)|\cdot\rho\big(f(x)\big)$. We conclude that

$$\bigcup_{N\in\mathcal{N}} C^N(X, \mathbb{S}^1) = \bigcup_{N\in\mathcal{N}} \rho\big(C^N(|G|, \mathbb{C} \setminus \{0\})\big) = \rho(A^{-1})$$

is dense in $\rho\big(C(|G|, \mathbb{C} \setminus \{0\})\big) = C(|G|, \mathbb{S}^1)$. Thus

$$(*) \qquad C(|G|, \mathbb{T}) = \overline{\bigcup_{N\in\mathcal{N}} C^N(|G|, \mathbb{T})}.$$

If X is a compact pointed space, let $K \subseteq C(X, \mathbb{T})$ denote the subgroup of all constant functions. Then $f \mapsto f(x_0): K \to \mathbb{T}$ is an isomorphism. Let $\mathrm{const}_t: X \to \mathbb{T}$ denote the constant function with value t. Then

$$f \mapsto (f(x_0), f - \mathrm{const}_{f(x_0)}): C(X, \mathbb{T}) \to \mathbb{T} \times C_0(X, \mathbb{T})$$

is an isomorphism with inverse $(t, f) \mapsto \mathrm{const}_t + f$. Applying the projection $C(|G|, \mathbb{T}) \to C_0(|G|, \mathbb{T})$ to $(*)$ we get the claim of the lemma. □

Exercise E8.11. The previous lemma holds more generally. Define the concept of a projective limit for compact pointed spaces, modelled after 1.27. Adjust the proof of the previous lemma to show the following result.

Lemma 8.55. *Let* $X = \lim_{j\in J} X_j$ *be a projective limit of compact pointed spaces. The limit maps* $f_j\colon X \to X_j$ *give morphisms* $C_0(f_j,\mathbb{T})\colon C_0(X_j,\mathbb{T}) \to C_0(X,\mathbb{T})$. *Then* $C_0(X,\mathbb{T}) = \overline{\bigcup_{j\in J} \operatorname{im} C_0(f_j,\mathbb{T})}$. □

Proposition 8.56. *Let* X *be a topological space and* G *a compact abelian group. Then there are isomorphisms of topological groups*

$$\alpha_{X,G}\colon C_0(X,G) \to \operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})\big), \quad \alpha_{X,G}(f)(\chi)(x) = \chi\big(f(x)\big).$$

and

$$\rho_{X,G}\colon [X,G] \to \operatorname{Hom}(\widehat{G}, [X,\mathbb{T}]).$$

Proof. Since $\widehat{G}$ is discrete, and $f \mapsto \chi\circ f\colon C_0(X,G) \to C_0(X,\mathbb{T})$ is continuous, $\alpha_{X,G}$ is continuous. Its inverse morphism is given by $\eta_G\big(\alpha^{-1}_{X,G}(\psi)(x)\big)(\chi) = \psi(\chi)(x)$ for $x \in X$, and $\chi \in \widehat{G}$. If $A = \widehat{G}$, then this morphism can be represented by $\beta\colon \operatorname{Hom}\big(A, C_0(X,\mathbb{T})\big) \to C_0(X,\widehat{A})$, $\beta(\psi)(x)(a) = \psi(a)(x)$. Since $\widehat{A}$ has the topology of pointwise convergence and $\psi \mapsto \psi(a)\colon \operatorname{Hom}\big(A, C_0(X,\mathbb{T})\big) \to C_0(X,\mathbb{T})$ is continuous, β and thus $\alpha^{-1}_{X,G}$ is continuous.

Next we prove the following

Claim. For a compact abelian group G and any compact connected space X, the identity component of $\operatorname{Hom}(\widehat{G}, C_0(X,\mathbb{T}))$ is $\operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})_0\big)$ and

$$\operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{R})\big) \to \operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})\big) \to \operatorname{Hom}(\widehat{G}, [X,\mathbb{T}]) \to 0$$

is exact.

For a proof we recall from Proposition 8.50 that $C_0(X,\mathbb{T}) = C_0(X,\mathbb{T})_0 \oplus B[X,\mathbb{T}]$ algebraically and topologically. Thus

$$\operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})\big) \cong \operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})_0\big) \oplus \operatorname{Hom}(\widehat{G}, B[X,\mathbb{T}])$$

algebraically and topologically. From 8.50 we know that $q\colon \mathbb{R} \to \mathbb{T}$ induces an isomorphism $C_0(X,\mathbb{R}) \to C_0(X,\mathbb{T})$. But $\operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{R})\big)$ is a real topological vector space and thus is connected, and $\operatorname{Hom}(\widehat{G}, [X,\mathbb{T}]) \subseteq [X,\mathbb{T}]^{\widehat{G}}$ is totally disconnected. Then the assertions of the claim follow.

From the claim we get a commutative diagram

$$\begin{array}{ccc}
C_0(X,G)_0 & \xrightarrow{(\alpha_{X,G})_0} & \operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})_0\big) \\
\text{incl}\big\downarrow & & \big\downarrow \operatorname{Hom}(\widehat{G},\text{incl}) \\
C_0(X,G) & \xrightarrow{\alpha_{X,G}} & \operatorname{Hom}\big(\widehat{G}, C_0(X,\mathbb{T})\big) \\
\text{quot}\big\downarrow & & \big\downarrow \operatorname{Hom}(\widehat{G},\text{quot}) \\
[X,G] & \xrightarrow[\rho_{X,G}]{} & \operatorname{Hom}(\widehat{G}, [X,\mathbb{T}])
\end{array}$$

in which $\rho_{X,G}$ is induced upon passing to quotients. Since $\alpha_{X,G}$ and $(\alpha_{X,G})_0$ are isomorphisms, so is $\rho_{X,G}$. □

Theorem 8.57 (Theorem on $[\mathbb{T}, G]$ and $[G, \mathbb{T}]$). (i) *Let G be a locally compact abelian group. The composition of functions*

$$\mathrm{Hom}(\mathbb{T}, G) \xrightarrow{\text{incl}} C_0(|\mathbb{T}|, G) \xrightarrow{\text{quot}} [|\mathbb{T}|, G] = \pi_1(G)$$

is an isomorphism; that is every homotopy class of continuous base point preserving functions $\mathbb{T} \to G$ contains exactly one homomorphism, adjoint to a morphism $\widehat{G} \to \mathbb{Z}$.

(ii) *Assume that G is compact and connected. Then the composition of functions*

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{T}) \xrightarrow{\text{incl}} C_0(|G|, \mathbb{T}) \xrightarrow{\text{quot}} [|G|, \mathbb{T}]$$

is an isomorphism; that is every homotopy class of continuous base point preserving continuous functions $G \to \mathbb{T}$ contains exactly one character of G and $C_0(|G|, \mathbb{T}) = \mathrm{Hom}(|G|, \mathbb{T}) \oplus C_0(|G|, \mathbb{T})_0$, algebraically and topologically, $C_0(|G|, \mathbb{T})_0 \cong C_0(|G|, \mathbb{R})$.

Proof. First we recall that $G = E \oplus H$ with $E \cong \mathbb{R}^n$ and H possessing a compact open subgroup by the Vector Group Splitting Theorem 7.57. Then $\mathrm{Hom}(\mathbb{T}, G) = \mathrm{Hom}(\mathbb{T}, E) \oplus \mathrm{Hom}(\mathbb{T}, H)$ and $C_0(X, G) = C_0(X, E) \oplus C_0(X, H)$ for all pointed topological spaces X. Also, the inclusion $\mathrm{comp}(G)_0 \to H$ induces an isomorphism $\mathrm{Hom}\big(\mathbb{T}, \mathrm{comp}(G)_0\big) \to \mathrm{Hom}(\mathbb{T}, H)$ and, provided X is connected, also an isomorphism $C_0\big(X, \mathrm{comp}(G)_0\big) \to C_0(X, H)$. If X is connected, then $[X, G] = [X, E] \oplus [X, H] = [X, \mathrm{comp}(G)_0]$. Taking $X = \mathbb{S}^1$ we see that it is no loss of generality to assume that G is compact connected which we will do from now on.

By 8.56 we have isomorphisms of topological groups

$$\alpha_{X,G}\colon C_0(X, G) \to \mathrm{Hom}\left(\widehat{G}, C_0(X, \mathbb{T})\right), \quad \alpha_{X,G}(f)(\chi)(x) = \chi\big(f(x)\big).$$

and

$$\rho_{X,G}\colon [X, G] \to \mathrm{Hom}(\widehat{G}, [X, \mathbb{T}]).$$

Now we let $X = |\mathbb{T}|$. Then there is an injective morphism $\Phi\colon \mathbb{Z} \to C_0(|\mathbb{T}|, \mathbb{T})$, $\Phi(n) = \mu_n$, $\mu_n(t) = n{\cdot}t$. Then the quotient map $\mathrm{quot}\colon C_0(|\mathbb{T}|, \mathbb{T}) \to [|\mathbb{T}|, \mathbb{T}]$ gives an isomorphism $\varphi \stackrel{\text{def}}{=} \mathrm{quot} \circ \Phi\colon \mathbb{Z} \to [|\mathbb{T}|, \mathbb{T}]$, $\varphi(n) = \mu_n + C_0(|\mathbb{T}|, \mathbb{T})_0$, $\mu_n \in C_0(|\mathbb{T}|, \mathbb{T})$, $\mu_n(t) = n{\cdot}t$ (See Exercise E8.10.)

Let $f\colon \mathbb{T} \to G$ be a morphism. Then $\widehat{f}\colon \widehat{G} \to \mathbb{Z}$ is given by $\widehat{f}(\chi) = n$ iff $\chi f = \mu_n$. Thus there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathbb{T}, G) & \xrightarrow{\text{incl}} & C_0(|\mathbb{T}|, G) & \xrightarrow{\text{quot}} & [|\mathbb{T}|, G] \\
{\scriptstyle f \mapsto \widehat{f}}\big\downarrow & & {\scriptstyle \alpha_{|\mathbb{T}|,G}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{|\mathbb{T}|,G}} \\
\mathrm{Hom}(\widehat{G}, \mathbb{Z}) & \xrightarrow{\mathrm{Hom}(\widehat{G}, \Phi)} & \mathrm{Hom}\left(\widehat{G}, C_0(|\mathbb{T}|, \mathbb{T})\right) & \xrightarrow{\mathrm{Hom}(\widehat{G}, \mathrm{quot})} & \mathrm{Hom}(\widehat{G}, [|\mathbb{T}|, \mathbb{T}]).
\end{array}$$

Since $\varphi = \mathrm{quot} \circ \Phi$ is an isomorphism, the lower row represents an isomorphism. Hence the top row represents an isomorphism, too. This completes the proof of the first part.

(ii) We consider the natural inclusion map

$$\mathrm{incl}\colon \widehat{G} = \mathrm{Hom}(G, \mathbb{T}) \to C_0(|G|, \mathbb{T})$$

and the quotient map $p\colon C_0(|G|, \mathbb{T}) \to [|G|, \mathbb{T}]$. The composition will be denoted

$$\alpha_G\colon \mathrm{Hom}(G, \mathbb{T}) = \widehat{G} \to [|G|, \mathbb{T}], \quad \alpha_G(\chi) = \chi + C_0(|G|, \mathbb{T}).$$

If $\chi \in \widehat{G}$ is such that $\alpha(\chi) = 0$, then $\mathrm{incl}(\chi) \in C_0(|G|, \mathbb{T})_0$ by 8.50(ii). Thus there is a $\widetilde{\chi}\colon G \to \mathbb{R}$ with $\chi = q \circ \widetilde{\chi}$. Set $\varphi(g_1, g_2) = \widetilde{\chi}(g_1 + g_2) - \widetilde{\chi}(g_1) - \widetilde{\chi}(g_2))$ for $g_1, g_2 \in G$. Then $q\big(\varphi(g_1, g_2)\big) = 0$; i.e. $\mathrm{im}\,\varphi \subseteq \ker q = \mathbb{Z}$. Thus $\varphi\colon G \times G \to \mathbb{Z}$ is a continuous function of a connected space into a discrete space and is, therefore, constant. Since all maps are base point preserving, its value is 0. Hence $\widetilde{\chi}\colon G \to \mathbb{R}$ is a morphism. But G is compact and the only compact subgroup of $\mathbb{R}$ is $\{0\}$. Hence $\widetilde{\chi} = 0$ and, therefore, $\chi = 0$. Thus $\mathrm{im}\,\mathrm{incl} \cap C_0(|G|, \mathbb{R}) = \{0\}$ and therefore α is injective.

We must now show that α_G is surjective. Now $C_0(|G|, \mathbb{R})$ is open in $C_0(|G|, \mathbb{T})$. It therefore suffices to show that $C_0(|G|, \mathbb{T})_0) + \mathrm{Hom}(G, \mathbb{T})$ is dense in $C_0(|G|, \mathbb{T})$. Let $\tau_N\colon G \to G/N$, $N \in \mathcal{N}$ ranges through all quotient morphisms of G onto Lie groups. Further $\alpha_{\mathbb{T}^n}\colon \widehat{\mathbb{T}^n} \overset{\cong}{\to} [|\mathbb{T}^n|, \mathbb{T}]$ is equivalent to $\alpha_{\mathbb{T}}^n\colon \widehat{\mathbb{T}}^n \to [|\mathbb{T}|, \mathbb{T}]^n$, and since $\alpha_{\mathbb{T}}$ is an isomorphism by Exercise E8.10 we conclude that $\alpha_{\mathbb{T}^n}$ is an isomorphism. We consider the commutative diagram

$$\begin{array}{ccc}
\widehat{G/N} & \xrightarrow{\alpha_{G/N}} & [|G/N|, \mathbb{T}] \\
{\scriptstyle \widehat{\tau}_N}\big\downarrow & & \big\downarrow{\scriptstyle [|\tau_N|, \mathbb{T}]} \\
\widehat{G} & \xrightarrow[\alpha_G]{} & [|G|, \mathbb{T}].
\end{array}$$

Since $G/N \cong \mathbb{T}^n$ for some n, and $\alpha_{\mathbb{T}^n}$ is an isomorphism as we have just observed, the top map is an isomorphism. Thus $\mathrm{im}\,\alpha_G$ contains the images of all maps $[|\tau_N|, \mathbb{T}]$. Another way of saying this is

$$\bigcup_{N \in \mathcal{N}} C_0^N(|G|, \mathbb{T}) \subseteq C_0(|G|, \mathbb{T}) + \mathrm{Hom}(G, \mathbb{T}).$$

By Lemma 8.54 we have

$$C_0(|G|, \mathbb{T}) = \overline{\bigcup_{N \in \mathcal{N}} C_0^N(|G|, \mathbb{T})}.$$

Hence $C_0(|G|, \mathbb{T}) + \mathrm{Hom}(G, \mathbb{T})$ is dense in $C_0(|G|, \mathbb{T})$ which we had to show. □

Comparing 8.57(ii) with 8.50(ii) we observe that, in the case of $X = |G|$ we do have a canonical complement for $C_0(X, \mathbb{T})_0$ in $C_0(X, \mathbb{T})$.

Theorem 8.58 (Torsion-Free Abelian Groups as $[X, \mathbb{T}]$). *Let A be a discrete torsion-free abelian group. Then there is a compact connected topological space X such that $A \cong [X, \mathbb{T}]$. The space X may be chosen to be the underlying space of a compact abelian group, and in this class of spaces, X is uniquely determined by $[X, \mathbb{T}]$ up to homeomorphism. In fact, if G is any compact connected abelian group such that $[|G|, \mathbb{T}] \cong A$, then $G \cong \widehat{A}$.*

Proof. We set $G = \widehat{A}$ and obtain a compact connected abelian group. Let $X = |G|$ denote the space underlying G. By 8.56 the composition

$$A \cong \widehat{G} \xrightarrow{\text{incl}} C_0(X, \mathbb{T}) \xrightarrow{\text{quot}} [X, \mathbb{T}]$$

is an isomorphism. Thus $[X, \mathbb{T}] \cong A$.

Let $Y = |H|$ be a compact connected abelian group such that $[Y, \mathbb{T}] \cong A$. Then $\widehat{H} \to C_0(Y, \mathbb{T}) \to [Y, \mathbb{T}]$ is an isomorphism by 8.56. Hence $\widehat{H} \cong A \cong \widehat{G}$, and therefore $H \cong G$. □

A compact connected abelian group is completely determined (up to isomorphism) by the topology of its underlying space as we shall formulate now. This is in striking contrast with other categories, e.g. that of Banach spaces where all infinite dimensional separable Banach spaces are homeomorphic ([23], p. 231, Corollary 9.1).

Theorem 8.59. *If compact connected abelian groups G_1 and G_2 are homeomorphic then they are isomorphic as topological groups.*

Proof. The abelian group $[X, \mathbb{T}]$ is a topological invariant of the compact connected pointed space X. The assertion therefore follows from 8.58. □

For example, an abelian topological group homeomorphic to a torus is a torus of the same dimension.

Definition 8.60. For compact pointed spaces X and Y let $X \wedge Y$ denote the space

$$(X \times Y)/\big((X \times \{y_0\}) \cup (\{x_0\} \times Y)\big),$$

with the class$(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as base point. For a compact space Y, the space $\mathbb{S}^1 \wedge Y$ is called *the suspension* ΣY of Y. (See e.g. [338], p. 41.) The recursive definition $\mathbb{S}^{n+1} = \Sigma \mathbb{S}^n$, $n = 0, 1, 2, \ldots$ gives the sequence of the n-spheres, and we define

$$\pi_n(Y) = [\mathbb{S}^n, Y], \quad n = 0, 2, \ldots. \qquad \square$$

For compact spaces X and Y, the function

$$F\colon C_0(X \wedge Y, \mathbb{T}) \to C_0\big(X, C_0(Y, \mathbb{T})\big),$$

$F(\varphi)(x)(y) = \varphi\big(\text{class}(x, y)\big)$ is a homeomorphism with inverse given by

$$F^{-1}(f)\big(\text{class}(x, y)\big) = f(x)(y).$$

Proposition 8.61. *Let G be a compact abelian group. Then the following conclusions hold.*

(i) *If X and Y are compact connected spaces, then $[X \wedge Y, G] = 0$.*
(ii) *$[\Sigma(X), G] = \{0\}$ for every compact connected space X.*
(iii) *$\pi_n(G) = 0$ for $n = 2, 3, \ldots$.*

Proof. (i) We note that for two compact connected pointed spaces X and Y the group $C_0\big(X, C_0(Y, \mathbb{T})\big)$ is naturally isomorphic to $C_0\big(X, C_0(Y, \mathbb{R})\big) \times C_0(X, [Y, \mathbb{T}]) \cong C_0\big(X, C_0(Y, \mathbb{R})\big)$ since $[Y, \mathbb{T}]$ is discrete by Proposition 2.92 and X is connected. This group is a topological $\mathbb{R}$-vector space and is, therefore, contractible. Hence $C_0(X \wedge Y, \mathbb{T})$ is a real topological vector space and

thus $[X \wedge Y, \mathbb{T}] = C_0(X \wedge Y, \mathbb{T})/C_0(X \wedge Y, \mathbb{T})_0 = \{0\}$.

By 8.60(ii) we then have $[X \wedge Y, G] \cong \mathrm{Hom}(\widehat{G}, [X \wedge Y, \mathbb{T}]) = \{0\}$.

(ii) As $\Sigma(X) = \mathbb{S}^1 \wedge X$, part (ii) is a special case of (i).

(iii) For $n = 2, 3, \ldots$ we have $\mathbb{S}^n = \Sigma(\mathbb{S}^{n-1})$. Thus (iii) is a special case of (ii). □

Exponential Function and Homotopy

The Exponential Function and Homotopy

Theorem 8.62. (i) *For every locally compact abelian group G there is an exact sequence of abelian groups*

$$0 \to \pi_1(G) \to \mathfrak{L}(G) \xrightarrow{\exp_G} G \to \pi_0(G) \to 0.$$

(ii) *If G is compact, then there are natural isomorphisms*

$$\pi_n(G) \cong \begin{cases} \mathrm{Ext}(\widehat{G}, \mathbb{Z}), & \text{for } n = 0, \\ \mathrm{Hom}(\widehat{G}, \mathbb{Z}), & \text{for } n = 1, \\ 0, & \text{for } n \geq 2. \end{cases}$$

If G is locally compact, this formula remains intact for $n \geq 1$, and if G/G_0 is compact it remains also intact for $n = 0$.

Proof. For all locally compact abelian groups, the exactness of the sequence

$$0 \to \mathrm{Hom}(\mathbb{T}, G) \to \mathfrak{L}(G) \xrightarrow{\exp} G \to \pi_0(G) \to 0$$

(with $\pi_0(G) \cong \mathrm{Ext}((\mathrm{comp}\, G)\widehat{}, \mathbb{Z})$ if G/G_0 is compact) was established in 7.66 and 8.30. By the Vector Group Splitting Theorem 7.57 we have $G = E \oplus H$ such that $E \cong \mathbb{R}^n$ and H has an open compact subgroup. Then for each compact connected pointed space X one gets $[X, G] = [X, E] \oplus [X, H] = [X, \mathrm{comp}(G)_0] = \mathrm{Hom}((\mathrm{comp}(G)_0)\widehat{}, [X, \mathbb{T}])$. From 8.57 and 8.61(iii) we now get the homotopy groups as claimed. □

This theorem complements information on the exponential function of locally compact abelian groups given in Theorems 7.66, 8.30, and 8.41.

Notice, in particular, that the fundamental group $\pi_1(G) \cong \mathrm{Hom}(\widehat{G}, \mathbb{Z})$ of a locally compact abelian group G is always torsion-free and agrees with the fundamental group of the maximal connected compact subgroup. In fact more is true: the entire homotopy of G is supported by $\overline{\mathrm{tor}\, G}$. Notice that $G/\overline{\mathrm{tor}\, G}$ is the largest torsion-free quotient of G.

Corollary 8.63. *If G is a compact connected abelian group, then the inclusion $\overline{\mathrm{tor}\, G} \to G$ induces an isomorphism $\pi_1(\overline{\mathrm{tor}\, G}) \to \pi_1(G)$.*

Proof. A divisible element of $\widehat{G}$ must have a trivial image in $\mathbb{Z}$ under any morphism $\widehat{G} \to \mathbb{Z}$. Hence the quotient map $\widehat{G} \to \widehat{G}/\operatorname{Div}(\widehat{G})$ induces an isomorphism $\mathrm{Hom}(\widehat{G}/\operatorname{Div}(\widehat{G}), \mathbb{Z}) \to \mathrm{Hom}(\widehat{G}, \mathbb{Z})$. By Propositions 8.2(i) and 8.3(ii), we know $\overline{\mathrm{tor}\, G} = (\mathrm{DIV}\, G)^\perp = (\operatorname{Div} G)^\perp$ and by the Annihilator Mechanism 7.64, accordingly $(\overline{\mathrm{tor}\, G})\widehat{\ } \cong \widehat{G}/\operatorname{Div}(\widehat{G})$. Then the corollary follows from Theorem 8.62. □

Naturally, the question arises: Which abelian groups are of the form $\mathrm{Hom}(A, \mathbb{Z})$ for a discrete abelian group A which we may assume to be torsion free? The entire last chapter of the book by Eklof and Mekler [99] (Chapter 14, "Dual Groups," pp. 420–452 is devoted to this issue.

The Fine Structure of Free Compact Abelian Groups

A free compact abelian group over a space X arises from dualizing the exact sequence

$$(9) \qquad 0 \to C_0(X, \mathbb{Z})_d \xrightarrow{C_0(X,j)} C_0(X, \mathbb{R})_d \xrightarrow{C_0(X,q)} C_0(X, \mathbb{T})_d \xrightarrow{\text{quot}} [X, \mathbb{T}] \to 0$$

where all groups are equipped with the discrete topology.

As the group $\mathbb{Z}$ is discrete, any function $f \in C_0(X, \mathbb{Z})$ is locally constant.

Definition 8.64. Let A denote an abelian group. For any pointed topological space X we set

$$\widetilde{H}^0(X, A) = C_0(X, A_d)_d,$$

the group of locally constant basepoint preserving functions into A with pointwise addition. □

In algebraic topology, $C_0(X, \mathbb{Z})_d$ is denoted $\widetilde{H}^0(X, \mathbb{Z})$ and called *the zeroth reduced Čech cohomology group.* (See [338], p. 168, p. 309.) In the light of an earlier remark, we may rewrite (9) as

$$(10) \qquad 0 \to \widetilde{H}^0(X, \mathbb{Z}) \to C_0(X, \mathbb{R})_d \to C_0(X, \mathbb{T})_d \to H^1(X, \mathbb{Z}) \to 0.$$

For the following we define the *reduced weight* $w_0(Y)$ of a topological space to be the weight $w(Y)$ if $w(Y)$ is infinite and $w(Y) - 1$ if $w(Y)$ is finite.

Lemma 8.65. *For any nonsingleton compact space X,*

$$\frac{C_0(X,\mathbb{R})_d}{C_0(X,\mathbb{Z})_d} \cong \widetilde{H}^0(X,\mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}^{(w(X)^{\aleph_0})},$$

$$\widetilde{H}^0(X,\mathbb{Q}/\mathbb{Z}) = \bigoplus_{p\ \text{prime}} \mathbb{Z}(p^\infty)^{(w_0(X/\text{conn}))},$$

where X/conn is the totally disconnected compact space of connected components, and

$$\text{card}[X,\mathbb{T}]^{\aleph_0} \leq w(X)^{\aleph_0}.$$

Proof. The abelian group $D \stackrel{\text{def}}{=} \frac{C_0(X,\mathbb{R})_d}{C_0(X,\mathbb{Z})_d}$ is a homomorphic image of a divisible group and is, therefore, divisible. By the Structure Theorem for Divisible Groups A1.42 we know that

$$D = \mathbb{Q}^{\text{rank}\, D} \oplus \bigoplus_{p\ \text{prime}} \mathbb{Z}(p^\infty)^{\text{rank}_p(D)}.$$

We have to determine the ranks of D.

First we identify the torsion group $\text{tor}\, D$. From 8.1 recall that $D[n] = \{d \in D \mid n{\cdot}d = 0\}$. Now an element $f + C_0(X,\mathbb{Z}) \in D[n]$ iff $n{\cdot}f \in C_0(X,\mathbb{Z})$ iff $f \in C_0\left(X, \frac{1}{n}\mathbb{Z}\right)$. We again write $C(n) = \left(\frac{1}{n}\mathbb{Z}\right)/\mathbb{Z}$ and conclude

$$D[n] \cong C_0\big(X, C(n)\big).$$

It follows that

$$\text{tor}\, D = \bigcup_{n\in\mathbb{N}} D[n] \cong \bigcup_{n\in\mathbb{N}} C_0\big(X, C(n)\big) = C_0(X, (\mathbb{Q}/\mathbb{Z})_d) = \widetilde{H}^0(X,\mathbb{Q}/\mathbb{Z}).$$

The p-primary component D_p is $C_0\big(X,\mathbb{Z}(p^\infty)\big) = \widetilde{H}^0(X,\mathbb{Z}(p^\infty)$. It remains to determine the ranks.

Now $\text{rank}(D) = \dim_{\mathbb{Q}} \mathbb{Q}\otimes D$. The exact sequence

$$0 \to C_0(X,\mathbb{Z}) \to C_0(X,\mathbb{R}) \to D \to 0$$

remains exact when it is tensored with $\mathbb{Q}$ by A1.45(v):

$$0 \to \mathbb{Q}\otimes C_0(X,\mathbb{Z}) \to \mathbb{Q}\otimes C_0(X,\mathbb{R}) \to \mathbb{Q}\otimes D \to 0.$$

As $C_0(X,\mathbb{R})$ is a $\mathbb{Q}$-vector space, $\mathbb{Q}\otimes C_0(X,\mathbb{R}) \cong C_0(X,\mathbb{R})$. Since all elements in $C_0(X,\mathbb{Z})$ are locally constant and thus take finitely many values, $\mathbb{Q}\otimes C_0(X,\mathbb{Z}) \cong C_0(X,\mathbb{Q}_d)$, and $\mathbb{Q}\otimes C_0(X,\text{incl}_{\mathbb{Z}\to\mathbb{R}}) = C_0(X,\text{incl}_{\mathbb{Q}\to\mathbb{R}})$. Thus

$$\mathbb{Q}\otimes D \cong \frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Q}_d)}.$$

By A4.9, $\text{card}\, C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$ and

$$\text{card}\, C_0(X,\mathbb{Q}_d) = \begin{cases} \aleph_0^n = \aleph_0 & \text{if } \text{card}(X/\text{conn}) = n+1, \\ w(X/\text{conn}) & \text{if } \text{card}(X/\text{conn}) \text{ is infinite.} \end{cases}$$

Thus $\mathrm{card}(\mathbb{Q} \otimes D) = w(X)^{\aleph_0}$. For infinite dimensional rational vector spaces dimension and cardinality agree. Thus this cardinal is $\mathrm{rank}\, D$.

Now let p be a prime number. Then $\mathrm{rank}_p\, D$ is the $\mathrm{GF}(p)$-dimension of the socle of $C_0(X, \mathbb{Z}(p^\infty)_d) \cong C_0(X/\mathrm{conn}, \mathbb{Z}(p^\infty)_d)$. By A4.9 again,

$$\mathrm{card}\, C_0(X, \mathbb{Z}(p^\infty)_d) = \begin{cases} \aleph_0^n = \aleph_0 & \text{if } \mathrm{card}(X/\mathrm{conn})X = n+1, \\ w(X/\mathrm{conn}) & \text{if } \mathrm{card}(X/\mathrm{conn}) \text{ is infinite.} \end{cases}$$

We conclude

$$\mathrm{rank}_p(D) = w_0(X/\mathrm{conn}).$$

Finally, $[X, \mathbb{T}]$ is a quotient group of $C_0(X, \mathbb{T})$. Thus

$$\mathrm{card}[X, \mathbb{T}] \leq \mathrm{card}\, C_0(X, \mathbb{T}) = w(X)^{\aleph_0}$$

by A4.9. The desired estimate then follows. □

Corollary 8.66. *Let X be a nonsingleton compact pointed space. Then*

$$C_0(X, \mathbb{T})_d = \mathbb{Q}^{(w(X)^{\aleph_0})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(w_0(X/\mathrm{conn}))} \oplus [X, \mathbb{T}].$$

Proof. We have $C_0(X, \mathbb{T}) \cong \frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Z})} \oplus [X, \mathbb{T}]$ by 8.50. Then 8.65 proves the claim. □

The Structure of Free Compact Abelian Groups

Theorem 8.67. *Let X be a nonsingleton compact space. Then*

$$F(X) \cong \widehat{\mathbb{Q}}^{w(X)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(X/\mathrm{conn})} \times [X, \mathbb{T}]\widehat{\ }.$$

Proof. This follows at once from 8.66. □

As an example, $F(\mathbb{S}^1) \cong \widehat{\mathbb{Q}}^{2^{\aleph_0}} \times \mathbb{T}$. This group contains a circle group which is not a free compact abelian group. Thus not all closed subgroups or even closed connected subgroups of a free compact abelian group are free compact abelian groups. But we shall see that the identity component of a free compact abelian group is a free compact abelian group (8.72).

Theorem 8.67 shows, in particular, that for a nonsingleton compact pointed space X the weight of the free compact abelian group $F(X)$ is at least $2^{\aleph_0}$ and so $F(X)$ is not metrizable by 8.43. Also, $\mathrm{card}\, F(X) \geq 2^{(2^{\aleph_0})}$.

If X is a connected compact pointed spaces which is *cohomologically trivial in dimension one*, i.e. satisfies $[X, \mathbb{T}] = \{0\}$ then

$$F(X) \cong \widehat{\mathbb{Q}}^{w(X)^{\aleph_0}}.$$

Thus in such a case, the structure of $F(X)$ is completely determined by the weight of X. So the free compact abelian groups over $[0,1]^n$, $n = 1, 2, \ldots, \omega$, or over $\mathbb{S}^n$, $n = 2, 3, \ldots$ are all isomorphic as topological groups.

Exercise E8.12. Determine those compact connected torsion-free abelian groups G for which $G \cong F(|G|)$. Show that there is a proper class with this property. □

For compact pointed spaces X and Y let $X \sqcup Y$ denote $(X \dot\cup Y)/\{x_0, y_0\}$, the disjoint sum space with base points glued together.

Lemma 8.68. *For compact pointed spaces X and Y the relation*

$$F(X \sqcup Y) = F(X) \times F(Y)$$

holds.

Proof. Exercise E8.13. □

Exercise E8.13. Prove 8.68.

[Hint. One way to prove the assertion is to observe, that finite products in the category of compact abelian groups are finite coproducts (with appropriate definition of coprojections) and that left adjoint functors preserve coproducts. (See Appendix 3, A3.52.)]

Exercise E8.14. Prove the following statement:

If X is a compact totally disconnected space, then $[X, \mathbb{T}] = 0$.

[Hint. We must show that every base point preserving continuous function $f: X \to \mathbb{T}$ lifts across $q: \mathbb{R} \to \mathbb{T}$. (See Appendix 2 for liftings and covering maps.) The pull-back

$$\begin{array}{ccc} E & \xrightarrow{f'} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & \mathbb{T} \end{array}$$

gives a covering $p: E \to X$. Cover X by finitely many disjoint open sets over each of which the covering is trivial. Use this to construct a continuous cross section $s: X \to E$ for p. Then $F \stackrel{\text{def}}{=} f' \circ s$ is the required lifting.] □

The Freeness Criterion

Theorem 8.69. *Let G be a compact abelian group and write $\widehat{G} = D \oplus A$ with $D = \operatorname{div} \widehat{G}$ and a reduced group A. Then the following statements are equivalent:*

(i) *G is free; i.e. there is a compact space X such that $G \cong F(X)$.*

(ii) *The following conditions are satisfied:*

   (a) *A is torsion-free,*

   (b) *the cardinals $\operatorname{rank}_p D$ agree for all primes,*

(c) $(\operatorname{rank} D)^{\aleph_0} = \operatorname{rank} D$, *and*
(d) $\max\{\operatorname{rank}_2 D, (\operatorname{card} A)^{\aleph_0}\} \leq \operatorname{rank} D$.

Proof. Assume $G \cong F(X)$ for a compact space X. By Theorem 8.67 we have

$$G \cong \widehat{\mathbb{Q}}^{w(X)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(X/\text{conn})} \times [X, \mathbb{T}]\widehat{\ }.$$

Then

$$D \cong \mathbb{Q}^{(w(X)^{\aleph_0})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(w_0(X/\text{conn}))} \oplus \operatorname{div}[X, \mathbb{T}]$$

and $A \cong [X, \mathbb{T}]/\operatorname{div}[X, \mathbb{T}]$. Then (a) follows from 8.50(ii) and the fact that $\operatorname{div}[X, \mathbb{T}]$ is direct summand of $[X, \mathbb{T}]$. Next, (b) follows from $\operatorname{rank}_p D = w_0(X/\text{conn})$ for all p, and since $(\operatorname{rank} D)^{\aleph_0} = \left(w(X)^{\aleph_0}\right)^{\aleph_0} = w(X)^{\aleph_0} = \operatorname{rank} D$, we have (c). By 8.65(iii) we know $(\operatorname{card}[X, \mathbb{T}])^{\aleph_0} \leq w(X)^{\aleph_0} \leq \operatorname{rank} D$; then a fortiori $(\operatorname{card} A)^{\aleph_0} \leq \operatorname{rank} D$, and so (d) holds.

Conversely assume that (ii) holds. Let

$$Y = \begin{cases} \{0, 1, \ldots, \operatorname{rank}_2 D\} & \text{if } \operatorname{rank}_2 D \text{ is finite,} \\ \{0, 1\}^{\operatorname{rank}_2(D)} & \text{otherwise} \end{cases}$$

where we consider $\{0, \ldots \operatorname{rank}_2 D\}$ as a discrete space with base point 0. We define $X = [0, 1]^{\operatorname{rank} D} \sqcup Y \sqcup |\widehat{A}|$. Then $F(X) \cong F([0, 1]^{\operatorname{rank} D}) \times F(Y) \times F(|\widehat{A}|)$ by 8.68.

Since $K \stackrel{\text{def}}{=} [0, 1]^{\operatorname{rank} D}$ is a contractible space of weight $\max\{\aleph_0, \operatorname{rank} D\}$ (see EA4.3) we have

$$\begin{aligned} \widehat{F(K)} &\cong C_0(K, \mathbb{T})_d \cong C_0(K, \mathbb{R})_d \cong \mathbb{Q}^{(w(K)^{\aleph_0})} \\ &= \mathbb{Q}^{(\max\{{\aleph_0}^{\aleph_0}, (\operatorname{rank} D)^{\aleph_0}\})} = \mathbb{Q}^{(\operatorname{rank} D)} \end{aligned}$$

because of (c), and thus $2 \leq \operatorname{rank} D$.

Similarly, Y is a totally disconnected compact space of weight $\operatorname{rank}_2 D$. By E8.14 the group $[Y, \mathbb{T}]$ is singleton. Thus $\widehat{F(Y)} =$

$$C_0(Y, \mathbb{T}) = \mathbb{Q}^{(w(Y)^{\aleph_0})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(w_0(Y))} = \mathbb{Q}^{(\operatorname{rank}_2 D)} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(\operatorname{rank}_2 D)}.$$

Finally, since A is torsion-free and thus $\widehat{A}$ is connected, we have $F(|\widehat{A}|)\widehat{\ } = C_0(|\widehat{A}|, \mathbb{T}) = \operatorname{Hom}(\widehat{A}, \mathbb{T}) \oplus C_0(|A|, \mathbb{T})_0 \cong A \oplus C_0(|A|, \mathbb{R}) = A \oplus \mathbb{Q}^{(w(\widehat{A})^{\aleph_0})} = A \oplus \mathbb{Q}^{\left((\operatorname{card} A)^{\aleph_0}\right)}$ using 8.57(ii) and 7.76(ii).

Putting things together we have

$$F(X)\widehat{\ } = \mathbb{Q}^{(\mathbf{a})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{\operatorname{rank}_2 D} \oplus A,$$

where

$$\mathbf{a} = \operatorname{rank} D + \operatorname{rank}_2 D + (\operatorname{card} A)^{\aleph_0}, \tag{11}$$

By (d) we conclude $\mathbf{a} = \operatorname{rank} D$. But then $\widehat{G} \cong \widehat{F(X)}$ and thus $G \cong F(X)$. □

We note that freeness of compact abelian groups is not guaranteed by a linear independence condition as in the case of abelian groups; it is an existence problem and is, therefore, more complicated.

Corollary 8.70. *Let G be a compact abelian group and assume $G \cong F(X)$ for some compact pointed space X. Then the following conditions are equivalent:*
(i) *X is connected.*
(ii) *G is connected.*

Proof. (i)⇒(ii) If X is connected, then $C_0(X, \mathbb{T}) \cong C_0(X, \mathbb{R}) \oplus [X, \mathbb{T}]$ by 8.50(iii) and thus the abelian group $C_0(X, \mathbb{T})_d$ is torsion-free. Hence its character group $F(X)$ is connected.

(ii)⇒(i) If G is connected, then the character group $C_0(X, \mathbb{T})_d$ of $F(X) \cong G$ is torsion-free. Hence $w_0(X/\mathrm{conn}) = 0$ by 8.67. Hence X is connected. □

For compact and *connected* abelian groups it is easier to recognize freeness. The following is a direct consequence of 8.69 and the fact that G is connected iff $\widehat{G}$ is torsion-free.

Corollary 8.71. *A compact connected abelian group G is a free compact abelian group if and only if the maximal divisible subgroup $\operatorname{div} \widehat{G}$ has infinite cardinality $\aleph$ satisfying $\aleph^{\aleph_0} = \aleph$ and if $\operatorname{card}(\widehat{G}/\operatorname{div} \widehat{G}) \leq \aleph$.* □

Corollary 8.72. *The identity component of a free compact abelian group is a free compact abelian group.* □

Also, the following result is readily proved from the main theorem:

Corollary 8.73. *If G is a free compact abelian group, then $G/\overline{\operatorname{tor}(G)}$ is a free compact abelian group.*

Proof. We recall that the annihilator of $\overline{\operatorname{tor}(G)}$ in $\widehat{G}$ is $\operatorname{div} \widehat{G}$ (see 8.4(8)). By the Annihilator Mechanism 7.64, the character group of $G_1 \stackrel{\text{def}}{=} G/\overline{\operatorname{tor} G}$ is $\operatorname{div} \widehat{G}$. Now $D = \operatorname{div} \widehat{G}$ satisfies the conditions (a)–(d) of 8.69(ii). But $D_1 \stackrel{\text{def}}{=} \operatorname{div}\left(\widehat{G_1}\right)$ and $\widehat{G_1} = D_1 \oplus A_1$, then $D_1 \cong D$ and $A_1 = \{0\}$. Thus $\widehat{G}_1$ satisfies (a)–(d) of 8.69(ii). Hence G_1 is a free compact abelian group by 8.69. □

If we apply Theorem 8.62 to $G = F(X)$ we get the canonical exact sequence

$$0 \to \pi_1\big(F(X)\big) \to \mathfrak{L}\big(F(X)\big) \xrightarrow{\exp_{F(X)}} F(X) \to \pi_0\big(F(X)\big) \to 0.$$

The following proposition clarifies the entries in this sequence.

Proposition 8.74. *Let X be a compact pointed space. Then*

(i) $\mathfrak{L}\big(F(X)\big) \cong \mathrm{Hom}(C_0(X,\mathbb{T}),\mathbb{R})$ *and* $\mathfrak{L}'\big(F(X)\big) \cong \frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Q}_d)} \oplus (\mathbb{Q}\otimes[X,\mathbb{T}])$.

(ii) $\mathfrak{K}\big(F(X)\big) = \ker\exp_{F(X)} \cong \pi_1\big(F(X)\big) \cong \mathrm{Hom}([X,\mathbb{T}],\mathbb{Z})$.

(iii) $\pi_0\big(F(X)\big) = F(X)/F(X)_a = \mathrm{Ext}\,\big(C_0(X,\mathbb{T}),\mathbb{Z}\big)$
$\cong \mathbb{Q}^{w(X)^{\aleph_0}} \oplus \prod_{p\ \text{prime}} \mathbb{Z}_p^{w_0(X/\text{conn})} \oplus \mathrm{Ext}([X,\mathbb{T}],\mathbb{Z})$.

Proof. (i) We compute $\mathfrak{L}\big(F(X)\big) = \mathrm{Hom}\,\big(\mathbb{R},F(X)\big) \cong \mathrm{Hom}(C_0(X,\mathbb{T}),R)$ since $F(X)$ and $C_0(X,\mathbb{T})$ are character groups of each other. From 7.66 we know that the dual $\mathfrak{L}'\big(F(X)\big)$ of $\mathfrak{L}\big(F(X)\big)$ is $\mathbb{Q}\otimes C_0(X,\mathbb{T})$. Since tensoring with $\mathbb{Q}$ preserves exactness,we have an exact sequence

$$0 \to \mathbb{Q}\otimes C_0(X,\mathbb{Z}) \to \mathbb{Q}\otimes C_0(X,\mathbb{R}) \to \mathbb{Q}\otimes C_0(X,\mathbb{T}) \to \mathbb{Q}\otimes[X,\mathbb{T}] \to 0.$$

We have noted earlier that $\mathbb{Q}\otimes C_0(X,\mathbb{Z}) \cong C_0(X,\mathbb{Q}_d)$ and $\mathbb{Q}\otimes C_0(X,\mathbb{R}) \cong C_0(X,\mathbb{R})$. Thus the kernel of $\mathbb{Q}\otimes C_0(X,\mathbb{T}) \to \mathbb{Q}\otimes[X,\mathbb{T}]$ is $\frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Q}_d)}$. The direct sum $C_0(X,\mathbb{T}) = C_0(X,\mathbb{T})_0 \oplus B[X,\mathbb{T}]$ gives a direct sum $\mathbb{Q}\otimes C_0(X,\mathbb{T}) = \big(\mathbb{Q}\otimes C_0(X,\mathbb{T})_0\big) \oplus \big(\mathbb{Q}\otimes[X,\mathbb{T}]\big)$. We conclude that $\mathbb{Q}\otimes C_0(X,\mathbb{T})_0 \cong \frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Q}_d)}$.

(iii) If A is any abelian torsion group, then $\mathrm{Hom}(A,\mathbb{R}) = 0$ and thus the sequence $0 \to \mathrm{Hom}(A,\mathbb{T}) \to \mathrm{Ext}(A,\mathbb{Z}) \to 0$ is exact and shows that $\widehat{A} \cong \mathrm{Ext}(A,\mathbb{Z})$ algebraically. The group $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})$ is torsion-free and divisible, hence a $\mathbb{Q}$-vector space. Because of $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p'}\mathbb{Z}(p^\infty)$ its cardinality is $2^{\aleph_0}$ and then so is its dimension. The group $\mathrm{Hom}(\mathbb{Q},\mathbb{Q})$ is isomorphic to $\mathbb{Q}$. The exact sequence $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \to 0$ then shows that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{Q}^{2^{\aleph_0}} \cong \mathbb{Q}^{\aleph_0}$. The functor $\mathrm{Hom}(-,A)$ on the category of abelian groups transforms direct sums into direct products. The exact sequence

$$0 \to \mathrm{Hom}(-,\mathbb{Z}) \to \mathrm{Hom}(-,\mathbb{R}) \to \mathrm{Hom}(-,\mathbb{T}) \to \mathrm{Ext}(-,\mathbb{Z}) \to 0$$

then shows that $\mathrm{Ext}(-,\mathbb{Z})$ transforms direct sums into direct products. The formula $C_0(X,\mathbb{T}) = \mathbb{Q}^{(w(X)^{\aleph_0})} \oplus \bigoplus_{p\ \text{prime}} \mathbb{Z}(p^\infty)^{\left(w_0(X/\text{conn})\right)} \oplus [X,\mathbb{T}]$ from 8.66 then proves the claim in view of $\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{Q}^{\aleph_0}$ and $\mathrm{Ext}(\mathbb{Z}(p^\infty),\mathbb{Z}) \cong \mathbb{Z}_p$.

Example 8.75. Let $X = \{x_0, x\}$, $x \neq x_0$. Then $C_0(X,\mathbb{T})_d = \mathbb{T}_d$, $F(X) = \widehat{\mathbb{T}_d}$. The morphism $\iota\colon\mathbb{Z} \to F(X) = (\widehat{\mathbb{Z}})_d\widehat{\ }$, $\iota(n)(\chi) = \chi(n)$, i.e. the adjoint of the identity morphism $\mathbb{T}_d \to \mathbb{T}$ is a dense injection. Note that $\iota(1) = \varepsilon_X(x)$.

The group $F(X)$ has weight $\operatorname{card}\mathbb{T} = 2^{\aleph_0}$. Thus $F(X)$ is not metric. (In fact no nontrivial free compact group is metric as we observed in a comment following Theorem 8.67.) On the other hand, since $\iota(Z) = \langle\iota(1)\rangle$ is dense, $F(X)$ is separable.

A compact abelian group contains a dense cyclic group if and only if its character group can be homomorphically injected into the circle group $\mathbb{T}_d$. A torsion-free abelian group can be embedded into $\mathbb{T}_d \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^{(2^{\aleph_0})}$ if and only if and only if its rank does not exceed the cardinality of the continuum. Hence a compact connected abelian group is topologically generated by one element if and only if its weight does not exceed $2^{\aleph_0}$. □

$F(\{x_0, x\}))$ is also called *the universal monothetic compact group.* Its homomorphic images are called *monothetic compact groups.*

Part 4: Aspects of Homological Algebra

Injective, Projective, and Free Compact Abelian Groups

In many categories relevant for homological algebra—such as the category of abelian groups—*projective* and *injective* objects play an important role. We have formulated and worked with these concepts in the category of discrete abelian groups in Appendix 1. We proceed to introduce the corresponding ideas for the category of compact abelian groups.

Definitions 8.76. (a) A compact abelian group I is called *injective* if for any injective morphism of compact abelian groups $f: A \to B$ and each morphism $j: A \to I$ of compact abelian groups there is a morphism $j': B \to I$ such that $j = j'f$.

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{id}_I} & I \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j'} \\ A & \xrightarrow[f]{} & B \end{array}$$

(b) A compact abelian group P is called *projective* if for any surjective morphism of compact abelian groups $f: A \to B$ and each morphism $p: P \to B$ of compact abelian groups there is a morphism $p': P \to A$ such that $p = fp'$.

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{id}_P} & P \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

□

Lemma 8.77. *Let $f: G \to H$ denote a morphism of compact abelian groups and $\widehat{f}: \widehat{H} \to \widehat{G}$ its adjoint. Then f is injective if and only if $\widehat{f}$ is surjective and f is surjective if and only if $\widehat{f}$ is injective.*

Proof. This follows at once from Proposition 7.65, since in the categories of discrete and of compact abelian groups $\operatorname{im}\widehat{f}$ and $\operatorname{im} f$, respectively, are closed. □

After this lemma it is clear that projectivity and injectivity in the categories of discrete abelian groups and compact abelian groups are dual properties.

Characterizations of injective objects and projective objects in the category of compact abelian groups are now readily available via duality:

Theorem 8.78 (Projective and Injective Compact Abelian Groups). (i) *A compact abelian group is projective if and only if it is torsion free if and only if it is of the* *form* $\widehat{\mathbb{Q}}^X \times \prod_{p\in\mathcal{P}} \mathbb{Z}_p^{X_p}$ *for the set* $\mathcal{P}$ *of prime numbers and suitable sets* X *and* X_p, $p \in \mathcal{P}$. *In particular, a compact* connected *abelian group is projective if and only if it is isomorphic to* $\widehat{\mathbb{Q}}^X$ *for some set* X.

(ii) *A compact abelian group is injective if and only if it is a torus, i.e. is isomorphic to* $\mathbb{T}^X$ *for some set* X.

Proof. (i) A compact abelian group G is projective if and only if its character group $\widehat{G}$ is injective, as follows readily from the definition and the duality of the category of compact abelian groups and that of discrete abelian groups. By Proposition A1.35, $\widehat{G}$ is injective if and only if it is divisible. This is the case if and only if G is torsion-free and is of the form asserted.

(ii) A compact abelian group G is injective if and only if its character group $\widehat{G}$ is projective. This is the case if and only if $\widehat{G}$ is free abelian in view of Proposition A1.14; that is if and only if there is a set X such that $\widehat{G} \cong \mathbb{Z}^{(X)}$ by Proposition A1.6. Duality implies that this is the case if and only if $G \cong \mathbb{T}^X$. □

In the category of discrete abelian groups, an object is free if and only if it is projective. This is not generally the case in the category of compact abelian groups. It depends on topological properties of the space X whether it is true or not.

Proposition 8.79. *Let* X *be any pointed topological space. Then the free compact abelian group* $F(X)$ *is projective if and only if* $[X, \mathbb{T}]$ *is divisible.*

Proof. We have seen in 8.78(i) that a compact abelian group G is projective if and only if its character group is divisible. The character group of a free compact abelian group $F(X)$ over a compact space X is $(C_0(X,\mathbb{T})_0)_d \oplus B[X,\mathbb{T}]$ by Proposition 8.50(i) and Theorem 8.53. Hence the maximal divisible subgroup of this group is $\operatorname{div}\big(C_0(X,\mathbb{T})_d\big) = (C_0(X,\mathbb{T})_0)_d \oplus \operatorname{div}[X,\mathbb{T}]$. Thus $F(X)$ is projective if and only if $[X,\mathbb{T}]$ is divisible. □

Recall for a better understanding of this fact that $[X,\mathbb{T}] \cong H^1(X,\mathbb{Z})$ if X is paracompact.

Exercise E8.15(i). Show that $F(\mathbb{S}^1)$ is not projective. □

We note that the group $[X,\mathbb{T}]$ is certainly divisible if it is zero, and this, in turn, is certainly the case if X is contractible. Thus for all cubes $C = [0,1]^J$ the group $F(C)$ is projective.

Definitions 8.80. For a compact abelian group G we define

$$\mathfrak{P}(G) = (\mathbb{Q} \otimes \widehat{G})\widehat{\ }.$$

The exact sequence

$$(1)\qquad 0\to \operatorname{tor}\widehat{G}\xrightarrow{\text{incl}}\widehat{G}\xrightarrow{\iota_{\widehat{G}}}\mathbb{Q}\otimes\widehat{G}\to T\to 0,\quad T=\frac{\mathbb{Q}\otimes\widehat{G}}{1\otimes\widehat{G}},$$

where $\iota_A\colon A\to\mathbb{Q}\otimes A$ is given by $\iota_A(a)=1\otimes a$, and where T is a divisible torsion group, gives rise to a dual exact sequence of compact abelian groups

$$(2)\qquad 0\to\Delta(G)\xrightarrow{\text{incl}}\mathfrak{P}(G)\xrightarrow{E_G}G\xrightarrow{\text{quot}}G/G_0\to 0,$$

with the adjoint $E_G\colon\mathfrak{P}(G)\to G$ of $\iota_{\widehat{G}}\colon\widehat{G}\to\mathbb{Q}\otimes\widehat{G}$, is called the *characteristic sequence* of G. We shall also call $\mathfrak{P}(G)$ *the projective cover* of G and E_G the *projective covering morphism.* □

Since the character group $\mathbb{Q}\otimes\widehat{G}$ of $\mathfrak{P}(G)$ is divisible and torsion-free, $\mathfrak{P}(G)$ is a torsion-free projective compact abelian group. We record that $\Delta(G)$, as the character group of T and annihilator in $\mathfrak{P}(G)=(\mathbb{Q}\otimes\widehat{G})\widehat{\ }$ of $1\otimes\widehat{G}$, is a torsion-free totally disconnected subgroup of $\mathfrak{P}(G)$. In view of 8.8, it is of the form $\prod_{p\text{ prime}}\Delta_p(G)$ where $\Delta_p(G)\cong\mathbb{Z}_p^{r_p}$ where $r_p=\operatorname{rank}_p T$.

We also point out that the projective covering morphism $E_G\colon\mathfrak{P}(G)\to G$ is not a covering map in the sense of Appendix 2 because its kernel $\Delta(G)$ is never discrete unless G is torsion-free projective (in which case it is singleton). It is, however, totally disconnected. Secondly, E_G is surjective if and only if G is connected. In that case,

$$0\to\Delta(G)\to\mathfrak{P}(G)\to G\to 0$$

is what in homological algebra is called a *projective resolution* in the category of compact abelian groups, and it is a canonical one.

Exercise E8.15(ii). Prove the following statement:

Proposition. *Every element in a compact connected abelian group is contained in a* connected *compact monothetic subgroup.*

[Hint. Let G be a compact connected abelian group and let $g\in G$. Since G is connected, $E_G\colon\mathfrak{P}(G)\to G$ is surjective. Thus we find an $x\in\mathfrak{P}(G)$ such that $E_G(x)=g$. Since $\mathfrak{P}(G)$, having a torsion-free and divisible character group, is a divisible and torsion free abelian group, the smallest pure subgroup

$$[x]=\{y\in\mathfrak{P}(G):(\exists n\in\mathbb{N})\,n{\cdot}y\in\langle x\rangle\}$$

containing x (see Proposition A1.25) is isomorphic to $\mathbb{Q}$. Then $\overline{[x]}$ is divisible as well (see Theorem 8.4) and hence is connected by 8.4. Since its density is $\aleph_0$ we know from Theorem 12.25 below that it is monothetic. Therefore $E_G(\overline{[x]})$ is a compact connected monothetic group containing $g=E_G(x)$.] □

The projective cover $\mathfrak{P}(G)$ is, algebraically, a $\mathbb{Q}$-vector space. We point out a certain parallel of the characteristic sequence to the exponential sequence

$$\begin{array}{ccccccccccc} 0 & \to & \Delta(G) & \xrightarrow{\text{incl}} & \mathfrak{P}(G) & \xrightarrow{E_G} & G & \to & G/G_0 & \to & 0, \\ 0 & \to & \mathfrak{K}(G) & \xrightarrow{\text{incl}} & \mathfrak{L}(G) & \xrightarrow{\exp_G} & G & \to & G/G_a & \to & 0. \end{array}$$

Here $\mathfrak{L}(G)$ is a weakly complete topological $\mathbb{R}$-vector space, $\mathfrak{K}(G) \cong \mathrm{Hom}(\widehat{G}, \mathbb{Z})$ is a totally disconnected subgroup algebraically isomorphic to $\pi_1(G)$, which is the annihilator in $\mathfrak{L}(G)$ of the subgroup $1 \otimes \widehat{G}$ of its dual $\mathbb{R} \otimes \widehat{G}$ (in the finest locally convex topology). Also $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields $\widehat{G} \cong \mathbb{Z} \otimes \widehat{G} \to \mathbb{Q} \otimes \widehat{G} \to (\mathbb{Q}/\mathbb{Z}) \otimes \widehat{G} \to 0$ exact, whence $\widehat{\Delta(G)} \cong T \cong (\mathbb{Q}/\mathbb{Z}) \otimes \widehat{G}$. Then $\Delta(G) \cong \mathrm{Hom}\big((\mathbb{Q}/\mathbb{Z}) \otimes \widehat{G}, \mathbb{T}) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathrm{Hom}(\widehat{G}, \mathbb{T})) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, G)$ which again illustrates a certain parallel to $\mathfrak{K}(G) \cong \mathrm{Hom}(\mathbb{T}, G) \cong \pi_1(G)$ (algebraically). Further, the cokernel of $\exp_G$ is G/G_a, the quotient group modulo the arc component.

Proposition 8.81. *Let G be a compact abelian group. Then*

$$\mathfrak{L}(E_G)\colon \mathfrak{L}\big(\mathfrak{P}(G)\big) \to \mathfrak{L}(G)$$

is an isomorphism, $\exp_{\mathfrak{P}(G)}\colon \mathfrak{L}\big(\mathfrak{P}(G)\big) \to \mathfrak{P}(G)$ *is injective, and* $\pi_0\big(\mathfrak{P}(G)\big) \cong \mathrm{Ext}(\mathbb{Q} \otimes \widehat{G}, \mathbb{Z}) \cong \mathbb{Q}^{\aleph_0 \cdot \mathrm{rank}\, \widehat{G}}$. *In particular, there is an exact sequence*

$$0 \to \mathfrak{L}(G) \to \mathfrak{P}(G) \to \mathbb{Q}^{\aleph_0 \cdot \mathrm{rank}\, \widehat{G}} \to 0.$$

Also, there are isomorphisms

$$\mathfrak{L}(G) \cong \mathbb{R}^{\dim G} \text{ and } \mathfrak{P}(G) \cong \widehat{\mathbb{Q}}^{\dim G}.$$

Proof. We recall from 7.66 the natural isomorphism $e_G\colon \mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G) \to \mathrm{Hom}(\widehat{G}, \mathbb{R})$, $e_G(X)(\chi) = \mathfrak{L}(\chi)(X) \in \mathbb{R} = \mathfrak{L}(\mathbb{T})$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{L}\big(\mathfrak{P}(G)\big) & \xrightarrow{\mathfrak{L}(E_G)} & \mathfrak{L}(G) \\ {\scriptstyle e_{\mathfrak{P}(G)}}\downarrow & & \downarrow{\scriptstyle e_G} \\ \mathrm{Hom}(\mathbb{Q} \otimes \widehat{G}, \mathbb{R}) & \xrightarrow{\mathrm{Hom}(\iota_{\widehat{G}}, \mathbb{R})} & \mathrm{Hom}(\widehat{G}, \mathbb{R}), \end{array}$$

where $\widehat{\mathfrak{P}(G)}$ and $\mathbb{Q} \otimes \widehat{G}$ are identified via duality. Since $\mathrm{Hom}(\iota_{\widehat{G}}, \mathbb{R})$ is an isomorphism of topological vector spaces, so is $\mathfrak{L}(E_G)$.

The kernel of $\exp_{\mathfrak{P}(G)}\colon \mathfrak{L}\big(\mathfrak{P}(G)\big) \to \mathfrak{P}(G)$ is isomorphic to $\mathrm{Hom}(\mathbb{Q} \otimes \widehat{G}, \mathbb{Z})$ by 7.66. But this last group is zero since $\mathbb{Q} \otimes \widehat{G}$ is divisible. Hence $\exp_{\mathfrak{P}(G)}$ is injective. From Theorem 8.30 we know that the cokernel of $\exp_{\mathfrak{P}(G)}$ is $\pi_0\big(\mathfrak{P}(G)\big) \cong \mathrm{Ext}(\mathbb{Q} \otimes \widehat{G}, \mathbb{Z})$. Now $\mathrm{rank}\, \widehat{G} = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \widehat{G})$ and thus $\mathbb{Q} \otimes \widehat{G} \cong \mathbb{Q}^{(\mathrm{rank}\, \widehat{G})}$. But $\mathrm{Ext}(-, \mathbb{Z})$ transforms direct sums into direct products. (Cf. proof of 8.74.) Also recall that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^{2^{\aleph_0}} = \mathbb{Q}^{\aleph_0}$. Thus $\mathrm{Ext}(\mathbb{Q} \otimes \widehat{G}, \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Q}, \mathbb{Z})^{\mathrm{rank}\, \widehat{G}} = (\mathbb{Q}^{\aleph_0})^{\mathrm{rank}\, \widehat{G}} = \mathbb{Q}^{\aleph_0 \cdot \mathrm{rank}\, \widehat{G}}$.

Finally, $\mathfrak{L}(G) \cong \mathrm{Hom}(\widehat{G}, R) \cong \mathbb{R}^{\mathrm{rank}\,\widehat{G}} = \mathbb{R}^{\dim G}$ in view of Theorem 7.66 and Definition 8.23. Likewise, $\mathfrak{P}(G) = (\mathbb{Q} \otimes \widehat{G})\widehat{\ } \cong (\mathbb{Q}^{(\mathrm{rank}\,\widehat{G})})\widehat{\ } \cong \widehat{\mathbb{Q}}^{\mathrm{rank}\,\widehat{G}} = \mathbb{Q}^{\dim G}$. □

Let X be a compact connected pointed space. Then

$$C_0(X, \mathbb{T})_d = (C_0(X, \mathbb{T})_0)_d \oplus B[X, \mathbb{T}]$$

and the group underlying the component is isomorphic to $(C_0(X, \mathbb{R})_0)_d$ and tensoring with $\mathbb{Q}$ produces an isomorphic group. Thus we get an exact sequence

$$0 \to C_0(X, \mathbb{T})_d \xrightarrow{\iota} \mathbb{Q} \otimes C_0(X, \mathbb{T})_d \to \frac{\mathbb{Q} \otimes [X, \mathbb{T}]}{1 \otimes [X, \mathbb{T}]} \to 0$$

where $1 \otimes [X, \mathbb{T}] \cong [X, \mathbb{T}]$ since $[X, \mathbb{T}]$ is torsion-free.

Proposition 8.82. *Let X be a compact connected pointed space. Then the characteristic sequence of $F(X)$ is*

$$0 \to \Delta\big(F(X)\big) \to \mathfrak{P}\big(F(X)\big) \xrightarrow{E_G} F(X) \to 0,$$

and the character group of $\Delta\big(F(X)\big)$ is given by the exact sequence

$$0 \to [X, \mathbb{T}] \xrightarrow{\iota_{[X,\mathbb{T}]}} \mathbb{Q} \otimes [X, \mathbb{T}] \to \Delta\big(F(X)\big)\widehat{\ } \to 0. \qquad \square$$

This gives, in the case of a connected compact pointed space X, a projective resolution of $F(X)$ which illustrates once again how far, in general, the free compact abelian group $F(X)$ is from being projective, i.e. from the projective covering morphism E_G being an isomorphism.

Part 5: Aspects of Algebraic Topology—Cohomology

Cohomology of Compact Abelian Groups

We use in this section firstly the cohomology groups of compact spaces in the sense of Čech, Alexander and Wallace. (Cf. e.g. [338].) We have noted before that for a compact space X we have $H^0(X, \mathbb{Z}) \cong C(X, \mathbb{Z})$ and $H^1(X, \mathbb{Z}) \cong [X, \mathbb{T}]$.

Let G be a compact abelian group and set $X = |G|$. Then $H^0(X, \mathbb{Z}) \cong C(|G/G_0|, \mathbb{Z})$ is the group of locally constant $\mathbb{Z}$-valued functions on $X = |G|$. By 8.58 we know that for *connected* G we have $H^1(X, \mathbb{Z}) \cong \widehat{G}$. If X is a totally disconnected compact space, then $H^0(X, \mathbb{Z}) = C(X, \mathbb{Z})$, the commutative ring of continuous functions from the space X into the discrete group $\mathbb{Z}$. (See [338], p. 168, p. 309. This is consistent with what we said in 8.64.) Notice that due to the discreteness of $\mathbb{Z}$ a function $X \to \mathbb{Z}$ is continuous if and only if it is locally constant. A *Hopf algebra* (see Appendix 3, A3.65) is a commutative ring R endowed with a coassociative ring morphism $R \to R \otimes R$ and a coidentity $R \to \mathbb{Z}$ (where the coassociativity and coidentity properties are obtained from the diagrams defining associativity and the identity property by reversing arrows; see also [198] or [338]).

The exterior algebra $\bigwedge A$ over an abelian group is a Hopf algebra in a natural fashion due to a natural isomorphism $\bigwedge(A \oplus B) \to (\bigwedge A) \otimes (\bigwedge B)$ and the morphism of graded rings $\bigwedge A \to \bigwedge(A \oplus A)$ induced by the diagonal morphism $A \to A \oplus A$. (In fact, a Hopf algebra has more structure. It is a group object in the monoidal symmetric category $(\mathbb{AB}, \otimes)$; but it would lead us too far at this point to go into these details.) For a compact spaces X, Y, the Künneth Theorem (see [338]), for any commutative ring R provides an exact sequence

$$\begin{aligned} 0 \to H^*(X,R) \otimes H^*(Y,R) &\xrightarrow{\alpha_{XY}} H^*(X \times Y, R) \\ &\to \operatorname{Tor}\big(H^*(X,R), H^*(Y,R)\big) \to 0, \end{aligned}$$

with a natural injective morphism α_{XY} of graded abelian groups. If G is a compact group, then from the diagram

$$G \xrightarrow{d} G \times G \xrightarrow{m} G$$

with the diagonal map d and the multiplication map m, by applying the contravariant functor $H^*(\cdot, R)$, we obtain a diagram

$$H^*(G,R) \xrightarrow{H^*(m,R)} H^*(G \times G, R) \xrightarrow{H^*(d,R)} H^*(G,R).$$

The morphism $\alpha_{GG} \circ H^*(d,R)\colon H^*(G,R) \otimes H^*(G,R) \to H^*(G,R)$ endows the abelian group $H^*(G,R)$ with the structure of a graded ring. (The group structure of G is not needed for this insight; this maintains for every compact space G.) If $\operatorname{Tor}\big(H^*(G,R), H^*(G,R)\big) = \{0\}$ (which is the case e.g. if R is a field) then α_{GG} is an isomorphism, and $\alpha_{GG}^{-1} \circ H^*(m,R)\colon H^*(G,R) \to H^*(G,R) \otimes H^*(G,R)$ endows $H^*(G,R)$ with the structure of a Hopf algebra. We shall prove in this section the following result.

The Space Cohomology of Compact Connected Abelian Groups

Theorem 8.83. *Let X be the underlying space of a compact connected abelian group G. Then $H^*(X, \mathbb{Z}) \cong \bigwedge \widehat{G}$ as graded Hopf algebras.*

Proof. The proof will require some basic category theory and some cohomology theory. A discussion of this topic is found in greater detail in [198]. We prove this assertion by a functorial argument. (The reader may consult [198] for further references.) We consider the category $\mathbb{AB}_{\mathrm{tf}}$ of torsion-free abelian groups, and the category $\mathbb{AB}^*$ of graded abelian groups $A^0 \oplus A^1 \oplus A^2 \oplus \cdots$ and gradation preserving morphisms of abelian groups. If H is a compact group then $|H|$ denotes the space underlying H. We consider two functors $F_j\colon \mathbb{AB}_{\mathrm{tf}} \to \mathbb{AB}^*$, $j = 1, 2$ given by

$$\begin{aligned} F_1(A) &= H^*(|\widehat{A}|, \mathbb{Z}), \\ F_2(A) &= \bigwedge A. \end{aligned}$$

We proceed through several steps.

Step 1. $F_1(\mathbb{Z}) \cong F_2(\mathbb{Z})$. Indeed $|\widehat{Z}| \cong \mathbb{S}^1$, whence

$$F_1(\mathbb{Z}) = H^*(\mathbb{S}^1, \mathbb{Z}) = H^0(\mathbb{S}^1, \mathbb{Z}) \oplus H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

in view of $H^*(\mathbb{S}^1,\mathbb{Z}) \cong [\mathbb{S}^1,\mathbb{T}] = \pi_1(\mathbb{S}^1) = \mathbb{Z}$. On the other hand, $F_2(\mathbb{Z}) = \bigwedge^0 \mathbb{Z} \oplus \bigwedge^1 \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ since $\operatorname{rank}\mathbb{Z} = 1$.

Step 2. Both F_1 and F_2 preserve directed colimits. The functor $X \mapsto H^*(X,\mathbb{Z})$ from the category of compact spaces and continuous maps to the category of graded abelian groups transforms projective limits of compact spaces into direct limits of graded abelian groups; this a fundamental property of Čech cohomology (see [338]). From this fact and the definition of F_1 we get the assertion for F_1.

The functor $\bigwedge\colon \mathbb{AB} \to \mathcal{AR}$ from the category of abelian groups into the category of graded anticommutative rings is left adjoint to the grounding functor $R = R^0 \oplus R^1 \oplus R^2 \oplus \cdots \mapsto R^1$, i.e. $\mathbb{AB}(A,R^1) \cong \mathcal{AR}(\bigwedge A, R)$ (cf. Appendix 3, A3.29ff.). Since left adjoints preserve colimits (see A3.52), the functor $\bigwedge$ preserves colimits. The inclusion functor $\mathbb{AB}_{\rm tf} \to \mathbb{AB}$ and the forgetful functor $\mathcal{AR} \to \mathbb{AB}^*$ preserve directed colimits. Hence F_2 preserved directed colimits.

Step 3. Both functors F_1 and F_2 transform $\oplus$ into $\otimes$. If A and B are (discrete) abelian groups, then $(A \oplus B)\widehat{\ } \cong \widehat{A} \times \widehat{B}$, naturally. If $B \cong \mathbb{Z}$, then $F_1(B) = H^*(|\widehat{B}|,\mathbb{Z}) = H^*(\mathbb{S}^1,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ is torsion-free. Hence $\operatorname{Tor}\big(R, H^*(|\widehat{B}|,\mathbb{Z})\big) = \{0\}$ for any graded abelian group R. Hence $\alpha_{|\widehat{A}|,|\widehat{B}|}\colon H^*(|\widehat{A}|,\mathbb{Z}) \otimes H^*(|\widehat{B}|,\mathbb{Z}) \to H^*(|\widehat{A} \times \widehat{B}|,\mathbb{Z})$ is an isomorphism. Since the tensor product of torsion-free groups is torsion-free, $H^*(|\widehat{A} \times \widehat{B}|,\mathbb{Z}) = F_1(A \oplus B)$ is torsion-free if $H^*(|\widehat{A}|,\mathbb{Z}) = F_1(A)$ is torsion-free. Hence by induction with respect to rank, it follows that $F_1(A)$ is torsion-free for a finitely generated free abelian group A. Since every torsion-free group is the direct union of all finitely generated subgroups and since F_1 preserves directed colimits by Step 2 above, $F_1(A)$ is torsion-free for all torsion-free abelian groups A. Hence $\operatorname{Tor}(H^*(|A|,\mathbb{Z}),R) = \{0\}$ for all graded abelian groups R. Thus for any pair of torsion-free abelian groups A and B the morphism $\alpha_{|\widehat{A},\widehat{B}|}\colon F_1(A) \otimes F_1(B) = H^*(|A|,\mathbb{Z}) \otimes H^*(|B|,\mathbb{Z}) \to H^*(|\widehat{A} \times \widehat{B}|,\mathbb{Z}) = F_1(A \oplus B)$ is an isomorphism and thus $F_1(A \oplus B) = (F_1A) \otimes (F_1B)$. The functor $\bigwedge\colon \mathbb{AB} \to \mathbb{AB}^*$ generally satisfies $\bigwedge(A \oplus B) = (\bigwedge A) \otimes (\bigwedge B)$.

Step 4. Assume that we are now given two functors $F_j\colon \mathbb{AB}_{\rm tf} \to \mathbb{AB}^*$, $j = 1,2$ which satisfy the following properties.

(1) There is an isomorphism $\beta\colon F_1(\mathbb{Z}) \to F_2(\mathbb{Z})$.

(2) For each pair A, B of objects of $\mathbb{AB}_{\rm tf}$ there is a natural isomorphism $\pi^{(j)}_{AB}\colon F_j(A \oplus B) \to (F_jA) \otimes (F_jB)$.

(3) The functors F_j, $j = 1,2$ preserve direct colimits. Then there is a natural isomorphism $\varepsilon\colon F_1 \to F_2$.

First we note that the subcategory $\mathbb{AB}_{\rm fin}$ of finitely generated free groups in $\mathbb{AB}_{\rm tf}$ is codense in the sense that every object of $\mathbb{AB}_{\rm tf}$ is a direct colimit of objects; in fact this happens canonically because every torsion-free A determines uniquely the directed set $\mathcal{F}(A)$ of finitely generated subgroups, and $A = \operatorname{colim}\mathcal{F}(A) = \bigcup \mathcal{F}(A)$. Then any natural isomorphism $\varepsilon\colon F_1|\mathbb{AB}_{\rm fin} \to F_2|\mathbb{AB}_{\rm fin}$ extends to a natural isomorphism $\overline{\varepsilon}\colon F_1 \to F_2$. (In view of the fact that F_j preserve colimits by **(3)**, this is an exercise in diagram chasing and the universal property of the colimit.)

Thus the problem is reduced to showing that on the category of finitely generated free abelian groups, two functors which satisfy **(1)** and **(2)** are naturally isomorphic. Since every category is equivalent to its skeleton (cf. Appendix 3, Exercise EA3.10), it suffices to prove this on the full subcategory containing the set of objects $\{\mathbb{Z}^n \mid n = 0, 1, 2, \ldots\}$.

For $A = \mathbb{Z}^n$ we have a natural isomorphism $\varphi_A^{(j)}\colon F_j(A) \to \bigotimes^n F_j(Z)$ arising from an iterated application of natural isomorphisms $\varphi_{AB}^{(j)}$ given by **(2)**. Define $\varepsilon_A\colon F_1(A) \to F_2(A)$ by $\varphi^{(2)} \circ \bigotimes^n \beta \circ \varphi_A^{(1)}$. This defines a family of isomorphisms. It remains to show that ε is natural. Thus let $\lambda\colon \mathbb{Z}^n \to Z^m$ be a morphism. Then λ is determined by an $m \times n$ matrix of integers such that

$$\lambda(x_1, \ldots, x_n) = \Big(\sum_{k=1}^n a_{1k}x_k, \ldots, \sum_{k=1}^n a_{mk}x_k\Big), \quad x_i \in \mathbb{Z},\, i = 1, \ldots, n.$$

Now we get multilinear maps

$$(\xi_1, \ldots, \xi_n) \to \sum_{k=1}^n a_{1k}\xi_k \otimes \cdots \otimes \sum_{k=1}^n a_{mk}\xi_k : F_j(\mathbb{Z})^n \to \bigotimes^m F_j(\mathbb{Z}), \quad j = 1, 2,$$

$\xi_i \in F_1(\mathbb{Z})$, $i = 1, \ldots, n$. The universal property the tensor product then yields a morphism

$$T_j\lambda : \bigoplus^n F_j(\mathbb{Z}) \to \bigoplus^m F_j(\mathbb{Z}), \quad j = 1, 2,$$

determined by

$$\xi_1 \otimes \cdots \otimes \xi_n \mapsto \sum_{k=1}^n a_{1k}\xi_k \otimes \cdots \otimes \sum_{k=1}^n a_{mk}\xi_k.$$

We must show that the diagram

$$\begin{array}{ccc} F_1A & \xrightarrow{\varepsilon_A} & F_2A \\ {\scriptstyle F_1\lambda}\big\downarrow & & \big\downarrow{\scriptstyle F_2\lambda} \\ F_1B & \xrightarrow[\varepsilon_B]{} & F_2B \end{array}$$

commutes. In view of the definition of ε (after some diagram chasing) this comes down to showing that the following diagram commutes.

$$\begin{array}{ccc} \bigotimes^n F_1\mathbb{Z} & \xrightarrow{\otimes^n\beta} & \bigotimes^n F_2\mathbb{Z} \\ {\scriptstyle T_1\lambda}\big\downarrow & & \big\downarrow{\scriptstyle T_2\lambda} \\ \bigotimes^m F_1\mathbb{Z} & \xrightarrow[\otimes^m\beta]{} & \bigotimes^m F_2\mathbb{Z}. \end{array}$$

But

$$(T_2\lambda \circ \bigotimes^n \beta)(\xi_1 \otimes \cdots \otimes \xi_n) = T_2\big(\beta(\xi_1) \otimes \cdots \otimes \beta(\xi_n)\big)$$

$$= \sum_{k=1}^n a_{1k}\beta(\xi_k) \otimes \cdots \otimes \sum_{k=1}^n a_{mk}\beta(\xi_k) = \bigotimes^m \beta\Big(\sum_{k=1}^n a_{1k}\xi_k \otimes \cdots \otimes \sum_{k=1}^n a_{mk}\xi_k\Big)$$

$$= \Big(\bigotimes^m \beta \circ T_1\Big)(\xi_1 \otimes \cdots \otimes \xi_n).$$

This proves the asserted commuting of the diagram. Step 4 is complete.

Step 5. The diagram $A \stackrel{d}{\to} A \oplus A \stackrel{m}{\to} A$ with the diagonal map d and the addition m of A gives a commuting diagram of $\mathbb{AB}^*$-morphisms

$$\begin{array}{ccccc} F_1(A) & \xrightarrow{F_1 d} & F_1(A \oplus A) & \xrightarrow{F_1 m} & F_1(A) \\ {\scriptstyle \varepsilon_A}\downarrow & & {\scriptstyle \varepsilon_{A\oplus A}}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ F_2(A) & \xrightarrow[F_2 d]{} & F_2(A \oplus A) & \xrightarrow[F_2 m]{} & F_2(A). \end{array}$$

Since we have natural isomorphisms $\pi^{(j)}_{AA}: F_j(A \oplus B) \to (F_jA) \otimes (F_jB)$, and the multiplication of $F_j(A)$ is given by

$$(F_jm) \circ \big(\pi^{(j)}_{AA}\big)^{-1}: (F_jA) \otimes (F_jA) \to F_jA$$

while the comultiplication is given by

$$\pi^{(j)}_{AA} \circ (F_jd): F_jA \to (F_jA) \otimes (F_jA),$$

the morphism ε is a natural isomorphism of Hopf algebras. (One checks that it preserves identities and coidentities correctly.)

Step 5 completes the proof of the theorem. □

Corollary 8.84 (The Space Cohomology of Compact Abelian Groups). *Let X be the underlying space of a compact abelian group G. Then*

$$H^*(X, \mathbb{Z}) \cong C(|G/G_0|, \mathbb{Z}) \otimes \bigwedge \widehat{G_0} \cong \bigwedge_{C(|G/G_0|,\mathbb{Z})} C(|G/G_0|, \mathbb{Z}) \otimes \widehat{G_0}$$

as graded rings.

Proof. Firstly, we shall see in Corollary 10.38, that for any compact group G, the spaces G and $G/G_0 \times G_0$ are homeomorphic. It then follows from the Künneth Theorem [338] that there is an exact sequence

$$0 \to H^*(G/G_0, \mathbb{Z}) \otimes H^*(G_0, \mathbb{Z}) \stackrel{\alpha}{\to} H^*(G/G_0 \times G_0, \mathbb{Z})$$
$$\to \operatorname{Tor}\big(H^*(G/G_0, \mathbb{Z}), H^*(G_0, \mathbb{Z})\big) \to 0,$$

with a morphism $\alpha = \alpha_{G/G_0,G_0}$ which is natural in G/G_0 and G_0. Since $H^*(G_0, \mathbb{Z})$ is torsion-free by Theorem 8.83, the torsion term in the exact sequences vanishes.

Therefore α is a natural isomorphism. Thus in view of Theorem 8.83 we get a natural isomorphy

$$H^*(G,\mathbb{Z}) \cong H^*(G/G_0,\mathbb{Z}) \otimes_{\mathbb{Z}} \bigwedge \widehat{G_0}.$$

Since G/G_0 is totally disconnected, $H^*(G/G_0,\mathbb{Z}) = H^0(G/G_0,\mathbb{Z}) \cong C(|G/G_0|,\mathbb{Z})$ as rings.

If A is an abelian group, and R a commutative ring, then the *ground ring extension* $R \otimes \bigwedge A$ of the Hopf algebra $\bigwedge A$ is isomorphic to the Hopf algebra $\bigwedge_R(R \otimes A)$, the exterior algebra of R-modules of the R-module $R \otimes A$. (Indeed, the universal property of $\bigwedge_R$ associates with the inclusion map $R \otimes \mathrm{incl}\colon R \otimes A \to \mathbb{R}\otimes\bigwedge_{\mathbb{Z}} A$ a morphism of R-modules $\bigwedge_R(R\otimes A) \to R\otimes\bigwedge_{\mathbb{Z}} A$ which is inverted by the morphism $R \otimes \bigwedge A \to \bigwedge_R(R \otimes A)$ obtained the natural map $j\colon \bigwedge A \to \bigwedge_R(R \otimes A)$ from the universal property of $\otimes$ via the bilinear map $(r,x) \mapsto r{\cdot}j(x)\colon R \times \bigwedge A \to \bigwedge_R(R \otimes A)$.) Thus we have

$$H^*(G,\mathbb{Z}) \cong \bigwedge_{C(|G/G_0|,\mathbb{Z})} C(|G/G_0|,\mathbb{Z}) \otimes \widehat{G_0},$$

for a natural isomorphism of graded rings. This is the assertion of the corollary. □

We have made it a point that the isomorphism in 8.84 is not an isomorphism of Hopf algebras but of graded rings. This is due to the fact that we used a *topological* splitting of G into a product of G_0 and G/G_0.

We further notice that Corollary 8.84 implies that G is finite dimensional if and only if the rank of the abelian group $H^*(X,\mathbb{Z})$ is finite.

Since $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Q}) = \{0\}$ as $\mathbb{Q}$ is torsion-free, the universal coefficient formula gives us a natural isomorphism $H^*(X,\mathbb{Z}) \otimes \mathbb{Q} \to H^*(X,\mathbb{Q})$. (See [338].)

Corollary 8.85. *Let X be the underlying space of a compact connected abelian group G and assume that $n \stackrel{\mathrm{def}}{=} \dim G = \operatorname{rank} \widehat{G} < \infty$. Then*

$$\operatorname{rank} H^m(X,\mathbb{Z}) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes H^m(X,\mathbb{Z}) = \dim_{\mathbb{Q}} H^m(X,\mathbb{Q}) = \binom{n}{m}$$

for $m = 0, 1, 2, \dots$.

In particular, $H^m(\mathbb{T}^n,\mathbb{Z}) = \mathbb{Z}^{\binom{n}{m}}$.

Proof. This follows straightforwardly from the preceding. □

Let K denote any field of characteristic 0. Then the universal coefficient formula and Theorem 8.83 yield a graded Hopf algebra isomorphism

$$(*) \qquad H^*(G,K) \cong K \otimes \bigwedge \widehat{G} \cong \bigwedge\nolimits_K (K \otimes \widehat{G})$$

for any compact connected abelian group G. If, in addition, G is a compact Lie group, i.e. a torus, then this corollary agrees with the Hopf–Samelson Theorem for Connected Compact Lie Groups 6.88.

Since $H^*(G,\mathbb{Z})$ is a torsion-free abelian group by 8.83 as $\widehat{G}$ is torsion-free by 8.5, we have $\mathrm{Tor}(H^*(G,\mathbb{Z}),K)=0$ for any abelian group K. Thus the formula $(*)$ above remains intact for any commutative ring K with identity.

Part 6: Aspects of Set Theory

Arc Components and Borel Sets

Definition 8.86. A subset B of a space X is a *Borel subset* if and only if it belongs to the smallest σ-algebra of sets containing the open subsets of X. □

Lemma 8.87. *Assume that X and Y are topological spaces and $B\subseteq X$. Assume that $y_0\in Y$ is such that $\{y_0\}$ is a Borel subset of Y. Then a subset $B\subset X$ is a Borel subset if and only if $B\times Y$ is a Borel subset of $X\times Y$.*

Proof. The projection $p\colon X\times Y\to X$ is continuous. Hence if B is a Borel subset then $B\times Y=p^{-1}(B)$ is a Borel subset of $X\times Y$. The function $x\mapsto(x,y_0):X\to X\times\{y_0\}$ is a homeomorphism, and thus B is a Borel subset of X iff $B\times\{y_0\}$ is a Borel subset of $X\times\{y_0\}$. Since $\{y_0\}$ is a Borel subset of Y, the set $X\times\{y_0\}$, being full inverse image of the Borel subset $\{y_0\}$ in Y under the projection $X\times Y\to Y$ is a Borel subset in $X\times Y$. Hence B is a Borel subset of X iff $B\times\{y_0\}$ is a Borel subset of $X\times Y$. Now assume that $B\times Y$ is a Borel subset of $X\times Y$. Then $B\times\{y_0\}=(B\times Y)\cap(X\times\{y_0\})$, as the intersection of two Borel subsets of $X\times Y$, is a Borel subset of $X\times Y$, and thus B is a Borel subset of X. □

Now consider a compact connected abelian group A. The union of all circle subgroups is contained in a unique fully characteristic smallest closed subgroup A_ℓ, containing all torus groups, called the *locally connected component* of A by Definition 8.40. We now refer to Theorem 8.46, and summarize:

Lemma 8.88. *Let A be a compact connected abelian group and assume that its locally connected component A_ℓ is metric. Then*

(i) *A_ℓ is a torus (i.e., a product of circle groups).*

(ii) *There is a closed connected subgroup H of A such that $A\cong A_\ell\times H$. Moreover,* $\exp\colon\mathfrak{L}(H)\to H$ *is injective.* □

Lemma 8.89. *Let A be a compact connected abelian group and T a torus subgroup (i.e. a product of circle groups). Then the following conditions are equivalent.*

(i) *A_a is a Borel subset of A.*

(ii) *$(A/T)_a$ is a Borel subset.*

Proof. From Theorem 8.78 we know that $A\cong T\times A/T$. Now $(T\times A/T)_a=T\times(A/T)_a$. The assertion now follows from Lemma 8.87. □

Lusin spaces

Here we refer particularly to [34], Chapter 9, which also appeared separately as Bourbaki, N., Topologie générale, Chap. 9, Utilisation des nombres réels, Hermann, Paris, 1958[2].

A *Polish space* is a completely metrizable second countable space (cf. loc. cit., §6 n° 1, Déf. 1.). This applies to second countable locally compact groups. We use the following characterisation of

([34], §6, n° 4, Déf. 6.): *A metrizable space is a Lusin space if and only if it is a bijective continuous image of a zero dimensional Polish space.* What is relevant is

Lemma 8.90. $\mathbb{R}$ *is a Lusin space.*

Proof. It is argued in Bourbaki, loc. cit. §6, n° 4, Lemme 2, that a metric space which is a countable disjoint union of Lusin spaces is a Lusin space. Now $\mathbb{R}$ is a countable disjoint union of singleton sets containing rational points and the set of irrationals. The latter one is a zero dimensional Polish space, hence is a Lusin space. Therefore $\mathbb{R}$ is a Lusin space. □

Since a countable product of zero dimensional Polish spaces is a zero dimensional Polish space we conclude at once:

Lemma 8.91. *A countable product of Lusin spaces is a Lusin space.* □

Hence $\mathbb{R}^n$, $n \in \mathbb{N}$ and $\mathbb{R}^{\mathbb{N}}$ are Lusin spaces.

Lemma 8.92. *If A is a metrizable compact abelian group, then $\mathfrak{L}(A)$ is a Lusin space.*

Proof. As a topological vector space, $\mathfrak{L}(A)$ is isomorphic to one of the vector spaces $\mathbb{R}^n$, $n \in \mathbb{N}$ or $\mathbb{R}^{\mathbb{N}}$. (Cf. Definition 5.7 and Proposition 7.36: $\mathfrak{L}(A) \cong \mathrm{Hom}(\widehat{A}, \mathbb{R}) \cong \mathbb{R}^F$ where F is a maximal free subset of the *countable* character group $\widehat{A}$.) □

Proposition 8.93. *Every bijective continuous image of a Polish space is a Lusin space.*

Proof. See [34], §6, n° 4, Proposition 12. □

Theorem 8.94. *The identity arc component A_a of a compact connected* metric *abelian group A is a Borel subset.*

Proof. By Lemma 8.88 A is isomorphic to the direct product of the torus A_ℓ and a metric compact subgroup H for which $\exp\colon \mathfrak{L}(H) \to H$ is injective. Now $\mathfrak{L}(H)$ is a Lusin space by Lemma 8.92. But then $H_a = \exp_H \mathfrak{L}(H)$ is the bijective image of the Lusin space $\mathfrak{L}(H)$ and thus is a Lusin space by Proposition 8.93. Then by

[34], §6, n$^{\text{o}}$ 7, Théorème 3, $(A/A_\ell)_a \cong H_a$ is a Borel subset of A/A_ℓ. Hence A_a is a Borel subset of A by Proposition 8.93. □

Note that all arc components are homeomorphic since A is homogeneous.

Let us now consider the following two propositions.

The Borel Set Proposition. *In* any *compact abelian group, the arc components are Borel subsets.*

The Anti-Borel Set Proposition. *There exists a compact abelian group G of weight $w(G) = \aleph_1$ such that G_a is not a Borel set.*

We shall argue that there is a model of set theory in which the Generalized Continuum Hypothesis and the Axiom of Choice hold and in which the Anti-Borel Set Proposition holds.

We do not know whether in a *constructible universe* the arc components of *all* compact abelian groups are Borel sets. A proof of this fact would show that the Borel Set Proposition is *undecidable in ZFC*.

It appears to be very hard to determine that a given subset of a topological space is *not* a Borel subset. In a compact group there is a classical trick that provides a sufficient condition.

Proposition 8.97. *Let C be a subgroup of a compact group G such that the set G/C of cosets gC is countably infinite. Then C is not Haar measurable and thus is not a Borel subset.*

Proof. Let m be normalized Haar measure on G and suppose that C is Haar measurable. Let $g_1, g_2, \ldots$ be a sequence of elements such that $\{g_nC : n \in \mathbb{N}\}$ is an enumeration of G/C. Then for each natural number N, using the invariance of m, we have

$$N{\cdot}m(C) = \sum_{n=1}^{N} m(g_nC) = m(g_1C \cup \cdots \cup g_NC) \le m(G) = 1.$$

Hence $m(C) \le 1/N$ for all $N = 1, 2, \ldots$ and thus $m(C) = 0$. But then $1 = m(G) = m(\bigcup_{n\in\mathbb{N}} g_nC) = \sum_{n\in\mathbb{N}} m(g_nC) = 0$: a contradiction which completes the proof. □

From 8.30 we know that *for any compact abelian group G and its character group A the (abstract) factor group G/G_a is isomorphic to* $\operatorname{Ext}(A,\mathbb{Z})$.

Thus Theorem 8.30 and Proposition 8.97 motivate us to formulate the following statement:

The Countability Proposition. *There is an abelian group such that* $\operatorname{Ext}(A,\mathbb{Z})$ *is countably infinite.*

By Theorem 8.30 and Proposition 8.97, *the Countability Proposition implies the Anti-Borel Set Proposition.*

The authors are indebted to Laszlo Fuchs who reported the following fact.

The Countability Proposition is undecidable in ZFC.

Indeed, first of all, if A is countable, but not free, then $\mathrm{Ext}(A,\mathbb{Z})$ is uncountable. For higher cardinalities, in Gödel's constructible universe it is either 0 or its torsion free part is uncountable (see e.g. the book [99], Chapter XII, in particular, Corollary 2.5).

On the other hand Shelah [330] has shown that even if the Generalized Continuum Hypothesis is assumed, for every countable divisible group D there is a model of set theory in which there is an abelian group A of cardinality $\aleph_1$ such that $\mathrm{Ext}(A,\mathbb{Z}) \cong D$.

Shelah's proof starts with any model V of ZFC+GCH, and he shows that it has an extension by forcing in which, for a suitable A, the group $\mathrm{Ext}(A,\mathbb{Z})$ is an arbitrarily prescribed countable divisible group.

In particular:

Theorem 8.99. *There is a model of set theory in which there is a compact abelian group G of weight $\aleph_1 = 2^{\aleph_0}$ such that arc component factor group is algebraically isomorphic to $\mathbb{Q}$ and hence has arc components which fail to be Borel subsets.* □

In the article [171] the theory is carried out for locally compact groups in place of compact groups. Given some standard structural information on locally compact groups, the generality is not much greater than that which we have presented in this section.

Postscript

While it is true that, in the face of duality, the structure of compact abelian groups can be thought of as purely in the realm of algebra, the questions which interest topological group theorists emphasize geometric aspects. The relevant geometry here concerns the various levels of connectivity. Specifically, we examine topological connectivity, arcwise connectivity, local connectivity, simple connectivity, homotopy and cohomology. It is, for example, remarkable that the underlying topology of a compact connected abelian group completely determines its structure as a topological group (8.59).

Topological connectivity of a compact abelian group is intimately tied up with group theoretical properties such as torsion and divisibility (8.4). As a consequence, a compact abelian group is connected if and only if it is divisible if and only if its character group is torsion-free.

Arcwise connectivity in a compact abelian group G is exposed in Theorem 8.30. The relevant fact here is that the arc component G_a of the identity is the image of the exponential function [86]. In other words, an element can be reached by an arc from the origin if and only if it can be reached by a one-parameter subgroup.

Local connectivity and arcwise connectivity are linked in the classical theory of metric spaces in so far as connected locally connected complete metric spaces are arcwise connected (see e.g. [101], p. 377, 6.3.14). This is reflected in Theorem 8.46 where it is noted that a compact connected metric group is locally connected if and only if it is arcwise connected if and only if it is a (finite or infinite) dimensional torus. In the absence of metrizability things are infinitely more complex. There is always a smallest closed subgroup containing all torus subgroups which is locally connected and which we call the locally connected component; we discuss it in Theorem 8.41. There are connected locally connected compact abelian groups which are not arcwise connected (8.36). One might reasonably conjecture that the Torus Proposition holds which says that every arcwise connected compact abelian group is a torus (statement preceding 8.48)—however, it turns out, amazingly, that this proposition or its negation may be adjoined to Zermelo–Fraenkel Set Theory plus Axiom of Choice without creating a contradiction (8.48). This is based on a theorem by Shelah [329], an exposition of which is given by Fuchs, reproduced in Appendix 1. A good reference for set theory is [221]. We saw in Theorem 8.94 that the identity arc component of a compact metric abelian group is a Borel subset and in Theorem 8.99 that there is a model of Zermelo–Fraenkel set theory in which the Axiom of Choice and the Generalized Continuum Theory hold and in which there is a compact abelian group of weight $\aleph_1$ such that $\pi_0(G) = G/G_a$ is isomorphic to $\mathbb{Q}$. Therefore, G_a cannot be a Borel subset (see Proposition 8.97). This provides yet another example in which the issue of arc connectedness leads us into subtle issues of set theory.

There is a very strong form of local connectivity of compact abelian groups described in Theorem 8.36bis. A compact connected group G has this property iff its character group $\widehat{G}$ is a torsion free group in which every finite rank pure subgroup is free and is a direct summand. Remarkably, this is equivalent to saying that the exponential function maps $\mathfrak{L}(G)$ openly onto the arc component G_a. Therefore $\mathfrak{L}(G)/\mathfrak{K}(G) \cong G_a$ where $\mathfrak{K}(G) = \ker\exp \cong \pi_1(G)$, and this also provides a striking example of a quotient group of a complete topological vector space $\mathfrak{L}(G)$ which is incomplete and indeed has a compact completion. More information is to be found in [193].

The fundamental group $\pi_1(G)$ of a compact abelian group, G, is a measure of how far G is from being simply connected. The group $\pi_1(G)$, and all other homotopy groups of G, are completely calculated in Theorem 8.62 ([8], [104]). Their close connection with the exponential function emerges there.
A lot of information is available about the question which torsion free abelian groups arise as group $\pi_1(G)$ for a compact connected abelian group G; but its presentation here would have required a substantial extension of our Appendix 1 on abelian groups. We therefore refer the interested reader to [99], Chapter 14, pp. 420–452.

The cohomology of a compact pointed space X is encountered in our presentation in the form of the group $[X, \mathbb{T}]$ of homotopy classes of continuous base point preserving functions $X \to \mathbb{T}$ into the circle. This group is known from algebraic topology to be naturally isomorphic to the first Čech–Alexander–Wallace cohomology group $H^1(X, \mathbb{Z})$. The key is that for a compact connected abelian group,

G, there is a natural isomorphism $\widehat{G} \to [G, \mathbb{T}]$ which arises out of the fact that every homotopy class of continuous base point preserving functions $G \to \mathbb{T}$ contains exactly one character (Theorem 8.57). This yields the previously mentioned result that the topology of G completely determines G as a topological group. The first cohomology suffices to yield the complete integral cohomology ring (and indeed Hopf algebra) of G with the aid of functorial arguments (see [198]).

A powerful instrument in analyzing the local structure of a compact abelian group, G, is a surprisingly general theorem which we call the Resolution Theorem 8.20, and which says that every compact abelian group G is a quotient group of a direct product of a compact totally disconnected subgroup of G and a weakly complete topological vector space, namely, the Lie algebra $\mathfrak{L}(G)$. This theorem becomes particularly lucid whenever the quotient map is a covering map as is the case whenever G is finite dimensional.

This calls for a discussion of the somewhat frustrating topic of topological dimension where a plethora of definitions exist. Happily, these coincide for locally compact groups [284]. Our approach in this chapter is to extract the key features of a well behaved, and algebraically defined dimension function on the class of locally compact abelian groups whose characteristic properties are satisfied by the various dimension functions (8.25 and 8.26).

A pervasive theme in any abelian category is homological algebra involving a classification of injectives, projectives, and free objects. The first two are easily dealt with via duality. Free compact abelian groups $F(X)$ over compact spaces X are more complicated and more interesting. They are defined in the usual categorical manner. But we have a complete description of the fine structure in Theorems 8.67 and 8.69. These results link the structure of $F(X)$ in a transparent fashion with the geometry of the generating space X. Indeed, $F(X)$ is completely determined by the weight, $w(X)$, of X, the weight, $w(X/\mathrm{conn})$, of the space of connected components of X, and the group $[X, \mathbb{T}] \cong H^1(X, \mathbb{Z})$.

We studied the cohomology of compact Lie groups in Chapter 6 (cf. 6.88ff.), at least for ground fields of characteristic 0. This required a fairly complete theory of graded commutative Hopf algebras over fields of characteristic 0 which we presented in Appendix 3 (cf. A3.65ff.). The reader will find a fairly self-contained approach to the space cohomology of compact connected abelian groups as the last item of the present chapter. In so far as we deal here with *commutative* groups, on the one hand we are in a more special situation than we were in Chapter 6; on the other hand, we have no restriction to Lie groups here and have a complete cohomology theory over $\mathbb{Z}$ rather than a field of characteristic zero; all other ground rings can be treated from here via the universal coefficient theory of cohomology when needed. For a compact connected abelian group G, the character group $\widehat{G}$ emerges precisely as the subgroup of *primitive* elements of the cohomology Hopf algebra $H^*(G, \mathbb{Z})$ (cf. A3.71).

We have encountered one of the standard graded Hopf algebras, namely, the exterior algebra $\bigwedge A$ for a torsion-free abelian group as the space cohomology of a compact connected abelian group $G = \widehat{A}$. The Hopf algebra structure of the exterior algebra is discussed in Appendix 3, see A3.67. Another standard Hopf algebra is the symmetric (or polynomial) algebra $\mathbf{S}A$ over a torsion-free abelian group A. For reasons that would become clear through the identification of this Hopf algebra in terms of algebraic topology one makes the assumption in this case that the elements of a inside $\mathbf{S}A$ are homogeneous of degree two, so that all nonzero homogeneous elements have even degree. This Hopf algebra turns out to be the Čech cohomology Hopf algebra of the classifying space $B(G)$ of the compact connected abelian group $G = \widehat{A}$. For the details we refer to [198]. On the subcategory of compact abelian Lie groups, Čech cohomology agrees with singular cohomology; however, on arbitrary compact groups they disagree. Some general information on the singular cohomology of a compact connected abelian group may be found in [198], p. 209.

References for this Chapter—Additional Reading

[8], [15], [23], [34], [46], [86], [100], [101], [103], [104], [109], [114], [115], [143], [147], [168], [171], [178], [198], [206], [210], [221], [230], [275], [276], [284], [286], [193], [264], [329], [330], [331], [338], [359], [363].

Chapter 9

The Structure of Compact Groups

In this chapter we present impressive and powerful theorems which describe the structure of compact groups. For example, every compact connected group G is a semidirect product of its commutator subgroup G' and an abelian topological subgroup isomorphic to G/G'. So, in particular, G is homeomorphic to the product of these two groups. Now G/G' is, of course, a compact connected abelian group, and the structure of such groups is known from Chapter 8. So it emerges that compact abelian groups are not simply examples of compact groups but basic ingredients of the structure of all compact groups. The group G' is a semisimple compact connected group. We will see that such groups are almost products of compact connected simple simply connected compact Lie groups. To be more precise, they are quotients of such groups with the kernels being central and totally disconnected. If we add to this the topological splitting Theorem 10.37, which we shall prove in the next chapter, we obtain: *If H is any compact group, then it is homeomorphic to the product of three groups: H/H_0, $(H_0)'$ and $H_0/(H_0)'$.* The first of these factors is a compact totally disconnected group and such groups are known to be homeomorphic to a product of two-point discrete spaces.

The second principal structure theorem for compact groups reduces in the connected case to: *If G is a compact connected group, then it is topologically isomorphic to the quotient group of the direct product $Z_0(G) \times G'$ obtained by factoring out a central totally disconnected compact group isomorphic to $Z_0(G) \cap G'$.* And, as noted above, G' can be expressed as a quotient group of a product of compact simple simply connected Lie groups with totally disconnected central kernel.

The third result we highlight here is the fact that compact connected groups have maximal pro-tori and the behavior and importance of these exactly parallels that of maximal tori in compact connected Lie groups.

The fourth result shows that the commutator group G' of a compact connected group is a semidirect factor, so that $G \cong G' \rtimes (G/G')$ with a compact connected abelian group G/G' whose structure we discussed extensively in Chapter 8. The fifth major structure theorem shows that every compact group G contains a totally disconnected group D such that $G = G_0 D$ and $G_0 \cap D$ is normal in G and central in G_0, we can even pick D in the normalizer of a preassigned maximal pro-torus.

These basic structure theorems are then applied to a theory of the fine structure of a compact group. Among many other things, we shall see structure theorems for the automorphism group of a compact group.

Prerequisites. This chapter requires no prerequisites beyond those in previous chapters, but familiarity with Chapter 2, Chapter 6 (which in turn rests on Chapter 5) and Chapter 8 (which requires Chapter 7) is essential.

Part 1: The Fundamental Structure Theorems of Compact Groups

Approximating Compact Groups by Compact Lie Groups

Historically questions about compact groups were answered in each case by seeking a reduction to the case of compact Lie groups. This can be avoided by using the structure theorems which we shall develop later in this chapter. The key lemma in avoiding the repeated use of projective limits is presented here.

For a topological group G let $\mathcal{N}(G)$ denote the set of all closed normal subgroups N of G such that G/N is a Lie group. Since the one element group G/G is a Lie group, trivially $G \in \mathcal{N}(G)$. In Chapter 2 we proved that every compact group is a strict projective limit of compact Lie groups (Corollary 2.43). By 1.33 this means that each compact group G is the projective limit of all compact Lie group quotients G/N. We shall use the following conclusions from this basic result.

Lemma 9.1. *Let G be a compact group. Then*

(i) *$\mathcal{N}(G)$ is closed under finite intersections. In particular, $\mathcal{N}(G)$ is a filter basis.*

(ii) $\bigcap \mathcal{N}(G) = \{1\}$.

(iii) *If H is a closed subgroup of G then $\bigcap_{N\in\mathcal{N}(G)} HN = H$.*

Proof. (i) If $M, N \in \mathcal{N}(G)$, then $f\colon G/(M\cap N) \to G/M \times G/N$, $f(g(M\cap N)) = (gM, gN)$ is injective. Since $G/M \times G/N$ is a compact Lie group, so is $L \overset{\text{def}}{=} f\big(G/(M\cap N)\big)$. Since G is compact, so are $G/(M\cap N)$ and L, and the corestriction of f to $G/(M\cap N) \to L$ is a homeomorphism. Hence $G/(M\cap N)$ is a compact Lie group.

(ii) By 2.39 and 1.33, G is a strict projective limit $G = \lim_{N\in\mathcal{N}(G)} G/N$. In particular, this implies that the quotient maps $G \to G/N$, $N \in \mathcal{N}(G)$ separate the points. This is equivalent to (iii).

(iii) Let $g \in \bigcap_{N\in\mathcal{N}(G)} HN$. Then for all $N \in \mathcal{N}(G)$ one has $g \in HN$, i.e. $Hg \cap N \neq \emptyset$. Then $\{Hg\cap N \mid N \in \mathcal{N}(G)\}$ is a filter basis of compact sets and then (iii) implies $Hg \cap \{1\} = Hg \cap \bigcap \mathcal{N}(G) = \bigcap_{N\in\mathcal{N}(G)}(Hg\cap N) \neq \emptyset$. Hence $1 \in Hg$; i.e. $g \in H$. □

Exercise E9.1. Formulate a sufficient condition for (ii) to be satisfied in an arbitrary topological group. Find sufficient conditions for (i) to be satisfied.

A filter basis $\mathcal{F}$ on a topological space G is said to *converge* to $g \in G$, written $g = \lim \mathcal{F}$, if for every neighborhood U of g there is an $F \in \mathcal{F}$ such that $F \subseteq U$. Assume that $\mathcal{N}(G)$ is a filter basis and that $G = \lim_{N\in\mathcal{N}(G)} G/N$. Prove

(iv) $1 = \lim \mathcal{N}(G)$. □

For more details on matters discussed in 9.1 and E9.1 see e.g. [194].

The Closedness of Commutator Subgroups

It is a remarkable fact that each element of the commutator subgroup of a compact connected group is a commutator, and it is an eminently useful consequence that the commutator subgroup is closed. Indeed one of the major results of this chapter will imply that every compact connected group G is homeomorphic to $G' \times G/G'$.

Gotô's Commutator Subgroup Theorem for a Compact Connected Group

Theorem 9.2. *Let G be a compact connected group. Then the function $(g, h) \mapsto ghg^{-1}h^{-1}: G \times G \to G'$ is surjective; that is every element of G' is a commutator.*

In particular, the algebraic commutator subgroup G' of a compact connected group is closed and thus is a compact connected group.

Proof. Let $g \in G'$. The function $\mathrm{comm}: G \times G \to G'$, $\mathrm{comm}(x, y) = xyx^{-1}y^{-1}$ is continuous. Hence for each $N \in \mathcal{N}(G)$, the set $C_N \stackrel{\mathrm{def}}{=} \{(x, y) \in G \times G \mid \mathrm{comm}(x, y) \in gN\}$ is closed in $G \times G$ and thus compact. If $N \subseteq M$ in $\mathcal{N}(G)$, then $C_N \subseteq C_M$. Now we apply Theorem 6.55 and conclude that for $N \in \mathcal{N}$ the element gN in the algebraic commutator group $G'N/N = (G/N)'$ of the connected Lie group G/N is a commutator. Thus there are elements xN, $yN \in G/N$ such that $\mathrm{comm}(x, y)N = xyx^{-1}y^{-1}N = (xN)(yN)(xN)^{-1}(yN)^{-1} = gN$. Hence $(x, y) \in C_N$. Thus $\{C_N \mid N \in \mathcal{N}(G)\}$ is a filter basis of compact subsets of $G \times G$ and therefore there is an element $(x, y) \in \bigcap_{N \in \mathcal{N}(G)} C_N$. Then $\mathrm{comm}(x, y) \in gN$ for all $N \in \mathcal{N}(G)$. But now we invoke 9.1(ii) and conclude $\bigcap_{N \in \mathcal{N}(G)} gN = \{g\}$. Thus $g = \mathrm{comm}(x, y)$. This proves the first assertion. The second is an immediate consequence because $G' = \mathrm{comm}(G \times G)$ is compact as the continuous image of a compact space. □

It is not true that the commutator subgroup of a compact group has to be closed. Indeed, Exercise E6.6 provides an example of a compact totally disconnected group whose commutator subgroup fails to be closed. Note however that we saw in Theorem 6.11 that the commutator subgroup of a compact Lie group is always closed.

The theorem above allows us to formulate a necessary and sufficient condition for the commutator subgroup of an arbitrary compact group to be closed.

Corollary 9.3. *For a compact group G, the algebraic commutator subgroup G' is closed if and only if the algebraic commutator group $(G/(G_0)')'$ is closed in $G/(G_0)'$.*

Proof. The group $(G_0)'$ is contained in G' and compact by 9.2. Since a subgroup H containing $(G_0)'$ is closed if and only if $H/(G_0)'$ is closed in $G/(G_0)'$, the assertion follows. □

From 6.10 we recall that an interesting characteristic subgroup of a compact group G is the group $\mathrm{comm}(G, G_0)$ algebraically generated by the elements $xyx^{-1}y^{-1}$, $x \in G$, $y \in G_0$. In 6.10 it was shown that this group is closed if G is a compact Lie group. We show in the following exercise, that it need not be closed in general.

Exercise E9.2. Review the circumstances in which the algebraic commutator subgroup is closed. (See Proposition 6.10, Theorem 6.11, Exercise E6.6, Theorem 6.18, Exercise E6.9, Theorem 6.55, Corollary 6.56, Theorem 9.2, Corollary 9.3. □

Semisimple Compact Connected Groups

The first proposition in this section has the noteworthy result that every compact connected group G satisfies $G'' = G'$. We call a compact connected group *semisimple* if the commutator subgroup is the whole group. In particular then the commutator subgroup of each compact connected group is semisimple. This section is devoted to a complete description of semisimple groups. This will be done by exposing the rather clear-cut structure of simply connected semisimple compact groups.

Proposition 9.4. *Let G be a compact connected group. Then $G'' = G'$.*

Proof. From 9.2 we know that G' and G'' are compact connected groups if G is compact connected. If $N \in \mathcal{N}(G)$, then $G''N/N = (G/N)'' = (G/N)' = G'N/N$ by 6.14(ii). Thus $G' \subseteq G''N$. Since $N \in \mathcal{N}(G)$ was arbitrary, 9.1(iii) proves $G' \subseteq G''$ which proves the claim since $G'' \subseteq G'$ is always true. □

The 8 element quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ has the 2 element commutator group $G' = \{1, -1\}$ while $G'' = \{1\}$. Thus 9.4 fails at once in the disconnected case.

Definition 9.5. A compact connected group G is called *semisimple* if $G = G'$. □

Corollary 6.16 shows that this definition agrees for compact connected Lie groups with Definition 6.17(ii).

Corollary 9.6. *The commutator subgroup of every compact connected group is semisimple.*

Proof. This follows from Proposition 9.4 and Definition 9.5. □

There is a profoundly interesting link between semisimple compact connected groups and simple connectivity. To expose this we need a sequence of technical lemmas.

Lemma 9.7. (i) *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and assume that* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{i} = \mathfrak{n} \oplus \mathfrak{j}$ *for ideals* $\mathfrak{n}$, $\mathfrak{i}$, *and* $\mathfrak{j}$ *of which* $\mathfrak{i}$ *and* $\mathfrak{j}$ *are semisimple. Then* $\mathfrak{i} = \mathfrak{j}$.

(ii) *Consider a diagram of morphisms of finite dimensional Lie algebras*

$$\begin{array}{ccccc} \mathfrak{s} & \xrightarrow{\text{id}} & \mathfrak{s} & \xrightarrow{\text{id}} & \mathfrak{s} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathfrak{a} & \xrightarrow[\varphi]{} & \mathfrak{b} & \xrightarrow[\psi]{} & \mathfrak{c}, \end{array}$$

and assume the following hypotheses:

(a) $\mathfrak{b}$ *is compact and* $\mathfrak{s}' = \mathfrak{s}$.
(b) $\operatorname{im}\alpha$ *and* $\operatorname{im}\beta$ *are ideals.*
(c) γ *is an isomorphism.*
(d) *The right square commutes, i.e.* $\gamma = \psi\beta$.
(e) *The outside rectangle commutes, i.e.* $\gamma = \psi\varphi\alpha$.
(f) φ *is surjective.*

Then the left square commutes; i.e. $\beta = \varphi\alpha$.

Proof. (i) The assumption implies $\mathfrak{n}' \oplus \mathfrak{i} = \mathfrak{n}' \oplus \mathfrak{i}' = \mathfrak{g}' = \mathfrak{n}' \oplus \mathfrak{j}' = \mathfrak{n}' \oplus \mathfrak{j}$, where $\mathfrak{n}'$ is semisimple, too, by 6.4(vi), (vii). Again by 6.4(vi), (vii) we have $\mathfrak{g}' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ for a unique family $\{\mathfrak{s}_1, \cdots, \mathfrak{s}_n\}$ of simple ideals, and $\mathfrak{n}'$ is a sum of a unique subfamily, say $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_k\}$ (after renaming, if necessary). Then once more by 6.4(vii) we have $\mathfrak{i} = \mathfrak{s}_{k+1} \oplus \cdots \oplus \mathfrak{s}_n = \mathfrak{j}$.

(ii) By (c) and (d) $\mathfrak{b} = \ker\psi \oplus \beta(\mathfrak{s})$. By (c) and (e) we have $\mathfrak{b} = \ker\psi \oplus (\varphi\alpha)(\mathfrak{s})$. By (f) and (b), both $(\varphi\alpha)(\mathfrak{s})$ and $\beta(\mathfrak{s})$ are ideals of $\mathfrak{b}$. By (a), Part (i) of the Lemma applies and shows that $(\varphi\alpha)(\mathfrak{s}) = \beta(\mathfrak{s})$. Denote this ideal of $\beta(\mathfrak{s})$ by $\mathfrak{i}$. Then by (c) and (d) the map $\psi|\mathfrak{i}\colon \mathfrak{i} \to \mathfrak{c}$ is an isomorphism. It now follows from (d) and (e) that for each $s \in \mathfrak{s}$ we have $\varphi\big(\alpha(s)\big) = (\varphi|\mathfrak{i})^{-1}\gamma(s) = \beta(s)$. $\square$

Lemma 9.8. *Consider a diagram of morphisms of compact connected Lie groups*

$$\begin{array}{ccccc} S & \xrightarrow{\text{id}} & S & \xrightarrow{\text{id}} & S \\ {\scriptstyle\alpha_*}\downarrow & & {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\gamma_*} \\ A & \xrightarrow[\varphi_*]{} & B & \xrightarrow[\psi_*]{} & C, \end{array}$$

and assume the following hypotheses:

(a) S *is semisimple.*
(b) $\operatorname{im}\mathfrak{L}(\alpha_*)$ *and* $\operatorname{im}\mathfrak{L}(\beta_*)$ *are ideals of* $\mathfrak{L}(B)$.
(c) γ_* *is a covering homomorphism.*
(d) *The right square commutes; i.e.* $\gamma_* = \psi_*\beta_*$.
(e) *The outside square commutes; i.e.* $\gamma_* = \psi_*\varphi_*\alpha_*$.
(f) φ_* *is surjective.*

Then the left square commutes; i.e. $\gamma_* = \varphi_*\alpha_*$.

Proof. We set $\mathfrak{a} = \mathfrak{L}(A)$, $\mathfrak{b} = \mathfrak{L}(B)$, $\mathfrak{c} = \mathfrak{L}(C)$, $\mathfrak{s} = \mathfrak{L}(S)$, and $L(\xi_*) = \xi$ for all morphisms in the diagram. Now the hypotheses of Lemma 9.7(ii) are satisfied and show $\mathfrak{L}(\varphi_*\alpha_*) = \varphi\alpha = \beta = \mathfrak{L}(\beta_*)$. Let $X \in \mathfrak{s}$. Then $(\varphi_*\alpha_*)(\exp_X) = \exp_B \mathfrak{L}(\varphi_*\alpha_*)(X) \exp_B \mathfrak{L}(\beta_*)(X) = \beta_*(\exp X)$. Thus the set $E \overset{\text{def}}{=} \{s \in S \mid (\varphi_*\alpha_*)(s) = \beta_*(s)\}$ is a closed subgroup of S containing the identity neighborhood $\exp_S \mathfrak{s}$ of S. Since the group S is connected, it is generated by $\exp_S \mathfrak{s}$ and thus $\varphi_*\alpha_* = \beta_*$ as asserted. (In fact, Theorem 6.30 shows that $\exp_S \mathfrak{s} = S$.) ☐

Lemma 9.9. *Let J be a directed set and assume that $\{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\ j \le k\}$ and $\{F_{jk}: H_k \to H_j \mid (j,k) \in J \times J,\ j \le k\}$ are two projective systems of compact groups and let G and H be their limits, respectively, with limit projections $f_j: G \to G_j$ and $F_j: H \to H_j$. Assume further that for each $j \in J$ there is a morphism $\omega_j: G_j \to H_j$ such that*

$$\begin{array}{ccc} G_j & \xleftarrow{f_{jk}} & G_k \\ {\scriptstyle \omega_j}\big\downarrow & & \big\downarrow{\scriptstyle \omega_k} \\ H_j & \xleftarrow[F_{jk}]{} & H_k \end{array}$$

commutes for all $j \le k$. Then there is a unique morphism $\omega: G \to H$ such that

$$\begin{array}{ccc} G_j & \xleftarrow{f_j} & G \\ {\scriptstyle \omega_j}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\ H_j & \xleftarrow[F_j]{} & H \end{array}$$

is commutative for all $j \in J$.

Proof. There is a morphism $\overline{\omega}: \prod_{j\in J} G_j \to \prod_{j\in J} H_j$ given by $\omega\big((g_j)_{j\in J}\big) = \big(\omega_j(g_j)\big)_{j\in J}$. Recall that $(g_j)_{j\in J} \in G = \lim_{j\in J} G_j$ iff $f_{jk}(g_k) = g_j$. Hence the hypotheses imply that $\overline{\omega}(G) \subseteq H$. The restriction and corestriction $\omega: G \to H$ of $\overline{\omega}$ satisfies the requirements. ☐

The preceding lemma is but one instance of a much more general category theoretical fact expressing the functoriality of the formation of a limit (cf. Appendix 3, A3.41ff., notably A3.42) .

Lemma 9.10. *Let $G = \lim_{j\in J} G_j$ be the limit of a strict projective system $\{f_{jk}: G_k \to G_j \mid j, k \in J, j \le k\}$ of connected Lie groups* (*see* 1.32, 1.33) *and assume that G is semisimple . For each $j \in J$ let $\pi_j: \widetilde{G}_j \to G_j$ denote the universal covering morphisms. Then there is a strict projective system $\{\widetilde{f}_{jk}: \widetilde{G}_k \to \widetilde{G}_j \mid j, k \in J, j \le k\}$. Let $\widetilde{G}$ denote its limit and $\widetilde{f}_j: \widetilde{G} \to \widetilde{G}_j$ its limit morphisms. Then there is a morphism $\pi: \widetilde{G} \to G$ such that the following diagram commutes.*

$$\begin{array}{ccccc} \widetilde{G}_j & \xleftarrow{\widetilde{f}_{jk}} & \widetilde{G}_k & \xleftarrow{\widetilde{f}_k} & \widetilde{G} \\ {\scriptstyle \pi_j}\big\downarrow & & \big\downarrow{\scriptstyle \pi_k} & & \big\downarrow{\scriptstyle \pi} \\ G_j & \xleftarrow[f_{jk}]{} & G_k & \xleftarrow[f_k]{} & G. \end{array}$$

The morphism π is surjective and has a totally disconnected central kernel.

Proof. Let $j \le k$ be in J. Since $\widetilde{G}_k$ is simply connected, the morphism $f_{jk}\pi_k\colon \widetilde{G}_k \to G_j$ has a unique lifting $\widetilde{f}_{jk}\colon \widetilde{G}_j \to \widetilde{G}_k$ across the covering $\pi_j\colon \widetilde{G}_j \to G_j$. This gives the commuting left square of the diagram. The image of $\widetilde{f}_{jk}$ is open and then agrees with $\widetilde{G}_j$ because this range is connected; i.e. $\widetilde{f}_{jk}$ is surjective. If $i \le j \le k$ in J we get a commutative diagram

$$\begin{array}{ccccc}
\widetilde{G}_i & \xleftarrow{\widetilde{f}_{ij}} & \widetilde{G}_j & \xleftarrow{\widetilde{f}_{jk}} & \widetilde{G}_k \\
{\scriptstyle \pi_i}\big\downarrow & & \big\downarrow{\scriptstyle \pi_j} & & \big\downarrow{\scriptstyle \pi_k} \\
G_i & \xleftarrow[f_{ij}]{} & Gj & \xleftarrow[f_{jk}]{} & G_k.
\end{array}$$

Then $\widetilde{f}_{ij} \circ \widetilde{f}_{jk}$ is the unique lifting of $f_{ij} \circ f_{jk} = f_{ik}$ and therefore agrees with $\widetilde{f}_{ik}$. Hence $\{\widetilde{f}_{jk}\colon \widetilde{G}_k \to \widetilde{G}_j \mid j, k \in J, j \le k\}$ is a strict projective system with a projective limit $\widetilde{G}$. Now Lemma 9.9 applies and yields a unique morphism $\pi\colon \widetilde{G} \to G$ such that $f_j \circ \pi = \pi_j \circ \widetilde{f}_j$. This completes the proof.

We show that π is surjective. Let $g \in G$. For each $j \in J$, we abbreviate $\widetilde{f}_j^{-1}\big[\pi_j^{-1}\big(f_j(g)\big)\big]$ by C_j. This set is compact and nonempty since π_j is surjective as a covering map and $\widetilde{f}_j$ is surjective as a limit projection in a strict projective limit (see 1.31). If $j \le k$ and $x \in C_k$ then $\pi_j \widetilde{f}_j(x) = \pi_j \widetilde{f}_{jk} \widetilde{f}_k(x) = f_{jk}\pi_k \widetilde{f}_k(x) = f_{jk} f_k(g) = f_j(g)$, implying $x \in C_j$. Thus $C_k \subseteq C_j$. Therefore $\{C_j \mid j \in J\}$ is a filter basis of compact subsets of $\widetilde{G}$ having an element $\widetilde{g}$ in its intersection. Then $f_j\pi(\widetilde{g}) = \pi_j \widetilde{f}_j(\widetilde{g}) = f_j(g)$ for all $j \in J$ and that implies $\pi(\widetilde{g}) = g$. We finally show that $\ker \pi$ is totally disconnected and thus central. For each $j \in J$, $\widetilde{f}_j(\ker \pi) \subseteq \ker \pi_j$. Since $\ker \pi_j$ is discrete as the kernel of a covering morphism, we have $\widetilde{f}_j\big((\ker \pi)_0\big) = \{1\}$. Since the $\widetilde{f}_j$ separate points, $(\ker \pi)_0 = \{1\}$ follows. Lemma 6.13 shows that $\ker \pi$ is central in $\widetilde{G}$. □

The following is, in addition to the role it plays in our proof, of independent interest. We shall illustrate that in Proposition 9.12. We use here the important fact that the universal covering group (cf. Appendix 2, A2.19) of a semisimple compact connected Lie group is likewise compact (5.77).

Lemma 9.11. *Assume that $G = \lim_{j\in J} G_j$ is the limit of a strict projective system $\{f_{jk}\colon G_k \to G_j \mid j, k \in J, j \le k\}$ of compact connected Lie groups. Let $f_j\colon G \to G_j$ denote the limit morphisms. Assume that for some $i \in J$ the group G_i is a semisimple Lie group. Let $\pi_i\colon \widetilde{G}_i \to G_i$ be the universal covering morphism. Then $\widetilde{G}_i$ is compact and there is a unique morphism $f_i'\colon \widetilde{G}_i \to G$ with a normal image such that $\pi_i = f_i \circ f_i'$.*

The group $G_i^* \overset{\text{def}}{=} f_i'(\widetilde{G}_i)$ is a compact normal Lie subgroup of G. The corestriction of f_i' is a universal covering morphism $\widetilde{G}_i \to G_i^*$ and $q_i|G_i^*\colon G_i^* \to G_i$ is a covering homomorphism.

If G_i is simply connected, then

$$G \cong G_i \times H, \quad H = \ker f_i.$$

Proof. Let $j \in J$, $i \le j$. By 6.4, the Lie algebra $\mathfrak{g}_j$ of G_j is the direct sum of the kernel of $\ker \mathfrak{L}(f_{ij})$ and its orthogonal complement. Then the hypotheses of the Supplement Theorem 5.78(ii) are satisfied and give a unique morphism $\pi_{ji}: \widetilde{G}_i \to G_j$ such that $f_{ij} \circ \pi_{ji} = \pi_i$ and

$$\mathfrak{g}_j = \mathfrak{L}(\ker f_{ij}) \oplus \mathfrak{L}\big(\pi_{ji}(\widetilde{G}_i)\big),$$

where $\oplus$ denotes the direct sum of ideals in the compact Lie algebra $\mathfrak{g}_j$ and where $\mathfrak{L}\big(\pi_{ji}(\widetilde{G}_i)\big)$ is a semisimple ideal. For $i \le j \le k$ in J we obtain a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{G}_i & \xleftarrow{\mathrm{id}} & \widetilde{G}_i & \xleftarrow{\mathrm{id}} & \widetilde{G}_i & \xleftarrow{\mathrm{id}} & \widetilde{G}_i \\
{\scriptstyle \pi_{ii}=\pi_i}\downarrow & & \downarrow{\scriptstyle \pi_{ji}} & & \downarrow{\scriptstyle \pi_{ki}} & & \\
G_i & \xleftarrow[f_{ij}]{} & G_j & \xleftarrow[f_{jk}]{} & G_k & \xleftarrow[f_k]{} & G.
\end{array}$$

Lemma 9.8 implies and yields $\pi_{ji} = f_{jk} \circ \pi_{ki}$. By Lemma 9.9 (or more directly by the universal property of the limit A3.41) we get a unique fill-in morphism $f_i': \widetilde{G}_i \to G$. Since the images of all π_{ji}, $j \in J$, are normal, the image of f_i' is normal.

By 5.77, $\widetilde{G}_i$ is a compact group which is locally isomorphic to the compact Lie group G_i. Hence it has no small subgroups and is therefore a compact Lie group by Definition 2.41. The image G_i^* is a quotient of $\widetilde{G}_j$ and is thus a compact Lie group by 6.7. Since $\pi = f_i f_i'$, the kernel $\ker f_i'$ of the morphism $\widetilde{G}_i \to G_i^*$ is contained in the kernel of π_i which is finite by 5.77. Hence it is a universal covering as $\widetilde{G}_i$ is simply connected. The kernel of the morphism $f_i|G_i^*$ is a homomorphic image of $\ker \pi_i$ and is therefore finite. Since it is surjective in view of the surjectivity of π_i, it is a covering morphism.

The last assertion, pertaining to the case that G_i is simply connected, follows. We take $\widetilde{G}_i = G_i$ and see that $f_i: G \to G_i$ is a homomorphic retraction with coretraction $f_i': G_i \to G$ such that $\operatorname{im} f_i'$ is normal. □

Exercise E9.3. Fill in the details of the last step in the proof of 9.11. □

Proposition 9.12. *Let G be a compact connected group and N a compact normal subgroup such that G/N is a semisimple compact Lie group. Then G has a semisimple compact normal Lie subgroup G^*, and there is a universal covering morphism $\widetilde{G/N} \to G^*$ from the universal covering of G/N such that the quotient morphism $q: G \to G/N$ induces a covering morphism $q|G^*: G^* \to G/N$.*

In particular, if G/N is simply connected, then

$$G \cong G/N \times H, \quad H = \ker q.$$

Proof. This is a consequence of Lemma 9.11. Let $J = \mathcal{N}(G)$ and $G_j = G/M$ for $j = M \in \mathcal{N}(G)$ with the obvious bonding morphisms and let $i = N$. Then $G \cong \lim_{M\in\mathcal{N}(G)} G/M$ and the proposition follows from 9.11. □

Every compact group is a strict projective limit of compact Lie groups. However, a compact group, even a compact connected abelian group, does not have to be a product of compact Lie groups. In the light of this remark, the next lemma is attractive.

Lemma 9.13. *Let $G = \lim_{j\in J} G_j$ be the limit of a strict projective system $\{f_{jk}\colon G_k \to G_j \mid j,k \in J, j \le k\}$ of simply connected semisimple compact Lie groups. Then $G \cong \prod_{a\in A} S_a$ for a family of simple simply connected compact Lie groups S_a.*

Proof. We apply Lemma 9.11 and note that for each $i \in J$ we may take $\widetilde{G}_i = G_i$ and $\pi_i = \mathrm{id}_{G_i}$. Then there is a unique morphism $f_i'\colon G_i \to G$ with a normal image such that $f_i \circ f_i' = \mathrm{id}_{G_i}$. Now we set $f_{kj}'\colon G_j \to G_k$, $f_{kj}' = f_k f_j'$. Then $f_{jk} f_{kj}' = f_{jk} f_k f_j' = f_j f_j' = \mathrm{id}_{G_j}$, and $f_k' f_{kj}' = f_k' f_k f_j' = f_j'$, and the image of f_{kj}' is normal. Thus G_k is the direct product of the normal subgroups $f_{jk}'(G_j)$ and $\ker f_{jk}$.

By 6.6, the simply connected compact semisimple Lie group G_j is isomorphic to $\prod_{\alpha\in A_j} S_\alpha$ for a unique family $\{H_\alpha \mid \alpha \in A_j\}$ of simply connected simple compact normal Lie subgroups H_α of G_j. Every connected closed normal subgroup of G_j is isomorphic to a partial product $\prod_{\alpha\in A} H_\alpha$, $A \subseteq A_j$ (see 6.4(vii), 5.49). Hence we may write $G_j = \prod_{\alpha\in A_j} H_\alpha$ for all $j \in J$. We know that G_j "is" a partial product of G_k, but we want to express this formally because of the set theoretical technicalities involved in infinite families.

Firstly, we consider the elements $\sigma \in H_j$ as functions $A_j \to \bigcup_{\alpha\in A_j} S_\alpha$. (The notation of writing σ as an A_j-tuple $\big(\sigma(\alpha)\big)_{\alpha\in A_j}$ is entirely equivalent.) Secondly, we may and shall assume that all A_j are disjoint (which we certainly may, otherwise we replace A_j by $A_j \times \{j\}$, an isomorphic set). Now we note that for $j \le k$ there is an injective function $\iota_{kj}\colon A_j \to A_k$ and isomorphisms $\zeta_\alpha\colon G_{\iota_{kj}(\alpha)} \to G_\alpha$ for $\alpha \in A_j$ such that for $\sigma_i \in H_i$ we have $f_{jk}(\sigma_k)(\alpha) = \zeta_\alpha(\sigma_k\big(\iota_{kj}(\alpha)\big)$ for $\alpha \in A_k$ and

$$f_{kj}'(\sigma_j)(\beta) = \begin{cases} \zeta_\alpha^{-1}\big(\sigma(\beta)\big) & \text{if } \iota_{kj}(\alpha) = \beta, \\ 1 & \text{otherwise} \end{cases}$$

for $\beta \in A_k$.

We note that for $j = k$ the ζ_α are identity maps and that

$$(*) \qquad \begin{aligned} (\forall i \le j \le i,\, j \in J) \quad & \iota_{ki} = \iota_{kj}\iota_{ji}, \\ (\forall j \in J) \quad & \iota_{jj} = \mathrm{id}_{A_j}\,. \end{aligned}$$

Now we consider the disjoint union $\mathbb{A} \overset{\text{def}}{=} \bigcup_{j\in J} A_j$ and define a subset $R \subseteq \mathbb{A}\times\mathbb{A}$ by

$$R = \{(\alpha,\beta) \in \mathbb{A}\times\mathbb{A} \mid (\exists i,j,k \in J)\, i,j \le k,\ \alpha \in A_i,\ \beta \in A_j, \text{ and } \iota_{ki}(\alpha) = \iota_{kj}(\beta)\}.$$

By its very definition, the relation R is symmetric, and by $(*)$ we see that R is reflexive and, by using the directedness of J, we verify that R is transitive. Thus R is an equivalence relation. We set $A = \mathbb{A}/R$. For $(\alpha, \beta) \in R$ we have isomorphisms $\zeta_\alpha: H_\alpha \to H_{\iota_{ki}(\alpha)}$ and $\zeta_\beta: H_\beta \to H_{\iota_{kj}(\beta)} = H_{\iota_{ki}(\alpha)}$. Thus for each $a \in A$ we find a compact simply connected simple Lie group S_a and isomorphisms $\lambda_\alpha: G_a \to G_\alpha$ for $\alpha \in a$ such that $\zeta_\alpha \lambda_\alpha = \lambda_{\iota_{ki}(\alpha)}$.

Now set $P = \prod_{a\in A} S_a$. We define $p_j: P \to G_j$, $p_j((s_x)_{x\in A})(\alpha) = \lambda_\alpha(s_a)$ for $\alpha \in a \cap A_j$. For $j \leq k$ in J we note that $\iota_{kj}(\alpha) \in a \cap A_k$ and compute $p_j((s_x)_{x\in A_j}) = \lambda_\alpha(s_a) = \xi_\alpha \lambda_{\iota_{kj}(\alpha)}(s_a) = f_{jk} p_k((s_x)_{x\in A})$. Thus p_j is in essence the projection of P onto a partial product $\prod_{a\in A,\, a\cap A_j \neq \emptyset} S_a \cong \prod_{\alpha\in A_j} H_\alpha$, and these projections are compatible with the bonding maps f_{jk}. By the universal property of the limit there is a unique morphism $\mu: P \to G$ such that $p_j = f_j \mu$. We claim that μ is surjective. Indeed, if $g = (g_j)_{j\in J} \in G$ we note that $g_j = (\sigma_j(\alpha))_{\alpha\in A_j} \in G_j = \prod_{\alpha\in A_j} H_\alpha$ such that for $j \leq k$ we have $g_j = f_{jk}(g_k)$, i.e. $f_{jk}(\sigma_k)(\alpha) = \xi_\alpha\big(\sigma_k(\iota_{kj}(\alpha))\big)$ for $\sigma_k \in G_k$, $\alpha \in A_j$. For $a \in A$ we find $j \in J$ and a unique $\alpha \in a \cap A_j$; define $s_a \in S_a$ by $s_a = \lambda_\alpha^{-1}(\sigma_j(\alpha))$. Let $\alpha \in a \cap A_j$; then $p_j((s_x)_{x\in A})(\alpha) = \lambda_\alpha(s_a) = \sigma_j(\alpha)$, i.e. $p_j\big(s_x)_{x\in A}\big) = g_j = f_j\big((g_i)_{i\in J}\big)$. Thus $\mu\big((s_x)_{x\in A}\big) = g$. Since the p_j separate the points of P, the morphism μ is injective. Hence it is an isomorphism. This completes the proof of the lemma. □

The following observation is an important link between the structure theory of compact groups and simple connectivity. It requires information on simple connectivity from Appendix 2. At a later stage we shall see that the restriction to Lie groups in part (iii) below is unnecessary (see 9.29).

Theorem 9.14. (i) *Let $\{S_j \mid j \in J\}$ be a family of topological groups, and assume that each S_j satisfies the following hypotheses:*

(a) *There are arbitrarily small open arcwise connected identity neighborhoods.*

(b) *There is at least one open identity neighborhood in which every loop at* **1** *is contractible.*

(c) *S_j is simply connected.*

Then $\prod_{j\in J} S_j$ is simply connected.

(ii) *If $\{S_j \mid j \in J\}$ is a family of simply connected linear Lie groups, then $\prod_{j\in J} S_j$ is simply connected.*

(iii) *The product of a family of simply connected compact Lie groups is simply connected.*

Proof. (i) We note that due to the connectedness of S_j (see (c)) every identity neighborhood U generates S_j. In particular, if U is arcwise connected (see (a)), then $(U \cup U^{-1})^n$ is arcwise connected, and thus $S_j = \bigcup_{n=1}^\infty (U \cup U^{-1})^n$ is arcwise connected. Therefore by hypotheses (a), (b), and (c), A2.11(iii) of Appendix 2 applies and proves the claim.

(ii) It suffices after (i) to convince ourselves that a linear Lie group satisfies hypotheses (a) and (b). But by Definition 5.32 of a linear Lie group, there are arbitrarily small identity neighborhoods homeomorphic to an open ball in a Banach space. Now any convex set in a topological vector space is arcwise connected and has the property that every loop at every point is contractible in it to this point (see A2.11). This secures the hypotheses of (i) and then (i) establishes the claim.

(iii) is a special case of (ii). □

The proof of Theorem 9.14 is simple given the information provided in Appendix 2 on simple connectedness. In particular it is shown there in Example A2.11(iii) that the product of any family of arcwise connected, locally arcwise connected, locally arcwise simply connected spaces is simply connected. That material is specifically provided for the application in 9.14. Our approach to simple connectivity in Appendix A2 otherwise is rather free of arc connectivity, a caution which is indicated to us in the context of compact groups that arc connectivity in compact groups is not a prevalent, and certainly a delicate, property (as we have seen in 8.30, 8.45, and 8.48). In the present chapter it becomes evident that in the case of semisimple compact connected groups, arc connectivity is not a problem. (see also 9.19 and 9.50 below).

Example 9.15. Let $G = \mathrm{SO}(3)^{\mathbb{N}}$ and $\widetilde{G} = \mathrm{SU}(2)^{\mathbb{N}}$. Let $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ the universal covering (cf. E1.2, E5.12). Then $\pi = q^{\mathbb{N}}\colon \widetilde{G} \to G$ is a surjective morphism from the simply connected compact group $\widetilde{G}$ with kernel $Z\big(\mathrm{SU}(2)\big)^{\mathbb{N}} \cong \mathbb{Z}(2)^{\mathbb{N}}$. The group G has fundamental group $\pi_1(G) \cong \mathbb{Z}(2)^{\mathbb{N}}$ but does not possess a universal covering group. It is not locally simply connected (cf. the definition preceding Lemma A2.12), and therefore the hypotheses of the Existence of the Universal Covering Theorem A2.14 are not satisfied. □

In view of Example 9.15, note that a compact connected semisimple group G will not necessarily have a universal covering $f\colon \widetilde{G} \to G$. We aim to show, however, that for any compact connected semisimple group G there is a simply connected semisimple compact group $\widetilde{G}$ and a natural morphism $\pi_G\colon \widetilde{G} \to G$ with a totally disconnected kernel and that this will have to serve as a substitute for a universal covering.

Lemma 9.16. (i) *Let (X, x_0) be a simply connected pointed space and G a compact group with a totally disconnected normal subgroup D. If $f\colon (X, x_0) \to G/D$ is a continuous map of pointed spaces and $q\colon G \to G/D$ the quotient morphism, then there is a unique continuous lifting of pointed spaces $F\colon (X, x_0) \to G$.*

(ii) *If X is a simply connected topological group then the lifting F is a morphism of groups.*

Proof. Let $N \in G$. Then q induces a quotient morphism $q_N\colon G/N \to G/ND \cong (G/N)/(ND/N)$, $q_N(gN) = gND$, of compact Lie groups whose kernel ND/N is a totally disconnected compact Lie group and thus a finite group. Consequently q_N is a covering map. Since (X, x_0) is simply connected then by the very definition A2.6 the continuous map of pointed spaces $x \mapsto f(x)ND\colon (X, x_0) \to G/ND$ has a unique lifting $F_N\colon (X, x_0) \to G/N$. If $M \subseteq N$ in $\mathcal{N}(G)$, then $q_M\big(F_M(x)\big)N =$

$q_N\big(F_N(x)\big)$, that is $F_M(x) \subseteq F_N(x)$. Thus $\{F_N(x) \mid N \in \mathcal{N}(G)\}$ is a filter basis of compact sets and thus has an element $F(x)$ in it which is as representative of each $F_N(x)$, i.e. $F(x)N = F_N(x)$. Now $q\big(F(x)\big)N = F(x)ND$. Thus $q\big(F(x)\big) \in \bigcap_{N\in\mathcal{N}(G)} F(x)DN = F(x)\bigcap_{N\in\mathcal{N}(G)} DN = F(x)D$ by 9.1(iii). Thus F is a lifting. Let U be an identity neighborhood of G. Then there is an $N \in \mathcal{N}(G)$ and an open identity neighborhood V of G such that $V = VN \subseteq U$. Since $F_N\colon X \to G/N$ is continuous there is a neighborhood W of x such that $F_N(W) \subseteq F(x)U/N$. Then $F(W) \subseteq F(x)U$. Thus F is continuous and is, therefore, the required lifting.

(ii) If X is a group and $N \in \mathcal{N}(G)$ then $F_N\colon X \to G/N$ is a morphism of topological groups by A2.32. Hence $F(x)F(y) \in F(x)F(y)N = F_N(x)F_N(y) = F_N(xy) = F(xy)N$. By 9.1(ii) the relation $F(x)F(y) = F(xy)$ follows. □

Lemma 9.17. *Assume that G is a semisimple connected compact group and that there is a simply connected semisimple compact group $\widetilde{G}$ and a surjective morphism $\pi_G\colon \widetilde{G} \to G$ with totally disconnected kernel.*

(i) *If $f\colon S \to G$ is a morphism for some simply connected compact group S, then there is a lifting morphism $\varphi\colon S \to \widetilde{G}$ such that $\pi_G\varphi = f$. If f is a quotient morphism with totally disconnected kernel* $\ker f$, *then φ is an isomorphism.*

(ii) *If $f\colon G \to H$ is any morphism of compact connected semisimple groups, and if $\pi_H\colon \widetilde{H} \to H$ is a quotient morphism with totally disconnected kernel then there is a unique morphism $\widetilde{f}\colon \widetilde{G} \to \widetilde{H}$ so that the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{H} \\ {\scriptstyle \pi_G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_H} \\ G & \xrightarrow{\ f\ } & H. \end{array}$$

Proof. (i) By Lemma 9.16 there is a unique morphism $\varphi\colon S \to \widetilde{G}$ of topological groups such that $\pi_G\varphi = f$. If f has a totally disconnected kernel then by 9.16 there is a unique morphism $\psi\colon \widetilde{G} \to S$ such that $f\psi = \pi_G$. Then $\pi(\varphi\psi) = f\psi = \pi = \pi\,\mathrm{id}_{\widetilde{G}}$. The uniqueness of the lifting yields $\varphi\psi = \mathrm{id}_{\widetilde{G}}$. Completely analogously we get $\psi\varphi = \mathrm{id}_S$. Thus $\psi = \varphi^{-1}$.

(ii) is a consequence of (i). □

Lemma 9.18. *Let $f\colon G \to H$ be a surjective morphism of compact groups. Then $f(G_0) = H_0$.*

Proof. Clearly $f(G_0) \subseteq H_0$. Since f maps $f^{-1}(H_0)$ onto H_0, it is no loss of generality to assume that H is connected and then to show that $f(G_0) = H$. For a proof let U be any open subgroup of G. Then since f is a quotient morphism by 1.10(iii), $f(U)$ is open in H, and thus $f(U) = H$. Let $\mathcal{U}$ denote the filter basis of all open, hence closed subgroups of G. By E1.13(iii), $\bigcap\mathcal{U} = H$. If $h \in H$, then $\{f^{-1}(h) \cap U \mid U \in \mathcal{U}\}$ is a filterbasis of compact sets. Let $g_h \in \bigcap\mathcal{U} = G_0$ be an element in its intersection. Then $f(g_h) = h$. Hence $f(G_0) = H$. □

An alternative proof may be instructive. Since f is surjective, $f(G_0)$ is a closed normal subgroup of H contained in H_0. Thus there is a morphism $F: G/G_0 \to H/f(G_0)$, $F(gG_0) = f(g)f(G_0) = f(gG_0)$, and F is surjective. But G/G_0 is totally disconnected, and so $H_0/f(G_0)$ is a connected compact group which is the totally disconnected compact group $F^{-1}\big(H/f(G_0)\big)$. But quotients of totally disconnected compact groups are totally disconnected by E1.13. Thus $H_0/f(G_0)$ is singleton and thus $f(G_0) = H_0$.

As we did in Chapter 6 we shall call a group G *centerfree* if its center $Z(G)$ is a singleton.

If $\prod_{j\in J} G_j$ is a product of groups, we shall say that a subgroup

$$\prod_{j\in J} H_j, \qquad H_j = \begin{cases} G_j & \text{if } j \in I, \\ \{1\} & \text{otherwise} \end{cases}$$

is a *partial product* with respect to the index set $I \subseteq J$. Such a partial product is clearly isomorphic to $\prod_{j\in I} G_j$.

At long last, we are able to give a complete description of the structure of semisimple compact connected groups. We fix some notation. We let $\mathcal{S}$ denote a set of simple compact Lie algebras containing for each simple compact Lie algebra exactly one member isomorphic to it and pick once and for all for each $\mathfrak{s} \in \mathcal{S}$ a simple simply connected Lie group $S_{[\mathfrak{s}]}$ and a simple centerfree compact connected Lie group $R_{[\mathfrak{s}]} = S_{[\mathfrak{s}]}/Z(S_{[\mathfrak{s}]})$ with $\mathfrak{L}(S_{[\mathfrak{s}]}) \cong \mathfrak{L}(R_{[\mathfrak{s}]}) \cong \mathfrak{s}$. The groups $S_{[\mathfrak{s}]}$ and $R_{[\mathfrak{s}]}$ are unique up to a natural isomorphism.

The Structure of Semisimple Compact Connected Groups

Theorem 9.19. *Assume that G is a semisimple compact connected group.*

(i) *There is a simply connected semisimple compact group $\widetilde{G}$ and a natural surjective morphism $\pi_G: \widetilde{G} \to G$ with a totally disconnected kernel. If $f: G \to H$ is a morphism of semisimple compact connected groups, then there is a unique morphism $\widetilde{f}: \widetilde{G} \to \widetilde{H}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\widetilde{f}} & \widetilde{H} \\ {\scriptstyle \pi_G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

Up to a natural isomorphism, $\widetilde{G}$ is uniquely determined. If C is a compact connected normal subgroup of G, then $\widetilde{C} \stackrel{\text{def}}{=} \big(\pi_G^{-1}(C)\big)_0$ is a compact connected normal subgroup of $\widetilde{G}$ mapping onto C with a totally disconnected kernel under the restriction of π_G.

(ii) *There is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups S_j such that*

$$\widetilde{G} \cong \prod_{j\in J} S_j.$$

The center $Z(G)$ is totally disconnected. Each compact connected normal subgroup of $\prod_{j\in J} S_j$ is isomorphic to a partial product

$$\widetilde{C}=\prod_{j\in J} G_j, \qquad G_j=\begin{cases} S_j & \text{if } j\in I,\\ \{1\} & \text{otherwise}\end{cases}$$

for a unique subset $I\subseteq J$.

In particular, every compact connected normal subgroup of G is of the form $\pi_G(\widetilde{C})$ for a suitable $I\subseteq J$.

(iii) *The prescription $\widetilde{\cdot}$ is a functor; i.e. $\widetilde{\mathrm{id}_G}=\mathrm{id}_{\widetilde{G}}$ and $\widetilde{ff'}=\widetilde{f}\widetilde{f'}$.*

(iv) *Set $R_j \stackrel{\mathrm{def}}{=} S_j/Z(S_j)$. Then $\prod_{j\in J} R_j$ is a compact connected centerfree group which is isomorphic to $G/Z(G)$.*

$$G \;\cong\; \left\{\begin{array}{c} \widetilde{G}\cong\prod_{j\in J} S_j \\ \Big| \\ Z(\widetilde{G})\cong\prod_{j\in J} Z(S_j) \\ \Big| \\ D \end{array}\right. \qquad \begin{array}{l} \big\}\;\cong\;\prod_{j\in J} R_j \\[2em] \big\}\;\cong\; Z(G) \end{array}$$

$$\begin{array}{c} D \\ \Big| \\ \{1\} \end{array}$$

Any compact normal subgroup of $\prod_{j\in J} R_j$ is connected and is a partial product isomorphic to $\prod_{j\in I} R_j$ for a unique subset I of J.

(v) *The family $\{S_j \mid j\in J\}$ is uniquely determined in the following sense: Let J be the set of all closed connected normal Lie subgroups of G which are simple Lie groups. For each $j\in J$ select a universal covering morphism $p_j\colon S_j\to j$ according to* A2.20 *and* 5.77. *Then $\{S_j \mid j\in J\}$ is such that $\widetilde{G}\cong\prod_{j\in J} S_j$, and if $\{S^*_k \mid k\in J^*\}$ is another such family, then there is a bijection $\beta\colon J\to J^*$ and a family of isomorphisms $\alpha_j\colon S_j\to S^*_{\beta(j)}$.*

(vi) *For each $\mathfrak{s}\in\mathcal{S}$ there are fully characteristic subgroups $\widetilde{G}_{\mathfrak{s}}$ of $\widetilde{G}$ and $G_{\mathfrak{s}}$ of G, respectively, and a unique cardinal $\aleph(\mathfrak{s},G)$ such that*

(a) *J is the disjoint union of subsets $J_{\mathfrak{s}}$ each of cardinality* $\operatorname{card} J_{\mathfrak{s}}=\aleph(\mathfrak{s},G)$, $\mathfrak{s}\in\mathcal{S}$.

(b) $\widetilde{G}_{\mathfrak{s}}\cong S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},G)}$.

(c) $\widetilde{G}\cong\prod_{\mathfrak{s}\in\mathcal{S}}\widetilde{G}_{\mathfrak{s}}$.

(d) $G_{\mathfrak{s}}=\pi_G(\widetilde{G}_{\mathfrak{s}})$.

(e) *G is generated by the union of the $G_{\mathfrak{s}}$, $\mathfrak{s}\in\mathcal{S}$ and there is a surjective morphism $\omega\colon\prod_{\mathfrak{s}\in S} G_{\mathfrak{s}}\to G$ for a totally disconnected central kernel.*

(f) *There is a surjective morphism* $\omega'\colon G \to \prod_{\mathfrak{s}\in\mathcal{S}} R_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)}$ *such that* $\ker\omega' = Z(G)$.

Proof. (i) and (ii) From Chapter 2 (see 2.43, cf. also Lemma 9.1 above) we know that $G = \lim_{N\in\mathcal{N}(G)} G/N$ for the strict projective system $\{f_{NM} = (gM \mapsto gN)\colon G/M \to G/N \mid N,\, M \in \mathcal{N}(G),\, M \subseteq N\}$. Since $G' = G$ we know $(G/N)' = G'N/N = G/N$. Hence all G/N are semisimple. Now Lemma 9.10 provides us with a compact connected group $\widetilde{G}$ which is the limit of a strict projective system $\{\widetilde{f}_{NM}\colon \widetilde{G}_M \to \widetilde{G}_N \mid N,\, N \in \mathcal{N}(G),\, M \subseteq N\}$ and with a surjective morphism $\pi_G\colon \widetilde{G} \to G$ with totally disconnected kernel such that the following diagram commutes:

$$\begin{array}{ccccc}
\widetilde{G}_N & \xleftarrow{\widetilde{f}_{NM}} & \widetilde{G}_M & \xleftarrow{\widetilde{f}_M} & \widetilde{G} \\
{\scriptstyle \pi_N}\big\downarrow & & \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow{\scriptstyle \pi} \\
G/N & \xleftarrow[f_{NM}]{} & G_M & \xleftarrow[f_M]{} & G.
\end{array}$$

Now from Lemma 9.13 we draw the conclusion that there is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups S_j such that $\widetilde{G} = \prod_{j\in J} S_j$. Thus we have the desired surjective morphism $f\colon \prod_{j\in J} S_j \to G$ with a totally disconnected central kernel D. The isomorphy theorem shows that $G \cong \prod_{j\in J} S_j/D$. Since $Z(\prod_{j\in J} S_j) = \prod_{j\in J} Z(S_j)$ and all $Z(S_j)$ are finite, $Z(\prod_{j\in J} S_j)$ is totally disconnected, i.e. has a basis of compact open identity neighborhoods, and thus $Z(G) \cong Z(\prod_{j\in J} S_j)/D$ has a basis of compact open identity neighborhoods and thus is totally disconnected, too.

From Lemma 9.17 we get that $\widetilde{G}$ is unique in the sense that for $\pi_G\colon \widetilde{G} \to G$ and $\pi_G^*\colon \widetilde{G}^* \to G$ with the same properties we obtain an isomorphism $\varphi\colon \widetilde{G} \to \widetilde{G}^*$ such that $\pi_G^*\varphi = \pi_G$. Further for any morphism $f\colon G \to H$ of semisimple compact connected groups we find a unique $\widetilde{f}\colon \widetilde{G} \to \widetilde{H}$ such that $\pi_H \widetilde{f} = f\pi_G$.

Let C be a compact connected normal subgroup of G. Since π_G is surjective, $\pi_G\big(\pi_G^{-1}(C)\big) = C$. If we set $\widetilde{C} \stackrel{\rm def}{=} \big(\pi_G^{-1}(C)\big)_0$, then Lemma 18 shows that $\pi_G(\widetilde{C}) = C$. We note that $\ker \pi_G|\widetilde{C} = \widetilde{C} \cap \ker \pi_G$ is totally disconnected.

Finally let $\widetilde{C}$ be a compact connected normal subgroup of $P \stackrel{\rm def}{=} \prod_{j\in J} S_j$. Let $F \subseteq J$ and set

$$P_F = \prod_{j\in J} G_j, \qquad G_j = \begin{cases} S_j & \text{if } j \in F,\\ \{1\} & \text{otherwise.}\end{cases}$$

Let $p_F\colon P \to P_F$ be the projection. Now assume that F is finite. Then by 6.18(iii) there is a finite subset $I_F \subseteq F$ such that $p_F(\widetilde{C}) = P_{I_F}$. If F_1 and F_2 are finite, then the uniqueness in 6.18(iii) allows us to conclude that $I_{F_1\cup F_2} = I_{F_1} \cup I_{F_2}$. Define $I = \bigcap\{I_F \mid F \subseteq J \text{ finite}\}$. Then clearly $p_F(\widetilde{C}) = p_F(P_I)$ for all finite subsets $F \subseteq J$. Since the projections p_F, $F \subseteq J$ finite, separate the points of P we conclude $\widetilde{C} = P_I$.

This completes the proof of parts (i) and (ii) of the theorem.

(iii) Exercise E9.4.

(iv) Let S be a compact connected semisimple group and Z its center. As S/Z is semisimple, the center $Z(S/Z) = N/Z$ is totally disconnected. Hence N is totally disconnected since Z is totally disconnected. Then N is central by 6.13 and thus $N \subseteq Z$. Hence S/Z is centerfree. Thus all groups R_j are centerfree and then $\prod_{j\in J} R_j$ is centerfree. The center of $\prod_{j\in J} S_j$ is $\prod_{j\in J} Z(S_j)$. Hence $D \subseteq \prod_{j\in J} Z(S_j)$. Since $\bigl(\prod_{j\in J} Z(S_j)\bigr)/D = Z\bigl((\prod_{j\in J} S_j)/D\bigr) \cong Z(G)$, the isomorphy theorem shows that $\prod_{j\in J} R_j \cong G/Z(G)$.

Now let N be a compact normal subgroup of $P \stackrel{\rm def}{=} \prod_{j\in J} R_j$. Then the identity component N_0 by (ii) above is a partial product

$$P_I \stackrel{\rm def}{=} \prod_{j\in J} G_j, \qquad G_j = \begin{cases} R_j & \text{if } j \in J, \\ \{1\} & \text{otherwise} \end{cases}$$

for some $I \subseteq J$. Then $P/N \cong P_{J\setminus I}$, and N/N_0 is a totally disconnected normal, hence central subgroup of the connected group P/N (cf. 6.13). But $P_{J\setminus I}$ is centerfree. Thus $N = N_0$. Thus assertion (iv) is proved completely.

(v) By (ii) we know that there is a family $\{S_k^* \mid k \in J^*\}$ and a quotient morphism $\pi\colon P \stackrel{\rm def}{=} \prod_{k\in J^*} S_k^* \to G$ with totally disconnected kernel. Consider the partial products

$$\widetilde{S_k} = \prod_{i\in J^*} G_i, \quad G_i = \begin{cases} S_k^* & \text{for } i = k, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $\pi(\widetilde{S_k}) \in J$ and we set $\beta\colon J^* \to J$, $\beta(k) = \pi(\widetilde{S_k})$.

Claim: β is injective. Assume $\beta(k_1) = \beta(k_2)$. Then $\pi(\widetilde{S_{k_1}}) = \pi(\widetilde{S_{k_2}})$. Denote this group by H. The map π restricts to universal coverings $p_m\colon \widetilde{S_{k_m}} \to H$, $m = 1, 2$. Then there is a unique morphism $\varphi\colon \widetilde{S}_{k_1} \to \widetilde{S}_{k_2}$ such that $p_1 = p_2\varphi$. Then $\pi\bigl((g^{-1}\varphi(g)\bigr) = p_1(g)^{-1}p_2\varphi(g) = 1$ for all $g \in \widetilde{S}_{k_1}$. Thus $g \mapsto g^{-1}\varphi(g)\colon S_{k_1} \to \ker\varphi$ is a continuous function from a connected space into a totally disconnected one and is, therefore constant. Thus $\varphi(g) = g$ for all $g \in \widetilde{S_{k_1}}$ which implies $k_2 = k_1$.

Claim: β is surjective. Let S be a simple compact connected normal subgroup of G. Set $H = \pi^{-1}(S)$. Then H is a compact normal subgroup of $\prod_{k\in J^*} S_k^*$ containing $\ker\pi \subseteq \prod_{k\in J^*} Z(S_k^*)$. Since the compact connected subgroup H_0 of H is characteristic, it is normal in the product, too. Then $\pi(H_0)$ is a connected compact normal subgroup of S and then is either S or singleton. Suppose the latter. Then S is a quotient of H/H_0, a totally disconnected and normal subgroup of the connected group P/H_0, and is therefore central (6.13). Hence H/H_0 is abelian and thus S is abelian. This is false and thus the supposition $S = \{1\}$ is refuted. Therefore $\pi|H_0\colon H_0 \to S$ is a morphism with totally disconnected kernel. The composition

$$\widetilde{H_0} \longrightarrow H_0 \xrightarrow{\pi|H_0} S$$

is a surjective morphism with totally disconnected kernel and by the proof of the injectivity of β the product representation of $\widetilde{H_0}$ contains only one factor. Hence H_0 itself is a simple compact Lie group.

Claim: There is a $k \in J^*$ such that $H_0 = \widetilde{S_k^*}$. Since H_0 is a compact Lie group there is an identity neighborhood U of P such that $H_0 \cap U$ does not contain any subgroup other than $\{1\}$. Then by the definition of the product topology on P there is a finite subset I^* of J^* such that the partial product

$$N \stackrel{\text{def}}{=} \prod_{i \in J^*} G_i, \quad G_i = \begin{cases} \{1\} & \text{for } i \in I^*, \\ S_i^* & \text{otherwise} \end{cases}$$

is contained in U. Then the projection $P \to \prod_{i \in I^*} S_i^*$ maps H_0 faithfully. From 6.18(iii) and the simplicity of H_0 we conclude that there is a $k \in I^*$ such that $H_0 N = \widetilde{S_k} N$. Since we may choose N arbitrarily small, 9.1(iii) shows $H_0 = \widetilde{S_k}$.

The restriction of the projection $P \to S_k^*$ to $\widetilde{S_k}$ is an isomorphism whose inverse we denote by $\iota_k \colon S_k^* \to \widetilde{S_k}$. The morphism $\gamma_k \colon S_k^* \to \beta(k)$, $\gamma_k = (\pi|\widetilde{S_k}) \circ \iota_k$ is a universal covering. Hence there is a unique morphism $\alpha_k \colon S_k^* \to S_{\beta(k)}$ such that $\gamma_k = p_k \circ \alpha_k$. Since p_k is likewise a universal covering, α_k is an isomorphism.

(vi) The proof of (vi) will be straightforward from the definitions that we are about to make. Indeed, by (ii) and (vi) there is a unique family $\{S_j \mid j \in J\}$ of simply connected simple groups S_j such that $\widetilde{G} = \prod_{j \in J} S_j$. Define $J_\mathfrak{s} = \{j \in J \mid \mathfrak{L}(S_j) \cong \mathfrak{s}\}$ and $\aleph(\mathfrak{s}, G) = \operatorname{card} J_\mathfrak{s}$. Then (a) is automatic. Next we set $\widetilde{G}_\mathfrak{s} = \prod_{j \in J_\mathfrak{s}} S_j$ considered in the obvious fashion as a subgroup of $\widetilde{G}$. Taking for S any simply connected compact simple group with $\mathfrak{L}(S) \cong \mathfrak{s}$ we have (b) and (c). Finally we set $G_\mathfrak{s} = \pi_G(\widetilde{G}_\mathfrak{s})$. This makes (d) automatic, and since $\widetilde{G}$ is generated topologically by the union of the $\widetilde{G}_\mathfrak{s}$, the group G is generated by the union of the $G_\mathfrak{s}$. This establishes the first part of (e). There is an obvious morphism $\alpha \colon \widetilde{G} = \prod_{\mathfrak{s} \in \mathcal{S}} \widetilde{G}_\mathfrak{s} \to \prod_{\mathfrak{s} \in \mathcal{S}} G_\mathfrak{s}$ defined in a componentwise fashion. Its kernel $\prod_{\mathfrak{s} \in \mathcal{S}} \ker(\pi_G|\widetilde{G}_\mathfrak{s})$ is contained in the kernel of π_G. Hence π_G factors as $\pi_G = \omega \circ \alpha$ with a unique ω.

Conclusion (f) follows from (iv) in view of the fact that for $j \in J_\mathfrak{s}$ we have $R_j \cong R_\mathfrak{s}$ and thus $\prod_{j \in J} R_j \cong \prod_{\mathfrak{s} \in \mathcal{S}} R_\mathfrak{s}^{\aleph(\mathfrak{s}, G)}$. □

The fully characteristic subgroups $\widetilde{G}_\mathfrak{s}$ of $\widetilde{G}$ and $G_\mathfrak{s}$ of G are called the *isotypic components of* $\widetilde{G}$, respectively, G *of type* $\mathfrak{s}$.

Exercise E9.4. Prove the functoriality of $G \mapsto \widetilde{G}$.

[Hint. Diagram chasing and uniqueness of the lifting.] □

Theorem 9.19 provides the first example of what we call a "Sandwich Theorem." It "sandwiches" the group G between two product groups by two surjective morphisms with totally disconnected central kernels.

The Sandwich Theorem for Semisimple Compact Connected Groups

Corollary 9.20. *Let G be a semisimple compact connected group. Then there is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups and there are surjective morphisms f and q of compact groups*

$$\begin{array}{ccccc}
\prod_{j\in J} S_j & \xrightarrow{f} & G & \xrightarrow{q} & \prod_{j\in J} S_j/Z(S_j)\\
\cong\Big\downarrow & & \mathrm{id}_G\Big\downarrow & & \Big\downarrow\cong\\
\prod_{\mathfrak{s}\in\mathcal{S}} S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},G)} & \longrightarrow & G & \longrightarrow & \prod_{\mathfrak{s}\in\mathcal{S}} R_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},G)},
\end{array}$$

such that $qf\colon \prod_{j\in J} S_j \to \prod_{j\in J} S_j/Z(S_j)$ is the product $\prod_{j\in J} p_j$ of the quotient morphisms $p_j\colon S_j \to S_j/Z(S_j)$.

Proof. This is a reformulation of 9.19(iv). □

Exercise E9.5. Verify the following facts.

In the circumstances of Theorem 9.19 *and Corollary* 9.20*, the cardinal numbers* $\operatorname{card} J$ *and* $\aleph(\mathfrak{s}, G)$*,* $\mathfrak{s} \in \mathcal{S}$ *are isomorphy invariants of* G*. Assume that* J *is infinite. Then* $\operatorname{card} J = w(G)$*, where* $w(G)$ *is the weight of* G*. Further,* G *contains a subspace homeomorphic to* $[0, 1]^{\operatorname{card} J}$*.*

G is sandwiched between the group $\prod_{\mathfrak{s}\in\mathcal{S}} \widetilde{G}_{\mathfrak{s}}$ and the group $\prod_{\mathfrak{s}\in\mathcal{S}} \underline{G}_{\mathfrak{s}}$ where $\underline{G}_{\mathfrak{s}} \stackrel{\text{def}}{=} \prod_{j\in J_{\mathfrak{s}}} R_j \cong R_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)}$, $R_j = S_j/Z(S_j)$.

Show that *every semisimple connected compact group is a dyadic space, i.e. a quotient space of a space $\{0, 1\}^I$ for some set I.*

[Hint. See Exercise EA4.3 in Appendix 4 for the weight of a product.

Use Alexandroff's Theorem which says that every compact metric space is dyadic (see Lemma A4.31).] □

The Levi–Mal'cev Structure Theorem for Compact Groups

For an optimal understanding of the main structure theorems on compact groups it helps to recall the concept of a semidirect product which was discussed in Definition 5.72 and the subsequent Exercise E5.27. For compact groups we can readily say the following.

Lemma 9.21 (Mayer-Vietoris). *Let G be a topological group, N a compact normal subgroup, and H a compact subgroup. Then NH is a compact subgroup of G. The morphism $\iota\colon H \to \operatorname{Aut} N$, $\iota(h)(n) = hnh^{-1}$ defines a semidirect product $N \rtimes_\iota H$, and there is a surjective morphism $\mu\colon N \rtimes_\iota H \to G$ whose image in NH, and there is an injective morphism $\delta\colon N \cap H \to N \rtimes_\iota H$, $\delta(d) = (d^{-1}, d)$ whose image is* $\ker \mu$*. The sequence of morphisms of topological groups*

(MV) $$\{1\} \to N \cap H \xrightarrow{\delta} N \rtimes_\iota H \xrightarrow{\mu} G \to \{1\}$$

is exact at all terms with the possible exception of G*; it is exact everywhere if and only if* $G = NH$*. There is an isomorphism* $NH \cong \frac{N \rtimes_\iota H}{D}$ *with* $D \stackrel{\text{def}}{=} \ker\mu = \operatorname{im}\delta \cong N \cap H$*.*

Proof. The function $(h,n) \mapsto \iota(h)(n) = hnh^{-1}\colon H \times N \to N$ is continuous, and thus $N \rtimes_\iota H$ is a well-defined semidirect product. It is straightforward to verify that μ and δ are morphisms with the asserted properties. □

We shall sometimes call the map μ the *Mayer-Vietoris morphism.*

Exercise E9.6. Prove the following proposition.

Let Z *denote a central subgroup of a compact group* G *and* H *a compact subgroup. Then there is a sequence*

$$\{1\} \to Z \cap H \stackrel{\delta}{\to} Z \times H \stackrel{\mu}{\to} ZH \to \{1\}.$$

In particular, $ZH \cong \frac{Z \times H}{D}$*, where* $D = \{(d^{-1}, d) \mid d \in Z \cap H\} \cong Z \cap H$*.* □

If G is a compact group, then its identity component G_0 equals $Z_0(G)(\overline{G'})_0$, where $Z_0(G)$ is the identity component of the center $Z(G)$, and the intersection $Z_0(G) \cap (\overline{G'})_0$ is trivial in the sense that it is totally disconnected [179].

The Structure Theorem for Compact Groups

Theorem 9.23. (i) *Let* G *be a compact group with center* $Z(G)$ *and closed commutator group* $\overline{G'}$*. Then* $Z(G)G'$ *contains* G_0 *and the intersection* $Z(G) \cap \overline{G'}$ *is totally disconnected.*

(ii) *There is an exact sequence*

$$\text{(MV)} \qquad \{1\} \to Z_0(G) \cap (\overline{G'})_0 \xrightarrow{\delta} Z_0(G) \times (\overline{G'})_0 \xrightarrow{\mu} G_0 \to \{1\}.$$

(iii) *Set* $G_A = G/\overline{G'}$ *and define* $\zeta\colon Z(G) \to G_A$ *by* $\zeta(g) = g\overline{G'}$*,* $\theta\colon G_A \to G/(Z(G)\overline{G'})$ *by* $\theta(g\overline{G'}) = gZ(G)\overline{G'}$*. Then* $(G_A)_0 \cong G_0\overline{G'}/\overline{G'} = Z_0(G)\overline{G'}/\overline{G'} \cong Z_0(G)/(Z_0(G) \cap \overline{G'})$*, and there are exact sequences of compact abelian groups*

$$\text{(A)} \quad 0 \to Z_0(G) \cap \overline{G'} \xrightarrow{\text{incl}} Z_0(G) \xrightarrow{\zeta|Z_0(G)} G_A \longrightarrow G_A/(G_A)_0 \to 0,$$

$$\text{(B)} \quad 0 \to Z(G) \cap \overline{G'} \xrightarrow[\text{incl}]{} Z(G) \xrightarrow[\zeta]{} G_A \xrightarrow[\theta]{} G/(Z(G)\overline{G'}) \to 0.$$

The sequence (A) *is the characteristic sequence of* G_A *in the sense of Definition* 8.80 *if and only if* $Z_0(G)$ *is torsion-free if and only if the character group* $\big(Z_0(G)\big)\widehat{\ }$ *is divisible.*

Proof. (i) For $N \in \mathcal{N}(G)$ let Z_N denote the full inverse image of the center $Z(G/N)$ of the Lie group G/N, that is, $N \subseteq Z_N$ and $Z_N/N = Z(G/N)$. It follows from the Structure Theorem on Compact Lie Groups 6.15 that $G_0N/N \subseteq (G/N)_0 \subseteq Z(G/N)(G/N)'$, that is, $G_0 \subseteq Z_NG'$, since $(G/N)' = G'N/N$. For each

$g \in G_0$ we therefore find elements $z_N \in Z_N$, $g_N \in G'$ such that $g = z_N g_N$, i.e. $gg_N^{-1} = z_N \in gG' \cap Z_N$. If $N \subseteq M$ in $\mathcal{N}(G)$ then $Z_N \subseteq Z_M$. Hence $\{gG' \cap Z_N \mid N \in \mathcal{N}(G)\}$ is a filter basis of compact sets. Let $x \in \bigcap_{N\in\mathcal{N}(G)} gG' \cap Z_N = gG' \cap Z$, $Z = \bigcap_{N\in\mathcal{N}(G)} Z_N$. Thus $g \in ZG'$.

We claim that $Z = Z(G)$. Clearly $Z(G) \subseteq Z_N$ for all $N \in \mathcal{N}(G)$ whence $Z(G) \subseteq Z$. Now let $z \in Z$ and $g \in G$. Then $\mathrm{comm}(z, g)N = \mathrm{comm}(xN, gN) \in \mathrm{comm}(Z_N/N, G/N) = \mathrm{comm}\big(Z(G/N), G/N)\big) = \{N\}$. Hence $\mathrm{comm}(z, g) \in N$. Thus $\mathrm{comm}(z, g) \in \bigcap \mathcal{N}(G) = \{1\}$ by 9.1(ii). Since g was arbitrary, $z \in Z(G)$.

We now claim that $Z(G) \cap \overline{G'}$ is totally disconnected. For a proof let C denote the identity component of $Z(G) \cap \overline{G'}$ and take $N \in \mathcal{N}(G)$. Then $\overline{(G/N)'} = (G/N)'$ by Theorem 6.11, and $(Z(G) \cap \overline{G'})N/N \subseteq Z(G/N) \cap \overline{(G/N)'} = Z(G/N) \cap (G/N)'$, and this latter group is finite by 6.15(iii). Hence $\big((Z(G) \cap \overline{G'})N/N\big)_0 = \{N\}$. Thus $(Z(G) \cap \overline{G'})_0 \subseteq N$. Hence $(Z(G) \cap \overline{G'})_0 \subseteq \bigcap \mathcal{N}(G) = \{1\}$ by 9.1(ii). Therefore $Z(G) \cap \overline{G'}$ is totally disconnected as asserted.

(ii) The function $\mu{:}\, P \stackrel{\mathrm{def}}{=} Z(G) \times \overline{G'} \to Z(G)\overline{G'}$, $\mu(z, g) = zg$ is a surjective morphism of compact groups. Since $G_0 \subseteq Z(G)\overline{G'}$, Lemma 9.18 yields $G_0 = \mu(Z_0(G) \cap (\overline{G'})_0$. The remainder then is a consequence of the Mayer-Vietoris Formalism 9.21.

(iii) We note that Corollary 7.73 and (i) above imply

$$(G_A)_0 = (G/\overline{G'})_0 = G_0\overline{G'}/\overline{G'} = Z_0(G)\overline{G'}/\overline{G'} \cong Z_0(G)/(Z_0(G) \cap \overline{G'}).$$

This shows, in particular, exactness of (A) at G_A. Exactness at the other places of (A) is obvious. Then exactness of (B) is clear.

Finally, the exact sequence (A) is the characteristic sequence of G_A by 8.80 if and only there is an isomorphism α such that the diagram

$$\begin{array}{ccc}
\mathfrak{P}(G_A) & \xrightarrow{E_G} & G_A \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{G_A}} \\
Z_0(G) & \xrightarrow[\zeta_0]{} & G_A
\end{array}$$

is commutative, where E_G is the adjoint of the morphism $\iota_{\widehat{G_A}}{:}\, \widehat{G_A} \to \mathbb{Q} \otimes \widehat{G}_A$, $\iota(\chi) = 1 \otimes \chi$, and where $\zeta_0 = \zeta|Z_0(G)$. Since $\mathrm{im}(\zeta_0) = (G_A)_0$ by the exactness of (A), the kernel of $\widehat{\zeta}_0$ is $\mathrm{tor}\,\widehat{G}_A = \ker \iota_{G_A}$ (see 7.65, 8.4(7), and A1.45); and since $\ker(\zeta_0)$ is totally disconnected by the exactness of (A), the cokernel of $\widehat{\zeta_0}$ is a torsion group; this means that $\widehat{\zeta}_0(\widehat{G_A})$ is a pure subgroup of the torsion-free abelian group $\big(Z_0(G)\big)\widehat{\ }$ (see 8.5[(i)⇔(iii)] and A1.24). Thus α exists if and only if $\big(Z_0(G)\big)\widehat{\ }$ is divisible. By 8.5[(1)⇔(2)] this the case iff $Z_0(G)$ itself is torsion-free. □

The characteristic sequence of G_A occurring in 9.23(iii)(A) will be of considerable significance for the structure of free compact groups, see Chapter 11 (cf. 11.11, 11.14, 11.16).

For the following major structure theorem recall from 9.6, that for a connected compact group G, the closed commutator subgroup G' is semisimple. Then from the Theorem on the Structure of Semisimple Connected Compact Groups 9.19, there is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups, unique up to isomorphism and permutation of the members, and a natural morphism $\pi_G : \widetilde{G'} \stackrel{\rm def}{=} \prod_{j\in J} S_j \to G'$ with totally disconnected central kernel.

The Levi–Mal'cev Structure Theorem for Connected Compact Groups

Theorem 9.24. *Let G be a connected compact group. Then the subgroup $\Delta \stackrel{\rm def}{=} Z_0(G) \cap G'$ is totally disconnected central and $G = Z_0(G)G'$. In particular,*

(i) *there is an exact sequence*

$$\{1\} \to \Delta \xrightarrow{\delta} Z_0(G) \times G' \xrightarrow{\mu} G \to \{1\}. \qquad \text{(MV)}$$

$$\textit{In particular,} \quad G \cong \frac{Z_0(G) \times G'}{\Delta_1}, \qquad \Delta_1 \cong \Delta.$$

(ii) *There is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups and there is a totally disconnected central subgroup D of $Z_0(G) \times \prod_{j\in J} S_j$ such that*

$$G \cong \frac{Z_0(G) \times \prod_{j\in J} S_j}{D}.$$

(iii) *There is a totally disconnected central subgroup C of G such that*

$$G/C \cong \frac{Z_0(G)}{\Delta} \times \prod_{j\in J} R_j, \quad R_j = \frac{S_j}{Z(S_j)}.$$

(iv) *The entire situation is captured in the following Hasse diagram:*

$$G \;\cong\; \left\{\begin{array}{c} Z_0(G) \times \prod_{j\in J} S_j \\ \Big| \\ Z_0(G) \times \prod_{j\in J} Z(S_j) \\ \Big| \\ D \end{array}\right. \quad \begin{array}{l} \left.\begin{array}{c}\\ \\ \end{array}\right\} \cong \prod_{j\in J} S_j/Z(S_j) \\ \left.\begin{array}{c}\\ \\ \end{array}\right\} \cong Z(G) \end{array}$$

$$\begin{array}{c} D \\ \Big| \\ \{1\}. \end{array}$$

Proof. If G is connected, then $\overline{G'} = G'$ by Theorem 9.2. Hence (i) is an immediate consequence of 9.23.

(ii) By 9.6, G' is semisimple. From 9.19(ii) we get the family $\{S_j \mid j \in J\}$ and the canonical surjective morphism $\pi_G\colon \prod_{j\in J} S_j \to G'$ with central totally disconnected kernel. We define $\mu_G\colon Z_0(G) \times \widetilde{G'} \to G$ by $\mu_G(z,s) = z\pi_G(s)$, $s = (s_j)_{j\in J}$. This morphism is clearly surjective, and an element (z,s) is in its kernel iff $z^{-1} = \pi(s)$ in $Z_0(G) \cap G'$. Thus the map $s \to (\pi_G(s)^{-1}, s)\colon \pi_G(\Delta) \to \ker \mu_G$ is an isomorphism. Since $\ker \pi_G$ is totally disconnected by 9.19 and Δ is totally disconnected by (i) we conclude that $(\ker \mu_G)_0$ is singleton and thus $\ker \mu_G$ is totally disconnected (and hence central by 6.13).

Finally, the morphism $\alpha\colon \frac{Z_0(G)}{\Delta} \to G/G'$, $\alpha(z\Delta) = zG'$ is an isomorphism of compact groups by (i). We also note that $G/Z(G) \cong G'/\Delta = G'/Z(G')$ by (i). Then from 9.19(iv) we have a surjective morphism $\beta\colon G \to \prod_{j\in J} S_j/Z(S_j)$ such that $\beta\mu_G(1,(s_j)_{j\in J}) = (s_jZ(S_j))_{j\in J}$. Now set $\nu_G(g) = \bigl(\alpha^{-1}(gG'), \beta(g)\bigr)$. Then with $s = (s_j)_{j\in J} \in \prod_{j\in J} S_j$ we have $\nu_G\mu_G(z,s) = \nu(zs) = \bigl(z\Delta, (s_jZ(S_j)_{j\in J}\bigr)$ in view of the definition of β. In particular, ν is surjective. Since $\mu^{-1}\ker\nu_G = \ker(\nu_G\mu_G)$ is totally disconnected and μ_G is surjective, $\ker\nu_G$ is totally disconnected, too. ☐

The proof of Theorem 9.24 yields the following information. For a connected compact group G, we shall set

$$\begin{array}{lcccl} G^* & \stackrel{\text{def}}{=} & Z_0(G) & \times & \prod_{j\in J} S_j, \\ G_* & \stackrel{\text{def}}{=} & \frac{Z_0(G)}{\Delta} & \times & \prod_{j\in J} \frac{S_j}{Z(S_j)}. \end{array}$$

The Sandwich Theorem for Compact Connected Groups

Corollary 9.25. *Let G be a connected compact group. Then there are morphisms*

$$\begin{array}{llllll} \mu_G: & G^* & \to & G, & \mu_G(z,s) = z\pi_G(s), & s = (s_j)_{j\in J}, \\ \nu_G: & G & \to & G_*, & \nu_G(g) = (z, (s_jZ(S_j))_{j\in J}, & g = z\pi_G\bigl((s_j)_{j\in J}\bigr) \end{array}$$

both with totally disconnected central kernels, satisfying

$$\nu_G\mu_G(z,(s_j)_{j\in J}) = \bigl(z\Delta, (s_jZ(S_j)\bigr)_{j\in J}. \qquad ☐$$

We derived the classical result 9.24 on connected compact groups from Theorem 9.23. At first pass it may seem as if, conversely, Theorem 9.23 might be a consequence of 9.24. This is not so. The example $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}(2)$, $\alpha(n+2\mathbb{Z})(t) = -1^n \cdot t$ shows that $Z(G) = \{(0,0), (\frac{1}{2}+\mathbb{Z}, 0)\}$ showing that $Z_0(G)$ is trivial and $G' = G_0$ while $Z(G_0) = G_0$ and $(G_0)'$ is trivial.

The Structure Theorem 9.24 describes the structure of a compact connected group G in terms of the $Z_0(G)$ and G'. The first component is a compact connected abelian group, and we discussed these in Chapter 8. The second component is a semisimple connected compact group whose structure we elucidated in this chapter. The subgroups $Z_0(G)$ and G' are fully characteristic, i.e. each of them is mapped into itself by all continuous endomorphisms. In fact, as we shall show now, they are preserved under morphisms.

Proposition 9.26. (First Theorem on Morphisms of Compact Groups). (i) If $f\colon G \to H$ is a morphism of compact groups, then $f(G_0) \subseteq H_0$, and if f is surjective, equality holds.

(ii) If $f\colon G \to H$ is a morphism of compact groups, then

$$(\dagger) \qquad f(G') \subseteq H'.$$

If f is surjective then

$$(\ddagger) \qquad f\big(Z(G_0)\big) \subseteq Z(H_0), \quad f\big(Z_0(G_0)\big) = Z_0(H_0) \quad \text{and} \quad f(G') = H'.$$

(iii) If $f\colon G \to H$ is a morphism of connected compact groups then there is a morphism $f^* = f|Z_0(G) \times \widetilde{f|G'}\colon G^* \to H^*$ involved in the commutative diagram

$$(*) \qquad \begin{array}{ccc} G^* & \xrightarrow{f^*} & H^* \\ {\scriptstyle \mu_G}\big\downarrow & & \big\downarrow{\scriptstyle \mu_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

Proof. (i) follows from Lemma 9.18.

(ii) Let $g = [g_1, g_2]$ be a commutator in G. Then $f(g) = [f(g_1), f(g_2)] \in H'$. Thus $(\dagger)$ is an immediate consequence.

If f is surjective, then every commutator of H is the image of a commutator of G and thus $f(G') = H'$. Also, $f(G_0) = H_0$ by (i) above. Hence in order to prove the statements about the centers, it is no loss of generality to assume that G is connected which we will do. Even then the remainder is a bit more difficult. Once we have established $f\big(Z(G)\big) = Z(H)$, then $f\big(Z_0(G)\big) = Z_0(H)$ follows from (i) above. Now we prove $Z(H) = f\big(Z(G)\big)$. We write $C \stackrel{\text{def}}{=} f\big(Z(G)\big)$. Since C is central, we have $C \subseteq Z(H)$ and we must show equality. The morphism f induces a surjection $F\colon G/Z(G) \to H/C$, $F\big(gZ(G)\big) = f(g)C$. By Theorem 9.24 we know that $\Gamma \stackrel{\text{def}}{=} \big(G/Z(G)\big) \cong G'/Z(G')$ is semisimple and centerfree. Hence by 9.19(iv) we may write $\Gamma = \prod_{j\in J} R_j$ with a family of centerfree simple compact connected Lie groups R_j. Suppose now that $Z(H/C) \neq \{1\}$. Then $D \stackrel{\text{def}}{=} F^{-1}\big(Z(H/C)\big)$ is a disconnected closed normal subgroup of Γ; let N be its identity component. By 9.19(ii) and (iii), N is a partial product of Γ isomorphic to $\prod_{j\in I} R_j$ for some $I \subseteq J$. It follows that D/N is isomorphic to a nondegenerate discrete normal and hence central subgroup of a partial product isomorphic to $\prod_{j\in J\setminus I} R_j$. However, since all R_j are centerfree, this product is centerfree and thus this is a contradiction. This proves that $Z(H/C)$ is singleton. This in turn implies that $C = Z(H)$ and this is what we had to show.

(iii) In order to show that $f(\widetilde{G'}) = \widetilde{H'}$, write $S = \widetilde{f}(\widetilde{H'})$ and $K = \ker \pi_H$. Then $\pi_H(S) = \pi_H\big(\widetilde{f}(\widetilde{H'})\big) = f\big(\pi_G(\widetilde{G'})\big) = f(G') = H'$. Thus $SK = \widetilde{H}$. Now $\widetilde{H}/S$ is semisimple as a homomorphic image of the semisimple compact connected group $\widetilde{H}$. But K is central, hence abelian. Thus $\widetilde{H}/S \cong K/(S \cap K)$ is abelian. Hence $\widetilde{H}/S$ is singleton, i.e. $\widetilde{H} = S$. The remainder of (iii) is then clear. □

We shall improve the result in 9.26(ii) by showing later in 9.47(iii) that for any surjective morphism $f: G \to H$ between (not necessarily connected) compact groups the equality $f\big(Z_0(G)\big) = Z_0(H)$ holds.

A surjective morphism $f: G \to H$ between groups obviously maps the center $Z(G)$ into the center $Z(H)$. In general $f\big(Z(G)\big) \neq Z(H)$ even if G is finite. Consider for instance the 8-element quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ whose center is $Z(G) = \{\pm 1\}$. The quotient map $f: G \to H \stackrel{\rm def}{=} G/Z(G)$ satisfies $f\big(Z(G)\big) = \{1\} \neq Z(H) = H$. However, in 9.28 we will show that $f\big(Z(G)\big) = Z(H)$ if G is a compact connected group and f is continuous. The proof of this useful proposition is less obvious than one might think at first glance. More specifically, for a group G and a normal subgroup we have trivially $Z(G)N/N \subseteq Z(G/N)$. The normal subgroup $M = \{x \in G \mid (\forall g \in G)\,\mathrm{comm}(x, g) \in N\}$ contains N and satisfies $M/N = Z(G/N)$. In general to $Z(G)N \neq M$ even if $N = Z(G)$ as the example above indicates.

Lemma 9.27. *If N is a normal subgroup of G and $G/NZ(G)$ is centerfree, then $Z(G/N) = NZ(G)/N$.*

Proof. Note $NZ(G) \subseteq M$ and that $M/NZ(G)$ is central in $G/NZ(G)$. Since this group is centerfree we conclude $M \subseteq NZ(G)$. This proves the lemma. □

Theorem 9.28. (Second Theorem on Morphisms of Compact Groups). Let $f: G \to H$ be a surjective morphism of compact connected groups. Then f maps the center $Z(G)$ of G onto the center $Z(H)$ of H.

Proof. Firstly, by Lemma 9.27, the assertion is true for the natural morphisms $\mu_G: Z_0(G) \times \widetilde{G'} \to G$ since the kernel of μ is contained in the center $Z_0(G) \times Z(\widetilde{G'})$ of its domain and since $\big(Z_0(G) \times \widetilde{G'}\big)/\big(Z_0(G) \times Z(\widetilde{G'})\big) \cong \widetilde{G'}/Z(\widetilde{G'})$ is centerfree by 9.19. By 9.26 we have $f\big(Z_0(G)\big) = Z_0(H)$ and $f(G') = H'$.

Consider the diagram $(*)$ in 9.26. We have seen that both vertical maps map the center onto the center. The bottom map therefore maps the center onto the center if that is true for the top map. The top function, however, maps the center onto the center if $\widetilde{f}\big(Z(\widetilde{G'})\big) = Z(\widetilde{H'})$.

It therefore suffices to prove the theorem for compact semisimple simply connected groups. Assume that $f: G \to H$ is a surjective morphism between semisimple, simply connected compact connected groups. Then H is a direct product $\prod_{j \in J} S_j$ of simple simply connected compact Lie groups. Hence H contains normal subgroups $S_N = \prod_{j \in F} S_j \times \{1\}$ and $N = \{1\} \times \prod_{j \in J \setminus F} S_j$ such that $H = S_N N$, a direct product, as F ranges through the finite subsets of J. Indeed, $\mathcal{N}(H)$ has a basis $\mathcal{B}$ of subgroups of the form N and the subgroups S_N are semisimple simply connected compact Lie group. Let $p_N: H \to S_N$ denote the projection. Then by Lemma 9.12, the morphism $p_N: G \to H/N$ splits, i.e. may be viewed as a projection of $G \cong S_N \times M$ onto S_N. But $Z(S_N \times M) = Z(S_N) \times Z(M)$, and thus the projection maps the center onto the center of $Z(S_N)$. Hence $p_N f\big(Z(G)\big) = Z(S_N)$, i.e. $f\big(Z(G)\big)N = Z(S_N)N$. Applying 9.1(iii) we find that the intersection of the left hand sides as N ranges through $\mathcal{B}$ is $f\big(Z(G)\big)$, while the intersection of the right hand sides is $Z(H)$. This completes the proof. □

The results 9.19 and 9.24 illustrate how the structure theory of compact connected groups falls into two parts—in a technical sense. We did not deal with simple connectivity for compact abelian groups in Chapter 8. The reason, as we shall now show, is that there are no simply connected compact abelian groups (while there are many compact connected abelian groups in which every loop at the identity element is contractible; in Theorem 8.62 we classified those).

Compact Groups and Simple Connectivity

Theorem 9.29. (i) *Every simply connected compact group is semisimple. In particular, each simply connected compact abelian group is singleton.*

(ii) *Any simply connected compact group is isomorphic to a product $\prod_{j\in J} S_j$ for a family of simply connected simple compact Lie groups S_j.*

(iii) *The product of any family of simply connected compact groups is simply connected.*

Proof. (i) Assume that G is simply connected. The morphism $\mu_G: Z_0(G)\times\widetilde{G'} \to G$ has a totally disconnected kernel. Hence by Lemma 9.17, the identity $\mathrm{id}_G: G \to G$ has a unique lifting; i.e. there is a unique cross section morphism $\sigma: G \to Z_0(G)\times\widetilde{G'}$ such that $\mu_G\sigma = \mathrm{id}_G$. Let $D = \ker\mu_G$ and $H = \sigma(G)$. Then $Z_0(G) \times \widetilde{G'}$ is the semidirect product $D \rtimes_\iota H$ (cf. Lemma 9.21). Since D is totally disconnected and $Z_0(G) \times \widetilde{G'}$ is connected, $D = \{1\}$ follows. Hence $G \cong Z_0(G) \times \widetilde{G'}$. Since each factor of a simply connected product space is simply connected (see A2.8(iv)), it follows that $Z_0(G)$ is simply connected. We must show that $Z_0(G)$ is singleton. Thus for the remainder of the proof of (i) we assume that G is abelian.

We claim that a simply connected compact (additively written) abelian group G is singleton. Suppose not. Then by 2.31 there is a nonconstant character $\chi: G \to \mathbb{T}$. The quotient morphism $q: \mathbb{R} \to \mathbb{T}$, $q(r) = r + \mathbb{Z}$ is a covering of $\mathbb{T}$ (see A2.3). Since G is simply connected, there is a unique lifting $\widetilde{\chi}: G \to \mathbb{R}$ with $\chi(0) = 0$. Then $\widetilde{\chi}$ is a morphism by A2.32. Then $\widetilde{\chi}(G)$ is a compact subgroup of $\mathbb{R}$ and is therefore $\{0\}$. Then $\chi = q\circ\widetilde{\chi}$ is constant, too, and this is a contradiction to our assumption.

(ii) follows immediately from (i) and Theorem 9.19.

(iii) If $\{S_j \mid j \in J\}$ is a family of simply connected compact groups, then each S_j is a product of a family of simple simply connected compact Lie groups by (ii) above. Hence $S \overset{\text{def}}{=} \prod_{j\in J} S_j$ is a product of some (generally larger) family of compact simply connected Lie groups. Then S is simply connected by 9.14(iii). □

Maximal Connected Abelian Subgroups

If A is a connected abelian subgroup of a topological group G, then the set of all connected abelian subgroups of G containing A is inductive; thus A is contained in a maximal connected abelian subgroup M. Since $\overline{M}$ is connected and abelian, $M = \overline{M}$. In a compact Lie group the maximal connected abelian subgroups are the maximal tori, and we have seen in Chapter 6 how important these were. Their analog for arbitrary compact groups is just as crucial.

Definitions 9.30. A *pro-torus* is a compact connected abelian group. If G is a compact group, then any maximal connected abelian subgroup of G is called a *maximal pro-torus.*

If T is a maximal connected abelian subgroup of G, and $N(T,G)$ the normalizer of T in G, set $\mathcal{W}(G,T) = N(T,G)/T$ and call this group *the Weyl group of G with respect to T.* □

Obviously, a maximal connected abelian subgroup T is closed, as $\overline{T}$ is abelian and connected and contains T. By 6.62, an element $g \in G$ is in $N(T,G)$ iff $gTg^{-1} \subseteq T$ iff $(\forall t \in T)\, gtg^{-1} \in T$. Hence $N(T,G)$ is closed and $\mathcal{W}(G,T)$ is a compact group.

Every compact connected abelian group, by 2.35, is a projective limit of finite dimensional tori. This accounts for the terminology[1].

Lemma 9.31. *Let $f\colon G \to H$ be a surjective morphism of compact groups and let $A \subseteq G$ be a pro-torus. Then the following statements are equivalent:*

(i) *$f(A) = f(S)$ for any maximal pro-torus S of G containing A.*

(ii) *$f(A)$ is a maximal pro-torus of H.*

Proof. (i)⇒(ii) Let T be a maximal pro-torus of H containing $f(A)$. The restriction of f to $f^{-1}(T)$ has T as image since f is surjective. As surjective morphisms of compact groups map components onto components (7.72) f maps the identity component C of $f^{-1}(T)$ onto T. If S is a maximal pro-torus containing A, then $f(S) = f(A) \subseteq T$ the pro-torus S is contained in C and then also maximal in C. Since T is abelian, $f(C') = \{1\}$. By Theorem 9.24 we have $C = Z_0(C)C'$ we have $T = f(C) = f\big(Z_0(C)\big)f(C') = f\big(Z_0(C)\big)$. But $Z_0(C)$ is contained in each maximal pro-torus of C. Thus $T = f\big(Z_0(C)\big) \subseteq f(A) \subseteq T$. Hence $f(A)$ is maximal as asserted.

1 A note on terminology and orthography may be in order. As a matter of principle we do not separate a prefix from the root of a word by a hyphen. We write, e.g. nonequal etc.; here we follow Webster rather than Oxford. We deviate from this practise in the particular instance of the pro-torus. Indeed this aid to reading the word correctly is required as 'proto'... is also a prefix occurring in such words as "prototype, Protorenaissance, ..." The word 'walrus,' German 'Walross' (of Scandinavian origin meaning something like whale-horse) shows that 'rus' (German 'Ross') is a word meaning horse (which is of the same root). Writing protorus may suggest to the reader that we talk about a primeval primitive horse.

(ii)⇒(i) Let S be any maximal pro-torus containing A. Then $f(A) \subseteq f(S)$. Since $f(S)$ is a pro-torus and $f(A)$ is maximal, equality follows. □

Theorem 9.32 will show the astonishing fact that maximal pro-tori in compact groups behave exactly like maximal tori do in Lie groups.

The Maximal Pro-Torus Theorem for Compact Connected Groups

Theorem 9.32. *Assume that G is a compact connected group.*

(i) *The maximal pro-tori of G are conjugate.*

(ii) *If T is a maximal pro-torus of G, then $G = \bigcup_{g\in G} gTg^{-1}$.*

(iii) *Let $g \in G$. Then $Z(g,G)_0 = \{x \in G \mid xg = gx\}_0$ is the union of all maximal pro-tori which contain g.*

(iv) *The center $Z(G)$ is the intersection of all maximal pro-tori.*

(v) *Let S be a connected abelian subgroup of G. Then*

$$Z(S,G) = \bigcup\{T \mid T \text{ is a maximal pro-torus of } G \text{ such that } S \subseteq T\}.$$

In particular, the centralizer of a connected abelian subgroup of G is connected.

(vi) *Each maximal pro-torus T of G is its own centralizer $Z(T,G)$.*

(vii) *Each maximal pro-torus of G is a maximal abelian subgroup.*

Proof. (i) Let T_1 and T_2 be two maximal pro-tori of G. Let $N \in \mathcal{N}(G)$ and define $C_N = \{g \in G \mid gT_1g^{-1}N = T_2N\}$. Then $N \subseteq M$ in $\mathcal{N}(G)$ implies $C_M \subseteq C_N$. We claim that $C_N \neq \emptyset$. By 9.31, T_jN/N is a maximal torus of G/N for $j = 1, 2$. Hence by the Transitivity Theorem 6.27 there is a $gN \in G/N$ such that $(gN)(T_1N/N)(gN)^{-1} = T_2N/N$, and then $gT_1g^{-1}N = T_2N$. Hence $g \in C_N$ as asserted. This implies $gT_1g^{-1} \subseteq T_2N$ and $g^{-1}T_2g \subseteq T_1N$. Now let g be in the intersection of the filter basis of compact sets C_N. Then $gT_1g^{-1} \subseteq T_2N$ for all $N \in \mathcal{N}(G)$. Then $gT_1g^{-1} \subseteq T_2$ and $g^{-1}T_2g \subseteq T_1$ by 9.1(iii). We conclude $gT_1g^{-1} = T_2$.

(ii) Let $N \in \mathcal{N}(G)$ and $x \in G$. By 9.31, the torus TN/N is maximal in G/N. By the Maximal Torus Theorem 6.30 there is a g such that $xN \in (gN)(TN/N)(gN)^{-1}$, that is, $g^{-1}xg \in TN$. Thus the set $F_N = \{g \in G \mid g^{-1}xg \in TN\}$ is not empty, and the F_N for a filter basis of compact sets containing an element g in its intersection. Then $g^{-1}xg \in TN$, i.e. $Tg^{-1}xg \cap N \neq \emptyset$ for all $N \in \mathcal{N}(G)$. Forming intersections and observing 9.1(iii) we again conclude $1 \in Tg^{-1}xg$. Thus $x \in gTg^{-1}$.

Proof of (iii) through (vii): Exercise E9.7. □

Exercise E9.7. Prove 9.25(iii) through (vii).

[Hint. (a) Show that, after 9.32(i) and (ii) and after replacing the word "torus" by the word "pro-torus," the proofs of 6.32 and 6.33 apply almost verbatim to the present more general situation.] □

Recall from E6.10 following 6.33 that the converse of 9.32(vii) fails even in compact connected Lie groups.

Corollary 9.33. *Let T be a maximal pro-torus of a compact group G. Then*

$$G = G_0 N(T,G), \text{ and}$$
$$G/G_0 \cong N(T,G)/(N(T,G)\cap G_0) = N(T,G)/N(T,G_0).$$

Proof. Exercise E9.8. □

Exercise E9.8. Prove 9.33.

[Hint. Use 9.25 in order to apply the proofs of 6.33(i), (ii) to prove (i) and (ii), respectively, and the proof of 6.35 to prove (iii).] □

Proposition 9.34. *Let T be a maximal pro-torus of a compact group G. Then $N(T,G)_0 = T$.*

Proof. Let $M \in \mathcal{N}(G)$. Recall from 9.9 that TN/N is a maximal torus of G/N. Then $N(TN/N, G/N)_0 = TN/N$. Since

$$gN \in N(TN/N, G/N) \quad \text{iff} \quad (gN)(TN/N)(gN)^{-1} = TN/N \quad \text{iff} \quad gTg^{-1} \subseteq TN$$

we know that $g \in N(T,G)$ implies $gN \in N(TN/T, G/N)$. It now follows that $N(T,G)_0 \subseteq TM$ for all $M \in \mathcal{N}(G)$. Then $N(T,G)_0 \subseteq T$ by 9.1(iv). □

Theorem 9.35 (Divisibility in Compact Groups). *For a compact group G the following conditions are equivalent:*

(i) *G is divisible, i.e. for each $g \in G$ and each natural number n there is an $x \in G$ such that $x^n = g$.*
(ii) *G is connected.*

Proof. (ii)⇒(i) Every compact connected abelian group (such as T in Theorem 9.32) is divisible by 8.5. The divisibility of G now follows from 9.32(ii).

(i)⇒(ii) Let $g \in G$ and assume that G is divisible. Recursively we find, using divisibility of G, a sequence of elements $x_1 = g, x_n, n = 2, \ldots$ such that $x_n^n = x_{n-1}$. Then we have a unique morphism of groups $f\colon \mathbb{Q} \to G$ such that $f(1) = g$ and $f(\frac{1}{n!}) = x_n$. Now $A = \overline{f(\mathbb{Q})}$ is a compact abelian group which is divisible (since the closure of a divisible subgroup of a compact group is divisible: Exercise E9.9). Hence by 8.5, A is connected. Hence $g \in G_0$. Since $g \in G$ was arbitrary, $G = G_0$. □

Exercise E9.9. Prove the following assertion.

The closure of a divisible subgroup of a compact group is divisible. □

The Structure of Maximal Pro-tori in Compact Connected Groups

Theorem 9.36. *Let G be a compact connected group and*

$$\mu_G\colon G^* \stackrel{\text{def}}{=} Z_0(G) \times \widetilde{G}' \to G, \quad \widetilde{G}' = \prod_{j\in J} S_j$$

the morphism given by $\mu_G(z,g) = z\pi_G(g)$, *where all* S_j *are simply connected simple compact Lie groups.*

(i) *Let* T_j *be a maximal torus of* S_j. *Then* $T^* \stackrel{\text{def}}{=} Z_0(G)\times\prod_{j\in J} T_j$ *is a maximal pro-torus of* G^*, *and every maximal pro-torus of* G^* *is so obtained.*

(ii) $T = \mu_G(T^*)$ *is a maximal pro-torus of* G, *and every maximal pro-torus of* G *is of this form with a suitable* T^*.

(iii) $N(T^*, G^*) = Z_0(G) \times \prod_{j\in J} N(T_j, S_j)$, *and*

$$\mathcal{W}(G^*,T^*) \cong \mathcal{W}(G,T) \cong \prod_{j\in J} \mathcal{W}(S_j,T_j).$$

(iv) *The homogeneous space* G/T *is homeomorphic to* $G^*/T^* \cong \prod_{j\in J} S_j/T_j$.

(v) *If* G *is a semisimple compact connected group, then any maximal pro-torus* T *is a torus, i.e. a product of circle groups.*

(vi) $w(G) = w(T)$.

(vii) *The homogeneous space* G/T *is simply connected.*

(viii) *For any infinite cardinal* $\aleph < w(G)$ *there is closed connected abelian subgroup of* G *of weight* $\aleph$.

(ix) *Every element of* G *is contained in a connected monothetic subgroup.*

Proof. (i) Let $T^\#$ be a connected abelian group containing T^*. The projection of $T^\#$ into $Z_0(G)$ is surjective. The projection of $T^\#$ into S_j is abelian, connected, and contains T_j, hence agrees with T_j. So $T^\# \subseteq Z_0(G) \times \prod_{j\in J} T_j = T^* \subseteq T^\#$.

(ii) After Lemma 9.31 this is a consequence of (i).

(iii) An element $g = (z,(s_j)_{j\in J}) \in G^*$ is in the normalizer of T^* iff $gT^*g^{-1} \subseteq T^*$ iff $s_jT_js_j^{-1} = T_j$ for all $j \in J$ iff $z \in Z_0(G)$ and $s_j \in N(T_j,S_j)$. This shows the asserted structure of $N(T^*,G^*)$. Then

$$\mathcal{W}(G^*,T^*) = \frac{Z_0 \times \prod_{j\in J} N(T_j,S_j)}{Z_0 \times \prod_{j\in J} T_j} \cong \prod_{j\in J} N(T_j,S_j)/T_j = \prod_{j\in J} \mathcal{W}(S_j,T_j).$$

Let $g \in G$. Then $g = \mu_G(g^*)$ for some $g^* \in G^*$. Now $g \in N(T,G)$ iff $\mu(g^*T^*(g^*)^{-1}) = gTg^{-1} \subseteq T = mu(T^*)$ iff $g^*T^*(g^*)^{-1} \subseteq T^* \ker\mu_G = T^*$ (since $\ker\mu_G \subseteq Z(G^*) \subseteq T^*$ by 9.25(iv)). This is the case iff $g^* \in N(T^*,G^*)$. Thus, recalling that $\ker\mu_G \subseteq T^*$ we get

$$\begin{aligned}\mathcal{W}(G,T) &= \frac{N(T,G)}{T} = \frac{\mu_G\big(N(T^*,G^*)\big)}{\mu_G(T^*)}\\ &\cong \frac{N(T^*,G^*)}{T^*} = \mathcal{W}(G^*,T^*) \cong \prod_{j\in J}\mathcal{W}(S_j,T_j).\end{aligned}$$

(iv) Consider the continuous map $\tau\colon G^*/T^* \to G/T$, $\tau(gT^*) = \mu_G(g)T$. Since μ_G is surjective, τ is surjective. The relation $\mu_G(gT^*) = \mu_G(hT^*)$ means $\mu_G(h^{-1}g)T = T$, i.e. $\mu_G(h^{-1}g) \in T$, i.e. $h^{-1}g \in \mu_G^{-1}(T) = T^*\ker\mu_G$. Since $\ker\mu_G \subseteq T^*$ by 9.32(iv), we have $T^* = \mu_G^{-1}(T)$. We conclude that $\mu_G(gT^*) = \mu_G(hT^*)$ implies $gT^* = hT^*$. Thus μ_G is bijective and therefore a homeomorphism. However,

$$\frac{G^*}{T^*} = \frac{Z_0(G)\times\prod_{j\in J} S_j}{Z_0(G)\times\prod_{j\in J} T_j} \cong \prod_{j\in J}\frac{S_j}{T_j}.$$

(v) If G is semisimple then $Z_0(G)=\{1\}$, $G=G'$, and $\mu_G=\pi_G\colon\widetilde{G}\to G$. Then $T=\pi_G(T^*)$ and $T^*=\prod_{j\in J} T_j$ As a product of tori, T^* is itself a torus. Quotients of tori are tori (since dually subgroups of free abelian groups are free). Thus T is a torus. All other pro-tori are conjugate to T by the Maximal Pro-Torus Theorem 9.10 and thus they are tori.

(vi) The relation $w(T)\le w(G)\le w(G^*)$ is trivial. Since $T\cong T^*/D$ for a totally disconnected group D we have an exact sequence

$$0\to\widehat{T}\xrightarrow{(\mu_G|T^*)\widehat{}}\widehat{T^*}\to\widehat{D}\to 0.$$

Since D is totally disconnected, $\widehat{D}$ is a torsion group (see 8.5). Hence $\widehat{T}$ is isomorphic to a pure subgroup of the torsion-free group $\widehat{T^*}$ (see A1.24). Hence $\mathbb{Q}\otimes\widehat{T}\cong\mathbb{Q}\otimes\widehat{T^*}$ and thus $w(T)=\operatorname{card}\widehat{T}=\operatorname{card}(\mathbb{Q}\otimes\widehat{T})=\operatorname{card}(\mathbb{Q}\otimes\widehat{T^*})=\operatorname{card}\widehat{T^*}=w(T^*)$. (See 7.76(ii).) It suffices now to verify $w(G^*)\le w(T^*)$. Since $w(G^*)=w(Z_0(G))+\sum_{j\in J} w(S_j)$ by EA4.34 and similarly $w(T^*)=w(Z_0(G))+\sum_{j\in J} w(T_j)$, and since $w(T_j)=\aleph_0=w(S_j)$ the relation $w(G^*)=w(T^*)$ follows. This completes the proof.

(vii) By (iv) above, the space G/T is homeomorphic to $\prod_{j\in J} S_j/T_j$; by E6.19 (following 6.97), each of the manifolds S_j/T_j is simply connected and orientable.

Then G/T is simply connected by A2.11(iii) of Appendix 2.

(viii) Let T again be a maximal pro-torus of G. Then $w(T)=w(G)$ by (vi). Then the claim follows from Exercise E8.3.

(ix) Every element is contained in a maximal pro-torus of G by Theorem 9.32(ii). Then the claim follows from Exercise E8.15(ii) preceding 8.81. □

The proof of E6.19 which we used in part (vii), invokes various facts from algebraic topology for which we had to refer to outside support.

The following Exercise shows that part (viii) of Theorem 9.36 has a considerable generalisation

Exercise E9.10. Prove the following assertion:

Proposition. *Let $\aleph$ be an infinite cardinal such that $\aleph < w(G)$ for a compact connected group G. Then G contains a closed* connected and normal *subgroup N such that $w(N)=\aleph$.*

[Hint. Following the Levi-Mal'cev Structure Theorem for Compact Connected Groups (Theorem 9.24) we have $G=G'Z_0(G)$ where the algebraic commutator subgroup G' is a characteristic compact (see Theorem 9.2) connected semisimple (see Corollary 9.6 and Theorem 9.19ff.) subgroup, and the identity component of the center $Z_0(G)$ is a characteristic compact connected abelian subgroup.

Case 1. $w(G')\le\aleph$. Then $w(G)=w(Z_0(G))$, and by Corollary 1.3, $Z_0(G)$ contains a connected closed subgroup N of weight $\aleph$; since it is central, it is normal.

Case 2. $w(G') > \aleph$. If we find a compact connected normal subgroup N of G', we are done, since the normalizer of N contains both G' and the central subgroup $Z_0(G)$, hence all of $G = G'Z_0(G)$. Thus it is no loss of generality to assume that $G = G'$ is a compact connected semisimple group.

Case 2a. $G = \prod_{j\in J} G_j$ for a family of compact connected (simple) Lie groups. Then $w(G_j) = \aleph_0$, and $\aleph < w(G) = \max\{\aleph_0, \operatorname{card} J\}$ (see EA4.3.). Since $\aleph$ is infinite and smaller than $w(G)$, we have $w(G) = \operatorname{card} J$. Then we find a subset $I \subseteq J$ such that $\operatorname{card} I = \aleph$, and set $N = \prod_{i\in I} G_i$. Then $w(N) = \operatorname{card} I = \aleph$.

Case 2b. By the Sandwich Theorem for Semisimple Compact Connected Groups (Corollary 9.20) there is a family of simply connected compact simple Lie groups S_j with center $Z(S_j)$ and there are surjective morphisms

$$\prod_{j\in J} S_j \xrightarrow{f} G \xrightarrow{q} \prod_{j\in J} S_j/Z(S_j)$$

such that qf is the product $\prod_{j\in J} p_j$ of the quotient morphisms $p_j\colon S_j \to S_j/Z(S_j)$. Now both products $\prod_{j\in J} S_j$ and $\prod_{j\in J} S_j/Z(S_j)$ have the same weight $\operatorname{card} J$ which agrees with the weight of the sandwiched group G. Define I as in Case 2a and set $N = f(\prod_{i\in I} S_i)$ and note that $q(N) = \prod_{i\in I} S_i/Z(S_i)$. Hence N is sandwiched between two products with weight $\operatorname{card} I = \aleph$ and hence has weight $\aleph$. This proves the existence of the asserted N in the last case.]

In the proof of the following proposition we need a lemma on compact Lie algebras

Lemma 9.37. *Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ be a direct sum of simple compact Lie algebras and $\mathfrak{t}_k$ be a Cartan subalgebra in $\mathfrak{s}_k$ for $k = 1, \dots, n$. Write $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_n$. Assume $\mathfrak{h}$ is a subdirect sum of $\mathfrak{g}$ containing $\mathfrak{t}$, that is $\mathfrak{t} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ and for each coordinate projection $p_k\colon \mathfrak{g} \to \mathfrak{s}_k$, $k = 1, \dots, n$, the relation $\mathfrak{s}_k = p_k(\mathfrak{h})$ holds. Then $\mathfrak{h} = \mathfrak{g}$.*

Proof. The algebra $\mathfrak{t}$ is a Cartan algebra of $\mathfrak{h}$. Since each direct summand $\mathfrak{s}_k$ is simple and the coordinate projection $p_k\colon G \to \mathfrak{s}_k$ of $\mathfrak{g}$ is surjective, the center of $\mathfrak{h}$ is annihilated by p_k. Since the p_k, $k = 1, \dots, n$ separate the points, the center of the compact Lie algebra $\mathfrak{h}$ is zero and thus $\mathfrak{h}$ is semisimple.

First assume that all $\mathfrak{s}_k$ are isomorphic to $\mathfrak{s}$. Then all simple factors of $\mathfrak{h}$ must be isomorphic to $\mathfrak{s}$ since $\mathfrak{h}$ is a subdirect product of $\mathfrak{g}$. Now $\mathfrak{g} \cong \mathfrak{s}^n$ and $\mathfrak{h} \cong \mathfrak{s}^m$ with $m \le n$. But isotypic semisimple Lie algebras are uniquely determined by their rank, that is, the dimension of their Cartan subalgebras. Since $\mathfrak{g}$ and $\mathfrak{h}$ have the common Cartan algebra $\mathfrak{t}$, we conclude $m = n$ and thus $\dim \mathfrak{h} = \dim \mathfrak{g}$, which implies $\mathfrak{h} = \mathfrak{g}$.

Now let the $\mathfrak{s}_k$ be arbitrary. We represent $\mathfrak{g}$ as a direct sum of isotypic ideals $\mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_p$ (that is, ideals each of which is a sum of isomorphic simple ideals) of different type. Then the projection $\pi_k\colon \mathfrak{g} \to \mathfrak{i}_k$ maps $\mathfrak{h}$ onto $\mathfrak{i}_k$ by the preceding part of the proof. This implies that the semisimple algebra $\mathfrak{h}$ contains an ideal $\mathfrak{j}_k \cong \mathfrak{i}_k$.

Since the simple components of different $\mathfrak{i}_k$ are not isomorphic, the ideals $\mathfrak{j}_k$ form a direct sum in $\mathfrak{h}$. Hence $\dim \mathfrak{g} = \sum_{k=1}^{p} \dim \mathfrak{i}_k = \sum_{k=1}^{p} \dim \mathfrak{j}_k \le \dim \mathfrak{h} \le \dim \mathfrak{g}$. Hence $\dim \mathfrak{h} = \dim \mathfrak{g}$, and thus $\mathfrak{h} = \mathfrak{g}$ finally follows. □

Theorem 9.38 (Generating Compact Connected Groups). *Let G be a compact connected group and T a maximal pro-torus, then there is an element $g \in G$ such that G is topologically generated by $T \cup \{g\}$, that is $G = \overline{\langle T \cup \{g\}\rangle}$.*

If the weight of G does not exceed $2^{\aleph_0}$, then $G = \overline{\langle t, g\rangle}$ for suitable elements $g, t \in G$.

Proof. In view of the fact that $G = Z_0(G)G'$ by the Structure Theorem 9.24 and that $Z_0(G)$ is contained in every maximal pro-torus, it suffices to consider the case that G is semisimple. As G is a quotient of a direct product of a family of simple compact Lie groups by 9.14, it suffices to assume that G itself is such a product. Thus we have $G = \prod_{j\in J} S_j$ and a maximal pro-torus $T = \prod_{j\in J} T_j$ with a maximal torus T_j of L_j for each $j \in J$.

By Auerbach's Generation Theorem 6.51, there are elements g_j and h_j such that S_j is topologically generated by $\{g_j, h_j\}$. Every element is contained in a maximal torus and the maximal tori are conjugate. Hence we may assume that $h_j \in T_j$. Thus G is topologically generated by $T_j \cup \{g_j\}$. We set $g = (g_j)_{j\in J}$ and claim that G is topologically generated by $T \cup \{g\}$. We now prove this claim. The group H topologically generated by $T \cup \{g\}$ in G contains T and is a subdirect product of $G = \prod_{j\in J} S_j$; i.e. $T \subseteq H \subseteq G$ and for each coordinate projection $p_j\colon G \to S_j$, $j \in J$, we have $S_j = p_j(H)$. For any subset $F \subseteq J$ let $p_F\colon G \to \prod_{j\in J} S_j$ be the partial projection. Assume momentarily that $p_F(H) = p_F(G)$ for all finite subsets F of J. Then H is dense in G by the definition of the product topology, and since H is compact, $H = G$ follows.

It suffices, therefore, to verify the claim $H = G$ if J is finite. The Lie algebras g of G and $\mathfrak{h}$ of H, however, satisfy the hypotheses of Lemma 9.37. Hence $\mathfrak{g} = \mathfrak{h}$ by 9.37. Thus H is open in G, and since G is connected, $G = H$ follows. The last remark of the proposition follows from the preceding and from Example 8.75, since according to 8.75 we have $T = \overline{\langle t\rangle}$ iff $w(T) \le 2^{\aleph_0}$. □

The Splitting Structure Theorem

In contrast with the Levi–Mal'cev Structure Theorem 9.24, the principal structure theorem [30, 319, 167] presented below in 9.39 expresses a semidirect splitting of a connected compact group over its (semisimple) commutator group. In a number of respects, certainly from the purely topologically point of view, this decomposition is superior to the Levi–Mal'cev decomposition. It has no natural generalisation to noncompact locally compact groups or even to linear Lie groups.

We recall the concept of a semidirect product from 5.72ff and from 9.21 above. If N is a closed normal subgroup of a compact group G and H a closed subgroup such that $N \cap H = \{1\}$ and $NH = G$ then we call N a *semidirect factor* and H a *cofactor*. (See also the general remarks preceding Lemma 6.37.)

The Borel–Scheerer–Hofmann Splitting Theorem

Theorem 9.39. (i) *Let G be a compact connected group. Then G' is a semidirect factor. In other words, there is a compact connected abelian subgroup A of G such that*

$$(g,a) \mapsto ga\colon G' \rtimes_\iota A \to G, \qquad \iota(a)(g) = aga^{-1}$$

is an isomorphism of compact groups.

Fix a maximal pro-torus T of G' and let $\mathrm{Hom}_{Z_0 \cap G'}(Z_0(G), T)$ denote the set of morphisms from $Z_0(G)$ to T restricting on $Z_0(G) \cap G'$ to the identity. Then for each $f \in \mathrm{Hom}_{Z_0(G)\cap G'}(Z_0(G), T)$ the set

$$A_f = \{f(z)^{-1}z \mid z \in Z_0(G)\}$$

is a cofactor to G', and every cofactor is conjugate to one of these.

(ii) *Each maximal pro-torus T of G is isomorphic to $T_{G'} \times A$ for a maximal torus $T_{G'}$ of G' and $A \cong G/G' \cong Z_0(G)/(Z_0(G) \cap G')$.*

Proof. (i) Let T be a maximal pro-torus of G'; then T is a torus by 9.36(v). The central group $Z_0(G) \cap G'$ is contained in T by the Maximal Pro-Torus Theorem 9.32(iv) and thus equals $Z_0(G) \cap T$. Since T is injective in the category of compact abelian groups by 8.78(ii), the identity morphism $Z_0(G) \cap T \to T$ extends to a morphism $Z_0(G) \to T$ whose coextension to G' we write $f\colon Z_0(G) \to G'$. We apply Lemma 6.37 with $N = G'$, $H = Z_0(G)$ with the trivial automorphic action of H on N since H is central. The group $A_f = \{f(z)^{-1}z \mid z \in Z_0(G)\}$ is a cofactor, and all cofactors arise from morphisms $f\colon Z_0(G) \to G'$ extending the identity map of $Z_0(G) \cap G'$ in this fashion by 6.37. If $f\colon Z_0(G) \to G'$ is one such morphism then $f\big(Z_0(G)\big)$ is a pro-torus and thus is contained in a maximal pro-torus T^*. By the Maximal Pro-Torus Theorem 9.32(i) there is a $g \in G$ such that $gT^*g^{-1} = T$. Then $\varphi(z) = gf(z)g^{-1}$ defines morphism $\varphi \in \mathrm{Hom}_{Z_0(G)\cap G'}(Z_0(G), G')$, and $gA_fg^{-1} = A_\varphi$.

(ii) The group $Z_0(G)T$ is a maximal pro-torus by 9.31 since it is a homomorphic image of the maximal pro-torus $Z_0(G) \times T$ under the surjective morphism $(z,g) \mapsto zg\colon Z_0(G) \times G' \to G$. We can write $Z_0(G)T = TA_f$ by (i), and $(t,a) \mapsto ta\colon T \times A_f \to Z_0(G)T$ is an isomorphism. ☐

Supplementing the Identity Component

We shall see later (cf. Corollary 10.37 below) that each compact group G contains a compact totally disconnected subset D such that $(g_0, d) \mapsto g_0d\colon G_0 \times D \to G$ is a homeomorphism. We also know that in general, D cannot be chosen to be a subgroup not even in the case that G is a nilpotent Lie group of class 2 (see Example E6.9) and not even in the case that G is abelian (see Example 8.11). It is therefore a useful fact that in any case, a totally disconnected subgroup D can be found which *supplements* G_0, and, notably, that it can be found such that $G_0 \cap D$ is pleasant.

Lemma 9.40. *Let G be a compact group and S a closed subgroup. Assume that a set $\mathcal{C}$ of closed subgroups of G is closed under the intersection of subsets which are totally ordered under inclusion. Then the subset $\mathcal{C}_S$ of all $H \in \mathcal{C}$ satisfying $SH = G$ contains minimal elements.*

Proof. We show that $(\mathcal{C}_S, \supseteq)$ is inductive. Let $\mathcal{T} \subseteq \mathcal{C}_S$ be a totally ordered subset. Then $C \stackrel{\text{def}}{=} \bigcap \mathcal{T}$ is a compact subgroup contained in $\mathcal{C}$ as $\mathcal{C}$ is closed under the intersection of chains. We must show $SC = G$. Let $g \in G$. Then $g \in SH$ for all $H \in \mathcal{T}$ since $\mathcal{T} \subseteq \mathcal{C}_S$. Equivalently, $Sg \cap H \neq \emptyset$. Thus $\{Sg \cap H \mid H \in \mathcal{T}\}$ is a filter basis of compact sets and there is an $x \in \bigcap_{H\in\mathcal{T}} Sg \cap H = Sg \cap \bigcap_{H\in\mathcal{T}} H = Sg \cap C$ whence $g \in SC$. Hence $\mathcal{C}_S$ is down-inductive as asserted, and thus, by Zorn's Lemma, contains minimal elements. □

Dong Hoon Lee's Supplement Theorem for compact Groups

Theorem 9.41. *Let G be a compact group and T an arbitrary maximal pro-torus of G. Then the following conclusions hold.*

(I) *$N(T, G) \subseteq G$ contains a compact subgroup S such that $G = \big((G_0)'\big)S$ and that $(G_0)' \cap S \subseteq Z((G_0)')$.*

(II) *There is a compact subgroup D with the following properties:*
- (i) *$G = G_0D$,*
- (ii) *$G_0 \cap D$ is normal in G,*
- (ii)′ *$G_0 \cap D \subseteq Z(G_0)$,*
- (iii) *$D \subseteq N(T, G)$, and*
- (iv) *D is totally disconnected.*

Proof. First we note that (ii) and (ii)′ are equivalent, as long as (i) and (iv) hold: By (ii) and (iv), $G_0 \cap D$ is a totally disconnected normal subgroup of the connected group G_0 and is therefore central in G_0 by 6.13. Conversely as was noted towards the end of the proof of 6.74, if (ii)′ holds then the centralizer and thus the normalizer of $G_0 \cap D$ contains G_0, and as G_0 is normal in G, the normalizer also contains D. Hence $G_0 \cap D$ is normal in $G_0D = G$ by (i).

We prove the theorem by a number of reductions.

(a) If G_0 is abelian, then $T = G_0$ and (I) is trivially satisfied with $S = G$. We claim that (II) holds. We need to find a subgroup D satisfying (i) and (iv); then (ii) is true since now G_0 and D are in the normalizer of $G_0 \cap D$, and (iii) is trivial. By Lemma 9.40 there is a minimal closed subgroup D such that $G = G_0D$. Let N be a compact normal subgroup of D such that D/N is a Lie group. Then $(D/N)_0 = D_0N/N$ by 9.18, and since $D_0 \subseteq G_0$, this group is abelian. Now by Theorem 6.10(i) there is a closed subgroup D_N of D containing N such that

$D/N = (D_0N/N)(D_N/N) = D_0D_N/N$ and D_N/N is finite. Hence $G = G_0D = G_0D_0D_N = G_0D_N$. Thus by the minimality of D we have $D_0D_N = D = D_N$. Hence $D_0 \subseteq D_N$, and since D_N/N is discrete, this implies $D_0 \subseteq N$. This holds for all $N \in \mathcal{N}(G)$ (cf. paragraph preceding 9.1), and thus by 9.1 we conclude $D_0 = \{1\}$. Therefore D is totally disconnected and (II) is proved in this case.

(b) We now show that the theorem is true if it is true whenever $Z(G_0) = \{1\}$. Indeed set $Z \stackrel{\text{def}}{=} Z(G_0)$ note that $(G/Z)_0 = G_0/Z$ by 9.26(i), and $Z(G_0/Z) = Z(G_0)/Z(G_0) = \{1\}$ by 9.23(ii). Thus the theorem applies to G by hypothesis. By 9.32(iv) we have $Z \subseteq T$, and T/Z is a maximal torus of G/Z by 9.31. Note that $G_0 = (G_0)'Z$ Hence we find a closed subgroup S of G containing the group Z such that $S/Z \subseteq N(T/Z, G/Z)$, $G/Z = \big((G/Z)_0\big)'(S/Z) = (G_0/Z)'(S/Z) = \big((G_0)'Z/Z\big)(S/Z) = (G_0)'S/Z$, and $\big((G_0)' \cap S\big)/\big((G_0)' \cap Z\big) \cong \big((G_0)' \cap S)Z/Z = \big((G_0)'Z \cap S)/Z = \frac{G_0}{Z} \cap \frac{S}{Z} = (G/Z)_0 \cap (S/Z) = Z((G_0/Z)') = \{Z\}$, because $G_0/Z(G_0)$ is centerfree by 9.24(iv). Hence $S \subseteq N(T, G)$, $G = (G_0)'S$, and $(G_0)' \cap S \subseteq (G_0)' \cap Z \subseteq Z\big((G_0)'\big)$. Then the group S satisfies the conclusions of (I).

Now $S_0 \subseteq G_0 \cap Z = Z$ is abelian. We apply (a) to S and find a totally disconnected subgroup $D \subseteq S \subseteq N(T, G)$ such that $S = S_0D$. Then $G = (G_0)'S \subseteq G_0S = G_0S_0D = G_0D \subseteq G$ and $G_0 \cap D \subseteq G_0 \cap S = Z(G_0)$. Hence (I) is proved. This completes the proof of reduction (b).

We now assume that $Z(G_0) = \{1\}$, i.e. that G_0 is a compact connected centerfree (hence semisimple) group. Accordingly, (I) and (II) are equivalent assertions. We may restrict out attention to (I). By 9.19(iv), the group G_0 is isomorphic to a product $\prod_{j\in J} R_j$ for a family of simple connected compact centerfree groups R_j. The maximal pro-torus T corresponds to a subgroup $\prod_{j\in J} T_j$ for a maximal torus T_j of R_j by 9.36.

(c) We let $\mathcal{S}$ be the set of all compact subgroups S of G such that

(i) $G = G_0S$,

(ii) $G_0 \cap S \trianglelefteq G$,

(iii) $(\forall s \in S) sTs^{-1} \subseteq T(G_0 \cap S)$,

are satisfied. This set is not empty because it contains G.

The members of $\mathcal{S}$ have rather special properties which we explore first. Indeed let $S \in \mathcal{S}$. Then $G_0 \cap S \trianglelefteq G_0$ by (ii) above. Hence $G_0 \cap S$ is connected by 9.19(iv) and thus is contained in $S_0 \subseteq G_0 \cap S$ and therefore

(1) $$S_0 = G_0 \cap S \trianglelefteq G.$$

In particular, S_0 is isomorphic to a partial product of $G_0 \cong \prod_{j\in J} R_j$. Hence

(2) $$Z(S_0) = \{1\}.$$

Since $S_0 \cong \prod_{j\in I} R_j$ there is a unique subgroup $G_1 \subseteq G_0$ which is a partial product isomorphic to $\prod_{j\in J\setminus I} R_j$ and is such that $G = G_1S_0$ is a direct product. For every automorphism α of G_0 the subgroup $\alpha(G_1)$ is a connected normal subgroup of G_0, hence is a partial product. Also G_0, being the direct product of the partial products G_1 and S_0, is the direct product of the partial products $\alpha(G_1)$ and $\alpha(S_0)$. Thus if $\alpha(S_0) = S_0$, then $\alpha(G_1) = G_1$ by the way that normal subgroups of G_0

are uniquely determined according to 9.19(iv). Applying this to the restrictions of inner automorphisms of G to G_0, the normality of S_0 implies the normality of G_1. Now $G = G_0S = G_1S_0S = G_1S$ and $G_1 \cap S \subseteq G_1 \cap (G_0 \cap S) = G_1 \cap S_0 = \{1\}$. Hence

$$G = G_1S \cong G_1 \rtimes S, \quad G_0 = G_1S_0 \cong G_1 \times S_0. \tag{3}$$

Since the maximal torus $T \subseteq S_0$ is isomorphic to $\prod_{j\in J} T_j \subseteq \prod_{j\in J} R_j \cong G_0$, moreover $G_1 \cong \prod_{j\in J\setminus I} R_j$ and $S_0 \cong \prod_{j\in I} R_j$, we find a maximal torus $T_1 \stackrel{\text{def}}{=} (T \cap G_1) \cong \prod_{j\in J\setminus I} T_j$ of G_1 and a maximal torus $S_0 \cap T \cong \prod_{j\in I} T_j$ of S_0 such that

$$\begin{gathered} T = T_1(T\cap S_0) \cong T_1 \times (T\cap S_0), \quad T\cap S_0 = T\cap S, \quad \text{and} \\ T(G_0\cap S) = TS_0 = T_1S_0 \cong T_1 \times S_0. \end{gathered} \tag{4}$$

From (iii) and (4) we see that $s \in S$ implies

$$sT_1s^{-1} \subseteq sTs^{-1} \cap sG_1s^{-1} = T(G_0 \cap S) \cap G_1 = T_1S_0 \cap G_1 = T_1.$$

Hence

$$S \subseteq N(T_1, G). \tag{5}$$

We claim that $\mathcal{S}$ contains minimal elements. By Lemma 9.40 it suffices to show that the set of all closed subgroups S satisfying (ii) and (iii) is closed under the intersection of chains. Let $\mathcal{T}$ be such a chain and let $S \stackrel{\text{def}}{=} \bigcap \mathcal{T}$. Obviously, S satisfies (ii). We claim that S satisfies (iii). For each $t \in T$, $s \in S$ and $R \in \mathcal{T}$ we have $s \in R$ and thus by $(\text{iii})_R$ there are elements $t_R \in T$ and $n_R \in G_0 \cap R$ such that $sts^{-1} = t_Rn_R$. If (t, n) is the limit of some converging subnet of the net $\big((t_R, n_R)\big)_{R\in\mathcal{T}}$ on the compact space $T \times G$, then $t \in T$ and $n \in G_0 \cap R$ for all $R \in \mathcal{T}$ and thus $sts^{-1} = tn \in T(G_0 \cap S)$. Hence (iii) is satisfied. Thus 9.40 applies and shows that there is a minimal element S satisfying (i), (ii), and (iii). We claim that $G_0 \cap S = \{1\}$; a proof of this claim will indeed finish the proof of the theorem.

Suppose that the claim is false. Then we find a closed normal subgroup M of G such that G/M is a Lie group and

$$M \cap S_0 = M \cap G_0 \cap S \neq G_0 \cap S \tag{6}$$

by 9.1(ii). We set $N \stackrel{\text{def}}{=} M \cap S_0$. Then $N \trianglelefteq G$ by (1), and $S_0/N = S_0/(S_0 \cap M) = S_0M/M$ is a Lie group. Since $T \cap S = T \cap S_0$ is a maximal pro-torus of S by (4) above, by 9.31, $(T \cap S)/N$ is a maximal torus of S/N.

We apply Lee's Theorem for Lie groups 6.74(I) to S/N and find a subgroup S_N of S containing N such that firstly, $S_0S_N/N = (S_0/N)(S_N/N) = (S/N)_0(S_N/N) = S/N$ (by 9.26(i)), secondly, $(S_0 \cap S_N)/N = (S_0/N) \cap (S_N/N) \subseteq Z(S_0/N) = Z(S_0)N/N = \{N\}$ by 9.26(ii) and (2) above, and thirdly, $(\forall s \in S_N)\ s(T\cap S)s^{-1} \subseteq (T \cap S)(S_0 \cap S_N)$. Hence $S = S_0S_N$ and $G_0 \cap S_N = S_0 \cap S_N = N \trianglelefteq G$. Further, by (5) above, $(\forall s \in S_N)\ sT_1s^{-1} \subseteq T_1$. Hence for all $s \in S_N$ we have $sTs^{-1} = sT_1s^{-1}s(T \cap S)s^{-1} \subseteq T_1(T \cap S)(S_0 \cap S_N) \subseteq T(G_0 \cap S_N)$. Thus S_N satisfies (i), (ii), and (iii) above and $S_N \subseteq S$. The minimality of S now implies

$S_N = S$ and thus $G_0 \cap M = N = G_0 \cap S_N = G_0 \cap S$, contradicting (5) above. This contradiction proves $G_0 \cap S = \{1\}$ and thereby completes the proof of the theorem. ☐

Note that the result for Lie groups was invoked in Section (a) and the very last paragraph of the proof.

Through 9.11 and 9.41 the supplementation of G_0 in G by a compact totally disconnected group takes place in $N(T, G)$ and therefore is largely a matter of compact groups with abelian identity component.

The Mayer-Vietoris formalism 9.4 now allows us the formulation of a sandwich theorem as follows.

Sandwich Theorem for Compact Groups

Corollary 9.42. *For any compact group G, there is a compact totally disconnected subgroup D of G providing a "sandwich situation"*

$$G_0 \rtimes_\iota D \xrightarrow{\mu} G \to G/(G_0 \cap D) \cong \frac{G_0}{G_0 \cap D} \rtimes \frac{D}{G_0 \cap D}$$

with $\iota\colon D \to \mathrm{Aut}(G_0)$ defined by $\iota(d)(g_0) = dg_0d^{-1}$, $\mu\colon G_0 \rtimes_\iota D \to G$ by $\mu(g_0, d) = g_0d$, and where the morphism $G \to G/(G_0 \cap D)$ is the quotient map. Both morphisms are surjective and have kernels isomorphic to the totally disconnected abelian group $G_0 \cap D$.

Proof. Exercise E9.11. ☐

Exercise E9.11. Prove Corollary 9.42.

[Hint. The proof follows from 9.41 in exactly the same fashion as Corollary 6.75 was proved from 6.74; what was the finite group E there is the totally disconnected compact group D here.] ☐

Again this corollary shows that a compact group is nearly a semidirect product of G_0 and G/G_0. This is not exactly true. Indeed, the example of the compact abelian group $G \stackrel{\text{def}}{=} \widehat{\nabla}$ of 8.11 shows that even in the abelian case a clean splitting of the identity component is not possible. Moreover, in Exercise E6.9(ii) we have seen a noncommutative Lie group in which a semidirect splitting of the identity component is impossible.

Corollary 9.43. *Let G be a compact group and T a maximal pro-torus. Then there is a compact totally disconnected subgroup D of $N(T, G)$ such that $G_0 \cap D \trianglelefteq G$, and a surjective morphism of compact groups*

$$\nu\colon (Z_0(G_0) \times \widetilde{(G_0)'}) \rtimes_\alpha D \to G, \quad \theta(z, g, d) = z\pi_G(g)d$$

for a suitable automorphic action of D on $(Z_0(G_0) \times \widetilde{(G_0)'})$. The kernel of ν is isomorphic to the totally disconnected subgroup $D \cap N(T, G_0)$.

Proof. By 9.4 there is a surjective morphism $\mu: G_0 \rtimes_\iota D \to G$ given by $\mu(n,d) = nd$, where $\iota(d)(n) = dnd^{-1}$. By Proposition 9.25, there is a natural $\mu_G: Z_0(G_0) \times \widetilde{(G_0)'} \to G_0$, $\mu_G(z,g) = z\pi_G(g)$. Now any endomorphism respects the fully characteristic subgroups $Z_0(G_0)$ and $(G_0)'$. This applies, in particular, to the restriction of the inner automorphisms $\iota(d)$, $d \in D$. By 9.19(iii), every $\iota(d)|(G_0)'$ lifts to a unique automorphism $\widetilde{\iota}(d): \widetilde{(G_0)'} \to \widetilde{(G_0)'}$. The function $(d,g) \mapsto \widetilde{\iota}(d)(g): D \times \widetilde{(G_0)'} \to \widetilde{(G_0)'}$ is continuous as is easily verified. Then we define

$$\alpha: D \to \mathrm{Aut}(Z_0(G_0) \times \widetilde{(G_0)'}) \text{ by } \alpha(d)(z,g) = \big(dzd^{-1}, \widetilde{\iota}(d)(g)\big).$$

Then $(d,z,g) \mapsto \alpha(d)(z,g): D \times Z_0(G_0) \times \widetilde{(G_0)'} \to Z_0(G_0) \times \widetilde{(G_0)'}$ is continuous. Thus the semidirect product in the corollary is defined, and θ is a morphism. □

Part 2: The Structure Theorems for the Exponential Function

The Exponential Function of Compact Groups

We now return to the theme of the exponential function and determine the Lie algebra of a compact group and its exponential function. The power of this tool will become evident.

Definition 9.44. A *weakly complete Lie algebra* $\mathfrak{g}$ is a weakly complete topological vector space (see 7.27) together with a continuous Lie bracket $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. (See 5.12.) A *morphism of weakly complete Lie algebras* is a continuous linear map $f: \mathfrak{g} \to \mathfrak{h}$ between weakly complete Lie algebras which preserves Lie brackets. □

The category of weakly complete Lie algebras defined in 9.44 is closed under the formation of arbitrary products and pull-backs, and is therefore closed under all limits. (See Appendix 3, A3.48, A3.49.)

Recall that in 5.7 we defined, for any topological group G, the topological space $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ with the compact open topology. By Proposition 7.38(iii), $\mathfrak{L}(\cdot)$ preserves projective limits. In particular, let $\{f_{jk}: G_k \to G_j \mid j \le k,\ j,k \in J\}$ be a projective system of compact Lie groups with limit G. Then $\{\mathfrak{L}(f_{jk}): \mathfrak{L}(G_k) \to \mathfrak{L}(G_j) \mid j \le k,\ j,k \in J\}$ is a projective system of finite dimensional compact Lie algebras and Lie algebra morphisms and $\mathfrak{L}(G)$ may be regarded as its limit; indeed if $(X_j)_{j\in J} \in \lim_{j\in J} \mathfrak{L}(G_j)$ with $f_{jk} \circ X_k = \mathfrak{L}(f_{jk})(X_k) = X_j$ for $j \le k$, then $\big(X_j(t)\big)_{j\in J} \in G$ and $t \mapsto \big(X_j(t)\big)_{j\in J}$ is in $\mathfrak{L}(G)$. All elements of $\mathfrak{L}(G)$ are so obtained. Since all $\mathfrak{L}(G_{jk})$ are finite dimensional Lie algebras, $\mathfrak{L}(G)$ is firstly a weakly complete vector space and secondly a Lie algebra with a continuous Lie bracket $[\cdot,\cdot]: \mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ and thirdly a G-module from an adjoint action via $\mathrm{Ad}: G \to \mathrm{Aut}\,\mathfrak{L}(G)$, $g(\exp X)g^{-1} = \exp \mathrm{Ad}(g)X$. We claim that indeed $\alpha_G: G \times \mathfrak{L}(G) \to L(G)$, $\alpha_G(g,X) = \mathrm{Ad}(g)(X)$ is continuous. Indeed the diagram

$$\begin{array}{ccc} G \times \mathfrak{L}(G) & \xrightarrow{\alpha_G} & G \\ {\scriptstyle q_N \times \mathfrak{L}(q_N)}\downarrow & & \downarrow{\scriptstyle q_N} \\ G/N \times \mathfrak{L}(G/N) & \xrightarrow[\alpha_{G/N}]{} & G/N \end{array}$$

is commutative for all $N \in \mathcal{N}(G)$, the vertical maps are limit maps; the horizontal maps are to be considered inside that category of pointed spaces and base point preserving continuous maps, and the maps $\alpha_{G/N}$ are the adjoint action map of compact Lie groups which are continuous. Thus α_G is a unique fill-in map for the limit $G = \lim_{N\in\mathcal{N}(G)} G/N$ in the category of pointed Hausdorff spaces according to Appendix 3, A3.43. In particular we see, recalling also the Definition 5.39 of the exponential function of any topological group,

Proposition 9.45. *If G is a compact group, then $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ is a weakly complete Lie algebra and a weakly complete G-module in a unique fashion such that each quotient map $q_N: G \to G/N$ for $N \in \mathcal{N}(G)$ induces a morphism of weakly complete Lie algebras $\mathfrak{L}(q_N): \mathfrak{L}(G) \to \mathfrak{L}(G/N)$ and G-modules with respect to adjoint actions.*

The function $\mathfrak{L}(G) \to G$ which assigns to a one-parameter subgroup $X: \mathbb{R} \to G$ the element $X(1) \in G$ is the exponential function $\exp_G: \mathfrak{L}(G) \to G$ *of the compact group G.* □

As a consequence of the duality Theorem 7.30 for real vector spaces, by the Theorem of Alaoglu, Banach, and Bourbaki ([4, 31]), a subset of $\mathfrak{L}(G)$ is weakly compact if and only if it is closed and bounded, and it is precompact if it is bounded. The topology of $\mathfrak{L}(G)$ is the topology of uniform convergence on compact sets of its dual according to 7.30. Compact subsets of the dual are contained in finite dimensional subspaces by 7.25(iv). Thus the topology of $\mathfrak{L}(G)$ agrees with the topology of pointwise convergence on the dual, i.e. with the weak topology. Condition 3.29 for feeble completeness is therefore satisfied by $\mathfrak{L}(G)$. The G-module theory of Chapter 4 therefore becomes available to us. In particular, by 3.52, the submodule $\mathfrak{L}(G)_{\mathrm{fin}}$ of almost invariant elements is algebraically the direct sum of summands $R_\varepsilon(G, \mathbb{R}) * \mathfrak{L}(G)$, $\varepsilon \in \widehat{G}$. In particular, if we set $\mathfrak{L}(G)_{\mathrm{fix}} = \{X \in \mathfrak{L}(G) \mid (\forall g \in G)\ \mathrm{Ad}(g)X = X\}$, then $\mathfrak{L}(G)_{\mathrm{fix}}$ is the trivial isotypic component obtained by applying the averaging operator

$$P_G: \mathfrak{L}(G) \to \mathfrak{L}(G), \quad P_G X = \int_G \mathrm{Ad}(g)(X)\, dg,$$

see 3.22f., notably Theorem 3.36.

We shall presently give a fairly explicit description of $\mathfrak{L}(G)$ and the exponential function $\exp_G: \mathfrak{L}(G) \to G$.

Proposition 9.46. *For a compact group G, the following conditions are equivalent.*

(i) *G is totally disconnected.*

(ii) *Each maximal pro-torus of G is singleton.*

(iii) *G has no nondegenerate one-parameter subgroup.*
(iv) $\mathfrak{L}(G) = \{0\}$.

Proof. By the definition of $\mathfrak{L}(G)$ as $\mathrm{Hom}(\mathbb{R}, G)$, conditions (iii) and (iv) are clearly equivalent. Every nondegenerate one-parameter subgroup is contained in a maximal pro-torus, and every nonsingleton compact connected abelian group T has non-degenerate one-parameter subgroups by 7.71. Thus (ii) and (iii) are equivalent conditions. Clearly (i) implies (ii). By the Maximal Pro-Torus Theorem 9.32(ii), if all maximal pro-tori are singleton, then G_0 is singleton. Hence (ii) implies (i).□

We are now in a position to use our knowledge of the exponential function to prove that if $f: G \to H$ is a surjective morphism between compact groups then $f\big(Z_0(G)\big) = Z_0(H)$.

Proposition 9.47. (i) *Let $f: G \to H$ be a surjective morphism of compact groups. Then f induces a morphism of weakly complete Lie algebras $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$, $\mathfrak{L}(f)(X) = f \circ X$. Let $j: N \to G$ denote the inclusion morphism of $N \stackrel{\mathrm{def}}{=} \ker f$ into G. Then we have a commuting diagram of exact sequences*

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{L}(K) & \xrightarrow{\mathfrak{L}(j)} & \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) & \to & 0 \\
 & & \Big\downarrow{\scriptstyle \exp_K} & & \Big\downarrow{\scriptstyle \exp_G} & & \Big\downarrow{\scriptstyle \exp_H} & & \\
0 & \to & K & \xrightarrow[j]{} & G & \xrightarrow[f]{} & H & \to & 0.
\end{array}$$

As a sequence of morphisms of weakly complete vector spaces, the top row splits, i.e. there is a morphism of weakly complete vector spaces $\sigma: \mathfrak{L}(H) \to \mathfrak{L}(G)$ such that $\mathfrak{L}(f) \circ \sigma = \mathrm{id}_{\mathfrak{L}(H)}$. In particular, $\mathfrak{L}(f)$ is a quotient morphism and therefore surjective and open.

(ii) $\mathfrak{L}(f)\big(\mathfrak{L}(G))_{\mathrm{fix}}\big) = \mathfrak{L}(H)_{\mathrm{fix}}$.

(iii) (Third Theorem on Morphisms of Compact Groups) $f\big(Z_0(G)\big) = Z_0(H)$.

Proof. (i) We know that $\mathfrak{L}(\cdot) = \mathrm{Hom}(\mathbb{R}, \cdot)$ is a limit preserving functor from the category of topological groups and continuous group morphisms into the category of pointed spaces (see 5.7 and 7.36ff. for the abelian case; see also Appendix 3, A3.23,) and from what we have recorded in Proposition 9.47, the prescription $\mathfrak{L}(\cdot)$ induces a functor from the category of compact groups and morphisms of compact groups to the category of weakly complete Lie algebras and continuous Lie algebra morphisms. The relation $\mathfrak{L}(\ker f) = \ker \mathfrak{L}(f)$ is readily verified (cf. 5.50 for linear Lie groups and 7.38(iii) for topological groups).

We claim that the surjectivity of f implies the surjectivity of $\mathfrak{L}(f)$. Indeed, let $Y: \mathbb{R} \to H$ be a one parameter subgroup of H. Then the subgroup $Y(\mathbb{R})$ of H is connected and abelian and is therefore contained in a maximal pro-torus T of H, abelian subgroup of H. Hence by 9.31, there is a maximal pro-torus S of G such that $f(S) = T$. Now $f|S: S \to T$ is a surjective morphism of compact abelian groups. Then $\mathfrak{L}(f|S): \mathfrak{L}(S) \to \mathfrak{L}(T)$ is surjective by 7.66(iii). Hence there is an $X \in \mathfrak{L}(S) \subseteq \mathfrak{L}(G)$ such that $\mathfrak{L}(f)(X) = \mathfrak{L}(f|S)(X) = Y$. Thus $\mathfrak{L}(f)$ is surjective, and the top sequence in the diagram is exact. The splitting now follows from 7.30(iv).

(ii) The morphism of topological vector spaces $\mathfrak{L}(f)$ is equivariant; that is $\mathfrak{L}(f)\big(\operatorname{Ad}(g)X\big) = \operatorname{Ad}\big(f(g)\big)\big(\mathfrak{L}(X)\big)$. Therefore it maps $\mathfrak{L}(G)_{\rm fix}$ into $\mathfrak{L}(H)_{\rm fix}$. We may therefore define a restriction and corestriction $\mathfrak{L}(f)_{\rm fix}$ to these trivial modules, respectively, and obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\ {\scriptstyle P_G}\Big\downarrow & & \Big\downarrow{\scriptstyle P_H} \\ \mathfrak{L}(G)_{\rm fix} & \xrightarrow[\mathfrak{L}(f)_{\rm fix}]{} & \mathfrak{L}(H)_{\rm fix}. \end{array}$$

Now $\mathfrak{L}(f)$ is surjective by (i). The averaging operator P_H is a linear retraction and is therefore surjective. Hence $\mathfrak{L}(f)_{\rm fix} \circ P_G = P_H \circ \mathfrak{L}(G)$ is surjective. This implies that $\mathfrak{L}(f)_{\rm fix}$ is surjective. This completes the proof of (ii).

(iii) The subgroup $Z_0(G)$ is the identity component of the center of G, i.e. the fixed point set of all inner automorphisms $I_g\colon G \to G$, $I_g(x) = gxg^{-1}$. So $\mathfrak{L}\big(Z_0(G)\big) = \{X \in \mathfrak{L}(G) \mid (\forall t \in \mathbb{R}) \quad I_g(\exp t{\cdot}X) = \exp t{\cdot}X\}$. But

$$\exp t{\cdot} \operatorname{Ad}(g)(X) = I_g(\exp t{\cdot}X),$$

whence

$$\mathfrak{L}\big(Z_0(G)\big) = \mathfrak{L}(G)_{\rm fix}.$$

From (ii) we get

$$(\dagger) \qquad \exp_H \mathfrak{L}(H)_{\rm fix} = \exp_H \mathfrak{L}(f)\big(\mathfrak{L}(G)_{\rm fix}\big) = f(\exp_G \mathfrak{L}(G)_{\rm fix}.$$

The groups $Z(G)$ and $Z(H)$ and thus the subgroups $Z_0(G)$ and $Z_0(H)$ are abelian. Hence from 7.71 we deduce $Z_0(G) = \overline{\exp_G \mathfrak{L}(G)_{\rm fix}}$ and $Z_0(H) = \overline{\exp_H \mathfrak{L}(H)_{\rm fix}}$. But if S is any subset of G then $\overline{S}$ is compact and thus $f(\overline{S})$ is compact, containing $f(S)$ densely. Thus $f(\overline{S}) = \overline{f(S)}$. Applying this with $S = \exp_G \mathfrak{L}(G)_{\rm fix}$, from (†) we obtain assertion (iii). ☐

Notice that, among other things, 9.47(iii) is a considerable sharpening of the same assertion proved in 9.26(ii) when G was *connected.*

Corollary 9.48. *Let $f\colon G \to H$ be a surjective morphism of compact groups, and $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$, $\mathfrak{L}(f)(X) = f \circ X$ the morphism induced between the Lie algebras. Then the following conditions are equivalent.*

(i) $\ker f$ *is totally disconnected.*

(ii) $\mathfrak{L}(f)$ *is an isomorphism of weakly complete topological Lie algebras.*

(iii) $\mathfrak{L}(f)$ *is injective, i.e.* $\ker \mathfrak{L}(f) = \{0\}$.

Proof. (i)⇒(ii) By 9.47, $\ker \mathfrak{L}(f) = \mathfrak{L}(\ker f) = \{0\}$ since $\ker f$ is totally disconnected. Hence $\mathfrak{L}(f)$ is injective. Also by 9.47, $\mathfrak{L}(f)$ is surjective and open, and thus $\mathfrak{L}(f)$ is an isomorphism of topological vector spaces.

Trivially, (ii) implies (iii), and by Proposition 9.46, (iii) implies (i). ☐

The Exponential Function of a Compact Connected Group

Theorem 9.49. *Let G be a compact connected group and $\mu_G\colon G^* \stackrel{\rm def}{=} Z_0(G) \times \prod_{j\in J} S_j \to G$ the surjective morphism of 9.25. Set $\mathfrak{z} = \mathfrak{L}\big(Z_0(G)\big)$ and $\mathfrak{s}_j = \mathfrak{L}(S_j)$, $j \in J$. Then*

(i) *the morphism $\mathfrak{L}(\mu_G)\colon \mathfrak{g} \stackrel{\rm def}{=} \mathfrak{z} \times \prod_{j\in J} \mathfrak{s}_j \to \mathfrak{L}(G)$ is an isomorphism of weakly complete Lie algebras and the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{z} \times \prod_{j\in J} \mathfrak{s}_j = \mathfrak{g} & \xrightarrow{\mathfrak{L}(\mu_G)} & \mathfrak{L}(G) \\ {\scriptstyle \exp_{Z_0(G)} \times \prod_{j\in J} \exp_{S_j}} \big\downarrow & & \big\downarrow {\scriptstyle \exp_G} \\ Z_0(G) \times \prod_{j\in J} S_j & \xrightarrow[\mu_G]{} & G. \end{array}$$

(ii) *For each $j \in J$, let $\mathfrak{t}_j$ be a Cartan subalgebra of $\mathfrak{s}_j$, and $\mathfrak{t}^* \stackrel{\rm def}{=} \mathfrak{z} \times \prod_{j\in J} \mathfrak{t}_j$. Let $T_j = \exp_{S_j} \mathfrak{t}_j$ be a maximal torus of S_j and set $T^* \stackrel{\rm def}{=} Z_0(G) \times \prod_{j\in J} T_j$ and $T = \mu_G(T^*)$. Then T is a maximal pro-torus of G and $\mathfrak{L}(\mu_G)|\mathfrak{t}^*\colon \mathfrak{t}^* \to \mathfrak{t}$ is an isomorphism of weakly complete Lie algebras. Further $G' \subseteq \exp_{\mathfrak{L}}(G)$, and $\exp_G$ is surjective if and only if $\exp_G |T\colon \mathfrak{t} \to T$ is surjective.*

(iii) *G' is always contained in the image of the exponential function.*

Proof. (i) By 9.48 the morphism $\mathfrak{L}(\mu_G)$ is an isomorphism of weakly complete Lie algebras. It suffices, therefore, to determine the Lie algebra and the exponential function of G^*. This is accomplished by the fact that the functor $\mathfrak{L}$ preserves products by 7.38(i).

(ii) Each $\exp_j\colon \mathfrak{s}_j \to S_j$ is surjective by the Maximal Torus Theorem 6.30 and its Corollary 6.31. From Theorem 9.36 we know that T^* and T are maximal pro-tori of G^* and G, respectively. Thus the diagram above shows that t^* and $\mathfrak{t}$ may be identified as the common Lie algebra of T^* and T, and that

$$G' = \mu_G(\{1\} \times \prod_{j\in J} S_j) \subseteq \operatorname{im}(\exp_G).$$

If $\exp_G(\mathfrak{t}) = T$, then $\exp \operatorname{Ad}(g)(\mathfrak{t}) = gTg^{-1}$ for all $g \in G$ and the Maximal Pro-Torus Theorem 9.32(ii) now shows that $G \subseteq \operatorname{im}(\exp_G)$. Conversely, assume that $\exp_G$ is surjective. Then $\exp_{G/G'}\colon \mathfrak{g}/\mathfrak{g}' \to G/G'$ is surjective. By the Borel–Scheerer–Hofmann Splitting Theorem 9.39 there is a closed connected abelian subgroup A of T such that $(t,a) \mapsto T_1 \times A \to T$, $T_1 = \mu_G(\{1\} \times \prod_{j\in J} T_j)$ is an isomorphism and that $a \mapsto aG'\colon A \to G/G'$ is an isomorphism. Thus $\exp_A\colon \mathfrak{a} \to A$ is surjective where $\exp_A = \exp_G |\mathfrak{a}$. Since $\exp_{T_1}\colon \mathfrak{t}_1 \to T_1$ is surjective regardless, $\exp_T\colon \mathfrak{t}_1 \oplus \mathfrak{a} \to T$, $\exp_T(X_{\mathfrak{t}_1} \oplus X_{\mathfrak{a}}) = (\exp_G X_{\mathfrak{t}_1})(\exp_G X_{\mathfrak{a}}) \in T_1A$ is surjective.

(iii) follows from (i) and the fact that $G' = \mu_G(\{1\} \times \prod_{j\in J} S_j)$. ☐

We draw attention to the fact that the surjectivity of the exponential function of a compact group turns out, in the final evaluation, to be an issue on compact connected abelian groups about which we have accumulated a considerable body of information in Chapter 8.

In the following corollary we retain the notation of Theorem 9.49.

The Classification of Connected Normal Subgroups

Corollary 9.50. *For a closed connected subgroup H of G, the set $\mathfrak{L}(H)$ is a closed subalgebra of $\mathfrak{L}(G)$ and the following conditions are equivalent:*

(i) *H is normal.*

(ii) *$\mathfrak{L}(H)$ is an ideal of $\mathfrak{L}(G)$.*

(iii) *There is a closed vector subspace $\mathfrak{c}$ of $\mathfrak{z}$ and a subset $I \subseteq J$ such that $\mathfrak{L}(H) = \mathfrak{L}(\mu_G)\big(\mathfrak{c} \times \prod_{j\in J} \mathfrak{s}_j(I)\big)$ where $\mathfrak{s}_j(I) = \begin{cases} \mathfrak{s}_j & \text{if } j \in I, \\ \{0\} & \text{otherwise.} \end{cases}$*

(iv) *There is a closed connected subgroup $C \subseteq Z_0(G)$ and a subset $I \subseteq J$ such that $H = \mu_G(C \times \prod_{j\in J} S_j(I)\big) = CS$ where $S = \pi_{G'}\big(\prod_{j\in J} S_j(I)\big)$ with $S_j(I) = \begin{cases} S_j & \text{if } j \in I, \\ \{1\} & \text{otherwise.} \end{cases}$*

Proof. Since $\mathfrak{L}(H) = \{X \in \mathfrak{L}(G) \mid X(\mathbb{R}) \subseteq H\}$ and H is closed, $\mathfrak{L}(H)$ is closed in the topology of pointwise convergence and then certainly in the finer topology of uniform convergence on compact sets.

(i)⇒(ii) If the subgroup H is normal then it is a kernel of the quotient map $p: G \to G/N$ and then $\mathfrak{L}(H) = \ker \mathfrak{L}(p)$ by 9.46 above and thus $\mathfrak{L}(H)$ is an ideal.

(ii)⇒(iii) Assume that $\mathfrak{L}(H)$ is an ideal of $\mathfrak{L}(G)$. Now let $\mathfrak{i}$ be an ideal of $\mathfrak{g} = \mathfrak{z} \times \prod_{j\in J} \mathfrak{s}_j$. Let $p_\mathfrak{z}: \mathfrak{g} \to \mathfrak{z}(\mathfrak{g}) = \mathfrak{z} \times \{0\}$ and $p_{\mathfrak{g}'}: \mathfrak{g} \to \mathfrak{g}' = \{0\} \times \prod_{j\in J} \mathfrak{s}_j$ denote the projections. Then $\mathfrak{j} \overset{\text{def}}{=} p_\mathfrak{j}(i) \oplus p_{\mathfrak{g}'}$ is an ideal of $\mathfrak{g}$. We claim $\mathfrak{i} = \mathfrak{j}$. For a proof it is no loss of generality to assume $\mathfrak{j} = \mathfrak{g}$ and to assume that $p_\mathfrak{z}$ and $p_{\mathfrak{g}'}$ are surjective. Then $p_{\mathfrak{g}'}([\mathfrak{i}, \mathfrak{i}]) = [p_{\mathfrak{g}'}(\mathfrak{i}), p_{\mathfrak{g}'}(\mathfrak{i})] = [\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'$. But also $\mathfrak{i}' = [\mathfrak{i}, \mathfrak{i}] \subseteq \mathfrak{g}'$. But $p_{\mathfrak{g}'}|\mathfrak{g}': \mathfrak{g}' \to \mathfrak{g}'$ is the identity of $\mathfrak{g}'$. We conclude that $\mathfrak{g}' = \mathfrak{i}' \subseteq \mathfrak{i}$. Then $\mathfrak{i} = \mathfrak{i} + \mathfrak{g}' = p_\mathfrak{z}^{-1} p_\mathfrak{z}(\mathfrak{i}) = p_\mathfrak{z}^{-1}(\mathfrak{z} \times \{0\}) = \mathfrak{g}$. This proves the claim. Thus any ideal $\mathfrak{i}$ of $\mathfrak{g}$ is of the form $\mathfrak{c} \times \mathfrak{s}$ with an ideal $\mathfrak{s} \trianglelefteq \prod_{j\in J} \mathfrak{s}_j$.

If $\mathrm{pr}_j: \prod_{i\in J} \mathfrak{s}_i \to \mathfrak{s}_j$ denotes the projection, $\mathrm{pr}_j(\mathfrak{s})$ is an ideal of the simple algebra $\mathfrak{s}_j$ and thus is either $\{0\}$ or $\mathfrak{s}_j$. Now set $I = \{j \in J \mid \mathrm{pr}_j(\mathfrak{s}) = \mathfrak{s}_j\}$. Let

$$\mathfrak{t} = \prod_{j\in J} \mathfrak{s}_j(I), \quad \mathfrak{s}_j(I) = \begin{cases} \mathfrak{s}_j & \text{if } j \in I, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{t}$ is an ideal containing $\mathfrak{s}$ such that $\mathrm{pr}_j(\mathfrak{s}) = \mathrm{pr}_j(\mathfrak{t})$ for all $j \in J$. Let $\mathcal{F}$ denote the directed set of all finite subsets $F \subseteq J$ and set

$$\mathfrak{t}_F = \{(s_j)_{j\in J} \in \mathfrak{t} \mid (\forall k \in J \setminus F)\, s_k = 1\}.$$

Then $(\mathfrak{s} + \mathfrak{t}_F)/\mathfrak{t}_F$ is an ideal of $\mathfrak{t}/\mathfrak{t}_F$ and the projections into the simple factors of these finite dimensional Hilbert Lie algebras agree. By 6.4(vii) the two agree. Hence $\mathfrak{t} = \mathfrak{s} + \mathfrak{t}_F$ for all $F \in \mathcal{F}$. An element $t = (t_j)_{j\in J} \in \mathfrak{t}$ is in $\bigcap_{F\in\mathcal{F}}(\mathfrak{s} + \mathfrak{t}_F)$ iff for each $F \in \mathcal{F}$ there is an element $s_F = \big(s_j(F)\big)_{j\in J} \in \mathfrak{s}$ such that $t_j = s_j(F)$ for all $j \in F$. Thus $t = \lim_{F\in\mathcal{F}} s_F \in \mathfrak{s}$ since $\mathfrak{s}$ is closed. Thus $\mathfrak{t} = \mathfrak{s}$. Hence $\mathfrak{i} = \mathfrak{c} \times \mathfrak{t}$ as asserted.

(iii)⇒(iv) Set $C = \overline{\exp_{Z_0(G)} \mathfrak{c}} \subseteq Z_0(G)$ and

$$S^* = \prod_{j \in J} S_j(I), \quad S_j(I) = \begin{cases} S_j & \text{if } j \in I, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $C \times S^*$ is compact connected normal in $Z_0(G) \times \prod_{j \in J} S_j$. Set $S = \mu_G(\{1\} \times S^*) = \pi_{G'}(S^*)$. Then $\mu_G(C \times S^*) = CS$ is a compact connected subgroup of G in which

$$\begin{aligned} \exp_G \mathfrak{L}(H) &= (\mathfrak{L}(\mu_G) \circ \exp_G)\big(\mathfrak{c} \times \prod_{j \in J} \mathfrak{s}_j(I)\big) \\ &= \big(\mu_G \circ (\exp_{Z_0(G)} \times \prod_{j \in J} \exp_{S_j})\big)\big(\mathfrak{c} \times \prod_{j \in J} \mathfrak{s}_j(I)\big) \end{aligned}$$

is dense. Since $H = \overline{\exp_H \mathfrak{L}(H)}$ by 7.71, we have $H = CS$.

(iv) trivially implies (i). Therefore the proof is complete. □

Next we generalize the Resolution Theorem 8.20 for compact abelian groups to the nonabelian case. We recall from Theorem 8.20 and the discussion which preceded it, notably from 8.15ff., that every compact abelian group A contains for a given totally disconnected subgroup D_1 at least one totally disconnected subgroup D containing D_1 such that A/D is a torus.

The Resolution Theorem for Compact Connected Groups

Theorem 9.51. *Assume that G is a compact connected group and that Δ is any compact totally disconnected subgroup of $Z_0(G)$ containing $Z_0(G) \cap G'$ such that $Z_0(G)/\Delta$ is a torus. Write $\mathfrak{L}(G) = \mathfrak{z} \times \mathfrak{g}'$, $\mathfrak{g}' = \prod_{j \in J} \mathfrak{s}_j$ as in 9.48 so that $X \in \mathfrak{L}(G)$ is of the form $X = (X_\mathfrak{z}, X')$, $X_\mathfrak{z} \in \mathfrak{z}$, $X' = (X_j)_{j \in J}$, $X_j \in \mathfrak{s}_j$. Let $\pi_G \colon \widetilde{G'} \to G'$ be as in 9.19, $\widetilde{G'} = \prod_{j \in J} S_j$ with simple simply connected compact Lie groups S_j. Then the following conclusions hold.*

(i) *The function*

$$\varphi_1 \colon \Delta \times \mathfrak{z} \times \prod_{j \in J} S_j \to G, \quad \varphi_1(d, X_\mathfrak{z}, g') = d(\exp X_\mathfrak{z})\mu_G(g')$$

is a quotient morphism whose kernel $\{(d, X_\mathfrak{z}, z) \mid d \exp X_\mathfrak{z} = \pi_G(z)^{-1},\ \pi_G(z) \in Z_0(G) \cap G'\}$ is totally disconnected.

(ii) *The function*

$$\varphi_2 \colon \Delta \times \mathfrak{z} \times \prod_{j \in J} \mathfrak{s}_j \to G, \quad \varphi_2(d, X) = d \exp X$$

is surjective and open at 0.

Proof. (i) The function $\varphi \colon \Delta \times \mathfrak{z} \to Z_0(G)$ given by $\varphi(d, X_\mathfrak{z}) = d \exp X_\mathfrak{z}$ is a quotient morphism whose properties were described in detail in Theorem 8.20. The map $\varphi_0 \colon \Delta \times \mathfrak{z} \times \widetilde{G'} \to Z_0(G) \times \widetilde{G'}$, $\varphi_0(d, X_\mathfrak{z}, g') = d \exp X_\mathfrak{z}, g')$ is therefore a quotient morphism whose kernel $K\times\{1\}$, $K=\ker\varphi$ is isomorphic to $\{X_{\mathfrak z}\in\mathfrak z \mid \exp X_{\mathfrak z}\in\Delta\}$, and this subset is mapped bijectively and continuously onto $D\supseteq Z_0(G)\cap G'$. The morphism $\mu_G\colon Z_0(G)\times\widetilde{G'}\to G$ is a quotient morphism with kernel $\{(\pi_G(z)^{-1},z)\mid z\in\pi_G^{-1}(Z_0(G)\cap G')\}$. Thus $\varphi_1=\mu_G\circ\varphi_0$ is a quotient morphism whose kernel $\varphi_0^{-1}(\ker\pi_G)=\{(d,X_{\mathfrak z},z)\mid d\exp X_{\mathfrak z}=\pi_G(z)^{-1}\mid \pi_G(z)\in Z_0(G)\cap G'\}$ is totally disconnected since $\pi_G^{-1}(Z_0(G)\cap G')\subseteq\prod_{j\in J}Z(S_j)$ is totally disconnected and $\ker\varphi$ is totally disconnected.

(ii) The function $\exp_j\colon\mathfrak s_j\to S_j$ is surjective by 6.31 and a local homeomorphism at 1. The filter of identity neighborhoods of $\prod_{j\in J}S_j$ has a basis of identity neighborhoods of the form $\prod_{j\in J}V_j$ where all V_j are identity neighborhoods of S_j which for all but a finite number of indices j agree with S_j. The zero neighborhoods of the weakly complete Lie algebra $\prod_{j\in J}\mathfrak s_j$ are of the form $\prod_{j\in J}U_j$ where all U_j are zero neighborhoods of $\mathfrak s_j$ which for all but a finite number of indices j agree with $\mathfrak s_j$ and otherwise are mapped homeomorphically onto an identity neighborhood of S_j by $\exp_{S_j}$. It then follows that $\prod_{j\in J}\exp_{S_j}\colon\prod_{j\in J}\mathfrak s_j\to\prod_{j\in J}S_j$ is surjective and open at zero. Since $\varphi_2=\varphi_1\circ(\{1\}\times\{0\}\times\prod_{j\in J}\exp_{S_j})$ the assertion follows. □

The Dimension of Compact Groups

We expand the dimension theory for compact abelian groups of Chapter 8, 8.28ff. to the general situation. We shall show in the next chapter (in 10.38) that *a compact group G is always homeomorphic to $G_0\times G/G_0$.* As far as dimension is concerned, we may therefore concentrate on G_0 which we shall do in the following. We recall from 9.2 and 9.19 and that for a compact connected group G the commutator group is a quotient of a product $\prod_{j\in J}S_i$ of a unique family of simple simply connected compact Lie groups S_j. The cardinal card J, in particular, is an isomorphy invariant by 9.19(v) (see also E9.5). We shall write

$$\aleph(G)=\operatorname{card}J=\sum_{\mathfrak s\in\mathcal S}\aleph(\mathfrak s,G).$$

The Structure of Finite Dimensional Compact Groups

Theorem 9.52. *For a compact connected group G the following conditions are equivalent.*

(i) $\operatorname{rank}(G/G')\widehat{}<\infty$ *and* $\aleph(G)<\infty$.

(ii) $\operatorname{rank}Z_0(G)\widehat{}<\infty$, *and* $\aleph(G)<\infty$.

(iii) $Z_0(G)$ *is finite dimensional and* G' *is a compact Lie group.*

(iv) G' *is a compact Lie group and* G/G' *is a finite dimensional compact abelian group.*

(v) *There are simple normal compact Lie subgroups* $S_1,\dots,S_k$ *and a totally disconnected subgroup* Δ *satisfying* $Z_0(G)\cap G'\subseteq\Delta\subseteq Z_0(G)$ *and* $Z_0(G)/\Delta\cong$

$\mathbb{T}^m$ such that $\mathfrak{z} = \mathfrak{L}\big(Z_0(G)\big) \cong \mathbb{R}^m$ and that there is a covering homomorphism

$$\varphi\colon \Delta \times \mathfrak{z} \times S_1 \times \cdots \times S_k \to G, \quad \varphi(d, X_\mathfrak{z}, s_1, \ldots, s_k) = d(\exp X_\mathfrak{z})s_1 \cdots s_k.$$

(vi) $\dim \mathfrak{L}(G) < \infty$. *In particular, if these conditions are satisfied, then G is locally isomorphic to $\Delta \times \mathbb{R}^m \times S_1 \times \cdots \times S_k$.*

A nonsingleton finite dimensional compact connected group has weight $\aleph_0$ and cardinality $2^{\aleph_0}$. In particular, it is metrizable.

Proof. Since $G = Z_0G'$ by 9.24(i), we have $G/G' \cong Z_0/(Z_0 \cap G')$. From 8.23 we recall $\dim Z_0(G) = \operatorname{rank}(Z_0(G)\widehat{\ }$ and $\dim G/G' = \operatorname{rank}(G/G')\widehat{\ }$. The compact group $Z_0 \cap G'$ is totally disconnected; hence by 8.24(iv) we have $\dim G/G' = \dim Z_0/(Z_0 \cap G') = \dim Z_0$. Therefore (i) is equivalent to (ii). The following diagrams illustrate the situation.

$$\begin{array}{cccc} & Z_0 & & \{0\} \\ G/G' \cong \Big\{ & \Big| & & \Big| \\ & Z_0 \cap G' & & (Z_0 \cap G')^\perp \cong (G/G')\widehat{\ } \\ & \Big| & & \Big| \\ & \{0\} & & \widehat{Z_0}. \end{array}$$

By 9.24, 9.19 we know that $\aleph(G)$ is finite if and only if G' is a compact Lie group. This shows that (i) through (iv) are equivalent.

Assume (i). Then by 9.19 we have $\widetilde{G'} = \widetilde{S}_1 \times \cdots \times S_k$ with $k = \aleph(G)$ and with universal covering groups $\widetilde{S}_j$ of normal simple subgroups S_j, $j = 1, \ldots, k$. Hence $\widetilde{G'}$ is a compact Lie group and therefore the locally isomorphic quotient G' is a compact Lie group. The group $Z_0(G)$ is a finite dimensional compact connected abelian group by 8.22. The morphism φ is a quotient morphism by 9.51, the kernel $\{(d, X_\mathfrak{z}, z) \mid d \exp X_\mathfrak{z} = z^{-1} \in Z_0 \cap G'\}$ is discrete because $\{(d, X_\mathfrak{z}) \mid d \exp X_\mathfrak{z} = 1\}$ is discrete and isomorphic to a discrete subgroup of $\mathfrak{z} \cong \mathbb{R}^n$ and because $Z_0 \cap G'$ is discrete. This proves (v).

Assume (v). Since $\ker \varphi$ is discrete and φ is surjective,

$$\mathfrak{L}(\varphi)\colon \mathfrak{L}(D \times \mathfrak{z} \times S_1 \times \cdots \times S_n) \to \mathfrak{L}(G)$$

is an isomorphism of weakly complete vector spaces by Proposition 9.47. The projection $p\colon D \times \mathfrak{z} \times S_1 \times \cdots \times S_n \to \mathfrak{z} \times S_1 \times \cdots \times S_n$ has a totally disconnected kernel and is surjective. Thus $\mathfrak{L}(p)$ is an isomorphism by 9.47. Hence $\dim \mathfrak{L}(G) = \dim \mathfrak{L}(\mathfrak{z} \times S_1 \times \cdots \times S_n) = \dim \mathfrak{z} + \dim \mathfrak{L}(S_1) + \cdots + \dim \mathfrak{L}(S_1) < \infty$. Thus (vi) is proved.

Finally assume (vi). By Theorem 9.48, $\mathfrak{L}(G) \cong \mathfrak{L}\big(Z_0(G)\big) \times \prod_{j\in J} \mathfrak{L}(\widetilde{S}_j)$ for a family of simple simply connected compact Lie groups $\widetilde{S}_j$ such that $\widetilde{G'} \cong \prod_{j\in J} \mathfrak{L}(\widetilde{S}_j)$; the family is unique up to isomorphism by 9.19. Now $\dim \mathfrak{L}(G) < \infty$ by (iv). Hence $\dim \mathfrak{L}\big(Z_0(G)\big) < \infty$ and $\aleph(G) = \operatorname{card} J < \infty$. By 8.22, $\dim \mathfrak{L}\big(Z_0(G)\big) = \operatorname{rank} \big(Z_0(G)\big)\widehat{\ }$. Thus (i) is proved.

Assume that G satisfies these conditions. Then the compact connected group G it is a quotient group of some group $\Delta \times \mathbb{R}^m \times S_1 \times \cdots \times S_k$ under a morphism inducing a local isomorphism, where Δ is a subgroup of $Z_0(G)$ whose character group has finite rank, hence is a subgroup of $\mathbb{Q}^n$ for some natural number n, and where the groups S_j are simple connected compact Lie groups. The assertions on the weight, the cardinality and the metrizability of G follow at once. (Cf. A4.16 for the metrizability of arbitrary topological groups.) □

Let us notice that the proof of the equivalence of (i) and (ii) in the preceding theorem shows that $\dim G/G' = \dim Z_0(G)$.

Definition 9.53. (i) Let V be a weakly complete topological vector space. Its dual V' is a real vector space (endowed with the finest locally convex topology) by 7.30. The dimension $\dim_{\mathbb{R}} V'$ as a real vector space is a cardinal, and we shall define

$$\dim V = \dim_{\mathbb{R}} V'.$$

(ii) Assume the G is a compact group. Then we set

$$\dim G = \dim \mathfrak{L}(G) = \dim_{\mathbb{R}} \mathfrak{L}(G)'. \qquad \square$$

9.54. Scholium. A) *Assume that* DIM *is a function defined on the class* $\mathcal{C}$ *of all locally compact spaces with values in* $\{0, 1, \ldots ; \infty\}$ *such that the Conditions* (Da), ..., (Dd) *of Scholium* 8.25 *are satisfied and that the following condition is also satisfied which, for homogeneous spaces, is somewhat stronger than* (Db).

(Db*) *If X admits a finite open cover each member of which is homeomorphic to $\mathbb{R}^n$, then* $\operatorname{DIM} X = n$.

Then for any compact group G,

$$\operatorname{DIM}(G) = \begin{cases} \dim G & \textit{if } \dim G \textit{ is finite,} \\ \infty & \textit{otherwise.} \end{cases}$$

B) *On the class of spaces underlying locally compact groups, small inductive dimension, local large inductive dimension, Lebesgue covering dimension, cohomological dimension, and sheaf theoretical dimension* (*for any ground ring*) *all agree and when finite, take the value* $\dim G = \dim G_0 = \dim_{\mathbb{R}} \mathfrak{L}(G)$. *The dimension of a compact group is a topological invariant; i.e. two homeomorphic compact groups have the same dimension.*

Proof. A) By 9.38(ii) a maximal pro-torus of G is a direct product TA of a maximal torus T of $(G_0)'$ and an abelian group $A \cong G_0/(G_0)' \cong Z_0(G)/(Z_0(G) \cap (G_0)'$. Thus $\dim TA = \dim T + \dim A = \dim T + \dim Z_0(G_0)$ by 8.34(iv). Consider $\pi_{G_0} : \widetilde{(G_0)}' \to (G_0)'$ be as in 9.19. We may write $\widetilde{(G_0)}' = \prod_{j\in J} S_j$ with a family of simple simply connected compact Lie groups S_j, $\operatorname{card} J = \aleph(G)$. Let T_j be a maximal torus of S_j. Then $\widetilde{T} \overset{\text{def}}{=} \prod_{j\in J} T_j$ is a maximal pro-torus of $\widetilde{G'}$ and by 9.36(v) it is a torus. Also, $\pi_G(\widetilde{T})$ is a torus and maximal pro-torus of $(G_0)'$ by 9.36. Since all maximal pro-tori are conjugate by the Maximal Pro-Torus Theorem

9.31(i), we may assume that $\pi_G(\widetilde{T})$ agrees with T. The kernel K of π_{G_0} is central by 9.19 and the center of $\widetilde{G'}$ is contained in $\widetilde{T}$ by 9.31(iv). Since K is totally disconnected, 8.24(iv) shows $\dim T = \dim \widetilde{T} = \prod_{j\in J} \dim T_j \geq \operatorname{card} J = \aleph(G)$. Thus $\dim TA \geq \aleph(G) + \dim Z_0(G_0)$. Then TA contains a subset homeomorphic to $[-1,1]^{\aleph(G)+\operatorname{rank}\left(Z_0(G)\right)\widehat{}}$ by Proposition 8.21. Thus if $\aleph(G)$ or $\dim Z_0(G_0)$ is infinite, then (Db) shows that $\operatorname{DIM}(G) = \infty$. We now assume that the conditions of 9.52 are satisfied. By 9.52 there is a covering morphism

$$\varphi: G^* \to G_0, \quad G^* = \Delta \times \mathfrak{z} \times S_1 \times \cdots \times S_k, \quad \mathfrak{z} = \mathfrak{L}\big(Z_0(G)\big),$$

$\varphi(\delta, X, s_1, \ldots, s_k) = \delta(\exp X)s_1 \cdots s_k$. Hence (Da) implies $\operatorname{DIM}(G) = \operatorname{DIM}(G^*)$. Since G^* contains a subset homeomorphic to $[-1,1]^{\dim_{\mathbb{R}} \mathfrak{L}(G)}$ we have $\dim G = \dim_{\mathbb{R}} \mathfrak{L}(G) = \dim[-1,1]^{\dim_{\mathbb{R}} \mathfrak{L}(G)} \leq \operatorname{DIM}(G^*) = \operatorname{DIM}(G)$ by (Db) and (Dc).

Now the compact Lie group $S_1 \times \cdots \times S_k$ has an open identity neighborhood V homeomorphic to $]-1,1[^{\dim \mathfrak{L}((G_0)')}$. Let D be a compact totally disconnected subset of G such that $(d,g) \mapsto dg: D \times G_0 \to G$ is a homeomorphism (see 6.81). Then the function

$$\Phi: D \times (\Delta \times \mathfrak{z} \times S_1 \times \cdots \times S_k) \to G, \quad \Phi\big(d, (\delta, X, s_1, \ldots, s_n)\big) = d\delta(\exp X)s_1 \cdots s_k,$$

is a covering map. Then $D \times G^*$ is covered by finitely many translates of the identity neighborhood $U = D \times (\Delta \times \mathfrak{z} \times V)$. Note that $\mathfrak{L}(G) = \mathfrak{z} \oplus \mathfrak{L}((G_0)')$. By (Dd) one has $\operatorname{DIM}(D \times \Delta \times [-1,1]^{\dim \mathfrak{L}(G)}) \leq \dim \mathfrak{L}(G) = \dim G$. The new postulate (Db*) then implies $\operatorname{DIM}(G) = \operatorname{DIM}(G^*) \leq \dim G$. Thus $\dim G = \operatorname{DIM} G$ is proved.

B) All concepts of topological dimension which we discussed before we stated Corollary 8.26 and for which we gave references to the literature, will assign to a manifold of local euclidean dimension n the topological dimension n; they therefore satisfy (Db*). As a consequence of Part A), we can therefore formulate conclusion B) as an extension of Corollary 8.26. □

Theorem 9.55. (i) *For a compact connected group G the following statements are equivalent:*

(1) *G is finite dimensional*
(2) *The Lie algebra $\mathfrak{L}(G)$ of G is the Lie algebra of a compact Lie group H.*
(3) *G' is a Lie group and G/G' is a finite dimensional compact connected abelian group see* (Theorem 8.22).
(4) *G is the semidirect product $G' \rtimes_\iota A$, where the commutator group G' is a semisimple compact Lie group and A is a finite dimensional compact connected abelian subgroup, and where $\iota(a)(g) = aga^{-1}$.*
(5) *G' is a Lie group and $Z_0(G)$ is a finite dimensional compact connected abelian group.*

(ii) (Cf. [196], p. 302, 2.12) *The n-th Čech cohomology group of an n-dimensional compact group G with integral coefficients in dimension n is torsion free and has rank $w(G/G_0)$.*

(iii) *Let G be finite dimensional. Then the following statements are equivalent*

(1) *G is a Lie group.*
(2) *G/G_0 is finite and $G_0/(G_0)'$ is a Lie group.*
(3) *G/G_0 is finite and $H^n(|G_0|,\mathbb{Z})$ is cyclic for Čech cohomology in dimension $n = \dim G$ of the space underlying the identity component.*

(iv) *If the dimension of a compact connected group G does not exceed 2 then G is abelian and its character group is a subgroup of $\mathbb{Q}^2$.*

Proof. (i) (1) $\iff$ (2): By Theorem 9.24, in view of 9.52 und 9.54, G is finite dimensional if and only if G is locally isomorphic to a group $\Delta\times\mathbb{R}^m\times S_1\times\cdots\times S_k$ for a totally disconnected compact abelian group Δ and simple compact connected Lie groups $S_1,\ldots,S_n$. This is the case if and only if the Lie algebra $\mathfrak{L}(G)$ is isomorphic to $\mathfrak{z}\oplus\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_n$ for an m-dimensional abelian Lie algebra $\mathfrak{z}$ and simple compact Lie algebras $\mathfrak{s}_k$, $k=1,\ldots,n$. Finally $\mathfrak{L}(G)$ is of this form if and only if it is the Lie algebra of the compact Lie group $\mathbb{T}^m\times S_1\times\cdots\times S_n$.

(2) $\iff$ (3): This is immediate by Theorem 9.52.

(3) $\iff$ (4): This follows from the Borel-Scheerer-Hofmann Splitting Theorem 9.39.

(3) $\iff$ (5): By Theorem 9.24, $G = Z_0(G)G'$. If (3) or (6) holds, then G' is a semisimple Lie group and therefore has a finite center, thus $Z_0(G)\cap G'$ is finite. Now $G = Z_0(G)G'$ implies $G/G' \cong Z_0(G)/(Z_0(G)\cap G')$; thus $Z_0(G)$ and G/G' are locally isomorphic. Hence if one of these two groups is finite dimensional, the other one is finite dimensional as well.

(ii) In Corollary 10.38 we shall see that G and $G_0\times G/G_0$ are homeomorphic groups and thus have the same cohomology. Recall that $|G|$ denoted the space underlying G. As we noted in the introduction to Part 5, Aspects of Algebraic Topology—Cohomology (following Proposition 8.82), $H^0(|G|,\mathbb{Z})\cong C(|G/G_0|\mathbb{Z})$, the group of locally constant $\mathbb{Z}$-valued functions on $|G|$, while $H^n(|G/G_0|,\mathbb{Z})=\{0\}$ for $n\ge 1$. From the Künneth Theorem (cf. [338], p. 360, E5)it then follows that $H^n(|G|,\mathbb{Z}) = H^n(|G_0|,\mathbb{Z})$ for $n\ge 1$. By (i)(5) above, we must compute $H^n(|(G_0)'|\times|A|;\mathbb{Z})G.$ with a closed abelian subgroup A isomorphic to $G_0/(G_0)'$. From 8.83 we know that $H^*(A,\mathbb{Z})\cong\bigwedge\widehat{A}$ as graded Hopf algebras, and if $q=\dim A$, then $H^q(A,\mathbb{Z})=\bigwedge^q\widehat{A}$. If $p=\dim(G_0)'$, then (G_0) being a compact orientable manifold, yields $H^p(|(G_0)'|,\mathbb{Z})\cong\mathbb{Z}$ and $H^{p'}(|(G_0)'|,\mathbb{Z})=\{0\}$ for $p'>p$ (see [96], p. 315, 6.8(iv)). By the Künneth Theorem, (cf. [338], p. 360, E5), since $\bigwedge A$ is torsion free by the torsion freeness of $\widehat{A}$ (cf. 8.5) and $n=p+q$ we have

$$H^n(|G_0|,\mathbb{Z})\cong H^p(|(G_0)'|,\mathbb{Z})\otimes H^q(A,\mathbb{Z})\cong\bigwedge^q\widehat{A}.$$

This abelian group is a torsion free group of rank 1. Also, $\operatorname{rank} C(|G/G_0|,\mathbb{Z}) = w(G/G_0)$. (See e.g. Theorem A4.9(ii), and note that for infinite rank torsion free abelian groups rank and cardinality agree.) Hence $\operatorname{rank} H^n(|G|,\mathbb{Z}) = w(|G/G_0|)$.

(iii) (1) $\iff$ (2): If G is a Lie group, then G/G_0 is discrete, and thus finite by compactness and $A\cong G_0/(G_0)'$ is a Lie group by Corollary 2.40, Definition 2.41 and Proposition 5.33(iii). Conversely, assume that G/G_0 is finite and A is a Lie group. Then G_0 is open and each maximal pro-torus T is a product of a maximal torus $T_{(G_0)'}$ of $(G_0)'$ and A by Theorem 9.93(ii). Hence T is a Lie group. Then the closed subgroup $Z(G)$ of T according to Theorem 9.32(iv) is a Lie group. Then $Z_0(G_0) \times (G_0)'$ is a Lie group and G_0 is a quotient group by Theorem 9.24. So, by Theorem 6.7, G_0 is a Lie group.

(1)⇒(3): If G is a Lie group, then G/G_0 is discrete and compact and therefore finite. Moreover, G_0 is an orientable compact manifold and thus has cyclic cohomology in the highest dimension (see [96], p. 315, 6.8(iv)).

(3)⇒(1): Assume now that

$$H^n(|G|, \mathbb{Z}) = H^p(|(G_0)'| \otimes \bigwedge^q \widehat{A} \otimes C(|G/G_0|, \mathbb{Z})$$

is cyclic. Then $\bigwedge^q \widehat{A}$ and $C(G/G_0, \mathbb{Z})$ are cyclic. In particular, $G = G_0$, i.e. G is connected. Let $0 \neq e \in \bigwedge \widehat{A}$ be one of the two generators. Then there is a free set $e_1, \ldots, e_q$ in $\widehat{A}$ such that $e_1 \wedge \cdots \wedge e_q = e$. We claim that $\widehat{A} = \bigoplus_{m=1}^q \mathbb{Z}{\cdot}e_m$. Indeed let $a = \sum_{m=1}^q q_m{\cdot}e_m$ with unique rational numbers q_m be an element of $\widehat{A}$. Then $a \wedge e_2 \wedge \cdots \wedge e_q = q_1{\cdot}e$, $e_1 \wedge a \wedge \cdots \wedge e_q = q_2{\cdot}e$ and so on, showing that $q_m \in \mathbb{Z}$ for all m and proving our claim. Thus $\widehat{A}$ is free and $A \cong G_0/(G_0)'$ is a torus. Therefore G is a Lie group by (ii) above. This completes the proof of (iii).

(iv) The dimension of a nonabelian compact Lie algebra is at least 3 by 6.53. Then by (i) above, $\mathfrak{L}(G) \cong \mathbb{R}^m$ with $m \leq 2$. Now 9.52 implies that G is abelian and that $\operatorname{rank} \widehat{G} \leq 2$. Thus $\mathbb{Q} \otimes \widehat{G} \cong \mathbb{Q}^2$ and since G is connected, $\widehat{G}$ is torsion-free by 8.5 and thus the natural morphism $\chi \mapsto 1 \otimes \chi: \widehat{G} \to \mathbb{Q} \otimes \widehat{G}$ is injective. □

We shall say that a topological space X *contains a cube* $\mathbb{I}^{\aleph}$, for a cardinal $\aleph$, where $\mathbb{I} = [0, 1]$ if there is a continuous injective map $\mathbb{I}^{\aleph} \to X$.

Proposition 9.56. (i) *Let G be a compact infinite dimensional group and T a maximal pro-torus. Then* $\dim G = \dim T = \operatorname{rank} \widehat{T}$.

(ii) *Every compact group G contains a cube* $\mathbb{I}^{\dim G}$.

(iii) *If a compact group contains a cube $\mathbb{I}^{\aleph}$ for some cardinal $\aleph$, then $\aleph \leq \dim G$.*

Proof. (i) By 9.49(ii) we may write $\mathfrak{L}(G) = \mathfrak{z} \times \prod_{j \in J} \mathfrak{s}_j$ and $\mathfrak{L}(T) = \mathfrak{z} \times \prod_{j \in J} \mathfrak{t}_j$. Then by Definition 9.53 we have $\dim G = \dim_{\mathbb{R}} \mathfrak{L}(G)' = \dim_{\mathbb{R}}(\mathfrak{z}' \oplus \bigoplus j \in J \mathfrak{s}'_j) = \dim_{\mathbb{R}} \mathfrak{z}' + \sum_{j \in J} \dim_{\mathbb{R}} \mathfrak{s}_j$ since $\dim_{\mathbb{R}} \mathfrak{s}'_j = \dim \mathfrak{s}_j$. Similarly, $\dim T = \dim_{\mathbb{R}} \mathfrak{z}' + \sum_{j \in J} \dim_{\mathbb{R}} \mathfrak{t}_j$. If J is finite, then $\dim_{\mathbb{R}} \mathfrak{z}'$ is infinite since G is infinite dimensional. Then $\dim G = \dim \mathfrak{z}' = \dim T$. If J is infinite, then $\dim_{\mathbb{R}} \sum_{j \in J} \mathfrak{s}_j = \operatorname{card} J$ and similarly with $\mathfrak{t}_j$ in place of $\mathfrak{s}_j$. Then $\dim G = \dim_{\mathbb{R}} \mathfrak{z}' + \operatorname{card} J = \dim T$. This completes the proof of $\dim G = \dim T$. The relation $\dim T = \operatorname{rank} \widehat{T}$ goes back to Definition 8.23.

(ii) . If $\dim G < \infty$, then G and $Z_0(G) \times G'$ are locally isomorphic. By Corollary 8.24(v), $Z_0(G)$ contains a cube whose dimension is the linear dimension of $\mathfrak{L}\big(Z_0(G)\big)$. The Lie group G' contains a cube whose dimension is the linear dimension of $\mathfrak{L}(G')$. Hence $Z_0(G) \times G'$ and therefore G contains a cube of dimension $\dim\big(\mathfrak{L}\big(Z_0(G)\big)\big) + \dim\big(\mathfrak{L}(G')\big) = \dim \mathfrak{L}(G) = \dim G$ in view of Scholium 9.54(B).

If $\dim G = \infty$, then $\dim G = \dim T$ and T contains a cube $\mathbb{I}^{\dim T} = \mathbb{I}^{\dim G}$ by Corollary 8.24(v).

(iii) We may assume that the cube $\mathbb{I}^{\aleph}$ is contained in G_0, and since $\dim G_0 = \dim G$ by Scholium 9.54(B), we may and will assume that G is connected. First assume that $\aleph$ is finite. Then $\mathrm{DIM}\,\mathbb{I}^{\aleph} = \aleph$ by the Euclidean Fundamental Theorem 8.25(Db). Further, by the Closed Subspace Theorem 8.25(Dc), $\mathrm{DIM}\,\mathbb{I}^{\aleph} \le \mathrm{DIM}\,G$. Thus $\aleph \le \dim G$ by 9.54(B). Next assume that $\aleph$ is infinite. Then $\aleph = w(\mathbb{I}^{\aleph})$ by Exercise EA4.3 following A4.8 in Appendix 4. Since the weight (see Definition A4.7) is monotone we have $w(\mathbb{I}^{\aleph}) \le w(G)$. By (i) above, we have $\dim G = \dim T$. By 8.24(ii), $\dim T = w(T)$. By Theorem 9.36, $w(T) = w(G)$, and so $w(G) = \dim G$. Hence $\aleph \le \dim G$, as asserted. □

The preceding Proposition shows the following proposition:

The set of cardinals $\aleph$ such that G contains cube $\mathbb{I}^{\aleph}$ contains a maximal one, namely, $\dim G$.

One can therefore define $\dim G$ as the sup of all cardinals $\aleph$ such that G contains a cube $\mathbb{I}^{\aleph}$ and assert, that the sup is attained.

A considerable amount of additional information is available in [185] where the dimension of quotient spaces of a compact group is discussed.

Locally Euclidean Compact Groups Are Compact Lie Groups

A Hausdorff space G is *locally euclidean* if it is the union of a set of open subsets U each of which is homeomorphic to an open subset of some space $\mathbb{R}^n$ (depending on U and on positive dimension). A topological group is clearly locally euclidean if there is an identity neighborhood which is homeomorphic to $\mathbb{R}^n$. In particular, such a group is locally connected and thus the identity component G_0 is open. We summarize these observation in the following result.

Hilbert's Fifth Problem for Compact Groups

Theorem 9.57. *A locally euclidean compact group is a compact Lie group.*

Proof. Let G be a locally euclidean group with compact identity component. By Scholium 9.54.B and the Euclidean Fundamental Theorem of Dimension Theory 8.25(Db) which is valid for all the dimension functions listed in 9.54.B above, we conclude that G is finite dimensional. Hence Theorem 9.52 shows that G is locally isomorphic to a group $\Delta \times \mathbb{R}^m \times S_1 \times \cdots \times S_k$. This group is locally connected if and only if the compact totally disconnected subgroup $\Delta \subseteq Z_0(G)$ is discrete. Since $\mathbb{Z}_0(G)/\Delta \cong \mathbb{T}^m$, we conclude that $Z_0(G)$, being locally isomorphic to $\mathbb{T}^m$, is a torus. By the Structure Theorem 9.24, G is a quotient of the group $Z_0(G) \times G'$ modulo a finite group and G' is a compact Lie group of the form $S_1 \times \cdots \times S_k$ modulo a finite group by 9.52. Thus G_0 is a compact Lie group. □

As a consequence of this theorem all information of Chapter 6 becomes available for locally euclidean compact groups. It remains true, more generally, that a locally euclidean topological group is a Lie group, but that is much harder to prove (see [263] (1955) or [229], (1971)). There are several variants of the proof of 9.57. By 9.39, G_0 is a semidirect product of the semisimple group $(G_0)'$ and a connected abelian group A isomorphic to $G_0/(G_0)'$. If either of the groups $(G_0)'$ or A fails to be finite dimensional, then it contains arbitrarily small subsets homeomorphic to a cube $[0,1]^{\mathbb{N}}$ by 9.19, respectively, 8.21, and such a set cannot be contained in a euclidean space. Thus it follows that G is finite dimensional in the sense of 9.52, and the argument continues from there. Thus no reference to topological dimension theory need be made except for the fact that a euclidean cell cannot contain a Hilbert cube $[0,1]^{\mathbb{N}}$.

Topological Characterisation of Compact Matrix Groups

Corollary 9.58. *A compact group is isomorphic* (*as a topological group*) *to a group of matrices if and only if it is locally euclidean.*

Proof. By 9.57, a compact group is a compact Lie group if and only if it is locally euclidean, and by Definition 2.41 it is isomorphic to a group of matrices if it is a compact Lie group. □

Elementary Geometric Properties of Compact Groups

Corollary 9.59. (i) *If the dimension of a locally euclidean compact connected group is less than* 3, *then it is a one-torus or a two-torus.*

(ii) *The only group on a compact surface is the two torus.*

(iii) *The only compact connected groups which contain an open set homeomorphic to euclidean three-space are: the three-torus, the group* SO(3) *of rotations of euclidean three-space, and the group* SU(2) *of isometries of two dimensional complex Hilbert space.*

(iv) (The Sphere Group Theorem) *The only groups on a sphere are: the two element group* $\mathbb{S}^0 = \{1, -1\}$, *the circle group* $\mathbb{S}^1$, *and the group* $\mathbb{S}^3$ *of quaternions of norm one* ($\cong$ SU(2)).

(v) *Let G be an n-dimensional compact connected group and assume that G has a homeomorphic copy K contained in $\mathbb{R}^{n+1}$. Then G is a Lie group and $\mathbb{R}^{n+1} \setminus K$ has two components U_b and U_u where matters can be arranged so that $U_b \cup K$ is compact while U_u is unbounded.*

Proof. (i) Let G be a compact connected group. If $\dim G < 3$, then by 9.55(iv), G_0 is abelian. If G is, in addition, locally euclidean, then it is a compact Lie group by 9.57 and is therefore a torus of dimension one or two.

(ii) follows from (i).

(iii) By 6.49(ii), (vii), the nonabelian compact connected three dimensional Lie groups are (up to isomorphy) SO(3) and SU(3).

(iv) The connected compact Lie groups on spheres were characterized in Theorem 6.95; but by 9.57 we know that any group on a sphere is a compact Lie group.

(v) By Alexander duality (see [338], p. 296, Theorem 16, we have $H^n(K,\mathbb{Z}) \cong \widetilde{H}_0(\mathbb{R}^{n+1}\setminus K,\mathbb{Z}) \cong \mathbb{Z}^{(c)}$, where c is the set of connected components of $\mathbb{R}^{n+1}\setminus K$ and where $\widetilde{H}_*$ is reduced singular homology. By Theorem 9.55(ii) the rank of $H^n(G,\mathbb{Z})$ is one. Therefore, the rank of $\widetilde{H}_0(\mathbb{R}^{n+1}\setminus K,\mathbb{Z})$ is one, that is, $\operatorname{card} c = 2$. Moreover, $\widetilde{H}_0(\mathbb{R}^{n+1}\setminus K,\mathbb{Z})$ is free. Hence $H^n(G,\mathbb{Z})$ is cyclic and thus G is a Lie group by Theorem 9.55(iii).

Now let $\mathbb{S}^n$ be the surface of a closed ball B^{n+1} in $\mathbb{R}^{n+1}$ containing K in its interior. Then $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}\setminus G$; since $\mathbb{S}^n$ is connected, there is one of the two components, say U_u which contains $\mathbb{S}^{n+1}$. Let U_b denote the other of the two members of c; then $U_b\cap\mathbb{S}^{n+1} = \emptyset$ and since U_b is connected, U_b is either contained in $B^{n+1}\setminus\mathbb{S}^{n-1}$ or in $\mathbb{R}^{n+1}\setminus B^{n+1}$. Since the boundary ∂U_b is contained in K and $K \subseteq B^{n+1}\setminus\mathbb{S}^{n-1}$, we conclude that $U_b\cup K \subseteq B^{n+1}$, and that $U_b\cup K$ is closed. Hence $U_b\cup K$ is compact, and, accordingly, U_u is unbounded. □

In Exercise E1.11 preceding Proposition 1.31 we saw the example of the dyadic solenoid giving us a 1-dimensional compact connected abelian group which is not embeddable in a 2-manifold by the preceding theorem, but is embeddable in $\mathbb{R}^3$.

If we speak here of "elementary geometric properties" we do not imply that the results listed here or their proofs are elementary, in fact they are not. Indeed the proof of a result like 9.59(iv) is much less elementary than its formulation might suggest. For Statement (v), see for instance [196], p. 303, 2.13. The present statement and its proof are improved versions. For information on embedding n-dimensional locally compact abelian groups into $\mathbb{R}^{n+1}$, see [70], p. 73: Theorem of Bognár, [25, 26, 27].

There is a construction which allows a compact n-dimensional Lie group to bound a compact manifold, namely, the following (cf. [199]).

Construction. Let G a compact connected Lie group of dimension n, and H a sphere subgroup, that is, $H \cong \mathbb{Z}(2)$, $\mathbb{T}$, or $\mathrm{SU}(2)$. On $G\times\mathbb{I}$ let R be the equivalence relation whose cosets $R(g,t)$ are singleton for $t>0$ and $Hg\times 0$ for $t=0$. Then $M^{n+1} \stackrel{\text{def}}{=} (G\times\mathbb{I})/R$ is a manifold (and a G-space with orbits $(G\times\{t\})/R$ and with $(G\times\{1\})/R$ as boundary. If H is normal then M^{n+1} is a compact topological monoid.

If two copies of M^{n+1} are glued together by identifying corresponding points of the respective boundaries, one obtains a manifold M_2^{n+1} of dimension $n+1$ into which G is embedded in such a fashion that G has a neighborhood which is homeomorphic to $G\times\mathbb{R}$.

Proof. We recall that $\mathbb{I}$ denotes the unit interval $[0,1]$ which may be equipped with the natural multiplication of real numbers if required. We set $\mathbb{I}_2 = [0,2]$. On $G\times\mathbb{I}_2$ we define the equivalence relation R_2 whose cosets $R_2(g,t)$ are singleton if $0 < t < 2$ and are $gH \times \{t\}$ if $t \in \{0,1\}$. We define $R \stackrel{\rm def}{=} R_2 \cap (G \times \mathbb{I})^2$, $R_G \stackrel{\rm def}{=} R_2 \cap (G \times \{1\})^2$ and set

$$M_2^{n+1} = (G \times \mathbb{I}_2)/R_2 \supseteq (G \times \{1\})/R_G \cong G,$$
$$M^{n+1} = (G \times \mathbb{I})/R_2 \supseteq (G \times \{1\})/R_G \cong G.$$

The space $G \times \mathbb{I}$ is a compact monoid. Both of these spaces are G spaces with respect to the action

$$\big(g, R(g', t)\big) \mapsto R(gg', t) : G \times M_2^{n+1} \to M_2^{n+1}.$$

it is straightforward to verify that the action is well-defined and continuous and that M^{n+1} is invariant and that the orbit spaces are naturally homeomorphic to $\mathbb{I}_2$, respectively, $\mathbb{I}$. The orbits $G{\cdot}R(1,0)$ and $G{\cdot}R(1,2)$ are both homeomorphic to G/H under the functions $gH \mapsto g{\cdot}R(1,t)$ with $t = 0$, respectively $t = 2$.

Now we show that every point $R(g,0)$ (and, analogously, every point $R(g,2)$) has an open neighborhood which is a manifold. For this purpose we pick a subset $U \in G$ containing 1 and being homeomorphic to an open m-cell, $m = \dim G/H$, such that the quotient map $p{:}\, G \to G/H$, $p(g) = gH$ maps U homeomorphically onto on open neighborhood of $p(1) = H$. Such a set exists by the Tubular Neighborhood Theorem for closed Subgroups 5.33(ii) (s. also the Local Cross Section Theorem 10.34 below). Then $(u,h) \mapsto uh : U \times U \times H \to UH$ is a homeomorphism onto an open H-saturated neighborhood of 1 in G. Now we define

$$S \stackrel{\rm def}{=} \bigcup_{0 \le t < 1} H{\cdot}R(1,t) = \{R(h,t) : h \in H\, 0 \le t < 1\}.$$

Then $H{\cdot}S = S$, $G{\cdot}R(1,0) \cap S = \{R(1,0)\}$, and

$$(u,s) \mapsto u{\cdot}s : U \times S \to U{\cdot}S \subseteq M^{n+1} \subseteq M_2^{n+1}$$

is a homeomorphism onto an open neighborhood of $R(1,0)$ in M^{n+1}.

Now S is an H space via the induced action such that the action is free with the sole exception at $R(1,0)$ where the orbit is singleton. Thus S is the open cone $(H \times [0,1[)/(H \times \{0\})$ over H. Since H is a sphere $\mathbb{S}^0$, $\mathbb{S}^1$ or $\mathbb{S}^3$ the space is homeomorphic to $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^4$. Accordingly, $U \times S$ is homeomorphic to $\mathbb{R}^{n+1}$.

This shows that $M^{n+1} \setminus \partial M^{n+1}$ and M_2^{n+1} are locally euclidean spaces of dimension $n+1$. If H is normal in G then R is a congruence and M^{n+1} is a compact monoid. □

If $\mathbb{S}^0 \cong \mathbb{Z}(2)$ is factored, we will in general arrive at a nonorientable manifold.

Notice that M^{n+1} is a mapping cylinder for the quotient map $G \mapsto G/H$ (cf. [338], p. 32). In particular, G/H and M^{n+1} are homotopy equivalent.

We are not addressing here the problem of determining which of these manifolds M_2^{n+1} are spheres. We record that $G = \mathbb{T}^n$ is embeddable into $\mathbb{R}^{n+1}$.

We notice that the Alexander Duality Theorem yields quite generally

$$\text{(A)} \qquad H^q(G,A) \cong \widetilde{H}_{n-q}(\mathbb{R}^{n+1} \setminus G, A)$$

for an arbitrary coefficient module A. Taking $A = \mathbb{Q}$, from the topology of connected compact Lie groups (6.88ff.) we know that $\bigoplus_{q=0}^{n} \widetilde{H}_{n-q}(\mathbb{R}^{n+1} \setminus G, \mathbb{Q}) \cong \bigwedge P\big(H(G,\mathbb{Q})\big)$ (s. the Hopf–Samelson Theorem 6.88), further $P\big(H(G,\mathbb{Q})\big) \cong \mathbb{Q} \otimes \widehat{T}$ for a maximal torus T (s. 6.93). Let $r = \dim T$ be the rank of G, let V_m denote the graded vector space $V_m^1 + \cdots + V_m^{d_m}$ where d_m is the $\mathbb{Q}$-dimension of the m-th homogeneous component of $P\big(H(G,\mathbb{Q})\big)$ and where $V_m^0, \ldots, V_m^{d_m}$ are one dimensional. Then

$$\bigoplus_{q=0}^{n} \widetilde{H}_{n-q}(\mathbb{R}^{n+1} \setminus G, \mathbb{Q}) \cong \bigwedge V_1^1 \otimes \bigwedge V_1^2 \otimes \cdots \otimes \bigwedge V_{2N-1}^{d_{2N-1}},$$

where $\bigwedge V_j^k = \mathbb{R} \oplus V_j^k$, $\dim \bigwedge V_j^k = 2$. Hence the reduced singular homology of each of the two complements of G in $\mathbb{R}^{n+1}$ is that of a product of odd dimensional spheres. A first illustration is given by the 2-torus $G = \mathbb{T}^2$ sitting in $\mathbb{R}^3$; the interior (that is, the bounded component of $\mathbb{R}^3 \setminus \mathbb{T}^2$) is homotopy equivalent to $\mathbb{S}^1$ as is the exterior. We recall $\mathrm{SU}(2) \cong \mathbb{S}^3 \subseteq \mathbb{H} = \mathbb{R}^4$ and note

$$\widetilde{H}_q^n(\mathbb{R}^4 \setminus \mathbb{S}^3) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, 4, \\ \{0\} & \text{if } q = 1, 2. \end{cases}$$

The general questions in the present context are

Problem A. *Which compact connected Lie groups bound compact connected manifolds?*

Problem B. *Which compact n-dimensional Lie groups can be embedded into $\mathbb{R}^{n+1}$?*

Part 3: The Connectivity Structure of Compact Groups

Arc Connectivity

We shall see that the issue of arc connectivity reduces, via the structure theory now available to us, to the abelian situation for which an extensive theory is at our disposal through 8.27–8.33. For example, a compact connected group G is arcwise connected if and only if G/G' is arcwise connected.

Recall that G_a denotes the arc component of the identity in a topological group and that $\pi_0(G) = G/G_a$ is the group of all arc components.

The Arc Component of a Compact Group

Theorem 9.60. *Assume that G is a compact group.*

(i) *The arc component G_a of G is $\exp \mathfrak{L}(G)$ and contains $(G_0)'$. The sequence*

$$\mathfrak{g}/\mathfrak{g}' \xrightarrow{X+\mathfrak{g}' \mapsto (\exp_G X)(G_0)'} G/(G_0)' \xrightarrow{p} \pi_0(G) \to 0,$$

$p(gG') = gG_a$ is exact.

(ii) *G is arcwise connected if and only if the exponential function is surjective if and only if G is connected and the (compact connected) abelian group G/G' is arcwise connected.*

(iii) *For a connected compact group G, the arcwise connectivity of the identity component $Z_0(G_0)$ of the center is sufficient for the arcwise connectivity of G.*

(iv) *If $f\colon G \to H$ is a surjective morphism of compact groups, then $f(G_a) = H_a$.*

(v) $\overline{G_a} = G_0$.

(vi) *An arcwise connected compact group is locally connected.*

(vii) (The Borel Set Reduction Theorem) *For a compact group G the following statements are equivalent:*

(1) *The arc components of G are Borel sets.*

(2) *The arc component of the identity $(G_0/(G_0)')_a$ in the compact connected abelian group $G_0/(G_0)'$ is a Borel subset.*

(viii) *The arc components of a compact group G are Borel sets if $G_0/(G_0)'$ is metric.*

Proof. (i) From 9.48 we know that $(G_0)' \subseteq \exp \mathfrak{g}$ and that $(G_0)' = \mu_G(\{1\} \times \prod_{j\in J} S_j \subseteq G_0$. From the Borel–Scheerer–Hofmann Splitting Theorem 9.39 we know that $G_0 \cong (G_0)' \rtimes_\iota A$, $A \cong G_0/(G_0)'$. The arc component of the identity in a product of topological groups is the product of the identity arc components. Hence $G_a = (G_0)' \rtimes_\iota A_a$, and the quotient morphism $p\colon G \to G/(G_0)'$ satisfies $p(G_a) = (G/(G_0)')_a$ and $p^{-1}\big((G/(G_0)')_a\big) = G_a$. From the Arc Component Theorem 8.30 for (locally) compact abelian groups we deduce that $A_a = \exp \mathfrak{a}$. $(G/(G_0)')_a = \exp_{G/(G_0)'} \mathfrak{g}/\mathfrak{g}' = (\exp_G \mathfrak{g})/(G_0)'$. Hence the full inverse image G_a of this group is $\exp_G \mathfrak{g}$.

Moreover,

$$\pi_0(G) = G/G_a \cong \frac{G/(G_0)'}{G_a/(G_0)'} \cong \frac{G/(G_0)'}{\exp \mathfrak{g}/(G_0)'}.$$

(ii) is a consequence of (i).

(iii) By 9.24 we have $G/G' = Z_0(G)/(Z_0(G) \cap G')$. Hence the arcwise connectedness of $Z_0(G)$ implies that of G/G' and thus that of G by (ii).

(iv) By (i)above, the arc component G_a of the identity in a compact group G is $\exp \mathfrak{L}(G)$. From 9.47 we conclude that

$$H_a = \exp_H \mathfrak{L}(H) = (\exp_H \circ \mathfrak{L}(f))\big(\mathfrak{L}(G)\big) = (f \circ \exp_G)\big(\mathfrak{L}(G)\big) = f(G_a).$$

(v) Clearly, $G_a \subseteq G_0$, and thus we may assume that G is connected. By (i) we have $G' \subseteq G_a$. By (iv) we have $(G/G')_a = G_a/G'$ and then $\overline{G_a}/G'$ is densely contained in $\overline{(G/G')_a}$ and is compact; hence the two groups agree. Since G/G' is abelian, (i) and 7.71 entail $\overline{G_a}/G' = \overline{(G/G')_a} = G/G'$, and this implies $\overline{G_a} = G$.

(vi) If G is an arbitrary connected compact group, then G is the semidirect product of G' and an abelian subgroup $A \cong G/G'$ by 9.39. Since G' is locally connected and arcwise connected by the preceding remark, G is locally connected, respectively, arcwise connected iff G/G' is locally connected, respectively, arcwise connected. Then by 8.36(iii) the arcwise connectivity of the compact group G implies its local connectivity.

(vii) The arc components of G are all homeomorphic to G_a; thus (i) amounts to saying that G_a is a Borel set. Since $G_a \subseteq G_0$ we may assume that G is connected.

As we have seen in the proof of (i), the arc component G_a is homeomorphic to $G' \times A_a$, and G' is closed and arcwise connected. Hence by Lemma 8.86, G_a is a Borel subset of G if and only if $G' \times A_a$ is a Borel subset of $G' \times A$ iff A_a is a Borel subset of A.

(viii) By the Reduction Theorem (vii) above, the assertion is true if and only if the arc component $(G/G')_a$ is a Borel subset of G/G'. Since G/G' is assumed to be metric, the assertion follows from Theorem 8.94. □

Of course there are numerous sufficient conditions which imply the hypothesis that G/G' is metric:

The Metrizability Supplement. For a compact group, each of the following conditions implies the next:
(i) G is metric.
(ii) G_0 is metric.
(iii) A maximal pro-torus T of G is metric.
(iv) The identity component $Z_0(G_0)$ of the center of G_0 is metric.
(v) The compact connected abelian group G/G' is metric.

Proof. Since $Z_0(G_0) \subseteq T \subseteq G_0 \subseteq G$, the first three implications are obvious. For (iv)⇒(v) we refer to 9.23 or 9.24 and conclude that $Z_0(G_0)/(Z_0(G_0) \cap K') \cong G_0/(G_0)'$. Thus $G_0/(G_0)'$ is a homomorphic image of $Z_0(G_0)$ and the implication follows. □

We should recall from Chapter 8, Part 6, Theorem 8.99 that there is a model of set theory in which there is a compact connected abelian group of continuum weight which has countably infinitely many arc components which are not Borel sets.

We emphasize that the arcwise connectedness of $Z_0(G)$ is not necessary for the arcwise nor the local connectedness of a compact group G. In fact, as we show in the following proposition, *any* compact connected abelian group can be the identity component of the center in an arcwise connected locally connected compact group.

We recall from Chapter 8, 8.23 that for a compact abelian group A we define the *dimension* by $\dim A = \operatorname{rank} \widehat{A} = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{A}$.

Proposition 9.61. *Let A be an arbitrary compact connected abelian group. Then the following conclusions hold.*
(i) *There is a compact totally disconnected subgroup D of A such that A/D is a torus isomorphic to $\mathbb{T}^{\dim A}$.*
(ii) *There is a family $\{n_j \mid j \in J\}$ of natural numbers satisfying* $\operatorname{card} J \leq \max\{\aleph_0, \operatorname{card} D\}$ *and an injective morphism* $j{:}\, D \to S \overset{\text{def}}{=} \prod_{j\in J} \mathrm{SU}(n_j)$.

In particular, if A is metric, then J may be taken to be finite or $\mathbb{N}$ and S is metric, too.

(iii) *There is a compact arcwise connected locally connected group G with $w(G) = w(A)$ such that the identity component $Z_0(G)$ of its center is isomorphic to A and that the factor group G/G' is isomorphic to $\mathbb{T}^{\dim A}$. Specifically, $G \cong \left(\prod_{j\in J} \mathrm{SU}(n_j)\right) \rtimes \mathbb{T}^{\dim A}$. If A is metric then G can be chosen to be metric.*

Proof. (i) We recall from 8.15 that there is a closed subgroup $D \subseteq A$ such that $\dim D = 0$ and that A/D is a torus whose dimension is $\dim A$.

(ii) The character group $\widehat{D}$ is a torsion group (see 8.5). Let $\mathcal{F}$ be the family of finite subgroups; then $\widehat{D} = \bigcup_{F\in\mathcal{F}} F$, a directed union, and $\operatorname{card}\mathcal{F} \leq \max\{\aleph_0, \operatorname{card}\widehat{D}\}$. The universal property of the direct sum (coproduct) yields a morphism $\Phi\colon \bigoplus_{F\in\mathcal{F}} F \to \widehat{D}$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\mathrm{copr}_F} & \bigoplus_{E\in\mathcal{F}} E \\ {\scriptstyle \mathrm{id}_F}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ E & \xrightarrow[\mathrm{incl}]{} & \widehat{D} \end{array}$$

commutes. Since the image $\operatorname{im}\Phi$ contains all $F \in \mathcal{F}$, it contains all of $\widehat{D}$ and the morphism Φ is surjective. Dually we get an injective morphism $D \to \prod_{F\in\mathcal{F}} \widehat{F}$ which is simply $\widehat{\Phi}$ if we identify $\prod_{F\in\mathcal{F}} \widehat{F}$ with the character group of $\bigoplus_{F\in\mathcal{F}} F$ (cf. 1.17). We have a nonnatural isomorphism $F \cong \widehat{F}$ (cf. 1.18). So by A1.11, each $\widehat{F}$ is a finite product of cyclic groups. Thus there is a family $\{n_j \mid j \in J\}$ of natural numbers with $\operatorname{card} J \leq \max\{\aleph_0, \operatorname{card}\mathcal{F}\} = \max\{\aleph_0, \operatorname{card}\widehat{D}\} \leq \max\{\aleph_0, \operatorname{card}\widehat{A}\}$, and an injective morphism

$$j'\colon D \to \prod_{j\in J} \mathbb{Z}(n_j).$$

If A is metric, then so is D and hence $\widehat{D}$ and $\mathcal{F}$ is countable. Thus J may be chosen countable.

For $n \in \mathbb{N}$ let $Z(n) \stackrel{\mathrm{def}}{=} Z\big(\mathrm{SU}(n)\big) = \{e^{2\pi m/n}\cdot\mathbf{1}_n \mid m = 0,\ldots,n-1\}$. Then there is an isomorphism $j_n\colon \mathbb{Z}(n) \to Z(n)$, and thus there is an injective morphism of compact groups

$$j\colon D \to S \stackrel{\mathrm{def}}{=} \prod_{j\in J} \mathrm{SU}(n_j).$$

(iii) We assume A to be additively written and define $\Delta \subseteq A \times S$ to be the subgroup of all pairs $\big(-a, j(a)\big) \in A \times S$. We set $G = \frac{A\times S}{\Delta}$. The center of $A \times S$ is $Z(A\times S) = A \times \prod_{j\in J} Z(n_j)$. The center of G, accordingly, is $Z(G) = Z(A\times S)/\Delta$, and by 7.73, $Z_0(G) = \frac{(A\times\{1\})\Delta}{\Delta} = \frac{A\times j(D)}{\Delta} \cong A$. Also $G' = \frac{\{0\}\times S}{\Delta} = \frac{D\times S}{\Delta} \cong S$. Further, $Z_0(G)/(Z_0(G)\cap G') \cong G/G' \cong \frac{A\times S}{D\times S} \cong A/D \cong \mathbb{T}^{\dim A}$ by (i). By the Borel–Scheerer–Hofmann Theorem 9.39, $G \cong G' \rtimes G/G' \cong S \rtimes \mathbb{T}^{\dim A}$, and this group is, topologically, a product of connected Lie groups and is therefore arcwise connected and locally connected. Clearly $w(A) \leq w(G) \leq \max\{\aleph_0\cdot \operatorname{card} J, w(\mathbb{T}^{\dim A})\} \leq w(A)$. (see Appendix 4, EA4.3). $\square$

A particular example may be instructive.

Exercise E9.12. Verify the details of the following special example:

Example 9.62. Let $\mathbb{Z}_p = \lim \mathbb{Z}(p^n)$ be the group of p-adic integers (see 1.28(i)). We may write $\mathbb{Z} \subseteq \mathbb{Z}_p$ with the integers forming a dense subgroup. We let $\mathbb{S}_p$ denote the p-adic solenoid which may constructed as a factor group

$$\mathbb{S}_p = \frac{\mathbb{R}\times\mathbb{Z}_p}{\Delta}, \quad \Delta = \{(n,-n) : n \in \mathbb{Z}\}.$$

The character group of $\mathbb{S}_p$ is $\frac{1}{p^\infty}\mathbb{Z}$, the additive group of all rational numbers which can be written with a denominator p^n. This group is countable but not free. Hence by 8.45 and 8.46, the solenoid $\mathbb{S}_p$ is metric connected but not arcwise connected and not locally connected. Moreover, $\mathbb{S}_p$ contains a copy Z_p of $\mathbb{Z}_p$ and $\mathbb{S}_p/Z_p$ is isomorphic to the circle group $\mathbb{T}$.

Next let S be the semisimple compact connected group $\mathrm{SU}(p) \times \mathrm{SU}(p^2) \times \mathrm{SU}(p^3) \times \cdots$. The center $Z(n) \stackrel{\rm def}{=} Z\big(\mathrm{SU}(n)\big) = \{e^{2\pi m/n}\cdot\mathbf{1}_n \mid m = 0,\dots,n-1\}$ of $\mathrm{SU}(n)$ is cyclic of order n. Let $\pi_n\colon Z(p^{n+1}) \to Z(p^n)$ be the surjective morphism given by $\pi_n(e^{2\pi/p^{n+1}}\cdot\mathbf{1}_{p^{n+1}}) = e^{2\pi/p^n}\cdot\mathbf{1}_{p^n}$. Define $C_p = \{(z_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} Z(p^n) : \pi_n(z_{n+1}) = z_n\} = \lim Z(p^n)$. Then there is an isomorphism $f\colon Z_p \to C_p$. given by $f\big((z_n + p^n\mathbb{Z})_{n\in\mathbb{N}}\big) = \big(\exp(2\pi z_n/p^n)\big)_{n\in\mathbb{N}}$, $z_n \in \mathbb{Z}$.

Finally set

$$G = \frac{\mathbb{S}_p \times S}{D}, \quad D = \{\big(z^{-1}, f(z)\big)\colon z \in Z_p\}.$$

Then G is a metric compact connected group and $G' = (\{0\}\times S)D/D \cong S$, while $Z_0(G) = (\mathbb{S}_p \times \{1\})D/D \cong \mathbb{S}_p$. Thus $Z_0(G)$ is not arcwise connected and not locally connected. But

$$G/G' \cong Z_0(G)/(Z_0(G)\cap G') = \frac{\mathbb{S}_p\times C_p}{Z_p \times C_p} \cong \mathbb{S}_p/Z_p \cong \mathbb{T}$$

is a circle group and

$$G \cong (\mathrm{SU}(p)\times\mathrm{SU}(p^2)\times\cdots)\rtimes\mathbb{T}. \qquad \square$$

The reduction of many structural features of a compact connected group to those of a compact connected abelian group can take three avenues: one is through $Z_0(G)$, one is through G/G', one is through a maximal pro-torus T. Sometimes the first is the one to take, but this example and the preceding discourse show that this is by no means always so; for topological properties it is almost always the second that leads to the conclusive answer. We have often seen the third one lead to success, such as in the case of studying the exponential function (9.48).

Local Connectivity

We shall exploit the knowledge accumulated on (locally) compact abelian groups and local connectivity in Chapter 8.34 (see 8.34ff.). We begin by observing a generalisation of 8.37. Recall that for a topological group G we denote by $\mathcal{T}(G)$ the set of all torus subgroups of G.

Proposition 9.63. *Every compact group G contains a characteristic arcwise connected subgroup*

$$G_T = \langle\bigcup\mathcal{T}(G)\rangle = \bigcup\mathcal{T}(G).$$

Proof. Every automorphism of G permutes the maximal pro-torus subgroups and thus leaves $\langle\bigcup\mathcal{T}(G)\rangle$ invariant. This group is obviously contained in G_0 and is characteristic in G_0. It is therefore no loss of generality if we assume that G is connected, i.e. that $G = G_0$. By 9.36(v), every maximal pro-torus of G' is a torus and thus is a member of $\mathcal{T}(G)$. By the Maximal Pro-Torus Theorem 9.31(i,ii), G' is the union of its torus subgroups. Hence $G' \subseteq G_T$.

We claim that, in generalisation of the abelian situation in the proof of 8.37, the arcwise connected subset $\bigcup\mathcal{T}(G)$ is a subgroup. This will complete the proof of the theorem. This set is certainly closed under passing to inverses. Let t_1 and t_2 be elements of this union. Then there are torus subgroups T_1 and T_2 of G such that $t_j \in T_j$, $j = 1, 2$. Then $G'T_1$ und $G'T_2$ are subgroups since G' is normal, and $G'/G'T_j$ is a torus as a homomorphic image of a torus. As G/G' is abelian, $G'T_1T_2$ is a subgroup of G which contains t_1t_2 and for which $G'T_1T_2/G'$ is a torus as a product of two tori $G'T_j/G'$, $j = 1, 2$. Let T be a maximal pro-torus of $G'T_1T_2$. By the Borel–Scheerer–Hofmann Splitting Theorem 9.38(ii) we have $T \cong T'_G \times A$ with a maximal torus T'_G and a pro-torus $A \cong G'T_1T_2/G'$ which is a torus, by what we just saw. Hence T is a torus. By the Maximal Pro-Torus Theorem 9.31 again, all maximal pro-tori of the group $G'T_1T_2$ are tori and cover it. Hence t_1t_2 is contained in a torus and thus in $\bigcup\mathcal{T}(G)$. This proves the claim. ☐

Note that in the abelian case 8.37 the collection $\mathcal{T}(G)$ was directed which allowed a quick proof in that situation. The example of the smallest compact connected nonabelian Lie group SO(3) shows that this fails miserably in the nonabelian case. This accounts for the variance in the proofs here and there.

Proposition 9.63 implies that the smallest closed subgroup

$$G_\ell \stackrel{\rm def}{=} \overline{\bigcup\mathcal{T}(G)} = \overline{G_T}$$

of G containing all torus subgroups of G is a fully characteristic closed connected subgroup of G. We shall presently justify the following definition.

Definition 9.64. In a compact group G we call G_ℓ the *locally connected component* (of the identity) of G. ☐

The Locally Connected Component of a Compact Group

Theorem 9.65. *In every compact group G the fully characteristic subgroup G_ℓ is the unique smallest closed subgroup containing all torus groups and is of the form $(G_0)' \rtimes_\iota A_\ell$, where A_ℓ is isomorphic to the locally connected component component $(G_0/(G_0)')_\ell$ of the compact connected abelian group $G_0/(G_0)'$. The group G_0 is locally connected iff $G_0/(G_0)'$ is locally connected.*

Proof. According to the Borel–Scheerer–Hofmann Splitting Theorem 9.38 we may write $G_0 = (G_0)'A \cong (G_0)' \rtimes_\iota A$ with a compact connected abelian group $A \cong G/G'$. By 9.63 we know $(G_0)' \subseteq G_T \subseteq G_\ell$. Then $G_\ell = (G_0)'A_\ell$ with $A_\ell = A \cap G_\ell$. Since then $G_\ell \cong (G_0)' \rtimes_\iota A_\ell$ we know that G_ℓ is locally connected since $(G_0)'$ is locally connected after the Structure Theorem of Compact Semisimple Groups 9.19 and A_ℓ is locally connected by the Theorem on the Largest Locally Connected Subgroup of a Compact Abelian Group 8.38.

Finally since $G_0 = (G_0)' \rtimes_\iota A$ by 9.38, and since $(G_0)'$ is locally connected as a consequence of Theorem 9.19(i), it follows that G_0 is locally connected iff A is locally connected. As $A \cong G_0/(G_0)'$, the last assertion follows. □

The structure of $(G_0)'$ is known after Theorem 9.19 and that of A_ℓ was described in detail in the Theorem on the Locally Connected Component of a Compact Abelian Group 8.41. In particular, specific information on $\mathfrak{L}(A_\ell)$ and the nature of the exponential function of $A_\ell \cong (G_0/(G_0)')_\ell$ was given.

Corollary 9.66. (i) *A compact group G is locally connected if G has finitely many components and if the compact connected abelian group $G_0/(G_0)'$ is locally connected if and only if every finite dimensional quotient group is a Lie group.*

(ii) *G is locally arcwise connected if and only if it has finitely many components and $G_0/(G_0)'$ is locally arcwise connected if and only if the character group of $G_0/(G_0)'$ is a* S*-group..*

Proof. (i) A locally connected space has open components. Hence the local connectivity of G implies that G_0 is open and the compactness of G then implies that G_0 has finite index in G. Conversely, if G_0 has finite index then it has inner points (by the Baire Category Theorem, if for no other reason) and thus it is open; in this circumstance G is locally connected if and only if G_0 is locally connected. That, however is the case if and only if $G_0/(G_0)'$ locally connected, by Theorem 9.65.

For a proof of the last equivalence, let G be a compact connected group, and $H = G/N$ a quotient group. By Theorem 9.52, H is finite dimensional iff $H' = G'N/N \cong G'/(G' \cap N)$ is a compact Lie group and $H/H' = (G/N)/(G'N/N) \cong G/G'N \cong (G/G')/(G'N/G')$ is a finite dimensional compact connected abelian group. Hence a finite dimensional quotient of G is a Lie group iff the induced quotient of G/G' is a torus.

Now a compact group G is locally connected iff G_0 is open and locally connected, that is, iff G_0 has finite index in G and is locally connected. We saw that G_0 is locally connected iff $G_0/(G_0)'$ is locally connected, and this is the case by

8.36(i) iff all of its finite dimensional quotients are Lie groups. By the preceding paragraph this is the case iff all finite dimensional quotients of G are Lie groups.

(ii) $(G_0)'$ is locally arcwise connected by 9.19, since $(G_0)'^{\sim}$ is locally arcwise connected. Hence if G_0 has finite index and $G_0/(G_0)'$ is locally arcwise connected, the group G, being isomorphic to a finite union of spaces homeomorphic to $G_0' \times (G_0/(G_0)'$ is locally arcwise connected. Conversely, if G is locally arcwise connected, it is locally connected, and thus G_0 has finite index by (i), and the topological direct factor $G_0/(G_0)'$ of the locally arcwise connected space G_0 is locally arcwise connected, and by Theorem 8.36bis this is the case if the character group is an S-group. □

Corollary 9.67. *Let G be a connected compact group. Then a sufficient condition for G to be locally connected is that the identity component $Z_0(G)$ of the center is locally connected. This condition is not necessary.*

Proof. Assume that $Z_0(G)$ is locally connected. Then $Z_0(G)\widehat{\ }$ is $\aleph_1$-free by Theorem 8.36 on the characterisation of local connectivity of (locally) compact abelian groups. And the subgroup $(Z_0 \cap G')^\perp$ is likewise $\aleph_1$-free. Thus $G/G' \cong \big((Z_0 \cap G')^\perp\big)\widehat{\ }$ is locally connected by 8.36. Therefore G is locally connected by 9.66. Example 9.62 shows that there are compact connected locally connected metric groups in which $Z_0(G)$ is not locally connected. □

Connectivity properties are much simpler in metric compact groups.

Connectivity in Compact Metric Groups

Theorem 9.68. (i) *Let G be a compact group and assume that the locally connected component G_ℓ (see Definition 9.64 and Theorem 9.65) is metric. This is the case, in particular, if G is metric. Then G_ℓ is a semidirect product of the metric semisimple group $(G_0)'$ and a metric torus, and there is a closed connected abelian torus free subgroup H such that $G_0 = G_\ell H \cong G_\ell \rtimes H$.*

(ii) *The following statements are equivalent for a compact metric group:*

(1) *$G = G_\ell$.*
(2) *G/G' is a torus.*
(3) *G is arcwise connected.*
(4) *G is connected and locally connected.*
(5) *$\exp_G\colon \mathfrak{L}(G) \to G$ is surjective.*

Proof. (i) Apply the Borel–Scheerer–Hofmann Splitting Theorem 9.38 and write $G_0 = (G_0)'A \cong (G_0)' \rtimes_\iota A$ with a closed connected abelian subgroup $A \cong G_0/(G_0)'$. By Theorem 9.61 we have $G_\ell = (G_0)'A_\ell$ and by Theorem 8.46 we have $A = A_\ell H \cong A_\ell \times H$ with a torus free compact connected abelian group H. Notice that $G_\ell \cap H = \{1\}$ and $G_\ell H = G_0$. Since G_ℓ is characteristic, hence normal, the function $(g, h) \mapsto gh\colon G_\ell \rtimes_\iota H \to G_0$ is an isomorphism of compact groups.

(ii) The equivalence of (1) and (4) follows from 9.65. By Corollary 9.66 this is tantamount to saying that G/G' is locally connected. Since G now is metric, this is equivalent to (2) by 8.46. By 9.60(ii) we know that (2), (3) and (5) are equivalent. ☐

Proposition 9.69. *A compact connected group G is torus free if and only if it is abelian and the equivalent conditions of* 8.47 *are satisfied.*

Proof. Assume that G is torus free. Then G' is torus free. By 9.36(v) each maximal pro-torus of G' is a torus, hence is singleton. Then by the Maximal Pro-Torus Theorem 9.32 the group G' is singleton itself. ☐

Compact Groups and Indecomposable Continua

Definition 9.70. A *continuum* is a compact connected space. It is called *decomposable* if it is the union of two proper subcontinua; otherwise, it is *indecomposable*. ☐

Indecomposable Group Continua

Theorem 9.71. *For a nonsingleton compact group G the following statements are equivalent:*

(i) *G is a compact connected abelian group of dimension one not isomorphic to a circle.*

(ii) *The underlying space of G is an indecomposable continuum.*

Proof. (i)⇒(ii). Assume that G is a compact connected group with $\dim G = 1$ and that G is not isomorphic to a circle and suppose that $G = C_1 \cup C_2$ with two proper nonsingleton continua C_1 and C_2. We shall derive a contradiction. The relation $\dim G = 1$ is equivalent to $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = \operatorname{rank} \widehat{G} = 1$ (8.22ff.). Since G is connected, $\widehat{G}$ is torsion-free (8.5) and thus the morphism $\chi \mapsto 1 \otimes \chi : \widehat{G} \to \mathbb{Q} \otimes \widehat{G}$ maps $\widehat{G}$ injectively. We may therefore assume that $\widehat{G}$ is a subgroup of $\mathbb{Q}$. It is no loss of generality to assume that $1 \in \widehat{G}$. Since G is not isomorphic to $\mathbb{T}$, the character group $\widehat{G}$ is not isomorphic to $\mathbb{Z} \supseteq \widehat{G}$. Let $\widehat{G} = \{g_1 = 1, g_2, \ldots\}$ be an enumeration of the countable set $\widehat{G}$ and choose recursively a sequence of natural numbers $n_1 < n_2 < n_3 \cdots$ such that $n_1 = 1$ and that, firstly $n_j | n_{j+1}$ for $j = 1, \ldots,$ and, secondly, $g_j | n_j$. Then

$$\bigcup_{j=1}^{\infty} \frac{1}{n_j}\mathbb{Z} = \widehat{G}.$$

Dually this means that

$$\lim(\mathbb{T} \xleftarrow{\mu_{n_1}} \mathbb{T} \xleftarrow{\mu_{n_2}} \mathbb{T} \xleftarrow{\mu_{n_3}} \cdots) = G,$$

where $\mu_n(x + \mathbb{Z}) = nx + \mathbb{Z}$. Let $f_m : L \to \mathbb{T}$ denote the limit projection onto the n-th component. Since the limit projections separate the points and C_1 and C_2 are proper subsets, there is an m such that $C_j' \stackrel{\text{def}}{=} f_m(C_j) \neq \mathbb{T}$ for both $j = 1$ and 2.

Consider the limit map $f_{m+1}: G \to \mathbb{T}$. Then $f_m = \mu_{n_{m+1}} \circ f_{m+1} = n_{m+1} \cdot f_{m+1}$. Since both C_j are continua, both continua C''_j are closed intervals of the circle. One of them must have length at least $\frac{1}{2}$, say C''_1. Then $C'_1 = n_{m+1} \cdot C''_1$ has length at least 1 since $n_{m+1} \geq 2$. Thus $C'_1 = \mathbb{T}$ contradicting the choice of m by which both C'_j were proper subcontinua. This contradiction completes this part of the proof.

(ii)⇒(i) We observe that the circle is decomposable. Therefore we must prove that $\dim G > 1$ implies the decomposability of G. A nonsingleton totally disconnected compact space is decomposable. Thus, if $G \neq G_0$, then G/G_0 is decomposable and therefore G is decomposable. So we assume G to be connected.

(a) We preface the proof by the following remark. Assume that N is a compact and *connected* normal subgroup N of G and that $p: G \to G/N$ is the quotient morphism. If $X \subseteq G/N$ is connected, then $p^{-1}(X)$ is connected. Thus, if G/N is decomposable, then G is decomposable. We assume henceforth that G is indecomposable and $\dim G > 1$ and aim for a contradiction.

(b) We apply this with $N = Z_0(G)$. Thus $S \stackrel{\text{def}}{=} G/Z_0(G) \cong G'/(Z_0(G) \cap G')$ is semisimple and indecomposable. We claim that S is singleton. Suppose this is not true. Let $\pi: \prod_{j\in J} S_j \to S$ be a surjective morphism as guaranteed by 9.19. Since S is not singleton, there is a $k \in J$ and we can form $P \stackrel{\text{def}}{=} \prod_{j\in J\setminus\{k\}} S_j$. Then P is connected and so $M \stackrel{\text{def}}{=} \pi(P)$ is a connected normal subgroup of S, and thus by (a) the quotient S/M is indecomposable. Let $p: S \to S/M$ be the quotient morphism. Define $\iota: S_k \to \prod_{j\in J} S_j$ by $\iota(s) = (s_j)_{j\in J}$ with $s_j = \begin{cases} s & \text{if } j = k, \\ 1 & \text{otherwise.} \end{cases}$. Then $p \circ \pi \circ \iota: S_k \to S/M$ is surjective. Then its kernel is a proper subgroup, and by the simplicity of S_j it has to be finite. Thus $S_k \to S/M$ is a covering morphism. Hence S/M is a simple compact Lie group. Then $\dim S/M = \dim S_k > 2$ by 6.53. So S/M is an at least 3-dimensional compact manifold and thus cannot be indecomposable (e.g. since it contains a euclidean ball whose complement is connected). This contradiction shows that $S = G/Z_0(G)$ is singleton, i.e. that G is abelian.

(c) Assume now that G is abelian. Since G is connected, $\widehat{G}$ is torsion-free. Since $\dim G > 1$ we have $\operatorname{rank} \widehat{G} \geq 2$. Hence we find a pure subgroup P of rank 2. (Indeed let F be a free subgroup of maximal rank, write it in the form $F_1 \oplus F_2$ with $\operatorname{rank} F_1 = 2$ and set $P = \{\chi \in \widehat{G} : (\exists n \in \mathbb{N})\, n\cdot\chi \in F_1\}$.) Let $C = P^\perp$, the annihilator of P in G. Then $\widehat{C} \cong \widehat{G}/P$ and $\widehat{G/C} \cong P$. Since P is pure, $\widehat{G}/P$ and thus $\widehat{C}$ is torsion-free. Hence C is connected. In view of Part (a) of the proof G/C is indecomposable Furthermore, $\operatorname{rank} P = 2$ implies $\dim G/C = 2$. In order to complete the proof it therefore suffices to prove that a 2-dimensional compact connected abelian group is decomposable. As $\dim G = 2$, by Theorem 8.49, the group G is metric. By 8.46(i,ii), G is torus free since G is clearly decomposable if G contains a circle group as a factor.

Using Theorem 8.22(7), we obtain a compact zero-dimensional subgroup Δ and a closed ε-ball neighborhood B of the origin in $\mathfrak{L}(G) \cong \mathbb{R}^2$ such that the covering morphism $\varphi: \Delta \times \mathbb{R}^2 \to G$, $\varphi(\delta, X) = \delta \exp X$, maps $\Delta \times B$ homeomorphically onto an identity neighborhood W of G. We set $D = \exp^{-1}\Delta \subseteq \mathfrak{L}(G)$ and note that $\exp X = \varphi(\delta, Y)$ with $\delta \in \Delta$ means $\delta = \exp(X - Y)$ and thus $X = Y + d$ with $d \in D$. Thus $\exp\colon \mathfrak{L}(G) \to G$ maps $D + B$ bijectively and continuously onto $W \cap \exp \mathfrak{L}(G)$. Let $U = \operatorname{int} B$ denote the manifold interior of B. By 8.20, the closed subgroup D of $\mathfrak{L}(G) \cong \mathbb{R}^2$ is a discrete lattice and therefore is countable. Thus $D + U$ is a countable disjoint union of open disks in the plane. Therefore its complement $E \stackrel{\text{def}}{=} \mathfrak{L}(G) \setminus \{D + U\}$ is connected. By 7.71, the set $\exp L(G)$ is dense in G and the complement $\exp L(G) \setminus W = \exp E$ is dense in the complement $A \stackrel{\text{def}}{=} G \setminus \varphi(\Delta \times U)$. Note that A is a compact subset of G since φ is an open map and that $\Delta A = A$. Also, A, being the closure of a connected set, is connected. Every arc component of W is of the form $\delta \exp B$, $\delta \in \Delta$, and its intersection with A is $\delta \exp \partial B$

Let K_1 be a proper compact open subgroup of Δ and set $K_2 = \Delta \setminus K_1$. Set $C_1 = A \cup \varphi(K_1 \times B)$ and $C_2 = A \cup \varphi(K_2 \times B)$. Since each arc component of the compact set $\varphi(K_j \times B_j)$, $j = 1, 2$ intersects the continuum A, each of the sets C_1 and C_2 is a continuum. Therefore, since C_1, C_2 are proper subcontinua of G and $G = C_1 \cup C_2$, the space G is decomposable as claimed. □

The indecomposable compact groups described in Theorem 9.71 are traditionally called *solenoids* or, historically more precisely the *solenoids of Vietoris and Van Dantzig*. For compact metric abelian groups, Theorem 9.71 is due to Van Heemert [359].

Part 4: Some Homological Algebra for Compact Groups

The Projective Cover of Connected Compact Groups

We observed in discussing the Theorem on Simple Connectivity in Compact Groups 9.27 that there are no simply connected compact abelian groups let alone the notion of a universal covering for connected compact abelian groups or even a "covering" of the kind we found in 9.19 for semisimple connected compact groups. However, we have already discussed in Chapter 8 a substitute for abelian compact groups, namely, the projective cover. This section is devoted to extending that concept. A natural setting for this is category theoretical. This strategy allows us to extend some homological algebra from the abelian situation to the general one, at least in the case of compact connected groups.

First we shall define the appropriate category to work in.

Lemma 9.72. *Let G, H, and K be compact connected groups and $\psi\colon G \to H$, and $\varphi\colon H \to K$ morphisms such that* $\operatorname{im}\psi$ *is normal in H and* $\operatorname{im}\varphi$ *is normal in K. Then* $\operatorname{im}(\varphi \circ \psi)$ *is normal in K.*

Proof. By 9.26 we get a commutative diagram

$$\begin{array}{ccccc} Z_0(G)\times\widetilde{G'} & \xrightarrow{\psi|Z_0(G)\times\widetilde{f|G'}} & Z_0(H)\times\widetilde{H'} & \xrightarrow{\varphi|Z_0(H)\times\widetilde{f|H'}} & Z_0(K)\times\widetilde{K'} \\ {\scriptstyle\mu_G}\Big\downarrow & & {\scriptstyle\mu_H}\Big\downarrow & & \Big\downarrow{\scriptstyle\mu_K} \\ G & \xrightarrow[\psi]{} & H & \xrightarrow[\varphi]{} & K. \end{array}$$

We note $\mu_H^{-1}\big(\psi(G')\big) = (Z_0(H)\cap H)\times\widetilde{\psi}(\widetilde{G'})$. Since the group $\psi(G') = \psi(G)'$ is characteristic in $\psi(G)$ it is normal in H. Then $\mu_H^{-1}\big(\psi(G')\big)$ is normal in $Z_0(H)\times\widetilde{H'}$. Then the identity component $\{1\}\times\widetilde{\psi}(\widetilde{H'})$ is normal, too and thus $\widetilde{\psi}(\widetilde{G'})$ is normal in $\widetilde{H'}$. By 9.19, $\widetilde{H'}$ is a direct product of simple simply connected groups. Then by 9.50, $\widetilde{\psi}(\widetilde{G'})$ is a partial product. Similarly, $\widetilde{\varphi}(\widetilde{H'})$ is a partial product in $\widetilde{K'}$. Since the group $\widetilde{\varphi}\widetilde{\psi}(\widetilde{G'})$ is normal in $\widetilde{\varphi}(\widetilde{H'})$ by 9.50, it is a partial product in $\widetilde{\varphi}(\widetilde{H'})$ and thus is a partial product in $\widetilde{K'}$. Hence it is normal in $\widetilde{K'}$. Thus $\varphi\psi(G') = \mu_G\big(\widetilde{\varphi}\widetilde{\psi}(\widetilde{G'})\big)$ is normal in K'. Since $K = Z_0(K)K'$ by 9.24, the group $\varphi\psi(G')$ is normal. Since $G = Z_0(G)G'$ and $\varphi\psi\big(Z_0(G)\big)\subseteq Z_0(K)$, the group $\varphi\psi(G)$ is normal in K. □

The class of connected compact groups together with the class of morphisms of compact connected groups with normal image form a category which we call the *category of compact connected groups and normal morphisms* and denote it by $\mathbb{CN}$. We note that the category $\mathbb{CN}$ contains the category of all compact connected abelian groups and their morphisms as a full subcategory.

In Appendix 3, A3.7 we explain the concepts of an isomorphism and of a retraction in a category, in A3.9 that of a monomorphism, and in A3.11 that of an epimorphism.

Proposition 9.73. *Assume that $f\colon A\to B$ is a morphism in $\mathbb{CN}$.*

(i) *f is a monomorphism in $\mathbb{CN}$ if and only if* $\ker f$ *is totally disconnected.*

(ii) *f is an epimorphism if and only if f is surjective.*

(iii) *f is a retraction with totally disconnected kernel if and only if it is an isomorphism.*

Proof. (i) Exercise. [The proof given in A3.10 3) for the category of connected Hausdorff topological groups works.]

(ii) If $e\colon A\to B$ is an epimorphism in $\mathbb{CN}$, then $\alpha\colon B\to B/e(A)$ is a morphism in $\mathbb{CN}$ since the compact subgroup $e(A)$ of B is normal. If $\beta\colon B\to B/e(A)$ is the constant morphism, then $\alpha e = \beta e$. Since e is an epimorphism, $\alpha = \beta$ follows, and this implies that $B/e(A)$ is singleton; i.e. $B = e(A)$. Thus e is surjective. The converse is always true: A surjective morphism is epic (see A3.12).

(iii) By (i) a morphism in $\mathbb{CN}$ is a monic iff its kernel is totally disconnected. Now in any category a monic retraction is an isomorphism. Indeed, let ρ be a retraction. Then there is a σ such that $\rho\sigma = \mathrm{id}$. Then $\rho\,\mathrm{id} = \rho = \rho\sigma\rho$, and since ρ is monic we conclude $\mathrm{id} = \sigma\rho$. Hence σ and ρ are inverses of each other and ρ is an isomorphism. □

We recall (e.g. from Appendix 1, A1.13 and the subsequent remark) that an object P in a category is a *projective* if for every epimorphism $e: A \to B$ and any morphism $p: P \to B$ there is a morphism $f: P \to A$ such that $p = e \circ f$.

We also recall from 8.80 that for a compact abelian group A we set $\mathfrak{P}(A) = (\mathbb{Q} \otimes \widehat{A})\widehat{\ }$. The adjoint of $\chi \mapsto 1 \otimes \chi: \widehat{A} \to \mathbb{Q} \otimes \widehat{A}$ is called $E_A: \mathfrak{P}(A) \to A$. The kernel $\Delta(A) \subseteq \mathfrak{P}(A)$ of E_G is the dual of $(\mathbb{Q} \otimes \widehat{G})/(1 \otimes \widehat{G})$.

Proposition 9.74. (i) *A compact connected group P is projective in* $\mathbb{CN}$ *if and only if $P \cong C \times \prod_{j\in J} S_j$ for a compact abelian group C such that $\widehat{C}$ is torsion-free divisible and all S_j are simple simply connected compact Lie groups.*

(ii) *A closed connected normal subgroup of a projective in* $\mathbb{CN}$ *is a direct factor.*

(iii) *A closed connected normal subgroup of a projective in* $\mathbb{CN}$ *is a projective.*

Proof. (i) First assume that $e: A \to B$ is surjective in $\mathbb{CN}$, and assume that $P \cong C \times \prod_{j\in J} S_j$ with $\widehat{C}$ torsion-free and divisible, and let $p: P \to B$ be a morphism. We want to find a morphism $f: P \to A$ such that $ef = p$. We consider the diagram

$$\begin{array}{ccc} A^* & \xrightarrow{e^*} & B^* \\ {\scriptstyle\mu_A}\big\downarrow & & \big\downarrow{\scriptstyle\mu_B} \\ A & \xrightarrow[e]{} & B. \end{array}$$

By hypothesis we know that $\mu_P: P^* \to P$ is an isomorphism. Assume momentarily that we find an $F: P^* \to A^*$ such that $e^*F = p^*$ then we are done, because $f = \mu_A \circ F \circ \mu_P^{-1}$ satisfies $ef = e\mu_A F\mu_P^{-1} = \mu_B e^* F\mu_P^{-1} = \mu_B p^* \mu_P^{-1} = p$. It is therefore no loss of generality if we assume henceforth that $A = A^*$, $B = B^*$, and $P^* = P$. Our problem, due to the structure of the groups A^* etc. (see 9.25) then falls into two separate cases, one abelian, one semisimple and simply connected. In the category of compact abelian groups, by 8.78, the group C is projective and thus the desired morphism exists in the abelian case. Finally assume that A, B, and P are compact, semisimple and simply connected. But then the surjective morphism $e: A \to B$ splits, since by 9.49 the kernel of e is a partial product of $A = \prod_{j\in J} S_j$. In other words, there is a morphism $\sigma: B \to A$ such that $e\sigma = \mathrm{id}_B$. Then $f = \sigma p$ satisfies $ef = e\sigma p = p$. This completes the proof of the first part.

Secondly, assume that P is projective in $\mathbb{CN}$. The quotient map $\mu_G: P^* \to P$ is surjective and hence an epic. Thus there is a morphism $f: P \to P^*$ such that $f\mu_P = \mathrm{id}_P$. Hence $P^* = (\ker \mu_G)f(P)$ and the product is semidirect. Since P^* is connected, $\ker \mu_P$ is singleton. Thus μ_P is bijective and thus is an isomorphism (due to compactness of P^*) (cf. also 9.73(iii)). Hence $P \cong Z_0(P) \times \prod_{j\in J} S_j$ with simple simply connected compact Lie groups S_j. Now $Z_0(G)$ is a retract of P in $\mathbb{CN}$, i.e. the codomain of a retraction in $\mathbb{CN}$ (see A3.7(b)). Now a retract of a projective is always projective in any category (Exercise E9.13 below). Thus the compact connected abelian group $Z_0(P)$ is projective in the category $\mathbb{CN}$. From 8.80 we have a surjective, hence epic morphism $E_G: \mathfrak{P}\big(Z_0(G)\big) \to Z_0(G)$ in $\mathbb{CN}$. Then E_G is a retraction (see Exercise E9.13 below). Then E_G is an isomorphism by 9.73(iii). Hence $Z_0(G)$ is of the form we asserted. This concludes the proof of (i).

(ii) Let G be a closed connected normal subgroup of $C \times \prod_{j\in J} S_j$ where $\widehat{C}$ is torsion-free and all S_j are compact simply connected simple Lie groups. By the Classification Theorem of Connected Normal Subgroups 9.50 there is a connected subgroup C_G of C and a subset $I \subseteq J$ such that $G = C_G \times \prod_{j\in I} S_j$. Now $\widehat{C}_G$ is divisible since C_G is a quotient of $\widehat{C}$ by the Annihilator Mechanism (see 7.64). Then it is a direct summand of $\widehat{C}$ (see A1.36). Hence C_G is a direct factor of C and assertion (ii) follows

(iii) By (ii), a compact connected subgroup of a projective is a direct factor and hence is a homomorphic retract. In Exercise E9.13 below we see that retracts of projectives are projective. Alternatively, the explicit form of a connected closed normal subgroup of a projective we derived in the proof of (ii) shows by (i) that the subgroup is projective. □

One may rephrase part (iii) of 9.74 by saying that $\mathbb{CN}$-*subobjects of projectives in* $\mathbb{CN}$ *are projectives.* Instead of referring to a "projective G in the category $\mathbb{CN}$" we shall also say that G is a *compact connected projective group.*

Exercise E9.13. Let $\rho: P \to R$ be an epic in a category and assume that P is projective. Then ρ is a retraction if and only if R is projective.

[Hint. First, assume that ρ is a retraction. This means that we have morphisms $\pi: P \to R$ and $\sigma: R \to P$ such that $\pi\sigma = \mathrm{id}_R$. Given an epic $e: A \to B$ and a morphism $p: R \to B$, using the projectivity of P get an $F: P \to A$ such that $eF = p\pi$. Now check that $f = F\sigma: R \to A$ satisfies $ef = p$.

Secondly, assume that R is projective. Apply the definition of projectivity of R to the epic $\rho: P \to R$ and the morphism $\mathrm{id}_R: R \to R$.] □

We now extend the definition of $\mathfrak{P}(\cdot)$ from the category of compact connected abelian groups to the category $\mathbb{CN}$. We also recall for a connected compact semisimple group G the unique morphism $\pi_G: \widetilde{G} \to G$ of 9.19.

Definition 9.75. For a compact connected group G we define

$$\mathfrak{P}(G) = \mathfrak{P}\big(Z_0(G)\big) \times \widetilde{G'},$$
$$E_G: \mathfrak{P}(G) \to G, \quad E_G(z, g) = E_{Z_0(G)}(z)\pi_{G'}(g).$$

We call $\mathfrak{P}(G)$ the *projective cover* of G and E_G the projective covering morphism. For $\mathfrak{s} \in \mathcal{S}$ according to the terminology of Theorem 9.14(vi) we shall now also write $\mathfrak{P}_\mathfrak{s}(G)$ in place of $(\widetilde{G'})_\mathfrak{s}$ and set $G_\mathfrak{s} \stackrel{\text{def}}{=} E_G\big(\mathfrak{P}_\mathfrak{s}(G)\big)$. The fully characteristic subgroups $\mathfrak{P}_\mathfrak{s}(G)$ of $\mathfrak{P}(G)$ and $G_\mathfrak{s}$ of G are called the *isotypic components of type* $\mathfrak{s}$ of $\mathfrak{P}(G)$ and G, respectively. □

The definition of the isotypic components extends our definition after Theorem 9.19 for the semisimple case.

By 9.74, the group $\mathfrak{P}(G)$ is indeed projective. This definition extends the respective definitions of 8.80 on the subcategory of compact connected abelian groups.

Theorem 9.76 (The Projective Cover). *Assume that G is a compact connected group.*

(i) *With the isotypic components $\mathfrak{P}_\mathfrak{s}(G) \cong \mathfrak{P}(G_\mathfrak{s})$ of $\mathfrak{P}(G)$ of type $\mathfrak{s}$ the following equation holds:*

$$\mathfrak{P}(G) \cong \mathfrak{P}\big(Z_0(G)\big) \times \prod_{\mathfrak{s}\in\mathcal{S}} \mathfrak{P}_\mathfrak{s}(G) \cong \widehat{\mathbb{Q}}^{\dim Z_0(G)} \times \prod_{\mathfrak{s}\in\mathcal{S}} \mathfrak{P}_\mathfrak{s}(G).$$

(ii) *The morphism E_G is surjective and the kernel* $\ker E_G = \{(z,g) \in \mathfrak{P}(G) \mid E_{Z_0(G)}(z)^{-1} = \pi_{G'}(g) \in Z_0(G) \cap G'\}$ *is isomorphic to a closed subgroup of*

$$\Delta\big(Z_0(G)\big) \times Z(\widetilde{G'})$$

and thus is totally disconnected.

(iii) $\mathfrak{L}(E_G)\colon \mathfrak{L}\big(\mathfrak{P}(G)\big) \to \mathfrak{L}(G)$ *is an isomorphism of weakly complete vector spaces.*

The morphism $E_G\colon \mathfrak{P}(G) \to G$ is an isomorphism iff G is projective in $\mathbb{CN}$.

(iv) *For a morphism $f\colon G \to H$ in $\mathbb{CN}$ we have one and only one morphism $\mathfrak{P}(f)\colon \mathfrak{P}(G) \to \mathfrak{P}(H)$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{P}(G) & \xrightarrow{\mathfrak{P}(f)} & \mathfrak{P}(H) \\ {\scriptstyle E_G}\big\downarrow & & \big\downarrow{\scriptstyle E_H} \\ G & \xrightarrow[f]{} & H \end{array}$$

is commutative.

(v) *The kernel of $\mathfrak{P}(f)$ is always connected. Further, $\mathfrak{P}(f)$ is injective if and only if* $\ker f$ *is totally disconnected, $\mathfrak{P}(f)$ is surjective if and only if f is surjective. In particular, $\mathfrak{P}(f)$ is an isomorphism if and only if* $\ker f$ *is totally disconnected and f is surjective.*

(vi) $w(\mathfrak{P}(G)) = w(G)$.

(vii) $\mathfrak{P}(G)$ *and G are dyadic in the sense of* Definition A4.30.

Proof. (i) is a consequence of Theorem 9.19(vi).

(ii) The image of E_G is $Z_0(G)G' = G$ by 9.24, and thus E_G is surjective. The kernel $\Delta(G)$ of $E_{Z_0(G)}$ is dual to the torsion group $(\mathbb{Q} \otimes Z_0(G)\widehat{\ })/(1 \otimes Z_0(G)\widehat{\ })$ (see 8.80). Since E_G is the composition

$$\mathfrak{P}\big(Z_0(G)\big) \times \widetilde{G'} \xrightarrow{E_{Z_0(G)} \times \pi_{G'}} Z_0(G) \times G' \xrightarrow{(z,g)\mapsto zg} G,$$

the kernel of E_G is $E_{Z_0(G)}^{-1}(Z_0(G) \cap G') = \pi_{G'}^{-1}(Z_0(G) \cap G')$ is contained in $\Delta(G) \times Z(\widetilde{G'})$.

(iii) The fact that $\mathfrak{L}(E_G)$ is an isomorphism follows from 9.48.

If E_G is an isomorphism then G is projective, trivially. Conversely, assume that G is projective. Then $Z_0(G) \cap G' = \{1\}$, and thus by 9.76(ii) for $(z,g) \in \ker E_G$, we have $E_{Z_0(G)}(z) = 1 = \pi_{G'}(g)$. By 9.74, $Z_0(G)$ is a projective abelian group and thus $E_{Z_0(G)}$ is an isomorphism. Similarly, G' is projective and thus $\pi_{G'}$ is an isomorphism. It follows that $z = g = 1$. Hence $\ker E_G = \{1\}$ and thus E_G is injective. From 9.76(ii) we conclude that E_G is an isomorphism. (iv) We define $\mathfrak{P}(f) = \mathfrak{P}(f|Z_0(G)) \times \widetilde{f|G'}: \mathfrak{P}\big(Z_0(G)\big) \times \widetilde{G'} \to \mathfrak{P}\big(Z_0(H)\big) \times \widetilde{H'}$. Then the commutativity of the diagram is immediate from the definitions. If $\varphi: \mathfrak{P}(G) \to \mathfrak{P}(H)$ is a morphism such that $E_H\varphi = fE_G = E_H\mathfrak{P}(f)$, then $g \mapsto \varphi(g)^{-1}\mathfrak{P}(f)(g): \mathfrak{P}(G) \to \ker EH$ is a well-defined continuous function from a connected space into a totally disconnected one and is therefore constant. Since for $g = 1$ it takes the value 1 we conclude $\varphi = \mathfrak{P}(f)$. Thus the diagram determines $\mathfrak{P}(f)$ uniquely.

(v) By the definition of $\mathfrak{P}(f)$ in the proof of (iv) above, $\ker \mathfrak{P}(f)$ is connected iff $\ker \mathfrak{P}\big(f|Z_0(G)\big)$ and $\ker \widetilde{f|G'}$ are both connected, i.e. if the assertion holds for abelian and for semisimple groups. If G and H are abelian, then $\mathfrak{P}(f): \mathfrak{P}(G) \to \mathfrak{P}(H)$ is the dual of $\mathbb{Q} \otimes \widehat{f}: \mathbb{Q} \otimes \widehat{H} \to \mathbb{Q} \otimes \widehat{G}$, and the character group of $\ker \mathfrak{P}(f)$ is isomorphic to $\operatorname{coker} \mathbb{Q} \otimes f$ (cf. 7.65). Since $\mathbb{Q} \otimes f$ is a $\mathbb{Q}$-linear map between $\mathbb{Q}$-vector spaces, this cokernel is a $\mathbb{Q}$-vector space and thus is, in particular, torsion-free. Hence $\ker \mathfrak{P}(f)$ is connected. Now assume that G and H are semisimple. Since $\mathfrak{P}(f)$ preserves isotypic components of type $\mathfrak{s} \in \mathcal{S}$, we may assume that for some simply connected compact Lie group S we have $\mathfrak{P}(f): S^X \to S^Y$ with a normal image. If $\mathrm{pr}_y: S^Y \to S$ denotes the y-th projection we have $\ker \mathfrak{P}(f) = \bigcap_{y \in Y} \ker(\mathrm{pr}_y \circ \mathfrak{P}(f))$. By Lemma 11.28 there is a function $\xi: Y \to X$ such that $\ker(\mathrm{pr}_y \circ \mathfrak{P}(f)) = S^{X \setminus \{\xi(y)\}}$ (identified with a partial product of S^X). Consequently, $\ker \mathfrak{P}(f) = \bigcap_{y \in Y} S^{X \setminus \{\xi(y)\}} = S^{X \setminus \{\xi(Y)\}}$, and this product is connected. This completes the proof of the assertion that $\mathfrak{P}(f)$ is always connected.

Next we investigate the injectivity of $\mathfrak{P}(f)$. By replacing f by the corestriction to its image we may assume that f is surjective. Then $\ker f$ is totally disconnected iff $\mathfrak{L}(f)$ is an isomorphism (by 9.48) iff $\mathfrak{L}\big(\mathfrak{P}(f)\big)$ is an isomorphism (by (iii) and (iv)) iff $\ker \big(\mathfrak{P}(f)\big)$ is totally disconnected (by 9.48 again); since $\ker \mathfrak{P}(f)$ is always connected, this is tantamount to the injectivity of $\mathfrak{P}(f)$.

Finally, f is surjective iff $f \circ E_G$ is surjective (by the surjectivity of E_G) iff $E_H \circ \mathfrak{P}(f)$ is surjective iff $\mathfrak{P}(H) = \operatorname{im}\big(\mathfrak{P}(f)\big) \ker E_H$. If $\mathfrak{P}(f)$ is surjective this is satisfied; conversely, if this relation holds, then $\operatorname{coker}\big(\mathfrak{P}(f)\big) = \mathfrak{P}(H)/\operatorname{im}\big(\mathfrak{P}(f)\big)$ is totally disconnected as a quotient of $\ker E_H$. On the other hand, it is connected as a quotient of $\mathfrak{P}(H)$, which is connected. Hence this cokernel must be singleton, which shows $\mathfrak{P}(H) = \operatorname{im}\big(\mathfrak{P}(f)\big)$.

(iv) From 9.36(vi) we have $w(T) = w(G)$ for a maximal pro-torus. It suffices therefore to compare the weights of T^* and T where T^* is a maximal pro-torus of $\mathfrak{P}(G)$ of mapping onto T under E_G. So we recall that $T \cong T^*/D$ where D is a totally disconnected group (in fact $D = \ker E_G$ because this kernel is central and T^* contains the center of $\mathfrak{P}(G)$). Thus, by duality, $\widehat{T}$ is a subgroup of $\widehat{T^*}$ with a torsion group as cokernel and is therefore isomorphic to a pure subgroup of $\widehat{T^*}$. Then $w(T) = \operatorname{card} T = \operatorname{card}(\mathbb{Q} \otimes \widehat{T}) = \operatorname{card}(\mathbb{Q} \otimes \widehat{T^*}) = \operatorname{card} \widehat{T^*} = w(T^*)$.

(vii) By (i), $\mathfrak{P}(G)$ is a product of a family of connected compact metric spaces and thus is dyadic by Corollary A4.32. Since E_G in Definition 9.75 is a continuous surjective map, G is dyadic as well. □

The information on the projective cover of compact abelian groups in 8.80 and on the constructions of 9.19 and 9.25 are relevant for projective covers.

The projective cover has a functorial aspect which is exposed if we recall the concept of a *left adjoint functor* as it is discussed in Appendix 3, A3.29.

Theorem 9.76bis. (The Projective Cover: Functorial Aspect) *Let* $\mathbb{PCN}$ *denote the full subcategory of* $\mathbb{CN}$ *containing all projective objects. Then* $\mathfrak{P}: \mathbb{CN} \to \mathbb{PCN}$ *is left adjoint to the inclusion functor* $\mathbb{PCN} \to \mathbb{CN}$.

Proof. In order to apply (the dual of) Theorem A3.28 of Appendix 3, we verify the universal property: Let P be a projective. Then $E_P: \mathfrak{P}(P) \to P$ is an isomorphism by 9.76(iii). If $f: P \to H$ is a morphism in $\mathbb{CN}$, then we define $f': P \to \mathfrak{P}(H)$ by $f' = \mathfrak{P}(f) \circ E_P^{-1}$. Then $f = E_H \circ f'$. If $f'': P \to \mathfrak{P}(H)$ satisfies $E_H \circ f'' = f = E_H \circ f'$, then the continuous function $F: P \to \mathfrak{P}(H)$, $F(p) = f''(p)f'(p)^{-1}$, maps the connected space P into the kernel of E_G, which is totally disconnected by 9.76(ii); therefore F is constant. Because of $f''(1) = 1 = f'(1)$ the constant value of F is 1, and thus $f'' = f'$. Thus by A3.28, $\mathfrak{P}$ extends to a functor, $E_G: G \to \mathfrak{P}(G)$ is a natural transformation, namely, the back adjunction for the left adjoint $\mathfrak{P}$ for the inclusion functor. □

In the spirit of Proposition A3.33 ff. in Appendix 3 we note that the assignment $f \mapsto f' : \mathbb{CN}(|P|, H) \to \mathbb{PCN}\big(P, \mathfrak{P}(H)\big)$ is a natural bijection (where $|P|$ simply denotes P considered as an object of $\mathbb{CN}$).

Theorem 9.77. *Let G be a compact group and M a compact normal subgroup such that G/M is connected. Then there is a compact connected subgroup N, normal in G_0, such that $G = MN$ and $M \cap N$ is totally disconnected. Moreover,*

(i) *whenever there is an N with these properties then there is a surjective morphism of compact connected groups $M \times N \to G$ with totally disconnected kernel and a surjective homomorphism $p: G \to N/(M \cap N)$ such that the composition $N \stackrel{i}{\to} G \stackrel{p}{\to} N/(M \cap N)$ of the inclusion map $i: N \to G$ with p is the quotient map.*

(ii) *The natural morphism $n(M \cap N) \mapsto nM: N/(M \cap N) \to G/M$ is an isomorphism.*

(iii) *If $f: G \to H$ is a morphism of compact groups onto a connected group, then there is a closed connected normal subgroup N of G_0, such that $f|N: N \to H$ is a surjective morphism with a totally disconnected kernel.*

Proof. By 9.26(i), $G_0M/M = G/M$ and thus $G_0M = G$. Hence $G_0/(G_0 \cap M) = G/M$. We note that $M_0 \subseteq G_0 \cap M$ and that $(G_0 \cap M)/M_0$ is totally disconnected. Assume for the moment that the assertion of the theorem is true when M and

G are connected. Then, applying the theorem to G_0 and M_0 we find a connected normal subgroup N such that $G_0 = M_0N$ and that $M_0 \cap N$ is totally disconnected. Then $G = G_0M = MG_0 = MM_0N = MN$ and $(M \cap N)_0 \subseteq M_0 \cap N$, and thus $(M \cap N)_0$ is singleton, i.e. $M \cap N$ is totally disconnected. Thus it is no restriction of the generality if we assume that M and G are connected. This we shall do from now on.

The identity component of $M^\# \stackrel{\rm def}{=} E_G^{-1}(M)$ satisfies $E_G(M^\#) = M$ and is a direct factor of $\mathfrak{P}(G)$ by 9.74(ii). Hence there is a complementary factor $N^\#$ such that $\mathfrak{P}(G) = M^\# N^\#$ and that this product is direct. Set $N = E_G(N)$. Then $G = MN$. We claim that $M \cap N$ is totally disconnected. Indeed the identity component $K^\#$ of $E_G^{-1}(M \cap N)$ maps onto $(M \cap N)_0$ (cf. 9.18) on the one hand and is contained in $E_G^{-1}(M)_0 \cap E_G^{-1}(N)_0 = M^\# \cap N^\# = \{1\}$ on the other. Thus $M \cap N = \{1\}$ and the claim is established.

(i) and (ii) The assertion about the surjective morphism $M \times N \to G$ is the standard Mayer-Vietoris argument.

By the isomorphy theorem and the compactness of the groups involved, the bijective morphism $j: N/(M\cap N) \to G/M$, $j\big(n(M\cap N)\big) = nN$, is an isomorphism. Thus the quotient morphism $q: G \to G/M$ yields a surjective morphism $p \stackrel{\rm def}{=} j^{-1}q: G \to M/(M \cap N)$. If $m \in M$, then $p(m) = j^{-1}(mM) = m(M \cap N)$.

(iii) We set $M = \ker f$ and apply the preceding results. □

The concept of an injective in a category is opposite (or dual) to that of a projective. The model of injectivity is that in the category of abelian groups which we discussed in Appendix 1, A1.1.34. The category $\mathbb{CN}$ is not particularly compatible with the concept of injectives due to the somewhat scurrilous nature of its monomorphisms (see 9.73(i)). From Exercise E9.13 we know that a monic with an injective domain is a coretraction. A good candidate for an injective object is $\mathbb{T}$ which is injective in the category of compact abelian groups. However, the monic $\mu_2: \mathbb{T} \to \mathbb{T}$, $\mu_2(t + \mathbb{Z}) = 2t + \mathbb{Z}$ is not a coretraction. In many categories it is therefore of some interest, to determine injectives *relative to a given class of monomorphisms*. In $\mathbb{CN}$ we call an object I *injective with respect to the class of embeddings* (or, equivalently, injective monomorphisms) if for any compact connected group B and any connected compact normal subgroup A, each morphism $p: A \to I$ extends to a morphism $f: B \to I$. Usually, considering *relative* injectives increases the supply of injective objects. There are limits here, however.

Exercise E9.14. The group $\mathbb{T}$ is not even a relative injective in $\mathbb{CN}$ with respect to embeddings.

[Hint. The inclusion morphism $Z\big(U(2)\big) \to U(2)$ is not a coretraction.] □

Nevertheless, we find some genuine injectives in the category $\mathbb{CN}$.

Let $\mathbb{ICN}$ denote the full subcategory of injective objects on $\mathbb{CN}$, set $\mathfrak{I}(G) = G/Z(G)$, and define $q_G: G \to \mathfrak{I}(G)$ to be the quotient morphism.

Proposition 9.78. (i) *A compact connected group G is injective in $\mathbb{CN}$ if and only if it is centerfree.*

(ii) *The assignment $\mathfrak{I}$ on objects extends to a functor $\mathfrak{I}: \mathbb{CN} \to \mathbb{ICN}$ which is left adjoint to the inclusion functor.*

Proof. (i) Firstly, we show that a centerfree member of $\mathbb{CN}$ is injective. By 9.24 and 9.19, G is centerfree iff it is isomorphic to $\prod_{j\in J} S_j$ with connected simple and centerfree compact Lie groups S_j. The product of (relative) injectives is a (relative) injective in any category (Exercise E9.15 below). Hence it suffices to prove that a simple simply connected compact Lie group S is an injective in $\mathbb{CN}$. Let A be a connected normal subgroup of B and $p: A \to S$ a monomorphism. Then $p\big(Z(A)\big) = \{1\}$ since S is centerfree. Then $p': A/Z(A) \to S$, $p'\big(aZ(A)\big) = p(a)$ is a morphism. Now $e': A/Z(A) \to B/Z(A)$ is injective, hence monic, and if we find a morphism $f': B/Z(A) \to S$ with $f'e' = p'$ we are done because $f: B \to S$, $f(b) = f'\big(bZ(A)\big)$ is the one we want. Thus we assume from here on that A is centerfree. But then we claim that A is a direct factor. Indeed from 9.51 we conclude that there is a connected normal subgroup N such that $B = AN$ and $A \cap N$ is totally disconnected. But A is centerfree and $A\cap N$, being totally disconnected and normal in A, is central in A (see 6.13). Hence $A \cap N = \{1\}$ and thus the claim is proved. Now let $\pi: B \to A$ be the projection with kernel N and set $f = p\pi$.

Secondly we show that the centerfree members of $\mathbb{CN}$ are the only injectives. Let G be injective. This means that every monic $j: G \to H$ in $\mathbb{CN}$ is a coretraction. In particular, G, whenever it is a normal subgroup of a compact connected group H is a direct factor. Set $H = \frac{G\times G'}{D}$, $D = \{(z^{-1}, z) \mid z \in Z(G')\}$. Then G has the isomorphic copy $\frac{G\times Z(G')}{D}$ which is a direct factor. By 9.51, the unique normal cofactor is $\frac{Z(G')\times G'}{D}$. Hence $\{D\} = \frac{G\times Z(G')}{D} \cap \frac{Z(G')\times G'}{D}$ which means that $\frac{Z(G')\times \mathbb{Z}'(G)}{D}$ is singleton. This is equivalent to $Z(G') = \{1\}$; i.e. G' is centerfree. Then $G = Z_0(G) \times G'$ by 9.24. By (the dual of) Exercise E9.13, a coretract of an injective is an injective. Hence $Z_0(G)$ is injective in $\mathbb{CN}$. In particular, it is a direct summand whenever it is a closed subgroup of a compact connected abelian group. Hence its character group is a projective in the category of torsion-free abelian groups. But then it is free (the proof of A1.14, (2)⇒(1) works in the subcategory of torsion-free groups). Hence $Z_0(G)$ is a torus. If it were nonsingleton, it would contain $\mathbb{T}$ as a factor which then would have to be injective by (the dual of) Exercise E9.13. But this is not the case by Exercise E9.14. Thus $G = G'$ and G is centerfree since we saw G' to be centerfree.

(ii) Again we apply Theorem A3.28 of Appendix 3. Let I be an injective object of $\mathbb{CN}$. Then $I = \prod_{j\in J} S_j$ with a family of centerfree compact connected simple Lie groups S_j by the proof of (i) above, and 9.23 and 9.19(v)(f). Assume that $f: G \to I$ is a morphism in $\mathbb{CN}$. Then by the definition of $\mathbb{CN}$, the image $f(G)$ is a compact connected normal subgroup of I and thus, by Corollary 9.50, is a partial product of I and thus is itself centerfree. By Theorem 9.28, we have $f\big(Z(G)\big) = Z\big(f(I)\big) = \{1\}$. Hence $Z(G) \subseteq \ker f$ and thus f factors through the quotient $G/Z(G) = \mathfrak{I}(G)$, that is, there is a unique morphism $f': \mathfrak{I}(G) \to I$ such that $f = f' \circ q_G$. This is the universal property required by Theorem A3.28 of Appendix 3 to prove the proposition. □

The assignment $f \mapsto f' : \mathbb{CN}(G, |I|) \to \mathbb{ICN}(\mathfrak{I}(G), I)$ is a natural bijection. Trivially, a compact connected group G is injective in $\mathbb{CN}$ iff q_G is an isomorphism.

Exercise E9.15. The product of a family of (relative) injectives is a (relative) injective.

[Hint. Consider a family I_j, $j \in J$ of (relative) injectives and a monomorphism $e: A \to B$ (in the prescribed class of monomorphisms). If $p: A \to \prod_{j\in J} I_j$, by injectivity of I_j there are morphisms $f_j: B \to I_j$ such that $\mathrm{pr}_j\, p = f_j e$. The universal property of the product (A3.43) gives an $f: B \to \prod_{j\in J} I_j$ such that $f_j = \mathrm{pr}_j\, f$. Verify $fe = p$.] □

Corollary 9.79. *Let G be a compact connected group. Then $G/Z(G)$ is injective in $\mathbb{CN}$.*

Proof. By 9.24 and 9.19, $G/Z(G)$ is centerfree. The assertion then follows from 9.78. □

Exercise E9.16. *Let G be a compact connected group. Then there is a totally disconnected central subgroup D such that G/D is the direct product of a torus and a product of simple centerfree connected compact Lie groups. Specifically, $G/D \cong \mathbb{T}^{\dim Z_0(G)} \times \prod_{\mathfrak{s}\in\mathcal{S}} R_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)}$.*

[Hint. Apply Corollary 8.18 to the factor $\frac{Z_0(G)}{\Delta}$ in Theorem 9.24(iii).] □

Part 5: The Automorphism Group of Compact Groups

Lemma 9.80. *Let G be a compact Lie group.*

(i) *The set of closed connected normal subgroups of G satisfies the ascending chain condition.*

(ii) *If N is a closed normal subgroup of G and an automorphism α of G satisfies $\alpha(N) \subseteq N$ then $\alpha(N) = N$.*

Proof. (i) Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of closed connected normal subgroups. Then as the Lie algebra $\mathfrak{L}(G)$ is finite dimensional, there is an $i \in \mathbb{N}$ such that $j \geq i$ implies $\mathfrak{L}(N_j) = \mathfrak{L}(N_i)$ and thus $N_j = N_i$ as $N_i = \exp \mathfrak{L}(N_i)$ for all i as all N_i are connected.

(ii) Assume that $\alpha(N) \subseteq N$ for some closed normal subgroup N. Clearly $\alpha(N_0) \subseteq N_0$. Inductively, $N_0 \subseteq \alpha^{-1}(N_0) \subseteq \alpha^{-2}(N_0) \subseteq \cdots$. By (i) above there is an i such that $\alpha^{-i-1}(N_0) = \alpha^{-i}(N_0)$. Then $\alpha^{-1}(N_0) = N_0$, i.e. $\alpha(N_0) = N_0$. Thus $\alpha(N)_0 = \alpha(N_0) = N_0$. Now N/N_0 is finite and isomorphic to $\alpha(N)/\alpha(N)_0 = \alpha(N)/N_0 \subseteq N/N_0$. We conclude $\alpha(N)/N_0 = \alpha(N)/\alpha(N)_0 = N/N_0$ and thus $\alpha(N) = N$. □

Exercise E9.17. Prove the following assertions.

(i) *In a compact Lie group G, the set of* all *closed normal subgroups satisfies the ascending chain condition modulo $Z_0(G_0)$, i.e. for every chain $N_1 \subseteq N_2 \subseteq \cdots$ of closed normal subgroups there is a $j \in \mathbb{N}$ such that $k \geq j$ implies $N_j = N_k Z_0(G_0)$.*

(ii) *The circle group $G = \mathbb{R}/\mathbb{Z}$ has a properly ascending tower of finite subgroups $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subseteq \cdots \subseteq \frac{1}{2^n}\mathbb{Z}/\mathbb{Z} \subseteq \cdots$.*

[Hint. (i) If $\{N_i \mid i \in \mathbb{N}\}$ is an ascending chain of closed normal subgroups, then $(N_j)_0$ is characteristic in N_j; hence we know $(N_j)_0 \trianglelefteq G$. By 9.80(i) there is an i such that $(N_{i+k})_0 = (N_i)_0$. It suffices to show that the ascending chain $N_{i+1}/N_i = N_{i+1}/(N_{i+1})_0 \subseteq N_{i+2}/N_i = N_{i+2}/(N_{i+2})_0 \subseteq \cdots$ of closed normal subgroups becomes stationary modulo $Z_0(G_0)N_i/N_i = Z_0(G_0N_i/N_i) = Z_0\big((G/N_i)_0\big)$ (cf. 9.26(i,ii)). In order to simplify notation we now assume that all N_i are discrete normal subgroups and that $Z_0(G_0) = \{1\}$, i.e. that G_0 is semisimple. Now $N_i \cap G_0 \subseteq Z(G_0)$ by 6.13, and $Z(G_0)$ is finite as a discrete Lie group. Hence $N_i \cap G_0$ is finite with bounded order. But $N_i/(N_i \cap G_0) \cong N_iG_0/G_0 \subseteq G/G_0$ is finite with bounded order, too, since G/G_0 is finite. Hence all N_i are finite with bounded order, and thus the chain $N_1 \subseteq \cdots$ is stationary from some point on. The verification of (ii) is immediate.] ☐

Lemma 9.81. *Let G be a compact group and M a closed subgroup. We set*

$$N\big(M, \mathrm{Aut}(G)\big) \stackrel{\mathrm{def}}{=} \{\alpha \in \mathrm{Aut}(G) \mid \alpha(M) = M\}.$$

(i) *Then there is a morphism $\nu_M\colon N(M, \mathrm{Aut}(G)) \to \mathrm{Aut}(G/M)$ given by $\nu_M(\alpha)(gM) = \alpha(g)M$.*

(ii) *If M is normal and G/M is a Lie group, then $\{\alpha \in \mathrm{Aut}(G) \mid \alpha(M) \subseteq M\} = N\big(M, \mathrm{Aut}(G)\big)$ and $N\big(M, \mathrm{Aut}(G)\big)$ is an open* (*and hence closed*) *subgroup of* $\mathrm{Aut}(G)$.

(iii) *Let $\mathcal{N}(G)$ denote the filter basis of all compact normal subgroups M of G such that G/M is a Lie group as in* 9.1. *Then*

$$\mathrm{Aut}(G)_0 \subseteq \bigcap_{M \in \mathcal{N}(G)} N\big(M, \mathrm{Aut}(G)\big).$$

Proof. The proof of (i) is Exercise E9.18 below.

(ii) We note from 9.80(ii) that $\alpha \in \mathrm{Aut}(G)$ and $\alpha(M) \subseteq M$ imply $\alpha(M) = M$. Hence $\{\alpha \in \mathrm{Aut}(G) \mid \alpha(M) \subseteq M\} = N\big(M, \mathrm{Aut}(G)\big)$. Clearly, $N\big(M, \mathrm{Aut}(G)\big)$ is a subgroup of $\mathrm{Aut}(G)$. In order to show that it is open in $\mathrm{Aut}(G)$ it suffices to show that a neighborhood of 1 in $\mathrm{Aut}(G)$ is contained in $N\big(M, \mathrm{Aut}(G)\big)$. The identity of the Lie group G/M has a neighborhood in which $\{1\}$ is the only subgroup (cf. 2.41). Thus we find an open neighborhood U of 1 in G satisfying $UM = MU = U$ such that for any subgroup H of H the relation $H \subseteq U$ implies $H \subseteq M$. Now the set $\mathcal{V}$ of all $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(x) \in Ux$ for all $x \in G$ is a neighborhood of the identity in $\mathrm{Aut}(G)$ (cf. the paragraphs preceding and following 6.62). If $\alpha \in \mathcal{V}$ then $\alpha(M) \subseteq UM = U$ whence $\alpha(M) \subseteq M$ and thus $\mathcal{V} \subseteq N\big(M, \mathrm{Aut}(G)\big)$. This shows that $N\big(M, \mathrm{Aut}(G)\big)$ is open in $\mathrm{Aut}(G)$ and thus proves the assertion.

(iii) is a direct consequence of (ii) above. □

Exercise E9.18. Verify the details of 9.81(i).

[Hint. Check that ν_M is algebraically a well-defined homomorphism; verify that if α is uniformly close to id_G then $\nu_M(\alpha)$ is uniformly close to $\mathrm{id}_{G/M}$.] □

The Iwasawa Theory of Automorphism Groups

In Chapter 6 we discussed at length the structure of the automorphism group of a compact Lie group. We now generalize some of those results to the study of the automorphism group of arbitrary compact groups.

We are ready to generalize Iwasawa's Theorem 6.66 and consider again the morphism $\iota: G \to \mathrm{Aut}(G)$ $\iota(g)(x) = gxg^{-1}$. Then the kernel of ι is $Z(G)$. We recall that $\mathrm{Inn}_0(G) = \iota(G_0)$; as the image $\iota(G_0)$ of the compact group G_0, this is a compact group.

Iwasawa's Automorphism Group Theorem for Compact Groups

Theorem 9.82. *Let G be a compact group. Then*

(i) $\mathrm{Aut}(G)_0 = \mathrm{Inn}_0(G)$*.* (Iwasawa [217])

(ii) $\mathrm{Inn}_0(G)$ *is a compact group isomorphic to* $G_0/\big(Z(G) \cap G_0\big)$*. If* $Z_0(G_0) \subseteq Z(G)$*, which is trivially the case if G is connected, then this group is a semisimple connected compact group. If G is connected then*

$$\mathrm{Aut}(G)_0 \cong \mathrm{Inn}(G) \cong G'/Z(G').$$

Proof. The proof of (ii) is simple and is in fact almost verbatim the same as that for 6.66(ii). We reproduce it only for easy reference: By our definitions, $\mathrm{Inn}_0(G) \cong G_0Z(G)/Z(G) \cong G_0/(Z(G) \cap G_0)$. Now $G_0 = Z_0(G_0)(G_0)'$ by 9.23. Assume $Z_0(G_0) \subseteq Z(G)$. Then $Z_0(G_0) \subseteq \ker\iota$ and thus $\mathrm{Inn}_0(G) = \iota(G_0) = \iota\big((G_0)'\big) = (G_0)'Z(G)/Z(G) \cong (G_0)'/\big((G_0)' \cap Z(G)\big)$, as asserted. By 9.26(ii) this connected compact group is its own commutator group and then is semisimple by 9.6. If G is connected, then $G = G_0$, $Z(G) \cap G' = Z(G')$ (since $Z(G')$ as a totally disconnected characteristic subgroup is central in the connected group G) and the notation simplifies. Thus part (ii) is proved.

Since $\mathrm{Inn}_0(G)$ is connected, for a proof of (i) we have to establish that $\mathrm{Aut}(G)_0 \subseteq \mathrm{Inn}_0(G)$.

By 9.81(iii), for each $M \in \mathcal{N}(G)$ we have $\mathrm{Aut}(G)_0 \subseteq N\big(M, \mathrm{Aut}(G)\big)$. We consider the morphism $\nu_M: N\big(M, \mathrm{Aut}(G)\big) \to \mathrm{Aut}(G/M)$ of 9.81(i) and note $\nu_M(\mathrm{Aut}(G)_0) \subseteq \mathrm{Aut}(G/M)_0 = \mathrm{Inn}_0(G/M)$ by the continuity of ν_M and by 6.66(i). Recall that $(G/M)_0 = G_0M/M$ by 9.18 and that $\mathrm{Inn}_0(G/M) = \iota_{G/M}(G_0M/M)$ Thus for each $\alpha \in \mathrm{Aut}(G)_0$ there is a $g_M \in G_0$ such that $\nu_M(\alpha) = \iota_{G/N}(g_MM)$, i.e. that $\alpha(x)M = \nu_M(\alpha)(xM) = \iota_{G/M}(g_MM)(xM) = (g_MM)(xM)(g_MM)^{-1} =$

$g_M x g_M^{-1} M = \iota_G(g_M)(x)M$ for all $x \in G$. Thus

$$(\forall \alpha \in \mathrm{Aut}(G)_0)(\forall M \in \mathcal{N}(G))(\exists g_M \in G_0)(\forall x \in G) \quad \alpha(x)^{-1}\iota(g_M)(x) \in M.$$

In the compact space G_0, the net $(g_M)_{M\in\mathcal{N}(G)}$ has a convergent subnet $(g_{M(j)})_{j\in J}$; say $g = \lim_{j\in J} g_{M(j)} \in G_0$. If $M \in \mathcal{N}(G)$ and $j \in J$ is such that $M(j) \subseteq M$, then

$$\alpha(x)^{-1}\iota(g_{M(j)})(x) \in M(j) \subseteq M.$$

Thus $\alpha(x)^{-1}\iota(g)(x) \in M$ for all $M \in \mathcal{N}(G)$. Now 9.1 implies $\alpha(x)^{-1}\iota(g)(x) = 1$ for all $x \in G$. Hence $\alpha = \iota(g)$. This shows $\mathrm{Aut}(G)_0 \subseteq \iota(G_0) = \mathrm{Inn}_0(G)$. □

Our next objective is to describe completely the automorphism group of a compact connected group which is projective in $\mathbb{CN}$.

First we introduce the topological group of permutations on a set X. Indeed let $P(X)$ denote the group of bijections $\sigma\colon X \to X$, and for each subset $Y \subset X$ write $P_Y(X) = \{\sigma \in P(X) \mid (\forall y \in Y)\, \sigma(y) = y\}$. Then $P_Y(X)$ is a subgroup. If $Y_1 \subseteq Y_2$ then $P_{Y_1}(X) \supseteq P_{Y_2}(X)$. If $\sigma \in P(X)$, then $\tau \in P_{\sigma(Y)}(X)$ iff $(\forall y \in Y)\tau\big(\sigma(y)\big) = \sigma(y)$ iff $(\forall y \in Y)\, (\sigma^{-1}\tau\sigma)(y) = y$ iff $\sigma^{-1}\tau\sigma \in P_Y(X)$ iff $\tau \in \sigma P_Y(X)\sigma^{-1}$. Hence

$$(1) \qquad P_{\sigma(Y)}(X) = \sigma P_Y(X)\sigma^{-1}.$$

Let $\mathrm{Fin}(X)$ denote the set of finite subsets of X; clearly $\mathrm{Fin}(X)$ is directed with respect to $\subseteq$. Now the filterbasis

$$\{P_F(X) \mid F \in \mathrm{Fin}(X)\}$$

is the basis of the filter of identity neighborhoods of a Hausdorff group topology on $P(X)$ (Exercise E9.19 below) and we shall henceforth assume that $P(X)$ is equipped with this topology. Every subgroup $P_F(X)$ is open and closed and thus $P(X)$ is totally disconnected; if X itself is finite, then $P(X)$ is discrete.

Exercise E9.19. Prove the following fact.

Let G be a group and $\mathcal{F}$ a filterbasis of subgroups with $\bigcap \mathcal{F} = \{1\}$ such that for each $M \in \mathcal{F}$ and each $g \in G$ there is an $N \in \mathcal{F}$ such that $gNg^{-1} \subseteq M$. Then there is a unique totally disconnected Hausdorff group topology on G such that $\mathcal{F}$ is a basis for the filter of identity neighborhoods. All $N \in \mathcal{F}$ are open and closed subgroups.

Let S be a simple connected compact Lie group. We define $G = S^X$. In determining the automorphism group of G we shall use general techniques which we introduced for dealing with the automorphism groups of isotypic semisimple Lie algebras around 6.58.

The group $\mathrm{Aut}(G)$ contains a subgroup N which is algebraically isomorphic to $\big(\mathrm{Aut}(S)\big)^X$ and contains precisely the elements $\widetilde{\Omega}$ defined by $\widetilde{\Omega}(g)(x) = \Omega(x)\big(g(x)\big)$ for $\Omega \in \big(\mathrm{Aut}(S)\big)^X$. By Theorem 6.61(v)and 6.63(vi) we have

$$(2) \qquad \mathrm{Aut}(S) = \mathrm{Inn}(S){\cdot}E(S) \cong e^{\mathrm{ad}\,\mathfrak{s}} \rtimes \mathrm{Out}(\mathfrak{s}),$$

where $\operatorname{Inn}(G) \cong e^{\operatorname{ad}\mathfrak{s}}$ is a compact subgroup of $\operatorname{Gl}(\mathfrak{s})$ and $E(G) \cong \operatorname{Out}(\mathfrak{s})$ is a finite subgroup meeting $\operatorname{Inn}(G)$ trivially. In particular, $\operatorname{Aut}(S)$ and thus $\big(\operatorname{Aut}(S)\big)^X$ is compact. Hence, on order to see that the function $\Omega \mapsto \widetilde{\Omega} : \big(\operatorname{Aut}(S)\big)^X \to N$ is an isomorphism of topological groups it suffices to understand that it is continuous. If $U = U^{-1}$ is a symmetric identity neighborhood of S, and F is a finite subset of X then

$$(3) \qquad V(F,U) \overset{\text{def}}{=} \{g \in G = S^X \mid g(F) \subseteq U\}$$

is an identity neighborhood of G, and every identity neighborhood contains one of these. Set

$$(4) \qquad \begin{aligned} \mathcal{W}(F,U) &= \{\alpha \in \operatorname{Aut}(G) \mid (\forall g \in G)\ \alpha(g),\, \alpha^{-1}(g) \in V(F,U)g\} \\ &= \{\alpha \in \operatorname{Aut}(G) \mid (\forall g \in G,\, x \in F)\ \alpha(g)(x),\, \alpha^{-1}(g)(x) \in Ug(x)\}. \end{aligned}$$

These sets form a basis for the identity neighborhoods of $\operatorname{Aut}(G)$. Let

$$U^* = \{\alpha \in \operatorname{Aut}(S) \mid (\forall s \in S)\ \alpha(s),\, \alpha^{-1}(s) \in Us\}.$$

Now the sets

$$(5) \qquad \begin{aligned} \mathcal{V}(F,U^*) &= \{\Omega \in \big(\operatorname{Aut}(S)\big)^X \mid \Omega(F) \subseteq U^*\} \\ &= \{\Omega \in \big(\operatorname{Aut}(S)\big)^X \mid (\forall x \in F,\, s \in S)\ \Omega(x)(s),\, \Omega(x)^{-1}(s) \in Us\} \end{aligned}$$

form a basis of the identity neighborhoods of $\big(\operatorname{Aut}(S)\big)^X$. If $\Omega \in \mathcal{V}(F,U^*)$ then $x \in F$ and $s \in S$ implies $\widetilde{\Omega}(g)g^{-1}(x) = \Omega(x)\big(g(x)\big)g(x)^{-1} \in U$ and $\widetilde{\Omega}^{-1}(g)g^{-1}(x) = \Omega(x)^{-1}\big(g(x)\big)g(x)^{-1} \in U$ whence $\widetilde{\Omega} \in \mathcal{W}(F,U)$. This shows the required continuity of $\Omega \mapsto \widetilde{\Omega} : \big(\operatorname{Aut}(S)\big)^X \to N$.

The group $\operatorname{Aut}(G)$ also contains the subgroup

$$H = \{\alpha \in \operatorname{Aut}(G) \mid \big(\exists \sigma \in P(X)\big)\, \alpha(g) = g \circ \sigma^{-1}\}.$$

For $\sigma \in P(X)$ we set $\widetilde{\sigma}(g) = g \circ \sigma^{-1}$, thus defining a function $\sigma \mapsto \widetilde{\sigma} : P(X) \to H$. We claim that this function is an isomorphism of topological groups. From the definitions is clear that it is algebraically an isomorphism of groups. We note that for all identity neighborhoods U of S and all $\sigma \in P(X)$, by (4) we have

$$(6) \qquad \widetilde{\sigma} \in \mathcal{W}(F,U) \Leftrightarrow (\forall g \in G,\ x \in F)\, g\big(\sigma(x)\big),\, g\big(\sigma^{-1}(x)\big) \in Ug(x).$$

If a basic identity neighborhood $\mathcal{W}(F,U)$ of $\operatorname{Aut}(G)$ is given according to (4), then $\sigma \in P_F(X)$ implies trivially, via (6), that $\widetilde{\sigma} \in \mathcal{W}(F,U)$. Thus the morphism $\sigma \mapsto \widetilde{\sigma}$ is continuous, and we must show that it is open. For this purpose let $P_F(X)$ be an identity neighborhood of $P(X)$; it will suffice to show that there is an identity neighborhood U of S such that $P_F(X)^\sim$ contains $\mathcal{W}(F,U) \cap H$. Thus it suffices to find a $g\colon X \to S$ such that

$$(7) \qquad \big((\forall x \in F)\, g\big(\sigma(x)\big),\, g\big(\sigma^{-1}(x)\big) \in Ug(x)\big) \Rightarrow \sigma \in P_F(X).$$

We pick a function $g\colon X \to S$ which maps F injectively into $S \setminus \{1\}$ and maps $X \setminus F$ to $\{1\}$. Then the set

$$E \stackrel{\text{def}}{=} \{g(x)g(y)^{-1} \mid x,\, y \in X,\ x \neq y,\ \{x,y\} \cap F \neq \emptyset\}$$

is finite and does not contain 1. Hence we find a symmetric identity neighborhood U such that $U \cap E = \emptyset$. Then $g(y)g(x)^{-1} \in U$ iff $g(y) = g(x)$ iff either both x and y are outside F or else $y = x$. Accordingly, for any $\sigma \in P(X)$ we have $g\big(\sigma(x)\big)g(x)^{-1} \in U$ iff either both $\sigma(x)$ and x are outside F or else $\sigma(x) = x$ iff either both $\sigma(x)$ and x are outside F or else $\sigma^{-1}(x) = x$. Thus (7) is satisfied for this g and we have completed the proof of our claim.

Now $\big(\widetilde{\sigma} \circ \widetilde{\alpha} \circ \widetilde{\sigma}^{-1}\big)(g)(x) = \widetilde{\sigma}(h)(x) = h\big(\sigma^{-1}(x)\big)$ with $h = \big(\widetilde{\alpha} \circ \widetilde{\sigma}^{-1}\big)(g) = \widetilde{\alpha}\big(\widetilde{\sigma}^{-1}(g)\big) = \widetilde{\alpha}(g \circ \sigma)$, i.e. $h(k) = \alpha(k)\big(g(\sigma(k))\big)$. Then

$$h\big(\sigma^{-1}(x)\big) = \alpha\big(\sigma^{-1}(x)\big)\big(g(x)\big) = (\alpha \circ \sigma^{-1})\widetilde{\ }(g)(x).$$

This shows that H is in the normalizer of N. An automorphism of $\mathrm{Aut}(G)$ is of the form $\widetilde{\alpha} = \widetilde{\sigma}$ iff $\alpha(x)\big(g(x)\big) = g\big(\sigma^{-1}(x)\big)$ for all g and all x; appropriately specializing g we see that only the identity automorphism satisfies this condition. Therefore NH is algebraically a semidirect product and thus the morphism $(\widetilde{\alpha}, \widetilde{\sigma}) \mapsto \widetilde{\alpha} \circ \widetilde{\sigma} : N \rtimes_I H \to N \circ H$ is an isomorphism of groups. We have seen, however, that $I_{\widetilde{\sigma}}(\widetilde{\alpha}) = \widetilde{\sigma} \circ \alpha \circ \widetilde{\sigma}^{-1} = (\alpha \circ \sigma^{-1})\widetilde{\ }$. Define the group homomorphism $\Sigma\colon P(X) \to \mathrm{Aut}\big([\mathrm{Aut}(S)]^X\big)$ by $\Sigma(\sigma)(\Omega) = \Omega \circ \sigma^{-1}$ for $\sigma\colon X \to X$, $\Omega\colon X \to \mathrm{Aut}(S)$. If U is an identity neighborhood of S and F a finite subset of X, then $V(F, U^*) = \{\Omega \in \big(\mathrm{Aut}(S)\big)^X \mid \Omega(F) \subseteq U^*\}$ is a basic identity neighborhood of $\big(\mathrm{Aut}(S)\big)^X$. The function $(\sigma, \Omega) \mapsto \Omega \circ \sigma^{-1}\colon P(X) \times \big(\mathrm{Aut}(S)\big)^X \to \big(\mathrm{Aut}(S)\big)^X$ maps $H_F(X) \times V(F, U^*)$ into $V(F, U^*)$. Hence $\big(\mathrm{Aut}(S)\big)^X \rtimes_\Sigma P(X)$ is a well defined topological group and the function $(\alpha, \sigma) \mapsto (\widetilde{\alpha}, \widetilde{\sigma})\colon \mathrm{Aut}(S)^X \rtimes_\Sigma P(X) \to N \rtimes_I H$ is an isomorphism of groups. Hence the function $\Phi\colon \mathrm{Aut}(S)^X \rtimes_\Sigma P(X) \to \mathrm{Aut}(G)$, $\Phi(\alpha, \sigma) = \widetilde{\alpha} \circ \widetilde{\sigma}$ is an injective morphism of topological groups.

Lemma 9.83. (i) *The morphism of topological groups*

$$\Phi\colon \mathrm{Aut}^*(G) \stackrel{\text{def}}{=} \mathrm{Aut}(S)^X \rtimes_\Sigma P(X) \to \mathrm{Aut}(G)$$

is bijective.

(ii) Φ *is an isomorphism of topological groups.*

(iii) *For each $\sigma \in P(X)$, the automorphism*

$$\Sigma(\sigma)\colon \big(\mathrm{Aut}(S)\big)^X \to \big(\mathrm{Aut}(S)\big)^X$$

leaves $E(S)^X$ invariant (*cf.* (2)) *and therefore induces an automorphism $\Sigma'(\sigma) \in \mathrm{Aut}(E(S)^X)$.*

(iv) $D \stackrel{\text{def}}{=} E(S)^X \rtimes_{s'} P(X)$ *is a totally disconnected subgroup of* $\mathrm{Aut}^*(G)$ *and* $\mathrm{Aut}^*(G)$ *is the semidirect product of* $\mathrm{Inn}(G)^X$ *and* D.

Proof. (i) We have to prove the surjectivity of Φ. Thus let α be an automorphism of $G = P(X)$. We follow closely the pattern of the proof of 6.58. Define the coprojection

$$\mathrm{copr}_x\colon S \to G \quad \text{by} \quad \mathrm{copr}_x(s) = (s_y)_{y\in X}, \quad s_y = \begin{cases} s & \text{if } y = x, \text{ and} \\ 1 & \text{otherwise;} \end{cases}$$

we define the projection $\mathrm{pr}_x\colon G \to S$ as usual and set $\alpha_{xy} \stackrel{\mathrm{def}}{=} \mathrm{pr}_x \circ \alpha \circ \mathrm{copr}_y$. By the simplicity of S, the morphism $\alpha_{xy}\colon S \to S$ is either the constant endomorphism c_S or is an automorphism. We claim that there is a bijection $\sigma \in S_n$ such that

$$(*) \qquad \alpha_{xy} \begin{cases} \in \mathrm{Aut}\, S & \text{if } x = \sigma(y), \\ = c_S & \text{otherwise.} \end{cases}$$

Indeed, for each $y \in X$ the isomorphic copy $\mathrm{copr}_y(S)$ of S is closed connected normal subgroup of G. By 9.19(ii), the automorphism α permutes the set $\{\mathrm{copr}_x(S) \mid x \in X\}$ of connected normal closed subgroups. Hence there is a unique $\sigma(y) \in X$ such that $\mathrm{copr}_{\sigma(y)}(S) = (\alpha \circ \mathrm{copr}_y)(S)$. The projection pr_x maps the direct factor $\mathrm{copr}_x(S)$ isomorphically onto S and maps all direct factors $\mathrm{copr}_{x'}(S)$ onto the singleton subgroup for $x' \neq x$. Thus $(*)$ follows. Now define $\beta \in (\mathrm{Aut}\, S)^X$, $\beta\colon X \to \mathrm{Aut}\, S$ by $\beta(x) = \alpha_{x,\sigma^{-1}(x)}$. Let $g \in G$, $g\colon X \to S$, and compute

$$\begin{aligned} \alpha(g)(x) = \mathrm{pr}_x\left(\alpha(g)\right) &= \alpha_{x\sigma^{-1}(x)}\big(g(\sigma^{-1}(x))\big) = \beta(x)\big(g(\sigma^{-1}(x))\big) \\ &= \beta(x)\big(\widetilde{\sigma}(g)(x)\big) = \widetilde{\beta}\big(\widetilde{\sigma}(g)\big)(x) = (\widetilde{\beta} \circ \widetilde{\sigma})(g)(x), \end{aligned}$$

and thus $\alpha = \widetilde{\beta} \circ \widetilde{\sigma} = \Phi(\beta, \sigma)$. This proves the surjectivity of Φ.

(ii) In order to show that Φ is an isomorphism of topological groups, we must show that the Mayer-Vietoris morphism $\mu\colon N \rtimes H \to \mathrm{Aut}(G)$, $\mu(\widetilde{\Omega}, \widetilde{\sigma}) = \widetilde{\Omega} \circ \widetilde{\sigma}$ is open. This can be accomplished by showing that

$$p \stackrel{\mathrm{def}}{=} \kappa^{-1} \circ \mathrm{pr}_H \circ \mu^{-1}\colon \mathrm{Aut}(G) \to P(X), \quad \kappa\colon P(X) \to H \text{ given by } \kappa(\sigma) = \widetilde{\sigma},$$

is continuous. Let F be a finite subset of X. We must find a finite subset F' of X and a symmetric open identity neighborhood of S such that $P_F(X) \supseteq p\big(\mathcal{W}(F', U)\big)$. We take $F' = F$ and select U according to (7) above. Then for $\sigma \in P(X)$ we have $\sigma \in P_F(X)$ if $\widetilde{\sigma} \in \mathcal{W}(F, U)$. From (i) we know that each $\alpha \in \mathrm{Aut}(G)$ is uniquely of the form $\alpha = \widetilde{\Omega} \circ \widetilde{\sigma}$ with $\Omega \in \big(\mathrm{Aut}(S)\big)^X$ and $\sigma = p(\alpha) \in P(X)$. Now $\alpha(g)(x) = \widetilde{\Omega}(g)\big(\sigma^{-1}(x)\big) = \Omega\big(\sigma^{-1}(x)\big)\big(g(\sigma^{-1}[x])\big)$. Let $F_1 \supseteq F$ and U_1 be such that $\mathcal{W}(F_1, U_1)^{-1}\mathcal{W}(F_1, U_1) \subseteq \mathcal{W}(F, U)$ and let $\alpha \in \mathcal{W}(F_1, U_1)$. Then for $x \in F_1$ and $g \in G$ we have $\widetilde{\Omega}(g)(x) = \widetilde{\Omega}(g)\big(\sigma^{-1}(x)\big) = \alpha(g)(x) \in U_1 g$, and $\widetilde{\Omega}(g)^{-1}(x) = \widetilde{\Omega}(g)^{-1}\big(\sigma^{-1}(x)\big) = \alpha(g)(x)^{-1} \in U_1 g$. Hence $\widetilde{\Omega} \in \mathcal{W}(F_1, U_1)$ and therefore $\widetilde{\sigma} = \widetilde{\Omega}^{-1}\alpha \in \mathcal{W}(F_1, U_1)^{-1}\mathcal{W}(F_1, U_1) \subseteq \mathcal{W}(F, U)$. Thus $\sigma \in P_F(X)$ by (7). This proves the continuity of p and thus Claim (ii) is established.

(iii) Let $\Omega \stackrel{\mathrm{def}}{=} \big(\varepsilon(x)\big)_{x\in X} \in E(S)^X$. Then from the definition of $\Sigma(\sigma)$ it follows that $\Sigma(\sigma)(\Omega) = \big(\varepsilon(\sigma^{-1}(x))\big)_{x\in X} \in E(S)^X$.

(iv) From (iii) we note that D is well-defined. Since D is compact totally disconnected and $P(X)$ is totally disconnected, it follows that $D \rtimes P(X)$ is totally disconnected. ☐

We recall $\mathrm{Inn}(S) \cong e^{\mathrm{ad}\,\mathfrak{s}} \subseteq \mathrm{Gl}(\mathfrak{s})$ and summarize:

Proposition 9.84. *Let X be a set and S a simply connected simple compact Lie group. Set $G = S^X$. Then*

$$\mathrm{Aut}(G) \cong (e^{\mathrm{ad}\,\mathfrak{s}} \rtimes \mathrm{Out}\,\mathfrak{s})^X \rtimes P(X) = (e^{\mathrm{ad}\,\mathfrak{s}})^X \rtimes \big(\big(\mathrm{Out}(\mathfrak{s})\big)^X \rtimes P(X)\big) \qquad \square$$

Parallel to this result we state and prove a proposition on the automorphism group of compact abelian groups. This requires that we deal at the same time with the automorphism group of discrete abelian groups.

Proposition 9.85. (i) *If A is a discrete abelian group, then the compact open topology on $\mathrm{Aut}(A)$ is the standard group topology for automorphism groups; it is induced on the subgroup $\mathrm{Aut}(A)$ of $P(A)$ by the topology of $P(A)$. The filter of identity neighborhoods therefore has a basis consisting of the subgroups $\mathrm{Aut}(A) \cap P_F(A)$, $F \in \mathrm{Fin}(A)$. In particular, $\mathrm{Aut}(A)$ is totally disconnected.*

(ii) *If G is a compact abelian group, then $\alpha \mapsto \widehat{\alpha} : \mathrm{Aut}(G) \to \mathrm{Aut}(\widehat{G})$ is an isomorphism of topological groups. In particular, $\mathrm{Aut}(G)$ is totally disconnected.*

Proof. (i) The refined compact open group topology on $\mathrm{Aut}(A)$ is given by the basic identity neighborhoods

$$V(K,U) = \{\alpha \in \mathrm{Aut}(A) \mid (\forall x \in K)\, \alpha(x)x^{-1},\, \alpha^{-1}(x)x^{-1} \in U\},$$

where K ranges through the compact subsets of A and U through the open identity neighborhoods of A (for compact groups cf. discussion preceding 6.63). As A is discrete, a subset K is compact iff K is finite, and $U = \{1\}$ is an open identity neighborhood of A. Hence $\{V(F,\{1\}) \mid F \in \mathrm{Fin}(X)\}$ is a basis of the identity neighborhoods of $\mathrm{Aut}(A)$. But $V(F,\{1\}) = \mathrm{Aut}(A) \cap P_F(A)$; therefore $\mathrm{Aut}(A)$ has the subgroup topology induced from $P(A)$. Since $P(A)$ is totally disconnected (cf. E9.19 and preceding discussion above), then so is $\mathrm{Aut}(A)$.

(ii) By 7.11(iii) for each pair of locally compact abelian groups A and B the function $\delta_{AB}\colon \mathrm{Hom}(A,\widehat{B}) \to \mathrm{Hom}(B,\widehat{A})$, $\delta_{AB}(f)(b)(a) = f(a)(b)$ is an isomorphism of topological abelian groups. By the Duality Theorem 7.63 (or, for compact abelian groups A, by 2.32), the evaluation map $\eta_A\colon A \to \widehat{\widehat{A}}$ is an isomorphism of topological abelian groups. Thus

$$\mathrm{Hom}(A,A) \xrightarrow{\mathrm{Hom}(A,\eta_A)} \mathrm{Hom}(A,\widehat{\widehat{A}}) \xrightarrow{\delta_{A\widehat{A}}} \mathrm{Hom}(\widehat{A},\widehat{A})$$

gives an isomorphism of topological groups. We claim that it agrees with $\alpha \mapsto \widehat{\alpha}$. Let $\alpha \in \mathrm{Hom}(A,A)$, $\chi \in \widehat{A}$, $a \in A$. Then

$$\begin{aligned}\big(\delta_{A\widehat{A}} \circ \mathrm{Hom}(A,\eta_A)\big)(\alpha)(\chi)(a) &= \big(\delta_{A\widehat{A}}(\eta_A \circ \alpha)\big)(\chi)(a)\\ &= (\eta_A \circ \alpha)(a)(\chi) = \eta_A\big(\alpha(a)\big)(\chi) = \chi\big(\alpha(a)\big) = \widehat{\alpha}(\chi)(a),\end{aligned}$$

and this proves the claim. Since $\widehat{G}$ is discrete, $\mathrm{Aut}(G) \cong \mathrm{Aut}(\widehat{G}))$ is totally disconnected. □

We shall assume henceforth that for a discrete abelian group A, the group $\mathrm{Aut}(A)$ has the topology induced from $P(A)$, the *topology of pointwise convergence.* If A is the additive group of a $\mathbb{Q}$-vector space, then $\mathrm{Aut}(A) = \mathrm{Gl}(A, \mathbb{Q})$ with the topology of pointwise convergence. If A is a finitely generated group, then $\mathrm{Aut}(A)$ is discrete.

Recall from the preamble to 9.19 that we let $\mathcal{S}$ denote a set of simple compact Lie algebras containing for each simple compact Lie algebra exactly one member isomorphic to it and pick once and for all for each $\mathfrak{s} \in \mathcal{S}$ a simple simply connected compact Lie group $S_{[\mathfrak{s}]}$ whose Lie algebra is isomorphic to $\mathfrak{s}$.

The Automorphism Group of a Compact Projective Group

Theorem 9.86. *Let G be a connected compact projective group, i.e. there are sets X and $X_\mathfrak{s}$, $\mathfrak{s} \in \mathcal{S}$ such that*

$$G \cong \widehat{\mathbb{Q}}^X \times \prod_{\mathfrak{s} \in \mathcal{S}} S_{[\mathfrak{s}]}^{X_\mathfrak{s}}, \tag{1}$$

where $S_{[\mathfrak{s}]}$ is the simlpy connected simple compact Lie group with $\mathfrak{L}\big(S_{[\mathfrak{s}]}\big) \cong \mathfrak{s}$. Let

$$T \cong \widehat{\mathbb{Q}}^X \times \prod_{\mathfrak{s} \in \mathcal{S}} T_{[\mathfrak{s}]}^{X_\mathfrak{s}}, \tag{2}$$

be a maximal pro-torus with a maximal torus $T_{[\mathfrak{s}]}$ in each $S_{[\mathfrak{s}]}$. Then

(i) $$\mathrm{Aut}(G) \cong \mathrm{Aut}(\widehat{\mathbb{Q}}^X) \times \prod_{\mathfrak{s} \in \mathcal{S}} \mathrm{Aut}(S_{[\mathfrak{s}]}^{X_\mathfrak{s}}),$$

(ii) $$\mathrm{Aut}(\widehat{\mathbb{Q}}^X) \cong \mathrm{Gl}(\mathbb{Q}^{(X)}, \mathbb{Q}),$$

(iii) $$\mathrm{Aut}\left(S_{[\mathfrak{s}]}^{X_\mathfrak{s}}\right) \cong \mathrm{Aut}\left(S_{[\mathfrak{s}]}\right)^{X_\mathfrak{s}} \rtimes P(X_\mathfrak{s}),$$

(iv) $$\mathrm{Aut}\left(S_{[\mathfrak{s}]}\right) = \big[\mathrm{Aut}\left(S_{[\mathfrak{s}]}\right)\big]_0 D_\mathfrak{s} \cong e^{\mathrm{ad}\,\mathfrak{s}} \rtimes \mathrm{Out}\left(S_{[\mathfrak{s}]}\right),$$
$$D_\mathfrak{s} \subseteq N\big(T_{[\mathfrak{s}]}, \mathrm{Aut}\left(S_{[\mathfrak{s}]}\right)\big) \stackrel{\mathrm{def}}{=} \{\alpha \in \mathrm{Aut}\left(S_{[\mathfrak{s}]}\right) \mid \alpha(T_{[\mathfrak{s}]}) = T_{[\mathfrak{s}]}\},$$

(v) $$\mathrm{Aut}(G) = \mathrm{Inn}(G)D \cong \mathrm{Inn}(G) \rtimes \mathrm{Out}(G), \quad D \subseteq N\big(T, \mathrm{Aut}(G)\big),$$

In particular, the totally disconnected subgroup $D \subseteq N\big(T, \mathrm{Aut}(G)\big) \subseteq \mathrm{Aut}(G)$ is of the form

(vi) $$D \cong \mathrm{Gl}(\mathbb{Q}^{(X)}) \times \prod_{\mathfrak{s} \in \mathcal{S}} \big(\mathrm{Out}(\mathfrak{s})^{X_\mathfrak{s}} \rtimes P(X_\mathfrak{s})\big).$$

Proof. (i) We write

$$G = \widehat{\mathbb{Q}}^X \times \prod_{\mathfrak{s} \in \mathcal{S}} S_{[\mathfrak{s}]}^{X_\mathfrak{s}}, \tag{1'}$$

and define subgroups

$$A = \widehat{\mathbb{Q}}^X \times \prod_{\mathfrak{s}\in\mathcal{S}} \{1\},$$

$$S_{\mathfrak{s}} = \{0\} \times \prod_{\mathfrak{s}'\in\mathcal{S}} S^*_{\mathfrak{s}'}, \quad S^*_{\mathfrak{s}'} = \begin{cases} S_{[\mathfrak{s}]}^{X_{\mathfrak{s}}} & \text{if } \mathfrak{s}' = \mathfrak{s}, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $A \cong \widehat{\mathbb{Q}}^X$ and $S_{\mathfrak{s}} \cong S_{[\mathfrak{s}]}^{X_{\mathfrak{s}}}$, and each of A and $S_{\mathfrak{s}}$ is a characteristic subgroup of G and thus is invariant under any automorphism. Hence each $\alpha \in \mathrm{Aut}(G)$ induces unique automorphisms $\alpha_A \in \mathrm{Aut}(A)$ and $\alpha_{\mathfrak{s}} \in \mathrm{Aut}(S_{\mathfrak{s}})$ such that

(2) $$f\colon \mathrm{Aut}(G) \to \mathrm{Aut}(A) \times \prod_{\mathfrak{s}\in\mathcal{S}} \mathrm{Aut}(S_{\mathfrak{s}}), \quad f(\alpha) = (\alpha_A, (\alpha_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{S}})$$

is an isomorphism of compact groups (Exercise E9.20 below). This proves (i).

(ii) It follows from Proposition 9.84 that

$$\mathrm{Aut}(A) \cong \mathrm{Aut}((\widehat{\mathbb{Q}}^X)\widehat{\ }) = \mathrm{Aut}(\mathbb{Q}^{(X)}) = \mathrm{Gl}(\mathbb{Q}^{(X)}, \mathbb{Q})$$

with the topology of pointwise convergence.

(iii) and (iv) follow directly from Proposition 9.84 above.

(v) and (vi) are consequences of 9.84 and (i) above. ☐

Exercise E9.20. Verify the details of the claim that f in line (2) of the proof of 9.86(i) is an isomorphism.

[Hint. By what was said in the proof of 9.85(i), the function f is well-defined, is a morphism, and is continuous as follows straightforwardly from the definition of the topology of the automorphism groups via the topology of uniform convergence. In order to verify the surjectivity of f, take $\beta \overset{\text{def}}{=} (\alpha^{(A)}, (\alpha^{(\mathfrak{s})})_{\mathfrak{s}\in\mathcal{S}}) \in \mathrm{Aut}(A)\times\prod_{\mathfrak{s}\in\mathcal{S}} S_{\mathfrak{s}}$ arbitrarily and define $\alpha\colon G \to G$ by

$$\alpha(a, (s_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{S}}) = (\alpha^{(A)}(a), \big(\alpha^{(\mathfrak{s})}(s_{\mathfrak{s}})\big)_{\mathfrak{s}\in\mathcal{S}}).$$

Then $\alpha \in \mathrm{Aut}(G)$ and $f(\alpha) = \beta$.] ☐

The Automorphism Group of a Compact Connected Group

Corollary 9.87. *Let G be a compact connected group. Then for a given maximal pro-torus T in G there is a totally disconnected closed subgroup D of* $\mathrm{Aut}\, G$ *contained in*

$$N(T, \mathrm{Aut}(G)) \overset{\text{def}}{=} \{\alpha \in \mathrm{Aut}(G) \mid \alpha(T) = T\}$$

such that

$$\mathrm{Aut}(G) = \mathrm{Inn}(G){\cdot}D, \quad \mathrm{Inn}(G) \cap D = \{1\}$$
$$\mathrm{Aut}(G) \cong \mathrm{Inn}(G) \rtimes \mathrm{Out}(G), \quad \mathrm{Out}(G) \cong D.$$

The group $\mathrm{Out}(G)$ *is totally disconnected, the group* $\mathrm{Inn}(G)$ *is compact, connected, semisimple, centerfree and isomorphic to $G'/Z(G')$.*

Proof. The isomorphism $\mathrm{Inn}(G) \cong G/Z(G) \cong G'/Z(G')$ is straightforward from the definitions and was observed in 9.82(ii). From 9.82(i) we know that $\mathrm{Inn}(G)$ is the identity component of $\mathrm{Aut}(G)$. Hence $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is totally disconnected. It remains to prove the main assertion of the theorem.

By Theorem 9.76 on the projective cover, notably 9.76(iv), there is a homomorphism of groups $f \mapsto \mathfrak{P}(f)\colon \mathrm{Aut}(G) \to \mathrm{Aut}\big(\mathfrak{P}(G)\big)$. This homomorphism is continuous (Exercise E9.21) and injective, since $\mathfrak{P}(f) = \mathrm{id}_{\mathfrak{P}(G)}$ implies $f \circ E_G = E_G$ (see commutating diagram in 9.76(iv)) and E_G is surjective. Also $\mathrm{Aut}(G)_0 = \mathrm{Inn}(G)$ is mapped onto $\mathrm{Aut}\big(\mathfrak{P}(G)\big)_0 = \mathrm{Inn}\big(\mathfrak{P}(G)\big)$ by 9.19 and 9.82. But by the Structure Theorem of the Automorphism Group of Projective Groups 9.86(v) above, the main assertion of the present theorem is true for the projective group $\mathfrak{P}(G)$. Hence it is true for G by Lemma 6.68. □

Exercise E9.21. Prove the continuity of the function

$$f \mapsto \mathfrak{P}(f) : \mathrm{Aut}(G) \to \mathrm{Aut}\big(\mathfrak{P}(G)\big).$$

[Hint. Note that $D \overset{\mathrm{def}}{=} \ker E_G$ is a totally disconnected compact subgroup of $\mathfrak{P}(G)$; then for every subgroup $N \in \mathcal{N}\big(\mathfrak{P}(G)\big)$ the groups G/N and $DN/N \cong D/(D \cap N)$ are Lie groups. As D is totally disconnected compact, DN/N is finite and we may pick a finite subset D_N such that $1 \in D_N$ and $(n, e) \mapsto ne\colon N \times D_N \to D$ is a homeomorphism. Let U be an identity neighborhood of $\mathfrak{P}(G)$. Then U contains an open identity neighborhood W_1 such that $W_1W_1 \subseteq U$; pick an $N \in \mathcal{N}(D)$ with $N \subseteq W_1$ and set $W = W_1N$. Then $WN = NW = W \subseteq U$; moreover make W so small that the sets W and $W(D_N \setminus \{1\})$ are disjoint; this is possible since D_NN/N is finite. Now $V \overset{\mathrm{def}}{=} E_G(W)$ is an identity neighborhood in G such that $V^{-1}V \subseteq W$. $E_G(V)$ is an identity neighborhood of G. If $f(g)g^{-1} \in V$ for all $g \in G$ then $\mathfrak{P}(f)(\widetilde{g})\widetilde{g}^{-1} \in WD = WND = WND_N = WD_N = W \,\dot{\cup}\, W(D_N \setminus \{1\})$. Since the set $\{\mathfrak{P}(f)(\widetilde{g})\widetilde{g}^{-1} \mid \widetilde{g} \in \mathfrak{P}(G)\}$ is connected, contains 1 and is contained in WD, it is contained in $W \subseteq U$. Draw the final conclusion.] □

Simple Compact Groups and the Countable Layer Theorem

The use of the word "simple" in group theory is a bit delicate whenever Lie groups are involved as we pointed out in a paragraph preceding Theorem 6.6. We call a Lie group G a *simple connected compact Lie group* iff its Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is a simple Lie algebra. This is equivalent to saying that any proper closed normal subgroup is discrete. We will use for groups in general the standard definition

Definition 9.88. (i) A group G is called a *simple group* iff $\{1\}$ and G are the only normal subgroups of G. A topological group G is called *simple* if the underlying group is simple.

(ii) A compact group G is called *strictly reductive* if it is isomorphic to a cartesian product of simple compact groups. □

Thus a compact connected Lie group which is a simple compact group is always a simple compact connected Lie group, but a simple connected compact Lie group is not always a simple compact group, because it may have a nontrivial finite central subgroup.

For further clarification the following central theorem of Yamabe-Gotô is useful; for its proof we refer to appropriate literature.

Lemma 9.89. (H. Yamabe, M. Gotô) *An arcwise connected subgroup of a Lie group is analytic.*

Proof. See [128], [41], Ch. III, §8, Exercice 4, p. 275, [153], p. 57ff., [154], p. 347ff. □

In particular, this theorem is true for linear and notably for compact Lie groups. The proof requires the Brouwer Fixed Point Theorem.

Theorem 9.90. *Let G be a compact group in which all* closed *normal subgroups are trivial. Then* all *normal subgroups of G are trivial, that is, G is a simple group.*

Proof. Let G be a compact group in which $\{1\}$ and G are the only closed normal subgroups. By Corollary 2.43 and Lemma 9.1, a compact group G has arbitrarily small closed normal subgroups $N \in \mathcal{N}(G)$. It follows that G must be a Lie group. The identity component G_0 is a normal subgroup, so it is either singleton or the whole group. In the first case G is discrete and we are done. In the second case G is a compact connected Lie group in which every closed normal subgroup is trivial. We now assume this case.

Let N be a nonsingleton normal subgroup of G; we must show that $N = G$. Since the closure $\overline{N}$ is normal, it must be G and thus N is dense. If N is arcwise totally disconnected, then N is central (because for each $n \in N$ the continuous function $g \mapsto gng^{-1}n^{-1}: G \to N$ maps a connected space into an arcwise totally disconnected one and is, therefore, constant; but the image contains he identity). Then $G = \overline{N}$ is a nonsingleton compact connected abelian group. Its character group is a torsion free nondegenerate abelian group by Corollary 8.5 which has nontrivial subgroups. Thus by the Annihilator Mechanism 7.64, G has nontrivial closed normal subgroups, which is not the case. Thus the identity arc component N_a of N is nontrivial, and as it is a characteristic subgroup of N, it is normal in G. It suffices to show that $N_a = G$; let us therefore assume that N is arcwise connected. By the Yamabe-Gotô Lemma 9.89 above an arcwise connected subgroup is analytic. Thus N is a dense analytic subgroup of G. By Proposition 5.62, a dense analytic subgroup N of a (linear) Lie group G contains the the commutator subgroup G'. By Gotô's Commutator Theorem 6.55, the commutator subgroup G' of a compact Lie group is closed. Thus G' is either singleton or agrees with G; the former case would again imply that G is abelian which is impossible. Thus $G = G' \subseteq N \subseteq G$, and this proves $N = G$ which we wanted to show. □

A relevant reference is [154], p. 356, Theorem 9.6.13.

Now that we know that a simple compact group is either a centerfree simple compact connected Lie group, or a discrete cyclic group of prime order, or a discrete simple (nonabelian) finite group, and that a strictly reductive compact group is a product of these we can present the following structure theorem of arbitrary compact groups. (Recall that $Z_0(G_0)$ denotes the identity component of the center of the identity component G_0 of G.)

Theorem 9.91. (Countable Layer Theorem) *Any compact group G has a canonical countable descending sequence $G = \Omega_0(G) \supseteq \cdots \supseteq \Omega_n(G) \supseteq \cdots$ of closed characteristic subgroups of G with the following two properties:*
(1) $\bigcap_{n=1}^{\infty} \Omega_n(G) = Z_0(G_0)$,
(2) $\Omega_{n-1}(G)/\Omega_n(G)$ *is a strictly reductive group for each* $n = 1, 2, \ldots$.

Proof. For a proof, we refer to [186]. ☐

We shall apply the Countable Layer Theorem 9.91 in Chapter 12. In the literature we have applied the Countable Layer Theorem to showing that, in many instances, a compact group G contains a closed abelian subgroup A such that $w(A) = w(G)$ (see [187]). The "Abelian Subgroup Conjecture", saying that this should hold for *all compact groups* is false as was shown by Herfort [142] by pointing out that the *free p-group* over the one point compactification of an uncountable discrete set is a counterexample. However, the following theorem out of Zelmanov's work [382] is a nontrivial result that was sought after for quite some time:

Theorem 9.91a. *Every infinite compact group contains an infinite abelian subgroup.* ☐

The Structure of Compact FC-Groups

Definition 9.92. In a group G, the set F of elements g whose conjugacy class is finite is called its FC-*center* . If $F = G$, then G is called an *FC-group* .

Exercise E9.22. Show that
(i) *the FC-center of a group is a characteristic subgroup,*
(ii) *the FC-center of a compact group is an F_σ, that is, a countable union of closed subsets.* ☐

By the structure theory of compact connected groups (see Theorem 9.23ff) the identity component G_0 of a compact FC-group G is abelian. Abbreviate the centralizer $Z(G_0, G)$ of G_0 by C. The action by inner automorphisms of G on G_0 induces an action of the profinite group G/C on the compact connected abelian group G_0 with finite orbits such that $gC*g_0 = gg_0g^{-1}$, $g \in G$, $g_0 \in G_0$, where each function $g_0 \mapsto g*g_0$ is an automorphism of G_0. An action $(g,x) \mapsto g{\cdot}x : G\times X \to X$ of a topological group G on a topological group X such that each $x \mapsto g{\cdot}x$ is an automorphism of X is called an *automorphic action*

As is Proposition 7.36 and Theorem 7.66, let the Lie algebra $\mathfrak{g} = \mathfrak{L}(G_0)$ of G_0 be defined as $\mathrm{Hom}(\mathbb{R}, G_0) \cong \mathrm{Hom}(\widehat{G_0}, \mathbb{R})$. The Lie algebra $\mathfrak{g}$ is a weakly complete vector space, which is isomorphic as a topological vector space to the complete locally convex space $\mathbb{R}^I$ for a suitable set I whose cardinality is the rank of the torsion-free abelian group $\widehat{G_0}$.

There is a natural morphism $\mathrm{Aut}\,G_0 \to \mathrm{Aut}\,\mathfrak{g}$: Indeed each automorphism α of G_0 induces an automorphism $\mathfrak{L}(\alpha)$ of $\mathfrak{g}$ as follows: Let $X\colon \mathbb{R} \to G_0$, $t \mapsto X(t)$, be a member of $\mathfrak{g}$, then $\mathfrak{L}(\alpha)(X)\colon \mathbb{R} \to G_0$ is defined by $\mathfrak{L}(\alpha)(X) = \alpha \circ X$. An element $g \in G$ induces an inner automorphism I_g on G_0 via $I_g(x) = gxg^{-1}$ and the function $g \mapsto I_g : G \to \mathrm{Aut}\,\mathfrak{g}$ factors through the quotient $G \to G/C$ giving us a chain of representations

$$G \to G/C \to \mathrm{Aut}\,G_0 \to \mathrm{Aut}\,\mathfrak{g},$$

whose composition yields a continuous representation

$$\pi\colon G/C \to \mathrm{Aut}\,\mathfrak{g}, \quad \pi(gC)(X) = I_g \circ X, \text{ that is, } \pi(gC)(X)(t) = gX(t)g^{-1}.$$

We claim that the fact, that every element of G has finitely many conjugates, implies for each $X \in \mathfrak{g}$ that $\pi(G/C)(X) \subseteq \mathfrak{g}$ is contained in a finite dimensional vector subspace of $\mathfrak{g}$. That is, we maintain that the G/C-module $\mathfrak{g}$ (see Definition 2.2) satisfies $\mathfrak{g}_{\mathrm{fin}} = \mathfrak{g}$. (See Definition 3.1.)

For an understanding of these facts and their proof below we recall that a subgroup of a topological group is called *monothetic* if it contains a dense cyclic subgroup. It is called *solenoidal* if it contains a dense one-parameter subgroup.

Lemma 9.93. *Let Γ be a compact group acting automorphically on a compact group G. Assume that all orbits of Γ on G are finite. Then the following conclusions hold:*

(i) *For each monothetic subgroup $M = \overline{\langle g \rangle}$ of G there is an open normal subgroup Ω of Γ which fixes the elements of M elementwise, and the finite group Γ/Ω acts on M with the same orbits as Γ.*

(ii) *The orbits of Γ on $\mathfrak{g} = \mathfrak{L}(G)$ for the induced automorphic action of Γ on $\mathfrak{L}(G)$ are finite.*

Proof. (i) Assume that $A = \overline{\langle g \rangle}$ is a monothetic subgroup. If $\alpha \in \Gamma$, then $\alpha \in \Gamma_g$ means $\alpha{\cdot}g = g$, that is, g belongs to the fixed point subgroup $\mathrm{Fix}(\alpha)$ of α. This is equivalent to $A \subseteq \mathrm{Fix}(\alpha)$, i.e., $\alpha \in \Gamma_a$ for all $a \in A$. So we have $\Gamma_g \subseteq \Gamma_a$ for all $a \in A$. Then the normal finite index subgroup $\Omega_g = \bigcap_{\gamma\in\Gamma} \gamma\Gamma_g\gamma^{-1}$ is contained in all Γ_a, $a \in A$. This establishes (i) as the remainder is clear.

(ii) A compact abelian group A is monothetic if and only if there is a morphism $f\colon \mathbb{Z} \to A$ with dense image, and in view of Pontryagin duality (see Chapters 7 and 8) this holds exactly when there is an injective morphism $\widehat{f}\colon \widehat{A} \to \widehat{\mathbb{Z}} \cong \mathbb{T}$. In the same spirit a compact abelian group is solenoidal if and only if there is a morphism $f:\mathbb{R}\to A$ with dense image exactly when, in view of Pontryagin duality again, there is an injective morphism $\widehat{f}:\widehat{A}\to\widehat{\mathbb{R}}\cong\mathbb{R}$. As $\widehat{A}$ is discrete, this happens if and only if $\widehat{A}$ is algebraically a subgroup of $\mathbb{R}$. Now $\mathbb{T}$ has a subgroup algebraically isomorphic to $\mathbb{R}$ (see Corollary A1.43). Thus if $\widehat{A}$ can be homomorphically injected into $\mathbb{R}$ it can be homomorphically injected into $\mathbb{T}$. Therefore

every solenoidal compact group is monothetic.

Thus let $X\in\mathfrak{g}$ be a one-parameter subgroup of G. Then by (i), the image $X(\mathbb{R})$ is contained in a monothetic subgroup. Hence by (i) above there is an open normal subgroup Ω of Γ such that each $\alpha\in\Omega$ satisfies $\alpha{\cdot}X(t)=X(t)$ for all $t\in\mathbb{R}$. Hence $\alpha{\cdot}X=X$ in $\mathfrak{g}$ with respect to the action of Γ induced on $\mathfrak{g}=\mathfrak{L}(G)$. Hence $\Gamma_X\supseteq\Omega$ has finite index for the action of Γ on $\mathfrak{g}$. □

Lemma 9.93(ii) completes the argument that the G-module $\mathfrak{g}$ in the case of a compact FC-group G satisfies $\mathfrak{g}_{\rm fin}=\mathfrak{g}$.

Now, using Exercise E4.8 here, we get the following result:

Lemma 9.94. *Let G be a compact* FC-*group. Then the centralizer $Z(G_0,G)$ of the identity component is open.*

Proof. By Lemma 9.93 we can apply the results of Exercise E4.8 with $\Gamma=G/Z(G_0,G)$ in place of G and the Γ-module $\mathfrak{g}=\mathfrak{L}(G)=\mathfrak{L}(G_0)$ in place of V. We conclude that Γ is finite. This proves the assertion. □

After Theorem 9.41 and Corollary 9.42 we can readily deduce the structure of a central extension of a compact abelian group by a profinite group:

Proposition 9.95. *Let G be a compact group such that G_0 is central. Then there is a profinite normal subgroup Δ such that $G=G_0\Delta$. In particular,*

$$G\cong\frac{G_0\times\Delta}{D},\quad D=\{(g,g^{-1}):g\in G_0\cap\Delta\}.$$ □

In order to pursue the structure of compact FC-groups further we cite Lemma 2.6, p. 1281 from the paper by Shalev [327], proved with the aid of the Baire Category Theorem:

Lemma 9.96. (Shalev's Lemma) *If G is a profinite* FC-*group then its commutator subgroup G' is finite.* □

In order to exploit this information, we need a further lemma:

Lemma 9.97. *If N is a compact nilpotent group of class ≤ 2 and N' is discrete, then the center $Z(N)$ has finite index in N.*

Proof. Since N is nilpotent of class at most 2, we have $N' \subseteq Z(N)$. So for each $y \in N$ the function $x \mapsto [x, y] : N \to N'$ is a morphism. Therefore, if we set $A = N/N'$ we have a continuous $\mathbb{Z}$-bilinear map of abelian groups $b\colon A \times A \to N'$, where $b(xN', yN') = [x, y]$. Since N' is discrete, $\{1\}$ is a neighborhood of the identity in N'. On the other hand, $b(\{1_A\} \times A) = \{1\}$. So for each $a \in A$ we have open neighborhoods U_a of 1_A and V_a of a, respectively, such that $b(U_a \times V_a) = \{1\}$ As A is compact, we find a finite set $F \subseteq A$ of elements such that $A = \bigcup_{a\in F} V_a$. Let $U = \bigcap_{a\in F} U_a$; then U is an identity neighborhood of A and $b(U \times A) = \{1\}$. Since b is bilinear, this implies $b(\langle U\rangle \times A) = \{1\}$. Then the full inverse image M of $\langle U\rangle$ in N under the quotient morphism $N \to A$ is an open subgroup of N satisfying $[M, N] = \{1\}$ and is, therefore, central. Thus the center $Z(N)$ of N is open and so has finite index in N. □

We remark that the proof of this lemma resembles that of Proposition 13.11, p. 574 of [188]. From the preceding two pieces of information we derive

Proposition 9.98. *Let G be a compact group whose commutator subgroup G' is finite, and let $Z(G', G)$ be the centralizer of G' in G. Then the center of $Z(G', G)$ is a characteristic abelian open subgroup $A_G \stackrel{\mathrm{def}}{=} Z(Z(G', G))$ of G.*

Proof. Let $f\colon G \to \mathrm{Aut}(G')$ denote the morphism defined by $f(g)(x) = gxg^{-1}$ for $x \in G'$. Since G' is finite by Lemma 9.96, so is Aut G'. Hence $f(G)$ is finite as well and so $G/\ker f \cong f(G)$ is finite. If follows that the centralizer $C \stackrel{\mathrm{def}}{=} Z(G', G)$ of G' in G, being equal to $\ker f$, has finite index in G and thus is open. The center $Z(G') = G' \cap C$ of G' is finite abelian, and so the isomorphism $C/Z(G') \to G'C/G' \subseteq G/G'$ shows that the commutator subgroup $C' \subseteq Z(G')$ is a finite subgroup of C. Since $G' \subseteq Z(Z(G', G), G) = Z(C, G)$, the subgroup $C' \subseteq G' \cap C$ is central in C. Hence C is nilpotent of class at most two with a finite commutator subgroup. Now Lemma 9.97 shows that $Z(C)$ is open in C, and since C is open in G we know that $Z(C)$, a characteristic subgroup of G, is open in G. □

By Lemma 9.96, this applies to all profinite FC-groups. Collecting our information on compact FC-groups, we can establish the following main structure theorem on the structure of compact FC-groups:

Theorem 9.99. (The Structure of Compact FC-groups) *Let G be a compact group. Then the following three statements are equivalent:*

(1) *G is an* FC*-group.*
(2) *$G/Z(G)$ is finite, that is, G is center by finite.*
(3) *The commutator subgroup G' of G is finite.*

Proof. (1)⇒(2): By Lemma 9.94, $Z(G_0, G)$ is open and so by Proposition 9.95 there is a profinite FC-group Δ commuting elementwise with G_0 such that $Z(G_0, G) = G_0\Delta$. By Lemma 9.96 and Proposition 9.98, A_Δ is an open characteristic abelian subgroup of Δ whence $A \stackrel{\text{def}}{=} G_0 A_\Delta$ is an open characteristic abelian subgroup of $G_0\Delta = Z(G_0, G)$. Since $Z(G_0, G)$ is normal in G, the subgroup A is normal in G. Since A is open in $Z(G_0, G)$ and this latter group is open in G by Lemma 9.94, A is an open normal subgroup of G. Thus G is an abelian by finite FC-group. We claim that G is therefore a center by finite group: Indeed, let $f\colon G/A \to \mathrm{Hom}(A, A)$ be defined by $f(gA)(a) = gag^{-1}a^{-1}$. For each coset $\gamma = gA$ the image of $f(\gamma)$ is finite since G is an FC-group, and so $Z(g, G) = \ker f(\gamma)$ has finite index. Therefore $Z(G) = \bigcap_{\gamma\in G/A} \ker f(\gamma)$ has finite index, and this proves the claim.

(2)⇒(1): Since $Z(G) \subseteq Z(g, G)$ for each g, by (2), the quotient $G/Z(g, G)$ is finite for all g. Hence G is an FC-group.

(1)⇒(3): Let G be an FC-group. By the equivalence of (1) and (2) we know that G_0, being contained in the open subgroup $Z(G)$, is central. By Proposition 9.95, there is a profinite subgroup Δ such that $G = G_0\Delta$. Then $[G, G] = [\Delta, \Delta]$. Now by Lemma 9.96, $[\Delta, \Delta]$ is finite.

(3)⇒(1): Set $C(g) = \{xgx^{-1} : x \in G\}$; then $C(g)g^{-1} \subseteq G'$, whence $C(g) \in G'g$. Thus the finiteness of G' implies that of $C(g)$ for all $g \in G$. □

In fact, a group G satisfying the equivalent conditions of Theorem 9.99 above is what has been called a BFC-*group*, that is, an FC-group with all conjugacy classes having a length bounded by a fixed number.

Let us remark that center by finite groups are subject to classical central extension theory. For instance, if $Z(G)$ happens to be divisible by $q = |G/Z(G)|$, then $G = Z(G)E$ for a finite subgroup E such that $Z(G)\cap E$ equals $\{z \in Z(G) : z^q = 1\}$. (Cf. method of proof of Theorem 6.10(i).)

Theorem 9.99 might suggest to the reader that a compact FC-group is necessarily a product of a finite subgroup and its center. However, this is definitely not the case as the following comparatively simple example of a class 2 nilpotent profinite pro-p-group indicates.

Example 9.100. Let H be the class 2 nilpotent compact group of all 3×3-matrices

$$M(a, b; z) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & a & z \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, z range through the ring $\mathbb{Z}_p$ of p-adic integers. Let Z be the closed central subgroup of H of all $M(0, 0; pz)$, $z \in \mathbb{Z}_p$. Then $G = H/Z$ is a compact nilpotent p-group of class 2 whose commutator group $G' = [H, H]/Z$ contains all elements $M(0, 0; z)Z$ and thus is isomorphic to $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$. Its center $Z(G)$ consists of all elements $M(a, b; z)Z$ with $a, b \in p\mathbb{Z}_p$, $z \in \mathbb{Z}_p$, whence $G/Z(G) \cong \mathbb{Z}(p)^2$. The factor group G/G' is isomorphic to $\mathbb{Z}_p^2$. The subgroup of all $M(a, 0; 0)$, $a \in \mathbb{Z}_p$, is isomorphic to $\mathbb{Z}_p$; it is topologically generated by $M(1, 0; 0)$. By Theorem 9.99 again, this example yields a compact FC-group which is not the product of a central group with a finite group. The only nonsingleton finite subgroup is $G' \cong \mathbb{Z}(p)$ □

The Commutativity Degree of a Compact Group

We now aim to combine our Structure Theorem 9.99 for compact FC-groups with the probability measure theory which we outlined in Appendix 5. Recall from the discussion there that a Borel probability measure μ on a compact group respects the members of a class $\mathcal{C}$ of Borel subgroups of G if $\mu(H) > 0$ for $H \in \mathcal{C}$ implies that H is open. Haar measure respects all Borel subgroups.

Theorem 9.101. *Let G be a compact group and F its* FC*-center. Further let μ_1 and μ_2 be two Borel probability measures on G and set $P = \mu_1 \times \mu_2$ and $D \stackrel{\text{def}}{=} \{(g,h) \in G \times G : [g,h] = 1\}$. Assume that μ_j respects closed subgroups for $j = 1, 2$ and that, if G is not a Lie group, μ_2 respects even Borel subgroups. Then the following conditions are equivalent:*

(1) *$P(D) > 0$.*
(2) *F is open in G.*
(3) *The characteristic abelian subgroup $Z(F)$ is open in G.*

Under these conditions, the centralizer $Z(F,G)$ of F in G is open, and the finite group $\Gamma \stackrel{\text{def}}{=} G/Z(F,G)$ is finite and acts effectively on F with the same orbits as G under the well defined action $\gamma{\cdot}x = gxg^{-1}$ for $(\gamma, x) \in \Gamma \times F$, $g \in \gamma$. The isotropy group Γ_x at $x \in F$ is $Z(x,G)/Z(F,G)$, and the set F_γ of fixed points under the action of γ is $Z(g,F)$ for any $g \in \gamma$.

Proof. (1)⇔(2): This is a part of Corollary A5.7.

(2)⇒(3): The FC-center F of G is an FC-group in its own right. Then Theorem 9.99 shows that $Z(F)$ is open in F. By (2), F is open in G. Then $Z(F)$ is open in G, and this establishes (3).

The implication (3)⇒(2) is trivial.

Now assume that these conditions are satisfied. Then the open subgroup $Z(F)$ is contained in the centralizer $Z(F,G) = \bigcap_{x\in F} Z(x,G)$, which is the kernel of the morphism $G \to \operatorname{Aut} F$ sending $g \in G$ to $x \mapsto gxg^{-1}$. The homomorphism $\pi\colon \Gamma = G/Z(F,G) \to \operatorname{Aut} F$ is therefore well-defined by $\pi(\gamma)(x) = gxg^{-1}$, $\gamma = gZ(F,G)$, independently of the choice of the representative $g \in \gamma$. If $x \in F$ and $\gamma \in \Gamma$, say $\gamma = gZ(F,G)$, then $\gamma \in \Gamma_x$ iff $\gamma{\cdot}x = x$ iff $gxg^{-1} = x$ iff $g \in Z(x,G)$ regardless of the choice of $g \in \gamma$. Thus $\Gamma_\gamma = Z(x,G)/Z(F,G)$. Similarly, we have $x \in F_\gamma$ iff $\gamma{\cdot}x = x$ iff $gxg^{-1} = x$ for $g \in \gamma$ iff $x \in Z(g,F)$. □

While it is reasonable to have this level of generality for the measures μ_1 and μ_2 (see for instance [203]), and then search for useful measures μ_1 and μ_2 respecting closed subgroups, we shall specialize μ_1 and μ_2 at once to Haar measure. This case represents the special class of examples in which $P(E)$ is the probability that two randomly picked elements commute. In this situation one has called $P(\{(x,y) \in G \times G : [x,y] = 1\})$ the *commutativity degree* $d(G)$ of G.

Here, by the Main Theorem 9.101, the finite group $\Gamma = G/Z(F,G)$ acts effectively on F so that its orbits are the G-conjugacy classes of elements of F and that for the isotropy groups and fixed point sets we have

$$\Gamma_x = Z(x,G)/Z(F,G) \quad \text{and} \quad (\forall g \in \gamma)\, F_\gamma = Z(g,F).$$

In particular, if ν_F is Haar measure of F, for the closed subgroup $Z(g,F)$ of F and $g \in \gamma \in \Gamma$ we conclude

$$\nu_F(F_\gamma) = \begin{cases} 0 & \text{if } F_\gamma \text{ has no inner points in } F, \\ |F/Z(g,F)|^{-1} & \text{if } F_\gamma \text{ is open in } F. \end{cases}$$

We now formulate and prove the following main result:

PROBABILITY THAT x AND y COMMUTE IN A COMPACT GROUP

Theorem 9.102. *Let G be any compact group and denote by $d(G)$ its commutativity degree.Then we have the following conclusions:*

PART (i) *The following conditions are equivalent:*
(1) $d(G) > 0$.
(2) *The center $Z(F)$ of the* FC-*center F of G is open in G.*

PART (ii) *Assume that these conditions of* (i) *are satisfied. Then there is a finite set of elements $g_1, \ldots, g_n \in G$, $n \le |G/Z(F,G)|$, such that*

$$d(G) = \frac{1}{|G/F| \cdot |G/Z(F,G)|} \cdot \sum_{j=1}^{n} |F/Z(g_j,F)|^{-1}.$$

PART (iii) *$d(G)$ is always a rational number.*

Proof. (i) follows directly from 9.101. For a proof of (ii) we let ν_G and ν_F be the Haar measures of G and F, respectively, and recall from A5.7 that

$$d(G) = P(D) = \int_{x \in F} \nu_G(Z(x,G))d\nu_G(x) = \int_{x \in F} |G/Z(x,G)|^{-1} d\nu_G(x).$$

We note that the ν_G-measure of F is $|G/F|^{-1}$; thus $\nu_F = |G/F| \cdot (\nu_G|F)$. Hence

$$d(G) = \frac{1}{|G/F|} \cdot \int_F |G/Z(x,G)|^{-1} d\nu_F(x).$$

Now

$$G/Z(x,G) \cong (G/Z(F,G))/(Z(x,G)/Z(F,G)) = \Gamma/\Gamma_x,$$

and so, letting

$$E_F = \{(\gamma, x) \in \Gamma \times F : \gamma \cdot x = x\}$$

and $P = \nu_\Gamma \times \nu_F$, by $(*)$ above, we get

$$d(G) = \frac{1}{|G/F|} \cdot \int_F |\Gamma/\Gamma_x|^{-1} d\nu_F = \frac{1}{|G/F|} \cdot \int_F \nu_\Gamma(\Gamma_x) d\nu_F = \frac{P(E_F)}{|G/F|}.$$

Next we select a set $\{g_1, \ldots, g_n\}$ of elements of G such that the cosets $g_j Z(F,G)$ are exactly those elements $\gamma \in \Gamma$ whose fixed point set F_γ is open in F. Then $\Gamma_{g_j Z(F,G)} = Z(g_j, F)$ for all $j = 1, \ldots, n$. We also recall $\Gamma = G/Z(F,G)$ and apply Lemma 2.5 to see that

$$P(E_F) = \frac{1}{|G/Z(F,G)|} \sum_{j=1}^{n} |F/Z(g_j, F)|^{-1}.$$

Therefore

$$d(G) = \frac{1}{|G/F| \cdot |G/Z(F,G)|} \cdot \sum_{j=1}^{n} |F/Z(g_j, F)|^{-1}.$$

This is what we claimed.

Part (iii) is now an immediate consequence of (ii) as $d(G)$ is trivially rational if it is 0. □

The main result says: *If the probablity that two randomly picked elements commute in a compact group is positive, then, no matter how small it is, the group is almost abelian.*

We draw attention to the fact that in 9.102(ii) the commutativity degree of G is expressed in purely arithmetic terms via the group theoretical data F, $Z(F,G)$, and $Z(g_j, F)$.

Postscript

This book is titled "The Structure of Compact Groups," and it is in this chapter that four principal structure theorems are proved: the Levi–Mal'cev Structure Theorem 9.24, the Maximal Pro-Torus Theorem 9.32, the Borel–Scheerer–Hofmann Splitting Structure Theorem 9.39, and the Dong Hoon Lee Supplement Theorem, the last one in a generality that has not been published before. As was said in the introduction to this chapter, these theorems are impressive and powerful. One might in fact also count the Theorem on the Structure of Semisimple Compact Connected Groups 9.19 among the basic structure theorems.

The basic structure theorems are used here to characterize connectedness, simple connectedness, local connectedness, local arcwise connectedness, arcwise connectedness, and indecomposability for compact groups.

While the exponential function is a standard tool in Lie group theory, its use in the study of general compact groups is not widespread. However, we see just how powerful a tool it is for compact abelian groups in Chapter 8 and for general compact groups in this chapter. The arc component of the identity in a compact group is identified as the image of the exponential function, and the identity component is the closure of this subgroup.

The closedness of the commutator subgroup of a connected compact group—indeed Gotô's Theorem that each of its elements is a commutator is fundamental and rests on nontrivial arguments on compact Lie groups. The Levi–Mal'cev Structure Theorem 9.24 for connected compact groups is a classic. Its basis 9.24(i) states that G is nearly a direct product of the connected center $Z_0(G)$ and the commutator subgroup G'. Its proof is relatively simple because it rests on the easily obtained structure of Hilbert Lie algebras which we exposed in Theorem 6.4. That portion of the proof of the Levi–Mal'cev Theorem which causes a considerable amount of technical work is the structure of the Levi–Mal'cev complement G' (see 9.19). The complications are partially caused by the fact that a finite direct product $G = G_1 \times G_2$ of compact groups can easily be internalized by writing $G = N_1 N_2$ with $N_1 = G_1 \times \{1\}$ and $N_2 = \{1\} \times G_2$, and by focusing on the simple algebraic properties of N_1 and N_2. The case of an infinite product $\prod_{j \in J} G_j$ is not readily internalized, and if one claims that a given group is isomorphic to an infinite product of this type, one is challenged to produce this cartesian product externally. Secondly, the technical intricacies are evidenced by the fact that the concepts of simple connectivity and of connected compact groups are a somewhat delicate blend. Also, the fact that a semisimple compact Lie group has a *compact* universal cover (see 5.76) is nontrivial and is proved in this book via the Vector Group Splitting Theorem 5.71 which involves compact groups and is interesting in its own right even though it is not a structure theorem on compact groups per se. The fact that Corollary 5.66 persists for arbitrary compact connected groups which are not necessarily Lie groups is treated in [344].

The Maximal Pro-Torus Theorem 9.32 is a direct extension of the Maximal Torus Theorem 6.30 for compact Lie groups whose proof in Chapter 6 was not trivial either. While the generalisation from the case of compact Lie groups to arbitrary compact groups is not profound, in combination with the other structure theorems, the Maximal Pro-Torus Theorem is amazingly effective. One application is the proof of the Borel–Scheerer–Hofmann Theorem 9.39 on the splitting of the commutator group of a connected compact group. This extremely useful theorem first appeared in [167]. The topological splitting was first proved in [319]. For connected compact Lie groups the topological decomposition over the commutator subgroup was noted in [30]. Many of the particular results on the structure of compact connected groups in the latter parts of this section rest on this theorem. It is noteworthy that the characterisation theorems for arcwise connectivity and for local connectivity rest on this theorem and not on the more classical Levi–Mal'cev Structure Theorem which usually comes to mind first. Proposition 9.61 and Example 9.62 illustrate why the Levi–Mal'cev Theorem fails for such purposes. Theorem 9.60 uses splitting in order to import the results of compact *abelian* group theory into the general situation and to give a good description of arc connectivity in compact connected groups. The reduction to the abelian situation is similarly visible in the discussion of local connectivity in Theorem 9.65 and Corollary 9.66.

In Proposition 9.55 we give practical characterisations for a compact group to be finite dimensional, and in Proposition 9.56 we see that the dimension of a compact group equals the least upper bound of the dimensions of the cubes that can be embedded homeomorphically into them; this persists into transfinite dimensions as we have used for compact groups. If an n-dimensional compact group can be embedded topologically into $\mathbb{R}^n$, then it is a Lie group, as we saw in Theorem 9.59(v), and a subsequent construction shows how such embeddings occur.

The concept of a projective in the category of compact connected groups and normal morphisms (cf. text preceding 9.73) is not only of general interest in the context of homological algebra and category theory but is quite relevant for the structure theory of compact connected groups. Every compact connected group is canonically a quotient of such a projective modulo a compact totally disconnected central subgroup, and the projectives are characterized by a countable set of cardinals (cf. 9.19, 9.70 and 9.73). This is used in the structure theorems of the topological automorphism group $\mathrm{Aut}(G)$ of compact connected groups (9.87) and will be used intensively in later chapters (see 11.14(ii), 11.15, 11.16, 11.49, 11.51, 11.54). Given the theory of the projective cover it is easy to show that every compact group is dyadic (9.76(vii)). The universal property of the projective cover is exposed in Theorem 9.76bis. The structure of the automorphism group of a compact Lie group was discussed at length in Chapter 6, but the theory of the automorphism group of an arbitrary compact group is not a straightforward generalisation as the reader might have observed in this chapter. The role which the Lie algebra played in the case of compact connected Lie groups and which allowed us there to identify the automorphism group with a linear group, i.e. a "group of matrices," is taken over here by the projective cover of a compact connected group. This material is new.

The injective objects in the category $\mathbb{CN}$ are less significant for the general theory, but they are interesting in themselves (see 9.75).

The adjective "simple" as applied to compact groups is somewhat ambiguous as we point out in the paragraph preceding Definition 9.88, where we distinguish between a simple group and a simple topological group as a group having no nontrivial *closed* normal subgroups. In the subsequent Theorem 9.90 we show that for compact groups the two concepts agree.

With the clarification of the concept of a simple compact group it is also unambiguously clear that a *strictly reductive* compact group is a product of simple compact groups (9.88(ii)). This allows the formulation of the Countable Layer Theorem 9.91 which is comparatively new, although predecessors have been observed by Varopoulos [361]. We shall use this theorem in Chapter 12 below in our dealing with cardinal invariants, and in Chapter 10, Theorem 10.40 for a proof that all compact groups are dyadic, topologically.

In the theory of abstract infinite groups, groups whose conjugacy classes are finite have an established position and are known under the name FC-groups. A complete theory for compact FC-groups is relatively recent [202], although in the context of profinite groups they have been looked at. In Theorem 9.99 we argue that infinite compact FC-groups are nearly abelian in a twofold sense. We apply these insights to the question how likely it is that a randomly picked pair of elements of a compact group commutes. In general this is not at all probable in an infinite compact group unless it is almost commutative in a sense made precise in Theorem 9.102. One finds a discourse on the historical background of such questions in [202].

References for this Chapter—Additional Reading

[6], [25], [26], 27], [30], [70], [101], [142], [167], [179], [202], [203], [186], [187], [194], [199], [216], [217], [229], [236], [263], [296], [319], [344], [361], [365], [359].

Chapter 10

Compact Group Actions

A central aim of this chapter is to prove that in any compact group G the identity component G_0 is topologically split; that is there exists a compact totally disconnected subspace D of G such that the map $(g, d) \mapsto gd : G_0 \times D \to G$ is a homeomorphism; so the compact groups G and $G_0 \times G/G_0$ are homeomorphic. This requires some basic transformation group theory, which we develop. The concept of a principal fiber bundle is introduced here; it emerges in many branches of mathematics and it does belong to a general structure theory. Our development includes the fact that if G is a compact Lie group acting on a locally compact space X with all isotropy groups conjugate, then X is a principal fiber bundle.

We have seen in the discourse around 1.9 through 1.12 that actions of a compact group on a space help us to understand even very basic concepts. The first part of the following discussion (through 10.32) can be followed on the level of understanding of compact groups which we had at the end of Chapter I. After this portion of the section, we do need substantial information about compact Lie groups. We alert the reader when this point is reached.

The second part of the chapter deals with morphisms $f\colon A \to B$ between compact groups for which there is a continuous cross section $s\colon B \to A$, $f \circ s = \mathrm{id}_B$. We call such morphisms *topologically split*. Considering a surjective morphism $f\colon A \to B$ and asking whether it has a cross section is tantamount to considering a compact normal subgroup N of A, the kernel of f and asking if there is a compact subspace C of A such that $(n, c) \mapsto nc\colon N \times C \to G$ is a homeomorphism. In traditional algebra such a C would be called a *system of coset representatives*, and in abstract group theory it always exists (by the Axiom of Choice). Therefore the issue of the existence of C does not arise in abstract group theory at all. We are dealing in fact with the question whether a topologically split morphism is actually split. An equivalent formulation of the question is this: If a compact normal subgroup is topologically a direct factor, does there exist a compact subgroup which is a cofactor? In abstract group theory this is not the case, since it would imply that all normal subgroup are semidirect factors—which is false. It is remarkable that, in fairly general circumstances, connected compact normal subgroups of connected compact groups are indeed semidirect factors if they are merely topological factors.

Prerequisites. This chapter demands no prerequisites beyond those in Chapter 1 and (later on) Chapter 6 on compact Lie groups, except in the last section. We shall use the language of nets and the formalism of proper maps in some general topology arguments. In the proof of the Tietze–Gleason Extension Theorem 10.33 we use the Tietze Extension Theorem, and in the proof of Theorem 10.39 on the existence of certain cross sections (a theorem which we shall not use later) we shall use the fact that fiber bundles over contractible paracompact base spaces are trivial. In the second part of the chapter, certain rudimentary facts on the classification of simple compact connected Lie groups are used for the construction of examples. We shall give references in the appropriate places.

A Preparation Involving Compact Semigroups

We begin with a technical lemma which we shall need and which is of independent interest in the context of compact groups. The proof rests on some simple basic facts on compact semigroups. (We encounter some compact semigroups in our proof of the existence of Haar measure on a compact group in 2.8 and E5.1 in Appendix 5.)

Lemma 10.1. (i) *Let S be a closed subsemigroup of a compact group G; i.e. S is a closed subset satisfying $SS \subseteq S$. Then S is a subgroup.*

(ii) *If H is a compact subgroup of G such that for some $g \in G$, $gHg^{-1} \subseteq H$, then $gHg^{-1} = H$ and g is in the normalizer $N(H,G)$ of H in G.*

Proof. (i) Let $g \in S$ and set $C_n = \overline{\{g^n, g^{n+1}, \ldots\}}$. Then $C = \bigcap_{n=1}^{\infty} C_n$ is a compact commutative subsemigroup of G. Since $gC_n = C_{n+1}$ (as $x \mapsto gx$ is a homeomorphism) we have $gC = C$, hence $g^nC = C$ for $n = 1, \ldots$. Thus the compact group $\{t \in G \mid tC = C\}$ contains C. Hence for a, $b \in C$ there is an $x \in C$ such that $ax = b$. Therefore C is a group and thus contains the inverse g^{-1} of g in G.

(ii) The assumption $gHg^{-1} \subseteq H$ implies $g^nHg^{-n} \subseteq gHg^{-1}$. The set $S \stackrel{\text{def}}{=} \{t \in G \mid tHt^{-1} \subseteq gHg^{-1}\}$ is a compact subsemigroup of G containing g. By (i), S is a group and thus contains g^{-1}. Hence $g^{-1}Hg \subseteq H$; i.e. $H \subseteq gHg^{-1}$. Thus $H = gHg^{-1}$. ☐

Part (i) is also proved in Proposition A4.34. But in order to keep this chapter self-contained we repeated a version of the proof. A comparison of the two proofs may be a helpful exercise.

Orbits, Orbit Space, and Isotropy

Definitions 10.2. Let G be a compact group.

(i) A *G-space* X is a Hausdorff space together with an action $(g,x) \mapsto g{\cdot}x\colon G \times X \to X$ (see Definitions 1.9 and Proposition 1.10).

(ii) Let X/G denote the *space of orbits* $G{\cdot}x$, $x \in X$, endowed with its quotient topology. Then the continuous map $p = p_{G,X}\colon X \to X/G$, $p(x) = G{\cdot}x$ is called the *orbit map*. ☐

Lemma 10.3. *Let the compact group G act on a Hausdorff space X. Then the following assertions hold.*

(i) *The orbit map $p\colon X \to X/G$ is continuous and open.*

(ii) *Every open neighborhood U of an orbit $G{\cdot}x$ contains an invariant neighborhood $U' \stackrel{\rm def}{=} \bigcap_{g\in G} g{\cdot}U$.*

(iii) *X/G is a Hausdorff space.*

(iv) *If C is a compact subspace of X/G then $p_{G,X}^{-1}C$ is a compact invariant subspace of X.*

Proof. Exercise E10.1. □

Exercise E10.1. Prove 10.3.

[Hint. (i) For an open subset U of X, the set $p^{-1}\big(p(U)\big)$ is $G{\cdot}U = \bigcup_{g\in G} g{\cdot}U$ and is, therefore, open in X.

(ii) Use the proofs of 1.11 as guide.

(iii) Assume that two orbits $G{\cdot}x$ and $G{\cdot}y$ are different and show, using their compactness, that there are two disjoint open neighborhoods U of $G{\cdot}x$ and V of $G{\cdot}y$. Form $U' \subseteq U$ and $V' \subseteq V$ as in (ii) and show that $p(U')$ and $p(V')$ are disjoint neighborhoods of $G{\cdot}x$ and $G{\cdot}y$ in X/G, respectively.

(iv) Let $\mathcal{U}$ denote an open cover of $p^{-1}(C)$. For each $\xi \in C$ there is a finite subset $\mathcal{F}_\xi$ such that $p^{-1}(\xi) \subseteq \bigcup \mathcal{F}_\xi$. Set $V_\xi = \{\gamma \in C \mid p^{-1}(\gamma) \subseteq \bigcup \mathcal{F}_\xi\}$. Form V'_ξ according to (ii). Use the compactness of C to find finitely many $\xi_1, \ldots, \xi_n$ such that $C \subseteq V'_{\xi_1} \cup \cdots \cup V'_{\xi_n}$. Finish by producing the required finite subcover of $\mathcal{U}$.]□

Recall that $G_x \stackrel{\rm def}{=} \{g \in G \mid g{\cdot}x = x\}$ is called the *isotropy subgroup* of G at x or the *stabilizer* of G at x. (See the discussion preceding 1.9.)

Lemma 10.4. *Assume that a group G acts on a set X. Then*

(i) *for all $g \in G$ and all $x \in X$ the equation $G_{g\cdot x} = gG_xg^{-1}$ holds. As a consequence, the isotropy groups of the elements of an orbit range through the entire conjugacy class of the isotropy group of a fixed point in this orbit.*

(ii) *The structure of an orbit $G{\cdot}x$ is determined by the bijective map*

$$\omega_x\colon G/G_x \to G{\cdot}x, \quad \omega_x(gG_x) = g{\cdot}x$$

which satisfies $g_1{\cdot}\omega_x(g_2G_x) = \omega_x(g_1g_2G_x)$ and is embedded in the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{g\mapsto g\cdot x} & X \\ {\scriptstyle\rm quot}\big\downarrow & & \big\uparrow{\scriptstyle\rm incl} \\ G/G_x & \xrightarrow[\omega_x]{} & G{\cdot}x. \end{array}$$

(iii) *If G is a compact group and X is a Hausdorff G-space, then all functions in the diagram are continuous closed maps and ω_x is a homeomorphism.*

(iv) *Let G be a compact group and X a Hausdorff G-space. Let $\dim G{\cdot}x$ be Lebesgue covering dimension if finite, and $w(G{\cdot}x)$ otherwise. Then $G{\cdot}x$ contains* *a cube homeomorphic to* $\mathbb{I}^{\dim G\cdot x}$, $\mathbb{I}=[0,1]$. *Moreover, if a homeomorphic copy of* $\mathbb{I}^{\aleph}$ *is contained in* $G\cdot x$, *then* $\aleph \le \dim G\cdot x$.

Proof. Exercise E10.2. □

Exercise E10.2. Prove 10.4. Attention: part (iv) is a nontrivial task. This information (and more) is contained in [185]. □

A typical action of a compact group without stable isotropy (as defined in 10.5) is given by the action of the circle group $G=\mathbb{S}^1$ on the complex unit disc $X=\mathbb{D}\overset{\text{def}}{=}\{z\in\mathbb{C}:|z|\le 1\}$ by multiplication. Then

$$G_z=\begin{cases}\{1\} & \text{if } z\ne 0,\\ G & \text{if } z=0.\end{cases}$$

The orbits are circles with the exception of the orbit of 0 which is singleton.

In view of 10.4 is readily understood why one says that two orbits $G\cdot x$ and $G\cdot y$ *have the same orbit type* if G_x and G_y are conjugate. Indeed, if $G_y=gG_xg^{-1}$ and if we write $I_g(hG_x)=ghG_xg^{-1}=ghg^{-1}G_y$, then we have a sequence of homeomorphisms

$$G\cdot x \xrightarrow{\omega_x^{-1}} G/G_x \xrightarrow{I_g} G/G_y \xrightarrow{\omega_y} G\cdot y.$$

Definitions 10.5. (i) We say that a group G acts *freely* on a set X if all isotropy groups are singleton.

(ii) The group G is said to act with *stable isotropy* or to act *with one single orbit type* if for two elements $x,y\in X$ there is an element $g\in G$ such that $G_x=gG_yg^{-1}$. If H is one of these isotropy groups we shall also say that G *acts with stable isotropy conjugate to* H. □

For a free action the functions $g\mapsto g\cdot x\colon G\to G\cdot x$ are bijective for all $x\in X$. A group acting freely obviously acts with stable isotropy. The latter concept is a generalization of the former.

The function $x\mapsto G_x$ from the space X to the set of closed subgroups of G has remarkable "continuity" properties as the following proposition shows. This proposition is not used directly in later proofs. Its proof is technical but elementary. Whatever is needed later in this direction will be provided independently.

Proposition 10.6 (Semicontinuity and Continuity of Isotropy). *Assume that the compact group* G *acts on the compact space* X *and let* $x\in X$. *Denote the filter of neighborhoods of* x *by* $\mathcal{U}$. *Then the following conclusions are valid.*

(i) $\bigcap_{U\in\mathcal{U}}\overline{\bigcup_{u\in U}G_u}\subseteq G_x$. *If* G *acts with stable isotropy, then equality holds.*

(ii) *For each identity neighborhood* V *of* G *there is a neighborhood* $U\in\mathcal{U}$ *of* x *in* X *such that* $u\in U$ *implies* $G_u\subseteq VG_x$. *If all isotropy groups are conjugate, then there is a* $U\in\mathcal{U}$ *such that*

$$(\forall u\in U)\quad G_u\subseteq VG_x \quad \textit{and} \quad G_x\subseteq VG_u$$

holds.

Proof. (i) For $U \in \mathcal{U}$ define

$$C_U = \overline{\bigcup_{u\in U} G_u}.$$

Note that $\{C_U \mid U \in \mathcal{U}\}$ is a filter basis of compact sets. Assume $g \in \bigcap_{U\in\mathcal{U}} C_U$. Then for each $U \in \mathcal{U}$ and each identity neighborhood V in G there is a $u \in U$ and $g_{U,V} \in G_u$ such that $g_{U,V} \in V^{-1}g$. Now $g_{U,V}{\cdot}u = u$. Thus

$$g{\cdot}u \in Vg_{U,V}{\cdot}u = V.u \subseteq V{\cdot}U.$$

Hence $g{\cdot}U \subseteq V{\cdot}U$ and since U and V were arbitrary, the continuity of the action implies $g{\cdot}x = x$; i.e. $g \in G_x$.

Now we assume that all isotropy groups are conjugate. For $U \in \mathcal{U}$ pick $u_U \in U$. By hypothesis, there is a $g_U \in G$ such that $g_U G_x g_U^{-1} = G_{u_U} \subseteq C_U$. Define $\Gamma_U \stackrel{\text{def}}{=} \overline{\{g_U \mid U \in \mathcal{U}\}}$. Then $\overline{\{\gamma h \gamma^{-1} \mid h \in G_x,\, \gamma \in \Gamma_U\}} \subseteq C_U \subseteq C_V$ for $U \subseteq V$. Since $\{\Gamma_U \mid U \in \mathcal{U}\}$ is a filter basis of compact sets, there is an element $g \in \bigcap_{U\in\mathcal{U}}$ in its intersection. We conclude $gG_xg^{-1} \subseteq C_U$ for all $U \in \mathcal{U}$. Thus $gG_xg^{-1} \subseteq C \stackrel{\text{def}}{=} \bigcap_{U\in\mathcal{U}} C_U \subseteq G_x$ by (i). From Lemma 1.40 we now derive $G_x = gG_xg^{-1}$ which implies $C = G_x$, and this proves (i).

(ii) Let V be an open identity neighborhood of G, then VG_x is an open neighborhood of G_x. We claim that (i) implies the existence of a $U \in \mathcal{U}$ such that $C_U \subseteq VG_x$ which will prove the claim. Suppose that our claim is false. Then $K_U \stackrel{\text{def}}{=} C_U \setminus VG_x$ is nonempty compact, and $\{K_U \mid U \in \mathcal{U}\}$ is a filter basis of compact sets. Its intersection therefore contains an element k. Then $k \notin VG_x$ and $k \in \bigcap_{U\in\mathcal{U}} C_U \subseteq G_x$ by (i). This is a contradiction.

Now we assume that all isotropy groups are conjugate. By hypothesis for each $u \in X$ there is a g_u such that $G_u = g_u G_x g_u^{-1}$. Thus in view of the first part of (ii), in order to finish the proof of (ii) we have to find $U \in \mathcal{U}$ such that $g_u G_x g_u^{-1} \subseteq VG_x$. Suppose this is not the case. Then for each $U \in \mathcal{U}$ there is a $u_U \in U$ and a $g_U \in G_x$ such that $g_{u_U} g_U g_{u_U}^{-1} \notin VG_x$. The net $(g_{u_U}, g_U)_{U\in\mathcal{U}}$ in the compact space $G \times G_x$ has a subnet which converges to an element $(g, g') \in G \times G_x$. (For the use of nets see e.g. [230], p. 62ff.) Since VG_x is open, we conclude that

$$(*) \qquad gg'g^{-1} \notin VG_x.$$

But for $U \subseteq V$ in $\mathcal{U}$ we have $g_{u_U} g_U g_{u_U}^{-1} \in G_{u_U} \subseteq C_V$. It follows that $gg'g^{-1} \in C_V$ for all $V \in \mathcal{U}$. Hence by (i) above,

$$(**) \qquad gg'g^{-1} \in \bigcap_{U\in\mathcal{U}} C_U \subseteq G_x.$$

But obviously $(*)$ and $(**)$ contradict each other. This completes the proof. □

Proposition 10.7. *Assume that the compact group G acts on the Hausdorff space X with stable isotropy. For some $x_0 \in X$ write $H \stackrel{\text{def}}{=} G_{x_0}$ and let N denote the normalizer $N(H,G) = \{g \in G \mid gHg^{-1} = H\}$ of H in G. Set $Y = \big\{x \in X \mid H{\cdot}x = \{x\}\big\}$. Then the following conclusions hold:*

(i) *Y is a closed subset of X invariant under the action of N.*
(ii) *For all $y \in Y$ the relation $G_y = H$ holds, and Y meets each orbit.*
(iii) *The compact group N/H acts on Y freely via $nH{*}y \stackrel{\rm def}{=} n{\cdot}x$.*
(iv) *If y, $g{\cdot}y \in Y$ for $g \in G$, then $g \in N$. Further, $Y \cap G{\cdot}y = N{\cdot}y$ for all $y \in Y$.*

Proof. (i) Since the subset Y of X is the intersection of the fixed point sets of the family of maps $x \mapsto h{\cdot}x$ for $h \in H$ it is certainly closed in X. The group H is normal in N. Thus $n \in N$, and $h \in H$ imply $n^{-1}hn \in H$, and if now $y \in Y$, then $h{\cdot}(n{\cdot}y) = n{\cdot}(n^{-1}hn){\cdot}y = n{\cdot}y$, since y is fixed by all elements of H by definition of Y. Thus Y is invariant under the action of N

(ii) For all $y \in Y$ we have $H \subseteq G_y$. Since all isotropy groups are conjugate, $G_y = g^{-1}Hg$ for some $g \in G$. Thus $gHg^{-1} \subseteq H$. By Lemma 10.1, equality holds and thus $G_y = H$ for all $y \in Y$. If $x \in X$ then by the stability of the isotropy there is a $g \in G$ such that $G_{g\cdot x} = gG_xg^{-1} = H$. Hence $g{\cdot}x \in Y$ and thus Y meets every orbit.

(iii) As a consequence of $N{\cdot}Y = Y$, the compact group N/H acts on Y via $nH{*}x = n{\cdot}x$ in such a fashion that nH is in the isotropy group $(N/H)_y$ at y iff $n{\cdot}y = (nH){*}y = y$ iff $n \in G_y = H$; that is all isotropy groups of the action of N/H on Y are trivial.

(iv) If y, $g{\cdot}y \in Y$ for some $g \in G$, then $G_x = G_{g\cdot y} = gHg^{-1}$. Since $x \in Y$ we have $G_x = H$. Hence $g \in N$. Since Y is invariant under N, clearly $N{\cdot}y \subseteq Y \cap G{\cdot}y$. Conversely, let $x \in Y \cap G{\cdot}y$. Then $x = g{\cdot}y \in Y$ and thus $g \in N$ by the preceding. Hence $x \in N{\cdot}y$. □

Equivariance and Cross Sections

Definitions 10.8. (i) If a group G acts on sets X and Y then a function $f\colon X \to Y$ is called *equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

(ii) Two actions of a topological group G on topological spaces X, respectively, Y, are called *isomorphic* if there is an equivariant homeomorphism $f\colon X \to Y$. We shall also call f an *isomorphism of G-spaces* in these circumstances. □

Definitions 10.9. Assume that a compact group G acts on a Hausdorff space X. A *function* $\sigma\colon X/G \to X$ is called a *cross section* if it is continuous and satisfies $p_{G,X} \circ \sigma = \mathrm{id}_{X/G}$. We say that the cross section *passes through* $x \in X$ if $\sigma\big(p_{G,X}(x)\big) = x$. A *subset* $S \subseteq X$ is called a *cross section* if it is closed and

$$X = G{\cdot}S \quad \text{and} \quad (\forall s \in S)\ \ S \cap G{\cdot}s = \{s\}. \qquad \square$$

We note that two different things are given the same name. This use is unfortunate, but it follows the tradition. (We observed a similar misnomer in the form of a *one parameter subgroup* in a topological group (Definition 5.7) which is not a "subgroup" in the usual sense, or in the form of a *compact Lie algebra* (Definition 6.1) which is not a compact space at all.)

Notice that if the action of G on X has a cross section $\sigma: X/G \to X$, then for each $g \in G$ the function $\xi \mapsto g{\cdot}\sigma(\xi): X/G \to X$ is likewise a cross section. In particular, for each $x \in X$ there is a cross section passing through x; indeed there is a $g \in G$ such that $x = g{\cdot}\sigma\big(p_{G,X}(x)\big)$.

The following theorem tells us that the two concepts of a cross section, at least in the context of the action of a *compact* group are equivalent.

Definition 10.10. An action of a compact group G on a compact space X is called *trivial* if there is a compact subgroup H of G and an equivariant homeomorphism $\Psi: G/H \times X/G \to X$ such that $p_{G,X}\big(\Psi(gH,\xi)\big) = \xi$ for all $gH \in G/H$ and $\xi \in X/G$. □

A trivial action is always an action with stable isotropy.

Let $\mathbb{S}^1$ again denote the complex unit circle. The group $G = \{1, -1\}$ acts on $X = \mathbb{S}^1$ by multiplication $(g, x) \mapsto gx: G \times X \to X$. This action is free (hence with stable isotropy) but is not trivial.

The Cross Section Theorem

Theorem 10.11. *Let the compact group G act on a Hausdorff space X. Consider the following conditions.*

(i) *There is an equivariant continuous surjective function $\Phi: G \times X/G \to X$ where the action on $G \times X/G$ is given by $g{\cdot}(g', \xi) = (gg', \xi)$ and where $\Phi(g, \xi) \in \xi$ for all $g \in G$ and $\xi \in X/G$.*

(ii) *There is a cross section $\sigma: X/G \to X$.*

(iii) *There is a cross section $S \subseteq X$.*

(iv) *The action is trivial.*

(v) *There is a closed subgroup H of G and there is a cross section $S \subseteq X$ such that $G_s = H$ for all $s \in S$.*

Then

$$\text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(i)} \Rightarrow \text{(iii)}$$

and if X is locally compact, then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)}.$$

Proof. Trivially (v) implies (i).

(iv)⇒(v) Assume (iv). By 10.10 we are given an equivariant homeomorphism $\Psi: G/H \times X/G \to X$ with $\Psi(gH, \xi) \in \xi$. The function $\sigma: X/G \to X$, $\sigma(\xi) = \Psi(H, \xi)$ satisfies the requirements of (v).

(v)⇒(iv) The function $\Phi: G \times X/G \to X$, $\Phi(g, \xi) = g{\cdot}\sigma(\xi)$ is continuous. Assume $\Phi(g_1, \xi_1) = \Phi(g_2, \xi_2)$ implies $\xi_1 = p\big(g_1{\cdot}\sigma(\xi_1)\big) = \Phi(g_1, \xi_1) = \Phi(g_2, \xi_2) = p\big(g_2{\cdot}\sigma(\xi_2)\big) = \xi_2 \stackrel{\text{def}}{=} \xi$. Then $g_1{\cdot}\sigma(\xi) = g_2{\cdot}\sigma(\xi)$ implies $g_2^{-1}g_1 \in G_{\sigma(\xi)} = H$. Thus, if $q: G \to G/H$ is the quotient map given by $q(g) = gH$, the function Φ factors through a unique bijective continuous function $\Psi: G/H \times X/G \to X$ so that $\Phi \stackrel{\text{def}}{=} \Psi \circ (q \times \mathrm{id}_{X/G})$. Since $\Psi(gH,\xi) = g\sigma(\xi)$ we have $\Psi\big(g{\cdot}(g'H,\xi)\big) = \Psi(gg'H,\xi) = gg'{\cdot}\sigma(\xi) = g{\cdot}\Psi(g'H,\xi)$; i.e. Ψ is equivariant.

The projection $\mathrm{pr}_2: G/H \times X/G \to X/G$ is proper (see [33], p. 77, Corollaire 5). Also, $\mathrm{pr}_2 = p_{X,G} \circ \psi$. Hence ψ is proper ([33], p. 73, Proposition 5d)) and thus is closed ([33], p. 72, Proposition 1). Hence it is a homeomorphism. By Definition 10.10, (iv) follows.

(i)⇒(ii) If $\Phi: G \times X/G \to X$ is an equivariant surjective continuous map, we have a commutative diagram

$$\begin{array}{ccc} G \times X/G & \xrightarrow{\Phi} & X \\ {\scriptstyle \mathrm{pr}_{X/G}}\downarrow & & \downarrow{\scriptstyle p_{G,X}} \\ X/G & \xrightarrow{\varphi} & X/G \end{array}$$

with a continuous surjective map φ given by $\varphi(\xi) = \Phi(G\times\{\xi\})$. Set $\sigma(\xi) = \Phi(1,\xi)$. Then $G{\cdot}\sigma(\xi) = G{\cdot}\Phi(1,\xi) = \Phi(G \times \{\xi\}) = \xi$ in view of the equivariance of Φ.

(ii)⇒(i) Define $\Phi: G\times X/G \to G$ by $\Phi(g,\xi) = g{\cdot}\sigma(\xi)$. Then $\Phi(g,\xi) \in G{\cdot}\sigma(\xi) = \xi$ and $g{\cdot}\Phi(g',\xi) = g{\cdot}\big(g'{\cdot}\sigma(\xi)\big) = gg'{\cdot}\sigma(\xi) = \Phi(gg',\xi)$.

(ii)⇒(iii) Assume (ii); set $S = \sigma(X/G)$. This set is a cross section.
Assume for the remainder of the proof that X is locally compact. Then X/G is a locally compact Hausdorff space by 10.3(i), (iii).

(iii)⇒(ii) The restriction $\pi \stackrel{\text{def}}{=} p|S: S \to X/G$ is continuous and bijective by (iii). Define $\sigma: X/G \to X$ by $\sigma(\xi) = \pi^{-1}(\xi)$. Then the function σ satisfies $p \circ \sigma = \mathrm{id}_{X/G}$. If C is a compact neighborhood of $G{\cdot}x \in X/G$, then $Y = p^{-1}(C)$ is a compact G-space by 10.3(iv) and $S \cap Y$ is a compact cross section. Hence the bijective continuous map $\pi|(S \cap Y): S \cap Y \to C$ is a homeomorphism, and thus $\xi \mapsto \sigma(\xi): C \to S \cap Y$, being its inverse, is continuous. Since C is a neighborhood of $G{\cdot}x$, σ is continuous at $G{\cdot}x$. Since $G{\cdot}x$ was an arbitrary element of X/G, σ is continuous and thus is a cross section. □

With the notation $Y = \big\{x \in X \mid H{\cdot}x = \{x\}\big\}$ of Proposition 10.7 we may reformulate condition (v) as

(v′) The $N(H,G)/H$-space Y has a cross section. Thus *under the hypotheses of* 10.7*, the G-space X is trivial if the N/H-space Y has a cross section.*

In conditions (i), (ii), (iii) on isotropy nothing is assumed, while in conditions (iv), (v) stable isotropy is built in; it is therefore no surprise that the five conditions of Theorem 10.11 are not equivalent. However, we shall see that in general, even if stable isotopy is guaranteed, the five conditions of Theorem 10.11 are not equivalent. (see Example 10.17). The next section will shed additional light on the question of equivalence of all five conditions.

Remark 10.12. Let the compact group G act on a Hausdorff space X. Then the following conditions are equivalent:

(i) There is a cross section $\sigma: X/G \to X$.

(ii) There is a continuous retraction $\kappa: X \to X$, $\kappa^2 = \kappa$, such that $\kappa(x) \in G{\cdot}x$.

Proof. (i)⇒(ii) The function $\kappa = \sigma \circ p_{G,X}$ satisfies the requirements.

(ii)⇒(i) For $x, y \in X$ we have $\kappa(x) = \kappa(y)$ iff there is a $g \in G$ such that $y = g{\cdot}x$. Thus the canonical factorisation theorem for continuous functions yields a continuous function $\sigma: X/G \to X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & X \\ {\scriptstyle p_{G,X}}\downarrow & & \| \\ X/G & \xrightarrow[\sigma]{} & X \end{array}$$

commutes. Then σ is the required cross section. □

Recall that an equivariant function $\psi: X \to Y$ of G-spaces induces a continuous function $\overline{\psi}: X/G \to Y/G$, $\overline{\psi}(G{\cdot}x) = G{\cdot}\psi(x)$. We say that ψ is *faithful on orbits*, if $\psi|G.x: G{\cdot}x \to G{\cdot}\psi(x)$ is injective for all $x \in X$. Notice that this map is automatically surjective and thus bijective under these circumstances. If G is compact then $\psi|G{\cdot}x$ is a homeomorphism.

Proposition 10.13. *Let $\psi: X \to Y$ an equivariant function of G-spaces for a topological group G. Then*

(i) *$G_x \subseteq G_{\psi(x)}$ and the following statements are equivalent:*
  (1) *$\psi|G{\cdot}x: G{\cdot}x \to G{\cdot}\psi(x)$ is injective.*
  (2) *$G_x = G_{\psi(x)}$.*

(ii) *If the action on Y is free then the action on X is free.*

(iii) *Assume that G is compact and that the equivariant continuous function $\psi: X \to Y$ is faithful on orbits.*

Assume firstly that X is locally compact and that $\tau: Y/G \to Y$ is a cross section. Then the function $\sigma: X/G \to X$, $\sigma(\xi) = (\psi_\xi)^{-1}\big(\sigma(\overline{\psi}(\xi))\big)$ is a cross section satisfying $(*)$ $\psi\big(\sigma(\xi)\big) = \tau\big(\overline{\psi}(\xi)\big)$; i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle \sigma}\uparrow & & \uparrow{\scriptstyle \tau} \\ X/G & \xrightarrow{\overline{\psi}} & Y/G. \end{array}$$

Assume secondly that $\overline{\psi}: X/G \to Y/G$ has a cross section ψ^*, that Y is locally compact and $\sigma: X/G \to X$ is a cross section. Then the function $\tau: Y/G \to Y$, $\tau = \psi \circ \sigma \circ \psi^*$ is a cross section satisfying $(*)$ above.

(iv) *Let G be a compact group acting on a locally compact Hausdorff space X and let $\psi: X \to Y$ be an equivariant continuous map into a trivial G-space which is faithful on orbits. Then X is a trivial G-space.*

(v) *Let G be a compact group acting on a locally compact Hausdorff space Y and let $\psi: X \to Y$ be an equivariant surjective continuous map from a trivial G-space which is faithful on orbits. Assume that $\overline{\psi}: X/G \to Y/G$ has a continuous cross section. Then Y is a trivial G-space.*

Proof. (i) Let $x \in G_x$. Then $g{\cdot}\psi(x) = \psi(g{\cdot}x) = \psi(x)$ and thus $x \in G_{\psi(x)}$.

(1)⇒(2) Conversely, $g \in G_{\psi(x)}$ implies $\psi(x) = g{\cdot}psi(x) = \psi(g{\cdot}x)$. Then (1) implies $g{\cdot}x = x$; i.e. $g \in G_x$. Thus (2) follows.

(2)⇒(1) Let ω_x and $\omega_{\psi(x)}$ be defined as in 10.4, and define $q\colon G/G_x \to G/G_{\psi(x)}$ by $q(gG_x) = gG_{\psi(x)}$. Then the following diagram commutes:

$$\begin{array}{ccc} G/G_x & \xrightarrow{\omega_x} & G{\cdot}x \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \psi|G_x} \\ G/G_{\psi(x)} & \xrightarrow[\omega_{\psi(x)}]{} & G{\cdot}\psi(x) = \psi(G{\cdot}x). \end{array}$$

Thus $\psi|G_x$ is injective iff q is injective, and that implies (2).

(ii) To say that the action on Y is free means that $G_y = \{1\}$ for all $y \in Y$. Then $G_x \in G_{\psi(x)} = \{1\}$ for all $x \in X$; i.e. the action on X is free.

(iii) The function is well defined; the identity $(*)$ and the relation $\sigma(\xi) \in \xi$ are consequences of the definition. We must show the continuity of σ. For this, for each $x \in X$ we show that for any net $(\xi_j)_{j\in J}$ converging to $\xi \stackrel{\text{def}}{=} G{\cdot}x$ in X/G we have $\sigma(\xi) = \lim_{j\in J}\sigma(\xi_j)$. Let C be a compact neighborhood of $\xi \in X/G$. Such a neighborhood exists since X is locally compact by 10.3(i). Then $X' = p_{G,X}^{-1}C$ is a compact neighborhood of $G{\cdot}x$ by 10.3(iv). Hence it suffices to show that any relation $x^* = \lim_{j\in J}\sigma(\xi_j)$ implies $x^* = \sigma(\xi)$. Now using $(*)$ and the continuity of ψ, τ, and ψ' we get $\psi(x*) = \lim_{j\in J}\psi(\sigma(\xi_j)) = \lim_{j\in J}\tau\big(\psi'(\xi_j)\big) = \tau\big(\psi'(\xi)\big) = \psi(\sigma(\xi)$. Since ψ is injective on fibers this entails $gH = f(b)$ and the continuity of σ and thus assertion (iii) are proved.

(iv) follows from (iii) and the Cross Section Theorem 10.11.

(v) According to hypothesis let $\psi^*\colon Y/G \to X/G$ be a cross section for $\overline{\psi}\colon X/G \to Y/G$. For any cross section $\sigma\colon X/G \to X$ define τ by $\psi \circ \sigma \circ \psi^*$ as in (iii) above. Then τ is a cross section by (iii) above. Since X is a trivial G-space, by the Cross Section Theorem 10.11 it has a cross section $\sigma\colon X/G \to X$ such that $G_{\sigma(\xi)} = H$ for all $\xi \in X/G$. For such a cross section σ, Condition (i) above yields $G_{\tau_\sigma(\eta)} = G_{\psi(\sigma(\tau(\eta)))} = G_{\sigma(\tau(\eta))} = H$. Thus by the Cross Section Theorem 10.11 again, Y is trivial. □

We observe in 10.13(iv) that something like a cross section for $X/G \to Y/G$ is necessary as the example in Exercise E10.3 shows.

Exercise E10.3. Verify the details of the following

Example. Let X be the 2-torus and $\mathbb{S}^1$-space $(\mathbb{S}^1)^2$ with $z{\cdot}(z_1, z_2) = (zz_1, z_2)$ and let the group $\Gamma = \{\mathrm{id}, \varepsilon\}$, $\varepsilon(z_1, z_2) = (-z_1, z_2^{-1})$ act on X. Then $z{\cdot}\varepsilon(z_1, z_2) = z{\cdot}(-z_1, z_2^{-1}) = (-zz_1, z_2^{-1}) = \varepsilon\big(z{\cdot}(z_1, z_2)\big)$. Thus the two actions commute, and so the Klein bottle $Y \stackrel{\text{def}}{=} X/\Gamma$ is an $\mathbb{S}^{-1}$-space via $z{\cdot}(z_1, z_2)\Gamma = (zz_1, z_2)\Gamma$. The Γ-orbit map $\psi\colon X \to Y$ is equivariant and faithful on $\mathbb{S}^1$-orbits. But X is a trivial $\mathbb{S}^1$ space and Y is not. The induced map $\overline{\psi}\colon X/\mathbb{S}^1 \to Y/\mathbb{S}^1$ is a double covering of the circle. □

Triviality of an Action

A trivial action clearly allows cross sections and has stable isotropy. The converse (in contrast to what is sometimes believed even in the literature) is not true. We shall see examples in this section; they arise out of a canonical construct attached to each action of a compact group with cross sections and stable isotropy and which allows the formulation of a precise triviality criterion.

The first proposition formulates information which is attached canonically to a pair (G, H) consisting of a topological group G and a closed subgroup H.

Proposition 10.14. *Assume that G is a topological group. Let H be a closed subgroup of G and $N = N(H,G)$ its normalizer $\{g \in G \mid gHg^{-1} = H\}$ in G. Denote by $\mathcal{P}$ the set of all closed nonempty subsets of G and by $\mathcal{C} \subseteq \mathcal{P}$ the class $\{gHg^{-1} \mid g \in G\}$ of conjugates of H. Then $N(H,G)$ is closed and G acts on $G/N(H,G)$ by $g{\cdot}\xi = gg'N$ for $\xi = g'N$. The following conclusions hold.*

(i) *The group G acts on $\mathcal{P}$ by $g{\cdot}A \stackrel{\text{def}}{=} gAg^{-1}$ and $\mathcal{C}$ is an orbit of this action. The isotropy group of this action at H is N, and the bijection $\beta\colon G/N \to \mathcal{C}$, $\beta(gN) = gHg^{-1}$ is a part of the commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{g\mapsto gHg^{-1}} & \mathcal{P} \\ {\scriptstyle\text{quot}}\big\downarrow & & \big\uparrow{\scriptstyle\text{incl}} \\ G/N & \underset{\beta}{\longrightarrow} & \mathcal{C}. \end{array}$$

(ii) *The group $\Gamma \stackrel{\text{def}}{=} H \rtimes N$ with multiplication $(h_1,n_1)(h_2,n_2) = (h_1n_1h_2n_1^{-1}, n_1n_2)$ acts on the right of $G \times G$ via $(g,x){\cdot}(h,n) = (gxhx^{-1}, xn)$. The orbit $(g,x){\cdot}\Gamma$ of (g,x) is the set $gxHx^{-1} \times xN$. The orbit space*

$$\mathcal{X} = \mathcal{X}(G;H) \stackrel{\text{def}}{=} (G \times G)/\Gamma$$

is a Hausdorff space if H and N are compact. The group G acts on the left on $(G \times G)$ via $g{\cdot}(g',x) = (gg',x)$, and this action commutes with that of Γ. Thus the action of G on the left on $G \times G$ induces an action of G on $\mathcal{X}$ on the left as follows. For $g \in G$ and $\mathbf{x} = (g',x){\cdot}\Gamma$ the set $g{\cdot}\mathbf{x} \stackrel{\text{def}}{=} (gg',x){\cdot}\Gamma$ is an element of $\mathcal{X}$ depending only on g and $\mathbf{x}$. Then the action $(g,\mathbf{x}) \mapsto g{\cdot}\mathbf{x} : G \times \mathcal{X} \to \mathcal{X}$ has the following properties.

(a) *For $\mathbf{x} = (g,x){\cdot}\Gamma$ the isotropy group $G_{\mathbf{x}}$ is $\beta(gxN) = gxH(gx)^{-1}$, and the orbit $G{\cdot}\mathbf{x}$ is $\{(g,x){\cdot}\Gamma \mid g \in G\}$ and if G is compact, then it is homeomorphic to $G/(gx)H(gx)^{-1}$ and to G/H.*

(b) *The projection $\mathrm{pr}_2\colon G \times G \to G$ induces a continuous function $p\colon \mathcal{X} \to G/N$, $p\big((g,x){\cdot}\Gamma\big) = xN$ which is equivalent to the orbit map $p_{G,\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}/G$ in the sense that there is a homeomorphism $\theta\colon G/N \to \mathcal{X}/G$, $\theta(xN) = G{\cdot}\big((1,x){\cdot}\Gamma\big) = \{(g,x){\cdot}\Gamma \mid g \in G\}$ such that $p_{G,\mathcal{X}} = \theta \circ p$.*

(c) *The function $\gamma\colon G/N \to \mathcal{X}$, $\gamma(xN) = (1,x){\cdot}\Gamma$ satisfies $p \circ \gamma = \mathrm{id}_{G/N}$ and $G_{\gamma(xN)} = \beta(xN) = xHx^{-1}$ for all $x \in G$.*

Proof. We note that $g \in N$ iff $gHg^{-1} \subseteq H$ and $g^{-1}Hg \in H$ iff $(\forall h \in H)\, ghg^{-1} \in H$ and $g^{-1}hg \in H$. Since the functions $g \mapsto ghg^{-1}$, $g^{-1}hg: G \to G$ are continuous and H is closed, it follows that $N(H,G)$ is closed. The action of G on G/N is clear. It is straightforward that G acts on $\mathcal{P}$ under conjugation, that the class of conjugates of H is an orbit and that, by the very definition of the normalizer, the isotropy group of this action at H is $N(H,G)$. The function β is well defined and the commutativity of the diagram is a consequence of 10.4(ii).

(ii) It is straightforward to verify that Γ acts on $G \times G$ on the right. An orbit space of a compact group acting on a Hausdorff space is Hausdorff by 10.3(iii). The verification that the left action of G on $G \times G$ by left multiplication on the left factor commutes with the right action of Γ and induces a left action on $\mathcal{X}$ is straightforward and is part of Exercise E10.4.

Property (a) An element $g' \in G$ is in the isotropy group $G_{\mathbf{x}}$, $\mathbf{x} = (g,x){\cdot}\Gamma$ of the action of G on $\mathcal{X}$ iff $(g,x){\cdot}\Gamma = \mathbf{x} = g'{\cdot}\mathbf{x} = (g'g,x){\cdot}\Gamma$ iff $gxHx^{-1} \times xN = g'gxHx^{-1} \times xN$ iff $gxHx^{-1}g^{-1} = g'gxHx^{-1}g^{-1}$ iff $g' \in (gx)H(gx)^{-1}$. Thus $G_{\mathbf{x}} = (gx)H(gx)^{-1}$. Also, $G{\cdot}\mathbf{x} = \{(g'g,x){\cdot}\Gamma \mid g' \in G\} = \{(g',x){\cdot}\Gamma \mid g' \in G\}$. By 10.4, if G is compact, this orbit is homeomorphic to $G/G_{\mathbf{x}}$ and thus to $G/(gx)H(gx)^{-1}$. The automorphism $y \mapsto (gx)y(gx)^{-1}: G \to G$ transforms G/H homeomorphically to $G/(gx)H(gx)^{-1}$.

Property (b) We have $\mathrm{pr}_2\big((g,x)\Gamma\big) = \mathrm{pr}_2(gxHx^{-1} \times xN) = xN$ whence p is well-defined as stated. Next we note that $x_1N = x_2N$ implies $(1,x_1){\cdot}\Gamma = x_1Hx_1^{-1} \times x_1N = x_2Hx_2^{-1} \times x_2N = (1,x_2){\cdot}\Gamma$. Thus θ is well defined. It has an inverse function given by $G{\cdot}\big((g,x)\Gamma\big) = \{gxHx^{-1} \times xN \mid g \in G\} \mapsto xN$ and is therefore a bijection. Since for $\mathbf{x} = (g',x)\Gamma$ we have $\theta\big(p(\mathbf{x})\big) = \theta(gN) = G{\cdot}\big((1,x)\,\Gamma\big)G{\cdot}\big((g',x)\,\Gamma\big) = G{\cdot}\mathbf{x}$.

Property (c) The verification of (c) is similarly straightforward. □

Exercise E10.4. Fill in the straightforward details of the proof of 10.14 which were omitted. □

Definition 10.15. The G-space $\mathcal{X}(G,H)$ is called *the G-space attached to the pair* (G,H). □

We note from 10.14 that *the G-space attached to* (G,H) *allows cross sections and has stable isotropy.* But is it trivial?

Theorem 10.16. (i) *Let H be a closed subgroup of a compact group G and write $N \stackrel{\text{def}}{=} N(H,G)$ for the normalizer of H in G. Let the topological group G act on the compact product space*

$$G/H \times G/N \quad \text{by} \quad g{\cdot}(g'H,\xi) = (gg'H,\xi).$$

Then the following conditions are equivalent:

(1) *There is an isomorphism of G-actions $\Phi: G/H \times G/N \to \mathcal{X}$. Notably, the G-space $\mathcal{X}(G,H)$ is trivial.*

(2) *There is a continuous cross section $\sigma: G/N \to \mathcal{X}$ such that $G_{\sigma(\xi)} = H$ for all $\xi \in G/N$.*

(3) *There is continuous cross section* $\rho: G/N \to G/H$ *for the quotient map* $\pi: G/H \to G/N$, $\pi(gH) = gN$; *i.e.* $\rho(\xi)N = \xi$.

Each of the following conditions is sufficient for Condition (1) *to hold:*

(4) H *is normal in* G.

(5) H *is its own normalizer.*

(6) G *is abelian.*

(ii) *Assume that* Ω *is a compact neighborhood of* N *in* G/N *and that* $\rho: \Omega \to G/N$ *is a continuous function such that* $\rho(\xi)N = \xi$ *for* $\xi \in \Omega$. *Then the* G*-space*

$$\mathcal{X}_\Omega(G,H) \stackrel{\text{def}}{=} p^{-1}_{G,\mathcal{X}(G,X)}(\Omega)$$

is trivial.

Proof. (i) The equivalence (1)⇔(2) follows from Theorem 10.11.

(2)⇒(3) We have two cross sections γ, $\sigma: G/N \to \mathcal{X}$. Thus for each $\xi \in G/N$ there is an element $g \in G$ such that $g{\cdot}\sigma(\xi) = \gamma(\xi)$. If $g' \in G$ also satisfies $g'{\cdot}\sigma(\xi) = \gamma(\xi)$, then $g^{-1}g'{\cdot}\sigma(\xi) = \sigma(\xi)$, i.e. $g^{-1}g' \in G_{\sigma(\xi)} = H$; i.e. $g' \in gH$. We set $\rho(\xi) = gH \in G/H$. Since $gHg^{-1} = gG_{\sigma(\xi)}g^{-1} = G_{g\cdot\sigma(\xi)} = G_{\gamma(\xi)}$. On the other hand, if $\xi = xN$, then $\gamma(\xi) = (1,x){\cdot}\Gamma$ by (c), and then $G_{\gamma(\xi)} = xHx^{-1}$ by (a). Thus $gHg^{-1} = xHx^{-1}$ and therefore $x^{-1}g \in N$. Hence $\pi\big(\rho(\xi)\big) = \pi(gH) = gN = xN = \xi$. Thus ρ is a cross section. The set $\{(\xi,g) \in G/N \times G \mid g{\cdot}\sigma(\xi) = \gamma(\xi)\}$ is closed in $G \times G/N$ by the continuity of all functions involved. By the compactness of G it is compact. The group H acts on the right on $G/N \times G$ by $(\xi,g){\cdot}h = (\xi,gh)$. The orbit space of this action is $G/N \times G/H$ and the subset $\{(\xi,gH) \mid g{\cdot}\sigma(\xi) = \gamma(\xi)\}$ is compact as the continuous image of a compact set. However this set is the graph of $\rho: G/N \to G/H$. Since G/H is compact, the Closed Graph Theorem for Compact Spaces E5.29 shows that ρ is continuous.

(3)⇒(1) This is a special case of Part (ii) of the theorem: take $\Omega = G/N$.

(4)⇒(3) If (4) is satisfied, then $N(H,G) = G$ and $G/N(H,G)$ is singleton.

(5)⇒(3) If (5) is satisfied, then $G/H \to G/N(H,G)$ is the identity map.

(6)⇒(4) If G is abelian, the H is trivially normal.

(ii) The cross section $\rho: \Omega \to G/H$ allows us to pick for each $x \in G$ with $xN \in \Omega$ an element $\tau(x) \in G$ such that $\tau(x)N = \rho(xN)$; since ρ is a cross section, $\tau(x)N = xN$; i.e. $\tau(x) = x\varepsilon(x)$ for some $\varepsilon(x) \in N$. (The functions $\varepsilon: G \to N$ and $\tau: G \to G$ determine each other, and neither exhibits any pleasant features at this stage.) Note that

$$(*) \qquad \tau(x)H\tau(x)^{-1} = x\varepsilon(x)H\varepsilon(x)^{-1}x^{-1} = xHx^{-1}.$$

We set $U \stackrel{\text{def}}{=} \{x \in G \mid xN \in \Omega\}$; then $G \times U$ is Γ-invariant and we define

$$\varphi: G \times U \to \mathcal{X}_\Omega \stackrel{\text{def}}{=} (G \times U)/\Gamma, \quad \varphi(g,x) = g\tau(x)^{-1}xHx^{-1} \times xN = (g\tau(x)^{-1},x){\cdot}\Gamma.$$

We claim that for $(h,n) \in H \times N$ we have $\varphi(gh,xn) = \varphi(g,x)$. Indeed, $\varphi(gh,xn) = gh\tau(xn)^{-1}(xn)H(xn)^{-1} \times xnN$. Then $xnN = xN$ and $(xn)H(xn)^{-1} = xHx^{-1}$ since n is in the normalizer of H.

Also $\tau(xn)H = \rho(xnN) = \rho(xN) = \tau(x)H$, and thus there is some $k \in H$ such that $\tau(xn) = \tau(x)k$. Then $gh\tau(xn)^{-1}(xn)H(xn)^{-1} = ghk^{-1}\tau(x)^{-1}\cdot xHx^{-1} = g\tau(x)^{-1}(\tau(x)hk^{-1}\tau(x)^{-1})\cdot xHx^{-1} = g\tau(x)^{-1}\cdot xHx^{-1}$ by $(*)$. This proves the claim. Thus there is a well-defined function

$$\Phi\colon \tfrac{G}{H} \times \Omega \to \mathcal{X}_\Omega, \quad \Phi(gH, xN) = (g\tau(x)^{-1}, x)\cdot\Gamma = g\tau(x)^{-1}xHx^{-1} \times xN$$

such that the following diagram is commutative.

$$\begin{array}{ccc} G \times U & \xrightarrow{\varphi} & \mathcal{X}_\Omega \subseteq \mathcal{X} \\ {\scriptstyle\text{quot}}\downarrow & & \| \\ \frac{G}{H} \times \Omega & \xrightarrow[\Phi]{} & \mathcal{X}_\Omega \subseteq \mathcal{X}. \end{array}$$

We consider an element $h \in H$ and compute $gxhx^{-1}\rho(xN) = gxhx^{-1}\tau(x)H = gxhx^{-1}x\varepsilon(x)H = gxhH\varepsilon(x) = gxH\varepsilon(x) = gx\varepsilon(x)H = g\tau(x)H = g\rho(xN)$. Therefore we can define a function $\Psi\colon \mathcal{X}_\Omega \to \frac{G}{H} \times \Omega$ by $\Psi\big((g,x)\cdot\Gamma\big) = \Psi(gxHx^{-1} \times xN) = (g\rho(xN), xN)$. We note $\Phi\Psi((g.x)\cdot\Gamma = \Phi(g\rho(xN), xN) = \Phi(g\tau(x)H, xN) = \big((g\tau(x))\tau(x)^{-1}, x)\cdot\Gamma = (g,x)\cdot\Gamma$. Also,

$$\begin{aligned}\Psi\Phi(gH, xN) &= \Psi\big(g\tau(x)^{-1}, x)\cdot\Gamma\big) = \big((g\tau(x)^{-1})\rho(xN), xN\big) \\ &= \big((g\tau(x)^{-1})\tau(x)H, xN) = (gH, xN).\end{aligned}$$

Hence Φ and Ψ are inverses of each other. The function $\psi\colon G \times \Omega \to \frac{G}{H} \times \Omega$, $\psi(g,\xi) = (g\rho(\xi), \xi)$ is continuous by the continuity of the cross section ρ. The commutativity of the diagram

$$\begin{array}{ccc} G \times \Omega & \xrightarrow{\psi} & \frac{G}{H} \times \Omega \\ {\scriptstyle\text{quot}}\downarrow & & \| \\ \frac{G}{H} \times \Omega & \xrightarrow[\Psi]{} & \mathcal{X} \end{array}$$

shows the continuity of Ψ. Then the graph of Ψ is closed and thus the graph of Φ is closed. Hence Φ is continuous by the Closed Graph Theorem for Compact Spaces E5.29.

Since

$$\begin{aligned} g\cdot\Phi(g'H, xN) &= g\cdot\big((g'\tau(x)^{-1}, x)\cdot\Gamma\big) = (gg'\tau(x)^{-1}, xN)\cdot\Gamma \\ &= \Phi(gg'H, xN) = \Phi\big(g\cdot(g'H, xN)\big), \end{aligned}$$

the homeomorphism Φ is equivariant. □

Example 10.17. Let G be a compact connected nonabelian Lie group and T a maximal torus. Let $N \stackrel{\text{def}}{=} N(T,G)$ be the normalizer of T in G; then $\mathcal{W}(T,G) = N/T$ is the Weyl group (cf. 6.22), a nontrivial finite group. In particular, $\pi\colon G/T \to G/N$ is a covering map with $\mathcal{W}(T,G)$ as group of covering transformations (see

Appendix 2, A2.17). The simplest example is

$$G = \mathrm{SO}(3), \quad T = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\} \quad N = T \cup \mathrm{diag}(1,-1,-1)T.$$

(Cf. Example 6.10.) Here the quotient map $G/T \to G/N$ is a covering map whose Poincaré group (see A2.17) $\mathcal{W}(T,G) = N/T$ operates simply transitively on the fibers. It therefore cannot have a cross section (cf. A2.9). In the example of $G = \mathrm{SO}(3)$ the space G/T may be identified with the 2-sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$, the boundary of the unit ball in euclidean 3-space. Then G/N becomes identified with the space $\mathbb{S}^2/\{1,-1\}$ obtained by identifying antipodal points, i.e. with the real projective plane. Thus $G/H \to G/N$ is a double covering in this case.

Now let $\mathcal{X} = \mathcal{X}(G,T)$ be the G-space attached to the pair (G,T). This G-space has the following properties:

(i) The orbit space $\mathcal{X}/G$ is homeomorphic to G/N.

(ii) The isotropy groups range through all conjugates of T, i.e. through all maximal tori.

(iii) There is a continuous cross section $\mathcal{X}/G \to \mathcal{X}$.

(iv) The action of G on $\mathcal{X}$ is not trivial, i.e. is not isomorphic to the action of G on $G/T \times G/N$ by $g{\cdot}(g'T, xN) = (gg'T, xN)$. □

Thus there are compact G-spaces which allow cross sections and have stable isotropy but which nevertheless are not trivial.

We now consider a compact group G acting on a Hausdorff space X with a cross section $\sigma: X/G \to X$ and with stable isotropy. We set $\kappa: X \to X$, $\kappa = \sigma \circ p_{G,X}$ and $S \stackrel{\mathrm{def}}{=} \kappa(X)$. For some $s_0 \in S$ we set $H = G_{s_0}$. Our aim is to find a useful connection between the given G-space X and the G-space $\mathcal{X}(G,H)$ attached to (G,H). Since the isotropy is stable, for each $s \in S$ there is a $g \in G$ such that $G_s = gHg^{-1}$. The Axiom of Choice secures a function $g(\cdot): S \to G$ such that

$$(\forall s \in S) \quad G_s = g(s)Gg(s)^{-1}.$$

Define $N = N(H,G)$ to be the normalizer of H in G. Then $n \in N$ implies $(g(s)n)H(g(s)n)^{-1} = g(s)Hg(s)^{-1} = G_s$. The coset $\mu(s) \stackrel{\mathrm{def}}{=} g(s)N \in G/N$ is then determined independently of the choice of g and thus determines a function $\mu: S \to G/N$. We define the function $\nu: X/G \to N/H$ by $\nu(G{\cdot}s) = g(s)N = \mu(s)$, $s \in S$. Then $\nu(G.x) = \mu\big(\sigma(G{\cdot}x)\big)$.

The canonical bijection $\beta: G/N \to \mathcal{C}$ of 10.14(ii) defines a unique compact topology on $\mathcal{C} = \{gHg^{-1} \mid g \in G\}$ such that β is a homeomorphism.

Lemma 10.18. (i) *The function $s \mapsto G_s: S \to \mathcal{C}$ is continuous.*

(ii) *$\mu: S \to G/N$ is continuous.*

(iii) *$\nu: X/G \to G/N$ is continuous.*

Proof. (i) Let U be a neighborhood of $g_0Hg_0^{-1} \in \mathcal{C}$. We claim that there is an open identity neighborhood V in G such that the relation $gHg^{-1} \subseteq Vg_0Hg_0^{-1}$ implies $gHg^{-1} \in U$. By definition of the topology on $\mathcal{C}$ we may assume that U is of the form $\beta(g_0WN/N) = \{g_0wHw^{-1}g_0^{-1} \mid w \in W\}$ with an open identity neighborhood W of G. Then $gHg^{-1} \in U$ means exactly that there is a $w \in W$ such that $(g_0^{-1}g)H(g_0^{-1}g)^{-1} = wHw^{-1}$ i.e. $g_0^{-1}g \in WN$. Suppose the claim is false. By 1.12 there is a basis of identity neighborhoods V which are invariant under inner automorphisms. Thus for every invariant identity neighborhood V there is a $g_V \in G$ such that $g_VHg_V^{-1} \subseteq Vg_0Hg_0^{-1}$ but $g_0^{-1}g_V \notin WN$. Since V is invariant the first of these two properties is equivalent to $(g_0^{-1}g)H(g_0^{-1}g)^{-1} \subseteq VH$. The net $(g_0^{-1}g_U)_{U\in\mathcal{U}}$ has a convergent subnet in the compact space G. Denote by $s \in G$ its limit. Since H is compact, $\overline{V}H$ is closed, and thus $sHs^{-1} \subseteq \overline{V}H$, for $V \in \mathcal{N}$ where $\mathcal{N}$ denotes the filter of identity neighborhoods of H. Thus $sHs^{-1} \subseteq \bigcap_{V\in\mathcal{N}} \overline{V}H = H$. Then Lemma 6.62 shows that $\beta(sN) = sHs^{-1} = H$. On the other hand, since WN is open, we have $s \notin WN$. Then $sN \notin WN/N$ and therefore $H = \beta(sN) \notin \beta(WN/N) = \bigcup_{w\in W} wHw^{-1}$, a contradiction which proves the claim.

(ii) Since $\beta\big(\mu(s)\big) = G_s$ and $\beta: G/N \to \mathcal{C}$ is a homeomorphism assertion (ii) is a consequence of (i).

(iii) This is an immediate consequence of (ii) since $s \mapsto G{\cdot}s: S \to X/G$ and $\xi \mapsto \sigma(\xi): X/G \to S$ are inverse homeomorphisms. ☐

In the following we shall identify G/N with the orbit space $\mathcal{X}(G,H)/G$ as in 10.14(b). Now we define $\gamma: G \times S \to G \times G$ by $\gamma(g,s) = \big(g, g(s)\big)$ and $\theta: G \times S \to \mathcal{X}(G,H) = \frac{G\times G}{\Gamma}$ by

$$\theta(g,s) = \big(g, g(s)\big)\Gamma = gg(s)Hg(s)^{-1} \times g(s)N = gG_s \times \mu(s).$$

If we set $\varphi: G \times S \to X$, $\varphi(g,s) = g{\cdot}s$, then $\varphi(g_1,s_1) = \varphi(g_2,s_2)$ means $g_1{\cdot}s_1 = g_2{\cdot}s_2$, whence $s_1 = \sigma\big(p_{X,G}(g_1.s_1)\big) = \sigma\big(p_{X,G}(g_2.s_2)\big) = s_2 \stackrel{\text{def}}{=} s$ and $g_2^{-1}g_1{\cdot}s = s$, i.e. $g_2^{-1}g_1 \in G_s$, equivalently $g_1G_s = g_2G_2$. Hence $\theta(g_1,s_1) = g_1G_s \times \mu(s) = g_2G_s \times \mu(s) = \theta(g_2,s_2)$. Therefore, there is a unique function $M: X \to \mathcal{X}(G,H)$ such that $M(g{\cdot}s) = (g,g(s)){\cdot}\Gamma = gG_s \times \mu(s)$ (unambiguously) and that

$$\begin{array}{ccc} G \times S & \xrightarrow{\ \gamma\ } & G \times G \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle p_{G,\mathcal{X}(G,H)}} \\ X & \xrightarrow[M]{} & \mathcal{X}(G,H) \end{array}$$

commutes. In this situation we will apply the following lemma:

Lemma 10.19. *Assume that A, B, C are topological spaces and that the function $f: A \to C$ is the composition $f = q \circ \gamma$ of a function $\gamma: A \to B$ and a continuous function $q: B \to C$ and assume that the following hypotheses are satisfied:*

(a) *B is compact,*

(b) *C is Hausdorff, and*

(c) *for each net* $(x_j)_{j\in J}$ *in* A *converging to* x *and each cluster point* y *of the net* $\big(\gamma(x_j)\big)_{j\in J}$ *the relation* $q(y) = f(x)$ *holds.*

Then f *is continuous.*

Proof. By the Closed Graph Theorem for Compact Spaces E5.29 it suffices to show that the graph $F \overset{\text{def}}{=} \{\big(x, f(x)\big) \mid x \in A\}$ is closed in $A \times C$. Let $(x, z) \in \overline{F}$. Then there is a net $(x_j)_{j\in J}$ in A such that $(x, z) = \lim_{j\in J}\big(x_j, f(x_j)\big)$. By (b) there is a subnet $(x_{j_k})_{k\in K}$ such that $y = \lim_{k\in K}\gamma(x_{j_k})$ exists. Now (c) implies $q(y) = f(x)$. But using the continuity of q we also get $z = \lim_{k\in K} f(x_{j_k}) = q\big(\lim_{k\in K}\gamma(x_{j_k})\big) = q(y)$. Hence $z = f(x)$ and thus $(x, z) \in F$. □

Lemma 10.20. $\theta\colon G \times S \to \mathcal{X}(G, H)$ *is continuous.*

Proof. We want to apply Lemma 10.18 with $A = G \times S$, $B = G \times G$, $C = \mathcal{X}(G, H)$ and $f = \theta$, $q = p_{\mathcal{X}(G,H)}$. Then (a) and (b) are satisfied and so we must verify (c) in order to complete the proof. Let $\big((g_j, s_j)\big)_{j\in J}$ be a net with limit (g, s) in $G \times S$ and assume that there is a subnet $\big((g_{j_k}, s_{j_k})\big)_{k\in K}$ such that $(g', g'') \overset{\text{def}}{=} \lim_{k\in K}\big(g_{j_k}, g(s_{j_k})\big)$ exists in $G \times G$. Then clearly $g' = g$, and $g(s)N = \mu(s) = \lim_{k\in K}\mu(s_{j_k}) = \lim_{k\in K} g(s_{j_k})N = g''N$ by the continuity of μ (see 10.17(ii)). Then $g(s)Hg(s)^{-1} = g''H(g'')^{-1}$ as $N = N(H, G)$. Thus $p_{\mathcal{X}(G,H)}(g', g'') = g'g''H(g'')^{-1} \times g''N = gg(s)Hg(s)^{-1} \times g(s)Np_{\mathcal{X}(G,H)}\big(g, g(s)\big) = \theta(g, s)$. □

For proper maps we refer e.g. to [33], p. 72ff.

Lemma 10.21. (i) $\varphi\colon G \times S \to X$ *is proper, hence closed.*

(ii) $M\colon X \to \mathcal{X}(G, H)$ *is continuous.*

Proof. (i) The composition $\kappa \circ \varphi\colon G \times S \to S$ agrees with the projection $\mathrm{pr}_2\colon G \times S \to S$ onto the second factor which is proper ([33], p. 77, Corollaire 5). Hence φ is proper ([33], p. 73, Proposition 5d)) and thus is closed ([33], p. 72, Proposition 1).

(ii) Let F be a closed subset of $\mathcal{X}(G, H)$. Then $M^{-1}(F) = \varphi\big(\theta^{-1}(F)\big)$ since φ is surjective. As θ is continuous, $\theta^{-1}(F)$ is closed, and since φ is closed, $\varphi\big(\theta^{-1}(F)\big)$ is closed. □

Lemma 10.22. *The function* $M\colon X \to \mathcal{X}(G, H)$ *is equivariant and is faithful on orbits.*

Proof. Let $x \in X$. Then $x = g_0{\cdot}s$ with $s\kappa(x)$ and $g \in G$ entails $g{\cdot}x = gg_0{\cdot}x$. Now $M(g{\cdot}x) = \big(gg_0, g(s)\big){\cdot}\Gamma = g{\cdot}\big[\big(g_0, g(s)\big){\cdot}\Gamma\big]g{\cdot}M(x)$. Thus M is equivariant.

In order to show that M is faithful on $G{\cdot}x$, by Proposition 10.13 we need to show that $G_x = G_{M(x)}$. Let again $x = g_0{\cdot}s$. Then $M(x) = \big(g_0, g(s)\big){\cdot}\Gamma$.

By 10.13(a), $G_{M(x)} = g_0g(s)Hg(s)^{-1}g_0^{-1} = g_0G_sg_0^{-1} = G_{g_0\cdot s} = G_x$ in view of 10.4(i). □

We summarize the preceding results as follows:

Theorem 10.23 (The Test Morphism Theorem for G-Spaces). *Let the compact group G act on a Hausdorff space X with a cross section $\sigma: X/G \to X$ and with all isotropy groups conjugate to H. Then there is an equivariant continuous function $M: X \to \mathcal{X}(G,H)$ into the G-space attached to (G,H) which is faithful on orbits. There is a continuous function $\nu: X/G \to N(H,G)/H$ such that, with the usual identification of the orbit space of $\mathcal{X}(G,H)/G$ with $G/N(H,G)$, $\nu(G\cdot s) = g(s)N(H,G)$, $s \in \sigma(X/G)$ and $g(s) \in G$ any element such that $G_s = g(s)Hg(s)^{-1}$. Also, $M(g\cdot s) = (g, g(s))\cdot\Gamma = gG_s \times g(s)N(H,G)$, unambiguously, and the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{M} & \mathcal{X}(G,H) \\ {\scriptstyle p_{G,X}}\Big\downarrow & & \Big\downarrow{\scriptstyle p_{G,\mathcal{X}(G,H)}} \\ X/G & \xrightarrow[\nu]{} & G/N(H,G). \end{array}$$

□

Notice that the right column of the diagram is determined by (G,H) alone. It should be remembered that we constructed M and ν using a fixed cross section of the G-space X.

Definition 10.24. Using the notation of Theorem 10.23 we define $B(G,X) = \nu(X/G) \subseteq G/N(H,G)$ and $E(G,X) = M(X)$. Let $p: E(G,X) \to B(G,X)$ be the restriction of the orbit map $p_{\mathcal{X}(G,H)}: \mathcal{X}(G,H) \to G/N(H,G)$ (with the identification of the orbit space of $\mathcal{X}(G,H)$ with $G/N(H,G)$ which we are using by 10.13(b). We say that the sub-G-space $E(G,X)$ of $\mathcal{X}(G,H)$ is *the test G-space attached to X* (and the given cross section σ). □

We are now ready for the second major result on cross sections and stable isotropy.

Triviality Theorem for Actions

Theorem 10.25. *Let G be a compact group and X a locally compact G-space and assume that X admits a cross section $\sigma: X/G \to G$ and has stable isotropy, i.e. each isotropy group G_x is conjugate to a closed subgroup H of G. Consider the following conditions:*

(i) *X is a trivial G-space.*

(ii) *The test G-space $E(G,X)$ attached to X and σ* (*see Definition* 10.24) *is trivial.*

Then (ii)⇒(i), *and if the corestriction $X/G \to B(G,X)$ of the function $\nu: X/G \to G/N(H,G)$* (*see Theorem* 10.23 *and Definition* 10.24) *has a cross section, then both conditions are equivalent.*

Proof. The theorem is a consequence of Theorem 10.23 and Proposition 10.13(iv) and (v). ☐

Corollary 10.26. *Let G be a compact group and X a locally compact G-space and assume that X admits a cross section and has stable isotropy with isotropy groups conjugate to $H \leq G$. Then X is trivial provided that the quotient map $G/H \to G/N(H,G)$, $gH \mapsto gN(H,G)$ has a cross section. This is the case when H is normal in G, for example if G is abelian, or if H agrees with its own normalizer, or if the action is free.*

Proof. Since $E(G,X)$ is a sub-G-space of the G-space $\mathcal{X}(G,H)$ by Definition 10.24, the G-space $E(G,H)$ is trivial if $\mathcal{X}(G,H)$ is trivial. By Theorem 10.16 this is the case if $G/H \to G/N(H,G)$ has a cross section. Sufficient conditions for this to happen were also listed in 10.16. If the action is free, then $H = \{1\}$, and H, in particular, is normal in G. ☐

In order to prove an important local version of the Triviality Theorem for Actions, we need a lemma. Recall that a subgroup N of a topological group *has a tubular neighborhood* if there is subset C of G such that $(c,n) \mapsto cn : C \times N \to CN$ is a homeomorphism onto a neighborhood of N in G. We set $\Omega_C \stackrel{\text{def}}{=} CN/N \subseteq G/N$. The Tubular Neighborhood Theorem for Subgroups of Linear Lie Groups 5.33(ii) says that *all* closed subgroups of a linear Lie group have a tubular neighborhood.

Lemma 10.27. *Let G be a topological group and assume that H is a closed subgroup whose normalizer $N \stackrel{\text{def}}{=} N(H,G)$ has a tubular neighborhood in G. Then the function $\rho\colon \Omega_C \to G/H$, $\rho(cN) = cH$, $c \in C$ is a well defined continuous local cross section for $G/H \to G/N$; i.e. ρ satisfies $\rho(cN) \subseteq cN$.*

Proof. We verify that ρ is a well-defined continuous map; it is then clear that it is a local cross section since $\rho(cN) \subseteq cN$ is trivial. Let $\mu\colon C \times H \to CH$ denote the homeomorphism given by $\mu(c,h) = ch$ and $p\colon C \times H \to C$, $p(c,h) = c$. The continuous function $g \mapsto p\big(\mu^{-1}(g)\big)\colon CN \to C$ identifies elements c_1n_1 and c_2n_2 in CN if and only if $c_1 = c_2$ and thus factors uniquely through the quotient map $CN \to \Omega$ with a unique continuous function $\pi\colon \Omega \to C$, $\pi(cN) = c$.

$$\begin{array}{ccc} CN & \xrightarrow{p\circ\mu^{-1}} & C \\ {\scriptstyle\text{quot}}\big\downarrow & & \big\| \\ \Omega = \frac{CN}{N} & \xrightarrow[\pi]{} & C. \end{array}$$

Let $q\colon C \to G/H$ be the restriction of the quotient map $g \mapsto gH\colon G \to G/H$. Obviously, q is continuous. Now $\rho\colon \Omega \to G/H$ is the composition $q \circ \pi$ and is therefore well-defined and continuous. ☐

We shall say that an action *admits local cross sections* if there is an open cover $\mathcal{U}$ of X/G such that for each $U \in \mathcal{U}$ there is a continuous function $\sigma: U \to X$ such that $p_{G,X}\big(\sigma(\xi)\big) = \xi$ for all $\xi \in U$.

Local Triviality Theorem for Actions

Theorem 10.28. *Let G be a compact group and X a locally compact G-space and assume that X admits local cross sections and has stable isotropy conjugate to H. Assume that the normalizer $N(H,G)$ of H in G has a tubular neighborhood in G. Then for each $x \in X$ the orbit $G{\cdot}x$ has an invariant neighborhood Y such that Y is a trivial G-space.*

Proof. Let $x \in X$. There is an open neighborhood U of $G{\cdot}x$ in X/G such that the G-space $Y \stackrel{\text{def}}{=} p_{G,X}^{-1}(U)$ is an open locally compact neighborhood of $G{\cdot}x$ and has a cross section. For the proof it is therefore no loss of generality if we now assume that X has a cross section $\sigma: X/G \to X$. The isotropy groups of points in the orbit $G{\cdot}x$ range through the conjugacy class of H; we may and shall assume that $G_x = H$. Let $\sigma: X/G \to X$ be a cross section. Then there is a $g \in G$ such that $g{\cdot}\sigma(G{\cdot}x) = x$. Then $\xi \mapsto g{\cdot}\sigma(\xi): X/G \to X$ is a cross section passing through x. We may and shall now assume that σ is a cross section of X with $G_{\sigma(G\cdot x)} = H$. We now consider the G-space $\mathcal{X}(G,H)$ attached to the pair (G,H) and the continuous functions $M: X \to \mathcal{X}(G,X)$ and $\nu: X/G \to G/N$, $N \stackrel{\text{def}}{=} N(H,G)$ of Theorem 10.23 constructed with the use of σ. We note that with our choice of σ we have $\nu(G\cdot) = N \in G/N$. By Lemma 10.27 we have a neighborhood Ω_C of N in G/N and a continuous function $\rho: \Omega_C: \Omega \to G/N$ such that $\rho(gN) \in gN$. Let Ω be a compact neighborhood of N in G/N which is contained in Ω_C and set $\mathcal{X}(G,H)_\Omega = p_{G,\mathcal{X}(G,H)}^{-1}(\Omega)$ (recall the identification of the orbit space of $\mathcal{X}(G,H)$ with G/N!). By Theorem 10.16(iii), the G-space $\mathcal{X}(G,H)_\Omega$ is trivial. Now $W \stackrel{\text{def}}{=} \nu^{-1}(\Omega)$ is a neighborhood of $G.x$ in X/G. Set $Y = p_{G,X}^{-1}(W)$. Then Y is a sub-G-space of X which is a neighborhood of Y. Note that $\sigma|W: W \to Y$ is a cross section and that the test G-space $E(G,Y)$ attached to Y and the cross section $\sigma|W$ is $E(G,Y) \cap E(G,X) \cap \mathcal{X}(G,H)_\Omega$ and thus is trivial. Hence by the Triviality Theorem for Actions 10.25, the G-space Y is trivial. □

This is the place to call on some terminology on fiber bundles.

Definitions 10.29. Consider a function $p: E \to B$ between topological spaces and a topological space F.

(i) The map $p: E \to B$ is called a *fiber bundle with fiber F* if there is an open cover $\{U_j \mid j \in J\}$ of B such that for each $j \in J$ there is a homeomorphism $h_j: F \times U_j \to p^{-1}(U_j)$ such that $p\big(h_j(f,u)\big) = u$ for all $f \in F$, $u \in U_j$.

(ii) Assume that, in addition, a topological group G acts on E and that $F = G/H$ for some closed subgroup H of G. Then $p: E \to B$ is called a *principal fiber bundle with fiber F* if the sets $p^{-1}(b)$, $b \in B$ are the orbits of the action and if, for the trivial action of G on $F \times U_j$, the homeomorphism $h_j: F \times U_j \to p^{-1}(U)$ may be chosen equivariant. □

For a discrete space F, Definitions 10.29(i) agrees with our definition of a covering map in Appendix 2, A2.1. More information is to be found e.g. in [338], p. 90ff.)

Corollary 10.30. *Under the hypotheses of the Local Triviality Theorem for Actions* 10.28 *the orbit map $p_{G,X}: X \to X/G$ is a principal fiber bundle with fiber G/H.* □

The action of the Poincaré group Γ of a space X (see A2.17) possessing a universal covering $\widetilde{p}: \widetilde{X} \to X$ makes $\widetilde{p}$ into a principal fiber bundle with fiber Γ in the sense of 10.29.

Quotient Actions, Totally Disconnected G-Spaces

Proposition 10.31 (Passage to Quotients). *Let N be a compact normal subgroup of the compact group G and assume that G acts on a Hausdorff space X. Then N acts on X and G/N acts on X/N by $(gN)*(N{\cdot}x) = N{\cdot}gx$. Denote these actions by $\alpha: G \times X \to X$ and $\beta: G/N \times X/N \to X/N$, the quotient homomorphism by $q: G \to G/N$, and the orbit map of the action of N on X by $\pi: X \to X/N$. Define $\iota: X/G \to (X/N)/(G/N)$ by $\iota(G.x) = (G/N)*(N{\cdot}x) = G{\cdot}x/N$. Then the following conclusions hold.*

(i) *ι is a homeomorphism and the following diagrams are commutative.*

$$\begin{array}{ccccccc}
G\times X & \xrightarrow{\alpha} & X & \qquad & X & \xrightarrow{\text{orbit map}} & X/G \\
{\scriptstyle q\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \iota} \\
G/N\times X/N & \xrightarrow[\beta]{} & X/N, & & X/N & \xrightarrow[\text{orbit map}]{} & (X/N)/(G/N).
\end{array}$$

(ii) *The isotropy groups satisfy the relation $(G/N)_{N\cdot x} = NG_x/N$ for all $x \in X$.*

(iii) *If G acts freely, respectively, with stable isotropy on X, then G/N acts freely, respectively with stable isotropy on X/N.*

Proof. Exercise E10.5. □

Exercise E10.5. Prove Proposition 10.31.

[Hint. (i) Verify the straightforward steps that are necessary. First show that the element $(gN){\cdot}(N{\cdot}x) = N{\cdot}gx$ is well defined, using normality and thus $gN = Ng$. Verify that this defines an action and that the left diagram commutes. Check continuity of the action. Verify the commutativity of the left hand diagram and that ι is a homeomorphism.

(ii) For $x \in X$ the orbit $G{\cdot}x$ is G-equivariantly homeomorphic to the homogeneous space G/G_x via $gG_x \mapsto g{\cdot}x$. In particular, this homeomorphism is N-equivariant and thus induces a homeomorphism of orbit spaces $\gamma\colon (G/G_x)/N \to (G{\cdot}x)/N$. Now $(G/G_x)/N = \{NgG_x \mid g \in G\} = (G/G_x)/(NG_x/G_x)$ is homeomorphic to G/NG_x and then to $(G/N)/(NG_x/N)$ in a G/N-equivariant fashion. Also $(G{\cdot}x)/N$ is homeomorphic to $(G/N)*(N{\cdot}x)$ in a G/N-equivariant fashion via $\delta\colon (G{\cdot}x)/N \to (G/N)*(N{\cdot}x)$, $\delta\big(N{\cdot}(g{\cdot}x)\big) = \delta(Ng{\cdot}x) = (gN)*(N{\cdot}x)$. On the other hand, this orbit is G/N-equivariantly homeomorphic to $(G/N)/(G/N)_{N\cdot x}$. Conclude $(G/N)_{N\cdot x} = NG_x/N$. Derive (iii) from (ii).] □

Proposition 10.32. *Assume that G is a compact group acting on a locally compact totally disconnected space. Then the orbit space X/G is totally disconnected.*

Proof. Exercise E10.6. □

Exercise E10.6. Prove 10.32.

[Hint. Let V be an open neighborhood of $G{\cdot}x$ in X/G find a compact open neighborhood U of x in X such that $p_{G,X}(U) \subseteq V$. Note that $G{\cdot}U$ is open and compact and that, therefore, $p_{G,X}(U)$ is an open and closed neighborhood of $G.x$ in X/G.]
□

This concludes the present section and that portion of the chapter which is elementary in the sense that it uses only point set topology as prerequisites. In the following sections Lie group theory will be applied.

Compact Lie Groups Acting on Locally Compact Spaces

In Theorem 6.7 we have observed that the quotient of a compact Lie group is a compact Lie group. In itself, this was not simple even though we were able to derive this at an early stage in this chapter. We will also use again the Tubular Neighborhood Theorem for Subgroups of Linear Lie Groups 5.33(ii). However, this information now allows us to complete a discussion of the action of compact groups on topological spaces which we started at the end of Chapter I. In the following result we also need the Tietze Extension Theorem which says that for a normal Hausdorff space X, and a continuous function $\varphi\colon Y \to \mathbb{R}^n$ for a closed subspace Y of X there exists a continuous extension $\widetilde{\varphi}\colon X \to \mathbb{R}^n$ ([101], p. 69, 2.1.8, [230], p. 115, p. 242).

Let us note that as an immediate consequence we obtain the following fact.

Extension Lemma. *Let C be a compact subspace of a locally compact Hausdorff space X and $\varphi\colon C \to \mathbb{R}^n$ a continuous function. Then there exists a continuous extension $\widetilde{\varphi}\colon X \to \mathbb{R}^n$.*

Proof. Recall that the one point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space is Hausdorff. Thus it is a normal space, and thus the Tietze Extension Theorem yields an extension $\widetilde{\varphi}: X \cup \{\infty\} \to \mathbb{R}^n$, and the restriction $\widetilde{\varphi}|X: X \to \mathbb{R}^n$ is the required extension. □

We recall that for the action of a compact group G on a space X we denote by $p = p_{G,X}: X \to X/G$ the orbit projection given by $p(x) = G\cdot x$. For the definition of a *trivial* action check Definitions 10.10.

Proposition 10.33. (The Tietze–Gleason Extension Theorem). Let G be a compact Lie group. Assume that G acts on a locally compact space X and assume that for some $x \in X$ the isotropy group $G_x = \{g \in G \mid g\cdot x = x\}$ at x is trivial; i.e. $G_x = \{1\}$. Then there is a compact neighborhood U of $p(x)$, $p = p_{G,X}$, such that the action of G on $X' \stackrel{\text{def}}{=} p^{-1}(U)$ is trivial.

Proof. By 2.40 we may assume that there is a finite dimensional real Banach space such that the Banach algebra $A = \operatorname{Hom}(V, V)$ containing G in its group of units. Since $G_x = \{1\}$, the map $g \mapsto g\cdot x: G \to G\cdot x$ yields a continuous function $\varphi: G\cdot x \to A$ mapping $G\cdot x$ homeomorphically onto G. The Extension Lemma gives us a continuous extension $\Phi: X \to A$. Now we define

$$\Psi: X \to A, \quad \Psi(x) = \int_G g\Phi(g^{-1}.x)\,dg$$

with Haar measure dg on G and the integral for vector valued functions. (Since A is finite dimensional this amounts to a simple extension of scalar integration. But we do have a general theory in Chapter 3, see 3.30, 3.31. In fact, Ψ is obtained by applying to Φ the averaging operator, see 3.32ff.) Now $g'\Psi(x) = \int_G g'g\Phi\big((g'g)^{-1}\cdot(g'\cdot x)\big)dg = \Psi(g'\cdot x)$ by the invariance of Haar integral. Hence Ψ is equivariant. By the Tubular Neighborhood Theorem for Subgroups 5.33(ii) there is a compact neighborhood A' of G in A^{-1} which satisfies $GA' = A'$ and on which G acts trivially. Let $X' = \Psi^{-1}(A')$. Then X' is a G-invariant neighborhood of $G\cdot x$ which, by Lemma 10.3(iv), is compact. By Proposition 10.13(iv), the action of G on X' is trivial. □

In this proof, once more, the averaging operator (see 3.32ff.) turns up as an extremely useful device.

We now are ready for a core theorem on the action of compact Lie groups. Recall from Definitions 10.5(ii) that G is said to act with stable isotropy if all isotropy groups are conjugate.

The Local Cross Section Theorem for Compact Lie Group Actions

Theorem 10.34. *Let G be a compact Lie group acting on a locally compact space X with stable isotropy conjugate to $H \leq G$. Then every point of X has a G-invariant neighborhood on which G acts trivially. In particular, X is a principal fiber bundle with fiber G/H.*

Proof. Let $x \in X$, set $H = G_x$ and consider $N \stackrel{\text{def}}{=} N(H,G)$. Define $Y = \{y \in X \mid H\cdot y = \{y\}\}$. Then Y is an N-invariant closed, hence locally compact subspace of X containing x and meets each orbit by 6.72(i), (ii). As a closed subgroup of G, the group N is a compact Lie group. The factor group N/H of a compact Lie group is a compact Lie group by Theorem 6.7. By 10.7(iii) the compact Lie group N/H acts freely on Y. Abbreviate $p_{G,X}$ by p. Then by the Tietze–Gleason Extension Theorem 10.33 there is a compact neighborhood U of $p(x)$ in X/G, such that N/H acts trivially on $p^{-1}(U) \cap Y$. By Lemma 10.3(iv) the G-space $p^{-1}(U) \subseteq X$ is compact and thus is a compact G-space with a cross section. Now the Local Triviality Theorem for Actions 10.28 shows that the action of G is trivial on some compact invariant neighborhood Y of the orbit $G\cdot x$ in $p^{-1}(U)$. Then Y is an invariant neighborhood of $G\cdot x$ in X and a trivial G-space and thus the theorem is proved. □

Triviality Theorems for Compact Group Actions

We now utilize the information that compact groups are projective limits of compact Lie groups in order to secure global cross sections for compact group actions under special assumptions on the orbit space.

Global Triviality Theorem for Totally Disconnected Base Spaces

Theorem 10.35. *Let G be a compact group acting on a compact space X with stable isotropy such that the orbit space X/G is totally disconnected; i.e. that every connected component of X is contained in an orbit. Then the action is trivial.*

Proof. By Theorem 10.11 we have to produce a cross section $S \subseteq X$ such that $G_s = G_t$ for all $s,\, t \in S$.

We let $\mathcal{N}$ denote the filter basis of *all* compact normal subgroups N of G and let $\mathcal{Z}$ denote the set of all pairs (Y, N) such that $N \in \mathcal{N}$ and Y is a compact subset of X such that the following conditions are satisfied:

(a) $p(Y) = X/G$ where $p = p_{G,X}$ is the orbit projection,
(b) $N\cdot Y = Y$,
(c) for all $y, y' \in Y$ with $y' \in G\cdot y$ we have $y' \in N\cdot y$, and
(d) for all $y, y' \in Y$ we have $G_y N = G_{y'} N$.

Certainly (X, G) is one such pair. We write $(Y_1, N_1) \leq (Y_2, N_2)$ iff $Y_2 \subseteq Y_1$ and $N_2 \subseteq N_1$. If $(Y_j, N_j)_{j\in J}$ is a directed family for some directed sets J, we define $Y = \bigcap_{j\in J} Y_j$ and $N = \bigcap_{j\in J} N_j$. Clearly N is a compact normal subgroup and Y is a compact subspace. We show that (a) is satisfied: for each $x \in X$, the set $Y_j \cap G\cdot x$ is not empty, hence the intersection of the filter basis $Y \cap G\cdot x = \bigcap_{j\in J}(Y_j \cap G\cdot x)$ is not empty. Hence $p(Y) = X/G$. We next claim that (b) is satisfied. The relation $n \in N$ implies $n \in N_j$ for all j and thus $y \in Y \subseteq Y_j$ implies $n\cdot y \in N_j\cdot Y_j \subseteq Y_j$ for all $j \in J$. Hence $n\cdot y \in \bigcap_{j\in J} Y_j = Y$. Finally we verify (c). Let $y,\, y' \in Y$ such that $y' \in G\cdot y$. For each $j \in J$ by (c) for (Y_j, N_j) there is an element $n_j \in N_j$

such that $y' = n_j{\cdot}y$. In the compact space G, the net $(n_j)_{j\in J}$ has a convergent subnet with a limit n. Since $j \le k$ implies $n_k \in N_k \subseteq N_j$ we see that $n \in N_j$ for all $j \in J$ and thus $n \in \bigcap_{j\in J} N_j = N$. Since the continuity of the action and the relations $y' = n_j{\cdot}y$ imply $y' = n{\cdot}y$ we have verified (c) for (Y, N). Finally we verify (d). Assume that y, $y' \in Y$. We must show that $G_y \subseteq G_{y'}N$, for then by symmetry, $G_{y'} \subseteq G_yN$ and the desired equality $G_yN = G_{y'}N$ will follow. Now for each $j \in J$ we have $G_yN_j = G_{y'}N_j$. Thus $G_y \subseteq \bigcap_{j\in J} G_{y'}N_j$. We claim that the last intersection is $G_{y'}N$; it is clear that this set is contained in the intersection; conversely let $g \in \bigcap_{j\in J} G_{y'}N_j$; then for each $j \in J$ there are elements $g_j \in G_{y'}$ and $n_j \in N_j$ such that $g = g_jn_j$. Some subnet of $\big((g_j, n_j)\big)_{j\in J}$ converges in the compact space $G_{y'} \times G$ to an element $(g', n) \in G_{y'} \times N$ (since $n \in N_j$ for each j). Thus $g = g'n \in G_{y'}N$ as we had to show.

Therefore $(Y, N) \in \mathcal{Z}$. Consequently $\mathcal{Z}$ is inductive with respect to $\le$ and therefore has maximal elements. Let (S, N) be a maximal element.

We shall now show that $N = \{1\}$. If that is accomplished, then S is a cross section because (c) for (S, N) implies $Y \cap G{\cdot}y = \{y\}$ for all $y \in Y$, and isotropy on S is constant.

We may simplify our notation and assume for the remainder of the proof that (X, G) is itself maximal in $\mathcal{Z}$; we must show that G is singleton.

Now let N be a normal subgroup of G such that G/N is a Lie group. Then G/N acts on X/N with stable isotropy by 10.31(iii). Thus each point $N{\cdot}x$ of X/N has a G/N invariant neighborhood U on which G/N acts trivially by The Local Cross Section Theorem for Compact Lie Group Actions 10.34. Now we use the hypothesis that X/G is totally disconnected and that $(X/N)/(G/N)$ is homeomorphic to X/G by 10.31(i). Accordingly we formulate a statement to which we refer back at a later point:

(†) The orbit space space $(X/N)/(G/N)$ is totally disconnected for any closed normal subgroup N of G such that G/N is a Lie group.

Thus we may assume that U is the inverse image of a compact open set in $(X/N)/(G/N)$ and so is compact and open itself. Since X/N is compact we may then assume that X/N is a finite disjoint union of G/N-invariant compact open sets on which G/N acts trivially; that is on each of these sets, there is a cross section with constant isotropy, and since each is open and closed, there is a cross section for G/N acting on X/N whose isotropy is constant on each of the disjoint open sets $U_1, \dots, U_k$ forming a cover of $(X/N)/(G/N)$, say G_mN on U_m with G_m conjugate to H for a fixed representative H of the isotropy of G on X (cf. 10.31(ii)). Let $S_m \subseteq X/N$ be the cross section above U_m. We find elements g_m such that $g_mG_mg_m^{-1} = H$. If $s = g_mN{\cdot}s'$ for $s' \in S_m$, then $h \in H$ implies the existence of a $g \in G_m$ such that $g_mgg_m^{-1} = h$ and thus $hN{\cdot}s = g_mgg_m^{-1}N{\cdot}g_mN{\cdot}s' = g_mgN{\cdot}s' = g_mN{\cdot}s' = s$. Hence $g_mN{\cdot}S_m$ is a cross section above U_m with constant isotropy HN/N. Therefore $S_N \stackrel{\text{def}}{=} \bigcup_{m=1}^k g_mN{\cdot}S_m$ is a global cross section for the action of G/N on X/N with isotropy $(G/N)_s = HN/N$ for all $s \in S_N$. In other words

(††) The action of the Lie group G/N on X/N is trivial.

Then $(S_N, N) \in \mathcal{Z}$. By the maximality of (X, G) in $\mathcal{Z}$ we conclude $S_N = X$ and $G = N$. But from Corollary 2.43 we know that the family of compact normal subgroups N such that G/N is a Lie group intersects in $\{1\}$. Thus $G = \{1\}$ which is what we had to show. □

Corollary 10.36. *Let G be a compact group and H a closed subgroup. Then there is a G_0-equivariant homeomorphism*

$$\frac{G_0}{G_0 \cap H} \times \frac{G}{G_0 H} \to \frac{G}{H},$$

where G_0 acts on $G_0/(G_0 \cap H)$ and on G/H by multiplication on the left. The space G/G_0H is totally disconnected compact.

Proof. Set $X = G/H$. Since G acts transitively on X, all connected components are homeomorphic under actions from G. Since the quotient map $G \to G/H$ is continuous, open and closed, clopen sets go to clopen sets. The intersection of all clopen neighborhoods of 1 in G is G_0. Since G is compact, $X_0 \stackrel{\text{def}}{=} G_0H/H$ is the intersection of all clopen neighborhoods of H in X and thus is the component of H in X.

The group G_0 acts on X under the restriction of the action of G on X. The orbit of $H \in X$ is $G_0 \cdot H = G_0H/H = X_0$. The element $g_0 \in G_0$ is in the isotropy group $(G_0)_{gH}$ iff $g_0gH = gH$ iff $g_0 \in gHg^{-1} \cap G_0 = g(H \cap G_0)g^{-1}$. Thus this action has stable isotropy, and the orbit space is $X/G_0 = \{G_0gH/H = gG_0H : g \in G\} = G/G_0H$, a compact totally disconnected space. Hence by Theorem 10.35, the action is trivial, that is, X is G_0-equivariantly homeomorphic to $\frac{G_0}{G_0 \cap H} \times \frac{G}{G_0 H}$. □

Corollary 10.37. *Let X be a compact group and G a closed subgroup, not necessarily normal, but containing X_0. Then there is a closed subset $S \subseteq X$ such that $m: G \times S \to X$, $m(g, s) = gs$ is a homeomorphism.*

Proof. This follows from 10.36 above. □

This result has an important application to the structure theory of compact groups; in fact this is one of the main motivations for us to present the basics of compact group actions in this chapter.

Splitting the Component of Compact Groups

Corollary 10.38. *Every compact group G contains a compact totally disconnected subspace D such that $(g, d) \mapsto gd : G_0 \times D \to G$ is a homeomorphism.*

The groups G and $G_0 \times G/G_0$ are homeomorphic.

Proof. This follows immediately from 10.37. □

Exercise E10.7. Prove the following corollary of 10.38:

Let G be an arbitrary compact group. Then the following conclusions hold:

(i) *The weight of G is calculated as follows:*

$$w(G) = \max\{w(G_0), w(G/G_0)\}.$$

(ii) *There is a profinite subgroup D of G such that*

$$w(G) = \max\{w(G_0), w(D)\}.$$

[Hint. In view of Exercise EA4.3 in the Appendix, (i) follows from Corollary 10.38 above.

For (ii) we obtain the existence of D from Lee's Supplement Theorem 9.41 so that $G = G_0D$. Hence $G/G_0 \cong D/(G_0 \cap D)$, and so $w(G/G_0) = w(D/(G_0 \cap D)) \leq w(D)$. Then from (i) above we conclude $w(G) = \max\{w(G_0), w(G/G_0)\} \leq \max\{w(G_0), w(D)\} \leq w(G)$, hence (ii).] □

We saw in the Borel–Scheerer–Hofmann Splitting Theorem 9.39 that the identity component G_0 of G is the semidirect product of the commutator subgroup $(G_0)'$ by an abelian subgroup isomorphic to $G_0/(G_0)'$. The preceding corollary will then yield the following noteworthy result.

The Topological Decomposition of Compact Groups

Corollary 10.39. *For any compact group G, the compact groups G and $(G_0)' \times G_0/(G_0)' \times G/G_0$ are homeomorphic.* □

Dyadicity of Compact Groups

Theorem 10.40. *Each compact group G with infinitely many components is homeomorphic to a product of G_0 and a Cantor cube. Every infinite compact group is dyadic.*

Proof. (a) First we show that an infinite totally disconnected compact group G is a Cantor cube, topologically. We apply the Countable Layer Theorem 9.91 to G and obtain an inverse system

$$\{1\} = G/\Omega_0 \xleftarrow{p_1} G/\Omega_1 \xleftarrow{p_2} G/\Omega_2 \xleftarrow{p_3} \cdots$$

where $\ker p_n \cong \Omega_{n-1}/\Omega_n$ is a strictly reductive totally disconnected group, that is, a product of groups of prime order or (nonabelian) finite simple groups. All groups G/Ω_{n-1} are totally disconnected. Thus, by Corollary 10.37, the hypothesis of Lemma A4.33 in Appendix 4 is satisfied, and thus we conclude that G and $\prod_{n=1}^{\infty} \Omega_{n-1}/\Omega_n$ are homeomorphic. Since each factor is itself a product of a family of finite sets, and G is infinite, G is homeomorphic to a product of an infinite family of finite sets. By Corollary A4.32, G is a Cantor cube.

(b) By Corollary 10.38 and (a) above, if G has infinitely many components, then G is homeomorphic to $G_0 \times \mathbb{Z}(2)^{w(G/G_0)}$, where $w(\mathbb{Z}(2)^{w(G/G_0)}) = w(G/G_0)$ by Exercise EA4.3 following A4.8.

(c) G_0 is dyadic by Theorem 9.76(vii). Then $G_0 \times \mathbb{Z}(2)^{w(G/G_0)}$ is dyadic by A4.32 if G/G_0 is infinite. There remains the case that G/G_0 is finite. Let G_0 be a continuous image of a Cantor cube C. Then G, being homeomorphic to $G \times G/G_0$ is a continuous image of $C \times G/G_0$ and this space is a Cantor cube by Corollary A4.32. □

Theorem 10.35 on Global Triviality Theorem for Totally Disconnected Base Spaces has an analog for contractible base spaces. Its proof requires information from the theory of fibrations. It is appropriate to mention it here because the methods of proof are very close to those used for the case of totally disconnected base spaces, given certain information on fibrations. We recall that a topological space T is called *contractible* if the identity map $T \to T$ is homotopic to a constant self-map $T \to T$, that is, there is a continuous function $F{:}\,T \times [0,1] \to T$ such that $(\forall x \in T)\, F(x,0) = x$ and $(\exists c \in T)(\forall x \in T)\, F(x,1) = c$.

Global Triviality Theorem for Contractible Base Spaces

Theorem 10.41. *Let G be a compact group acting with stable isotropy conjugate to H on a compact space X such that X/G is contractible. Then the action is trivial.*

Proof. As in the proof of Theorem 10.35, we consider any compact normal subgroup N of G and use the hypothesis that X/G is contractible. We recall that $(X/N)/(G/N)$ is homeomorphic to X/G by 10.31(i) and conclude that, in particular for the case that G/N is a compact Lie group, we have:

(‡) The orbit space $(X/N)/(G/N)$ is contractible, for any closed normal subgroup N of G such that G/N is a Lie group.

The Local Cross Section Theorem for Compact Lie Group Actions 10.34 shows that for a compact Lie group G/N, a compact space X/N on which G/N acts with stable isotropy conjugate to HN/N is a (locally trivial) fiber bundle. If the orbit space $(X/N)/(G/N)$ is contractible, then the action is trivial by [342], p. 53, Corollary 11.6:

(‡‡) The action of the Lie group G/N on X/N is trivial.

Now we inspect the proof of Theorem 10.35. We observe that it applies here when we replace statements (†) and (††) in that proof by the statements (‡) and (‡‡) here and that it therefore shows that the action of G on X is trivial. □

The proof (†) ⇒(††) in the proof of 10.35 was done directly and in a self-contained fashion, while in the proof of Theorem 10.40 we invoked Steenrod's book [342] as a standard source for proving (‡) ⇒(‡‡).

Example 10.17 yields a plenitude of actions of a compact connected Lie group on a compact manifold with stable isotropy (which is therefore locally trivial by 10.34) and with a global cross section but which are not trivial.

We remark that we encountered another global cross section theorem in the form of the Topological Splitting Theorem for Vector Subgroups 5.70.

Exercise E10.8. Recall the definition of a principal fiber bundle in Definition 10.29 and prove the following immediate consequence of Theorem 10.41.

Proposition. (Actions of compact groups with orbits having contractible neighborhoods in the orbit space) *Let G be a compact group acting with stable isotropy on a compact space X such that every point in X/G has a contractible neighborhood. Then the orbit map $p: X \to X/G$ is a principal fiber bundle with fiber G/H.*

[Hint. Let U be a contractible neighborhood of some orbit in X/G. Then Theorem 10.41 applies to $X_U \stackrel{\text{def}}{=} p^{-1}(U)$ and shows that $p|X_U : X_U \to U$ is a trivial G-space; according to Definition 10.29 this established the claim.] □

The triviality results for actions proved in the preceding sections call for a definition singling out a particular class of surjective morphisms between compact groups.

Split Morphisms

Definitions 10.42. A morphism of compact groups $f: A \to B$ is said to be *topologically split* or is said *to split topologically* if there is a continuous function $\sigma: B \to A$ preserving identity elements and satisfying $\sigma f = \mathrm{id}_A$. Also we shall say that σ is *a continuous cross section for f*. A morphism $f: A \to B$ is called *split* or is said *to split* or *to split algebraically* if there is a morphism $s: B \to A$ of compact groups such that $fs = \mathrm{id}_A$. □

A morphism $f: A \to B$ is topologically split if and only if there is a compact subspace X of A such that the map $(n,h) \mapsto nh: N \times X \to A$, $N = \ker f$, is a homeomorphism. Then X is homeomorphic to B under $f|X: X \to B$ and A is topologically a product of N and B. Thus the groups A and $N \times B$ are homeomorphic.

Likewise, $f: A \to B$ is split if and only there is a compact subgroup $H \cong B$ in A such that A is the semidirect product NH; we shall review this situation in greater detail in the last section of this chapter below.

Notice that topologically split morphisms are surjective. They are not as rare as one might think at first:

Remark. Assume that $f: A \to B$ is a surjective morphism such that B is totally disconnected. Then f is topologically split.

Proof. The group $H \stackrel{\text{def}}{=} f^{-1}(\{1\})$ contains the identity component A_0 of A. Then Corollary 10.37 proves the claim. □

On the other hand we recall that there are even abelian compact groups such that the quotient morphism $G \to G/G_0$ modulo the identity component is not split (see Example 8.11), while it is topologically split by the preceding remarks.

Homological algebra has emphasized the significance of projectivity and we have discussed versions of this idea in Chapters 8 and 9 and in Appendix 1. We want to understand projectivity for the category of compact groups, and this section presents an additional contribution to this topic.

Let us begin with a closer look at the relationship of freeness and projectivity in the categories most familiar, e.g. that of [abelian] groups. In Appendix 1, A1.14 we show, using the Axiom of Choice that a free abelian group is projective. We saw in Chapter 8 that this fails in general in the category of compact abelian groups (see 8.79). What we use in the case of abelian groups is that for a surjective morphism $e: A \to B$ there is a *function*, i.e. a morphism in the category of sets, $s: B \to A$ which is a cross section, i.e. satisfies $e \circ s = \mathrm{id}_B$. However, a continuous surjective morphism $e: A \to B$ in the category of compact abelian groups, is not in general topologically split, i.e. we do not find a *continuous* cross section $s: B \to A$. Indeed we saw throughout this chapter that the existence of continuous cross sections is a delicate matter. It is the defect of morphisms failing to be split which accounts for free compact abelian groups failing to be projective in the sense of the basic definition of projectivity. One quick remedy would be that we focus, in the definition of projectivity, on topologically split morphisms $e: A \to B$.

Let us formulate the appropriate category theoretical language

Definition 10.43. Let $\mathcal{E}$ denote a class of epics in a category $\mathcal{C}$. An object P in $\mathcal{C}$ is called an *$\mathcal{E}$-projective* if for each $f: A \to B$ from $\mathcal{E}$ and each morphism $\mu: P \to B$ there is a $\nu: P \to A$ such that $\mu = f\nu$. □

This generalizes the definition of a *projective* object which we obtain if $\mathcal{E}$ is the class of *all* epics.

Exercise E10.9. Prove the following remark.

Any morphism $f: A \to P$ from $\mathcal{E}$ to an $\mathcal{E}$-projective P is a retraction; i.e. there is a morphism $g: G \to A$ with $fg = \mathrm{id}_P$.

The connection with free objects FX we shall take up in the next chapter on free compact groups; but here, in a chapter on compact group actions and cross sections, we pursue the topic on split morphisms further to prepare the ground for that discussion since we shall take for $\mathcal{E}$ the class of split morphisms in $\mathbb{CN}$.

We shall now investigate circumstances under which semidirect product decompositions are respected by morphisms. If $G = NH$ is a semidirect product with normal factor N and cofactor H we shall also refer to these data as a semidirect splitting of G.

Proposition 10.44. *If G_1 is a connected normal subgroup of G_2 and $G_1'A_1$ is a semidirect splitting of G_1, then there is a closed subgroup A_2 in G_2 containing A_1 such that $G_2'A_2$ is a semidirect splitting of G_2.*

Proof. There is a morphism $c_1: Z_0(G_1) \to G_1'$ extending the identity on $G_1' \cap Z_0(G_1)$ such that $A_1 = \{c_1(z)^{-1}z \mid z \in Z_0(G_1)\}$. Let $D = G_2' \cap Z_0(G_2)$ and let $dz = d'z' \in DZ_0(G_1)$ with $d, d' \in D$, $z, z' \in Z_0(G_1)$. We notice that $G_2' \cap G_1 = G_1'$: This is true for Lie groups where it is readily verified on the Lie algebra level; it follows by approximation in the general case. Therefore, $c_1(z')c_1(z)^{-1} = c_1(z'z^{-1}) = c_1(d'^{-1}d)$. Since

$$\begin{aligned} d'^{-1}d = z'z^{-1} &\in D \cap Z_0(G_1) \\ &= G_2' \cap Z_0(G_2) \cap Z_0(G_1) = G_2' \cap G_1 \cap Z_0(G_1) \\ &= {G_1}' \cap Z_0(G_1), \end{aligned}$$

we have $d'^{-1}d = c(d'^{-1}d) = c_1(z')c_1(z)^{-1}$, i.e. $dc_1(z) = d'c_1(z')$. Hence c extends to a morphism $c: DZ_0(G_1) \to G_1'$ via $c(dz) = dc(z)$ and $c(d) = d$ for $d \in D$. Now let T be any maximal torus of G_2 containing $c_1\big(Z_0(G_1)\big)$. Since all maximal tori of G_2 contain D, we have $\operatorname{im} c \subseteq T$. Since T as a maximal pro-torus in a semisimple compact connected group is a torus by 9.36(v) and since T is injective in the category of compact abelian groups by Theorem 8.78(ii), the corestriction $c: DZ_0(G_1) \to T$ extends to a morphism $c_2: Z_0(G_2) \to T \subseteq G_2'$ which agrees with the identity on D. Now $A_2 = \{c_2(z)^{-1}z \mid z \in Z_0(G_2)\}$ is the desired group. □

Proposition 10.45. *If $f: G_1 \to G_2$ is a surjective morphism of compact connected groups, then for every semidirect decomposition $G_2 = G_2'A_2$ of G_2 there is a semidirect decomposition $G_1 = G_1'A_1$ of G_1 with $f(A_1) = A_2$. (Of course, $f(G_1') = G_2'$ is automatic.)*

Proof. Let $N = G_1' \cap ker f$. If G/N decomposes semidirectly into

$$(G_1'/N)(A/N) \quad \text{with} \quad f(A) = A_2,$$

then $A = A_0N$. As a normal subgroup of the semisimple compact connected group G_1', the group N is of the form N_0Z with some compact abelian totally disconnected group Z which is central in G_1 and a semisimple compact connected group N_0.

We write $A_0 = N_0Z_0(A_0)$ and have $A = N_0ZZ_0(A_0)$ with an abelian group $ZZ_0(A_0)$ which is central in A. Then we find an abelian compact group A^* in A such that $A = N_0A^*$ is semidirect. Let A_1 be the identity component of A^*. Now $N = N_0(N \cap A^*)$ semidirectly, and $A = NA_1$ semidirectly. Now $G = G_1'A_1$ and $G_1' \cap A_1 \subseteq G_1' \cap A \cap A_1 = N \cap A_1 = \{\mathbf{1}\}$. Hence G_1 is decomposed semidirectly in the form $G_1'A_1$ and $f(A_1) = f(NA_1) = f(A) = A_2$. Thus we may assume from here on that $G_1 \cap \ker f = \{1\}$, i.e., that $f|G_1': G_1' \to G_2'$ is an isomorphism.

Now let $A_1 = f^{-1}(A_2)$. If $g \in G_1' \cap A_1$, then $f(g) \in f(G_1') \cap f(A_1) = G_2' \cap A_2 = \{\mathbf{1}\}$. Hence $g \in G_1 \cap \ker f = \{\mathbf{1}\}$, and thus $G = G_1'A_1$ is a semidirect product and $f(A_1) = A_2$. □

Our preceding results allow us to draw the following conclusion on topologically split morphisms of compact connected groups.

Proposition 10.46. *For a morphism $f\colon G_1 \to G_2$ of compact connected groups the following conditions are equivalent.*

(1) *f splits topologically.*

(2) *The semisimple part*

$$f'\colon G_1' \to G_2', \quad f'(g) = g$$

and the abelian part

$$F\colon G_1/G_1' \to G_2/G_2', \quad F(gG_1') = f(g)G_2'$$

both split topologically.

Proof. (2)⇒(1) A topological split morphism is surjective. Thus (2) implies that f' and F are surjective. Let $g_2 \in G_2$. Then the surjectivity of F implies the existence of a $g_1 \in G_1$ such that $f(g_1)G_2' = F(g_1G_1') = g_2G_2'$, i.e., $f(g_2) = f(g_1)g_2'$ for some $g_2' \in G_2'$. Then the surjectivity of f' implies the existence of an $f_1' \in G_1'$ such that $f(g_1') = g_2'$. then $f(g_2) = f(g_1)f(g_2') \in f(G_1)$. Thus f is surjective and we may apply Proposition 10.45 to get Borel–Scheerer–Hofmann splittings $G_j = G_j'A_j$, $j = 1, 2$ such that $f(A_1) \subseteq f(A_2)$. Then by (2), $(f|G_1')\colon G_1' \to G_2'$ and $(f|A_1)\colon A_1 \to A_2$ are topologically split, and then f is topologically split.

(1)⇒(2) We have semidirect decompositions $G_j = G_j' \rtimes A_j$, $j = 1, 2$, with $f(A_1) = A_2$ by Proposition 10.45. Let $\sigma\colon G_2 \to G_1$ denote a topological cross section for f. Define $\sigma'\colon G_2' \to G_1'$ and $\alpha\colon A_2 \to A_1$ by $\sigma'(g) = \mathrm{pr}_{G_1'}\sigma(g)$ for $g \in G_2$ and $\alpha(a) = \mathrm{pr}_{A_1}\sigma(a)$ for $a \in A_2$. Then $f\sigma'(g) = f\,\mathrm{pr}_{G_1'}\sigma(g) = \mathrm{pr}_{G_2'} f\sigma(a) = \mathrm{pr}_{G_2'}(g) = g$ and $f\alpha(a) = f\,\mathrm{pr}_{A_1}\sigma(a) = \mathrm{pr}_{A_2} f\sigma(a) = \mathrm{pr}_{A_2}(a) = a$. Hence σ' and α are the desired topological cross sections. □

The situation unfortunately is more complicated for splitting in the group sense. Assume that $f\colon G_1 \to G_2$ is a split morphism of compact groups with a homomorphic cross section $s\colon G_2 \to G_1$. Then $s(G_2') \subseteq G_1'$, and the restriction $f'\colon G_1' \to G_2'$ and corestriction $s'\colon G_2' \to G_1'$ satisfy $f's' = \mathrm{id}_{G_2}$. Hence f' splits. We let $F\colon G_1/G_1' \to G_2/G_2'$ and $S\colon G_2/G_2' \to G_1/G_1'$ be the induced morphisms. Then $FS = \mathrm{id}_{G_2/G_2'}$ and thus F splits, too. So this direction is simple.

However, the converse may be false. In order to understand more clearly what happens we prove a lemma:

Lemma 10.47. *Let N denote a semisimple compact connected normal subgroup of a compact connected group G. There is a unique compact connected normal subgroup M such that $G' = NM$ and $N \cap M$ is totally disconnected and central* *in G. Let $Z(M)$ and $Z(N)$ be the centers of M and N, respectively, and write $\Delta = Z_0 \cap G'$. Then*

(i) *the following conditions are equivalent:*

(1) *G is a semidirect product NB, $N \cap B = \{\mathbf{1}\}$.*

(2) *There is a morphism $\alpha: MZ_0 \to N$ extending the inclusion map $Z(N) \cap Z(M)\Delta$.*

(ii) *If α exists as in* (2) *and is surjective, then*

$$M \cap N \cap Z_0 = Z(M) \cap Z(N) \cap \Delta = \{\mathbf{1}\}.$$

Proof. The existence of M follows from Theorem 9.74. We note that $N \cap (MZ_0)$ is totally disconnected in view of the structure theory, and so is central. Hence $N \cap (MZ_0) \subseteq Z(N) \cap Z(MZ_0) = Z(N) \cap Z(M)Z_0$. If $n = mz$ with $n \in N$, $m \in M$, and $z \in Z_0$, then $z = m^{-1}n \in MN \cap Z_0 = G' \cap Z_0 = \Delta$. Thus $N \cap (MZ_0) \subseteq Z(N) \cap Z(M)\Delta$, and the reverse inclusion is trivial, so equality holds. The equivalence of (1) and (2) is now simply a consequence of Lemma 6.37. Thus (i) is proved. For a proof of (ii) note that the surjective morphism $\alpha: MZ_0 \to N$ must map Z_0 to the identity. Hence $\alpha(M \cap Z_0) = \{\mathbf{1}\}$. On the other hand, $\alpha(m) = m$ for $m \in Z(N) \cap Z(M)\Delta$ by (2). Hence (ii) follows. □

Example 10.48. There is a connected compact Lie group G with the following properties:

(i) $G/G' \cong \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

(ii) There is a compact connected normal subgroup N contained in G' such that N is a semidirect factor in G' but not in G.

(iii) If $f: G \to G/N$ denotes the quotient morphism, then f' splits and the induced morphism $G/G' \to (G/N)/(G/N)'$ is an isomorphism.

(iv) f splits topologically but not algebraically.

(v) The smallest example G of this kind has dimension 7 with $\dim G/N = 4$.

Proof. Let L denote a simple simply connected Lie group with cyclic center Z of order n with generator z. Denote with $t \in \mathbb{T}$ an element of order n. In the group $L \times L \times \mathbb{T}$ consider the central subgroup D generated by the elements $(z, z, 0)$ and $(\mathbf{1}, z, t)$. Set $G = (L \times L \times \mathbb{T})/D$. Then $G' = (L \times L \times \{0\})D/D \cong (L \times L)/\{(c, c) \mid c \in Z\}$ contains the normal subgroups $N = (L \times \{\mathbf{1}\} \times \{0\})D/D \cong L$ and $M = (\{\mathbf{1}\} \times L \times \{0\})D/D \cong L$, and the subgroup $C = \{(d, d, 0) \mid d \in L\}D/D \cong L/Z$. We claim that G' is the semidirect product of N and H. Clearly $NC = G'$. In order to show $N \cap C = \{\mathbf{1}\}$ consider $(u, v, w)D \in N \cap C$. Then $(u, v, w) \in (L \times \{\mathbf{1}\} \times \{0\})D \cap \{(d, d, 0) \mid d \in L\}D = \{(x, z^m, p{\cdot}t) \mid x \in L,\ m, p \in \mathbb{Z}\} \cap \{(d, dz^p, p{\cdot}t) \mid d \in L,\ p \in \mathbb{Z}\} = D$ which proves the claim.

Also

$$\begin{aligned} N \cap Z_0 &= (L \times \{\mathbf{1}\} \times \{0\})D/D \cap (\{\mathbf{1}\} \times \{\mathbf{1}\} \times \mathbb{T})D/D \\ &= (L \times Z \times \mathbb{Z}{\cdot}t \cap Z \times Z \times \mathbb{T})/D \\ &= (Z \times Z \times \mathbb{Z}{\cdot}t)/D \cong Z. \end{aligned}$$

Thus $N \cap MZ_0 \supseteq N \cap Z_0 \neq \{\mathbf{1}\}$ if $Z \neq \{\mathbf{1}\}$.

Likewise

$$\begin{aligned} M \cap Z_0 &= \big((\{\mathbf{1}\} \times L \times \{0\})D/D \cap (\{\mathbf{1}\} \times \{\mathbf{1}\} \times \mathbb{T})D\big)/D \\ &= \big(Z \times L \times \mathbb{Z}{\cdot}t \cap \mathbb{Z} \times Z \times \mathbb{T}\big)/D \\ &= (Z \times Z \times \mathbb{Z}{\cdot}t)/D = N \cap Z_0. \end{aligned}$$

Thus $M \cap N \cap Z_0 \cong Z \neq \{1\}$.

If we let $f\colon G \to G/N$ denote the quotient morphism, then $f'\colon G' \to G'/N$ splits by the preceding discussion. Now $G/G' \cong (G/N)/(G'/N) = (G/N)/(G/N)'$ by the first isomorphy theorem. In particular, f splits topologically since $G = G'A$ semidirectly with a suitable abelian group $A \cong G/G'$ by the Borel–Scheerer–Hofmann Splitting Theorem 10.39, and thus topologically $G = N \times C \times A$.

However, N does not admit a semidirect group complement in G. For a proof we note that by Lemma 10.47 all subgroups of G complementary to N are classified by a morphism $\alpha\colon MZ_0 \to N \cong L$ extending the identity map of $N \cap MZ_0 \cong Z$. Then α cannot be constant if $Z \neq \{\mathbf{1}\}$. In this case α is necessarily surjective. Hence by Lemma 10.47(ii) we have $M \cap M \cap Z_0 = \{\mathbf{1}\}$, a contradiction.

The smallest example is given by $L = \mathrm{SU}(2)$ with $Z = \{\mathbf{1}, -1\}$ of order 2. In this case $\dim G = 3 + 3 + 1 = 7$. □

This example shows, in particular, that there are morphisms which are topologically split but are not split. One might surmise that this may not occur with morphisms between semisimple groups. However this is not the case as the following example illustrates:

Example 10.49. There is a surjective morphism $f\colon G_1 \to G_2$ of semisimple isotypical compact connected Lie groups which splits topologically but not algebraically. One example is given by $G_2 = \mathrm{PSU}(6)$ and G_1 locally isomorphic to $\mathrm{SU}(6)^2$.

Proof. We let L again denote a simple simply connected Lie group with nontrivial cyclic center Z. We consider $\Delta = \{(z^a, z) \mid z \in Z\} \subseteq L^2$ with a natural number a. We set $G_1 = L^2/\Delta$, $G_2 = L/Z$. We let $f\colon G_1 \to G_2$ denote the morphism induced by the projection $p\colon L^2 \to L$ onto the last component. The kernel N then equals $(L \times \{\mathbf{1}\})\Delta/\Delta = (L \times Z)/\Delta \cong Z$. The unique supplementary normal subgroup is $M = (\{\mathbf{1}\} \times L)\Delta/\Delta = (Z^a \times L)/\Delta \cong L/Z[a]$ where $Z[a] = \{x \in Z \mid x^a = \mathbf{1}\}$. Moreover, $N \cap M = (Z^a \times Z)/\Delta \cong Z/Z[a]$. By Lemma 6.37, the complements for N in G_1 are characterized by morphisms $\alpha\colon M \to N$ extending the inclusion $N \cap M \to N$. Since $N \cap M$ is nontrivial if $Z[a] \neq Z$, any such morphism is nontrivial. But a nontrivial morphism $L/Z[a] \to L$ must be an isomorphism, and $Z[a] = \{\mathbf{1}\}$. Hence if $\{\mathbf{1}\} \neq Z[a] \neq Z$, such an α cannot exist. Whenever the order of Z is not a prime, a number a with this property exists. An example is $L = \mathrm{SU}(6)$.

On the other hand let us consider the continuous function $\widetilde{\sigma}\colon L \to L^2$ given by $\widetilde{\sigma}(v) = (v^a, v)$. Then $p \circ \widetilde{\sigma} = \mathrm{id}_L$. Also, if $z \in Z$, then $\widetilde{\sigma}(zv) = \big((zv)^a, zv\big) = (v^a, v)(z^a, z) \in \widetilde{\sigma}(v)\Delta$ since z is central. Hence $\widetilde{\sigma}$ induces a base point preserving continuous function $\sigma: G_2 \to G_1$ given by $\sigma(vD) = (v^a, v)\Delta$ which is a continuous cross section for f. □

In view of Exercise E10.9, the group G_2 is an example of a compact connected group which is not $\mathcal{E}$-projective for the class $\mathcal{E}$ of split morphisms.

However, the abelian situation is radically different:

Proposition 10.50. *A topologically split morphism $f: A \to B$ of compact connected abelian groups splits.*

Proof. Let $\sigma: B \to A$ denote the topological cross section. We denote with $H^*(X) = H^*(X, \mathbb{Z})$ the integral Čech cohomology of a compact space X. The relation $f\sigma = \mathrm{id}_B$ induces the relation

$$H^*(\sigma)H^*(f) = H^*(f\sigma) = H^*(\mathrm{id}_B) = \mathrm{id}_{H^*(B)}: H^*(B) \to H^*(A)$$

in Čech-cohomology over the integers. We specialize to dimension one and obtain a split exact sequence

$$0 \to \ker H^1(\sigma) \xrightarrow{\mathrm{incl}} H^1(B) \underset{f^*}{\overset{\sigma^*}{\rightleftarrows}} H^1(A) \to 0$$

with the notation $\varphi^* = H^1(\varphi)$, designating a morphism of discrete torsion-free abelian groups . The dual is a split sequence of compact connected abelian groups

$$(\dagger) \qquad 0 \to H^1(A)\widehat{\ } \underset{\widehat{f^*}}{\overset{\widehat{\sigma^*}}{\rightleftarrows}} H^1(B)\widehat{\ } \xrightarrow{\widehat{\mathrm{incl}}} \big(\ker H^1(\sigma)\big)\widehat{\ } \to 0.$$

There is, however, a natural isomorphism $\widehat{G} \to H^1(G)$ between the character group of a compact connected group and its first integral cohomology group. (See 8.57(ii), 8.83. Further details are to be found in [198].) Consequently, by Pontryagin duality, there is a natural isomorphism $H^1(A)\widehat{\ } \to A$ and $H^1(B)\widehat{\ } \to B$ by which $H^1(f)\widehat{\ }$ becomes identified with $f: A \to B$. The split exact sequence (†) therefore proves the proposition. □

Before we generalize this result, we need a reduction:

Lemma 10.51. (i) *If $f: A \to B$ is a surjective morphism of compact groups and A_1 is a subgroup of A with $f(A_1) = B$, then f splits* [*topologically*] *if $f|A_1: A_1 \to B$ splits* [*topologically*].

(ii) *If $f: A \to B$ is a morphism of compact groups onto a connected group and $f_0: A_0 \to B$ denotes the restriction, then f_0 splits topologically if f splits topologically and f splits if f_0 splits.*

Proof. (i) Let $K = \ker f$. It suffices to find a compact subgroup [subspace] H of A such that $A = KH$ and $(k, h) \mapsto kh: K \times H \to A$ is a homeomorphism, for then $f|H: H \to B$ is an isomorphism [homeomorphism] and $s = j(f|H)^{-1}: B \to A$ with the inclusion $j: H \to A$ is the required homomorphic [continuous] cross section. Now let $H \subseteq A_1$ denote a [topological] complement for $K \cap A_1$ in A_1; then $K \cap H = (K \cap A_1) \cap H = \{1\}$ and $f(A_1) = B$ implies $KH = A$. Thus $kh = k'h'$ implies $k'^{-1}k = h'h^{-1} \in H \cap K = \{\mathbf{1}\}$, i. e., $k = k'$ and $h = h'$. Hence H is a [topological] complement for K in A.

(ii) If f splits topologically, then any base point preserving cross section $\sigma: B \to A$ maps B into A_0, whence f_0 splits topologically. The second assertion follows from (i) above. □

Lemma 10.52. *If $f: G_1 \to G_2$ is a morphism with a topological base point preserving cross section $\sigma: G_2 \to G_1$ and if H_2 is any subgroup of G_2, then the restriction and corestriction $f^{-1}(H_2) \to H_2$ is a topologically split morphism.*

Proof. Since σ is a base point preserving cross section of f we know that $\sigma(H_2) \subseteq f^{-1}(H_2)$. Hence σ restricts and corestricts to a base point preserving map $H_2 \to f^{-1}(H_2)$ which is a cross section for the morphism $f^{-1}(H_2) \to H_2$ induced by f.□

Theorem 10.53. (i) *A topologically split morphism from a compact group $f: G_1 \to G_2$ onto a connected abelian group splits.*

(ii) *If $f: G_1 \to G_2$ is any topologically split morphism of compact groups and T is a connected abelian subgroup of G_2 then the restriction and corestriction $T_1 \to T$, $T_1 = f^{-1}(T)$ splits. Moreover, if S is any maximal connected abelian subgroup of T_1, then the restriction $S \to T$ splits.*

Proof. By Lemma 10.52(ii) it is no loss of generality to assume that G_1 is connected. Then by 9.39, the group G_1 is a semidirect product $G'A$ with an abelian group A with $f(A) = G_2$ since f is surjective and G_2 is abelian. As the induced morphism $f|A: A \to G_2$ is equivalent to the induced morphism $G_1/G_1' \to G_2/G_2'$ and thus splits topologically by Proposition 10.46, it splits by Proposition 10.50. Hence there is a compact subgroup B of A such that $A = (N \cap A)B$ is direct with $N = \ker f$. Now $NB = N(N \cap A)B = NA = G$ and $N \cap B = N \cap A \cap B = \{\mathbf{1}\}$. Hence NB is a semidirect decomposition and the inverse of the isomorphism $f|B: B \to G_2$ produces the required homomorphic cross section.

(ii) By Lemma 10.52, the restriction and corestriction $T_1 \to T$ is topologically split. Hence it splits by (i). Thus $T_1 = NA$ with $N = \ker f$ and an abelian group $A \cong T$. Let S be a maximal connected abelian subgroup of T_1 containing A. Then $S = (S \cap N)A$ is a direct decomposition, and thus the induced morphism $S \to T$ splits. Since all maximal connected abelian subgroups of T_1 are conjugate, the assertion follows. □

In the remainder of the section we shall prove that the topological splitting of a morphism of compact connected groups $f: A \to B$ reduces to the topological splitting of the isotypical components $\mathfrak{P}_\mathfrak{s} f_0: \mathfrak{P}_\mathfrak{s} A_0 \to \mathfrak{P}_\mathfrak{s} B$. Moreover we shall precisely describe the topological and the algebraic splitting of the isotypical components.

Throughout the following discussion we let $\sigma: B \to A$ denote a continuous cross section for f.

Since the abelian case is settled, we now turn to the general semisimple case.

Proposition 10.54. *Let $f: A \to B$ be a topologically split homomorphism with a continuous cross section $\sigma: B \to A$, and assume that $A = A'$ and $B = B'$ are semisimple. Then there is a continuous base point preserving map $\varphi: \mathfrak{P}(B) \to \mathfrak{P}(A)$ such that*

(i) $\mathfrak{P}(f\sigma) = \mathrm{id}_{\mathfrak{P}(B)}$.

(ii) $\sigma\tau_B = \tau_A\varphi$.

(iii) $\varphi(tx) = \varphi(t)\varphi(x)$ *for all* $x \in \mathfrak{P}(B)$, $t \in \ker \tau_B$, *and there is a commuting diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \Delta_B & \xrightarrow{\text{incl}} & \mathfrak{P}(B) & \xrightarrow{\tau_B} & B & \to & 0 \\ & & \psi\big\downarrow & & \big\downarrow\varphi & & \big\downarrow\sigma & & \\ 0 & \to & \Delta_A & \xrightarrow{\text{incl}} & \mathfrak{P}(A) & \xrightarrow{\tau_A} & A & \to & 0 \end{array}$$

where $\Delta_X = \ker \tau_X$ and where $\psi = \varphi|\Delta_B$ is the restriction and corestriction of φ.

(vi) *The morphism $\pi = \mathfrak{P}(f)|\Delta_A \to \Delta_B$ splits and $\pi\psi = \mathrm{id}_{\Delta_B}$. In particular Δ_A is a direct product* $(\ker \pi)(\mathrm{im}\, \psi)$.

Proof. (i) Recall from 9.19 that for $\mathfrak{s} \in \mathcal{S}$ we picked a simple simply connected Lie group $S_\mathfrak{s}$ with $\mathfrak{L}(S_\mathfrak{s}) \cong \mathfrak{s}$ and set $R_\mathfrak{s} = S_\mathfrak{s}/Z(S_\mathfrak{s})$. Let $q_a: A' \to A/Z(A)$ denote the natural homomorphism $a \mapsto aZ(A)$. Then $\pi_A = q_A\tau_A: \mathfrak{P}(A) \to A/Z(A)$ is equivalent to the morphism

$$\prod_{\mathfrak{s}\in\mathcal{S}} S_\mathfrak{s}^{\aleph(\mathfrak{s},A)} \to \prod_{\mathfrak{s}\in\mathcal{S}} R_\mathfrak{s}^{\aleph(\mathfrak{s},A)}$$

induced by the universal covering morphisms $p_j: S_\mathfrak{s} \to R_\mathfrak{s}$ for $j \in J$ in the appropriate simultaneous index set for both products. We identify π_A with this morphism $\prod_{j\in J} p_j$ and consider the continuous base point preserving function $\psi = q_A\sigma\tau_B: \mathfrak{P}(B) \to A/Z(A)$ with the topological cross section $\sigma: B \to A$. As a product of simply connected spaces L_s, the space $\mathfrak{P}(B)$ is simply connected. Hence every morphism $\psi_j = p_j\psi: \mathfrak{P}(B) \to S_\mathfrak{s}$ lifts uniquely to a base point preserving map $\varphi_j: \mathfrak{P}(B) \to L_s$ satisfying $p_j\varphi_j = \psi_j$. Hence there is a unique base point preserving map $\varphi: \mathfrak{P}(B) \to \mathfrak{P}(A)$ with $\pi_A\varphi = \psi$. Now we compute

$$\begin{aligned} \pi_B\big(\mathfrak{P}(f)\big)\varphi &= \big(f/Z(A)\big)\pi_A\varphi = \big(f/Z(A)\big)\psi \\ &= \big(f/Z(A)\big)q_A\sigma\tau_B = q_B f\sigma\tau'_B \\ &= q_B\tau_B = \pi_B. \end{aligned}$$

Thus $g \mapsto g^{-1}\big(\mathfrak{P}(f)\big)\varphi(g): \mathfrak{P}(B) \to B/Z(B)$ is a base point preserving continuous function mapping the connected space $\mathfrak{P}(B)$ into the totally disconnected kernel

$\ker \pi_B = Z(B)$. Hence it is constant. Thus

$$\big(\mathfrak{P}(f)\big)\varphi = \mathrm{id}_{\mathfrak{P}(B)}\,.$$

(ii) The base point preserving continuous maps $\alpha \stackrel{\rm def}{=} \tau_A\varphi$ and $\beta \stackrel{\rm def}{=} \sigma\tau_B$ satisfy $q_A\alpha = \psi = q_A\beta$ by the definition of φ. Hence $g \mapsto \alpha(g)^{-1}\beta(g)\colon PB \to \ker q_A = Z(A)$ is a well defined base point preserving map from a connected space into a totally disconnected one and is therefore constant. Hence

$$\tau_A\varphi = \sigma\tau_B.$$

(iii) The relation $\tau_A\varphi(tx) = \sigma\tau_B(tx) = \sigma\tau_B(x) = \tau_A\varphi(x)$ shows that

$$\varphi(tx)\varphi(x)^{-1} \in \ker \tau_A = \Delta_A.$$

For each fixed $t \in \Delta_B$ the continuous base point preserving function

$$x \mapsto \varphi(tx)\varphi(x)^{-1} : \mathfrak{P}(B) \to \Delta_A$$

from a connected space to a totally disconnected one is necessarily constant. The base point $\mathbf{1}$ is mapped to $\varphi(t\mathbf{1})\varphi(\mathbf{1})^{-1} = \varphi(t)$. Thus $\varphi(tx)\varphi(x)^{-1} = \varphi(t) = \psi(t)$ for all $t \in \Delta_B$ and $x \in \mathfrak{P}(B)$. Hence (iii) is proved.

(vi) The relation $\mathfrak{P}(f) \circ \varphi = \mathrm{id}_{\mathfrak{P}(B)}$ implies $\pi \circ \psi = \mathrm{id}_{\Delta_B}$ by simple restriction and corestriction. Thus $\Delta_A = (\ker \pi)(\mathrm{im}\,\psi)$ is a semidirect product decomposition. But Δ_A is central in PA, and thus we have a direct product decomposition. □

Lemma 10.55. *For a topologically split morphism $f\colon A \to B$ of compact connected groups there is a continuous cross section $\varphi_{\mathfrak{s}}\colon \mathfrak{P}_{\mathfrak{s}}\big(\mathfrak{P}(B)\big) \to \mathfrak{P}_{\mathfrak{s}}\big(\mathfrak{P}(A)\big)$ for the isotypic $\mathfrak{s}$-component $\mathfrak{P}_{\mathfrak{s}}(f)$ and a continuous cross section $\sigma_{\mathfrak{s}}\colon \mathfrak{P}_{\mathfrak{s}}(B) \to \mathfrak{P}_{\mathfrak{s}}(A)$ for the $\mathfrak{s}$ component $\mathfrak{P}_{\mathfrak{s}}(f)\colon \mathfrak{P}_{\mathfrak{s}}(A) \to \mathfrak{P}_{\mathfrak{s}}(B)$ such that $\sigma_{\mathfrak{s}}\big(\mathfrak{P}_{\mathfrak{s}}(\tau_B)\big) = \big(\mathfrak{P}_{\mathfrak{s}}(\tau_A)\big)\varphi_{\mathfrak{s}}$.*

Proof. The morphism $\mathfrak{P}(f)\colon \mathfrak{P}(A) \to \mathfrak{P}(B)$ respects the isotypic $\mathfrak{s}$-components and may be uniquely written in the form

$$\prod_{\mathfrak{s}\in\mathcal{S}} f_{\mathfrak{s}}\colon \prod_{\mathfrak{s}\in\mathcal{S}} S_{\mathfrak{s}}^{\aleph(\mathfrak{s},A)} \to \prod_{\mathfrak{s}\in\mathcal{S}} S_{\mathfrak{s}}^{\aleph(\mathfrak{s},B)}.$$

If $\mathrm{copr}_{\mathfrak{s}}\colon S_{\mathfrak{s}}^{\aleph(\mathfrak{s},B)} \to PB$ is the natural embedding then

$$\varphi_{\mathfrak{s}} \stackrel{\rm def}{=} \mathrm{pr}_{\mathfrak{s}}\,\varphi\,\mathrm{copr}_{\mathfrak{s}}\colon L^{\aleph(\mathfrak{s},B)} \to L^{\aleph(\mathfrak{s},A)}$$

is a well defined base point preserving continuous map and

$$\mathfrak{P}_{\mathfrak{s}}(f)\varphi_{\mathfrak{s}} = \mathrm{id}\colon L^{\aleph(\mathfrak{s},B)} \to L^{\aleph(\mathfrak{s},B)}$$

as we see from the commutative diagram

$$\begin{array}{ccccc}
L^{\aleph(\mathfrak{s},B)} & \xrightarrow{\quad\varphi_{\mathfrak{s}}\quad} & L^{\aleph(\mathfrak{s},A)} & \xrightarrow{\quad P_{\mathfrak{s}}(f)\quad} & L^{\aleph(\mathfrak{s},B)} \\
{\scriptstyle \mathrm{copr}_{\mathfrak{s}}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{pr}_{\mathfrak{s}}} & & \big\uparrow{\scriptstyle \mathrm{pr}_{\mathfrak{s}}} \\
\prod_{\mathfrak{s}\in\mathcal{S}} L^{\aleph(\mathfrak{s},B)} & \xrightarrow{\quad\varphi\quad} & \prod_{\mathfrak{s}\in\mathcal{S}} L^{\aleph(\mathfrak{s},A)} & \xrightarrow{\quad f\quad} & \prod_{\mathfrak{s}\in\mathcal{S}} L^{\aleph(\mathfrak{s},B)}.
\end{array}$$

We now abbreviate $A_{\mathfrak{s}} = \mathfrak{P}_{\mathfrak{s}}(A) = \tau_A(S_{\mathfrak{s}}^{\aleph(\mathfrak{s},B)})$ and $B_{\mathfrak{s}}$ accordingly so that $\mathfrak{P}_{\mathfrak{s}}(f) = \mathfrak{P}_{\mathfrak{s}}(f)\colon A_{\mathfrak{s}} \to B_{\mathfrak{s}}$. Let $j_A\colon A_{\mathfrak{s}} \to A$ denote the inclusion. We define $\sigma'_{\mathfrak{s}} = \sigma j_B\colon B_{\mathfrak{s}} \to A$ and let $\tau_{B,\mathfrak{s}}\colon S_{\mathfrak{s}}^{\aleph(\mathfrak{s},B)} \to B_{\mathfrak{s}}$ denote the corestriction of $\tau_B \circ \mathrm{copr}_{\mathfrak{s}}$. Then $\sigma'_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = \sigma\tau_B\,\mathrm{copr}_{\mathfrak{s}}^{(B)} = \tau_A\varphi\,\mathrm{copr}_{\mathfrak{s}}^{(B)} = \tau_A\,\mathrm{copr}_{\mathfrak{s}}^{(A)}\,\varphi_{\mathfrak{s}} = j_A\tau_{A,\mathfrak{s}}\varphi_{\mathfrak{s}}$, and this shows that the image of $\sigma'_{\mathfrak{s}}$ is contained in $A_{\mathfrak{s}}$. Hence there is a well defined corestriction $\sigma_{\mathfrak{s}}\colon B_{\mathfrak{s}} \to A_{\mathfrak{s}}$ such that

$$\sigma_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = \tau_{A,\mathfrak{s}}\varphi_{\mathfrak{s}} \quad \text{and} \quad j_A\sigma_{\mathfrak{s}} = \sigma j_B,$$

where the second equation follows from $j_A\sigma_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = \sigma'_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = \sigma j_B\tau_{B,\mathfrak{s}}$ and the surjectivity of $\tau_{B,\mathfrak{s}}$. Now

$$j_b f_{\mathfrak{s}}\sigma_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = f j_A\sigma_{\mathfrak{s}}\tau_{B,\mathfrak{s}} = f\sigma j_B\tau_{B,\mathfrak{s}} = j_B\tau_{B,\mathfrak{s}},$$

and thus the surjectivity of $\tau_{B,\mathfrak{s}}$ and injectivity of j_B show

$$f_{\mathfrak{s}}\sigma_{\mathfrak{s}} = \mathrm{id}_{B_{\mathfrak{s}}}\,. \qquad \square$$

We now concentrate on topologically split morphisms of isotypical groups.

Lemma 10.56. *Let $\chi\colon L^Y \to L^U$ denote a morphism. Then the morphism $\chi \in \mathrm{Hom}(L^Y, L^U)$ is characterized by a partial function $\nu\colon V \to Y$, $V \in U$ and a function $\alpha\colon V \to \mathrm{Aut}\,L$, $(\alpha_u)_{u\in V} \in (\mathrm{Aut}\,L)^V$ which determine $\chi \overset{\mathrm{def}}{=} \chi_{\nu,\alpha}$ according to*

$$(*) \qquad \chi\big((a_y)_{y\in Y}\big) = (b_u)_{u\in U} \quad \textit{with} \quad b_u = \begin{cases} \alpha_u(a_{\nu(u)}) & \textit{if } u \in V, \\ \mathbf{1} & \textit{if } u \in U \setminus V. \end{cases}$$

Each one of these induces a morphism $Z^Y \to Z^U$ by restriction and corestriction.

Proof. For each $u \in U$ the composition $\mathrm{pr}_u \circ \chi\colon L^Y \to L$ is either the constant morphism 0 or else there is a unique element $\nu(u) \in Y$ and an automorphism $\alpha_u\colon L \to L$ such that $\alpha_u = \mathrm{pr}_u \circ \chi \circ \mathrm{copr}_{\nu(u)}$. In a diagram:

$$\begin{array}{ccc} L^Y & \xrightarrow{\quad\chi\quad} & L^U \\ {\scriptstyle \mathrm{copr}_{\nu(u)}}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{pr}_u} \\ L & \xrightarrow[\quad\alpha_u\quad]{} & L. \end{array}$$

Let us set $V = \{u \in U \mid \mathrm{pr}_u \circ \chi \neq 0\}$. Then $(*)$ holds. $\square$

Proposition 10.57. (i) *Let $f\colon A \to B$ be a surjective morphism of isotypical compact connected groups. Then $\mathfrak{P}(A) = L^X$ and $\mathfrak{P}(B) = L^Y$ with a simple simply connected Lie group L with Lie algebra $\mathfrak{s}$ and sets $X = \aleph(\mathfrak{s}, A)$ and $Y = \aleph(\mathfrak{s}, B)$ such that X is a disjoint union $U \dot\cup Y$ such that we can write $\mathfrak{P}(A) = L^U \times L^Y$*

and have exact sequences

$$0 \to \Delta_A \xrightarrow{\text{incl}} L^U \times L^Y \xrightarrow{\tau_A} A \to 0$$

$$0 \to \Delta_B \xrightarrow{\text{incl}} L^Y \xrightarrow{\tau_B} B \to 0.$$

There is a family $(\alpha_y)_{y\in Y} \in (\operatorname{Aut} L)^Y$ *and a bijection* $\rho\colon Y \to Y$ *such that*

$$\mathfrak{P}(f)\big((g_u)_{u\in U}, (g_y)_{y\in Y}\big) = (\alpha_y g_{\rho(y)})_{y\in Y} \stackrel{\text{def}}{=} f_2\big((g_u)_{y\in Y}\big)$$

and $\mathfrak{P}(f)(\Delta_A) \subseteq \Delta_B$.

(ii) f *has a homomorphic cross section* $s\colon B \to A$ *if and only if there is a subset* $V \subseteq U$, *a function* $(\alpha'_u)_{u\in V} \in (\operatorname{Aut} L)^V$ *and a function* $\nu\colon V \to Y$ *such that*

$$\big(\mathfrak{P}(s)\big)\big((g_y)_{y\in Y}\big) = \big(\chi_{\nu,\alpha'}\big((g_y)_{y\in Y}\big), \alpha^{-1}_{\rho^{-1}(y)}(g_{\rho^{-1}(y)})_{y\in Y}\big) = (h_u)_{u\in Y},$$

with

$$h_u = \begin{cases} \alpha'_u(g_{\nu(u)}) & \text{if } u \in V, \\ 0 & \text{if } u \in U \setminus V, \end{cases}$$

defines a homomorphic cross section for Pf, *and that* $\big(\mathfrak{P}(s)\big)(\Delta_B) \subseteq \Delta_A$.

(iii) f *has a continuous cross section* $\sigma\colon B \to A$ *if and only if there is a continuous base point preserving function* $\kappa\colon L^Y \to L^U$ *such that the function* $\varphi\colon L^Y \to L^U \times L^Y$ *defined by* $\varphi(x) = \big(\kappa(x), f_2^{-1}(x)\big)$ *satisfies* $\varphi\big(\Delta_B\big) = \Delta_A$.

Proof. (i) Since $A = A_\mathfrak{s}$ and $B = B_\mathfrak{s}$ are isotypic the abbreviations $L = S_\mathfrak{s}$ and $\aleph(\mathfrak{s}, A) = X$, $\aleph(\mathfrak{s}, B) = Y$ yield $PA = L^X$ and $\mathfrak{P}(B) = L^Y$. If Z is the finite center of L, then $\delta_A \subseteq Z^X$ and $\delta_B \subseteq Z^Y$. Every surjective morphism from a projective object splits. Hence we now write $X = U \mathbin{\dot\cup} Y$ and $L^X = L^U \times L^Y$ where $L^U = \ker Pf$ and thus, writing the elements of $\mathfrak{P}A$ as pairs $(a_1, a_2) \in L^U \times L^Y$, we have $\mathfrak{P}(f)(a_1, a_2) = f_2(a_2)$ with an isomorphism $f_2\colon L^Y \to L^Y$ which by Lemma 10.56, is necessarily of the form $(a_y)_{y\in Y} \mapsto \big(\alpha_y(a_{\rho(y)})\big)_{y\in Y}$ with a bijection ρ of Y and automorphisms α_y of L. It is clear from the naturality of τ that $Pf(\Delta_A) \subseteq \Delta_B$. Note that $f_2^{-1}\big((b_y)_{y\in Y}\big) = \big(\alpha^{-1}_{\rho^{-1}(y)} h_{\rho^{-1}(y)}\big)$

(ii) If $s\colon B \to A$ is a homomorphic cross section for f, then $\mathfrak{P}(s)$ is a homomorphic cross section for $\mathfrak{P}(f)$. If $\big(\mathfrak{P}(s)\big)(x) = (a_1, a_2)$ then $x = \big(\mathfrak{P}(f)\big)\big(\mathfrak{P}(s)\big)(x) = \big(\mathfrak{P}(f)\big)(a_1, a_2) = f_2(a_2)$, whence $a_2 = f_2^{-1}(x)$. Also, $a_1 = \chi(x)$ for some morphism $\chi\colon L^Y \to L^U$. It follows from Lemma 10.56 that $\big(\mathfrak{P}(s)\big)(\Delta_B) \subseteq \Delta_A$. It is clear that any morphism $x \mapsto \big(\chi(x), f_2^{-1}(x)\big)$ respecting the Δ will be a morphism $\mathfrak{P}s$ for a homomorphic cross section $s\colon B \to A$ for f.

(iii) Assume that f has a continuous cross section. By Proposition 10.55 there is a continuous cross section φ for $\mathfrak{P}(f)$. Then as in the proof of (ii) we conclude that $\varphi(x) = \big(\kappa(x), f_2^{-1}(x)\big)$ with a base point preserving continuous function $\kappa\colon L^Y \to L^U$.

From 3.11 we know $\varphi(tx) = \varphi(t)\varphi(x)$ for $t \in \Delta_B$ and $x \in L^Y$, and this yields

$$\begin{aligned} \big(\kappa(tx), f_2^{-1}(tx)\big) &= \varphi(tx) = \varphi(t)\varphi(x) \\ &= \big(\kappa(t), f_2^{-1}(t)\big)\big(\kappa(x), f_2^{-1}(x)\big) = \big(\kappa(t)\kappa(x), f_2^{-1}(tx)\big). \end{aligned}$$

Thus

$$(**) \qquad \kappa(tx) = \kappa(t)\kappa(x) \quad \text{for all} \quad t \in \Delta_B \subseteq Z^Y \text{ and } x \in L^Y,$$

where

$$\big(\kappa(t), f_2^{-1}(t)\big) \in \Delta_A \quad \text{for all} \quad t \in \Delta_B.$$

Conversely, any base point preserving continuous map which is of the form $x \mapsto \big(\kappa(x), f_2^{-1}(x)\big)$ and respects the Δ's will be the φ for a continuous cross section $\sigma: B \to A$ for f. This completes the proof. □

If f has a continuous cross section, then the group $\Delta_A \subseteq Z^U \times Z^Y$ is a direct product of the subgroups $\ker \pi = \Delta_A \cap (L^U \times \{\mathbf{1}\})$ and $\operatorname{im} \psi = \{\big(\kappa(x), f_2^{-1}(x)\big) \mid x \in \Delta_B\}$.

The problem of converting a topological splitting of a morphism of isotypic groups into an algebraic splitting, after 10.57, is the following:

For a continuous function $\kappa: L^Y \to L^U$ satisfying $(**)$ find a partial function $\nu: V \to Y$, $V \subseteq U$ and a function $\alpha: V \to \operatorname{Aut}(L)$ such that $\kappa|\Delta_B = \chi_{\nu,\alpha}|\Delta_B$.

Our Example 10.48 shows that this is not always possible. We shall now render this example more precise. Let Y and U be singleton. Let Δ_B be a central subgroup of L and set $B = L/\Delta_B$. Let a denote a natural number and define $\kappa: L \to L$ by $\kappa(x) = x^a$. Then (6) is satisfied. Set $\Delta_A = \{\big(\kappa(z), z\big) \mid z \in \Delta_B\}$ and $A = (L \times L)/\Delta_A$. The projection $L \times L \to L$ onto the second factor induces a topologically split morphism $f: A \to B$. In order for it to be split we must find a morphism $\chi: L \to L$ with

$$(\dagger) \qquad \chi(z) = \kappa(z) = z^a \quad \text{for} \quad z \in \Delta_B.$$

We shall now search for groups L for which there is a natural number a such that (†) can be satisfied for suitable endomorphism χ of L. If $\Delta_B^a = \{\mathbf{1}\}$, then the constant χ will satisfy (†). If $z^a = z$, for all z, i. e., $\Delta_B^{a-1} = \{\mathbf{1}\}$, then $\chi = \mathrm{id}_L$ satisfies (†). We now look for those L such that for each nonconstant endomorphism $\chi \neq \mathrm{id}_L$ there is a natural number $a > 1$ such that (†) is satisfied. Then χ must be an isomorphism because of the simplicity of L. In particular, $z \mapsto z^a$ has to be a nonidentity automorphism of Δ_B. The automorphisms induced on the center of a simple simply connected Lie group are known and catalogued (see e. g. [353]). They come from automorphisms of the Dynkin diagram; if the center is nontrivial the automorphism groups induced on the center have order 2 or are (in the case D_4) isomorphic to S_3. Thus we now inspect the list whether we find outer automorphisms χ of L which on the center induce a map of the form $z \mapsto z^a$.

Type A_{n-1} represented by $L = \mathrm{SU}(n)$, $\Delta_B \cong \mathbb{Z}(n)$, $\chi(z) = z^{-1} = z^a$ with $a \equiv -1 \pmod n$. If $n > 3$ then we always find natural numbers $a > 1$ with $a \not\equiv -1 \pmod n$.

Types B and C have no outer automorphisms.

Type D_n, $n = 4, 5 \ldots$ represented by $\mathrm{Spin}(n)$,

$$\Delta_B \cong \begin{cases} \mathbb{Z}(2)^2 & \text{if } n \equiv 0 \pmod 2, \\ \mathbb{Z}(4) & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

In the first case, the outer automorphisms χ do not satisfy (†) for any a. In the second case $\chi(z) = z^3$ for any outer automorphism. If $a = 2$ then (†) is not satisfied.

Types G_2, F_4, E_7, E_8 have no outer automorphisms. The compact simply connected form L of E_6 has a center $\Delta_B \cong \mathbb{Z}(3)$ and thus has outer nontrivial automorphisms χ with $\chi(z) = z^2$.

These remarks allow the following observation:

Example 10.58. Assume that G is a simple connected but not simply connected compact Lie group which is not isomorphic to $\mathrm{SO}(3)$, $\mathrm{PSU}(3)$, E_6/Z, $\mathrm{SO}(2m)$ or a double covering of $\mathrm{SO}(2m)$. Then there is a topologically split morphism $f\colon A \to G$ with A locally isomorphic to G^2 such that f does not split. □

We summarize:

Theorem 10.59. *Let $f\colon A \to B$ be a topologically split morphism of compact groups and let B be connected. Let $f_0\colon A_0 \to B$ denote the restriction to the identity component of A. Then*

(i) *f_0 is topologically split.*

(ii) *The induced morphism $F\colon A_0/A_0' \to B/B'$ is split.*

(iii) *Each isotypic $\mathfrak{s}$-component $\big(\mathfrak{P}_\mathfrak{s}(f_0)\big)\colon \mathfrak{P}_\mathfrak{s}(A_0) \to \mathfrak{P}_\mathfrak{s}(B)$ is topologically split. The morphism $f'\colon A' \to B'$ induced on the commutator groups is topologically split.*

The conditions (i), (ii), *and* (iii) *do not imply that f is split. In particular, a topologically split morphism between isotypic compact connected semisimple groups need not be split. Its splitting and topological splitting is characterized in Proposition* 10.57 *in terms of the projective cover.*

(iv) *Assume that T is any connected abelian subgroup of B. Then the homomorphism $f^{-1}(T) \to T$ induced by f splits, as does its restriction to any maximal connected abelian subgroup of $f^{-1}(T)$.*

Proof. This is just a summary of what was discussed before. □

Actions of Compact Groups and Acyclicity

Let R be a commutative ring with identity, which we fix for no other purpose than to have a general coefficient ring for cohomology of compact spaces. Most of the time we take $R = \mathbb{Z}$ (as e.g. in Theorem 8.83ff.) or any field such as $R = \mathbb{Q}$ (as in

Theorems 6.88ff. or A3.90ff.) or $R = \mathrm{GF}(2)$. In this subsection we envisage again Čech cohomology theory over the coefficient ring R on topological spaces, notably on compact spaces. According to historical preferences this topology is also called Alexander-Spanier-Wallace cohomology.

Definition 10.60. A compact space X is called *acyclic* over R (or simply *acyclic* if the coefficient ring R is understood) iff it is nonempty and its cohomology $H^*(X,R)$ is isomorphic to that of a singleton space, that is, iff

$$H^n(X,R) = \begin{cases} R & \text{if } n = 0, \\ \{0\} & \text{if } n > 0. \end{cases} \qquad \square$$

Now we recall the *Strong Homotopy Axiom* for Čech cohomology. Cohomology will be Čech cohomology over R. A good overview of the topic can be found in [340].

Proposition 10.61. (Generalized Homotopy Axiom) *Let X be a compact space and C a compact connected space. For two points $c_1, c_2 \in C$ set $e_j: X \to X \times C$, $e_j(x) = (x, c_j)$ for $j = 1, 2$. Then $H^*(e_1) = H^*(e_2) : H^*(X \times C) \to H^*(X)$.* □

The more classical version has $C = \mathbb{I} = [0,1]$, the unit interval and $c_1 = 0$, $c_1 = 1$.

One says that two continuous functions $f_j: X \to Y$, $j = 1, 2$ are *weakly homotopic*, if there is a compact connected space C with two points $c_j \in C$ and a continuous function $F: X \times C \to Y$ such that $f_j(x) = F(x, c_j)$, $j = 1, 2$. If this is true with $C = \mathbb{I}$, $c_1 = 0$, $c_2 = 1$, then f_1 and f_2 are said to be *homotopic*.

Corollary 10.62. *If $f_1, f_2: X \to Y$ are two weakly homotopic continuous functions between compact spaces, then $H^*(f_1) = H^*(f_2): H^*(Y) \to H^*(X)$.*

Proof. Exercise. □

Exercise E10.10. Prove Corollary 10.61.

[Hint. From the definition of weak homotopy we have a compact connected space and a continuous function $F: X \times C \to Y$ such that $f_j(x) = F(x, c_j)$, $x \in X$, $j = 1, 2$. Define $e_j: X \to X \times C$ by $e_j(x) = (x, c_j)$. Then $f_j = F \circ e_j$ and thus $H^*(f_j) = H^*(e_j) \circ H^*(F)$. Since $H^*(e_1) = H^*(e_2)$ by 10.61, the assertion follows.] □

Recall that a space X is called *contractible* respectively, *weakly contractible* (to a point $x \in X$) if the identity map of X and the constant self-map of X with value x are homotopic, respectively, weakly homotopic. The required function $F: X \times C \to X$ is called a *contraction*, respectively, *weak contraction*.

The following then is immediate from the definitions:

Proposition 10.63. *Every compact weakly contractible space is acyclic over any coefficient ring.* □

Fixed Points of Compact Abelian Group Actions

Now we record an impressive result on group actions (see [197], Theorem 3.21 or [162], Proposition 1.10).

The Fixed Point Theorem for Compact Abelian Group Actions

Theorem 10.64. *Let G be a compact connected abelian group acting on a compact space X which is acyclic over $\mathbb{Q}$. Then the subspace F of fixed points and the orbit space X/G are acyclic.* □

In particular, such a group action has a fixed point, and the fixed point set is connected. In the absence of connectivity this result fails due to counterxamples. (For more information see e.g. [271], p. 91f.)

The first corollary concerns the structure theory of compact groups.

Corollary 10.65. *Let G be a compact connected group and H a closed subgroup such that the quotient space G/H is acyclic over $\mathbb{Q}$, then H contains a maximal connected compact abelian subgroup of G.*

Proof. Let T be a maximal pro-torus of G (see Theorem 9.32). Then T is a compact connected abelian group acting on the rationally acyclic space G/H via $t{\cdot}gH = tgH$. By Theorem 10.64, there is a fixed point, say $g^{-1}H$, that is $tg^{-1}H = g^{-1}H$ for all $t \in T$. Then $gTg^{-1} \subseteq H$, and that is the assertion according to Theorem 9.32(i). □

One uses the expression that a closed subgroup H of a compact connected group G containing a maximal pro-torus *has maximal rank*.

Corollary 10.66. *If G and H satisfy the hypotheses of Corollary 10.65, then H contains the center $Z(G)$ of G.*

Proof. This follows from Corollary 10.65 and and Theorem 9.32(iv). □

The deeper structure theory of compact connected monoids demands significant applications of the Fixed Point Theorem.

Corollary 10.67. *Let S be a compact connected topological monoid with zero element 0 and let T be a compact connected abelian group of units, that is, invertible elements. Then the centralizer $Z(T, G)$ is a closed connected submonoid containing T and 0.*

Proof. A compact connected monoid with zero 0 is weakly contractible to 0 via its multiplication $S \times S \to S$ as weak contraction. Now T acts on S under inner automorphisms: $t{\cdot}s = tst^{-1}$. The centralizer $Z(T, S)$ is precisely the fixed point set of this action. Then Theorem 10.64 proves the claim. □

The existence of a zero element is no essential sacrifice of generality for a nongroup compact monoid, because every such has a closed minimal ideal which we may collapse to a single point thereby obtaining a monoid with zero element. For the minimal ideal $\mathcal{M}(S)$ of a compact semigroup see the paragraph preceding Proposition A4.34. For the general theory of compact semigroups see for instance [197] and [56].

In the theory of compact monoids, Corollary 10.67 is an essential tool in the proof of a fundamental theorem:

Theorem 10.68. *Every compact connected monoid S contains a connected abelian submonoid A meeting the minimal ideal $\mathcal{M}(S)$, that is, A contains* 1 *and points of $\mathcal{M}(S)$.* □

For a proof and the significance of 10.68 we refer to [197].

Transitive Actions of Compact Groups

Assume that a compact group G acts transitively on a Hausdorff space X. For an element $x \in X$ let G_x again denote the isotropy group $\{g \in G : g{\cdot}x = x\}$. We write $H \stackrel{\text{def}}{=} G_x$. Transitivity of the action means $X = G{\cdot}x$. Thus Lemma 10.4 shows that there is an equivariant homeomorphism $gH \mapsto g{\cdot}x : G/H \to X$. Conversely, if H is any closed subgroup of G, then G acts transitively on the compact space G/H so that H is the isotropy subgroup at the point $H \in G/H$. Accordingly, the theory of transitive actions of a compact group is completely equivalent to structure theory of the quotient spaces together with the multiplication of G on the left of G/H.

In the present context we discuss what happens if G/H is acyclic. Recalling Example E6.10, we consider the matrices

$$M(t, e) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ \sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \in \mathrm{SO}(3), \quad t \in [0, 1[,\ e = \pm 1,$$

and define $T = \{M(t, e) : t \in [0, 1[,\ e = 1\}$, $H = N\big(T, \mathrm{SO}(3)\big) = \{M(t, e) : t \in [0, 1[,\ e = \pm 1\}$. Then G/T is homeomorphic to the 2-sphere $\mathbb{S}^2$ which is a 2 sheeted universal covering space of $X \stackrel{\text{def}}{=} G/H$ which is the real projective plane $P\mathbb{R}^2$ obtained from $\mathbb{S}^2$ by identifying antipodal points. We have

$$H^n(X, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } n = 0, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$H^n(X, \mathrm{GF}(2)) = \begin{cases} \mathrm{GF}(2) & \text{if } n = 0, 1, 2, \\ \{0\} & \text{otherwise.} \end{cases}$$

Thus X is rationally acyclic but is not contractible. As we noted above, acyclicity over *every* coefficient ring is a consequence of the assumption that G/H is (weakly) contractible. In that case, in particular, G/H is connected. The quotient function $g \mapsto gH : G \to G/H$ maps components onto components. Hence $G/H = G_0H/H \cong G_0/(G_0 \cap H)$. We may therefore assume that G is itself connected. If N is a closed normal subgroup of G, then $\frac{G/N}{H/N} \cong G/H$. Thus we may assume that H does not contain any nontrivial normal subgroups of G. Then G is a compact connected centerfree group according to Corollary 10.67. Hence by Theorem 9.24 there is a family of centerfree simple compact connected Lie groups G_j, $j \in J$ such that $G = \prod_{j\in J} G_j$.

On the basis of these results, A. Borel shows the following result which one finds proved in [197], pp. 308–310.

Theorem 10.69. [A. Borel, 1964-65] *Let G be a compact connected group with a closed subgroup H such that G/H is rationally acyclic and that H contains no nondegenerate normal subgroup of G. Then there exists*

(i) *a family of centerfree simple connected groups G_j, $j \in J$,*

(ii) *for each $j \in J$ a closed subgroup H_j of G_j and a maximal torus T_j of G_j such that*

(a) $G = \prod_{j\in J} G_j$ *and* $H = \prod_{j\in J} H_j$,

(b) $(\forall j \in J)\, T_j \subseteq H_j$,

(c) $(\forall j \in J)\, H_i = (H_j)_0{\cdot}N(T_j, G_j)$ *for the normalizer* $N(T_j, G_j)$ *of* T_j *in* G_j, *and, finally, that*

(c) G/H *is homeomorphic to*

$$\prod_{j\in J} \frac{G_j}{(H_j)_0{\cdot}N(T_j, G_j)}.$$

In particular, a rationally acyclic quotient space of a compact connected group is a product of compact manifolds. □

Now, a product of compact manifolds cannot be acyclic modulo 2. One therefore obtains the following result.

A. Borel's Degeneracy Theorem

Theorem 10.70. *Let H be a closed subgroup of a compact group such that G/H is acyclic over $\mathbb{Q}$ and* GF(2)*. Then $H = G$.* □

For an alternate proof see [257] (2011).

Corollary 10.71. *A (weakly) contractible quotient of a compact group is singleton.*

In particular, a contractible compact group is a point. □

Szenthe's Theory of Transitive Actions of Compact Groups

We are finally led to a recasting of Janoś Szenthe's far reaching theory on transitive actions of compact (and locally compact) groups on locally contractible space. We discuss a modern version of his theory.

1. Finite Dimensional Quotient Spaces

Accordingly we shall discuss pairs consisting of a compact group G and closed subgroup H, giving rise to a quotient space $X = G/H$.

Lemma 10.72. *There is a totally disconnected subspace $D \subseteq G$ and D homeomorphic to $G/(G_0H)$ such that*

$$(d, x) \mapsto dx : D \times G_0H \to G$$

and

$$(d, xH) \mapsto dxH : D \times (G_0H/H) \to G/H$$

are natural homeomorphisms. Also

$$g(G_0 \cap H) \mapsto gH : \frac{G_0}{G_0 \cap H} \to G_0H/H$$

is a natural homeomorphism.

Proof. The quotient space of G/G_0H is compact totally disconnected. Thus the action of the compact group G_0H on G by multiplication has a totally disconnected orbit space. Now Theorem 10.35 applies and shows that G is homeomorphic to $D \times G_0H$ with a totally disconnected compact space D (homeomorphic to $G/(G_0H)$) in such a fashion that the action of G_0H is by multiplication on the second factor on the right. Therefore G/H is naturally homeomorphic to $D \times G_0H/H$.

The last homeomorphism is standard. □

This Lemma shows that $X = G/H$ and the connected component $X_0 = G_0H/H \cong G_0/(G_0 \cap H)$ of the point $x_0 = H$ in X differ only by a totally disconnected compact topological factor.

The following conclusion is straightforward from Lemma 10.72:

Lemma 10.73. *The following conditions are equivalent:*
1. *X is locally connected.*
2. *G_0H has finite index in G.*
3. *X_0 is open.*
4. *X and $G_0/(G_0 \cap H)$ are locally homeomorphic.*

Proof. By Lemma 10.72, assertion (1) holds if and only if $D \cong G/(G_0H)$ is finite, iff (2) holds, and $G/(G_0H)$ is finite iff $X_0 = G_0H/H$ is open in X, i.e., (2) $\iff$ (3). By Lemma 10.72, X_0 and $G_0/(G_0 \cap H)$ are homeomorphic. Thus (4) holds iff X and X_0 are locally homeomorphic. Since X is homeomorphic to $X_0 \times D$ by 10.72, we conclude that this holds iff X_0 is open in X. □

For the moment, we shall be primarily interested here in the case that the dimension of X is finite. It was shown in 8.25 and 9.54, that a few axioms on a function DIM on the class of compact spaces taking values in $\{0, 1, 2, \ldots, \infty\}$ characterize it uniquely on the class of the underlying spaces of compact groups, and that it agrees on these with, say, the Lebesgue covering dimension. Such a function was called an *admissible dimension*, and it was shown that for a compact group G we have $\operatorname{DIM} G = \dim_{\mathbb{R}} \mathfrak{L}(G)$ where $\mathfrak{L}(G)$ is the Lie algebra of G (see Scholium 9.54), and where $\dim_{\mathbb{R}}$ is the dimension of a weakly complete real topological vector space, agreeing with the linear dimension of the dual $\mathfrak{L}(G)'$.

We denote by $N_H \stackrel{\text{def}}{=} \bigcap_{g\in G} gHg^{-1}$ the largest normal subgroup of G contained in H. The function

$$gH \mapsto (gN_H){\cdot}(H/N_H) : G/H \to \frac{G/N_H}{H/N_H}$$

is a natural homeomorphism.

Now we shall extend the discussion of dimension to quotient spaces $X = G/H$. Our paper [185] was devoted to the dimension theory of quotient spaces $X = G/H$ on the basis of the theory provided in this book, and we quote the following result from it.

Proposition 10.74. *Let G be a compact group and H a closed subgroup and let $X = G/H$. Denote by* DIM *any admissible dimension function on the class of compact spaces. Assume that* $\operatorname{DIM} X < \infty$. *Then we have the following conclusions:*

(i) $\operatorname{DIM} X = \dim_{\mathbb{R}} \mathfrak{L}(G)/\mathfrak{L}(H) = \dim \mathfrak{L}(G_0)/\mathfrak{L}(H_0)$.

(ii) *Both G/N_H and H/N_N are finite dimensional compact groups and* $G/H \cong \frac{G/N_H}{H/N_N}$.

(iii) *Assume that G is connected and $N_H = \{1\}$. Then G contains a totally disconnected central subgroup D such that G/D is a Lie group, and the base point $x_0 = H$ of G/H has a neighborhood homeomorphic to*

$$D \times \frac{\mathfrak{L}(G)}{\mathfrak{L}(H)}.$$

Proof. For the proofs we refer to [185], 1.11, and 3.1. □

We shall find the following corollary relevant:

Corollary 10.75. *Let G be a compact group and H a closed subgroup, and set $G/H = X$. Then the following conditions are equivalent:*

(1) *X is locally euclidean.*

(1′) $\frac{G/N_H}{H/N_H}$ *is locally euclidean.*

(2) *G/N_H is a Lie group.*
(3) *X is finite dimensional and locally connected.*

Proof. Since

$$gH \mapsto (gN_H)\cdot(H/N_H) : G/H \to \frac{G/N_H}{H/N_H}$$

is a natural homeomorphism, (1) is equivalent to (1′). For proving the equivalence of (1) and (2) it is therefore no loss of generality to assume $N_H = \{1\}$; then (2) is saying that G is a Lie group.

The implication (2) ⇒(1) is clear: Indeed if G is a Lie group, so is H (see Proposition 5.33(iii)). We apply Proposition 10.74(iii) to G_0 and $G_0 \cap H$); if G_0 is a Lie group, the space D is finite and thus $G_0/(G_0 \cap H)$ is locally euclidean; accordingly, G_0H/H is locally euclidean. But G_0 and so G_0H is open since G is a Lie group. Thus G_0H/H is open in G/H and so G/H is locally eulidean.

Proof of (1)⇒(2): First assume that G is connected. Again we invoke Proposition 10.74(iii): We know that X is locally euclidean, and so D is necessarily finite, and since G/D is a Lie group, G is a Lie group. Now allow G to be disconnected. Since X is locally euclidean, so is $X_0 \cong G_0/(G_0 \cap H)$. Hence by what we just saw, G_0 is a Lie group. Then $\mathfrak{L}(G_0) = \mathfrak{L}(G)$ is finite dimensional. Let $\varphi\colon H \to \mathrm{Aut}(\mathfrak{L}(G_0))$ be the linear representation obtained by restricting the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{L}(G))$. Set $Z \overset{\mathrm{def}}{=} Z(H, G_0)$. Then $Z = \ker\varphi$ and so $H/Z \cong \mathrm{im}\, p$ is a Lie group. The normalizer of Z in G contains G_0 and H, and thus the open subgroup G_0H. It therefore has finite index in G. Thus the largest normal subgroup N_Z of G, being a finite intersection of conjugates of Z, has finite index in Z. Since H does not contain any nontrivial normal subgroups of G, we know that Z has finite order. Since H/Z is a Lie group, H is a Lie group, and so G_0H is a Lie group which has finite index in G. Thus G is a Lie group which proves (2).

Trivially (1) implies (3). Now assume (3). Then we are in the situation of Proposition 10.73 with X_0 open. By Proposition 10.74(iii), X_0 is locally homeomorphic to $\Delta \times (\mathfrak{L}(G_0))/(\mathfrak{L}(G_0 \cap H))$ for a totally disconnected normal subgroup Δ of G_0. Since this space is locally connected, it follows that Δ is finite and the space is locally euclidean. Thus X_0 and therefore X is locally euclidean. □

In particular, under these circumstances $X = G/H$ is locally euclidean if and only if it is a real analytic manifold.

We would like a topological condition on X which is implied by being locally euclidean. An example of such a condition is that X has an open set which is contractible to a point. We want to exploit this idea.

2. Piecewise Contractible Spaces

We abbreviate the unit interval $[0, 1]$ by $\mathbb{I}$.

Definition 10.76. A subset U of a topological space X will be called *X-contractible*, or *contractible in X*, if there is a continuous function

$$(t, u) \mapsto f_t(u) : \mathbb{I} \times U \to X$$

such that f_0 is the inclusion map $\mathrm{incl}_{X,U}$ of U into X and f_1 is a constant function of U into X. A space is *contractible* if it is contractible in itself. A topological space X will be called *piecewise contractible* if it has an X-contractible open subset. □

An X-contractible set is nonempty, and any nonempty subset of an X-contractible set is X-contractible.

If $\mathbb{D}$ is the closed complex unit disc and U is the punctured disc $\mathbb{D} \setminus \{0\}$, then U is $\mathbb{D}$-contractible (to 0), yet is not contractible.

Exercise E10.10. (i) If a product $X \times Y$ is piecewise contractible, then X is piecewise contractible.

(ii) A homogeneous space X is piecewise contractible iff X has an open cover consisting of X-contractible sets.

(iii) If x is any point in a piecewise contractible homogeneous space X, then all sufficiently small neighborhoods of x are X-contractible.

(iv) If X is a piecewise contractible homogeneous space and V is any open and closed subset of X, then V is piecewise contractible.

[Hint. The proof of (i) is straightforward. The proofs of (ii) and (iii) are easy from the definitions. Proof of (iv): Assume that the nonempty open set U is contractible in X. By homogeneity we may assume that $V \cap U \neq \emptyset$. Let $h_t: U \to X$ be a homotopy contracting U to u. Since V is open and closed, $v \in V \cap U$ implies that the connected subspace $h_{\mathbb{I}}(u)$ is contained in V. By definition of h_t, it is also contained in U. Thus $h_{\mathbb{I}}(V \cap U) \subseteq V \cap U$ and so $V \cap U$ is contractible to a point in $V \cap U$. Thus V is piecewise contractible.] □

Exercise E10.11. If $X = G/H$ is piecewise contractible, for a topological group G and a closed subgroup H, then the component X_0 of $H \in X$ is open and closed in X and therefore is piecewise contractible.

[Hint. Due to the homogeneity of X we may assume that some open set U is contractible to $x_0 = H \in X$. Then U is contained in the arc component X_a of x_0 in X. Since $X_a \subseteq X_0$, this means that X_0 is open, hence closed, in the homogeneous space X. Now by Exercise E10.10(iv) above, X_0 is piecewise contractible.] □

For showing that X is locally euclidean, it therefore suffices to show that X_0 is locally euclidean.

If G is compact, then $G_0H/H = X_0$. From $G_0/(G_0 \cap H) \cong G_0H/H \subseteq G/H$ we know that G/H is piecewise contractible iff $G_0/(G_0 \cap H)$ is piecewise contractible; likewise $G_0/(G_0 \cap H)$ is locally euclidean if X is locally euclidean.

Thus for an eventual proof that piecewise contractibility implies that X is locally euclidean, we need only consider *connected* compact groups G which we may assume in most of the remainder of the discussion of the case of compact groups G.

The following lemma is a technical step used in the next partial result.

Lemma 10.77. *If H/H_0 is finite and G/H is piecewise contractible then G/H_0 is piecewise contractible.*

Proof. The quotient map $G/H_0 \to G/H$ is a finite covering. If $U = UH$ is an open subset of G such that U/H is contractible in G/H to a point, we may assume that U/H is evenly covered, that is, U is a disjoint union $U' \cup U'h_2 \cup \cdots \cup U'h_n$ with $1 \in U' = U'H_0$. Then $U'H/H \cong U/H$ is contractible in G/H, and by the homotopy lifting property of coverings, U'/H_0 can be homotopically contracted in G/H_0 to a finite set contained in a fiber of the covering. Then there is an open subset U'' of U' containing 1 and satisfying $U''H_0 = U''$ such that U''/H_0 is contractible to a point in G/H_0. □

We recall from Theorem 9.2, that the algebraic commutator subgroup G' of a connected compact group G is closed. The structure of G' is known from Proposition 9.4 through Corollary 9.20. In particular, G' is finite dimensional if and only if it is a Lie group.

Lemma 10.78. *Let G be a finite dimensional connected compact group, H a closed subgroup, and assume that $X = G/H$ is piecewise contractible. Then G/H is locally euclidean.*

Proof. Since $(G/N_H)' = G'N_H/N_H$ is finite dimensional if G' is finite dimensional we may assume that H does not contain any nondegenerate normal subgroup of G and show that G is a Lie group.

By the Levi-Mal'cev Theorem for compact connected groups Theorem 9.24 the centralizer $Z(G', G)$ is central in G. The representation π of H into $\mathrm{Aut}(G')$ by inner automorphisms has the kernel $Z(G', H) = Z(G', G) \cap H = Z(G) \cap H$, and since this group is normal in G, and since H does not contain nondegenerate normal subgroups of G, it must be singleton. So we know that π is injective and thus that H is a Lie group since G' and so $\mathrm{Aut}\, G'$ is a Lie group. Hence $G'H$ is a Lie group.

From the Borel-Hofmann-Scheerer Splitting Theorem 9.39 we know that we can write $G = G' \rtimes A$ with a compact connected abelian group A. The projection of $G'H_0$ into A is a connected Lie subgroup of A, that is, a torus. Hence it splits as a direct factor of A (see Theorem 8.78(ii)). That is, there is a closed subgroup B of A such that we can write $G = G'H_0 \times B$. Accordingly, $G/H_0 \cong (G'H_0/H_0) \times B$.

Recall that H is a Lie group. Thus H/H_0 is finite. By Lemma 10.77, G/H_0 is piecewise contractible. Then by Exercise E10.10(i) the group B is piecewise contractible. Therefore the arc component of the identity is open; since B is connected it is arcwise connected. Since the subgroup B of G is finite dimensional, by 8.22, its torsion-free character group $\widehat{B}$ is of finite rank and thus is countable. Since B

is arcwise connected, $\widehat{B}$ is a Whitehead group (see 8.30(iv)). A countable Whitehead group is free by Pontryagin's Theorem (see A1.62). Thus B is a torus as the character group of a free abelian group of finite rank. Hence $G = G'H_0 \times B$ is a Lie group as we had to show. □

3. Piecewise Contractibility of Quotient Spaces

The following concept is not used in our discussion of compact groups but is widely used in the literature:

Definition 10.79. A space X is called *locally contractible* if for all points $x \in X$ and every neighborhood V of x there is a V-contractible neighborhood U of x in V. □

Any locally euclidean space is locally contractible. A locally contractible space is locally arcwise connected and piecewise contractible. On the other hand, it is easy to envisage contractible but not locally arcwise connected continua in the plane; these will be piecewise contractible but not locally contractible (for instance: the union of straight lines in the unit square $[0,1] \times [0,1]$ connecting $v = (\frac{1}{2}, 1)$ and the points $(c, 0)$ as c ranges through the standard Cantor set in the unit interval $[0,1]$ (see paragraph following Example 1.19); this space is called the Cantor fan and is compact contractible, but locally connected at no point different from v.)

Our main goal is a proof of the following theorem:

Szenthe's Theorem Revisited

Theorem 10.80. *Let G be a compact group, H a closed subgroup, N_H the largest normal subgroup of H. Set $X = G/H$. Then the following statements are equivalent:*

(1) *X is piecewise contractible.*
(2) *X is locally contractible.*
(3) *X is locally connected and finite dimensional.*
(4) *X is locally euclidean.*
(5) *X is a real analytic finite dimensional manifold.*
(6) *X is the quotient space of a Lie group modulo a closed subgroup.*
(7) *G/N_H is a Lie group.*

After our preparations, what is missing is (1)⇒(4), that is, we must prove

Theorem A. *If the quotient space $X = G/H$ is piecewise contractible for a compact group G and a closed subgroup H, then it is locally euclidean.*

Exercise E10.12. Prove that Theorem A and the information already provided would complete the proof of the Theorem 10.80. □

The background and history of this result is interesting in its own right, and we shall comment on it in the Postscript to this chapter. Notice that even the case that $H = \{1\}$ and $X = G$ is of interest for the structure theory of compact groups as it is discussed in this book:

A compact group G is piecewise contractible iff it is locally contractible iff it is locally euclidean iff it is a Lie group.

Hilbert's Fifth Problem for compact groups was discussed in Theorem 9.57, and local connectedness, which is much weaker than local contractibility was analyzed in various places (cf. 8.36, 9.66, 9.68).

4. The Homotopy Theoretical Backgrouns

Our proof of Theorem A is based on some basic results on homotopy theory and fibrations.

Definition 10.81. We say that a map $q\colon E \to B$ has the *homotopy lifting property*, if for any space U, any homotopy

$$(x,t) \mapsto h_t(x) : U \times \mathbb{I} \to B$$

and for any continuous function $f\colon U \to E$ such that $q \circ f = h_0$, there is a homotopy

$$(x,t) \mapsto \widetilde{h} : U \times \mathbb{I} \to E$$

such that $p \circ \widetilde{h} = h$, that is,

$$(1) \qquad (\forall t \in \mathbb{I}) \quad \begin{array}{ccc} U & \xrightarrow{\ \widetilde{h}_t\ } & E \\ {\scriptstyle \mathrm{id}\,U}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[\ h_t\ ]{} & B \end{array} \quad \text{commutes.}$$

Frequently a map with the homotopy lifting property is called a *fibration.* □

Exercise E10.13. (i) Let $p\colon E \to B$ be a fibration and $x \in E$. Then p maps the arc component of x onto the arc component of $p(x)$ in B.

(ii) If, for a topological group G and a closed subgroup H, the quotient map $p\colon G \to X = G/H$ is a fibration, then the arc component X_a of $x_0 = H$ in X is G_aH/H.

[Hint. (i) Clearly arc components are mapped into arc components. Conversely, let $f\colon \mathbb{I} \to B$ be an arc with $f(0) = p(x)$. Then the homotopy lifting property of p yields an arc $\widetilde{f}\colon \mathbb{I} \to E$ such that $p \circ \widetilde{f} = f$.

(ii) This follows immediately from (i) above.] □

Lemma 10.82. (The Homotopy Lifting Theorem of Madison-Mostert-Skljarenko). *For a compact group G, the map $p\colon G \to G/H$ is a fibration with the homotopy lifting property.*

Proof. See [249], Theorem 2. □

Corollary 10.83. (i) *Let G_a denote the arc component of* 1 *in the compact group* G*. Then the arc component X_a of $x_0 = H \in X = G/H$ is G_aH/H.*

(ii) *If X is piecewise contractible, then G_0H is open and closed in G and $G_0/(G_0 \cap H)$ is piecewise contractible.*

Proof. (i) The claim follows immediately from Exercise E10.13 and Lemma 10.82.

(ii) By Exercise E2.2, $X_0 = X_a$ is open and closed in X. By (i) above, $G_aH/H = X_a = X_0$. Since G is compact, $X_0 = G_0H/H$. Thus G_aH is open and closed in G, and so $G_0H \subseteq G_aH \subseteq G_0H$. Now the natural map $G_0/(G_0 \cap H) \to G_0H/H$ is a homeomorphism and $G_0H/H = X_0$ is piecewise contractible by Exercise E10.11, the assertion follows. □

Recall that G_a is dense in G_0 by Theorem 9.60(v).

After Corollary 10.83(ii), in showing that the piecewise contractibility of G/H makes it locally euclidean, it will be no loss of generality to assume that G is connected.

Definition 10.84. We shall call a self-homotopy $(x, t) \mapsto h_t(x) : A \times \mathbb{I} \to A$ of a space A a *compression* if $h_0 = \mathrm{id}_X$ and h_1 is not surjective. We shall call a space X *incompressible* if it does not support any compression. □

Madison [249] calls such spaces *irreducible*, but we hesitate to adopt this terminology.

Lemma 10.85. (The Madison Incompressibility Theorem) *If G is a compact group, then G is incompressible.*

Proof. In [249], Theorem 1, this is proved for connected G. By Corollary 10.38, an arbitrary compact group G is homeomorphic to $G_0 \times G/G_0$. A self-homotopy of a product projects to a self-homotopy of a factor. The totally disconnected space G/G_0 has no self-homotopy except the constant one. Thus a self-homotopy of G leaves each component invariant, and each component is homeomorphic to G_0. But G_0 does not allow any compression by Madison's Theorem. Hence G does not allow any compression. □

In [249], Madison proves Lemma 10.85 more generally for "limit manifolds". Every compact connected group is a limit manifold. He applies his results to argue that for a compact group G and a closed subgroup H, the quotient space G/H, if it is connected, is incompressible. See also [176].

Later we shall use the following corollary:

Lemma 10.86. *Let K be a compact group and $R \subseteq S$ closed subgroups of K with quotient maps $q: K \to K/R$ and $r: K \to K/S$. Let $F: K/R \to K/S$ be defined by $F(kR) = kS$. Assume that there is a homotopy*

$$(\xi, t) \mapsto h_t(\xi) : K/R \times \mathbb{I} \to K/S$$

such that $h_0 = F$. *Then* h_1 *is surjective.*

Proof. Then by Lemma 10.82 we get a lifting

$$(g,t) \mapsto \widetilde{H}_t(k) : K \times \mathbb{I} \to K,$$

such that the following diagram commutes

$$\begin{array}{ccc} K & \xrightarrow{\quad \widetilde{H}_t \quad} & K \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ K/R & \xrightarrow[\quad h_t \quad]{} & K/S \end{array}$$

and $\widetilde{H}_0 = \mathrm{id}_K$. Suppose that h_1 is not surjective. Then $\widetilde{H}_1(K)$ being contained in $r^{-1}\big(h_1(K/R)\big)$ is a proper subset of K. But this contradicts the Incompressibility Theorem 10.85. $\square$

This Lemma was also proved in [176] (see Corollary 1.9).

5. The Proof of Theorem A for compact groups

We assume throughout this subsection that

G is a compact connected group and H a closed subgroup such that $X = G/H$ is piecewise contractible.

Since G/H is piecewise contractible, there is an open subset $U \subseteq G$ such that $1 \in U = UH$ and that there is a homotopy

$$(\xi, t) \mapsto f_t(\xi) : U/H \to G/H, \;\; f_0 = \mathrm{incl}_{G/H, U/H}$$

and

$$f_1(U/H)) = \{\xi_o\}, \;\; \xi_0 = H \in G/H.$$

With the aid of Wallace's Lemma (see e.g. Proposition A4.29) let us find a closed normal subgroup N of G such that

(a) $NH \subseteq U$, and

(b) G/N is a Lie group.

If we define $k_t = f_t|NH/H$, then

$$(*) \qquad (\xi, t) \mapsto k_t(\xi) : NH/H \times \mathbb{I} \to G/H$$

is a homotopy such that

$$k_0 = \mathrm{incl}_{G/H, NH/H} \;\text{ and } k_1(NH/H) = \{\xi_0\}, \xi_0 = H \in G/H.$$

We conclude from Theorem 9.77(iii) the following consequence:

Lemma 10.87. *Let $f\colon G \to C$ be a surjective morphism of compact connected groups. Then there is a connected closed normal subgroup Γ of G such that $f|\Gamma : \Gamma \to C$ is a surjective morphism with a totally disconnected kernel.* $\square$

We apply this lemma with $C = G/N$ and the quotient morphism. Hence there is a closed connected normal subgroup Γ of G such that $N\Gamma/N = G/N$, that is, $N\Gamma = G$ and we have a morphism $F: G \to N/(N \cap \Gamma) \cong G/\Gamma$. Also, $\Gamma/(\Gamma \cap N)$ is a Lie group, while $\Gamma \cap N$ is totally disconnected. This implies $\dim \Gamma < \infty$.

Lemma 10.88. $\Gamma H = G$.

Proof. The quotient morphism $F: G \to N/(N \cap \Gamma)$ induces a surjective map $f: G/H \to N/(N \cap \Gamma H) \neq \{1\}$, and the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\quad F \quad} & \frac{N}{N\cap\Gamma} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \text{quot}} \\ \frac{G}{H} & \xrightarrow[\quad f \quad]{} & \frac{N}{N\cap\Gamma H}. \end{array}$$

We note that $f|NH : \frac{NH}{H} \to \frac{N}{N\cap\Gamma H}$ is the natural quotient map via $\frac{N}{N\cap H}$. Now we apply Lemma 10.86 with $K = N$, $R = N \cap H$, $q: N \to N/(N \cap H)$ the quotient map, $S = N \cap \Gamma H$ and $r: N \to N/(N \cap \Gamma H)$ the quotient map; moreover, let $X = G/H$ and let $f: X \to N/(N \cap \Gamma H)$ be as above; finally, define $h_t: N/(N \cap H) \times \mathbb{I} \to N/(N \cap \Gamma H)$ as the composition

$$\frac{N}{N \cap H} \xrightarrow{\cong} \frac{NH}{H} \xrightarrow{k_t} \frac{G}{H} \xrightarrow{f} \frac{N}{N \cap \Gamma H} \quad \text{with } k_t \text{ as in } (*).$$

Then Lemma 10.86 implies $N \cap \Gamma H = N$, that is, $N \subseteq \Gamma H$. In view of $G = N\Gamma$ this means $G = \Gamma H$ as asserted. □

Now $\Gamma H = G$ implies $G/H \cong \Gamma/(\Gamma \cap H)$, and $\dim \Gamma < \infty$. Therefore we are left to deal with the finite dimensional case. But in this direction we have Lemma 10.78. If $\dim G < \infty$ then $\dim G' < \infty$. Then Lemma 10.78 shows that $G/H \cong \Gamma/(\Gamma \cap H)$ is locally euclidean. This completes the proof of Theorem A and thus the proof of Theorem 10.80 for compact groups. □

6. The reduction of the case of locally compact groups to the case of compact ones

Since we got that far with the case of compact group actions, let us comment that, given certain pieces of information on the structure of locally compact groups, it is not hard to generalize the main result to the action of locally compact groups where, admittedly we allow a slightly stronger hypothesis of contractibility on the quotient space. Thus let us consider a locally compact group G and a closed subgroup H; again we write $X = G/H$.

Let us first argue, that it is no loss to assume that G is *almost connected*, that is, that the factor group G/G_0 of G modulo the identity component is compact. Recall that G contains an open almost connected subgroup G_1 (see e.g. [263]). Then G_1H/H is open, hence closed in the homogeneous space $X = G/H$ and therefore is piecewise contractible if X has this property (see Exercise E10.10(iv). The natural bijective continuous map $G_1/(G_1 \cap H) \to G_1H/H$ is a homeomorphism by the Open Mapping Theorem, since G_1H/H is locally compact, and thus a Baire space and G_1 is σ-compact. (We are dealing with a special case of the Open Mapping Theorem in Exercise EA1.21.)

Thus we now assume that G/G_0 is almost connected. Locally compact almost connected groups are pro-Lie groups (see [263], p. 157). In [217], p. 547, Theorem 11, Iwasawa proved

Iwasawa's Local Splitting Theorem. *Let G be a locally compact connected pro-Lie group. Then G has arbitrarily small neighborhoods which are of the form NC such that N is a compact normal subgroup and C is an open n-cell which is a local Lie group commuting elementwise with N and is such that $(n, c) \mapsto NC : N \times C \to NC$ is a homeomorphism.* □

In such sources as [126], [185], [189] and [190] we find the basis for the following result:

Theorem 10.89. *Let G be a locally compact group. Then there exists an open, almost connected subgroup G_1 such that for every identity neighborhood U there is a compact normal subgroup N of G_1 contained in $U \cap G_1$, a (simply) connected Lie group L, and an open and continuous surjective morphism $\varphi\colon N \times L \to G_1$ with discrete kernel such that $\varphi(n, 1) = n$ for all $n \in N$.* □

Lemma 10.90. (The Reduction Lemma) *Let G be a locally compact almost connected group and H a closed subgroup. Then G contains arbitrarily small compact normal subgroups N such that the following conditions hold:*

(i) *If G/H is locally contractible then $N/(N \cap H)$ is piecewise contractible.*

(ii) *If $N/(N \cap H)$ is locally euclidean, then G/H is locally euclidean.*

Proof. We let $\varphi\colon N \times L \to G$ be a morphism whose existence is guaranteed by Theorem 10.89, and set $G_2 = N \times L$ and $H_2 = \varphi^{-1}(H)$. Now G_2/H_2 is naturally homeomorphic to $G_1H/H \cong G_1/(H \cap G)$, an open quotient subspace of G/H. Thus if we prove the Lemma for (G_2, H_2) in place of (G, H) we are done. Therefore we may and will assume $G = NL$ for a compact normal subgroup N and a normal Lie subgroup L such that the product is direct. Since N is compact, $G^* \stackrel{\text{def}}{=} NH$ is closed, and $N/(N \cap H) \cong G^*/H$. Let $H_L \stackrel{\text{def}}{=} G^* \cap L$ be the projection of G^* (and H) into L. Then H_L is a Lie subgroup of L and $G/G^* \cong L/H_L$ is a Lie group quotient.

The following diagram describes the situation:

$$\begin{array}{rcl} N \times L = & G & \\ & | & \Big\} \cong L/H_L \\ N \times H_L = & G^* & \\ & | & \Big\} \cong G^*/H \\ & H & \end{array} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\} \cong \quad G/H$$

Since L is a Lie group, the quotient map $G \to G/G^* \cong L/H_L$ is a locally trivial principal bundle. In particular, the identity in G has a neighborhood homeomorphic to $G^* \times B$ where B is a cell homeomorphic to $\mathbb{R}^{\dim G/G_2}$. Accordingly, G/H has an open neighborhood W of G^*/H homeomorphic to $(G^*/H) \times B$.

Now we prove (i) and assume that G/H is locally contractible. Then we find an open neighborhood U of H in G/H that is contractible in W/H. Accordingly, G^*/H is piecewise contractible by Exercise E10.10(i). Since $G^*/H \cong N/(N \cap H)$, assertion (i) is proved.

Now we prove (ii). Assume that $N/(N \cap H)$ is a manifold. Then $G^*/H \cong N/(N \cap H)$ is a manifold. Since G/H is locally homeomorphic to $(G^*/H) \times B$, we conclude that G/H is a manifold. This establishes claim (ii) and concludes the proof of the lemma. □

As a consequence of the Reduction Lemma 10.90 and Theorem A (see Main Theorem 10.80) we have the following:

Theorem 10.91. *Let G be a locally compact group with a closed subgroup H. If the quotient space G/H is locally contractible, then G/H is locally euclidean.* □

Corollary 10.92. *Let G be a locally compact group with a closed subgroup H such that the quotient space G/H is locally contractible. Then there is an almost connected open subgroup G_1 of G such that H contains a normal subgroup M of G_1 for which G_1/M is a Lie group. In particular,*

$$G_1H/H \cong G_1/(G_1 \cap H) \cong \frac{G_1/M}{(G_1 \cap H)/M}$$

is a Lie group quotient and an open submanifold of G/H.

Proof. We let G_1 and $\varphi\colon N \times L \to G_1$ be as in Theorem 10.89. Then G_1/N is a Lie group and by Lemma 10.90, $N/(N \cap H)$ is a manifold. Let M be the largest normal subgroup of N contained in $N \cap H$. Then by Theorem 10.80, N/M is a Lie group. Since $G_1 = \varphi(N \times L)$, the group $M = \varphi(M \times L)$ is normal in G_1. Since G_1/N and N/M are Lie groups, G_1/M is a Lie group. This was the first part of the assertion and the rest is straightforward. □

The special case $H = \{1\}$ is still of interest. As an immediate consequence we have the following result:

Corollary 10.93. *A locally compact locally contractible group is a Lie group.* □

Postscript

For a compact Lie group G it is trivial that the two Lie groups G and $G_0 \times G/G_0$ are homeomorphic. However, in the case of an arbitrary compact group G, in contrast with the Lie group case, it is not trivial to decide, whether *topologically* G is a direct product of G_0 with a subspace homeomorphic to G/G_0. However, we have seen in this chapter that this is true. Therefore, even though transformation group theory does not belong to the structure theory of compact groups in the strict sense of the word, it is appropriate to look at some basic facets of its foundations. These include the Cross Section Theorem 10.11, elucidating the presence of a cross section, the Triviality Theorem 10.25 and the perhaps even more important Local Triviality Theorem for Compact Actions 10.28 giving an apparatus for deciding whether an action having stable isotropy and a cross section is actually a trivial action. All of this background is provided with a purely point set topological background and could have been treated in Chapter I as far as background knowledge is concerned. For other crucial parts, compact Lie group theory is needed. This applies to the Local Cross Section Theorem for Compact Lie Group Actions stating that a locally compact G-space with stable isotropy for a compact Lie group G is a principal fiber bundle. We actually need this theorem for a proof of the two global cross section theorems we prove in this section; one of these, the Global Triviality Theorem for Totally Disconnected Base Spaces 10.35 is the key to the topological splitting of the identity component in a compact group mentioned above. A second triviality theorem is formulated and proved as well: The Global Triviality Theorem for Contractible Base Spaces. Its proof is largely parallel to that of the former theorem. We have already seen other cross section theorems in a preceding chapter (5.70).

A significant motivation for the presentation in this chapter is that of stable isotropy. If we had been satisfied with proving the structure result on splitting topologically the identity component in a compact group, we would have gotten by with a theory of free actions in which the technical complications on triviality theorems which we discussed in the section are not necessary. However, in view of fiber bundle theory and the theory of compact transformation groups in general, the hypothesis of stable isotropy is natural. There appear to be gaps in the literature, one example is [196], p. 317, where in 1.12 the triviality of certain actions with stable isotropy is asserted but where only the existence of a global cross section is proved. The device of the G-space $\mathcal{X}(G,H)$ which we attached to a compact group G and a closed subgroup H is original in this text as is the equivariant mapping $M\colon X \to \mathcal{X}(G,H)$ from a G-space X with stable isotropy conjugated to H to this device (Theorem 10.23).

In Chapters 6 and 9 we saw how important semidirect decompositions are in the structure theory of compact groups. Semidirect products of compact groups can be looked at in terms of split morphisms (see 10.41). A weaker form of split morphisms is that of topologically split morphisms (10.41). In structural terms this means the following. Let G be a compact group and N a compact normal subgroup. The statement that the quotient morphism $G \to G/N$ is topologically split is equivalent to the existence of a compact subspace C of G such that the map $(n,c) \mapsto nc\colon N \times C \to G$ is a homeomorphism, i.e. to the fact that N is topologically a direct factor. This issue clearly belongs to the more general question of cross sections for compact group actions and thus we have treated this topic here, although some of this material could have been included in Chapter 9. Our treatment of topologically split morphisms includes the result that a topologically split morphism of a compact group onto a compact connected abelian group is always split, and a precise characterization of topologically split morphisms of isotypical semisimple compact connected groups. By 9.19 these groups are the building blocks from which arbitrary compact semisimple groups are constructed. These results are taken from [183] and are published in a book here for the first time.

The survey of the action of compact groups and acyclicity covers material belonging to compact transformation group theory and the application of cohomology. While most of it dates to the sixties and seventies, recent interest was stirred by the discovery in 2011 by Sergey Antonyan [9] that the original proof of the much used Theorem 10.80, dating back to 1974, contains a serious gap. Still, for some of the results recorded in this subsection, we have to refer to original sources. The original record is known to specialists from tenuously connected origins like [197]. Such matters have recently attracted renewed attention by A. Adel George Michael, who contributed novel proofs [257].

The cohomology theory applied to transformation groups uses two functors E and B from the categories of topological groups (in our case compact groups only) to the categories of k-spaces (see the paragraph preceding Theorem 7.7). For a compact group G, the k-space $E(G)$ is contractible and contains G. Moreover, for two compact groups G_1 and G_2 the topological spaces $E(G_1 \times G_2)$ and $E(G_1)\times E(G_2)$ are naturally isomorphic, that is, the functor E is multiplicative in the sense of Definition A3.66. In particular, $E(G)$ is a topological group by Proposition A3.68. Under these circumstances, $E(G)$ is called a universal space. Since G is a subgroup of $E(G)$, we can pass to the quotient space $B(G) \stackrel{\rm def}{=} E(G)/G$ which is called a *classifying space* of G. The theory of E and B has been studied extensively in the context of both fibration and of transformation group theory. For a compact group G the Čech cohomology $H^*(B(G), R)$ has been studied and used as a useful version of what has been called the *algebraic cohomology theory.* For instance, if G is a compact connected abelian group, then the Čech cohomology algebra $H^*(B(G), \mathbb{Z})$ is the polynomial algebra generated by the abelian group $\widehat{G}$, the character group of G, contained in the homogeneous component $H^2(B(G), \mathbb{Z})$ of degree 2 (up to natural isomorphism). This was proved by Hofmann and Mostert in [198], see p. 206, Theorem 1.9, and is parallel to what we see in the cohomology algebra $H^*(G, \mathbb{Z})$ of a compact connected abelian group in Theorem 8.83 in Chapter 8 above. The existence of $E(G)$ and $B(G)$ was not yet available to A. Borel in his famous thèse “Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groups de Lie compacts” (Ann. of Math. **57** (1953), 115–207), the results of which enter into the proof of his Acyclicity Theorem 10.69, nor when he organized the Princeton Seminar on Transformation Groups [31], so one had to use the artificial limit arguments still present in the proof of Theorem 10.69 in about 1964-65 as presented in [197], pp. 308f. or in pp. 321ff. for the proof of the Fixed Point Theorem for Compact Abelian Group Actions 10.64. A proof of this theorem using $E(G)$ and $B(G)$ is presented in [162], pp. 15–27, a source that is not easily available.

The proof valididating Szenthe's Local Contractibility Theorem 10.80 (respectively, its locally compact version 10.91) is recent. This proof belongs to the circuit of ideas of Hilbert's Fifth Problem [263]. In 1900, David Hilbert asked whether a locally euclidean topological group supports the structure of a Lie group. By the middle of the century, this was settled affirmatively by Gleason, Montgomery, and Zippin [263]. In 1974, Szenthe realized that it suffices to know that the underlying space is locally contractible. In 2009, Hofmann and Neeb [200] observed that the hypothesis of local contractibility caused a pro-Lie group to be a Lie group (modelled on a locally convex topological vector space), which recovers Szenthe's version on the basis of the solution to Hilbert's Fifth Problem.

The information on the "linearization of dimension theory" for quotient spaces of compact groups contained in Proposition 10.74 is from the authors' article [185].

Janoś Szenthe's influential paper [347] was published in 1974, containing in essence the stronger Theorems 10.80 and 10.91. Clearly belonging to the circle of ideas of Hilbert's Fifth Problem (see e.g. [217], [263]) it was overdue some 20 years after the solution of this problem. The statement provided a result which was needed and applied in various areas, notably in the domain of topological geometry, (see e.g. [311]).

We have seen that the essential aspect of the problem pertains to compact groups and thus has its place in this book as well. Concerning this aspect, it is relevant to point out that, in his 1954 textbook [295], §47, Satz 74 and Satz 57 (p. 121), L. S. Pontryagin showed the equivalence of the statements (3), (4), and (7) of our central Theorem 10.80.

While Szenthe's important result remained unproved, and Antonyan demoted it to the status of a conjecture in 2011, by the end of 2012 several proofs were proposed: one by A. A. George Michael [121], one by Sergey Antonyan himself in collaboration wit Dobrowolski [11], and one by K. H. Hofmann and L. Kramer [172], a variant of which is presented here. An essential portion of the homotopy theory of compact quotient groups such as Lemma 10.86 was already treated by K. H. Hofmann and M. W. Mislove in 1975 [176]; this lemmas was also used in [11].

It had been discovered in the mid-nineties in the circuit of [311] and was recorded by Bickel [24] in 1995, that Szenthe's method of approximating a locally compact group G by Lie groups forces G to be metric, that is, first countable. Thus apart from the gap in Szenthe's proof discovered by Antonyan there is a second problem limiting the applicability of his original version. Szenthe's approach to approximating a compact group by Lie groups is in all likelihood a misinterpretation of a method introduced by Pontryagin (see [295]); a correct interpretation however reemerged in some recent literature such as [11]. In the end, Szenthe's vision laid down in [347] and its many applications are confirmed and corroborated.

References for this Chapter—Additional Reading

[9], [11], [33], [31], [48], [183], [175], [162], [196], [197], [198], [185], [257], [283], [338], [347], [353].

Chapter 11

The Structure of Free Compact Groups

In Chapter 8 we completely described the structure of free compact abelian groups. For example we showed that if X is an infinite compact connected metric space, then the dual group of the free compact abelian group on X, which we shall now call $F_{\rm ab}X$ (in order to distinguish it from the free compact group on X, called FX) is topologically isomorphic to the direct sum $H^1(X,\mathbb{Z})\oplus G$, where G is a rational vector space of dimension $2^{\aleph_0}$ and $H^1(X,\mathbb{Z})\cong[X,\mathbb{S}^1]$ denotes the first Alexander–Spanier cohomology group.

In this chapter we describe the structure of the free compact group FX on X. We begin by investigating its center ZFX and by determining how far it is from the free compact abelian group $F_{\rm ab}X\cong FX/\overline{(FX)'}$, where $F'X$ denotes the commutator subgroup of FX. We prove the powerful result that for X any infinite compact connected space, FX is topologically isomorphic to $F_{\rm ab}X\times F'X$ if and only if $H^1(X,\mathbb{Z})$ is a divisible group. So, for example if $H^1(X,\mathbb{Z})=\{0\}$ or more particularly if X is a contractible space, then FX is topologically isomorphic to $F_{\rm ab}X\times F'X$.

For any infinite compact space X, we show that the center ZFX is always contained in the identity component F_0X and F_0X is a semidirect factor of FX.

The identity component of the center of the free compact group is shown to be naturally isomorphic to the projective cover of the free compact abelian group. The weight $w(FX)$ of any free compact group FX is shown to be equal to $(w(X))^{\aleph_0}$.

We will be able to clarify to a large extent (although inconclusively) the structure of the commutator group $F'X$ of a free compact group if X is connected; its projective cover will be completely described. In the process we invent the concept of a free semisimple compact group and the free compact group F_SX with respect to a given simple compact connected Lie group S.

In the last section of this chapter we examine relative projectivity in the category of compact groups.

Prerequisites. In this chapter we shall more frequently use category theoretical concepts than before. Whatever we use is presented in Appendix 3.

The Category Theoretical Background

We begin with the definition of a free compact group on a pointed space.

Definition 11.1. A *free compact group on a topological space* X with base point x_0 is a compact topological group, FX, together with a continuous function

$e_X: X \to FX$ mapping x_0 to the identity 1 of FX such that the following universal property is satisfied: For every base point preserving continuous function $f: X \to G$ into a compact topological group G (with 1 as its base point) there is a unique continuous group morphism $f': FX \to G$ satisfying $f = f'e_X$. □

We recall that all topological spaces and topological groups we consider are assumed to be Hausdorff unless explicitly stated otherwise. If $\mathbb{TOP}_0$ denotes the category of Hausdorff topological spaces with base points and base point preserving maps, and if $\mathbb{KG}$ is the category of compact groups, then $F: \mathbb{TOP}_0 \to \mathbb{KG}$ is the left adjoint of the grounding functor $|\ |: \mathbb{KG} \to \mathbb{TOP}_0$ associating with a compact group its underlying space $|G|$ with base point 1. (See Appendix 3, A3.29ff.)

Our primary interest will be with connected spaces X; under these circumstances FX will be connected. Some of the results do pertain to the general situation.

To bring our strategy into focus we now proceed to some category theoretical remarks and then to general observations on the structure of compact connected groups.

Several adjoint situations are nearby. Firstly, denote by $\mathbb{KTOP}_0$ the category of compact spaces with base points and base point preserving continuous functions.

Lemma 11.2. *The inclusion functor* $\mathbb{KTOP}_0 \to \mathbb{TOP}_0$ *has a left adjoint* β: $\mathbb{TOP}_0 \to \mathbb{KTOP}_0$, *called the Stone–Čech compactification. The functor* β *is a retraction; i.e. if* X *is a compact Hausdorff space, then* βX *and* X *are naturally isomorphic.*

Proof. Exercise E11.1. □

Exercise E11.1. Prove the existence of the Stone–Čech compactification.

[Hint. Verify the hypotheses of the Adjoint Functor Existence Theorem A3.60. Prove the claim that β is a retraction either by establishing the appropriate category theoretical lemma or by a simple direct argument, verifying the characteristic universal property.] □

Lemma 11.3 ([179], 1.4.2). *The functor* $F: \mathbb{TOP}_0 \to \mathbb{KG}$ *factors through* β. *In fact, if* $b_X: X \to \beta X$ *is the front adjunction or unit of the Stone–Čech compactification* (*see Appendix* 3, A3.37), *then there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{e_X} & FX \\ {\scriptstyle b_X}\downarrow & & \downarrow{\scriptstyle F(b_X)} \\ \beta X & \xrightarrow[e_{\beta X}]{} & F(\beta X) \end{array}$$

such that $F(b_X)$ *is an isomorphism and* $e_{\beta X}$ *is a homeomorphic embedding.*

Proof. By the universal property of β, the function $e_X: X \to FX$ factors uniquely through b_X; that is there is a unique map $f_X: \beta X \to FX$ such that

$$\begin{array}{ccc} X & \xrightarrow{b_X} & \beta X \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle f_X} \\ X & \xrightarrow[e_X]{} & FX \end{array}$$

commutes. By the universal property 11.1 of FX there is a unique morphism of compact groups $(f_X)': F(\beta X) \to FX$ such that

$$\begin{array}{ccc} \beta X & \xrightarrow{e_{\beta X}} & F(\beta X) \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle (f_X)'} \\ FX & \xrightarrow[\mathrm{id}_{FX}]{} & FX \end{array}$$

commutes. Then

$$\mathrm{id}_{FX}\, e_X = f_X b_X = (f_X)' e_{\beta X} b_X = (f_X)'\big(F(b_X)\big)e_X = \big((f_X)'F(b_X)\big)e_X.$$

So by the uniqueness property in 11.1 we get $(f_X)' \circ F(b_X) = \mathrm{id}_{FX}$. Since b_X and $F(b_X)$ are epic we conclude that $F(b_X)$ is an isomorphism. Since the continuous base point preserving functions $f: \beta X \to \mathbb{T}$ from $C_0(\beta X, \mathbb{T})$ separate points, there is a continuous base point preserving injection $\beta X \to \mathbb{T}^{C_0(\beta X,\mathbb{T})}$ into a compact (abelian) group, $\xi \mapsto \big(f(\xi)\big)_{f \in C(\beta X,\mathbb{T})}$. By compactness of βX it is an embedding and by the universal property of FX in 11.1 it factors through e_β. Hence $e_{\beta X}$ is an embedding. □

Exercise E11.2. Prove the corresponding lemma for the free compact abelian group $F_{\mathrm{ab}}X$ in place of FX.

[Hint. The proof of 11.3 applies verbatim.] □

After Lemma 11.3, without loss of generality, we can and shall restrict our attention to pointed compact spaces. Further, we shall regard X as a subspace of FX.

Lemma 11.4. (i) *If G' denotes the commutator subgroup of the compact group, G, then the functor $G \mapsto G_A \overset{\mathrm{def}}{=} G/\overline{G'}$ is left adjoint to the inclusion functor $\mathbb{KAB} \to \mathbb{KG}$ of the category of abelian groups into the category of compact groups. The group $F_{\mathrm{ab}}X \overset{\mathrm{def}}{=} FX/\overline{(FX)'}$ is naturally isomorphic to the free compact abelian group on the space X (see Definition* 8.51*).*

(ii) *If G_0 denotes the identity component of the compact group, G, then the functor $G \mapsto G/G_0$ is left adjoint to the inclusion functor $\mathbb{KZG} \to \mathbb{KG}$ of the category of compact* 0*-dimensional groups into the category of compact groups. The functor $X \mapsto FX/F_0X: \mathbb{TOP}_0 \to \mathbb{KZG}$, $F_0X = (FX)_0$, is left adjoint to the forgetful functor from the category of compact totally disconnected groups into the category of pointed topological spaces.*

Proof. (i) is Exercise E11.3 and (ii) is Exercise E11.4. □

Exercise E11.3. Prove 11.4.

[Hint. Verify the universal property A3.28 for the commutator factor group functor and for the functor $X \mapsto F_{\rm ab}X = FX/\overline{F'X}$.] □

Exercise E11.4. Prove 11.5.

[Hint. Verify the universal property A3.28 for the component factor group functor and for the functor $X \mapsto FX/F_0X$.] □

The group FX/F_0X is called the *free compact zero-dimensional group on* X or the *free profinite group* on X and is denoted F_zX (see Theorem 1.34).

For a compact space X, let X/conn be the zero-dimensional space of all components of X. Then $X \mapsto X/\operatorname{conn}$ extends to a functor from the category of compact spaces to the category of compact totally disconnected spaces, and this functor is left adjoint to the inclusion functor. Indeed, the quotient map $\gamma_X\colon X \to X/\operatorname{conn}$ is readily seen to satisfy the required universal property: For each continuous function $f\colon X \to Y$ into a compact totally disconnected space Y there is a unique continuous map $f'\colon X/\operatorname{conn} \to Y$ such that $f = f'\gamma_X$. (Cf. A3.28ff.)

Proposition 11.5. *For a compact space X, the natural map $\gamma_X\colon X \to X/\operatorname{conn}$ induces an isomorphism $F_z\gamma_X\colon F_zX \to F_z(X/\operatorname{conn})$.*

Proof. Let $\eta_X\colon X \to F_zX$ be the front adjunction. Since F_zX is a zero-dimensional space, η_X factors through $\gamma_X\colon X \to X/\operatorname{conn}$; that is there is a continuous function $\varphi\colon /\operatorname{conn} \to F_zX$ such that $\eta_X = \varphi\gamma_X$. By the universal property of $F_z(X/\operatorname{conn})$ there is a unique morphism $\varphi'\colon F_z(X/\operatorname{conn}) \to F_z(X/\operatorname{conn})$ such that $\varphi'\gamma_{X/\operatorname{conn}} = \varphi$. Now $\eta_X = \varphi\gamma_X = \varphi'\eta_{X/\operatorname{conn}}\gamma_X = \varphi'(F\gamma_X)\eta_X$. By the uniqueness in the universal property, this implies $\varphi'(F\gamma_X) = \operatorname{id}$ and since $F\gamma_X$ is surjective, this proves the claim. □

For free profinite groups there is a rather detailed theory largely due to Mel'nikov [254, 255, 256, 379]. We shall frequently, but by no means exclusively, be concerned in this chapter with the opposite situation, namely, that of a connected pointed space X. If X is connected, then $F_zX = \{1\}$.

We accepted the existence of the free compact group FX on a compact space X on the basis of category theoretical principles such as the Adjoint Functor Existence Theorem (see A3.29ff.). It serves a useful purpose to give a more or less explicit construction of FX; it will at any rate be explicit enough to allow us to compute the weight of FX as soon as we have a little more information on the center. Moreover, this construction does give us a certain idea about the way by which X is embedded into FX; in spirit this way is not entirely dissimilar to the route taken in the abelian situation.

Proposition 11.6. *Let X be a compact pointed space.*

(i) *Let P be a compact group and $e: X \to P$ a continuous function of pointed spaces such that for every continuous function of pointed spaces $f: X \to \mathrm{U}(n)$ into a unitary group there is a function $f': F \to \mathrm{U}(n)$ such that $f = f' \circ e$, then $FX \stackrel{\mathrm{def}}{=} \overline{\langle e(X)\rangle}$ together with the corestriction $e_X: X \to FX$ of e is the free compact group on X.*

(ii) *Let $C_0\big(X, \mathrm{U}(n)\big)$ the set of all continuous and base point preserving functions $X \to \mathrm{U}(n)$. Set $G = \prod_{n=1}^{\infty} \mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$. We define $e: X \to G$ as follows. Let $x \in X$, and define an element $e_n(x): C\big(X, \mathrm{U}(n)\big) \to \mathrm{U}(n)$ of $\mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$ by $e_n(x)(f) = f(x)$. Finally define $e(x) = \big(e_n(x)\big)_{n\in\mathbb{N}}$. Then $e: X \to G$ has the universal property of* (i) *above.*

(iii) *The weight $w(FX)$ of the free compact group on X is estimated above by $w(FX) \le w(X)^{\aleph_0}$.*

Proof. (i) Let H be a compact group and $f: X \to H$ a base point preserving continuous function. By Corollary 2.29, we may assume that $H \subseteq \prod_{j\in J} \mathrm{U}(n_j)$ for a family $\{n_j \mid j \in J\}$ of natural numbers. For each $j \in J$ the restriction of the j-th projection $\mathrm{pr}_j|H: H \to \mathrm{U}(n_j)$ gives a continuous function $(\mathrm{pr}_j|H)\circ f: X \to \mathrm{U}(n_j)$. By hypothesis there is a morphism $f_j': G \to \mathrm{U}(n_j)$ such that $(\mathrm{pr}_j|H) \circ f = f_j' \circ e$ holds. Let $\psi': G \to \prod_{j\in J} \mathrm{U}(n_j)$ be defined by $\psi'(g) = \big(f_j'(g)\big)_{j\in J}$. Let $x \in X$. Then $\psi'\big(e(x)\big) = \big(f_j'\big(e(x)\big)\big)_{j\in J} = \big(\mathrm{pr}_j|H\big(f(x)\big)\big)_{j\in J} \in H \subseteq \prod_{j\in J} \mathrm{U}(n_j)$. Hence we can set $FX \stackrel{\mathrm{def}}{=} \overline{\langle e(X)\rangle}$, let $e_X: X \to FX$ be defined as the corestriction of e, and define $f': FX \to H$ by $f'(g) = \psi'(g)$. Then $f = f' \circ e_X$. Moreover, if $f'': FX \to H$ also satisfies $f = f'' \circ e_X$, consider the equalizer $E \stackrel{\mathrm{def}}{=} \{g \in FX \mid f'(g) = f''(g)\}$. Then E is a closed subsemigroup containing $e(x)$ for all $x \in X$. Since $FX = \overline{\langle e(X)\rangle}$ we conclude $E = FX$. Thus $f'' = f'$. Thus FX has the universal property which after 1.1 is characteristic for the free compact group on X.

(ii) Let $f: X \to \mathrm{U}(m)$ be a base point preserving continuous function. Then $f \in C_0\big(X, \mathrm{U}(m)\big)$. Let $\mathrm{pr}_f: G \stackrel{\mathrm{def}}{=} \prod_{n=1}^{\infty} \mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$ be the projection onto $\mathrm{U}(n)$ determined by the index f; i.e. if $\varphi_n \in \mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$ then $\mathrm{pr}_f\,\big((\varphi_n)_{n\in\mathbb{N}}\big) = \varphi_n(f)$. Accordingly, $f' \stackrel{\mathrm{def}}{=} \mathrm{pr}_f: G \to \mathrm{U}(n)$ satisfies $f'\big(e(x)\big) = pr_f\big(\big(e_n(x)\big)_{n\in\mathbb{N}} = e_n(x)(f) = f(x)$.

(iii) By Theorem A4.9 of Appendix 4, $\mathrm{card}\, C_0\big(X, \mathrm{U}(n)\big) = w(X)^{\aleph_0}$. Then, again by Appendix 4, notably Exercise EA4.3, $w(G) = w(\prod_{n=1}^{\infty} \mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}) = \sum_{n=1}^{\infty} \mathrm{card}\, C_0\big(X, \mathrm{U}(n)\big) = w(X)^{\aleph_0}$. By (i) and (ii) we have $FX \subseteq G$, and thus $w(FX) \le w(X)^{\aleph_0}$. ☐

Chapter 8 contains a complete description of the free compact *abelian* group. In the case of a connected pointed space X, resulting in a compact connected group FX, we have a good insight into the general structure of FX due to the structure theorems in Chapter 9. In particular, we know $\overline{F'X} = F'X$ from Theorem 9.2 and $FX = ZFX{\cdot}F'X$ with a totally disconnected central group $ZFX \cap F'X$

from Theorems 9.23 and 9.24 and $FX \cong F'X \rtimes F_{\rm ab}X$ from Theorem 9.39. We are therefore challenged to determine the building blocks $F'X$ and $ZFX \cap F'X$. Specifically, we are faced quite generally with the extension problems

$$0 \to F_0X \to FX \to F_zX \to 0,$$

$$0 \to \overline{F'X} \to FX \longrightarrow F_{\rm ab}X \to 0.$$

We shall address the first one in the next section and deal with the second one later.

Splitting the Identity Component

Let G_0 denote the identity component of the topological group G. From the Theorem on Splitting the Components of Compact Groups 10.37 we know that

For any compact group G the quotient morphism $p\colon G \to G/G_0$ has a continuous cross section $\sigma\colon G/G_0 \to G$,

that is a continuous map satisfying $p \circ \sigma = {\rm id}_{G/G_0}$. This was an immediate consequence of the Global Cross Section Theorem for Totally Disconnected Base Spaces 10.35. From Lee's Theorem for Compact Groups 9.41 we know that G always contains a compact zero-dimensional subgroup D such that $G = G_0D$ while, in general, $G_0 \cap D$ is not singleton. Neither of these important results says that G_0 is a semidirect factor. In fact this is not even true in the abelian case as we saw in Example 8.11(iv). It is therefore interesting to observe that G_0 is a semidirect factor whenever G is a free compact group.

The Component Splitting Theorem for Free Compact Groups

Theorem 11.7. *The identity component F_0X of any free compact group FX is a semidirect factor. More specifically, there are closed totally disconnected subgroups D of FX with $FX = (FX)_0D$, and every such group D contains a totally disconnected closed subgroup $T \cong F_zX$ of FX such that the function $(c,t) \mapsto ct\colon (FX)_0 \rtimes T \to FX$ is an isomorphism of compact groups.*

Proof. By Lee's Theorem 9.41, FX contains a compact zero-dimensional subgroup D such that $FX = (FX)_0D$. The surjective morphism

$$\pi\colon D \to F_zX, \qquad \pi(d) = dF_0X \in F_zX = FX/F_0X$$

allows a topological cross section $\sigma\colon F_zX \to D$ by the Cross Section Theorem 10.35. Thus the canonical map $\varepsilon\colon X \to F_zX$, $\varepsilon(x) = e_X(x)F_0X$ gives a map $\sigma\circ\varepsilon\colon X \to D$ which, by the universal property of F_zX, factors through a morphism $\lambda\colon F_zX \to D$ satisfying $\pi \circ \lambda = {\rm id}_{F_zX}$. Then $\lambda(F_zX)$ is the required group T. □

Here the extension problem of F_0X by F_zX is comparatively simple. Every representation $FX = F_0X{\cdot}D \cong F_0X \rtimes F_sX$ provides us with a projection ${\rm pr}_0\colon FX \to$

F_0/X, $\mathrm{pr}_0(gd) = g$, $g \in F_0X$, $d \in D$. The continuous function $\mathrm{pr}_0 \circ e_X : X \to F_0X$ gives us a morphism $p: FX \to F_0X$ such that $p \circ e_X = \mathrm{pr}_1 \circ e_X$. We know $p(F_0X) = F_0X$ from 9.26(i) and thus $FX = F_0X \cdot D_1$ with $D_1 = \ker p \trianglelefteq FX$. The method of proof of Theorem 11.7 will produce a totally disconnected subgroup $D_2 \leq D_1$ such that $FX = F_0X \cdot D_2$, $F_0X \cap D_2 = \{1\}$. This seems to get us back to the point we were in 11.7. Except that we know that the group generated by the conjugates of D_2 is contained in D_1, and $D_1 \neq FX$, because $D_1 = FX$ implies $F_0X \cap \ker p = F_0X$, while $\operatorname{card} X > 1$ implies $\{1\} \neq F_0X \cong F_0X/\ker(p|F_0X) = F_0X/(F_0X \cap \ker p)$.

The Center of a Free Compact Group

We consider a compact group G and recall $G_A = G/\overline{G'}$ and the morphisms $\zeta: Z(G) \to G_A$, $\zeta(g) = g\overline{G'}$, $\zeta_0: Z_0(G) \to G_A$, $\zeta_0 = \zeta|G_0$. Then by Theorem 9.23(iii) there is an exact sequence

$$\text{(A)} \qquad 0 \to Z_0(G) \cap \overline{G'} \to Z_0 \stackrel{\zeta_0}{\to} G_A \to G_A/(G_A)_0 \to 0.$$

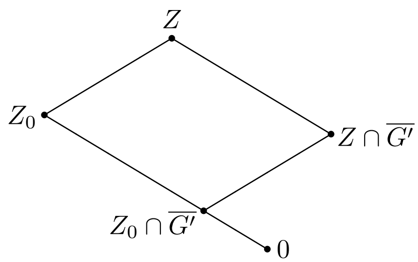
Thus, roughly speaking, the identity component $Z_0(G)$ of the center of any compact group, G, differs from the abelianization $G_A = G/G'$ of G just by groups of dimension zero. After the next proposition we shall prove that the center of a free compact group is contained in the identity component.

Proposition 11.8. *Let G be a compact group and assume that $Z(G) \subseteq G_0$. Then*

(i) *$Z(G)\overline{G'} = G_0\overline{G'}$; that is $(G_A)_z = G/Z(G)\overline{G'}$. The identity component can be written $G_0 = Z(G)(G_0 \cap \overline{G'})$.*

(ii) *$Z(G) = Z_0(G)(Z(G) \cap \overline{G'})$.*

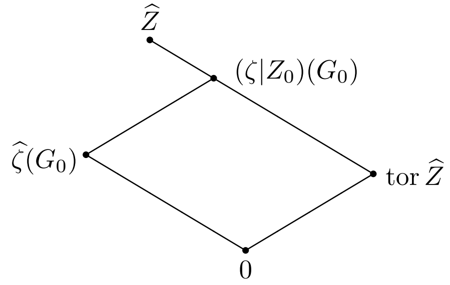
That is we have a lattice diagram



In particular, $Z_z(G) = (Z \cap \overline{G'})/(Z_0(G) \cap \overline{G'})$.

(iii) *Let $\zeta: Z(G) \to G_A$ be given by $\zeta(z) = z\overline{G'}$. In the character group $\widehat{Z}(G)$ of $Z(G)$ we have $Z_0(G)^\perp = \operatorname{tor} \widehat{Z}(G)$, $(Z(G) \cap \overline{G'})^\perp = \widehat{\zeta}(G)$ and $(Z_0(G) \cap \overline{G'})^\perp =$*

$\widehat{\zeta}(G_0)$. *Thus the dual diagram to that in* (ii) *is*



Proof. (i) By Theorem 9.23, we have $G_0 \subseteq Z(G)\overline{G'}$. By hypothesis, $Z(G)\overline{G'} \subseteq G_0\overline{G'}$, so $G_0\overline{G'} = Z(G)\overline{G'}$. Hence $(G_A)_0 = (Z(G)\overline{G'})/\overline{G'}$ by Theorem 9.23(iii), and this implies $(G_A)_z = G_A/(G_A)_0 = (G/\overline{G'})/(Z(G)\overline{G'})/\overline{G'} \cong G/(Z(G)\overline{G'})$. Clearly $Z(G) \subseteq G_0$ implies $Z(G)(G_0 \cap \overline{G'}) \subseteq G_0$. The relation $G_0 \subseteq Z(G)\overline{G'}$ implies the reverse inclusion.

(ii) Clearly, $Z_0(G)(Z(G) \cap \overline{G'}) \subseteq Z(G)$. Now let $z \in Z(G)$. Then $z \in G_0 \subseteq Z_0(G)\overline{G'}$ by 9.23(i). Hence, $z = z_0 g$ with $z_0 \in Z_0(G)$, $g \in \overline{G'}$, whence $g = zz_0^{-1} \in Z(G) \cap \overline{G'}$ and thus $z \in Z_0(G)(Z(G) \cap \overline{G'})$. Trivially, $Z_0(G) \cap \overline{G'} = Z_0(G) \cap (Z(G) \cap \overline{G'})$.

(iii) The annihilator of $Z_0(G)$ in the character group $\widehat{Z}(G)$ of $Z(G)$ is tor $Z(G)$ by Theorem 8.4(7). Since $Z(G) \cap \overline{G'} = \ker \zeta$ by 9.23(iii)(B), we have

$$(Z(G) \cap \overline{G'})^\perp = (\ker \zeta)^\perp = \operatorname{im} \widehat{\zeta} = \zeta(G)$$

by 7.65. Similarly, by 9.23(A)(iii)(A) we note $Z_0(G) \cap \overline{G'} = \ker \zeta_0$, $\zeta_0 = \zeta|Z_0(G)$, whence

$$(Z_0(G) \cap \overline{G'})^\perp = \big(\ker(\zeta_0)\big)^\perp = \operatorname{im}(\widehat{\zeta_0}) = \zeta(G_0). \qquad \square$$

The following result appears in [174]. Its proof is completely self-contained from first principles. The work of Mel'nikov [254] contains a more general result for a more special class of spaces, namely, one point compactifications of discrete spaces.

Theorem 11.9. *Let X be a compact pointed space with at least three points. Then the center Z of the free compact group FX is contained in the identity component F_0X.*

Proof. First we note that it suffices to show that the center of the free compact zero-dimensional group F_zX is singleton; for ZF_0X/F_0X is contained in the center of F_zX. The proof of this claim is by contradiction. We suppose that F_zX contains a central element $z_0 \neq 1$. Then we find a finite quotient $q: FX \to G$ with $z = q(z_0) \neq 0$. If we denote $q(e(X)$ by Y, then Y is a generating set of G. The next step utilizes a familiar lifting technique based on the universal property of F_zX.

Claim 1. Let E be a finite group and $p: E \to G$ a quotient morphism. If $s: Y \to E$ is any function satisfying $ps = \mathrm{id}_Y$, then there is a unique morphism

$Q: FX \to E$ with $pQ = q$. The subgroup $\operatorname{im} Q$ of E is generated by $s(Y)$ and centralizes $Q(z_0)$.

Proof of Claim 1. By the universal property, the function $sqe: X \to E$ determines a unique morphism $Q: FX \to E$ with $Qe = sqe$. Then $pQe = psqe = qe$, and by the uniqueness aspect of the universal property, $pQ = q$ follows. Since FX is topologically generated by $e(X)$ and since E is discrete, the group $\operatorname{im} Q$ is generated by $Q(e(X)) = sqe(X) = s(Y)$. Since z_0 is central in FX, the element $z = Q(z_0)$ is central in $\operatorname{im} Q$. This completes the proof of Claim 1.

Our objective is to obtain a contradiction by choosing the parameters E and s. For this purpose we consider a finite-dimensional vector space M over a finite field K and assume that G operates linearly on M on the left. Then M is a G- and a $K(G)$-module, where $K(G)$ is the group ring of G over K. We take for E the semidirect product $E = M \times G$; that is the cartesian product with the multiplication $(m, g)(n, h) = (m + gn, h)$. The first application results from taking $M = K(G)$ with multiplication on the left as action.

Claim 2. For each $y \in Y$ there is a natural number a with $y^a = z$.

Proof of Claim 2. We apply Claim 1 with $E = K(G) \times_s G$, and $p = \mathrm{pr}_2: E \to G$; furthermore we take $s: Y \to E$ to be defined by $s(Y) = (1, y)$ for all $y \in Y$. If we write $Q(z_0) = (c, z)$ and recall that (c, z) is centralized by $\operatorname{im} Q$, hence in particular commutes with all elements $(1, y)$, we obtain $(1 + yp, yz) = (1, y)(c, z) = (c, z)(1, y) = (c + z, zy)$, whence $1 + yc = c + z$. This may be written as

$$(1) \qquad 1 - z = (1 - y)c \quad \text{for all } y \in Y.$$

In the group ring $K(G)$ the element c is of the form $\sum r_g \cdot g$ with $r_g = r(g) \in K$ and the summation extended over all $g \in G$. One observes that the relations (1) then are equivalent to the equations

$$(2) \qquad \begin{aligned} &\text{(i)} \quad r(y^{-1}) = r(1) - 1, \\ &\text{(ii)} \quad r(y^{-1}z) = r(z) + 1, \\ &\text{(iii)} \quad r(y^{-1}g) = r(g) \qquad \text{for all } g \neq 1,\ y \in Y. \end{aligned}$$

Suppose now that the claim is false. Then we would find a $y \in Y$ such that no power of y equals z. We consider the coefficients $r(g)$ for g in the cyclic subgroup$\{1, y^{-1}, y^{-2}, \ldots, y^{-n+1}\}$, where n is the order of y. From (2)(i) and (iii), we conclude inductively that $r(y^{-m}) = r(1) - 1$ for $m = 1, \ldots, n$ and thus arrive at the contradiction $r(1) = r(y^{-n}) = r(1) - 1$. This proves Claim 2.

In order to conclude with a contradiction, after Claim 2 it suffices to find a finite quotient map $Q: FX \to H$ such that $Q(z_0)$ is contained in the subgroup generated by some $y \in Q(e(X))$.

Claim 3. The group FX is nonabelian and $q: FX \to G$ is a nonabelian finite quotient with $z = q(z_0) \neq 1$. Apply Claim 1 with $E = K(G) \times_s G$ and $p = \mathrm{pr}_2: E \to G$ and with $s: Y \to E$ defined by $s(y) = (0, y)$ for all $y \in Y$ with the exception of one $u \in Y$ for which we define $s(u) = (u, u)$. Then $H = \operatorname{im} Q$ is a finite quotient of FX with the property that for at least one generator $Q(e(X))$ of H the element $Q(z_0)$ is not in the subgroup generated by $Q(e(X))$.

Proof of Claim 3. Since FX is nonabelian, there are finite nonabelian quotients G of the required sort. Moreover, the generating set Y has to contain, outside 1 and u, at least one other element v. Suppose now, by way of contradiction, that $Q(z_0) = (c,z)$ equals $s(v)^a = (0,v)^a$ for some natural number a. By Claim 1, then $(u,u) = s(u)$ commutes with $(c,z) = (0,v^a)$, whence $(u,uv^a) = (u,u)(0,v^a) = (0,v^a)(u,u) = (v^a u, v^a u)$. This implies $v^a u = u$ and thus $v^a = 1$, which yields $z = 1$ in contradiction with the choice of q. Thus any x with $e(x) = v$ satisfies the conclusion of Claim 3.

By the remark preceding Claim 3, and since X has at least three points and FX is therefore nonabelian, the proof of the theorem is now complete. □

By Proposition 11.8, we now obtain the following result.

Theorem 11.10. *For any compact space X, the following conclusions hold:*

(i) $Z(FX)\overline{F'X} = F_0X\overline{F'X}$*; that is* $(F_{\rm ab}X)_z = FX/Z(FX)\overline{F'X}$. *Further* $F_0X = Z(FX)(F_0X \cap \overline{F'X})$.

(ii) $Z(FX) = Z_0(FX)(Z(FX) \cap \overline{F'X})$. *In particular,*

$$Z_z(FX) = (Z(FX) \cap \overline{F'X})/(Z_0(FX) \cap \overline{F'X}).$$

(iii) *Let* $\zeta\colon Z(FX) \to F_{\rm ab}X$ *be given by* $\zeta(z) = z\overline{F'X}$. *In the character group* $\widehat{Z}(FX)$ *of* $Z(FX)$ *we have* $Z_0(FX)^\perp = \operatorname{tor}\widehat{Z}(FX)$, $(Z(FX) \cap \overline{F'X})^\perp = \widehat{\zeta}(FX)$ *and* $(Z_0(FX) \cap \overline{F'X})^\perp = \widehat{\zeta}(F_0X)$. □

We shall now aim for precise information on $Z_0(FX)$ and $Z_0(FX)\cap\overline{F'X}$. Recall that $\mathrm{U}(1) = \mathbb{S}^1$ is the circle group $\{z \in \mathbb{C} : |z| = 1\}$. In the following proposition, let $p_n\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be the endomorphism given by $p_n(z) = z^n$.

Proposition 11.11. *For a compact group G the following conditions are equivalent.*

(i) *The exact sequence* (A) *is the characteristic sequence of* G_A.

(ii) *The group* $\big(Z_0(G)\big)^\frown$ *is divisible.*

(ii)′ *The group* $Z_0(G)$ *is torsion-free.*

(iii) *For each* $\chi \in \operatorname{im}\widehat{\zeta_0} \subseteq \big(Z_0(G)\big)^\frown$ *and each natural number, n, the element has an n-th root in* $\big(Z_0(G)\big)^\frown$.

(iii)′ *The subgroup* $\widehat{\zeta_0}(\widehat{G_A})$ *of* $\big(Z_0(G)\big)^\frown$ *is pure.*

(iv) *For each continuous morphism* $\psi\colon G \to \mathrm{U}(1)$ *and each natural number, n, there is a character* $\varphi\colon Z_0(G) \to \mathrm{U}(1)$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} Z_0(G) & \xrightarrow{\ \Phi\ } & \mathrm{U}(1) \\ {\scriptstyle \rm incl}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ G & \xrightarrow[\ \psi\ ]{} & \mathrm{U}(1). \end{array}$$

Proof. The equivalence of (i), (ii), and (ii)′ was proved in 9.23(iii).

Trivially, (ii) implies (iii).

For the equivalence of (iii) and (iii)′ see A1.24.

We now show that (iii)′ implies (ii). Let $\chi \in \widehat{Z_0}$ and let n be a natural number. By (iii)′ there is a natural number m such that $\chi^m \in \operatorname{im}\widehat{\zeta_0}$. By Condition (iii), there is a $\varphi \in \big(Z_0(G)\big)\widehat{}$ such that $\varphi^{mn} = \chi^m$. Since $\big(Z_0(G)\big)\widehat{}$ is torsion-free, $\varphi^n = \chi$ follows.

Now we show that the conditions (iii) and (iv) are equivalent. If $q\colon G \to G_A = G/G'$ denotes the quotient map, then every $\psi\colon G \to \mathrm{U}(1)$ factors uniquely through q, since $\mathrm{U}(1)$ is abelian. Because $q \circ \mathrm{incl} = \zeta$ with incl: $Z_0(G) \to G$, condition (iv) is equivalent to saying that for every character $\chi\colon G_A \to \mathrm{U}(1)$ and every natural number n there is a character $\varphi \in Z_0(G)$ such that $\chi \circ \zeta_0 = \varphi^n$. But this statement is exactly condition (iii). □

Proposition 11.12. *The conditions* (i) *through* (iv) *of Proposition* 11.11 *are implied by the following condition*

(iii)* *For each $\chi \in \operatorname{im}\widehat{\zeta} \subseteq \big(Z(G)\big)\widehat{}$ and each natural number, n, the element has an n-th root in $\big(Z(G)\big)\widehat{}$.*

This condition is in turn implied by

(v) *For every morphism $\psi\colon G \to \mathrm{U}(1)$ and each natural number n, there is an irreducible representation $\pi\colon G \to \mathrm{U}(n)$ such that $\psi(g) = \det \pi(g)$ for all $g \in G$.*

Proof. (iii)* implies (iii). We abbreviate $\big(Z_0(G)\big)\widehat{}$ by B, and $\big(Z(G)\big)\widehat{}$ by A. Then the inclusion $j\colon Z_0(G) \to Z(G)$ gives a quotient morphism $q = \widehat{j}\colon A \to B$ with kernel tor A. Set $I \stackrel{\text{def}}{=} \operatorname{im}\widehat{\zeta} \subseteq A$. Then $q(I) = \operatorname{im}(\zeta \circ j)\widehat{} = \operatorname{im}(\widehat{j} \circ \widehat{\zeta}) = \operatorname{im}\widehat{\zeta_0}$. Then Condition (iii)* says $I \in \operatorname{Div}(A)$ (see A1.29). Now $q\big(\operatorname{Div}(A)\big) \subseteq \operatorname{Div}(B)$. Hence $q(I) \subseteq \operatorname{Div}(B)$ which is exactly (iii).

Exactly as in the proof of the equivalence of (iii) and (iv) in 11.11 we see that (iii)* is equivalent to

(iv)* For each continuous morphism $\psi\colon G \to \mathrm{U}(1)$ and each natural number, n, there is a character $\varphi\colon Z(G) \to \mathrm{U}(1)$ such that the following diagram is commutative:

$$\begin{array}{ccc} Z(G) & \xrightarrow{\ \Phi\ } & \mathrm{U}(1) \\ {\scriptstyle\mathrm{incl}}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ G & \xrightarrow[\ \psi\ ]{} & \mathrm{U}(1). \end{array}$$

Now we show that (v) implies (iv)*. This will prove the proposition. Assume that a morphism $\psi\colon G \to \mathrm{U}(1)$ and a natural number n are given. Let $\pi\colon G \to \mathrm{U}(n)$ be the irreducible representation according to (v). If $g \in Z$, then by the irreducibility of π and Schur's Lemma, $\pi(g)$ is of the form $\varphi(g)E_n$ with the $n \times n$ identity matrix E_n and a complex number $\varphi(g)$. Now $\varphi\colon Z \to \mathrm{U}(1)$ is a character, and for $g \in Z$ we have $\det \varphi(g) = \varphi(g)^n$, as asserted in (iv). □

The situation of condition (v) is illustrated in the following diagram:

$$\begin{array}{ccc} Z(G) & \xrightarrow{\pi|Z(G)} & \mathrm{U}(1) \\ {\scriptstyle\mathrm{incl}}\downarrow & & \downarrow{\scriptstyle\mathrm{diag}} \\ G & \xrightarrow{\pi} & \mathrm{U}(n) \\ {\scriptstyle\mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle\det} \\ G & \xrightarrow{\psi} & \mathrm{U}(1) \end{array}$$

with $p_n = \det \circ \operatorname{diag}$, $\operatorname{diag}(z) = z{\cdot}E_n$.

In reference to the proof of (iii)$^*\Rightarrow$(iii) in 11.12 we note in passing that the surjective homomorphism $p\colon \nabla \to \mathbb{Q}$ of A1.32 has $\operatorname{tor}\nabla$ as kernel by A1.32(iv), but $p(\operatorname{Div}\nabla) = \mathbb{Z}$ by A1.32(vi) while $\operatorname{Div}(\mathbb{Q}) = \mathbb{Q}$.

We now verify condition (v) of Proposition 11.12 for the free compact group $G = FX$.

Lemma 11.13. *Let $G = FX$ be the free compact group on a compact space X with a base point and at least two further distinct points a and b. Then condition* (v) *of Proposition* 11.12 *is satisfied.*

Proof. Assume that $\psi\colon FX \to \mathrm{U}(1)$ is given. For each $n = 1, 2, \ldots$ we have to produce an irreducible representation $\pi\colon FX \to \mathrm{U}(n)$ such that $\psi(g) = \det \pi(g)$ for all $g \in FX$. Since $FX = \overline{\langle X\rangle}$ (by the uniqueness of the universal property, since $\overline{\langle X\rangle}$ has the universal property of 11.1) it suffices to find π so that $\det \pi(x) = \psi(x)$ for all $x \in X$.

Now let $s\colon \mathrm{U}(1) \to \mathrm{U}(n)$ be defined by $s(z) = \operatorname{diag}(z, 1, \ldots, 1)$. Thus (i) $\det s(z) = z$. By 6.51, the group $\mathrm{SU}(n)$ is topologically generated by two elements g and h. Since X is compact there is a continuous function from X into $[-1, 1]$ with a going to a, the base point to 0, and b to 1. Now $\mathrm{SU}(n)$ is path-connected (cf. E1.2 for $\mathrm{SU}(2)$ and E6.5 for the fact that every element of $\mathrm{SU}(n)$ is contained in a subgroup isomorphic to $\mathrm{SU}(2)$). Thus there is a continuous function from $[-1, 1]$ to $\mathrm{SU}(n)$ sending -1 to $gs\big(\psi(a)\big)^{-1}$, 0 to $\mathbf{1}$, and 1 to $hs\big(\psi(b)\big)^{-1}$. Hence there is a continuous function $f_0\colon X \to \mathrm{SU}(n)$ such that (ii) $f_0(a) = gs(\psi(a))^{-1}$, $f_0(b) = hs(\psi(b))^{-1}$. Now let $f(x) = f_0(x)s(\psi(x))$. Then (iii) $\det f(x) = \psi(x)$ by (i), and (iv) $f(a) = g$, $f(b) = h$ by (ii). By the universal property of F, there is a representation $\pi\colon FX \to \mathrm{U}(x)$ with (v) $\pi|X = f$. Then $\pi(FX)$ contains $\mathrm{SU}(n)$, since $\{f(a), f(b)\}$ topologically generates $\mathrm{SU}(n)$. Hence π is irreducible. By (iii) and (iv), however, $\det \circ \pi$ and ψ agree on X, which was to be shown. □

The Center of a Free Compact Group

Theorem 11.14. (i) *Let X be an arbitrary compact pointed space X. If* $\operatorname{card} X \neq 2$*, then the center $Z(FX)$ of the free compact group FX on X is contained in the* *identity component* F_0X *of* FX*, and if* $\operatorname{card} X = 2$ *then* $Z(FX) = FX$ *is the universal compact monothetic group* (*Example* 8.75)*.*

Irrespective of cardinality, the following sequence is the characteristic sequence of the free compact abelian group, $F_{\rm ab}X$*:*

$$0 \to Z_0(FX) \cap \overline{F'X} \to Z_0(FX) \xrightarrow{\zeta} F_{\rm ab}X \to (F_{\rm ab}X)_z \to 0,$$

where $Z_0(FX)$ *is the identity component of the center of* FX*, where* $F'X$ *denotes the commutator subgroup of* FX*, where* $F_{\rm ab}X = FX/\overline{F'X}$ *and* $(F_{\rm ab}X)_z = F_{\rm ab}X/(F_{\rm ab}X)_0$*, and where* $\zeta(g) = g\overline{F'X}$*.*

(ii) *Assume that* X *is connected. Let* $\pi_{F'X}\colon \widetilde{F'X} \to F'X$ *denote the natural morphism of the Structure Theorem for Semisimple Compact Connected Groups* 9.19 *with a unique simply connected compact domain* $\widetilde{F'X}$ *and a totally disconnected central kernel. Then* $Z_0(FX) \times \widetilde{F'X}$ *is the projective cover* $\mathfrak{P}(FX)$ *of* FX *and the projective covering morphism is given by*

$$E_{FX}\colon Z_0(FX) \times \widetilde{F'X} \to FX, \quad E_{FX}(z,g) = z\pi_{F'X}(g).$$

Its kernel is isomorphic to a closed subgroup of $Z(\widetilde{F'X})$*, and*

$$\mathfrak{L}(E_G)\colon \mathfrak{L}\big(Z_0(FX)\big) \times \mathfrak{L}(\widetilde{F'X}) \to \mathfrak{L}(FX),$$
$$\mathfrak{L}(\zeta)\colon \mathfrak{L}\big(Z_0(FX)\big) \to \mathfrak{L}(F_{\rm ab}X)$$

are isomorphisms of weakly complete vector spaces.

Proof. (i) If X has at least three points, then by Theorem 11.10, Propositions 11.11 and 11.12 together with Lemma 11.13 the proof is completed. If X has one point, then all groups in sight are singleton, and the assertion is true by default. There remains the case of two points. Then FX is abelian and thus agrees with the free compact abelian group $F_{\rm ab}X$. In this case, $F'X = \{0\}$ and $Z(FX) = FX$, whence $Z_0(FX) = (F_{\rm ab}X)_0$, and the sequence in question becomes

$$(*) \qquad 0 \to (F_{\rm ab}X)_0 \to F_{\rm ab}X \to (F_{\rm ab}X)_z \to 0.$$

Its dual sequence is $0 \to \mathbb{Q}/\mathbb{Z} \to (\mathbb{R}/\mathbb{Z})_d \to (\mathbb{R}/\mathbb{Q})_d \cong \mathbb{R}_d \to 0$ which is equivalent to

$$0 \to \operatorname{tor} \mathbb{T}_d \to \mathbb{T}_d \xrightarrow{\iota_{\mathbb{T}_d}} \mathbb{Q} \otimes \mathbb{T}_d \to 0.$$

Thus $(*)$ is the characteristic sequence of $F_{\rm ab}X$ by 8.80. (Cf. 8.75.)

(ii) By (ii) we have $\mathfrak{P}\big(Z_0(FX)\big) = Z_0(FX)$. Then the assertion is an immediate consequence of Definition 9.72 and Theorem 9.73 where we apply 9.73(iii) first to FX and, secondly, to $F_{\rm ab}X$. ☐

In Chapter 8 we have a complete structure theory of $F_{\rm ab}X$ and all terms of the characteristic sequence of $F_{\rm ab}X$ in 8.82 for a compact and connected X. (For an arbitrary compact space X, see [179], 2.2.4.) Now we have obtained the characteristic sequence of $F_{\rm ab}X$ starting from the free compact group FX. The isomorphisms on the Lie algebras will give us a complete description of the Lie algebra of a free compact group.

Theorem 11.14 gives, in particular,

Corollary 11.15 (The Connected Center of a Free Compact Group). *For any compact space X, the identity component $Z_0(FX)$ of the center of FX is (naturally isomorphic to) the projective cover $\mathfrak{P}(F_{\rm ab}X) = (\mathbb{Q}\otimes\widehat{F_{\rm ab}X})\widehat{\ }$ of $F_{\rm ab}X$ (according to* 8.80*), and the morphism $\zeta\colon Z_0(FX)\to F_{\rm ab}X$, $\zeta(g)=g\overline{F'X}$ is (equivalent to) the projective covering morphism $E_{F_{\rm ab}X}\colon\mathfrak{P}(F_{\rm ab}X)\to F_{\rm ab}X$.*

The group $Z_0(FX)\cap\overline{F'X}$ is naturally isomorphic to $\Delta(F_{\rm ab}X)=\ker E_{F_{\rm ab}}$.

Proof. This is a consequence of 11.14 and Definition 8.80 and the observations following 8.80. ☐

These results concern compact abelian groups. So one wishes to identify them in terms of their character groups.

Corollary 11.16. *The character groups of the terms of the characteristic sequence of $F_{\rm ab}X$ in Theorem* 11.14 *are given by*

(i) $$\widehat{F_{\rm ab}X}\cong\frac{C_0(X,\mathbb{R})_d}{C_0(X,\mathbb{Z})_d}\oplus H^1(X,\mathbb{Z}),$$

(ii) $$\frac{C_0(X,\mathbb{R})}{C_0(X,\mathbb{Z})}\cong\widetilde{H}^0(X,\mathbb{Q}/\mathbb{Z})\oplus\mathbb{Q}^{(w(X)^{\aleph_0})}.$$

(iii) $$\mathrm{tor}(\widehat{F_{\rm ab}X})\cong\widetilde{H}^0(X,\mathbb{Q}/\mathbb{Z})\cong\bigoplus_{p\ \mathrm{prime}}\mathbb{Z}(p^\infty)^{\left(w_0(X/\mathrm{conn})\right)},$$

(iii)′ $$\big((F_{\rm ab}X)_z\big)\widehat{\ }\cong\mathrm{tor}(\widehat{F_{\rm ab}X}),$$

(iv) $$Z_0(FX)\widehat{\ }\cong\mathbb{Q}^{(\omega(X)^{\aleph_0})},$$

(v) $$(Z_0(FX)\cap\overline{F'X})\widehat{\ }\cong\frac{\mathbb{Q}\otimes H^1(X,\mathbb{Z})}{1\otimes H^1(X,\mathbb{Z})}\cong(\mathbb{Q}/\mathbb{Z})\otimes H^1(X,\mathbb{Z}).$$

Proof. (i) was shown in 8.50 and (ii),(iii) in 8.65. Relation (iii)′ is a consequence of 7.69(ii). For a proof of (iv) we note from 11.15 that $Z_0(FX)\cong\mathfrak{P}(F_{\rm ab}X)$. The character group of $\mathfrak{P}(F_{\rm ab}X)$ is $\mathbb{Q}\otimes\widehat{F_{\rm ab}X}$ by 8.80. By (i) and (ii) this group is isomorphic to $\mathbb{Q}^{(w(X)^{\aleph_0})}\oplus\big(\mathbb{Q}\otimes H^1(X,\mathbb{Z})\big)$. Now $[X,\mathbb{T}]\cong H^1(X,\mathbb{Z})$ is a quotient group of $C_0(X,\mathbb{T})$ whose cardinality is $w(X)^{\aleph_0}$. Thus the cardinality of $\mathbb{Q}\otimes H^1(X,\mathbb{Z})$ does not exceed that of $\mathbb{Q}^{(w(X)^{\aleph_0})}$. Hence $\mathbb{Q}^{(w(X)^{\aleph_0})}\oplus\big(\mathbb{Q}\otimes H^1(X,\mathbb{Z})\big)\cong\mathbb{Q}^{(w(X)^{\aleph_0})}$. The assertion (iv) follows.

(v) By 11.15 above, $Z_0(FX)\cap\overline{F'X}=\Delta(FX)=\ker E_{F_{\rm ab}X}$. By 8.80, the character group of $\Delta(F_{\rm ab}X)$ is $\mathrm{coker}\,\iota_{\widehat{F_{\rm ab}X}}\cong\frac{\mathbb{Q}\otimes\widehat{F_{\rm ab}X}}{1\otimes\widehat{F_{\rm ab}X}}\cong\frac{\mathbb{Q}\otimes H^1(X,\mathbb{Z})}{1\otimes H^1(X,\mathbb{Z})}$ (in view of (ii)). (Cf. 8.82.) This proves the first isomorphy. A proof of the second requires a generalisation of some information on the tensor product which we presented in

A1.45. Indeed, if we tensor the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

by $H^1(X, \mathbb{Z})$, then

$$\mathbb{Z} \otimes H^1(X,\mathbb{Z}) \cong H^1(X,\mathbb{Z}) \to \mathbb{Q} \otimes H^1(X,\mathbb{Z}) \to (\mathbb{Q}/\mathbb{Z}) \otimes H^1(X,\mathbb{Z}) \to 0$$

remains exact. □

We now have enough information to compute the weight of a free compact group on a compact connected pointed space.

Weight of a Free Compact Group

Theorem 11.17. *Let X be a compact connected space. Then*

$$w(FX) = w(X)^{\aleph_0}.$$

Proof. By 11.6 we have $w(FX) \le w(X)^{\aleph_0}$. From Corollary 11.16(iv) we know that $Z_0(FX) \cong \widehat{Q}^{w(X)^{\aleph_0}}$. Thus $w(X)^{\aleph_0} = w(\widehat{Q}^{w(X)^{\aleph_0}}) = w\big(Z_0(FX)\big) \le w(FX)$. From both estimates we get the desired equality. □

Theorem 11.18. *Let X be a compact pointed space. Then F_0X is a direct product of $Z_0(FX)$ and $(F_0X\overline{F'X})$ if and only if $H^1(X,\mathbb{Z})/\operatorname{tor}\big(H^1(X,\mathbb{Z})\big)$ is divisible.*

Proof. We observe that $H^1(X,\mathbb{Z})/\operatorname{tor}\big(H^1(X,\mathbb{Z})\big)$ is divisible if and only if the map $\iota_{H(X,\mathbb{Z})}\colon H^1(X,\mathbb{Z}) \to \mathbb{Q}\otimes H^1(X,\mathbb{Z})$ is surjective. By 11.16(v) this is equivalent to $Z_0(FX) \cap \overline{F'X} = \{1\}$. Now $F_0(X) = Z_0(FX)(\overline{F'X})_0$ by 11.10(i). □

The most striking consequence is the following.

Free Compact Group Direct Product Theorem

Theorem 11.19. *Let X be a compact connected pointed space. Then the free compact group is the direct product of the free compact abelian group $F_{\mathrm{ab}}X$ and its commutator subgroup $F'X$ if and only if $H^1(X,\mathbb{Z})$ is divisible.*

Proof. Since X is connected, $F_0X = FX$. Then the commutator subgroup $F'X$ is closed by 9.2. Thus the divisibility of $H^1(X,\mathbb{Z})$ is equivalent to FX being the direct product of the two characteristic closed subgroups $Z_0(FX)$ and $F'X$. But this implies that $Z_0(FX) \cong FX/F'X = F_{\mathrm{ab}}X$. Conversely, $FX \cong F_{\mathrm{ab}}X \times F'X$, then the connected compact factor $F_{\mathrm{ab}}X \times \{1\}$ is contained in the identity component of the center, and since the center of $F'X$ is totally disconnected by 9.19 and 9.24, it agrees with the identity component of the center. □

Corollary 11.20. *If X is a compact connected space such that $H^1(X,\mathbb{Z}) \cong [X,\mathbb{T}] = \{0\}$, then $FX \cong F_{\mathrm{ab}}X \times F'X$.* □

Notice that every contractible space certainly satisfies $[X, \mathbb{T}] = \{0\}$.

We have not fully exploited Lemma 11.13. Now we utilize this lemma for a proof of the following result:

Theorem 11.21. *Let X be a compact space and let $\zeta\colon ZFX \to F_{\rm ab}X$ be again defined by $\zeta(g) = g\overline{F'X}$. Then $\overline{\rm tor}\big(Z(G)\big) \subseteq Z(G) \cap \overline{F'X}$. Moreover, if X is connected, then equality holds, i.e. $\overline{\rm tor}\big(Z(G)\big) = Z(FX) \cap F'X$.*

Proof. A character $\chi\colon Z \to U(1)$ of Z is in $\widehat{\zeta}(F_{\rm ab}X)$ if and only if there is a morphism $\psi\colon FX \to U(1)$ such that $\chi = \psi|Z(G)$. Condition (v) of Proposition 11.12 applied to $G = FX$ states that every element of $\widehat{\zeta}(F_{\rm ab}X)$ is a divisible element in $\widehat{Z}(G)$. Thus $\widehat{\zeta}(F_{\rm ab}X) \subseteq {\rm div}(\widehat{Z}(G))$. By the exactness of the characteristic sequence of $F_{\rm ab}X$ in Theorem 11.14 on the center of the free compact group, we have $\widehat{\zeta}(F_{\rm ab}X) \subseteq (Z(G) \cap \overline{F'X})^\perp$. Consequently, we have $(Z(G) \cap \overline{F'X})^\perp \subseteq {\rm div}\,\big(\widehat{Z}(G)\big)$. Passing to annihilators we get $\big({\rm div}\,\widehat{Z}(G)\big)^\perp \subseteq (Z(G) \cap \overline{F'X})^{\perp\perp} = Z(G) \cap \overline{F'X}$ by 7.64(iv). By 8.4(8) $\big({\rm div}\,\widehat{Z}(G)\big)^\perp = \overline{{\rm tor}\, Z(G)}$. This proves the first part of the theorem.

Now assume that X is connected. Then FX is connected and $F'X$ is a semisimple closed characteristic subgroup. By the Structure Theorem of Semisimple Connected Compact Groups 9.19 there is a quotient morphism with totally disconnected central kernel $\pi : S \to F'X$, where $S = \prod_{j\in J} S_j$ and the S_j are simple

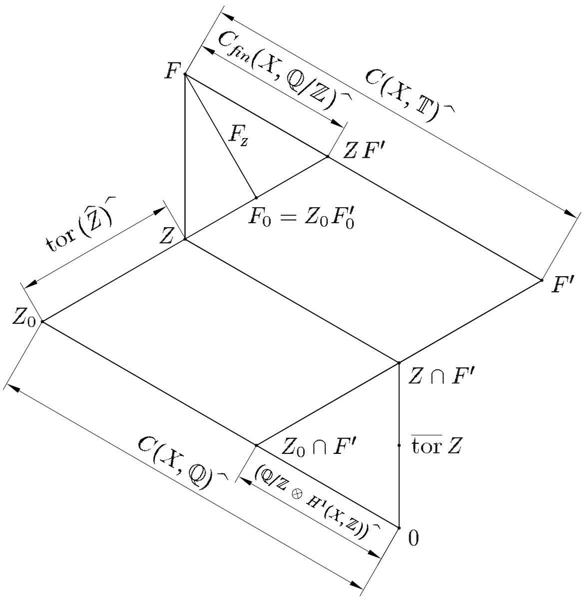


Figure 11.1: The free compact group and its center

simply connected compact Lie groups. Since the center of a semisimple compact Lie group is finite, $Z(S)$ has a dense torsion subgroup. Consequently, $Z(F'X) = Z(FX) \cap F'X$ has a dense torsion subgroup. Thus $Z(G) \cap F'X \subseteq \overline{\mathrm{tor}} Z(G)$. Since the converse containment has been proved in the first part of the proof, the theorem is proved. □

We have summarized all of our results on the center of a free compact group on a compact pointed space X in the lattice diagram shown in the figure on the previous page.

The Commutator Subgroup of a Free Compact Group

So far we know little about $F'X$. This section is devoted to the structure theory of this portion of the free compact group. For compact connected spaces X, this group is a semisimple compact connected group, and by Theorem 9.19 one has general information on the structure of such groups. The approach we take is to investigate the basic "molecules" from which $F'X$ is made up. Simple compact Lie groups serve as an index set for these molecules. (A simple Lie group is by definition nonsingleton.) We shall therefore fix a compact connected simple Lie group G and define a new type of free compact group as follows

Definitions 11.22. (i) A basepoint preserving function $f: X \to G$ from a pointed space into a topological group is said to be *essential* if it is basepoint preserving and G is topologically generated by $f(X)$, that is G is the smallest closed subgroup containing $f(X)$.

(ii) For any compact group G, and any compact pointed space X, the *essential G-free compact group* $F_G X$ (if it exists) is a compact group together with a natural map $e_X^{(G)}: X \to F_G X$ such that for every essential function $f: X \to G$ mapping the basepoint of X to the identity of G there is a unique continuous homomorphism $f': F_G X \to G$ such that $f = f' \circ e_X$. If no confusion is possible we shall simply write e_X in place of $e_X^{(G)}$.

(iii) The *free semisimple compact connected group* on a pointed space X is a compact connected semisimple group $F_{\mathrm{ss}} X$ together with a base point preserving continuous function $i_X: X \to F_{\mathrm{ss}} X$ such that for every base point preserving essential continuous function $f: X \to S$ into a compact connected semisimple group S there is a unique morphism $f': F_{\mathrm{ss}} X \to S$ such that $f = f' \circ i_X$. □

Notice that $F_G X = \{1\}$ if $\operatorname{card} X \leq 2$ and G is nonabelian, since one needs at least 2 nonidentity elements to topologically generate a nonabelian compact group. However, we shall see that for connected compact simple Lie groups G and spaces X with at least 3 points, e_X is an embedding so that f' is simply a homomorphic extension of a continuous function on the subspace X of $F_G X$ (after natural identification).

We shall abbreviate $F'_0 X \stackrel{\text{def}}{=} ((FX)'_0)$ and $Z_0 FX \stackrel{\text{def}}{=} \big(Z\big((FX)_0\big)\big)_0$.

Proposition 11.23. *For a compact connected pointed space X the free semisimple compact connected group exists and equals $F'X/(Z_0(FX)\cap F'X)$. There is a commutative diagram*

$$\begin{array}{ccccc}
X & \xrightarrow{e_X} & FX & \xrightarrow{\mathrm{id}_{FX}} & FX \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{quot}} \\
X & \xrightarrow[i_X]{} & F_{\mathrm{ss}}X = \frac{F_0'X}{Z_0FX\cap F'X} & \xrightarrow{j} & FX/Z_0(FX)
\end{array}$$

with an isomorphism j.

Proof. Let $f\colon X\to S$ be an essential base point preserving continuous map into a semisimple compact connected group. By the universal property 11.1 of the free compact group FX there is a unique morphism $f^*\colon FX\to S$ such that $f=f^*\circ e_X$. Since f is essential, $S=\overline{\langle f(X)\rangle}=\overline{\langle f^*(e_X(X))\rangle}=f^*\big(\overline{\langle e_X(X)\rangle}\big)=f^*(FX)$. Thus f^* is surjective. Hence by 9.26(iii) we have $f^*\big(Z_0(FX)\big)=Z_0(S)$. Since S is semisimple compact connected, $Z_0(S)=\{1\}$ (see 9.19(ii)). Thus there is a unique morphism $f'\colon FX/Z_0(FX)\to S$ such that $f\big(gZ_0(FX)\big)=f^*(g)$. Thus $FX/Z_0(FX)$ has the universal property of $F_{\mathrm{ss}}X$ which characterizes it uniquely up to natural isomorphism. Since $FX=Z_0(FX)F'X$, the morphism

$$j\colon F'X/(Z_0(FX)\cap F'X)\to FX/Z_0(FX),\quad j\big(g(Z_0(FX)\cap F'X)\big)=gZ_0(FX),$$

is an isomorphism. □

This proposition is the reason why we shall restrict our attention to essential functions in the present context.

We shall give a complete structure theory of F_GX and the way X is embedded into F_GX and shall see that, remarkably, *if X contains at least 4 points outside the basepoint, the structure of F_GX is that of the power $G^{w(X)^{\aleph_0}}$ where $w(X)$ denotes the weight of X.*

1. *Homomorphically simple groups.*

Definition 11.24. A compact group G will be called *homomorphically simple* if each endomorphism of G is either constant or an automorphism.

Lemma 11.25. *If $f\colon G\to G$ is an endomorphism of a connected Lie group G with finite fundamental group $\pi_1(G)$, and if the morphism $\mathfrak{L}(f)$ induced on the Lie algebra is an isomorphism then f is an isomorphism.*

Proof. Since $\mathfrak{L}(f)$ is an isomorphism, the morphism $\widetilde{f}\colon\widetilde{G}\to\widetilde{G}$ induced by f on the simply connected covering group $\widetilde{G}$ of G is an isomorphism. If $p\colon\widetilde{G}\to G$ denotes the covering morphism then $p\widetilde{f}=fp$ by the definition of the lifting $\widetilde{f}$. If $K=\ker f$ then $\widetilde{f}(K)\subseteq K$ follows. Now $K\cong\pi_1(G)$, whence K is finite by hypothesis. Since $\widetilde{f}$ is an isomorphism this implies that $\widetilde{f}|K\colon K\to K$ is an isomorphism, and, since p is in particular a quotient morphism, this entails that $f\colon G\to G$ is an isomorphism, too. □

It may be of interest to note in passing that a surjective endomorphism $f\colon G \to G$ of a connected Lie group is an open mapping as a consequence of the Open Mapping Theorem EA1.21 as G, like any connected locally compact group is a countable union of compact subsets. Hence the endomorphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(G)$ of Lie algebras is surjective and thus is an isomorphism as a morphism between finite dimensional vector spaces.

Lemma 11.26. (i) *Every compact connected simple group and every finite simple group is homomorphically simple.*

(ii) *Conversely, a connected homomorphically simple compact Lie group is semisimple or singleton.*

Proof. (i) Assume that G is a compact connected simple group and $f\colon G \to G$ a nonconstant endomorphism. Then $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(G)$ is a nonzero morphism of simple Lie algebras and is, therefore an isomorphism. Since G is a simple connected compact Lie group, $\pi_1(L)$ is finite by Theorem 5.77. Hence Lemma 1.2 applies and shows that f must be an isomorphism. This proves the claim in the first case, and the second case is trivial.

(ii) Let G denote a connected nonsingleton homomorphically simple compact Lie group. If G is not semisimple, then G/G' is a nondegenerate torus, hence there is nontrivial character $\chi\colon G \to \mathbb{T}$ onto the circle group. Now let $X\colon \mathbb{T} \to G$ be any morphism with a finite nontrivial kernel; such morphisms exist since every nondiscrete compact Lie group has circle subgroups. Then $f = X \circ \chi$ is a nonconstant endomorphism which is not injective. □

The proof of Lemma 11.26 is rather direct, but not elementary. It is instructive to note that a covering morphism $\mathbb{T} \to \mathbb{T}$ in general is by no means an isomorphism even though it induces an isomorphism $\mathbb{R} \cong \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{T}) \cong \mathbb{R}$.

A homomorphically simple compact Lie group G may not be simple.

Example 11.27. Let $Z \cong \mathbb{Z}(7)$ denote the center of $\mathrm{SU}(7)$ with generator z. Define the subgroup D of $Z \times Z$ by $D = \{(z, z^3)\colon z \in Z\}$. Then $G = \big(\mathrm{SU}(7) \times \mathrm{SU}(7)\big)/D$ is homomorphically simple but not simple.

Proof. G is the product of two elementwise commuting subgroups

$$A = \big(\mathrm{SU}(7) \times \{1\}\big)D/D \quad \text{and} \quad B = \big(\{1\} \times \mathrm{SU}(7)\big)D/D.$$

A nonconstant endomorphism f of G induces an isomorphism $\mathfrak{L}(f)$—in which case it is an isomorphism by Lemma 1.2— or has $\mathfrak{L}(A)$ or $\mathfrak{L}(B)$ as kernel. In the first case $\ker f$ is locally isomorphic to A; but $G/A \cong B/(A \cap B) \cong \mathrm{SU}(7)/Z \stackrel{\mathrm{def}}{=} \mathrm{PSU}(7)$. Since $\mathrm{PSU}(7)$ is centerfree, $A = \ker f$ and $\mathrm{im}\, f \cong \mathrm{PSU}(7)$. However, the Lie subgroups of G which are locally isomorphic to $\mathrm{SU}(7)$ but are different from A, B and G are all of the form $S_\alpha = \{\big(g, \alpha(g)\big)D \mid g \in \mathrm{SU}(7)\}$ for an automorphism $\alpha \in \mathrm{Aut}\,\big(\mathrm{SU}(7)\big)$. If K is the kernel of the morphism $g \mapsto \big(g, \alpha(g)\big) : \mathrm{SU}(7) \to S_\alpha$, then $S_\alpha \cong \mathrm{SU}(7)/K$. But $k \in K$ if and only if $\big(k, \alpha(k)\big) \in D$, that is if there is a

$z \in Z$ such that $\big(k, \alpha(k)\big) = (z, z^3)$. This means $k \in Z$ with $\alpha(k) = k^3$. Now an automorphism α of SU(7) either fixes every element of Z or else is of order 2, and thus $\alpha(k) = k^{-1}$. Therefore $k \in K$ if and only if k is in Z and satisfies $k = k^3$ or $k^{-1} = k^3$, that is $k^2 = 1$ or $k^4 = 1$. In both cases we conclude $k = 1$. Thus all subgroups of G which are locally isomorphic to SU(7) are isomorphic to SU(7), hence cannot be the image of f. Analogously, $\ker \mathfrak{L}(f) = \mathfrak{L}(B)$ is impossible, too. Hence any nonconstant endomorphism of G is an automorphism. □

It is an instructive exercise to show that, for instance, the group

$$\big(\mathrm{SU}(2) \times \mathrm{SU}(2)\big)/D \text{ with the diagonal } D \text{ of the center } Z \times Z$$

is not homomorphically simple. (Note that all nonnormal Lie subgroups of this group which are locally isomorphic to SU(2) are isomorphic to SO(3).)

Lemma 11.28. *Let G be a connected homomorphically simple compact Lie group and J any set. Then any nonconstant morphism $G^J \to G$ is a projection followed by an automorphism of G.*

Proof. Let $f\colon G^J \to G$ be any morphism. Since G is a Lie group there is an identity neighborhood V in G containing no subgroup other than $\{1\}$. Then any subgroup of G^J contained in the identity neighborhood $f^{-1}(V)$ is in the kernel of f. By the definition of the product topology on G^J, there is a cofinite subset I of J such that the partial product G^I (identified with the obvious subgroup of G^J) is annihilated by f. Hence f factors through the projection $G^J \to G^{J\setminus I}$. We may therefore assume that J is finite. If the kernel N of f meets any of the factors G inside G^J, it must contain this factor since the restriction of f to this factor is either constant or an isomorphism. Now $\mathfrak{L}(N)$ is an ideal in $\mathfrak{L}(G^J) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with $\mathfrak{g}_j \cong \mathfrak{L}(G)$. Since G is semisimple by Lemma 11.26(ii), $\mathfrak{L}(G) = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$ with simple ideals $\mathfrak{s}_k$, whence $\mathfrak{g}_j = \mathfrak{s}_{j1} \oplus \cdots \oplus \mathfrak{s}_{jp}$ with $s_{j\mathfrak{k}} \cong \mathfrak{s}_k$. Now the ideal $\mathfrak{L}(N)$ is a sum of the $\mathfrak{s}_{jk}$ and if it contains $\mathfrak{s}_{jk}$ then it contains $\mathfrak{g}_j$. It follows that $\mathfrak{L}(N)$ is a sum of the $\mathfrak{g}_j$. Consequently, $\mathfrak{L}(f)\colon \mathfrak{L}(G)^J \to \mathfrak{L}(G)$ is a projection, whence $f\colon G^J \to G$ is a projection. □

2. *The G-free compact group over X.*

The concept of subdirect products belongs to universal algebra, we have used it for Lie algebras in 9.37. We formulate it for the category of compact groups in which we work.

Definition 11.29. A closed subgroup S of a product $P = \prod_{j\in J} G_j$ of a family of compact groups is called *a subdirect product of this family* if $G_j = \mathrm{pr}_j(S)$ for each projection $\mathrm{pr}_j\colon P \to G_j$. □

Typically, the diagonal in any power G^J of a compact group G is a subdirect product and is itself isomorphic to G. For any given family, the subdirect products are of a great diversity. Accordingly, the usefulness of this concept depends significantly on the family whose subdirect products we consider. For instance, *every* compact connected abelian group is a subdirect product of a family of circle groups $\mathbb{T}$; indeed the evaluation injection $G \to \mathbb{T}^{\widehat{G}\setminus\{0\}}$ defines such a subdirect product. This information cannot be of much value. The situation is different with subdirect products of powers of simple compact connected groups. One might conjecture that a subdirect product inside a power of a simple connected compact Lie group is itself isomorphic to a power of this group. Some examples are instructive:

Example 11.30. Let G be a simple connected compact Lie group with nontrivial center Z. Let D denote the diagonal in G^n for $n > 1$. Then $S = DZ^n \subseteq G^n$ is a subdirect product in G^n which is not isomorphic to G^m. □

This shows that without connectivity assumptions on S the conjecture is false. The following example is more interesting:

Example 11.31. Let S be a simple group with center $\langle z, z'\rangle$, where $\langle z\rangle \cong \langle z'\rangle \cong \mathbb{Z}(2)$ such that there is an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(S)$ with $z' = \widetilde{\alpha}(z)$. Such groups exist, for instance $S = \mathrm{Spin}(2m)$ with $m > 2$. Now we consider the quotient morphisms $\pi\colon S \to S/\langle z\rangle$ and $\pi'\colon S \to G \stackrel{\mathrm{def}}{=} S/\langle z'\rangle$. Then $\widetilde{\alpha}$ induces an isomorphism $\alpha\colon S/\langle z\rangle \to G$. Define $\delta\colon S \to G^2$ by $\delta(g) = \big(\alpha\big(\pi(g)\big), \pi'(g)\big)$. Then $\delta(S)$ is a subdirect product in G^2. □

This example shows that a simply connected simple compact Lie group may be a subdirect product in a power of simple Lie groups which are not simply connected. Hence the conjecture is invalid unless further hypotheses on the global geometry of G are imposed. On the infinitesimal level, however, the conjecture is true:

Remark 11.32. In the category of finite dimensional real Lie algebras and Lie algebra morphisms, a subdirect product $\mathfrak{s}$ inside a finite power $\mathfrak{g}^n = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $\mathfrak{g}_j \cong \mathfrak{g}$ of a simple algebra $\mathfrak{g}$ is isomorphic to a power of $\mathfrak{g}$. If $\mathfrak{s}$ is subdirect in $\mathfrak{g}^n$ then there is an ideal $\mathfrak{n}$ of $\mathfrak{g}^n$ so that $\mathfrak{g}$ is the semidirect sum of $\mathfrak{n}$ and $\mathfrak{s}$.

Proof. Since the projections onto the simple factors separate the points, the radical of $\mathfrak{s}$ must be zero. Hence $\mathfrak{s}$ is semisimple, that is, is isomorphic to a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of simple ideals, and every ideal of $\mathfrak{s}$ is a sum of a subset of these summands. Since the homomorphisms onto $\mathfrak{g}$ separate points, $\mathfrak{s}_j \cong \mathfrak{g}$ follows for all $j = 1, \ldots, m \leq n$. Reorder indices so that $\mathfrak{s}_j$ projects faithfully onto the summand $\mathfrak{g}_j$, $j = 1, \ldots, m$. Then $\mathfrak{n} = \sum_{m+1}^{n} \mathfrak{g}_k$ is the required ideal. □

Remark 11.33. Let G denote a connected simple compact Lie group and n a natural number. If $S \subseteq G^n$ is a subdirect product, and if S is connected, then $S \cong G^m$ for some $m = 1, \ldots, n$, provided that G is centerfree.

Proof. By the preceding remarks, $\mathfrak{L}(S) \cong \mathfrak{L}(G)^m$. Hence S and G^m are locally isomorphic. If G is centerfree, then so are G^m and S since the surjective morphisms $S \to G$ separate points. The isomorphy of $L(S)$ and $L(G^m)$ then implies the isomorphy of S and G^m. $\square$

In the following exercise we shall generalize this remark. However, we shall not need it for the pursuit of our main objective. In this line we shall find that sometimes other information may allow us to conclude that a subdirect product of a power of a simple group is itself a power of this group.

Exercise E11.5. Prove the following generalisation of Remark 11.33.

Let G be a connected simple compact Lie group which is centerfree and J an arbitrary set. If S is a subdirect product in G^J and if S is connected, then there is a surjective function $\sigma: J \to I$ and a function $j \mapsto \alpha_j: J \to \mathrm{Aut}(G)$ such that the morphism

$$\varphi: G^I \to G^J, \qquad \varphi\big((x_i)_{i\in I}\big) = \big(\alpha_j(x_{\sigma(j)})\big)_{j\in J}$$

maps G^I isomorphically onto S.

[Hint. We write $\delta: S \to G^J$ for the inclusion and define a binary relation $\sim$ on J by writing $j \sim k$ if and only if there is an $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \circ \mathrm{pr}_j \circ \delta = \mathrm{pr}_k \circ \delta$. One observes at once that $\sim$ is an equivalence relation. This allows us to set $I = J/\sim$ and to define $\sigma: J \to I$ as the quotient map. For each $i \in I$ let $G_i = G^{\sigma^{-1}(i)}$ with projections $\mathrm{pr}_{ji}: G_i \to G$ for $j \in i$, and let S_i denote the projection of S into the partial product G_i with $\delta_i: S_i \to G_i$ the inclusion. Then S_i is a subdirect product in G_i since S is a subdirect product in G^J.

We claim that each morphism $\mathrm{pr}_{ji} \circ \delta_i : S_i \to L$ with $j \in i$ is an isomorphism; that is S_i behaves in G_i like a diagonal. Indeed by the definition of $\sim$, for each $k \in i$ we know $k \sim j$, and thus there is a $\beta_k \in \mathrm{Aut}(G)$ such that $\mathrm{pr}_k \circ \delta = \beta_k \circ \mathrm{pr}_j \circ \delta$. The morphism $\psi_j: L \to G_i$ defined by $\psi_j(x) = \big(\beta_k^{-1}(x)\big)_{k\in I}$ is injective. Further, if $(x_k)_{k\in i} \in S_i$, then $x_k = \beta_k(x_j)$ for all $k \in i$ and $\psi_j\big(\mathrm{pr}_{ji}\{(x_k)_{k\in i}\}\big) = \psi_j(x_j) = \big(\beta_k^{-1}(x_j)\big)_{k\in i} = (x_k)_{k\in i}$. Since $\mathrm{pr}_{ji}(S_i) = G$, the image of ψ is S_i, and its corestriction to this image is an inverse of $\mathrm{pr}_{ji} \circ \delta_i$. This proves the claim.

In particular, each S_i is isomorphic to G. In other words, there is an isomorphism $\gamma_i: L \to S_i$. For each $j \in J$ we define

$$\alpha_j: G \to G, \qquad \alpha_j = \mathrm{pr}_{j\sigma(j)} \circ \gamma_{\sigma(j)}.$$

The map $\varphi: G^I \to$ given by

$$\varphi\big((x_i)_{i\in I}\big) = \big(\alpha_j(x_{\sigma(j)}\big)_{j\in J} = \big(\mathrm{pr}_{j\sigma(j)}\{\gamma_{\sigma(j)}(x_{\sigma(j)})\}\big)_{j\in J}$$

is clearly an injective morphism of compact groups. Hence it is an isomorphism onto its image.

We claim that this image is S. A proof of this claim will finish the proof of the proposition. Firstly, we observe that $\mathrm{im}\,\varphi = \prod_{i\in I} S_i$ where we have identified G^J and $\prod_{i\in I} G_i$ in the obvious fashion. Clearly $S \subseteq \prod_{i\in I} S_i$, and we have to show equality. This product is subdirect by the definition of S_i and each S_i is isomorphic to G. If $i \neq i'$ in I, then there is no isomorphism $\rho: S_i \to S_{i'}$ such that $\mathrm{pr}_{i'}|S = \rho \circ \mathrm{pr}_i|S$, for such a ρ would give us for $j \in i$ and $j' \in i'$ an $\alpha \in \mathrm{Aut}(G)$ given by $\alpha = (\mathrm{pr}_{ji}|S_i)^{-1}\rho(\mathrm{pr}_{j'i'})$ such that $\mathrm{pr}_{j'}|S = \mathrm{pr}_{j'i'}(\mathrm{pr}_{i'}|S) = \mathrm{pr}_{j'i'}\rho(\mathrm{pr}_i|S) = \alpha\,\mathrm{pr}_{ji}(\mathrm{pr}_i|S) = \alpha(\mathrm{pr}_j|S)$, and this would imply $j \sim j'$, that is $i = i'$ contrary to the assumption.

Hence in order to prove our last claim, in simplified notation, we have to prove that for a subdirect product $S \subseteq G^J$ we have $S = G^J$ if the relation $\sim$ on J is equality. Let E denote any finite subset of J and S_E the projection of S into G^E. Then S_E is a subdirect product of G^E and the relation $\sim$ on E is likewise trivial. If these circumstances imply $S_E = G^E$ for all finite subsets E of J, then $S = G^J$ follows, since G^J is the projective limit of its projections onto all finite partial products.

Therefore it suffices to prove the claim when J is finite. We suppose that the claim is false and derive a contradiction. Let us suppose that $S \subseteq G^n$ is a counterexample to the claim with a minimal natural number n. Evidently, $n \geq 2$. The projection S^* into G^{n-1} (after identifying G^n with $G^{n-1} \times G$) satisfies all hypotheses and cannot be a counterexample. Thus $S^* = G^{n-1}$. If we denote by $F \cong G$ the subgroup $\{1\} \times G$ in $G^{n-1} \times G$, then we have an exact sequence

$$1 \to S \cap F \to S \xrightarrow{\pi} G^{n-1} \to 1.$$

Since $S \cong G^m$ for some m by Lemma 2.5, $m \geq n-1$. If $m = n$, then $\dim S = \dim G^n$ and $S = G$ contrary to our supposition. Thus $S \cong G^{n-1}$. Accordingly, $\mathfrak{L}(\pi)$ is an isomorphism and thus π a covering morphism. Now Lemma 1.2 implies that π is an isomorphism and thus $S \cap F = \{1\}$. Now $S \subseteq G^{n-1} \times G$ is the graph of a morphism $\theta: G^{n-1} \to G$ and since S is subdirect, θ is surjective. We use Lemmas 1.3 and 1.5 to conclude, that θ is a projection of G^{n-1} followed by an automorphism α of G; let us say that the projection maps onto the last factor of G^{n-1}. This means that $S = \{(1, \ldots, 1, s, \alpha(s)) \mid s \in G\} \subseteq G^n$. If $p: G^n \to G$ is the projection of G^n onto the next to last factor and q the projection on the last factor, then $\alpha \circ p|S = q|S$ in violation of our assumption. This contradiction completes the proof.] □

Exercise E11.6. Prove the following observation:

If $S \subseteq P = \prod_{j \in J} G_j$ is a subdirect product of compact connected semisimple groups G_j, then so is the identity component S_0 of S.

[Hint. If $\mathrm{pr}_j: P \to G_j$ is the projection, then $N_j = \mathrm{pr}_j(S_0)$ is a normal subgroup of G_j. If $N_j \neq G_j$, then N_j is contained in the finite center Z_j of G_j. Then the surjective projection $\mathrm{pr}_j|S: S \to G_j$ gives us a surjective morphism $S/S_0 \to G_j/Z_j$. Since S/S_0 is compact zero dimensional, the image of this map is a zero dimensional subgroup of a compact Lie group hence is finite. But if G_j/Z_j is infinite since $N_j \neq G_j$. This contradiction shows that $N_j = G_j$.] □

Definition 11.34. Let X be a pointed compact space and G any compact group. We denote by $E(X,G)$ the set of all essential functions $f: X \to G$. The *evaluation map* $\mathrm{ev}_X: X \to G^{E(X,G)}$ is given by $\mathrm{ev}_X(x)(f) = f(x)$ for $f \in E(X,G)$.

Lemma 11.35. *For any essential function $f: X \to G$ there is a morphism f': $G^{E(X,G)} \to G$ of compact groups such that $f = f' \circ \mathrm{ev}_X$.*

Proof. In view of the definition of the product topology, that is the topology of simple convergence on $G^{E(X,G)}$, the function ev_X is continuous, and if $f \in E(X,G)$, then $f(x) = \mathrm{ev}_X(x)(f) = (\mathrm{pr}_f \circ \mathrm{ev}_X)(x)$ and thus $f' = \mathrm{pr}_f: G^{E(X,G)} \to G$ satisfies the requirement. □

Lemma 11.36. *If G is an arcwise connected compact nonsingleton group whose weight is at most $2^{\aleph_0}$, and if X contains at least 4 points then the evaluation map $\mathrm{ev}_X: X \to G^{E(X,G)}$ is a topological embedding of the compact pointed space X.*

Proof. The base point preserving functions of $C_0(X,[-1,1])$ separate the points of X. Thus the evaluation map $\eta: X \to G^{C_0(X,[-1,1])}$ is injective. If the compact group G is a compact connected group whose weight does not exceed the cardinality of the continuum, then it is topologically generated by two points a and b (see 9.38).

Since G is arcwise connected by hypothesis, there is a homeomorphism h of the interval $[-1,1]$ into G with $h(0) = 1$, $h(-1) = a$ and $h(1) = b$. Since X contains at least four points $x_0 =$ basepoint, x_1, x_2, and x, the set $C_*(X,[-1,1])$ of functions f with $f(x_0) = 0$, $f(x_1) = -1$, $f(x_2) = 1$ still separates the points of X. The homeomorphism h induces an injection $C_0(X,h)$ of $C_*(X,[-1,1])$ into $C_0(X,G)$ such that all functions f in the image satisfy $a,b \in f(X)$ and thus are essential. Hence we have an embedding of $C_*(X,\mathbb{I})$ into $E(X,G)$. This shows that the functions of $E(X,G)$ separate the points of X. Hence the evaluation $\mathrm{ev}_X: X \to G^{E(X,G)}$ is injective. Since X is compact and $G^{E(X,G)}$ Hausdorff, it is an embedding. □

This information now readily allows us a first identification of the essential G-free compact group.

Proposition 11.37. *Let X be any compact pointed space and G any compact group. Let $F_G X$ denote the compact subgroup generated by $\mathrm{ev}_X(X)$ in $G^{E(X,G)}$ and $e_X: X \to F_G X$ the essential map obtained by corestricting ev_X. Then $F_G X$ is the essential G-free compact group over X and e_X the universal mapping of X into it. Moreover, the following statement holds:*

(i_4) *If G is arcwise connected and of weight at most $2^{\aleph_0}$ and if X has at least 4 points, then e_X is a topological embedding.*

Proof. Lemma 11.35 secures the universal property, including uniqueness, and Lemma 11.36 establishes that e_X is a topological embedding. □

Corollary 11.38. *The group $F_G X$ is a subdirect product in $G^{E(X,G)}$.*

Proof. If $f \in E(X, G)$, then $\mathrm{pr}_f(F_G X)$ is the subgroup topologically generated by $f(X)$ in G, hence is G. □

We now proceed to describe this subdirect product $F_G X$ accurately if G is a homomorphically simple compact Lie group. We begin with the observation that the automorphism group $\mathrm{Aut}\, G$ acts on the set $E(X, G)$ on the left by

$$(\alpha, f) \mapsto \alpha \circ f : \mathrm{Aut}\, G \times E(X, G) \to E(X, G).$$

We shall denote the orbit space $E(X, G)/\mathrm{Aut}\, G$ by $A(X, G)$ and write $[f]$ for the orbit $\{\alpha \circ f \mid \alpha \in \mathrm{Aut}\, G\}$.

For two functions $f, f' \in E(X, G)$ we shall write $f \sim f'$ if and only if there is an automorphism $\alpha \in \mathrm{Aut}\, G$ such that $\mathrm{pr}_{f'} | F_X G = \mathrm{pr}_f | F_G X$, and this is tantamount to $\mathrm{pr}_{f'} \circ \mathrm{ev}_X = \alpha \circ \mathrm{pr}_f \circ \mathrm{ev}_X$, that is to $f' = \alpha \circ f$. Thus $f \sim f'$ is equivalent to $[f] = [f']$. Hence $\sim$ is none other than the orbit equivalence of the action of $\mathrm{Aut}\, G$ on $E(X, G)$.

For each $F \in A(X, G)$ we select a function $s_F \in F \subseteq E(X, G)$. Thus $[s_F] = F$ and $f \sim s_{[f]}$. Hence for each $f \in A(X, G)$ there is at least one $\alpha_f \in \mathrm{Aut}\, G$ such that

$$f = \alpha_f \circ s_{[f]}. \tag{1}$$

We define $\varepsilon_X : X \to G^{A(X,G)}$ by

$$\varepsilon_X(x)(F) = s_F(x). \tag{2}$$

Then (1) implies

$$\big(f(x)\big)_{f \in E} = \mathrm{ev}_X(x) = \alpha_{\mathrm{ev}_X(x)} \circ s_{[\varepsilon_X(x)]} = \big(\alpha_f\{s_{[f]}(x)\}\big)_{f \in E}.$$

If we define

$$\varphi_X : G^{A(X,G)} \to G^{E(X,G)}, \quad \varphi_X\big(g_F\big)_{F \in A(X,G)} = \big(\alpha_f(g_{[f]})\big)_{f \in E(X,G)} \tag{3}$$

then

$$\mathrm{ev}_X = \varphi_X \circ \varepsilon_X. \tag{4}$$

In particular, $\varepsilon_X : X \to G^{A(X,G)}$ is an embedding whenever ev_X is an embedding.

Now assume that $f : X \to G$ is an essential map. Set $f^* : G^{A(X,G)} \to G$, $f^*\big((g_F)_{F \in A(X,G)}\big) = \alpha_f(g_{[f]})$. Then

$$f^*\big(\varepsilon_X(x)\big) = f^*\big\{\big((s_F(x)\big)_{F \in A(X,G)}\big\} = \alpha_f\big(s_{[f]}(x)\big) = f(x)$$

in view of (2) and (1). We notice that the following diagram is commutative.

$$\begin{array}{ccccc}
X & \xrightarrow{\varepsilon_X} & G^{A(X,G)} & \xrightarrow{\varphi_X} & G^{E(X,G)} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'} \\
G & \xrightarrow[\mathrm{id}_G]{} & G & \xrightarrow[\mathrm{id}_G]{} & G
\end{array}$$

Now assume that $\mu_j: G^{A(X,G)} \to G$, $j = 1, 2$ are two morphisms such that $\mu_1 \circ \varepsilon_X = \mu_2 \circ \varepsilon_X$. At this point we assume that G is homomorphically simple; then Lemma 11.28 implies that there are automorphisms $\beta_j \in \operatorname{Aut} G$, $j = 1, 2$ such that $\mu_j = \beta_j \circ \mathrm{pr}_{F_j}$. Now $\beta_j \, \mathrm{pr}_{F_j}\big(\varepsilon_X(x)\big) = \beta_j\big(s_{F_j}(x)\big)$ by (2). Hence $s_{F_2} \circ \mathrm{ev}_X = (\beta_2^{-1}\beta_1) s_{F_1} \circ \mathrm{ev}_X$ and this implies $s_{F_1} \sim s_{F_2}$. Thus $F_1 = [s_{F_1}] = [s_{F_2}] = F_2$. But now, setting $F = F_1$, via (2) we find $s_F = \varepsilon_X(F) = \mathrm{pr}_F \, \varepsilon_X = (\beta_2^{-1}\beta_1) \, \mathrm{pr}_F \, \mathrm{ev}_X = (\beta_2^{-1}\beta_1) s_F$. Since $s_F \in E(X, G)$, the subset $s_F(X)$ topologically generates G. Thus $\beta_1 = \beta_2$ follows, and we have $\mu_1 = \mu_2$. Therefore, the following universal property of $G^{A(X,G)}$ is proved.

Lemma 11.39. *Let G be a connected homomorphically simple compact Lie group, further X a compact pointed space, and $\varepsilon_X: X \to G^{A(X,G)}$ the function defined by* (2)*. Then for each essential function $f: X \to G$ there is a unique morphism $f^*: G^{A(X,G)} \to G$ such that $f = f^* \circ \varepsilon_X$. Moreover:*

(i$_4$) *If X has at least* 4 *points, then ε_X is an embedding.* □

Hence $G^{A(X,G)}$ is in fact the essential G-free compact group over X. This immediately entails the following principal result for whose formulation we use the continuous function $\varepsilon_X: X \to G^{A(X,G)}$ of (2) and the injective morphism $\varphi_X: G^{A(X,G)} \to G^{E(X,G)}$ of (3).

Theorem 11.40. *Let G be a connected homomorphically simple compact Lie group and X a compact pointed space. Then φ_X corestricts to an isomorphism $\varphi_X: G^{A(G,X)} \to F_G X$ such that $\varphi_X \circ \varepsilon_X = e_X$. Moreover, the following statement is true:*

(i$_4$) *If X has at least* 4 *points, then ε_X is an embedding.* □

We now apply the information contained in this theorem to compute the weight $w(F_G X)$ of the essential G-free compact group.

Lemma 11.41. *If G is a connected nonsingleton compact Lie group and X a compact pointed space with at least* 4 *points, then* $\operatorname{card} E(X, G) = w(X)^{\aleph_0}$.

Proof. If $C_0(X, G)$ denotes the set of basepoint preserving continuous functions $X \to G$, then $E(X, G) \subseteq C_0(X, G)$. Now $\operatorname{card} C_0(X, G) = w(X)^{\aleph_0}$ (see Appendix 4, A4.9). Thus $\operatorname{card} E(G, X) \leq w(X)^{\aleph_0}$. We have seen in Theorem 6.51 that G is topologically generated by two points a and b. We noted in the proof of Lemma 11.36 that there is a homeomorphism h from the interval $[-1, 1]$ into G which induces an injection of $C_*(X, [-1, 1])$ into $E(X, G)$ such that all functions f in the image satisfy $a, b \in f(X)$ and thus are essential. Hence we have an embedding of $C_*(X, [-1, 1])$ into $E(X, G)$ where $C_*(X, [-1, 1])$ denotes the set of functions $f: X \to [-1, 1]$ with $f(x_0) = 0$, $f(x_1) = -1$, $f(x_2) = 1$. Hence $\operatorname{card} C_*(X, [-1, 1]) \leq \operatorname{card} E(X, G)$. Let **8** denote the figure 8 obtained by collapsing in $[-1, 1]$ the points -1, 0 and 1 into one point and X_* the space obtained from X by collapsing x_0, x_1, and x_2 into the basepoint. Then there is a surjection

$C_*(X,[-1,1])$ onto $C_0(X_*,\mathbf{8})$ and thus

$$\operatorname{card} C_*(X,[-1,1]) \geq \operatorname{card} C_0(X_*,\mathbf{8}) w(X_*)^{\aleph_0} = w(X)^{\aleph_0},$$

if X has at least 5 points, and thus X_* has at least 2 points. However, if X has 4 points, then X_* has 1 point, and therefore $\operatorname{card} C_0(X_*,\mathbf{8}) = \operatorname{card}\mathbf{8} = 2^{\aleph_0} = (\operatorname{card} X)^{\aleph_0} = w(X)^{\aleph_0}$. Hence $w(X)^{\aleph_0} \leq \operatorname{card} E(X,G)$. □

Lemma 11.42. *If X is a compact space with at least 4 points, then*

$$\operatorname{card} A(X,G) = w(X)^{\aleph_0}.$$

Proof. Since the orbit map is a surjective function $E(X,G) \to A(X,G)$ we have $\operatorname{card} A(X,G) \leq \operatorname{card} E(X,G) = w(X)^{\aleph_0}$. Now $\operatorname{Aut} G$ is a compact Lie group and contains all inner automorphisms of G, whence $\operatorname{card}(\operatorname{Aut} G) = 2^{\aleph_0}$. The orbits $[f]$ therefore have at most continuum cardinality. Thus $\operatorname{card} E(X,G) \leq \max\{2^{\aleph_0}, \operatorname{card} A(X,G)\}$. We finish the proof by showing that $2^{\aleph_0} \leq \operatorname{card} A(X,G)$ because then $w(X)^{\aleph_0} = \operatorname{card} E(X,G) \leq \operatorname{card} A(X,G)$. Now $\operatorname{Aut} G$ is a finite extension of the group $\operatorname{Inn} G$ of inner automorphisms, and G has a continuum of conjugacy classes, hence a continuum of $\operatorname{Aut} G$-orbits F. Let $x \in X$ be any point different from the basepoint x_0. Now for each $\operatorname{Aut} G$-orbit F on G there is an $f_F \in E(X,G)$ such that $f_F(x) \in F$. Then $\{[f_F] \mid F \in G/\operatorname{Aut} G\}$ is a subset of $A(X,G)$ of continuum cardinality, whence $\operatorname{card} A(X,G) \geq 2^{\aleph_0}$, which we asserted. □

Proposition 11.43. *Let G be a compact connected homomorphically simple compact Lie group and X a compact pointed space with at least 4 points. Then $w(F_G X) = w(X)^{\aleph_0}$.*

Proof. We have $w(F_G X) = w(G^{A(X,G)}) = \operatorname{card} A(X,G) = w(X)^{\aleph_0}$ by Lemma 11.42. □

Remark 11.44. For each infinite cardinal $\aleph_\nu$ there is a compact connected semisimple group G containing a subset Y not containing 1 with the following properties:

(a) $G = \overline{\langle Y \rangle}$ and Y is discrete and closed in $G \setminus \{1\}$,
(b) $\operatorname{card} Y = \aleph_\nu$, and
(c) $w(G) = (\aleph_\nu)^{\aleph_0} = (\operatorname{card} Y)^{\aleph_0}$.

Proof. Let $X = D \dot\cup \{\infty\}$ be the one point compactification of a discrete space D of cardinality $\aleph_\nu$ based at ∞, set $G = F_{\mathrm{SO}(3)} X$ and define $Y = e_X(X) \setminus \{1\}$. But then G is topologically generated by Y and 11.36 shows that Y is homeomorphic to D, whence Y is discrete and $\overline{Y} = Y \cup \{1\}$. Thus (a) holds. Also $\operatorname{card} Y = \operatorname{card} D = \aleph_\nu$, whence (b) is satisfied. From Proposition 11.43 we know $w(G) = w(X)^{\aleph_0} = \aleph_\nu^{\aleph_0}$. □

We observe that in the preceding construction, G may contain a subset Y' satisfying (a) with Y' in place of Y and
(b′) $\operatorname{card} Y' < \aleph_\nu$.
Indeed, let $\nu = 0$. Then $w(F_{\mathrm{SO}(3)}X) = w(Y)^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. But by 9.38, G is generated by two elements a and b different from 1 and so $Y' = \{a, b\}$ satisfies (a) and $\operatorname{card} Y' \leq 2 < \aleph_0$.

From 11.26 we know that every connected simple compact Lie group is homomorphically simple. So from Theorem 11.40 and Proposition 11.43 we get the following result for whose formulation we must recall definitions preceding 11.39, notably that of $A(X,G) = E(X,G)/\operatorname{Aut} G$ and the projection $\varphi\colon G^{E(X,G)} \to G^{A(X,G)}$. Essential G-free groups were defined in 11.22(ii).

The Structure of Essential S-Free Compact Groups

Theorem 11.45. *Let S be a connected simple compact Lie group and X a compact pointed space. Then the essential S-free compact group $F_S(X)$ is $S^{A(X,S)}$ with a suitable natural function $e_X\colon X \to S^{A(X,S)}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\ e_X\ } & S^{E(X,S)}, \\ {\scriptstyle \mathrm{id}_S}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[\ \varepsilon_X\ ]{} & F_SX = S^{A(X,S)}. \end{array} \qquad e_X(x)(f) = f(x),$$

Moreover, the following statement holds:

(i$_4$) *If X has at least 4 points, then ε_X is an embedding, and the cardinality of $A(X,G)$ and the weight of F_SX both equal $w(X)^{\aleph_0}$.* □

Note that as a compact group, F_SX has two structural invariants which determine the structure up to an isomorphism of compact groups, namely, S and $w(X)^{\aleph_0} = w(F_SX)$. It is instructive to pause for a moment and to consider the universal covering morphism $p\colon \widetilde{S} \to S$ with the simply connected universal covering group (which is compact by 9.19). Then $F_{\tilde{S}}X = \widetilde{S}^{A(X,\widetilde{S})}$ and $F_SX = S^{A(X,S)}$.

Remark 11.46. There is a unique surjective morphism $F_pX\colon F_{\widetilde{S}}X \to F_SX$ such that the following diagram is commutative.

$$\begin{array}{ccccc} X & \xrightarrow{\ \varepsilon_X^{(\tilde{S})}\ } & F_{\tilde{S}}X & \xrightarrow{\ \mathrm{id}_{F_{\tilde{S}}X}\ } & F_{\tilde{S}}X \\ {\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle F_pX}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{pr}_F \circ e_X^{(S)})'} \\ X & \xrightarrow[\ \varepsilon_X^{(S)}\ ]{} & F_SX & \xrightarrow[\ \mathrm{pr}_F\ ]{} & S, \end{array} \qquad F \in A(X,S).$$

Let $\pi_{F_SX}\colon \widetilde{F_SX} \to F_SX$ denote the natural morphism of Theorem 9.19. Then there is a morphism $\widetilde{F}_pX\colon F_{\tilde{S}}X \to \widetilde{F_SX}$ such that $F_pX = p_{F_SX} \circ \widetilde{F}_pX$.

If X is simply connected, the map

$$E(X,p)\colon E(X,\widetilde{S}) \to E(X,S), \quad E(X,p)(f) = f \circ p$$

is bijective, and if the unique lifting assignment $\alpha \mapsto \widetilde{\alpha}\colon \mathrm{Aut}(S) \to \mathrm{Aut}(\widetilde{S})$ is surjective, we may identify $A(X,\widetilde{S})$ and $A(X,S)$ in which case $F_pX\colon \widetilde{S}^{A(X,S)} \to S^{A(X,S)}$ is calculated componentwise as $F_pX\big((s_F)_{F\in A(X,S)}\big) = \big(p(s_F)\big)_{F\in A(X,S)}$ and $\widetilde{F}_pX$ is an isomorphism.

Proof. For each $F \in A(X,S)$, the universal property of $\varepsilon_X^{(\tilde{S})}\colon X \to F_{\tilde{S}}X$ yields a unique morphism $(\mathrm{pr}_F \circ \varepsilon^{(S)x})'\colon F_{\tilde{S}}X \to S$ such that the outside of the diagram commutes. Then the universal property of the product (see A3.41 and A3.43) provides the fill-in morphism $F_pX\colon F_{\tilde{S}}X \to F_SX$. Since $e_X^{(S)}$ is essential, this fill-in is surjective.

If X is simply connected, then by the very Definition A2.6 of simple connectivity, every continuous base point preserving map $f \in E(X,S)$ has a unique lifting to a base point preserving continuous map $\widetilde{f}\colon X \to \widetilde{S}$ such that $p \circ \widetilde{f} = f$ and that $f \mapsto \widetilde{f}$ and $p^*\colon E(X,\widetilde{S}) \to E(X,S)$, $p^*(f') = p \circ (f')$, are inverses of each other. Moreover, the map p^* is equivariant in the following sense. By hypothesis, every automorphism of $\widetilde{S}$ is the lifting of exactly one automorphism $\alpha \in \mathrm{Aut}\,S$; then one has $p^*(\widetilde{\alpha} \circ f') = \alpha \circ (p \circ f')$. This allows us to identify $A(X,\widetilde{S}) = E(X,\widetilde{S})/\,\mathrm{Aut}(\widetilde{S})$ and $A(X,S) = E(X,S)/\,\mathrm{Aut}(S)$ and the assertion follows. □

We note that if X is not simply connected, the morphism F_pX is no longer obtained in a very obvious fashion on the level of the products. In particular, the morphism $F_sX\colon F_{\tilde{S}}X \to F_SX$ in general is not equivalent to the "covering" $\pi_{F_SX}\colon \widetilde{F_SX} \to F_SX$ of 9.19.

Example 11.47. Let $S = \mathrm{SO}(3)$ and let X be the space underlying S. Then $\widetilde{S} \cong \mathrm{SU}(2) \cong \mathbb{S}^3$, and the identity map $j\colon X \to S$ is in $E(X,S)$ but does not lift to an element in $E(X,\widetilde{S})$. Since every automorphism of $\widetilde{S}$ fixes necessarily the 2-element center (even elementwise), the map $\alpha \mapsto \widetilde{\alpha}\colon \mathrm{Aut}\,S \to \mathrm{Aut}\,\widetilde{S}$ is an isomorphism. The function $\mathrm{pr}_{[j]} \circ \varepsilon_X^{(S)} = \mathrm{pr}_{[j]} \circ F_pX \circ \varepsilon_X^{(\tilde{S})}$ does not factor through any projection of a factor of $F_{\tilde{S}}X = A^{A(X,\tilde{S})}$. Hence $\widetilde{F}_pX\colon F_{\tilde{S}}X \to \widetilde{F_SX}$ cannot be an isomorphism and thus $\ker F_pX$ is not totally disconnected. □

Equipped with this information we shall now have another look at the structure of $F'X$ the commutator subgroup of the free compact group FX. The formulation of the following theorem is facilitated by the terminology which we introduce in the following definition.

Definition 11.48. A morphism $\kappa\colon G \to H$ of compact groups is said to be *nearly split*, respectively, *nearly normally split*, if G contains a closed subgroup N, respectively, closed *normal* subgroup N such that the restriction $\kappa|N\colon N \to H$ is surjective and that $\ker(\kappa|N) = N \cap \ker\kappa$ is totally disconnected. □

We notice that $G = (\ker\kappa)N$, that $(\ker\kappa)\cap N$ is totally disconnected, and that there is an exact sequence

$$\{1\} \to \ker(\kappa|N) \xrightarrow{\ d\ } \ker\kappa \rtimes N \xrightarrow{\ m\ } G \to \{1\},$$

$d(k) = (k^{-1}, k)$, $m(k,n) = kn$. From 9.74 we know that *a morphism $\kappa: G \to H$ is nearly split if H is connected and that the near factor N is normal in G_0.* If $f: G \to H$ is nearly split then the isomorphism $q: N/\ker(f|N) \to G/\ker f$, $q(n\ker(f|N) = n\ker f$ permits a factorisation of f as follows: Firstly, the canonical decomposition of $f|N: N \to H$ provides an isomorphism $\overline{f}: N/\ker(f|N) \to H$, $\overline{f}(n\ker(f|N)) = f(n)$. Secondly, there is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & H \\ {\scriptstyle\text{quot}}\big\downarrow & & \big\uparrow{\scriptstyle\overline{f}} \\ G/\ker f & \xrightarrow[q^{-1}]{} & N/\ker(f|N). \end{array}$$

We shall write $\underline{f}: G \to N/\ker(f|N)$, $\underline{f}(g)q^{-1}(gN)$ and thus have $f = \overline{f}\circ\underline{f}$,

$$G \xrightarrow{\ \underline{f}\ } G/\ker(f|N) \xrightarrow{\ \overline{f}\ } H.$$

We shall call this *the natural decomposition of the nearly split morphism f*. As in Theorem 9.19 we let $\mathcal{S}$ be a system of representatives for the class of simple compact Lie algebras. We consider a simple connected compact Lie group S and let $\mathfrak{s}$ be that element in $\mathcal{S}$ which satisfies $\mathfrak{s} \cong \mathfrak{L}(S)$. We recall that $S_{[\mathfrak{s}]}$ denotes a simply connected compact simple group whose Lie algebra is $\mathfrak{s}$. There is a covering morphism $f_S: S_{[\mathfrak{s}]} \to S$. We recall that FX is abelian if $\operatorname{card} X \le 2$ and thus has no essential morphisms onto a simple connected compact Lie group. With the notation that we chose we obtain the following sandwich theorem.

Theorem 11.49. *Let X be a compact space containing at least 2 points and S a simple connected compact Lie group. Consider the natural morphisms $\kappa_S: FX \to F_S X = S^{A(X,S)}$. Then there is a normal closed connected semisimple subgroup $\Phi_S(X)$ of FX such that the restriction $\rho_S \overset{\text{def}}{=} \kappa_S|\Phi_S(X): \Phi_S(X) \to S^{A(X,S)}$ is surjective and has a totally disconnected kernel; i.e. κ_S is nearly normally split.*

Moreover, the projective cover $\mathfrak{P}\big(\Phi_S(X)\big)$ may be identified with $S_{[\mathfrak{s}]}^{A(X,S)}$ such that the composition

$$S_{[\mathfrak{s}]}^{A(X,S)} \xrightarrow{E_{\Phi_S(X)}} \Phi_S(X) \xrightarrow{\ \rho_S\ } S^{A(X,S)}$$

is $f_S^{A(X,S)}: S_{[\mathfrak{s}]}^{A(X,S)} \to S^{A(X,S)}$.

Proof. The universal property of FX gives us a natural surjective morphism $\kappa_S: FX \to F_S X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & FX \\ {\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_S} \\ X & \xrightarrow[\varepsilon_X^{(S)}]{} & F_S X \end{array}$$

is commutative. By Theorem 9.74, there is a connected normal subgroup $\Phi_S(X)$ of $F_0X \stackrel{\rm def}{=} (FX)_0$ such that $\rho_S \stackrel{\rm def}{=} \kappa_S|\Phi_S(X)\colon \Phi_S(X) \to S^{\mathbb{A}(X,S)}$ is surjective and has a totally disconnected kernel. By the preceding remarks we have for $\kappa_S\colon FX \to S^{A(X,S)}$ a natural decomposition

$$FX \xrightarrow{\kappa_S} \Phi_S(X)/\ker \rho_S \xrightarrow{\overline{\kappa_S}} S^{A(X,S)}.$$

Then from the functorial properties of the projective cover of connected compact groups there is a commutative diagram

$$\begin{array}{ccccc} \mathfrak{P}(\Phi_S(X)) & \xrightarrow{\mathfrak{P}(\rho_S)} & \mathfrak{P}(S^{A(X,S)}) & \xrightarrow{\alpha} & S_{[\mathfrak{s}]}^{A(X,S)} \\ {\scriptstyle E_{\Phi_S(X)}}\big\downarrow & & {\scriptstyle E_{S^{A(X,S)}}}\big\downarrow & & \big\downarrow{\scriptstyle f_S^{A(X,S)}} \\ \Phi_S(X) & \xrightarrow{\rho_S} & S^{A(X,S)} & = & S^{A(X,S)} \end{array}$$

with an isomorphism α. Since ρ_S is surjective so is $\mathfrak{P}(\rho_S)$; since ρ_S has a totally disconnected kernel, $\mathfrak{P}(\rho_S)$ is injective and thus is an isomorphism (see 9.73). Therefore, we have a sandwich situation

$$S_{[\mathfrak{s}]}^{A(X,S)} \xrightarrow{E_{\Phi_S(X)}\circ\mathfrak{P}(\rho_S)^{-1}\circ\alpha^{-1}} \Phi_S(X) \xrightarrow{\rho_S} S^{A(X,S)}.$$

It remains to show that $\Phi_S(X)$ is fully characteristic in F_0X and therefore normal in FX.

We write $F_0X = (FX)_0$ and $F_0'X \stackrel{\rm def}{=} \big((FX)_0\big)'$. Let $\mathfrak{P}(F_0'X) = \widetilde{F_0'X} = \prod_{j\in J} S_j$ with simply connected simple compact Lie groups S_j according to Theorem 9.19(ii) and $E_{F_0X}\colon \mathfrak{P}(F_0X) = \mathfrak{P}\big(Z_0(F_0X)\big)\times\widetilde{F_0'X} \to F_0X$ the projective cover of Theorem 11.14(ii). We have

$$\mathfrak{P}(F_0X) = \mathfrak{P}\big(\Phi_S(X)\big) \times K, \quad K = \ker E_{F_0X} \circ (\kappa_S|F_0)$$

and indeed this decomposition defines $\Phi_S(X)$ in the first place as projection of the first factor in this decomposition. We notice that $E_{F_0X}(K) = \ker(\kappa_S|F_X) = F_0X \cap \kappa_S$. Thus the subgroup $E_{F_0X}(K)$ of F_0X is normal in FX, i.e. it is invariant under all automorphisms α of F_0X which are restrictions $I_g|F_0X$ of inner automorphisms I_g of FX, $g \in FX$. In the projective group $\mathfrak{P}(F_0X)$, however, any automorphism which fixes the factor K as a whole fixes the unique normal complementary factor $\mathfrak{P}\big(\Phi_S(X)\big)$ as a whole (cf. 9.50). This applies, in particular, to the automorphism $\mathfrak{P}(\alpha)$ of $\mathfrak{P}(F_0X)$. But then $\mathfrak{P}(\alpha)\big(\mathfrak{P}\big(\Phi_S(X)\big)\big) = \mathfrak{P}\big(\Phi_S(X)\big)$ implies $g\Phi_S(X)g^{-1} = \alpha\big(\Phi_S(X)\big) = \Phi_S(X)$ for all $g \in G$ and thus $\Phi_S(X)$ is normal in FX. □

We recall from 9.19 that $\mathcal{S}$ describes a set of representatives of all compact simple Lie algebras and that every compact connected semisimple group G determines for each $\mathfrak{s}$ a unique cardinal number $\aleph(\mathfrak{s}, G)$ such that $\mathfrak{P}(S) = \widetilde{G} = G \cong \prod_{\mathfrak{s}\in\mathcal{S}} S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},G)}$ where $S_{[\mathfrak{s}]}$ is a representative of the class of isomorphic simply connected simple groups with Lie algebra isomorphic to $\mathfrak{s}$. By Theorem 9.19(iv), and Theorem 11.14(ii), the projective cover of the identity component F_0X of the free compact group FX on a compact pointed space X is

$$\text{(P)} \qquad \mathfrak{P}F_0X \cong Z_0(F_0X) \times \prod_{\mathfrak{s}\in\mathcal{S}} S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},F_0'X)}.$$

Thus the structure of $\mathfrak{P}F_0X$ is known if the cardinals $\aleph(\mathfrak{s}, F_0'X)$ are known.

The projective cover $S_{[\mathfrak{s}]}^{A(S,X)}$, $\mathfrak{s} \cong \mathfrak{L}(S)$ of $\Phi_S(X)$ may be identified with a subproduct of $\mathfrak{P}_{\mathfrak{s}}(FX) \cong S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},F_0'X)}$. (cf. 9.73(i)). Hence

$$\operatorname{card} A(S,X) = w(S_{[\mathfrak{s}]}^{A(X,S)}) \le w(S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},F_0'X)}) = \aleph(\mathfrak{s}, F_0'X).$$

Also, $w(S_[\mathfrak{s}]^{\aleph(\mathfrak{s},F_0'X)}) = w\big(\big(S_{[\mathfrak{s}]}/Z(S_{[\mathfrak{s}]})\big)^{\aleph(\mathfrak{s},F_0'X)}\big) \le w(F_0'X)$ (9.20), and $w(F_0'X) \le w(FX) = w(X)^{\aleph_0}$ by 11.6(iii). Hence $\operatorname{card} A(X,S) \le \aleph(\mathfrak{s}, F_0'X) \le w(X)^{\aleph_0}$. Thus we have

Corollary 11.50. *In the circumstances of Theorem* 11.49 *the common projective cover of* $\Phi_S(X)$ *and* F_SX *is isomorphic to* $S_{[\mathfrak{s}]}^{A(X,S)}$. *In particular,* $\operatorname{card} A(X,S) \le \aleph(\mathfrak{s}, FX) \le w(X)^{\aleph_0}$ *for* $\mathfrak{s} \cong \mathfrak{L}(S)$. *If* X *has at least* 4 *points then* $\aleph(\mathfrak{s}, F_0'X) = w(X)^{\aleph_0}$. □

The group S is simply connected if and only if $f_S\colon S_{[\mathfrak{s}]} \to S$ is an isomorphism if and only if $f_S\colon S_{[\mathfrak{s}]}^{A(X,S)} \to S^{A(X,S)}$ is an isomorphism. Then Theorem 11.49(ii) yields at once the following corollary.

Corollary 11.51. *In the circumstances of Theorem* 11.49, *if* S *is simply connected, then* $\Phi_S(X) \cong S^{A(X,S)} = F_SX \cong \mathfrak{P}(F_SX)$. □

The group S is centerfree if and only if $f_S\colon S_{[\mathfrak{s}]} \to S$ has kernel $Z(S_{[\mathfrak{s}]})$ (cf. 26(ii)) if and only if $f_S\colon S_{[\mathfrak{s}]}^{A(X,S)} \to S^{A(X,S)}$ has kernel $Z(S_{[\mathfrak{s}]})^{A(X,S)}$. Assuming that S is centerfree, let $F_{\mathfrak{s}}X$ denote the isotypic component of $F_0'X$ of type $\mathfrak{s}$ according to Theorem 9.19 and a subsequent Definition. Then there is a sandwich situation

$$S_{[\mathfrak{s}]}^{\aleph(\mathfrak{s},F_0'X)} = \mathfrak{P}(F_{\mathfrak{s}}X) \xrightarrow{E_{F_{\mathfrak{s}}X}} F_{\mathfrak{s}}X \xrightarrow{\omega_{\mathfrak{s}}'} S^{\aleph(\mathfrak{s},F_0'X)}$$

according to 9.19(vi) or 9.20, and $\omega_{\mathfrak{s}}'$ induces an isomorphism $q_s\colon F_{\mathfrak{s}}X/Z(F_{\mathfrak{s}}X) \to S^{\aleph(\mathfrak{s},F_0'X)}$.

Theorem 11.52. *Assume that* X *is connected. If* S *is a centerfree connected simple compact Lie group then the restriction of* $\kappa_S\colon FX \to F_SX$ *to* $F_{\mathfrak{s}}X$ *induces* *an isomorphism $F_{\mathfrak{s}}X/Z(F_{\mathfrak{s}}X) \cong F_SX$; that is we have a sandwich situation*

$$S_{[\mathfrak{s}]}^{w(X)^{\aleph_0}} = \mathfrak{P}(F_{\mathfrak{s}}X) \xrightarrow{E_{F_{\mathfrak{s}}X}} F_{\mathfrak{s}}X \to F_SX \cong S^{w(X)^{\aleph_0}},$$

i.e. two surjective morphisms whose composition is $f_S^{w(X)^{\aleph_0}}$.

Proof. The center $Z(F_{\mathfrak{s}}X)$ is characteristic, hence normal in FX and thus is central in FX since it is totally disconnected (cf. 6.13). Therefore $Z(F_{\mathfrak{s}}X) = F_{\mathfrak{s}}X \cap Z(FX)$. By 9.74 there is a compact central subgroup $K_{\mathfrak{s}}$ of $F_{\mathfrak{s}}X$ such that there is a surjective morphism $FX \to F_{\mathfrak{s}}X/K_{\mathfrak{s}}$. Hence there is a surjective morphism $r_{\mathfrak{s}}: FX \to F_{\mathfrak{s}}X/Z(F_{\mathfrak{s}}X)$ such that $r_{\mathfrak{s}}|F_{\mathfrak{s}}X: F_{\mathfrak{s}}X \to F_{\mathfrak{s}}X/Z(F_{\mathfrak{s}}X)$ is the quotient map. This yields, via 11.1, an essential continuous map

$$X \xrightarrow{\varepsilon_X} FX \xrightarrow{q_{\mathfrak{s}}r_{\mathfrak{s}}} S^{\aleph(\mathfrak{s},F'X)}.$$

Hence the universal property of F_SX gives us a natural surjective morphism $\alpha_{\mathfrak{s}}: F_SX \to S^{\aleph(\mathfrak{s},F'X)}$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{\varepsilon_X^{(S)}} & F_SX & = & S^{A(X,S)} \\ {\scriptstyle\varepsilon_X}\downarrow & & \downarrow{\scriptstyle\alpha_{\mathfrak{s}}} & & \\ FX & \xrightarrow[r_{\mathfrak{s}}]{} & F_{\mathfrak{s}}X/Z(F_{\mathfrak{s}}X) & \xrightarrow[q_{\mathfrak{s}}]{} & S^{\aleph(X,F'X)}. \end{array}$$

The morphism $\kappa_S: FX \to F_SX = S^{A(X,S)}$ of 11.49 annihilates the center $Z(FX)$ (since $S^{A(X,S)}$ is centerfree) and all $F_{\mathfrak{t}}X$ with $\mathfrak{s} \neq \mathfrak{t} \in \mathcal{S}$. Thus, by 9.19 and 9.24, $\kappa_S|F_{\mathfrak{s}}X: F_{\mathfrak{s}}X \to F_SX$ is surjective and factors through a surjective morphism $\beta_{\mathfrak{s}}: F_{\mathfrak{s}}X/Z(F_sX) \to F_SX$ such that the following diagram is commutative:

$$\begin{array}{ccc} FX & \xrightarrow{q_{\mathfrak{s}}} & F_{\mathfrak{s}}/Z(F_{\mathfrak{s}}X) \\ {\scriptstyle\kappa_S}\downarrow & & {\scriptstyle\beta_{\mathfrak{s}}}\downarrow \\ F_SX & \xrightarrow[\mathrm{id}_{F_SX}]{} & F_SX. \end{array}$$

Since $\varepsilon_X^{(S)} = \kappa_S \circ \varepsilon_X$ the two diagrams taken together yield $\varepsilon_X^{(S)} = \beta_{\mathfrak{s}}\alpha_{\mathfrak{s}}\varepsilon_X^{(S)}$. The uniqueness in 11.22 gives $\mathrm{id}_{F_SX} = \beta_{\mathfrak{s}}\alpha_{\mathfrak{s}}$. The surjectivity of α then shows $\beta_{\mathfrak{s}} = \alpha_{\mathfrak{s}}^{-1}$. □

Corollary 11.53. *If X is connected compact and S is centerfree then $\Phi_S(X) = F_{\mathfrak{s}}X$.*

Proof. By Theorem 11.52 the restriction $\kappa' \overset{\text{def}}{=} (\kappa_S|F_{\mathfrak{s}}X): F_{\mathfrak{s}}X \to F_SX$ has the kernel $Z(F_{\mathfrak{s}}X)$. By Theorem 11.49, $\kappa'\big(\Phi_S(X)\big) = F_SX$. The groups $F_{\mathfrak{s}}X$ and $\Phi_S(X) \le F_SX$ are connected and $F_{\mathfrak{s}}X = \Phi_S(X)Z(F_{\mathfrak{s}}X)$. Since $Z(F_{\mathfrak{s}}X)$ is totally disconnected we conclude $\Phi_S(X) = (F_{\mathfrak{s}}X)_0 = F_SX$. □

Theorem 11.54. *Let X be a compact pointed space with at least 4 points, then the projective cover $\mathfrak{P}(F_0X)$ of the identity component F_0X of the free compact* *group on X is isomorphic to*

$$\left(\widehat{\mathbb{Q}} \times \prod_{\mathfrak{s}\in\mathcal{S}} S_{[\mathfrak{s}]}\right)^{w(X)^{\aleph_0}}.$$

The projective cover $\mathfrak{P}(F_0'X)$ of the commutator group $F_0'X = \big((FX)_0\big)'$ of the identity component of the free compact group on X is isomorphic to

$$\left(\prod_{\mathfrak{s}\in\mathcal{S}} S_{[\mathfrak{s}]}\right)^{w(X)^{\aleph_0}}.$$

Proof. By 9.73(i) we have

$$\mathfrak{P}(F_0X) = \mathfrak{P}\big(Z_0(FX)\big) \times \prod_{\mathfrak{s}\in\mathcal{S}} \mathfrak{P}_{\mathfrak{s}}(FX).$$

By 11.15, the group $Z_0(FX)$ is projective and thus agrees with $\mathfrak{P}\big(Z_0(FX)\big)$. From 11.16(iv) and duality we conclude $\mathfrak{P}\big(Z_0(FX)\big) \cong \mathbb{Q}^{w(X)^{\aleph_0}}$. From Corollary 11.50 we derive $\mathfrak{P}_{\mathfrak{s}}(FX) = \mathfrak{P}(F_{\mathfrak{s}}X) \cong S_{[\mathfrak{s}]}^{w(X)^{\aleph_0}}$. This completes the proof of the theorem, since by the definition of the projective cover we have $\big(\mathfrak{P}(G)\big)' = \mathfrak{P}(G')$. ☐

Freeness Versus Projectivity

We have noted the relationship between freeness and projectivity in the category of abelian groups in Appendix 1 and analyzed the more complex situation in the category of compact abelian groups in Chapter 8. Let us carefully review the issue of projectivity once more in category theoretical terms. Assume that $U\colon \mathcal{C} \to \mathcal{B}$ is a faithful "grounding functor." In our concrete situation $\mathcal{C}$ would be the category of [abelian] groups and $\mathcal{B}$ the category of sets with the forgetful functor U. We say that a morphism $f\colon A \to B$ in $\mathcal{C}$ is *$\mathcal{B}$-split* if there is a morphism $\sigma\colon UB \to UA$ in $\mathcal{B}$ such that $(Uf)\sigma = \mathrm{id}_{UB}$. Trivially, the retraction Uf is epic and since U is faithful, f is epic. Let $\mathcal{E}$ denote the class of $\mathcal{B}$-split morphisms. If $\mathcal{C}$ is the category of [abelian] groups and $\mathcal{B}$ the category of sets, and if the axiom of choice applies, then *every* surjective morphism is $\mathcal{B}$-split. Now the general background for our initial discussion is the following:

Proposition 11.55. *Assume that $F\colon \mathcal{B} \to \mathcal{C}$ is left adjoint. Then FX is $\mathcal{E}$-projective for the class $\mathcal{E}$ of $\mathcal{B}$-split morphisms.*

Proof. Let $\eta_X\colon X \to UFX$ denote the front adjunction and set

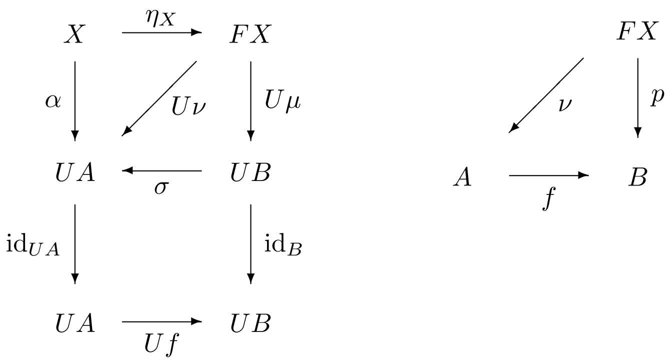
$$\alpha = \sigma(U\mu)\eta_X\colon X \to UA.$$

The universal property of adjoints yields a unique morphism $\nu\colon FX \to A$ such that $\alpha = (U\nu)\eta_X$. Now we have

$$(U\mu)\eta_X = (Uf)\sigma(U\mu)\eta_X = (Uf)\alpha = (Uf)(U\nu)\eta_X = U(f\nu)\eta_X.$$

The uniqueness in the universal property of the adjoint now yields $\mu = f\nu$. This proves the proposition. □

The following diagrams may be helpful:



We note that *retracts of $\mathcal{E}$-projectives are $\mathcal{E}$-projective.* Indeed if we have morphisms $\pi\colon P \to Q$ and $\rho\colon Q \to P$ with $\pi\rho = \mathrm{id}_Q$, and if P is $\mathcal{E}$-projective, then Q is also an $\mathcal{E}$- projective; for if $\mu'\colon Q \to B$ is given, we set $\mu = \mu'\pi$, apply Proposition 10.44 to get a $\nu\colon FX \to A$ with $f\nu = \mu = \mu'\pi$ and obtain $\nu' = \nu\rho\colon Q \to B$ such that $f\nu' = \mu'\pi\rho = \mu'$. It then follows from Proposition 11.44 that any retract of a free $\mathcal{C}$-object FX is an $\mathcal{E}$-projective. On the other hand, by a fundamental property of adjoints, for every object C of $\mathcal{C}$ we have the back adjunction $\varepsilon_C\colon FUC \to C$ which is $\mathcal{E}$-split since $(U\varepsilon_C)\eta_{UC} = \mathrm{id}_{UC}$. Hence if C is an $\mathcal{E}$-projective, then there is a $\nu\colon C \to FUC$ such that $\varepsilon_C\nu = \mathrm{id}_C$. Thus every $\mathcal{E}$-projective is a retract of a free object. We summarize:

Corollary 11.56. *Let $F\colon \mathcal{B} \to \mathcal{C}$ denote a left adjoint of a faithful grounding functor $U\colon \mathcal{C} \to \mathcal{B}$. An object P of the category $\mathcal{C}$ is an $\mathcal{E}$-projective for the class of $\mathcal{B}$-split epics if and only if it is a retract of a free object FX.* □

We are interested in the following application: Let $\mathcal{C} = \mathbb{CG}$ denote the category of compact groups and $\mathcal{B}$ the category of pointed spaces and base point preserving maps. Then the $\mathcal{B}$-split morphisms $f\colon A \to B$ are exactly the split morphisms of compact groups as introduced in Definition 10.41.

After these remarks, for compact groups, we have to consider two kinds of projectives: (i) The $\mathcal{E}$-projectives for the class of topologically split morphisms, and (ii) the projectives, period.

We have a pretty good idea of the latter after Chapter 9 (see in particular 9.70ff.). Our understanding of the $\mathcal{E}$-projectives, however, depends in large measure on our understanding of topologically split morphisms. Thus category theoretical considerations led to an internal problem on the structure of compact groups and their morphisms, namely, the study of topologically split and split morphisms which we presented in Chapter 10 on compact group actions and cross sections. We shall now exploit the insights gathered there.

Theorem 11.57. *Every compact connected abelian group is an $\mathcal{E}$-projective for the class of topologically split epics in the category $\mathbb{CG}$ of compact groups. In particular, every such group is a homomorphic retract of a free compact abelian group.*

Proof. Assume that P is any compact connected abelian group and $f: P \to B$ a morphism of compact groups. Let $e: A \to B$ be a topologically split morphism. Then $e_1: e^{-1}\big(f(P)\big) \to f(P)$ is a topologically split morphism by Lemma 10.52. Hence it splits by Theorem 10.53(i). If $s: f(P) \to e^{-1}\big(f(P)\big)$ is a homomorphic cross section for e_1, then $F: P \to A$, $F(g) = s\big(f(p)\big)$ satisfies $ef = f$. This proves that P is an $\mathcal{E}$-projective. The last assertion is a consequence of Corollary 11.56.□

Proposition 11.58. *There is a connected compact Lie group L with $L/L' \cong \mathbb{T}$ which is not $\mathcal{E}$-projective for the class of topologically split morphisms.*

Proof. We let G be the group of Example 10.48 and N compact connected normal subgroup of 10.48(ii). Then the quotient morphism $f: G \to L$ splits topologically but not algebraically. Hence the group $L \overset{\text{def}}{=} G/N$ is not an $\mathcal{E}$-projective since an $\mathcal{E}$-morphism onto an $\mathcal{E}$-projective splits algebraically. □

Proposition 11.59. *Let G be simple connected but not simply connected compact Lie group which is not isomorphic to* $\mathrm{SO}(3)$*,* $\mathrm{PSU}(3)$*,* E_6/Z*,* $\mathrm{SO}(2m)$ *or a double covering of* $\mathrm{SO}(2m)$*. Then G is not $\mathcal{E}$-projective for the class of topologically split morphisms.*

Proof. By Example 10.58 there is a topologically split morphism $f: A \to G$ with A locally isomorphic to G^2 such that f does not split. If G were $\mathcal{E}$-projective, then f would have to be split, which it is not. □

It is an open question whether an $\mathcal{E}$-projective simple compact Lie group *has to* be simply connected. This is still possible after 11.59. But our methods in 10.58, notably the choice of $\kappa: L \to L$, $\kappa(g) = g^a$, will not carry us further than stated in Example 10.58 and Proposition 11.59 above.

Postscript

In this chapter we have presented the structure theory of free compact groups which has been developed since 1979 when the authors of this book began their joint study of these groups. Much of this material has appeared in the papers [178], [179], [180], [182] and [183].

While we presented, in Chapter 8, a complete description of the structure of free compact abelian groups, the state of knowledge of the structure of free compact groups is not complete. Nevertheless the known structural results are substantial and some are surprising.

The Schreier subgroup theorem for (algebraic) free groups says that every subgroup of a free group is a free group. Hence the free group on an infinite set X is never isomorphic to the direct product of the free abelian group on X and the commutator subgroup of the free group. So it is indeed surprising and interesting to find that an analog is true for free compact groups. The Free Compact Group Direct Product Theorem 11.19 says that for any compact connected pointed space X the free compact group is the direct product of the free compact abelian group $F_{\text{ab}}X$ and its commutator subgroup $F'X$ if and only if $H^1(X,\mathbb{Z})$ is divisible. So, for example if $H^1(X,\mathbb{Z}) = \{0\}$ or more particularly if X is a contractible space, then FX is topologically isomorphic to $F_{\text{ab}}X \times F'X$.

The Component Splitting Theorem 11.14 is another surprising result. It says that the identity component F_0X of any free compact group FX is a semidirect factor of FX. And the Center Theorem 11.14 says that if $\operatorname{card} X \neq 2$ then the center $Z(FX)$ of FX is contained in the identity component F_0X of FX.

Corollary 11.15 says that for any compact space X, the identity component $Z_0(FX)$ of the center of FX is naturally isomorphic to the projective cover of $F_{\text{ab}}X$, that is to $(\mathbb{Q} \otimes \widehat{F_{\text{ab}}X})\widehat{\ }$. Theorem 11.54 describes the projective cover of the identity component of FX.

Theorem 11.17 shows that if X is a compact connected space the weight $w(FX)$ of FX equals $(w(X))^{\aleph_0}$.

For *connected compact groups* there is a certain tendency for topologically split morphisms to be split. This is correct (although nontrivial) for compact abelian groups. As a consequence, we saw that *every* compact connected abelian group is an $\mathcal{E}$-projective in the category of compact groups. This is in contrast with the rather special structure of the abelian connected projectives which are exactly the duals of rational vector groups. In the context of free compact groups, $\mathcal{E}$-projectivity of all compact connected abelian groups implies that every such group is a homomorphic retract of some free compact group according to Corollary 11.57. For semisimple compact Lie groups there are topologically split morphisms which are not split. The examples discussed in Chapter 10 are rather instructive. Perhaps they tell the story even better than the theorems. The obstruction seems to be in the first homotopy, but this is not sufficient to explain everything. In particular we saw that for a simple connected compact Lie group G to be $\mathcal{E}$-projective it is necessary that the fundamental group $\pi_1(G)$ be trivial or has exponent 2 or 3. In particular, this means that a simple connected compact Lie group which is not simply connected and whose fundamental group does not have exponent 2 or 3 cannot be a homomorphic retract of a free compact group.

References for this Chapter—Additional Reading

[38], [174], [178], [179], [180], [182], [183], [254], [256], [379].

Chapter 12

Cardinal Invariants of Compact Groups

Compact groups may be very large. What this means precisely can be described only in terms of cardinal invariants. We have encountered several cardinal invariants associated with compact groups; examples are the weight $w(G)$, first introduced for locally compact abelian groups in Chapter 7, notably 7.75 (see also Appendix 4, A4.7ff.), and the (fine) dimension $\dim(G)$, first discussed in Chapter 8 for compact abelian groups and then for arbitrary compact groups in Chapter 9, notably 9.53ff. The theory of free compact groups allows us now to pursue the topic of cardinal invariants further. None of these invariants is restricted to compact groups; they extend, notably, to locally compact groups. But on the class of compact groups we are in a position to be very explicit.

Prerequisites. This chapter rests, in spirit and content, on Chapter 11 and therefore invokes the prerequisites for that chapter. In one place (the proof of Lemma 12.10) we use a theorem of Varopoulos [361], for whose proof we refer to the original source.

Suitable sets

Definition 12.1. A subset X of a topological group is called *suitable* if
(i) G is the smallest closed subgroup of G containing X; that is, X topologically generates G.
(ii) The identity element $1 \notin X$ and X is discrete and closed in $G \setminus \{1\}$; that is 1 is the only possible accumulation point of X in G. □

Observe that in a compact group, if X is suitable, then $X \cup \{1\}$ is compact. Indeed, $X \cup \{1\}$ is the one-point-compactification of the discrete space X.

The result that every compact group has a suitable set, is not a priori clear, but its proof is our first main objective.

The following simple remark will be helpful.

Lemma 12.2. *Let G be a Hausdorff space and A a closed subset. For a relatively compact subset X of G contained in $G \setminus A$ the following conditions are equivalent:*
(1) *X is discrete and closed in $G \setminus A$.*
(2) *For each open subset U of G containing A the set $X \setminus U$ is finite.*

Proof. Exercise E12.1. □

Exercise E12.1. Prove Lemma 12.2. □

Lemma 12.3. *If G is a Hausdorff topological group which is the product NH of two subgroups N and H each of which has a suitable set, then G has a suitable set.*

Proof. If X and Y are suitable sets of N and H, respectively, then $X \cup Y$ is discrete and closed in $G \setminus \{1\}$ and generates G topologically, hence is a suitable set of G. □

Of course, this lemma generalizes to the case of any finite number of subgroups.

Lemma 12.4. *Let $f: G \to H$ be a morphism of topological groups with dense image. If G has a relatively compact suitable set X, then H has a suitable set $f(X) \setminus \{1\}$.*

Proof. Let X be a relatively compact suitable set of G and set $Y = f(X) \setminus \{1\}$. Since X topologically generates G and $f(G)$ is dense in H, the subset Y topologically generates H. Also, since $X \cup \{1\}$ is compact, the set $Y \cup \{1\}$ is compact. In order to show that Y is suitable, we verify condition (2) of Lemma 12.2. Let V denote an open neighborhood of 1 in H. Then $U \stackrel{\text{def}}{=} f^{-1}(V)$ is an open neighborhood of 1 in G. Now $f(G \setminus U) = H \setminus V$. But $X \setminus U$ is finite as X is suitable, and so $Y \setminus V = f(X \setminus U)$ is finite. Hence Y is suitable. □

Lemma 12.5. *Let $f: G \to H$ be a surjective morphism of compact groups and Y a suitable set of H. Let $C = f^{-1}(\overline{Y})$ and note $\overline{Y} = Y \cup 1_H$. Then the function $f|C: C \to \overline{Y}$ has a continuous base point preserving cross section $\sigma: \overline{Y} \to C$, i.e. a continuous function such that $f\big(\sigma(y)\big) = y$ for all $y \in Y$ and $\sigma(1_H) = 1_G$. The set $X \stackrel{\text{def}}{=} \sigma(Y)$ is discrete and closed in $G \setminus \{1\}$.*

Proof. The compact group $K \stackrel{\text{def}}{=} \ker f$ acts on the compact space C via $(k, c) \mapsto kc$. Let $\pi: C \to C/K \stackrel{\text{def}}{=} \{Kc \mid c \in C\}$ be the orbit map. The map $f_C: C/K \to \overline{Y}$, $f_C(cK) = f(c)$ is a unique homeomorphism such that $f|C = f_C \circ \pi$. The space $\overline{Y}$ is compact and totally disconnected; indeed the only accumulation point is 1_H. Now we apply the Global Cross Section Theorem for Totally Disconnected Base Spaces 10.35 and find the desired cross section $\sigma: \overline{Y} \to C$ for which we may assume $\sigma(1_H) = 1_G$ (for if not, then $y \mapsto \sigma(1_H)^{-1}\sigma(y): \overline{Y} \to C$ is a cross section of the desired kind).

Then $X = \sigma(Y)$ is contained in $G \setminus K \subseteq G \setminus \{1_G\}$ and since $\overline{X} = X \cup \{1\}$ is homeomorphic to $\overline{Y}$ under the map $f|\overline{X}: \overline{X} \to \overline{Y}$ (whose inverse is the corestriction of σ to its image), X is closed and discrete in $G \setminus \{1_G\}$. □

Proposition 12.6. *Let $f: G \to H$ be a surjective morphism of compact groups and assume that G is connected and* $\ker f$ *is totally disconnected. Then for each* *suitable set Y in H there is a suitable set X in G such that $f|\overline{X}\colon\overline{X}\to\overline{Y}$ is a homeomorphism.*

Proof. Let Y be a suitable set of H. By Lemma 12.5, there is a subset X of $G\setminus\{1\}$ which has 1 as its only point of accumulation and is such that $f|\overline{X}\colon\overline{X}\to\overline{Y}$ is a homeomorphism. We claim that X is suitable in G, i.e. that $G_1\stackrel{\rm def}{=}\overline{\langle X\rangle}$ is G. Now $f(G_1)=\overline{\langle f(X)\rangle}=\overline{\langle Y\rangle}=H$. We write $D\stackrel{\rm def}{=}\ker f$ and note that $f(G_1)=H$ implies $G=G_1D$. But $G/G_1\cong D/(G_1\cap D)$ is totally disconnected as a quotient of a totally disconnected compact group. On the other hand, since G is connected, G/G_1 is connected. Hence G/G_1 is singleton, i.e. $G_1=G$. □

Lemma 12.7. *Every direct product of any family of topological groups with suitable sets has a suitable set.*

Proof. Let $\{G_j\mid j\in J\}$ be a family of topological groups, each with a suitable set X_j and let P be the direct product of the G_j. Now let $e_j\colon G_j\to P$ be defined by

$$e_j(g)=(g_k)_{k\in J},\quad g_k=\begin{cases}g & \text{if } k=j,\\ 1 & \text{otherwise.}\end{cases}$$

Then $X=\bigcup_{j\in J}e_j(X_j)$ is the required suitable set of P as is readily verified with the aid of Lemma 12.2. □

Lemma 12.8. *Every connected compact group G has a suitable set.*

Proof. Let $E_G\colon\mathfrak{P}(G)\to G$ be the projective covering morphism according to 9.72. If $\mathfrak{P}(G)$ has a suitable set, then G has a suitable set by Lemma 12.4. Thus we may assume that G is projective in $\mathbb{CN}$. Then $G=\widehat{\mathbb{Q}}^I\times\prod_{j\in J}S_j$ with a set I and a family of simple simply connected compact Lie groups S_j by 9.70(i). The group $\widehat{\mathbb{Q}}$ is monothetic by 8.75 and therefore has a suitable one element set. Each S_j is generated by two elements by 6.51 and thus has a two element suitable set. Now Lemma 12.7 shows that G has a suitable set. □

By Lee's Theorem 9.41, any compact group is the product of G_0 and some totally disconnected subgroup D. Then, in view of Lemmas 12.3 and 12.8, in order to show that every compact group has a suitable set it now suffices to show that the claim is true for totally disconnected groups.

Therefore we now have to deal with the case of totally disconnected compact groups. Douady [91] reports on a proof of Tate that every compact totally disconnected group has a suitable set. This proof is extremely condensed. We give here a different proof which depends on the Countable Layer Theorem 9.91

Lemma 12.9. *Assume that G is compact and that there is a descending series of compact groups $G=G_1\rhd G_2\rhd\cdots\rhd G_n\rhd\cdots$ such that*

(i) $\bigcap G_n=\{1\}$.
(ii) *For each $n=1,2,\dots,$ the quotient group G_n/G_{n+1} has a suitable set.*

(iii) *For each* $n = 1, 2, \ldots,$ *there is a compact subspace* $Y_n \subseteq G_n$ *containing* 1 *such that* $(y, g) \mapsto yg : Y_n \times G_{n+1} \longrightarrow G_n$ *is a homeomorphism.*

Then G *has a suitable set.*

Proof. For $n = 1, 2, \ldots$ let $X_n \subseteq Y_n$ be such that $(X_n G_{n+1})/G_{n+1}$ is suitable in G_n/G_{n+1}. Then for every $x \in X_n$, the set xG_{n+1} is isolated in $X_n G_{n+1}/G_{n+1}$, hence in G/G_{n+1}, and so $\{x\}$ is isolated in X_n. Moreover, if $g \in \overline{X_n} \setminus X_n$, then $gG_{n+1} \in (X_n G_{n+1} \cup G_{n+1})/G_{n+1}$, whence $g \in X_n G_{n+1} \cup G_{n+1}$. But since each point of X_n is isolated, we may conclude that $g \in X_n \cup G_{n+1}$. Because $g \notin X_n$, we finally have $g \in G_{n+1}$. On the other hand, we know $g \in \overline{X_n} \subseteq Y_n$ and thus $g \in Y_n \cap G_{n+1} = \{1\}$. Therefore $g = 1$ and X_n is discrete in $G \setminus \{1\}$. Now the set $Z_n = X_1 \cup \cdots \cup X_n$ is discrete in G. We set $X = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} Z_n$. Let $U \subseteq G$ be open and contain 1. Since $\bigcap_{n=1}^{\infty} G_n = \{1\}$, by the compactness of G, we find an n such that $G_{n+1} \subseteq U$. Now $X \setminus U \subseteq (Z_n \cup G_{n+1}) \setminus U = Z_n \setminus U$ is finite. Hence X is discrete $G \setminus \{1\}$ by Lemma 1.2. Finally, we claim that X topologically generates G. Indeed, let H be the closed subgroup topologically generated by X. Further, let N be an arbitrary compact normal subgroup such that G/N is a Lie group. Since Lie groups satisfy the descending chain condition, we find an n such that $G_{n+1} \subseteq N$. Since $X_k G_{k+1}/G_{k+1}$ topologically generates G_k/G_{k+1}, we conclude that $G_{n-j}/N \subseteq HN/N,\ j = 1, \ldots, n-1$ and thus that $G \subseteq HN$. Since N was arbitrary and $G = \lim G/N$, we have $G = H$ as we had claimed. □

Lemma 12.10. *Every totally disconnected compact group* G *has a suitable set.*

Proof. By the Countable Layer Theorem 9.91 there is a sequence $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright \cdots$ with $\bigcap G_n = \{1\}$ and such that G_n/G_{n+1} is a direct product of finite simple groups. Hence G_n/G_{n+1} has a suitable set by Lemma 12.7. Condition 12.9(iii) is satisfied by 10.37. The assertion now follows from the preceding Lemma 12.9. □

Suitable Set Theorem

Theorem 12.11. *Every compact group* G *has a suitable set.*

Proof. By Lee's Theorem 9.41 there is a compact totally disconnected subgroup D in G such that $G = G_0 D$. The assertion now follows from Lemmas 12.10, 12.8, and 12.3. □

An alternative proof would have been to use 12.8, 12.10 and the Countable Layer Theorem 9.91 directly. Further alternative proofs not using the structure theory of compact groups were given by Dikranjan and Shakhmatov, see [84] and [326].

Our concern in this book is with compact groups. However, we mention that it is also known that every locally compact group has a suitable set [181] as does every metrizable topological group and every countable topological group

[73]. But not all topological groups have suitable sets (see [73] and also work of D. Dikranjan, O. G. Okunev, M. Tkachenko and V. V. Tkachuk).

The following proposition shows that, for compact connected groups, the question of suitable sets is largely an issue of suitable sets of compact connected abelian groups.

Proposition 12.12. *Let G be a compact connected group and T a maximal pro-torus in G. If X is a suitable set of T, then there is a $g \in G$ such that $X \cup \{g\}$ is a suitable set of G. If $w(G) \leq 2^{\aleph_0}$, then we may assume* $\operatorname{card} X = 1$.

Proof. From Proposition 9.38 it follows that there is a $g \in G$ such that $G = \overline{\langle T \cup \{g\} \rangle}$. Since $X \cup \{g\}$ is discrete and closed in $G \setminus \{1\}$, the assertion follows.□

Definition 12.13. For an infinite cardinal n let D_n denote a discrete space of cardinality n and let n^* denote the pointed compact space obtained as the one point compactification of D_n, the base point being ∞. For a finite number $n = 1, 2 \ldots$ let $D_n = \{1, \ldots, n, \infty\}$ with base point ∞ and set $D_0 = \{\infty\}$. Let $\beta^* n$ for infinite n be the Stone–Čech compactification of D_n, the base point being a point in D_n and for finite n the space $n^* = \{1, \ldots, n, \infty\}$ with base point ∞. We define $F[n] = F(n^*)$ and $F[[n]] = F(\beta^* n)$, the free compact group on the pointed space n^*, respectively, $\beta^* n$. For a finite cardinal $n = 0, 1, 2, \ldots$ we set $F[n] = F[[n]] \cong F\{0, 1, 2, \ldots, n\}$ (with 0 as base point). We let $F_0[n]$ denote the identity component of $F[n]$ and $F_{\rm ab}[n]$ the free compact abelian group $F[n]/F'[n]$ on n^*. The compact groups $F[[n]]$, $F_0[[n]]$, $F_{\rm ab}[[n]]$ are defined analogously. □

We note $F[0] = F[[0]] = \{1\}$, $F[1] = F[[1]] = C_0(\{\infty, 0\}, \mathbb{T})_d\widehat{\ } = \widehat{\mathbb{T}_d}$ the universal monothetic group.

Corollary 12.14. *For every compact group G there are cardinals m and n for which there are surjective morphisms $p: F[m] \to G$ and $q: F[[n]] \to G$. If the group G is connected then $G = p(F_0[m]) = q(F_0[[n]])$. If it also satisfies $w(G) \leq 2^{\aleph_0}$ then there is a surjective morphism $p: F_0[2] \to G$.*

Proof. (a) By Theorem 12.13, G has a suitable set X. By its very definition, there is a cardinal $m = \operatorname{card} X$ and a continuous map of pointed spaces $p_0: m^* \to X \cup \{1\}$. Now by the universal property of the free compact group $F[m]$, there is a unique morphism of compact groups $p: F[m] \to G$ such that $p|m^* = p_0$. The image $p(F[m])$ is a closed subgroup of G containing X, but since $G = \overline{\langle X \rangle}$ we have $G = p(F[m])$. Thus p is surjective. From 9.18 we know that $G_0 = f(F_0[m])$. If G is connected and satisfies $w(G) \leq 2^{\aleph_0}$ then G has suitable sets of 2 elements or less by 12.13.

(b) Let Y be the pointed space obtained by considering the discrete topology on any dense subset of G containing 1 as the base point. The inclusion map $Y \to G$ extends to a continuous function $q_0: \beta Y \to G$ which is surjective since $q_0(Y)$ is dense in G, and the universal property of $F(\beta Y)$ yields a surjective morphism $q\colon F(\beta Y) \to G$ which we may write as $q\colon F[[n]] \to G$ for $n = \operatorname{card} Y$. $\square$

Generating Rank and Density

Our Theorem 12.11 enables us to introduce a new cardinal invariant for compact groups representing a kind of rank. (For profinite groups, this cardinal was formulated by Mel'nikov [254].)

Definitions 12.15. (i) For a compact group G we set

$$s(G) = \min\{\aleph \mid \text{there is a suitable set } X \text{ with } \operatorname{card} X = \aleph\}.$$

We call $s(G)$ the *generating rank of* G. A suitable set X of G with $\operatorname{card} X = s(G)$ will be called a *special generating subset.*

(ii) Let G be a topological space. The *density of* G is defined as

$$d(G) = \min\{\aleph \mid \text{there is a dense subset } X \text{ of } G \text{ with } \operatorname{card} X = \aleph\}. \quad \square$$

The obvious fact that the density and the weight are purely topological invariants has some immediate consequences for compact groups on the basis of information we already have. Indeed, by Corollary 10.38 and Theorem 10.40 and its proof,

(a) a compact group G is homeomorphic to $G_0 \times G/G_0$ and,
(b) in the case G_0 does not have finite index in G, the factor group G/G_0 is homeomorphic to $\mathbb{Z}(2)^{w(G/G_0)}$.
(c) For the case of a finite G, trivially, $d(G) = \operatorname{card}(G) = w(G)$.

These facts allow some immediate conclusions as follows:

Lemma 12.15a. (i) *For any infinite compact group G, its density and its weight satisfy*

$$(*) \qquad d(G) = \max\{d(G_0), d(G/G_0)\}, \text{ and}$$

$$(**) \qquad w(G) = \max\{w(G_0), w(G/G_0)\}.$$

(ii) *If G/G_0 is finite, then $d(G) = d(G_0)$, $w(G) = w(G_0)$.*

(iii) *If G is totally disconnected, there is an infinite cardinal n such that*

$$d(G) = d(\mathbb{Z}(2)^n) \text{ and } w(G) = w(\mathbb{Z}(2)^n) = n$$

Proof. All of these statements are immediate consequences of the preceding citations. $\square$

Proposition 12.16. *Let G be a topological group and X a suitable subset. Then the following conclusions hold:*

(i) $\operatorname{card} X \leq w(G)$.
(ii) $d(G) \leq \max\{\aleph_0, \operatorname{card} X\}$.
(iii) $s(G) \leq w(G)$.
(iv) *If $s(G)$ is infinite, then $d(G) \leq s(G) \leq w(G)$.*
(v) *If $s(G) = w(G)$, then $\operatorname{card} X = w(G)$. In particular, all suitable sets have the same cardinality.*

Proof. Let $\langle X\rangle$ denote the subgroup generated by X and note firstly, that by Definitions 12.1 and 12.15, we trivially have

$$(*)\qquad s(G) \leq \operatorname{card} X \quad\text{and}\quad d(G) \leq \operatorname{card}\langle X\rangle.$$

(i) Let $\mathcal{B}$ be a basis of the topology of G of cardinality $w(G)$. Since X is discrete in $G \setminus \{1\}$, for every element $x \in X$ there is an element $U(x) \in \mathcal{B}$ with $U(x) \cap S = \{x\}$. Then $x \mapsto U(x) : S \to \mathcal{B}$ is an injective function and thus $\operatorname{card} S \leq \operatorname{card} \mathcal{B} = w(G)$.

(ii) As $\langle X\rangle = \bigcup_{n=1}(X\cup X^{-1})^n$, and so $\operatorname{card}\langle X\rangle \leq \aleph_0\cdot \operatorname{card} X = \max\{\aleph_0, \operatorname{card} X\}$, Claim (ii) follows from $(*)$.

(iii) The claim follows from $(*)$ and (i).

(iv) If $s(G)$ is infinite so is $\operatorname{card} X$ by $(*)$, whence $d(G) \leq \operatorname{card} X$ by (ii). Then, taking for X a special generating set, we have $s(G) = \operatorname{card} X$ and derive the claim, using (i).

(v) If $s(G) = w(G)$, then $(*)$ and (i) imply the claim. □

Item (v) will become relevant for profinite groups in Proposition 12.28 below.

In view of 12.15 we note right away that $s(G)$ is the smallest cardinal n for which there is a surjective morphism $F[n] \to G$. If G is compact connected, respectively, compact abelian, respectively, compact connected abelian group then $s(G)$ is the smallest cardinal n for which there is a surjective morphism $F_0[n] \to G$, respectively, $F_{\mathrm{ab}}[n] \to G$, respectively, $(F_{\mathrm{ab}}[n])_0 \to G$.

If D is a dense subset of G of cardinality $d(G)$ containing 1, then we obtain a surjective morphism $F[[d(G)]] \to G$. Conversely, if we have a surjective morphism $q\colon F[[n]] \to G$, $F[[n]] = F\bigl(\beta(D_d)\bigr)$ for the discrete pointed space D_d of cardinality n, then the subgroup $\langle q(D_d)\rangle$ is dense in G and

$$\operatorname{card}\langle q(D_d))\rangle = \begin{cases} \operatorname{card} G & \text{if } G \text{ is finite,}\\ \aleph_0\cdot \operatorname{card} D_d = \aleph_0\cdot n & \text{if } G \text{ is infinite.}\end{cases}$$

Thus, for an infinite compact group G, the cardinal $d(G)$ is the smallest infinite cardinal n such that there is a surjective morphism $F[[n]] \to G$. For a finite group G we have, trivially, $d(G) = w(G) = \operatorname{card} G$, and the number $s(G)$ is what it is: the smallest natural number n such that G has a generating set of n elements.

If X is a completely regular Hausdorff space, then X is dense in the Stone–Čech compactification βX. If D is dense in X and $\operatorname{card} D = d(X)$, then D is dense in βX, whence $d(\beta X) \leq d(X)$. The space X is open in βX iff it is locally compact;

if it is, then for each dense D in βX with $\operatorname{card} D = d(\beta X)$, the set $D \cap X$ is dense in X, whence $d(X) \le \operatorname{card}(D \cap X) \le \operatorname{card} D \le d(\beta X)$. Thus in this case $d(X) = d(\beta X)$.

If B is a compact totally disconnected space then every basis closed under finite intersections and finite unions contains the set $\mathcal{CO}(B)$ of compact open subsets of B. If B is infinite, this implies $w(B) = \operatorname{card} \mathcal{CO}(B)$. If $B = \beta X$ for a discrete space X, then the function $A \mapsto \overline{A}$ from the set of subsets of X to the set of compact open subsets of B is a bijection. Hence $w(B) = 2^{\operatorname{card} X}$ if X is infinite. Thus we record $d(\beta X) = \operatorname{card} X$ and $w(\beta X) = 2^{\operatorname{card} X}$ in this case.

Let X be an arbitrary completely regular Hausdorff pointed space. Then we may write $X \subseteq \beta X \subseteq F(\beta X)$ (see 11.1 and 11.3). We notice that $w(n^*) = w(\beta^* n) = n + 1$ if n is finite, and $w(n^*) = n$ and $w(\beta^* n) = 2^n$ if n is infinite.

If G is a compact abelian group, then every surjective morphism $F[n] \to G$ factors through a surjective morphism $F_{\rm ab}[n] \to G$, and every surjective morphism $F[[n]] \to G$ factors through a surjective morphism $F_{\rm ab}[[n]] \to G$. Note that $(2^n)^{\aleph_0} = 2^{n\aleph_0} = 2^{\max\{n,\aleph_0\}}$. By the Structure Theorem of Free Compact Abelian Groups 8.67 we have (assuming n infinite in the case of $F_{\rm ab}[[n]]$)

$$F_{\rm ab}[n] \cong \widehat{\mathbb{Q}}^{w(n^*)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(n^*)} \cong \widehat{\mathbb{Q}}^{(n+1)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{n},$$

$$F_{\rm ab}[[n]] \cong \widehat{\mathbb{Q}}^{w(\beta^* n)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(\beta^* n)} \cong \widehat{\mathbb{Q}}^{2^n} \times \prod_{p\,\text{prime}} \mathbb{Z}_p^{2^n} \cong \Big(\widehat{\mathbb{Q}} \times \prod_{p\,\text{prime}} \mathbb{Z}_p\Big)^{2^n}.$$

Further,

$$\begin{aligned} w(F_{\rm ab}[n]) &= (n+1)^{\aleph_0}, & \operatorname{card}\langle n^* \rangle &= \max\{\aleph_0, n\}, \\ w(F_{\rm ab}[[n]]) &= 2^{\max\{n,\aleph_0\}}, & \operatorname{card}\langle D_n \rangle &= \max\{\aleph_0, n\}, \end{aligned}$$

where $D_n \subseteq \beta^* n$ is as in the definition of $\beta^* n$, having cardinality $n + 1$.

We shall now clarify the generating rank and the density of compact abelian groups and for this purpose must now recall the definition of the rank and the p-rank of an abelian group for a prime number p: see Appendix 1, Definitions following A1.7 and A1.21.

We begin by introducing a function from the class of cardinals to itself which provides a convenient terminology.

Definition 12.16a. For any cardinal n we set

$$\mathbf{log}\, n = \begin{cases} n & \text{if } n \le \aleph_0, \\ \aleph_0 & \text{if } \aleph_0 < n < 2^{\aleph_0}, \\ \min\{m : n \le 2^m\} & \text{if } 2^{\aleph_0} \le n. \end{cases}$$

Notice that $m < 2^m$ and so $\mathbf{log}\, m \le m$ for all cardinals and that the interval $[m, 2^m]$ contains $2^{\mathbf{log}\, m}$. The Generalized Continuum Hypothesis postulates that

$[m, 2^m] = \{m, 2^m\}$. We shall not use it here. However, for any cardinal n of the form $n = 2^m$ we have $m \leq \mathbf{log}\, n \leq m$, that is,

$$(\forall m)\ \mathbf{log}\, 2^m = m < 2^m.$$

In particular, for the cardinality $\mathfrak{c} \stackrel{\text{def}}{=} 2^{\aleph_0}$ of the continuum we have

$$\mathbf{log}\, \mathfrak{c} = \aleph_0 < \mathfrak{c}.$$

For the following we need the subsequent lemma:

Lemma 12.17a. (i) *Let G be a compact totally disconnected group, Then $d(G) = \mathbf{log}\, w(G)$*

(ii) *If G is an arbitrary compact group whose identity component satisfies $d(G_0) = \mathbf{log}\, w(G_0)$, then G itself satisfies $d(G) = \mathbf{log}\, w(G)$.*

Proof. (i) If G is finite, the assertion is true by the definition of $\mathbf{log}$. Now assume that G is totally disconnected and infinite. Then we may assume by Lemma 12.15a(iii) that $G = \mathbb{Z}(2)^m$ for an infinite cardinal $m = w(G)$. Then $d(G)$ is the smallest possible dimension n of GF(2)-vector spaces of a dense vector subspace $V \subseteq \mathbb{Z}(2)^m$, $V \cong \mathbb{Z}(2)^{(n)}$. By duality this is equivalent to $\mathbb{Z}(2)^{(m)}$ being algebraically isomorphic to an m-dimensional GF(2)-vector subspace of $\widehat{V}$, which is a compact abelian group of exponent 2, that is, $\widehat{V} \cong \mathbb{Z}(2)^n$ for a smallest possible infinite cardinal $n = w(\widehat{V})$. Since we know that $\dim_{GF(2)} \mathbb{Z}(2)^n = 2^n$ we have

$$n = \min\{n' : m \leq 2^{n'}\} = \mathbf{log}\, m.$$

That is $d(G) = \mathbf{log}\, w(G)$, which we had to show in this case.

(ii) Let G be an arbitrary infinite compact group such that $d(G_0) = \mathbf{log}\, w(G_0)$. Then by (i) above,

$$d(G) = \max\{d(G_0), d(G/G_0)\} = \max\{\mathbf{log}\, w(G_0), \mathbf{log}\, w(G/G_0)\}$$

$$= \mathbf{log}\,(\max\{w(G_0), w(G/G_0)\}) \text{ since } \mathbf{log} \text{ is nondecreasing } = \mathbf{log}\, w(G)$$

by Lemma 12.15a(**). So the lemma is proved. □

Proposition 12.17. (i) *For a compact abelian group G*

$$\begin{aligned} \dim G &= \operatorname{rank} \widehat{G}, \\ w(G) &= \operatorname{card} \widehat{G}, \\ s(G) &= \min\{n \mid \operatorname{rank} \widehat{G} \leq (n+1)^{\aleph_0} \textit{ and } (\forall p \textit{ prime}) \operatorname{rank}_p \widehat{G} \leq n\}, \\ d(G) &= \max\{d(G_0), d(G/G_0)\}. \end{aligned}$$

(ii) *For a nonsingleton compact* connected *abelian group G, the generating rank $s(G)$ is the smallest cardinal n such that the following relation holds:*

$$\dim G \leq (n+1)^{\aleph_0} = \begin{cases} 2^{\aleph_0} & \textit{for } n = 1, \\ n^{\aleph_0} & \textit{for } n > 1. \end{cases}$$

(iii) *For the density of an arbitrary compact abelian group we have*

$$d(G) = \mathbf{log}\, w(G).$$

Proof. (i) The statement on the weight we know since 7.76(ii); it is recorded here for the sake of completeness. By the preceding remarks and duality, $s(G)$ is the smallest of all cardinals n such that $\widehat{G}$ is isomorphic to a subgroup of

$$\Delta \stackrel{\rm def}{=} \mathbb{Q}^{((n+1)^{\aleph_0})} \oplus \bigoplus_{p\,\rm prime} \mathbb{Z}(p^\infty)^{(n)}.$$

By a remark following Appendix 1, A1.39, by the Structure Theorem of Divisible Abelian Groups A1.42, and by A1.36(ii), the divisible hull D_A of an abelian group A is uniquely characterized by the cardinals $\operatorname{rank} D_A = \operatorname{rank} A$, $\operatorname{rank}_p D_A = \operatorname{rank}_p A$. Since $\widehat{G}$ is embeddable into Δ iff $D_{\widehat{G}}$ is embeddable into Δ, the assertion follows.

The formula for the density was observed in equation $(*)$ in Lemma 12.15a(i).

(ii) We remember that $\operatorname{rank}\widehat{G} = \dim G$. For a connected G we have $\operatorname{rank}_p \widehat{G} = 0$, and thus (ii) follows from (i) for $s(G)$.

(iii) We still have to show that for a nonsingleton compact abelian group G we have $d(G) = \mathbf{log}\, w(G)$.

Firstly, if G is finite, the claim follows trivially from $\mathbf{log}\, n = n$ for finite n.

Secondly, assume that G is connected. We know that $w(G) = \operatorname{card}\widehat{G}$ from (i), and we have observed that $d(G)$ is the smallest cardinal n such that G is a quotient of $\big(\widehat{\mathbb{Q}} \times \prod_{p\,\rm prime} \mathbb{Z}_p\big)^{2^n}$. But since G is connected, $d(G)$ is the smallest cardinal n such that G is quotient of $\widehat{\mathbb{Q}}^{2^n}$, and dually this means that $\widehat{G}$ is isomorphic to subgroup of $\mathbb{Q}^{(2^n)}$ where $\operatorname{rank}\widehat{G} = w(G)$; this is equivalent to saying that $d(G) = n$ is the smallest cardinal such that $w(G) \le 2^n$, that is $d(G) = \mathbf{log}\, w(G)$ as asserted.

Thirdly, if G is any compact abelian group, we have $d(G_0) = \mathbf{log}\, w(G_0)$ by what we just saw. So the claim follows from Lemma 12.17a(ii). □

We record a special case as an example

Example. For an arbitrary cardinal n set $G = F_{\rm ab}[[n]]$. Then

(1) $$w(G) = w(G_0) = 2^{\max\{n,\aleph_0\}},$$

(2) $$s(G) = s(G_0) = \begin{cases} \aleph_0 & \text{if } n \text{ is finite,} \\ n & \text{if } n \text{ is infinite.} \end{cases}$$

(3) $$d(G) = d(G_0) = \left.\begin{cases} \operatorname{card} G & \text{if } n \text{ is finite,} \\ \min\{m \mid 2^n \le m^{\aleph_0}\} & \text{if } n \text{ is infinite,} \end{cases}\right\} \le 2^n$$

Lemma 12.18. *Let $f\colon G \to H$ be a surjective morphism of compact groups. Then $s(H) \le s(G)$ and $d(H) \le d(G)$.*

Proof. (a) Let X be a suitable set of G with $\operatorname{card} X$. By 12.4, the image H has a suitable set $f(X) \setminus \{1\}$. Clearly, $s(H) \leq \operatorname{card}(f(X) \setminus \{1\}) \leq \operatorname{card} X$.

(b) The assertion on the densities is straightforward. □

Proposition 12.19. (i) *Let $f: G \to H$ be a surjective morphism of compact groups. Then $s(H) \leq s(G) \leq s(\ker f) + s(H)$.*

(ii) *If, in addition, G is connected and* $\ker f$ *is totally disconnected, then $s(G) = s(H)$.*

(iii) *If G is connected and* $\ker f$ *is totally disconnected, then $w(G) = w(H)$.*

Proof. (i) By Lemma 12.18 we have $s(H) \leq s(G)$. We must show $s(G) \leq s(\ker f) + s(H)$. Let Y be a special generating set of H. Then $\operatorname{card} Y = s(H)$. By Lemma 12.5 there is a subset X of $G \setminus \{1\}$ which is closed and discrete in $G \setminus \{1\}$ and mapped bijectively onto Y by f. Let Z be a special generating set of $\ker f$. Then $X \cup Z$ is discrete and closed in G and $\overline{\langle X \cup Z \rangle}$ contains $\ker f$ and maps onto H under f, hence is G. Thus $X \cup Z$ is suitable in G and therefore $s(G) \leq \operatorname{card}(X \cup Z) \leq \operatorname{card} X + \operatorname{card} Z = s(Y) + s(\ker f)$.

(ii) If G is connected and $\ker f$ is totally disconnected, then 12.6 applies and shows that X as constructed in (i) is suitable in G. Hence $s(G) \leq \operatorname{card} X = \operatorname{card} Y = s(H)$.

(iii) A totally disconnected normal subgroup of a connected group is central (see A4.26). Thus the center of G is contained in every maximal connected abelian subgroup T of G (see 9.32(iv)). Thus $N \subseteq T$. We know $w(G) = w(Y)$ (and $w(G/N) = w(T/N)$ by 9.63(iv). Thus the claim is reduced to the case that G is commutative. Now $w(G) = |G|$ and $w(G/N) = |\widehat{G/N}|$ by 7.75. The annihilator mechanism of duality provides $\widehat{G/N} \cong N^\perp$ and $\widehat{N} \cong \widehat{G}/N^\perp$ (see Theorem 7.20). Moreover, $\widehat{G}$ is torsion free and $\widehat{N}$ is a torsion group (see 8.5). Thus, the claim is equivalent to the following assertion on abelian groups.

Let B be a subgroup of a torsionfree abelian group A such that A/B is a torsion group. Then $|B| = |A|$.

The fact that A/B is a torsion group is equivalent to saying that the pure subgroup [B] generated by B is A (see A1.25). Since $|B| = |[B]|$ this implies $|B| = |A|$. □

The Cardinal Invariants of Connected Compact Groups

In this section we describe what we may call the *Descent Procedure*, namely the procedure to read off all cardinal invariants at least for infinite dimensional compact connected groups G from those of a maximal pro-torus T. In other words, in the realm of compact connected groups, for cardinal invariants, the abelian theory suffices.

Lemma 12.21. *For a compact connected group G with a maximal pro-torus T the relation $s(G) \leq s(T) + 1$ holds. If $s(T)$ is infinite, then $s(G) \leq s(T) = \min\{n \mid \dim T \leq n^{\aleph_0}\}$.*

Proof. This is a consequence of 12.12 and 12.17(ii) above. □

Theorem 12.22. *Let G be a compact connected group and T a maximal pro-torus. If $w(G) \leq 2^{\aleph_0}$ then*

$$s(G) = \begin{cases} s(T) = 1 & \text{if } G \text{ is abelian,} \\ s(T) + 1 = 2 & \text{if } G \text{ is nonabelian.} \end{cases}$$

If $w(G) > 2^{\aleph_0}$, then

$$s(G) = s(T) = \min\{n \mid w(G) \leq n^{\aleph_0}\}.$$

Proof. Assume $w(G) \leq 2^{\aleph_0}$. If G is abelian, i.e. $G = T$, then G, being connected, is monothetic. Hence $s(G) = s(T) = 1$. If G is nonabelian, then $s(T) = 1$ and $s(G) = 2$ by 9.38.

Now assume $w(G) > 2^{\aleph_0}$. After 12.21 we must show $s(T) \leq s(G)$. By 9.52, the group G cannot be finite dimensional. Then from 9.56 we know $\dim G = \dim T$ and from 9.36(vi) we have $w(G) = w(T)$. Further $w(T) = \operatorname{card} \widehat{T} = \operatorname{rank}(\widehat{T})$, the last equality holding because $\operatorname{card} \widehat{T} = w(T) = w(G) \geq 2^{\aleph_0}$. Since $\operatorname{rank}(\widehat{T}) = \dim T$, putting everything together, we have $\dim G = w(G)$ in our situation. By 12.16 we have $s(T) = \min\{n \mid \dim T \leq n^{\aleph_0}\}$, i.e. by the preceding remark $s(T) = \min\{n \mid w(G) \leq n^{\aleph_0}\}$. Therefore we must show that

$$(*) \qquad w(G) \leq s(G)^{\aleph_0}.$$

By a remark following 12.16, $s(G)$ is the smallest cardinal n for which there is a surjective morphism of compact groups $F_0[n] \to G$. Thus in order to prove $(*)$ we must show that (for $2^{\aleph_0} < w(G)$)

$$(**) \qquad (\forall n)\big((\exists \pi \text{ surjective})\ \pi\colon F_0[n] \to G\big) \Rightarrow (w(G) \leq n^{\aleph_0}).$$

Because $2^{\aleph_0} < w(G) \leq w(F_0[n])$, the cardinal n is necessarily infinite. Hence from Theorem 11.54 we know $w(F_0[n]) = n^{\aleph_0}$. Therefore, $w(G) \leq n^{\aleph_0}$ is in fact the same as $w(G) \leq w(F_0[n])$ which is a consequence of the existence of a surjective morphism $\pi\colon F_0[n] \to G$. Thus $(**)$ is satisfied and the theorem is proved. □

In 9.36(vi) we proved the equality $w(T) = w(G)$ for a compact connected group and any maximal pro-torus T in G. We did not have the concept of a density at that stage. For the following results we need a lemma which could have been discussed in the context of the Maximal Pro-Torus Theorem 9.32.

Lemma 12.23. *Let G be a compact connected group, T a maximal pro-torus and S a closed subgroup of T such that $G = \bigcup_{g\in G} gSg^{-1}$. Then $S = T$.*

Proof. (a) We first show that the lemma is true if it holds for compact Lie groups. Assume that $S \neq T$. Then by 9.1(iii) there is an $N \in \mathcal{N}(G)$ such that $SN \neq T$. Assuming that the lemma is true for compact Lie groups, we get $G/N \neq \bigcup_{g\in G}(gN)(SN/N)S(gN)^{-1}$. But this implies that $\bigcup_{g\in G} gSg^{-1} \subseteq \bigcup_{g\in G} gSNg^{-1} \neq G$. This proves our first claim.

(b) Now assume that G is a compact Lie group. We find an element t in T such that $T = \overline{\langle t\rangle}$ (see 1.24(v), 8.75ff.). Because $G = \bigcup_{g\in G} gSg^{-1}$ there are elements $s \in S$ and $g \in G$ such that $t = gsg^{-1}$. Then $s = g^{-1}tg$. Being a generator of a maximal torus is a property which is preserved under automorphisms, notably under inner automorphisms. Thus $\overline{\langle s\rangle}$ is a maximal torus of G. On the other hand $\overline{\langle s\rangle} \subseteq S \subseteq T$. This implies $\overline{\langle s\rangle} = T$ and thus $S = T$. □

Theorem 12.24. *For a compact connected group G and a maximal pro-torus T the equality $d(T) = d(G)$ holds.*

Proof. We assume that G is not singleton. Then T is not singleton (see 9.32). Hence $d(T) \geq \aleph_0$. Let X be a dense subset of T with $\operatorname{card} X = d(T)$. By 9.38 there is a $g \in G$ such that $\overline{\langle X \cup \{g\}\rangle} = \overline{\langle T \cup \{g\}\rangle} = G$. Now $\operatorname{card}\langle X \cup \{g\}\rangle = \aleph_0{\cdot}(\operatorname{card} X + 1) = \operatorname{card} X$ since $\operatorname{card} X$ is infinite. Hence $d(G) \leq \operatorname{card}\langle X\cup\{g\}\rangle = \operatorname{card} X = d(T)$.

Now let D be a dense subgroup of G with cardinality $d(G)$. We construct recursively an ascending sequence $D_1, D_2, \ldots$ of subgroups as follows. First, using the Maximal Pro-Torus Theorem 9.32 and the Axiom of Choice, we select a function $\gamma\colon G \to G$ such that $x \in \gamma(x)T\gamma(x)^{-1}$. Then we set $D_1 = D$ and

$$D_{n+1} = \langle D_n \cup \gamma(D_n) \cup \textstyle\bigcup_{d\in D_n} \gamma(d)(D_n)\gamma(d)^{-1}\rangle \leq G.$$

Since $D_1 \subseteq D_2 \subseteq \cdots$ and $\aleph_0 \leq D_1$, we have

$$\operatorname{card} D_n \leq \operatorname{card} D_{n+1} \leq \aleph_0{\cdot}\big(\operatorname{card} D_n + \operatorname{card} D_n \operatorname{card}(D_n)^2\big) = \operatorname{card} D_n.$$

Now we set $\widetilde{D} = \bigcup_{n=1}^{\infty} D_n$. Since $\gamma(D_n) \subseteq D_{n+1}$ we have

$$\gamma(\widetilde{D}) \subseteq \widetilde{D}.$$

We claim

(i) $\operatorname{card} \widetilde{D} = \operatorname{card} D = d(G)$,

(ii) $\widetilde{D}^- = G$,

(iii) $\widetilde{D} = \bigcup_{d\in\widetilde{D}} d(T \cap \widetilde{D})d^{-1}$.

The proofs of (i) and (ii) are clear. The right side of (iii) is contained in the left side. If $g \in \widetilde{D}$, then there is an n such that $g \in D_n$. Now $g \in \gamma(g)T\gamma(g)^{-1}$ and $\gamma(g) \in \widetilde{D}$. Then $g \in \widetilde{D} = \gamma(g)\widetilde{D}\gamma(g)^{-1}$, and thus $g \in \gamma(g)(T \cap \widetilde{D})\gamma(g)^{-1}$. Hence $g \in \bigcup_{d\in\widetilde{D}} d(T \cap \widetilde{D})d^{-1}$. Now set $S = \overline{\langle T \cap \widetilde{D}\rangle}$. Then by (iii) we have $\widetilde{D} \subseteq \bigcap_{g\in G} gSg^{-1}$ and the right side, being the continuous image of $G \times S$ under $(g, s) \mapsto gsg^{-1}$ is compact. By (ii), the group $\widetilde{D}$ is dense in G. Hence

$$G = \bigcup_{g\in G} gSg^{-1}.$$

By Lemma 12.19 this implies $S = T$. Thus T has a dense subset $T \cap \widetilde{D}$ of cardinality $\operatorname{card}(T \cap \widetilde{D}) \leq \operatorname{card} \widetilde{D} = d(G)$ by (i). Hence $d(T) \leq d(G)$. □

Cardinal Invariants of Connected Compact Groups

Theorem 12.25. *Let G be a nonsingleton compact connected group. Then*

$$w(G) = \max\{\aleph_0, \dim G\}, \qquad d(G) = \begin{cases} \aleph_0 & \textit{if } \dim G < \infty, \\ \min\{n \mid \dim G \leq 2^n\} & \textit{if } \dim G \geq \aleph_0; \end{cases}$$

$$s(G) = \begin{cases} 1 & \textit{if } \dim G \leq 2^{\aleph_0} \textit{ and } G \textit{ is abelian}, \\ 2 & \textit{if } \dim G \leq 2^{\aleph_0} \textit{ and } G \textit{ is not abelian}, \\ \min\{n \mid \dim G \leq n^{\aleph_0}\} & \textit{if } \dim G > 2^{\aleph_0}. \end{cases}$$

In particular, a compact connected abelian group is separable, that is, has a countable dense subset, if and only if it is monothetic.

Proof. Assume that G is connected and nonsingleton. We have $w(G) = w(T)$ by Theorem 9.36(vi), further $s(G) = 1$, respectively, $s(T) = 2$ if $w(G) \leq 2^{\aleph_0}$ and G is abelian, respectively, not abelian. Otherwise $s(G) = s(T)$ by Theorem 12.22; finally $d(G) = d(T)$ by Theorem 12.24. It therefore suffices to verify the assertion for a compact connected abelian group G with $w(G) > 2^{\aleph_0}$.

Since G is connected, $\widehat{G}$ is torsion-free. By 7.76(ii), $w(G) = \operatorname{card} \widehat{G}$. By 8.26, $\dim G = \operatorname{rank} \widehat{G}$. If A is an abelian torsion-free group with $\operatorname{rank} A \geq \aleph_0$, then $\operatorname{rank} A = \operatorname{card} A$. Hence we have $2^{\aleph_0} < w(G) = \operatorname{card} \widehat{G} = \operatorname{rank} \widehat{G} = \dim G$. From 12.22 we obtain the calculation of $s(G)$, from 12.24 and 12.17 the calculation of $d(G)$.

Finally, let G be nonsingleton compact connected abelian and have density $\aleph_0$. Then $\dim G \leq 2^{\aleph_0}$ and so $s(G) = 1$, that is, G is monothetic. Conversely, if G is monothetic, it contains a dense copy of a cyclic group which must be $\mathbb{Z}$ since G is nonsingleton connected; thus G is separable. □

This theorem shows that for compact connected groups the invariant $\dim G$ is the most useful; for large compact groups all the others are expressible in an explicit way through $\dim G$. For finite dimensional compact groups, it discriminates between groups which cannot be distinguished by the other cardinal invariants. For very large compact connected groups, i.e. compact connected groups whose weight exceeds the cardinality of the continuum, dimension and weight agree. This is relevant because the weight is a very robust and well-behaved cardinal. The theorem also illustrates once more the observation we made repeatedly that many structural features of a compact connected group G can be described in terms of associated compact connected abelian groups, here again in terms of the maximal pro-torus groups T.

Cardinal Invariants in the Absence of Connectivity

Let us record and summarize the behavior of our cardinal invariants under surjective homomorphisms, i.e. under passing to quotients.

Proposition 12.26. *Assume that* $f\colon G \to H$ *is a surjective morphism of compact groups. Then the inequalities* $\dim H \le \dim G$, $w(H) \le w(G)$, $s(H) \le s(G)$, *and* $d(H) \le d(G)$ *hold.*

If G *is connected and* $\ker f$ *is totally disconnected, then equality holds in all cases.*

Proof. Exercise E12.2. □

Exercise E12.2. Prove 12.26.

[Hint. For the generating rank $s(\cdot)$ refer to 12.4, 12.5, and 12.20. For dimension recall the definition and note that by 9.47, $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective.] □

In the absence of connectivity, a few observations are in order. By 10.38 we note that G and $G_0 \times G/G_0$ are homeomorphic whence

$$w(G) = \max\{w(G_0), w(G/G_0)\} \quad \text{and} \quad d(G) = \max\{d(G_0), d(G/G_0)\}.$$

Thus the computation of the topological cardinal invariants is completely reduced to the calculation of these for totally disconnected groups.

Proposition 12.27. *For any compact group* G *the estimates* $s(G) \le w(G) \le s(G)^{\aleph_0}$ *hold.*

Proof. The first estimate was noted in 12.16, and we must prove the second. There is a surjective homomorphism $F[s(G)] \to G$. Hence $w(G) \le w(F[s(G)]) \le s(G)^{\aleph_0}$ by 11.6(iii). □

Proposition 12.28. *Let* G *be an infinite totally disconnected compact, that is, a profinite group. Then the following conclusions hold:*

(i) $w(G) = \max\{\aleph_0, s(G)\}$.

(ii) *If* S *is any infinite suitable subset of* G, *then* $\operatorname{card} S = w(G)$. *In particular, all infinite suitable subsets of* G *have the same cardinality, namely, the weight of* G.

Proof. (i) From 12.16 we know that always $s(G) \le w(G)$. We must show that for an infinite generating rank $s(G)$, the reverse inequality holds. For this purpose let F be the free group on a special generating set D of G of cardinality $s(G)$ and let $\varphi\colon F \to G$ denote the algebraic homomorphism uniquely defined by $\varphi(x) = x$ for $x \in D$. Let $\mathcal{N}_{\mathrm{fin}}(F)$ be the filter basis of normal subgroups of F with finite index such that $D \setminus N$ is finite. We note $\bigcap_{N \in \mathcal{N}_{\mathrm{fin}}(F)} N = \{1\}$. Then the finite factor groups F/N, $N \in \mathcal{N}_{\mathrm{fin}}(F)$ form, together with the obvious quotient maps $F/N \to$

F/M for $M \subseteq N$ in $\mathcal{N}_{\rm fin}(F)$, an inverse system. Let $F^* = \lim_{N\in\mathcal{N}_{\rm fin}(F)} F/N \subseteq \prod_{N\in\mathcal{N}_{\rm fin}(F)} F/N$ and let $j\colon F \to F^*$ be the natural homomorphism defined by $j(x) = (xN)_{N\in\mathcal{N}_{\rm fin}(F)}$. Its kernel is $\ker j = \bigcap_{N\in\mathcal{N}_{\rm fin}(F)} N = \{1\}$. Let $p_N\colon F^* \to F/N$ denote the limit maps. Then $N^* \stackrel{\rm def}{=} \ker p_N = \overline{j(N)}$ is such that p_N induces an isomorphism $F^*/N^* \to F/N$. Let M be a compact open normal subgroup of G. Then $D \setminus M$ is finite; thus $\varphi^{-1}(M) \in \mathcal{N}_{\rm fin}(F)$, and φ induces an isomorphism $F/\varphi^{-1}(M) \to G/M$. Hence we have a morphism $\gamma_M\colon F^* \to G/M$ with kernel $\varphi^{-1}(M)^*$ inducing an isomorphism $F^*/\varphi^{-1}(M)^* \to G/M$ such that $\gamma_M\big(j(x)\big) = \varphi(x)M$ for $x \in F$. If $M \supseteq M'$ then the natural morphism $\pi_{MM'}\colon G/M' \to G/M$ satisfies $\gamma_M = \pi_{MM'} \circ \gamma_{M'}$. Then, since $G \cong \lim_{M\in\mathcal{N}(G)} G/M$, by the universal property of the limit, there is a morphism of compact groups $\gamma\colon F^* \to G$ with $\gamma_M(\xi) = \gamma(\xi)M \in G/M$ for all $\xi \in F^*$. Then $\gamma \circ j = \varphi$ and thus $D = \varphi(D) \subseteq \gamma(F^*)$. Since $G = \overline{\langle D\rangle}$ we conclude that γ is surjective. Hence $w(G) \le w(F^*)$. Now $w(F^*) = w(\lim_{N\in\mathcal{N}_{\rm fin}(F)} F/N) \le w(\prod_{N\in\mathcal{N}_{\rm fin}(F)} F/N) = \operatorname{card}\mathcal{N}_{\rm fin}(F)$ by EA4.3. But $\operatorname{card}\mathcal{N}_{\rm fin}(F) = \operatorname{card} D = s(G)$. Thus $w(G) \le s(G)$.

(ii) Now let S be an arbitrary suitable subset of G. If $w(G)$ is uncountable then $s(G) = w(G)$ by (i) above. Thus 12.16(v) implies the claim. Now assume that $w(G) = \aleph_0$. Then 12.16(i) implies $\aleph_0 \le \operatorname{card} S \le w(G) = \aleph_0$, and the claim is true in this case as well. □

In fact, if we set $X = D \cup \{1\} \subseteq G$, then in the form of F^* we have constructed the group $F(X)/F(X)_0$, the free compact zero-dimensional group on X. If we let βD denote the pointed space with a base point in D, in order to construct $F(\beta D)/F(\beta D)_0$ in an analogous fashion in place of $\mathcal{N}_{\rm fin}(F)$ we would have had to consider the filter basis of *all* normal subgroups of finite index in F. The cardinality of this filterbasis is $2^{\operatorname{card} D} = 2^{s(G)}$.

Comparing the cases of connected groups in Theorem 12.25 and of totally disconnected groups in 12.28 we notice that the bound for the weight in terms of the generating rank is lower in the totally disconnected situation.

Proposition 12.29. *Let G be a compact group. Then $s(G) \le s(G_0) + s(G/G_0)$. If $s(G/G_0)$ is infinite, then $s(G) \le \min\{n \mid \dim G \le n^{\aleph_0}\} + w(G/G_0)$.*

Proof. The first assertion arises from 12.20(i). The second then follows from 12.25 and 12.28 above. □

Proposition 12.30. *Let G be a compact group with $w(G) \le 2^{\aleph_0}$. Then the following cases can occur:*

(i) $s(G) = 0$ *if* $G = \{1\}$.

(ii) $s(G) = 1$ *if G is monothetic, in particular, if G is connected and abelian.*

(iii) $s(G) = 2$ *if G is connected and nonabelian.*

(iv) $\max\{\aleph_0, s(G)\} = w(G/G_0)$ *if G has infinitely many components.*

(v) $s(G/G_0) \le s(G) \le s(G/G_0) + 2$ *if G has at least 2, but finitely many components.*

Proof. (iv) From Proposition 12.29 we know $s(G) \leq s(G_0) + s(G/G_0)$. If G/G_0 is infinite, then $\max\{\aleph_0, s(G/G_0)\} = w(G/G_0)$ by Lemma 12.28. By (i)–(iii), assertion (iv) follows. Finally, in order to prove (v), let $2 \leq \mathrm{card}(G/G_0) < \aleph_0$. Then (v) follows from (ii), (iii) above and Lemma 12.29. □

Exercise E12.3. Prove the following piece of information:

Let G be a profinite group of uncountable weight and let $\aleph < w(G)$ be an infinite cardinal. Then there is a closed subgroup H such that $w(H) = \aleph$.

[Hint. Let T be a suitable subset of G with $\mathrm{card}\, T = w(G)$ according to Proposition 12.28. Then T contains a subset S of cardinality $\aleph$. We set $H = \overline{\langle S \rangle}$. Now S is discrete in $H\backslash\{1\}$ since T is discrete in $G\backslash\{1\}$. Hence S is an infinite suitable subset of the profinite group H. Hence, by Proposition 12.16(v), $w(H) = \mathrm{card}(S) = \aleph$ follows.] □

Theorem 12.31. *Let $\aleph$ be an infinite cardinal and G a compact group such that $\aleph < w(G)$. Then there is a closed subgroup H such that $w(H) = \aleph$, and if G is connected, H may be chosen normal and connected.*

Proof. Let G_0 denote the identity component of G. The case that $w(G) = w(G_0)$ is handled in Exercise E9.10. So we assume $w(G_0) < w(G)$. By Exercise E10.7(ii) we may assume that G is totally disconnected, that is, profinite. Then Exercise E12.3 proves the assertion of the theorem. □

The Relation of Density and Weight

Theorem 12.31a. *For a compact group G, its topological invariants density and weight are linked by the equation*

$$(\#) \qquad d(G) = \mathbf{log}\, w(G).$$

Proof. By Theorem 12.25, equation (#) holds if G is connected. By Lemma 12.17a(ii) then the assertion of the theorem follows. □

In view of the somewhat difficult concept of density, this theorem provides seemingly simple, yet tricky consequences, for instance the following:

Corollary 12.31b. *In a compact group G, the density of a subgroup of G never exceeds the density of G itself.*

Proof. It is a well understood property of the weight, that the weight of a closed subgroup of a topological group never exceeds that of the group itself. The assertion is then an immediate consequence of Theorem 12.31a. □

It was proved by Gerald Itzkowitz in [216] that a subgroup of a compact separable group is separable. For a survey of such results see [240]

On the Location of Special Generating Sets

We shall discuss particular locations of special generating subsets of a compact group. Notably we shall prove the following assertion:

(A) *If G is a compact connected group, then the arc component of the identity $G_a = \exp \mathfrak{L}(G)$ in G contains a special generating subset of G.*

The proof will proceed through several reductions. Until further notice, G will always denote a compact connected group.

Lemma 12.32. (i) *Assume that* (A) *is true for all abelian groups G. Then* (A) *is true in general.*

(ii) *Statement* (A) *is true for all abelian G with $w(G) \leq 2^{\aleph_0}$.*

Proof. Proof of (i): Let G be a compact connected nonabelian group and T a maximal pro-torus (cf. 9.30ff.) By hypothesis, we can find a special generating subset X of T in T_a. By Proposition 12.12, there is a $g \in G$ such that $X \cup \{g\}$ is a suitable subset of G. Since G is the union of the conjugates of T by the Maximal Pro-Torus Theorem 9.32, there is an $h \in G$ such that $g \in hTh^{-1}$. Clearly G is topologically generated by $T \cup hTh^{-1}$ and thus by $Y = X \cup hXh^{-1}$. Since X is suitable so is Y by Lemma 12.2. Also, $Y \subseteq T_a \cup hT_a h^{-1} \subseteq G_a$. If $\aleph_0 \leq w(G) \leq 2^{\aleph_0}$, then $\operatorname{card} X = 1$ by Proposition 9.38. Thus, since G is not abelian, $\operatorname{card} Y = 2 = s(G)$ and so Y is special. If $2^{\aleph_0} < w(G)$, then $\aleph_0 \leq \operatorname{card} X$ by 9.38. Then, again by 9.38, $\aleph_0 \leq s(G) \leq \operatorname{card} X + 1 = \operatorname{card} X \leq \operatorname{card} Y = \operatorname{card} X = s(G)$, and hence Y is special. This completes the proof of part (i) of the lemma.

Proof of Part (ii): After Lemma 12.31, the task is reduced to the abelian case. All compact connected abelian groups of weight not exceeding the cardinality of the continuum are monothetic. Hence we must show that each connected monothetic G has a generator in G_a. Now the hypothesis that G is connected monothetic means that $\widehat{G}$ is torsion-free and of rank $\leq 2^{\aleph_0}$. Let $\mathbb{T}$ denote $\mathbb{R}/\mathbb{Z}$ and $p: \mathbb{R} \to \mathbb{T}$ the quotient homomorphism. The group $\mathbb{T}$ is algebraically isomorphic to $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{R}$. Hence there is an injective morphism, $j: \widehat{G} \to \mathbb{R}$ such that $p \circ j: \widehat{G} \to \mathbb{T}$ remains injective. Hence the dual $\widehat{j} \circ \widehat{p}: \mathbb{Z} \to G = \widehat{\widehat{G}}$ has dense image and factors through $\widehat{p}: \mathbb{Z} \to \mathbb{R}$. Thus $\widehat{j}\widehat{p}(1)$ is a generator of the arc component of the identity. □

Let us consider the following assertion

(B) *Let G be a compact connected abelian group with $w(G) > 2^{\aleph_0}$. There is a suitable set Y of $\mathfrak{L}(G)$ with $Y \cup \{0\}$ compact and $s(\mathfrak{L}(G)) \leq \operatorname{card} Y = s(G)$.*

Before we prove Assertion (B) in several steps, we observe, that it will finish the proof of Assertion (A), the main result of this section. Indeed, let Y be a suitable set of $\mathfrak{L}(G)$. The fact that the exponential function is a morphism with dense image, implies that $\exp Y$ topologically and algebraically generated G. In order to show that $\exp Y$ is suitable, we verify condition (2) of Lemma 12.2. Let V denote an open neighborhood of 1 in G. Then $U \stackrel{\text{def}}{=} \exp^{-1}(V)$ is an open neighborhood of 0 in $\mathfrak{L}(G)$. Then $X \setminus U$ is finite as X is relatively compact suitable in $\mathfrak{L}(G)$ (cf. 12.2), and so $\exp(Y) \setminus V = \exp(X \setminus U)$ is finite. Hence $\exp Y$ is a suitable set of G contained in $G_a = \exp L(G)$ and $s(G) \leq \operatorname{card}(\exp Y) \leq \operatorname{card} Y = s(G)$. This establishes Claim (A).

The proof of Claim (B) requires several further lemmas and is the bulk of the argument.

The first of these lemmas is proved by diagram chasing.

Lemma 12.33 (Diagram Lemma). *Consider the commutative diagram of abelian groups with exact columns. If the first two rows are exact, then the third row is exact.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \to & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \sigma_1} & & \\
0 & \to & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 & \to & 0 \\
 & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \sigma_2} & & \\
0 & \to & A_3 & \xrightarrow{\alpha_3} & B_3 & \xrightarrow{\beta_3} & C_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

□

If X is a pointed compact space and $\mathbb{K} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{T}\}$, we shall write $C_0(X, \mathbb{K})$ for the abelian group of all base point preserving continuous functions under pointwise addition. Further, if A is a subgroup of $\mathbb{K} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{T}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}\}$, then $C_{\text{fin}}(X, A)$ will denote the subgroup of $C_0(X, \mathbb{K})$ consisting of all functions taking only finitely many values in A. Finally, $[X, \mathbb{T}]$ is the group of all homotopy classes of continuous base point preserving functions $X \to \mathbb{T}$. We recall $[X, \mathbb{T}] \cong H^1(X, \mathbb{Z})$.

Lemma 12.34. *For a compact pointed space X such that $[X, \mathbb{T}] = 0$ we have*

$$C_0(X, \mathbb{R})/C_{\text{fin}}(X, \mathbb{Q}) \cong C_0(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}).$$

Proof. The exact sequence

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{p} \mathbb{T} \to 0$$

induces an exact sequence

$$0 \to C_0(X, \mathbb{Z}) \xrightarrow{j^*} C_0(X, \mathbb{R}) \xrightarrow{p^*} C_0(X, \mathbb{T}) \to [X, \mathbb{T}] \to 0.$$

We now assume that $[X, \mathbb{T}] = \{0\}$. We set $B^* = C_0(X, \mathbb{R})/C_{\text{fin}}(X, \mathbb{Q})$ and $B = C_0(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z})$. Then we have a commutative diagram with exact columns whose first two rows are exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_{\rm fin}(X,\mathbb{Z}) & \to & C_{\rm fin}(X,\mathbb{Q}) & \to & C_{\rm fin}(X,\mathbb{Q}/\mathbb{Z}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_0(X,\mathbb{Z}) & \to & C_0(X,\mathbb{R}) & \to & C_0(X,\mathbb{T}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & B^* & \to & B & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By the Diagram Lemma 12.33, the assertion follows. □

Lemma 12.35. *Let X denote a compact space. Then, as rational vector spaces, $C_0(X,\mathbb{R}) \cong C_0(X,\mathbb{R})/C_{\rm fin}(X,\mathbb{Q})$.*

Proof. Write $R = \mathbb{Q} \oplus E$ for a suitable $\mathbb{Q}$-vector space complement E for $\mathbb{Q}$ in $\mathbb{R}$. Then $C_{\rm fin}(X,\mathbb{Q}) \cap C_{\rm fin}(X,E) = \{0\}$ and thus there is a vector space complement $\mathcal{F}$ of $C_{\rm fin}(X,\mathbb{Q})$ in $C_0(X,\mathbb{R})$ containing $C_{\rm fin}(X,E)$. We note that $E \cong \mathbb{Q}^{(\mathfrak{c})}$ and thus $C_{\rm fin}(X,E) \cong C_{\rm fin}(X,\mathbb{Q})^{(\mathfrak{c})}$, and $\mathcal{F}$ contains a vector subspace $\mathcal{V} \cong C_{\rm fin}(X,\mathbb{Q})^{(\mathfrak{c})}$. We write $\mathcal{F} = \mathcal{V} \oplus \mathcal{W}$. Therefore,

$$C_0(X,\mathbb{R}) \cong C_{\rm fin}(X,\mathbb{Q}) \oplus \mathcal{F} = C_{\rm fin}(X,\mathbb{Q}) \oplus \mathcal{V} \oplus \mathcal{W} \cong \mathcal{V} \oplus \mathcal{W} = \mathcal{F}.$$

Since $\mathcal{F} \cong C_0(X,\mathbb{R})/C_{\rm fin}(X,\mathbb{Q})$ the assertion follows. □

Lemma 12.36. (i) *If X is a compact pointed space such that* $\dim_{\mathbb{R}} C_0(X,\mathbb{R}) \ge \mathfrak{c}$ *then, for every subgroup A of $C_0(X,\mathbb{R})$, there is an injective $\mathbb{R}$-linear map $\mathbb{R}\otimes_{\mathbb{Z}} A \to C_0(X,\mathbb{R})$.*

(ii) *If X is a compact pointed space with $w(X) > 2^{\aleph_0}$ then* $\dim_{\mathbb{R}} C_0(X,\mathbb{R}) > 2^{\aleph_0}$ *and so Part* (i) *applies.*

Proof. (i) The inclusion $j\colon A \to C_0(X,\mathbb{R})$ induces an injective $\mathbb{R}$ linear map $\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} j\colon \mathbb{R} \otimes_{\mathbb{Z}} A \to \mathbb{R} \otimes_{\mathbb{Z}} C_0(X,\mathbb{R})$ because $\mathbb{R}$ is torsion free. The assertion will be proved if we show that the $\mathbb{R}$-vector spaces $\mathbb{R} \otimes_{\mathbb{Z}} C_0(X,\mathbb{R})$ and $C_0(X,\mathbb{R})$ are isomorphic. For this it suffices to show that their $\mathbb{R}$-dimensions are equal.

Let S denote a set. Then, as $\mathbb{Q}$-vector spaces, $\mathbb{R}^{(S)} \cong (\mathbb{Q}^{(\mathfrak{c})})^{(S)} \cong \mathbb{Q}^{(\mathfrak{c}\cdot S)}$. Thus $\mathrm{card}\,\mathbb{R}^{(S)} = \mathfrak{c}\cdot \mathrm{card}\, S$. If V is a real vector space, then

$$(*) \qquad \mathrm{card}\, V = \mathfrak{c}\cdot \dim_{\mathbb{R}} V$$

and if $\dim_{\mathbb{R}} V \ge \mathfrak{c}$, then $\dim_{\mathbb{R}} V = \mathrm{card}\, V$.

Now

$$\mathbb{R} \otimes_{\mathbb{Z}} C_0(X,\mathbb{R}) \cong \mathbb{R}^{(\dim_{\mathbb{Q}} C_0(X,\mathbb{R}))} = \mathbb{R}^{(\mathrm{card}\, C_0(X,\mathbb{R}))}$$

because $\dim_{\mathbb{Q}} C_0(X,\mathbb{R})$ is infinite. Further, $\mathrm{card}\, C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$ (Appendix 4). Thus $\mathbb{R} \otimes_{\mathbb{Z}} C_0(X,\mathbb{R}) \cong \mathbb{R}^{(w(X)^{\aleph_0})}$. Hence $\dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$ and

$\operatorname{card} C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$. Since $\dim_{\mathbb{R}} C_0(X,\mathbb{R})$ was assumed to be $\geq \mathfrak{c}$ we conclude

$$\dim_{\mathbb{R}} C_0(X,\mathbb{R}) = w(X)^{\aleph_0}.$$

This gives the desired equality of dimensions.

(ii) For infinite X we know $\operatorname{card} C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$. Thus $w(X) > \mathfrak{c}$ implies $\operatorname{card} C_0(X,\mathbb{R}) > \mathfrak{c}$. If $\dim C_0(X,\mathbb{R}) \leq \mathfrak{c}$, then

$$\operatorname{card} C_0(X,\mathbb{R}) = \mathfrak{c}\cdot \dim_{\mathbb{R}} C_0(X,\mathbb{R}) \leq \mathfrak{c}.$$

Therefore $\dim_{\mathbb{R}} C_0(X,\mathbb{R}) > \mathfrak{c}$, as asserted. □

Lemma 12.37. *Let A denote an abelian torsion group, B a torsion-free abelian group and C a torsion-free subgroup of $A \oplus B$. Then the projection $p\colon A \oplus B \to B$ maps C injectively into B.*

Proof. Since $\ker p = A$ we have $\ker(p|C) = A \cap C$. As A is a torsion group and C is torsion-free we have $A \cap C = \{0\}$. Thus $p|C$ is injective. □

Lemma 12.38. *Let A be a subgroup of $C_0(X,\mathbb{T})$ for a compact space X with $w(X) > 2^{\aleph_0}$ and with $[X,\mathbb{T}] = 0$. Then there is an injective linear map $\mathbb{R} \otimes_{\mathbb{Z}} A \to C_0(X,\mathbb{R})$.*

Proof. Since $[X,\mathbb{T}] = 0$ the group $C_0(X,\mathbb{T})$ is a quotient of $C_0(X,\mathbb{R})$ and thus is divisible. Hence its torsion subgroup $C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})$ is a direct summand. Thus Lemma 12.37 applies and shows that A is isomorphic to a subgroup of the factor group $C_0(X,\mathbb{T})/C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})$. This latter group is isomorphic to $C_0(X,\mathbb{R})$ by Lemmas 12.34 and 12.35. Thus A is isomorphic to a subgroup of $C_0(X,\mathbb{R})$. But then Lemma 12.36 applies and proves the claim. □

Lemma 12.39. *Let E be a real topological vector space and X a subset of E such that E is the closed linear span of X. Then as an additive topological group, $E = \langle X \cup \sqrt{2}X\rangle$. If X is discrete and closed in $E \setminus \{0\}$, then $X \cup \sqrt{2}X$ is suitable.*

Proof. For each $x \in X$, the group $\langle X \cup \sqrt{2}X\rangle$ contains $\mathbb{R}\cdot x = \langle \mathbb{Z} + \sqrt{2}X\rangle$, hence it contains the linear span of X.

If X is discrete, then $X \cup \sqrt{2}X$ is discrete, and if X is closed in $E \setminus \{0\}$ then so is $X \cup \sqrt{2}X$. □

Lemma 12.40. *Let V be a real vector space and V^* the algebraic dual with the topology of pointwise convergence. Denote by $(V^*)'$ the topological dual of V^*. Then $e\colon V \to (V^*)'$, $e(v)(\alpha) = \alpha(v)$ is an isomorphism of $\mathbb{R}$-vector spaces.*

Proof. This is a consequence of 7.5(iii) and the Duality Theorem of Real Vector Spaces 7.30. □

Exercise E12.4. Give a direct proof of Lemma 12.40.

[Hint. Since V^* separates the points of V, clearly e is injective. Let $\Omega: V^* \to \mathbb{R}$ be a continuous linear functional. Let $U = \Omega^{-1}(]-1,1[)$. Then by the definition of the topology of pointwise convergence on V^*, there are vectors $v_1, \ldots, v_n \in V$ and there is an $\varepsilon > 0$ such that $|\alpha(v_j)| < \varepsilon$, $j = 1, \ldots, n$, implies $\alpha \in U$; that is $|\Omega(\alpha)| < 1$. Let F denote the span of the v_j and $A = F^\perp$ the vector space of all $\alpha \in V^*$ vanishing on all v_j. Then $\Omega(A)$ is a vector subspace contained in $]-1,1[$ and is, therefore $\{0\}$. Thus Ω induces a linear functional ω on $V^*/A \cong F^*$; that is $\omega \in F^{**}$. Hence, by the duality of finite dimensional vector spaces, there is a $w \in F$ such that $\omega(\alpha + A) = \alpha(w)$. It follows that $\Omega(\alpha) = \alpha(w)$ and thus $\Omega = e(w)$. Thus e is surjective, too.] □

Lemma 12.41. *The closed* $\mathbb{R}$*-linear span of* $\eta(X')$ *is* $C_0(X', \mathbb{R})$.

Proof. Set $E = \langle \mathbb{R}\cdot\eta(X') \rangle$, the closed $\mathbb{R}$-linear span of $\eta(X')$ in $C_0(X', \mathbb{R})^*$. We claim that $E = C_0(X', \mathbb{R})^*$. If not, then there is a nonzero continuous linear functional $\Omega: C_0(X', \mathbb{R})^* \to \mathbb{R}$ vanishing on E by the Hahn–Banach Theorem. Now we apply Lemma 12.40 with $V = C_0(X', \mathbb{R})$ and find that there is an $f \in C_0(X', \mathbb{R})$ such that $\Omega(\alpha) = \alpha(f)$. Hence $E(f) = \{0\}$. In particular, $f(x) = \eta(x)(f) = 0$ for all $x \in X'$. Thus $f = 0$ and therefore $\Omega = 0$, a contradiction. Thus $E = C_0(X', \mathbb{R})$ is proved. □

Now we are ready for a proof of Assertion (B). Thus we consider a compact connected abelian group G with weight $w(G) > 2^{\aleph_0}$. We know that $s(G)^{\aleph_0} = w(G)^{\aleph_0}$. If we had $s(G) \le 2^{\aleph_0}$, then

$$w(G) \le w(G)^{\aleph_0} = s(G)^{\aleph_0} \le (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0},$$

in contradiction to our hypothesis. Thus G contains a special subset X of cardinality $s(G) > 2^{\aleph_0}$ such that $X' = X \cup \{\mathbf{1}\}$ is compact. For infinite suitable sets X we have $w(X') = \operatorname{card} X$. Thus $w(X') > 2^{\aleph_0}$. Since the pointed space X' is generating, the natural morphism $f: FX' \to G$ from the free compact abelian group FX' on X' to G satisfying $f(x) = x$ for $x \in X$ is surjective. Hence $\widehat{f}: \widehat{G} \to \widehat{FX'}$ is injective. But $\widehat{FX'} \cong C_0(X', \mathbb{T})$. By Lemma 12.38 we thus have an injective $\mathbb{R}$-linear map $j: \mathbb{R} \otimes_{\mathbb{Z}} \widehat{G} \to C_0(X', \mathbb{R})$. Its dual $\operatorname{Hom}_{\mathbb{R}}(j, \mathbb{R}): \operatorname{Hom}_{\mathbb{R}}(C_0(X', \mathbb{R}), \mathbb{R}) \to \operatorname{Hom}(\mathbb{R} \otimes_{\mathbb{Z}} \widehat{G}, \mathbb{R})$ is a surjective continuous $\mathbb{R}$-linear map between topological vector spaces. But $\operatorname{Hom}_R(\mathbb{R} \otimes_{\mathbb{Z}} \widehat{G}, \mathbb{R}) \cong \operatorname{Hom}(\widehat{G}, \mathbb{R}) \cong L(G)$. Thus we have produced a continuous surjective $\mathbb{R}$-vector space morphism $j^*: C_0(X', \mathbb{R})^* \to L(G)$, where E^* denotes the algebraic dual of a real vector space E endowed with the topology of pointwise convergence. The natural map $\eta: X' \to C_0(X', \mathbb{R})^*$, $\eta(x)(f) = f(x)$ is a topological embedding since the continuous functions on a compact space separate the points, since the topology of $C_0(X', \mathbb{R})^*$ is that of pointwise convergence, and since X' is compact. By Lemma 12.2 we know that $Z = j^*(\eta(X') \setminus \{0\})$ is discrete and closed in $L(G) \setminus \{0\}$ and is such that $Z \cup \{0\}$ is compact. By Lemma 12.41, the closed $\mathbb{R}$-linear span of $\eta(X')$ is $C_0(X', \mathbb{R})$. Hence the closed $\mathbb{R}$-linear span of Z is $L(G)$. Then by Lemma 12.39, the set $Y = Z \cup \sqrt{2}Z$ is suitable in $L(G)$. By Lemma 12.2 we know that $\exp Y$ is suitable in G. Hence

$s(G) \leq \operatorname{card}(\exp Y) \leq \operatorname{card} Y \leq \operatorname{card} X = s(G)$. So $\exp Y$ is a special subset of G. Since Y is a suitable set of $L(G)$ we have $s(L(G)) \leq \operatorname{card} Y = s(G)$.

This completes the proofs of Assertion (B). Thus we have proved the following theorem.

Theorem 12.42 (Special Generating Sets in Connected Compact Groups). *The arc component G_a of the identity in a compact connected group G contains a special generating subset of G.* □

Postscript

Cardinal invariants of topological spaces have a long standing tradition in general topology and in topological group theory (see e.g. [67], [68], [70]); examples are the weight, density, and various topological dimensions. In topological group theory other invariants emerge that take the group structure into account. An example is the generating rank $s(G)$ which was introduced by the authors in [181]. They showed it to be applicable to compact groups by proving that suitable sets exist in every locally compact group. This extended earlier work by Mel'nikov [254] on profinite groups. In [73] suitable sets are shown to exist for all metrizable groups and all countable topological groups. There is also work of D. Dikranjan, O. G. Okunev, D. Shakhmatov, M. Tkachenko and V. V. Tkachuk on the existence and nonexistence of suitable sets in topological groups.

In the study of the structure of compact groups of large cardinality, it is convenient to have cardinal invariants as a coarse tool of discriminating amongst them. We have seen in this book that the weight is a particularly convenient topological invariant. In Chapters 8 and 9 we also introduced the dimension dim as topological group invariant for compact groups which is in line with our emphasis of the exponential function as a major structural component.

The linking of the cardinal invariants weight, density, generating rank and dimension is not trivial. For compact *connected* groups the dimension is revealed to be the most important one because all the others are expressed in terms of the dimension (Theorem 12.25). One of the most interesting phenomena in the context of connected compact groups is the "Descent Procedure" by which the cardinal invariants of the group are reduced to that of a maximal pro-torus (see 12.11, 12.24, 9.36(vi)).

The class of connected compact and that of totally disconnected compact groups exhibit significantly different behavior, as is illustrated by a juxtaposition of 12.25 and 12.28. The results in 12.27ff show that for topological group cardinal invariants, a reduction to the cases of compact connected and compact totally disconnected groups is not trivially in sight. The results of Chapter 10 leading into the Topological Decomposition of Compact Groups Theorem 10.39 show that for topological invariants, a compact group is of the form $A \times S \times T$ for a compact connected *abelian* group A, a compact connected *semisimple* group (which is very nearly a product of simple compact Lie groups as we explained e.g. in the Structure of Semisimple Compact Connected Groups Theorem 9.19), and a compact totally disconnected group T. (Recall from [147], Theorem 9.15, p. 95, that every infinite compact totally disconnected group is homeomorphic to a power of the two point space.) This shows for the study of topological cardinal invariants, knowing them for *connected* compact groups suffices. Remarkably, pursuing these observations for the simply accessible cardinal invariant of *density* allows us to note in Theorem 12.31a that it depends on the weight in an order preserving fashion. The fact that $d(\mathbb{Z}(2)^{2^n}) = n$ for all infinite cardinals may be counterintuitive at first sight.

References for this Chapter—Additional Reading

[68], [71], [147], [73], [91], [181], [184], [254], [361].

Appendix 1

Abelian Groups

In this appendix we shall record background material on abelian groups. We begin by fixing standard notation and by listing elementary examples. Basically we stay in the domain of algebra—occasional excursions into analysis notwithstanding. Certain simple constructions of new abelian groups from given ones will be recorded and we shall discuss the concepts of free, projective and injective groups and their characterizations. In the end we shall turn to the homological algebra of abelian groups and give a complete discussion of the Whitehead problem and Shelah's solution of it [329]. This material is important for an understanding of arcwise connectivity of compact abelian groups in Chapter 8.

By and large we shall write abelian groups additively; for some examples, the multiplicative notation is natural. Frequently in this appendix, we shall simply refer to an abelian group as a group; if non-abelian groups are meant we will say so.

Prerequisites. An understanding of the concepts of a subgroup and a factor group modulo a subgroup are taken for granted as is a grasp of the idea of a homomorphism $f: A \to B$ and its canonical decomposition

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \mathrm{incl}} \\ A/\ker f & \xrightarrow[f']{} & \mathrm{im}\, f \end{array}$$

in which $f': A/\ker f \to \mathrm{im}\, f$ is an isomorphism.

Usually, exercises will ask for a proof of a statement. This will not always be repeated. Thus unless stated otherwise, all assertions made in the exercises are to be proved.

For some exercises on the homological algebra of abelian groups one may wish to consult such sources as MacLane's book on homology [245]. The last section on Whitehead groups will place more demands on the reader's willingness to follow arguments in the line of axiomatic set theory and of ordinal and cardinal arithmetic and their applications in abelian group theory. A good reference for these developments is the book by Eklof and Mekler [99] and a good source for set theory is the book by Jech [221].

Examples

We begin with the examples most closely at hand.

Examples A1.1. (i) The additive group of real numbers is called $\mathbb{R}$; in it we find the subgroup $\mathbb{Q}$ of all rational numbers and the group $\mathbb{Z}$ of all integers. All of these groups also carry a multiplication; the first two are fields, the last is a ring.

Every infinite *cyclic* (i.e. singly generated) group is isomorphic to $\mathbb{Z}$. Indeed, if g is an element of a multiplicatively written group G then

$$e\colon \mathbb{Z} \to G, \quad e(n) = g^n$$

is a homomorphism and the group $\langle g\rangle$ generated by g is the image $\operatorname{im} e$ and $\langle g\rangle \cong \mathbb{Z}/\ker e$ by canonical decomposition. But if K is a nonzero subgroup of $\mathbb{Z}$, then $\mathbb{Z}/K$ is finite. (Exercise EA1.1(i).) Thus, if $\langle g\rangle$ is infinite, then $\langle g\rangle \cong \mathbb{Z}$.

(ii) If $m \in \mathbb{N}$, we write $\mathbb{Z}(m) = \mathbb{Z}/m\mathbb{Z}$. If G is cyclic of order m, then the surjective homomorphism e has the kernel $m\mathbb{Z}$ (Exercise 1(i)) and thus induces an isomorphism $e'\colon \mathbb{Z}(m) \to G$ via the canonical decomposition of e.

Since $m\mathbb{Z}$ is an ideal of the ring $\mathbb{Z}$, the factor group $\mathbb{Z}(m)$ is in fact a ring.

(iii) The factor group $\mathbb{R}/\mathbb{Z}$ of *reals modulo one* is written $\mathbb{T}$ and is called the 1-*torus*. It contains the subgroup $\mathbb{Q}/\mathbb{Z}$ as the subgroup of all elements of finite order. If m is a natural number, then $\frac{1}{m}\cdot\mathbb{Z}/\mathbb{Z}$ is the unique subgroup of order m.

(iv) In the field $\mathbb{C}$ of complex numbers, the set $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is multiplicatively closed and is closed under inversion. Hence it is a multiplicative group called *the circle group*. The exponential function $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot z^n$ yields a homomorphism $e\colon \mathbb{R} \to \mathbb{S}^1$ via $e(t) = \exp 2\pi i t$. Its kernel is $\mathbb{Z}$. Hence the canonical decomposition of e yields an isomorphism $e'\colon \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$. Thus the 1-torus and the circle group are isomorphic. The group of *rotations* of the euclidean plane $\mathbb{R}^2$ is given by the group of matrices

$$\mathrm{SO}(2) = \left\{ \left.\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} \right| t \in \mathbb{R} \right\}.$$

The function

$$t + \mathbb{Z} \mapsto \exp 2\pi t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} : \mathbb{T} \to \mathrm{SO}(2)$$

is an isomorphism. The 1-torus thus also manifests itself as the group of rotations of the euclidean plane. □

Exercise EA1.1. (i) Every subgroup of $\mathbb{Z}$ is cyclic. If $0 < m$, then $\mathbb{Z}/m\mathbb{Z}$ has m elements.

(ii) Every closed proper subgroup of $\mathbb{R}$ is cyclic.

(iii) $\mathbb{R}$ has uncountably many subgroups which are not cyclic.

[Hint. Regarding (ii): Let A be a closed proper subgroup of $\mathbb{R}$. Assume $A \neq \{0\}$. Set $a = \inf\{r \in A \mid 0 < r\}$. Use the closedness of A to show $a \in A$. If $a = 0$ show $A = \mathbb{R}$. If $0 < a$ show $A = \mathbb{Z}a$.

Regarding (iii): Show that $\mathbb{Q}$ is not cyclic and consider $r\mathbb{Q}$ for $r \notin \mathbb{Q}$.] □

Exercise EA1.2. (i) A function $f:\mathbb{R}\to X$ of the group $\mathbb{R}$ into a set is said to be *periodic with period* p if

$$(\forall x\in\mathbb{R})\quad f(x+p)=f(x).$$

The set of all periods of a function $f:\mathbb{R}\to X$ is a subgroup of $\mathbb{R}$. If X is a topological space and f is continuous, then this subgroup is closed.

(ii) Let $q:\mathbb{R}\to\mathbb{T}$ denote the quotient homomorphism given by $q(x)=x+\mathbb{Z}$ and let $e:\mathbb{T}\to\mathbb{S}^1$ denote the isomorphism given by $e(t+\mathbb{Z})=\exp 2\pi i t$. Then the following statements are equivalent for a function $f:\mathbb{R}\to\mathbb{R}$:

(1) f has period 1.
(2) There is a function $F:\mathbb{T}\to\mathbb{R}$ such that $f=F\circ q$.
(3) There is a function $\varphi:\mathbb{S}^1\to\mathbb{R}$ such that $f=\varphi\circ e\circ q$.

In this sense, periodic functions on $\mathbb{R}$ with period 1 and real-valued functions on $\mathbb{T}$ and real-valued functions on $\mathbb{S}^1$ are one and the same thing.

In the elementary theory of analytic functions we learn that for a function $\varphi:U\to\mathbb{C}$ which is analytic on an open set in $\mathbb{C}$ containing $\mathbb{S}^1$ there are real numbers $0<r<1<R$ such that the ring domain $V=\{z\in U\mid r<|z|<R\}$ is contained in U and φ has an expansion $\sum_{n=-\infty}^{\infty}a_nz^n$ into a Laurent series in V. Assume that $z=e(t+\mathbb{Z})$ and express $\varphi(z)$ as a series of scalar multiples of the numbers $\sin 2\pi t$ and $\cos 2\pi t$.

These observations are at the root of the theory of Fourier series expansions of functions which are periodic with period 1. □

Exercise EA1.3. (i) If a multiplication $(x,y)\mapsto xy$ makes $\mathbb{Q}/\mathbb{Z}$ into a ring, then all products are zero.

(ii) The circle group $\mathbb{T}$ has a natural topology such that the following conditions are satisfied:

(a) The quotient map $t\mapsto t+\mathbb{Z}:\mathbb{R}\to\mathbb{T}$ is continuous and open.
(b) The addition $(x,y)\mapsto x+y$ and the inversion $x\mapsto -x$ are continuous.
(c) $e:\mathbb{T}\to\mathbb{S}^1$, $e(t+\mathbb{Z})=e^{2\pi i t}$ is a homeomorphism (where $\mathbb{S}^1$ inherits its topology from that of $\mathbb{C}$).

The subgroup $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$ is dense.

If a *continuous* multiplication makes $\mathbb{T}$ into a ring, then all products are 0. □

The most elementary constructions are the forming of the direct product and the direct sum.

Lemma A1.2. (i) *If $\{A_j\mid j\in J\}$ is a family of abelian groups, then the cartesian product*

$$\prod_{j\in J}A_j=\{(a_j)_{j\in J}\mid(\forall k\in J)\ a_k\in A_k\}$$

is an abelian group under componentwise operations.

(ii) *If* $\operatorname{supp}\big((a_j)_{j\in J}\big)$ *denotes the set* $\{j \in J \mid a_j \neq 0\}$, *then* $\operatorname{supp}(g+h) \subseteq \operatorname{supp}(g) \cup \operatorname{supp}(h)$ *and* $\operatorname{supp}(-g) = \operatorname{supp}(g)$. *Hence*

$$\bigoplus_{j\in J} A_j \stackrel{\text{def}}{=} \Big\{g \in \prod_{j\in J} A_j \mid \operatorname{card}\operatorname{supp}(g) < \infty\Big\}$$

is a subgroup.

(iii) *If all* $A_j \subset A$ *for a group* A, *then the function* $f\colon \bigoplus_{j\in J} A_j \to A$, $f\big((a_j)\big) = \sum_{j\in J} a_j$ *is a well-defined homomorphism whose image is the subgroup* $\Big\langle \bigcup_{j\in J} A_j \Big\rangle$ *of* A *generated by all* A_j.

Proof. Exercise EA1.4. □

Exercise EA1.4. Prove A1.2.

[Hint. Assume that A_j is a family of subgroups of one and the same group A. If the support of $(a_j)_{j\in J}$ is finite, say $\{j_1, \dots, j_n\}$, then $\sum_{j\in J} a_j \stackrel{\text{def}}{=} a_{j_1} + \cdots + a_{j_n}$ is a well-defined element, and the function

$$(a_j)_{j\in J} \mapsto \sum_{j\in J} a_j \colon \bigoplus_{j\in J} A_j \to A$$

is a homomorphism whose image is the subgroup generated by $\bigcup_{j\in J} A_j$.] □

Definition A1.3. The group $\prod_{j\in J} A_j$ is called *the direct product of the groups* A_j and $\bigoplus_{j\in J} A_j$ is called their *direct sum*. If $A_j = A$ for all $j \in J$ we write

$$\prod_{j\in J} A_j = A^J \qquad \text{and} \qquad \bigoplus_{j\in J} A_j = A^{(J)}.$$ □

In these special cases it may be advantageous to write the elements of A^J as functions $f\colon J \to A$.

In particular, the group $\mathbb{Z}^{(X)}$ contains a copy of the set X via the inclusion map $j_X\colon X \to \mathbb{Z}^{(X)}$ given by

$$j_X(x)(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

The simple construction of the direct product and the direct sum allow us to construct abelian groups of arbitrarily large cardinality from the elementary examples given in the beginning. But even in elementary situations these constructions are indispensable. We shall prove in the next section the fundamental theorem of finitely generated abelian groups saying that any such group is a direct sum of cyclic groups. A simple special case is considered in the following easy exercise:

Exercise EA1.5. If $m = p_1^{n_1} \cdots p_k^{n_k}$ is the prime decomposition of a natural number then

$$\mathbb{Z}(m) \cong \mathbb{Z}(p_1^{n_1}) \oplus \cdots \oplus \mathbb{Z}(p_k^{n_k}).$$ □

Free Abelian Groups

Definition A1.4. Let X denote a set and $F(X)$ a group together with a function $j: X \to F(X)$. We say that $F(X)$ is *a free abelian group over the set* X (via j) if for every function $f: X \to A$ into an abelian group there is a unique morphism $f': F(X) \to A$ such that $f = f' \circ j$.

$$\begin{array}{ccccc}
\multicolumn{3}{c}{\textbf{Sets}} & & \textbf{Abgroups} \\
\hline
X & \xrightarrow{j} & F(X) & \qquad & F(X) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f'} \\
A & \xrightarrow[\mathrm{id}_A]{} & A & & A
\end{array}$$

We say that an abelian group is *free* if it is isomorphic to a free abelian group over some set.

A subset X of a group G is said to be *free* if the subgroup $\langle X\rangle$ generated in G by X is a free group over X (via the inclusion map $X \to G$). □

This is a good example of a definition expressed in terms of a universal property. At first glance such definitions appear to be roundabout. They turn out to be very effective. We shall soon say what free abelian groups look like.

Remark A1.5. (i) The map j in the definition of a free abelian group is injective.

(ii) If for $n = 1, 2$ the two functions $j_n: X \to F_n(X)$ define free abelian groups over X, then there is a unique isomorphism $i: F_1(X) \to F_2(X)$ such that $j_2 = i \circ j_1$.

Proof. Exercises EA1.6. □

Exercise EA1.6. Prove (i), (ii). □

In view of Remark A1.5(i) we may assume that X is in fact a subset of $F(X)$, and that the morphism f' extends the function f. Remark A1.5(ii) shows that the free abelian group over X is unique (up to a natural isomorphism).

We shall now explicitly describe the structure of the free abelian group over a given set X. First we define a function $x \mapsto e_x: X \to \mathbb{Z}^{(X)}$ by

$$e_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition A1.6. *The group $\mathbb{Z}^{(X)}$ is the free abelian group over X via $x \mapsto e_x$.*

Proof. Let $f: X \to A$ be a function. Define $f': \mathbb{Z}^{(X)} \to A$ by $f'\big((z_x)_{x\in X}\big) = \sum_{x\in X} z_x \cdot f(x)$. Notice $(z_x)_{x\in X} = \sum_{x\in X} z_x \cdot e_x$. Now f' is a morphism satisfying $f(x) = f'(e_x)$, and it is unique with respect to this property. □

Remark A1.7. (i) A subset X of an abelian group G is free if and only if any relation $\sum_{x\in X} n_x{\cdot}x = 0$, $n_x \in \mathbb{Z}$ implies $n_x = 0$ for all $x \in X$.

(ii) For two sets X and Y we have $F(X) \cong A \leq F(Y)$ if and only if $\operatorname{card} X \leq \operatorname{card} Y$. (The cardinality of a set X is denoted by $\operatorname{card} X$.)

(iii) $F(X) \cong F(Y)$ if and only if $\operatorname{card} X = \operatorname{card} Y$.

Proof. Exercise EA1.7. □

Exercise EA1.7. Prove A1.7.

[Hint. (i) The family $(e_x)_{x\in X}$ in $\mathbb{Z}^{(X)}$ certainly does not satisfy a nontrivial linear relation. Conversely, if a subset $X \subseteq G$ satisfies no nontrivial linear relation, define $\varphi\colon \mathbb{Z}^{(X)} \to G$ by $\varphi(e_x) = x$ and show that it has zero kernel.

(ii) If $F(X) \cong A \subseteq F(Y)$, then $\mathbb{Z}^{(X)} \cong B \subseteq \mathbb{Z}^{(Y)}$ and hence $\mathbb{Q}^{(X)} \cong \operatorname{span}_{\mathbb{Q}} B \subseteq \mathbb{Q}^{(Y)}$. Now it is a question of linear algebra to observe that the dimension of a vector subspace does not exceed the dimension of the containing space.

If $\operatorname{card} X \leq \operatorname{card} Y$ then $\mathbb{Z}^{(X)}$ may be identified with a subgroup of $\mathbb{Z}^{(Y)}$.

(iii) is a consequence of (ii).] □

From Remark A1.7(iii) we see that $\operatorname{card} X$ is an invariant of any free abelian group $G \cong F(X)$; it is called *the rank of* G and is written $\operatorname{rank} G$.

Proposition A1.8. (i) *Every abelian group is a quotient group of a free abelian group.*

(ii) *Every abelian group A is a quotient group of the free abelian group $F(A)$ over the set underlying A.*

(iii) *Every countable abelian group is a quotient group of a countable free abelian group.*

Proof. Let A denote an abelian group. Then the universal property of Definition A1.4 provides a surjective morphism $F(A) \to A$ arising from $X = A$ and $f = \mathrm{id}_A$. This proves (i) and (ii) from which (iii) immediately follows. □

If we realize the free abelian group over A as $\mathbb{Z}^{(A)}$, then the surjective morphism $\mathbb{Z}^{(A)} \to A$ is explicitly given by $\sum_{a\in A} z_a{\cdot}e_a \mapsto \sum_{a\in A} z_a{\cdot}a$.

There is an important result which is harder. We recall that a *well-order* $<$ on a set X is a total order such that every non-empty subset has a minimal element. An ordered set whose order is a well-order is called *well-ordered*. The *Well-Ordering Theorem* states:

(WOT) *Every set can be well-ordered.*

This statement is equivalent to the *Axiom of Choice*:

(AC) *For each surjective function $f\colon X \to Y$ there is a function $\sigma\colon Y \to X$ such that $f \circ \sigma = \mathrm{id}_Y$.*

It is also equivalent to Zorn's Lemma (see e.g. [136] or [375]):

(ZL) *A partially ordered set has maximal elements provided it is inductive* (*i.e. every totally ordered subset is bounded above*).

The Subgroup Theorem for Free Abelian Groups

Theorem A1.9. *Every subgroup of a free abelian group is a free abelian group.*

Proof. (AC)[1] Let F be a free abelian group and G a subgroup. Let X be a free set of generators of F and, by the (WOT) find a well-order $<$ on X. The coordinate projection $p_x: F \to \mathbb{Z}$ given by $p_x(\sum_{y\in X} n_y{\cdot}y) = n_x$ is a homomorphism. Let $F_x = \langle y \in X \mid y \le x\rangle$ denote the subgroup spanned by the $y \in X$ with $y \le x$. The subgroup $p_x(F_x \cap G)$ of $\mathbb{Z}$ is generated by one element, say z_x. By (AC) we pick $g_x \in F_x \cap G$ so that $p_x(g_x) = z_x$ and that $g_x = 0$ if $z_x = 0$. We write $G_x = \{g_y \mid y \le x \text{ and } g_y \ne 0\}$ and $Y \overset{\text{def}}{=} \{x \mid x \in X \text{ and } g_x \ne 0\}$. Note that $G_\infty = \bigcup_{x\in X} G_x = \{g_y \mid y \in Y\}$. We claim

(a) G_∞ generates G,

(b) G_∞ is free,

which will prove the theorem. First, in order to prove (a), we consider the set X^* of all $x \in X$ such that

$$\langle G_x\rangle = F_x \cap G.$$

We claim $X^* = X$. Assume momentarily that this claim were established. If $g \in G$ then $g = \sum_{y\in X} m_y{\cdot}y$, and if $x = \max\{y \mid m_y \ne 0\}$, then $g \in F_x \cap G$. Then g is a linear combination of the g_y since $\langle G_x\rangle = F_x \cap G$. Hence (a) follows.

Now we shall prove that for an arbitrary $x \in X$ the assumption $y < x \Rightarrow y \in X^*$ implies $x \in X^*$. If this is established we have $X^* = X$ for if not, then the complement $X \setminus X^*$ has a minimal element x; then the predecessors of x in X are in X^* and by what we assume this implies $x \in X^*$—a contradiction. Therefore, assume that $y \in X^*$ for all $y < x$. Consider $g \in F_x \cap G$. Then $p_x(g) = nz_x$ with some $n \in \mathbb{Z}$. Hence $p_x(g - n{\cdot}g_x) = p_x(g) - np_x(g_x) = 0$, and thus $g - n{\cdot}g_x = \sum_{y<x} m_y{\cdot}g_y$. Thus there is an $x' < x$ with $g - n{\cdot}g_x \in F_{x'} \cap G$. By assumption, $F_{x'} \cap G = \langle G_{x'}\rangle$, and thus $g - n{\cdot}g_x \in \langle G_{x'}\rangle$. Then $g \in \langle G_x\rangle$ which completes the proof of (a).

Now we prove (b). Let $\sum_{y\in Y} m_y{\cdot}g_y = 0$. We must show $m_y = 0$ for all $y \in Y$. Suppose this is not true. Since at most finitely many of the m_y are nonzero, the element $x = \max\{y \mid m_y{\cdot}g_y \ne 0\}$ exists. Now $0 = p_x(\sum_{y\le x} m_y{\cdot}g_y) = p_x(\sum_{y<x} m_y{\cdot}g_y) + p_x(m_x{\cdot}g_x)$. Now let $x' = \max\{y \mid m_y{\cdot}g_y \ne 0 \text{ and } y < x\} < x$. Since $x' < x$ and $G_{x'} \subseteq F_{x'}$ and since $p_x(F_{x'}) = \{0\}$ we know that $p_x(\sum_{y<x} m_y{\cdot}g_y) = 0$. But $m_x p_x(g_x) = m_x z_x$. Since $x \in Y$ implies $g_x \ne 0$ we have $z_x \ne 0$. Hence $m_x = 0$ a contradiction to the definition of x. This completes the proof of (b) and thereby the proof of the theorem. □

If X is a finite set then we can say much more about the subgroups of $F(X)$. Recall that we have on $\mathbb{N}$ a partial order $a|b$ ("a divides b") iff $\mathbb{Z}{\cdot}b \subseteq \mathbb{Z}{\cdot}a$ iff there is an $a' \in \mathbb{N}$ such that $aa' = b$.

1 We mark proofs in this appendix that use the Axiom of Choice in this fashion.

The Elementary Divisor Theorem

Theorem A1.10. (A) *Let F be a free abelian group over a finite set and G a nonzero subgroup. Then there exist, firstly, a free generating set $X = \{x_1, \ldots, x_n\}$ of F, secondly a natural number $1 \le d \le n$, and thirdly natural numbers $m_1|m_2|\cdots|m_d$ such that $\{m_1{\cdot}x_1, \ldots, m_d{\cdot}x_d\}$ is a free generating set of G.*

(B) *If P is a subgroup of G such that F/P has no elements of finite order, then there is a nonnegative integer c, $0 \le c \le d$ such that, in the case that $0 < c$, the elements $x_1, \ldots, x_c$ may be chosen to be a free generating set of P and that $m_1, \ldots, m_c = 1$.*

Proof. We consider the set $\mathcal{X}$ of all free generating sets X of F. For $0 \ne g \in G$ and $X \in \mathcal{X}$ we write $g = \sum_{x\in X} n_x{\cdot}x$ in a unique fashion and set $n(g,X) = \min\{|n_x| \mid x \in X,\, 0 \ne n_x\}$. We define

$$\mathbb{N}(G,F) = \{n(g,X) \in \mathbb{N} \mid (g,X) \in G \times \mathcal{X}\}.$$

Then $m = \min \mathbb{N}(G,F)$ is a positive natural number and we find a pair $(g,X) \in G \times \mathcal{X}$ such that $m = n(g,X)$, and since $n(g,X) = n(-g,X)$ we may assume $g = m{\cdot}x + \sum_{y\in X\setminus\{x\}} n_y{\cdot}y$. We claim that $m|n_y$. Proof of the claim: We find integers q_y and r_y such that $n_y = q_y m + r_y$ with $0 \le r_y < m$. Setting $x' = x + \sum_{y\in X\setminus\{x\}} q_y{\cdot}y$ we have $Y = \{x'\} \cup (X \setminus \{x\}) \in \mathcal{X}$, and $g = m{\cdot}x' + \sum_{y\in X\setminus\{x\}} r_y{\cdot}y$. By the minimality of m we conclude $r_y = 0$ for all $y \in X \setminus \{x\}$. This proves the claim. We note that $m{\cdot}x' = g$. The projection $p_{x'}\colon F \to \mathbb{Z}$, defined by $f = p_{x'}(f){\cdot}x' + \sum_{y\in X\setminus\{x\}} n_y{\cdot}y$, maps G onto a subgroup of $\mathbb{Z}$ with minimal element m, i.e. on $\mathbb{Z}{\cdot}m$. Thus, for $h \in G$ we compute $p_{x'}(h - \frac{p_{x'}(h)}{m}{\cdot}g) = 0$ and thus we find $h - \frac{p_{x'}(h)}{m}{\cdot}g \in G \cap \langle X \setminus \{x\}\rangle$. Therefore, setting $F' = \langle X \setminus \{x\}\rangle$ and $G' = G \cap \langle X \setminus \{x\}\rangle$ we get

$$F = \mathbb{Z}{\cdot}x' \oplus F',$$
$$G = \mathbb{Z}{\cdot}g \oplus G'.$$

Moreover, if X' is any free generating set of F' and $g' = \sum_{y'\in X'} n_{y'}{\cdot}y' \in G'$, then $m|n_{y'}$ for all $n_{y'}$ by the definition of m as we see by the same proof as for the claim above.

The proof of the theorem now follows by induction on $\operatorname{rank}(G)$. □

The (uniquely determined) natural numbers $m_1, \ldots, m_d$ are called the *elementary divisors* of G in F.

The Fundamental Theorem of Finitely Generated Abelian Groups

Theorem A1.11. *Let A be a finitely generated abelian group.*

(i) *Then there is a unique sequence of natural numbers $1 < m_1|m_2\cdots|m_d$ and a natural number m_0 such that*

(1) $$A \cong \mathbb{Z}(m_1) \oplus \cdots \oplus \mathbb{Z}(m_d) \oplus \mathbb{Z}^{m_0}.$$

(ii) *For each prime* p *there is a subgroup*

$$A_p \cong \mathbb{Z}(p)^{k(p,1)} \oplus \mathbb{Z}(p^2)^{k(p,2)} \oplus \mathbb{Z}(p^3)^{k(p,3)} \oplus \cdots \tag{2}$$

such that

$$A = \mathbb{Z}^{m_0} \oplus \bigoplus A_p \tag{3}$$

$$= \mathbb{Z}^{m_0} \oplus \bigoplus_{\substack{p \text{ prime} \\ n\in\mathbb{N}}} \mathbb{Z}(p^n)^{k(p,n)}, \tag{4}$$

where $k(p,n) = 0$ *for all but finitely many pairs* (p,n).

Proof. (i) By A1.8 there is a finite set X and a quotient morphism $q: F(X) \to A$. Let $G = \ker q$. By the Elementary Divisor Theorem A1.10 there is an ordered sequence of natural numbers $1,\dots,1,m_1,\dots,m_d$, $1 < m_1|\cdots|m_d$ and a free generating set

$$\{e_1,\dots,e_m,x_1,\dots,x_d,x_{d+1},\dots,x_{d+m_0}\}$$

such that

$$\{e_1,\dots,e_m,m_1{\cdot}x_1,\dots,m_d{\cdot}x_d\}$$

is a free generating set of G. Then

$$A \cong \frac{\mathbb{Z}^{\{e_1,\dots,e_m,x_1,\dots,x_d,x_{d+1},x_{d+m_0}\}}}{\mathbb{Z}^{\{e_1,\dots,e_m,x_1,\dots,x_d\}}} \cong \mathbb{Z}(m_1) \oplus \cdots \oplus \mathbb{Z}(m_d) \oplus \mathbb{Z}^{m_0}.$$

(ii) By Exercise EA1.5, for each natural number $m = p_1^{n_1}\cdots p_k^{n_k}$ (p_j prime) we have

$$\mathbb{Z}(m) = \mathbb{Z}(p_1^{n_1}) \oplus \cdots \oplus \mathbb{Z}(p_k^{n_k}). \tag{5}$$

All we need to do now is to substitute (5) into (1) and rearrange direct summands to get (4). Finally, (2) and (3) are just a consequence of regrouping (4). □

Some remarks are in order on the preceding two theorems.

First a general remark which we shall pursue a bit more systematically later (cf. A1.14). If a subgroup B of an abelian group A is such that A/B is free, then there is a free subgroup C of A such that $A = B \oplus C$. Indeed, let $\{x_j + B \mid j \in J\}$ denote a free generating set of A/B. By the universal property of free groups there is a unique homomorphism $f: A/B \to A$ such that $f(x_j + B) = x_j$. Set $C = f(A/B)$. Then $f(a+B) + B = a + B$, equivalently, $a \in f(a+B) + B$ for all $a \in A$. Thus $A = B + C$. Let $c \in B \cap C$, say $c = f(a+B)$. Then $a \in f(a+B)+B = c+B = B$, whence $c = f(B) = 0$. Thus $A = B \oplus C$ as asserted. Since $C \cong A/B$, the subgroup C is free.

Assume the circumstances of the Elementary Divisor Theorem A1.10 and assume, in addition that P is a subgroup of G such that F/P is a finitely generated abelian group without nonzero elements of finite order. Then, since the factor group F/P is finitely generated, it is free by the Fundamental Theorem A1.11. Hence $F = P \oplus C$ by the preceding observation and $G = P \oplus (G \cap C)$. We can therefore fix a free generating set $\{x_1, \ldots, x_c\}$ of P and applying the Elementary Divisor Theorem A1.10 to C and $G \cap C$ actually produce a free generating set $X = \{x_1, \ldots, x_c, x_{c+1}, \ldots, x_d, x_{d+1}, \ldots, x_n\}$ of F whose existence is asserted in A1.10 in such a fashion that $m_1 = \cdots = m_c = 1$ and that the elements $x_1, \ldots, x_c$ generate P.

It is important to realize that *the group G may contain subgroups S such that $P \cap S = \{0\}$ which are not contained in any direct summand C of F that is complementary to P.* A simple example is illustrated by the case of $F = \mathbb{Z}^2 \subseteq \mathbb{Q}^2$, $P = \mathbb{Z} \times \{0\}$, $S = \mathbb{Z}\cdot(5,3)$. If S were contained in a direct summand C complementary to P, then, as a rank one subgroup, it would have to be contained in $\mathbb{Z}^2 \cap \mathbb{Q}\cdot(5,3) = \mathbb{Z}\cdot(5,3)$. But since $P \oplus \mathbb{Z}\cdot(5,3) = \mathbb{Z} \times 3\mathbb{Z}$, this is not possible.

If, in this example, we set $A = F/\mathbb{Z}\cdot(0,3)$, then $\operatorname{tor} A \cong \{0\} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}(3)$. The subgroup $Z \overset{\text{def}}{=} \big(S + \mathbb{Z}\cdot(0,3)\big)/\mathbb{Z}\cdot(0,3) = (5\mathbb{Z} \times 3\mathbb{Z})/(\{0\} \times 3\mathbb{Z}) \cong 5Z$ is not contained in any direct summand complementary to $\operatorname{tor} A$. Thus *a finitely generated abelian group A may contain a free subgroup Z but no direct summand C of A complementary to* $\operatorname{tor} A$ *contains Z.*

It is a fact that any discrete subgroup of $\mathbb{R}^n$ is a free abelian group of finite rank, i.e. a copy of $\mathbb{Z}^m$ for some m. Therefore the following theorem, even though it exhibits topological aspects, belongs here:

Closed Subgroups and Quotient Groups of $\mathbb{R}^n$

Theorem A1.12. (i) *If S is a closed subgroup of $\mathbb{R}^n$, then there is a basis $e_1, \ldots, e_n$ and there are natural numbers p, q such that*

$$S = \mathbb{R}\cdot e_1 \oplus \cdots \oplus \mathbb{R}\cdot e_p \oplus \mathbb{Z}\cdot e_{p+1} \oplus \cdots \oplus \mathbb{Z}\cdot e_{p+q}.$$

(ii) *The factor group $\mathbb{R}^n/S$ is isomorphic to $\mathbb{T}^q \oplus \mathbb{R}^m$ with $m = n - p - q$.*

(iii) *If E is a vector subspace of $\mathbb{R}^n$ containing q linearly independent elements from* $\operatorname{span}_{\mathbb{R}}\{e_{p+1}, \ldots, e_{p+q}\}$, *then $\frac{S+E}{S} \cong E/(S \cap E)$ is a closed subgroup of $\mathbb{R}^n/S$.*

Proof. Let V denote the largest vector subspace of $\mathbb{R}^n$ contained in S. Every vector subspace is closed. We let $e_1, \ldots, e_p$ denote a basis of V and record $V = \mathbb{R}\cdot e_1 \oplus \cdots \oplus \mathbb{R}\cdot e_p$. Now we can write $\mathbb{R}^n = V \oplus W$ with an arbitrarily chosen vector space complement $W \cong \mathbb{R}^{n-p}$ of V. Since $V \subseteq S$ we have $S = V = \oplus(W \cap S)$ (for if $s \in S$, write $s = v \oplus w$ and note that $w = -v + s \in W \cap S$). If we can prove the assertion for the subgroup $S \cap W$ in W, we have clearly proved assertion (i) in general. Thus without losing generality we assume that S contains no vector subgroups. First we claim that S is discrete. For a proof consider any norm on $\mathbb{R}^n$, e.g. the one given by the standard scalar product. If S is not discrete then there is a sequence of nonzero elements $s_n \in S$ converging to 0. Since the unit sphere is compact, after picking a converging subsequence we may assume that $e = \lim_{n\to\infty} \|s_n\|^{-1}\cdot s_n$ exists. If $r \in \mathbb{R}$, then $d_n \overset{\text{def}}{=} r\|s_n\|^{-1} - [r\|s_n\|^{-1}]$ is in $[0,1[$

(where $[x] = \max\{z \in \mathbb{Z} \mid z \leq x\}$). Thus $\lim_{n\to\infty} d_n{\cdot}s_n = 0$ whence

$$\begin{aligned}\|r{\cdot}e - [\frac{r}{\|s_n\|}]{\cdot}s_n\| &\leq \|r{\cdot}e - \frac{r}{\|s_n\|}{\cdot}s_n\| + \|\frac{r}{\|s_n\|}{\cdot}s_n + [\frac{r}{\|s_n\|}]{\cdot}s_n\| \\ &= |r|\|e - \|s_n\|^{-1}{\cdot}s_n\| + d_n\|s_n\| \to 0\end{aligned}$$

for $n \to \infty$. Since $[\frac{r}{\|s_n\|}]{\cdot}s_n \in S$ and S is closed, we conclude that $r{\cdot}e \in S$. Thus $\{0\} \neq \mathbb{R}{\cdot}e \subseteq S$, and this contradicts our assumption that S does not contain nontrivial vector subgroups. Now we claim that S is generated by a finite $\mathbb{R}$-linearly independent subset of $\mathbb{R}^n$: For a proof, let $f_1, \ldots, f_r$ denote any basis of the real vector subspace $\operatorname{span}_{\mathbb{R}} S$ spanned by the elements of S. Then $\operatorname{span}_{\mathbb{R}} S/\langle f_1, \ldots, f_r\rangle$ is isomorphic to $(\mathbb{R}/\mathbb{Z})^r$, and therefore is a compact group. The image $S/\langle f_1, \ldots, f_r\rangle$ of S is closed and discrete, hence is finite. Thus S is generated by a finite set F, say. For each $f \in F$ there is a natural number m such that $m{\cdot}f \in \langle f_1, \ldots, f_r\rangle$. Hence there is a natural number M such that $M{\cdot}F \subseteq \langle f_1, \ldots, f_r\rangle$. Hence $S = \langle F\rangle \subseteq \langle \frac{1}{M}{\cdot}f_1, \ldots, \frac{1}{M}{\cdot}f_q\rangle$. By the Elementary Divisor Theorem A1.10, there exist numbers $m_1|\cdots|m_q$, $q \leq r$ such that

$$S = \frac{m_1}{M}\mathbb{Z}{\cdot}f_1 \oplus \cdots \oplus \frac{m_k}{M}\mathbb{Z}{\cdot}f_q.$$

The elements $e_j = \frac{m_j}{M}$, $j = 1, \ldots, q$ are $\mathbb{R}$-linearly independent as asserted. (In fact, since $f_j \in S$ for $j = 1, \ldots, r$ we may conclude $q = r$.) This completes the proof of assertion (i).

The proof of assertion (ii) is immediate from (i) since $\mathbb{R}^n = \mathbb{R}{\cdot}e_1 \oplus \cdots \oplus \mathbb{R}{\cdot}e_n$ and the direct sum decomposition of S given in (i) fits this direct sum decomposition of $\mathbb{R}^n$.

(iii) We can write $E = E_1 \oplus E_2$ with $E_2 = \operatorname{span}_{\mathbb{R}}\{e_{p+1}, \ldots, e_{p+q}\}$ and $E_1 \subseteq F \stackrel{\text{def}}{=} \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_p, e_{p+q+1}, \ldots, e_n\}$. Then $\frac{E+S}{S} \cong F/E_1 \oplus E_2/(S \cap E_2)$. This reduces the proof to the case that $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ and $E = E_2 = \mathbb{R}^n$ in which case $\frac{E+S}{S} = \mathbb{R}^n/S$ and the assertion is trivial. □

Exercise EA1.8. (i) Prove a common generalisation of the Subgroup Theorem for $\mathbb{R}^n$ and the Elementary Divisor Theorem in the following form:

Assume that S is a closed subgroup of $\mathbb{R}^m \oplus \mathbb{Z}^n$. Then there are basis elements $e_1, \ldots, e_m$ of $\mathbb{R}^m$ and $e_{m+1}, \ldots, e_{m+n}$ of $\mathbb{Z}^n$ and natural numbers $n_1|n_2|\ldots|n_k$, $k \leq n$ such that

$$S = \mathbb{R}{\cdot}e_1 \oplus \cdots \oplus \mathbb{R}{\cdot}e_p + \oplus\mathbb{Z}{\cdot}e_{p+1} \oplus \cdots \oplus \mathbb{Z}{\cdot}e_{p+q} \oplus n_1\mathbb{Z}{\cdot}e_{m+1} \oplus \cdots \oplus n_k\mathbb{Z}{\cdot}e_{m+k}.$$

(ii) Determine all quotient groups of $\mathbb{R}^m \oplus \mathbb{Z}^n$ modulo closed subgroups.

(iii) Determine all closed subgroups and quotient groups modulo closed subgroups of groups of the form $\mathbb{R}^a \times \mathbb{T}^b \times \mathbb{Z}^c \times \mathbb{Z}(m_1) \times \cdots \times \mathbb{Z}(m_j)$. □

Projective Groups

We have two equivalent characterizations of free abelian groups: one by a universal property, one by giving an explicit description of the structure of a free abelian group. We are aiming for another characterization which will be expressed in terms of the existence of morphisms.

Definition A1.13. An abelian group P is called *projective* if for every surjective morphism $e: A \to B$ and every morphism $p: P \to B$ there is a morphism $f: P \to A$ with $p = e \circ f$.

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{id}_P} & P \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[e]{} & B \end{array}$$

□

We remark in passing that in the same fashion one can define projective modules over rings other than $\mathbb{Z}$, indeed one can define projective objects in an arbitrary category where surjective morphisms have to be replaced by epimorphisms.

Proposition A1.14. *For an abelian group G the following conditions are equivalent:*
(1) *G is free.*
(2) *G is projective.*

Proof. (1)⇒(2) (AC). Assume that $j: X \to G$ is free over X via j. Let $e: A \to B$ be surjective. Then the Axiom of Choice (AC) secures a function $\sigma: B \to A$ with $e \circ \sigma = \mathrm{id}_B$. If now $f: G \to B$ is a morphism, then for the function $\sigma \circ p \circ j: X \to A$ we find a unique morphism $f': G \to A$ with $f' \circ j = \sigma \circ p \circ j$ by the universal property of free abelian groups (A1.5). The uniqueness in the universal property of G then shows $f' = \sigma \circ p$. This implies $e \circ f' = e \circ \sigma \circ p = p$.

(2)⇒(1). Assume that G is projective. By Proposition A1.8 there is a surjective morphism $e: F \to G$ from a free abelian group F. If we take for $f: G \to G$ the identity map of G, the defining property of a projective group yields a morphism $f': G \to F$ with $e \circ f' = f = \mathrm{id}_G$. Thus f' is certainly injective and G is isomorphic to a subgroup of F. Then the Subgroup Theorem A1.9 shows that G is free. □

Proposition A1.15. *If $f: A \to F$ is a morphism of an abelian group A into a free abelian group, then A has a free subgroup $C \cong f(A)$ such that $A = \ker f \oplus C$. If f is surjective, then $f|C: C \to F$ is an isomorphism.*

Proof. (AC) By the Subgroup Theorem A1.9 the group $f(A)$ is free. Hence it is projective by Proposition A1.14. Thus there is a morphism $f': f(A) \to A$ such that $f\big(f'(x)\big) = x$ for all $x \in f(A)$. We let $C = f'\big(f(A)\big)$. This group satisfies all requirements: If $a \in \ker f \cap C$, then $a = f'(x)$ and $0 = f(a) = f\big(f'(x)\big) = x$ whence $a = f'(x) = 0$. If $a \in A$, then $a = \big[a - f'\big(f(a)\big)\big] + f'\big(f(a)\big)$, and $f\big[a - f'\big(f(a)\big)\big] =$

$f(a) - f(a) = 0$. Thus $\ker f \cap C = \{0\}$ and $A = \ker f + C$. The last assertion is clear since $f|C$ is injective. □

Torsion Subgroups

The subset of all elements of finite order in an abelian group is closed under addition and passing to the inverse.

Definition A1.16. (i) If A is an abelian group, then

$$\operatorname{tor}(A) = \{a \in A \mid (\exists n \in \mathbb{N})\ n{\cdot}a = 0\}$$

is called *the torsion subgroup* of A. A group is called *torsion-free* if $\operatorname{tor}(A) = \{0\}$.

(ii) A subgroup C of a (not necessarily abelian) group G is called *characteristic*, respectively, *fully characteristic* if for every automorphism, respectively endomorphism, α of G we have $\alpha(C) \subseteq C$. □

Remark A1.17. (i) The factor group $A/\operatorname{tor}(A)$ is torsion-free.

(ii) Let A be an abelian group, B a torsion-free group, and assume that $f\colon A \to B$ is a homomorphism. Then

a) $\operatorname{tor} A \subseteq \ker f$.

b) There is a unique morphism $F\colon A/\operatorname{tor} A \to B$ such that $F(a + \operatorname{tor} A) = f(a)$.

In particular, any subgroup G of A such that A/G is torsion-free contains $\operatorname{tor}(A)$.

(iii) The torsion subgroup is a fully characteristic subgroup.

Proof. Exercise EA1.9. □

Exercise EA1.9. Prove the preceding Remark A1.17. □

Definition A1.18. For a natural number p, a group A is called a *p-group* if for each element $a \in A$ there is an $n \in \mathbb{N}$ with $p^n{\cdot}a = 0$. For a group A we set $A_p = \{a \in A \mid (\exists n \in \mathbb{N})\ p^n{\cdot}a = 0\}$ and call it the *p-primary component of* A or the *p-Sylow subgroup* of A. □

Of course, to justify such a designation we have to verify that A_p is indeed a subgroup.

Theorem A1.19. *For an abelian group A, the p-primary component A_p is a fully characteristic subgroup, and* $\operatorname{tor}(A) = \bigoplus_{p \text{ prime}} A_p$.

Proof. One notices quickly that A_p is a fully characteristic subgroup. The morphism f given by $(a_p)_{p \text{ prime}} \mapsto \sum_{p \text{ prime}} a_p : \bigoplus_{p \text{ prime}} A_p \to A$ maps into the torsion group. Assume that $\sum_{p \text{ prime}} a_p = 0$. The set $F = \{p \mid a_p \neq 0\}$ is finite and the element $a = \sum_{p \in F} a_p$ is contained in the subgroup G generated by the elements

a_p, $p \in F$. Hence G is finite. Thus Theorem A1.12 implies that $G = \bigoplus_{p\in F} G_p$ and then $a_p \in G_p$. Hence $a = 0$ in G implies $a_p = 0$ for all p. Conversely, if $a \in \mathrm{tor}(A)$ is given, then the cyclic group $\langle a \rangle$ generated by a is finite and thus is a direct sum of cyclic p-groups, hence is contained in $\sum_{p \text{ prime}} A_p$. Thus f is an isomorphism onto the torsion group $\mathrm{tor}(A)$. □

Definition A1.20. For an abelian group A and a prime p we set $S_p(A) = \{a \in A \mid p{\cdot}a = 0\}$. This set is a subgroup called the *p-socle*. If A is already a p-group, we write $S(A) = S_p(A)$. □

Since the map $x \mapsto p{\cdot}x\colon A \to A$ is an endomorphism of A, the p-socle of A is exactly the kernel of this endomorphism.

Remark A1.21. (i) The p-socle $S_p(A)$ of an abelian group is the additive group of a vector space over the field of p elements $\mathrm{GF}(p)$.

(ii) $S_p(A)$ is a fully characteristic subgroup of A such that for each $0 \neq x \in A_p$ there is a natural number n such that $0 \neq p^n{\cdot}g \in S_p(A)$.

Proof. Exercise EA1.10. □

As a consequence, $S_p(A)$ has a basis over $\mathrm{GF}(p)$; its unique cardinality is called *the p-rank* written $\mathrm{rank}_p(A)$.

Exercise EA1.10. Prove Remark A1.21. □

Pure Subgroups

Definition A1.22. A subgroup P of an abelian group A is called *pure* if

$$(\forall p \in P,\, a \in A,\, n \in \mathbb{N}) \quad n{\cdot}a = p \;\Rightarrow\; (\exists x \in P) \quad n{\cdot}x = p. \qquad \square$$

If we denote by μ_n the endomorphism $x \mapsto n{\cdot}x$ of A, then *a subgroup P is pure iff $P \cap \mu_n(A) \subseteq \mu_n(P)$ iff $P \cap \mu_n(A) = \mu_n(P)$*. If $A = B \oplus C$, then $B \cap \mu_n(A) = B \cap (\mu_n(B) \oplus \mu_n(C)) = \mu_n(B)$ for each n. Thus *every direct summand is a pure subgroup.* If P is a pure subgroup of A and Q is a pure subgroup of P, then for each natural n we have $P \cap \mu_n(A) = \mu_n(P)$ and $Q \cap \mu_n(P) = \mu_n(Q)$. Hence $\mu_n(Q) = Q \cap P \cap \mu_n(A) = Q \cap \mu_n(A)$. Thus Q is pure in G. *Purity is transitive.*

Lemma A1.23. (i) *The torsion subgroup* $\mathrm{tor}(A)$ *is a pure subgroup of A.*

(ii) *The p-primary component A_p is pure in A.*

Proof. Since purity is transitive and A_p is a direct summand of $\mathrm{tor}\, A$ by A1.19, it suffices to show (i). Assume that $m{\cdot}a = 0$ with $0 < m \in \mathbb{N}$ minimal and that $n{\cdot}g = a$. Thus $mn{\cdot}g = 0$ and then $g \in \mathrm{tor}\, A$. This already completes the argument. □

Proposition A1.24. (i) *For a subgroup G of a torsion-free abelian group A the following conditions are equivalent:*

(1) *G is pure.*
(2) *The factor group A/G is torsion-free.*
(3) $(\forall n \in \mathbb{N}, a \in A) \quad n{\cdot}a \in G \Rightarrow a \in G$.
(4) $(\forall n \in \mathbb{N}) \quad \mu_n^{-1}(G) \subseteq G$.

(ii) *If these conditions are satisfied and A/G is finitely generated, then $A = F \oplus G$ with a F finitely generated free abelian group.*

Proof. (i) Note that $n{\cdot}a \in G$ and $n{\cdot}(a+G) = 0$ in A/G are equivalent statements. This implies the equivalence of (1) and (2). Further (1) is equivalent to

$$(\forall n \in \mathbb{N})(\forall a \in A) \quad \mu_n(a) = n{\cdot}a \in G \;\Rightarrow\; (\exists x \in G) \quad \mu_n(x) = \mu_n(a).$$

Since μ is injective, this condition is equivalent to

$$(\forall n \in \mathbb{N})(\forall a \in A) \quad \mu_n(a) = n{\cdot}a \in G \Rightarrow a \in G,$$

i.e. to (3). This condition is clearly equivalent to (4).

(ii) (AC) Assume that G is pure in A. Then A/G is torsion-free by (i). If A/G is also finitely generated, then A/G is free by A1.11. But then the assertion follows from A1.15. □

The subgroup $\mathbb{Z}$ of $\mathbb{Q}$ is not pure. (The expression *impure* does not seem to be customary.)

Proposition A1.25. *In a torsion-free abelian group A, every subset X is contained in a unique smallest pure subgroup $[X]$ given via*

$$[X] = \{a \in A \mid (\exists x_1, \ldots, x_k \in X, n_0, n_1, \ldots, n_k \in \mathbb{Z})\ n_0{\cdot}a = \sum_{j=1}^{k} n_j{\cdot}x_j\}.$$

If X is a subgroup, then $[X] = \{a \in A \mid (\exists n \in \mathbb{N})\ n{\cdot}a \in X\}$.

Proof. If H is any subgroup of A, then the set of all $a \in A$ for which there is a natural number n with $n{\cdot}a \in H$ is a subgroup $[H]$. We claim that it is pure: If $h \in H$ and $m{\cdot}x = h$ then there is a natural number n with $n{\cdot}h \in H$, and thus $mn{\cdot}x \in H$, whence $x \in [H]$. Thus $[H]$ is pure. Assume that P is a pure subgroup containing H. If $x \in [H]$, then $n{\cdot}x \in H$ for some $n \in \mathbb{N}$. Since P is pure and $n{\cdot}x \in H \subseteq P$ there is a $p \in P$ with $n{\cdot}p = n{\cdot}x$. Then $n{\cdot}(x-p) = 0$ and since A is torsion-free we conclude $x = p \in P$. Thus $[H] \subseteq P$. This shows that $[H]$ is the unique smallest pure subgroup containing H. Now we let $H = \langle X \rangle$ be the subgroup generated by X, i.e. the set of all integral linear combinations $\sum_{j=1}^{k} n_j{\cdot}x_j$. Then $[X] = [H]$ is the required pure subgroup.

The last assertion of the proposition is clear. □

We shall call $[X]$ *the pure subgroup generated by* X. In the countable group $\mathbb{Q}$, like in any torsion-free abelian group, every finitely generated subgroup is free. But the only pure subgroups are $\{0\}$ and $\mathbb{Q}$.

The following proposition is another characterisation of free abelian groups as long as they are countable. (We shall use this later in the proof of Theorem A1.62 below.)

Proposition A1.26. *A countable torsion-free abelian group is free if* (*and only if*) *every pure subgroup generated (as a pure subgroup) by a finite set is free.*

Proof. Firstly, since G is countable, there is an ascending sequence P_n $n \in \mathbb{N}$ of finitely generated pure subgroups with $G = \bigcup_{n\in\mathbb{N}} P_n$. Indeed, if $G = \{g_1, g_2, \ldots\}$, it suffices to take $P_n = [g_1, \ldots, g_n]$. We may assume that $P_n \neq P_{n+1}$. Assume that we have found an ascending family of subsets $X_k \subseteq P_k$ for $k = 1, \ldots, n$ such that P_k is free over X_k. By hypothesis, P_{n+1} is free. We may identify this group with $\mathbb{Z}^{(X)}$ for some set X, and it is then contained in the $\mathbb{Q}$-vector space $V \stackrel{\text{def}}{=} \mathbb{Q}^{(X)}$. Then $V = \mathbb{Q}{\cdot}g_1 + \cdots + \mathbb{Q}{\cdot}g_{n+1}$ since P_{n+1} is the pure subgroup generated by $\{g_1, \ldots, g_{n+1}\}$. Hence $\operatorname{card} X = \dim_{\mathbb{Q}} V < \infty$. Now Proposition A1.24(ii) applies and yields $P_{n+1} = P_n \oplus F_n$ with a group F_n which is free over X'_n. Then P_{n+1} is free over $X_{n+1} = X_n \cup X'_n$. Now we set $X = \bigcup_{n\in\mathbb{N}} X_n$. Then $\langle X\rangle = \sum_{n\in\mathbb{N}}\langle X_n\rangle = \sum_{n\in\mathbb{N}} P_n = G$. Then the morphism $(n_x)_{x\in X} \mapsto \sum_{x\in X} n_x{\cdot}x : \mathbb{Z}^{(X)} \to G$ is surjective. But it is injective, for if $\sum_{x\in X} n_x{\cdot}x = 0$, then there is a finite subset $X' \subseteq X$ with $n_x = 0$ for $x \notin X'$. Then we find an n such that $X' \subseteq X_n$. But then $\sum_{x\in X_x} n_x{\cdot}x = 0 \in P_n$ and P_n was free over X_n. Hence $n_x = 0$ for $x \in X' \subseteq X_n$. We have shown $G \cong \mathbb{Z}^{(X)}$. ☐

Free Quotients

Let A be an abelian group. Let $\mathcal{K}$ denote the set of all subgroups K of A such that A/K is free. Then $\mathcal{K}$ is a filter basis; for if K_1, $K_2 \in \mathcal{K}$ then $K = K_1 \cap K_2$ is the kernel of the map $a \mapsto (a + K_1, a + K_2) \to A/K_1 \times A/K_2$. The image of this homomorphism is a subgroup of a free group and thus is free by the Subgroup Theorem A1.9. We can form the characteristic subgroup

$$K_\infty = K_\infty(A) = \bigcap \mathcal{K}.$$

(Cf. [108].) Then all morphisms into free groups factor through $A \to A/K_\infty$ and the homomorphisms $A/K_\infty \to \mathbb{Z}$ separate the points. In particular, A/K_∞ is torsion-free and $\operatorname{tor} A \subseteq K_\infty$. Notably, K_∞ is a pure subgroup. An abelian group A with $K_\infty(A) = \{0\}$ is also called *torsionless.*

Lemma A1.27. *If* $K_\infty(A) = \{0\}$, *i.e. if* A *is torsionless, then every finite rank pure subgroup of* A *is free.*

Proof. Let P be a finite rank pure subgroup of A and F a maximal rank free subgroup contained in P. Then $P = [F]$. Since $K_\infty = \{0\}$, there is a subgroup $K \in \mathcal{K}$ such that $F \cap K = \{0\}$. It follows that $P \cap K = \{0\}$, for if $p \in P \cap K$ then there is an $m \in \mathbb{N}$ such that $m{\cdot}p \in F \cap K = \{0\}$, whence $p = 0$ since A is torsion-free. The map $x \mapsto x + K : P \to A/K$ is therefore injective. But A/K is free by the definition of $\mathcal{K}$, and thus P is free by the Subgroup Theorem A1.9, as we wanted to show. □

Proposition A1.28. *Let A be an abelian group such that $A/K_\infty(A)$ is countable. Then A contains a free subgroup F such that $A = F \oplus K_\infty(A)$. Moreover, $K_\infty(A)$ does not have any nondegenerate free quotient groups.*

Proof. The group $A/K_\infty(A)$ is torsion-free, countable, and the morphisms into free groups separate the points. Hence $K_\infty\big(A/K_\infty(A)\big) = \{0\}$. Thus from Lemma A1.27 we know that every finite rank pure subgroup is free. Then, by Proposition A1.26, the quotient $A/K_\infty(A)$ is free. Since free groups are projective, this implies the existence of F as asserted. Again any free quotient of $K_\infty(A)$ splits, so $K_\infty(A) = F' \oplus K$ with a free F' isomorphic to the free quotient. But then $F \oplus F'$ is free and thus $K \in \mathcal{K}$. It follows that $K_\infty(A) \subseteq K$ and that, as a consequence, F' is degenerate. □

The statement that $K_\infty(A) = \{0\}$ is tantamount to saying that the morphisms of $\mathrm{Hom}(A, \mathbb{Z})$ separate the points of A. There is a vast class of groups for which this is the case but which are not free (for an example see A1.65 below).

Divisibility

Recall that in an abelian group we write $\mu_n(x) = n{\cdot}x$.

Definition A1.29. Let A be an abelian group. An element $a \in A$ is called *divisible* if for each natural number n there is an $x \in A$ with $n{\cdot}x = a$. The set $\bigcap_{n\in\mathbb{N}} \mu_n(A)$ of all divisible elements is called $\mathrm{Div}(A)$. The group A is called *divisible* if $A \subseteq \mathrm{Div}(A)$. We say A is *reduced* if $\{0\}$ is the only divisible subgroup of A. □

We notice quickly that $\mathrm{Div}(A)$ is a characteristic subgroup.

Definition A1.30. If p is a prime, we write $\frac{1}{p^\infty}\mathbb{Z}$ for the group of all rational numbers which can be written m/p^n for an integer m and a natural number n. We define $\mathbb{Z}(p^\infty) = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$. □

Exercise EA1.11. (i) The additive group of a vector space over $\mathbb{Q}$ (or any field of characteristic 0) is divisible.

(ii) $\frac{1}{p^\infty}\mathbb{Z}$ is not divisible.

(iii) If m and n are relatively prime integers and $m \cdot a = 0$ in A then there is an x such that $n \cdot x = a$.

(iv) $\mathbb{Z}(p^\infty)$ is divisible.

(v) If $\{A_j \mid j \in J\}$ is a family of divisible groups, then $\bigoplus_{j \in J} A_j$ and $\prod_{j \in J} A_j$ are divisible.

(vi) Every homomorphic image of a divisible group is divisible.

(vii) Every pure subgroup of a divisible group is divisible. □

In particular, every direct sum of copies of $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$, p prime, is divisible. We shall see shortly, namely, in A1.41 below, that all divisible groups are so obtained.

Proposition A1.31. *Each abelian group contains a unique largest divisible subgroup* $\operatorname{div}(A)$.

Proof. If $\{B_j \mid j \in J\}$ is a family of divisible subgroups, then $\bigoplus_{j \in J} B_j$ is divisible by EA1.11(v) and $\sum_{j \in J} B_j$ is the homomorphic image of this group under the homomorphism $(b_j)_{j \in J} \mapsto \sum_{j \in J} b_j$ (see A1.2(iii)). Hence it is divisible by EA1.11(vi). Now we apply this to the family of *all* divisible subgroups. Then their sum $\operatorname{div}(A)$ is divisible, and this group contains all divisible subgroups. □

We call $\operatorname{div}(A)$ the *largest divisible subgroup* of A. Notice that $\operatorname{div}(A)$ is a fully characteristic subgroup and that $\operatorname{div}(A) \subseteq \operatorname{Div}(A)$. We shall show in the following theorem that the converse inclusion may fail. This theorem deals with *one single* group; but this one is eminently important and complicated enough to be looked at carefully.

In $\mathbb{Z}^{(\mathbb{N})}$ we consider the elements e_n given as in A1.6 by $e_n(m) = 1$ for $m = n$ and 0 otherwise. We define W to be the subgroup of $\mathbb{Z}^{(\mathbb{N})}$ generated by the elements $e_1 - n \cdot e_n$, $n \in \mathbb{N}$ and set $\nabla = \mathbb{Z}^{(\mathbb{N})}/W$. Also we shall write $g_n = e_n + W \in \nabla$. Roots b_n of an element b_1 in a group B, $n \cdot b_n = b_1$ are called *consecutive* if for all pairs (m, n) of natural numbers we have $m \cdot b_{mn} = b_n$.

We now describe an example that illustrates well the complications arising with the concept of divisibility.

Theorem A1.32. *The group* ∇ *has the following properties:*

(i) *If* B *is any abelian group with a divisible element* b_1*, then for every set of roots* $b_n \in B$ *with* $n \cdot b_n = b_1$*, there is a unique homomorphism* $d\colon \nabla \to B$ *with* $d(g_n) = b_n$.

(ii) *There is a surjective homomorphism* $p\colon \nabla \to \mathbb{Q}$ *defined by* $p(g_n) = 1/n$. *If in* (i) *above the roots are consecutive, then* $d\colon \nabla \to B$ *factors through* p, *i.e. there is a unique* $q\colon \mathbb{Q} \to B$ *with* $d = qp$.

(iii) $\mathbb{Z} \cdot g_n \cong \mathbb{Z}$ *for all* n *and* $g_1 = n \cdot g_n$.

(iv) $\ker p = \operatorname{tor}(\nabla)$. *That is,* $0 \to \operatorname{tor}(\nabla) \to \nabla \xrightarrow{p} \mathbb{Q} \to 0$ *is exact.*

(v) $\nabla / \mathbb{Z} \cdot g_1 \cong \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$ *and* $\operatorname{tor}(\nabla) \cap \mathbb{Z} \cdot g_1 = \{0\}$. *If* G *is any subgroup with* $G \cap \mathbb{Z} \cdot g_1 = \{0\}$, *then* $G \subseteq \operatorname{tor}(\nabla)$.

(vi) $\operatorname{div}(\nabla) = \{0\}$, *i.e.* ∇ *is reduced. Also,* $\operatorname{Div}(\nabla) = \mathbb{Z} \cdot g_1$.

(vii) $\operatorname{tor}(\nabla)$ *is not a direct summand of* ∇.

(viii) *The group* $S \stackrel{\text{def}}{=} \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$ *contains a subgroup* $K \cong \operatorname{tor}(\nabla)$ *such that* $S/K \cong \mathbb{Q}/\mathbb{Z}$. *The subgroup* K *of* S *is not a direct summand.*

(ix) *Define* $\mathbb{P}_p = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)$, $\mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. *Then*

$$S = \bigoplus_{p\,\text{prime}} \mathbb{P}_p,$$

and if K_p *is the* p*-primary component of* K*, there is an exact sequence*

$$0 \to K_p \xrightarrow{\text{incl}} \mathbb{P}_p \xrightarrow{\gamma_p} \mathbb{Z}(p^\infty) \to 0.$$

In particular, the p*-rank of* K_p *and thus the* p*-rank of* ∇ *is infinite.*

(x) $\operatorname{tor}(\nabla) \cong \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$, *and there is an exact sequence*

$$0 \to \bigoplus_{n=2}^{\infty} \mathbb{Z}(n) \to \nabla \xrightarrow{p} \mathbb{Q} \to 0.$$

Proof. We prove (i) by using the freeness of $\mathbb{Z}^{(\mathbb{N})}$ over $\{e_n \mid n \in \mathbb{N}\}$, and by considering the unique morphism $f\colon \mathbb{Z}^{(\mathbb{N})} \to \nabla$ given by $f(e_n) = b_n$. We observe that $f(e_1 - n{\cdot}e_n) = f(e_1) - n{\cdot}f(e_n) = b_1 - n{\cdot}b_n = 0$. Hence f vanishes on W. Thus f induces a unique morphism $d\colon \nabla = \mathbb{Z}^{(\mathbb{N})}/W \to B$ with $d(x+W) = f(x)$. In particular, $d(g_n) = f(e_n) = b_n$.

The first part of (ii) is an immediate consequence of (i). Now let $m/n = m'/n'$ with $n, n' \in \mathbb{N}$. Then $m{\cdot}b_n = mn'{\cdot}b_{nn'} = m'n{\cdot}b_{nn'} = m'{\cdot}b_n$. Hence we can unambiguously define $q\colon \mathbb{Q} \to B$ by $q(m/n) = m{\cdot}b_n$. Also $q(m_1/n + m_2/n) = q\big((n_1+n_2)/m\big) = (n_1+n_2){\cdot}b_n = n_1{\cdot}b_n + n_2{\cdot}b_n = q(m_1/n) + q(m_2/n)$; thus q is a morphism. Now $1/n = p(g_n)$. Hence $d(g_n) = b_n = q(1/n) = q\big(p(g_n)\big)$. Since the g_k generate ∇, we conclude $d = q \circ p$. Since p is surjective, q is uniquely determined by this equation.

(iii) Since $p(g_n) = \frac{1}{n} \in \mathbb{Q}$ and this element generates an infinite group, the group $\mathbb{Z}{\cdot}g_n$ is infinite.

(iv) Next we take $a \in \nabla$ with $p(a) = 0$. We have $a = \sum_{n\in\mathbb{N}} z_n{\cdot}g_n$ with $0 = p(a) = \sum_{n\in\mathbb{N}} z_n/n$. If $N = \max\{n \mid z_n \neq 0\}$, then $\sum_1^N (N!/n)z_n = 0$, or, equivalently,

$$N!z_1 = -\sum_{n=2}^{N} \frac{N!}{n} z_n.$$

Consequently,

$$\begin{aligned} N!{\cdot}\sum_{n=1}^{N} z_n{\cdot}e_n &= N!z_1{\cdot}e_1 + \sum_{n=2}^{N} N!z_n{\cdot}e_n \\ &= -\sum_{n=2}^{N} \frac{N!}{n} z_n{\cdot}(e_1 - n{\cdot}e_n) \in W. \end{aligned}$$

Thus $a \in \mathrm{tor}(\nabla)$. Since $\mathbb{Q}$ is torsion-free, $\ker p \supseteq \mathrm{tor}(\nabla)$. Thus $\ker p = \mathrm{tor}(\nabla)$.

(v) Since $\mathbb{Z}\cdot g_1$ is infinite by (iii) we have $\mathbb{Z}\cdot g_1 \cap \mathrm{tor}(\nabla) = \{0\}$. Also, $\nabla/\mathbb{Z}\cdot g_1 \cong \mathbb{Z}^{(\mathbb{N})}/\langle e_1, 2\cdot e_2, 3\cdot e_3, \ldots\rangle \cong \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$. It follows that $\mathrm{tor}(\nabla)$ is isomorphic to a subgroup of $\bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$

If $a = \sum_{n\in\mathbb{N}} z_n\cdot g_n$ is any element of ∇ with $N = \max\{n \mid z_n \neq 0\}$, then $N!\cdot g_n = \frac{N!}{n} n\cdot g_n = \frac{N!}{n}\cdot g_1$, whence $N!\cdot a \in \mathbb{Z}\cdot g_1$. Thus if G is any subgroup of ∇ not contained in $\mathrm{tor}(\nabla)$, then G must contain an element a of infinite order and we have $0 \neq N!\cdot a \in G \cap \mathbb{Z}\cdot g_1 \neq \{0\}$.

(vi) Claim: The group $S \stackrel{\text{def}}{=} \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$ contains no divisible elements. Hence it is reduced. Indeed, if a were a nonzero divisible element, then $S = F \oplus F'$ with a finite group $F \cong \bigoplus_{n=1}^{N} \mathbb{Z}(n)$ containing a and with $F' = \bigoplus_{n=N+1}^{\infty} \mathbb{Z}(n)$. Homomorphisms preserve divisibility, whence a, being the image of the projection onto F is divisible in F. But $|F|\cdot x = 0$ for all $x \in F$, and so $a = 0$, a contradiction. This proves the claim.

Hence the image of $\mathrm{Div}(\nabla)$ is trivial in $\nabla/\mathbb{Z}\cdot g_1$. Thus $\mathrm{Div}(\nabla) \subseteq \mathbb{Z}\cdot g_1$. But $\mathbb{Z}\cdot g_1$ is reduced, and thus $\mathrm{div}(\nabla) \subseteq \mathrm{Div}(\nabla)$ is zero. Since $e_1 - n\cdot e_n \in W$ we know $g_1 = n\cdot g_n$. Thus $\mathbb{Z}\cdot g_1 \subseteq \mathrm{Div}(\nabla)$. Hence $\mathbb{Z}\cdot g_1 = \mathrm{Div}(\nabla)$ follows.

(vii) If $\mathrm{tor}(\nabla)$ were a direct summand, then ∇ would contain a subgroup A such that $\nabla = \mathrm{tor}(\nabla) \oplus A$ and $A \cong \nabla/\mathrm{tor}(\nabla) \cong \mathbb{Q}$. But this would be a contradiction to the fact that ∇ is reduced.

(viii) Set $\nabla_1 = \mathbb{Z}\cdot g_1 + \mathrm{tor}(\nabla)$. Since $\mathrm{tor}(\nabla) \cap \mathbb{Z}\cdot g_1 = \{0\}$ by (v), in view of (iii), the functions $x \mapsto x + \mathbb{Z}\cdot g_1 \colon \mathrm{tor}(\nabla) \to \nabla_1/\mathbb{Z}\cdot g_1$ and $n \mapsto n\cdot g_1 + \mathrm{tor}(\nabla) \colon \mathbb{Z} \to \nabla_1/\mathrm{tor}(\nabla)$ are isomorphisms. By (iv) we have $\nabla/\mathrm{tor}(\nabla) \cong p(\nabla) = \mathbb{Q}$, and for every infinite cyclic subgroup $\mathbb{Z}q$ of $\mathbb{Q}$, we have $\mathbb{Q}/\mathbb{Z}q \cong \mathbb{Q}/\mathbb{Z}$. As a consequence,

$$\frac{\mathbb{Q}}{\mathbb{Z}} \cong \frac{\nabla/\mathrm{tor}(\nabla)}{\nabla_1/\mathrm{tor}(\nabla)} \cong \frac{\nabla}{\nabla_1} \cong \frac{\nabla/\mathbb{Z}\cdot g_1}{\nabla_1/\mathbb{Z}\cdot g_1}.$$

Since there is an isomorphism $\kappa \colon \bigoplus_{n=2}^{\infty} \mathbb{Z}(n) \to \nabla/\mathbb{Z}\cdot g_1$ by (v), the first assertion follows if we define

$$K \stackrel{\text{def}}{=} \kappa^{-1}(\nabla_1/\mathbb{Z}\cdot g_1).$$

In order to prove the second, suppose that K were a direct summand of S. This would imply that ∇ would contain a subgroup A such that $A + \nabla_1 = \nabla$ and $A \cap \nabla_1 = \mathrm{DIV}(\nabla)$. Then we would conclude $\nabla = A + \mathrm{tor}(\nabla)$ and $A \cap \mathrm{tor}(\nabla) = \{0\}$ which would contradict the fact that $\mathrm{tor}(\nabla)$ is not a direct summand of ∇.

(ix) Let $\mathbb{N}_0$ be the additive monoid $\{0, 1, 2, 3, \ldots\}$. Then we have a morphism $\mu \colon \mathbb{N}_0^{(\mathbb{N})} \to \mathbb{N}$ into the multiplicative monoid $\mathbb{N}$ of natural numbers defined as follows: Let $\nu = (n_1, n_2, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$ and let $(p_1, p_2, p_3, \ldots)$ be the sequence of prime numbers $(2, 3, 5, \ldots)$. Then $\mu(\nu) = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$ in an obvious sense due to the fact that all but a finite number of the n_j are 0. Then the theorem on the unique prime decomposition of natural numbers is exactly the statement that

μ is an isomorphism of monoids.

Now

$$\mathbb{Z}(\mu(\nu)) = \mathbb{Z}(p_1^{n_1}) \oplus \mathbb{Z}(p_2^{n_2}) \oplus \mathbb{Z}(p_3^{n_3}) \oplus \cdots,$$

which clarifies the p-primary decomposition of S.

Specifically, we may consider $\mathbb{Z}(p^\infty)$ as the colimit of the direct system

$$\cdots \mathbb{Z}(p^n) \xrightarrow{\eta_n} \mathbb{Z}(p^{n+1}) \cdots, \quad \eta_n(1 + p^n\mathbb{Z}) = p + p^{n+1}\mathbb{Z},$$

and where γ_p is the map from the coproduct to the colimit. If the generator $1+p^n\mathbb{Z}$ of $\mathbb{Z}(p^n)$ is abbreviated by ε_n, then K_p is generated by the set $\{p{\cdot}\varepsilon_{n+1} - \varepsilon_n : n = 2, 3, \ldots\}$.

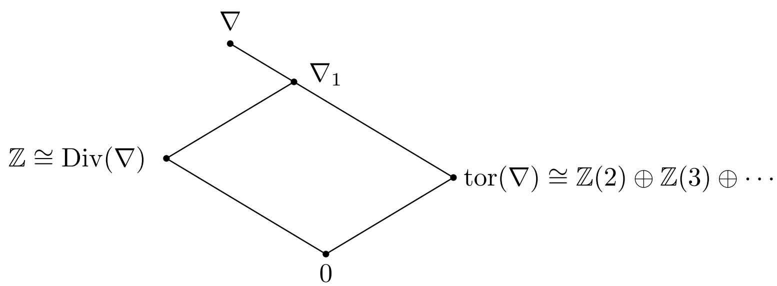
(x) The second part follows from the first through (ii) and (iv). For proving the first part, we show that there is an injective endomorphism $\eta\colon S \to S$ with image K.

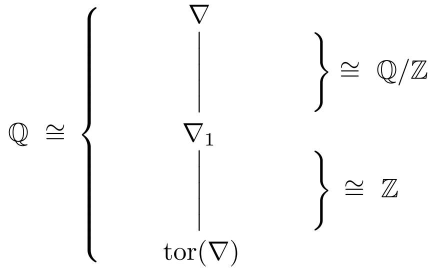
For this purpose we invoke (ix) to see that it is sufficient to show that for each prime p, there is an exact sequence

$$(E) \qquad 0 \to \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n) \xrightarrow{\eta_p} \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n) \to \mathbb{Z}(p^\infty) \to 0.$$

Indeed we define $\eta_p(\varepsilon_n) = p{\cdot}\varepsilon_{n+1} - \varepsilon_n$, $n = 1, 2, \ldots$. Since clearly $p{\cdot}\varepsilon_{n+1} - \varepsilon_n$ has order p^n, we have to show that η_p is injective. For a proof let $a = \sum_{n=1}^{\infty} z_n{\cdot}\varepsilon_n$ with a finite support sequence of elements $z_n \in \mathbb{Z}$ such that $\eta_p(a) = 0$. Then $0 = \sum_{n=1}^{\infty} z_n{\cdot}\varepsilon_n - \sum_{n=1}^{\infty} pz_n{\cdot}\varepsilon_{n+1} = \sum_{n=1}^{\infty} z_n{\cdot}\varepsilon_n - \sum_{n=2}^{\infty} pz_{n-1}{\cdot}\varepsilon_n = z_1.\varepsilon_1 + \sum_{n=2}^{\infty}(z_n - pz_{n-1}){\cdot}\varepsilon_n$. In the direct sum of the subgroups $\mathbb{Z}(p^n)$, this implies, successively, $z_1 = 0$ (modulo p), $z_2 - pz_1 = 0$ (modulo p^2), $z_3 - pz_2 = 0$ (modulo p^3), and so on. Inductively, this shows $z_n = 0$ (modulo p^n), n=1,2,..., and so $a = 0$. This completes the proof of the injectivity of η_p, and the remaining statements involved in the exactness of the sequence (E) are routine. □

As the numerous statements of the theorem indicate, this example is complex and surprising. It is very useful to have in our bag of tricks. Let us visualize its structure in diagrams





Notice that, in particular, a group with a nonzero divisible element b is infinite. For if $d(\nabla)$ is a finite group of order N, then $0 = N{\cdot}d(g_N) = d(g_1) = b$, a contradiction. Also observe that conclusion (ii) yields for every element b_1 in a divisible group B a homomorphism $q\colon \mathbb{Q} \to B$ with $q(1) = b_1$.

The following result is a counterpart to the statement that every abelian group is the quotient of a free one.

Proposition A1.33. *For each abelian group A there is a divisible group D and an injective morphism $A \to D$. In this sense, every abelian group is a subgroup of a divisible one.*

Moreover, D can be chosen so that $A \subseteq D$ and that

(i) $\operatorname{card} D \le \max\{\aleph_0, \operatorname{card} A\}$, *where equality holds if $A \neq \{0\}$.*

(ii) (AC) *Any nonzero subgroup of D meets A nontrivially.*

Proof. By Propositions A1.7 and A1.8 there is a set X and a surjective homomorphism $p\colon \mathbb{Z}^{(X)} \to A$. We may assume that $A = \mathbb{Z}^{(X)}/K$ with $K = \ker p$ and that $\operatorname{card} X = \operatorname{card} A$. Now $K \subseteq \mathbb{Z}^{(X)} \subseteq \mathbb{Q}^{(X)}$. Then $A = \mathbb{Z}^{(X)}/K \subseteq \mathbb{Q}^{(X)}/K$, and $D = \mathbb{Q}^{(X)}/K$ is divisible. The statement (i) on the cardinals is now straightforward since $\operatorname{card} \mathbb{Q}^{(X)} = \operatorname{card} \mathbb{Z}^{(X)}$. Assertion (ii) is proved as follows: The set of subgroups S of D with $S \cap A = \{0\}$ is inductive with respect to "$\subseteq$". Hence by Zorn's Lemma, we find a maximal one, say M. Then $M \cap A = \{0\}$ implies that the quotient map $q\colon D \to D/M$ restricts to an injective morphism $q|A\colon A \to D/M$, and D/M is divisible as a homomorphic image of a divisible group. Now let S be a subgroup of D containing M such that S/M meets $(A+M)/M$ trivially. Then S meets A trivially, and by maximality of S we have $M = S$, i.e. S/M is singleton. Thus we may replace A by $(A+M)/M$ and D by D/M and then the conclusion of (ii) is satisfied. □

If the inclusion $A \subseteq D$ satisfies A1.33(ii) we say that D is *a divisible hull of A*.

The idea of projectivity yielded a characterization of free abelian groups. Now we encounter the dual concept of injectivity.

Definition A1.34. An abelian group I is called *injective* if for every injective morphism $i\colon A \to B$ and every morphism $j\colon A \to I$ there is a morphism $f\colon B \to I$ with $j = f \circ i$.

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{id}_I} & I \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & B \end{array}$$

□

One may rephrase injectivity in the following convenient fashion: *An abelian group I is injective if and only if any homomorphism $j\colon A \to I$ of a subgroup A of a group B extends to a homomorphism $f\colon B \to I$ on the whole group.*

Proposition A1.35. *For an abelian group G the following conditions are equivalent:*

(1) *G is divisible.*
(2) *G is injective.*

Proof. (1)⇒(2). (AC) Assume that A is a subgroup of B and that a homomorphism $j\colon A \to G$ is given. We must extend j to a morphism $f\colon B \to G$. We consider the set of all morphisms $\varphi\colon C \to G$ with $A \subseteq C \subseteq B$ and $\varphi|A = j$. This set is partially ordered by inclusion of domains and extension of mappings (i.e. $\varphi \le \varphi'$ if $C \subseteq C'$ and $\varphi'|C = \varphi$). One verifies quickly that this set is inductive, hence by Zorn's Lemma contains a maximal element $\mu\colon M \to G$. We must show $M = B$. Let $b \in B$. Then $M \cap \mathbb{Z}{\cdot}b$ is a cyclic group, say $n\mathbb{Z}{\cdot}b$. Since G is divisible, there is an element $d \in G$ such that $n{\cdot}d = \mu(n{\cdot}b)$. Assume now that $m_1 + z_1{\cdot}b = m_2 + z_2{\cdot}b$. Then $m_2 - m_1 = (z_1 - z_2){\cdot}b \in M \cap \mathbb{Z}{\cdot}b = n\mathbb{Z}{\cdot}b$. In particular, the kernel of $m \mapsto m{\cdot}b\colon \mathbb{Z} \to G$ is contained in $n\mathbb{Z}$. Thus there is a $z \in \mathbb{Z}$ with $(z_1 - z_2 - zn){\cdot}b = 0$, and thus $z_1 - z_2 - zn = z'n$ for some $z' \in \mathbb{Z}$. Hence $\mu(m_2 - m_1) = \mu\big((z_1 - z_2){\cdot}b\big) = \mu\big((z + z')n{\cdot}b\big) = (z + z')n{\cdot}d = (z_1 - z_2){\cdot}d$ and thus $\mu(m_1) + z_1{\cdot}d = \mu(m_2) + z_1{\cdot}d$. Therefore we define unambiguously a function $\mu'\colon M' \to I$, $M' = M + \mathbb{Z}{\cdot}b$ by $\mu'(m + z{\cdot}b) = \mu(m) + z{\cdot}d$ satisfying $\mu'|M = \mu$. It is easy to verify that μ is a morphism. Hence $\mu \le \mu'$. By the maximality of μ we have $\mu' = \mu$ and thus $M' = M$. Hence $b \in M$. Thus $M = B$.

(2)⇒(1). By Proposition A1.33 there is a divisible group D with $G \subseteq D$. Since G is injective there is a morphism $f\colon D \to G$ such that $f|G = \mathrm{id}_G$. Hence G is a homomorphic image of a divisible group and is, therefore, divisible. □

Corollary A1.36. (i) *A divisible subgroup D of an abelian group is a direct summand; i.e. there is a subgroup D' such that $A = D \oplus D'$.*

(ii) *If $A \subseteq D_j$, $j = 1, 2$ are two divisible hulls of A then there is an isomorphism $f\colon D_1 \to D_2$ such that $f|A = \mathrm{id}_A$.*

(iii) *If D is a divisible hull of A, then* $\mathrm{rank}_p D = \mathrm{rank}_p A$ *for all primes p, and if X is a maximal free subset of A (i.e. one for which $\langle X \rangle$ is a free subgroup of A*

freely generated by X, and for which X is maximal with respect to this property), then X is a maximal free subset of D.

Proof. (i) By Proposition A1.35, the group D is injective, and thus there is a morphism $f: A \to D$ with $f|D = \mathrm{id}_D$. Set $D' = \ker f$. If $a \in D \cap D'$, then $a \in D'$ implies $0 = f(a)$ and $a \in D$ implies $f(a) = a$. Thus $a = 0$. If $a \in A$, then $a = f(a) + \big(a - f(a)\big)$; here $f(a) \in D$ and $a - f(a) \in \ker f = D'$. Thus $A = D \oplus D'$.

(ii) By the injectivity of D_2 there is an $f: D_1 \to D_2$ such that $f(a) = a$ for all $a \in A$. The kernel K of f is a subgroup of D_1 meeting A trivially; since D_1 is a divisible hull, $K = \{0\}$. Thus f is injective. The subgroup $f(D_1)$ of D_2 is divisible; hence by (i) above there is a subgroup T of D_2 such that $D_2 = f(D_1) \oplus T$. Since $A \subseteq f(D_1)$ we conclude $A \cap T = \{0\}$. Since D_2 is a divisible hull of A we have $T = \{0\}$, i.e. $f(D_1) = D_2$. Thus f is bijective.

(iii) The p-socle $S_p(A)$ of A (see A1.20) is contained in $S_p(D)$. Since $S_p(A)$ is a vector space over $\mathrm{GF}(p)$, there is a subgroup $T \subseteq S_p(D)$ such that $S_p(D) = S_p(A) \oplus T$. This implies $A \cap T = \{0\}$. Since D is a divisible hull of A it follows that $T = \{0\}$, i.e. $S_p(A) = S_p(D)$, whence $\mathrm{rank}_p A = \mathrm{rank}_p D$ (see the definition following A1.21). Now let X be a maximal free subset of A. Assume that Y is a free subset of D containing X. Since $\langle Y\rangle$ is freely generated by Y we have $\langle Y\rangle = \langle X\rangle \oplus \langle Y \setminus X\rangle$, and from the maximality of X in A we conclude that $A \cap \langle Y \setminus X\rangle = \{0\}$. Since D is a divisible hull, it follows that $\langle Y \setminus X\rangle = \{0\}$ and this implies $X = Y$. Now $D = \mathrm{tor}\, D \oplus \Delta$ with a torsion-free divisible subgroup $\Delta \cong D/\mathrm{tor}\, D$ since $\mathrm{tor}\, D$ is divisible and divisible subgroups are direct summands by (i). Then X is a maximal free subset of D if and only if its projection into Δ is maximal free. □

The last observation will allow us shortly to introduce the torsion-free rank of an abelian group as the cardinality of a maximal free subset; we have to have the means to argue that all such sets have the same cardinality.

Theorem A1.37. *Every abelian group A is isomorphic to* $\mathrm{div}(A) \oplus A/\mathrm{div}(A)$ *and* $A/\mathrm{div}(A)$ *is reduced.*

Proof. By Corollary A1.36 there is a subgroup A' such that $A = \mathrm{div}(A) \oplus A'$. We have $A' \cong A/\mathrm{div}(A)$. Since $\mathrm{div}(A)$ contains all divisible subgroups of A, any complementary summand A' must be reduced. □

Proposition A1.38. *If A is a divisible group, then* $A = \mathrm{tor}(A) \oplus A'$ *and all primary components A_p are divisible.*

Proof. In view of Corollary A1.36, this follows from the fact that $\mathrm{tor}(A)$ and A_p are pure subgroups of A by Lemma A1.23. □

For a complete description of divisible groups it is now required that we determine the structure of *torsion-free divisible groups* and of *divisible p-groups.*

Proposition A1.39. *A divisible torsion-free abelian group is the additive group of a rational vector space.*

A subset X of a divisible abelian group D is free if and only if it is linearly independent over $\mathbb{Q}$. It is a maximal free subset if and only if it is a $\mathbb{Q}$ vector space basis of D. Its cardinality then is $\dim_{\mathbb{Q}} D$.

Proof. Let $a \in A$. The morphism $f\colon \mathbb{Z} \to A$, $f(z) = z{\cdot}a$ extends to a morphism $F\colon \mathbb{Q} \to A$. (See also Theorem A1.32(ii)!) We claim that this extension is unique: If $F'\colon \mathbb{Q} \to A$ is likewise an extension of f, then $q = m/n \in \mathbb{Q}$ implies $n{\cdot}\bigl(F'(q) - F(q)\bigr) = F'(nq) - F(qn) = F'(m) - F(m) = m{\cdot}a - m{\cdot}a = 0$. Since A is torsion-free, $F'(q) - F(q) = 0$ follows.

We set $q{\cdot}a = F(q)$. The functions $q \mapsto q{\cdot}a + q{\cdot}a'$ and $q \mapsto q(a + a')$ both extend $n \mapsto n{\cdot}a + n.a' = n{\cdot}(a + a')\colon \mathbb{Z} \to A$. Hence they agree by uniqueness. Thus $(q, a) \mapsto q{\cdot}a : \mathbb{Q} \times A \to A$ is bilinear. The proof of $q{\cdot}(q'{\cdot}a) = qq'{\cdot}a$ follows from $n{\cdot}(n'{\cdot}a) = nn'{\cdot}a$.

By A1.7, a subset $X \subseteq D$ is free if and only if every relation $\sum_{x\in X} n_x{\cdot}x = 0$ with $n_x \in \mathbb{Z}$ implies $n_x = 0$ for all $x \in X$. Every such set is linearly independent over $\mathbb{Q}$, because for any relation $\sum_{x\in X} q_x{\cdot}x = 0$ with $q_x \in \mathbb{Q}$ only finitely many of the q_x are nonzero, and thus there is a natural number m such that $mq_x \in \mathbb{Z}$ for all $x \in X$ yielding $q_x = 0$. Trivially, every $\mathbb{Q}$-linearly independent set is free according to A1.7. The remainder is now clear. ☐

If A is a torsion-free group and D is a divisible hull, then $A \cap \operatorname{tor} D = \{0\}$ implies $\operatorname{tor} D = \{0\}$. By A1.36(iii) and the preceding proposition, a subset of A is maximal free iff it is a vector space basis of D and thus its cardinal is $\dim_{\mathbb{Q}} D$ and thus is an isomorphy invariant of A, called the *rank of* A, written $\operatorname{rank} A$. The rank of an arbitrary abelian group A is defined to be $\operatorname{rank}(A/\operatorname{tor} A)$. Since a subset X of A is maximal free if and only if its image $(X + \operatorname{tor} A)/\operatorname{tor} A$ is maximal free in $A/\operatorname{tor} A$ we know that $\operatorname{rank} A$ is the cardinal of any maximal free subset of A. Accordingly, by A1.33 and A1.36, *if D is a divisible hull of an abelian group A, then* $\operatorname{rank} D = \operatorname{rank} A$ *and* $\operatorname{rank}_p D = \operatorname{rank}_p A$ *for all primes p.*

Before we elucidate the structure of a divisible p-group we prove a lemma. We recall that $S_p(A) = \{x \in A : p{\cdot}x = 0\}$ denotes the p-socle. Further recall that a sum $\sum_{j\in J} A_j$ of a family of subgroups of a group A is called *direct* if the function $(a_j)_{j\in J} \mapsto \sum_{j\in J} a_j : \bigoplus_{j\in J} A_j \to A$ is injective.

Lemma A1.40. *Let A be a p-group and $\{A_j \mid j \in J\}$ a family of subgroups. The sum $\sum_{j\in J} A_j$ is direct if and only if $\sum_{j\in J} S_p(A_j)$ is direct.*

Proof. It is clear that the sum of the socles $S_p(A_j)$ is direct if the sum of the A_j is direct.

Now we prove the converse. Let $\sigma\colon \bigoplus_{j\in J} A_j \to A$ be the morphism defined by $\sigma\bigl((a_j)_{j\in J}\bigr) = \sum_{j\in J} a_j$. We assume that the restriction of σ to $\bigoplus_{j\in J} S_p(A_j)$ is injective. Now let $g = (a_j)_{j\in J}$ be in the kernel of σ. Define $F(g) = \{j \in J : a_j \neq 0\}$ and recall that $n(g) = \operatorname{card} F(g)$ is finite. We claim that $g = 0$. Suppose that this is not the case; we shall derive a contradiction. Let $o(j)$ denote the order of a_j and $m = \max\{\nu_j : p^{\nu_j} = o(a_j), j \in F(g)\}$. Set $I(g) = \{j \in F(g) : o(j) = p^m\}$. Then $0 = p^{m-1}{\cdot}\sigma(g) = \sigma(p^{m-1}{\cdot}g) = \sum_{j\in I(p)} p^{m-1}{\cdot}a_j$. But now $p^{m-1}{\cdot}a_j \in S_p(A_j)$ for all $j \in J$. So by hypothesis, $p^{m-1}{\cdot}a_j = 0$ for all $j \in I(g)$, but then $o(a_j) = p^{m-1}$ for $j \in I(g)$, a contradiction. □

Proposition A1.41. *A divisible p-group is isomorphic to $\mathbb{Z}(p^\infty)^{(X)}$ for some set X.*

Proof. (AC) The socle $S(A)$ is a vector space over $\mathrm{GF}(p)$ by Remark A1.21. Hence there is a basis $X \subseteq S(A)$. For $x \in X$ we define $x_1 = x$ and recursively $p{\cdot}x_n = x_{n-1}$ for $n = 2, 3, 4, \ldots$. If $m/p^n = m'/p^{n'}$ with, say $n \le n'$, then $mp^{n'-n} = m'$ and thus $m'x_{n'} = mp^{n'-n}x_{n'} = mx_n$. Thus we define a function $F: \frac{1}{p^\infty}\mathbb{Z} \to A$ by $F(m/p^n) = m{\cdot}x_n$. Then $F(1) = F(p/p) = p{\cdot}x_1 = p{\cdot}x = 0$. We claim that $\ker F = \mathbb{Z}$. Indeed if not, then there exist nonzero integers m relatively prime to p, and n such that $F(m/p^n) = 0$. Then $m{\cdot}x_n = 0$. We write $1 = sm + tp^{n+1}$ and compute $0 = sm{\cdot}x_n = x_n - tp^{n+1}{\cdot}x_n = x_n$. But $p^n x_n = x \ne 0$, and thus $x_n \ne 0$. This contradiction proves the claim $\ker F = \mathbb{Z}$. Hence there is a unique isomorphism $f: \mathbb{Z}(p^\infty) \to \operatorname{im} F$ with $f(q + \mathbb{Z}) = F(q)$. Thus we have a subgroup $C(x) \cong \mathbb{Z}(p^\infty)$ in A containing x.

Now we define a morphism

$$\varphi: \bigoplus_{x\in X} C(x) \to A$$

by $\varphi\big((c_x)_{x\in X}\big) = \sum_{x\in X} c_x$. The image is divisible, hence a direct summand containing the entire socle. Hence the socle of the complementary summand must be trivial, and this means that this complement is zero. Hence the function is surjective. By Lemma A1.40, however, φ is injective. □

We now put all this information together for the following structure theorem:

The Structure Theorem for Divisible Subgroups

Theorem A1.42. *Let A be an abelian group and D a divisible subgroup, for instance, $D = \operatorname{div}(A)$. Then A is a direct sum of*

(i) *unique direct summands $D_p = D \cap A_p \cong \mathbb{Z}(p^\infty)^{\left((X(p)\right)}$ with sets $X(p)$ of a unique cardinality $\operatorname{rank}_p(D)$,*

(ii) *a direct summand $D_0 \cong D/\operatorname{tor}(D) \cong \mathbb{Q}^{(X)}$ with a set X of a unique cardinality $\operatorname{rank}(D)$,*

(iii) *a direct summand $A' \cong A/D$ (reduced if $D = \operatorname{div}(A)$).*

In particular,

$$D = D_0 \oplus \bigoplus_{p \text{ prime}} D_p \cong \mathbb{Q}^{(\operatorname{rank} D)} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(\operatorname{rank}_p(D))}. \qquad \square$$

Corollary A1.43. *Let $\mathfrak{c}$ denote the cardinality of $\mathbb{R}$. The following list gives the structure of familiar groups in terms of the Structure Theorem* A1.42 *above:*
(i) $\mathbb{R} \cong \mathbb{Q}^{(\mathfrak{c})} \cong \mathbb{C}$.
(ii) $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{Q}^{(\mathfrak{c})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty) \cong \mathbb{R} \times \mathbb{T} \cong (\mathbb{C} \setminus \{0\})^\times$. □

Exercise EA1.12. *Let A be an uncountable abelian group and $\aleph_0 \le \aleph < |A|$. Then A contains a subgroup B such that $(A : B) = \aleph$.*

Moreover, if $(A : \operatorname{tor} A)$ is at least $\aleph$ then B may be picked so that B is pure and A/B is torsion free.

In particular, every uncountable abelian group has a proper subgroup of index $\aleph_0$.

[Hint. Let D be a divisible hull of A according to Proposition A1.33. If $D = D_1 \oplus D_2$ is any direct decomposition, and $\mathrm{pr}_1 \colon D \to D_1$ is the projection onto the first summand of D, then $A/(A \cap D_2) \cong (A + D_2)/D_2 \cong \mathrm{pr}_1(A)$. Now any subgroup of D_1 is a subgroup of the divisible hull of A and therefore meets A and thus $\{0\} \ne S \cap A \cap D_1 \subseteq \mathrm{pr}_1(A)$ (see Proposition A1.33); therefore D_1 is a divisible hull of $\mathrm{pr}_1(A)$. Hence either $\mathrm{pr}_1(A)$ is finite or else $\operatorname{card} D_1 = \operatorname{card} \mathrm{pr}_1(A) = \operatorname{card}(A/(A \cap D_2))$ by Proposition A1.33(i).

Since we control $\operatorname{card} D_1$ by choosing D_1 appropriately, we aim to set $B = A \cap D_2$ and thereby prove our first assertion. We thus have to exclude the possibility that $\mathrm{pr}_1(A)$ might turn out to be finite by an inappropriate choice of D_1. We now let $\operatorname{tor} A$ denote the torsion subgroup of A. Then $\operatorname{tor} D$ is a divisible hull of $\operatorname{tor} A$, and $D \cong (\operatorname{tor} D) \times (D/\operatorname{tor} D)$ by Proposition A1.38. We now distinguish two cases:

(a) Case $\operatorname{card}(\operatorname{tor} A) = \operatorname{card} A$. Since A is uncountable, one of the p-primary components of $\operatorname{tor} A$, as p ranges through the countable set of primes, say $A(p)$, satisfies $\operatorname{card} A(p) = \operatorname{card}(\operatorname{tor} A) = \operatorname{card} A$. In particular $\operatorname{card} A(p)$ is uncountable, that is, its p-rank $\operatorname{card} A$ is uncountable and agrees with the p-rank of $D(p)$ (see Corollary A1.36(iii)). In view of $D(p) \cong (\mathbb{Z}(p^\infty)^{(\operatorname{card} A)})$ by Theorem A1.42(iii) above, we find a direct summand D_1 of $D(p)$ of p-rank $\aleph$, giving us a direct summand of $\operatorname{tor} D$ and thus yielding a direct sum decomposition $D = D_1 \oplus D_2$. Since the p-rank $\aleph$ of D_1 is infinite, and D_1 is the divisible hull of $\mathrm{pr}_1(A)$ we know that $\mathrm{pr}_1(A)$ cannot be finite, whence $\aleph = \operatorname{card} D_1 = \operatorname{card}(A/(A \cap D_2))$. Our first assertion then follows with $B = A \cap D_2$.

(b) Case $\operatorname{card}(A/\operatorname{tor} A) = \operatorname{card} A$. Then the (torsion free) rank of D is $\operatorname{card} A$ (see Corollary A1.36(iii)). By the structure theorem of divisible groups (see Theorem A1.42]) and elementary cardinal arithmetic, we can write $D = D_1 \oplus D_2$ with a torsionfree subgroup D_1 of cardinality $\aleph$. Then $\mathrm{pr}_1(A) \subseteq D_1$ cannot be finite, and as in the first case, we let $B = A \cap D_2$ and have $\aleph = \operatorname{card} D_1 = \operatorname{card} A/B$ as in our first assertion.

It remains to inspect the case that $\operatorname{card}(A/\operatorname{tor} A) \ge \aleph$. Then we may assume $\operatorname{tor} A \subseteq D_2$ and D_1 torsion free. Since $A \cap D_2$ is torsion as D_2 is torsion, we have

$\operatorname{tor} A = A \cap D_2 = B$. So, firstly, A/B is torsion free, and, secondly, B, as the torsion group of A is pure in A.]

Some Homological Algebra

We have seen the direct product and the direct sum as basic constructions to create abelian groups from given ones. Other basic constructions arise from homomorphisms.

(i) If A and B are abelian groups, then the set

$$\operatorname{Hom}(A, B) \stackrel{\text{def}}{=} \{f \mid f\colon A \to B \quad \text{is a homomorphism}\} \subseteq B^A$$

is a subgroup of B^A. We shall refer to the abelian group $\operatorname{Hom}(A, B)$ as a *hom-group.*

(ii) If A, B, and C are abelian groups, then a function $f\colon A \times B \to C$ is called *bilinear* if $a \mapsto f(a, b_0) : A \to A$ and $b \mapsto f(a_0, b) : B \to B$ are morphisms for all $b_0 \in B$ and $a_0 \in A$. □

Proposition A1.44. (i) *For abelian groups A and B there is a group $A \otimes B$, unique up to isomorphism, with a function $(a, b) \mapsto a \otimes b : A \times B \to A \otimes B$ such that for every bilinear map $f\colon A \times B \to C$ there is a unique homomorphism $f'\colon A \otimes B \to C$ such that $f'(a \otimes b) = f(a, b)$.*

(ii) *The groups*

$$\operatorname{Hom}(A \otimes B, C), \quad \operatorname{Hom}\big(A, \operatorname{Hom}(B, C)\big), \quad \textit{and} \quad \operatorname{Hom}\big(B, \operatorname{Hom}(A, C)\big)$$

are naturally isomorphic and are isomorphic to the group of all bilinear maps in $C^{A \times B}$.

Proof. (i) In order to prove existence we consider the free group $\mathbb{Z}^{(A \times B)}$ and define the subgroup U spanned by all elements $e(a + a', b) - e(a, b) - e(a', b)$ and $e(a, b + b') - e(a, b) - e(a, b')$ with $a, a' \in A$ and $b, b' \in B$. Then we set $A \otimes B = \mathbb{Z}^{(A \times B)}$ and we set $a \otimes b = e(a, b) + U$. Then $A \otimes B$ has the required property; the uniqueness follows from the universal property.

The details and the proof of part (ii) are left as an exercise. (See Exercise EA1.13.) □

Exercise EA1.13. Prove part (ii) of Proposition A1.44. □

The abelian group $A \otimes B$ which allows us to reduce bilinear maps on $A \times B$ to linear ones on $A \otimes B$ is called the *tensor product of A and B.*

The following observation applies tensor products to divisibility theory.

Proposition A1.45. *Let A denote an abelian group and $\delta\colon A \to \mathbb{Q} \otimes A$ the morphism given by $f(a) = 1 \otimes a$. Then*

(i) *every element in* $\mathbb{Q}\otimes A$ *is of the form* $q\otimes a$. *In particular,* $(\mathbb{Q}\otimes A)/(1\otimes A)$ *is a torsion group.*

(ii) *The function*

$$(q, q'\otimes a)\mapsto qq'\otimes a\colon \mathbb{Q}\times(\mathbb{Q}\otimes A)\to\mathbb{Q}\otimes A$$

makes $\mathbb{Q}\otimes A$ *into a rational vector space. Hence* $\mathbb{Q}\otimes A$ *is a torsion-free divisible group, and no nonzero subgroup of* $\mathbb{Q}\otimes A$ *meets* $1\otimes A$ *trivially.*

(iii) $\ker\delta=\operatorname{tor}(A)$.

(iv) *The embedding* $A/\operatorname{tor}(A)\to\mathbb{Q}\otimes A$ *is the unique embedding of a torsion-free abelian group into a smallest divisible one, and* $\mathbb{Q}\otimes\big(A/\operatorname{tor}(A)\big)\cong\mathbb{Q}\otimes A$.

(v) *If* A *is a subgroup of* B*, then the inclusion map* $j\colon A\to B$ *induces an injection* $\mathrm{id}_{\mathbb{Q}}\otimes j\colon\mathbb{Q}\otimes A\to\mathbb{Q}\otimes B$. *The quotient map* $p\colon B\to B/A$ *induces a surjection* $\mathrm{id}_{\mathbb{Q}}\otimes p\colon\mathbb{Q}\otimes B\to\mathbb{Q}\otimes(B/A)$ *whose kernel is* $\operatorname{im}(\mathrm{id}_Q\otimes j)$.

Proof. (i) If $t=\sum_{j\in J}q_j\otimes a_j$ is a typical element of the tensor product $Q\otimes A$ with a finite set J. Then we can write $q_j=m_j/n$ and $q_j\otimes a_j=\frac{1}{n}\otimes m_j{\cdot}a_j$. Hence $t=\frac{1}{n}\otimes\sum_{j\in J}m_j{\cdot}a_j$.

If $q=\frac{m}{n}$ then $n{\cdot}(q\otimes a)=\frac{mn}{n}\otimes a=m{\cdot}(1\otimes a)$, whence $(\mathbb{Q}\otimes A)/(1\otimes A)$ is a torsion group.

(ii) $\mathbb{Q}\otimes A$ is a $\mathbb{Q}$-vector space with respect to a scalar multiplication given by $q{\cdot}(q'\otimes a)=qq'\otimes a$. In particular, it is a torsion-free divisible group. If $0\neq q\otimes a\in S\leq\mathbb{Q}\otimes A$, with $q=\frac{m}{n}$ then $0\neq n{\cdot}(q\otimes a)\in S\cap(1\otimes A)$.

(iii) Any homomorphism into a torsion-free group kills the torsion group, hence $\operatorname{tor}(A)\subseteq\ker f$. Conversely, let $a\notin\operatorname{tor}A$, i.e. $z\mapsto z{\cdot}a:\mathbb{Z}\to A$ is injective, and thus, by the injectivity of $\mathbb{Q}$, the inclusion $\mathbb{Z}\to\mathbb{Q}$ "extends" to a morphism $F\colon A\to\mathbb{Q}$ with $F(z{\cdot}a)=z$ for $z\in\mathbb{Z}$. The bilinear map $(q,a)\mapsto qF(a)$ factors through $\mathbb{Q}\otimes A$ with $\varphi\colon\mathbb{Q}\otimes A\to\mathbb{Q}$, $\varphi(q\otimes a)=qF(a)$. In particular, $\varphi(1\otimes a)=F(a)=1$. Hence $1\otimes a\neq 0$.

(iv) By (iii), $A/\operatorname{tor}(A)$ is injected into $\mathbb{Q}\otimes A$ and the image is $1\otimes A$. The smallest divisible subgroup of $\mathbb{Q}\otimes A$ containing $1\otimes A$ is $\mathbb{Q}{\cdot}(1\otimes A)=\mathbb{Q}\otimes A$. If A is torsion-free and $j\colon A\to D$ is an injection into a divisible group D, then there is a unique morphism $i\colon\mathbb{Q}\otimes A\to D$ with $j(a)=i(1\otimes a)$. It is readily seen to be injective, since $i(q\otimes a)=0$ implies $0=i(1\otimes a)=j(a)$ and thus $a=0$. In this sense $\mathbb{Q}\otimes A$ is the smallest divisible group into which $A/\operatorname{tor}(A)$ is embedded.

(v) If $q\in\mathbb{Q}$, then $(\mathrm{id}_Q\otimes j)(q\otimes_A a)=\mathfrak{q}\otimes_B a$. This element vanishes if either $q=0$ or a is a torsion element in B by (iv) above. But this is the case iff $q\otimes_A a{=}0$. Thus $\mathrm{id}_Q\otimes j$ is injective. If $q\otimes_{B/A}(b+A)$ is given, then it is the image of $q\otimes_B b$ under $\mathrm{id}_Q\otimes p$. Assuming $q\neq 0$ we have $q\otimes_{B/A}(b+A)=0$ iff there is an n such that $n{\cdot}b\in A$ by (iv) above. Then $q\otimes_B b=(q/n)\otimes_B n{\cdot}b\in\mathbb{Q}\otimes_B A$ which is the image of $\mathrm{id}_Q\otimes j$. □

Assertion A1.45(v) is a very special case of a more general fact; indeed the assertion remains true if $\mathbb{Q}$ is replaced by any torsion-free group.

After A1.45(i,ii) the group $\mathbb{Q}\otimes A$ is a divisible hull of $1\otimes A$. The embedding of an abelian group A into a divisible hull D according to Proposition A1.33 is not functorial in the sense that a morphism $A_1 \to A_2$ extends canonically to a morphism $D_1 \to D_2$ of divisible hulls. The embedding of a *torsion-free* abelian group into a minimal divisible torsion-free abelian group, however, *is* functorial, and the injection is given by $A \to \mathbb{Q}\otimes A$.

In particular, after A1.45 we know that $\operatorname{rank} A = \dim_{\mathbb{Q}} \mathbb{Q}\otimes A$ (see the definition following A1.39).

Proposition A1.46. (i) *If D is a divisible group and A is any abelian group, then $D\otimes A$ is divisible.*

(ii) *If D is divisible and A is a torsion group, then $D\otimes A = \{0\}$.*

(iii) *An abelian group A is torsion-free and divisible if and only if $A \mapsto \mathbb{Q}\otimes A$ is an isomorphism.*

Proof. Exercise EA1.14(i). □

Exercise EA1.14. (i) Prove Proposition A1.46. (ii) Prove that for a torsion-free group F and any subgroup A of an abelian group B, the group $F\otimes A$ may be identified with a subgroup of $F\otimes B$. □

Proposition A1.47. *The tensor product is commutative and additive in each argument: $A\otimes B \cong B\otimes A$ and $A\otimes\bigoplus_{j\in J} B_j \cong \bigoplus_{j\in J} A\otimes B_j$.*

Proof. Exercise EA1.15(i). □

We shall now provide some links between the hom-sets and direct products and direct sums. For a family $\{A_j \mid j\in J\}$ of abelian groups the projections $\operatorname{pr}_j\colon \prod_{i\in J} A_i \to A_j$ are defined by $\operatorname{pr}_j\big((a_i)_{i\in J}\big) = a_j$ and the coprojections copr_j: $A_j \to \bigoplus_{i\in J} A_i$ by $\operatorname{copr}_j(a) = (x_i)_{i\in J}$ with $x_j = a$ and $x_i = 0$ otherwise.

Proposition A1.48. *For a fixed abelian group G and a family $\{A_j \mid j\in J\}$ of abelian groups we have the following conclusions:*

(i) *There is an isomorphism $\operatorname{Hom}(G, \prod_{j\in J} A_j) \to \prod_{j\in J}\operatorname{Hom}(G, A_j)$ which assigns to $\varphi\colon G\to\prod_{j\in J} A_j$ the element $(\operatorname{pr}_j\circ\varphi)_{j\in J}$. Its inverse mapping assigns to an element $(\varphi_j)_{j\in J}\in\prod_{j\in J}\operatorname{Hom}(G, A_j)$ the morphism*

$$g \mapsto \big(\varphi_j(g)\big)_{j\in J}\colon G \to \prod_{j\in J} A_j.$$

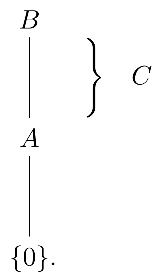
(ii) *There is an isomorphism $\prod_{j\in J}\operatorname{Hom}(A_j, G) \to \operatorname{Hom}(\bigoplus_{j\in J} A_j, G)$ which assigns to an element $(\varphi_j)_{j\in J}$ the morphism $(a_j)_{j\in J} \mapsto \sum_{j\in J}\varphi_j(a_j)$: $\bigoplus_{j\in J} A_j \to G$ and whose inverse map assigns to a morphism $\varphi\colon\bigoplus_{i\in J} A_i \to G$ the element $(\varphi\circ\operatorname{copr}_i)_{i\in J}$.*

Proof. Exercise EA15(ii). □

Exercise EA1.15. (i) Prove Proposition A1.47.
(ii) Prove Proposition A1.48. □

Exact Sequences

It is frequently convenient to think about the data given through an abelian group B and a subgroup A in various ways. Firstly, we have the quotient morphism $p: B \to C$ with $C = B/A$ and the inclusion map $j: A \to B$. The purely order theoretical aspects are well visualized in a lattice diagram of the type indicated below, known as a *Hasse diagram*:



Another way, sometimes preferable, is the use of the exact sequence. A finite or infinite sequence $\mathcal{C}$ of morphisms of abelian groups

$$\cdots \to A_{n-1} \overset{f_n}{\to} A_n \overset{f_{n+1}}{\to} A_{n+1} \to \cdots$$

is called *a cochain complex* if $f_{n+1} \circ f_n = 0$ for all n. This means, of course, that $\operatorname{im} f_n \subseteq \ker f_{n+1}$, and the factor groups $H^n(\mathcal{C}) \overset{\text{def}}{=} \ker f_{n+1}/\operatorname{im} f_n$ are called the *cohomology groups* of the complex. The sequence $\mathcal{C}$ is called *exact* (*at* A_n) if $\operatorname{im} f_n = \ker f_{n+1}$. In particular, a cochain complex is exact iff its cohomology groups vanish.

With the aid of this concept our data can be represented in terms of the exact sequence

$$0 \to A \overset{j}{\to} B \overset{p}{\to} C \to 0. \tag{1}$$

We now need to discuss some homological algebra of abelian groups. For a given abelian group G we need to discuss the *functor* $\operatorname{Hom}(G, -)$, that is the assignment of the abelian group $\operatorname{Hom}(G, A)$ to an abelian group A and the homomorphism $\operatorname{Hom}(G, f): \operatorname{Hom}(G, A) \to \operatorname{Hom}(G, B)$ defined by $\operatorname{Hom}(G, f)(\varphi) = f \circ \varphi$ for $f: A \to B$ and $\varphi: G \to A$. We shall have cause to apply $\operatorname{Hom}(G, -)$ to the exact sequence (1).

Lemma A1.49. *If* (1) *is exact, then the sequence*

$$0 \to \operatorname{Hom}(G, A) \xrightarrow{\operatorname{Hom}(G,j)} \operatorname{Hom}(G, B) \xrightarrow{\operatorname{Hom}(G,p)} \operatorname{Hom}(G, C) \tag{2}$$

is exact.

Proof. Exercise EA1.16. □

Exercise EA1.16. We observe that $\mathrm{Hom}(G,p)$ need not be surjective, thus exactness would fail at $\mathrm{Hom}(G,C)$ if we extended the sequence (2) by the zero morphism at the right end. We need a measure of the degree of failure of $\mathrm{Hom}(G,p)$ to be surjective. □

We shall consider the class of all exact sequences (1) with fixed groups A and C and call them *extensions of A by C*. We shall call

$$(1_n)\qquad E_n = (0 \to A \overset{j_n}{\to} B_n \overset{p_n}{\to} C \to 0)$$

for $n = 1, 2$ *equivalent* if and only if there is an isomorphism $f: B_1 \to B_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \overset{j_1}{\to} & B_1 & \overset{p_1}{\to} & C & \to & 0 \\ \downarrow & & {\scriptstyle \mathrm{id_A}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id_C}} & & \downarrow \\ 0 & \to & A & \overset{j_2}{\to} & B_2 & \overset{p_2}{\to} & C & \to & 0 \end{array}$$

commutes. Since the cardinality of B is the product of the cardinalities of A and C, the class of equivalence classes $[E]$ of all extensions of A by C is a set, called $\mathrm{Ext}(C,A)$.

We shall now denote by O the *trivial extension* $B = A \oplus C$ with the coprojection $j: A \to A \oplus B$ onto the first summand and the projection $p: A \oplus C \to C$ onto the second summand. Now let E_1 and E_2 be two extensions. We shall create an extension $E_1 + E_2$ whose equivalence class will only depend on the equivalence classes $[E_1]$ and $[E_2]$. First we form the subgroup $\Delta_A = \{\big(j_1(a), -j_2(a)\big) \mid a \in A\} \subseteq B_1 \oplus B_2$. The subgroup of all $(b_1, b_2) \in B_1 \oplus B_2$ with $p_1(b_1) = p_2(b_2)$ contains Δ_A. So we can define a subgroup $B = \{(b_1, b_2) + \Delta_A \mid p_1(b_1) = p_2(b_2)\}$ in $(B_1 \oplus B_2)/\Delta_A$ and define $j: A \to B$ by $j(a) = (j_1(a), 0) + \Delta_A = \big(0, j_2(a)\big) + \Delta_A$. This homomorphism is injective. We can specify $p: B \to C$ by $p\big((b_1, b_2) + \Delta_A\big) = p_1(b_1) = p_2(b_2)$. Clearly, p is surjective, and $pj = 0$, i.e. $\mathrm{im}\, j \subseteq \ker p$. An element $(b_1, b_2) + \Delta_A$ is in the kernel of p if and only if $p_1(b_1) = p_2(b_2) = 0$, that is if and only if there are elements $a, a' \in A$ such that $(b_1, b_2) = \big(j_1(a), j_2(a')\big) \in (j_1(a + a'), 0) + \Delta_A j(a'')$, where $a'' = a + a' \in A$. We thus have $\mathrm{im}\, j = \ker p$. Thus we have an exact sequence

$$(*)\qquad E_1 + E_2 = (0 \to A \overset{j}{\to} B \overset{p}{\to} C \to 0).$$

The class $[E_1 + E_2]$ depends only on $[E_1]$ and $[E_2]$ and we can set $[E_1] + [E_2] \overset{\mathrm{def}}{=} [E_1 + E_2]$.

Lemma A1.50. *For fixed groups A and C we have the following conclusions:*
(i) $[E_1] + [E_2] = [E_2] + [E_1]$.

(ii) $[E] + [O] = [E]$.

(iii) *Assume that*

$$E = (0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0).$$

Set

$$-E = (0 \to A \xrightarrow{-j} B \xrightarrow{p} C \to 0).$$

Then $[E] + [-E] = [O]$.

(iv) *Addition of equivalence classes of extensions is associative.*

Proof. Exercise EA1.17. □

Exercise EA1.17. Prove Lemma A1.50. □

Definition A1.51. The abelian group constructed in Lemma A1.50 is called *the group of extensions of A by C* and is written $\mathrm{Ext}(C, A)$. (The first argument C is the factor group, the second argument A is the subgroup!)

Lemma A1.52. *If $f: A \to A'$ is a morphism, then there is a morphism* $\mathrm{Ext}(C, f)$: $\mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A')$ *which assigns to the class of*

$$E = (0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0)$$

the class of the exact sequence

$$E' = (0 \to A' \xrightarrow{j'} B' \xrightarrow{p'} C \to 0)$$

with $B' = (A' \oplus B)/\Delta_A$ with $\Delta_A = \{(f(a), -j(a)) \mid a \in A\}$, $j'(a') = (a', 0) + \Delta_A$, $p'((a', b) + \Delta_A) = p(b)$.

Proof. Exercise EA1.18. □

Exercise EA1.18. Prove Lemma A1.52. □

Proposition A1.53. *Let G be an abelian group. Then G is injective if and only if* $\mathrm{Ext}(X, G) = 0$ *for all abelian groups X, and G is projective if and only if* $\mathrm{Ext}(G, X) = 0$ *for all abelian groups X.*

Proof. If G is injective, then $\mathrm{Ext}(X, G) = 0$ by Proposition A1.35 and Corollary A1.36. In order to prove the converse, assume that $j: A \to B$ is injective and that $f: A \to G$ is given. Then we have an exact sequence

$$E = (0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0)$$

and, by the preceding lemma (and its proof) also a sequence

$$E = (0 \to G \xrightarrow{j'} B' \xrightarrow{p'} C \to 0)$$

Let $f' \colon B \to B'$ be given by $f'(b) = (0, b) + \Delta_A$. Then $f'j = j'f$. If now $\operatorname{Ext}(C, G) = 0$ then the second exact sequence splits, and there is a homomorphism $\sigma \colon B' \to G$ with $\sigma j' = \mathrm{id}_G$. We set $\varphi \colon B \to G$ $\varphi = \sigma \circ f'$. Then $\varphi \circ j = \sigma \circ f' \circ j = \sigma \circ j' \circ f = f$. This shows that G is injective.

The proof of the second part is Exercise EA1.19. □

Exercise EA1.19. Prove the second part of Proposition A1.53. □

Lemma A1.54. (i) *For an exact sequence*

$$E = (0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0)$$

and for any abelian group G, there is a morphism

$$\delta \colon \operatorname{Hom}(G, C) \to \operatorname{Ext}(G, A)$$

which associates with a morphism $f \colon G \to C$ the class of the exact sequence

$$E_f = (0 \to A \xrightarrow{j_f} B_f \xrightarrow{p_f} G \to 0)$$

with $B_f = \{(g, b) \in G \times B \mid f(g) = p(b)\}$, $j_f(a) = \big(0, j(a)\big)$, $p_f(g, b) = g$.

(ii) $\ker \delta = \{f \in \operatorname{Hom}(G, C) \mid (\exists g \in \operatorname{Hom}(G, B)) \quad f = p \circ g\}$.

(iii) $\operatorname{im} \delta = \ker \operatorname{Ext}(G, j)$.

Proof. (i) We have to show that E_f is exact. Firstly, j_f is injective since j is injective, and p_f is surjective since p is surjective.

Next, $p_f j_f(a) = pj(a) = 0$; hence $\operatorname{im} j_f \subseteq \ker p_f$. Now assume that $p_f(g, b) = 0$, i.e. $g = 0$. Then $p(b) = f(g) = 0$. By the exactness of E there is an $a \in A$ with $j(a) = b$, and thus $j_f(a) = \big(0, j(a)\big) = (g, b)$. This completes the proof of the exactness of E_f. We must show that $E_{f_1+f_2}$ is equivalent to $E_{f_1} + E_{f_2}$; we leave this as an exercise.

(ii) We begin by taking an $f \colon G \to C$ with $\delta(f) = 0$ and show that f lifts, i.e. there is a $\varphi \colon G \to B$ such that $f = p\varphi$. Now $\delta(f) = 0$ means that E_f splits, i.e. that there is an $s \colon G \to B_f$ with $p_f \circ s = \mathrm{id}_G$, and that says that $s(g) = \big(g, \varphi(g)\big)$ with $\varphi \colon G \to B$. Now $\big(g, \varphi(g)\big) \in B_f$ means $f(g) = p\big(\varphi(g)\big)$.

The next step is to assume that $f = p \circ \varphi$ with a $\varphi \colon G \to B$. We must show that E_f splits. We set $s \colon G \to B_f$, $s(g) = \big(g, \varphi(g)\big)$. Indeed $f(g) = p\big(\varphi(g)\big)$ and thus s is well-defined. But $p_f s(g) = p_f\big(g, \varphi(g)\big) = g$, and thus s is the desired splitting morphism.

(iii) Firstly, we claim that $\operatorname{im} \delta \subseteq \ker \operatorname{Ext}(G, j)$. This means that for E_f the following sequence splits:

$$E' = (0 \to B \xrightarrow{i} (B \oplus B_f)/\Delta_A \xrightarrow{r} G \to 0)$$

where $\Delta_A = \{\big(j(a), (0, -j(a))\big) \in B \oplus B_f \mid a \in A\}$ and where $i(b) = \big(b, (0, 0)\big) + \Delta_A$, $r\big((b, (g, b')\big) + \Delta_A = g$. For $g \in G$ there is a $b \in B$ with $f(g) = p(b)$. If also $p(b') = f(g)$ then $p(b - b') = 0$, and thus there is an $a \in A$ with $b - b' = j(a)$, that is $b' = b - j(a)$. Then the elements $\big(-b,(g,b)\big)$ and $\big(-b',(g,b')\big) = \big(-b,(g,b)\big) + \big(j(a),(0,-j(a))\big)$ are congruent modulo Δ_A. Thus we may unambiguously define $s(g) = \big(-b,(g,b)\big)+\Delta_A$. Then $s\colon G \to (B\oplus B_f)/\Delta_A$ is a morphism with $rs(g) = g$. Hence E' splits.

Secondly, let us assume that an exact sequence

$$E = (0 \to A \xrightarrow{e} X \xrightarrow{q} G \to 0)$$

satisfies $\mathrm{Ext}(G,j)[E] = 0$. We must show that E and E_f are equivalent for some $f\colon G \to C$. The hypothesis means that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\xi} & X & \xrightarrow{\eta} & G & \to & 0 \\ \downarrow & & \mathrm{j}\downarrow & & q\downarrow & & \downarrow\mathrm{id_G} & & \downarrow \\ 0 & \to & B & \xrightarrow{\mathrm{copr}_B} & B\oplus G & \xrightarrow{\mathrm{pr}_G} & G & \to & 0. \end{array}$$

The morphism $q\colon X \to B\oplus G$ is of the form $q(x) = \big(\varphi(x),\eta(x)\big)$ with $\varphi\colon X \to B$ such that $\varphi\big(\xi(a)\big) = j(a)$. Now $p\varphi\xi(a) = pj(a) = 0$. Hence there is a unique $f\colon G \to C$ with $f\big(\eta(x)\big) = p\varphi(x)$. It remains to show that there is an isomorphism $\psi\colon X \to B_f$ such that $\mathrm{pr}_G\,\psi = \eta$ and $\mathrm{pr}_B\,\psi = \varphi$ holds. Accordingly, we set $\psi(x) = \big(\eta(x),\varphi(x)\big)$. If $\psi(x) = 0$, then $\eta(x) = 0$ and thus there is an $a \in A$ with $x = \xi(a)$. Now $0 = \varphi(x) = \varphi\xi(a) = j(a)$ and thus $a = 0$ since j is injective. Further, let $(g,b) \in B_f$. Then $f(g) = p(b)$. Now since η is surjective, we find an $x' \in X$ with $\eta(x') = g$. Now $p(b) = f\eta(x') = p\varphi(x')$. Hence $b-\varphi(x') \in \ker p = \operatorname{im} j$ and thus there is an $a \in A$ such that $j(a) = b - \varphi(x')$. Set $x = x' + \xi(a)$. Then $\varphi(x) = \varphi(x') + \varphi\xi(a) = \varphi(x') + j(a) = b$. Thus ψ is bijective, and the claim is proved. □

This is the raw material for an understanding of the following basic theorem of homological algebra:

The Exact Hom-Ext Sequence

Theorem A1.55. *For an exact sequence*

$$0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$$

and any abelian group G we obtain an exact sequence

$$0 \to \mathrm{Hom}(G,A) \xrightarrow{\mathrm{Hom}(G,j)} \mathrm{Hom}(G,B) \xrightarrow{\mathrm{Hom}(G,p)} \mathrm{Hom}(G,C)$$

$$\xrightarrow{\delta} \mathrm{Ext}(G,A) \xrightarrow{\mathrm{Ext}(G,j)} \mathrm{Ext}(G,B) \xrightarrow{\mathrm{Ext}(G,p)} \mathrm{Ext}(G,C) \to 0.$$

Proof. The first half of the long exact sequence was already observed in Lemma A1.49. Exactness at $\mathrm{Hom}(G,C)$ and $\mathrm{Ext}(G,A)$ was proved in Lemma A1.54. We propose the proof of the exactness at the remaining terms as an exercise. □

Exercise EA1.20. Prove the exactness of the Hom-Ext-sequence.

[Hint. As a reference one may consult [245], pp. 74ff.] □

The following is an important corollary for our purposes. We apply the theorem to the exact sequences

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{p} \mathbb{T} \to 0$$

and

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \to 0.$$

In both instances, B is divisible, hence injective, and thus $\mathrm{Ext}(G, B) = 0$ in both instances. Hence we have the following result:

Theorem A1.56. *For any abelian group G we have exact sequences*

$$0 \to \mathrm{Hom}(G,\mathbb{Z}) \xrightarrow{\mathrm{Hom}(G,j)} \mathrm{Hom}(G,\mathbb{R}) \xrightarrow{\mathrm{Hom}(G,p)} \mathrm{Hom}(G,\mathbb{T})$$

$$\xrightarrow{\delta} \mathrm{Ext}(G,\mathbb{Z}) \to 0,$$

and

$$0 \to \mathrm{Hom}(G,\mathbb{Z}) \xrightarrow{\mathrm{Hom}(G,j)} \mathrm{Hom}(G,\mathbb{Q}) \xrightarrow{\mathrm{Hom}(G,p)} \mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$$

$$\xrightarrow{\delta} \mathrm{Ext}(G,\mathbb{Z}) \to 0.$$ □

The first of these sequences is of crucial significance for the structure theory of compact abelian groups. We notice that $\mathrm{Hom}(G, \mathbb{R})$ is always an $\mathbb{R}$-vector subspace of the topological vector space $\mathbb{R}^G$ with its product structure.

Corollary A1.57. *For an arbitrary abelian group G and for the quotient map $p\colon \mathbb{R} \to \mathbb{T}$, the following two statements are equivalent:*

(1) $\mathrm{Hom}(G, p)\colon \mathrm{Hom}(G, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{T})$ *is surjective.*

(2) $\mathrm{Ext}(G, \mathbb{Z}) = 0$*; that is every exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{i} X \xrightarrow{q} G \to 0$$

(*for some abelian group X*) *splits.* □

The preceding ideas also give an indication how one might use the long exact sequence to calculate $\mathrm{Ext}(G, A)$. If G and A are given then we consider an embedding $j\colon A \to B$ into a divisible group such that we have an exact sequence

$$E = (0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0)$$

with a divisible torsion group C. Then $\mathrm{Ext}(G, B) = 0$ by Proposition A1.53, since B is injective, and thus we have an exact sequence

$$\mathrm{Hom}(G,B) \xrightarrow{\mathrm{Hom}(G,p)} \mathrm{Hom}(G,C) \xrightarrow{\delta} \mathrm{Ext}(G,A) \to 0.$$

In other words, we have

Proposition A1.58. *If G and A are abelian groups and A is considered as a subgroup of a divisible group B according to Proposition* A1.33 *and if $q: B \to B/A$ is the quotient morphism, then*

$$\mathrm{Ext}(G, A) \cong \mathrm{Hom}(G, B/A)/\,\mathrm{im}\big(\mathrm{Hom}(G, q)\big).$$

The group B may be chosen so that B/A is a torsion group, and if A is torsion-free, we take $B = \mathbb{Q} \otimes A$. □

Definition A1.59. If A is an abelian group, then $\dim_{\mathbb{Q}} \mathbb{Q} \otimes A$ is called the *rank* of A written $\mathrm{rank}\, A$. □

Note that $\mathbb{Q} \otimes A \cong \mathbb{Q} \otimes (A/\,\mathrm{tor}\, A)$ by A1.45. Therefore

$$\mathrm{rank}\, A = \mathrm{rank}(A/\,\mathrm{tor}\, A).$$

In the proof of Lemma A1.61, we can effectively utilize some information on continuous homomorphic images of $\mathbb{R}^n$ such as are collected in Remark A1.60 below. In its proof we use a result which we shall first formulate in an exercise:

Exercise EA1.21. Prove the following theorem.

The Open Mapping Theorem for Locally Compact Groups. *If $f: G \to H$ is a surjective continuous morphism between locally compact groups and G is a countable union of compact subsets, then f is open.*

[Hint. We must show that for each identity neighborhood U of G the image $f(U)$ is an identity neighborhood. Observe that it suffices to show that for each identity neighborhood U the set $f(U)$ has nonempty interior. Using the fact that G is a countable union of compact sets, show the existence of a sequence $g_n \in G$, $n \in \mathbb{N}$ such that $G = \bigcup_{n=1}^{\infty} U g_n$. Take U compact. Then $H = \bigcup_{n=1}^{\infty} f(U)f(g_n)$ is a countable union of closed sets. Since H is locally compact, H is a Baire space. (This is one portion of the Baire Category Theorem. For more details see e.g. the sources given in Chapter 2 preceding Theorem 2.3; a reference also yielding the Open Mapping Theorem is Hewitt and Ross [147], p. 42, 5.28 and 5.29). If a Baire space is a countable union of closed sets, one of them has nonempty interior. Conclude that $f(U)$ has nonempty interior.] □

Remark A1.60. (i) If S is a closed subgroup of $\mathbb{R}^n$ and if the quotient $\mathbb{R}^n/S$ is compact, then S contains an $\mathbb{R}$-basis of $\mathbb{R}^n$.

(ii) If $f: \mathbb{R}^n \to K$ is a surjective morphism onto a compact group, then $\ker f$ contains an $\mathbb{R}$-basis of $\mathbb{R}^n$.

Proof. (i) is clear from Theorem A1.12.

(ii) Set $S = \ker f$. In order to apply (i) it suffices to show that f is open, Since $\mathbb{R}^n$ is a countable union of compact sets this follows from the Open Mapping Theorem for Locally Compact Groups. □

Lemma A1.61. *If a finite rank abelian group G satisfies* $\mathrm{Ext}(G,\mathbb{Z})=0$ *then it is free.*

Proof. From Corollary A1.57 we know that the vanishing of $\mathrm{Ext}(G,\mathbb{Z})$ is equivalent to the information that every morphism $f\colon G\to\mathbb{T}$ lifts to a morphism $\varphi\colon G\to\mathbb{R}$, $f=p\circ\varphi$. In particular, each morphism $f\colon G\to\mathbb{T}$ annihilates the torsion subgroup. If Z is a cyclic subgroup of G of finite order, then there is an injection into $\mathbb{Q}/\mathbb{Z}$, hence into $\mathbb{T}$, and by Proposition A1.35 this injection extends to a morphism $G\to\mathbb{T}$. We saw that every such morphism annihilates torsion subgroups. Hence $Z=0$. Thus G is torsion-free.

Now recall that $\dim_{\mathbb{Q}}\mathbb{Q}\otimes G=n<\infty$. Note that $f\mapsto f(1):\mathrm{Hom}(\mathbb{Q},\mathbb{R})\to\mathbb{R}$ is an isomorphism and that therefore, in view of Proposition A1.44(ii) we have

$$\begin{aligned}\mathrm{Hom}(G,\mathbb{R})&\cong\mathrm{Hom}\big(G,\mathrm{Hom}(\mathbb{Q},\mathbb{R})\big)\\&\cong\mathrm{Hom}(\mathbb{Q}\otimes G,\mathbb{R})\cong\mathrm{Hom}(\mathbb{Q}^n,\mathbb{R})\cong\mathbb{R}^n.\end{aligned}$$

Thus we obtain an exact sequence

$$(*)\qquad 0\to\mathrm{Hom}(G,\mathbb{Z})\xrightarrow{\mathrm{Hom}(G,j)}\mathrm{Hom}(G,\mathbb{R})\cong\mathbb{R}^n\xrightarrow{\mathrm{Hom}(G,p)}\mathrm{Hom}(G,\mathbb{T})\to 0.$$

The group $\widehat{G}\overset{\mathrm{def}}{=}\mathrm{Hom}(G,\mathbb{T})$ is a compact group as a closed subgroup of $\mathbb{T}^G$. The map $\mathrm{Hom}(G,p)$ is a continuous homomorphism if we give $\mathrm{Hom}(G,\mathbb{R})$ and $\mathrm{Hom}(G,\mathbb{T})$ the topology of pointwise convergence inherited from $\mathbb{R}^G$, respectively, $\mathbb{T}^G$. With respect to this topology, $\mathrm{Hom}(G,\mathbb{R})$ is a topological $\mathbb{R}$-vector subspace of $\mathbb{R}^G$. Since $\mathrm{Hom}(G,\mathbb{R})$ is a finite dimensional real topological vector space, it admits only one vector space topology. Thus the isomorphism $\mathrm{Hom}(G,\mathbb{R})\cong\mathbb{R}^n$ is an isomorphism of topological vector spaces. Thus $\mathrm{Hom}(G,p)$ implements a continuous surjective homomorphism from $\mathbb{R}^n$ onto the compact group $\widehat{G}$ whose kernel we denote with S. By the preceding Remark A1.60(ii) it follows that $\mathrm{span}_{\mathbb{R}}S=\mathbb{R}^n$. If we apply this to $(*)$, we notice that $\mathrm{Hom}(G,\mathbb{Z})$, when identified with a subgroup of $\mathrm{Hom}(G,\mathbb{R})$ via $\mathrm{Hom}(G,j)$, must contain a basis $e_1,\dots,e_n$ of $\mathrm{Hom}(G,\mathbb{R})$, where $e_j\colon G\to\mathbb{Z}\subseteq\mathbb{R}$, $j=1,\dots,n$. If $e_j(g)=0$ for all j, then $g=0$ since every morphism $G\to\mathbb{R}$ is a linear combination of the e_j. Hence $g\mapsto\big(e_1(g),\dots,e_n(g)\big):G\to\mathbb{Z}^n$ is injective. Hence G is isomorphic to a subgroup of $\mathbb{Z}^n$ and is therefore free by Theorem A1.9 (or A1.10). □

We have seen in Proposition A1.53 that G is free if $\mathrm{Ext}(G,X)=0$ for *all* abelian groups X. The following result is remarkable in many respects.

Theorem A1.62 (Pontryagin). *Let G be a countable abelian group. Then the following conditions are equivalent:*

(1) *G is free.*
(2) $\mathrm{Ext}(G,\mathbb{Z})=0$.

Proof. The implication (1)⇒(2) is clear from Proposition A1.53. Therefore we must prove (2)⇒(1): The information $\mathrm{Ext}(G,\mathbb{Z})=0$ means that every morphism

$f\colon G \to \mathbb{T}$ lifts. However, if every morphism $f\colon G \to \mathbb{T}$ lifts to a morphism $\varphi\colon G \to \mathbb{R}$, then this is certainly also the case for every morphism $f_1\colon G_1 \to \mathbb{T}$, for any subgroup $G_1 \subseteq G$, since f_1 extends to a morphism $f\colon G \to \mathbb{T}$ by the divisibility of $\mathbb{T}$ (Proposition A1.35). Thus $\mathrm{Ext}(G_1, \mathbb{Z}) = 0$ for all subgroups G_1 of G. Thus every finite rank subgroup of G is free by Lemma A1.61. In particular, G is torsion-free. Since G is countable, G is free by A1.26. This completes the proof of Theorem A1.62. □

It is now convenient to have the following definitions.

A subgroup G of an abelian group G is said to *split* if there is a subgroup H of A such that $A = G \oplus H$. Likewise we say that a short exact sequence

$$(E) \qquad 0 \to G \overset{j}{\to} A \to B \to 0$$

is *split* if $j(G)$ is a split subgroup of A. This is tantamount to saying that the equivalence class of (E) in $\mathrm{Ext}(B, G)$ is zero. (See Definition A1.51). A subgroup G of an abelian group A splits automatically if it is divisible or G/A is free (See Corollary A1.36 and Proposition A1.15).

Definition A1.63. (i) An abelian group is called $\aleph_1$-*free* if every finite rank pure subgroup is free.

(ii) An abelian group A is called a *Whitehead group* if $\mathrm{Ext}(A, \mathbb{Z}) = 0$.

(iii) An abelian group A is called an S-*group* if for each $a \in A$ the pure subgroup $[\mathbb{Z}{\cdot}a]$ is free and splits.

Notice that every free group is both an $\aleph_1$-free and a Whitehead group.

Proposition A1.64. *The following conditions are equivalent for an abelian group G:*

(1) *G is $\aleph_1$-free.*
(2) *Every countable subgroup of G is free.*

Also, the following statements are equivalent:

(i) *A is an* S-*group.*
(ii) *Every rank one pure subgroup is free and splits.*
(iii) *Every finite rank pure subgroup is free and splits.*

Proof. (1)⇒(2): Firstly, if G is $\aleph_1$-free, then G is torsion-free, since the torsion subgroup has rank zero and hence is free, and thus zero. Let H be a countable subgroup of G. Then $[H]$ is still countable by Proposition A1.25. By A1.26 and Condition (1) we know that $[H]$ is free. Then H is free by Theorem A1.10.

(2)⇒(1): If P is a finite rank pure subgroup of G, then it is countable as a subgroup of a finite dimensional rational vector space. Then by Condition (2) it is free.

For the equivalence of (i), (ii) and (iii) see [193]. □

We have the following proper containments of torsion free classes.

$\aleph_1$-free groups

|

S- groups

|

free groups

Example A1.65. The group $\mathbb{Z}^{\mathbb{N}}$ is an $\aleph_1$-free abelian group (and thus an S-group) which is not a Whitehead group. The subgroup $\mathbb{Z}^{(\mathbb{N})}$ is a countable free subgroup which does not split.

Proof. (i) The group $\mathbb{Z}^{\mathbb{N}}$ is a subgroup of $\mathbb{Q}^{\mathbb{N}}$. For a finite dimensional vector space V of functions $\mathbb{N} \to \mathbb{Q}$ there is a finite subset $F \subseteq \mathbb{N}$ such that $f \mapsto f|F : V \to \mathbb{Q}^F$ is an injection. If V is the $\mathbb{Q}$-span of a finite rank subgroup G of $\mathbb{Z}^{\mathbb{N}}$, then this function maps G injectively into $\mathbb{Z}^F$, a free group. Hence G is free by Theorem A1.9. This shows that $\mathbb{Z}^{\mathbb{N}}$ is a $\aleph_1$-free.

(ii) We have to show that $\operatorname{Ext}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \neq 0$. By Corollary A1.57 it suffices to show that there is a morphism $\chi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{T}$ which does not lift to a morphism $\varphi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$.

We shall denote by $\beta\mathbb{N}$ the Stone–Čech compactification [369, 124] of the discrete set $\mathbb{N}$. We may assume that $\mathbb{N} \subseteq \beta\mathbb{N}$. The universal property of $\beta\mathbb{N}$ gives us for any function $f : \mathbb{N} \to X$ into a compact space a unique continuous function $f' : \beta\mathbb{N} \to X$ with $f'|\mathbb{N} = f$. Thus, in particular, there is a natural bijection $f \mapsto f' : \mathbb{T}^{\mathbb{N}} \to C(\beta\mathbb{N}, \mathbb{T})$ with inverse $g \mapsto g|\mathbb{N}$ which implements, in fact, an isomorphism of abelian groups. Now let $\xi \in \beta\mathbb{N} \backslash \mathbb{N}$. The evaluation $f \mapsto f'(\xi) : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}$ defines a morphism $\alpha \in \operatorname{Hom}(\mathbb{T}^{\mathbb{N}}, \mathbb{T})$ which has the following property:

(P) *For every $f \in \mathbb{T}^{\mathbb{N}}$ which takes the value t on all but finitely many elements of $\mathbb{N}$ we have $\alpha(f) = t$.*

We now define $\eta : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\eta(f)(m) = \frac{1}{2} f(m)$. Then $\mathbb{Z}^{\mathbb{N}} \subseteq \eta(\mathbb{Z}^{\mathbb{N}})$. We note $\eta^{-1}(\mathbb{Z}^{\mathbb{N}}) = (2\mathbb{Z})^{\mathbb{N}}$ and let $2 : \mathbb{Z}^{\mathbb{N}} \to (2\mathbb{Z})^{\mathbb{N}}$ denote the isomorphism inverting the restriction and corestriction of η.

Next we define $\chi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{T}$ by $\chi = \alpha \circ p^{\mathbb{N}} \circ \eta$ with the quotient map $p : \mathbb{R} \to \mathbb{T}$. We assume that there is a lifting $\varphi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$ so that $\chi = p \circ \varphi$ and we shall derive a contradiction and thereby complete the proof.

If $i : (2\mathbb{Z})^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ denotes inclusion, we have $\chi \circ i \circ 2 = \alpha \circ p^{\mathbb{N}} \circ j^{\mathbb{N}} = 0$ where $j : \mathbb{Z} \to \mathbb{R}$ is the inclusion. Hence there is a map $\zeta : H \to \mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}^{\mathbb{N}} & \xrightarrow{j^{\mathbb{N}}} & \mathbb{R}^{\mathbb{N}} & \xrightarrow{p^{\mathbb{N}}} & \mathbb{T}^{\mathbb{N}} & \to & 0 \\
 & & {\scriptstyle 2}\big\downarrow & & {\scriptstyle \eta}\big\uparrow & & \big\downarrow{\scriptstyle \alpha} & & \\
 & & (2\mathbb{Z})^{\mathbb{N}} & \xrightarrow[i]{} & \mathbb{Z}^{\mathbb{N}} & \xrightarrow[\chi]{} & \mathbb{T} & & \\
 & & \big\downarrow{\scriptstyle \zeta} & & {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{T}}} & & \\
0 & \to & \mathbb{Z} & \xrightarrow[j]{} & \mathbb{R} & \xrightarrow[p]{} & \mathbb{T} & \to & 0.
\end{array}$$

The mapping $\xi = \zeta \circ 2 = 2\zeta : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ has finite support, i.e. there is a finite subset $F \subseteq \mathbb{N}$ such that $\xi(\mathbb{Z}^{\mathbb{N} \setminus F}) = \{0\}$, where we have identified $\mathbb{Z}^{\mathbb{N}\setminus F}$ with the subgroup of $\mathbb{Z}^{\mathbb{N}}$ of functions $\mathbb{N} \to \mathbb{Z}$ vanishing on F. We define a function $f: \mathbb{N} \to \mathbb{Z}$ by

$$(*) \qquad (\eta f)(m) = \frac{1}{2} f(m) = \begin{cases} 0 & \text{for } m \in F, \\ \frac{1}{2} & \text{for } m \notin F. \end{cases}$$

Then $\chi(f) = \alpha p^{\mathbb{N}} \eta(f) = \frac{1}{2} + \mathbb{Z}$ in view of property (P). But $\eta(2f) \in \mathbb{Z}^{\mathbb{N}\setminus F} \subseteq \ker \xi$. In particular, $\eta(2f) \in \mathbb{Z}^{\mathbb{N}}$. Hence $\varphi(2f) = j\zeta i^{-1}(2f) = j\xi\eta(2f) = 0$. Then $0 = \frac{1}{2}\varphi(2f) = \varphi(f)$, whence $\chi(f) = 0$, a contradiction. □

Proposition A1.66. (i) *Every abelian group A contains a characteristic pure subgroup $K_\infty(A)$ such that $A/K_\infty(A)$ is $\aleph_1$-free and that $K_\infty(A)$ is the largest subgroup annihilated by all morphisms $A \to \mathbb{Z}$. If $A/K_\infty(A)$ is countable, then $A = F \oplus K_\infty$ with a countable free direct summand F, and K_∞ has no nontrivial free quotient groups.*

(ii) (Laszlo Fuchs, personal communication, February 26, 1996) *There is an $\aleph_1$-free abelian group A such that $K_\infty(A) \cong \mathbb{Z}$ and $A/K_\infty \cong \mathbb{Z}^{\mathbb{N}}$.*

Proof. (i) The assertions follow from Definition A1.63, Lemma A1.27 and Proposition A1.28.

(ii) We shall produce an abelian group A containing a subgroup $Z \cong \mathbb{Z}$ such that $G/Z \cong \mathbb{Z}^{\mathbb{N}}$ and such that all morphisms $f: A \to \mathbb{Z}$ have Z in their kernel. This will prove the claim.

Lemmas A and B below are contained in [116] and provide an argument for the existence of such a group.

Lemma A. *Let $E = [0 \to Z \hookrightarrow A \to B \to 0]$ be any extension of $Z \cong \mathbb{Z}$ by an abelian group B. Then there is a homomorphism $f: A \to \mathbb{Z}$ whose restriction to Z is nontrivial if and only E represents an element of finite order in* $\mathrm{Ext}(B, \mathbb{Z})$.

Proof. Assume E has order n in $\mathrm{Ext}(B, Z)$. Then the extension $n{\cdot}E = [0 \to Z \to A^* \to B \to 0]$ induced from E by $\mu_n: Z \to Z$, $\mu_n(x) = n{\cdot}x$ splits. Thus there is an $f: A \to Z$, namely, $A \to A^*$ followed by the projection from A^* to $Z \cong \mathbb{Z}$. Conversely, assume that there is a homomorphism $f: A \to \mathbb{Z}$ which is nontrivial on Z. Without losing generality we can assume that f is surjective. Clearly, $f|Z$ is multiplication by an integer n; it induces an extension $n{\cdot}E = [0 \to Z \to A^* \to$

$B \to 0]$ which splits in view of the existence of f. This completes the proof of Lemma A. □

For the following see the discussion of algebraically compact groups in [114], p. 159ff. The group $C \overset{\text{def}}{=} \mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is a torsion-free algebraically compact group ([114], p. 176, 42.2); it is not divisible, so it contains a copy of the group $\mathbb{Z}_p$ of p-adic integers (for a definition of $\mathbb{Z}_p$, see Chapter 1, 1.28(i)) for each p as a direct summand ([114], p. 169, 40.4).

Lemma B. $\mathrm{Ext}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ *contains* $2^{(2^{\aleph_0})}$ *elements of infinite order.*

Proof. The exact sequence $0 \to \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}^{\mathbb{N}} \to C \to 0$ induces the exact sequence

$$\mathrm{Hom}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}) \to \mathrm{Ext}(C, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}) = \{0\}.$$

Since $\operatorname{card} \mathbb{Z}^{\mathbb{N}} = 2^{\aleph_0}$, it suffices to show that $\mathrm{Ext}(C, \mathbb{Z})$ contains the direct sum of more than $2^{\aleph_0}$ copies of $\mathbb{Q}$. We are done if we can prove that already $\mathrm{Ext}(\mathbb{Z}_p, \mathbb{Z})$ contains such a direct sum. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}_p \to D \to 0$, where D is the direct sum of a torsion-free divisible group of cardinality $2^{\aleph_0}$ and the Prüfer group $\mathbb{Z}(p^\infty)$, we derive the exact sequence $0 \to \mathbb{Z} \to \mathrm{Ext}(D, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}_p, \mathbb{Z}) \to 0$. Since $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is a torsion-free divisible group of cardinality $2^{\aleph_0}$, we observe $\operatorname{card}\big(\mathrm{Ext}(D, \mathbb{Z})\big) = 2^{(2^{\aleph_0})}$, and we are done. □

Whitehead's Problem

In this section we closely follow notes by Fuchs [117] which he kindly supplied to us together with permission to use them in this place. We also thank him expressly for discussing the final version of this section with us.

Whitehead's Problem asks whether the following statement is true.

Proposition W. *If an abelian group A satisfies* $\mathrm{Ext}(A, \mathbb{Z}) = \{0\}$*, then it is free.*

In short: Whitehead groups are free. We have seen that Proposition W holds for countable groups A (see A1.62).

In order to address the truth or falsehood of Proposition W we need to record some axioms of set theory. The first of these axiom requires some concepts in ordinal and cardinal set theory. Let κ be a well ordered set without maximal element. A subset $C \subseteq \kappa$ is closed and unbounded (*a cub*) if it has no upper bound in κ and $X \subseteq C$, $\sup X \in \kappa$ implies $\sup X \in C$. A subset S of κ is called *stationary* if it intersects every cub in κ. We consider cardinals as those ordinals whose cardinality is bigger than that of all preceding ordinals. A family $\{X_\alpha\}_{\alpha<\kappa}$ of subsets of a set X, or subgroups of an abelian group, indexed by the ordinals below $\kappa \overset{\text{def}}{=} \operatorname{card} X$ is called a *continuous chain* if the following conditions hold:

(Fi) $\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$,

(Fii) $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for each limit ordinal $\lambda < \operatorname{card} X$.

The family is called a *filtration* of X if, in addition, the following conditions hold:
(Fiii) $\operatorname{card} X_\alpha < \operatorname{card} X$,
(Fiv) $X = \bigcup_{\alpha < \operatorname{card} X} X_\alpha$.

A cardinal κ is called *regular*, if it agrees with its *cofinality*, i.e. the smallest cardinal α such that κ contains a subset X with $\operatorname{card} X = \alpha$ and $\sup X = \kappa$. We note that every subset of X of cardinality below κ is contained in some member X_α of a filtration of X whenever κ is regular. Now we can formulate the

Diamond Principle $\diamond$ (Jensen [222]). *Let E be any stationary subset of the set of predecessors of an uncountable regular cardinal κ. Let X be a set with $\operatorname{card} X = \kappa$ and $\{X_\alpha \mid \alpha < \kappa\}$ a filtration of X. Then there is a family $\{S_\alpha \mid \alpha \in E\}$ with $S_\alpha \subseteq X_\alpha$ such that for any $Y \subseteq X$ the set $E' \stackrel{\text{def}}{=} \{\alpha \in E \mid Y \cap X_\alpha = S_\alpha\}$ is a stationary subset of the set of predecessors of κ.* $\square$

Since Gödel's Axiom of Constructibility "$V = L$" (saying that the model V of set theory we are working in is the constructible universe L) according to Gödel is consistent with ZFC, and since Gödel's Axiom implies $\diamond$ we know that $\diamond$ is consistent with ZFC.

We now turn to the second axiom to which we will refer. A partially ordered set P is said to *satisfy the countable antichain condition* if the subset $S \subseteq P$ is countable whenever no pair of elements of S has an upper bound in P.

Martin's Axiom MA. *Let P be a partially ordered set satisfying the countable antichain condition. Then for every family $\{C_j \mid j \in J\}$ of cofinal subsets of P such that $\operatorname{card} J < 2^{\aleph_0}$ there is a directed subset D of P such that $D \cap C_j \neq \emptyset$ for each $j \in J$.* $\square$

Zermelo–Fraenkel Set Theory and the Axiom of Choice (ZFC) and $\aleph_1 < 2^{\aleph_0}$ and Martin's Axiom are consistent [337].

In the discourse which follows below we shall establish the following results.

Theorem A1.67. *Assume that the axioms of ZFC and $\diamond$ hold. Then all Whitehead groups A with $\operatorname{card} A \leq \aleph_1$ are free.*

Proposition A1.68. *Assume that the axioms of ZFC, MA and $\aleph_1 < 2^{\aleph_0}$ hold. Then there exists a Whitehead group A with $\operatorname{card} A = \aleph_1$ which is not free.*

This result entails at once the following more general theorem.

Theorem A1.69. *Assume that the axioms of ZFC, MA and $\aleph_1 < 2^{\aleph_0}$ hold. Then for any cardinal $\aleph \geq \aleph_1$ there exists a Whitehead group A with $\operatorname{card} A = \aleph$ which is not free.*

Proof. Using Proposition A1.68 above, we find a nonfree Whitehead group A_1 with $\operatorname{card} A_1 = \aleph_1$. Then $A = A_1 \oplus \mathbb{Z}^{(\aleph)}$ is a nonfree Whitehead group with $\operatorname{card} A = (\operatorname{card} A_1)\cdot(\operatorname{card} \mathbb{Z}^{(\aleph)}) = \aleph$. □

The Continuum Hypothesis (CH) $\aleph_1 = 2^{\aleph_0}$ is independent from ZFC by Cohen's Theorem [63, 64]. We therefore have the following result [329].

Shelah's Independence Theorem

Theorem A1.70. ([329]) *If ZFC is consistent, then ZFC + Proposition* W *and ZFC + ¬Proposition* W *are consistent; i.e. Proposition* W *is undecidable in ZFC.* □

We now proceed to prove Theorem A1.67 and Proposition A1.68 in this order and begin by discussing some ordinal and cardinal number arguments.

Exercise EA1.22. The intersection of two cubs in an uncountable regular cardinal κ is a cub.

[Hint. If A and B are cubs, then the intersection $C = A \cap B$ is closed; one has to show that it is unbounded. If $\alpha_1 \in A$ then, since B is unbounded, there is a $\beta_1 \in B$ with $\alpha_1 < \beta_1$. Then there is an $\alpha_2 \in A$ with $\beta_1 < \alpha_2$. And so on. The chain $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ has a supremum in C exceeding α_1. Since A is unbounded, so is C.] □

We index the ordinals in a cub in κ by the ordinals below κ. Therefore a cub C in κ is order-isomorphic to κ and thus there is a monotone bijection $f\colon \kappa \to C$.

Lemma A1.71. *Let κ be an uncountable regular cardinal. Then for two filtrations $\{X_\alpha\}_{\alpha<\kappa}$ and $\{Y_\alpha\}_{\alpha<\kappa}$ of a set X of cardinality κ, the set $C = \{\alpha < \kappa \mid X_\alpha = Y_\alpha\}$ is a cub in κ.*

Proof. The set C is closed because the filtrations are continuous (cf. (Fii)). We prove that C is unbounded. For each $\alpha < \kappa$ conditions (Fiii) and (Fiv) imply that there is a β with $\alpha \le \beta < \kappa$ such that $X_\alpha \subseteq Y_\beta$. Thus for any $\alpha_1 \in C$ we find a β_1 with $\alpha_1 \le \beta_1 < \kappa$ and $X_{\alpha_1} \subseteq Y_{\beta_1}$. Analogously, we get an α_2 such that $\beta_1 \le \alpha_2 < \kappa$ and $Y_{\beta_1} \subseteq X_{\alpha_2}$. Thus, recursively, we find a sequence $\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \cdots$ such that $X_{\alpha_1} \subseteq Y_{\beta_1} \subseteq X_{\alpha_2} \subseteq Y_{\beta_2} \subseteq \cdots$. Then $\gamma \stackrel{\text{def}}{=} \sup_n \alpha_n = \sup_n \beta_n$, by continuity (Fii), produces $X_\gamma = \bigcup_n X_{\alpha_n} = \bigcup_n Y_{\beta_n} = Y_\gamma$. Thus $\alpha_1 \le \gamma \in C$. Hence C is unbounded in κ. □

Lemma A1.72. ($\diamond$) *If E is a stationary subset of the uncountable regular cardinal κ, and if $\{X_\alpha\}_{\alpha<\kappa}$ is a filtration of a set X of cardinality κ, then for any countable set Y there is a family $(g_\alpha)_{\alpha\in E}$ of functions $g_\alpha\colon X_\alpha \to Y \times X_\alpha$ such that, for any function $g\colon X \to Y \times X$, the set $E' \stackrel{\text{def}}{=} \{\alpha \in E : g|X_\alpha = g_\alpha\}$ is stationary in κ.*

Proof. Define $X' = X \times (Y \times X)$ and $X'_\alpha = X_\alpha \times (Y \times X_\alpha)$; now apply $\diamond$ to the filtration $\{X'_\alpha\}_{\alpha<\kappa}$ of X'. We find subsets $S_\alpha \subseteq X_\alpha \times Y \times X_\alpha$ with the property specified in $\diamond$. Fix a $y_0 \in Y$. Define $g_\alpha: X_\alpha \to Y \times X_\alpha$ to have S_α as graph if S_α is the graph of a function, and via $g_\alpha(x) = (y_0, x)$ otherwise. Now consider a function $g: X \to Y \times X$. Let S be its graph. Set $E' = \{\alpha \in E : g|X_\alpha = g_\alpha\}$. Then $\diamond$ yields that E' is stationary in κ. □

Lemma A1.73. *Let* $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ *be an exact sequence of torsion-free groups in which* B *is a Whitehead group but* C *is not. Then the following statements hold.*

(i) *There exists a morphism* $\varphi: A \to \mathbb{Z}$ *which does not extend to a morphism* $B \to \mathbb{Z}$, *i.e. for which there is no* $\Phi: B \to \mathbb{Z}$ *with* $\varphi = \Phi \circ \alpha$.

(ii) *Define a morphism* $\chi: \mathbb{Z} \oplus A \to \mathbb{Z} \oplus B$ *by* $\chi(n, a) = \big(n + \varphi(a), \alpha(a)\big)$ *with* $\varphi: A \to \mathbb{Z}$ *as in* (i) *above. Then there is no morphism* $\tau: B \to \mathbb{Z} \oplus B$ *such that the following diagram commutes:*

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{\ \mathrm{copr}_A\ } & \mathbb{Z} \oplus A \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\chi} \\ B & \xrightarrow[\ \tau\ ]{} & \mathbb{Z} \oplus B. \end{array}$$

Proof. (i) In the exact sequence

$$\mathrm{Hom}(B, \mathbb{Z}) \xrightarrow{\mathrm{Hom}(\alpha,\mathbb{Z})} \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Ext}(C, \mathbb{Z}) \xrightarrow{\mathrm{Ext}(\beta,\mathbb{Z})} \mathrm{Ext}(B, \mathbb{Z}) = \{0\}$$

we have $\mathrm{Ext}(C, \mathbb{Z}) \neq \{0\}$ since C is not a Whitehead group. Thus $\mathrm{Hom}(\alpha, \mathbb{Z})$ is not surjective and this implies the assertion.

(ii) From the definition of χ we get

$$(**) \qquad \mathrm{pr}_{\mathbb{Z}} \circ \chi \circ \mathrm{copr}_A = \varphi: A \to \mathbb{Z}.$$

Suppose that, contrary to the assertion of (ii), a τ exists making $(*)$ commutative. Then we set $\Phi = \mathrm{pr}_{\mathbb{Z}} \circ \tau$. Now $\Phi \circ \alpha = \mathrm{pr}_{\mathbb{Z}} \circ \tau \circ \alpha = \mathrm{pr}_{\mathbb{Z}} \circ \chi \circ \mathrm{copr}_A = \varphi$ by $(*)$ and $(**)$, contrary to the choice of φ. □

Lemma A1.74. ($\diamond$) *Let* κ *be an uncountable regular cardinal and* A *a torsion-free group of cardinality* κ. *Assume that* A *has a filtration* $\{A_\alpha\}_{\alpha<\kappa}$ *such that*

(i) A_α *is a pure free subgroup of* A *for each* $\alpha < \kappa$,

(ii) $\mathrm{card}\, A_\alpha < \kappa$ *for each* $\alpha < \kappa$,

(iii) *the set* $E \overset{\mathrm{def}}{=} \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha$ *is not a Whitehead group*$\}$ *is stationary in* κ.

Then A *is not a Whitehead group.*

Proof. By Lemma A1.72 there is a family $(g_\alpha)_{\alpha\in E}$ of functions $g_\alpha: A_\alpha \to \mathbb{Z} \times A_\alpha$ such that for every function $g: A \to \mathbb{Z} \times A$, the set $E' \overset{\mathrm{def}}{=} \{\alpha \in E : g|A_\alpha = g_\alpha\}$ is stationary in κ. We will construct, by transfinite induction, a direct system $\varphi_{\alpha\beta}: G_\alpha \to G_\beta$, $\alpha < \beta < \kappa$, and morphisms $\pi_\alpha: G_\alpha \to A_\alpha$ such that with the inclusion morphisms $\psi_{\alpha,\beta}: A_\alpha \to A_\beta$, we have commuting diagrams of split exact sequences

$$(\dagger)\qquad \begin{array}{ccccccccccc} S_\alpha: & 0 & \to & \mathbb{Z} & \xrightarrow{j_\alpha} & G_\alpha & \xrightarrow{\pi_\alpha} & A_\alpha & \to & 0 \\ & & & \| & & \downarrow{\scriptstyle\varphi_{\alpha\beta}} & & \downarrow{\scriptstyle\psi_{\alpha\beta}} & & \\ S_\beta: & 0 & \to & \mathbb{Z} & \xrightarrow[j_\beta]{} & G_\beta & \xrightarrow[\pi_\beta]{} & A_\beta & \to & 0. \end{array}$$

We claim further that we can implement the construction in such a fashion that for the colimits $G = \operatorname{colim}_{\alpha<\kappa} G_\alpha$ and $A = \operatorname{colim}_{\alpha<\kappa} A_\alpha = \bigcup_{\alpha<\kappa} A_\alpha$ we obtain an exact sequence

$$S : 0 \to \mathbb{Z} \xrightarrow{j} G \xrightarrow{\pi} A \to 0$$

which *does not split.*

Now we begin the induction. There is nothing to prove for $\alpha = 0$. Assume that for some $\gamma < \kappa$ the sequence S_α has been defined for all $\alpha < \gamma < \kappa$ so that the diagrams $(\dagger)$ commute for all $\alpha < \beta < \gamma$. Since all S_α are split for $\alpha < \gamma$ we have isomorphisms of groups $\varepsilon_\alpha: \mathbb{Z} \times A_\alpha \to G_\alpha$. We have to consider several possibilities.

Case 1. Assume that γ is a limit ordinal. Set $G_\gamma = \operatorname{colim}_{\alpha<\gamma} G_\alpha$ and obtain the exact sequence S_γ as the colimit of the sequences S_α. Since A_γ is free, this sequence splits and yields an isomorphism $\varepsilon_\gamma: \mathbb{Z} \times A_\gamma \to G_\gamma$.

Case 2. Assume that $\gamma = \beta + 1$ for some β. Here we consider two subcases:

2a. $\beta \notin E$ or $g_\beta: A_\beta \to \mathbb{Z} \times A_\beta$ is not a cross section homomorphism for the morphism $\pi_\beta \circ \varepsilon_\beta = \operatorname{pr}_{A_\beta}: \mathbb{Z} \times A_\beta \to A_\beta$.

In these circumstances let

$$S_\gamma: \quad 0 \to \mathbb{Z} \xrightarrow{j_\gamma} G_\gamma \xrightarrow{\pi_\gamma} A_\gamma \to 0$$

be any split extension, giving rise to an isomorphism $\varepsilon_\gamma: \mathbb{Z} \times A_\gamma \to G_\gamma$, and find $\varphi_{\beta\gamma}$ via

$$\begin{array}{ccc} \mathbb{Z} \times A_\beta & \xrightarrow{\varepsilon_\beta} & G_\beta \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}} \times \psi_{\beta\gamma}}\downarrow & & \downarrow{\scriptstyle\varphi_{\beta,\gamma}} \\ \mathbb{Z} \times A_\gamma & \xrightarrow[\varepsilon_\gamma]{} & G_\gamma. \end{array}$$

Then the following diagram of split exact sequences commutes:

$$\begin{array}{ccccccccc} S_\beta: & 0 \to & \mathbb{Z} & \xrightarrow{j_\beta} & G_\beta & \xrightarrow{\pi_\beta} & A_\beta & \to 0 \\ & & \| & & \downarrow{\scriptstyle\varphi_{\beta\gamma}} & & \downarrow{\scriptstyle\psi_{\beta\gamma}} & \\ S_\gamma: & 0 \to & \mathbb{Z} & \xrightarrow[j_\gamma]{} & G_\gamma & \xrightarrow[\pi_\gamma]{} & A_\gamma & \to 0. \end{array}$$

Since (†) is assumed to commute for all $\alpha < \beta < \gamma$ a simple diagram chase yields that

$$\begin{array}{lccccccc}
S_\alpha: & 0 \to & \mathbb{Z} & \xrightarrow{j_\alpha} & G_\alpha & \xrightarrow{\pi_\alpha} & A_\alpha & \to 0 \\
 & & \| & & \varphi_{\alpha\gamma}\downarrow & & \downarrow\psi_{\alpha\gamma} & \\
S_\gamma: & 0 \to & \mathbb{Z} & \xrightarrow[j_\gamma]{} & G_\gamma & \xrightarrow[\pi_\gamma]{} & A_\gamma & \to 0
\end{array}$$

commutes for any $\alpha < \gamma$.

2b. $\beta \in E$ and $g_\beta: A_\beta \to \mathbb{Z} \times A_\beta$ is a cross section morphism for $\pi_\beta \circ \varepsilon_\beta$. Now we apply Lemma A1.73 with $A = A_\beta$, $B = A_\gamma$, $C = A_{\beta+1}/A_\beta$ and obtain a morphism $\chi_\beta: \mathbb{Z} \times A_\beta \to \mathbb{Z} \times A_\gamma$, an isomorphism $\varepsilon_\gamma: A_\gamma \to \mathbb{Z} \times A_\gamma$, and an exact sequence S_γ in a commutative diagram

$$\begin{array}{lccccccc}
S_\beta: & 0 \to & \mathbb{Z} & \xrightarrow{j_\beta} & G_\beta & \xrightarrow{\pi_\beta} & A_\beta & \to 0 \\
 & & \| & & \varphi_{\beta\gamma}\downarrow & & \downarrow\psi_{\beta\gamma} & \\
S_\gamma: & 0 \to & \mathbb{Z} & \xrightarrow[j_\gamma]{} & G_\gamma & \xrightarrow[\pi_\gamma]{} & A_\gamma & \to 0
\end{array}$$

with $\varphi_{\beta\gamma} = \varepsilon_\gamma^{-1}\chi_\beta\varepsilon_\beta$, such that there is no morphism $\tau: A_\gamma \to \mathbb{Z} \times A_\gamma$ which makes the following diagram commutative:

$$(*) \qquad \begin{array}{ccccc}
A_\beta & \xrightarrow{\mathrm{copr}_{A_\beta}} & \mathbb{Z} \times A_\beta & \xrightarrow{\varepsilon_\beta^{-1}} & G_\beta \\
\psi_{\beta\gamma}\downarrow & & \chi_\beta\downarrow & & \downarrow\varphi_{\beta\gamma} \\
A_\gamma & \xrightarrow{\tau} & \mathbb{Z} \times A_\beta & \xrightarrow{\varepsilon_\gamma^{-1}} & G_\gamma.
\end{array}$$

This concludes the inductive construction.

The ε_α are not to be expected to be cofinally compatible with the inclusion maps $\psi_{\alpha\beta}: A_\alpha \to A_\beta$; therefore we cannot conclude the existence of a limit isomorphism $\varepsilon: G \to A \times \mathbb{Z}$. On the contrary, we claim that the colimit exact sequence

$$S: \quad 0 \quad \to \quad \mathbb{Z} \quad \xrightarrow{j} \quad G = \mathrm{colim}_{\alpha<\kappa} G_\alpha \quad \xrightarrow{\pi} \quad A = \bigcup_{\alpha<\kappa} A_\alpha \quad \to \quad 0$$

does not split. By way of contradiction suppose that it did and that there is a cross section morphism $g': A \to G$ such that $\pi \circ g' = \mathrm{id}_A$. If $\varphi_\alpha: G_\alpha \to G$ denotes the colimit morphism and $\psi_\alpha: A_\alpha \to A$ the inclusion morphism, we have $\pi\varphi_\alpha = \psi_\alpha\pi_\alpha$ and thus $\varphi_\alpha = g'\pi\varphi_\alpha = g'\psi_\alpha\pi_\alpha$, whence $g'(A_\alpha) \le \varphi_\alpha(G_\alpha)$. Thus, since the corestriction $\varphi'_\alpha: G_\alpha \to \varphi_\alpha(G_\alpha)$ of φ_α is an isomorphism, the morphism $g'_\alpha \stackrel{\mathrm{def}}{=} (\varphi'_\alpha)^{-1}g'\psi_\alpha: A_\alpha \to G_\alpha$ is a cross section homomorphism for π_α, i.e. $\pi_\alpha g'_\alpha = \mathrm{id}_{A_\alpha}$ and

$$(**) \qquad \begin{array}{ccc}
A_\alpha & \xrightarrow{j_\alpha} & G_\alpha \\
\| & & \downarrow\varepsilon_\alpha \\
A_\alpha & \xrightarrow[\mathrm{copr}_{A_\alpha}]{} & \mathbb{Z} \times A_\alpha
\end{array}$$

is commutative.

From the exact sequence S we do find a bijection (not a morphism!) $\varepsilon: G \to \mathbb{Z} \times A$ which we fix. Define the function $g: \mathbb{Z} \to A \times G$ by $g \stackrel{\text{def}}{=} \varepsilon \circ g'$. By the initial choice of the functions $g_\alpha: A_\alpha \to \mathbb{Z} \times A_\alpha$, there is at least one $\gamma \in E'$ (indeed cofinally many of them below κ) such that $g|A_\gamma = g_\gamma$. For all $\alpha < \kappa$ let us write $g^*_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha^{-1} \circ g_\alpha: A_\alpha \to G_\alpha$. Then we have $\varphi_\gamma g^*_\gamma = \varphi_\gamma \varepsilon_\gamma^{-1} g_\gamma = \varphi_\gamma \varepsilon_\gamma^{-1}(g|A_\gamma) = \varphi_\gamma \varepsilon_\gamma^{-1} g \psi_\gamma = \varphi_\gamma g' \psi_\gamma = g'_\gamma$. In other words

$$\varepsilon_\gamma^{-1} g_\gamma: A_\gamma \to G_\gamma$$

is a cross section morphism for $\pi_\gamma: G_\gamma \to A_\gamma$.

Therefore $S_{\gamma+1}$ was constructed according to Case 2b. In view of $(**)$ we have a commutative diagram

$$\begin{array}{ccccc} A_\gamma & \xrightarrow{\operatorname{copr}_{A_\gamma}} & \mathbb{Z} \times A_\gamma & \xrightarrow{\varepsilon_\gamma^{-1}} & G_\gamma \\ {\scriptstyle \psi_{\gamma(\gamma+1)}}\downarrow & & {\scriptstyle \chi_\gamma}\downarrow & & \downarrow{\scriptstyle \varphi_{\gamma(\gamma+1)}} \\ A_{\gamma+1} & \xrightarrow{\operatorname{copr}_{A_{\gamma+1}}} & \mathbb{Z} \times A_{\gamma+1} & \xrightarrow{\varepsilon_{\gamma+1}^{-1}} & G_{\gamma+1}. \end{array}$$

This is a commutative diagram of type $(*)$ above which according to our construction in Case 2b could not exist. This contradiction proves the claim that the sequence S does not split. Hence it provides a nonzero element of $\operatorname{Ext}(A, \mathbb{Z})$ and thus A is not a Whitehead group, as asserted in the lemma. □

Lemma A1.75. *Assume that A is a Whitehead group such that* $\operatorname{card} A$ *is an uncountable regular cardinal. Then every subgroup $B \leq A$ with* $\operatorname{card} B < \operatorname{card} A$ *is contained in a pure subgroup C with* $\operatorname{card} C < \operatorname{card} A$ *such that for every subgroup $D \leq A$ with $C \leq D$ and* $\operatorname{card} D < \operatorname{card} A$ *the factor group D/C is a Whitehead group.*

Proof. Suppose that the lemma is false and that there is a pure subgroup $B \leq A$ with $\operatorname{card} B < \operatorname{card} A$ such that for every pure subgroup $C \geq B$ of cardinality below $\operatorname{card} A$ we find a subgroup $D \geq C$ of cardinality below $\operatorname{card} A$ such that D/C is not a Whitehead group. Set $\kappa \stackrel{\text{def}}{=} \operatorname{card} A$. By transfinite induction, we will construct a continuous well-ordered ascending chain $\{A_\alpha\}_{\alpha<\kappa}$ of pure subgroups of A. Set $A_0 = B$, and if $\beta < \kappa$ and A_α is defined for all $\alpha < \beta$, define A_β as follows. If β is a limit ordinal, set $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$; if $\beta = \alpha + 1$ then by assumption we find a pure subgroup $A_\beta \leq A$ containing A_α such that $\operatorname{card} A_\beta < \kappa$ and A_β/A_α is not a Whitehead group. Since subgroups of Whitehead groups are Whitehead groups, $A' \stackrel{\text{def}}{=} \bigcup_{\alpha<\kappa} A_\alpha$ is a Whitehead group. However, Lemma A1.74 now applies to A' with $E = \kappa$ and proves that A' is not a Whitehead group. This contradiction proves the lemma. □

Only the first and simple part of the following lemma is needed for the next theorem; the second, more refined part will be useful later.

Lemma A1.76. (i) *If $\{A_\alpha\}_{\alpha<\kappa}$ is a continuous chain of pure subgroups of an abelian group A and $A = \bigcup_{\alpha<\kappa} A_\alpha$, and if $A_{\alpha+1}/A_\alpha$ is free for all $\alpha < \kappa$, then A is free.*

(ii) *Let κ be an uncountable regular cardinal and*

$$\{0\} = P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \cdots,$$

$\alpha < \kappa$, a chain of pure subgroups of an abelian group A such that
(Ai) *$\bigcup_{\alpha<\lambda} P_\alpha = P_\lambda$ for every limit ordinal $\lambda < \kappa$,*
(Aii) *P_α is a free abelian group of cardinality $< \kappa$ for all $\alpha < \kappa$, and*
(Aiii) *$A = \bigcup_{\alpha<\kappa} P_\alpha$.*
Then the following two conditions are equivalent:

(1) *A is free.*
(2) *$E \stackrel{\text{def}}{=} \{\alpha < \kappa \mid (\exists \beta > \alpha)\, \frac{A_\beta}{A_\alpha}$ is not free$\}$ is not stationary in κ.*

Proof. (i) For each $\alpha < \kappa$ we find a free subgroup F_α of $A_{\alpha+1}$ such that $A_{\alpha+1} = A_\alpha \oplus F_\alpha$. Let X_α be a free generating set of F_α. Then $X \stackrel{\text{def}}{=} \bigcup_{\alpha<\kappa} X_\alpha$ is a free generating set of A. Thus A is free.

Proof of (ii). (1)⇒(2) Since A is free we can choose a filtration $\{A_\alpha\}_{\alpha<\kappa}$ of A whose members A_α are direct summands. The set $C = \{\alpha < \kappa \mid P_\alpha = A_\alpha\}$ is a cub by Lemma A1.71. Thus $\{P_\alpha\}_{\alpha\in C}$ is a filtration whose members are direct summands. Hence A/P_α is free for $\alpha \in C$, and this implies that $C \cap E = \emptyset$. Hence E is not stationary.

(2)⇒(1) Since E is not stationary, there is a cub $C \subseteq \kappa$ with $C \cap E = \emptyset$. Then $\{P_\alpha\}_{\alpha\in C}$ is a filtration of A. In order to simplify notation we may rename the P_α and assume that we have a filtration $\{A_\alpha\}_{\alpha<\kappa}$ such that all factor groups $A_{\alpha+1}/A_\alpha$, $\alpha < \kappa$, are free and $A_0 = \{0\}$. Then (i) proves that A is free and thus completes the proof. □

Since countable Whitehead groups are free by Pontryagin's Theorem A1.62 and a subgroup of a Whitehead group is a Whitehead group, all Whitehead groups are $\aleph_1$-free. Hence the following theorem will prove Theorem A1.67 for Whitehead groups of cardinalities $\leq \aleph_1$.

Theorem A1.77. ($\diamond$) *Let κ be an uncountable regular cardinal and assume that A is a Whitehead group of cardinality κ. If all Whitehead groups of cardinality $< \kappa$ are free, then A is free.*

Proof. The conclusions of Lemma A1.75 apply with the additional information that D/C is free as a Whitehead group with $\operatorname{card} D/C < \kappa$. We construct a filtration $\{A_\alpha\}_{\alpha<\kappa}$ of A. Choose a maximal free subset $\{e_\alpha \mid \alpha < \kappa\}$ in A. Set $A_0 = \{0\}$ and assume that for some $\beta < \kappa$ and all $\alpha < \beta$ groups A_α have been constructed such that (i) $\operatorname{card} A_\alpha < \kappa$, (ii) $\{A_\alpha\}_{\alpha<\beta}$ is a continuous chain, and (iii) $e_\rho \in A_\alpha$ for $\rho < \alpha$. Now we set $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ if β is a limit ordinal, and if $\beta = \alpha + 1$, we let B be the pure subgroup generated by $A_\alpha \cup \{e_\alpha\}$ and take A_β to be a pure subgroup C guaranteed by Lemma A1.75. This construction indeed produces a filtration. Since A is a Whitehead group, by Lemma A1.74, the set $E \stackrel{\text{def}}{=} \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha \text{ is not free}\}$ fails to be stationary in κ. Hence there is a cub F in κ which does not intersect E. Let us index F in the form $F = \{\nu_\alpha \mid \alpha < \kappa\}$. Then $\{A_{\nu_\alpha}\}_{\alpha<\kappa}$ is a filtration of A. We claim that $A_{\nu_{\alpha+1}}/A_{\nu_\alpha}$ is free; this claim will finish the proof by A1.76(i). By our recursive construction, $A_{\nu_\alpha+1}/A_{\nu_\alpha}$ is free. Moreover, in our application of Lemma A1.75 we obtained $A_{\nu_\alpha+1}$ in the inductive construction as C from A_{ν_α} as B; thus the group $A_{\nu_{\alpha+1}}$ can be taken as a D in Lemma A1.75. Hence $A_{\nu_{\alpha+1}}/A_{\nu_\alpha+1}$ is free, and this proves the claim. □

We will prove the full version of A1.67 later, but now we aim for a proof for A1.68. For the line of argument presented here see [98]. We recall that ω_1 denotes the first uncountable ordinal.

Lemma A1.78. (ZFC + MA + ¬CH) *Let $\{A_\alpha\}_{\alpha<\omega_1}$ be a well-ordered continuous chain of subgroups of an abelian group A of cardinality $\aleph_1$ such that*

(i) $\operatorname{card} A_\alpha = \aleph_0$ *and A_α is free for all $\alpha < \omega_1$,*
(ii) $A = \bigcup_{\alpha<\omega_1} A_\alpha$,
(iii) *$A_{\alpha+1}$ is a direct summand of A_β for each $\alpha < \beta < \omega_1$.*

Then A is a Whitehead group.

Proof. Let $0 \to \mathbb{Z} \xrightarrow{j} G \xrightarrow{\pi} A \to 0$ be an exact sequence. We have to show that it splits. Thus we are looking for a morphism $\Delta: A \to G$ such that $\pi\Delta = \mathrm{id}_A$. In search of such a Δ consider the set $\mathcal{A}$ of all morphisms $\delta: S \to G$ such that S is a finitely generated pure subgroup of A and $\pi\delta = \mathrm{id}_S$. Then $\mathcal{A}$ is partially ordered by the extension of functions. For every finitely generated pure subgroup S, the restriction $\pi|\pi^{-1}(S): \pi^{-1}(S) \to S$ is a surjective morphism onto a free group and thus it splits by A1.15. Hence there is a $\delta: S \to G$ in $\mathcal{A}$. The morphism δ is determined by its action on the finitely many generators of S and has $\pi^{-1}(S)$ as range; since $\ker\pi \cong \mathbb{Z}$, this range is a finitely generated free group. Thus for a given S there are only countably many $\delta: S \to G$ in $\mathcal{A}$. We now proceed through several steps.

Claim 1. For every $\delta: S \to G$ in $\mathcal{A}$ and every finite $F \subseteq A$ there is a $\delta': S' \to G$ in $\mathcal{A}$ such that $S \cup F \subseteq S'$ and $\delta'|S = \delta$.

Clearly, by (ii), the finitely generated group $\langle S \cup F\rangle$ is contained in a pure free subgroup and thus there is a finitely generated pure subgroup S' of A containing $S \cup F$. Since S is pure, S'/S is finitely generated torsion-free and thus by A1.11 and A1.15 we have $S' = S \oplus S_*$ with a finitely generated pure subgroup S_* of A. We find a $\delta_*: S_* \to G$ in $\mathcal{A}$ and define $\delta'(s + s_*) = \delta(s) + \delta_*(s)$.

Claim 2. Assume that $\mathcal{B}$ is an uncountable subset of $\mathcal{A}$. Then there is a free pure subgroup F of A and an uncountable subset $\mathcal{B}'$ of $\mathcal{B}$ such that for $\delta: S \to G$ in $\mathcal{B}'$ we have $S \leq F$.

It is enough to consider the case that $\operatorname{card}\mathcal{B} = \aleph_1$. We write $\mathcal{B} = \{\delta_\alpha\}_{\alpha<\omega_1}$. For each $\alpha < \omega_1$ the domain S_α of δ_α has finite rank, and thus there is an integer

m such that $\{\alpha \mid \operatorname{rank} S_\alpha = m\}$ is uncountable. It is therefore no loss of generality to assume that $\operatorname{rank} S_\alpha = m$ for all $\alpha < \omega_1$. Among all pure subgroups contained in uncountably many S_α let T be one of highest rank. Then, as we observed earlier, $\{\delta_\alpha|T \mid \alpha < \omega_1\}$ is countable. So once again we may assume without losing generality that $T \leq S_\alpha$ for all $\alpha < \omega_1$ and that all $\delta_\alpha|T$ agree. We set $F_0 = T$ and assume that for some $\beta < \omega_1$ we have defined a continuous chain $\{F_\alpha\}_{\alpha<\beta}$ of countable subgroups of A such that $S_\alpha \leq F_\alpha$. If β is a limit ordinal, set $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$. If $\beta = \alpha+1$, find a $\gamma' < \omega_1$ such that $F_\alpha \subseteq A_{\gamma'}$. Set $\gamma = \gamma'+1$. Then $F_\alpha \subseteq A_\gamma$. Moreover, $T = F_0 \subseteq F_\alpha \subseteq A_\gamma$. Since A_γ is countable, the set $\{S_\nu \cap A_\gamma \mid \nu < \omega_1\}$ is countable; thus one of these groups, say, S' is such that $\{\nu < \omega_1 \mid S_\nu \cap A_\gamma = S'\}$ is uncountable. Since $T \subseteq S'$ and since S' is pure, as all S_ν are pure by definition and A_γ is pure by (iii), the definition of T shows $T = S'$. Let μ be such that $T = S_\mu \cap A_\gamma$. Let F_β be the smallest pure subgroup containing $F_\alpha + S_\mu$. We claim that $F_\beta \cap A_\gamma = F_\alpha$. Obviously the right hand side is contained in the left. Now let $a \in F_\beta \cap A_\gamma$. Then there is natural number n such that $n{\cdot}a = b+s$ with $b \in F_\alpha$ and $s \in S_\mu$. Then $s = -b + n{\cdot}a \in S_\mu \cap A_\gamma = T = F_0 \subseteq F_\alpha$, whence $n{\cdot}a \in F_\alpha$, and since F_α is pure, we have $a \in F_\alpha$ and thus the claim is established. Now $F_\beta/F_\alpha = F_\beta/(F_\beta \cap A_\gamma) \cong (F_\beta + A_\gamma)/A_\gamma$. Since $F_\beta + A_\gamma$ is countable, there is an $\eta < \omega_1$ such that $F_\beta + A_\gamma \subseteq A_\eta$. Since A_η is free by (i) and A_γ is a direct summand of A_η by (iii), the group A_η/A_γ is free, and thus F_β/F_α is free. If we set $F = \bigcup_{\alpha<\omega_1} F_\alpha$ then F is a pure subgroup of A since all F_α are pure subgroups of A, and by A1.76(i), the group F is free. Finally, set $\mathcal{B}' = \{\delta_\nu \in \mathcal{B} : S_\nu \leq F\}$. This completes the proof of Claim 2.

Claim 3. The partially ordered set $\mathcal{A}$ satisfies the antichain condition.

Let $\mathcal{B}$ be an uncountable subset of $\mathcal{A}$. Choose F and $\mathcal{B}'$ as in Claim 2. We will show that $\mathcal{B}'$ contains a two element set with an upper bound in $\mathcal{A}$. It suffices to show that there are two elements in $\mathcal{A}$ which have an upper bound in $\mathcal{A}$ and dominate elements in $\mathcal{B}'$. Select a basis X of F containing a basis Y of T. Claim 1 permits us to enlarge each element of $\mathcal{B}$ to a member $\delta: S \to G$ of $\mathcal{A}$ such that S is generated by a finite subset of X. We write $\mathcal{B}''$ for the set of these δ. Now select any $\delta_{\nu_1}: S_{\nu_1} \to G$ from $\mathcal{B}''$ such that $T \neq S_{\nu_1}$. There are only countably many pure subgroups T' of A such that $T < T' \leq S_{\nu_1}$. By the construction of T, for each of these T', the set $\mathcal{B}''(T') \overset{\text{def}}{=} \{\delta_\nu: S_\nu \to G \mid \delta_\nu \in \mathcal{B}'', \quad T' \leq S_\nu\}$ is countable, and thus $\bigcup_{T'} \mathcal{B}''(T')$ is countable. Thus the complement of this union in $\mathcal{B}''$ therefore is uncountable. Let $\delta_{\nu_2}: S_{\nu_2} \to G$ be a member of this complement. Then $S_{\nu_1} \cap S_{\nu_2} = T$. Now $S_* \overset{\text{def}}{=} S_{\nu_1} + S_{\nu_2}$ is a pure subgroup F as it is generated by a subset of X. Hence it is a pure subgroup of A. The morphisms $\delta_{\nu_j}: S_{\nu_j} \to G$, $j = 1, 2$ agree on T.

Thus they have a common extension $\delta_*: S_* \to G$ which is a member of $\mathcal{A}$ and is an upper bound for the two different elements δ_{ν_j}, $j = 1, 2$. This establishes Claim 3.

In order to complete the proof of the lemma, for each $a \in A$, we let $\mathcal{A}_a = \{\delta: S \to G \mid \delta \in \mathcal{A},\ a \in S\}$. Then for each $a \in A$ the set $\mathcal{A}_a$ is cofinal in $\mathcal{A}$, and the cardinality of the family $\{\mathcal{A}_a\}_{a\in A}$ does not exceed the cardinality of A, which is $\aleph_1$

by hypothesis. Since we assume $\neg$(CH), we have $\operatorname{card}\{\mathcal{A}_a\}_{a\in A} < 2^{\aleph_0}$. Now Claim 3 and Martin's Axiom yield a directed subset $\mathcal{D} \subseteq \mathcal{A}$ such that $\mathcal{D}\cap\mathcal{A}_a \neq \emptyset$ for all $a \in A$. We define a function $\Delta\colon A \to G$ as follows. Let $a \in A$. Pick a $\delta_\nu\colon S_\nu \to G$ in $\mathcal{D}\cap\mathcal{A}_a$. Then $a \in S_\nu$. Assume that $\delta_\rho\colon S_\rho \to G$ is also in $\mathcal{D}\cap\mathcal{A}_a$. Then $a \in S_\rho$ and since $\mathcal{D}$ is directed, there is a $\delta_\sigma\colon S_\sigma \to G$ which is an upper bound for δ_μ and δ_ρ. Hence $\delta_\nu(a) = \delta_\sigma(a) = \delta_\rho(a)$. Hence we can unambiguously define $\Delta(a) \stackrel{\text{def}}{=} \delta_\nu(a)$. If $a_j \in A$, $j = 1, 2$, we find $\delta_{\nu_j}\colon S_{\nu_j} \to G$ in $\mathcal{D}\cap\mathcal{A}_{a_j}$, $j = 1, 2$.

Since $\mathcal{D}$ is directed, there is an upper bound $\delta_\rho\colon S_\rho \to G$ of $\{\delta_{\nu_1}, \delta_{\nu_2}\}$. Then $a_j \in S_{\nu_j} \subseteq S_\rho$, whence $a_1 + a_2 \in S_\rho$. Then $\Delta(a_1+a_2) = \delta_\rho(a_1+a_2) = \delta_\rho(a_1)+\delta_\rho(a_2) = \delta_{\nu_1}(a_1) + \delta_{\nu_2}(a_2) = \Delta(a_1) + \Delta(a_2)$. Hence $\Delta\colon A \to G$ is a morphism. Moreover, for $a \in A$ let $\delta_\nu\colon S_\nu \to G$ in $\mathcal{D}\cap\mathcal{A}_a$. Then we have $\pi\big(\delta(a)\big) = \pi\big(\delta_\nu(a)\big) = a$ because $\delta_\nu\colon S_\nu \to G$ was a cross section morphism for $\pi|\delta_\nu^{-1}(S_\nu) : \delta_\nu^{-1}(S_\nu) \to S_\nu$. Thus $\Delta\colon A \to G$ is the required cross section morphism for $\pi\colon G \to A$. □

Lemma A1.79. *There is a torsion-free group A of cardinality $\aleph_1$ which is not free and has a chain of subgroups as described in Lemma* A1.78.

Proof. For each ordinal $\nu < \omega_1$ we set $Z_\nu = \mathbb{Z}$ and form $G \stackrel{\text{def}}{=} \prod_{\nu<\omega_1} Z_\nu$ and $H \stackrel{\text{def}}{=} \bigoplus_{\nu<\omega_1} Z_\nu$. Define $G_\mu \stackrel{\text{def}}{=} \prod_{\nu<\mu} Z_\nu$ considered as a direct summand of G and $H_\mu \stackrel{\text{def}}{=} \bigoplus_{\nu<\mu} Z_\nu$. For any subgroup $X \le G$ set $X_\mu = X \cap G_\mu$, similarly for H in place of G. Every limit ordinal $\lambda < \omega_1$ has cofinality ω and thus we can choose a sequence of ordinals $\nu_1 < \nu_2 < \cdots$ converging to λ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\operatorname{pr}_{\nu_m}(x_n) = \begin{cases} \frac{m!}{n!} & \text{for } m \ge n,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$(\forall m \ge n) \quad n \operatorname{pr}_{\nu_m}(x_n) = \operatorname{pr}_{\nu_m}(x_{n-1}), \quad n = 2, 3, \ldots.$$

Set $X_\lambda = \langle x_1, x_2, \ldots\rangle$. Define $Y_\lambda \stackrel{\text{def}}{=} X_\lambda/(X_\lambda \cap H) \cong (X_\lambda + H)/H$ and set $\xi_n = x_n + (X_\lambda \cap H)$; then $n\xi_n = \xi_{n-1}$, $n = 2, 3, \ldots$. Thus we have a morphism $\varphi\colon Y_\lambda \to \mathbb{Q}$ given by $\varphi(\xi_n) = \frac{1}{n!}$. Since Y_λ is a rank 1 divisible torsion-free group, φ is an isomorphism.

Let Λ be the set of all limit ordinals $\lambda < \omega_1$. Define $A \le G$ by $A = H + \sum_{\lambda\in\Lambda} X_\lambda$. Then A is a pure subgroup of cardinality $\aleph_1$. Set

$$A_\nu = H + \sum_{\lambda\in\Lambda,\,\lambda<\nu} X_\lambda, \quad \nu \in \Lambda.$$

The family $\{A_\lambda\}_{\lambda\in\Lambda}$ is a filtration of A. The group G is $\aleph_1$-free; hence A_λ, being countable, is free for all $\lambda \in \Lambda$. We want to apply A1.76(ii) and consider $E = \{\alpha < \omega_1 \mid A_{\alpha+1}/A_\alpha \text{ is not free}\}$. If λ is a limit ordinal, then $A_{\lambda+1} = X_\lambda + A_\lambda$. Thus $A_{\lambda+1}/A_\lambda \cong X_\lambda/(X_\lambda \cap A_\lambda)$ is a nonzero quotient of $Y_\lambda \cong \mathbb{Q}$ and thus is not free. Hence $\Lambda \subseteq E$, and thus E is stationary in ω_1. Hence by A1.76(ii) the group A is not free.

We claim that $A_\beta/A_{\alpha+1}$ is free for all $\alpha < \beta < \omega_1$. But $A_\beta/A_{\alpha+1}$ is a countable subgroup of $A/A_{\alpha+1} = A/(A \cap G_{\alpha+1}) \cong (A + G_{\alpha+1})/G_{\alpha+1} \leq G/G_{\alpha+1}$ which is an $\aleph_1$-free group. Thus $A_\beta/A_{\alpha+1}$ is free. □

The preceding lemmas yield a proof of Proposition A1.68.

Proposition A1.80 = A1.68. (ZFC + MA + ¬CH) *There are Whitehead groups of cardinality $\aleph_1$ which are not free.*

Proof. This follows at once from Lemmas A1.78 and A1.79. □

We have already noted in Theorem A1.69 that Proposition A1.80 maintains for all uncountable cardinals in place of $\aleph_1$.

It remains now to prove A1.67, which we have established for Whitehead groups of cardinality $\leq \aleph_1$, for Whitehead groups of any cardinality. We want to do this by transfinite induction. Theorem A1.77 allows us to pass through regular cardinals. We need a theorem which allows us to pass through singular cardinals, i.e. cardinals which fail to be regular.

Let κ be any cardinal. A subgroup P of an abelian group A is said to be *κ-pure* if P is a direct summand in every subgroup $B \subseteq A$ containing P such that $\mathrm{card}(B/P) < \kappa$. In generalisation of Proposition A1.64 we shall say that an abelian group A is *κ-free* if every subgroup whose rank is smaller than κ is free. If κ is regular we say that A is *strongly κ-free* if it is κ-free and every subgroup of cardinality below κ is contained in some κ-pure subgroup of cardinality below κ.

Shelah's Singular Compactness Theorem

Theorem A1.81 ([329]). *Let λ be a singular cardinal and assume that A is an abelian group of cardinality λ. If all subgroups of cardinality $< \lambda$ in A are free, then A is free.*

Proof. Let $\kappa < \lambda$ be the cofinality of λ and consider an increasing well-ordered set $\{\kappa_\nu \mid \nu < \kappa\}$ of cardinals such that $\nu \mapsto \kappa_\nu$ preserves sups with $\lambda = \sup_{\nu<\kappa} \kappa_\nu$ and let $\{A_\nu\}_{\nu<\kappa}$ be a continuous chain of pure subgroups of A with union A such that $\mathrm{card}\, A_\nu = \kappa_\nu$. We shall construct a continuous chain of subgroups $B_\nu \supseteq A_\nu$ such that $B_0 \subseteq B_1 \subseteq \cdots$, and that all factor groups $B_{\nu+1}/B_\nu$ are free. Then A1.75(i) shows that $A = \bigcup_{\nu<\lambda} A_\nu \subseteq \bigcup_{\nu<\lambda} B_\nu \subseteq A$ is free.

The construction of the B_ν is accomplished by a somewhat elaborate recursion. We let μ^+ denote the first cardinal which is larger than μ.

$$\mathcal{P}_\nu = \{B \leq A \mid B \text{ is } \kappa_\nu^+\text{-pure in } A \text{ and } \mathrm{card}\, B \leq \kappa_\nu\}.$$

Since the group A is α-free for all $\alpha < \lambda$, it is strongly α-free for all $\alpha < \lambda$. Thus every subgroup $B \leq A$ with $\mathrm{card}\, B \leq \kappa_\nu$ is contained in a member of $\mathcal{P}_\nu$. For all $\nu < \kappa$ we shall recursively define subgroups B_ν^n and subsets $X_\nu^n \subseteq B_\nu^n$, subject to the following properties:

(i) $B_\nu^n \in \mathcal{P}_\nu$ for $\nu < \kappa$, $n \in \mathbb{N}_0$,
(ii) B_ν^n is a free abelian group with a basis X_ν^n for all $\nu < \kappa$, $n \in \mathbb{N}_0$,
(iii) $A_\nu < B_\nu^0$, and $X_\nu^m \subset X_\nu^n$ for each $\nu < \kappa$, $m < n$ in $\mathbb{N}$,
(iv) $B_\nu^{n-1} \le \langle B_\nu^n \cap X_{\nu+1}^{n-1}\rangle$ for each $\nu < \kappa$, $n \in \mathbb{N}$,
(v) if $\mu < \kappa$ is a limit ordinal, then X_μ^n is the union of a chain of subsets $Y_{\alpha\mu}^n$ with $\operatorname{card} Y_{\alpha\mu}^n = \kappa_\alpha$ and $Y_{\alpha\mu}^n \subseteq B_\alpha^{n+1}$ for all $\alpha < \mu$.

Step 1. Construction of B_ν^0 by induction on ν. Pick B_0^0 to be any member of $\mathcal{P}_0$ containing A_0. If for some $\mu < \kappa$, the group B_ν^0 is defined for all $\nu < \mu$, then select with the aid of the cardinality assumptions $B_\mu^0 \in \mathcal{P}_\mu$ such that it contains $A_\mu + \sum_{\nu<\mu} B_\nu^0$. This gives a chain $B_0^0 < B_1^0 < \cdots < B_\nu^0 < \cdots$ such that $\operatorname{card} B_\nu^0 = \kappa_\nu$. Next we select a basis X_ν^0 of B_ν^0 for each $\nu < \kappa$. If μ is a limit ordinal below κ we can represent X_μ^0 as the union of a chain of subsets $Y_{\alpha\mu}^0$, each of cardinality κ_α.

Step 2. Induction with respect to $n \in \mathbb{N}_0$. Let $n \in \mathbb{N}$. Assume that the groups B_ν^m and the subsets X_ν^m have been defined for all $\nu < \kappa$ and all $m < n$, and that, moreover, for all limit ordinals $\nu < \kappa$, the sets $Y_{\alpha\nu}^m$ are defined for $\alpha < \nu$, $m < n$. Now choose $B_\mu^n \in \mathcal{P}_\mu$ containing B_μ^{n-1} and such that (iv) is satisfied, further so that B_μ^n contains firstly, B_ν^n for all $\nu < \mu$, and secondly, the sets $Y_{\alpha\nu}^{n-1}$ for all limit ordinals $\nu < \mu$ and all $\alpha < \nu$. Since B_μ^{n-1} is a direct summand of B_μ^n we can select a basis X_μ^n of B_μ^n containing X_μ^{n-1}. If μ is a limit ordinal, since $\kappa_\mu < \kappa$ we succeed in choosing a family $Y_{\alpha\mu}^n$, $\alpha < \mu$ of subsets of X_μ^n such that (v) holds. With these choices, conditions (i), …,(v) are satisfied, and the induction is complete.

Step 3. We claim that the subgroups $B_\nu \stackrel{\text{def}}{=} \bigcup_{n=0}^\infty B_\nu^n$, $\nu < \kappa$ form a continuous chain such that all factor groups $B_{\nu+1}/B_\nu$ are free. In order to show that $\nu \mapsto B_\nu$ preserves sups, we let $\mu < \kappa$ be a limit ordinal and compute

$$B_\mu = \bigcup_{n=0}^\infty B_\mu^n = \bigcup_{n=0}^\infty \langle X_\mu^n\rangle = \bigcup_{n=0}^\infty \bigcup_{\alpha<\mu} \langle Y_{\alpha\mu}^n\rangle \le \bigcup_{n=0}^\infty \bigcup_{\alpha<\mu} B_\alpha^{n+1} = \bigcup_{\alpha<\mu} B_\alpha \le B_\mu.$$

This proves the asserted continuity.

Let us set $X_\nu = \bigcup_{n\in\mathbb{N}_0} X_\nu^n$. Then X_ν is a basis of B_ν. Note that by (iv), we have

$$\begin{aligned} B_\nu = \bigcup_{n=1}^\infty B_\nu^{n-1} &\le \bigcup_{n=1}^\infty \langle B_\nu^n \cap X_{\nu+1}^{n-1}\rangle = \Big\langle \bigcup_{n=1}^\infty (B_\nu^n \cap X_{\nu+1}^{n-1})\Big\rangle \\ &\subseteq \Big\langle \bigcup_{n=1}^\infty (B_\nu \cap X_{\nu+1}^{n-1})\Big\rangle = \langle B_\nu \cap X_{\nu+1}\rangle \subseteq B_\nu. \end{aligned}$$

Thus the group B_ν is generated by $B_\nu \cap X_{\nu+1}$. Accordingly, $B_{\nu+1}/B_\nu$ is free, as asserted.

By our initial remarks this completes the proof. □

This theorem holds in ZFC and is certainly of independent interest as we note in Chapter 8, Exercise E8.8; but in the present discourse it allows us to finish the proof of Theorem A1.67.

Theorem A1.82=A1.67. *Assume that the axioms of ZFC and $\diamond$ hold. Then all Whitehead groups are free.*

Proof. We prove the assertion "A Whitehead group A with $\operatorname{card} A = \aleph_\alpha$ is free" by transfinite induction with respect to α. By Pontryagin's Theorem A1.62 all countable Whitehead groups are free and so the assertion holds for $\alpha = 0$. Assume now that for some $\beta > 0$ the assertion has been established for all $\alpha < \beta$. Case 1: $\aleph_\beta$ is regular. Then Theorem A1.77 applies and shows that A is free; so the assertion is true for $\aleph_\beta$. Case 2: $\aleph_\beta$ is singular. Let B be a subgroup of A with $\operatorname{card} B < \operatorname{card} A$. Then B is a Whitehead group of cardinality $< \aleph_\beta$. Hence it is free by induction hypothesis. Now Shelah's Singular Compactness Theorem A1.81 applies and shows that A is free. Thus the assertion holds for $\operatorname{card} A = \aleph_\beta$ in this case, too. The transfinite induction is complete. □

Postscript

This appendix includes a short introductory course in abelian group theory. The material presented is primarily that which is essential for the structure theory of compact abelian groups. This motive accounts for the inclusion of certain elements of homological algebra and, in the end the presentation of the solution of the Whitehead Problem.

Some parts of our presentation are not prominent in standard sources, for example, the detailed discussion of the reduced group ∇ in A1.32. We encounter frequently the phenomenon that countability often opens up structural results which otherwise are not available, e.g. in A1.26, A.1.28, A1.62.

The material on Whitehead's Problem is motivated by our discussion of the connectivity properties of compact abelian groups in Chapter 8. The essential result is Saharon Shelah's Theorem that the assertion of freeness of Whitehead groups is independent from ZFC (see A1.70). The proof presented here is more in the spirit of abelian group theory. We gratefully acknowledge Laszlo Fuchs' permission to present material from unpublished course notes of his. This material very convincingly shows that in compact abelian group theory, leaving the metric situation means getting deeply into set theory and logic.

References for this Appendix—Additional Reading

[63], [64], [98], [99], [108], [113], [114], [115], [116], [117], [124], [134], [136], [147], [221], [222], [228], [245], [250], [295], [329], [337], [369], [375].

Appendix 2

Covering Spaces and Groups

In this appendix we summarize, for easy reference, material on covering maps of topological spaces and the concept of simple connectivity as well as some background material on topological groups which is related to these matters and is useful in any approach to Lie group theory.

All topological spaces considered in this appendix are assumed to be Hausdorff spaces.

Covering Spaces and Simple Connectivity

Definitions A2.1. A function $f{:}\,X \to Y$ is called a *covering map* or simply *a covering* if Y has an open cover $\{U_j \mid j \in J\}$ such that for each $j \in J$ there is a nonempty discrete space F_j and a homeomorphism $h_j{:}\,F_j \times U_j \to f^{-1}(U_j)$ such that the following diagram commutes:

$$\begin{array}{ccc} F_j \times U_j & \xrightarrow{h_j} & f^{-1}(U_j) \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle f|f^{-1}(U_j)} \\ U_j & \xrightarrow[\mathrm{id}_{U_j}]{} & U_j. \end{array}$$

We will briefly say that $f^{-1}(U_j)$ *is compatibly homeomorphic to* $F_j \times U_j$. We call F_j the *fiber* over U_j and Y the *base space* of the covering.

A function $f{:}\,X \to Y$ between topological spaces is said to induce *a local homeomorphism* at x if there are open neighborhoods U of x in X and V of $f(x)$ in Y such that $f|U{:}\,U \to V$ is a homeomorphism. It is said to *induce local homeomorphisms* if it induces local homeomorphisms at all points. (Many authors say in these circumstances that f *is a local homeomorphism.*)

A function $f{:}\,G \to H$ between topological groups is said to be a *covering morphism* if, algebraically, it is a homomorphism and if it is a covering of topological spaces. $\square$

Since coverings are clearly continuous and open, a covering morphism is always an open morphism of topological groups.

It is noted immediately that coverings are surjective, continuous and open maps. We recall at this point that a morphism of topological groups is always understood to be a continuous group homomorphism.

Remark A2.2. Every covering induces local homeomorphisms. The converse fails in general. □

The assertions in the following examples are left as an exercise.

Examples A2.3. (i) Let G be a topological group and H a discrete subgroup. Let G/H denote the space of all cosets gH, $g \in G$ endowed with the quotient topology and let $p: G \to G/H$, $p(g) = gH$ be the quotient map. Then p is a covering.

(ii) If $f: G \to H$ is a morphism of topological groups then f is a covering if and only if the following conditions are satisfied:

(a) $\ker f$ is discrete.
(b) f is open.
(c) f is surjective.

(iii) Let $(g, x) \mapsto g{\cdot}x : G \times X \to X$ be an action of a finite discrete group G on a Hausdorff space such that all $x \mapsto g{\cdot}x$ are continuous and that the action is free, i.e. that $g{\cdot}x = x$ implies $g = 1$. Then the orbit map $q: X \to X/G = \{G{\cdot}x \mid x \in X\}$ is a covering when X/G is given the quotient topology.

(iv) By (i), the homomorphism $p: \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $p(r) = r + \mathbb{Z}$ is a covering. Its restriction to $\left]0, 1\frac{1}{2}\right[$ induces local homeomorphisms but is not a covering. □

Exercise EA2.1. Verify the claims of Examples A2.3.

[Hint for (iii). Let $x \in X$. Find an open neighborhood U of x in X such that $(\forall g \in G \setminus \{1\})\, g{\cdot}U \cap U = \emptyset$; indeed if that were not possible, then for each U there would be x_U, $y_U \in U$ and a $g_U \in G$ such that $g_U{\cdot}x_U = y_U$. Since G is finite, we may assume that for a basis of neighborhoods V of x we have $g_V = g \in G \setminus \{1\}$. But x_V, $y_V \to x$; thus $g{\cdot}x = x$ by the continuity of $z \mapsto g{\cdot}z$; a contradiction to the freeness of the action. Now the function $(g, u) \mapsto G \times U \to G{\cdot}U = q^{-1}q(U)$ is a homeomorphism.] □

One can construct new coverings from given ones as the following proposition shows.

Proposition A2.4. (i) *If $f_j: X_j \to Y_j$, $j = 1, 2$ are coverings, then $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ is a covering. In short: Finite products of coverings are coverings.*

(ii) *If $p: E \to B$ is a covering, $f: X \to B$ any continuous function, and if*

$$\begin{array}{ccc} P & \xrightarrow{f^*} & E \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

is a pullback diagram (i.e. $P = \{(x, e) \in X \times E \mid f(x) = p(e)\}$, $f^*(x, e) = e$, $p^*(x, e) = x$), then $p^*: P \to X$ is a covering. In short: Pullbacks of coverings are coverings.

(iii) *If* $f: X \to Y$ *is a covering and* $Y' \subseteq Y$*, then* $f': X' \to Y'$ *is a covering where* $X' = f^{-1}(Y')$ *and* $f' = f|X'$*. In short: Restrictions of coverings are coverings.*

(iv) *Assume that* $p: E \to B$ *is a covering,* B *is connected, and that* B *admits a cover of connected open sets* U_j*,* $j \in J$ *such that* $p^{-1}(U_j)$ *is compatibly homeomorphic to* $F \times U_j$*. Then for every connected component* E' *of* E *the restriction* $p|E': E' \to B$ *is a covering.*

Proof. The proofs are largely straightforward from the definition of a covering:

(i) Assume that $\{U_j \mid j \in J\}$ is an open cover of Y_1 such that for each $j \in J$, the space $p^{-1}(U_j)$ is compatibly homeomorphic to $F_j \times U_j$, and $\{V_k \mid k \in K\}$ is an open cover of Y_2 such that $f_2^{-1}(V_k)$ is compatibly homeomorphic to $G_k \times V_k$. Then $\{U_j \times V_k \mid (j,k) \in J \times K\}$ is an open cover of $Y_1 \times Y_2$ such that $(f_1 \times f_2)^{-1}(U_j \times V_k)$ is compatibly homeomorphic to $(F_j \times G_k) \times (U_j \times V_k)$.

(ii) Assume that $\{U_j \mid j \in J\}$ is an open cover of B such that for each $j \in J$, the space $f_1^{-1}(U_j)$ is compatibly homeomorphic to $F_j \times U_j$. Then $\{f^{-1}(U_j) \mid j \in J\}$ is an open cover of X such that for each j the space $(p^*)^{-1}\big(f^{-1}(U_j)\big) = \{(x,e) \in X \times E \mid f(x) = p(e) \in U_j\}$ is compatibly homeomorphic to $F_j \times f^{-1}(U_j)$.

In fact this proof shows that in pullbacks the fibers are pulled back.

The proof of (iii) is quite straightforward.

(iv) For each $j \in J$ there is a homeomorphism $h_j: F_j \times U_j \to p^{-1}(U_j)$ such that $ph_j(x,u) = u$. We consider $e \in p^{-1}(U_i) \cap E'$. Then $h_j\big(x, p(e)\big) = e$ for some $x \in F_j$, and $h_j(\{x\} \times U_j)$ is a connected open subset of E containing e. Hence it is contained in E'. If we set $F'_j = \{x \in F_j \mid h_j(\{x\} \times U_j) \neq \emptyset\}$, then $(p|E)^{-1}(U_j) = p^{-1}(U_j) \cap E'$ is compatibly homeomorphic to $F'_j \times U_j$. □

Even though in the context of topological groups the great generality in which coverings are defined is justified, the most viable context is that of connected spaces and of pointed spaces. A *pointed space* is a pair (X,x) of a space and a *base point* $x \in X$; a *morphism of pointed spaces* $f: (X,x) \to (Y,y)$ is a continuous function $f: X \to Y$ such that $f(x) = y$. It is also called a *base point preserving continuous map.* Often pointed spaces occur quite naturally; e.g. all topological groups have their identity as a natural base point, and homomorphisms are automatically base point preserving.

A *covering of pointed spaces* is a covering between pointed spaces which is base point preserving.

If $p: (E,e) \to (B,b)$ is a covering of pointed spaces and $f: (X,x) \to (B,b)$ is a morphism of pointed spaces, then a function $F: X \to E$ is called *a lifting of* f *across* p if it is a morphism of pointed spaces and $f = p \circ F$.

$$\begin{array}{ccc} (X,x) & \xrightarrow{F} & (E,e) \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle p} \\ (X,x) & \xrightarrow[f]{} & (B,b). \end{array}$$

Proposition A2.5. (i) *Assume that X is a connected space, $x_0 \in X$ and that $\varphi, \psi: X \to Y$ are continuous functions such that $\varphi(x_0) = \psi(x_0)$. Assume further that for some continuous function $\rho: Y \to Z$ which induces local homeomorphisms the compositions $\rho \circ \varphi$ and $\rho \circ \psi$ agree. Then $\varphi = \psi$.*

(ii) *A lifting of a morphism f of pointed spaces across a covering of pointed spaces is unique if the domain of f is connected.* □

Exercise EA2.2. Prove Proposition A2.5.

[Hint. (i) Define $X' = \{x' \in X \mid \varphi(x') = \psi(x')\}$. Since all spaces considered are assumed to be Hausdorff spaces, X' is closed. Note that $x_0 \in X'$ and prove that X' is open in X using the fact that p induces local homeomorphisms. Use the connectivity of X to conclude the assertion. Derive (ii) from (i).] □

Defining Simple Connectivity

Definition A2.6. A topological space X is called *simply connected* if it is connected and has the following universal property: For any covering map $p: E \to B$ between topological spaces, any point $e_0 \in E$ and any continuous function $f: X \to B$ with $p(e_0) = f(x_0)$ for some $x_0 \in X$ there is a continuous map $\widetilde{f}: X \to E$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(x_0) = e_0$.

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B. \end{array}$$

□

The lifting $\widetilde{f}$, if it exists, is automatically unique by A2.5(ii).

The definition we give is particularly useful in the context of topological groups and transformation groups because it specifies directly the property one uses most often. It is noteworthy that it does not depend on arcwise connectedness.

The conventional definition in the context of arcwise connected spaces is more geometric but coincides on this class of spaces with our definition. We shall deal with the equivalence of the two concepts for arcwise connected pointed spaces in Proposition A2.10 and Exercise EA2.6 below.

One notices that Definition A2.6 is best phrased in terms of the category of pointed topological spaces and base point preserving continuous maps. Then it simply says that a pointed space is simply connected if any morphism into the base space of a covering lifts across the covering. In this category, simply connected spaces are, for those who know category theoretical elementary concepts, exactly the connected relative projectives with respect to the class of epics containing exactly the coverings.

Notice also that the definition of a simply connected pointed space (X, x_0) can also be expressed as follows.

Whenever

$$\begin{array}{ccc} (P, p_0) & \xrightarrow{F} & (E, e) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ (X, x_0) & \xrightarrow[f]{} & (B, b) \end{array}$$

is a pullback, then there is a subspace (P', p_0) of (P, p_0) such that $\pi|(P', p_0)$ is bijective. Indeed, this restriction being a covering by A2.4.(ii), (iii), its inversion is continuous and gives rise to the required lifting; the necessity is clear.

This raises at once the question of the existence of simply connected spaces. We shall first give examples and later exhibit a far ranging existence theorem.

Example A2.7. Assume that X is a totally ordered space, i.e. a set with a total order and a topology generated by the set of all open intervals $]a, b[\stackrel{\text{def}}{=} \{x \in X \mid a < x < b\}$, and assume that X is connected. Then X is simply connected. Examples are the space of real numbers and all of its intervals. □

Exercise EA2.3. Prove the claim in Example A2.7.

[Hint. Let $x_0 \in X$ and $f: X \to B$ a continuous map for a covering $p: E \to B$ and let $e_0 \in E$ be such that $p(e_0) = f(x_0)$. Let $\mathcal{U}$ be the set of all functions $\varphi: U \to E$ such that U is an open interval of X with $x \in U$ and that $\varphi(x_0) = e_0$ and $p\big(\varphi(u)\big) = f(u)$ for $u \in U$. Consider an open neighborhood V of $f(x_0)$ and a discrete set F and a homeomorphism $h: F \times V \to E$ such that $\operatorname{im} h = p^{-1}(V)$ and $p\big(h(y, v)\big) = v$. Let $h\big(y_0, f(x_0)\big) = e_0$. There is an open interval U around x_0 in X with $f(U) \subseteq V$ and define $\varphi: U \to E$ by $\varphi(u) = h\big(y_0, f(u)\big)$. Verify $\varphi \in \mathcal{U}$. Show that $\mathcal{U}$ is inductive with respect to extension of functions as partial order. Let $\widetilde{f}: W \to E$ be a maximal element in $\mathcal{U}$ using Zorn's Lemma. Finish the proof by showing that $W = X$; if not then there is an $x \in X$ with $u < x$ (say) and $x \notin W$. Set $W_1 = \{x \in X \mid (\exists w \in W)\, x \le w\}$. Since W is open, so is W_1. Set $W_2 = \{x \in X \mid (\forall w \in W)\, w < x\}$. For $x \in W_2$ use the covering property around $f(x) \in B$ to show that there is a whole neighborhood of x contained in W_2. Thus W_2 is open. Show that $X = W_1 \cup W_2$ and note that this is a contradiction to the connectivity of X.] □

Proposition A2.8. (i) *Assume that (X, x_0) and (Y, y_0) are simply connected pointed spaces and that $p: (E, e) \to (B, b)$ is a covering. Let $f: \big(X \times Y, (x_0, y_0)\big) \to (B, b)$ be a morphism of pointed spaces. Assume finally that Y is locally connected at y_0. Then f has a lifting $\widetilde{f}: \big(X \times Y, (x_0, y_0)\big) \to (E, e)$ across p.*

(ii) *If X and Y are simply connected, and if Y is locally connected at the base point, then $X \times Y$ is simply connected.*

(iii) *All spaces $\mathbb{R}^n$, $[0, 1]^n$ (i.e. all open and all closed n-cells), $n \in \mathbb{N}$, are simply connected.*

(iv) *Each retract of a simply connected space is simply connected. In particular, if a product of spaces is simply connected, then each factor is simply connected.*

Proof. Exercise EA2.4. □

Exercise EA2.4. Prove Proposition A2.8.

[Hint. (i) The proof of the asserted lifting requires some subtle hypotheses and arguments which we outline in their entirety. We begin with three independent lemmas.

Lemma A. *Let $p: E \to B$ be a continuous function defined on a Hausdorff space E such that it induces at each point of E a local homeomorphism. Let $f, g: Z \to E$ be two continuous functions satisfying $p \circ f = p \circ g$. Then $D \stackrel{\text{def}}{=} \{z \in Z : f(z) = g(z)\}$ is open and closed in Z.*

In particular, if $f(z) = g(z)$, then f and g agree on the connected component of z in Z.

Proof. Define $\varphi: Z \to B \times B$ by $\varphi(z) = (f(z), g(z))$. Since B is a Hausdorff space, the diagonal Δ in $B \times B$ is closed and so the equalizer $D = \varphi^{-1}(\Delta)$ is closed. Now let $z \in D$ and set $e = f(z) = g(z)$. There is an open neighborhood U of e in E and an open neighborhood V of $p(e)$ in B such that $p|U : U \to V$ is a homeomorphism. Let W be an open neighborhood of z in Z such that $f(W) \subseteq U$ and $g(W) \subseteq U$. Now $w \in W$ implies $f(w) = (p|U)^{-1}(p|U)f(w) = (p|U)^{-1}(p|U)g(w) = g(w)$ and so $W \subseteq D$. Therefore D is open in Z. □

A similar argument yields

Lemma B. *Under the hypotheses of* Lemma A *on $p: E \to B$, let $F: Z \to E$ be any function such that $p \circ F : Z \to B$ is continuous. Then the set of points of continuity of F is open.*

Proof. Let z be a point of continuity of F and let U be an open neighborhood of $F(z)$ such that $f|U: U \to V$ is a homeomorphism onto an open neighborhood of $p(F(z))$. Since F is continuous at z there is an open neighborhood of z in Z such that $F(W) \subseteq U$. Then $F|W = (p|V)^{-1} \circ (p \circ F)|W$ and thus $F|W$ is continuous. □

Lemma C. *Let $p: E \to B$ be a covering, and let $F: Z \to E$ be any function such that $p \circ F : Z \to B$ is continuous. Then the set of points of continuity of F is open and closed.*

Proof. Since the openness is a consequence of Lemma B we must prove closedness. Let $C \subseteq Z$ be the set of points at which F is continuous and let $c \in \overline{C}$. Let V be an open neighborhood of $b \stackrel{\text{def}}{=} p(F(c)) \in B$ such that $p^{-1}(V)$ is compatibly homeomorphic to $F_b \times V$. Let $U = h_b(\{\xi\} \times V)$ be that open subset of $p^{-1}(V) \subseteq E$ for which $h_b(\xi, b) = F(c)$, and so U is an open neighborhood of $F(c)$ in E. Also $p|U: U \to V$ is a homeomorphism. Let W be an open neighborhood of c in Z such that $F(W) \subseteq U$. Then we find a point of continuity z of F in $W \cap C$. Now let

$w \in W$; then $F(w) = \mathrm{id}_U(F(w)) = (p|U)^{-1}(p|U)(F(w)) = (p|U)^{-1}(p \circ F)(w)$. The functions $(p \circ F)|W\colon W \to V$ and $(p|U)^{-1}\colon V \to U$ are continuous, hence $F|W = (p|U)^{-1} \circ (p \circ F)|W$ is continuous. Hence the open neighborhood W of c is contained in C and so $c \in C$. This proves $\overline{C} = C$. □

Now we prove Part (i) of Proposition A2.8.
Step 1: Local continuous lifting at (x_0, y_0):
Let V be an open neighborhood of $b = f(x_0, y_0) \in B$ such that $p^{-1}(V)$ is compatibly homeomorphic to $F \times V$. Let $U = h_b(\{\xi\} \times V)$ be that open subset of E for which $h_b(\xi, f(x_0, y_0)) = \widetilde{f}(x_0, y_0)$. Then $p|U\colon U \to V$ is a homeomorphism. Let W be an open neighborhood of (x_0, y_0) in $X \times Y$ such that $f(W) \subseteq V$. The function

$$\widetilde{f}_0 \stackrel{\mathrm{def}}{=} (p|U)^{-1} \circ f|W : W \to V$$

is a continuous lifting of $f|W\colon W \to U$ across $p|V$.

Step 2: Global lifting, continuous at (x_0, y_0):
Since (X, x_0) is simply connected, the continuous morphism $f|(X \times \{y_0\}) : (X \times \{y_0\}, (x_0, y_0)) \to (B, b)$ has a unique lifting

$$\varphi\colon (X \times \{y_0\}, (x_0, y_0)) \to (E, e)$$

across p. Since $\varphi(x_0, y_0) = e = f_0(x_0, y_0)$ and φ is continuous and U is an open neighborhood of e in E, there is an open neighborhood $W_1$of x_0 in $W \subseteq X \times Y$ such that $\varphi(W_1 \times \{y_0\}) \subseteq U$. Then, noting the restriction

$$f|(W_1 \times \{y_0\}) : W_1 \times \{y_0\} \to V,$$

we obtain that $\varphi_0 \stackrel{\mathrm{def}}{=} \varphi|((W_1 \times \{y_0\}) : W_1 \times \{y_0\} \to U$ satisfies

$$\varphi_0 = (p|U)^{-1} \circ (f|(W_1 \times \{y_0\})).$$

We recall that $\widetilde{f}_0 : W \to U$ was given by $\widetilde{f}_0 = (p|U)^{-1} \circ (f|W)$, we observe $\widetilde{f}_0|(W_1 \times \{y_0\}) = (p|U)^{-1} \circ f|(W_1 \times \{y_0\})$. Therefore $\widetilde{f}_0|(W_1 \times \{y_0\}) = \varphi_0 = \varphi|(W_1 \times \{y_0\})$, that is, $\widetilde{f}(x, y_0) = \varphi(x, y_0)$ for all $x \in W_1$.

Now, since Y is locally connected at y_0, we select a connected open neighborhood W_2 of y_0 in Y such that $W_1 \times W_2 \subseteq W$. Let us simplify notation by assuming $W = W_1 \times W_2$. Since Y is simply connected, for each $x \in X$, there is a unique lifting $\psi_x\colon (\{x\} \times Y, (x, y_0)) \to (E, \varphi(x))$ of

$$f|(\{x\} \times Y) : (\{x\} \times Y, (x, y_0)) \to (B, f(x, y_0))$$

across p. We define $\widetilde{f}\colon (X \times Y, (x_0, y_0)) \to (E, e)$ by $\widetilde{f}(x, y) = \psi_x(y)$; as $\varphi(x_0, y_0) = e$ we indeed have $\widetilde{f}(x_0, y_0) = \psi_{x_0}(y_0) = \varphi(x_0, y_0) = e$ as required. Moreover, for each $(x, y) \in X \times Y$ we get $p(\widetilde{f}(x, y)) = p(\psi_x(y)) = f(x, y)$. Thus $\widetilde{f}$ is a lifting of f across p.

Now for each $x \in W_1$, the function

(1) $$\widetilde{f}|(\{x\} \times W_2) : (\{x\} \times W_2, (x, y_0)) \to (E, \varphi(x, y_0))$$

is a continuous lifting of

$$f|(\{x\} \times W_2) : (\{x\} \times W_2, (x, y_0)) \to (B, f(x, y_0)) \tag{2}$$

across p. Likewise,

$$\widetilde{f_0}|(\{x\} \times W_2) : (\{x\} \times W_2, (x, y_0)) \to (E, \varphi(x, y_0)) \tag{3}$$

is a unique continuous lifting of (1) across p as well. Now since W_2 is **connected**, Lemma A shows that (1) and (3) agree for all $x \in X$. Hence $\widetilde{f}|W = \widetilde{f_0}|W$. In other words, $\widetilde{f}: X \times Y \to E$ is continuous at every point of $W = W_1 \times W_2$.

Step 3. Continuity of $\widetilde{f}$.
Let C denote the set of all $(x, y) \in X \times Y$ at which $\widetilde{f}$ is continuous. By Step 2 we have $(x_0, y_0) \in W \subseteq C$, that is, $C \neq \emptyset$. Now Lemma C applies to show that C is open closed in $X \times Y$. Since X and Y are connected, $X \times Y$ is connected, and so $X \times Y = C$. Thus $\widetilde{f}$ is continuous as asserted.

(ii) is a consequence of (i) and (ii) implies (iii).

(iv) Let $X \subseteq Y$ and $r: Y \to X$ be a retraction. If $p: E \to B$ is a covering and $f: X \to B$ a continuous function, then $fp: Y \to B$ has a lifting $F: Y \to E$. Then $F|X: X \to E$ is the required lifting of f.] ☐

Since this proof is somewhat delicate, we point out that the local connectivity of Y at one point, namely y_0, was used merely in showing that the fiberwise lifting $\widetilde{f}$ is continuous at least at one point (x_0, y_0).

On a more elementary level, the proof that a product of connected spaces is connected requires a bit of circumspection which is remotely reflected in the present proof.

Proposition A2.9. *For a connected space X, the following statements are equivalent:*

(i) *X is simply connected.*

(ii) *Whenever $f: E \to X$ is a covering and E_0 is a connected component of E, then $f|E_0: E_0 \to X$ is a homeomorphism.*

Proof. (i)⇒(ii) Let $f: E \to X$ be a covering and assume (i). Pick $e_0 \in E_0$ and set $x_0 = f(e_0)$. Then the identity map $i: X \to X$ has a lifting $\widetilde{i}: X \to E$ such that $f\big(\widetilde{i}(x_0)\big) = x_0$ and $f \circ \widetilde{i} = i$. We claim that the image $\widetilde{i}(X)$ is open in E. Indeed let $x \in X$ and find an open neighborhood U of x in X such that for some discrete set F and some homeomorphism $h: F \times U \to f^{-1}(U)$ we have $f\big(h(y, u)\big) = u$. Let $h(y_0, x) = \widetilde{i}(x)$. Then $W \stackrel{\text{def}}{=} h(\{y_0\} \times U)$ is an open neighborhood of $\widetilde{i}(x)$. Since $\widetilde{i}$ is continuous there is an open neighborhood V of x in U such that $\widetilde{i}(V) \subseteq W$. Since $f|W: W \to U$ is a homeomorphism, $\widetilde{i}(V)$ is a neighborhood of $\widetilde{i}(x)$. Hence W and thus $\widetilde{i}(X)$ is a neighborhood of $\widetilde{i}(x)$. Now $\widetilde{i} \circ f: E \to E$ is a retraction with image $\widetilde{i}(X)$ and the image of retractions in Hausdorff spaces are closed, $\widetilde{i}(X)$ is a connected open closed subset of E containing e_0. It therefore agrees with E_0 and the assertion follows.

(ii)⇒(i) Assume (ii) and consider a covering $p: E \to B$ and a continuous function $f: X \to B$ such that $f(x_0) = p(e_0)$ for suitable $(x_0, e_0) \in X \times E$. Now we consider the pullback

$$\begin{array}{ccc} P & \xrightarrow{f^*} & E \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B. \end{array}$$

Let $p_0 \in P$ denote the unique point with $p^*(p_0) = x_0$ and $f^*(p_0) = e_0$. Then p^* is a covering by A2.3(ii). Let P_0 denote the component of p_0 in P. Then by (ii) the restriction $p^*|P_0: P_0 \to X$ is a homeomorphism. Denote the inclusion map $P_0 \to P$ by j and set $\widetilde{f} \stackrel{\text{def}}{=} f^* \circ j \circ (p^*|P_0)^{-1} : X \to E$. Then $\widetilde{f}(x_0) = f^*(p_0) = e_0$ and $p \circ \widetilde{f} = p \circ f^* \circ j \circ (p^*|P_0)^{-1} = f \circ p^* \circ j \circ (p^*|P_0)^{-1} = f$. This completes the proof. □

Sometimes simple connectivity is defined by condition (ii).

We say that two continuous functions f, $g: (X, x_0) \to (Y, y_0)$ of pointed spaces are *homotopic* if there is a continuous function $H: [0,1] \times X \to Y$ such that $H(0, x) = f(x)$, $H(t, x_0) = y_0$ and $H(1, x) = g(x)$ for all $t \in [0,1]$ and $x \in X$. Let $\mathbb{I}$ denote the pointed unit interval $([0,1], 0)$. A continuous function $f: (\mathbb{S}^1, 1) \to (Y, y_0)$ is called *a loop* at y_0. It is said to be *contractible*, if it is homotopic to the constant morphism of pointed spaces. We note that the contractibility of loops in X at x_0 is the same as saying that every continuous function $\partial D \to X$ from the boundary of the unit square $D = [0,1]^2$ into X (mapping $(0,0)$ to x_0) extends to a continuous function $D \to X$, and that, in turn, means that two paths α, $\beta: \mathbb{I} \to (X, x_0)$ starting at x_0 and ending at the same point $\alpha(1) = \beta(1)$ are homotopic. Homotopy is an equivalence relation on the set $C_0(X, Y)$ of base point preserving functions from a pointed space X to a pointed space Y.

For each point x in an arcwise connected pointed space (X, x_0) we associate a discrete set $F(x)$, namely the set of homotopy classes $[\alpha]$ of arcs $\alpha: \mathbb{I} \to X$ from $x_0 = \alpha(0)$ to $x = \alpha(1)$. Write $\widetilde{X} = \bigcup_{x \in X} F(x)$ and set $p: \widetilde{X} \to X$, $p([\alpha]) = \alpha(x)$. Now assume that X has an open cover $\{U_j \mid j \in J\}$ such that each U_j is arcwise connected and every loop in every U_j is contractible; we call such spaces *locally arcwise simply connected*. For each $j \in J$ pick a $u_j \in U_j$. For each $f = [\alpha] \in F_j \stackrel{\text{def}}{=} F(u_j)$ and $u \in U_j$ we connect u_j and u by an arc ε in U_j. Every other arc from u_j to u is homotopic to ε by assumption on U_j. Let β denote the arc obtained by going from x_0 to u_j by α and from u_j to u by ε. Write $h_j(f, u) = [\beta] \in F(u)$. Then $p\big(h_j(f, u)\big) = u$. Thus $h_j: F_j \times U_j \to p^{-1}(U_j)$ is a well-defined function. If $[\gamma] \in p^{-1}(U_j)$, then $u = p([\gamma]) = \gamma(1)$ and there is an arc η in U_j from u to u_j, unique up to homotopy. The arc δ is obtained by going from x_0 to u by γ and from $u \to u_j$ by η. Then $f \stackrel{\text{def}}{=} [\delta]$ is an element of $F(u_j) = f_j$, and $(f, u) = h^{-1}(u)$. Thus h_j is bijective. There is a unique topology on $\widetilde{X}$ which induces on $p^{-1}(U_j)$ that topology which makes $h_j: F_j \times U_j \to p^{-1}(U_j)$ a homeomorphism. Then $p: \widetilde{X} \to X$ is a covering map.

Proposition A2.10. *For an arcwise connected locally arcwise connected pointed Hausdorff space* (X, x_0) *consider the following conditions:*

(i) *All loops at* x_0 *are contractible.*

(ii) (X, x_0) *is simply connected.*

Then (i) *implies* (ii). *If* X *is also locally arcwise simply connected, then both conditions are equivalent.*

Proof. (i)⇒(ii) Let $p: (E, e) \to (X, x_0)$ be a covering which we assume to be connected by A2.4(iv). By the simple connectivity of $\mathbb{I}$, every arc $\alpha: \mathbb{I} \to (X, x_0)$ lifts to a unique arc $\widetilde{\alpha}: \mathbb{I} \to (E, e)$, and by the simple connectivity of D, homotopic arcs lift to homotopic arcs. Define $\sigma: (X, x_0) \to (E, e)$ by $\sigma(x) = \widetilde{\alpha}(1)$ for any member α of the class of homotopic arcs from x_0 to x. Then $p\sigma(x) = p\widetilde{\alpha}(1) = \alpha(1) = x$. Let $\{U_j \mid j \in J\}$ be an open cover of X consisting of arcwise connected open sets such that for each U_j there is a discrete space F_j and a homeomorphism $h_j: F_j \times U_j \to p^{-1}(U_j)$ such that $p\big(h_j(f, u)\big) = u$ for all $(f, u) \in F_j \times U_j$. Let $x \in U_j$. Elements y nearby in U_j can be reached by a small arc ε from x to y, giving an arc via α from x_0 to x and from there to y; call this arc β. There is a unique $f \in F_j$ such that $h_j(f, \alpha(x)) = \widetilde{\alpha}(x) = \sigma(x)$. Then $t \mapsto h_j\big(f, \varepsilon(t)\big)$ is a small arc in $h_j(\{f\} \times U_j)$ from $\sigma(x)$ to a unique point in the set above y, which is necessarily the endpoint of $\widetilde{\beta}$. This point is $\sigma(y)$. It follows that $\sigma(u) = h_j(f, u)$ for $u \in U_j$. In particular, σ is continuous, induces local homeomorphisms, and satisfies $p\sigma = \mathrm{id}_X$. Then $\sigma(X)$ is an open subspace of E such that for all $j \in J$ the relation $h_j(\{f\} \times U_j) \cap \sigma(X) \neq \emptyset$ implies $h_j(\{f\} \times U_j) \subseteq \sigma(X)$. Hence $p|\sigma(X): \sigma(X) \to X$ is a covering map and the complement of $\sigma(X)$ in E is open, too. Since E is connected, $\sigma(X) = E$. Then $\sigma = p^{-1}$. That is, p is a homeomorphism. Then X is simply connected by A2.9.

(ii)⇒(i) Let $p: \widetilde{X} \to X$ be the covering constructed in the paragraph preceding the proposition. Since X is simply connected, p is bijective by A2.9. By the definition of $\widetilde{X}$ this means that two arcs linking x_0 with a point x in X are homotopic, and this is equivalent to (i). ☐

Example A2.11. (i) All continuous functions $f: (X, x_0) \to (C, c_0)$ preserving base points into a convex subset C of any real topological vector space E are contractible. Hence all convex subsets of any real topological vector space are simply connected.

(ii) All spheres $\mathbb{S}^n$ are simply connected spaces with the exception of the zero- and one-dimensional ones. In particular $\mathbb{S}^3 \cong \mathrm{SU}(3)$ is a simply connected compact topological group.

(iii) Let $\{S_j \mid j \in J\}$ be a family of simply connected, arcwise connected, locally arcwise connected and locally arcwise simply connected pointed spaces. Then the product space $\prod_{j \in J} S_j$ is simply connected.

Proof. Exercise EA2.5. □

Exercise EA2.5. Prove the assertions of the examples in A2.11.

[Hint. (i) Work with the function $H(r,x) = (1-r)\cdot f(x) + r\cdot c_0$.

(ii) Show that in all spheres of dimension 2 or more each loop is contractible. Observe that $\mathbb{S}^0$ fails to be connected. Show that for the covering $p: \mathbb{R} \to \mathbb{S}^1$, $p(t) = e^{it}$ the identity map $f: \mathbb{S}^1 \to \mathbb{S}^1$ does not lift to a continuous function $\widetilde{f}: \mathbb{S}^1 \to \mathbb{R}$; if it did, the map $\widetilde{f}$ and then f would be contractible, but the coextension $f: \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$ has winding number one, and contractible loops in $\mathbb{C} \setminus \{0\}$ would have winding number 0. (A little elementary complex analysis is used here!)

(iii) The product $S = \prod_{j\in J} S_j$ is arcwise and locally arcwise connected by the definition of the product topology. If $\alpha: \mathbb{S}^1 \to S$ is a loop, then $\alpha(t) = \big(\alpha_j(t)\big)_{j\in J}$ and $\alpha_j: \mathbb{S}^1 \to S_j$ is a loop in S_j. Then by A2.10, since A_j is simply connected, there is continuous extension $A_j: \mathbb{D} \to S_j$ to the complex unit disc $\mathbb{D}$. Then $A: \mathbb{D} \to S$, $A(t) = \big(A_j(t)\big)_{j\in J}$ is a continuous extension of α. Hence every loop in S is contractible and thus S is simply connected by A2.10.] □

In the Definition A2.1 of a covering, the open cover $\{U_j : j \in J\}$ plays a somewhat volatile role; there is however, a class of spaces in which such a cover may be chosen in a canonical fashion. Indeed, a space X will be called *locally simply connected* if the set $\mathcal{S}(X)$ of simply connected open subsets of X covers X. (In the constructions of A2.10 we have used a similar hypothesis.)

Lemma A2.12. *Let X be a locally simply connected space. Then for each covering $p: E \to X$ there is a family $(F_S)_{S\in\mathcal{S}(X)}$ of discrete spaces and a family of homeomorphisms $(h_S)_{S\in\mathcal{S}(X)}$, $h_S: F_S \times S \to f^{-1}(S)$ such that $f\big(h_S(y,s)\big) = s$ for all $s \in S$.*

Proof. Let $S \in \mathcal{S}(X)$. The restriction $f|f^{-1}(S) : f^{-1}(S) \to S$ is a covering by A2.4(iii). Let F_S denote the set of connected components of $f^{-1}(S)$. By A2.9 each restriction $f|T: T \to S$ for $T \in F_S$ is a homeomorphism. Define $h_S: F_S \times S \to f^{-1}(S)$ by $h_S(T,s) = (f|T)^{-1}(s)$. Then $f\big(h_S(T,s)\big) = f\big((f|T)^{-1}(s)\big) = s$. □

We fix a connected and locally simply connected space X and consider the class $\mathcal{C}(X)$ of all coverings $p: E \to X$, denoted (E,p), together with the maps $f: E_1 \to E_2$ for objects (E_j,p_j), $j = 1$, 2 satisfying $p_2 \circ f = p_1$. This class forms a category with these maps as morphisms $f: (E_1,p_1) \to (E_2,p_2)$.

We assume that X is connected and consider the subclass $\mathcal{C}_0(X)$ of *connected* coverings (meaning, of course those coverings (E,p) for which E is connected. We claim that there is an upper bound to the cardinality of E depending on X only. We define an equivalence relation R on E consisting of all pairs $(x,y) \in E \times E$ such that there is a finite sequence of open subsets $U_1, \ldots, U_n$ in E such that

(i) $p(U_j) \in \mathcal{S}(X)$ for $j = 1, \ldots, n$,

(ii) $U_{j-1} \cap U_j \neq \emptyset$, $j = 2, \ldots, n$,

(iii) $x \in U_1$ and $y \in U_n$.

Undoubtedly R is an equivalence relation, and obviously its cosets are all open. But each coset of an equivalence relation whose cosets are open is closed (as the complement of the union of all the other cosets). Since E is connected, there is only one equivalence class. At this point we pass to pointed spaces and fix an $x_0 \in X$ and consider each covering (E, p) of X to be equipped with a base point $e_0 \in E$ such that $p(e_0) = x_0$. For each x we find a chain $U_1, \ldots, U_n$ satisfying (i), (ii) and (iii)$_0$ $e_0 \in U_1$ and $x \in U_n$.

Then we set $V_j \stackrel{\text{def}}{=} p(U_j)$, $j = 1, \ldots, n$ and notice

(a) $V_j \in \mathcal{S}(X)$ for $j = 1, \ldots, n$,

(b) $V_{j-1} \cap V_j \neq \emptyset$, $j = 2, \ldots, n$,

(c) $x_0 \in V_1$ and $p(x) \in V_n$.

We observe that every such chain V_j and every choice of an element $y \in V_n$ gives rise to only one lifting to a chain of sets U_j satisfying (i), (ii), and (iii) and the selection of exactly one $x \in U_n$ such that $p(x) = y$. The cardinality of the set of all finite chains Vj is not bigger than the cardinality of finite sequences of the infinite set of all subsets of X and is therefore not bigger than $2^{\operatorname{card} X}$. The function assigning to $(V_1, \ldots, V_n; y)$ where $(V_1, \ldots, V_n)$ satisfies (a), (b), and (c) and $y \in V_n$ the unique $x \in U_n$ with the unique lifting $(U_1, \ldots, U_n)$ and $f(x) = y$ is surjective. Hence

$$\operatorname{card} E \leq \operatorname{card} X \cdot 2^{\operatorname{card} X} = 2^{\operatorname{card} X}.$$

It follows that there is a *set* J of coverings $\big((E, e_0), p)\big)$ of (X, x_0) with a connected covering space E such that every isomorphy class of $\mathcal{C}(X)$ contains exactly one member of J, and we may assume that (X, id_X) is one of them. We say that a covering $j_1 = \big((E_1, e_{10}), p_1\big)$ is *above* a covering $j_2 = \big((E_2, e_{20}, p_2\big)$ iff there is a morphism of coverings of pointed spaces $f: j_1 \to j_2$ and write $j_2 \leq j_1$. Since a base point preserving morphism is a lifting of the base point preserving covering $p_1: E_1 \to X$, it is unique by Proposition A2.5(ii). Hence J is a partially ordered set with respect to the "above" relation. Due to the pullback construction in 2.4(ii) this partially ordered set is directed, since for any two coverings there will be one which is above the two.

We propose to show that J contains a maximal element $(\widetilde{X}, \widetilde{p})$ which is above all others. If there is such an element then $\widetilde{X}$ will be simply connected by Proposition A2.9, and up to isomorphisms of coverings, it will be unique.

Definition A2.13. A covering $(\widetilde{X}, \widetilde{p})$, $\widetilde{p}: \widetilde{X} \to X$ is called a *universal covering* if $\widetilde{X}$ is simply connected. □

As an example consider the one-sphere $\mathbb{S}^1$. We take $x_0 = 1$ as base point. Among the coverings we have the following morphisms

1) All maps $\mu_n: \mathbb{S}^1 \to \mathbb{S}^1$, $n \in \mathbb{Z}$. The fiber over any simply connected open set in $\mathbb{S}^1$ (here being homeomorphic to an interval) is isomorphic to $\ker \mu_n = \{e^{2\pi i m/n} \mid m = 0, \ldots, n-1\}$,

2) The map $\exp: \mathbb{R} \to \mathbb{S}^1$, $\exp r = e^{2\pi i r}$. The fiber over any simply connected open set in $\mathbb{S}^1$ is $\ker \exp \mathbb{Z}$.

Since $\mathbb{R}$ is simply connected by Proposition A2.7 the covering in 2) is universal, and indeed given any other one in 1) there is a covering from the universal one to it. The issue is now: Do we always find a universal covering?

The construction of an inverse limit in Chapter 1, notably 1.25ff. (for which category theory provides sweeping generalisations) suggests that we construct a limit. For this purpose, we let M denote the set of all morphisms $f: (E^f, p^f) \to (E_f, p_f)$ between the coverings of X in J and consider, in the category of pointed spaces, the projective limit

$$L(X) \stackrel{\text{def}}{=} \{(x_E)_{(E,p)\in J} \in \textstyle\prod_{(E,p)\in J} E \mid (\forall f \in M)\ f(x_{E^f}) = x_{E_f}\}.$$

We define the map $\overline{p}: L(X) \to X$, $\overline{p}\big((x_E)_{(E,p)\in J}\big) = x_X$ recalling (X, id_X) to be the minimal element of J. Set $e = (e_j)_{j\in J}$ where e_j is the base point of E where $j = (E,p) \in J$. Then $\overline{p}(e) = x_0$. Now let $\xi = (x_j)_{j\in J} \in L(X)$ and set $x = \overline{p}(\xi)$. Let $S \in \mathcal{S}(X)$ be a simply connected open neighborhood of x in X. Then for each $j = (E,p) \in J$ there is a unique cross section $\sigma_j: S \to E$ of pointed spaces such that $p \circ \sigma_j = \mathrm{id}_S$ and $\sigma_j(x) = x_j$ see A2.9). Then $\sigma(x') \stackrel{\text{def}}{=} (\sigma_j(x'))_{j\in J} \in \prod_{(E,p)\in J} E$ is seen to be in $L(X)$ for all $x' \in S$ by the uniqueness of liftings (A2.5(ii)). Hence $\sigma_{(\xi,S)}: X \to L(X)$, is a cross section satisfying $\sigma_{(\xi,S)}(\overline{p}|\overline{p}^{-1}(S)) = \mathrm{id}_S$ and $\sigma_{(\xi,S)}(x) = \xi$. In particular, if x is in the image of $\overline{p}$ then every simply connected neighborhood S of X is in the image I of $\overline{p}$. This shows that I is open. If $x \in \overline{I}$ then some simply connected neighborhood S of x meets I. Hence $x \in S \subseteq I$. Thus $x \in I$ and I is also closed. Since X is connected, $I = X$ and thus $\overline{L}(X) \to X$ is surjective. The set $\mathcal{B}$ of all open subsets U of X for which there is an $S \in \mathcal{S}(X)$ with $U \subseteq S$ form a basis for the topology of X. The set of all $V_{\xi,U} \stackrel{\text{def}}{=} \sigma_{\xi,S}(U)$ for any $S \in \mathcal{S}(X), U \subseteq S$ is a basis for a topology $\mathcal{O}$ on $L(X)$ such that the components of $\overline{p}^{-1}(S)$ are exactly the sets $\sigma_{\xi,S}(S)$. For $S \in \mathcal{S}$ we let F_S denote the set of these components and define $h_S: F_S \times S \to \overline{p}^{-1}(S)$ by $h_S(\sigma_{\xi,S}(S), y) = \sigma_{\xi,S}(y)$. Then $\overline{p}\big(h_S(C,y)\big) = y$. This shows that $\overline{p}: \big(L(X), \mathcal{O}\big) \to X$ is a covering of pointed spaces. Let $\widetilde{X}$ be the connected component of e in $\big(L(X), \mathcal{O}\big)$, and let $\widetilde{p}$ be the restriction $\overline{p}|\widetilde{X}$. Then $\widetilde{p}: (\widetilde{X}, e) \to (X, x_0)$ is a covering by A2.4(iv). Thus $\big((\widetilde{X}, e), \widetilde{p}\big) \in J$.

For each $k \in J$ we have a limit map $\overline{p}_k: L(X) \to E$ for $k = (E,p)$ given by $\overline{p}\big((x_j)_{j\in J}\big) = x_k$. The space E is locally simply connected; e.g. the connected components of $p^{-1}(S)$, $S \in \mathcal{S}(X)$ are homeomorphic to S by A2.4(iv) and A2.9. Then just as in the case of the minimal $k = (X, \mathrm{id}_X)$ we see that $\overline{p}_k: \big(L(X), \mathcal{O}\big) \to (E,p) = k$ is a covering. By A2.4(iv), accordingly, write $\widetilde{x} = (x_j)_{j\in J}$ for an element in $\widetilde{X}$ and note that $p\big(\overline{p}_k(\widetilde{x})\big) = p(x_k) = x_{(X,\mathrm{id}_X)} = \widetilde{p}\big((x_j)_{j\in J}\big) = \widetilde{p}(\widetilde{x})$. Thus the map $\overline{p}_k|\widetilde{X}: (\widetilde{X}, e) \to (E, e_j)$ is a morphism of pointed coverings of (X, x_0). Thus $(\widetilde{X}, \widetilde{p})$ is maximal in J and $\widetilde{p}: (\widetilde{X}, e) \to (X, x_0)$ is a universal covering.

We have now proved the following existence theorem:

Existence of Universal Coverings

Theorem A2.14. *Every connected locally simply connected Hausdorff space has a universal covering.* □

Since each open n-ball in $\mathbb{R}^n$ is simply connected (see EA2.8(iii)) every locally euclidean space (i.e. every space having an open cover consisting of sets homeomorphic to an open ball of $\mathbb{R}^n$) is locally simply connected. By A2.10 and A2.11(i) all open balls in a Banach space are simply connected. Thus every space covered by a family of open sets each homeomorphic to an open ball in some Banach space is locally simply connected. Let us call such spaces *topological manifolds.*

Universal Coverings of Manifolds

Corollary A2.15. *Every connected topological manifold has a universal covering.* □

From hindsight the somewhat lengthy proof of Theorem A2.14 exhibits a curiosity as far as limit constructions go: After we were all through we discovered that the limit was none other than a member of the inverse system itself because the index set turned out to have a maximal element. The example of the one-sphere mentioned above illustrates this fact: The limit of the coverings listed under 1) alone is a genuine solenoid; if we include the covering under 2), the limit degenerates to the universal covering itself.

The Group of Covering Transformations

Let X denote a space possessing a universal covering $\widetilde{p}: \widetilde{X} \to X$. Recall that a morphism $f: (E_1, p_1) \to (E_2, p_2)$ of coverings of X is a continuous function $f: E_1 \to E_2$ such that $p_2 f = p_1$, equivalently, that f is a lifting of p_1 across p_2 (see A2.1). Let Γ denote the group of all automorphisms $\gamma: \widetilde{X} \to \widetilde{X}$ of the covering $\widetilde{p}: \widetilde{X} \to X$.

Fix a point $x_0 \in X$ and write $F = \widetilde{p}^{-1}(x_0)$. Now fix a point $\widetilde{x}_0 \in F$.

Proposition A2.16. *For each $\widetilde{x} \in F$ there is a unique automorphism $\gamma_{\tilde{x}}$ of (unpointed) covering spaces of $\widetilde{p}: \widetilde{X} \to X$ such that $\gamma_{\tilde{x}}(\widetilde{x}_0) = \widetilde{x}$. The map $\widetilde{x} \mapsto \gamma_{\tilde{x}}: F \to \Gamma$ is a bijection.*

Proof. By Definition A2.6 of simple connectivity there is a lifting $\gamma: \widetilde{X} \to \widetilde{X}$ across $\widetilde{p}: \widetilde{X} \to X$ of the map $\widetilde{p}: X \to X$ such that $\gamma_{\tilde{x}}(\widetilde{x}_0) = \widetilde{x}$. Likewise there is a lifting γ' satisfying $\gamma'(\widetilde{x}) = \widetilde{x}_0$. Now $\gamma'\gamma_{\tilde{x}}$, $\gamma_{\tilde{x}}\gamma'$, and $\mathrm{id}_{\widetilde{X}}$ are all liftings fixing x_0, and thus by the uniqueness of liftings (see Proposition A2.5(ii)) they all agree. Hence $\gamma_{\tilde{x}}$ is invertible and thus a member of Γ. By the uniqueness of liftings, $\widetilde{x} \mapsto \gamma_{\tilde{x}}: F \to \Gamma$ is injective. If $\gamma \in \Gamma$, then $\gamma_{\gamma(\widetilde{x}_0)} = \gamma$ by uniqueness again. □

Definition A2.17. The members of Γ are called *covering transformations* or *deck transformations* and Γ is called the *Poincaré* group of X. If the dependence of X is to be made evident we write $\Gamma(X)$ for the Poincaré group of X. □

The underlying set of Γ is a prototype for the fibers of the universal covering of X. It is clear that X is simply connected if and only if Γ is singleton.

The theory of simple connectivity can be based on homotopy theory as was already indicated in A2.10. The discourse in the following pursues this idea in identifying $\Gamma(X)$ with the so called *fundamental group* $\pi_1(X)$ of an arcwise connected space.

Exercise EA2.6. A morphism $\alpha\colon \mathbb{I} \to (X, x_0)$ lifts uniquely to a morphism $\widetilde{\alpha}\colon \mathbb{I} \to \widetilde{X}$ across a universal covering $\widetilde{p}\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. If $\alpha(1) = x_0$, then α represents a loop in X and $\widetilde{\alpha}(1) \in F \stackrel{\rm def}{=} \widetilde{p}^{-1}(x_0)$. If X is arcwise connected, then so is $\widetilde{X}$. For every path $\beta\colon \mathbb{I} \to (\widetilde{X}, \widetilde{x}_0)$ with $\beta(1) \in F$ the path $\alpha = \widetilde{p} \circ \beta$ satisfies $\alpha(1) = x_0$, and β is the unique lifting of α across $\widetilde{p}$. If two loops α, $\alpha'\colon \mathbb{I} \to X$ are homotopic via a homotopy leaving the initial and end point fixed then the homotopy lifts and yields homotopic liftings $\widetilde{\alpha}$, $\widetilde{\alpha'}\colon \mathbb{I} \to \widetilde{X}$ such that $\widetilde{\alpha}(1)$, $\widetilde{\alpha'} \in F$; then $\widetilde{\alpha}(1) = \widetilde{\alpha'}(1)$. Conversely, if two liftings $\widetilde{\alpha}$ and $\widetilde{\alpha'}$ of two closed paths end at the same point, then $\widetilde{\alpha}$ and $\widetilde{\alpha'}$ are homotopic, and thus α and α' are homotopic. Let $\pi_1(X)$ denote the set of homotopy classes of loops in X based at x_0. Then the function $\pi_1(X) \to \Gamma(X)$ which associates with a homotopy class of a closed path α at x_0 the transformation $\gamma_{\widetilde{\alpha}(1)}$ is a bijection.

Explain the group structure transported to $\pi_1(X)$ by this map in terms of the loops and operations on them. □

Exercise EA2.6 which puts the sets $\Gamma(X)$ and $\pi_1(X)$ into bijective correspondence shows, in particular, that an arcwise connected space X is simply connected if and only if $\Gamma(X)$ is singleton if and only if $\pi_1(X)$ is singleton if and only if every loop in X based at x_0 is contractible.

In view of this exercise, the Poincaré group is often written as $\pi_1(X)$.

Universal Covering Groups

Proposition A2.18. *Let G be a topological group and assume that the underlying space of G admits a universal covering $\widetilde{p}\colon \widetilde{G} \to G$. Then for each $\widetilde{\mathbf{1}} \in \widetilde{p}^{-1}(\mathbf{1})$ there exists a unique topological group structure on $\widetilde{G}$ with $\widetilde{1}$ as the identity element and that $\widetilde{p}$ is a group homomorphism which, as a consequence, induces a local isomorphism at $\mathbf{1}$.*

Proof. The continuous function $(\widetilde{g}, \widetilde{h}) \mapsto \widetilde{p}(\widetilde{g})\widetilde{p}(\widetilde{h}) : \widetilde{G} \times \widetilde{G} \to G$ mapping $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\mathbf{1}$ has a unique lifting across $\widetilde{p}\colon \widetilde{G} \to G$ mapping $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\widetilde{\mathbf{1}}$ which we shall write $(\widetilde{g}, \widetilde{h}) \mapsto \widetilde{g}\widetilde{h}$. Then $\widetilde{p}(\widetilde{g})\widetilde{p}(\widetilde{h}) = \widetilde{p}(\widetilde{g})\widetilde{p}(\widetilde{h})$. The functions $(\widetilde{g}, \widetilde{h}, \widetilde{k}) \mapsto (\widetilde{g}\widetilde{h})\widetilde{k})$, $\widetilde{g}(\widetilde{h}\widetilde{k})$:

$\widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ both map $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ to $\mathbf{1}$, and their compositions with $\widetilde{p}: \widetilde{G} \to G$ agree. Hence by the uniqueness of the lifting they agree. Hence the multiplication on $\widetilde{G}$ is associative. The maps $\widetilde{x} \mapsto \widetilde{x}\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}\widetilde{x}: \widetilde{X} \to \widetilde{X}$ as well as the identity map of $\widetilde{X}$ all map $\mathbf{1}$ to $\mathbf{1}$ and agree when followed by $\widetilde{p}$. Hence they agree. Thus $\widetilde{G}$ is a monoid with $\widetilde{\mathbf{1}}$ as identity. Finally, by the same procedure we lift the map $\widetilde{x} \mapsto \widetilde{p}(\widetilde{x})^{-1}: \widetilde{G} \to G$ across the universal covering mapping $\widetilde{\mathbf{1}}$ to itself, and then we see again that the lifting is inversion on $\widetilde{G}$ making $\widetilde{G}$ a topological group in precisely the fashion asserted. □

Definitions A2.19. If the underlying space of a topological group G permits a universal covering, then the simply connected topological group $\widetilde{G}$ according to Proposition A2.19 is called a *universal covering group*, and the morphism $\widetilde{p}: \widetilde{G} \to G$ is called a *universal covering morphism.* □

Exercise EA2.7. Prove the following proposition:

Universal covering groups are determined uniquely up to an isomorphism of covering morphisms.

[Hint. If $\widetilde{p}_j: \widetilde{X}_j \to X$, $j = 1, 2$ are two universal covering morphisms, then by the simple connectivity of X_1 the morphism $\widetilde{p}_1$ lifts across the covering $\widetilde{p}_2$ to a unique morphism of pointed spaces $\alpha: \widetilde{X}_1 \to X_2$. Show the existence of an inverse of α by reversing the roles of the two universal coverings.] □

Existence of Universal Covering Groups

Theorem A2.20. *Every Hausdorff topological group G which is connected and possesses a simply connected open identity neighborhood has a universal covering morphism $\widetilde{p}: \widetilde{G} \to G$.*

Proof. If S is a simply connected open identity neighborhood then gS is a simply connected neighborhood of g in G. Thus G is locally simply connected. (See the definition immediately preceding A2.12.) Hence the underlying pointed space of G has a universal covering by the Existence of Universal Coverings Theorem A2.14. By Proposition A2.18 above the assertion now follows. □

Corollary A2.21. *Every connected topological group with an identity neighborhood homeomorphic to an open ball in a Banach space has a universal covering group. In particular, every linear Lie group and indeed every topological group locally isomorphic to a linear Lie group has a universal covering group.* □

Example A2.22. (i) The quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ (which is also the exponential map of the n-torus) is a universal covering homomorphism.

(ii) The homomorphism $p: \mathbb{S}^3 \to \mathrm{SO}(3)$ of E1.2(ii) is a universal covering. □

Proposition A2.23. *Let $\widetilde{p}: \widetilde{G} \to G$ be a universal covering morphism of topological groups. Then* $\ker \widetilde{p}$ *is a discrete central subgroup of* $\widetilde{G}$ *and for each* $\widetilde{z} \in \ker \widetilde{p}$ *the* *function* $T_{\widetilde{z}}\colon \widetilde{G} \to \widetilde{G}$ *given by* $T_{\widetilde{z}}\widetilde{g} = \widetilde{z}\widetilde{g} = \widetilde{g}\widetilde{z}$ *is a covering transformation and the function* $z \mapsto T_z\colon \ker \widetilde{p} \to \Gamma(G)$ *is an isomorphism of groups. In particular, the Poincaré group of every topological group is abelian.*

Proof. Since $\ker \widetilde{p}$ is the fiber of a covering, it is a discrete normal subgroup. Hence it is central (cf. 6.13). Now $T_{\widetilde{z}}$ (well defined by centrality!) is a homeomorphism of $\widetilde{G}$ satisfying $\widetilde{p} \circ T_{\widetilde{z}} = \widetilde{p}$. Hence it is a member of $\Gamma(G)$. Clearly, $T_{\widetilde{z}}$ is the identity of $\widetilde{G}$ if and only if $\widetilde{z} = \widetilde{\mathbf{1}}$. Thus $\widetilde{z} \mapsto T_{\widetilde{z}}$ is injective. If $\gamma \in \Gamma(G)$, then $\widetilde{z} \stackrel{\text{def}}{=} \gamma(\mathbf{1})$ is in $\ker \widetilde{p}$. Then both γ and $T_{\widetilde{z}}$ are elements of $\Gamma(G)$ mapping $\widetilde{\mathbf{1}}$ to $\widetilde{z}$. Hence they agree because the Poincaré group operates *simply* transitively on the fibers. Thus $\widetilde{z} \mapsto T_{\widetilde{z}}$ is surjective. It is straightforward to verify that it is a morphism. The remainder is then clear. □

Groups Generated by Local Groups

This section deals with generating groups from local data in topological groups. Dealing with local topological groups is always messy. It is unfortunate that each author has a definition different from all other ones. The situation is a little better in the case of the idea of a local group *within a given group*. It is this situation we are dealing with here. In fact we shall consider a group G and a subset K supporting a topology τ_K satisfying the following conditions, to be augmented as we proceed:

(i) $\mathbf{1} \in K$.

(ii) $(\forall x, y \in K,\ V \in \tau_K) \quad xy \in V \Rightarrow (\exists U \in \tau_K)\ y \in U$ and $xU \subseteq V$.

(iii) The set $D \stackrel{\text{def}}{=} \{(x,y) \in K \times K \mid xy \in K\}$ is a neighborhood of $(\mathbf{1},\mathbf{1})$ in $K \times K$, and multiplication $(x,y) \mapsto xy : D \to K$ is continuous at $(\mathbf{1},\mathbf{1})$.

(iv) $K^{-1} = K$.

(v) Inversion $x \mapsto x^{-1}\colon K \to K$ is continuous at $\mathbf{1}$.

(vi) $(\forall y \in K,\ V \in \tau_K) \quad y \in V \Rightarrow (\exists U \in \tau_K)\ \mathbf{1} \in U$ and $Uy \subseteq V$.

We define a subset of the set of subsets of G as follows:

$$\tau_G = \{W \subseteq G \mid (\forall w \in W)(\exists U \in \tau_K)\ \mathbf{1} \in U \text{ and } wU \subseteq W\}.$$

It follows immediately from the definition that τ_G is a topology on G and that it is invariant under all left translations, i.e. that all left translations L_g, $L_g(x) = gx$ are τ_G-homeomorphisms. If we apply (ii) with $y = 1$ and consider the definition of τ_G we obtain at once that every $V \in \tau_K$ is a member of τ_G:

$$(\tau) \qquad \tau_K \subseteq \tau_G.$$

In particular, $K \in \tau_G$ (i.e. K is open in G) and $\tau_G|K = \tau_K$ (i.e. the topology of G induces the given one on K).

Lemma A2.24. *Assume that G is a group and that $K \subseteq G$ satisfies conditions* (i), …, (vi). *Then there is a unique maximal τ_G-open subgroup H of G such that $(H, \tau_G|H)$ is a topological group. In particular the connected component G_0 of* $\mathbf{1}$

in G is topological, and if K is connected, then G_0 is the subgroup $\langle K\rangle$ generated by K.

Proof. Since $\tau_K \subseteq \tau_G$ multiplication and inversion of G are continuous at $(\mathbf{1},\mathbf{1})$ and $\mathbf{1}$, respectively, by (iii) and (v).

As a first step we shall construct H. Let $\mathcal{U}$ denote the neighborhood filter of the identity in (G,τ_G). The group G acts on the set of all filters $\mathcal{F}$ on G via $(g,\mathcal{F}) \mapsto g\mathcal{F}g^{-1} = \{gFg^{-1} \mid F \in \mathcal{F}\}$. We set $H = \{g \in G \mid g\mathcal{U}g^{-1} = \mathcal{U}\}$, the stabilizer of $\mathcal{U}$ for this action. Then H is a subgroup. By (iii) there is an identity neighborhood $U \in \tau_G$ such that $UU \subseteq K$. By (iii) once more we find an identity neighborhood $V \in \tau_G$ such that $VV \subseteq U$ and by (v) we find a $W \in \tau_G \cap \mathcal{U}$ such that $W \cup W^{-1} \subseteq V$. As a consequence we have $WWW^{-1} \subseteq K$. Thus the function $y \mapsto yw^{-1}: W \to K$ is defined and by (vi) it is continuous at $\mathbf{1}$. Therefore, the function $x \mapsto wxw^{-1}: W \to K$ is defined for all $w \in W$ and is continuous at $\mathbf{1}$, since all left translations are continuous. As a consequence, $w\mathcal{U}w^{-1} = \mathcal{U}$. Thus $W \subseteq H$. Therefore H contains all hW, $h \in H$ and thus is open. By definition, all inner automorphisms I_h of H, $I_h(x) = hxh^{-1}$ are continuous. Then H is a group in which left translations and all inner automorphisms are continuous, multiplication is continuous at $(\mathbf{1},\mathbf{1})$ and inversion is continuous at $\mathbf{1}$. We claim that a group with these properties is topological: We note that the right translations $R_g = I_{g^{-1}}L_g$ are continuous and that inversion ι, $\iota(x) = x^{-1}$ is continuous at each g because ι is continuous at $\mathbf{1}$ and $\iota = R_{g^{-1}} \circ \iota \circ L_{g^{-1}}$. Finally, multiplication μ, $\mu(x,y) = xy$ is continuous at each (g,h) because μ is continuous at $(\mathbf{1},\mathbf{1})$ and $R_h \circ L_g \circ \mu \circ (L_{g^{-1}} \times R_{h^{-1}}) = \mu$.

As a second step we show that H is the largest open topological subgroup of G. Let A be a subgroup of G which is τ_G-open and is topological with respect to $\tau_G|A$. Since A is open, the neighborhood filter $\mathcal{U}_A$ of the identity in A generates $\mathcal{U}$. If $a \in A$, since $(A,\tau_G|A)$ is topological, $a\mathcal{U}_Aa^{-1} = \mathcal{U}_A$, and thus $a\mathcal{U}a^{-1} = \mathcal{U}$. Then $a \in H$ by the definition of H. Therefore $A \subseteq H$.

Thirdly we observe that $G_0 \subseteq H$. Since H is open, this will be shown if we prove that every open subgroup U of G is also closed and thus must contain the identity component. Now each left translations L_g of (G,τ_G) is continuous and thus, having the inverse $L_{g^{-1}}$, is a homeomorphism. Hence gU is open for all $g \in G$. Thus $U = G \setminus \bigcap_{g\notin U} gU$ is closed.

Finally assume that K is connected. Then K contains $\mathbf{1}$ by (i) and is connected as a subspace of (G,τ_G) since $\tau_G|K = \tau_K$. Hence $K \subseteq G_0$ and thus $\langle K\rangle \subseteq G_0$. From $\tau_K \subseteq \tau_G$ we know that K is open in (G,τ_G). Hence $K^n = \bigcup_{k_1,\dots,k_{n-1}\in K} k_1\cdots k_{n-1}K$ is open. Since $K^{-1} = K$ we have $\langle K\rangle = \bigcup_{n\in\mathbb{N}} K^n$ and so this group is open. Then $\langle K\rangle$ contains G_0 as we have seen in the previous paragraph. □

We summarize the essence of this discussion in the following theorem.

Groups Generated by Local Subgroups

Theorem A2.25. *Let K be a symmetric subset ($K = K^{-1}$) of a group G containing $\mathbf{1}$. Assume that K is a connected topological space such that*
(i) *$x, y, xy \in K$, with $xy \in V$ for an open subset V of K imply the existence of open neighborhoods U_x and U_y of x and y such that $xU_y \cup U_x y \subseteq V$,*
(ii) *$\{(x, y) \in K \times K \mid x, y, xy \in K\}$ is a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $K \times K$, and multiplication is continuous at $(\mathbf{1}, \mathbf{1})$,*
(iii) *inversion is continuous at $\mathbf{1}$.*
Then there is a unique topology on the subgroup $H = \langle K \rangle$ generated by K which induces on K the given topology and makes H a topological group such that K is an open identity neighborhood of H. □

With the information we provided on simple connectivity and with this result we can derive a powerful result on the extension of local morphisms.

Extending Local Homomorphisms

Corollary A2.26. *Let S be a simply connected topological group and G an arbitrary (not necessarily topological) group. Let U be an open connected symmetric identity neighborhood of S and $f: U \to G$ a function such that $x, y, xy \in U$ implies $f(x)f(y) = f(xy)$. Then there is a unique group homomorphism extending f.*

Proof. On the group $S \times G$ we consider $K \stackrel{\text{def}}{=} \{(x, f(x)) \mid x \in U\}$ and equip K with that topology which makes the bijection $x \mapsto (x, f(x)): U \to K$ a homeomorphism. It is easily seen that the hypotheses of the Generation Theorem A2.25 are satisfied, giving us on the subgroup $H = \langle K \rangle$ a unique topology making G a topological group such that the projection $\mathrm{pr}_S: S \times G \to S$ induces a covering morphism $\mathrm{pr}_S |H: H \to S$. Since S is simply connected, Proposition A2.9 implies that $\mathrm{pr}_S |H$ is a homeomorphism. If $\iota: H \to S \times G$ is the inclusion morphism, then

$$F \stackrel{\text{def}}{=} \mathrm{pr}_G \circ \iota \circ (\mathrm{pr}_S |H)^{-1}: S \to G$$

is the required morphism whose uniqueness with respect to extending f is secured by the fact that the connected group S is generated by the identity neighborhood U. (Cf. the proof of 5.4(iii).) □

Corollary A2.27. *If, under the conditions of Corollary* A2.26, *G is a topological group and f is continuous at* 1, *then f is a morphism of topological groups. If f is open, so is F.*

Proof. Since F extends f and f is defined on an open neighborhood of 1 in S, the group homomorphism F is continuous at 1 and is, therefore, continuous. The remainder is similar. □

Universality of the Universal Covering Group

Corollary A2.28. *Let G be a topological group with a universal covering morphism $\widetilde{p}\colon \widetilde{G} \to G$. Assume that H is a connected topological group and that U and V are open connected symmetric identity neighborhoods of G and H, respectively, such that there is a homeomorphism $\varphi\colon U \to V$ satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all x, y, $xy \in U$. Then there exists an open symmetric identity neighborhood $\widetilde{W}$ of $\widetilde{G}$ and a unique covering morphism $\widetilde{q}\colon \widetilde{G} \to H$ such that*

(i) $\widetilde{p}(\widetilde{W}) \subseteq U$,

(ii) $\widetilde{q}(w) = \varphi\big(\widetilde{p}(w)\big)$ *for all* $w \in \widetilde{W}$.

Proof. By the continuity of the covering morphism $\widetilde{p}$ we find an open symmetric identity neighborhood $\widetilde{W}$ of $\widetilde{G}$ such that $\widetilde{p}(\widetilde{W}) \subseteq U$. Define $f\colon \widetilde{W} \to H$ by $f(w) = \varphi\big(\widetilde{p}(w)\big)$ for all $w \in \widetilde{W}$. Then $f(ww') = f(w)f(w')$ for all w, w', $ww' \in \widetilde{W}$. Since $\widetilde{G}$ is simply connected, the Extending Local Homomorphisms Corollary A2.26 applies and furnishes a unique extension $\widetilde{q}\colon \widetilde{G} \to H$ of f. Since $\widetilde{p}$ is a local homeomorphism at $\widetilde{1}$ and φ is a homeomorphism, the map $\widetilde{q}$ is a local homeomorphism at $\mathbf{1}$. For a morphism of topological groups implies that its kernel $\ker \widetilde{q}$ is discrete and that it is open. In particular, $\widetilde{q}(\widetilde{G})$ is an open, hence closed subgroup of H. (Cf. the third step of the proof of A2.24.) Since H is closed this shows that $\widetilde{q}$ is surjective. Hence it is a covering morphism by A2.3(ii). □

Remark A2.29. If the topological group G has a universal covering group $\widetilde{G}$ then $\widetilde{G}$ classifies the connected topological groups which are locally isomorphic to G. Each of them is a quotient group of $\widetilde{G}$ modulo a discrete normal (hence central) subgroup. □

Example A2.30. The connected topological groups locally isomorphic to SO(3) are isomorphic to either SO(3) or $\mathbb{S}^3 \cong \mathrm{SU}(2)$.

Proof. The morphism $\mathbb{S}^3 \to \mathrm{SO}(3)$ of E1.2(ii) is a universal covering homomorphism (by A2.11 (ii)) whose kernel $\mathbb{S}^0 = \{1, -1\}$ is the full center of $\mathbb{S}^3$. The assertion the follows. □

Corollary A2.31 (Generating Subgroups of Topological Groups). *Let G be a topological group and K a symmetric connected subspace containing the identity such that the following condition is satisfied:*

(∗) $\{(x,y) \in K \times K : xy \in K\}$ *is a neighborhood of* $(\mathbf{1},\mathbf{1})$ *in* $K \times K$, *and* x, y, $xy \in K$, *with* $xy \in V$ *for an open subset* V *of* K *imply the existence of open neighborhoods* U_x *and* U_y *of* x *and* y *such that* $xU_y \cup U_x y \subseteq V$.

Then there is a topological group H and an injective morphism of topological groups $f\colon H \to G$ such that for some open symmetric identity neighborhood V of H we have

(i) $H = \langle V \rangle$,

(ii) $f(V) = K$ *and* $f|V\colon V \to K$ *is a homeomorphism.*

Hypothesis $(*)$ *is satisfied if there is an open symmetric identity neighborhood* U *in* G *and* K *is a connected symmetric subset of* U *containing the identity such that*

$$KK \cap U \subseteq K.$$

Proof. The hypotheses of A2.25 are quite clearly satisfied. Hence the subgroup $\langle K\rangle$ has a unique topology τ making it into a topological group H and inducing on K the same topology as does that of G such that $V \stackrel{\text{def}}{=} (K, \tau|K)$ is open and generates H. The inclusion map $f: H \to G$ then satisfies the requirements.

The last claim of the corollary is straightforward. □

This is in fact the topological group version of what we do in the Recovery of Subalgebras 5.52, whose proof we base directly on Theorem A2.25. The connectivity of K is not absolutely necessary for the mechanism of this subsection to work; in the absence of this hypothesis the proof of A2.24 shows that there is an open symmetric identity neighborhood W of K such that $\langle W\rangle$ has a group topology inducing on W the given one. Thus W will then play the role of K.

Proposition A2.32 (Lifting Homomorphisms). *Let* $p: E \to B$ *be a covering morphism of topological groups and suppose that* $f: S \to B$ *is a morphism of topological groups from a simply connected group* S *to* B*. Then the unique lifting* $\widetilde{f}: S \to E$ *according to Definition* 2.6 *is a morphism of topological groups.*

Proof. Let $g \in S$ and set $\varphi: S \to B$, $\varphi(x) = f(gx)$. Then $\widetilde{\varphi}: S \to E$, $\widetilde{\varphi}(x) = \widetilde{f}(gx)$ and $\widetilde{\psi}: S \to E$, $\widetilde{\psi}(x) = \widetilde{f}(g)\widetilde{f}(x)$ satisfy $q\big(\widetilde{\varphi}(x)\big) = q\big(\widetilde{f}(gx)\big) = f(gx) = f(g)f(x) = q\big(\widetilde{f}(g)\big)q\big(\widetilde{f}(x)\big) = q\big(\widetilde{f}(g)\widetilde{f}(x)\big) = q\big(\widetilde{\psi}(x)\big)$ and $\widetilde{\varphi}(1) = \widetilde{f}(g) = \widetilde{\psi}(1)$. Then $\widetilde{\varphi} = \widetilde{\psi}$ by A2.5. Since g is arbitrary, $\widetilde{f}(gh) = \widetilde{f}(g)\widetilde{f}(h)$ for all g, $h \in S$ follows. □

Postscript

Our approach to covering spaces and to simple connectivity through the Poincaré group and covering groups is inspired by Chevalley's presentation which appeared in the first modern book on Lie group theory in 1946 [58]. We develop this approach further and formulate it in an even more operational fashion; more specifically, our definition of simple connectedness of a space is immediately ready for application in the context of topological groups. The concept of a covering independent of special properties of the base space is modelled after fiber space theory and was used in this context by Tits in the lecture notes which circulated in the early sixties and which much later appeared in book form [354]. The emphasis on pullbacks is the fruit of category theory and of fiber space oriented thinking. The proof of the existence of universal covering spaces is a bit more general than it appears in most other sources.

Chevalley's proof of the Extension Theorem of Local Homomorphisms (Corollary A2.26) was rather complicated, whereas Bourbaki presents a comparatively direct proof ([41], Chap. 3, §6, n^o 2, Lemme 1; see also Corollary A2.26 above). Our blending of covering theory with the local theory of topological groups is from the article by Hofmann [166], see also [165]. Among other things it yields a smooth and lucid proof of the Extension Theorem of Local Homomorphisms. (For other fruits of this approach see e.g. the proof of 5.52.)

References for this Appendix—Additional Reading

[41], [58], [118], [165], [166], [338], [354].

Appendix 3

A Primer of Category Theory

Categorical and functorial thinking are an undercurrent in most of this book. Therefore, in this appendix we provide an introduction to those features of category theory which are essential for a better understanding of the structure theory of compact groups. Prerequisites in the usual sense are not required—basic material such as would be presented in courses such as *Linear Algebra* and *Introduction to Algebra* are presupposed.

A certain maturity in abstract reasoning will help. In illustrating the working of category theory we shall cite many examples of mathematical subspecialties not all of which may be familiar to every reader. This should not deter the reader in this appendix; nothing will be lost if unfamiliar examples are simply skipped. While this appendix is self-contained as far as the presentation of category theory is concerned, the discussion of examples will frequently draw on knowledge stemming from other areas. In particular, the last section on commutative monoidal categories and graded commutative Hopf algebras does require experience and facility with multilinear algebra over a field.

Categories, Morphisms

The first task is to motivate categories, the second to define them formally. The basic philosophy of category theory is to observe the functions and transformations which are at large in any given field of mathematical investigation and to focus on these more sharply than on the objects of the field themselves. From this viewpoint, in *set theory*, functions look more important than sets, in *linear algebra*, linear maps are more important than vector spaces, and matrices more important than n-tuple spaces.

The ancestor of all categories is the category of all sets. We perceive of the sets themselves as *objects* and of the functions between them as the relevant transformations or *morphisms*. The elements of the sets are, for the moment, totally ignored. We observe that this collection of objects is not a set, but rather a proper class (cf. e.g. [230, 305]). Appropriate care needs to be taken in dealing with proper classes.

Guided by the example of sets and functions we attempt a definition.

Definition A3.1. A *category* $\mathcal{C}$ is a collection of data satisfying conditions which are listed in the following.

Data:

1) A class of objects $\mathrm{ob}(\mathcal{C})$.
2) For each ordered pair (X, Y) of objects, a set $\mathcal{C}(X, Y)$ of *morphisms* or *arrows from X to Y*. The statement $f \in \mathcal{C}(X, Y)$ is equivalently written $f\colon X \to Y$ or $X \xrightarrow{f} Y$, if the category $\mathcal{C}$ is understood.
3) For each object X a morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$.
4) For each ordered triple (X, Y, Z) of objects a function

$$(f, g) \mapsto f \circ g\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

called *composition*. (This function is the empty function if one or both of $\mathcal{C}(X, Y)$ and $\mathcal{C}(Y, Z)$ are empty.)

Properties:

I) For each morphism $f\colon X \to Y$ we have $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$.

II) For each triple (f, g, h) of morphisms $h\colon X \to Y$, $g\colon Y \to Z$, $f\colon Z \to A$ the equation $(f \circ g) \circ h = f \circ (g \circ h)$ holds. □

The sets $\mathcal{C}(X, Y)$ are called *hom-sets*. The class $\bigcup\{\mathcal{C}(X, Y) \mid X, Y \in \mathrm{ob}(\mathcal{C})\}$ is called the *class of morphisms*, written $\mathrm{morph}(\mathcal{C})$. Sometimes it is also postulated that $(X, Y) \neq (X', Y')$ implies $\mathcal{C}(X, Y) \cap \mathcal{C}(X', Y') = \emptyset$. In most categories arising in nature this is automatically satisfied; a category not satisfying this condition is easily converted into one satisfying it by simple set-theoretical gimmicks.

Proposition A3.2. *In any category $\mathcal{C}$, all sets $\mathcal{C}(X, X)$, $X \in \mathrm{ob}(\mathcal{C})$, are monoids with respect to composition with* id_X *as identity.*

Proof. By Definition A3.1 3) and 4), the composition of any pair of elements in $\mathcal{C}(X, X)$ is secured, and by Definition A3.1 II) composition is associative. Definition A3.1 I) says that id_X is an identity. □

One might say that a monoid and a one-object category are one and the same thing; for, given a monoid M, we set $\mathrm{ob}\,\widetilde{M} = \{1\}$, $\mathrm{morph}\,\widetilde{M} = M$, and, for $f, g \in M$ also $f \circ g = fg$ with the multiplication in M. In this fashion we have defined a one-object category $\widetilde{M}$.

The elements of $\mathcal{C}(X, X)$ are called the *endomorphisms of X*.

If $f\colon X \to Y$ is a morphism in a category then it is called an *isomorphism* if there is a $g\colon Y \to X$ with $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$; i.e. if f is invertible. Two objects X and Y are called *isomorphic* if there is an isomorphism $f\colon X \to Y$. A morphism which is simultaneously an endomorphism and an isomorphism is called an *automorphism*. A category $\mathcal{G}$ such that $\mathrm{morph}\,\mathcal{G}$ consists of isomorphisms only is called a *groupoid*. A one-element groupoid is one and the same thing as a group in the same way as a one-element category is the same thing as a monoid.

It may be considered as clear when a subclass $\mathcal{C}' \subseteq \mathcal{C}$ endowed with a subset $\mathcal{C}'(X', Y') \subset \mathcal{C}(X', Y')$ for each pair $(X', Y') \in \mathcal{C}' \times \mathcal{C}'$ is called a *subcategory* of

$\mathcal{C}$. We shall call a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ *full*, if $\mathcal{C}'(X', Y') = \mathcal{C}(X', Y')$ for each pair $(X', Y') \in \mathcal{C}' \times \mathcal{C}'$. (See also Exercise EA3.10 below.)

Now we illustrate the definition with a long list of examples. The verification of the conditions of Definition A3.1 are usually straightforward; in the case they are not, we pose an exercise.

Examples A3.3. The following are examples of categories:

1. The category $\mathbb{S}$ of sets

Objects: sets X,Y etc.
Morphisms: functions $f: X \to Y$. Usual notation: $\mathbb{S}(X, Y) = Y^X$.
Isomorphisms: bijections.
Automorphisms: permutations.
Endomorphisms: $\mathbb{S}(X, X) = X^X$.

A small but significant modification is sometimes relevant: the *category* $\mathbb{S}_0$ *of pointed sets*:

Objects: pairs (X, x_0) consisting of a set X and a distinguished point $x_0 \in X$.
Morphisms: $f: (X, x_0) \to (Y, y_0)$; i.e. functions $f: X \to Y$ with $f(x_0) = y_0$.

2. The category $\mathbb{V}_K$ of *K*-vector spaces

Let K be any field.

Objects: K-vector spaces V, W etc.
Morphisms: linear maps $L: V \to W$. Usual notation: $\mathbb{V}_K(V, W) = \mathrm{Hom}(V, W)$
Endomorphisms: $\mathbb{V}_K(V, V) = \mathrm{End}(V)$.
Automorphisms: $\mathrm{Aut}(V) = \mathrm{Gl}(V)$ invertible linear self-maps of V (if $\dim V < \infty$: endomorphisms with nonzero determinant)

3. The category $\mathbb{AB}$ of abelian groups

Objects: abelian groups (preferably written additively) A, B etc.
Morphisms: homomorphisms $f: A \to B$. Usual notation: $\mathbb{AB}(A, B) = \mathrm{Hom}(A, B)$.
Endomorphisms: $\mathbb{AB}(A, A) = \mathrm{End}(A)$.

The category of (not necessarily abelian) groups is defined similarly.

Common generalization: the category $\mathbb{AB}_R$ of *R*-left-modules

Objects: R-left-modules M, N, etc. for a fixed ring R with identity (commutative or not).
Morphisms: R-module homomorphisms $f: M \to N$. Usual notation: $\mathbb{AB}_R(M, N) = \mathrm{Hom}_R(M, N)$.

Clearly, Examples 2 and 3 are subsumed under this idea with $\mathbb{V}_K = \mathbb{AB}_K$ and $\mathbb{AB} = \mathbb{AB}_{\mathbb{Z}}$.

Next to the category $\mathbb{S}$ of sets, the categories $\mathbb{AB}_R$ are ancestors of category theory, too. Since major parts of this book deal with abelian groups, the category $\mathbb{AB} = \mathbb{AB}_{\mathbb{Z}}$ is of particular relevance for us; Appendix 1 is a self-contained introduction to it.

4. The category $\mathbb{M}_K$ of *K*-matrices

Objects: $\mathrm{ob}(\mathbb{M}_K) = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
Morphisms: $\mathbb{M}_K(n, m) = M_{mn}(K) =$ set of all $m \times n$-matrices over K if $m, n \geq 1$. $\mathbb{M}_K(0, m) = \{0_{m0}\}$, $\mathbb{M}_K(n, 0) = \{0_{0n}\}$.
Automorphisms: $\mathrm{Gl}(n, K)$ invertible matrices, matrices with nonzero determinant.
Composition: matrix multiplication and the rules $0_{0m}A = 0_{0n}$ and $A0_{n0} = 0_{m0}$ for $A \in \mathbb{M}_K(n, m)$. For $n \in N_0$ we have $\mathrm{id}_n = E_n$, the $n \times n$-identity matrix. Usual notation: $\mathbb{M}_K(n, m) = M_{mn}(K)$.
Endomorphisms: $\mathbb{M}_K(n, n) = M_{nn}(K) =$ set of $n \times n$-matrices.

Note that objects play a subordinate role and that morphisms are not functions. Further note that the class of objects as well as the class of morphisms are sets. Categories in which the class of objects is a set are called *small.* However, we also observe that there is a bijection of $\mathrm{ob}(\mathbb{M}_K)$ onto $\big\{\{0\} = K^0, K^1, K^2, \ldots\big\}$ given by $n \mapsto K^n$ and also a bijection $\mathrm{morph}(\mathbb{M}_K)$ onto $\bigcup_{m,n\in\mathbb{N}_0} \mathrm{Hom}_K(K^n, K^m)$ sending an $m \times n$-matrix $A = (a_{jk})_{\substack{j=1,\ldots,m \\ k=1,\ldots,n}}$ to the unique linear map $K^n \to K^m$ given by $(x_1, \ldots, x_n) \mapsto (\sum_{k=1}^n a_{1k}x_k, \ldots, \sum_{k=1}^n a_{mk}x_k)$ and every 0_{0m} and 0_{n0} to the respective constant morphism.

5. Partially ordered sets as categories

Let $(X, \leq)$ denote a set with a reflexive, transitive and antisymmetric relation.

Objects: $x\,, x \in X$.
Morphisms: if $x \not\leq y$ then $\hom(x, y) = \emptyset$; if $x \leq y$ then $\mathrm{card}\big(\hom(x, y)\big) = 1$. Thus the existence of an arrow $x \to y$ is equivalent to $x \leq y$, and this arrow is unique.
Law of composition: if $x \to y$ and $y \to z$ are given, then $x \leq y$ and $y \leq z$; thus transitivity yields $x \leq z$; i.e. there is a unique arrow $x \to z$ which we define to be $(y \to z) \circ (x \to y)$. Associativity of composition, where defined, is an immediate consequence of the uniqueness of arrows.
Existence of the identities: if x is given, then reflexivity gives $x \leq x$ and thus a unique arrow $x \to x$ with the required properties.

This example, just as the preceding one, shows that categories need not be made up of structured sets with structure preserving functions as morphisms.

6. Topological spaces $\mathbb{TOP}$

Objects: topological spaces X, Y etc.
Morphisms: continuous functions $f\colon X \to Y$. Usual notation for the set of morphisms: $\mathbb{TOP}(X, Y) = C(X, Y)$.
Isomorphisms: homeomorphisms.

An important variation of this theme is the category of pointed spaces $\mathbb{TOP}_0$:

Objects: pointed spaces (X, x_0) where X is a space and $x_0 \in X$ a base point.
Morphisms: base point preserving continuous functions.

7. The homotopy category $[\mathbb{TOP}]$

Objects: topological spaces; i.e. $\mathrm{ob}([\mathbb{TOP}]) = \mathrm{ob}(\mathbb{TOP})$.
Morphisms: we say that two functions f, $g: X \to Y$ between topological spaces are *homotopic*, written $f \sim g$, if there is a family $\{F_t \mid F_t: X \to Y, \quad t \in [a, b]\}$, $a \leq b$ in $\mathbb{R}$ of continuous functions such that $F_a = f$, $F_b = g$ and that $(x, t) \mapsto F_t(x): X \times [a, b] \to Y$ is continuous.

Exercise EA3.1. Homotopy is an equivalence relation on $\mathrm{morph}(\mathbb{TOP})$. It is a congruence for composition, i.e. $f_1 \sim f_2$ implies $f_1 \circ e \sim f_2 \circ e$ and $g \circ f_1 \sim g \circ f_2$, assuming the compositions are defined. □

The equivalence class of f is denoted $[f]$. For $X \overset{\alpha}{\to} Y \overset{f}{\to} Z$ we therefore define unambiguously $[f] * [\alpha] = [f \circ \alpha]$. We set $\mathrm{id}_X = [\mathrm{id}_X]$. The required properties for the identity and composition are clear. The usual notation for $[\mathbb{TOP}](X, Y)$ is $[X, Y]$.

Isomorphisms: homotopy equivalences.

Variation of the same theme: the pointed homotopy category $[\mathbb{TOP}_0]$.

Two maps $f, g: (X, x_0) \to (Y, y_0)$ of pointed spaces are *homotopic* if $f, g: X \to Y$ are homotopic via F_t such that $F_t(x_0) = y_0$ through the homotopy. Clearly, $[\mathbb{TOP}_0]\big((X, x_0), (Y, y_0)\big) \subseteq [\mathbb{TOP}](X, Y)$, and both sets are equal if in both X and Y, given two points, there is a continuous self map mapping one to the other and being homotopic to the identity. Special cases lead to a particular notation:

$$[(\mathbb{S}^n, \mathrm{southpole}), (X, x_0)] = \pi_n(X, x_0).$$

If X is arcwise connected, $\pi_n(X, x_0) = [\mathbb{S}^n, X]$, and one writes $\pi_n(X)$.

We notice that the homotopy category illustrates the existence of categories whose objects are structured sets (here: topological spaces), but whose morphisms are not functions.

8. The path groupoid of a space

Let X be a topological space.

Objects: x, $x \in X$.
Morphisms: two curves $\gamma_j: [a_j, b_j] \to X$, $j = 1, 2$ from $x = \gamma_1(a_1) = \gamma_2(a_2)$ to $y = \gamma_1(b_1) = \gamma_2(b_2)$ are equivalent if there is a monotonically increasing homeomorphism $f: [a_1, b_1] \to [a_2, b_2]$ with $\gamma_1 = \gamma_2 \circ f$. This indeed defines an equivalence relation. The equivalence class $\Gamma = [\gamma]$ of a curve from x to y is called a *path* from x to y, written $\Gamma: x \to y$. Let $\Omega: y \to z$ be a path from y to z. If $\Gamma = [\gamma]$ and $\Omega = [\omega]$, then one can define a concatenation of curves $\omega \# \gamma : [0, 1] \to X$ from $x \to z$ and define $\Omega + \Gamma : x \to z$ by $[\omega \# \gamma]$. The resulting category $\Pi(X)$ is called the *path category of* X.

One defines, without difficulty, the concept of *homotopy of paths*. Accordingly, one obtains the category of homotopy classes of paths on the set of points of X as objects. It is, in fact, a groupoid (cf. [54]) called the *path groupoid* $[\pi(X)]$ of X.

Endomorphisms if X is arcwise connected: $[\Pi(X)](x,x) = \pi_1(X)$, the fundamental group of X.

Exercise A3.2. Fill in the missing details of the example of the path groupoid.□

9. The category of posets

Objects: posets $(X, \leq)$; short notation X.
Morphisms: order preserving maps $f: X \to Y$. Usual notation $[X \to Y]$.

10. The category of metric spaces and contractions

Objects: metric spaces (X, d_X), short notation X.
Morphisms: contractions $f: X \to Y$; i.e. functions satisfying

$$d_Y\big(f(x_1), f(x_2)\big) \leq d_X(x_1, x_2).$$

Isomorphisms: isometries.

11. The category $\mathbb{BAN}$ of Banach spaces

Objects: Banach spaces E.
Morphisms: bounded linear maps $T: E \to F$. The usual notation is $\mathbb{BAN}(E,F) = L(E,F)$.
Isomorphisms: invertible bounded operators; by the Open Mapping Theorem, (see [94]) every bijective bounded operator is an isomorphism.
Endomorphisms: $L(E)$, the set of bounded linear operators on E.

An important variation of this theme: the category $\mathbb{BAN}_1$ of Banach spaces with linear contractions:

Objects: Banach spaces E (as before).
Morphisms: linear contractions $T: E \to F$; i.e. linear maps with $\|Tx\| \leq \|x\|$. The sets $\mathbb{BAN}_1(E,F) = L_1(E,F)$ are the unit balls of the Banach spaces $L(E,F)$.
Isomorphisms: linear isometries.
Automorphisms in the case that E is a Hilbert space: unitary operators.
Endomorphisms: $L_1(E)$, the unit ball of $L(E)$.

12. The category attached to a variety (equational class)

Objects: algebras
$f_j: X \underbrace{\times \cdots \times}_{n_j \text{ times}} X \to X$, $j \in J$ with n_j-ary operations $n_j \in \mathbb{N}_0$, satisfying equations.
(Example: Monoids; $J = \{1, 2\}$, $f_1: G^0 = * \to G$, $f(*) = e$ (nullary operation), $f_2: G \times G \to G$, $f_2(g,h) = gh$ multiplication. Equations: $eg = ge = g$, $g(hk) = (gh)k$.)
Morphisms: functions $\varphi: X \to Y$ with

$$f_j^Y\big(\varphi(x_1), \ldots, \varphi(x_{n_j})\big) = \varphi\big(f_j^X(x_1, \ldots, x_{n_j})\big)$$

for all $j \in J$. □

We shall use the following definition in an informal way:

Definition A3.4. A category $\mathcal{C}$ will be called *set-based*, if the objects in $\text{ob}\,\mathcal{C}$ are sets with additional structure and the morphisms in $\text{morph}\,\mathcal{C}$ are functions preserving structure. □

One should be aware that the precision of this definition leaves much to be desired since we appeal to consensus when we speak of "additional structure." The examples, however, illustrate what is meant. Clearly such categories as groups, abelian groups, modules, topological spaces are set-based. Our formulation is not invariant under "isomorphisms" of categories (which we have not defined yet anyhow). The category $\mathbb{M}_K$ of matrices over K (see Example A3.3 (4)) is *not* set-based; the category of all vector spaces K^m, $m \in \mathbb{N}_0$ with all linear maps between them is set-based; these two categories would have to be called "isomorphic" under any reasonable concept of isomorphy of categories. The homotopy category (Example A3.3 (7)) is not set-based due to the nature of its morphisms.

These informal discussions perhaps motivate an equivalent definition of the concept of a category which does not appeal very closely to our intuition of set-based categories.

Let us consider a category $\mathcal{C}$. Let us call a morphism $e \in \text{morph}\,\mathcal{C}$ *an identity* if it is a member of $\mathcal{C}(X,X)$ for some object and if $e = \text{id}_X$. This is tantamount to saying that $e \circ e$ is defined and that $f \circ e = f$ for each $f \in \text{morph}\,\mathcal{C}$ for which $f \circ e$ is defined, and, likewise, that $e \circ g = g$ for each $g \in \text{morph}\,\mathcal{C}$ for which $e \circ g$ is defined. Thus we have two functions

$$D, R\colon \text{morph}\,\mathcal{C} \to \text{morph}\,\mathcal{C}$$

called *domain* and *range projections* such that for any morphism $f\colon X \to Y$, the morphism Df is the identity id_X of the *domain* X of f and Rg is the identity id_Y of the *range* Y of f. As is usual we write $D^2 f = D(Df)$ and so on and quickly observe

$$D^2 = D, \quad R^2 = R, \quad DR = R, \quad RD = D, \tag{1}$$

that is, as function under composition, the set $\{D, R\}$ is a *right zero semigroup* of two elements. We write $C = \text{morph}\,\mathcal{C}$ and define

$$C \times_{DR} C = \{(f,g) \in C \times C \mid Df = Rg\}.$$

Then we have a composition:

$$(f,g) \mapsto fg\colon C \times_{DR} C \to C \tag{2}$$

with $fg = f \circ g$ such that

$$\begin{aligned} &(Df, Df), (Rf, Rf), (f, Df), (Rf, f) \in C \times_{DR} C \quad \text{and} \\ &(Df)^2 = Df,\ (Rf)^2 = Rf,\ f(Df) = f,\ (Rf)f = f. \end{aligned} \tag{3}$$

Clearly, a further consequence of Definition A3.1 is

$$\begin{aligned} &(fg, h), (f, gh) \in C \times_{DR} C \text{ and } (fg)h = f(gh) \quad \text{whenever} \\ &(f,g), (g,h) \in C \times_{DR} C. \end{aligned} \tag{4}$$

Also we record:

$$(5) \qquad (\forall f \in C) \quad \{g \in C \mid Dg = Df \text{ and } Rg = Rf\} \text{ is a set.}$$

Proposition A3.5. *If $\mathcal{C}$ is a category, then for $C = \operatorname{morph}\mathcal{C}$ there are two functions $D, R: C \to C$ satisfying* $(1), \ldots, (5)$.

Conversely, assume C is a class with two self-functions D and R satisfying $(1), \ldots, (5)$. *Then $E = \operatorname{im} D = \operatorname{im} R$ is a well-defined class, and $\mathcal{C} = (E, \mathcal{C}(\bullet, \bullet), \operatorname{id}, \circ)$ with $\mathcal{C}(e_1, e_2) = \{f \in C \mid Df = e_1 \text{ and } Rf = e_2\}$ for all $(e_1, e_2) \in E^2$, $\operatorname{id}_e = e$ for $e \in E$, and $f \circ g = fg$ for all $(f, g) = C \times_{DR} C$ is a category with $\operatorname{ob}\mathcal{C} = E$, $\operatorname{morph}\mathcal{C} = C$.*

Proof. Exercise EA3.4. □

Exercise EA3.4. Prove Proposition A3.5. □

The preceding proposition shows that there is a definition of categories which is completely equivalent to Definition A3.1, and which is completely based on the concept of morphism. This definition, apart from set-theoretical concerns, is in the spirit of an algebraic structure: A category is a universal partial algebra with two unary operations and one partial binary operation. The issue whether or not a category is set-based does not arise here. A very good example fitting this second formal definition is the category $\mathbb{M}_K$ of K-matrices where the objects were an artificial decoration to begin with.

It is instructive to realize a visualization of the new definition of a category. Let us consider a square $P = M \times M$ of a set M. If we identify the diagonal $E = \{(x, x) \mid x \in M\}$ with M under $x \mapsto (x, x)$, then we can identify the two projections $d', r': P \to M$, $d'(x, y) = x$, $r'(x, y) = y$ with two retractions d, $r: M \times M \to E$, $d(x, y) = (x, x)$, $r(x, y) = (y, y)$. Now picture a subset $C \subseteq P$ containing E and let $D = d|C$, $R = r|C$, and write $C \times_{DR} C = \{(f, g) \mid f = (y, z), g = (x, y), f, g \in C\}$ and set $fg = (x, z)$. Then C, D, and R satisfy conditions (1)–(5). Now imagine for each pair $(x, y) \in C$ a whole set $C_{(x,y)}$ and replace C by the union of these sets, assuming $(x, x) \in C_{(x,x)}$. For $f \in C_{(x,y)}$ write $Df = (x, x)$ and $Rf = (y, y)$. Now we have the raw material to visualize a C endowed with the structure of a category. An instructive case is obtained by taking any subset $N \subseteq \mathbb{R} \times \mathbb{R}$ and to try $(\mathbb{R}^2 \setminus N) \times \mathbb{R}$ with a composition obtained by restricting the multiplication $(y, z, f)(x, y, g) = (x, z, f + g)$.

We note that once the concept of a category C expressed through the data D,R, and the associative partial multiplication $m: C \times_{DR} C \to C$ is understood, it would be no problem at all to ask, for instance, that C be a topological space (or, for that matter, a smooth manifold) such that D, R, and m are continuous (respectively, smooth). In this sense this definition opens up the possibility to speak of *topological categories*, or *smooth categories* and so on. The latter case is of considerable recent interest in the case of groupoids [244].

The more abstract formulation of a category in A3.5 illustrates very well the fact that, sometimes, it is not clear how one would recover the particular nature of an object from the data of a category. Sometimes it is possible to recover at least a portion of this structure, e.g. the underlying set: In the category $\mathbb{S}$ of sets, the hom-set $\mathbb{S}(\{*\}, X) = X^{\{*\}}$ is in natural bijective correspondence with X. In the category $\mathbb{G}$ of groups, the hom-set $\mathbb{G}(\mathbb{Z}, G) = \mathrm{Hom}(\mathbb{Z}, G)$ is in bijective correspondence with the underlying set of G, but that, certainly is all that can be recovered. In the case of $\mathbb{AB}$, in this regard, the situation is much better, but that is the precise topic of Appendix 1.

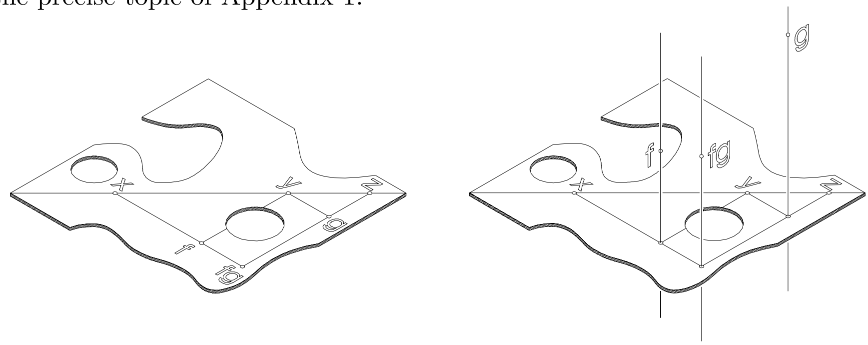


Figure A3.1: Image of a category

Pointed Categories

As an exercise and because of their practical significance, let us consider the following concepts:

Definition A3.6. If $\mathcal{C}$ is a category then an object N is called
(i) *initial* if $\mathcal{C}(N, X)$ has exactly one element 0_{NX} for all $X \in \mathrm{ob}\, X$,
(ii) *terminal* if $\mathcal{C}(X, N)$ has exactly one element 0_{XN} for all $X \in \mathrm{ob}\, X$,
(iii) *null* or *a null object* if it is both initial and terminal. A category $\mathcal{C}$ is called *pointed* if it has null objects. □

If $\mathcal{C}$ is a pointed category and N a null object, then each hom-set $\mathcal{C}(X, Y)$ contains a unique morphism $0 = 0_{XY}\colon X \to Y$ given by $0_{XY} = 0_{NY} \circ 0_{XN}$.

0_{XY} does not depend on N. Further $0f = f0 = 0$ for all morphisms f and the appropriate 0-morphisms.

There is no great harm in denoting the unique morphisms in the sets $\mathcal{C}(N, X)$, respectively, $\mathcal{C}(X, N)$, by 0.

Exercise EA3.5. Verify the assertions made on pointed categories. □

Typically $\mathbb{AB}_R$ is pointed with any one-element module (vector space, abelian group) as null objects. The categories $\mathbb{S}_0$ and $\mathbb{TOP}_0$ of pointed sets and pointed spaces are pointed categories with singletons as null objects.

Exercise EA3.6.

(i) Does $\mathbb{S}$, the category of sets, have initial objects; does it have terminal objects? Does it have null objects?

(ii) If $(X, \leq)$ is a quasiordered set, what (if any) are the initial (terminal) objects? Are there null objects? □

Types of Morphisms

We know from all the set-based categories with which we are familiar that the concept of an isomorphism is crucial. This is an entirely categorical idea.

In the category of sets it is important that we can speak of injective and surjective functions. First we recall, for completeness, the concept of an isomorphism:

Definition A3.7. (a) If $f: X \to Y$ is a morphism in a category then it is called an *isomorphism* if there is a $g: Y \to X$ with $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$; i.e. if f is invertible. Two objects X and Y are called *isomorphic* if there is an isomorphism $f: X \to Y$. A morphism which is simultaneously an endomorphism and an isomorphism is called an *automorphism.*

(b) A morphism $f: X \to Y$ is called a *retraction* (respectively, *coretraction*) if there is a morphism $g: Y \to X$ such that $fg = \mathrm{id}_Y$ (respectively, $gf = \mathrm{id}_X$).

(c) An endomorphism $p: X \to X$ is a *projection* if $p^2 = p$. □

Obviously, the ideas of retractions and coretractions arise from "splitting" the definition of an isomorphism. Indeed f *is an isomorphism if and only if it is both a retraction and a coretraction.*

Compositions of isomorphisms are isomorphisms. If f is an isomorphism, then its *inverse* f^{-1} is uniquely determined. If $f: X \to Y$ is a retraction with $fg = \mathrm{id}_Y$, then $p = gf$ is a projection of X.

Exercise EA3.7. Prove these assertions. □

Examples A3.8. (1) Let $f: A \to B_1$ denote a morphism of R-modules. Then f is a retraction if and only if there is a submodule B of A such that $A = B \oplus \ker f$ and $B \cong B_1$, if and only if there is a submodule B of A such that $f|B: B \to B_1$ is an isomorphism. If R is a field K (i.e. in the category $\mathbb{V}_K$ of K-vector spaces) every surjective morphism is a retraction. The quotient morphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \neq \pm 1$ is never a retraction in $\mathbb{AB}$.

(2) A morphism $f: G \to H_1$ of groups (topological groups) is a retraction if and only if there is a closed subgroup H such that with $N = \ker f$ the map $(n, h) \mapsto nh: N \rtimes_\alpha H \to G$ is an isomorphism of (topological) groups. That is, G is a semidirect product of the kernel of f and a complementary subgroup. (The multiplication on the semidirect product $N \rtimes_\alpha H$ is given by $(n, h)(n', h') = \big(n\big(\alpha(h)(n')\big), hh'\big)$ with $\alpha(h)(n) = hnh^{-1}$; also, $\alpha: H \to \mathrm{Aut}(N)$ is a morphism of groups. In the case of topological groups, the definition of a group topology on

Aut(N) is somewhat delicate, but this does not affect the inherent simplicity of the definition of the semidirect product; after all, this information on α is just an additional piece of information.)

(3) If $f: X \to Y_1$ is a continuous function between topological spaces, then f is a retraction if and only if there is a continuous self-map $p: X \to X$ with $p^2 = p$; i.e. a projection, with image Y such that $f|Y: Y \to Y_1$ is a homeomorphism and $f = (f|Y) \circ p$. Thus a retraction is equivalent to a projection and a projection is a continuous self map p of X onto a subspace Y such that $p(y) = y$ for all $y \in Y$. Such maps are called retractions in the classical terminology of topology.

A space Y_1 is then called *a retract* of a space X if there is a retraction $f: X \to Y_1$.

(4) In the homotopy category $[\mathbb{TOP}]$ a retraction $[f]: X \to Y$ is called a *homotopy retraction* and Y is called a *homotopy retract* of X. If $[f]$ is an isomorphism, then f (or $[f]$) is called a *homotopy equivalence.* Two isomorphic spaces in this category are called *homotopy equivalent.* □

Exercise EA3.8. Assume that one is given that the $n-1$-dimensional unit sphere $\mathbb{S}^{n-1}$ is not a retract of the n-dimensional unit ball B^n. With this information at hand Brouwer's Fixed Point Theorem [92] is quite elementary. □

Exercise EA3.9. All initial (terminal, null) objects are isomorphic. □

Exercise EA3.10. (i) *If $\mathcal{C}$ is a category and $\mathcal{A}$ an arbitrary class of objects in $\mathcal{C}$, then this class together with all morphisms between them is again a category.* It is called the *full subcategory* of the objects in $\mathcal{A}$, and is often (and unambiguously) denoted $\mathcal{A}$.

(ii) Isomorphy is an equivalence relation on ob $\mathcal{C}$. Assume that one is allowed a "Big Axiom of Choice" and that one can select in each isomorphy class one object. Let $\mathcal{C}_{\cong}$ denote the full subcategory of these objects, it is called a *skeleton* of $\mathcal{C}$.

In the category $\mathbb{V}_K(\text{fin})$ of finite dimensional vector spaces the full subcategory of all objects $\mathbb{K}^n$, $n = 0, 1, 2, \ldots$ is a skeleton of $\mathbb{V}_K(\text{fin})$. If one lets n range through the class of all cardinal numbers then the full subcategory of all vector spaces $K^{(n)}$ of direct sums of n copies of K and all linear maps between them is a skeleton of $\mathbb{V}_K$. □

Exercise EA3.11. Let $\mathcal{F}_p$ denote the category of fields of characteristic p and all morphisms of rings between them. Are there initial, terminal, null objects? □

In the category of sets, functions $f: X \to Y$ are classified according as they are injective, surjective or both. Injectivity, as everyone knows means that $f(x) = f(y)$ implies $x = y$. Trying to express this in terms of arrows we readily see that for any two functions $a, b: U \to X$ with $fa = fb$ we have $a = b$. Conversely, taking for U a one element set, and for a and b the functions taking this one element to x and y respectively, we recognize that this condition implies injectivity.

Definition A3.9. A morphism $f: X \to Y$ in a category is called a *monomorphism* or *monic* if the relation $fa = fb$ for any two morphisms $a,\ b: R \to X$ in the category for any object R implies $a = b$. □

We have just seen that in the category $\mathbb{S}$ of sets, an injective morphism is a monomorphism. In all set-based categories, injective morphisms are always monomorphisms. The converse is frequently, but not always, true as we shall see at once:

Examples A3.10. 1) Let $f: A \to B$ be a morphism in $\mathbb{AB}_R$. Set $K = \ker f$ and consider the inclusion morphism $a = \text{inc}: K \to A$ and the zero morphism $b = 0: K \to A$. Then $fa = fb$. If f is a monomorphism, this implies $a = b$, that is $\text{inc} = 0$, which means $K = \{0\}$. This is equivalent to the injectivity of f. Thus, in $\mathbb{AB}_R$ the monics are exactly the injective morphisms.

2) In the category of groups, topological groups, or Hausdorff topological groups, the same argument shows that monomorphisms are injective morphisms.

3) Let $\mathcal{C}$ denote the category of connected Hausdorff topological groups. If $f: G \to H$ is a monic in this category, then in general the kernel $\ker f$ is not connected and is, therefore, not an object in the category. However, $K = (\ker f)_0$, the identity component of the kernel, is such an object and we can consider $a, b: K \to G$ with $a = \text{inc}$ and $b = 0$. Then $fa = fb$, and thus again $a = b$ and $(\ker f)_0 = \{\mathbf{1}\}$. In other words, $\ker f$ is totally disconnected, i.e. has singleton connected components. Conversely, if $\ker f$ is totally disconnected and $a, b: A \to G$ are morphisms such that $fa = fb$, then we consider the continuous function $\varphi: A \to G$ given by $\varphi(x) = a(x)^{-1}b(x)$. Then $\varphi(\mathbf{1}) = \mathbf{1}$, and $\varphi(A)$ is connected as a continuous image of a connected space. Hence $\varphi(A) \subseteq (\ker f)_0 = \{\mathbf{1}\}$, and this means $a = b$. Thus *in the category of all connected Hausdorff topological groups and continuous morphisms, a monomorphism is a continuous homomorphism with totally disconnected kernel.* □

From this example we learn, in particular, that *even in set-based categories, monomorphisms need not be injective.*

The dual concept of that of a monomorphism is that of an epimorphism.

Definition A3.11. A morphism $f: X \to Y$ is an *epimorphism* or *epic* if the relation $af = bf$ for any two morphisms $a, b: Y \to C$ and an arbitrary object C implies $a = b$. □

Of course, epimorphisms should model surjectivity of functions. We have seen that in set-based categories, monomorphisms are usually, but not always, injective.

Remark A3.12. In any set based category, surjective morphisms are epimorphisms.

Proof. Let $f: X \to Y$ denote a surjective morphism in a set-based category. If α, $\beta: Y \to Z$ are morphisms with $\alpha f = \beta f$, let $y \in Y$ be arbitrary. Since f is surjective, there is an $x \in X$ with $y = f(x)$. Then $\alpha(y) = \alpha\big(f(x)\big) = \beta\big(f(x)\big) = \beta(y)$, and thus $\alpha = \beta$. □

Remark A3.13. (i) If fg is a monic, then g is monic.

(ii) If fg is epic, then f is epic.

(iii) If $fg = 1$ and f is monic or g is epic, then f and g are inverse isomorphisms. In other words, a monic retraction and an epic coretraction are both isomorphisms.

Proof. Exercise EA3.12. □

Exercise EA3.12. Prove A3.13. □

Now the question arises, whether in a set-based category, epimorphisms are surjective. This is a question which is frequently much deeper than the question when are monomorphisms injective. In fact it leads very quickly to mathematical difficulties which appear insurmountable.

Examples A3.14. (i) In the category of sets, every epimorphism is surjective.

(ii) In the category $\mathbb{TOP}$ of topological spaces, every epimorphism is surjective.

(iii) In the category $\mathbb{TOP}_2$ of Hausdorff spaces and continuous maps, a function $f: X \to Y$ is an epimorphism if and only if it has a dense image.

Proof. (i) Let $f: X \to Y$ be an epimorphism. Assume that f is not surjective. Let R denote the equivalence relation on Y whose cosets are the two (non-empty!) sets $f(X)$ and $Y \setminus f(X)$. Let $\alpha: Y \to Y/R$ denote the quotient map and $\beta: Y \to Y/R$ the constant function with value $f(X)$. Then αf and βf are both the constant functions with value $f(X)$ and therefore agree. Hence $\alpha = \beta$ which is inconsistent with the fact that α is not constant.

(ii) Use the same technique (Exercise EA3.13).

(iii) Unfortunately, this technique leads nowhere in the category $\mathbb{TOP}_2$ of Hausdorff spaces and continuous maps. We need another idea:

If $\overline{f(X)} = Y$, then all pairs of functions α, $\beta: Y \to Z$ agreeing on the dense set $f(X)$ must agree since Z is Hausdorff. (Consider the continuous function $\varphi: Y \to Z \times Z$ given by $\varphi(y) = \big(\alpha(y), \beta(y)\big)$. Since the diagonal Δ_Z of $Z \times Z$ is closed on account of the Hausdorff separation in Z, the inverse image $\varphi^{-1}(\Delta_Z)$ is closed in Y and therefore contains $\overline{f(X)} = Y$. Thus $\alpha = \beta$.)

Conversely, assume that $f: X \to Y$ is an epimorphism. We want to show that $\overline{f(X)} = Y$. Let $f = jf_1$ where $f_1: X \to \overline{f(X)}$ is the corestriction of f and $j: \overline{f(X)} \to Y$ is the inclusion. Then j is an epic by Remark A3.13(ii). It is therefore no loss in generality if we assume that X is a closed subspace of Y and f is the inclusion. Then we consider $Z_1 = Y \times \{0, 1\}$ and α_1, $\beta_1: Y \to Z_1$ given by $\alpha_1(y) = (y, 0)$ and $\beta_1(y) = (y, 1)$. Now we define on Z_1 an equivalence relation R where $(y_1, s_1)\, R\, (y_2, s_2)$ if $y_1 = y_2 \in X$. We set $Z = Z_1/R$ and let $q: Z_1 \to Z$ denote the quotient map. If $R(y_1, s_1) \neq R(y_2, s_2)$ then $y_1 \neq y_2$ or $y_1 = y_2 \notin X$ and

$s_1 \neq s_2$. In the first case find disjoint open neighborhoods U_j of y_j, $j = 1, 2$; then $q(U_j)$ are disjoint open neighborhoods of $R(y_j, s_j)$. In the second case let U be an open neighborhood of $y_1 = y_2$ with $U \cap X = \emptyset$. Then $q(U \times \{0\})$ and $q(U \times \{1\})$ are disjoint open neighborhoods of $R(y_1, s_1)$, and $R(y_2, s_2)$, respectively. Hence Z is a Hausdorff space, and $\alpha = q \circ \alpha_1$ and $\beta = q \circ \beta_1$ are continuous functions which agree on X. Since the inclusion $X \to Y$ is an epimorphism, $\alpha = \beta$ follows. But this is readily seen to imply $X = Y$. □

Exercise EA3.13. Prove the assertion in Example A3.14(ii). □

Notice that the technique of the previous examples is suitable for treating epics in the category of (pointed) sets, (pointed) topological spaces, (pointed) regular spaces.

Remark A3.15. In the category of R-modules every epimorphism is surjective.

Proof. Let $f: A \to B$ be an epimorphism. Let $\alpha: B \to B/f(A)$ denote the quotient morphism and $\beta: B \to B/f(A)$ the constant morphism. Then $\alpha f = \beta f$. It follows that $\alpha = \beta$ and thus $B/f(A)$ must be singleton, i.e. $f(A) = B$. □

The quotient $B/f(A)$ is called *the cokernel* of f.

Exercise EA3.14. Prove the following assertion.

In the category of Hausdorff abelian topological groups a morphism is an epimorphism if and only if it has a dense image. □

Recall for the following that $\mathbb{N}_0 = \{0, 1, \ldots\}$ and that $n + \mathbb{N}$ is an additive semigroup for all non-negative integers n.

Example A3.16. The inclusion morphism $f: X \to \mathbb{Z}$ for any additive subsemigroup X of $\mathbb{Z}$ which generates $\mathbb{Z}$ as a group is an epimorphism in the category of semigroups. In particular, the inclusion $\mathbb{N} \to \mathbb{Z}$ is an epimorphism.

Proof. Let α, $\beta: \mathbb{Z} \to S$ be morphisms of semigroups with $\alpha f = \beta f$. Then $G = \alpha(\mathbb{Z})$ and $H = \beta(\mathbb{Z})$ are subgroups of the semigroup S as homomorphic images of a group. Their intersection $G \cap H$ contains $\alpha(X) = \beta(X)$; in particular, it is not empty. Let g be an element in it and let e denote the identity of G. Denote the largest subgroup containing e by $H(e)$; indeed this is the set of all elements $s \in S$ for which there is an s' with $es' = s'e = s'$ and $ss' = s's = e$. Then $G \subseteq H(e)$. We claim that the identity u of H agrees with e. Firstly, let g' denote the inverse of g in G. Then $eu = g'gu = g'g = e$ since $gu = g$ as $g \in H$. Likewise $ue = ugg' = gg' = e$. But by symmetry, we also have $ue = u = eu$. This proves the claim. Hence G and H are subgroups of the group $H(e)$. However, since $\mathbb{Z}$, as a group, is generated by X, both groups G and H are generated (as groups) by $G \cap H$. Hence $G = H$. Let α', $\beta': \mathbb{Z} \to G$ denote the corestrictions of α, β to their common image. As semigroup morphisms between groups they are morphisms of groups, and they agree on a group generating set; hence they agree. But then $\alpha = \beta$. $\square$

Thus epimorphisms of semigroups are not surjective by a long way.

Exercise EA3.15. Consider the semigroup S consisting of 5 elements e, f, x, y and 0 such that $e^2 = e$, $f^2 = f$, $ex = x = xf$, $fy = y = ye$, $xy = e$, $yx = f$, and such that all other products are 0. Prove that the inclusion of the semigroup $\{e, f, x, 0\}$ into S is an epimorphism. $\square$

This observation shows, that epimorphisms in the category of finite semigroups are not surjective. The preceding Example A3.16 showed also that epimorphisms in the category of commutative semigroups are not surjective.

It is not trivial to show that *in the category of finite commutative semigroups, epimorphisms are surjective* (Isbell [212]).

What is the situation in the category of groups? Here as in similar situations in other categories, the methods of proof are more interesting than the actual results.

Exercise EA3.16. In the category of groups, every epimorphism is surjective.

[Hint. Let G denote a group, H an epimorphically embedded subgroup. We have to show that $H = G$; this clearly suffices.

Consider a G-module E. Form the semidirect product $E \rtimes G$ with multiplication $(x, g)(y, h) = (x + g{\cdot}y, gh)$. Let $f\colon G \to E$ denote a function. Then the set $\{(f(g), g) \mid g \in G\}$ is a subgroup if and only if $f(gg') = f(g) + g{\cdot}f(g')$. Such functions are called 1-*cocycles*. Assume that H is a proper subgroup of G. If we succeed in finding a non-constant cocycle which is 0 on all elements of H, then the two morphisms $g \mapsto (0, g)$, $(f(g), g)\colon G \to E \rtimes G$ agree on H, but are different. Thus H cannot be epimorphically embedded.

One source of cocycles are the so-called *coboundaries* f defined by $f(g) = -m + g{\cdot}m$ for some $m \in E$. Indeed, $f(gg') = -m + gg'{\cdot}m = (-m + g{\cdot}m) + g{\cdot}(-m + g'{\cdot}m) = f(g) + g{\cdot}f(g')$. Therefore, we are done if we find an element $m \in E$ such that $H{\cdot}m = \{m\}$ and that there is some g with $g{\cdot}m \neq m$.

We take an arbitrary abelian group A, e.g. $A = \mathbb{Z}(p)$ for some $p \neq 1$ and take $E = A^G$ with $g{\cdot}\varphi(g') = \varphi(g'g)$. If $H \neq G$ then we find a function $m\colon G \to A$ which is constant on the cosets gH, $g \in G$ but satisfies $m(\mathbf{1}) \neq m(g_0)$ for some $g_0 \notin H$. Then $h{\cdot}m(g') = m(g'h) = m(g')$, but $g_0{\cdot}m(\mathbf{1}) = m(g_0) \neq m(\mathbf{1})$.] $\square$

We notice that this proof shows us that epimorphisms in the category of all finite groups (or, to give another example, the category of all finite p-groups) are surjective. Observation A3.16 is not a trivial result. Notice under which circumstances it can be used to show that in some preassigned category of groups, epimorphisms are surjective: We need, in the category in question, a G-module E, i.e. an abelian group object E in the category and an automorphic group action $G \times E \to E$ such that (1) the semidirect product $E \rtimes G$ exists in the category,

(2) there is a fixed element $m \in E$ for H which is not fixed under all of G. It is a good exercise to try out whether this method has a chance to work for, say, Hausdorff topological groups. (Warning: It does not work!) Also, the method of proof gives additional information which other proofs do not yield. We can in fact choose the function $m: G \to A$ in the proof in such a fashion that $(\forall g \in G)\ m(gh) = m(g)$ if and only if $h \in H$. Then we obtain the following (sharper) result.

Given a subgroup H of a group G we find a group S containing G and another isomorphic copy G_2 of G in such a fashion that $G \cap G_2 = H$.

The method of proof presented here, however, is of no help in proving that epimorphisms of compact groups are surjective.

Exercise EA3.17 (Poguntke [294]). *Epimorphisms of compact groups are surjective.*

[Hint. It suffices again to show that an epimorphically embedded closed subgroup of a compact group is the whole group. Thus consider a closed subgroup H of a compact group G such that the inclusion $H \to G$ is an epimorphism.

Claim 1. Every irreducible G-module is an irreducible H-module.

Proof of Claim 1. Let E denote an irreducible G-module over $\mathbb{C}$. We may assume that E is a unitary module (Weyl's Trick 2.10). By compactness of H, each H-submodule is an orthogonal direct summand. If $E_1 \oplus E_2$ is an orthogonal direct sum of H-submodules and $p_j: E \to E$ is the projection onto E_j, then the function $\varphi: E \to E$ given by $\varphi(v) = p_1(v) - p_2(v)$ is a unitary automorphism of the vector space E with precise fixed point set E_1. If $h \in H$, then $\varphi(h{\cdot}v) = p_1(h{\cdot}v) - p_2(h{\cdot}v) = h{\cdot}p_1(v) - h{\cdot}p_2(v)$ since both projections are H-equivariant; and this last element equals $h{\cdot}\varphi(v)$ by the linearity of the action. Hence φ is H-equivariant, i.e. an H-module automorphism. Let $\pi: G \to \mathrm{U}(E)$ denote the unitary representation associated with the G-module E. This map as well as the map $g \mapsto \varphi\pi(g)\varphi^{-1}: G \to \mathrm{U}(E)$ are morphisms of compact groups. Now $\pi(h)\varphi = \varphi\pi(h)$, i.e. $\pi(h) = \varphi\pi(h)\varphi^{-1}$ for all $h \in H$. Since the inclusion $H \to G$ is an epimorphism, we conclude $\pi(g) = \varphi\pi(g)\varphi^{-1}$ for all $g \in G$. Thus φ is a G-module automorphism. Since E is an irreducible G-module, we conclude $\varphi = c{\cdot}\mathrm{id}_E$ for some $c \in \mathbb{C}$ by Schur's Lemma. But the definition of φ is compatible with this only if $c = 1$ and $E_2 = \{0\}$. Hence E is an irreducible H-module as asserted.

Claim 2. If E is an arbitrary feebly complete G-module (see 3.29; for instance, a Banach G-module will do), then the fixed point modules $E_{\mathrm{fix}}(G)$ and $E_{\mathrm{fix}}(H)$ and the effective submodules $E_{\mathrm{eff}}(G)$ and $E_{\mathrm{eff}}(H)$ with respect to H and G agree (see 3.34).

Proof of Claim 2. By the Big Peter–Weyl Theorem 3.51, the algebraic sum of all nontrivial irreducible G-modules (resp., H-modules) is dense in $E_{\mathrm{eff}}(G)$ (resp., $E_{\mathrm{eff}}(H)$). By Claim 1, we therefore have $E_{\mathrm{eff}}(G) \subseteq E_{\mathrm{eff}}(H)$. Trivially, we have $E_{\mathrm{eff}}(H) \subseteq E_{\mathrm{eff}}(G)$. Now we have canonical direct sum decompositions

$$E = E_{\mathrm{fix}}(G) \oplus E_{\mathrm{eff}}(G) = E_{\mathrm{fix}}(H) \oplus E_{\mathrm{eff}}(G).$$

The asserted equalities follow.

Claim 3. The projection operators P_G and P_H given by $P_G v = \int_G g{\cdot}v\,dg$ and $P_H v = \int_H h{\cdot}v\,dh$ with the respective normalized Haar integrals agree. This claim follows at once from Claim 2, since $\ker P_G = E_{\rm eff}(G)$ and $\operatorname{im} P_G = E_{\rm fix}(G)$; the analogous relations hold for H in place of G.

Now we apply Claim 3 to the module $E = C(G,\mathbb{C})$. Then $P_G f = \int_G f(g)\,dg$ and $P_H f = \int_H f(h)\,dh$. If $H \neq G$, then we find a continuous function $f\colon G \to [0,1]$ with $f(H) = \{0\}$ and $f(g) = 1$ for some $g \in G$. Then $0 = \int_H f(h)\,dh = \int_G f(g)\,dg > 0$, a contradiction. Hence $H = G$.] □

For a long time, the following problem was unsettled (sometimes referred to as Hofmann's Epimorphism Problem for Hausdorff Groups). Prove the following assertion or refute it with a counterexample: *If the inclusion map of a closed subgroup H into a Hausdorff topological group G is an epimorphism in the category of Hausdorff topological groups, then $H = G$.* □

Eventually a counterexample was found:

Exercise EA3.18. *Let G denote the group of all self-homeomorphisms of $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and H the closed proper subgroup of all $f \in G$ with $f(1) = 1$. Then the inclusion map $H \to G$ is an epimorphism.*

(See Uspenskiĭ [357, 358].) □

After Uspenskiĭ's Theorem was established, people thought that similar answers would be found for categories not too remote from this one. Vladimir Pestov asked whether in the category of C*-algebras and *-preserving algebra morphisms there exist examples showing that epimorphisms were not necessarily surjective.

Exercise EA3.19 (Hofmann and Neeb [200]). *In the category of C*-algebras all epimorphisms are surjective.* □

Let us see some really simple exercises:

(i) *Any retraction is an epic, every coretraction is a monic.*

(ii) *Any isomorphism is both a monic and an epic. The converse is not true in general.*

In the category of Hausdorff topological groups, the function $f\colon \mathbb{R}_d \to \mathbb{R}$, $\mathbb{R}_d =$ additive group of real numbers with the discrete topology, $f(r) = r$, is bijective and thus is both a monic and an epic. It is not an isomorphism.

In the category $\mathbb{AB}_R$ a morphism is an isomorphism if it is both a monic and an epic.

Functors

Functors are structure preserving maps between categories, and natural transformations are structure preserving "maps" between functors. One prototype of functor is a functor forgetting structure. For instance, if we consider the category

Sem of semigroups and semigroup morphisms and the category $\mathbb{S}$ of sets, then we can assign to any semigroup S the underlying set $U(S)$ and to any morphism $f: S \to S'$ of semigroups the underlying function $U(f): U(S) \to U(S')$. Then we certainly assign to any identity morphism $\mathrm{id}_S: S \to S$ the identity function of $U(S)$, that is, $U(\mathrm{id}_S) = \mathrm{id}_{U(S)}$. If $f'': S' \to S''$ and $f': S \to S'$ are semigroup morphisms, clearly we have $U(f''f') = U(f'')U(f')$.

Such an assignment we shall call a *functor*.

Definition A3.17. A *functor* $F: \mathcal{C}_1 \to \mathcal{C}_2$ of a category $\mathcal{C}_1$ to a category $\mathcal{C}_2$ is a function $\mathrm{ob}\,\mathcal{C}_1 \to \mathrm{ob}\,\mathcal{C}_2$ which assigns to an object X of $\mathcal{C}_1$ an object FX of $\mathcal{C}_2$ and a function $\mathrm{morph}\,\mathcal{C}_1 \to \mathrm{morph}\,\mathcal{C}_2$ which maps $\mathcal{C}_1(X, Y)$ into $\mathcal{C}_2(FX, FY)$ in such a fashion that the following conditions are satisfied:
(i) $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ for all objects X in $\mathcal{C}_1$.
(ii) $F(fg) = (Ff)(Fg)$ whenever the domain of f equals the range of g. □

Remark A3.18. Assume that C_j, for each of $j = 1, 2$, is a class endowed with functions $D_j, R_j: C_j \to C_j$ and compositions $C_j \times_{D_j R_j} C_j \to C_j$, both satisfying conditions 1.4.(1), (3), (4), and (5), defining categories $\mathcal{C}_j$ in the sense of Proposition A3.5. Then a function $F: C_1 \to C_2$ is a functor if and only if $FD_1 = D_2F$ and $FR_1 = R_2F$ and $F(fg) = (Ff)(Fg)$ for all $(f, g) \in C_1 \times_{D_1 R_1} C_1$.

Proof. Exercise EA3.20. □

Exercise EA3.20. Prove A3.18. □

We shall record a sample of functors to familiarize ourselves with the concept.

Example A3.19 (Grounding functors, forgetful functors). Assume that $\mathcal{C}$ is a set-based category. Then the assignment $U: \mathcal{C} \to \mathbb{S}$ which maps an object X of $\mathcal{C}$ to the underlying set $U(X)$, and which maps a morphism $f: X \to Y$ to the underlying function $U(f): U(X) \to U(Y)$ is a functor. It is called the *underlying, grounding* or *forgetful functor* (it forgets the structure!).

There are other forgetful functors which forget a portion of the structure:

Let $\mathbb{TOPG}$ denote the category of (Hausdorff) topological groups and continuous morphisms, $\mathbb{TOP}_0$ the category of pointed (Hausdorff) topological spaces and $\mathbb{G}$ the category of groups. Consider the assignment $V: \mathbb{TOPG} \to \mathbb{TOP}_0$ which assigns to a topological group G the underlying space with the identity as base point, and which assigns to a morphism of topological groups the underlying base point preserving continuous function. Then V is a functor, as is the assignment $W: \mathbb{TOPG} \to \mathbb{G}$ which maps G to the underlying (abstract) group and each morphism of topological groups to the underlying homomorphism of groups. There are grounding functors $\mathbb{TOP}_0 \to \mathbb{S}_0 \to \mathbb{S}$ and $\mathbb{G} \to \mathbb{S}_0 \to \mathbb{S}$. There is no limit to the imagination in observing grounding functors. □

Typically, the grounding functors have functors in the reverse direction.

Example A3.20 (Free functors. Cf. Appendix 1, A1.4ff.). Let R denote a ring and $\mathbb{AB}_R$ the category of R-modules and R-module morphisms. If X is a set we consider the R-module $R^{(X)} \subseteq R^X$ of all functions $f\colon X \to R$ with finite support. For $x \in X$, we define $e_x\colon X \to R$ by $e_x(y) = \delta_{xy}$ (the Kronecker-Delta). Then every element $f \in R^{(X)}$ may be written as a linear combination $f = \sum_{x\in X} f(x){\cdot}e_x$. We write $FX = R^{(X)}$ and identify an element $x \in X$ with the function $e_x\colon X \to R$. Then the elements of FX are the linear combinations $\sum_{x\in X} r_x{\cdot}x$ with $x \mapsto r_x$ in $R^{(X)}$. If $\varphi\colon X \to Y$ is a function, then the prescription $F\varphi\colon FX \to FY$ given by $(F\varphi)(\sum_{x\in X} r_x{\cdot}x) = \sum_{x\in X} r_x.\varphi(x)$ makes $F\colon \mathbb{S} \to \mathbb{AB}_R$ into a functor.

The identification of x and e_x may seem to be too daring. We can say at any rate that we have a function $x \mapsto e_x\colon X \to FX$. If we insist on writing arrows for morphisms in well-defined categories, this function is not an arrow, since X is an object of $\mathbb{S}$ and FX is an object of $\mathbb{AB}_R$. This is easily remedied with the help of the grounding functor $U\colon \mathbb{AB}_R \to \mathbb{S}$. We can now write

$$\eta_X\colon X \to U(FX), \quad \eta_X(x) = e_x.$$

The module FX is called the *free module* over X, the function η_X the natural embedding of X, and F is called *the free functor from* $\mathbb{S}$ *to* $\mathbb{AB}_R$. Of course, the entire construction specializes to the case $\mathbb{AB}$ of abelian groups ($R = \mathbb{Z}$) and the case $\mathbb{V}_K$ of K-vector spaces ($R = K$ a field).

Proposition A3.21 (Universal Property of Free Functors). *Every function f: $X \to A$ where X is a set and A an R-module extends uniquely to an R-module homomorphism $f'\colon FX \to A$ given by $f'(\sum_{x\in X} r_x{\cdot}x) = \sum_{x\in X} r_x{\cdot}f(x)$. More formally: Let $U\colon \mathbb{AB}_R \to \mathbb{S}$ denote the grounding functor. Then the free functor $F\colon \mathbb{S} \to \mathbb{AB}_R$ has the following universal property.*

Given any function $f\colon X \to U(A)$ (where A is any R-module), there is a unique morphism of R-modules $f'\colon FX \to A$ such that $f = (Uf') \circ \eta_X$.

$$\begin{array}{ccc|c}
 & \mathbb{S} & & \mathbb{AB}_R \\
\hline
X & \xrightarrow{\eta_X} & UFX & FX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Uf'} & \downarrow{\scriptstyle f'} \\
UA & \xrightarrow[\mathrm{id}_{UA}]{} & UA & A
\end{array}$$

Proof. Exercise EA3.21. □

Exercise EA3.21. Prove A3.21. □

Let us notice that the universal property gives us a bijection

$$f \mapsto f'\colon \mathbb{S}(X, UA) \to \mathbb{AB}_R(FX, A).$$

In terms of more traditional algebra, one could write this in fact as an isomorphism

$$A^X \cong \mathrm{Hom}_R(FX, A)$$

of abelian groups (and indeed of R-modules if R is commutative). The free functor allows the representation of a function module A^X as a Hom-module.

Many functors arise from identity functors of a category through the selection of canonically defined subobjects.

Example A3.22 (Subfunctors). Consider the category $\mathbb{AB}_R$ of R-modules for any integral domain R. (Again: For $R = \mathbb{Z}$ we obtain abelian groups, for R a field K we obtain K-vector spaces!) For each R-module A we select a submodule $TA = \{a \in A \mid (\exists r \in R \setminus \{0\})\ r{\cdot}a = 0\}$, the so-called *torsion submodule* (cf. Appendix 1, A1.16ff.). If $f\colon A \to B$ is a morphism of R-modules, then $f(TA) \subseteq TB$. Hence the restriction and corestriction $Tf\colon TA \to TB$ is well-defined. If $\operatorname{tor}\mathbb{AB}_R$ denotes the full subcategory of torsion modules over R, then $T\colon \mathbb{AB}_R \to \operatorname{tor}\mathbb{AB}_R$ is a functor. Of course, there is an inclusion functor $I\colon \operatorname{tor}\mathbb{AB}_R \to \mathbb{AB}_R$.

This pair of functors also has a close connection. Firstly, for each module A there is an inclusion morphism $\varepsilon_A\colon ITA \to A$ of the submodule TA into A, but in as much as TA is in the other category of torsion modules, we should write ITA when we refer to the morphism ε_A in the category $\mathbb{AB}_R$. Now assume that B is any torsion module, and that we have a morphism $f\colon IB \to A$ in $\mathbb{AB}_R$. Then $f(B)$ is a torsion module and a submodule of A. Hence $f(B) \subseteq TA$. We can consider the corestriction $f'\colon B \to TA$ of f and then note that this f' is a unique $\operatorname{tor}\mathbb{AB}_R$-morphism such that $f = \varepsilon_A \circ If'$.

$$
\begin{array}{c|ccc}
\operatorname{tor}\mathbb{AB}_R & & \mathbb{AB}_R & \\
\hline
TA & ITA & \xrightarrow{\varepsilon_A} & A \\
{\scriptstyle f'}\uparrow & \uparrow{\scriptstyle If'} & & \uparrow{\scriptstyle f} \\
B & IB & \xrightarrow[\mathrm{id}_{IB}]{} & IB
\end{array}
$$

The function $f \mapsto f'\colon \mathbb{AB}_R(IB, A) \to \operatorname{tor}\mathbb{AB}_R(B, TA)$ implements an isomorphism of sets—and indeed R-modules.

It may appear that we have chosen an awfully roundabout way of saying that every homomorphism from an abelian torsion group B into an abelian group A factors through the torsion subgroup TA of A. However, in doing so we have discovered a striking parallel to the set-up of the free functors in Proposition A3.21. There we had an isomorphism

$$\mathbb{S}(X, UA) \cong \mathbb{AB}_R(FX, A).$$

Here we have an isomorphism

$$\operatorname{tor}\mathbb{AB}_R(B, TA) \cong \mathbb{AB}_R(IB, A).$$

This cannot be an accident. We have to discover why. Note, in passing, that for R a field we have $TA = \{0\}$. Vector spaces simply do not have a torsion theory. All vector spaces are free, as we have seen.

The previous example arose from picking subobjects in a natural way. Now we attempt the same for quotient objects in set-based categories.

Example A3.23. (Quotient functors). (a) For an R-module A in $\mathbb{AB}_R$ we form the quotient module $FA = A/TA$. This module is torsion-free; i.e. $T(FA) = \{0\}$.

Any morphism $f: A \to B$ induces $Tf: TA \to TB$, and thus a morphism $Ff: FA \to FB$ given by $(Ff)(a+TA) = f(a)+TB$. If $\mathrm{tfr}\,\mathbb{AB}_R$ denotes the category of torsion-free R-modules, then $F: \mathbb{AB}_R \to \mathrm{tfr}\,\mathbb{AB}_R$ is a functor. Of course, there is an inclusion functor $U: \mathrm{tfr}\,\mathbb{AB}_R \to \mathbb{AB}_R$. For every module, let $\eta_A: A \to UFA$ denote the quotient homomorphism. Every morphism $f: A \to UB$ from A into a torsion free module B is annihilated on all torsion elements. Hence there is a unique morphism $f': FA \to B$ such that $f = (Uf') \circ \eta_A$. The function $f' \mapsto Uf' \circ \eta_A$ implements an isomorphism

$$\mathrm{tfr}\,\mathbb{AB}_R(FA, B) \cong \mathbb{AB}_R(A, UB).$$

(b) For a topological space X let R_X be the equivalence relation which associates with a point x the equivalence class $R_X(x)$ which is the intersection of all open-closed (*"clopen"*) neighborhoods of x. Then the quotient space $Y = X/R_X$ is R_Y-trivial; i.e. $R_Y(y) = \{y\}$ for all $y \in Y$. This is a consequence of the fact that every open closed subset of X is R_X-saturated, i.e., is a union of R_X-cosets. If we set $FX = X/R_X$, then F defines a functor $\mathbb{TOP} \to \mathbb{TOP}^\top$ into the full subcategory of R_X-trivial spaces X. The inclusion functor $\mathbb{TOP}^\top \to \mathbb{TOP}$ is denoted U.

Every continuous function $f: X \to UY$ into an R_Y-trivial space Y equalizes R_X-equivalent points, hence factors through the natural quotient morphism $\eta_X: X \to UFX$, i.e. there is a unique continuous map $f': FX \to Y$ such that $f = (Uf') \circ \eta_X$. Again

$$\mathbb{TOP}^\top(FX, Y) \cong \mathbb{TOP}(X, UY). \qquad \square$$

The R_X-classes are called the *quasicomponents* of X. Every connected component is contained in a quasicomponent. The converse may not be true. However, this *is* true in the category of compact spaces. In this category a space X is R_X-trivial if and only if X is totally disconnected, and in general, for a compact space X, the space FX is the space of connected components.

Exercise EA3.23. Show that a similar procedure works for topological groups.□

Example A3.24 (Hom-functors). (a) If $A \in \mathrm{ob}(\mathbb{S})$, then we define a functor $H: \mathbb{S} \to \mathbb{S}$ by $H(X) = \mathbb{S}(A, X) = X^A$ for all objects $X \in \mathrm{ob}(\mathbb{S})$ and by $H(f): X^A \to Y^A$, $(Hf)(\varphi) = f \circ \varphi: A \to Y$ for $f: X \to Y$ and $\varphi \in X^A$.

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle (Hf)(\varphi)} \\ X & \xrightarrow{f} & Y. \end{array}$$

The functor H is also called a *covariant* hom-*functor*.

Let us also define a functor $P: \mathbb{S} \to \mathbb{S}$ by $P(X) = X \times A$ on objects and $Pf: X \times A \to Y \times A$ by $(Pf)(x, a) = (f(x), a)$. The verification that our prescriptions are functors is straightforward.

Now observe that we have a morphism $\eta_X: X \to (X \times A)^A = HP(X)$ given by $\eta_X(x)(a) = (x, a)$. If $f: X \to Y^A$ is a function then we have a function $f': X \times A \to Y$ given by $f'(x, a) = f(x)(a)$. This function satisfies $f(x)(a) = f'(x, a)$, and $\big(H(f') \circ \eta_X\big)(x) = H(f')\big(\eta_X(x)\big) = f' \circ \eta_X(x)$, whence $\big(H(f') \circ \eta_X\big)(x)(a) = f'\big(\eta_X(x)(a)\big) = f'(x, a) = f(x)(a)$. Thus $f(x) = \big(H(f') \circ \eta_X\big)(x)$ and thus $f = H(f') \circ \eta_X$.

$$\begin{array}{ccccc} & \mathbb{S} & & & \mathbb{S} \\ \hline X & \xrightarrow{\eta_X} & HPX & & PX \\ {\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle Hf'} & & \big\downarrow{\scriptstyle \exists! f'} \\ HY & \xrightarrow[\mathrm{id}_{HY}]{} & HY & & Y. \end{array}$$

In this diagram and in all following ones of an analogous type, the symbols $\exists! f'$ means that there exists a *unique* morphism f' such that the rectangle on the left commutes. The function $f \mapsto f': \mathbb{S}(X, HY) \to \mathbb{S}(PX, Y)$ is a bijection inverted by $g \mapsto (Hg) \circ \eta_X$. In other words,

$$\mathbb{S}(X, Y^A) \cong \mathbb{S}(X \times A, Y).$$

Sometimes this isomorphism, which can also be written in the form

$$Y^{X \times A} \cong \big(Y^A\big)^X$$

is called the *exponential law* for the hom-functor. Of course this exponential law shows the same formal relationship between the functors H and P which we have seen before.

Exponential laws are more prevalent than one thinks.

(b) Let R be a commutative ring (such as $\mathbb{Z}$ or a field) and $\mathbb{M} = \mathbb{AB}_R$ the category of R-modules. For a fixed R-module A, the set $\mathbb{M}(A, X)$ of all module homomorphisms $f: A \to X$ is a submodule of HX, the module X^A of all functions $f: A \to X$ with the pointwise operations. If $f: X \to Y$ is a module homomorphism, then we obtain a module homomorphism $Hf: HX \to HY$ via $(Hf)(\varphi) = f \circ \varphi$ as in example (a) above. In this fashion we obtain a self-functor $H: \mathbb{M} \to \mathbb{M}$.

On the other hand we have a self-functor $T: \mathbb{M} \to \mathbb{M}$ given by $TX = X \otimes_R A$ on objects and by the module morphisms $Tf: TX \to TY$ which is uniquely characterized by the condition $(Tf)(x \otimes a) = f(x) \otimes a$. (Cf. Appendix 1, A1.44ff.)For each $X \in \operatorname{ob} \mathbb{M}$ we set $\eta_X: X \to HT(X)$, $\eta_X(x)(a) = x \otimes a$. Then every morphism $f: X \to HY$ yields a unique $f': TX \to Y$ such that $f = (Hf') \circ \eta_X$, namely, the one given through $f'(x \otimes a) = f(x)(a)$.

$$\begin{array}{ccccc} & \mathbb{M} & & & \mathbb{M} \\ \hline X & \xrightarrow{\eta_X} & \mathbb{M}(A, X \otimes A) & & X \otimes A \\ {\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle Hf'} & & \big\downarrow{\scriptstyle \exists! f'} \\ \mathbb{M}(A, Y) & \xrightarrow[\mathrm{id}_{\mathbb{M}(A,Y)}]{} & \mathbb{M}(A, Y) & & Y. \end{array}$$

As before, we obtain an isomorphism

$$\mathbb{M}(X \otimes A, Y) \cong \mathbb{M}\big(X, \mathbb{M}(A, Y)\big),$$

where $\mathbb{M}(A, Y) = HY$ is understood to carry the natural R-module structure.

(c) If $\mathbb{H}$ is the category of Hausdorff topological spaces, then the set $\mathbb{H}(K, Y)$ supports a Hausdorff topology, namely, the compact-open topology, making it into a Hausdorff space $C(K, Y)$. If K is locally compact and X and Y are arbitrary spaces the formalism indicated before provides an isomorphism of sets

$$\mathbb{H}\big(X, C(K, Y)\big) \cong \mathbb{H}(X \times K, Y),$$

and indeed a homeomorphism

$$C\big(X, C(K, Y)\big) \cong C(X \times K, Y).$$

(d) There is a version of this formalism for the category $\mathbb{H}_0$ of pointed Hausdorff spaces. The set $\mathbb{H}_0(K, Y)$ of base point preserving maps from a locally compact pointed space K to Y becomes a Hausdorff space $C_0(K, Y)$ with the compact open topology. If we set $X \divideontimes K = (X \times K)/\big((X \times \{k_0\}) \cup \{x_0\} \times K\big)$, the quotient space obtained from $X \times K$ by collapsing the subspace $\big((X \times \{k_0\}) \cup \{x_0\} \times K\big)$ to a point, and if we define $\eta_X\colon X \to C_0(K, X \divideontimes K)$ by $\eta_X(x)(k) = [x, k]$, the class of (x, k) in the quotient space, then for every morphism $f \in \mathbb{H}_0\big(X, C_0(K, Y)\big)$ there is a unique $f'\colon X \divideontimes K \to Y$ with $f = C_0(K, f') \circ \eta_X$, namely the one given by $f'([x, k]) = f(x)(k)$.

$$\begin{array}{ccccc}
 & \mathbb{H}_0 & & & \mathbb{H}_0 \\
\hline
X & \xrightarrow{\ \eta_X\ } & \mathbb{H}_0(K, X \divideontimes K) & & X \divideontimes K \\
{\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle \mathbb{H}_0(K, f')} & & \downarrow{\scriptstyle \exists ! f'} \\
\mathbb{H}_0(K, Y) & \xrightarrow[\ \mathrm{id}\ ]{} & \mathbb{H}_0(K, Y) & & Y.
\end{array}$$

In particular,

$$C_0(X \divideontimes K, Y) \cong C_0\big(X, C_0(K, Y)\big) :$$

But note also that, if X is locally compact, these same spaces are homeomorphic to the following

$$C_0(K \divideontimes X, Y) \cong C_0\big(K, C_0(X, Y)\big).$$

If $K = \mathbb{S}^1$ (with a base point), then there are special names in use:

$\Omega(Y) = C_0(\mathbb{S}^1, Y) =$ loop space of Y,
$\Sigma(X) = X \divideontimes \mathbb{S}^1 =$ suspension of X.

With this notation we have at once

$$\begin{array}{ccccc}
 & \mathbb{H}_0 & & & \mathbb{H}_0 \\
\hline
X & \xrightarrow{\ \eta_X\ } & \Omega\Sigma(X) & & \Sigma X \\
{\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle \Omega(f')} & & \downarrow{\scriptstyle \exists ! f'} \\
\Omega Y & \xrightarrow[\ \mathrm{id}\ ]{} & \Omega Y & & Y,
\end{array}$$

and

$$C_0(\Sigma X, Y) \cong C_0(X, \Omega Y).$$

If X is locally compact, we also have

$$C_0(\Sigma X, Y) \cong C_0\big(S^1, C_0(X, Y)\big) = \Omega\big(C_0(X, Y)\big).$$

Passage to the set of arc components of $C_0(A, B)$ yields $[A, B]$, the hom-set in the homotopy category of pointed spaces, and thus we have at once

$$[\Sigma X, Y] \cong [X, \Omega Y],$$

and if X is also locally compact, then this set is the set $\big[\Omega\big(C_0(X, Y)\big)\big]$ of arc components of $\Omega\big(C_0(X, Y)\big)$. Note that $\pi_1(Z) = [\Omega(Z)]$, whence $[\Sigma X, Y] \cong [X, \Omega Y] \cong \pi_1\big(C_0(X, Y)\big)$.

Very frequently, functors "reverse the direction of arrows." For a category $\mathcal{A}$ we let $\mathcal{A}^{\mathrm{op}}$ denote the *opposite category* which has the same objects but for which $\mathcal{A}^{\mathrm{op}}(A, B) = \mathcal{A}(B, A)$ and the composition is given by $f \circ^{\mathrm{op}} g = g \circ f$. One verifies readily that these prescriptions define a category.

Definition A3.25. If $\mathcal{A}$ and $\mathcal{B}$ are categories, then a functor $F\colon \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ is called a *contravariant functor* $F\colon \mathcal{A} \to \mathcal{B}$. □

Thus a contravariant functor is a prescription F assigning to each object A of $\mathcal{A}$ an object FA and to each morphism $f\colon A_1 \to A_2$ a morphism $(Ff)\colon FA_2 \to FA_1$ such that $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ and that $F(fg) = (Fg)(Ff)$ whenever fg is defined.

The most typical contravariant functors are the contravariant hom-functors.

Example A3.26 (Contravariant hom-functors). (a) Let A denote a set. Then the prescription $DX = \mathbb{S}(X, A) = A^X$ gives a set, and if $f\colon X \to Y$ is a function, then $Df\colon DY \to DX$, $(Df)(\varphi) = \varphi \circ f$ gives a well-defined function.

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle (Df)(\varphi)}\uparrow & & \uparrow{\scriptstyle \varphi} \\ X & \xrightarrow{f} & Y. \end{array}$$

It is readily verified that $D\colon \mathbb{S} \to \mathbb{S}^{\mathrm{op}}$ is a functor.

Now let us observe that we have a morphism $\eta_X\colon X \to DDX = A^{(A^X)}$ given by $\eta_X(x)(\alpha) = \alpha(x)$ for $\alpha\colon X \to A$. For each $f\colon X \to DY = A^Y$ we define a map $f'\colon Y \to DX = A^X$ such that $f'(y)(x) = f(x)(y)$. Now $\big((Df') \circ \eta_X)(x) = (Df')\big(\eta_X(x)\big) = \eta_X(x) \circ f'$, whence $\big[\big((Df') \circ \eta_X\big)(x)\big](y) = [\eta_X(x) \circ f'](y) = \eta_X(x)\big(f'(y)\big) = f'(y)(x) = f(x)(y)$. It follows that $f = (Df') \circ \eta_X$; and one verifies readily that only one function f' has this property.

$$\begin{array}{ccccc}
 & \mathbb{S} & & & \mathbb{S} \\
\hline
X & \xrightarrow{\eta_X} & D^2(X) & & DX \\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle (Df')} & & \big\uparrow{\scriptstyle \exists! f'} \\
DY & \xrightarrow[\text{id}]{} & DY & & Y.
\end{array}$$

The function $f \mapsto f'$ is an isomorphism

$$\mathbb{S}(X, DY) \cong \mathbb{S}(Y, DX) = \mathbb{S}^{\text{op}}(DX, Y).$$

Needless to emphasize that, once again, we have a pairing of functors—this time a "self-pairing" $D\colon \mathbb{S} \to \mathbb{S}^{\text{op}}$ and $D\colon \mathbb{S}^{\text{op}} \to \mathbb{S}$.

(b) This formalism applies to many categories $\mathcal{C}$ in such a fashion that the hom-set $\mathcal{C}(X, A)$ can be endowed with the structure of an object DX in $\mathcal{C}$.

Example: $\mathcal{C} = \mathbb{AB}_R$ for a commutative ring. Then $DX = \mathbb{AB}_R(X, A) \subseteq A^X$ inherits the module structure from A^X. The map $\eta_X\colon X \to D^2X$ is the well-known evaluation morphism in the case of vector space duality; if $R = K$ is a vector space and $A = K$, then $DX = \text{Hom}(X, K) = \widehat{X}$ is the dual vector space, and $D^2X = \widehat{\widehat{X}}$ is the bidual. The duality theorem for finite dimensional vector spaces expresses the fact that η_X is an isomorphism if and only if $\dim X$ is finite. If $R = \mathbb{Z}$ and $A = \mathbb{R}/\mathbb{Z}$, then $DX = \text{Hom}(X, \mathbb{R}/\mathbb{Z})$ is the character group of X.

(c) In the category of topological spaces and the associated homotopy categories this formalism works. Let us consider $\mathbb{H}_0$ and inspect the contravariant functor $X \mapsto DX = C_0(X, A)$ for some fixed Hausdorff space A. If A happens to be a topological group, then DX is a topological group with respect to pointwise operations, as is the "bidual" D^2X. We have

$$C_0\big(X, C_0(Y, A)\big) = C_0(X, DY) \cong C_0(Y, DX) = C_0\big(Y, C_0(X, A)\big).$$

If X is locally compact we know that these spaces are homeomorphic to

$$C_0(X \divideontimes Y, A) = D(X \divideontimes Y).$$

If $A = \mathbb{S}^1$ one defines $H^1(X, \mathbb{Z}) = [C_0(X, \mathbb{S}^1)]$; this is an abelian group, namely, the factor group of the topological group $C_0(X, \mathbb{S}^1)$ modulo the arc component of the origin. (In fact, if X is compact, we define A to be the real Banach algebra $C(X, \mathbb{R})$. Then $G \stackrel{\text{def}}{=} C_0(X, \mathbb{S}^1)$ is a closed multiplicative subgroup of A^{-1} and $\mathfrak{g} = iC_0(X, \mathbb{R}) = C_0(X, i\mathbb{R})$ is mapped to G under exp, $\exp(if)(x) = e^{if(x)}$, in a locally homeomorphic fashion so as to make G into a linear Lie group.) The group $H^1(X, \mathbb{Z})$ is also called the *Bruschlinsky group*; for paracompact spaces X it is the same as the first integral Čech cohomology group [338].

Let us briefly consider pointed Hausdorff spaces U, V. Let $U \vee V$ denote the disjoint union with the base points identified. We may certainly identify U and V with their images in this quotient space. Then $C_0(U \vee V, A) \cong C_0(U, A) \times C_0(V, A)$ under $f \mapsto (f|U, f|V)$.

Now we consider two homeomorphic copies U and V of $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ and denote with $c_1, c_2: \mathbb{T} \to \mathbb{T} \vee \mathbb{T}$ the two embeddings. Then the pinching map

$$p: \mathbb{T} \to \mathbb{T} \vee \mathbb{T} \qquad p(r + \mathbb{Z}) = \begin{cases} c_1(2r + \mathbb{Z}), & \text{if } 0 \le r \le \frac{1}{2}, \\ c_2(2r + \mathbb{Z}), & \text{if } \frac{1}{2} < r < 1 \end{cases}$$

defines, by contravariance, a function

$$C_0(\mathbb{T}, A) \times C_0(\mathbb{T}, A) \xleftarrow{\cong} C_0(\mathbb{T} \vee \mathbb{T}, A) \xleftarrow{C_0(p,A)} C_0(\mathbb{T}, A),$$

that is, a continuous function

$$m: \Omega(A) \times \Omega(A) \to \Omega(A).$$

Exercise EA3.24. Prove the following lemma.

Lemma. *Let $\sim$ denote the relation of being connected by an arc; (i.e. $f \sim g$ in the loop space ΩA if and only if f is homotopic to g.) Then $\sim$ is a congruence for the multiplication m, and the multiplication induced on $\pi_1(A) = [\Omega(A)] = \Omega(A)/\sim$ makes $\pi_1(A)$ into a group. In other words, m is homotopy associative, has the constant loop as a homotopy identity, and $\check{f}$ given by $\check{f}(r + \mathbb{Z}) = f(-r + \mathbb{Z})$ is a homotopy inverse of f.* □

There is one good reason for using the homotopy category instead of the category of spaces and continuous maps: $\Omega(A)$ has a binary multiplication which does not have any particularly nice algebraic properties, but $[\Omega(A)] = [\mathbb{T}, A]$ is a group. Notice that both $[\mathbb{T}, A]$ and $[A, \mathbb{T}]$ are groups, and that their behavior is quite different. The fundamental group is a covariant functor and the Bruschlinsky group is a contravariant functor.

The relations $\pi_1\big(C_0(X, Y)\big) = \big[\Omega\big(C_0(X, Y)\big)\big] \cong [\Sigma X, Y] \cong [X, \Omega Y]$ for locally compact X show that $[\Sigma X, Y]$ is always a group. It follows recursively that all $[\Sigma^n X, Y]$ are groups. With $X = S^1$ and $S^n = \underbrace{S^1 \divideontimes \cdots \divideontimes S^1}_{n \text{ times}}$ we have that all $\pi_n(Y) = [S^n, Y] = [\Sigma(S^{n-1}), Y]$ are groups for $n = 1, 2 \ldots$. These are the *homotopy groups of* Y.

We look now at an elementary but very useful example of functors.

Example A3.27 (Partially ordered sets). Let P and Q be partially ordered sets. Each is a category with the existence of an arrow $x \to y$ if and only if $x \le y$. A functor $g: P \to Q$ is exactly a monotone function, i.e. a function such that $x \le y$ in P implies $g(x) \le g(y)$ in Q. Often we find a monotone map $d: Q \to P$ such that $dx \le y$ if and only if $x \le gy$ for all $x \in Q$ and $y \in P$. This of course is tantamount to saying that the hom-sets $P(dx, y)$ and $Q(x, gy)$ are isomorphic. Also since $P(dx, dx)$ contains an element because of the reflexivity of $\le$, then also $Q(x, gdx)$ contains an element, and thus $x \le gdx$. Whenever $x \le gy$ for an $x \in Q$ and $y \in P$; that is, an arrow $f: x \to gy$, then $dx \le y$ by definition, i.e. there is an arrow $f': dx \to y$ and by the monotonicity of g we get $gdx \leq gy$.

$$\begin{array}{ccccc} & Q & & & P \\ \hline x & \leq & gd(x) & & d(x) \\ {\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle g(f')} & & \downarrow{\scriptstyle \exists! f'} \\ g(y) & = & g(y) & & y \end{array}$$

A pair of monotone functions (d, g) between posets like the one we described is called a (*covariant*) *Galois connection* between P and Q. The function g is called *the upper adjoint* and d *the lower adjoint*.

We remark in this context that a *Heyting algebra* $(H, \vee, \wedge, \Rightarrow)$ (see e.g. [122], p. 25, Definition 3.17) is an algebra with three binary operations such that $(H, \vee, \wedge)$ is a lattice and that for all $a \in H$ we have $y \leq (a \Rightarrow x)$ if and only if $(a \wedge y) \leq x$, i.e. that $x \mapsto (a \Rightarrow x)$ is a upper adjoint of $y \mapsto (a \wedge y)$. Such lattices are distributive. Every Boolean lattice is a Heyting algebra if we set $(a \Rightarrow x) = (\neg a \vee x)$.

We conclude this discussion of examples of functors with one of the most important definitions of elementary category theory and begin by formalizing the concept of universal property.

The Universal Property

Theorem A3.28. *Let $\mathcal{A}$ and $\mathcal{B}$ denote two categories and $U: \mathcal{B} \to \mathcal{A}$ a functor. Assume that $F: \operatorname{ob} \mathcal{A} \to \operatorname{ob} \mathcal{B}$ is a function of objects and that for each object $A \in \operatorname{ob} \mathcal{A}$ there is a morphism $\eta_A: A \to UFA$ such that for all morphisms $f: A \to UB$ in $\mathcal{A}$ there is a unique morphism $f': FA \to B$ such that $f = (Uf') \circ \eta_A$.*

$$\begin{array}{ccccc} & \mathcal{A} & & & \mathcal{B} \\ \hline A & \xrightarrow{\eta_A} & UFA & & FA \\ {\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle Uf'} & & \downarrow{\scriptstyle \exists! f'} \\ UB & \xrightarrow[\mathrm{id}]{} & UB & & B \end{array}$$

Then the following conclusions hold:

(i) *The function F extends in a unique fashion to a functor $F: \mathcal{A} \to \mathcal{B}$ such that for each morphism $\varphi: A_1 \to A_2$ one has $(UF\varphi) \circ \eta_{A_1} = \eta_{A_2} \circ \varphi$, i.e. that the following diagram commutes:*

$$\begin{array}{ccc} A_1 & \xrightarrow{\eta_{A_1}} & UFA_1 \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle UF\varphi} \\ A_2 & \xrightarrow[\eta_{A_2}]{} & UFA_2. \end{array}$$

(ii) *For each pair of objects $(A, B) \in \operatorname{ob} \mathcal{A} \times \operatorname{ob} \mathcal{B}$ there is an $\mathbb{S}$-isomorphism*

$$\mu_{A,B}: \mathcal{B}(FA, B) \to \mathcal{A}(A, UB), \quad \mu_{A,B}(g) = (Ug) \circ \eta_A, \quad \mu_{A,B}^{-1}(f) = f'$$

such that $\mu_{A,FA}(\mathrm{id}_{FA}) = \eta_A$.

Proof. (i) For a morphism $\varphi: A_1 \to A_2$ we set

(1) $$F\varphi: FA_1 \to FA_2, \quad F\varphi = (\eta_{A_2} \circ \varphi)'.$$

The universal property makes this definition possible. According to this property, we now have $(UF\varphi) \circ \eta_{A_1} = (\eta_{A_2} \circ \varphi)' \circ \eta_{A_1} = \eta_{A_2} \circ \varphi$.

(2) $$\begin{array}{ccccc} A_1 & \xrightarrow{\eta_{A_1}} & UFA_1 & & FA_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle UF\varphi} & & \downarrow{\scriptstyle F\varphi = (\eta_{A_2} \circ \varphi)'} \\ A_2 & \xrightarrow[\eta_{A_2}]{} & UFA_2 & & FA_2. \end{array}$$

If $\varphi = \mathrm{id}_A$, then $\mathrm{id}_{FA}: FA \to FA$ is a morphism such that

$$\eta_A = \mathrm{id}_{UFA} \circ \eta_A = (U\,\mathrm{id}_{FA}) \circ \eta_A,$$

and by the uniqueness in the universal property we deduce $F\,\mathrm{id}_A = (\eta_A)'$.

Now we consider $\varphi_j: A_j \to A_{j+1}$, $j = 1, 2$. We must show that $F(\varphi_2 \circ \varphi_1) = F\varphi_2 \circ F\varphi_1$. For this purpose we observe by piling up two diagrams like (2), that $F\varphi_2 \circ F\varphi_1$ is the right fill-in morphism for $F(\varphi_2 \circ \varphi_1)$. The uniqueness in the universal property then shows the required equality.

$$\begin{array}{ccccc} A_1 & \xrightarrow{\eta_{A_1}} & UFA_1 & & FA_1 \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle UF\varphi_1} & & \downarrow{\scriptstyle F\varphi_1 = (\eta_{A_2} \circ \varphi_1)'} \\ A_2 & \xrightarrow[\eta_{A_2}]{} & UFA_2 & & FA_2 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle UF\varphi_2} & & \downarrow{\scriptstyle F\varphi_2 = (\eta_{A_3} \circ \varphi_2)'} \\ A_3 & \xrightarrow[\eta_{A_3}]{} & UFA_3 & & FA_3. \end{array}$$

This proves that F is a functor. Assuming that $f': FA_1 \to FA_2$ was another morphism satisfying $(Uf') \circ \eta_{A_1} = \eta_{A_2} \circ \varphi$ in place of $F\varphi$, then the uniqueness postulate in the universal property shows $f' = F\varphi$. Thus (i) is proved.

(ii) By the universal property, $f \mapsto f': \mathcal{A}(A, UB) \to \mathcal{B}(FA, B)$ is a well-defined function ν such that $\mu_{A,B} \circ \nu = \mathrm{id}_{\mathcal{A}(A,UB)}$. The uniqueness, however, also shows that $\nu \circ \mu_{A,B} = \mathrm{id}_{\mathcal{B}(FA,B)}$. This proves the first assertion of (ii). The conclusion $\eta_A = \mu_{A,FA}(\mathrm{id}_{FA})$ is immediate from the definitions. □

This theorem now permits the following definition:

The Definition of Adjoint Functors

Definition A3.29. Functors $F: \mathcal{A} \to \mathcal{B}$ and $U: \mathcal{B} \to \mathcal{A}$ are called *adjoint* if for every object $A \in \mathrm{ob}\mathcal{A}$ there is a morphism $\eta_A: A \to UF(A)$ in $\mathcal{A}$ such that for all morphisms $f: A \to UB$ in $\mathcal{A}$ there is a *unique* morphism $f': FA \to B$ in $\mathcal{B}$ such that $f = (Uf') \circ \eta_A$ and the functor F is determined as in Theorem A3.28.

$$\begin{array}{ccccc} & \mathcal{A} & & & \mathcal{B} \\ \hline A & \xrightarrow{\eta_A} & UFA & & FA \\ {\scriptstyle\forall f}\downarrow & & \downarrow{\scriptstyle Uf'} & & \downarrow{\scriptstyle \exists! f'} \\ UB & \xrightarrow[\text{id}]{} & UB & & B \end{array}$$

The functor F is called a *left adjoint* and U is called a *right adjoint.* □

We have already observed that $f \mapsto f' \colon \mathcal{A}(A, UB) \to \mathcal{B}(FA, B)$ is a bijection inverted by $g \mapsto (Ug) \circ \eta_A$. The isomorphy of sets

$$\mathcal{A}(A, UB) \cong \mathcal{B}(FA, B)$$

justifies the choice of the adjectives "left" and "right."

Definition A3.30. We also say that two contravariant functors $F\colon \mathcal{A} \to \mathcal{B}$ and $U\colon \mathcal{B} \to \mathcal{A}$ are *adjoint on the right* if there is a morphism $\eta_A\colon A \to UFA$ for each $A \in \operatorname{ob} \mathcal{A}$ such that for each morphism $f\colon A \to UB$ there is a unique morphism $f'\colon B \to FA$ such that $f = (Uf') \circ \eta_A$ and if for $\varphi\colon A_1 \to A_2$ we have $F\varphi = (\eta_{A_2} \circ \varphi)'\colon FA_2 \to FA_1$.

$$\begin{array}{ccccc} & \mathcal{A} & & & \mathcal{B} \\ \hline A & \xrightarrow{\eta_A} & UFA & & FA \\ {\scriptstyle\forall f}\downarrow & & \downarrow{\scriptstyle Uf'} & & \uparrow{\scriptstyle \exists! f'} \\ UB & \xrightarrow[\text{id}]{} & UB & & B \end{array}$$

□

Natural Transformations

In almost all branches of algebra and topology, but also in other areas of mathematics such as functional analysis, one speaks frequently of "natural isomorphisms" or "canonical isomorphisms." For instance, the isomorphism of a finite dimensional vector space with its bidual is natural; an isomorphism between a finite dimensional vector space and its dual is not. The formalism of category theory provides a precise setting for natural morphisms.

Definition A3.31. Let $S, T\colon \mathcal{A} \to \mathcal{B}$ be two functors. A *natural transformation* $\alpha\colon S \to T$ is a function which assigns to each object $A \in \operatorname{ob} \mathcal{A}$ a $\mathcal{B}$-morphism $\alpha_A\colon SA \to TA$ such that the following diagram commutes for all $f\colon A \to A'$ in $\mathcal{A}$.

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SA' \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_{A'}} \\ TA & \xrightarrow[Tf]{} & TA' \end{array}$$

□

If all α_A are isomorphisms, then α is called a natural isomorphism.

We begin with examples from the elementary areas.

Example A3.32. (1) Let $\mathbb{V}$ denote the category of finite dimensional vector spaces over a field K (i.e. $\mathbb{V} = \mathbb{AB}_K$). Let $DV = \mathrm{Hom}_K(V, \mathbb{K})$ denote the dual, and let I denote the identity functor of $\mathbb{V}$. Now

$$\eta_V : IV \to DDV, \quad \eta_V(v)(\omega) = \omega(v)$$

is a natural isomorphism $\eta: I \to D^2$.

It is noteworthy to observe, that V and DV are isomorphic, because they are vector spaces of the same dimension. However, this isomorphism in reality does not count for anything, because it depends on the choice of some basis in both vector spaces. Also, the two functors I and D are not suitable for comparison because I is covariant while D is contravariant. However, the well-known isomorphism of a finite dimensional vector space and its double dual is indeed a natural isomorphism worthy of this name.

(2) In $\mathbb{AB}_R$ for an integral domain R, let I denote the identity functor and tor the self-functor associating with A its torsion submodule $\mathrm{tor}\, A$. Then the inclusion morphism $\eta_A: \mathrm{tor}\, A \to A$ is a natural transformation $\eta: \mathrm{tor} \to I$.

(3) Let $\mathbb{TOPG}$ denote the category of Hausdorff topological groups. Let U: $\mathbb{TOPG} \to \mathbb{TOP}_0$ be the forgetful functor into pointed Hausdorff topological spaces. For a topological group G set $LG = \mathbb{TOPG}(\mathbb{R}, G)$ with the compact open topology. (Cf. 5.7.) Then L is a hom-set functor $\mathbb{TOPG} \to \mathbb{TOP}_0$. For a given $G \in \mathrm{ob}\, \mathbb{TOPG}$ set

$$\exp_G : LG \to UG, \quad \exp X = X(1).$$

Then $\exp: L \to U$ is a natural transformation, called the *exponential function.* When restricted to linear Lie groups, LG can be given the structure of a Lie algebra and UG that of a pointed real analytic manifold. (Cf. 5.41 and 5.36). Then exp is a natural transformation from the functor L to the functor U where L goes from Lie groups to pointed analytic manifolds (underlying real vector spaces) and U from Lie groups to pointed analytic manifolds. As a morphism of pointed analytic manifolds it is real analytic.

One will notice from these examples that natural transformations are very prevalent. They provide the correct formalism for giving a meaning to the frequently used word "canonical" or "natural" in the context of isomorphisms or homomorphisms. Natural transformations are typical for the universal property defining adjoint functors.

Proposition A3.33. *If $F: \mathcal{A} \to \mathcal{B}$ is left adjoint to U then the morphism $\eta_A: A \to UFA$ is a natural transformation $\eta: \mathrm{id}_{\mathcal{A}} \to UF$. The isomorphism*

$$\mu_{A,B} : \mathcal{B}(FA, B) \to \mathcal{A}(A, UB), \quad \mu_{A,B}(g) = (Ug) \circ \eta_A$$

is a natural transformation

$$\mu_{A,\bullet} : \mathcal{B}(FA, \bullet) \to \mathcal{A}(A, U\bullet)$$

for each $A \in \operatorname{ob}\mathcal{A}$ *and*

$$\mu_{\bullet,B}: \mathcal{B}(F\bullet, B) \to \mathcal{A}(\bullet, UB)$$

for each $B \in \operatorname{ob}\mathcal{B}$.

Before we prove this proposition it is feasible to establish a lemma.

Lemma A3.34. *If* $F: \mathcal{A} \to \mathcal{B}$ *and* $U: \mathcal{B} \to \mathcal{A}$ *are two functors and*

$$\mu_{A,B}: \mathcal{B}(FA, B) \to \mathcal{A}(A, UB)$$

is a function for each pair of objects $A \in \operatorname{ob}\mathcal{A}$ *and* $B \in \operatorname{ob}\mathcal{B}$, *then the following conditions are equivalent:*

(1) *For each* $A \in \operatorname{ob}\mathcal{A}$ *and for each* $B \in \operatorname{ob}\mathcal{B}$, *both*

$$\mu_{A,\bullet}: \mathcal{B}(FA, \bullet) \to \mathcal{A}(A, U\bullet),$$

and

$$\mu_{\bullet,B}: \mathcal{B}(F\bullet, B) \to \mathcal{A}(\bullet, UB)$$

are natural transformations.

(2) *For each* $f: A_1 \to A_2$ *in* $\mathcal{A}$, *each* $g: B_1 \to B_2$ *in* $\mathcal{B}$, *each* $\varphi: FA_2 \to B$, *and each* $\psi: FA \to B_1$ *the conditions*

$$(*) \qquad \mu_{AB_2}(g \circ \psi) = Ug \circ \mu_{AB_1}(\psi)$$

and

$$(**) \qquad \mu_{A_1B}(\varphi \circ Ff) = \mu_{A_2B}(\varphi) \circ f$$

hold.

Proof. We have to note that the naturality conditions asserted in (1) mean exactly that the following diagrams of sets and functions commute:

$$\begin{array}{ccc}
\mathcal{B}(FA, B_1) & \xrightarrow{\mathcal{B}(FA,g)} & \mathcal{B}(FA, B_2) \\
{\scriptstyle \mu_{A,B_1}}\downarrow & & \downarrow{\scriptstyle \mu_{A,B_2}} \\
\mathcal{A}(A, UB_1) & \xrightarrow[\mathcal{A}(A,Ug)]{} & \mathcal{A}(A, UB_2),
\end{array}$$

$$\begin{array}{ccc}
\mathcal{B}(FA_2, B) & \xrightarrow{\mathcal{B}(Ff,B)} & \mathcal{B}(FA_1, B) \\
{\scriptstyle \mu_{A_2,B}}\downarrow & & \downarrow{\scriptstyle \mu_{A_1,B}} \\
\mathcal{A}(A_2, UB) & \xrightarrow[\mathcal{A}(f,UB)]{} & \mathcal{A}(A_1, UB).
\end{array}$$

In view of the fact that $\mathcal{B}(FA, g)(\varphi) = g \circ \varphi$ and $\mathcal{B}(Ff, B)(\varphi) = \varphi \circ Ff$ etc. it is straightforward to verify that these commutativity conditions are exactly the conditions expressed in (2). □

Proof of Proposition A3.33. The first assertion on the naturality of η follows from the definition of naturality and from Theorem A3.28.

It remains to show that, firstly, $\mu_{A,\bullet}\colon \mathcal{B}(FA,\bullet) \to \mathcal{A}(A, U\bullet)$ is natural. Let $g\colon B_1 \to B_2$ be a morphism in $\mathcal{B}$ and let $\varphi \in \mathcal{B}(FA, B_1)$. Then $\mu_{A,B_2}(g \circ \varphi) = U(g \circ \varphi) \circ \eta_A = Ug \circ [(U\varphi) \circ \eta_A] = \mathcal{A}(A, Ug)\big(\mu_{A,B_1}(\varphi)\big)$.

Secondly, we have to show the naturality of $\mu_{\bullet,B}$. So let $\varphi \in \mathcal{B}(FA_2, B)$. Then $\mu_{A_1,B}(\varphi \circ Ff) = U(\varphi \circ Ff) \circ \eta_{A_1} = U\varphi \circ [UFf \circ \eta_{A_1}] = U\varphi \circ [\eta_{A_2} \circ f] = [(U\varphi) \circ \eta_{A_2}] \circ f = \mu_{A_2,B}(\varphi) \circ f = \mathcal{A}(f, UB)\big(\mu_{A_2,B}(\varphi)\big)$. Here we have used the naturality of η.

The preceding lemma shows the required naturality conditions. □

Now we consider the converse of Proposition A3.33.

Proposition A3.35. *Assume that $F\colon \mathcal{A} \to \mathcal{B}$ and $U\colon \mathcal{B} \to \mathcal{A}$ are functors and $\mu_{A,B}\colon \mathcal{B}(FA, B) \to \mathcal{A}(A, UB)$ is a bijective function for each A and B such that the functions*

$$\mu_{A,\bullet}\colon \mathcal{B}(FA,\bullet) \to \mathcal{A}(A, U\bullet)$$

and

$$\mu_{\bullet,B}\colon \mathcal{B}(F\bullet, B) \to \mathcal{A}(\bullet, UB)$$

are natural transformations for each $A \in \operatorname{ob}\mathcal{A}$ and for each $B \in \operatorname{ob}\mathcal{B}$. Then $\eta_A = \mu_{A,FA}(\mathrm{id}_{FA})\colon A \to UFA$ is a natural transformation so that F is left adjoint to U through the universal property defined by η_A.

Proof. First we observe the naturality of η_A: If $f\colon A_1 \to A_2$ is a morphism then we have to show that $UFf \circ \eta_{A_1} = \eta_{A_2} \circ f$. But $UFf \circ \eta_{A_1} = UFf \circ \mu_{A_1,FA_1}(\mathrm{id}_{FA_1}) = \mu_{A_1,FA_2}(Ff)$ by $(*)$, and this last expression is $\mu_{A_2FA_2}(\mathrm{id}_{FA_2}) \circ f = \eta_{A_2} \circ f$ by $(**)$.

Next we establish the universal property. Existence: Let $f\colon A \to UB$ be given. Set $f' = \mu_{AB}^{-1}(f)\colon FA \to B$. Then

$$Uf' \circ \eta_A = U\mu_{AB}^{-1}(f) \circ \mu_{A,FA}(\mathrm{id}_{FA}) = \mu_{AB}(\mu_{AB}^{-1}(f)) = f.$$

Uniqueness: If also $Uf'' \circ \eta_A = f$, then $f = \mu_{AB}(f'')$ by (*) and thus $f'' = \mu_{AB}^{-1}(f) = f'$. □

Thus F is left adjoint to U if there is a natural isomorphism

$$\mu_{A,B}\colon \mathcal{B}(FA, B) \to \mathcal{A}(A, UB).$$

Then

$$\nu_{B,A} \overset{\text{def}}{=} \mu_{A,B}^{-1}\colon \mathcal{A}^{\mathrm{op}}(UB, A) \to \mathcal{B}^{\mathrm{op}}(B, FA)$$

is a natural isomorphism. Thus $U\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ is left adjoint to F. Hence by the preceding, there is a natural transformation $\varepsilon_B\colon B \to FUB$ in $\mathcal{B}^{\mathrm{op}}$ relative to which the universal property holds. If we reinterpret this in the original categories, we get:

Proposition A3.36. *Assume that $F\colon \mathcal{A} \to \mathcal{B}$ and $U\colon \mathcal{B} \to \mathcal{A}$ are functors. Then F is left adjoint to U if and only if there is a natural transformation $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{B}}$*

such that for every morphism $g: FA \to B$ *there is a unique morphism* $g': A \to UB$ *with* $\varepsilon_B \circ (Fg') = g$. □

Definition A3.37. The natural transformations η and ε are called *the front adjunction* and *the back adjunction*, respectively. One uses also the terms *unit* and *counit.* □

Proposition A3.38. *Assume that* $F: \mathcal{A} \to \mathcal{B}$ *and* $U: \mathcal{B} \to \mathcal{A}$ *are functors and* $\eta: \mathrm{id}_{\mathcal{A}} \to UF$ *and* $\varepsilon: FU \to \mathrm{id}_{\mathcal{B}}$ *are natural transformations. Then the following statements are equivalent:*

(1) F *is left adjoint to* U *and* η *and* ε *are the front adjunction and the back adjunction, respectively.*
(2) $(\varepsilon F)(F\eta) = F$ *and* $(U\varepsilon)(\eta U) = U$, *that is for all* $A \in \mathrm{ob}\,\mathcal{A}$ *we have* $(\varepsilon_{FA}) \circ (F\eta_A) = \mathrm{id}_{FA}$ *and for all* $B \in \mathrm{ob}\,\mathcal{B}$ *we have* $U(\varepsilon_B) \circ (\eta_{UB}) = \mathrm{id}_{UB}$.

Proof. Exercise EA3.24. □

Exercise EA3.24. Prove Proposition A3.38. □

As a typical example, if $D: \mathcal{A} \to \mathcal{B}$ is a contravariant functor adjoint to itself on the right with the "evaluation" front adjunction $\eta_A: A \to D^2A$, then

$$(\eta_{DA})(D\eta_A) = \mathrm{id}_{DA}$$

under all circumstances. In other words, the objects $DA \in \mathcal{B}$ always have duality.

Equivalence of Categories

Definition A3.39. (i) Two categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if there are adjoint functors $F: \mathcal{A} \to \mathcal{B}$ and $U: \mathcal{B} \to \mathcal{A}$ such that front and back-adjunctions are isomorphisms. Briefly:

$$\mathrm{id}_{\mathcal{A}} \cong UF \qquad \text{and} \qquad FU \cong \mathrm{id}_{\mathcal{B}}\,.$$

(ii) A *skeleton* of a category $\mathcal{A}$ is a category $\mathcal{S}$ satisfying

(a) $\mathcal{S}$ is equivalent to $\mathcal{A}$.
(b) If A, A' are isomorphic in $\mathcal{S}$ then $A = A'$. □

Somewhat more informally, we discussed skeletons in EA3.10.

Proposition A3.40. (i) *Let* $\mathcal{A}$ *be a category. Assume that there is a subclass* $\mathcal{C}$ *of* $\mathrm{ob}\,\mathcal{A}$ *which meets each isomorphy class of* $\mathrm{ob}\,\mathcal{A}$ *in precisely one object. Then the full subcategory* $\mathcal{S}$ *whose objects are those of* $\mathcal{C}$ *is a skeleton of* $\mathcal{A}$.

In particular, if we allow a "Big Axiom of Choice," then every category has a skeleton.

(ii) *Assume that the functor* $F: \mathcal{A} \to \mathcal{B}$ *is left adjoint to the functor* $U: \mathcal{B} \to \mathcal{A}$, *and let* $\mathcal{A}_0$ *denote the full subcategory of* $\mathcal{A}$ *containing all objects* A *such that* $\eta_A: A \to UF(A)$ *is an isomorphism and* $\mathcal{B}_0$ *the full subcategory of* $\mathcal{B}$ *containing all objects* B *such that* $\varepsilon_B: FU(B) \to B$ *is an isomorphism. Then* F *restricts and corestricts to a functor* $F_0: \mathcal{A}_0 \to \mathcal{B}_0$ *and* U *to a functor* $U_0: \mathcal{B}_0 \to \mathcal{A}_0$ *such that* F_0 *is left adjoint to* U_0 *and that* $\eta_A: A \to F_0U_0(A)$ *is an isomorphism for all* $A \in \mathrm{ob}(\mathcal{A}_0)$ *and* $\varepsilon_B: U_0F_0(B) \to B$ *for all* $B \in \mathrm{ob}(\mathcal{B}_0)$. *In particular,* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *are equivalent categories.*

Proof. Exercise EA3.25. □

Exercise EA3.25. Prove Proposition A3.40. [Hint for (ii): If $A \in \mathrm{ob}(\mathcal{A}_0)$, then $\eta_A: A \to UF(A)$ is an isomorphism, hence $F(\eta_A): FA \to FUF(A)$ is an isomorphism. By A3.38(2) we have $\varepsilon_{FA} \circ (F\eta_A) = \mathrm{id}_{FA}$. Hence $\varepsilon_{FA}: FUFA \to FA$ is an isomorphism. Therefore $FA \in \mathrm{ob}(\mathcal{B}_0)$. This shows the existence of the restriction and corestriction $F_0: \mathcal{A}_0 \to \mathcal{B}_0$ of F. The proof of the existence of the restriction and corestriction U_0 of U is analogous and the verification of the other assertions are straightforward.] □

Limits

In this section we deal with functors $D: J \to \mathcal{C}$ which we call *diagrams* if J is a *small* category.

For fixed J and $\mathcal{C}$, the class $\mathcal{C}^J$ of all diagrams together with all natural transformations $\alpha: D_1 \to D_2$ between D_1 and D_2 as hom-set $\mathcal{C}^J(D_1, D_2)$ is a "generalized" category: All axioms of a category are satisfied with the possible exception of that which demands that hom-sets be sets rather than proper classes; if J is small, this condition is satisfied as well. For each object A of $\mathcal{C}$ we obtain the *constant* diagram $\mathrm{const}(A): J \to \mathcal{C}$ mapping all objects j of J to A and all morphisms of J to id_A. A natural transformation $\alpha : \mathrm{const}(A) \to D$ is called a *cone* with vertex A. Any morphism $f: A \to B$ in $\mathcal{C}$ gives a natural transformation of diagrams $\mathrm{const}(f): \mathrm{const}(A) \to \mathrm{const}(B)$. The assignment const is a functor $\mathrm{const}: \mathcal{C} \to \mathcal{C}^J$ of generalized categories.

The Definition of Limits

Definition A3.41. We say that a diagram $D: J \to \mathcal{C}$ has a *limit* $\lim D \in \mathrm{ob}\,\mathcal{C}$ if there is a cone $\lambda: \mathrm{const}(\lim D) \to D$ such that for each cone $\alpha: \mathrm{const}\,A \to D$ there is a unique morphism $\alpha': A \to \lim D$ such that $\alpha = \lambda \circ \mathrm{const}(\alpha')$. □

Proposition A3.42. *The assignment*

$$\nu: \mathcal{C}(A, \lim D) \to \mathcal{C}^J(\mathrm{const}(A), D), \quad \nu(f) = \lambda \circ \mathrm{const}(f): \mathrm{const}(A) \to D$$

is a bijection with inverse $\nu^{-1}(\alpha) = \alpha'$.

Proof. Exercise EA3.26. □

Exercise EA3.26. Prove Proposition A3.41. □

We shall call a category J *discrete* if it has no morphisms except the identity morphisms. We may say that a discrete category has only objects and no morphisms (with the exception of those which it has to have).

Definition A3.43. (i) If J is a discrete category, then a diagram $D: J \to \mathcal{C}$ is nothing but a family of objects $\{D(j) \mid j \in J\}$. The limit $\lim D$ of D is called *the product* of the family, written $P = \prod_{j\in J} A_j$, $A_j = D(j)$, and the morphisms $\lambda_j: P \to A_j$ are called *projections*, frequently written pr_j. If $D = \{A, B\}$, we write $\lim D = A \times B$ and pr_A, pr_B for the projections. If $\alpha: X \to A$ and $\beta: X \to B$ are morphisms, we write $(\alpha, \beta): X \to A \times B$ for the unique fill-in morphism of the product.

(ii) If f, $g: A \to B$ are two morphisms in $\mathcal{C}$, then an object E is called an *equalizer* $\mathrm{eq}(f, g)$ of f and g if there is a morphism $e: E \to A$ such that $fe = ge$ and for every morphism $\varphi: X \to A$ with $f\varphi = g\varphi$ there is a unique morphism $\varphi': X \to E$ with $e\varphi' = \varphi$.

(iii) If $f: A \to C$ and $g: B \to C$ are two morphisms in $\mathcal{C}$, then an object P is called a *pullback* or *fibered product* of f and g written $P = A \times_C B$ (somewhat incompletely, since P depends on f and g), if there are morphisms $p: P \to A$ and $q: P \to B$ such that $fp = gq$ and that for each pair of morphisms $\alpha: X \to A$ and $\beta: X \to B$ with $f\alpha = g\beta$ there is a unique morphism $\xi: X \to P$ such that $\alpha = p\xi$ and $\beta = q\xi$.

We also call the entire diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

a *pullback* if it has the universal property described above. If both f and g are monics, then the pullback P is called an *intersection*. □

Exercise EA3.27. Show that equalizers and pullbacks are limits.

Determine the products, equalizers and pullbacks in the category of sets. Why does an intersection deserve its name? □

Proposition A3.44. (Pullbacks via Products and Equalizers). Let $f: A \to C$ and $g: B \to C$ be morphisms such that the product $A \times B$ and the equalizer $e: P = \mathrm{eq}(f\,\mathrm{pr}_A, g\,\mathrm{pr}_B) \to A \times B$ exist. Then P is the pullback of f and g with $p = \mathrm{pr}_A\, e$ and $q = \mathrm{pr}_B\, e$.

Proof. Verify the universal property of the pullback by diagram chasing. Exercise EA3.28. □

Exercise EA3.28. Prove Proposition A3.44. □

Proposition A3.45 (Equalizers via Products and Intersections). *Assume that we have morphisms $f, g: A \to B$, and that the product $A \times B$ and the pullback E of $(\mathrm{id}_A, f): A \to A \times B$ and $(\mathrm{id}_A, g): A \to A \times B$ exist. If $p, q: E \to A$ are the two pullback morphisms, then $p = q$ and E is the equalizer of f and g with $e = p = q$.*

Proof. Notice that (id_A, f) and (id_A, g) are coretractions, hence certainly monics, whence the pullback is in fact an intersection. Verify the universal property of the equalizer by diagram chasing. See Exercise EA3.29 below. □

Exercise EA3.29. Complete the proof of A3.45. □

Let us say, that a category has *intersections of retracts* if two coretractions have a pullback. The preceding proposition yields the following corollary.

Corollary A3.46. *A category has pullbacks if it has finite products and intersections of retracts.* □

Arbitrary Limits through Equalizers and Arbitrary Products

Theorem A3.47. *Let $D: J \to \mathcal{C}$ denote an arbitrary diagram.*

Hypotheses: Assume that the following products exist:

(i) $P = \prod_{j \in \mathrm{ob}\, J} D(j)$,

(ii) $Q = \prod_{f \in \mathrm{morph}\, J} D(\mathrm{ran}\, f)$.

For each $f \in \mathrm{morph}\, J$ define two morphisms $\alpha_f, \beta_f: P \to D(\mathrm{ran}\, f)$ by $\alpha_f = (Df) \circ \mathrm{pr}_{\mathrm{dom}\, f}$ and $\beta_f = \mathrm{pr}_{\mathrm{ran}\, f}$. By the universal property of the product Q there exist unique morphisms

$$\alpha', \beta': P \to Q \quad \text{with} \quad \alpha_f = \mathrm{pr}_f \circ \alpha' \quad \text{and} \quad \beta_f = \mathrm{pr}_f \circ \beta'.$$

Assume that the following equalizer exists:

(iii) $L = \mathrm{eq}(\alpha', \beta')$, $e: L \to P$.

Conclusions: The prescription $\lambda_j = \mathrm{pr}_j \circ e: L \to D(j)$ defines a cone $\lambda: \mathrm{const}(L) \to D$ and L is the limit of D with respect to this cone.

Proof. (a) For each $f: j \to k$ in J we must show that $\lambda_k = (Df)\lambda_j$, i.e. $\mathrm{pr}_k\, e = (Df)\,\mathrm{pr}_j\, e$. This means $\alpha_f e = \beta_f e$ which is equivalent to $\mathrm{pr}_f\, \alpha' e = \mathrm{pr}_f\, \beta' e$, and since e is the equalizer of α' and β' this condition is satisfied.

(b) Let $\xi: \mathrm{const}\, X \to D$ be a cone. By the universal property of P, there is a unique morphism $\xi'': X \to P$ such that $\xi_j = \mathrm{pr}_j\, \xi''$ for all j. Let $f: j \to k$ in J. Then $\mathrm{pr}_f\, \alpha' \xi'' = \alpha_f \xi'' = (Df)\,\mathrm{pr}_j\, \xi'' = (Df)\xi_j = \xi_k = \mathrm{pr}_k\, \xi'' = \beta_f \xi'' = \mathrm{pr}_f\, \beta' \xi''$. By the uniqueness of the universal property of Q we conclude $\alpha' \xi'' = \beta' \xi''$. Since L is the equalizer of α' and β' we conclude the existence of a unique $\xi': X \to L$ such that $\xi'' = e\xi'$. Then $\xi_j = \mathrm{pr}_j\, \xi'' = \mathrm{pr}_j\, e\xi' = \lambda_j \xi'$. The uniqueness of ξ' follows from its construction. □

The preceding results yield at once:

The Limit Existence Theorem

Theorem A3.48. *If a category $\mathcal{C}$ has arbitrary products and intersections of retracts, then it has arbitrary limits.* □

This theorem is of the greatest importance in determining the existence of limits. It reduces the verification of the existence of limits to that of intersections and products. In general, these are readily checked or refuted.

Definition A3.49. A category is said to be *complete* if it has arbitrary limits.

Exercise EA3.30. Give examples of complete categories. Try $\mathbb{S}$, $\mathbb{TOP}$, compact Hausdorff spaces, $\mathbb{AB}_R$, any algebraic variety such as semigroups, abelian semigroups, semilattices, rings. □

The category $\mathbb{BAN}$ of Banach spaces and bounded operators as morphisms is not complete. However, the category $\mathbb{BAN}$contr of Banach spaces and linear contractions is complete. The products in this category are remarkable: *If* $\{B_j \mid j \in J\}$ *is a family of Banach spaces and* $\prod^{\mathbb{S}}_{j\in J} B_j$ *is the product in* $\mathbb{S}$, *i.e. the cartesian product, then* $\prod_{j\in J} B_j = \{(x_j)_{j\in J} \in \prod^{\mathbb{S}}_{j\in J} B_j \mid \sup_{j\in J} \|x_j\| < \infty\}$ *is the product in* $\mathbb{BAN}$contr. The category C^* of all C^*-algebras and *-morphisms between them is complete (since all of these morphisms are in $\mathbb{BAN}$contr).

The categories of locally compact Hausdorff spaces (groups etc.) and the category of Hilbert spaces are notoriously incomplete. (It should be clear what morphisms are meant. No sensible choice of morphisms between Hilbert spaces produces a complete category.)

Exercise EA3.31. Investigate a partially ordered set $(X, \leq)$ as a category with $j \to k$ iff $j \leq k$. What are limits? What does completeness mean? □

Now we consider the relation between functors and limits.

The Definition of Continuous Functors

Definition A3.50. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to *preserve limits* or to be *continuous* if for every diagram $D\colon J \to \mathcal{A}$ with a limit $\lim D$, $\lambda\colon \mathrm{const}(\lim D) \to D$ the diagram $FD\colon J \to \mathcal{B}$ has the limit $F(\lim D)$, $F\lambda\colon F\big(\mathrm{const}(\lim D)\big) \to FD$. □

In a considerable compactification of the circumstances we can write: "F is continuous iff $\lim FD \cong F(\lim D)$."

Proposition A3.51. *The following conditions are equivalent for a functor U from a complete category to an arbitrary category:*

(i) *U preserves arbitrary limits, i.e. U is continuous.*

(ii) *U preserves intersections and arbitrary products.*
(iii) *U preserves equalizers and arbitrary products.*

Proof. Exercise EA3.32. □

Exercise EA3.32. Prove Proposition A3.51.
[Hint. Use Theorem A3.47 and Propositions A3.44 and A3.45.] □

The Continuity of Adjoints

It is of great importance to note that right adjoint functors preserve limits.

The Continuity of Adjoints

Theorem A3.52. *If a functor $U: \mathcal{B} \to \mathcal{A}$ has a left adjoint, then it preserves all limits.*

Proof. Let $F: \mathcal{A} \to \mathcal{B}$ denote the left adjoint of U. The naturality of the back adjunction $\varepsilon_B: FUB \to B$ yields a commutative diagram of natural transformations

$$\begin{array}{c|ccc}
\mathcal{A} & & \mathcal{B} & \\
\hline
UD(j) & FUD(j) & \xrightarrow{\varepsilon_{D(j)}} & D(j) \\
{\scriptstyle U\lambda_j}\uparrow & \uparrow{\scriptstyle FU\lambda_j} & & \uparrow{\scriptstyle \lambda_j} \\
U(\lim D) & FU(\lim D) & \xrightarrow[\varepsilon_{\lim D}]{} & \lim D.
\end{array}$$

We shall operate in the framework of this diagram. Let $\alpha_j: A \to UD(j)$ denote a cone. We have to find a morphism $\alpha': A \to U(\lim D)$ uniquely such that $\alpha_j = (U\lambda_j)\alpha'$. (This happens on the left side of the diagram.) We define $\beta_j: FA \to D(j)$ by $\beta_j = \varepsilon_{D(j)}(F\alpha_j)$. By the universal property of the limit there exists a unique morphism $\beta': FA \to \lim D$ such that $\beta_j = \lambda_j\beta'$. By the adjunction there is a unique $\alpha': A \to U(\lim D)$ such that $\beta' = \varepsilon_{\lim D}(F\alpha')$. Now we start diagram chasing on the right side: $\varepsilon_D(j)F\big((U\lambda_j)\alpha'\big) = \varepsilon_{D(j)}(FU\lambda_j)(F\alpha') = \lambda_j\varepsilon_{\lim D}(F\alpha') = \lambda_j\beta' = \beta_j = \varepsilon_{D(j)}(F\alpha_j)$. The uniqueness in the universal property of the adjunction now shows that $\alpha_j = (U\lambda_j)\alpha'$—which is what we had to show. □

The Left Adjoint Existence Theorem

It is almost true that, conversely, a continuous functor $U: \mathcal{B} \to \mathcal{A}$ defined on a complete category has a left adjoint. We shall begin by considering a useful construct aiding us in the main existence proof.

The Definition of Comma Category

Definition A3.53. Let A denote an object in a category $\mathcal{A}$ and let $U\colon \mathcal{B} \to \mathcal{A}$ denote a functor. The *comma category* (A, U) has as objects all pairs (f, B) of morphisms $f\colon A \to UB \in \mathcal{A}(A, UB)$ for all $B \in \operatorname{ob}\mathcal{B}$, and the hom-set

$$(A, U)\big((f_1, B_1), (f_2, B_2)\big)$$

consists of all morphisms $\varphi \in \mathcal{B}(B_1, B_2)$ with $f_2 = (U\varphi)f_1$. □

We verify very quickly that this is a category.

Remark A3.54. An initial object (η_A, FA) in this category is a morphism $\eta_A\colon A \to UFA$ such that for each morphism $f\colon A \to UB$ there is a unique morphism $f'\colon FA \to B$ such that $f = (Uf') \circ \eta_A$. □

This is immediate from the definitions and shows that, according to Proposition A3.28, the existence of a left adjoint F of U is equivalent to the existence of an initial element in each of the comma categories (A, U) as A ranges through the objects of $\mathcal{A}$.

Lemma A3.55. *Assume that A is an object of $\mathcal{A}$ and that $U\colon \mathcal{B} \to \mathcal{A}$ is a functor. Let the functor $D\colon (A, U) \to \mathcal{B}$ be given by $D(f, B) = B$ on objects and $D\varphi = \varphi$ on morphisms. Consider the following two statements:*

(1) *The comma category (A, U) has an initial object (η_A, FA).*

(2) *The diagram $D\colon (A, U) \to \mathcal{B}$ (which is large in general) has a limit $\lim D$ with the limit cone $\lambda_{(f,B)}\colon \lim D \to B$.*

Then (1) *implies* (2), *and if U preserves limits then the conditions are equivalent.*

Proof. (1) ⇒(2) Assume that the initial object (η_A, FA) of (A, U) exists. We claim that $\lambda_{(f,B)}\colon FA \to D(f, B)$, $\lambda_{(f,B)} = f'\colon FA \to B$ is a limit cone in $\mathcal{A}$.

Firstly, the family $\lambda_{(f,B)}$ is a cone: Assume that $\varphi\colon (f_1, B_1) \to (f_2, B_2)$ is a morphism in (A, U). Then $\varphi\colon B_1 \to B_2$ is a $\mathcal{B}$-morphism such that $(U\varphi)f_1 = f_2$. By the initiality of (η, FA) we get *unique* morphisms $\lambda_{(f_j,B_j)} = f'_j\colon FA \to B_j$ such that $(Uf'_j)\eta_A = f_j$ for $j = 1, 2$. The upper part of the following diagram commutes:

$$\begin{array}{ccccc}
A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A \\
{\scriptstyle \eta_A}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{id}_A} \\
UFA & \xrightarrow[Uf_1']{} & UB_1 & \xleftarrow{f_1} & A \\
{\scriptstyle \mathrm{id}_{UFA}}\big\downarrow & & \big\downarrow{\scriptstyle U\varphi} & & \big\downarrow{\scriptstyle \mathrm{id}_A} \\
UFA & \xrightarrow[Uf_2']{} & UB_2 & \xleftarrow[f_2]{} & A
\end{array}$$

$$\begin{array}{ccc}
FA & \xrightarrow{f_1'} & B_1 \\
{\scriptstyle \mathrm{id}_{FA}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\
FA & \xrightarrow[f_2']{} & B_2.
\end{array}$$

Hence both $\varphi f_1'\colon FA \to B_2$ and $f_2'\colon FA \to B_2$ define morphisms $(\eta_A, FA) \to (f_2, B_2)$. By the uniqueness in the initial object property,

$$\lambda_{(f_2,B_2)} = f_2' = \varphi f_1' = (D\varphi)\lambda_{(f_1,B_1)}$$

follows. And this proves the claim that $\lambda_{(f,B)}\colon FA \to D(f,B)$ is a cone.

Secondly we verify the universal property. First the existence part: Assume that $\alpha_{(f,B)}\colon B_0 \to D(f,B) = B$ is a cone. We set $\alpha' = \alpha_{(\eta_A,FA)}\colon B_0 \to FA$ and note that $\lambda_{(f,B)} \circ \alpha' = f' \circ \alpha_{(\eta_A,FA)} = \alpha_{(f,B)}$ by the naturality of α. Next the uniqueness part: Assume that $\alpha''\colon B_0 \to D(f,B) = B$ is a $\mathcal{B}$-morphism such that $f'\alpha'' = \lambda_{(f,B)}\alpha'' = \alpha_{(f,B)} = \lambda_{(f,B)}\alpha'$ for all $(f,B) \in \mathrm{ob}(A,U)$. In particular, this applies to $(f,B) = (\eta_A, FA)$. But the initial object $\eta_A(FA)$ has only one endomorphism, namely, the identity $\mathrm{id}_{FA}\colon FA \to FA$, and thus $\eta_A'\colon FA \to FA$ agrees with id_A. Therefore, $\alpha'' = \eta_A'\alpha'' = \eta_A'\alpha' = \alpha'$. Thus α' is unique with respect to the fill-in property. Thus $FA = \lim D$ and $\lambda_{(f,B)}\colon FA \to D(f,B) = B$ is a limit cone as asserted.

(2)⇒(1) For the proof of the converse we assume now that U preserves limits and assume that $\lim D$ exists with limit cone $\lambda_{(f,B)}\colon \lim D \to D(f,B) = B$. Then we set $FA = \lim D$ and know that $U\lambda_{(f,B)}\colon UFA \to UB$ is a limit cone since U preserves limits. Now $\alpha_{(f,B)} = f\colon A \to UD(f,B) = UB$ is a cone by the definition of the comma category (A,U), and by the limit property there exists a unique $\eta_A\colon A \to UFA$ such that

$$(*) \qquad \big(\forall (f,B) \in \mathrm{ob}(A,U)\big) \quad f = \alpha_{(f,B)} = \big(U\lambda_{(f,B)}\big)\eta_A.$$

We consider $\lambda_{(\eta_A,FA)}\colon \lim D \to D(\eta_A, FA) = FA$. Our first aim is to show that

$$(**) \qquad \lambda_{(\eta_A,FA)} = \mathrm{id}_{FA}\,.$$

By the definition of $\eta_A\colon A \to U(\lim D)$ as the universal fill-in morphism for the limit cone $U\lambda_{(f,B)}\colon U\lim D \to UD(f,B) = UB$ we have $\big(U\lambda_{(f,B)}\big)\eta_A = f$ for all (f,B). Therefore $\lambda_{(f,B)}\colon (\eta_A, FA) \to (f,B)$ is an (A,U)-morphism and $\lambda_{(f,B)} = D\lambda_{(f,B)}$.

It follows that

$$\lambda_{(f,B)}\lambda_{(\eta_A,FA)} = \big(D\lambda_{(f,B)}\big)\lambda_{(\eta_A,FA)} = \lambda_{(f,B)} = \lambda_{(f,B)}\,\mathrm{id}_{FA}$$

by the naturality of λ.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & FA \\ {\scriptstyle \mathrm{id}_A}\big\downarrow & & \big\downarrow{\scriptstyle U\lambda_{(f,B)}} \\ A & \xrightarrow[f]{} & UB \end{array}$$

$$\begin{array}{ccc} FA & \xrightarrow{\mathrm{id}_{FA}} & FA \\ {\scriptstyle \lambda_{(\eta_A,FA)}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_{(f,B)}} \\ D(\eta_A,FA) & \xrightarrow[D\lambda_{(f,B)}]{} & D(f,B). \end{array}$$

Thus we have two fill-in maps $\lambda_{(\eta_A,FA)}: FA \to \lim D$ and $\mathrm{id}_{FA}: FA \to \lim D$ for the cones $\lambda_{(f,B)}: FA \to D(f,B)$ and $\lambda_{(f,B)}: \lim D \to D(f,B)$. By the uniqueness in the limit property, $\lambda_{(\eta_A,FA)} = \mathrm{id}_{FA}$ follows as asserted in $(**)$.

We now claim that (η_A, FA) is an initial element of (A, U). For a proof of the claim let (f, B) be an arbitrary object of (A, U). We have to find a unique morphism $\varphi: (\eta_A, FA) \to (f, B)$, i.e. a unique morphism $\varphi: FA \to B$ in $\mathcal{B}$ such that $(U\varphi)\eta_A = f$ in $\mathcal{A}$. First the existence: Set $\varphi = \lambda_{(f,B)}: (\eta_A, FA) \to (f, B)$. Then $f = \big(U\lambda_{(f,B)}\big)\eta_A$ by $(*)$. Now we have to show uniqueness: Assume that $\psi: (\eta_A, FA) \to (f, B)$ is a morphism in (A, U). Then $\psi = \psi\,\mathrm{id}_{FA} = (D\psi)\lambda_{(\eta_A,FA)} = \lambda_{(f,B)} = \varphi$ by the naturality of λ.

$$\begin{array}{ccc} \lim D & \xrightarrow{\mathrm{id}_{\lim D}} & \lim D \\ {\scriptstyle \lambda_{(\eta_A,FA)}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_{(f,B)}} \\ D(\eta_A,FA) & \xrightarrow[D\psi]{} & D(f,B). \end{array}$$

The uniqueness of φ is established and the proof is complete. □

In order to reduce the problem of the existence of "large" limits to "small limits" we next discuss the concept of cofinality.

Definition A3.56. A functor $C: K \to J$ is called *cofinal* if it satisfies the following two conditions:

(i) $(\forall j \in \mathrm{ob}\, J)(\exists f)\ f \in J(Ck, j)$. A simple diagram:

$$j \xleftarrow{f} Ck.$$

(ii) $(\forall f_i \in J(Ck_i, j),\ i = 1,2)(\exists g_i \in J(k, k_i))\ f_1(Cg_1) = f_2(Cg_2)$. A simple diagram:

$$\begin{array}{ccc} Ck_1 & \xleftarrow{\ Cg_1\ } & Ck \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle Cg_2} \\ j & \xleftarrow[f_2]{} & Ck_2. \end{array}$$

□

Theorem A3.57. *Assume that $C: K \to J$ is a cofinal functor and $D: J \to \mathcal{C}$ a diagram. If $\lim DC$ exists with limit cone $\lambda: \lim DC \to DC$, then D has the limit $\lim DC$ with a suitable limit cone $\widetilde{\lambda}: \lim DC \to D$ such that $\widetilde{\lambda}_{Ck} = \lambda_k$.*

Proof. First the construction of $\widetilde{\lambda}$: If $j \in \operatorname{ob} J$, then by A3.56(i) there is an $f: Ck \to j$ and we set $\lambda_{j,f} = (Df)\lambda_k: \lim DC \to D(j)$. Now assume that $f_i: Ck_i \to j$, $i = 1,2$ are given. Then by A3.56(ii) we find $g_i: k \to k_i$ with $f_1(Cg_1) = f_2(Cg_2)$. Now $\lambda_{j,f_1} = (Df_1)\lambda_{k_1} = (Df_1)(DCg_1)\lambda_k = D\big(f_1(Cg_1)\big)\lambda_k = D(f_2Cg_2)\lambda_k = \cdots = \lambda_{j,f_2}$. Thus $\lambda_{j,f}$ does not depend on the particular choice of f and we may set $\widetilde{\lambda}_j = \lambda_{j,f}$ unambiguously. We note that $\widetilde{\lambda}_{Ck} = \lambda_{k,\mathrm{id}_{Ck}} = \lambda_k$. With A3.56(ii) again we verify that $\widetilde{\lambda}: \lim DC \to D$ is a cone.

We claim it has the universal property. Let $\alpha: A \to D$ be a cone. Then $\alpha_{Ck}: A \to DC(k)$ is a cone, and by the limit property there is a unique morphism $\alpha': A \to \lim DC$ such that $\alpha_{Ck} = \lambda_k\alpha'$. If now $j \in \operatorname{ob} J$ by A3.56(i) we find an $f: Ck \to j$, and by the definition of $\widetilde{\lambda}$ we have $\widetilde{\lambda}_j = (Df)\lambda_k$. Now $\alpha_j = (Df)\alpha_{Ck}$ (since α is natural) $= (Df)\lambda_k\alpha' = \widetilde{\lambda}_j\alpha'$. We note the uniqueness of α' as a fill-in for the cone $\alpha_j: A \to Dj$: Let $\alpha'': A \to \lim DC$ be a morphism such that $\widetilde{\lambda}_j\alpha'' = \widetilde{\lambda}_j\alpha'$ for all $j \in \operatorname{ob} J$. Let $k \in \operatorname{ob} K$. Then $\lambda_k\alpha'' = \widetilde{\lambda}_{Ck}\alpha'' = \widetilde{\lambda}_{Ck}\alpha'$, and the uniqueness of α' as a fill-in for the cone $\alpha_{Ck}: A \to DCk$ shows $\alpha'' = \alpha'$.

The proof is now complete. □

Definition A3.58. We say that the functor $U: \mathcal{B} \to \mathcal{A}$ *satisfies the solution set condition* if for each $A \in \operatorname{ob} \mathcal{A}$ there is a *set* $S(A)$ of pairs (φ, M), $\varphi: A \to UM$ such that for every pair (f, B), $f: A \to UB$ there is some $(\varphi, M) \in S(A)$ with some factorisation $f = (Uf_0)\varphi$ and $f_0: M \to B$. □

Lemma A3.59. *Assume that $\mathcal{B}$ has pullbacks and that U preserves pullbacks and satisfies the solution set condition. Then the full subcategory $\langle S(A)\rangle$ generated in the comma category (A, U) by the objects of $S(A)$ is cofinal in (A, U); i.e. the inclusion functor is cofinal.*

Proof. First we show A3.56(i). If $(f, B) \in \operatorname{ob}(A, U)$ then by Definition A3.58 there is a $(\varphi, M) \in \operatorname{ob}\langle S(A)\rangle$ and an $f_0 \in (A, U)\big((\varphi, M), (f, B)\big)$.

Next we show A3.56(ii). Let $f_i \in (A,U)\big((\varphi_i, M_i),(f,B)\big)$, $i=1,2$ be given. Since $\mathcal{B}$ has pullbacks, we can form the pullback of $f_i\colon M_i \to B$, say, $p_i\colon P \to M_j$:

$$\begin{array}{ccc} P & \xrightarrow{p_1} & M_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ M_2 & \xrightarrow[f_2]{} & B. \end{array}$$

Since U preserves pullbacks, $Up_j\colon UP \to UM_j$ is a pullback of $Uf_i\colon UM_j \to UB$. Since $(Uf_1)\varphi_1 = f = (Uf_2)\varphi_2$, the pull back property gives $\pi\colon A \to UP$ such that $\varphi_i = (Up_i)\pi$, i.e. that $p_i \in (A,U)\big((\pi,P),(\varphi_i,M_i)\big)$. By the solution set condition, there is a $(\varphi_0, M) \in \mathrm{ob}(A,U)$ and an $f_0 \in (A,U)\big((\varphi_0,M),(\pi,P)\big)$. Set $g_i = p_i f_0 \in (A,U)\big((\pi,P),(\varphi_i,M_i)\big)$, $i=1,2$. Then $f_1 g_1 = f_2 g_2$ and the assertion A3.56(ii) is established. □

We have now arrived at a major result in category theory.

The Adjoint Functor Existence Theorem

Theorem A3.60. *Assume that the functor $U\colon \mathcal{B} \to \mathcal{A}$ satisfies the solution set condition and that $\mathcal{B}$ is complete. Then the following conditions are equivalent:*
(1) *U has a left adjoint $F\colon \mathcal{A} \to \mathcal{B}$.*
(2) *U preserves limits.*

Proof. In A3.52 we have seen that (1) implies (2). For each $A \in \mathrm{ob}\,\mathcal{A}$ we set $J = \langle S(A)\rangle$ and define $D\colon J \to \mathcal{B}$ by $D(\varphi, M) = M$ (and accordingly for morphisms). Since $\mathcal{B}$ is complete, $FA = \lim D$ exists. Since $\langle S(A)\rangle$ is cofinal in (A,U), then FA is the limit of the functor $(f,B) \mapsto B\colon (A,U) \to \mathcal{B}$ according to Theorem A3.57. Since U preserves limits, Lemma A3.55 applies. Thus (A,U) has an initial element (η_A, FA). But then by Remark A3.54 and by Theorem A3.28, this implies the existence of the desired left adjoint F. □

In checking the concrete occurrences of the situation of the Adjoint Functor Existence Theorem one observes that the Solution Set Condition practically never causes problems, and all the other conditions are readily verified.

As a rule, the question of free and universal objects can therefore be disposed of with the following sentence.

By the Adjoint Functor Existence Theorem, the required universal objects exist functorially.

Standard examples are all free objects, also such things as free topological groups, free compact groups, free compact abelian groups, free compact semigroups, free compact spaces (that is, Stone–Čech compactifications), almost periodic and weakly almost periodic compactifications of topological groups and so on ad infinitum.

Commutative Monoidal Categories and their Monoids

The concept of commutative monoidal categories and the multiplicative functors between them is at the root of Hopf algebras, at least wherever they arise naturally such as group algebras, universal enveloping algebras of Lie algebras (accordingly also polynomial algebras), exterior algebras, cohomology algebras of compact groups, dual objects of compact groups, dual objects of algebraic groups. Most of these avenues are not pursued in this book; the issue of the cohomology of compact groups alone is motivation enough to present the category theoretical background in this appendix. It is inherent in the technical complexity of the subject that this subsection is not as self-contained as are the preceding sections.

Part 1: The Quintessential Diagram Chase

We consider a category $\mathbb{A}$ which supports a functor $\otimes\colon \mathbb{A}\times\mathbb{A}\to\mathbb{A}$.

Let us look at examples; the simplest examples are the categories having finite products.

Examples A3.61. (i) Let $\mathbb{S}$ denote the category of sets and functions. Define $X\otimes Y \overset{\text{def}}{=} X\times Y$.

(ii) Let $\mathbb{CTOP}$ denote the category of compact spaces and continuous functions. Define $X\otimes Y \overset{\text{def}}{=} X\times Y$.

(iii) Let R be a commutative ring with identity and $\mathbb{AB}_R$ the category of left modules over R. Set $V\otimes W \overset{\text{def}}{=} V\otimes_R W$, the tensor product of the R-modules V and W.

(iv) Let $\mathbb{AB}^*_R$ be the category of (nonnegatively) graded R-modules $V = \bigoplus_{n\in\mathbb{N}_0} V^n$ and gradation preserving linear maps. Set

$$V\otimes^* W = \bigoplus_{n\in\mathbb{N}_0}(V\otimes^* W)^n, \quad \text{where} \quad (V\otimes^* W)^n = \bigoplus_{p+q=n} V^p\otimes_R W^q. \qquad \square$$

If in the Examples (iii) and (iv) we let R be a field K, then $\mathbb{AB}_K$ is the category of K-vector spaces and $\mathbb{AB}^*_K$ is the category of graded K-vector spaces. As we progress we will soon specialize to the case that $R=K$ is a field of characteristic 0. Thus the reader will not lose much in understanding of the remainder of this appendix if R is taken to be a field of characteristic 0 right away.

Since later in this section we shall have a lot to do with graded R-modules $A=\bigoplus_{n\in\mathbb{N}_0} A^n$ let us recall some standard terminology: The direct summand A^n is called the *homogeneous component of A of degree n*. An element $x\in A$ is called *homogeneous* if there is an $n\in\mathbb{N}_0$ such that $x\in A^n$ and if $x\neq 0$ then the *degree* $\deg(x)$ is n.

Now we attempt to express the fact that $\otimes$ is associative and assume that there is a natural isomorphism

$$\alpha_{ABC}: A \otimes (B \otimes C) \to (A \otimes B) \otimes C.$$

In the category $\mathbb{S}$ of sets or the category $\mathbb{CTOP}$ of compact spaces we may take $\alpha_{XYZ}\big(\xi, (\eta, \zeta)\big) = \big((\xi, \eta), \zeta\big)$; in the category $\mathbb{AB}_R$ of R-modules with the tensor product we have indeed a natural isomorphism α_{UVW} which is characterized by the identity $\alpha_{UVW}\big(u \otimes (v \otimes w)\big) = \big((u \otimes v) \otimes w\big)$. Correspondingly, we have a natural isomorphism α_{UVW} implementing the associativity of the tensor product of graded vector spaces.

We can form a diagram which may or may not be commutative.

$$\begin{array}{ccc}
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\mathrm{id}_{(A\otimes B)\otimes(C\otimes D)}} & (A \otimes B) \otimes (C \otimes D) \\
{\scriptstyle \alpha_{AB,C\otimes D}}\Big\uparrow & & \Big\downarrow{\scriptstyle \alpha_{A\otimes B,CD}} \\
A \otimes \big(B \otimes (C \otimes D)\big) & & \big((A \otimes B) \otimes C\big) \otimes D \\
{\scriptstyle \mathrm{id}_A \otimes \alpha_{BCD}}\Big\downarrow & & \Big\uparrow{\scriptstyle \alpha_{ABC} \otimes \mathrm{id}_D} \\
A \otimes \big((B \otimes C) \otimes D\big) & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & \big(A \otimes (B \otimes C)\big) \otimes D.
\end{array}$$

In our basic examples, this diagram does commute. Because of its five different vertices we shall call it the *pentagon diagram.*

Next we focus on the commutativity of multiplication and consider a natural isomorphism $\kappa_{AB}: A \otimes B \to B \otimes A$ with $\kappa_{AB}^{-1} = \kappa_{BA}$. In our basic examples one readily identifies such isomorphisms: In the case of the cartesian product in $\mathbb{S}$ or $\mathbb{CTOP}$ we have $\kappa_{XY}(\xi, \eta) = (\eta, \xi)$, in the case of the tensor product of R-modules there is indeed an isomorphism κ_{VW} characterized by $\kappa_{VW}(v \otimes w) = w \otimes v$.

An interesting situation arises when we define isomorphisms κ that are suitable for the tensor product of graded R-modules: If $V = \bigoplus_{n\in\mathbb{N}_0} V^n$ and $W = \bigoplus_{n\in\mathbb{N}_0} W^n$ are graded R-modules, consider elements $v = \sum_{n=0}^{\infty} v_n \in V$ and $w = \sum_{n=0}^{\infty} w_n \in W$. We define two involutive isomorphisms

$$\kappa_{VW},\, \kappa'_{VW}: V \otimes^* W \to W \otimes^* V$$

as follows: Write $v \otimes^* w = \sum_{n=0}^{\infty} (v \otimes^* w)^n$ and define

$$[\kappa] \qquad \big(\kappa_{VW}(v \otimes^* w)\big)^n = \sum_{p+q=n} (-1)^{pq} w_q \otimes v_p \in (W \otimes^* V)^n,$$

$$[\kappa'] \qquad \big(\kappa'_{VW}(v \otimes^* w)\big)^n = \sum_{p+q=n} w_q \otimes v_p \in (W \otimes^* V)^n.$$

Using α's and κ's we can form the following diagram which may or may not commute for all $A, B, C \in \mathrm{ob}\,\mathbb{A}$:

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\alpha_{ABC}} & (A \otimes B) \otimes C & \xrightarrow{\kappa_{A\otimes B,C}} & C \otimes (A \otimes B) \\ {\scriptstyle \mathrm{id}_A \otimes \kappa_{BC}} \downarrow & & & & \downarrow {\scriptstyle \alpha_{CAB}} \\ A \otimes (C \otimes B) & \xrightarrow[\alpha_{ACB}]{} & (A \otimes C) \otimes B & \xrightarrow[\kappa_{AC} \otimes \mathrm{id}_B]{} & (C \otimes A) \otimes B \end{array}$$

This diagram is called *the hexagon diagram.*

A multiplication, like in a commutative monoid, should have a neutral element. Therefore, we finally assume that the category $\mathbb{A}$ has an object E and natural isomorphisms $\iota_A: E \otimes A \to A$, $\iota'_A: A \otimes E \to A$. We may derive ι' from ι via κ if κ is available.

In the category of sets or compact spaces we may take for E a one element set, e.g. $\{\emptyset\}$, and for ι and ι' the respective projections; in the category of modules over the commutative ring R we may take for E the module underlying R. Then the scalar multiplication $(r, v) \mapsto r{\cdot}v: R \times V \to V$ gives an isomorphism $\iota_V: R \otimes V \to V$.

In the prototypical examples the following diagram commutes:

$$\begin{array}{ccc} E \otimes A & \xrightarrow{\kappa_{EA}} & A \otimes E \\ {\scriptstyle \iota_A} \downarrow & & \downarrow {\scriptstyle \iota'_A} \\ A & \xrightarrow[\mathrm{id}_A]{} & A. \end{array}$$

We call it the *triangle diagram.* The object E is only relevant up to natural isomorphism. E.g. $E \otimes E$ and E are naturally isomorphic via the very definition of ι.

Definition A3.62. A *commutative monoidal category* is a 7-tuple

$$(\mathbb{A}, \otimes, \alpha, \kappa, E, \iota, \iota')$$

with natural isomorphisms α, κ, ι and ι' such that the pentagon diagram, the hexagon diagram, and the triangle diagram commute. □

Frequently a commutative monoidal category is called a *symmetric monoidal category.* It is sloppy but convenient to speak of $(\mathbb{A}, \otimes)$ or even of $\mathbb{A}$ as a commutative monoidal category if the other ingredients are specified or understood. Commutative monoidal categories are quite prevalent. In this book we shall consider only a small selection.

Proposition A3.63. *The following are commutative monoidal categories.*

(A) *The category* $(\mathbb{CTOP}, \times)$ *with the cartesian product as multiplication, a one element object* $E = \{\emptyset\}$*, and the standard isomorphisms* α*,* κ*,* ι *and* ι'*.*

(B) *The category* $(\mathbb{AB}_R, \oplus)$ *of* R*-modules for a commutative ring* R *and the direct sum as multiplication, the object* $E = \{0\}$ *as* R*-module, and the standard isomorphisms* α*,* κ*,* ι *and* ι'*.*

(B*) *The category* $(\mathbb{AB}_R^*, \oplus^*)$ *of graded* R*-modules for a commutative ring* R *and the graded direct sum as multiplication, the object* $E = \{0\}$ *as a graded* R*-module, and the standard isomorphisms* α, κ, ι *and* ι'.

(C) *The category* $(\mathbb{AB}_R, \otimes)$ *of* R*-modules for a commutative ring* R *and the tensor product over* R *as multiplication, the object* $E = R$ *as* R*-module, and the standard isomorphisms* α, κ, ι *and* ι'.

(C*) *The category* $(\mathbb{AB}_R^*, \otimes^*)$ *of graded* R*-modules for a commutative ring* R *and the graded tensor product as multiplication, the object* $E = \bigoplus_{n\in\mathbb{N}_0} E^n$ *as graded* R*-module with* $E^0 = R$, $E^n = \{0\}$ *for* $n > 0$, *and the standard isomorphisms* α, ι *and* ι'. *For* κ *we take the isomorphism given in* $[\kappa]$.

(D) *The category* $(\mathbb{AB}_R^*, \otimes^*)'$ *of graded* R*-modules for a commutative ring* R *and the graded tensor product as multiplication, the object* $E = \bigoplus_{n\in\mathbb{N}_0} E^n$ *as graded* R*-module with* $E^0 = R$, $E^n = \{0\}$ *for* $n > 0$, *and the standard isomorphisms* α, ι *and* ι'. *For* κ *we take the isomorphism given in* $[\kappa']$.

Proof. Exercise EA3.33. □

The case (D) will only occur in examples.

Exercise EA3.33. Prove Proposition A3.63. □

On the abstract level, the theory of monoidal symmetric categories is based on a fact which MacLane called *coherence*: With the available natural isomorphisms, a countably infinite family of correctly formed diagrams can be drawn which may or may not be commutative. MacLane's Coherence Theorem states that all of these commute. The proof proceeds by a technically complicated set of proofs of judiciously chosen assertions by induction. The practitioners have a tendency to ignore this necessity. They believe that, in the symmetric monoidal categories occuring in concrete situations, coherence is "obvious," and that the commutativity of the few additional diagrams needed is verifiable ad hoc and directly. Moreover, the proof of the coherence theorems does not elucidate the much more pressing concrete problems of dealing with, say, group objects in commutative monoidal categories and their transformation under multiplicative functors. We adopt the practitioners' viewpoint only in as much as we do not present the proof of the Coherence Theorem but rather refer to MacLane's text [247], p. 157ff., or to the survey [246], p. 75ff., or to other sources [170].

A typical natural isomorphism which must be defined uniquely is the so-called *middle four exchange* μ_{ABCD}:

$$\begin{array}{ccc}
(A\otimes B)\otimes(C\otimes D) & \xrightarrow[\cong]{\text{[associativity]}} & A\otimes\big((B\otimes C)\otimes D\big)\\
{\scriptstyle \mu_{ABCD}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_A\otimes(\kappa_{BC}\otimes\mathrm{id}_D)}\\
(A\otimes C)\otimes(B\otimes D) & \xleftarrow[\text{[associativity]}]{\cong} & A\otimes\big((C\otimes B)\otimes D\big).
\end{array}$$

We are now ready for introducing a concept of utmost importance in concrete situations.

Definition A3.63a. An *associative multiplication*on an object A of a symmetric monoidal category is a morphism $m\colon A \otimes A \to A$ such that the following diagram is commutative:

$$\begin{CD}
A \otimes (A \otimes A) @>{\mathrm{id}_A \otimes m}>> A \otimes A \\
@V{\alpha_{AAA}}VV @VV{\mathrm{id}_{A\otimes A}}V \\
(A \otimes A) \otimes A @. A \otimes A \\
@V{m \otimes \mathrm{id}_A}VV @VV{m}V \\
A \otimes A @>>{m}> A.
\end{CD}$$

The multiplication is called *commutative* if the following diagram commutes:

$$\begin{CD}
A \otimes A @>{\kappa_{AA}}>> A \otimes A \\
@V{m}VV @VV{m}V \\
A @>>{\mathrm{id}_A}> A.
\end{CD}$$

In an analogous fashion, an associative (respectively, commutative) *comultiplication* $c\colon A \to A \otimes A$ is defined by reversing arrows in the associativity defining diagram.

We say that there is an *identity* for a multiplication $m\colon A \otimes A \to A$ if there is a morphism $u\colon E \to A$ such that the following diagram is commutative:

$$\begin{CD}
E \otimes A @>{u \otimes \mathrm{id}_A}>> A \otimes A @<{\mathrm{id}_A \otimes u}<< A \otimes E \\
@V{\iota_A}VV @V{m}VV @VV{\iota'_A}V \\
A @>>{\mathrm{id}_A}> A @<<{\mathrm{id}_A}< A.
\end{CD}$$

Similarly, we define a *coidentity* $k\colon A \to E$ for a comultiplication. An object with an associative multiplication and an identity is called a *monoid* in the commutative monoidal category $\mathbb{A}$. Analogously, a *comonoid* is defined. □

Note that the diagonal map $x \mapsto (x, x)\colon X \to X \times X$ is an associative and commutative comultiplication in $\mathbb{S}$ or $\mathbb{CTOP}$.

If objects A and B have multiplications and identities then a morphism $f\colon A \to B$ of $\mathbb{A}$ is a *morphism of monoids* if the following diagram commutes:

$$\begin{CD}
A \otimes A @>{m_A}>> A @<{u_A}<< E \\
@V{f \otimes f}VV @VV{f}V @VV{\mathrm{id}_E}V \\
B \otimes B @>>{m_B}> B @<<{u_B}< E.
\end{CD}$$

A monoid in a commutative monoidal category $\mathbb{AB}_R$ (see A3.63(C)) is an associative R-algebra with identity element. A monoid in the commutative monoidal category $\mathbb{AB}^*_R$ (see A3.63(C*)) is a graded associative R-algebra with identity. (Because of the particular commutation isomorphism κ we prefer to consider in this category, these algebras are also called *anticommutative*.) And so on.

With a little diagram chasing, using coherence, one can show that the product $A \otimes B$ of two monoids is again a monoid; e.g. the multiplication is defined via the middle four exchange by

$$m_A \bullet m_B = \{(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mu_{ABAB}} (A \otimes A) \otimes (B \otimes B) \xrightarrow{m_A \otimes m_B} A \otimes B\},$$

and the identity is defined by

$$u_A \bullet u_B = \{E \xrightarrow[\cong]{\iota_E^{-1}} E \otimes E \xrightarrow{u_A \otimes u_B} A \otimes B\}.$$

Of, course, one has to verify, as an exercise, that in this way,

$$(A \otimes B, m_A \bullet m_B, u_A \bullet u_B)$$

is a monoid. The fact that for a monoid A also $A \otimes A$ is monoid is used crucially in the following definition.

Definition A3.64. (i) A *bimonoid* in a commutative monoidal category $(\mathbb{A}, \otimes)$ is an object A with a multiplication $m_A: A \otimes A \to A$ and an identity $u_A: E \to A$ making A a monoid, further a comultiplication $c_A: A \to A \otimes A$ and a coidentity $k_A: A \to E$ making A a comonoid such that the monoid and the comonoid are linked by the assumption that c_A is a monoid morphism $A \to A \otimes A$. A morphism of bimonoids $f: A \to B$ in the commutative monoidal category $(\mathbb{A}, \otimes)$ is an $\mathbb{A}$-morphism which is both a monoid and comonoid morphism. In the following, for the sake of brevity we shall write m, u, c, k in place of m_A, u_A, c_A, and k_A.

(ii) A *group object* or simply a *group* in a commutative monoidal category $(\mathbb{A}, \otimes)$ is a bimonoid A whose comonoid is commutative together with a morphism $\sigma: A \to A$, called *inversion* such that the following diagrams commute with $p \stackrel{\text{def}}{=} uk: A \to A$:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\sigma \otimes \mathrm{id}_A} & A \otimes A \\ {\scriptstyle c}\uparrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[p]{} & A, \end{array} \qquad \begin{array}{ccccc} E & \xleftarrow{k} & A & \xrightarrow{c} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_E} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma \otimes \sigma} \\ E & \xleftarrow[k]{} & A & \xrightarrow[c]{} & A \otimes A. \end{array}$$

A *morphism of groups* or *group morphism* is a bimonoid morphism of the underlying bimonoids. □

The hypothesis that m is assumed to be a comonoid morphism is equivalent to the commuting of the following diagrams:

$$\begin{array}{ccccccc}
E & \xleftarrow{k} & A & \xrightarrow{\mathrm{id}_A} & A & \xleftarrow{u} & E \\
{\scriptstyle \iota_E}\big\uparrow & & {\scriptstyle m}\big\uparrow & & \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle \iota_E^{-1}} \\
E\otimes E & \xleftarrow[k\otimes k]{} & A\otimes A & & A\otimes A & \xleftarrow[u\otimes u]{} & E\otimes E \\
& & {\scriptstyle c\otimes c}\big\downarrow & & \big\uparrow{\scriptstyle m\otimes m} & & \\
& & (A\otimes A)\otimes(A\otimes A) & \xrightarrow[\mu_{AAAA}]{} & (A\otimes A)\otimes(A\otimes A), & &
\end{array}$$

$$c \bullet c = \mu_{AAAA} \circ (c \otimes c) \quad \text{and} \quad m \bullet m = (m \otimes m) \circ \mu_{AAAA}.$$

In particular, the hypothesis is symmetric: m is a comonoid morphism if and only if the diagram above commutes if and only if c is a monoid morphism.

The following remark, simple as it may sound, is nevertheless relevant: *In any pointed category $\mathbb{A}$ with finite products, every monoid A in the commutative monoidal category $(\mathbb{A}, \times)$ is automatically a bimonoid when given the diagonal morphism $c_A: A \to A \times A$ as comultiplication and the unique morphism $k_A: A \to E$ to the terminal object as a coidentity.* It is for this reason that bimonoids are very prevalent without being recognized as such.

It can be verified directly that for a group object the left hand diagram also commutes with $\mathrm{id}_A \otimes \sigma$ in place of $\sigma \otimes \mathrm{id}_A$.

Exercise EA3.34. *A morphism of groups automatically preserves inversion.* □

Each commutative monoidal category $(\mathbb{A}, \otimes)$ defines a category of bimonoids $\mathbf{Bimon}(\mathbb{A}, \otimes)$ and a category of groups $\mathbf{Gr}(\mathbb{A}, \otimes)$. The category of groups is full in the category of bimonoids. If A and B are bimonoids (groups) then $A \otimes B$ is a bimonoid (group), too. Thus the categories of bimonoids, respectively, groups become commutative monoidal categories $\big(\mathbf{Bimon}(\mathbb{A}, \otimes), \otimes\big)$, respectively, $\big(\mathbf{Gr}(\mathbb{A}, \otimes), \otimes\big)$. The multiplication $m: A \otimes A \to A$ of a *commutative monoid* is a morphism of monoids. Conversely, if the multiplication $m: A \otimes A \to A$ of a monoid is a monoid morphism, then m is commutative as is deduced from the definition of $m \bullet m = (m \otimes m) \circ \mu_{AAAA}$ (Hilton's Lemma). In this sense, the category of commutative groups in $(\mathbb{A}, \otimes)$ may be identified with the category of groups in $(\mathbf{Gr}(\mathbb{A}, \otimes), \otimes)$.

A group object in the symmetric monoidal category $\mathbb{S}$ is a group in the traditional sense with the diagonal map $c: G \to G \times G$, the constant function $k: G \to \{1\}$, and $\sigma(x) = x^{-1}$. In the symmetric monoidal category of compact topological spaces, a group object is a compact (topological) group. Algebraic geometry provides a category of varieties with a (not entirely obvious) finite product $(V, W) \mapsto V \times W$; a group object in this category is called an algebraic group. A Lie group may be considered as a group object in the category of smooth manifolds although in this book we have opted for a different approach pedagogically.

We notice that in the definition of a group object we have allowed for a slight asymmetry in this definition in so far as we have postulated the commutativity of the comultiplication; this condition is always satisfied for the groups in concrete categories where the multiplication is given by a cartesian product.

Definition of Hopf Algebras

Definition A3.65. Let R be a commutative ring with identity.

(i) A *Hopf algebra* over R is a bimonoid in the commutative monoidal category $(\mathbb{AB}_R, \otimes_R)$ of R-modules.

(ii) A *graded Hopf algebra* over R is a bimonoid in the commutative monoidal category $(\mathbb{AB}^*_R, \otimes^*_R)$ of graded R-modules.

(iii) Let $(\mathbb{A}, \otimes)$ be a symmetric monoidal category. A *symmetric Hopf algebra in* $(\mathbb{A}, \otimes)$ is a group object in $(\mathbb{A}, \otimes)$. A *morphism of Hopf algebras, respectively, morphism of symmetric Hopf algebras* is a morphism of bimonoids, respectively, group objects in the category $(\mathbb{AB}^*_R, \otimes^*)$, respectively, $(\mathbb{A}, \otimes)$. □

A large portion of the following discourse will deal with graded commutative Hopf algebras. This is justified by their eminent role in homological algebra. Therefore it is important to recall that the multiplication of commutative monoids in the category $(\mathbb{AB}^*_R, \otimes^*)$ (i.e. the multiplication of graded commutative algebras) is sometimes called *anticommutative*. If $\mathbb{A}$ denotes one of the commutative monoidal categories $(\mathbb{AB}_R, \otimes_R)$, $(\mathbb{AB}^*_R, \otimes^*_R)$, or $(\mathbb{AB}^*_R, \otimes^*_R)'$, then the category $\mathbf{Hopf}(\mathbb{A})$ of graded Hopf algebras in $\mathbb{A}$ is simply the category of bimonoids $\mathbf{Bimon}(\mathbb{A}^{\rm op})^{\rm op}$ in $\mathbb{A}$.

The concept of a Hopf algebra appears in the literature in different variations. Frequently the word "Hopf algebra" means what we call "symmentric Hopf algebra". A frontal approach to the definition of a Hopf algebra requires the specification of a variety of maps and their properties. The category theoretical definition has the great advantage that the definition is compact, and one understands directly what it means and where it comes from; this motivation will be corroborated in the following discussion. Indeed the most prominent classical Hopf algebras arise in a functorial context which we discuss next.

Definition A3.66. Let $\mathbb{A}_j$, $j = 1, 2$ be commutative monoidal categories. A functor $F\colon \mathbb{A}_1 \to \mathbb{A}_2$ is called *multiplicative* if there are natural isomorphisms

$$\mu_{AB}\colon F(A \otimes_1 B) \to FA_1 \otimes_2 FA_2 \quad \text{and} \quad \xi\colon FE_1 \to E_2$$

such that the following diagrams are commutative:

$$\begin{array}{ccccc} F\big(A \otimes (B \otimes C)\big) & \xrightarrow{\mu_{A,B\otimes C}} & FA \otimes F(B \otimes C) & \xrightarrow{\mathrm{id}_{FA}\otimes\mu_{BC}} & FA \otimes (FB \otimes FC) \\ \Big\downarrow{\scriptstyle F(\alpha^{(1)}_{ABC})} & & & & \Big\downarrow{\scriptstyle \alpha^{(2)}_{FA,FB,FC}} \\ F\big((A \otimes B) \otimes C\big) & \xrightarrow[\mu_{A\otimes B,C}]{} & F(A \otimes B) \otimes FC & \xrightarrow[\mu_{AB}\otimes\mathrm{id}_{FC}]{} & (FA \otimes FB) \otimes FC, \end{array}$$

$$\begin{array}{ccccc} F(E_1\otimes A) & \xrightarrow{\mu_{E_1A}} & FE_1\otimes FA & \xrightarrow{\xi\otimes \mathrm{id}_{FA}} & E_2\otimes FA\\ \Big\downarrow F(\iota_A^{(1)}) & & & & \iota_{FA}^{(2)}\Big\downarrow\\ FA & \xrightarrow[\mathrm{id}_{FA}]{} & FA & \xrightarrow[\mathrm{id}_{FA}]{} & FA, \end{array}$$

$$\begin{array}{ccc} F(A\otimes B) & \xrightarrow{\mu_{AB}} & FA\otimes FB\\ F\kappa_{AB}^{(1)}\Big\downarrow & & \Big\downarrow \kappa_{FA,FB}^{(2)}\\ F(B\otimes A) & \xrightarrow[\mu_{BA}]{} & FB\otimes FA. \end{array}$$

Let us look at examples. In a ring A let $\langle X\rangle$ denote the two sided ideal generated by a subset X of A. Recall the tensor algebra $\bigotimes V = \bigoplus_{n\in N_0}\bigotimes^n V$ of an R-modules V, where $\bigotimes^0 V = R$. For an R-module V let $\mathbf{S}V$ denote the graded R-module underlying the symmetric algebra over V, i.e.

$$\mathbf{S}V = \bigotimes V/\langle v\otimes w - w\otimes v \mid v,\, w\in V\rangle,$$

and let $\bigwedge V$ denote graded R-module underlying the exterior algebra over V, i.e.

$$\bigwedge V = \bigotimes V/\langle v\otimes w + w\otimes v \mid v,\, w\in V\rangle.$$

For a graded R-module $V = \bigoplus_{n\in\mathbb{N}} V^n$ (with degree null component being zero) let $\mathbf{H}V$ be the graded R-module underlying the graded commutative algebra:

$$\mathbf{H}V = \bigotimes V/\langle v^p\otimes w^q - (-1)^{pq} w^q\otimes v^p \mid v^p\in V^p, w^q\in W^q, \quad p,q\in\mathbb{N}\rangle.$$

Note that indeed $\mathbf{H}V$ is a graded R-algebra.

If all even homogeneous components of $V = \bigoplus_{n\in\mathbb{N}} V^n$ are zero, then $\mathbf{H}V \cong \bigwedge V$ with an appropriate gradation; if all odd homogeneous components of V are zero, then $\mathbf{H}V = \mathbf{S}V$ with an appropriate gradation. More generally, if we write $V_0 = \bigoplus_{m=1}^{\infty} V^{2m}$ and $V_1 = \bigoplus_{m=1}^{\infty} V^{2m-1}$, then

$$\mathbf{H}V = \mathbf{S}V_0 \otimes^* \bigwedge V_1$$

with the appropriate gradation. Thus the algebras $\mathbf{H}V$ generalize simultaneously exterior algebras and polynomial algebras; they play a crucial role in the remainder of this section and we assume them as well understood.

For our purposes the following examples are of immediate interest.

Proposition A3.67. (i) *The assignments*

$$\mathbf{S}\colon (\mathbb{AB}_R,\oplus)\to(\mathbb{AB}_R^*,\otimes^*)',$$
$$\bigwedge\colon (\mathbb{AB}_R,\oplus)\to(\mathbb{AB}_R^*,\otimes^*), \quad \textit{and}$$
$$\mathbf{H}\colon (\mathbb{AB}_R^*,\oplus^*)\to(\mathbb{AB}_R^*,\otimes^*)$$

are multiplicative functors.

(ii) *Let K be a field and for a compact space X let $H(X)$ denote the graded K-vector space $\bigoplus_{n\in\mathbb{N}_0} H^n(X,K)$ Alexander–Čech–Spanier–Wallace cohomology (see* *e.g.* [338], p. 306ff.). *Then the assignment*

$$H\colon (\mathbb{CTOP}, \times) \to (\mathbb{AB}^*_K, \otimes^*)^{\rm op}$$

is a multiplicative functor.

Proof. The assertions on $\mathbf{S}$ and $\bigwedge$ are standard multilinear algebra in as much as they code the natural isomorphisms $\mathbf{S}(V \oplus W) \cong \mathbf{S}V \otimes \mathbf{S}W$ and $\bigwedge(V \oplus W) \cong \bigwedge V \otimes \bigwedge W$. The assertion on $\mathbf{H}$ generalizes these facts and can be directly derived from them via $\mathbf{H}V = \mathbf{S}V_0 \otimes^* \bigwedge V_1$.

The assertion of (ii) is the Künneth Theorem for cohomology over fields. (Cf. [245], p. 166, [338], p. 360.) □

Proposition A3.68. *A multiplicative functor maps bimonoids to bimonoids and groups to groups.*

Proof. Exercise EA3.35. □

Exercise EA3.35. Prove A3.68. □

We say that a graded Hopf algebra (respectively, bimonoid) $A = \bigoplus_{n\in\mathbb{N}_0} A^n$ is *connected* if $A^0 = R{\cdot}1$. The full subcategory of connected graded Hopf algebras is written $\mathbf{Hopf}_0(\mathbb{AB}^*_R, \otimes^*)$, and the full subcategory in $\mathbb{AB}^*_R$ of graded R-modules V whose homogeneous component V^0 is zero will be written $(\mathbb{AB}^*_R)_0$. These modules we call *graded R-modules without zero component.*

Corollary A3.69. *For any R-module V, the symmetric algebra* $\mathbf{S}V$*, the exterior algebra* $\bigwedge V$ *and, in the case that V is graded without zero term, the object* $\mathbf{H}V$*, all are connected graded commutative Hopf algebras, the graded Hopf algebra* $\mathbf{S}V$ *being a strictly commutative one in the sense that* $ab = ba$ *for all* $a, b \in \mathbf{S}V$*.*

Proof. This is an immediate consequence of Proposition A3.68 in view of Proposition A3.67(i) and the fact, that $(\mathbb{AB}_R, \oplus)$ and $(\mathbb{AB}^*_R, \oplus^*)$ may be identified with the categories of commutative group objects. □

We rightfully take the position that the *Hopf* algebra structure of the graded commutative algebras $\mathbf{H}V$ is also well understood.

Corollary A3.70. *For a compact topological monoid G and a field K, the cohomology*

$$H(G) = \bigoplus_{n\in\mathbb{N}_0} H^n(G, K) \quad over \quad K$$

is a graded commutative Hopf algebra, and if G is connected, then H(G) is connected. If G is a compact group then H(G) is a cogroup object in $(\mathbb{AB}^*_K, \otimes^*)$*.*

Proof. This is an immediate consequence of Proposition A3.68 in view of Proposition A3.67(ii) and the fact that a compact space X is connected iff $H^0(X, K) = K$. □

It is quite remarkable, that these conclusions emerge from little more than diagram chasing or what, by some, is called "general nonsense," meaning the disciplined and consistent application of category theoretical thinking. The concrete input, in the background, is multilinear algebra, the machinery of Čech cohomology and the homological algebra of the Künneth formula. The further success of this approach depends on how much we will be able to elucidate the *concrete* situation of graded commutative Hopf algebras over a field K.

Part 2: Connected Graded Commutative Hopf Algebras

For a graded R-module $A = \bigoplus_{n\in\mathbb{N}_0} A^n$ we let $A^+ = \{0\} \oplus \bigoplus_{n\in\mathbb{N}} A^n$ be the graded submodule with $(A^+)^0 = \{0\}$. We denote the inclusion morphism $A^+ \to A$ by i_A and the projection $A \to A^+$ with kernel A^0 by κ_A. Let $m: A \otimes^* A \to A$ be the multiplication of a graded algebra over R with identity $u_A: R \to A$. Then A^+ is an ideal and we have a multiplication

$$m^+ \stackrel{\text{def}}{=} (A^+ \otimes^* A^+ \xrightarrow{i_A \otimes^* i_A} A \otimes^* A \xrightarrow{m} A \xrightarrow{\kappa_A} A^+).$$

Technically, in terms of elements, m^+ just represents the restriction of the ring multiplication of A to the ideal A^+. The image $S(A) \stackrel{\text{def}}{=} \operatorname{im} m^+ = m^+(A^+ \otimes^* A^+)$ is the set of all elements in A^+ which may be represented as the finite linear combinations of products ab with $a, b \in A^+$, i.e. of products ab of homogeneous elements a and b of positive degree. Thus $S(A)$ is an ideal of A. Then

$$A/S(A) \cong A^0 \oplus Q(A), \quad Q(A) \stackrel{\text{def}}{=} A^+/\operatorname{im} m^+$$

is a graded commutative algebra satisfying $Q(A)^2 = \{0\}$. The graded module $Q(A) \subseteq A^0 \oplus Q(A)$ is called the *set of indecomposable elements of A*.

Exercise EA3.36. If A is the underlying algebra of $\mathbf{H}V$, then the quotient morphism $\mathbf{H}V \to R \oplus Q(A)$ maps $V \subseteq \mathbf{H}V$ isomorphically onto $Q(A)$. □

For graded Hopf algebras the concept of a primitive element is crucial. We shall return to the concept of primitive and grouplike elements again at Definition A3.95ff. For a comonoid A in a commutative monoidal category $(\mathbb{A}, \otimes)$ we set

$$p_1 \stackrel{\text{def}}{=} \iota'_A \circ (\mathrm{id}_A \otimes k_A): A \otimes A \to A, \quad \text{and}$$
$$p_2 \stackrel{\text{def}}{=} \iota_A \circ (k_A \otimes \mathrm{id}_A): A \otimes A \to A.$$

By the definition of a coidentity, both of these morphisms are retractions since $p_j \circ c = \mathrm{id}_A$, $j = 1, 2$. Dually, for a monoid A we have two coretractions

$$\lambda_A = \big(A \xrightarrow{\iota_A^{-1}} E \otimes A \xrightarrow{u \otimes \mathrm{id}_A} A \otimes A\big)$$
$$\rho_A = \big(A \xrightarrow[{\iota'_A}^{-1}]{} A \otimes E \xrightarrow[\mathrm{id}_A \otimes u]{} A \otimes A\big).$$

If $\mathbb{A} = \mathbb{AB}_R^*$ and A is a graded commutative Hopf algebra, then $A \otimes^* A$ contains several isomorphic copies of the graded commutative algebra A as module direct summands, namely, $c(A)$, $\lambda_A(A) = \bigoplus_{n \in \mathbb{N}_0} 1 \otimes A^n$, and $\rho_A(A) = \bigoplus_{n \in \mathbb{N}_0} A^n \otimes 1$. Assuming that $\mathbb{A}$ has finite products, we can form the diagonal morphism $d_A \colon A \to A \times A$; then we have a morphism $(\rho_A \times \lambda_A) \circ d_A \colon A \to (A \otimes A) \times (A \otimes A)$. If $\mathbb{A} = \mathbb{AB}_R^*$ (or if $\mathbb{A}$ is any other category whose objects allow an abelian group addition $\mathrm{add}\colon X \times X \to X$ on their objects which is a morphism), then we have a morphism $\mathrm{add}_{A \otimes A} \colon (A \otimes A) \times (A \otimes A) \to A \otimes A$ giving us a unique $\mathbb{A}$-morphism

$$\pi_A \stackrel{\mathrm{def}}{=} \mathrm{add}_{A \otimes A} \circ (\rho_A \times \lambda_A) \circ d_A \colon A \to A \otimes A.$$

In the case of $\mathbb{A} = \mathbb{AB}_R^*$ we have

$$(\forall a \in A) \quad \pi_A(a) = a \otimes 1 + 1 \otimes a.$$

If A is a comonoid in $(\mathbb{A}, \otimes)$ with comultiplication $c_A \colon A \to A \otimes A$, since $a = p_1 c_A(a) = p_2 c_A(a)$ we have $c_A(a) - \pi_A(a) \cap A \otimes 1 + 1 \otimes A = \{0\}$. If equalizers exist, then we can form the equalizer

$$P(A) \stackrel{e}{\to} A \underset{\pi_A}{\overset{c_A}{\rightrightarrows}} A \otimes A.$$

Recall that $\kappa_A \colon A \to A^+$ denotes the projection of graded modules with kernel A^0 and $i_A \colon A^+ \to A$ the inclusion map. Then $\kappa_A \otimes^* \kappa_A \colon A \otimes^* A \to A^+ \otimes^* A^+$ has the kernel $A \otimes^* 1 + 1 \otimes^* A = A^0 \otimes A^0 + A^+ \otimes^* 1 + 1 \otimes^* A^+$. If we can form

$$c^+ \stackrel{\mathrm{def}}{=} (A^+ \xrightarrow{i_A} A \xrightarrow{c} A \otimes^* A \xrightarrow{\kappa_A \otimes^* \kappa_A} A^+ \otimes^* A^+),$$

then

$$\ker c^+ = A^+ \cap c^{-1}(A^0 \otimes A^0 + A^+ \otimes^* 1 + 1 \otimes^* A^+) = c^{-1}(A^+ \otimes^* 1 + 1 \otimes^* A^+).$$

Definition A3.71. Let A denote a graded Hopf algebra over a commutative ring R with comultiplication $c \colon A \to A \otimes^* A$. Set

$$P(A) = \{a \in A \mid c(a) = a \otimes 1 + 1 \otimes a\} \subseteq A.$$

The elements of $P(A)$ are called the *primitive elements* of the Hopf algebra and $P(A)$ the *primitive submodule* of A.

If A is the smallest Hopf subalgebra of A containing $P(A)$ then A is called *primitively generated.* □

We shall later completely clarify the structure of primitively generated connected graded commutative Hopf algebras over fields of characteristic 0.

The restriction of the algebra morphism $A \to A^0 \oplus Q(A)$ induces a morphism of graded R-modules $q_A \colon P(A) \to Q(A)$.

Lemma A3.72. (i) *The set $P(A)$ is a graded R-submodule of A.*

(ii) *If $f\colon A \to B$ is a morphism of graded Hopf algebras, then the restriction and corestriction of f defines a morphism of graded R-modules $P(f)\colon P(A) \to P(B)$.*

(iii) *If $r \in R$ then $r{\cdot}1 \in P(A)$ iff $r = 0$.*

(iv) *Write $A^+ = \ker k_A = \bigoplus_{n\in\mathbb{N}} A^n$. Then*

$$P(A) = c^{-1}(A^+ \otimes^* 1 + 1 \otimes^* A^+).$$

(v) $P(A) = \ker c^+ \subseteq A^+$.

(vi) *The morphism $q_A\colon P(A) \to Q(A)$ is injective if and only if $P(A) \cap S(A) = \{0\}$ and q_A is surjective if and only if $P(A) + S(A) = A^+$. It is an isomorphism of graded R-modules if and only if $A^+ = P(A) \oplus S(A)$.*

(vii) *If $d = \min\big\{n \in \mathbb{N} \mid A^n \neq \{0\}\big\}$, then $A^d \subseteq P(A)$.*

Proof. (i) The functions $c\colon A \to A \otimes^* A$, $\gamma\colon A \to A \otimes^* A$, $\gamma(a) = a \otimes 1 + 1 \otimes a$ are morphisms of graded R-modules. As the equalizer $\{a \in A \mid c(a) = \gamma(a)\}$ of these two morphisms, $P(A)$ is a graded R-module.

(ii) Since $c\big(f(a)\big) = (f \otimes^* f)c(a)$ as f is a morphism of comonoids (in $(\mathbb{AB}^*_R, \otimes^*)$), and since $\gamma\big(f(a)\big) = f(a) \otimes 1 + 1 \otimes f(a) = (f \otimes^* f)(a \otimes 1 + 1 \otimes a)$ as f preserves identities, we see that $f\big(P(A)\big) \subseteq P(B)$ so that the restriction of f to $P(A)$ and the corestriction to $P(B)$ is a well-defined morphism of graded R-modules.

(iii) Assume now that $r{\cdot}1 \in P(A)$. Since c preserves identities,being a monoid morphism, we have $r{\cdot}(1 \otimes 1) = c(r{\cdot}1) = r{\cdot}1 \otimes 1 + 1 \otimes r{\cdot}1$, i.e. $r{\cdot}(1 \otimes 1) = 0$ which implies $r = 0$.

(iv) Assume that $c(a) \in A^+ \otimes^* 1 + 1 \otimes^* A^+$. Thus there are elements a_1, $a_2 \in A^+$ such that $c(a) = a_1 \otimes 1 + 1 \otimes a_2$. Then $a_1 = p_1(a_1 \otimes 1 + 1 \otimes a_2) = p_1\big(c(a)\big) = a$, similarly $a_2 = a$.

(v) This follows from (iv) and the structure of $\ker c^+$ computed prior to A3.71.

(vi) By definition $S(A)$ is the kernel of the quotient map of graded modules $A^+ \to Q(A)$, whose restriction to $P(A) \subseteq A^+$ is q_A. The assertions are then immediate.

(vii) If d is as stated, then $m \in \{1, \ldots, d-1\}$ implies $A^{d-m} \otimes A^m = \{0\}$. Thus $c(A^d) \in A^d \otimes 1 \oplus 1 \otimes A^d$. Hence $A^d \subseteq P(A)$ by (iv) above. □

The proof is easily converted into a proof using arrows only.

Since the composition $R \xrightarrow{u_A} A \xrightarrow{k_A} R$ is the identity, for objects A and B, the following compositions represent identity morphisms also:

$$\begin{array}{ccccc} R \otimes B & \xrightarrow{u_A \otimes \mathrm{id}_B} & A \otimes B & \xrightarrow{k_A \otimes \mathrm{id}_B} & R \otimes B \\ A \otimes R & \xrightarrow[\mathrm{id}_A \otimes u_B]{} & A \otimes B & \xrightarrow[\mathrm{id}_A \otimes k_B]{} & A \otimes R. \end{array}$$

In particular, the compositions

$$\begin{array}{ccccccc} B & \xrightarrow{\iota_B^{-1}} & R \otimes B & \xrightarrow{u_A \otimes \mathrm{id}_B} & A \otimes B & \subseteq & A \otimes B \\ A & \xrightarrow[(\iota'_A)^{-1}]{} & A \otimes R & \xrightarrow[\mathrm{id}_A \otimes u_B]{} & A \otimes B & \subseteq & A \otimes B \end{array}$$

are injective and induce, in view of $P(A) \otimes 1 \cap 1 \otimes P(B) = \{0\}$ in $A \otimes B$, isomorphisms

$$\begin{array}{ccccc} P(A) & \to & P(A) \otimes 1 & \subseteq & A \otimes A \\ P(B) & \to & 1 \otimes P(B) & \subseteq & B \otimes B \\ P(A) \oplus P(B) & \to & P(A) \otimes 1 + 1 \otimes P(B) & \subseteq & A \otimes B. \end{array}$$

Now assume that R is a field K.

Proposition A3.73. *The assignment*

$$P\colon \mathbf{Hopf}_0(\mathbb{AB}_K^*, \otimes^*) \to (\mathbb{AB}_K^*, \oplus^*)_0$$

is a multiplicative functor.

Proof. Let A and B be two graded Hopf algebras. Then $A \otimes^* B$ is a graded Hopf algebra (since the product of two bimonoids in a commutative monoidal category is a bimonoid in this category). We have seen $P(A) \oplus P(B) \cong P(A) \otimes 1 + 1 \otimes P(B)$ and now claim that $P(A \otimes^* B) = P(A) \otimes 1 + 1 \otimes P(B)$, which will prove the proposition.

Firstly let $a \in P(A)$ and $b \in P(B)$. Then

$$\begin{aligned} c_{A\otimes B}(a \otimes 1 + 1 \otimes b) &= \mu\big((c \otimes c)(a \otimes 1 + 1 \otimes b)\big) \\ &= \mu\big(c(a) \otimes (1 \otimes 1)\big) + \mu\big((1 \otimes 1) \otimes c(b)\big) \\ &= \mu\big((a \otimes 1 + 1 \otimes a) \otimes (1 \otimes 1)\big) \\ &\quad + \mu\big((1 \otimes 1) \otimes (b \otimes 1 + 1 \otimes b)\big) \\ &= \mu\big((a \otimes 1) \otimes (1 \otimes 1)\big) + \mu\big((1 \otimes a) \otimes (1 \otimes 1)\big) \\ &\quad + \mu\big((1 \otimes 1) \otimes (b \otimes 1)\big) + \mu\big((1 \otimes 1) \otimes (1 \otimes b)\big) \\ &= (a \otimes 1) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (a \otimes 1) \\ &\quad + (1 \otimes b) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (1 \otimes b) \\ &= (a \otimes 1 + 1 \otimes b) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (a \otimes 1 + 1 \otimes b). \end{aligned}$$

Thus $P(A) \otimes 1 + 1 \otimes P(B) \subseteq P(A \otimes^* B)$.

Secondly let $x = \sum_{n \in \mathbb{N}_0} x_n \in P(A \otimes^* B)$. Then

$$(*) \qquad c_{A\otimes B}(x_n) = x_n \otimes (1 \otimes 1) + (1 \otimes 1) \otimes x_n.$$

By the definition of $A \otimes^* B$, and since A and B and thus $A \otimes^* B$ are connected, we have $x_0 = 0$ and the element x is of the form

$$x = a \otimes 1 + 1 \otimes b + \sum_{\substack{j \in J \\ p,q \in \mathbb{N}}} a_{jp} \otimes b_{jq}, \quad a \in A^+, b \in B^+$$

and we will first show that all a_{jp} and b_{jq} vanish. In view of $c_{A\otimes B} = \mu\circ(c_A \otimes c_B)$ we get

$$\begin{aligned}
c_{A\otimes B}(a_{jp}\otimes b_{jq}) &= \mu\big(c(a_{jp})\otimes c(b_{jq})\big)\\
&= \mu\big((a_{jp}\otimes 1 + 1\otimes a_{jp})\otimes(b_{jq}\otimes 1 + 1\otimes b_{jq})\big)\\
&= \mu\big((a_{jp}\otimes 1)\otimes(b_{jq}\otimes 1) + (a_{jp}\otimes 1)\otimes(1\otimes b_{jq})\\
&\quad + (1\otimes a_{jp})\otimes(b_{jq}\otimes 1) + (1\otimes a_{jp})\otimes(1\otimes b_{jq})\big)\\
&= (a_{jp}\otimes b_{jq})\otimes(1\otimes 1) + (1\otimes 1)\otimes(a_{jp}\otimes b_{jq})\\
&\quad + [(a_{jp}\otimes 1)\otimes(1\otimes b_{jq}) + (1\otimes b_{jq})\otimes(a_{jp}\otimes 1)].
\end{aligned}$$

Together with $(*)$ this implies

$$(**)\qquad \sum_{\substack{j\in J\\ p,q\in\mathbb{N}}} [(a_{jp}\otimes 1)\otimes(1\otimes b_{jq}) + (1\otimes b_{jq})\otimes(a_{jp}\otimes 1)] = 0.$$

Since K is a field we may assume the a_{jp} and b_{jq}, $j\in J$, $p,q\in\mathbb{N}$ to be linearly independent families. Then

$$\big((a_{jp}\otimes 1)\otimes(1\otimes b_{jq}),\,(1\otimes b_{jq})\otimes(a_{jp}\otimes 1)\big)_{j\in J,\,p,q\in\mathbb{N}}$$

is a linearly independent family. By $(**)$ this family can only contain the zero vector and thus $a_{jp} = b_{jq} = 0$ for all $j\in J$ and all p, $q\geq 1$. Hence $x = a\otimes 1 + 1\otimes b$ with $a\in A^+$ and $b\in B^+$. Using $(*)$ and (the computation leading to) $(**)$ we straightforwardly verify that $a\in P(A)$ and $b\in P(B)$. □

For any graded vector R-module $V = V^1\oplus V^2\oplus\cdots$ we consider V as embedded into

$$\mathbf{H}V = \bigotimes V/I,\quad I = \langle a^p\otimes b^q - (-1)^{pq}b^q\otimes a^p \mid a^p\in V^p,\ b^q\in V^q\rangle$$

via $v\mapsto v+I$.

Lemma A3.74. $P\big(\mathbf{H}V\big) = V$.

Proof. We have a natural isomorphism $\mu_{VW}\colon \mathbf{H}(V\oplus W)\to \mathbf{H}V\otimes^* \mathbf{H}W$, which is uniquely determined by its action on the elements $(v,w)\in V\times W\cong V\oplus W\subseteq \mathbf{H}(V\oplus W)$ $\mu(v,w) = v\otimes 1 + 1\otimes w$.

The comultiplication c of $A \overset{\text{def}}{=} \mathbf{H}V$ is induced by the diagonal morphism $d\colon V\to V\oplus V$ as

$$A\xrightarrow{\mathbf{H}d}\mathbf{H}(V\oplus V)\xrightarrow{\mu_{VV}}A\otimes^* A.$$

Thus $c(v) = \mu(v,v) = v\otimes 1 + 1\otimes v$, whence $V\subseteq P(A)$.

We note $\mathbf{H}V = \bigoplus_{n\in\mathbb{N}_0}\mathbf{H}^n(V)$ of A, $\mathbf{H}^n(V) = \mathrm{span}_K \underbrace{V\cdots V}_{n\text{ times}}$, pick a basis $\{x_i \mid i\in I\}$ of homogeneous elements in V on whose index set we may assume a total order, and observe $x_ix_j = (-1)^{ij}x_jx_i$. Then we can represent each element

a of A in the form

$$a = r\cdot 1 + \sum_{\substack{k\in\mathbb{N}\\ i_1\le\ldots\le i_k}} \alpha(i_1,\ldots,i_k)\cdot x_{i_1}\cdots x_{i_k}$$

with $\alpha(i_1,\ldots,i_k)\in K$, $i_j\in I$. If $a\in P(A)$ then $r=0$ and

$$\begin{aligned}\sum \alpha(i_1,\ldots,i_k)\cdot(x_{i_1}\cdots x_{i_k}\otimes 1 + 1\otimes x_{i_1}\cdots x_{i_k}) &= a\otimes 1 + 1\otimes a\\ = c(a) = \sum \alpha(i_1,\ldots,i_k)\cdot(x_{i_1}\otimes 1 + 1\otimes x_{i_1})&\cdots(x_{i_k}\otimes 1 + 1\otimes x_{i_k})\end{aligned}$$

since c is an algebra morphism and $c(w) = w\otimes 1 + 1\otimes w$ for $w\in V$. For $J = \{j_1,\ldots,j_p\}\subseteq\{1,\ldots,k\}$, $j_1<\cdots<j_p$, we write $x_J = x_{i_{j_1}}\cdots x_{i_{j_p}}$. Then $(x_{i_1}\otimes 1 + 1\otimes x_{i_1})\cdots(x_{i_k}\otimes 1 + 1\otimes x_{i_k}) = \sum_{J\subseteq\{1,\ldots,k\}} x_J\otimes x_{J'}$, where J' denotes the complement of J in $\{1,\ldots,k\}$. It follows that

$$\sum_{\substack{k=2,3,\ldots\\ i_1\le\ldots\le i_k}} \alpha(i_1,\ldots,i_k)\cdot x(i_1,\ldots,i_k) = 0 \quad\text{where}$$

$$x(i_1,\ldots,i_k) = \sum_{\emptyset\ne J\subset\{1,\ldots,k\}} x_J\otimes x_{J'}$$

and where $\subset$ means proper containment. Since the $x(i_1,\ldots,i_k)$ are linearly independent, it follows that $\alpha(i_1,\ldots,i_k)=0$ for all tuples $(i_1,\ldots,i_k)$ with $i_1\le\ldots\le i_k$ for $k\ge 2$. This means that $a = \sum_{i\in I}\alpha(i)\cdot x_i$, i.e. $a\in V$. □

Lemma A3.75. (i) *Let A be a connected commutative graded algebra and V a graded R-module with $V^0=\{0\}$. Let $f\colon V\to A$ be a morphism of graded R-modules. Then there is a unique morphism $f'\colon \mathbf{H}V\to A$ of graded algebras extending f.*

(ii) *Assume, in addition that A is a connected graded commutative Hopf algebra. Then the inclusion map $P(A)\to A$ of graded modules induces a morphism of graded commutative Hopf algebras $\varepsilon_A\colon \mathbf{H}\big(P(A)\big)\to A$.*

Proof. (i) By the universal property of the tensor algebra there is a unique morphism of graded R-algebras $\varphi\colon\bigotimes V\to A$ extending the morphism of graded modules $f\colon V\to A$. If $v\in V^m$ and $w\in V^n$, then $wv = (-1)^{mn}vw$. Hence φ vanishes on $I = \langle a^p\otimes b^q - (-1)^{pq}b^q\otimes a^p \mid a^p\in V^p,\ b^q\in V^q\rangle$, hence induces a morphism of graded commutative R-algebras $f'\colon\mathbf{H}V\to A$ extending the inclusion $f\colon V\to A$ (where we have assumed, as we may, that $V\subseteq\mathbf{H}V$.

(ii) Now assume that A is a Hopf algebra and apply (i) to the inclusion $P(A)\to A$ and obtain a morphism of algebras $\varepsilon_A\colon\mathbf{H}\big(P(A)\big)\to A$. The diagram

$$\begin{array}{ccc} P(A) & \xrightarrow{P(c_A)} & P(A\otimes^* A)\\ {\scriptstyle\text{incl}}\big\downarrow & & \big\downarrow{\scriptstyle\text{incl}}\\ A & \xrightarrow[c_A]{} & A\otimes^* A\end{array}$$

commutes. Then by the universal property of $\mathbf{H}$ there is a commutative diagram

$$\begin{array}{ccc} \mathbf{H}\big(P(A)\big) & \xrightarrow{\mathbf{H}\big(P(c_A)\big)} & \mathbf{H}\big(P(A\otimes^* A)\big)\\ {\scriptstyle\varepsilon_A}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon_{A\otimes^* A}}\\ A & \xrightarrow[c_A]{} & A\otimes^* A. \end{array}$$

Since $\mathbf{H}$ and P are multiplicative functors there is a commutative diagram

$$\begin{array}{ccc} \mathbf{H}\big(P(A\otimes^* A)\big) & \xrightarrow{\cong} & \mathbf{H}\big(P(A)\big)\otimes^*\mathbf{H}\big(P(A)\big)\\ {\scriptstyle\varepsilon_{A\otimes^* A}}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon_A\otimes^*\varepsilon_A}\\ A\otimes^* A & \xrightarrow[\mathrm{id}_A]{} & A\otimes^* A. \end{array}$$

The two preceding diagrams together show that

$$\begin{array}{ccc} \mathbf{H}\big(P(A)\big) & \xrightarrow{\mathbf{H}\big(P(c_A)} & \mathbf{H}\big(P(A)\big)\otimes^*\mathbf{H}\big(P(A)\big)\\ {\scriptstyle\varepsilon_A}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon_A\otimes^*\varepsilon_A}\\ A & \xrightarrow[c_A]{} & A\otimes^* A \end{array}$$

commutes. In a similar fashion we observe that ε_A respects identity, coidentity and inversion. □

We write $\mathbf{Hopf}_0^{\mathrm{comm}}(\mathbb{AB}_K^*,\otimes^*)$ for the full category of commutative graded Hopf algebras. Our findings may then be rephrased as follows.

Proposition A3.76. *The functor* $\mathbf{H}\colon(\mathbb{AB}_K^*,\oplus^*)_0\to\mathbf{Hopf}_0^{\mathrm{comm}}(\mathbb{AB}_K^*,\otimes^*)$ *is left adjoint to the functor* $P\colon\mathbf{Hopf}_0^{\mathrm{comm}}(\mathbb{AB}_K^*,\otimes^*)\to(\mathbb{AB}_K^*,\oplus^*)_0$.

Proof. Exercise EA3.37. □

Exercise EA3.37. Prove A3.76.

[Hint. Verify the universal property

$$\begin{array}{cccc} (\mathbb{AB}_K^*,\oplus^*)_0 & & & \mathbf{Hopf}_0^{\mathrm{comm}}(\mathbb{AB}_K^*,\otimes^*)\\ \hline PA & \mathbf{H}\big(P(A)\big) & \xrightarrow{\varepsilon_A} & A\\ {\scriptstyle P(f)}\big\uparrow & \big\uparrow{\scriptstyle\mathbf{H}\big(P(f)\big)} & & \big\uparrow{\scriptstyle f}\\ V=P\big(\mathbf{H}V\big) & \mathbf{H}V & \xrightarrow[\mathrm{id}_{\mathbf{H}V}]{} & \mathbf{H}V.] \end{array}$$

□

We must now look more closely into the linear algebra of graded commutative Hopf algebras. In particular, one wants to understand the nature of singly generated Hopf subalgebras. This appears as a fairly delicate situation. We shall heavily use the fact that the multiplication is commutative. A graded Hopf subalgebra B of a graded commutative Hopf algebra A has very special properties. Recalling the notation $B^+ = \sum_{n\in\mathbb{N}} B^n$ introduced above we shall associate with B the smallest ideal $I(B)$ of A containing B^+, specifically, $I(B) = \operatorname{span}_K AB^+ = \sum_{b\in B^+} Ab$. Note that $I(B) = \bigoplus_{n\in\mathbb{N}_0} I^n(B)$ with $I^n(B) = \sum_{q=1}^n \operatorname{span}_K A^{n-q}B^q$. Hence

$$A/\!\!/B \stackrel{\text{def}}{=} A/I(B)$$

is a graded commutative algebra and there is a quotient morphism $q: A \to A/\!\!/B$ of graded commutative algebras. The morphism of graded algebras $(q \otimes^* q) \circ c: A \to (A/\!\!/B) \otimes^* (A/\!\!/B)$ has the kernel $c^{-1}\bigl(\ker(q \otimes^* q)\bigr) = c^{-1}(A \otimes^* I(B) + I(B) \otimes^* A)$. Now $c(AB^+) = c(A)c(B^+) \subseteq (A \otimes^* A)(B^+ \otimes^* B^+) \subseteq AB^+ \otimes^* AB^+ \subseteq A \otimes^* I(B) + I(B) \otimes^* A$. Hence $I(B) \subseteq \ker(q \otimes^* q) \circ c$ and thus $(q \otimes^* q) \circ c$ induces a comultiplication $\overline{c}: A/\!\!/B \to (A/\!\!/B) \otimes^* (A/\!\!/B)$. With the aid of the surjectivity of $q: A \to A/\!\!/B$ and the associativity of c one verifies that $\overline{c}$ is associative. Similarly one obtains $\overline{u}$, $\overline{k}$, and $\overline{\sigma}$ so that $A/\!\!/B$ is a graded commutative Hopf algebra. For any homogeneous element $x \in A$ we set $B[x] \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}_0} Bx^n$. Then $B[x]$ is the graded subalgebra generated in A by $B \cup \{x\}$. Further we write $B_A \stackrel{\text{def}}{=} B^0 + I(B) \supseteq B$; in particular, $B_A[x] = \sum_{n\in\mathbb{N}_0} B_A x^n$. Note that $c(B_A) \subseteq c(B) + c\bigl(I(B)\bigr) \subseteq B \otimes^* B + \operatorname{span}_K c(A)c(B^+) \subseteq B \otimes^* B + \operatorname{span}_K (A \otimes^* A)(B \otimes^* B) = B \otimes^* B + \operatorname{span}_K AB^+ \otimes^* AB^+) \subseteq B \otimes^* B + I(B) \otimes^* I(B) \subseteq B_A \otimes^* B_A$, whence B_A is a Hopf subalgebra.

In $A/\!\!/B$ we write $\overline{0} = q(0)$ and $\overline{1} = q(1)$.

For any homogeneous element x in A we write once and for all

$$J(x) = \begin{cases} \{0,1\} \subseteq \mathbb{N}_0 & \text{if } d \text{ is odd} \\ \mathbb{N}_0 & \text{if } d \text{ is even.} \end{cases}$$

For any subset X of a graded commutative algebra over K we denote by $K[X]$ the subalgebra generated by X. Remember that a subalgebra is understood to contain the identity element.

Proposition A3.77. *Let A be a connected commutative graded Hopf algebra and B a graded Hopf subalgebra. Assume that x is a homogeneous element of degree d contained in $A \setminus B$ such that*

(min) $\qquad d = \min\{\deg(y) : y \in A \setminus B \text{ and } y \text{ is homogeneous}\}.$

Then

(i) *$x \notin I(B)$, i.e., $q(x) \neq 0$. In particular, $x \notin B_A$.*
(ii) *$B_A[x]/\!\!/B = K[q(x)]$ is a Hopf subalgebra of $A/\!\!/B$ such that $q(x) \in P(A/\!\!/B)$.*
(iii) *$B_A[x]$ is a Hopf subalgebra of A containing $B[x]$ such that*

$$c(x) \in x \otimes 1 + 1 \otimes x + A \otimes I(B) + I(B) \otimes A.$$

(iv) *Assume $\operatorname{char} K = 0$. Then $B_A[x]$ is a free B_A-module with $\{x^m \mid m \in J(x)\}$ as a basis, i.e., $B_A[x] = \bigoplus_{m\in J(x)} B_A x^m$ with $B_A x^m \cong B_A$ for $m \in J(x)$.*

(v) *The natural morphism*

$\mu\colon B_A \otimes^*_K K[x] \to B_A[x]$, $\mu(b \otimes c) = bc$, *and its restriction* $B \otimes^* K[x] \to B[x]$

are isomorphisms of graded algebras. Moreover, $\mu^{-1}\big(I(B)\big) = B_A \otimes^* 1$ *and* $\mu^{-1}(B) = B \otimes^* 1$.

(vi) *If* $B = K$, *and if* $A = K[x]$, *then* $A = \bigwedge K{\cdot}x$ *if* $\deg x$ *is odd and* $A = \mathbf{S}(K{\cdot}x)$, *the polynomial algebra over* K *in one variable* x, *if* $\deg x$ *is even.*

Proof. (i) Suppose $x \in I(B) = \sum_{m \in \mathbb{N}_0,\, b_n \in B^n,\, n \in \mathbb{N}} A^m b^n$. We select a basis $(e_{jp})_{j \in J_p}$ of A^p such that for a subset $I_p \subseteq J_p$ the family $(e_{ip})_{i \in I_p}$ is a basis of B^p. Then

$$x = \sum_{\substack{p=1,\dots,d \\ j \in J_{d-p},\, i \in I_p}} r_{jip}{\cdot}e_{j,d-p}e_{ip}$$

with $r_{jip} \in K$. Since $x \notin B$ there is at least one $p \in \{1, \dots, d\}$ and a pair $(j,i) \in I_{d-p} \times I_p$ such that $e_{j,d-p}e_{ip} \notin B$. Since $e_{ip} \in B$ this implies $e_{j,d-p} \in A^{d-p} \setminus B$. Now $p \geq 1$ whence $\deg(e_{j,d-p}) = d - p < d = \deg(x)$ which contradicts (min). This contradiction proves claim (i).

(ii) As the ground ring is a field, we may write

$$B_A[x]/\!\!/B = \sum_{n \in \mathbb{N}_0} K{\cdot}q(x)^n = K[q(x)] \subseteq A/\!\!/B.$$

From this we conclude two things: Firstly,

$$(B_A[x]/\!\!/B)^p = \{\overline{0}\} \qquad \text{for} \qquad p = 1, \dots, d-1, \text{ i.e.,}$$

$$(\dagger) \qquad B_A[x]^p \subseteq I(B) \qquad \text{for} \qquad p = 1, \dots, d-1.$$

Now

$$(*)\quad c(x) = x \otimes 1 + 1 \otimes x + \sum_{p=1}^{d-1} a_p \quad \text{with} \quad a_p \in A^p \otimes A^{d-p} \quad \text{for } p = 1, \dots, d-1.$$

We claim $a_p \in A^p \otimes I(B)^{d-p} + I(B)^p \otimes A^{d-p}$. For if not, then writing $a_p = \sum_{k=1}^m \alpha_k \otimes \beta_k$ with $\alpha_k \in A^p$ and $\beta_k \in A^{d-k}$ would yield at least one $\alpha_k \in A^p \setminus I(A)^p$ or one $\beta_k \in A^{d-p} \setminus I(B)^{d-p}$; in view of (i) above this would be a contradiction to the definition of d in (min). Hence

$$(\dagger\dagger) \qquad c(x) - (x \otimes 1 + 1 \otimes x) \in A \otimes^* I(B) + I(B) \otimes^* A.$$

Secondly,

$$\big((A/\!\!/B) \otimes^* (A/\!\!/B)\big)^d \supseteq K{\cdot}q(x) \otimes \overline{1} + \overline{1} \otimes K{\cdot}q(x).$$

Then in view of A3.72(iv) we have

$$(\ddagger) \qquad \overline{c}\big(q(x)\big) = q(x) \otimes \overline{1} + \overline{1} \otimes q(x), \quad \text{i.e.} \quad q(x) \in P(A/\!\!/B).$$

Then $\overline{c}(x^n) = \big(q(x) \otimes \overline{1} + \overline{1} \otimes q(x)\big)^n \in B_A[x] \otimes^* B_A[x]$. Thus $B_A[x]/\!\!/B$ is a Hopf subalgebra of $A/\!\!/B$.

(iii) This is a consequence of (ii).

(iv) We now define the graded algebra homomorphism $\varphi: A \to A \otimes^* (A/\!\!/B)$ by $\varphi = (\mathrm{id}_A \otimes^* q) \circ c$. Then for each $b \in B_A$, then $b = r{\cdot}1 + b^+$ with $b^+ \in I(B)$ and thus $c(b) = r{\cdot}(1 \otimes 1) + c(b^+) = r{\cdot}(1 \otimes 1) + (b^+ \otimes 1) + \sum_{j \in J} b'_j \otimes b''_j$ with $b'_j \in B_A$ and $b''_j \in I(B)$. Hence $\varphi(b) = r{\cdot}(1 \otimes \overline{1}) + b^+ \otimes \overline{1} = b \otimes \overline{1}$.

From (††) we get $\varphi(x) = x \otimes \overline{1} + 1 \otimes q(x)$. Therefore

$$\begin{aligned}\varphi(x^2) &= x^2 \otimes \overline{1} + (x \otimes \overline{1})\big(1 \otimes q(x)\big) + \big(1 \otimes q(x)\big)(x \otimes \overline{1}) + 1 \otimes q(x)^2 \\ &= x^2 \otimes \overline{1} + (1 + (-1)^{d^2})\big(x \otimes q(x)\big) + 1 \otimes q(x)^2.\end{aligned}$$

If d is odd, then $x^2 = (-1)^{d^2} x^2 = -x^2$ and thus $x^2 = 0$ if $\mathrm{char}\, K \neq 2$. If d is odd, for $b \in B_A$ and $n = 1$, or if d is even and $n \in \mathbb{N}$ we get

$$\begin{aligned}\varphi(bx^n) &= \varphi(b)\varphi(x)^n = (b \otimes \overline{1})(x \otimes \overline{1} + 1 \otimes q(x))^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (b \otimes \overline{1})(x^j \otimes q(x)^{n-j}) \\ &= \sum_{j=0}^{n} \binom{n}{j} (bx^j \otimes q(x)^{n-j}).\end{aligned} \tag{L0}$$

Now assume $\mathrm{char}\, K = 0$ and suppose that there is a nontrivial linear relation

$$\sum_{m=0}^{N} b_m x^m = 0, \quad b_m \in B_A \quad \text{with minimal} \quad N > 0. \tag{L1}$$

We shall derive a contradiction.

If d is odd, then $N = 1$. At any rate, with the aid of (L0) we compute

$$\begin{aligned}0 = \varphi\Big(\sum_{m=0}^{N} b_m x^m\Big) &= \sum_{0 \le j \le m \le N} \binom{m}{j} (b_m x^{m-j} \otimes q(x)^j) \\ &= \sum_{j=0}^{N} \frac{1}{j!} \Big(\sum_{m=j}^{N} m(m-1)\cdots(m-j+1){\cdot}b_m x^{m-j} \Big) \otimes q(x)^j.\end{aligned}$$

The first summand with $j = 0$ vanishes because of (L1). Applying to the remaining sum the projection

$$A \otimes^* (A/\!\!/B) = \bigoplus_{m \in \mathbb{N}_0} A \otimes (A/\!\!/B)^m \longrightarrow A \otimes (A/\!\!/B)^d, \quad d = \deg x,$$

we get

$$\sum_{m=1}^{N} m{\cdot}b_m x^{m-1} \otimes q(x) = 0. \tag{L2}$$

Since $q(x) \neq 0$ by (i), it follows from (L2) that

$$\text{(L3)} \qquad \sum_{m=1}^{N} m{\cdot}b_m x^{m-1} = 0.$$

Now (L1) and (L3) are linear relations of the same type, but (L3), which follows from (L1) contradicts the minimality of N in (L1), and this contradiction completes the proof of the fact that there cannot be a relation $\sum_{m=0}^{N} b_m x^m = 0$ with $b_N x^N \neq 0$. In particular, $bx^m = 0$ with $m \geq 1$, $b \in B_A$ and $x^m \neq 0$ implies $b = 0$. It now follows that $B_A[x] = \bigoplus_{m\in\mathbb{N}_0} B_A x^m$ and $B_A x^m \cong B_A$ if $B_A x^m \neq 0$. For d odd we have $B_A[x] = K \oplus B_A x$. If d is even, $B_A[x] = \bigoplus_{n\in\mathbb{N}_0} B_A x^n$. Thus (iv) is proved.

(v) By (iv) above, $B_A x^m \cong B_A \otimes K{\cdot}x^m$ for all m and

$$B_A[x] = \bigoplus_{m\in J(x)} B_A x^m.$$

But now the natural morphism $\mu\colon B_A \otimes^* K[x] \to B_A[x]$ is given by $\mu(\sum_{m\in J(x)} b_m \otimes x^m) = \sum_{m\in J(x)} b_m x^m$ and we recognize that it is an isomorphism in view of these facts. The remainder is then straightforward.

(vi) This is a direct consequence of (iv) above. □

The arguments in the proof of Proposition A3.77 are in some sense elementary, but its architecture is fairly delicate.

Proposition A3.78. *Let A be a connected commutative graded Hopf algebra over a field K.*

(i) *If $V \subseteq P(A)$, then the subalgebra $K[V]$ generated by V is a primitively generated Hopf subalgebra and the morphism of graded algebras $i'\colon \mathbf{H}V \to A$ induced by the inclusion $i\colon V \to A$ by* A3.75(i) *is a morphism of Hopf algebras onto $K[V]$. In particular, $K[P(A)]$ is the largest primitively generated subalgebra of A and $\varepsilon_A\colon \mathbf{H}\big(P(A)\big) \to A$ is a morphism of Hopf algebras onto $K[P(A)]$.*

(ii) *Set $d = \min\big\{n \in \mathbb{N} \mid A^n \neq \{0\}\big\}$. Then $A^d \subseteq P(A)$.*

(iii) *A is primitively generated if and only if the underlying algebra is generated by $P(A)$.*

Proof. (i) We must show $c_A(K[V]) \subseteq K[V] \otimes^* K[V]$. Since c is an algebra morphism, it suffices to check this for algebra generators $x \in V$. Since $x \in P(A)$ we have $c_A(x) = x \otimes 1 + 1 \otimes x \in K[V] \otimes^* K[V]$. In $\mathbf{H}V \supseteq V$ we have $c_{\mathbf{H}V}(x) = x \otimes 1 + 1 \otimes x$. Thus, since $i'|V = i$,

$$\begin{aligned}\big((i' \otimes^* i') \circ c_{\mathbf{H}V}\big)(x) &= i'(x) \otimes 1 + 1 \otimes i'(x) = x \otimes 1 + 1 \otimes x \\ &= c_A(x) = c_A\big(i'(x)\big) = (c_A \circ i')(x).\end{aligned}$$

Since $(i' \otimes^* i') \circ c_{\mathbf{H}V}$ and $c_A \circ i'$ are morphisms of algebras, their equalizer is a subalgebra; but it contains V which is a set of algebra generators of $\mathbf{H}V$. Hence they agree and i' therefore is a morphism of Hopf algebras. It is readily checked that i' is compatible with coidentities.

(ii) Let $x \in A^d$. If $x = 0$ there is nothing to prove. If $x \neq 0$ then we can apply Proposition A3.77 with $B = K$ and $A = A/\!\!/B$. Then A3.77(ii) proves the claim.

(iii) Assume that A is primitively generated. Then $A = K[P(A)]$ by (i), and thus A is generated as an algebra by $P(A)$. The converse is trivial. □

Proposition A3.79. (i) *Let B be a Hopf subalgebra of a connected graded commutative Hopf algebra A over a field of characteristic* 0 *and let V be a graded vector subspace of $P(A)$ such that $V \cap B = \{0\}$. Let $i'\colon \mathbf{H}V \to A$ be the algebra morphism induced by the inclusion $i\colon V \to A$ according to* A3.75(i). *Then the morphism*

$$\psi\colon B \otimes^* \mathbf{H}V \to A, \quad \psi(b \otimes h) = bi'(h),$$

is an isomorphism of graded algebras onto the image $B[V]$.

(ii) *In particular, for any connected commutative graded Hopf algebra A over a field K of characteristic* 0 *the morphism $\varepsilon_A\colon \mathbf{H}\big(P(A)\big) \to A$ of graded Hopf algebras is injective and thus is an isomorphism of Hopf algebras onto $K[P(A)]$.*

Proof. (i) By A3.78(i) the subalgebra $K[V]$ is a Hopf subalgebra of A and $i'\colon \mathbf{H}V \to K[V]$ is a surjective morphism of Hopf algebras. The vector space V is the directed union of the set $\mathcal{F}$ of all finite dimensional subspaces $W \subseteq V$. Accordingly $\mathbf{H}V$ is the directed union of the set of all $\mathbf{H}W$, $W \in \mathcal{F}$. The function ψ is injective if and only if $\psi|B \otimes^* \mathbf{H}W$ is injective for all $W \in \mathcal{F}$. It suffices therefore to prove the claim for the case that V is finite dimensional.

In that case we prove the claim by induction with respect to $m = \dim_K V$. For $m = 0$ the assertion holds trivially since then $B \otimes^* \mathbf{H}\{0\} = B \otimes^* K$ and ψ is (essentially) the inclusion morphism $B \to A$. Assume that $\dim_K V = m > 0$ and that the claim has been proved for all graded vector subspaces $W \subseteq P(A)$ with $\dim_K W < m$ and $B \cap W = \{0\}$. Assume $V \subseteq P(A)$, $\dim_K V = m$, and $B \cap V = \{0\}$. We consider an $m-1$-dimensional graded vector subspace W of V and let $B' \overset{\text{def}}{=} B[W]$. Then $\psi|B \otimes^* \mathbf{H}W\colon B \otimes^* \mathbf{H}W \to B'$ by induction hypothesis is an isomorphism of Hopf algebras. Now pick $x \in V \setminus B'$ homogeneous with minimal degree. Then $V = W \oplus K{\cdot}x$. Now $K[x] = \bigoplus_{n \in J(x)} K{\cdot}x^n$. By Proposition A3.77(v), the natural morphism $m\colon B' \otimes^* K[x] \to A$ characterized by $m(b \otimes c) = bc$ is an isomorphism of graded algebras onto $B'[x] = B[V]$. By A3.77(v) we know that $\varepsilon_{K[x]}\colon \mathbf{H}(K{\cdot}x) \to K[x]$ is an isomorphism. Thus $B \otimes^* \mathbf{H}W \otimes^* \mathbf{H}(K{\cdot}x) \to B[V]$, $b \otimes h \otimes k \mapsto \mathbf{b}i'_V(h) \otimes i'_{K \cdot x}(k)$ is an isomorphism. From the commutativity of the diagram

$$\begin{array}{ccc}
\mathbf{H}W \otimes^* \mathbf{H}(K{\cdot}x) & \xrightarrow{\varepsilon_{K[W]} \otimes^* \varepsilon_{K[x]}} & K[W] \otimes^* K[x] \\
{\scriptstyle M}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
\mathbf{H}(W \oplus K{\cdot}x) & \xrightarrow[\varepsilon_K[V]]{} & K[W \oplus K{\cdot}x]
\end{array}$$

we conclude that $\psi\colon B \otimes^* \mathbf{H}V \to B[V] \subseteq A$ is an isomorphism of graded algebras.

(ii) This is a special case of (i) with $B = K$ and $V = P(A)$. □

Proposition A3.80. *Let A be a connected graded commutative Hopf algebra over a field of characteristic* 0*. Then the morphism $q_A: P(A) \to Q(A)$ is injective.*

Proof. By 3.72(vi) we have to show that $P(A) \cap S(A) = \{0\}$. By A3.78(ii) we have $P(A) \neq \{0\}$ and by A3.79(ii) we know $K[P(A)] \cong \mathbf{H}\big(P(A)\big)$ and we further know that $P\big(\mathbf{H}V\big) \cap S\big(\mathbf{H}V\big) = \{0\}$. Hence $K[P(A)]$ is a member of the set $\mathcal{B}$ of graded subalgebras B of A containing $K[P(A)]$ such that $P(A) \cap S(B) = \{0\}$. Then this set is inductive. Let B be a maximal element. We claim $B = A$. If not then there is a homogeneous element $x \in A \setminus B$ of minimal degree. By A3.77(v), $B \otimes^* K[x] \to B[x]$ is an isomorphism of algebras. Now $P(B \otimes^* K[x]) = P(A) \otimes 1$ and $S(B \otimes^* K[x]) = B^+ \otimes K[x]x + S(B) \otimes 1 + K \otimes K[x]x^2$. Thus $P(B \otimes^* K[x]) \cap S(B \otimes^* K[x]) = P(A) \otimes 1 \cap S(B) \otimes 1 = \{0\}$ since $B \in \mathcal{B}$. Hence $P(A) \cap S(B[x]) = \{0\}$ and thus $B[x] \in \mathcal{B}$. This contradicts the maximality of B and thus proves $B = A$ and thus the proposition. □

Let A be a connected graded commutative Hopf algebra over a field of characteristic 0 and set $A(n) = K[\sum_{m=1}^{n} A^m]$. Clearly, $A = \bigcup_{n=1}^{\infty} A(n)$. Recall that $P(A)$ is a graded submodule $\bigoplus_{n\in\mathbb{N}} P(A)^n$, and each $A(n)$ is a graded subalgebra $A(n) = \bigoplus_{m\in\mathbb{N}_0} A(n)^m$. Every element in $A(n)^{n+1}$ is a linear combination of nondegenerate products of homogeneous elements $x \in A$ of degree $\deg x \in \{1, 2, \ldots, n\}$, and every element of $A^{n+1} \cap S(A)$ is necessarily of this form. Hence $A(n)^{n+1} = A^{n+1} \cap S(A)$. It then follows from Proposition A3.80 that $P(A) \cap A(n)^{n+1} = \{0\}$.

We decompose the vector space A^{n+1} as a direct sum $A(n)^{n+1} \oplus P(A)^{n+1} \oplus W(n+1)$. Note that the vector space $W(n+1)$ is not uniquely determined.

Proposition A3.81. *In the circumstances of the preceding paragraph the following conclusions hold:*

(i) *$A(n)$ is an ascending sequence of Hopf subalgebras.*

(ii) *The inclusions $j: P(A)^{n+1} \to A(n+1)$ and $j_W: W(n+1) \to A(n+1)$ by* A3.75(i) *induce unique morphisms of graded algebras $j': \mathbf{H}\big(P(A)^{n+1}\big) \to A(n+1)$ and $j'_W: \mathbf{H}\big(W(n+1)\big) \to A(n+1)$ such that the morphism of algebras $\psi: A(n) \otimes^* \mathbf{H}\big(P(A)^{n+1}\big) \otimes^* \mathbf{H}\big(W(n+1)\big) \to A(n+1)$ given by $\psi(a \otimes^* v \otimes^* w) = aj'(v)j'_W(w)$ is an isomorphism of algebras. The restriction to $A(n) \otimes^* \mathbf{H}\big(P(A)^{n+1}\big) \otimes^* K$ is a morphism of Hopf algebras.*

(iii) *The following statements are equivalent.*
  (1) *$q_A: P(A) \to Q(A)$ is an isomorphism of graded modules.*
  (2) $(\forall n = 2, 3, \ldots)$ $W(n) = \{0\}$.
  (3) *A is primitively generated.*

Proof. (i) We prove the assertion by induction. For $n = 1$ we have $A^1 \subseteq P(A)$ by Proposition A3.72(vii). Thus $A(1)$ is a Hopf subalgebra by A3.78(i). Note that

$$A^m = (A(n))^m \quad \text{for } m = 0, \ldots, n \quad \text{and that}$$

$$c(A^{n+1}) \subseteq A^{n+1} \otimes 1 + \sum_{m=1}^{n-1} A^m \otimes A^{n-m} + 1 \otimes A^{n+1} \subseteq A(n+1) \otimes^* A(n+1).$$

Now assume that $A(n)$ is a Hopf subalgebra. Then $c(A^m) \subseteq A(n) \otimes^* A(n) \subseteq A(n+1) \otimes^* A(n+1)$ for $m = 1, \ldots, n$. Since $A(n+1)$ is generated as an algebra by $K + A^1 + \cdots + A^{n+1}$ and c is a morphism of algebras, we conclude that $c\big(A(n+1)\big) \subseteq A(n+1) \otimes^* A(n+1)$. Thus the induction is complete.

(ii) The existence of the vector subspaces $P(A)^{n+1}$ and $W(n+1)$ is clear. The subalgebra $K[P(A)^{n+1}]$ is a primitively generated Hopf subalgebra by A3.78 and the restriction of ψ to $A(n) \otimes^* \mathbf{H}\big(P(A)^{n+1}\big) \otimes^* K$ is an morphism of graded Hopf algebras by A3.78(i). Consider the set $\mathcal{V}$ of graded vector subspaces $W \subseteq W(n+1)$ such that the inclusion $j_W\colon W \to A(n+1)$ induces an algebra morphism $j'_W\colon \mathbf{H}W \to A(n+1)$ in such a fashion that the induced algebra morphism $\psi_W\colon A(n) \otimes^* \mathbf{H}(P(A)^{n+1}) \otimes^* \mathbf{H}W \to A(n+1)$ is injective and that

$$(*) \qquad \operatorname{im} \psi_W = A(n)[(P(A)][W]$$

is a Hopf subalgebra of A. Since the tensor product commutes with direct limits, the set $\mathcal{V}$ is inductive and contains $\{0\}$. Let W be maximal in $\mathcal{V}$. We claim that $W = W(n+1)$. This will prove that ψ is an isomorphism of algebras.

Suppose not. Then there is a homogeneous element $x \in W(n+1) \setminus W$. Set $B' \stackrel{\text{def}}{=} \operatorname{im} \psi_W$ according to $(*)$, a Hopf subalgebra of A. Then $x \notin B'$ and $\deg x$ is minimal w.r.t. this property. By A3.77(v), the natural morphism of graded algebras

$$B' \otimes^* K[x] \to B'[x] = A(n)[P(A)^{n+1}][W + K{\cdot}x], \quad b' \otimes y \mapsto b'y$$

is an isomorphism of graded algebras. By A3.77(iv) the morphism of graded algebras $j'\colon \mathbf{H}(K{\cdot}x) \to K[x]$ is bijective. It follows that $B' \otimes^* \mathbf{H}(K{\cdot}x) \to B'[x]$, $b' \otimes h = b'j'(h)$ is bijective. In view of the isomorphism $\mathbf{H}(W \oplus K{\cdot}x) \to \mathbf{H}W \otimes^* \mathbf{H}(K{\cdot}x)$ we obtain the injectivity of the restriction of ψ to $A(n) \otimes^* \mathbf{H}\big(P(A)^{n+1}\big) \otimes^* \mathbf{H}(W \oplus K{\cdot}x)$. Furthermore, since for $m = 1, \ldots, n$ we have $A^m = A(n) \subseteq B'$, we conclude $c(x) \in x \otimes 1 + 1 \otimes x + \sum_{m=1} A^m \otimes A^{n-m} \subseteq B'[x] \otimes B'[x]$. Hence $B'[x]$ is a Hopf subalgebra of A. Thus $W + K{\cdot}x \in \mathcal{V}$, and this is a contradiction to the maximality of W. This contradiction proves the claim of (ii).

(iii) (1)⇒(2). In view of A3.72(vi) and A3.80, condition (1) is equivalent to $P(A) + S(A) = A^+$. Because of $A^{n+1} \cap S(A) = A(n)^{n+1}$ this is equivalent to $A^{n+1} = A(n)^{n+1} + P(A)^{n+1}$ for $n = 1, 2 \ldots$. By the definition of $W(n+1)$ this means $W(n+1) = \{0\}$ for $n = 1, 2 \ldots$. This proves (2).

(2)⇒(3). Condition (2) is equivalent to $A^n = P(A)^n \oplus A(n-1)^n$ for $n = 1, 2 \ldots$. Thus if $A(n-1)$ is primitively generated, so is $A(n)$. Since $A(1)$ is primitively generated by A3.78(ii), by induction, $A(n)$ is primitively generated for all $n \in \mathbb{N}$. As $A = \bigcup_{n \in \mathbb{N}} A(n)$, the whole algebra A is primitively generated.

(3)⇒(1). If A is primitively generated, then $A \cong \mathbf{H}\big(P(A)\big)$ by A3.79(ii). For the Hopf algebras $\mathbf{H}V$, however, we know $P\big(\mathbf{H}V\big) = V$ and that V is mapped isomorphically onto $Q\big(\mathbf{H}V\big) \cong \mathbf{H}V^+/S\big(\mathbf{H}V\big)$ under the quotient morphism. Thus (1) is proved. □

Corollary A3.82. *Let A be a connected graded commutative Hopf algebra over a field of characteristic 0 and define the graded vector subspace $W_1 = \{0\}$, $W_n = W(2) \oplus W(3) \oplus \cdots \oplus W(n)$, $n = 2, \ldots$.*

(i) *Then the morphism of graded algebras*

$$\nu_n\colon \mathbf{H}(P(A)^1 \oplus P(A)^2 \oplus \cdots \oplus P(A)^n) \otimes^* \mathbf{H}W_n \to A(n), \quad \nu_n(p \otimes h) = ph$$

is an isomorphism of graded algebras.

(ii) ν_n *is a morphism of Hopf algebras if and only if* $W_n = \{0\}$ *for* $n \in \mathbb{N}$.

Proof. (i) We prove the claim by induction. By A3.78(ii) we have $A^1 = P(A)^1$ and thus $\mathbf{H}(P(A)^1) \to K[P(A)^1] = A(1)$ is an isomorphism and the assertion is true for $n = 1$. Suppose it is true for $n-1$ with $n > 1$. From A3.81(ii) the morphism of graded algebras $A(n-1) \otimes^* \mathbf{H}\big(P(A)^n\big) \otimes^* \mathbf{H}\big(W(n)\big) \to A(n)$ is an isomorphism. By induction hypothesis and the multiplicativity of the functor $\mathbf{H}$ the morphism $\mathbf{H}\big(P(A)^1 \oplus \cdots \oplus P(A)^{n-1} \oplus P(A)^n\big) \otimes^* \mathbf{H}\big(W_{n-1} \oplus W(n)\big) \to A(n)$ is an isomorphism of graded algebras. Since $W_{n-1} \oplus W(n) = W_n$, the induction is complete.

(ii) For $n = 1$ there is nothing to prove. If $W_{n-1} = \{0\}$, i.e., $\mathbf{H}\big(P(A)^1 \oplus \cdots \oplus P(A)^{n-1}\big) \cong A(n-1)$, then $\mathbf{H}(P(A)^1 \oplus P(A)^2 \oplus \cdots \oplus P(A)^n) \otimes^* \mathbf{H}\big(W(n)\big) \to A(n)$ is an isomorphism of Hopf algebras if and only if $K[W(n)]$ is a Hopf subalgebra. Since $\mathbf{H}\big(W(n)\big) \cong K[W(n)]$ this is the case iff $W(n) = P(K[W(n)]) \subseteq P(A)$ iff $W(n) = \{0\}$. □

Theorem for Connected Graded Commutative Hopf Algebras

Theorem A3.83. *Let* K *be a field of characteristic* 0.

(i) *There is a graded vector subspace* W *with* $W^0 = W^1 = \{0\}$ *such that the unique morphism of graded algebras* $\nu_A\colon \mathbf{H}\big(P(A)\big) \otimes^* \mathbf{H}W \to A$ *extending the morphism* $P(A) \otimes^* 1 \oplus 1 \otimes^* W \to P(A) \oplus W \subseteq A$ *given by* $p \otimes 1 + 1 \otimes w = p + w$ *is an isomorphism of graded commutative algebras.*

(ii) *The quotient morphism* $A \to A/S(A)$ *maps the graded submodule* $P(A) + W$ *isomorphically onto* $Q(A)$.

(iii) *The following statements are equivalent:*

(1) $q_A\colon P(A) \to Q(A)$ *is an isomorphism of graded modules.*

(2) $W = \{0\}$.

(3) A *is primitively generated.*

(4) ν_A *is an isomorphism of Hopf algebras.*

(iv) *If* $\dim_K A < \infty$ *then* $Q(A)^{2m} = \{0\}$ *for* $m \in \mathbb{N}$. *Moreover,*

$$\nu_A\colon \bigwedge P(A) \otimes^* \bigwedge W \to A$$

is an isomorphism of graded commutative algebras.

Proof. (i) We let $W = W(2) \oplus W(3) \oplus \cdots$ with $W(n) \subseteq A(n)$ constructed inductively as in A3.81. Since tensor products commute with direct limits the claim then follows from A3.82.

(ii) The inverse image of $P(A) + W$ under ν_A is $P(A) \otimes^* 1 + 1 \otimes^* W$ in $\widetilde{A} \stackrel{\text{def}}{=} \mathbf{H}\big(P(A)\big) \otimes^* \mathbf{H}W$. Also, $S(\widetilde{A}) = \mathbf{H}\big(P(A)\big)^+ \otimes^* \mathbf{H}W^+ S\big(\mathbf{H}\big(P(A)\big)\big) \otimes^* 1 + 1 \otimes^* S\big(bfH(W)\big)$. It follows that the quotient morphism $\widetilde{A} \to \widetilde{A}/S(\widetilde{A})$ maps $P(A) \otimes^* 1 + 1 \otimes^* W$ isomorphically onto $\widetilde{A}^+/S(\widetilde{A}) = Q(\widetilde{A})$

(iii) The equivalence of (1), (2) and (3) follows from A3.81(iii) upon passing to direct limits. Clearly (3)⇒(4) by A3.79(ii). We prove (4)⇒(2): If ν_A is a morphism of Hopf algebras, then $\mathbf{H}\big(P(A)\big) \otimes^* \mathbf{H}W$ is a Hopf algebra. As $\mathbf{H}$ is a multiplicative functor by A3.67 and thus this Hopf algebra is isomorphic to $\mathbf{H}\big(P(A) \oplus W\big)$. The module of primitive elements of this Hopf algebra is $P(A) \oplus W$ by A3.74. Since the isomorphism v_A of Hopf algebras transports primitive elements, the submodule of primitive elements of A is $P(A) \oplus W$. This is equivalent to $W = \{0\}$. This is (2).

(iv) The subalgebras $K[P(A)^{2m}] \cong \mathbf{H}(P(A)^{2m})$ and $K[W^{2m}] \cong \mathbf{H}(W^{2m})$ are finite dimensional if and only if both $P(A)^{2m}$ and W^{2m} vanish, because otherwise they are isomorphic to polynomial algebras over a set of commuting variables. By (ii) above this is equivalent to $Q^{2m} = \{0\}$ for $m \in \mathbb{N}$. Furthermore, if all homogeneous components of even degree in $P(A)$ and W vanish, then $\mathbf{H}\big(P(A)\big) = \bigwedge P(A)$ and $\mathbf{H}W = \bigwedge W$. □

We let $\mathbf{Hopf}_0^{\text{pgc}}(\mathbb{AB}_K^*, \otimes^*)$ denote the full subcategory of primitively generated connected graded commutative Hopf algebras over K in $\mathbf{Hopf}(\mathbb{AB}_K^*, \otimes^*)$).

By Lemma A3.74 all Hopf algebras $\mathbf{H}V$ are primitively generated, and accordingly this holds for the special cases $\bigwedge V$ and $\mathbf{S}(V)$. Recall the concept of the equivalence of categories from Definition A3.39.

In the following corollary we continue to assume that K is a field of characteristic 0.

Corollary A3.84. (i) *The functors*

$$\mathbf{H}\colon (\mathbb{AB}_K^*, \oplus^*)_0 \to \mathbf{Hopf}_0^{\text{pgc}}(\mathbb{AB}_K^*, \otimes^*) \quad \textit{and} \quad P\colon \mathbf{Hopf}_0^{\text{pgc}}(\mathbb{AB}_K^*, \otimes^*) \to (\mathbb{AB}_K^*, \oplus^*)_0$$

implement an equivalence of categories.

(ii) *Every primitively generated connected graded commutative Hopf algebra over K is isomorphic to a Hopf algebra $\mathbf{H}V$ for a graded K-vector space V without degree 0 component.*

(iii) *Every finite dimensional primitively generated connected graded commutative Hopf algebra over K is isomorphic to $\bigwedge V$ for some finite dimensional graded K-vector space all of whose homogeneous components of even degree are zero.*

(iv) *Every finite dimensional primitively generated connected graded commutative Hopf algebra A over K is a tensor product of finitely many singly generated Hopf algebras $\bigwedge K{\cdot}v$ where v is an element of odd degree.*

Proof. (i) By Lemma A3.74 we have $P \circ \mathbf{H} = \mathrm{id}_{(\mathbb{AB}_K^*)_0}$ (assuming that we write, as we may, $V \subseteq \mathbf{H}V$; if we do not make this assumption, the equality is replaced by a natural isomorphism). By Proposition A3.79(ii), the natural morphism $\varepsilon_A\colon \mathbf{H}\big(P(A)\big) \to A$ is injective; if A is primitively generated, then ε_A is surjective, because ε_A is a morphism of graded Hopf algebras, whence the image of ε_A is a

Hopf subalgebra of A containing $P(A)$, while A is the smallest Hopf subalgebra of A containing $P(A)$. Hence ε_A is a natural isomorphism and the claim is proved by Lemma A3.76 and Definition A3.39.

(ii) This is an immediate consequence of (i).
(iii) This is a consequence of Theorem A3.83(iii).
(iv) now follows from (iii) via A3.67. □

This structure theorem says in effect that primitively generated connected commutative graded Hopf algebras over a field of characteristic 0 are the same thing as graded vector spaces with no degree zero components. It is implicit in the structure theorem that these Hopf algebras have a commutative comultiplication. For fields of finite characteristic separate efforts are necessary because of our use of the binomial formula in Proposition A3.77.

Proposition A3.85. *Let A be a primitively generated connected graded Hopf algebra such that $P(A)^{2m} = \{0\}$ for $m \in \mathbb{N}$. Then A is commutative.*

Proof. For two homogeneous elements x and y of A we write $[x,y] \overset{\text{def}}{=} xy - (-1)^{\deg x \deg y} yx$ and extend this definition by bilinearity to a bilinear function $[\cdot,\cdot]\colon A \times A \to A$. If x, $y \in P(A)$ are homogeneous primitive elements, then

$$\begin{aligned} c([x,y]) &= c(x)c(y) - (-1)^{\deg x \deg y} c(y)c(x) \\ &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) \\ &\quad - (-1)^{\deg x \deg y}(y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\ &= (xy - (-1)^{\deg x \deg y} yx) \otimes 1 + 1 \otimes (xy - (-1)^{\deg x \deg y} yx) \\ &= [x,y] \otimes 1 + 1 \otimes [x,y]. \end{aligned}$$

Thus $[x,y]$ is a homogeneous primitive element of degree $\deg x + \deg y$. This number is even. Then $P(A)^{\deg x + \deg y} = \{0\}$ by hypothesis. Therefore $xy = (-1)^{\deg x \deg y} yx$ for all primitive homogeneous elements. Then this holds when x and y are arbitrary products of homogeneous primitive elements. Since A is primitively generated it is generated by $P(A)$ as an algebra by A3.78(iii). It thus follows that the multiplication of the graded algebra A is commutative. □

Part 3: Duality of Graded Hopf Algebras

One of the nice features of a bimonoid A in a commutative monoidal category $(\mathbb{A}, \otimes)$ is that it is also a bimonoid in the opposite category $(\mathbb{A}^{\text{op}}, \otimes)$ obtained from $\mathbb{A}$ by simply "reversing arrows."

This is of particular interest under circumstances when dual categories are concretely realized. So let K be a field and $\mathbb{AB}^*_{K,\text{fin}}$ the category of finite dimensional graded vector spaces. For each such object V the vector space dual V' is naturally graded. Thus $V \mapsto V'\colon \mathbb{AB}^*_{K,\text{fin}} \to \mathbb{AB}^{*\text{op}}_{K,\text{fin}}$ is an equivalence of categories; the category is "autodual". If V and W are finite dimensional graded vector spaces, then $(V \otimes^* W)'$ is naturally isomorphic to $V' \otimes^* W'$ and we may identify these two graded vector space in such a way that $\alpha \in V'$, $\beta \in W'$, $v \in V$, $w \in W$ implies $\langle \alpha \otimes \beta, v \otimes w \rangle = \langle \alpha, v \rangle \langle \beta, w \rangle$.

Lemma A3.86. (i) *Let A be a connected graded Hopf algebra with data*

$$A \xrightarrow{c} A \otimes^* A \xrightarrow{m} A, \quad K \xrightarrow{u_A} A, \quad A \xrightarrow{k_A} K.$$

Then

$$A' \xleftarrow{c'} A' \otimes^* A' \xleftarrow{m'} A, \quad K \xleftarrow{u'_A} A', \quad A' \xleftarrow{k'_A} K$$

are the data of a graded connected Hopf algebra.

(ii) *The exact sequences*

$$0 \to P(A) \xrightarrow{j} A^+ \xrightarrow{c^+} A^+ \otimes^* A^*$$

and

$$A^+ \otimes^* A^+ \xrightarrow{m^+} A^+ \xrightarrow{p} Q(A) \to 0$$

yield exact sequences

$$0 \leftarrow P(A)' \xleftarrow{j'} (A')^+ \xleftarrow{(c')^+} (A')^+ \otimes^* (A')^+$$

and

$$(A')^+ \otimes^* (A')^+ \xleftarrow{(m')^+} (A')^+ \xleftarrow{p'} Q(A)'.$$

(iii) *If $Q(A)'$ is identified with a submodule of A' it is the submodule $P(A')$ of primitive elements, and $P(A)'$ may be identified with the module $Q(A')$ of indecomposable elements. The morphism $q'_A\colon Q(A)' \to P(A)'$ thereby may be identified with $q_{A'}\colon P(A') \to Q(A')$.*

Proof. Exercise EA3.38. □

The proof is direct.

Exercise EA3.38. Prove Lemma A3.86. □

Structure of Finite Dimensional Graded Hopf Algebras

Theorem A3.87. *Let A be a* 1) *connected,* 2) *graded,* 3) *commutative Hopf algebra A, defined* 4) *over a field with characteristic* 0 *which is, in addition,* 5) *finite dimensional. Then the following conclusions hold.*

(i) *A is primitively generated.*
(ii) $P(A)^{2m} = \{0\}$ *for* $m \in \mathbb{N}$.
(iii) *The natural map $\varepsilon_A\colon \bigwedge P(A) \mapsto A$ is an isomorphism of graded Hopf algebras.*

Proof. (i) Since $\dim A$ is finite dimensional, $Q(A)^{2m} = \{0\}$ for $m \in \mathbb{N}$ by A3.83(iv). Then by Lemma A3.86(iii) we have $P(A')^{2m} = \{0\}$ for all $m \in \mathbb{N}$.

Now by A3.85, the multiplication of A' is commutative. Thus $q_{A'}: P(A') \to Q(A')$ is injective by A3.80. Hence by Lemma A3.86(iii) again, $q_A: P(A) \to Q(A)$ is surjective and thus bijective in view of A3.80. Now A3.83(iii) implies that A is primitively generated.

(ii) This follows from $P(A) \cong Q(A)$ and $Q(A)^{2m} = \{0\}$ for $m \in \mathbb{N}$.

(iii) From (i) above and Corollary A3.83 we know that $\varepsilon_A: \mathbf{H}\big(P(A)\big) \to A$ is an isomorphism. By (ii) we have $\mathbf{H}\big(P(A)\big) = \bigwedge P(A)$. □

Corollary A3.88. *A finite dimensional connected graded commutative Hopf algebra A over a field with characteristic 0 is uniquely and functorially determined by the graded vector space $P(A)$ of its primitive elements.* □

Corollary A3.89. *Let A be a connected graded commutative Hopf algebra over a field with characteristic 0 satisfying the following hypothesis:*

5′) *The collection $\mathcal{A}$ of finite dimensional Hopf subalgebras is $\leq$-directed and $A = \bigcup \mathcal{A}$.*

Then the conclusions (i), (ii), *and* (iii) *of* Theorem A3.87 *hold.*

Proof. Exercise EA3.39. □

Exercise EA3.39. Prove Corollary A3.89.

[Hint. The properties (i), (ii), (iii) are preserved under direct limits.] □

Part 4: An Application to Compact Monoids

Let us consider a compact connected topological monoid G such that for a field K of characteristic 0, the cohomology vector space $H(G) \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{N}_0} H^n(G,K)$ is finite dimensional. By Corollary A3.70, $H(G)$ is a graded commutative Hopf algebra. We continue to use Čech cohomology.

Theorem A3.90 (H. Samelson). *If G is a compact connected topological monoid G whose cohomology Hopf algebra $H(G)$ over a field K of characteristic 0 is finite dimensional, then*

(i) *there are natural isomorphisms of graded Hopf algebras*

$$\begin{aligned} H(G) &\cong \bigwedge P\big(H(G)\big) \\ &\cong \bigwedge P\big(H(G)\big)^1 \otimes \bigwedge P\big(H(G)\big)^3 \otimes \cdots \otimes \bigwedge P\big(H(G)\big)^{2N-1}. \end{aligned}$$

(ii) *The graded vector space $P\big(H(G)\big)$ of primitive elements of $H(G)$ has only odd dimensional nonvanishing homogeneous components and determines $H(G)$ uniquely and functorially.*

(iii) (H. Hopf) *If* $d_{2j-1} = \dim P\big(H(G)\big)^{2j-1}$, $j = 1, 2, \ldots, N$, *define* S *to be the product*

$$S \stackrel{\text{def}}{=} (\mathbb{S}^1)^{d_1} \times (\mathbb{S}^3)^{d_3} \times \cdots \times (\mathbb{S}^{2N-1})^{d_{2N-1}}.$$

Then the graded commutative K*-algebras* $H(G)$ *and* $H(S)$ *are isomorphic.*

Proof. (i) and (ii) are consequences of A3.70 and A3.87.

(iii) The functor

$$H\colon (\mathbb{CTOP}, \times) \to (\mathbb{AB}_K^*, \otimes^*)^{\mathrm{op}}$$

of Proposition A3.67(ii) is a multiplicative functor. Hence there is an algebra isomorphism

$$(**) \qquad H(S) \cong \bigotimes^{d_1} H(\mathbb{S}^1) \otimes \cdots \otimes \bigotimes^{d_{2N-1}} H(\mathbb{S}^{2N-1}).$$

Now

$$H^m(\mathbb{S}^p, K) = \begin{cases} K & \text{if } m = 0 \text{ or } m = p \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence

$$\bigotimes^{d_{2j-1}} H(\mathbb{S}^{2j-1}) = \bigwedge V^{2j-1},$$

where V^{2j-1} is a vector space of dimension d_{2j-1} and all of its nonzero elements have degree $2j-1$. But then

$$\bigwedge V^{2j-1} \cong \bigwedge P\big(H(G)\big),$$

and the assertion follows. □

We should hasten to add, that for every compact manifold (locally euclidean space) M of dimension n we have $\dim H^*(M, K) < \infty$. (See e.g. [338], in particular p. 292ff.)

Corollary A3.91. *Assume that* G *is a compact connected topological monoid* G *which is a projective limit* $\lim_{j\in J} G_j$ *of an inverse system of compact topological monoids such that* $\dim_K H^*(G_j, K) < \infty$ *for a field* K *of characteristic* 0 *and all* $j \in J$. *Then*

(i) *there are natural isomorphisms of graded Hopf algebras* $H(G) \cong \bigwedge P\big(H(G)\big)$.

(ii) *The graded vector space* $P\big(H(G)\big)$ *of primitive elements of* $H(G)$ *has only odd dimensional nonvanishing homogeneous components and determines* $H(G)$ *uniquely and functorially.*

Proof. The contravariant functor

$$H\colon (\mathbb{CTOP}, \times) \to (\mathbb{AB}_K^*, \otimes^*)$$

of A3.67(ii) converts projective limits into direct limits. (See e.g. [338], pp. 318, 319.) Therefore $H(G) \cong \operatorname{colim}_{j\in J} H(G_j)$. The Theorem of Hopf–Samelson A3.90

applies to each $H(G_j)$ and each of its homomorphic images in $H(G)$. The family of images of the $H(G_j)$ is an upwards directed family $\mathcal{A}$ satisfying the hypothesis $5')$ of Corollary A3.89. The assertion then follows from Corollary A3.89. □

The main application aims at compact groups. From Chapter 1 we know that every compact group is a projective limit of Lie groups (see 2.43). The underlying space of a compact Lie group is locally euclidean (see Chapters 5 and 6). Hence A3.91 applies to every compact group G in particular.

Corollary A3.92. *Suppose that G is a compact connected group. Then*

(i) *there is a natural isomorphism of graded Hopf algebras $H(G) \cong \bigwedge P\big(H(G)\big)$.*

(ii) *The graded vector space $P\big(H(G)\big)$ of primitive elements of $H(G)$ has only odd dimensional nonvanishing homogeneous components and determines $H(G)$ uniquely and functorially.*

Proof. The proof is immediate from A3.91. □

It may be puzzling at first sight that compact connected monoids and compact groups have the same type of cohomology and that the cohomology Hopf algebra is in fact a cogroup in the category $(\mathbb{AB}^*_K, \otimes^*_K)$ of graded K-vector spaces. However this is a consequence of a fact known from the theory of compact monoids, that every maximal subgroup of its minimal ideal (cf. Exercise E5.1(e), see e.g. [196]) is a (weak) homotopy retract of the monoid and therefore the two have isomorphic cohomology.

Part 5: Symmetric Hopf Algebras over $\mathbb{R}$ and $\mathbb{C}$

We shall now restrict our attention to the ground fields $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. In order to define the concept of a weakly complete Hopf algebra over $\mathbb{K}$, we summarize its definition which is based on our category theoretical discussions on Definitions A3.62 through A3.65 applied to the symmetric monoidal category $\mathcal{W}$ of weakly complete $\mathbb{K}$-vector spaces presented in Appendix 7 from Definition A7.8 through A7.19. The concept of a weakly complete unital K-algebra is defined in Definition A7.32, and we consider the definition of a unital (associative) $\mathbb{K}$-algebra as well understood. While our earlier definitions should make it clear what a Hopf algebra and a symmetric Hopf algebra (over $\mathbb{K}$) are, for the convenience of the reader we repeat here the definition in the case of the weakly complete Hopf algebras.

Definition A3.93. A *weakly complete* $\mathbb{K}$-*Hopf algebra* is a weakly complete unital $\mathbb{K}$-algebra space equipped with

(i) a coassociative comultiplication $c: A \to A \otimes_{\mathcal{W}} A$ which is an algebra morphism,

(ii) a coidentity $k: A \to \mathbb{K}$, which is an algebra morphism, such that with the identity $u: \mathbb{K} \to A$, $k(t) = t{\cdot}1$ and the multiplication $m: A \otimes_{\mathcal{W}} A \to A$,

$m(a \otimes b) = ab$ the conditions of a bimonoid in the symmetric monoidal category $(\mathcal{W}, \otimes_W)$ are satisfied (see Definitiion A3.64).

A weakly complete $\mathbb{K}$-Hopf algebra A is called *symmetric* if in addition there is

(iii) a $\mathcal{W}$-morphism $\sigma: A \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes_{\mathcal{W}} A & \xrightarrow{\sigma \otimes \mathrm{id}_A} & A \otimes_W A \\ c\uparrow & & \downarrow m \\ A & \xrightarrow[u \circ k]{} & A. \end{array}$$

□

Theorem A3.94. *The categories of symmetric $\mathbb{K}$-Hopf algebras and Hopf algebra morphisms and that of weakly complete $\mathbb{K}$-Hopf algebras and morphisms of weakly complete Hopf algebra morphisms are dual to each other.*

Proof. The proof is staightfoward from the definitions and the duality of $\mathbb{K}$-vector spaces and weakly complete $\mathbb{K}$-vector spaces in Theorem A7.9. □

In any theory of Hopf algebras it is common to single out two types of special elements, and we review them in the case of weakly complete Hopf algebras.

Definition A3.95. Let A be a weakly complete Hopf algebra with comultiplication c and coidentity k. Then an element $a \in A$ is called *grouplike* if $k(a) = 1$ and $c(a) = a \otimes a$. The set of grouplike elements we call $\mathbb{G}(A)$. □

In Definition A3.71 we already said that an element $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$ and that we denoted the set of primitive elements of A by $\mathbb{P}(A)$.

For any $a \in A$ with $c(a) = a \otimes a$, the conditions $a \neq 0$ and $k(a) = 1$ are equivalent.

Remark A3.96. In at least one source on bialgebras in earlier contexts, the terminology conflicts with the one introduced here which is now commonly accepted. In [163], p. 66, Definition 10.17, the author calls a grouplike element in a coalgebra *primitive*. Thus some caution is in order concerning terminology. Primitive elements in the sense of Definition A3.95 do not occur in [163].

Lemma A3.97. *The set G of grouplike elements of a weakly complete Hopf algebra A is a closed submonoid of $(A, \cdot)$ and the set L of primitive elements of A is a closed Lie subalgebra of A_{Lie}. If A is a symmetric Hopf algebra, then G is a closed subgroup of A^{-1}.*

Proof. The proof is straightforward. □

Let us briefly consider one aspect of the duality in Theorem A3.94. For a morphism $f: W_1 \to W_2$ of weakly complete vector spaces let $f' = \mathcal{W}(f, \mathbb{K}): W_2' \to$

W_1' denote the dual morphism of vector spaces. Again by the sheer duality of the categories $\mathbb{V}$ of $\mathbb{K}$-vector spaces and the category $\mathcal{W}$ of weakly complete $\mathbb{K}$-vector spaces, for a weakly complete coalgebra A let $A' = \mathcal{W}(A, \mathbb{K})$ be the dual of A. Then A' is an algebra: If $c\colon A \to A \otimes A$ is the comultiplication of A then $c'\colon A' \otimes A' \to A'$ is the multiplication of A'. For a unital algebra R and a weakly complete coalgebra A in duality let $(a, g) \mapsto \langle a, g\rangle : R \times A \to \mathbb{K}$ denote the pairing between R and A, where for $f \in R = \mathrm{Hom}(A, \mathbb{K})$ and $a \in A$ we write $\langle f, a\rangle = f(a)$.

Definition A3.98. Let R be a unital algebra over $\mathbb{K}$. Then a *character* of R is a morphism of unital algebras $R \to \mathbb{K}$. The set of characters is subset of $\mathbb{K}^R$ and so inherits the topology of pointwise convergence from $\mathbb{K}^R$. This topological space of characters is called the *spectrum* of R and is denoted $\mathrm{Spec}(R)$.

An element $d \in \mathcal{V}(R, \mathbb{K})$ is called a *derivative* (sometimes also called a *derivation* or *infinitesimal character* of R (with respect to k) if it satisfies

$$(\forall x, y \in R)\, d(xy) = d(x)k(y) + k(x)d(y).$$

The set of all derivatives of R is denoted $\mathrm{Der}(R)$. □

Now let R be a unital algebra and $A \overset{\text{def}}{=} R^*$ its dual weakly complete coalgebra with comultiplication c such that $ab = c'(a \otimes b)$ for all $a, b \in R$. In these circumstances we have:

Proposition A3.99. *Let $g \in A$. Then the following statements are equivalent:*
(i) $g \in \mathbb{G}(A)$.
(ii) $g \in \mathrm{Spec}(R)$.

Proof. The dual of $A \otimes A$ is $R \otimes R$ in a canonical fashion such that for $r_1, r_2 \in R$ and $h_1, h_2 \in A$ we have

$$\langle r_1 \otimes r_2, h_1 \otimes h_2\rangle = \langle r_1, h_1\rangle\langle r_2, h_2\rangle.$$

The set of linear combinations $L = \sum_{j=1}^n a_j \otimes b_j \in A \otimes A$ is dense in $A \otimes A$. So two elements $x, y \in R \otimes R$ agree if and only if for all such linear combinations L we have

$$\langle x, L\rangle = \langle y, L\rangle,$$

and this clearly holds if and only if for all $a, b \in A$ we have

$$\langle x, a \otimes b\rangle = \langle y, a \otimes b\rangle.$$

We apply this to $x = c(g)$ and $y = g \otimes g$ and observe that (i) holds if and only if

$$\text{(0)} \qquad (\forall r, s \in R)\, \langle r \otimes s, c(g)\rangle = \langle r \otimes s, g \otimes g\rangle.$$

Now

$$\text{(1)} \qquad g(rs) = \langle rs, g\rangle = \langle m(r \otimes s), g\rangle = \langle r \otimes s, c(g)\rangle.$$

(2) $$g(r)g(s) = \langle r, g\rangle\langle s, g\rangle = \langle r \otimes s, g \otimes g\rangle.$$

So in view of (0),(1) and (2), assertion (i) holds if and only if $g(rs) = g(r)g(s)$ for all $r, s \in R$ is true. Since g is a linear form on A, this means exactly that a nonzero g is a morphism of weakly complete algebras, i.e., $g \in \mathrm{Spec}(R)$. □

Let us return to the primitive elements of a unital bialgebra. For this purpose assume that R is not only a unital algebra, but a Hopf algebra over $\mathbb{K}$, which implies that its dual $A \stackrel{\mathrm{def}}{=} R^*$ is a Hopf algebra over $\mathbb{K}$ by Theorem A3.94.

Proposition A3.100. *Let R be a unital Hopf algebra and $d \in A$. Then the following statements are equivalent:*
(i) $d \in \mathbb{P}(A)$
(ii) $d \in \mathrm{Der}(R)$. □

The procedure of the proof of Proposition A3.99 allows us to leave the explicit proof of this proposition as an exercise.

Definition A3.101. For any weakly complete Hopf algebra A let $\mathbf{S}(A)$ denote the closed linear span of $\mathbb{G}(A)$ in A. We say that A is *group-saturated* if $A = \mathbf{S}(A)$. □

Since $\mathbb{G}(A)$ is a submonoid of A^{-1}, the linear span of $\mathbb{G}(A)$ is a unital subalgebra, and so $\mathbf{S}(A)$ is a closed weakly complete subalgebra of A. Moreover, if $f\colon A \to B$ is a morphism of weakly complete Hopf algebras, then $f(\mathbb{G}(A)) \subseteq \mathbb{G}(B)$, and so $f(\mathbf{S}(A)) \subseteq \mathbf{S}(B)$. Therefore $c(\mathbf{S}(A)) \subseteq \mathbf{S}(A \otimes_{\mathcal{W}} A)$, where the comultiplication c of A replaces f.

If A is a weakly complete symmetric Hopf algebra then $\mathbb{G}(A)$ is a closed subgroup of A^{-1} which is an almost connected pro-Lie group (cf. A7.22ff.) according to Proposition A7.37, and whose Lie algebra is $\mathfrak{L}(A^{-1}) = A_{\mathrm{Lie}}$, the Lie algebra whose underlying vector space is A and whose Lie bracket is $[x, y] = xy - yx$. Therefore $\mathbb{G}(A)$ is a pro-Lie group. The exponential function $\exp_{A^{-1}}\colon \mathfrak{L}(A^{-1}) \to A^{-1}$ is the exponential function of A by Appendix 7, Theorem A7.41. Let us now identify its Lie algebra and its exponential function $\exp_{\mathbb{G}(A)}\colon \mathfrak{L}(\mathbb{G}(A)) \to \mathbb{G}(A)$. (See Appendix 7 for some basic information on pro-Lie groups.)

Theorem A3.102. *Let A be a symmetric $\mathbb{K}$-Hopf algebra. Then $\mathbb{G}(A)$ is a pro-Lie group whose Lie algebra may be identified with the closed Lie subalgebra $\mathbb{P}(A)$ of primitive elements in A_{Lie}. The exponential function of $\mathbb{G}(A)$ is the restriction and corestriction of the exponential function of A.*

Proof. It was noticed by R. Dahmen in [76] that for a primitive element $x \in \mathbb{P}(A)$ we compute $c(\exp x) = \exp c(x) = \exp(x \otimes 1 + 1 \otimes x) = (\exp(x \otimes 1))(\exp(1 \otimes x)) = ((\exp x) \otimes 1)(1 \otimes (\exp x)) = (\exp x) \otimes (\exp x)$. Therefore, $\exp x \in \mathbb{G}(A)$. Thus $\exp \mathbb{P}(A) \subseteq \mathbb{G}(A)$. Hence for each primitive element $x \in \mathbb{P}(A)$, the one parameter subgroup $\{\exp t{\cdot}x : t \in \mathbb{R}\}$ is contained in $\mathbb{G}(A)$. We have to verify the converse: So let $\exp t{\cdot}x \in \mathbb{G}(A)$ for all $t \in \mathbb{R}$ and some $x \in A$. We have to show that $x \in \mathbb{P}(A)$. Now

$$\exp t{\cdot}c(x) = \exp c(t{\cdot}x) = c(\exp t{\cdot}x) = (\exp t{\cdot}x) \otimes (\exp t{\cdot}x)$$
$$= \big((\exp t{\cdot}x) \otimes 1\big)\big(1 \otimes (\exp t{\cdot}x)\big) = \exp(t{\cdot}x \otimes 1)\exp(1 \otimes t{\cdot}x)$$
$$= \exp((t{\cdot}x \otimes 1) + (1 \otimes t{\cdot}x)) = \exp t{\cdot}(x \otimes 1 + 1 \otimes x),$$

for all $t \in \mathbb{R}$, and this implies:

$$c(x) = x \otimes 1 + 1 \otimes x$$

which indeed means $x \in \mathbb{P}(A)$. □

Let us look more closely at the dual $\mathbb{K}$-Hopf algebra of a weakly complete symmetric $\mathbb{K}$-Hopf algebra A. For the group $G = \mathbb{G}(A)$ of grouplike elements, the underlying weakly complete vector space of A is a topological left and right G-module A with the module operations

$$\begin{array}{llll}(g,a) \mapsto g{\cdot}a: & G \times A \to A, & g{\cdot}a := ga, & \text{and} \\ (a,g) \mapsto a{\cdot}g: & G \times A \to A, & a{\cdot}g := ag. & \end{array}$$

In Appendix 7 we denoted by $\mathbb{I}(A)$ the filterbasis of closed two-sided ideals I of A such that A/I is a finite dimensional algebra and that $A \cong \lim_{I \in \mathbb{I}(A)} A/I$. We can clearly reformulate Corollary A7.35 in terms of G-modules as follows:

Lemma A3.103. *For the topological group $G = \mathbb{G}(A)$, the G-module A has a filter basis $\mathbb{I}(A)$ of closed two-sided submodules $I \subseteq A$ such that* $\dim(A/J) < \infty$ *and that $A = \lim_{I \in \mathbb{I}(A)} A/I$ is a strict projective limit of finite dimensional G-modules. The filter basis $\mathbb{I}(A)$ in A converges to* $0 \in A$. □

For a $J \in \mathbb{I}(A)$ let $J^\perp = \{f \in A' : (\forall a \in J)\, \langle f, a\rangle = 0\}$ denote the annihilator of J in the dual A' of A. We compare the "Annihilator Mechanism" from Proposition 7.62 and observe the following configuration:

$$\begin{array}{ccl} A & \{0\} & \\ | & | & \Big\} \cong (A/I)' \\ I & I^\perp & \\ | & | & \Big\} \cong I' \\ \{0\} & A'. & \end{array}$$

In particular we recall the fact that $I^\perp \cong (A/I)'$ showing that $I^\perp$ is a finite-dimensional G-module on either side. By simply dualizing Lemma 3.103 we obtain

Lemma A3.104. *For the topological group $G = \mathbb{G}(A)$, the dual G-module $R \stackrel{\text{def}}{=} A'$ of the weakly complete G-module A has an up-directed set $\mathbb{D}(R)$ of finite-dimensional two-sided G-submodules (and $\mathbb{K}$-coalgebras!) $F \subseteq R$ such that R is the direct limit*

$$R = \operatorname*{colim}_{F \in \mathbb{D}(R)} F = \bigcup_{F \in \mathbb{D}(R)} F. \qquad \square$$

The colimit is taken in the category of (abstract) G-modules, i.e. modules without any topology.
This means that for the topological group $G = \mathbb{G}(A)$, every element ω of the dual of A' is contained in a finite dimensional left- and right-G-module (and $\mathbb{K}$-subcoalgebra).

We record this in the following form:

Lemma A3.105. *Consider $\omega \in A'$. Then the vector subspaces* $\operatorname{span}(G{\cdot}\omega)$ *and* $\operatorname{span}(\omega{\cdot}G)$ *of both the left orbit and the right orbit of ω are finite dimensional, and both are contained in a finite dimensional $\mathbb{K}$-subcoalgebra of A'.* $\square$

For any $\omega \in A'$ the restriction $f \stackrel{\text{def}}{=} \omega|G : G \to \mathbb{K}$ is a continuous function such that each of the sets of translates f_g, $f_g(h) = f(gh)$, respectively, ${}_gf$, ${}_gf(h) = f(hg)$ forms a finite dimensional vector subspace of the space $C(G, \mathbb{K})$ of the vector space of all continuous $\mathbb{K}$-valued functions f on G.

In Definition 3.3., for an arbitrary topological group G we defined $R(G, \mathbb{K}) \subseteq C(G, \mathbb{K})$ to be that set of continuous functions $f : G \to \mathbb{K}$ for which the linear span of the set of translations ${}_gf$, ${}_gf(h) = f(hg)$, is a finite dimensional vector subspace of $C(G, \mathbb{K})$. The functions in $R(G, \mathbb{K})$ were called *representative functions.*

In Lemma A3.105 we saw that for a weakly complete symmetric $\mathbb{K}$-Hopf algebra A and its dual A' (consisting of continuous linear forms) we have a natural linear map

$$\tau_A : A' \to R(\mathbb{G}(A), \mathbb{K}),\ \tau_A(\omega) = \omega|\mathbb{G}(A).$$

By Proposition A3.99, for each $g \in \mathbb{G}(A)$ the function $\omega \mapsto \omega(g) : A' \to \mathbb{K}$ is a character, i.e. is multiplicative which is also clear from the fact that it is a point evaluation on $R(\mathbb{G}(A), \mathbb{K})$. An element $\omega \in A'$ is in the kernel of τ_A if and only if $\omega(\mathbb{G}(A)) = \{0\}$ if and only if $\omega\big(\mathbf{S}(A)\big) = \{0\}$ if and only if $\omega \in \mathbf{S}(A)^\perp$. We therefore observe:

Lemma A3.106. *There is an exact sequence of $\mathbb{K}$-vector spaces*

$$0 \to \mathbf{S}(A)^\perp \xrightarrow{\text{incl}} A' \xrightarrow{\tau_A} R(\mathbb{G}(A), \mathbb{K}). \qquad \square$$

Using the terminology in Definition A3.101 we can formulate a conclusion that more intuitively represents the connection beween the dual A' of a weakly complete symmetric Hopf algebra A and the representation ring $R(\mathbb{G}(A), \mathbb{K})$ of its pro-Lie group $\mathbb{G}(A)$ of grouplike elements. Recall that A is called group-saturated iff $\mathbf{S}(A) = A$ iff $\mathbf{S}(A)^\perp = \{0\}$.

Corollary A3.107. *Let A be a weakly complete, symmetric Hopf algebra. Then the natural morphism $\tau_A: A' \to R(\mathbb{G}(A), \mathbb{K})$, $\tau_A(\omega) = \omega|\mathbb{G}(A)$ is injective if and only if A is group-saturated.* □

The issue of surjectivity is clear up to a point:

Remark A3.108. The morphism $\tau_A: A' \to R(\mathbb{G}(A), \mathbb{K})$ is surjective if and only if every representative function $f: \mathbb{G}(A) \to \mathbb{K}$ extends to some continuous linear functional $F: A \to \mathbb{K}$. □

If G is any topological group and $f \in R(G, \mathbb{K})$, then from Proposition 3.34 (whose proof does not depend on the compactness of the group G) there is a finite dimensional G-module V, an element ω in its dual V', and an element $v \in V$ such that $f(g) = \langle \omega, \pi(g)(v)\rangle$ with the representation π of G belonging to the G-module V. Then $\pi: G \to \mathrm{End}(V)$ where $\pi(G) \subseteq \mathrm{Aut}(V) = (\mathrm{End}(V)^{-1}$.

Now assume that $G = \mathbb{G}(A)$ as in Remark A3.108. If we assume that π has an extension to an algebra morphism $\overline{\pi}: A \to \mathrm{End}(V)$, then indeed $F(a) = \langle \omega, \overline{\pi}(a)(v)\rangle$ defines a linear form extending f and the surjectivity of τ is secured. If A happens to be the weakly complete group algebra of a compact group G, then $\mathbb{G}(A) \cong G$ by the results of the third part of Chapter 3, the extension $\overline{\pi}$ and, therefore, F exist. Thus τ is indeed surjective in such a situation.

Postscript

This appendix is a short introductory course in category theory. Our presentation of category theory aims for making understood a few basic concepts: morphisms in their various forms, functors, natural transformations, adjoint pairs of functors, limits and how these interrelate. However, it is a principal goal of our discussion to fill these concepts with mathematical life from the beginning and to illustrate them with a wealth of examples stemming from various mathematical domains, notably from algebra and topology. We opted for this mode of illustration even though it breaks with a principle we have adhered to in other parts of the book, namely, to make everything self-contained and to prove all assertions made in this book. As far as basic category theory itself is concerned to the extent it is presented here, it is completely self-contained. Since categories of modules are one major class of examples, Appendices 1 and 3 complement each other.

The last section on commutative monoidal categories is located between category theory in general and multilinear algebra and therefore requires familiarity with multilinear algebra and some maturity in handling tensor products. With this proviso, complete proofs are given in the section on graded Hopf algebras. The literature on this subject is not abundant. This is the reason why we felt we had to give a self-contained account of this type of Hopf algebras. Finally, the new section on weakly complete symmetric Hopf algebras over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is needed in the third part of Chapter 3 on weakly complete group $\mathbb{K}$-algebras which, among other results leads to a proof of Tannaka duality. The duality between $\mathbb{K}$-vector spaces and weakly complete $\mathbb{K}$ vector spaces that is required in this context is comprehensively presented in Appendix 7.

References for this Appendix—Additional Reading

[3], [21], [54], [78], [94], [92], [112], [122], [129], [146], [151], [153], [170], [196], [200], [205], [212], [230], [244], [245], [246], [247], [259], [294], [305], [313], [338], [357], [358].

Appendix 4

Selected Results on Topology and Topological Groups

In this appendix we gather together material on the arc component topology, weight, and metrizability of topological groups which we need.

The Arc Component Topology

A space X is said to be *locally arcwise connected* if the topology of X has a basis of arcwise connected open sets.

Lemma A4.1. *The set of arc components of open sets of X is a basis for a topology $\mathcal{O}^\alpha(X)$ making the underlying set of X into a topological space X^α such that the following statements hold.*

(i) *The identity map $\varepsilon_X: X^\alpha \to X$ is continuous and has the following universal property: If $f: Z \to X$ is a continuous map from any locally arcwise connected space Z to X, then it factors through ε_X; i.e. the underlying function of f defines a continuous map $f': Z \to X^\alpha$ such that $f = \varepsilon_X \circ f'$.*

(ii) *A function $f: \mathbb{I} \to X$ from the unit interval $\mathbb{I} = [0,1]$ to X is continuous if and only if the function $f': \mathbb{I} \to X^\alpha$, $f'(r) = f(r)$ is continuous.*

(iii) *For each $x \in X$ the arc components X_x and $(X^\alpha)_x$ agree. In particular, X^α is locally arcwise connected.*

(iv) *For a continuous function $f: X \to Y$ the function $f^\alpha = f: X^\alpha \to Y^\alpha$ is also continuous. The assignment $X \mapsto X^\alpha$ is a functor from the category of Hausdorff spaces into the category of locally arcwise connected Hausdorff spaces, right adjoint to the forgetful functor.* (*See Appendix* A3, *Definition* A3.29)

(v) *The relation $X = X^\alpha$ holds if and only if X is locally arcwise connected.*

Proof. (i) For any subset $S \subseteq X$ and an element $s \in S$ we let S_s denote the arc component of s in S. If U and V are open in X and $x \in U \cap V$, then $(U \cap V)_x \subseteq U_x \cap V_x$. Hence the set $\{U_u \mid U \in \mathcal{O}(X),\ u \in U\}$ is a basis of a topology $\mathcal{O}^\alpha(X)$. The topological space $\big(X, \mathcal{O}^\alpha(X)\big)$ will be denoted X^α. The identity map $\varepsilon_X: X^\alpha \to X$ is obviously continuous.

Now let $f: Z \to X$ be a continuous function and assume that Z is locally arcwise connected. Let $z \in Z$ and let V be an open neighborhood of $f(z)$ in X^α. By the definition of $\mathcal{O}^\alpha$ there is an open neighborhood W of $f(z)$ in X containing V such that the arc component $W_{f(z)}$ of $f(z)$ is contained in V. Since f is continuous and Z is locally arcwise connected, there is an arcwise connected neighborhood U

of z such that $f(U) \subseteq W$. Since continuous functions preserve arc connectivity, $f(U) \subseteq W_{f(z)}$. Thus $f(U) \subseteq V$ and hence f is continuous as a function $Z \to X^\alpha$. This is what is asserted in (i).

(ii) Let $f: \mathbb{I} \to X$ be a function. If $f': \mathbb{I} \to X^\alpha$ is continuous, then clearly $f = \varepsilon_X \circ f': \mathbb{I} \to X$ is continuous. Conversely, let $f: \mathbb{I} \to X$ be continuous. Since the unit interval $\mathbb{I}$, its topology having a basis of intervals, is locally arcwise connected, the function $f': X \to X^\alpha$ is continuous.

(iii) Any arc component X_x of X is arcwise connected in X^α by (ii) and thus $X_x \subseteq (X^\alpha)_x$. The relation $(X^\alpha)_x \subseteq X_x$ is trivial. Since the arc components of X form a basis of the topology of X^α and these are arcwise connected in X^α, it follows that X^α is locally arcwise connected.

(iv) Assume that $f: X \to Y$ is a continuous map between Hausdorff spaces. Then $f \circ \varepsilon_X: X^\alpha \to Y$ is a continuous map from a locally arcwise connected space to Y. Hence by (i) it factors through $\varepsilon_Y: Y^\alpha \to Y$; i.e. there is a continuous map $f^\alpha X^\alpha \to Y^\alpha$, $f^\alpha(x) = f(x)$ such that $\varepsilon_Y \circ f^\alpha = f \circ \varepsilon_X$. $f(U_x) \subseteq W$. Thus $f: X^\alpha \to Y^\alpha$ is continuous.

The assignment $X \mapsto X^\alpha$ is clearly a functor, and its universal property formulated in (i) shows that it is right adjoint to the forgetful functor.

(v) is a simple corollary of (iii) and the definition of X^α. □

A few comments are in order. We could have refined the given topology by taking connected components in place of the arc components. The universal property (i) then holds for locally connected spaces Z. However, the argument that the space with the refined topology is locally connected breaks down. This is the reason why the refinement mechanism does not work properly for connectivity.

Exercise EA4.1. Let $\mathbb{T}_2$ denote the dyadic solenoid. @See Chapter 1, Example 1.28(ii).) Set $X = (\mathbb{T}_2)^{\mathbb{N}}$. Discuss $(\mathbb{T}_2)^\alpha$ and X^α and investigate the space obtained on the underlying set of X by considering the connected components of all open sets as a basis for a topology. □

The argument for Assertion (iv) in A4.1 is a concrete version of a category theoretical argument deriving functoriality from the universal property (see Appendix 3, A3.28).

Definition A4.2. The topology $\mathcal{O}^\alpha(X)$ is called the *arc component topology* on X and X^α is called *the space on X with the canonical locally arcwise connected topology.* □

Lemma A4.3. *If X and Y are Hausdorff spaces, then $(X \times Y)^\alpha = X^\alpha \times Y^\alpha$.*

Proof. Exercise E4.2. □

Exercise EA4.2. Prove A4.3. □

Lemma A4.4. *Let G be a topological group. Then G^α is a locally arcwise connected topological group whose filter of identity neighborhoods $\mathcal{U}^\alpha$ has a basis of open identity neighborhoods U_1 where U ranges through the open members of $\mathcal{U}$, the filter of identity neighborhoods of G.*

Proof. In view of A4.3 and A4.1(iv), multiplication $(x, y) \mapsto xy: G^\alpha \times G^\alpha = (G \times G)^\alpha \to G^\alpha$ and inversion $x \mapsto x^{-1}: G^\alpha \to G^\alpha$ are continuous. Thus G^α is a topological group.

The sets U_1, $U \in \mathcal{U}$ are certainly open sets of G^α. If $g \in G$ and $g \in V \in \mathcal{O}(G)$, then V_g is a basic open set of G^α. Then $g^{-1}V \in \mathcal{U}$ and $g^{-1}V_g = (g^{-1}V)_g \in \mathcal{U}^\alpha$. The assertion follows. □

There is an application of these concepts of topological group theory which is relevant in Lie group theory. We refer to Appendix 2, Corollary A2.31 on the Generating of Subgroups of Topological Groups. The gist of that corollary was that a subset K of a topological group G generates a subgroup with very well controlled properties provided K satisfies certain conditions linking group structure and topology on K. Therefore the following proposition sheds new light on this idea.

Proposition A4.5. *Assume the hypotheses of Corollary* A2.31 *and assume in addition that the subspace K of G is arcwise connected and locally arcwise connected. Let $H_* = \langle K \rangle$ denote the subgroup generated in G by K given the induced topology. Then the topological group H of* A2.31 *may be identified with $(H_*)^\alpha$ and f with the inclusion map.*

Proof. By A2.31, the group H has an open identity neighborhood V which is homeomorphic to K, and is therefore arcwise connected and locally arcwise connected. Thus the topological group H is locally arcwise connected, and, being generated by V, it is arcwise connected, too. The continuous bijection $f': H \to H_*$ obtained from corestricting f to its image factors through $\varepsilon_{H_*}: (H_*)^\alpha \to H$ by A4.1(i) with a bijective continuous morphism $F: H \to (H_*)^\alpha$, $f' = F \circ \varepsilon_{H_*}$. Since K is locally arcwise connected, the topology $\mathcal{O}$ of H_* and the arc component topology $\mathcal{O}^\alpha$ on H_* induce on K the same topology (cf. A2.33(i), applied to the inclusion $K \to H_*$). Since $f'|V: V \to K$ is a homeomorphism, $F|V: V \to K \subseteq (H_*)^\alpha$ is a homeomorphism, too. Thus F is a bijective morphism of topological groups which maps an open identity neighborhood of its domain onto an open identity neighborhood of its image. Therefore it is also open and thus ε is an isomorphism of topological groups. □

Proposition A4.6. *Assume that $p: G \to G/N$ is a quotient morphism of a topological group which has arc lifting* (*see* 8.27)*. Then $p^\alpha = p: G^\alpha \to (G/N)^\alpha$ is a quotient morphism.*

Proof. A subset $S \in G$ will be called *saturated* iff $SN = NS = S$. If $S' \subseteq G$, then $S \stackrel{\text{def}}{=} S'N$ is the smallest saturated set containing S'. If S is saturated we write $p(S) = S/N$. The open sets of G/N are exactly the sets U/N with U ranging through all saturated *open* subsets of G. If U is a saturated open identity neighborhood of G, then $U_1 \in \mathcal{U}^\alpha$ and $p^{-1}p(U_1) = U_1N = \bigcup_{n\in N} U_1 n$ is open in G^α. The quotient group G^α/N is therefore a locally arcwise connected topological group and thus the identity map $G^\alpha/N \to (G/N)^\alpha$ is continuous by A8.1(i) We have to show that it is open. Let U be a saturated open identity neighborhood of G. The spaces U and U/N are pointed at 1 and N. Then $p^{-1}\big((U/N)_1\big) = \{g \in U \mid (\exists\gamma \in C_0(\mathbb{I}, U/N))\, \gamma(1) = gN\}$ and $U_1N = \{g \in G \mid (\exists\gamma \in C_0(\mathbb{I}, U))\, \gamma(1)N = gN\}$ Clearly the second of these two sets is contained in the first. For a proof of the reverse containment consider a $g \in p^{-1}\big((U/N)_1\big)$ and let $\gamma\colon \mathbb{I} \to U/N \subseteq G/N$ be a pointed arc such that $\gamma(1) = gN$. Since $p\colon G \to G/N$ has arc lifting, there is an arc $\widetilde{\gamma}\colon \mathbb{I} \to G$ such that $\widetilde{\gamma}(r)N = \gamma(r)$ for all $r \in \mathbb{I}$. Thus $\widetilde{\gamma}(\mathbb{I})p^{-1}(U/N) = U$ and $\widetilde{\gamma}(1) \in gN$. Hence

$$p^{-1}\big((U/N)_1\big) = U_1N.$$

Thus the identity map $G^\alpha/N \to (G/N)^\alpha$ is also open and therefore a homeomorphism. It follows that $g \mapsto gN\colon G^\alpha \to G^\alpha/N$ is a quotient map which has arc lifting. □

The Weight of a Topological Space

Definition A4.7. Let X be a topological space and $\mathcal{O}$ its topology. The set $\{\operatorname{card}\mathcal{B} : \mathcal{B} \text{ is a basis for } \mathcal{O}\}$ is well ordered and thus has a minimal element. This cardinal is called the *weight* of X and is written $w(X)$. □

Clearly if $X \subseteq Y$, or if $f\colon Y \to X$ is surjective, continuous, and open, then $w(X) \le w(Y)$. For a discrete space X one has $w(X) = \operatorname{card} X$. The spaces with $w(X) \le \aleph_0$ are said *to satisfy the Second Axiom of Countability.* All separable metric spaces have this property. We shall encounter separable compact spaces of weight $2^{\aleph_0}$ shortly. Every Cantor space $\mathbf{2}^J$, $\mathbf{2}$ denoting the discrete two element space $\{0, 1\}$, has weight $w(\mathbf{2}^J) = \operatorname{card} J$; in particular, every cardinal occurs as a weight of a compact Hausdorff space. We shall prove this and indeed a more general statement in the next exercise below.

Lemma A4.8. *Assume X is a space with an infinite topology $\mathcal{O}$. Let $\mathcal{S}$ be a subbasis for $\mathcal{O}$ (i.e. $\mathcal{S} \subseteq \mathcal{O}$ such that the collection of finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{O}$). Then $w(X) \le \operatorname{card}\mathcal{S}$.*

Proof. This is a consequence of the fact that the set of finite subsets of an infinite set has the same cardinality as this set. □

Exercise EA4.3. Prove the following assertion.

If X_j, $j \in J$ is a family of topological spaces of weights $w_j = w(X_j)$, such that the topology of $X \stackrel{\text{def}}{=} \prod_{j\in J} X_j$ is infinite. Then

$$w(X) = \sum_{j\in J} w_j = \sup(\{\operatorname{card} J\} \cup \{w_j \mid j \in J\}).$$

[Hint. (a) The equality

$$\sum_{j\in J} w_j = \sup(\{\operatorname{card} J\} \cup \{w_j \mid j \in J\})$$

for infinite left hand side is a simple exercise in cardinal arithmetic. Indeed, the right hand side is clearly dominated by the left hand side, and if $\aleph$ is the right hand side, then the left hand is dominated by $(\operatorname{card} J)(\sup_{j\in J} w_j) = \aleph^2 = \aleph$.

(b) Let $\mathcal{B}_j$ be a basis for the topology of X_j of cardinality w_j. For $W \in \mathcal{B}_k$, $k \in J$ write $U_k(W) = \{(x_j)_{j\in J} \in X \mid x_k \in W\}$. Then $\{U_j(W) \mid W \in \mathcal{B}_j,\, j \in J\}$ is a subbasis of the product topology on X and its cardinality is $\sum_{j\in J} w_j$. Thus $w(X) \le \sum_{j\in J} w_j$ by Lemma A4.8.

(c) Next we claim that $\sup(\{\operatorname{card} J\} \cup \{w_j \mid j \in J\}) \le w(X)$. In view of (a) this will complete the proof. Let $\mathcal{B}$ be a basis of the topology on X of cardinality $w(X)$. The projections $\mathrm{pr}_j\colon X \to X_j$ are continuous and open. Hence $\mathrm{pr}_j(\mathcal{B})$ is a basis of the topology of X_j, whence $w_j \le w(X)$. Hence $\sup\{w_j,\, j \in J\} \le w(X)$. If J is finite, we are done. Assume that J is infinite. Let $\mathrm{Fin}(J)$ denote the set of finite subsets of J. If $C \subseteq \mathrm{Fin}(J)$ is cofinal, i.e. for $F \in \mathrm{Fin}(J)$ there is an $F' \in C$ with $F \subseteq F'$, we define retractive function $\gamma\colon \mathrm{Fin}(J) \to C$ by picking for each $F \in \mathrm{Fin}(J)$ a minimal $\gamma(F) \in C$ containing F. In particular, γ is surjective; and since $\gamma^{-1}(F')$ is finite for every $F' \in C$ we conclude $\operatorname{card} C = \operatorname{card} \mathrm{Fin}(J) = \operatorname{card} J$. If $U \in \mathcal{B}$, then there is a minimal $\omega(U) \in \mathrm{Fin}(J)$ such that for some $(u_j)_{j\in J} \in U$ the set $\{(x_j)_{j\in J} \mid j \in F \Rightarrow x_j = u_j\}$. Since the image of the function $\omega\colon \mathcal{B} \to \mathrm{Fin}(J)$ is cofinal and thus has cardinality $\operatorname{card} J$ by the preceding observations, we deduce $w(X) = \operatorname{card} \mathcal{B} \ge \operatorname{card} J$. Thus $w(X) \ge \sup(\{\operatorname{card} J\} \cup \{w_j, \mid j \in J\})$ as asserted.]□

Recall (cf. Definition 12.15(ii)) that if X is a topological space, the *density of* X is defined to be

$$d(X) = \min\{\aleph \mid \text{there is a dense subset } Y \text{ of } X \text{ with } \operatorname{card} Y = \aleph\}.$$

Exercise EA4.4 ([67]). Prove the following assertions.

(i) Let B be a nonseparable normed vector space. Consider the subsets S of the open unit sphere $S(1)$ with the property that $\|x-y\| \ge \frac{1}{2}$, for $x, y \in S$, $x \ne y$. Partially order these subsets by set-theoretic inclusion and, by using Zorn's Lemma pick a maximal subset A among them. Then $\operatorname{card} A = d(B)$.

(ii) (Kruse–Schmidt–Stone Theorem) ([67], [68], [233], [323], [345]) *If B is a Banach space then*

$$\operatorname{card} B = (\operatorname{card} B)^{\aleph_0} = (d(B))^{\aleph_0}.$$

[Hint. (i) Let D be a dense subset of B of cardinality $d(B)$. Each of the card A disjoint open balls of radius $\frac{1}{4}$ with midpoints $a \in A$ has to contain an element of D. Thus the Axiom of Choice yields an injective function $A \to D$. It follows that $\operatorname{card} A \leq d(B)$. The reverse inequality is proved by finding a dense subset D of B of cardinality card A, as follows.

Put $D = \{\sum_{i=1}^{k} q_i a_i \mid k \in \mathbb{N},\ q_i \in \mathbb{Q},\ a_i \in A\}$, and observe that $\operatorname{card} D = \operatorname{card} A$, since A is infinite. To show D is dense, prove that for each $x \in B$ there is a sequence of points s_n, $n \in \mathbb{N}$, in D converging to x. Let $M \in \mathbb{N}$ be such that $\|x\| < M$. Then $\frac{1}{M}x \in S(1)$ and since A is maximal, there exists an element $a_1 \in A$ such that $\|\frac{1}{M}x - a_1\| < \frac{1}{2}$. Putting $s_1 = Ma_1$ we see that $s_1 \in C$ and $\|x - s_1\| < \frac{M}{2}$. Complete the proof by inductively defining the other terms of the sequence; indeed, assume $s_n \in C$ is such that $\|x - s_n\| < \frac{M}{2^n}$. Then $\frac{2^n}{M}x - \frac{2^n}{M}s_n \in S(1)$. So there exists $a_{n+1} \in A$ such that $\|(\frac{2^n}{M}x - \frac{2^n}{M}s_n) - a_{n+1}\| < \frac{1}{2}$. Putting $s_{n+1} = s_n + \frac{M}{2^n}a_{n+1}$ it follows that $\|x - s_{n+1}\| < \frac{M}{2^{n+1}}$.]

(ii) If B is separable, then $\operatorname{card} B = 2^{\aleph_0}$ and the result is clear. So assume B is nonseparable. Let A be as in (i), so that $\operatorname{card} A = d(B)$. Define a map $f\colon A^{\mathbb{N}} \to B$ by $f(a) = \sum_{j=1}^{\infty} \frac{1}{6^j} a_j$ for $a = (a_j)_{j \in \mathbb{N}}$. As B is a complete metric space and $\|a_j\| < 1$, this series converges absolutely and $\|f(a)\| < \sum_{j=1}^{\infty} \frac{1}{6^j} = \frac{1}{5} < 1$. Thus $f(A^{\mathbb{N}}) \subseteq S(1)$. Verifying that f is one-to-one will show that $\operatorname{card} B \geq (\operatorname{card} A)^{\aleph_0} = (d(B))^{\aleph_0}$. The reverse inequality, $\operatorname{card} B \leq (d(B))^{\aleph_0}$, follows from the fact that every point in B is the limit of a sequence in a dense subset. So $\operatorname{card} B = (d(B))^{\aleph_0}$ and hence $\operatorname{card} B = (\operatorname{card} B)^{\aleph_0}$.

Finally, prove that f is one-to-one by considering distinct elements $a, b \in A^{\mathbb{N}}$, namely $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$, and showing that their images under f are distinct. Now $a_m \neq b_m$, for some $m \in \mathbb{N}$, and $a_j = b_j$, for $j \leq m$. So

$$\begin{aligned} \|f(a) - f(b)\| &\geq \left\|\frac{a_m - b_m}{6^m}\right\| - \sum_{j=m+1}^{\infty} \left\|\frac{a_j - b_j}{6^j}\right\| \\ &\geq \frac{1}{2}\cdot\frac{1}{6^m} - 2\sum_{j=m+1}^{\infty} \frac{1}{6^j} = \frac{1}{2\cdot 6^m}\Big(1 - \frac{4}{5}\Big) > 0 \end{aligned}$$

and so the images are distinct, as required.] □

The Kruse–Schmidt–Stone Theorem is a remarkable result. Among other things it tells us that the cardinality $\operatorname{card} B = \aleph$ of any Banach space B has the property that $\aleph = \aleph^{\aleph_0}$.

Let X be a pointed compact Hausdorff space, G an infinite abelian topological group, $C(X, G)$ the group of all continuous functions from X into G, and $C_0(X, G)$ the group of all continuous basepoint preserving functions from X into G. If G is $\mathbb{R}$ or $\mathbb{T}$ we consider the topology of uniform convergence on $C_0(X, G)$ given by the complete metric $\|f - g\| = \sup\{|f(x) - g(x)| : x \in X\}$, for $f, g \in C_0(X, G)$. Further, $C_0(X, \mathbb{R})$ is a Banach space when $\|f\|$ is defined to be $\sup\{|f(x)| : x \in X\}$, for each $f \in C_0(X, \mathbb{R})$.

Theorem A4.9 (The Weight of Function Spaces). (i) *Let X be a compact Hausdorff space. Then*

$$\operatorname{card} C_0(X,\mathbb{R}) = \operatorname{card} C_0(X,\mathbb{T}) = \operatorname{card} C_0(X,U(n)) = w(X)^{\aleph_0}.$$

(ii) *If A is a discrete abelian group, then*

$$\operatorname{card} C_0(X,A) = \begin{cases} (\operatorname{card} A)^n & \text{if } \#X = n+1, \\ \max\{w(X/\mathrm{conn}), \operatorname{card} A\} & \text{if } \#X \text{ is infinite.} \end{cases}$$

Proof. Exercises EA4.5 and EA4.6. □

Exercise EA4.5. Prove Theorem A4.9(i) by proceeding through the following steps.

(i) Let X_1, X_2 and X_3 be compact Hausdorff pointed spaces.

If $f\colon X_1 \to X_2$ is a continuous injection and $g\colon X_2 \to X_3$ is a continuous surjection, then

$$\operatorname{card} C_0(X_1,\mathbb{R}) \le \operatorname{card} C_0(X_2,\mathbb{R}) \quad \text{and} \quad \operatorname{card} C_0(X_3,\mathbb{R}) \ge \operatorname{card} C_0(X_2,\mathbb{R}).$$

(ii) $\operatorname{card} C_0(X,\mathbb{R}) = \operatorname{card} C_0(X,\mathbb{T})$.

[Hint. Note that $\mathbb{T}$ is a quotient group of $\mathbb{R}$ and so $\operatorname{card} C_0(X,\mathbb{T}) \ge \operatorname{card} C_0(X,\mathbb{R})$. The reverse inequality follows by observing that $\mathbb{T}$ is homeomorphic to a subspace of $\mathbb{R}^2$ and $\operatorname{card} C_0(X,\mathbb{R}^2) = (\operatorname{card} C_0(X,\mathbb{R}))^2$.]

(iii) Let $\mathcal{F}$ be a dense subset of $C_0(X,\mathbb{R})$ such that $\operatorname{card} \mathcal{F} = d(C_0(X,\mathbb{R}))$. Then the family $\mathcal{F}$ separates points (that is for $x \ne y$ in X there is an $f \in \mathcal{F}$ such that $f(x) \ne f(x)$). The map $x \mapsto \big(f(x)\big)_{f\in\mathcal{F}} : X \to \mathbb{R}^{\mathcal{F}}$ is continuous and is injective since $\mathcal{F}$ separates points. As X is compact, it is an embedding of X into $\mathbb{R}^{d(C_0(X,\mathbb{R}))}$. So

$$w(X) \le d(C_0(X,\mathbb{R})).$$

[Hint. By Exercise EA4.3, $w(\mathbb{R}^{d(C_0(X,\mathbb{R}))}) = d(C_0(X,\mathbb{R}))$.]

(iv) $(w(X))^{\aleph_0} \le \operatorname{card} C_0(X,\mathbb{R})$.

[Hint. Use (iii) and the Kruse–Schmidt–Stone Theorem of EA4.4(ii).]

(v) If X is infinite and 0-dimensional, then $\operatorname{card} C_0(X,\mathbb{Z}(2)) = w(X)$, where $\mathbb{Z}(2)$ is the discrete cyclic group of order 2.

[Hint. The cardinality of $C_0(X,\mathbb{Z}(2))$ equals that of $C(X,\mathbb{Z}(2))$ and this is equal to that of the set L of all compact open subsets of X. Let $\mathcal{B}$ be a basis for the open sets with $\operatorname{card}\mathcal{B} = w(X)$. Now form the set $\mathcal{B}'$ of finite unions of sets in $\mathcal{B}$. Then $\operatorname{card}\mathcal{B}' = \operatorname{card}\mathcal{B} = w(X)$. But then $L \subseteq \mathcal{B}'$. Thus $\operatorname{card} L \le w(X)$. But as L is a basis for the topology of the compact 0-dimensional space X, $w(X) \le \operatorname{card} L$. So $\operatorname{card} C_0(X,\mathbb{Z}(2)) = w(X)$.]

(vi) If X is infinite and 0-dimensional, then $\operatorname{card} C_0(X,\mathbb{R}) = w(X)^{\aleph_0}$.

[Hint. Let $\mathbb{K}$ be the compact abelian topological group $(\mathbb{Z}(2))^{\aleph_0}$, which is homeomorphic to the Cantor space (the subspace $\{2\sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid (a_1,a_2,\dots) \in \{0,1\}^{\mathbb{N}}\}$ of the unit interval $\mathbb{I} = [0, 1]$). Noting that $C_0(X, \mathbb{K}) \cong C_0(X, \mathbb{Z}(2))^{\mathbb{N}}$ it follows from (iv) that $\operatorname{card} C_0(X, \mathbb{K}) = w(X)^{\aleph_0}$.

Let $k\colon \mathbb{K} \to \mathbb{I}$ be the continuous surjective Cantor–Carathéodory function defined by

$$k\Big(2\sum_{n=1}^{\infty}\frac{a_n}{3^n}\Big) = \sum_{n=1}^{\infty}\frac{a_n}{2^n}.$$

As $\mathbb{T}$ is a quotient space of $\mathbb{I}$, there is a continuous surjective map m of $\mathbb{K}$ onto $\mathbb{T}$. Then $C_0(X, m)\colon C_0(X, \mathbb{K}) \to C_0(X, \mathbb{T})$ is surjective. To see this, note that since X is totally disconnected and thus has a basis of compact open subsets, the set of locally constant and finitely valued functions in $C_0(X, \mathbb{T})$ is uniformly dense. Each such function lifts to a function in $C_0(X, \mathbb{K})$ as m is surjective. Thus the image of $C(X, m)$ is uniformly dense. Since it is also closed in the topology of uniform convergence, the claim follows. Hence $\operatorname{card} C_0(X, \mathbb{K}) \leq \operatorname{card} C_0(X, \mathbb{T})$.

Let j be an embedding of $\mathbb{K}$ in $\mathbb{T}$. Then $C_0(X, j)\colon C_0(X, \mathbb{K}) \to C_0(X, \mathbb{T})$ is injective, and thus $\operatorname{card} C_0(X, \mathbb{T}) \leq \operatorname{card} C_0(X, \mathbb{K})$.

So $\operatorname{card} C_0(X, \mathbb{T}) = \operatorname{card} C_0(X, \mathbb{K})$. Thus $\operatorname{card} C_0(X, \mathbb{T}) = w(X)^{\aleph_0}$. Finally apply (ii).]

(vii) $\operatorname{card} C_0(\mathbb{I}^A, \mathbb{R}) = (\operatorname{card} A)^{\aleph_0}$ for any infinite set A.

[Hint. Noting that $\mathbb{K}$ is a subspace of $\mathbb{I}$ and $\mathbb{I}$ is a quotient space of $\mathbb{T}$, apply (i) to show $\operatorname{card} C_0(\mathbb{I}^A, \mathbb{R}) = C_0(\mathbb{K}^A, \mathbb{R})$. Then apply (vi) and EA4.3.]

(viii) The space X can be embedded into $\mathbb{I}^{w(X)}$.

[Hint. For finite X this is trivial so assume that X is infinite. Let B be a basis for the open sets of X with $\operatorname{card} B = w(X)$. Set $J \stackrel{\text{def}}{=} \{(U, V) \in B \times B \mid \overline{U} \cap \overline{V} = \emptyset\}$. Then $w(X) \leq \operatorname{card} J \leq w(X^2) = w(X)$ since $w(X)$ is infinite. For each $j = (U, V) \in J$ select a continuous function $f_j\colon X \to \mathbb{I}$ with $U \subseteq f^{-1}(0)$ and $V \subseteq f^{-1}(1)$. The function

$$x \mapsto \big(f_j(x)\big)\colon X \to \mathbb{I}^J$$

is an embedding. Finally, $\operatorname{card} J = \operatorname{card} w(X)$.]

(ix) $\operatorname{card} C_0(X, \mathbb{R}) \leq (w(X))^{\aleph_0}$ for any infinite compact space.

[Hint. Apply (i), (vii) and (viii).]

(x) For any compact Hausdorff pointed space,

$$\operatorname{card} C_0(X, \mathbb{R}) = w(X)^{\aleph_0}.$$

[Hint. If X is infinite, this follows from (iv) and (ix). If X contains $n > 1$ points, then $C_0(X, \mathbb{R}) = \mathbb{R}^{n-1}$, whence $\operatorname{card} C_0(X, \mathbb{R}) = 2^{\aleph_0}$. On the other hand, $w(X) = n > 1$ and thus $w(X)^{\aleph_0} = n^{\aleph_0} = 2^{\aleph_0}$. If X is singleton, then $C_0(X, \mathbb{R})$ is singleton and $w(X) = 1$, whence $w(X)^{\aleph_0} = 1$.]

(xi) For any compact Hausdorff pointed space,

$$\operatorname{card} C_0(X, U(n)) = \operatorname{card} C_0(X, \mathbb{R}).$$

[Hint. Use (i), that $\mathbb{R}$ can be embedded as a subspace of $U(n)$, that $U(n)$ can be embedded as a subspace of $\mathbb{R}^{n^2}$, and that

$$\operatorname{card} C_0(X, \mathbb{R}^{n^2}) = (\operatorname{card} C_0(X, \mathbb{R}))^{n^2} = (w(X)^{\aleph_0})^{n^2} = w(X)^{\aleph_0}.]$$

An alternative to going through these steps in order to prove Part (i) of Theorem 4.9 is to use Smirnov's Theorem ([67], [335]) which, with minor modification, tells us that for any infinite compact Hausdorff space X we have $d(C_0(X, \mathbb{R})) = w(X)$.] □

Exercise EA4.6. Prove Theorem A4.9(ii). □

Metrizability of Topological Groups

A metric d on a group G is called *left invariant* if $d(gx, gy) = d(x, y)$ for all g, x, $y \in G$.

We may cast the presence of a left invariant metric into different guises involving functions.

Lemma A4.10. *For any Hausdorff topological group G, the following statements are equivalent.*

(i) *There exists a left invariant metric d on G defining the topology of G.*

(ii) *There exists a continuous function $\|\cdot\| : G \to \mathbb{R}^+ = [0, \infty[$ such that*

  (1) *$\|x\| = 0$ if and only if $x = 1$.*

  (2) *$\|x^{-1}\| = \|x\|$ for all $x \in G$.*

  (3) *$\|xy\| \leq \|x\| + \|y\|$ for all x, $y \in G$.*

  (4) *For each identity neighborhood U there is an $n \in \mathbb{N}$ such that $\|g\| < \frac{1}{n}$ implies $g \in U$.*

(iii) *There exists a function $p: G \to [0, 1]$ such that*

  (1) *$p(1) = 0$ and for each identity neighborhood U there is an $n \in \mathbb{N}$ such that $p(g) < \frac{1}{n}$ implies $g \in U$.*

  (2) *For all $n \in \mathbb{N}$ there is an identity neighborhood U such that for all $g \in G$ and $u \in U$ the relation $p(gu) \leq p(g) + \frac{1}{n}$ holds.*

If these conditions are satisfied, then $\|\cdot\|$ may be chosen to arise from d, p from $\|\cdot\|$, and d from p, as follows.

$$\begin{aligned} \|x\| &= d(x, 1), \\ p(x) &= \min\{\|x\|, 1\}, \\ d(x, y) &= \sup\{|p(gy) - p(gx)| : g \in G\}. \end{aligned}$$

Proof. (i)⇒(ii). Set $\|x\| \stackrel{\text{def}}{=} d(x, 1)$. Then (ii)(1) follows from the positive definiteness of the metric. Further $|x| = d(x^{-1}, 1) = d(xx^{-1}, x) = d(1, x) = d(x, 1) = |x|$ by left invariance and symmetry. Thus (ii)(2) holds. Finally, $|xy| = d(xy, 1) = d(y, x^{-1}) \leq d(y, 1) + d(1, x^{-1}) = d(y, 1) + d(x^{-1}, 1) = \|y\| + \|x^{-1}\| = \|x\| + \|y\|$ by left invariance, triangle inequality, and (ii)(2). This shows (ii)(3) holds.

(ii)⇒(iii). Set $p(x) = \min\{\|x\|, 1\}$ for all $x \in G$. Then $p(1) = 0$ is clear. By (ii)(4), for every identity neighborhood U there is an $n \in \mathbb{N}$ such that $\|g\| < \frac{1}{n}$ implies $g \in U$. This is (iii)(1). Next (ii)(3) and the continuity of $\|\cdot\|$ give (iii)(2).

(iii)⇒(i). Set $d(x,y) \stackrel{\text{def}}{=} \sup\{|p(gy) - p(gx)| : g \in G\}$. Since p is bounded, there is no problem with the existence of the least upper bound. Then $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $(\forall g \in G)\ p(gy) = p(gx)$, and this holds only if $0 = p(1) = p(y^{-1}y) = p(y^{-1}x)$ for all $x, y \in G$. By (iii)(1), and since G is Hausdorff, this implies $y^{-1}x = 1$, since 1 is the only element contained in each identity neighborhood. So $x = y$. Conversely, if $x = y$ then trivially $d(x,y) = 0$. Hence d is definite. The symmetry of d is immediate from the definition.

Also $d(gx, gy) = \sup\{|p(hgx) - p(hgy)| : h \in G\} = d(x,y)$. Thus d is left invariant.

Finally $|p(gx) - p(gz)| \leq |p(gx) - p(gy)| + |p(gy) - p(gz)| \leq d(x,y) + d(y,z)$ for all $g \in G$ whence $d(x,z) \leq d(x,y) + d(y,z)$. Thus d is indeed a left invariant metric. It remains to show that d defines the topology. Because of left invariance, it suffices to show that the sequence of sets $U_n \stackrel{\text{def}}{=} \{x \in G \mid d(x,1) < \frac{1}{n}\}$ for $n = 1, 2, \ldots$ forms a basis for the filter of identity neighborhoods. First we show that all U_n are identity neighborhoods. Consider an $n \in \mathbb{N}$. By(iii)(2), there is an identity neighborhood $U = U^{-1}$ such that for all $g \in G$ we have $p(gu) \leq p(g) + \frac{1}{2n}$ and $p(g) = p(guu^{-1}) \leq p(gu) + \frac{1}{2n}$ for all g whence $|p(gu) - p(g)| \leq \frac{1}{2n}$ for all $g \in G$ and thus $d(u,1) \leq \frac{1}{2n} < \frac{1}{n}$. Thus $U \subseteq U_n$. Now let an open identity neighborhood U be given. Then by (iii)(1) we find an $n \in \mathbb{N}$ such that $x \notin U$ implies $\frac{1}{n} \leq p(x) = |p(1x) - p(1)| \leq \sup\{|p(gx) - p(g)| : g \in G\} = d(x,1)$. □

Lemma A4.11. *Assume that d, $\|\cdot\|$, and p are linked as in Lemma* A4.10 *and that Γ is a set of automorphisms of the topological group G. Then the following conditions are equivalent:*

(4) $\|\gamma(x)\| = \|x\|$ *for all* $x \in G$, $\gamma \in \Gamma$,

(4′) $p\big(\gamma(x)\big) = p(x)$ *for all* $x \in G$, $\gamma \in \Gamma$,

(4″) $d\big(\gamma(x), \gamma(y)\big) = d(x,y)$ *for all* $x \in G$, $\gamma \in \Gamma$.

If Γ is the group of inner automorphisms, then these conditions are also equivalent to

(4‴) $d(xg, yg) = d(x,y)$ *for all* g, x, $y \in G$.

Proof. The proofs of (4)⇒(4′)⇒(4″)⇒(4) are straightforward from the definitions.

Assume now that Γ is the group of inner automorphisms. We note that $d(xg, yg) = d(g^{-1}xg, g^{-1}yg)$ by left invariance. Thus invariance of the metric under right translations and invariance under inner automorphisms are equivalent for any left invariant metric. □

Condition (4‴) is equivalent to the additional *right invariance* of d. A metric which is both left and right invariant is called *biinvariant*.

In conjunction with Lemma A4.10, a left invariant metric defining a topology can also be translated into terms of certain families of identity neighborhoods.

Lemma A4.12. *For a topological group G the following two conditions are equivalent.*

(iii) *There exists a function $p: G \to [0,1]$ such that conditions* (1) *and* (2) *of* A4.10(iii) *are satisfied.*

(iv) *There is a function $r \mapsto U(r):]0,\infty[\to \mathcal{P}(G)$ into the set of subsets of G containing* 1 *such that the following conditions are satisfied:*

(A) $(\forall r > 1)\, U(r) = G$.

(B) $(\forall 0 < s)\, \bigcup_{r<s} U(r) = U(s)$.

(C) *For each identity neighborhood U there is an $n \in \mathbb{N}$ such that $U(\frac{1}{n}) \subseteq U$.*

(D) *For each $n \in \mathbb{N}$ there is an identity neighborhood U such that $U(r)U \subseteq U(r+\frac{1}{n})$ holds.*

Moreover, the two concepts are related by

$$p(g) = \inf\{r \in]0,1] \mid g \in U(r)\} \quad \textit{and} \quad U(r) = \{g \in G \mid p(g) < r\}.$$

Proof. (iii)⇒(iv) For $0 < r$ define $U(r) \overset{\text{def}}{=} \{g \in G \mid p(g) < r\}$. If $1 < r$, then $p(g) \le 1 < r$ for all $g \in G$, and so $g \in U(r)$. Now (A) follows from the fact that $p(g) \le 1$ for all $g \in G$.

Proof of (B). Let $g \in \bigcup_{r<s} U(r)$. Then there is an $r < s$ such that $g \in U(r)$ and then by definition $p(g) < r$. Then $p(g) < s$, i.e. $g \in U(s)$. Now let, conversely, $g \in U(s)$. Then $p(g) < s$ by definition. Set $t = \frac{p(g)+s}{2}$. Then $p(g) < t < s$, and thus $g \in U(t) \subseteq \bigcup_{r<s} U(r)$.

Proof of (C). For a given U choose n as in A4.10(iii)(1). Then $g \in U(\frac{1}{n})$ implies $p(g) < \frac{1}{n}$ and thus $g \in U$.

Proof of (D). By (iii)(2) for a given $n \in \mathbb{N}$ we find an identity neighborhood such that $p(gu) < p(g) + \frac{1}{n}$ for all $g \in G$ and $u \in U$. So for a $g \in U(r)$ and $u \in U$ we have $p(gu) < t_{g,u} < p(g) + \frac{1}{n} < r + \frac{1}{n}$ so that $gu \in \bigcup_{s<r+\frac{1}{n}} U(s) = U(r+\frac{1}{n})$.

Finally, p is retrieved from $U(\cdot)$ via $p(g) = \inf\{r \in [0,1] \mid g \in U(r)\}$; indeed let the right side be denoted by m. If $g \in U(r)$, then by definition $p(g) < r$, and so $p(g)$ is a lower bound for the set $\{r \mid g \in U(r)\}$. Hence $p(g) \le m$. Now let $p(g) < r$. Then $g \in U(r)$ and thus $m \le r$. It follows that $m \le p(g)$ and $p(g) = m$ is proved.

(iv)⇒(iii) For $g \in G$ define $p(g) = \inf\{r \in]0,1] \mid g \in U(r)\}$. This definition is possible by (A). Clearly, $0 \le p(g) \le 1$. Since $1 \in U(r)$ for all $r > 0$ by hypothesis on $r \mapsto U(r)$, we have $p(1) = 0$.

Proof of (C)⇒(iii)(1). Let U be given. Find n so that $U(\frac{1}{n}) \subseteq U$. If $p(g) < \frac{1}{n}$, then $g \in U(\frac{1}{n}) \subseteq U$.

Proof of (D)⇒(iii)(2). Let $n \in \mathbb{N}$. Then by (D) there is an identity neighborhood U such that $U(r)U \subseteq U(r+\frac{1}{n})$. Now let $g \in G$ and $u \in U$. Take any r with $g \in U(r)$. Then $gu \in U(r)U \subseteq U(r+\frac{1}{n})$ and thus $p(gu) < r + \frac{1}{n}$. We conclude $p(gu) \le p(g) + \frac{1}{n}$.

Finally, $U(\cdot)$ is retrieved from p via $U(r) = \{g \in G \mid p(g) < r\}$. Indeed, let $g \in U(r)$, then by (B) there is an $s < r$ with $g \in U(s)$. Then $p(g) \le s < r$; thus the left hand side is contained in the right hand side. Conversely, assume that

$p(g) < r$. Since $p(g) = \inf\{s \mid g \in U(s)\}$, there is an s with $p(g) \leq s < r$ such that $g \in U(s)$. Then, a fortiori, $g \in U(r)$. So both sides are equal. □

Lemma A4.13. *The metric d corresponding to the p in Lemma* A4.12 *is biinvariant if and only if $gU(r)g^{-1} = U(r)$ for all $g \in G$ and all $r \in [0,1]$. More generally, d is invariant under the members of a set Γ of automorphisms of G if and only if all sets $U(r)$ are invariant under the automorphisms from Γ.*

Proof. This is immediate from A4.11 and the connection between $r \mapsto U(r)$ and p in A4.12. □

The function $U(\cdot)$ now permits an access to metrizability theorems on a purely algebraic level. A subset D of a set X endowed with a partial order $\leq$ is called *a directed set* if it is not empty and each nonempty finite subset of D has an upper bound in D.

Definition A4.14. A *semigroup with a conditionally complete order* is a semigroup S together with a partial order $\leq$ such that the following conditions are satisfied:

(i) $(\forall s,\, t,\, x)\ s \leq t \Rightarrow sx \leq tx$ and $xs \leq xt$.

(ii) $(\forall s,\, t)\ s \leq st$.

(iii) Every directed subset of S has a least upper bound. Further S has a (semigroup) zero which is the largest element of S. □

The set of identity neighborhoods of a topological group G is a semigroup with the conditionally complete order $\subseteq$.

Lemma A4.15. *Let S be a semigroup with a conditionally complete order. Assume that there is a sequence of elements u_n, $n = 1, 2\ldots$ in S satisfying the following condition: $(\surd)$ $u_{n+1}^2 \leq u_n$.*
Then there is a function $F\colon]0,\infty] \to S$ such that
(I) $(\forall r > 1)\ F(r) = \max S$.
(II) $(\forall 0 < s)\ \sup_{r<s} F(r) = F(s)$.
(III) $(\forall n \in \mathbb{N})\ F(\frac{1}{2^n}) \leq u_n$, *and*
(IV) $(\forall r > 0,\ n \in \mathbb{N})\ F(r)u_{n+1} \leq F(r + \frac{1}{2^n})$.
Moreover, F takes its values in the smallest subsemigroup containing

$$\{u_1, u_2, \ldots; \max S\}$$

which is closed under the formation of directed suprema.

Proof. (a) Note that $u_{n+1} \leq u_{n+1}^2 \leq u_n$ by A4.13(ii) and $(\surd)$. Thus

$$(\#) \qquad (\forall m,\, n \in \mathbb{N},\, m \leq n) \quad u_n \leq u_m.$$

We shall first define a function $f\colon J \to S$ on the set J of dyadic rationals $r = m/2^n$, $m,\, n \in \mathbb{N}$ with values in the subsemigroup $T \stackrel{\text{def}}{=} \langle u_1, u_2, \ldots; \max S\rangle$. Once and for all we set $f(r) = \max S = 0 \in T$ for all $1 \le r \in J$. To get started in earnest, we set $f(1/2) = u_1 \in T$. The next step is to define $f(r)$ for $r \in \{\frac{1}{4}, \frac{3}{4}\}$; note that $f(\frac{2}{4}) = u_1$ is already defined. We set $f(\frac{1}{4}) = u_2 \in T$ and $f(\frac{3}{4}) = f(1/2)u_2 = u_1 u_2 \in T$. This indicates our strategy of producing a recursive definition. We set

$$J_n = \left\{ \frac{m}{2^n} \mid m = 1, \dots, 2^n \right\}, \quad n = 0, 1, 2, \dots$$

and note $J_0 = \{1\} \subseteq J_1 = \{\frac{1}{2}, 1\} \subseteq J_2 = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\} \subseteq J_3 \subseteq \cdots$ and $J = (J \cap [1, \infty[) \cup \bigcup_{n \in \mathbb{N}} J_n$. Assume that f is defined on J_n with $f(J_n) \subseteq T$ in such a way that $f(\frac{1}{2^m}) = u_m$ for $m = 1, \dots, n$ and that

$$(\dagger_n) \qquad (\forall r \in J_n)\ f(r)u_n \le f(r + \frac{1}{2^n})$$

holds. We note that $f(r) \le f(r)u_m$ by A4.14(ii) and that therefore $(\dagger_n)$ implies that f *is monotone on* J_n, that is

$$(\#\#) \qquad (\forall r, s \in J, r \le s)\, f(r) \le f(s).$$

We must define $f(r)$, $r = m/2^{n+1}$. If m is even, then $r \in J_n$ and $f(r) \in T$ is defined. If $r = \frac{1}{2^{n+1}}$, we set $f(r) = u_{n+1} \in T$; if $r \in J_n$ we set $f(r + \frac{1}{2^{n+1}}) = f(r)u_{n+1} \in TT \subseteq T$. We must show that $(\dagger_{n+1})$ holds.
Case 1. $r \in J_n$, $m = n + 1$. Then $f(r)u_{n+1} = f(r + \frac{1}{2^{n+1}})$ by definition.
Case 2. $r \in J_{n+1} \setminus J_n$. Then $r = r_0 + \frac{1}{2^{n+1}}$ with $r_0 \in \{0\} \cup J_n$. Now

$$\begin{aligned} f(r)u_{n+1} = &\left\{ \begin{matrix} u_{n+1}u_{n+1} \le u_n & \text{if } r_0 = 0 \\ f(r_0)u_{n+1}u_{n+1} \le f(r_0)u_n & \text{if } r_0 > 0 \end{matrix} \right\} \le f(r_0 + 1/2^n) \\ &= f(r_0 + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}) = f\big(r + (1/2^{n+1})\big) \end{aligned}$$

by the definition of f on J_{n+1} and A4.14(i), by $(\surd)$, and by the induction hypothesis $(\dagger_n)$.

The induction is complete, and we have defined $f \colon J \to S$ by recursion in such a fashion that (##)) and $(\dagger_n)$ are satisfied. Now we extend f to a function $F \colon]0, \infty] \to S$ by $F(r) = \max S$ for $r > 1$ and

$$F(r) = \sup_{s \in J,\, s < r} f(s) \quad for 0 < r \le 1.$$

This least upper bound exists by A4.14(iii). If $\overline{T}$ denotes the smallest subsemigroup of S containing T and being closed under the formation of directed sups, then $\operatorname{im} F \subseteq \overline{T}$.

Clearly (I) is satisfied. Proof of (II). We compute

$$\sup_{0<s<r} F(s) = \sup_{0<s<r} \Big(\sup_{u \in J,\, u<s} f(u) \Big) = \sup_{\{(u,s) \mid u \in J,\, u<s<r\}} f(u) = \sup_{u \in J,\, u<r} f(r) = F(r)$$

since J is order dense in $]0, 1]$.

Proof of (III): We have $F(1/2^n) = \sup_{r \in J,\, r < \frac{1}{2^n}} f(r) \le u_n$ since $r < \frac{1}{2^n}$ implies $f(r) \le f(\frac{1}{2^n})$ by (##), and $f(\frac{1}{2^n}) = u_n$ by the construction of f.

Proof of (IV). Fix an $n \in \mathbb{N}$ and consider an $r \in]0,1]$. If $1-(1/2^{n+1}) \leq r$, then $F(r+(1/2^n)) = \max S \geq F(r)u_{n+1}$. So assume that $r < 1-(1/2^{n+1})$ and let s be the first element of J_{n+1} such that $r \leq s$. Then (##) and the definition of F implies

(α) $$F(r) = \sup_{q\in J,\, q<r} f(q) \leq f(s),$$

and since the element $s+(1/2^{n+1}) = \big(s-(1/2^{n+1})\big)+(1/2^n) < r+(1/2^n)$ belongs to J, we have

(β) $$f\big(s+(1/2^{n+1})\big) \leq F\big(r+(1/2^n)\big).$$

From $(\dagger_{n+1})$ we get

(γ) $$f(s)u_{n+1} \leq f\big(s+(1/2^{n+1})\big).$$

Now (α), (β), and (γ) and 2.11(i) together imply

$$F(r)u_{n+1} \leq f(s)u_{n+1} \leq F\big(r+(1/2^n)\big),$$

and this is what we had to show. The proof of the lemma is now complete. □

Theorem A4.16 (Characterisation of Left Invariant Metrizability). (a) *For a topological group G, the following conditions are equivalent:*

(1) *The topology of G is defined by a left invariant metric.*

(2) *The filter of identity neighborhoods* (*equivalently, that of any point in G*) *has a countable basis.*

(b) *Also, the following conditions are equivalent:*

(3) *The topology of G is defined by a biinvariant metric.*

(4) *The filter of identity neighborhoods has a countable basis each member of which is invariant under inner automorphisms.*

(c) *For a locally compact group G conditions* (1) *and* (2) *are equivalent to the following condition.*

(5) *There is a countable family of identity neighborhoods intersecting in* $\{1\}$.

Proof. Clearly, (1)⇒(2) and (3)⇒(4).

We assume (2) and show (1). In order to prove (4)⇒(3) at the same time we consider a set Γ of automorphisms of the topological group G, e.g. $\Gamma = \{\mathrm{id}\}$, or the group of all inner automorphisms. Let O_n, $n \in \mathbb{N}$ be a family of Γ-invariant identity neighborhoods which form a basis for the filter of identity neighborhoods. We define recursively a new basis U_n by setting $U_1 = O_1$. Assume that $U_1, \ldots, U_n$ is defined so that all U_m are Γ-invariant and satisfy $U_m^2 \subseteq U_{m-1} \cap O_{m-1}$, $m = 2, 3, \ldots, n$. There is an identity neighborhood V such that $VV \subseteq U_n \cap O_n$. Since the O_m form a basis for the identity neighborhoods there is an index $j(n)$ such that $O_{j(n)} \in V$. Set $U_{n+1} = O_{j(n)}$. The recursion is complete and yields a basis of Γ-invariant identity neighborhoods U_n with $(U_{n+1})^2 \subseteq U_n$.

Now we let S denote the semigroup of all Γ-invariant identity neighborhoods under multiplication of subsets of G. Containment $\subseteq$ endows S with a conditionally complete order (see A4.13). Then Lemma A4.15 applied with $u_n = U_n$ yields a function $r \mapsto U(r)\colon]0,\infty[\to \mathcal{P}(G)$ such that Conditions A4.15(I)–(IV) are satisfied with U in place of F. We claim that (A)–(D) from A4.12 are satisfied. We have (A) $\iff$ (I) and (B) $\iff$ (II). In order to prove (C) let U be any identity neighborhood. Since the U_k form a basis for the identity neighborhoods we find an m such that $U_m \subseteq U$. We set $n = 2^m$ and see that $U(1/n) = U(1/2^m) \subseteq U_m \subseteq U$ by (III). In order to verify (D) we let n be given. Pick an $m \in \mathbb{N}$ such that $n \leq 2^m$. Then set $U = U_{m+1}$. Then for each $t > 0$ we have $U(t)U = U(t)U_{m+1} \subseteq U\big(t + (1/2^m)\big) \leq U\big(t + (1/n)\big)$ by (IV) and the monotonicity of $s \mapsto U(s)$, secured by (II). Thus $s \mapsto U(s)$ satisfies conditions (A)–(D) of A4.12. Then Lemmas A4.10 through 13 show that G has a Γ-invariant metric defining its topology.

It is obvious that (1) implies (5). We now assume that G is locally compact and prove that (5) implies (1). Assume $\{1\} = \bigcap_{n\in\mathbb{N}} U_n$ for a family of identity neighborhoods U_n. We may assume that $\overline{U_n}$ is compact for all n and that $U_{n+1} \subseteq U_n$. Let U be an open identity neighborhood in G. Claim: There is an N such that $U_N \subseteq U$. Suppose not, then $\overline{U_n} \setminus U$ is a filter basis of compact sets. Its nonempty intersection is contained in $\{\mathbf{1}\}$ on the one hand and in $G \setminus U$ on the other. This contradiction proves the claim. □

We remark that the preceding theorem allows us to conclude that a topological group with a metrizable identity neighborhood is left invariantly metrizable. This is the case if some identity neighborhood is homeomorphic to an open ball in some Banach space. In particular we obtain the following corollary.

Corollary A4.17. *A linear Lie group has a left invariant metric. It has a biinvariant metric if and only if it has arbitrarily small identity neighborhoods which are invariant under inner automorphisms if and only if the Lie algebra has arbitrarily small zero neighborhoods invariant under the adjoint representation.* □

We notice that in Theorem A4.16 we have proved a little more:

Corollary A4.18. *Assume that G is a topological group and Γ a set of automorphisms. If G has a countable basis of Γ-invariant identity neighborhoods, then the topology of G is defined by a left-invariant metric satisfying $d\big(\gamma(x), \gamma(y)\big) = d(x, y)$ for all automorphisms from the group $\langle\Gamma\rangle$ generated by Γ.* □

Corollary A4.19. *The topology of every compact group with a countable basis of identity neighborhoods is defined by a biinvariant metric.*

Proof. By Corollary 1.12, every compact group has a basis of identity neighborhoods which are invariant under inner automorphisms. The assertion then follows from A4.18 with the group of all inner automorphisms Γ. □

Exercise EA4.7. (i) The power semigroup $\mathcal{P}(G)$ of a topological group G has various subsemigroups which are conditionally complete in the containment order. Examples:

(a) The semigroup of all normal subgroups.

(b) The semigroup of all open closed normal subgroups.

Note that all elements in these semigroups are idempotent. What are the consequences for the metric constructed according to Theorem A4.16 from a countable basis U_n for the filter of identity neighborhoods consisting of open normal subgroups?

(ii) Show that, for a metric group G there is a family $r \mapsto U(r):]0,\infty[\to \mathcal{P}(G)$ which, in addition to the conditions (A)–(D) of Lemma A4.12 also satisfies the following condition:

(E) $(\forall 0 < s,\, t)\ U(s)U(t) \leq U(s+t)$.

[Hint for (ii). Consider $p(x) = \min\{\|x\|, 1\}$ for a function $\|\cdot\|$ satisfying the conditions A4.10(ii)(1)–(4) and define $U(r) = \{g \in G \mid p(g) < r\}$.] ☐

Regarding Exercise EA4.7(ii) it is not known whether a semigroup theoretical proof exists to construct a function F such as in Lemma A4.15 with the additional property that $F(s)F(t) \leq F(s+t)$. In the presence of certain additional conditions such a proof was given in [161].

Exercise EA4.8. Recall that a *pseudometric* satisfies all axioms of a metric with the possible exception of the postulate that $d(x,y) = 0$ implies $x = y$. Use the tools at our disposal in order to prove the following result.

In a topological group G let U_n be a sequence of identity neighborhoods satisfying $(U_{n+1})^2 \subseteq U_n$. Then there is a continuous left invariant pseudometric d such that for any n the identity neighborhood U_n contains some open d-ball around the identity. ☐

Notice that in a group with a left invariant pseudometric the set of elements with distance 0 from the identity is a subgroup.

Proposition A4.20. (i) *Let G be a metric subgroup of a product $\prod_{j\in J} G_j$ of topological groups. Then there is a countable subset J_1 of such that the projection onto the partial product $\prod_{j\in J_1} G_j$ maps G continuously and bijectively onto its image.*

(ii) *Let G be a subgroup without small subgroups contained in a product $\prod_{j\in J} G_j$ of topological groups. Then there is a finite subset J_1 of such that the projection onto the partial product $\prod_{j\in J_1} G_j$ maps G continuously and bijectively onto its image.*

Proof. Exercise EA4.9. ☐

Exercise EA4.9. Prove A4.20. Apply Part (ii) to prove the following result:

If the additive topological group of a Banach space G is a subgroup of the product $\prod_{j\in J} G_j$ of topological groups, then a continuous bijective copy of it is contained in a finite subproduct of it.

[Hint. For Part (i), $J' \subseteq J$ set $H_{J'} \stackrel{\text{def}}{=} \prod_{j\in J} G_j^*$,

$$G_j^* = \begin{cases} \{\mathbf{1}\} & \text{for } j \in J', \\ G_j & \text{otherwise.} \end{cases}$$

Let $\{U_n \mid n \in \mathbb{N}\}$ be a countable family of identity neighborhoods of $\prod_{j\in J} G_j$ such that $\{G \cap U_n \mid n \in \mathbb{N}\}$ is a basis for the filter of identity neighborhoods of G. For each n there is a finite subset F_n of J such that H_{F_n} is contained in U_n. Then $J_1 \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} F_n$ is a countable subset of J and $G \cap H_{J_1} = \{1\}$. Then G is mapped faithfully into H_J/H_{J_1}. For Part (ii) take an open identity neighborhood U of the product such that $G \cap U$ does not contain a nonsingleton subgroup. Proceed in the spirit of the proof above.] □

Duality of Vector Spaces

In Chapter 7, in comments preceding Proposition 7.25 we introduce for a real vector space E the finest locally convex vector space topology $\mathcal{O}(E)$. We note here that there is a finest vector space topology $\mathcal{O}'(E)$ and obviously, $\mathcal{O}(E) \subseteq \mathcal{O}'(E)$. For vector spaces of countable dimension the two topologies agree (see e.g. [40], p. 136, Ex. 16.)

Proposition A4.21. *Let E be a real vector space whose dimension is uncountable. Then*

(i) $\mathcal{O}(E) \neq \mathcal{O}'(E)$.

(ii) $\big(E, \mathcal{O}'(E)\big)'' = \big(E, \mathcal{O}(E)\big)'' = \big(E, \mathcal{O}(E)\big)$.

(iii) *The underlying abelian topological group of $\big(E, \mathcal{O}'(E)\big)$ is semireflexive but not reflexive* (*see Definitions* 7.8).

Proof. Exercise EA4.10. □

Exercise EA4.10. Prove A4.21.

[Hint. (i) Let $\{e_j \mid j \in J\}$ be a basis of E and fix a number p, $0 < p < 1$. Show that the following defines a left invariant metric according to A4.10: $|\sum_{j\in J} r_j{\cdot}e_j| = \sum_{j\in J} |r_j|^p$. Show that the ball $B = \{x \in E : |x| < 1\}$ does not contain any identity neighborhood of $\mathcal{O}(E)$ (Suppose that V is a balanced absorbing convex set with $V \subseteq B$. For each $j \in J$, set $\lambda_j = \sup\{|r| : r{\cdot}e_j \in V\}$. Then $0 < \lambda_j \leq 1$. Let $J_n = \{j \in J \mid \lambda_j > \frac{1}{n}\}$. Since J is uncountable, for some n the set J_n is uncountable. Hence there is an uncountable subset $I \subseteq J$ and a positive number ε such that $\varepsilon{\cdot}x_i \in V \subseteq B$ for all $i \in I$. For each finite subset $F = \{s_1, \ldots, s_N\}$, the convex combination $x_F = \frac{1}{N}\sum_{n=1}^{N} \varepsilon{\cdot}e_{s_n}$ is contained in V. But $|y_F| = N{\cdot}(\frac{\varepsilon}{N})^p = \varepsilon^p N^{1-p}$. As N becomes large, $|y_F|$ tends to infinity. Thus y_F is not in B for large enough N. This is a contradiction.

(ii) Show that $\big(E, \mathcal{O}(E)\big)$ and $\big(E, \mathcal{O}'(E)\big)$ have the same compact sets and the same functionals (see 7.25(iv)). Hence their duals agree as topological vector spaces and thus their biduals agree. For $\big(E, \mathcal{O}(E)\big)'' = \big(E, \mathcal{O}(E)\big)$ see 7.5 and 7.30(i).

(iii) is a consequence of (ii).] □

The above argument was communicated to us by Arkady Leiderman. We shall say much more about the duality of the category $\mathbb{V}$ of vector spaces E over $\mathbb{R}$ or

$\mathbb{C}$ in Appendix 7, notably, on the case of the finest locally convex topology $\mathcal{O}(E)$ on E.

Subgroups of Topological Groups

There are a number of facts concerning subgroups of topological groups which are used frequently and most of which are quite elementary. We collect them here for easy reference.

Definition A4.22. Let G be a topological group and H a subgroup. Then H is called *locally closed* if there is an open set U of G and a subset W of U such that for some $h_0 \in W \cap H$ the set W is a neighborhood of h_0 in G and that $\overline{W \cap H} \cap U \subseteq H$. □

Proposition A4.23. *Let G be a topological group and H a subgroup. Then the subgroup H is closed in G if and only if it is locally closed.*

Proof. Trivially, a closed subgroup is locally closed; we must prove the reverse. The left translation $\lambda_{h_0}: G \to G$, $\lambda_{h_0}(x) = h_0 x$ is a homeomorphism and $\lambda_{h_0}^{-1}$ maps h_0 to 1. Now $U' = h_0^{-1}U$ is an open subset of G and $W' = h_0^{-1}W$ is an identity neighborhood in G such that $\overline{W' \cap H} \cap U' \subseteq H$. We may and shall therefore assume from here on that $h_0 = 1$.

Let W_0 denote the interior of W. If $g \in \overline{H}$, then there is an $h \in gW_0^{-1} \cap H$. Hence $g \in hW_0 \subseteq hU$. Let $\mathcal{U}$ denote the filter basis of identity neighborhoods V of G such that $gV \subseteq hW_0$. Then $V \in \mathcal{U}$ implies $H_V \stackrel{\text{def}}{=} gV \cap H \neq \emptyset$ and $H_V \subseteq hW \cap H = hW \cap hH = h(W \cap H)$. Now $g \in \bigcap_{V \in \mathcal{U}} \overline{H_V} \subseteq \overline{h(W \cap H)} \cap hU = h\big(\overline{W \cap H} \cap U\big) = k(W \cap H)$ since $W \cap H$ is closed in U by hypothesis. Then $g \in hH = H$. □

Corollary A4.24. *Every locally compact subgroup of a Hausdorff topological group is closed.*

Proof. Let G be a Hausdorff topological group and H a locally compact subgroup. Take an identity neighborhood W such that $W \cap H$ is compact and let U be the interior of W. Since $W \cap H$ is compact and G is Hausdorff, the set $W \cap H$ is closed in G and thus $\overline{W \cap H} \cap U = W \cap H \cap U = H \cap U \subseteq H$. Then Proposition A4.23 shows that H is closed in G. □

Proposition A4.25. *Let G be a topological group. Then*

(i) *a subgroup H is open if and only if it contains a nonempty open subset of G.*

(ii) *Any open subgroup H of G is closed.*

(iii) *Any subset with nonempty interior, in particular any identity neighborhood of G, generates an open closed subgroup.*

(iv) *The identity component G_0 is contained in any open subgroup.*

Proof. (i) If H contains a nonempty open subset U and $h \in H$, let $u \in U$ and note that $hu^{-1}U$ is an open subset of G containing h and contained in H. Thus $H = \bigcup_{h\in H} hu^{-1}U$ is open.

(ii) If H is open, then gH is open for all $g \in G$. Since G is the disjoint union of the cosets gH, $g \in G$, we observe that $H = G \setminus \bigcup_{g\in G\setminus H} gH$ is the complement of an open set and is therefore closed.

(iii) If U is a nonempty open subset of G then $U \subseteq H \stackrel{\text{def}}{=} \langle U\rangle$, and then H is open by (i).

(iv) Let H be an open subgroup. Then $G = H \dot\cup (G \setminus H)$ is a decomposition of G in two open and closed subsets and $1 \in H$. Since G_0 cannot be decomposed into a disjoint union of two nonempty open subsets and $1 \in G_0$ we must have $G_0 \cap (G \setminus H) = \emptyset$; i.e. $G_0 \subseteq H$.

Corollary A4.26. *A connected topological group is generated by any nonempty open subset and, in particular, by any of its identity neighborhoods.*

Proof. This is an immediate consequence of A4.25(iii), (iv). □

Proposition A4.27. *A totally disconnected normal subgroup of a connected topological group is central.* (Cf. 6.13.)

Proof. Let N be a normal totally disconnected subgroup of the connected topological group G. Let $n \in N$. The function $g \mapsto \text{comm}(g, n) = gng^{-1}n^{-1}: G \to N$ is continuous. Since G is connected and N is totally disconnected, the image is singleton, and it contains $1 = \text{comm}(1, n)$. Hence $\text{comm}(g, n) = 1$ for all g, and this proves the claim. □

Proposition A4.28. *Let G be a topological group and D a discrete normal subgroup and let $q: G \to G/D$ be the quotient morphism. Then G contains an open symmetric identity neighborhood U such that for the image $V \stackrel{\text{def}}{=} UD/D$ in G/D the following assertions hold.*

(i) *q induces a homeomorphism $\varphi \stackrel{\text{def}}{=} q|U: U \to V$.*
(ii) *If g_1, g_2, $g_1g_2 \in U$, then $\varphi(g_1)\varphi(g_2) = \varphi(g_1g_2)$.*
(iii) *If γ_1, γ_2, $\gamma_1\gamma_2 \in V$, then $\varphi^{-1}(\gamma_1)\varphi^{-1}(\gamma_2) = \varphi^{-1}(\gamma_1\gamma_2)$.*

Proof. Since D is discrete, there is an open identity neighborhood W such that $W \cap D = \{1\}$. We find an open identity neighborhood U_0 such that $U_0U_0^{-1} \subseteq W$. Then $q|U_0$ is injective; for if $q(u) = q(u')$ for u, $u' \in U_0$, then $u'u^{-1} \in U_0U_0^{-1} \cap \ker q \subseteq W \cap D = \{1\}$. Hence $u = u'$. Now let U be a symmetric open identity neighborhood of G such that $U^2 \subseteq U_0$. Since q is open, $V \stackrel{\text{def}}{=} q(U)$ is an open symmetric identity neighborhood of G/D. Then $\varphi = q|U: U \to V$ is continuous, open, and bijective. Thus (i) is proved. (ii) is obvious since φ is the restriction of a morphism. We show (iii). Let $x = \varphi^{-1}(\gamma_1)\varphi^{-1}(\gamma_2)$ and $y = \varphi^{-1}(\gamma_1\gamma_2)$. Then

$x \in UU$, $y \in U \subseteq UU$, and $q(x) = q\big(\varphi^{-1}(\gamma_1)\varphi^{-1}(\gamma_2)\big) = \gamma_1\gamma_2 = q\big(\varphi^{-1}(\gamma_1\gamma_2)\big) = q(y)$. But $UU \subseteq U_0$ and $q|U_0$ is injective. Hence $x = y$, and the assertion in (iii) follows. □

Wallace's Lemma

The following is a very simple, but also very useful tool, based on first principles.

Proposition A4.29 (Wallace's Lemma). *Let A be a compact subspace of X and B a compact subspace of Y, and assume that there is an open subset U of $X \times Y$ containing $A \times B$. Then there are open neighborhoods V of A in X and W of B in Y such that $V \times W \subseteq U$.*

Proof. Exercise. □

Exercise EA4.11. Prove Wallace's Lemma.

[Hint. For the proof, assume first that B is singleton. Conclude that in the general case for each $b \in B$ there is an open set V_b containing A and an open neighborhood W_b of b in Y such that $V_b \times W_b \subseteq U$. Cover $A \times B$ by the $V_b \times W_b$.]

Cantor Cubes and Dyadic spaces

Let $\mathbf{2}$ denote the discrete space $\{0, 1\}$ of two elements.

Definition A4.30. A *Cantor cube* is space which is homeomorphic to a product space $\mathbf{2}^{\aleph}$ for some infinite cardinal $\aleph$. A dyadic space is a continuous image of a Cantor cube. □

The Cantor cube $\mathbf{2}^{\aleph_0}$ is (homeomorphic to) the standard Cantor set.

Lemma A4.31 (Alexandroff's Theorem on Dyadic Spaces). *Every compact metric space is dyadic and is, in fact, a continuous image of the standard Cantor set.*

A compact metric space is homeomorphic to $\mathbf{2}^{\mathbb{N}}$ iff it is totally disconnected and has no isolated points.

Proof. See e.g. [101], 4.5.9(b), p. 291) and [101], 6.2.A(c), p. 370. □

Corollary A4.32. *Every cartesian product of a family of compact metric spaces is dyadic.*

Every product of Cantor cubes is a Cantor cube.

Every infinite product of finite sets is a Cantor cube.

Proof. Exercise. □

Exercise EA4.12. Prove Corollary A4.32.

[Hint. Let $X=\prod_{j\in J}M_j$ and assume that M_j is compact metric for each $j\in J$, respectively a Cantor cube. Then M_j is a continuous image of $\mathbf{2}^{\aleph_0}$ by Lemma A4.31, respectively, a homeomorphic image of $\mathbf{2}^{A_j}$. Thus X is a continuous image of $(\mathbf{2}^{\aleph_0})^J=2^{\max\{\aleph_0,J\}}$, respectively, a homeomorphic image of $\mathbf{2}^{\bigcap_{j\in J}A_j}$.

If J is an infinite set and $X=\prod_{j\in J}F_j$ with finite sets F_j, then by the Well-Ordering Theorem we can write J as the disjoint union $\bigcup_{k\in K}J_k$ of countably infinite sets J_k. Then $X=\prod_{k\in K}X_k$ where $X_k-\prod_{j\in J_k}F_j$. Each X_k is a standard Cantor set $\mathbf{2}^{\mathbb{N}}$, Hence X is a Cantor cube.] □

Lemma A4.33. *Assume that we are given an inverse system of compact spaces*

$$X_1\xleftarrow{p_1}X_2\xleftarrow{p_2}X_3\cdots$$

and that there are homeomorphisms $f\colon X_n\to X_{n-1}\times Y_n$, $X_0=\{1\}$ *singleton and* $Y_1=X_1$, *such that the following diagram commutes for* $n=1,2,\ldots$*:*

$$(1_n)\qquad \begin{array}{ccc} X_n & \xleftarrow{p_n} & X_{n+1}\\ {\scriptstyle \mathrm{id}_{X_n}}\big\downarrow & & \big\downarrow{\scriptstyle f_{n+1}}\\ X_n & \xleftarrow[pr_{X_n}]{} & X_n\times Y_{n+1}.\end{array}$$

Set $X\stackrel{\mathrm{def}}{=}\lim_n\{\cdots X_n\xleftarrow{p_n}X_{n+1}\cdots\}$ *and* $Y\stackrel{\mathrm{def}}{=}\prod_n Y_n$.

Then

(i) X *and* Y *are homeomorphic, and*

(ii) *If all* Y_n *are Cantor cubes, then* X *is a Cantor cube.*

Proof. (i) Let us recall that the product of a family $\{Z_j : j\in J\}$ is the set of all functions $\alpha\colon J\to U\stackrel{\mathrm{def}}{=}\bigcup_{j\in J}Z_j$ such that $\alpha(j)\in X_j$ for all $j\in J$. For $I\subseteq J$, the function $\alpha\mapsto\alpha|I:\prod_{j\in J}Z_j\to\prod_{i\in I}Z_i$ will be denoted p_I. We write $\mathbb{N}_n\stackrel{\mathrm{def}}{=}\{1,\ldots,n\}$.

Claim 1. For each n there is a homeomorphism $F_n\colon X_n\to\prod_{m\in\mathbb{N}_n}Y_n$ such that the following diagram commutes

$$(2_n)\qquad \begin{array}{ccc} X_n & \xleftarrow{p_n} & X_{n+1}\\ {\scriptstyle F_n}\big\downarrow & & \big\downarrow{\scriptstyle F_{n+1}}\\ \prod_{m\in\mathbb{N}_n}Y_m & \xleftarrow[p_{\mathbb{N}_n}]{} & \prod_{m\in\mathbb{N}_{n-1}}Y_m.\end{array}$$

We prove this claim by induction. Claim (2_1) is true by (1_1). Assume that (2_{n-1}) has been proved for $n\ge 2$. We must construct $F_n\colon X_{n-1}\times Y_n$. By induction hypothesis we have a homeomorphism $F_{n-1}\colon X_{n-1}\to X_{n-2}\times Y_{n-1}$ satisfying (2_{n-1}). We identify $\prod_{m\in\mathbb{N}_{n-1}}Y_m\times Y_n$ and $\prod_{m\in\mathbb{N}_n}Y_m$ and define F_n to be the composition

$$X_n\xrightarrow{f_n}X_{n-1}\times Y_n\xrightarrow{F_{n-1}\times\mathrm{id}_{Y_n}}\prod_{m\in\mathbb{N}_{n-1}}Y_m\times Y_n=\prod_{m\in\mathbb{N}_n}Y_m.$$

Then we have a commutative diagram

$$\begin{array}{ccc}
X_{n-1} & \xleftarrow{\quad p_n \quad} & X_n \\
{\scriptstyle \mathrm{id}_{X_{n-1}}}\big\downarrow & (8_n) & \big\downarrow{\scriptstyle f_n} \\
X_{n-1} & \xleftarrow[\mathrm{pr}_{X_{n-1}}\; X_{n-1}\times Y_n]{\qquad\qquad} & X_{n-1}\times Y_n \\
{\scriptstyle F_{n-1}}\big\downarrow & & \big\downarrow{\scriptstyle F_{n-1}\times \mathrm{id}_{\mathrm{Y_n}}} \\
\prod_{m\in\mathbb{N}_{n-1}} Y_m & \xleftarrow[p_{\mathbb{N}_{n-1}}]{\quad} & \prod_{m\in\mathbb{N}_n} Y_m.
\end{array}$$

after our identification. Since the vertical map on the right is F_n by definition, we have (2_n). This proves Claim 1.
Claim 2. We have a commutative diagram of inverse systems in which the rows are limit diagrams, where we abbreviate $P_n = \prod_{m\in\mathbb{N}_n} Y_m$, and where $F\colon X \to L$ is the induced morphism:

$$\begin{array}{ccccccccc}
X_1 & \xleftarrow{p_1} & X_2 & \xleftarrow{p_2} & X_3 & \cdots & \longleftarrow & X \\
{\scriptstyle F_1}\big\downarrow & & {\scriptstyle F_2}\big\downarrow & & & \cdots & & \big\downarrow{\scriptstyle F} \\
P_1 & \xleftarrow[p_{\mathbb{N}_1}]{} & P_2 & \xleftarrow[p_{\mathbb{N}_2}]{} & P_3 & \cdots & \longleftarrow & L.
\end{array}$$

Since all F_n are homeomorphisms, F is also a homeomorphism.
Claim 3. $L = Y$: We may assume that

$$L = \{\big((y_m^{(n)})_{m\in\mathbb{N}_n}\big)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}\prod_{m\in\mathbb{N}_n} Y_n : p_{\mathbb{N}_n}(y_m^{(n+1)})m \in \mathbb{N}_{n+1}) = (y_m^{(n)})m \in \mathbb{N}_n\}.$$

By Induction it follows that $y_m^{(n)} = y_m^{(n+1)}$, $m \in \mathbb{N}_n$. We set $y_m = y_m^{(n)}$ for $n \geq m$. Thus the function

$$(y_m)_{m\in\mathbb{N}} \mapsto \big((y_m)_{m\in\mathbb{N}_n}\big)_{n\in\mathbb{N}} : Y \to L$$

is a homeomorphism.

This completes the proof of the (i).

Assertion (ii) follows immediately from (i) and Corollary A4.32. □

Some Basic Facts on Compact Monoids

In the next Appendix we will present an independent and self-contained proof of the existence and uniqueness of Haar measure. An essential part of it rests on a few basic facts from the theory of compact topological monoids which we select and present here to have them available then.

If S is a compact topological semigroup, then the set of ideals $I \subseteq S$, that is, of subsets satisfying $SI \cup IS \subseteq S$ is a filter basis (note $IJ \subseteq I \cap J$ for ideals!) in which the filter basis of *compact* ideals is cofinal ($s \in I$ implies $SsS \subseteq I$!). This implies that there is a unique minimal ideal $\mathcal{M}(S)$ which is compact. If S is abelian then $A = \mathcal{M}(S)$ is a compact abelian semigroup satisfying $Aa = A$ for all $a \in A$. Hence A is a group. In particular, A contains an idempotent e. If $s \in S$ in an arbitrary compact semigroup, then $\Gamma(s) = \overline{\{s, s^2, s^3, \ldots\}}$ is a compact abelian semigroup. Its minimal ideal $\mathcal{M}\big(\Gamma(s)\big)$ is a group whose identity is an idempotent $e(s)$. In this minimal ideal, the element $se(s)$ has an inverse s'.

Proposition A4.34. *A closed subsemigroup in a compact group is a subgroup.*

Proof. If S is a closed subsemigroup of a compact group G and $s \in S$, then $e(s) = \mathbf{1}$ since there is only one idempotent in G. Therefore $s' \in S$ is the inverse s^{-1} of s in G. Hence S is a group. □

Now we consider the set of closed left ideals of S which contains S and is downward inductive: Indeed if $\mathcal{T}$ is a tower of closed left ideals, then in particular it is a filterbasis of compact sets and so $\bigcap \mathcal{T}$ is nonempty and closed; its being a left ideal implies that it is a lower bound of $\mathcal{T}$ proving the claim. Hence by Zorn's Lemma

the compact topological semigroup S has minimal closed left ideals.

Let L be one of these and let X be its (nonempty!) set of idempotent elements. If $e \in X$, then $Se \subseteq L$ is a closed left ideal of S and thus by minimality of L we have $L = Se$. Then for all $\in L$ we note $x = se$ for some $s \in S$ and so $xe = (se)e = se = x$, that is e is a right identity of L. In particular for any pair of idempotents $x, y \in X$ we have $xy = x$.

Thus X is what in semigroup theory is called a *left zero semigroup*. If the two element set $\{0, 1\}$ is endowed with its left zero multiplication, then $\{0\}$ and $\{1\}$ are two disjoint left ideals. In retrospect this makes it clear that the Axiom of Choice appears to be inevitable in general for a proof of the existence of minimal closed left ideals.

Definition A4.35. An involution $x \mapsto x^*$ on a semigroup S is a self map satisfying $x^{**} = x$ and $(xy)^* = y^*x^*$. A semigroup S is called *involutive* if it possesses an involution. □

Proposition A4.36. *The minimal ideal of an involutive compact topological semigroup is a group if its involution fixes idempotents.* □

Proof. If L is a minimal left ideal and e and f are idempotents of L, then $e = ef$ and $e = e^* = (ef)^* = f^*e^* = fe = f$. Thus L contains only one idempotent e and $L = Se = S^*e^* = eS$. Thus e is an identity of L. If $s \in L$, then $e(s) = e$ and s' is therefore an inverse of s with respect to e, and so L is a group. Since $L = eS$ is also a right ideal, L is an ideal and thus is the minimal ideal. □

We continue with a purely topological and geometric definition: An *affine space* X is a convex subset of a real (but not necessarily topological!) vector space together with a topology on X such that the function

$$(r, s, t) \mapsto r \cdot s + (1 - r) \cdot t \colon [0, 1] \times X \times X \to X,$$

is continuous, where "$\cdot$" denotes the scalar multiplication of the surrounding vector space.

A continuous map $f\colon X \to Y$ between affine spaces is *affine* if

$$(\forall x, x' \in X, \forall r \in [0,1]) \quad f(r\cdot x + (1-r)\cdot x') = r\cdot f(x) + (1-r)\cdot f(x'),$$

Definition A4.37. A topological semigroup S is *affine* if

(1) S is an affine space, and

(2) the self maps $s \mapsto su$ and $s \mapsto us$ of S are affine for all $u \in S$.

A *morphism* $f\colon S \to T$ *of affine semigroups* is a continuous mapping between affine semigroups that preserves both the semigroup operation and is an affine map. □

The unit interval, int $\stackrel{\text{def}}{=}$ $[0,1]$ in the usual topology and with the usual product and affine structure, is the canonical example of a compact affine monoid. It is noteworthy that the open half-line $((0,\infty),\cdot)$ under multiplication and the affine structure induced by $\mathbb{R}$ is a locally compact affine topological *group*.

Later, after we have developed a sufficient body of measure theory we shall see that the probability measures on a compact group form a compact affine semigroup.

In a compact affine semigroup we have a significant ergodic theorem which is an affine parallel to the statement that in every compact semigroup S and every $x \in S$ the compact subsemigroup $\Gamma(x)$ has a compact abelian monothetic group as a minimal ideal with idempotent $e(x)$.

Proposition A4.38. (Chow's Lemma [61]) *For each element x of a compact affine semigroup S, the closed convex hull $C(x) \stackrel{\text{def}}{=} \overline{\text{conv}}(\Gamma(x))$ of $\Gamma(x)$ is a compact commutative affine subsemigroup with a zero z, where*

$$z = \lim_{n\to\infty} \frac{1}{n}\cdot(x + x^2 + \cdots + x^n).$$

Proof. It is straightforward to show that the convex hull of a commutative subsemigroup in an affine topological semigroup is a commutative subsemigroup, and since the closure of a commutative, affine subsemigroup is another such, it follows that $C(x)$ is a compact commutative affine subsemigroup of S for each $x \in S$.

Fix an $x \in S$, and define $x_n \stackrel{\text{def}}{=} \frac{1}{n}\cdot(x + x^2 + \cdots + x^n)$ for $n = 1, 2, \ldots$. In the compact space $\Gamma(x) \times C(x)$, the sequence $(x^n, x_n)_{n\in\mathbb{N}}$, has a cluster point (h, z). Thus there is a directed set J and a cofinal function $j \mapsto n(j) : J \to \mathbb{N}$ such that such that $\lim_{j\in J}(x^{n(j)}, x_{n(j)}) = (h, z)$. Now

$$\text{(i)} \qquad x_{n+1} = \frac{1}{n+1}\cdot(n\cdot x_n + x^{n+1}) = \frac{n}{n+1}\cdot x_n + \frac{1}{n+1}\cdot x^{n+1}.$$

Also,

$$\text{(ii)} \qquad x_{n+1} = \frac{1}{n+1}\cdot(x + n\cdot xx_n) = \frac{1}{n+1}\cdot x + \frac{n}{n+1}\cdot xx_n, \quad xx_n = x_n x$$

In (i) and (ii) we replace n by $n(j)$ and pass to the limit as j ranges through J. Then, since G is a compact affine semigroup, from (i) we obtain

$$\lim_{j\in J} x_{n(j)+1} = z$$

and from (ii) we get

$$\lim_{j\in J} x_{n(j)+1} = xz = zx.$$

Therefore, $x = xz = zx$. Since the set of finite convex combinations of the powers of x is dense in $C(x)$, the element z is a zero of $C(x)$.

In the second part of the proof we argue that z is actually the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ by claiming that z is the only cluster point of $(x_n)_{n\in\mathbb{N}}$. So let z' be an arbitrary cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$. Then there is a subnet $(x_{m(i)})_{i\in I}$ converging to z'. Now by the compactness of $\Gamma(x)$, there is a cofinal function α from a directed set K to I such that $(x^{m(\alpha(k))}, x_{m(\alpha(k))})_{k\in K}$ converges to (h', z'). Then the argument in the first part of the proof shows that z' is a zero of $C(x)$. But zeros are unique; hence $z' = z$ and this proves the claim. □

(For further comments see [177], p. 137.)

Theorem A4.39. *A compact affine group G is singleton.*

Proof. Let $g \in G$. By Chow's Lemma A4.38, the compact affine subsemigroup $C(g)$ has a zero z; but the identity e of G is the only idempotent of G whence $z = e$ and thus $g = ge = e$. So $G = \{e\}$. □

The example of the positive half-line shows that this theorem fails for locally compact affine groups.

Corollary A4.40. *Let S be a compact involutive affine semigroup in which the involution fixes the idempotents. Then S has a zero.*

Proof. By Proposition A4.36 the minimal ideal of S is a group G with identity e. The map $f\colon S \to G$, given by $f(x) = xe$ is a surjective affine morphism. Thus G is a compact affine group. Hence it is singleton by Theorem A4.39. But then e is a zero of S. □

Postscript

The first section deals with the *arc component topology* which is a more or less classical, but not much referenced, tool of point set topology (see [378]); it is appropriate to have it here since by Theorem 5.52 it is that topology which distills a Lie group topology from an analytic subgroup even if the latter is not closed in the induced topology. It also serves us well in the fine structure theory of compact abelian groups as is exemplified by Corollary 8.31. The arc component topology may be considered as the topological analog of the Riemann–Rinow metric attached to a metric space, i.e. the metric derived from a given metric by measuring distance along rectifiable curves (to the extent these exist) ([302], p. 118ff.).

The concept of *weight* of a topological space is a well-known tool from the arsenal of cardinal invariants attached to a space ([67, 68]). One important fact which we use in the determination of the structure of free compact abelian groups is that, in A4.9, for all compact Hausdorff spaces X we have $\operatorname{card} C_0(X,\mathbb{T}) = w(X)^{\aleph_0}$. To obtain this we include a proof of the Kruse–Schmidt–Stone Theorem in EA4.4, which says that the cardinal of any Banach space B is $(\operatorname{card} B)^{\aleph_0} = \big(d(B)\big)^{\aleph_0}$ with the density $d(B)$ of B. A sidelight is that any cardinal number $\aleph$ which occurs as the cardinality of some Banach space satisfies $\aleph^{\aleph_0} = \aleph$.

Metrizability of topological groups is an established topic in this area. The standard texts have quite satisfactory treatments, see e.g. [147], p. 68ff., or [263], p. 28ff., 34ff., and p. 58ff. Bourbaki [34] devotes a whole section to metrizable groups (Chapter 9, §3, n$^{\text{o}}$ 1), but there the treatment is based on an extensive study of uniform spaces which the author rightfully presents at an early stage. Our treatment is not based on uniform spaces; we give a discourse directly suited to topological groups which is appropriate for our needs as it allows us to construct left invariant metrics with additional invariance properties, as we certainly wish to do in the context of compact groups. Our emphasis here is on grouping various functional descriptions of left invariant metrics together (Lemmas A4.10, A4.11) and then to establish an equivalent formulation in terms of families $r \mapsto U(r)$, $0 < r \in \mathbb{R}$ of identity neighborhoods which are to be created from sequences $n \mapsto U_n$, $n = 1, 2, \ldots$ forming a basis for the identity neighborhoods. The construction of such a family $r \mapsto U(r)$, however, is in reality a simple algebraic and order theoretical construction in ordered semigroups (Lemma A4.15). This approach was observed by Hofmann in [161] for the context of compact spaces and special metrics.

The last portion covers some basic material on compact and compact affine monoids needed in our proof of the existence of Haar measure in the next appendix. It contains a remarkably direct and simple Lemma due to Chow [61] whose proof we slightly modified allowing it to apply to the slightly more general definition of a compact affine monoid given here, avoiding, as it were, the embedding into a locally convex topological vector space. For more detailed comments we refer to [177], a source illustrating renewed interest in this subject matter.

References for this Appendix—Additional Reading

[40], [61], [67], [68], [84], [143], [147], [161], [176], [177], [178], [233], [263], [302], [323], [334], [335], [345], [378].

Appendix 5

Measures on Compact Groups

In this appendix we present some results that are used in the structure theory of compact groups. The most important fact in this regard is the existence and uniqueness of normalized Haar measure on a compact group.

The Definition of Haar Measure

For the moment let G denote a compact Hausdorff space and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $C(G, \mathbb{K})$ denote the Banach space of continuous $\mathbb{K}$-valued functions on G. An element μ of the topological dual—that is the vector space of all continuous linear functionals—of the Banach space $C(G, \mathbb{K})$ is a ($\mathbb{K}$-valued) *integral* or *measure*. The number $\mu(f)$ is also written $\langle \mu, f \rangle$ or indeed $\int f \, d\mu = \int_G f(g) \, d\mu(g)$. What we need is the uniqueness and existence of a particular measure on a compact group G; such a measure is familiar from the elementary theory of Fourier series as Lebesgue measure on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The formulation of the existence (and uniqueness theorem) is easily understood. Here we shall give a proof through a sequence of smaller steps.

For a semigroup G, an element $g \in G$, and a function $f\colon G \to \mathbb{K}$ we recall ${}_gf(x) = f(xg)$. If μ is a measure, we define μ_g by $\mu_g(f) = \mu({}_gf)$.

Definition A5.1. Let G denote a compact group. A measure μ is called *invariant* if $\mu_g = \mu$, that is, $\mu({}_gf) = \mu(f)$ for all $g \in G$ and $f \in E = C(G, \mathbb{K})$. It is called *positive* if it satisfies $\mu(f) \geq 0$ for all $f \geq 0$. A measure is called a *Haar measure* if it is invariant and positive. The measure μ is called *normalized* if $\mu(1) = 1$ where 1 also indicates the constant function with value 1. A normalized positive measure is also called a *probability measure.* □

We now state the Existence and Uniqueness Theorem on Haar Measure. We shall provide one of its numerous proofs in the following; this one uses compact semigroups and thus is also of independent interest.

Theorem A5.2. (The Existence and Uniqueness of Haar Measure) *For each compact group G there is one and only one normalized Haar measure.*

The Required Background of Radon Measure Theory

First we note in a self-contained fashion, some basic features of the measure theory we use. By definition, for a compact Hausdorff space X, the space $M(X, \mathbb{K})$ of $\mathbb{K}$-

valued *measures* is the topological dual $M(X,\mathbb{K}) = C(X,\mathbb{K})'$ of the Banach space of continuous $\mathbb{K}$-valued functions on X. If $\varphi: Y \to X$ is a continuous map between compact spaces, then it induces a contractive linear map $f \mapsto f \circ \varphi: C(X,\mathbb{K}) \to C(Y,\mathbb{K})$ (contravariant!) which is also called $C(\varphi)$ and induces in turn a contractive linear adjoint operator $M(\varphi): M(Y,\mathbb{K}) \to M(X,\mathbb{K})$ (covariant!) via $M(\varphi)(\mu)(f) = \mu(f \circ \varphi)$. In the literature, $M(\varphi)$ is sometimes denoted $\varphi^\star$, but we reserve this notation for the adjoint of an element in an involutive semigroup. If φ is the inclusion map of a closed subset Y into X then $M(\varphi)(\mu)$ is the extension of μ to a measure μ_X, that is, $\mu_X(f) = \mu(f|Y)$ for $f \in C(X,\mathbb{K})$.

Product Measures

We need an understanding of how measures on products of spaces are to be treated. Indeed for two measures μ_1 on a compact space X_1 and μ_2 on a compact space X_2, define the *product measure* $\mu_1 \otimes \mu_2$ on $X_1 \times X_2$ in the following discussion.

If $f \in C(X_1 \times X_2, \mathbb{K})$ we note that f is uniformly continuous on the compact space $X_1 \times X_2$ and so $x_2 \mapsto f(-, x_2): X_2 \to C(X_1, \mathbb{K})$ is continuous. The details are left to the following exercise.

Exercise EA5.1. Prove that for $f \in C(X_1 \times X_2, \mathbb{K})$ for compact spaces X_1 and X_2 the function $x_2 \mapsto f(-, x_2): X_2 \to C(X_1, \mathbb{K})$ is continuous. □

With this information, from the continuity of μ_2, we conclude that

$$x_2 \mapsto \langle \mu_1, f(-, x_2)\rangle = \int_{X_1} f(x_1, x_2)\, d\mu_1(x_1)$$

is a member of $C(X_2, \mathbb{K})$. Therefore

$$f \mapsto \int_{X_2} \left(\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2)$$

is a member of $M(X_1 \times X_2, \mathbb{K})$. We provisionally denote it by $\mu_1 \otimes_1 \mu_2$, that is,

$$\langle \mu_1 \otimes_1 \mu_2, f\rangle = \int_{X_1 \times X_2} f\, d(\mu_1 \otimes_1 \mu_2) = \int_{X_2} \left(\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2).$$

Quite analogously we define

$$\langle \mu_1 \otimes_2 \mu_2, f\rangle = \int_{X_1 \times X_2} f\, d(\mu_1 \otimes_2 \mu_2) = \int_{X_1} \left(\int_{X_2} f(x_1, x_2)\, d\mu_2(x_2) \right) d\mu_1(x_1).$$

Now consider two functions $f_j, \in C(X_j, \mathbb{K})$, $j = 1, 2$ and define $f_1 \otimes f_2: X_1 \times X_2 \to \mathbb{K}$ by $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2)$. The finite linear combinations of these functions form a dense subalgebra

$$C(X_1, \mathbb{K}) \otimes C(X_2, \mathbb{K}) \quad \text{of} \quad C(X_1 \times X_2, \mathbb{K}).$$

Therefore a continuous linear functional from $M(X_1 \times X_2, \mathbb{K})$ on $C(X_1 \times X_2, \mathbb{K})$ is uniquely determined by its values on $C(X_1, \mathbb{K}) \otimes C(X_2, \mathbb{K})$.

Now we compute

$$\begin{aligned}\langle \mu_1 \otimes_1 \mu_2, f_1 \otimes f_2 \rangle &= \int_{X_2} \left(\int_{X_1} (f_1 \otimes f_2)(x_1, x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2) \\ &= \int_{X_2} \left(\int_{X_1} f_1(x_1) \cdot f_2(x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2) \\ &= \int_{X_2} \left(\int_{X_1} f_1(x_1)\, d\mu_1(x_1) \cdot f_2(x_2) \right) d\mu_2(x_2) \\ &= \mu_2(\mu_1(f_1) \cdot f_2) \\ &= \mu_1(f_1)\mu_2(f_2).\end{aligned}$$

In the same spirit we calculate

$$\langle \mu_1 \otimes_2 \mu_2, f_1 \otimes f_2 \rangle = \mu_1(f_1)\mu_2(f_2).$$

We conclude that $\otimes_1 = \otimes_2$ and that

$$\langle \mu_1 \otimes \mu_2, f_1 \otimes f_2 \rangle = \mu_1(f_1)\mu_2(f_2).$$

Moreover, we have, for $f \in C(X_1 \times X_2, \mathbb{K})$, the (small) *Fubini Theorem*

$$\begin{aligned}\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \otimes \mu_2)(x_1, x_2) &= \int_{X_2} \left(\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2) \\ &= \int_{X_1} \left(\int_{X_2} f(x_1, x_2)\, d\mu_2(x_2) \right) d\mu_1(x_2).\end{aligned}$$

This concludes the definition of the product measure $\mu_1 \otimes \mu_2$.

The Support of a Measure

Let μ be a positive measure on a compact space G. An open subset U of G is a μ-null set if for every positive $f \in C(G, \mathbb{K})$ such that $\{x \in G \mid f(x) > 0\} \subseteq U$ we have $\mu(f) = 0$. The support $\mathrm{supp}(\mu)$ of a positive measure is the complement of the largest open μ-null set.

Proposition A5.3. *Let X and Y be compact Hausdorff spaces.*

(1) *Let μ be a positive measure on X. Then $x \notin \mathrm{supp}(\mu)$ iff there is a non-negative continuous function f such that*
  (a) *$\langle \mu, f \rangle = 0$, and*
  (b) *$f(x) > 0$.*

(2) *Assume $\varphi: X \to Y$ is a continuous map and that $\mu \in M(X, \mathbb{K})$ is a positive measure on X. Then an open set V in Y is a $M(\varphi)(\mu)$-null set iff $\varphi^{-1}(V)$ is a μ-null set in X.*

(3) *In the circumstances of* (2) *we have* $\mathrm{supp}(M(\varphi)(\mu)) = \varphi\big(\mathrm{supp}(\mu)\big)$.

Proof. For (1), if $x \notin \mathrm{supp}\,\mu$ use complete regularity of X to find a continuous $f: X \to [0, 1]$ such that $f(x) = 1$ and $f\big(\mathrm{supp}(\mu)\big) \subseteq \{0\}$; then the definition of $\mathrm{supp}(\mu)$ implies $\langle \mu, f \rangle = 0$. Conversely, assume that there is an f satisfying (a) and (b). By (b) find a compact neighborhood K of x such that

$f(K) \subseteq [f(x)/2, \max f(X)]$. Then $\varphi \stackrel{\text{def}}{=} \min\{f, f(x)/2\}$ is nonnegative and satisfies $\varphi\big(\operatorname{supp}(\mu)\big) \subseteq \{0\}$ and $\varphi(K) = \{f(x)/2\}$. Now from (a) we have $0 \leq \langle \mu, \varphi\rangle \leq \langle \mu, f\rangle = 0$. Now let V be the interior of K and take any nonnegative continuous function F such that $F(X \setminus V) \subseteq \{0\}$. Since F is bounded there is a $k \geq 0$ such that $F \leq k{\cdot}\varphi$. Hence $\langle \mu, F\rangle \leq k{\cdot}\langle \mu, \varphi\rangle = 0$. Thus V is an open μ-null set containing x and so $x \notin \operatorname{supp}(\mu)$ by the definition of $\operatorname{supp}(\mu)$.

For (2), let f be a positive member of $C(Y, \mathbb{K})$ such that $f(g) > 0$ implies $g \in V$. Then $\langle M(\varphi)(\mu), f\rangle = \langle \mu, f \circ \varphi\rangle$ and so an open set V in Y is a $M(\varphi)(\mu)$-null set iff $\varphi^{-1}(V)$ is a μ-null set; note that (1) is helpful for the harder implication.

For (3), we have

$$\varphi^{-1}\big(Y \setminus \operatorname{supp}(M(\varphi)(\mu))\big) \subseteq X \setminus \operatorname{supp}(\mu),$$

$$\text{i.e., } \operatorname{supp}(\mu) \subseteq \varphi^{-1}\big(\operatorname{supp}(M(\varphi)(\mu))\big) \text{ by (2), and so}$$

$$(*) \qquad \varphi\big(\operatorname{supp}(\mu)\big) \subseteq \operatorname{supp}\big(M(\varphi)(\mu)\big);$$

but $V \stackrel{\text{def}}{=} Y \setminus \varphi\big(\operatorname{supp}(\mu)\big)$ is an open set since $\varphi\big(\operatorname{supp}(\mu)\big)$ is compact as a continuous image of a compact set, and $\varphi^{-1}(V) \cap \operatorname{supp}(\mu) = \emptyset$, whence V is a $M(\varphi)(\mu)$-null set by (2) and thus is contained in $X \setminus \operatorname{supp}\big(M(\varphi)(\mu)\big)$ by the definition of the support. Thus equality holds in $(*)$. □

Proposition A5.4. *Let X be a compact space, μ a positive measure, and $f \in C(X, \mathbb{K})$. Assume that $f(x) \geq 0$ for all $x \in \operatorname{supp}(\mu)$. Then*

(1) *$\langle \mu, f\rangle \geq 0$, and*
(2) *if there is an $s \in \operatorname{supp}(\mu)$ such that $f(s) > 0$, then $\langle \mu, f\rangle > 0$.*
(3) *If $g \in C(X, \mathbb{K})$ agrees with f on $\operatorname{supp}(\mu)$, then $\langle \mu, g\rangle = \langle \mu, f\rangle$.*

Proof. As a first step of the proof define $F = \max\{f, 0\}$. Then F is a continuous nonnegative function such that, by hypothesis on f, we have $F(x) = f(x)$ for $x \in \operatorname{supp}(\mu)$ and $F(x) \geq f(x)$ for $x \in X \setminus \operatorname{supp}(\mu)$. Then

$$(F - f)(x) \left\{ \begin{matrix} = 0 & \text{for } x \in \operatorname{supp}(\mu) \\ \geq 0 & \text{for } x \in X \setminus \operatorname{supp}(\mu) \end{matrix} \right\} \geq 0.$$

Then, by the definition of the support of μ, we have $\langle F - f, \mu\rangle = 0$. Thus $\langle F, \mu\rangle = \langle f, \mu\rangle$ and the positivity of μ implies $\langle f, \mu\rangle \geq 0$ and that is the proof of (1).

For a proof of (2), assume $f(s) > 0$ for an $s \in \operatorname{supp}(\mu)$. We must argue that $\langle \mu, f\rangle > 0$. We proceed by contradiction and assume that $\langle \mu, f\rangle = 0$.

Let U be an open neighborhood of s such that $f(u) \geq f(s)/2 > 0$ for all $u \in U$. Now let φ be any nonnegative continuous function such that $\varphi(X \setminus U) = \{0\}$. As a continuous function, φ is bounded and after multiplying φ with a positive number, we assume without loss of generality that $\varphi(X) \subseteq [0, f(s)/2]$. Then

$$0 \leq \varphi(x) \left\{ \begin{matrix} \leq f(s)/2 \leq f(x) & \text{for } x \in U \\ = 0 & \text{for } x \in X \setminus U \end{matrix} \right\} \leq f(x),$$

since $f \geq 0$. Thus $0 \leq \langle\mu,\varphi\rangle \leq \langle\mu,f\rangle = 0$ by assumption. But then, by the definition of open μ-null set U is an open μ-null set. By the definition of the support, this implies $U \cap \operatorname{supp}(\mu) = \emptyset$. Yet this contradicts $s \in U \cap \operatorname{supp}(\mu)$. This contradiction completes the proof of (2), and (3) is an immediate consequence of (2) applied to $\max\{f-g,0\}$ and $\max\{g-f,0\}$. ☐

Proposition A5.5. *If $\mu_1 \in M(X_1,\mathbb{K})$ and $\mu_2 \in M(X_2,\mathbb{K})$ are positive measures, then*

$$(**) \qquad \operatorname{supp}(\mu_1 \otimes \mu_2) = \operatorname{supp}(\mu_1) \times \operatorname{supp}(\mu_2).$$

Proof. If U is a μ_1-null set in X_1, then we claim that the product $U \times X_2$ is an open $(\mu_1 \otimes \mu_2)$-null set. For let $F\colon X_1 \times X_2 \to \mathbb{K}$ be a positive function vanishing outside $U \times X_2$, then by Fubini's Theorem we have

$$\begin{aligned}\langle \mu_1 \otimes \mu_2, F\rangle &= \int_{X_1\times X_2} F(x_1,x_2)\; d(\mu_1 \otimes \mu_2)(x_1,x_2)\\ &= \int_{X_2}\Big(\int_{X_1} F(x_1,x_2)\; d\mu_1(x_1)\Big)\, d\mu_2(x_2) = 0,\end{aligned}$$

since $F(-,y)\colon X \to \mathbb{K}$ vanishes outside U for each y and U is a μ-null set. Thus the claim is established. In particular, this applies to $U = X_1 \setminus \operatorname{supp}(\mu_1)$. So

$$X_1 \times X_2 \setminus \operatorname{supp}(\mu_1) \times \operatorname{supp}(\mu_2) = \big(\big(X_1 \setminus \operatorname{supp}(\mu_1)\big) \times X_2\big) \cup \big(X_1 \times \big(X_2 \setminus \operatorname{supp}(\mu_2)\big)\big)$$

is an open $\mu_1 \otimes \mu_2$-null set and thus $S \stackrel{\text{def}}{=} \operatorname{supp}(\mu_1 \otimes \mu_2) \subseteq \operatorname{supp}(\mu_1) \times \operatorname{supp}(\mu_2)$. By way of contradiction, suppose the containment is proper and pick $x_k \in \operatorname{supp}(\mu_k)$, $k = 1,2$ with $(x_1,x_2) \notin S$. Find $f_k \geq 0$ in $C(X_k,\mathbb{K})$ with $f_k(x_k) > 0$ such that $\operatorname{supp}(f_1 \otimes f_2)$ does not meet the closed set S. Then $\int f_k d\mu_k > 0$, $k = 1,2$ by Proposition A5.4. Thus $\int (f_1 \otimes f_2)\, d(\mu_1 \otimes \mu_2) = \langle \mu_1 \otimes \mu_2, f_1 \otimes f_2\rangle = \langle \mu_1, f_1\rangle\langle \mu_2, f_2\rangle > 0$ on the one hand, and $\int (f_1 \otimes f_2)\, d(\mu_1 \otimes \mu_2) = 0$ on the other, since $\operatorname{supp}(f_1 \otimes f_2) \cap S = \emptyset$. This contradiction shows that $S = \operatorname{supp}(\mu_1) \times \operatorname{supp}(\mu_2)$ as asserted. ☐

Measures on Compact Groups: Convolution

For the following discussion we fix a compact group G. Let $m\colon G \times G \to G$ denote the multiplication of G. Then we have the operator

$$M(m)\colon M(G \times G) \to M(G).$$

For $\mu_1, \mu_2 \in M(G)$ we set $\mu_1 * \mu_2 = M(m)(\mu_1 \otimes \mu_2)$,

$$\begin{aligned}\langle \mu_1 * \mu_2, f\rangle = \langle \mu_1 \otimes \mu_2, C(m)(f)\rangle &= \int_{G\times G} f(g_1g_2)\, d(\mu_1 \otimes \mu_2)(g_1,g_2)\\ &= \int_G \Big(\int_G f(gh)\, d\mu_1(g)\Big)\, d\mu_2(h)\\ &= \int_G \Big(\int_G {}_h f\, d\mu_1\Big)\, d\mu_2(h).\end{aligned}$$

The product $\mu_1 * \mu_2$ is called the *convolution* of the measures μ_1 and μ_2.

For $f \in C(G,\mathbb{K})$ and $\mu \in M(G,\mathbb{K})$, the function $g \mapsto \mu({}_g f)\colon G \to \mathbb{K}$ is in $C(G,\mathbb{K})$.

Exercise EA5.2. Prove directly that $g \mapsto \int_G f(xg)\,d\mu(x): G \to \mathbb{K}$ is continuous.

[Hint. We do this for $\mathbb{K} = \mathbb{R}$ and derive the assertion for $\mathbb{K} = \mathbb{C}$ from this result. Note that ${}_gf$ is *uniformly* continuous on G. As G is a compact group, this means that, given $\varepsilon > 0$ there is an identity neighborhood U of G such that $|{}_gf(xu) - {}_gf(x)| < \varepsilon$ for all $u \in U$. Write the inequality in the form ${}_gf(x) - \varepsilon < {}_gf(xu) < {}_gf(x) + \varepsilon$. The positivity of μ yields

$$\mu({}_gf) - \varepsilon\cdot\mu(1) < \mu({}_gf) < \int_G f(xgu)\,d\mu(x) < \mu({}_gf) + \varepsilon\cdot\mu(1).$$

It is not hard to complete the proof.] □

We recall that $M(G,\mathbb{K})$ has a natural norm, the *dual norm*, that is, the *operator norm* on continuous functionals on $C(G,\mathbb{K})$ defined by $\|\mu\| = \sup_{\|f\|\le 1} \|\mu(f)\|$. If G is a compact group, then inversion $\sigma: G \to G$, $\sigma(g) = g^{-1}$ is an involution that is, satisfies $\sigma(gh) = \sigma(h)\sigma(g)$ and $\sigma^2 = \mathrm{id}_G$. We shall abbreviate $M(\sigma)(\mu)$ by μ^*.

Lemma A5.6. *Let G be a compact semigroup. Then convolution $(\mu, \nu) \mapsto \mu * \nu$ makes $M(G,\mathbb{K})$ into a Banach algebra with respect to the dual norm on $M(G,\mathbb{K})$. If G is a group, then $\mu^{**} = \mu$ and $(\mu * \nu)^* = \nu^* * \mu^*$, that is, $M(G,\mathbb{K})$ is an involutive Banach algebra.*

Proof. This is the topic of the following exercise. □

Exercise EA5.3. Prove Lemma A5.6.

[Hint. Show the bilinearity and associativity of $*$ and verify $\|\mu * \nu\| \le \|\mu\|\cdot\|\nu\|$. Check the condition on the involution in the group case.] □

Lemma A5.7.

(1) *For positive measures μ, $\nu \in M(G,\mathbb{K})$ one has*

(†) $$\mathrm{supp}(\mu * \nu) = \mathrm{supp}(\mu)\mathrm{supp}(\nu).$$

(2) *The support of an idempotent probability measure is a compact subsemigroup of G.*

Proof. (1) The (semi)group multiplication $m: G \times G \to G$ induces an operator

$$M(m): M(G \times G,\mathbb{K}) \to M(G,\mathbb{K}) \quad \text{satisfying} \quad M(m)(\mu \otimes \nu) = \mu * \nu.$$

Proposition A5.3 (3) implies $\mathrm{supp}(\mu * \nu) = \mathrm{supp}\big(M(m)(\mu \otimes \nu)\big) = m\big(\mathrm{supp}(\mu \otimes \nu)\big)$; by Proposition A5.5 $(**)$ we have $\mathrm{supp}(\mu \otimes \nu) = \mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$. Thus $\mathrm{supp}(\mu * \nu) = m\big(\mathrm{supp}(\mu) \times \mathrm{supp}(\nu)\big) = \mathrm{supp}(\mu)\,\mathrm{supp}(\nu)$.

(2) This is immediate from (1). □

There are two significant topologies on $M(G,\mathbb{K})$: firstly, the dual norm topology, endowing $M(G,\mathbb{K})$ with the structure of a Banach algebra, and, secondly, the topology of pointwise convergence of functionals, that is, the topology induced from the inclusion $M(G,\mathbb{K}) \subseteq \mathbb{K}^{C(G,\mathbb{K})}$. This latter one is called the *weak $*$-topology*, endowing $M(G,\mathbb{K})$ with a locally convex algebra topology. It is the weak $*$-topology we are interested in here.

Recall that a measure $\mu \in M(G,\mathbb{K})$ is called a *probability measure* if it is positive and normalized. The topological space of all probability measures equipped with the weak $*$-topology will be denoted $P(G) \subseteq M(G,\mathbb{K})$.

Lemma A5.8. *For a compact semigroup G, endowed with the weak $*$-topology, the space $P(G)$ has the following properties.*

(1) *$P(G)$ is a compact and convex subset of $M(G,\mathbb{R})$.*
(2) *$P(G)$ is a compact topological semigroup with respect to convolution.*
(3) *For $\mu_1, \mu_2, \nu \in P(G)$ and $0 \le r_1, r_2$, $r_1 + r_2 = 1$ we have*

$$\begin{aligned} \nu * (r_1{\cdot}\mu_1 + r_2{\cdot}\mu_2) &= r_1{\cdot}(\nu * \mu_1) + r_2{\cdot}(\nu * \mu_2),\\ (r_1{\cdot}\mu_1 + r_2{\cdot}\mu_2) * \nu &= r_1{\cdot}(\mu_1 * \nu) + r_2{\cdot}(\mu_2 * \nu). \end{aligned}$$

(4) *If G is a group, the involution $\mu \mapsto \mu^*$ leaves $P(G)$ invariant.*

Proof. For (1) we have to show compactness, convexity, and for (2) that $P(G)$ is closed under convolution, and that the multiplication

$$(\mu, \nu) \mapsto \mu * \nu : P(G) \times P(G) \to P(G)$$

is weak $*$–continuous. Assertion (3) is due to the fact that multiplication on $P(G)$ is the restriction of an algebra multiplication to a convex subset. Claim (4) is straightforward. The details are left to the following exercise. □

Exercise EA5.4. Provide the missing details of the proof of Lemma A5.8.

[Hint. (i) Show convexity directly. Show that $P(G)$ is weak-$*$-closed and bounded; then apply the Theorem of Bourbaki-Banach-Alaoglu to prove compactness.

(ii) For proving continuity of a function $\alpha\colon X \to P(G)$, recall that this amounts to showing that the functions $\alpha \mapsto \langle \alpha(x), f\rangle : X \to \mathbb{K}$ are continuous for all $f \in C(G,\mathbb{K})$.

(iii) In a real algebra, multiplication is linear in each argument separately and this implies the assertion.] □

By Definition, an *affine semigroup S* is a topological semigroup satisfying

$$u(r_1{\cdot}v + r_2{\cdot}w) = r_1{\cdot}uv + r_2{\cdot}uw \quad \text{ and } \quad (r_1{\cdot}u + r_2{\cdot}v)w = r_1{\cdot}uw + r_2{\cdot}vw,$$

for all $u, v, w \in S$ and real numbers $0 \le r_1, r_2$, $r_1 + r_2 = 1$.

Also by Definition, an *involutive semigroup S* has an involution $s \mapsto s^*$ satisfying $s^{**} = s$ and $(st)^* = t^*s^*$. Accordingly, we can say that

Corollary A5.9. *For any compact group G, the space $P(G)$ of probability measures on G is a locally convex, affine, compact involutive semigroup.* □

Recall that the *point-mass* δ_g concentrated at $g \in G$ is the probability measure defined by $\delta_g(f) = f(g)$. The function $g \mapsto \delta_g : G \to P(G)$ is an injective morphism of compact topological semigroups. The element $\delta_{\mathbf{1}}$ is the identity of $P(G)$.

Exercise EA5.5.
(1) Discover the group $H\big(P(G)\big)$ of elements of $P(G)$ which are invertible with respect to δ_1.
(2) Prove that $\mu_g = \mu * \delta_g$.

[Hint. Let $\mu, \nu \in P(G)$ be such that $\mu * \nu = \delta_1$. Then $\operatorname{supp}(\mu)\operatorname{supp}(\nu) = \operatorname{supp}(\delta_1) = 1$. This implies that μ and ν are Dirac measures.
The remainder is straightforward.] □

Semigroup Theoretical Characterization of Haar Measure

Recall from the introduction to the section on the definition of Haar measure that $\langle \mu_g, f\rangle = \langle \mu, {}_g f\rangle$; likewise we now define $\langle {}_g\mu, f\rangle = \langle \mu, f_g\rangle$ for ${}_g f(x) = f(xg)$ and $f_g(x) = f(gx)$ and call $\gamma_g = \gamma$ the *right invariance* and ${}_g\gamma = \gamma$ the *left invariance* of γ. We observe that

$$(+) \qquad \mu_g = \mu * \delta_g \quad \text{and} \quad {}_g\mu = \delta_g * \mu.$$

Proposition A5.10. *For a probability measure $\gamma \in P(G)$ the following statements are equivalent:*
(1) *γ is a left- and right-invariant measure of G.*
(2) *$\gamma * \mu = \mu * \gamma = \gamma$ for all $\mu \in P(G)$, that is, γ is the zero element of the compact semigroup $P(G)$.*

Moreover, if these conditions are satisfied, then $\operatorname{supp}(\gamma) = G$.

Proof. This is the subject of the following exercise. □

Exercise EA5.6. Prove (+) and Proposition A5.10.

[Hint. (+) is straightforward, e.g.

$$\langle \mu * \delta_g, f\rangle = \int_G \int_G f(xy)\, d\delta_g(y) d\mu(x) = \int_G f(xg)\, d\mu(x) = \langle \mu, {}_g f\rangle = \langle \mu_g, f\rangle.$$

(1)⇒(2). By definition of the convolution, $(\gamma * \mu)(f) = \int_G \gamma({}_g f)\, d\mu(g)$. Now $\gamma({}_g f) = \gamma(f)$ by right invariance of γ. Proceed.

Next $(\mu * \gamma)(f) = \int_G \mu({}_g f)\, d\gamma(f) = \int_G \big(\int_G f(xg)\, d\mu(x)\big)\, d\gamma(g)$. By the Fubini Theorem we can invert the order of integration:

$$(\mu * \gamma)(f) = \int_G \left(\int_G f(xg)\, d\gamma(g)\right) d\mu(x).$$

We have $\int_G f(xg)\, d\gamma(g) = \gamma(f)$. So $\mu * \gamma = \gamma$ follows.

(2)⇒(1). Via (+) this is straightforward by taking $\mu = \delta_g$.

Moreover, assume (2) satisfied. Then from Proposition A5.10 we derive

$$g\cdot \operatorname{supp}(\gamma) = \operatorname{supp}(\delta_g * \gamma) = \operatorname{supp}(\gamma)$$

and since G acts transitively under left translation and $\operatorname{supp}(\gamma) \neq \emptyset$, we have $\operatorname{supp}(\gamma) = G$.] ☐

Proposition A5.10 shows that there is at most one probability measure which is both left and right invariant, since a zero of a semigroup is unique. If an arbitrary compact space X is endowed with the semigroup multiplication $gh = g$ (the so-called *left zero multiplication*) then *every* measure is right invariant, but if X has at least two elements, a point measure δ_g is not left invariant. In fact, the proof of Proposition A5.10 shows that a right invariant measure on G is a left zero of $P(G)$.

After Proposition A5.10 the Existence and Uniqueness of a two-sided-invariant probability measure of a compact group is equivalent to the assertion that

for a compact group G the compact topological semigroup $P(G)$ has a zero.

In particular, Haar measure is an idempotent in $P(G)$, if it exists. The element $\delta_{\mathbf{1}}$ is an idempotent. One needs to understand the idempotents of $P(G)$.

Idempotent Probability Measures on a Compact Group

We now return to our study of measures on a compact group.

Proposition A5.11. *Assume that μ is an idempotent probability measure on a compact group G. Then the following conclusions hold:*

(1) *The support* $\operatorname{supp}(\mu)$ *is a closed subgroup of G.*
(2) *If $g \in \operatorname{supp}(\mu)$ then $\mu_g = \mu$, that is $\int {}_gf\,d\mu = \int f\,d\mu$ for all $f \in C(G,\mathbb{K})$.*
(3) *If $\nu \in P(G)$ and* $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\mu)$, *then $\mu * \nu = \nu * \mu = \mu$.*
(4) *$\mu^* = \mu$.*

Proof. (1) Lemma A5.7(2) and Proposition A4.34. imply the claim.

(2) It is sufficient to prove (2) for positive f. So we assume that f is positive. We must show that the function $F\colon G \to \mathbb{K}$ defined by $F(g) = \langle \mu, {}_gf\rangle$ is constant on the compact subgroup $\operatorname{supp}(\mu)$. Since supp is compact and F is continuous, there is an $m \in \operatorname{supp}(\mu)$ such that $F(m) = \max F(\operatorname{supp}(\mu))$. Then ${}_mF$ attains its maximum on $\operatorname{supp}(\mu)$ in the identity 1, and it is no loss of generality if we replace F by ${}_mF$ and assume now that F attains its maximum on $\operatorname{supp}(\mu)$ in 1. Thus $F(1) - F$ is a continuous function on G which is nonnegative on $\operatorname{supp}(\mu)$. Then Proposition A5.4(1) allows us to conclude $\langle \mu, F(1) - F\rangle \geq 0$, that is $\langle \mu, F\rangle \leq \langle \mu, F(1)\rangle = F(1)$. Now we calculate

$$\begin{aligned} F(1) &= \langle \mu, f\rangle = \langle \mu * \mu, f\rangle = \int_G \int_G f(xy)\,d\mu(x)d\mu(y) \\ &= \int_G \left(\int_G {}_yf(x)\,d\mu(x)\right) d\mu(y) = \int_G F(y)\,d\mu(y) = \langle \mu, F\rangle \leq F(1). \end{aligned}$$

Thus equality holds and therefore we have, for the continuous function $F(1)-F: G \to \mathbb{R}$, which takes nonnegative values on $\mathrm{supp}(\mu)$, the relation $\int_G (F(1)-F)\,d\mu = 0$. Now Proposition A5.4(2) implies $(F(1)-F)|\,\mathrm{supp}(\mu) \equiv 0$ and this proves that F is constant on $\mathrm{supp}(\mu)$.

(3) We compute $(\mu*\nu)(f) = \int_G \mu({}_gf)\,d\nu(g)$. Now $\mu({}_gf) = \mu(f)$ for $g \in \mathrm{supp}(\mu)$ and so certainly for $g \in \mathrm{supp}(\nu)$. So Proposition A5.4(3) implies $(\mu*\nu)(f) = \int_G \mu({}_gf)\,d\nu = \int_G \langle \mu, f\rangle\,d\nu = \mu(f)$. The proof of the relation $\nu * \mu = \mu$ is similar.

(4) By Proposition A5.3(3) applied with $\varphi(g) = g^{-1}$ we have $\mathrm{supp}(\mu) = \mathrm{supp}(\mu^*)$ and therefore $\mu * \mu^* = \mu$ by (3). But we also have $\mu^* * \mu^* = \mu^*$ and thus we may apply the results of (3) to μ^* and find $\mu * \mu^* = \mu^*$. Thus $\mu^* = \mu$ follows. □

Corollary A5.12. *$P(G)$ has a zero.*

Proof. From Proposition A5.11 we know that $P(G)$ is a compact affine involutive semigroup in which all idempotents are involutive. The assertion then follows from Corollary A4.40. □

In view of Proposition A5.10 this implies the existence and uniqueness of Haar measure on a compact group. Thus Theorem A5.2 is proved.

Actions and Product Measures

In the discussions leading to the Structure Theorem 9.102 on the probability that two elements commute in a compact group we need some measure theory beyond the necessities required to prove existence (and uniqueness) of Haar measure. The following discussions provide for this material.

Let $(g,x) \mapsto g{\cdot}x : G \times X \to X$ be a continuous action α of a compact group G on a compact space X (see Definition 1.9). As usual in this book all spaces in sight are assumed to be Hausdorff. We specify a Borel probability measure P on $G \times X$ and discuss the probability that a group element $g \in G$ fixes a phase space element $x \in X$ for a pair (g,x), randomly picked from $G \times X$, that is, that $g{\cdot}x = x$. We define

$$E \stackrel{\text{def}}{=} \{(g,x) \in G \times X : g{\cdot}x = x\},$$

that is, E is the equalizer of the two functions $\alpha, \mathrm{pr}_X : G \times X \to X$ and is therefore a closed subset of $G \times X$.

Let $G_x = \{g \in G : g{\cdot}x = x\}$ be the isotropy (or stability) group at x and let $X_g = \{x \in X : g{\cdot}x = x\}$ be the set of points fixed under the action of g. We note that $G_{g\cdot x} = gG_xg^{-1}$. The function $g \mapsto g{\cdot}x : G \to G{\cdot}x$ induces a continuous equivariant bijection $G/G_x \to G{\cdot}x$ which, due to the compactness of G, is a homeomorphism (see Proposition 1.10). We have

$$\begin{aligned} E &= \{(g,x) : g \in G_x,\, x \in X\} = \bigcup_{x\in X} G_x \times \{x\} \\ &= \{(g,x) : g \in G,\, x \in X_g\} = \bigcup_{g\in G} \{g\} \times X_g \subseteq G \times X\cdot \end{aligned}$$

Now we assume that μ and ν are Borel probability measures on G and X, respectively, and that $P = \mu \times \nu$ is the product measure. For information on measure theory the reader may refer to [37]. Let $\chi_E : G \times X \to \mathbb{R}$ be the characteristic function of E. We define the function $m : X \to \mathbb{R}$, $m(x) = \mu(G_x)$. Then by the Theorem of Fubini we compute

$$(*) \qquad \begin{aligned} P(E) &= \int_{G\times X} \chi_E(g,x) dP = \int_X \left(\int_G \chi_E(g,x) d\mu(g) \right) d\nu(x) \\ &= \int_X \mu(G_x) d\nu(x) = \int m\, d\nu. \end{aligned}$$

Likewise

$$(**) \qquad \begin{aligned} P(E) &= \int_{G\times X} \chi_E(g,x) dP = \int_G \left(\int_X \chi_E(g,x) d\nu(x) \right) d\mu(g) \\ &= \int_G \nu(X_g) d\mu(g). \end{aligned}$$

We now see from $(*)$ that $P(E) > 0$ implies, firstly, that there is at least one $x \in X$ such that $\mu(G_x) = m(x) > 0$ holds, and that, secondly, the set of all x for which $m(x) > 0$ has positive ν-measure. Likewise, there is at least one $g \in G$ such that $\nu(X_g) > 0$ and that the set of all of these g has positive μ-measure. At this point we introduce a terminology which we shall retain and use.

Definition A5.13. Let $\mathcal{C}$ be a set of subgroups of a compact group G, such as the set of all closed subgroups or all subgroups whose underlying set is a Borel subset of G. We shall say that a Borel probability measure σ on G *respects $\mathcal{C}$-subgroups* if every subgroup $H \in \mathcal{C}$ with $\sigma(H) > 0$ is open. □

Recall that an open subgroup H of a topological group G, being the complement of all the cosets gH for $g \notin H$, is closed and that it contains the *identity component* G_0 of G. If G is compact, then H has finite index in G.

We claim that

Haar measure μ on a compact group G respects Borel subgroups.

Indeed, assume that H is a Borel subgroup of G with $\mu(H) > 0$. Then by [147], p. 296, Corollary 20.17, $H = HH$ contains a nonvoid open set and thus is open.

If, in the cirumstances discussed here, μ respects closed subgroups, and $\mu(G_x) > 0$ then the subgroup G_x is open in G and hence contains G_0. If G is a compact Lie group, then the condition $G_0 \subseteq G_x$ is also sufficient for the openness of the subgroup G_x of G, since the identity component of a Lie group is open. If G_x is open, then $G{\cdot}x \cong G/G_x$ is discrete and compact, hence finite.

Let now

$$(\dagger)\qquad F \stackrel{\text{def}}{=} \{x \in X : |G{\cdot}x| < \infty\} = \{x \in X : G_x \text{ is open}\}.$$

These remarks require that henceforth whenever the measure μ occurs we shall assume that μ respects closed subgroups.

Lemma A5.14. *Let G be a compact group acting on a compact space X.*

(i) *If G is a Lie group, then for each $x \in X$ there is an open invariant neighborhood U_x of x such that all isotropy groups of elements in U are conjugate to a subgroup of the isotropy subgroup G_x.*

(ii) *Under these circumstances, m takes its maximum on U_x in x. That is, $m^{-1}(]-\infty, m(x)])$ is a neighborhood of x. In particular, m is upper semicontinuous.*

(iii) *If G is a Lie group, then the subspace F of X is compact.*

(iv) *If G is an arbitrary compact group, then the subspace F of X is an F_σ, that is, a countable union of closed subsets and thus is a Borel subset.*

(v) *There is a compact commutative monoid X with compact group G of invertible elements such that for the action of G on X by multiplication, the set $F = X \setminus G$ is a nonclosed F_σ.*

Proof. Assertion (i) is a consequence of the Tube Existence Theorem (see e.g. [48], p. 86, Theorem 5.4, or [356], p. 40, Theorem 5.7).

(ii) Immediate from (i) and from $m(u) = \mu(G_u)$.

(iii) We recall that $y \in F$ if and only if $m(y) > 0$. Hence F is the complement of $m^{-1}(0)$. By conclusion (ii), however, $m^{-1}(0)$ is open. Thus F is closed and therefore compact.

(iv) For a natural number $n \in \mathbb{N}$ we set $F(n) = \{x \in X : |G{\cdot}x| \le n\}$. We claim that $F(n)$ is closed in X for all $n \in \mathbb{N}$. Since $F = \bigcup_{n=1}^{\infty} F(n)$, this claim will prove assertion (iv).

We prove the claim by contradiction and suppose that there is an $n \in \mathbb{N}$ such that $\overline{F(n)}$ contains an $x' \notin F(n)$. Then there exist elements $g_1, \dots, g_{n+1} \in G$ such that $|\{g_1{\cdot}x', \dots, g_{n+1}{\cdot}x'\}| = n+1$. Now we find a compact normal subgroup N of G such that G/N is a Lie group and that $Ng_j{\cdot}x' \cap Ng_k{\cdot}x' = \emptyset$ for all $j \ne k$ in $\{1, \dots, n+1\}$.

The Lie group G/N acts on $X/N = \{N{\cdot}x : x \in X\}$ via $(gN)\circ(N{\cdot}x) = N{\cdot}(g{\cdot}x)$. By what we just saw $|(G/N)\circ(N{\cdot}x_0)| \ge n+1$. On the other hand, $F_N(n) = \{N{\cdot}x \in X/N : |(G/N)\circ(N{\cdot}x)| \le n\}$ is closed by (ii) above. Since the orbit map $\pi_N\colon X \to X/N$ is continuous and $\pi(F(n)) \subseteq F_N(n)$ we have $N{\cdot}x_0 = \pi(x_0) \subseteq \overline{F_N(n)} = F_N(n)$. Thus, by the definition of $F_N(n)$ we have $|(G/N)\circ(N{\cdot}x_0)| \le n$. This contradiction proves the claim.

(v) Let $\mathbb{I}$ denote the unit interval $[0,1]$ under ordinary multiplication, a compact connected topological monoid. Set $\mathbb{I}_0 = [0, 1-1/p]$, $\mathbb{I}_1 =]1-1/p, 1-1/p^2]$,

$\ldots,\quad \mathbb{I}_n =]1-1/p^n, 1-1/p^{n+1}], \ldots, \mathbb{I}_\infty = \{1\}$. We form the compact topological product monoid $S = \mathbb{I} \times \mathbb{Z}_p$ for the additive group $\mathbb{Z}_p$ of p-adic integers. For $r \in \mathbb{I}$ set

$$J_r = \begin{cases} \mathbb{Z}_p, & \text{if } r \in \mathbb{I}_0, \\ p^n\mathbb{Z}, & \text{if } r \in \mathbb{I}_n,\ n = 1, 2, \ldots, \\ \{0\}, & \text{if } \in \mathbb{I}_\infty. \end{cases}$$

The binary relation R, whose cosets are $R(t, z) = \{t\} \times (z + J_t)$ is a closed congruence relation. Therefore, $X \stackrel{\text{def}}{=} S/R$ is a compact connected abelian monoid with zero $R(0, 0)$ whose group of units is $G \stackrel{\text{def}}{=} (\{1\} \times \mathbb{Z}_p)/R \cong \mathbb{Z}_p$. For $t < 1$ in $\mathbb{I}$ and $x = R(t, z)$ we have $G{\cdot}x = \big(\{t\} \times (z + \mathbb{Z}_p/I_p)\big) \cong \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ for $t \in \mathbb{I}_n$, $n = 0, 1, \ldots$.

In particular, considering X as a G-space under multiplication, the space of finite orbits F is $([0, 1[\times G)/R$ is not closed in X. □

Statement A5.14(v) shows that A5.14(iv) cannot be improved to read that F is closed.

For $p = 2$, the space X is the standard binary tree with G as the Cantor set of leaves. Compact monoids like X above were considered in [196] rather generally under the name *cylindrical semigroups*; for our construction see in particular D-2.3.3ff on p. 241.

Recall that we assume that μ respects closed subgroups. Now that we know that $F \subseteq X$ is a Borel set, hence is ν-measurable, we can state that, regardless of any particular property of ν, the function m satisfies

$$P(E) = \int_X m d\nu = \int_X \chi_H{\cdot}m d\nu = \int_{x \in F} m(x) d\nu(x).$$

Here $x \in F$ implies $0 < m(x) = \mu(G_x) = 1/|G/G_x| \leq 1$.

Lemma A5.15. *If μ respects closed subgroups and $P(E) > 0$, then $0 < P(E) \leq \nu(F)$. In particular, $F \neq \emptyset$.*

Proof. We have seen that $P(E) = \int_F m\, d\nu$ since F is Borel measurable. The Lemma then follows from this fact and $m(x) \leq 1$ for $x \in F$. □

We shall say that the group G *acts automorphically* on X if X is a compact group and $x \mapsto g{\cdot}x : X \to X$ is an automorphism for all $g \in G$.

Lemma A5.16. *Assume that G and X are compact groups and assume the following hypotheses:*

(a) *G acts automorphically on X.*
(b) *μ respects closed subgroups.*
(c) *ν respects Borel subgroups or else X is a Lie group and ν respects closed subgroups.*
(d) *$P(E) > 0$.*

Then F is an open, hence closed subgroup of X.

Proof. Let $x, y \in F$. Then $G{\cdot}x$ and $G{\cdot}y$ are finite sets by the definition of F. Now $G(xy^{-1}) = \{g{\cdot}(xy^{-1}) : g \in G\} = \{(g{\cdot}x)(g{\cdot}y)^{-1} : g \in G\} \subseteq \{(g{\cdot}x)(h{\cdot}y)^{-1} : g, h \in G\} = (G{\cdot}x)(G{\cdot}y)^{-1}$, and the last set is finite as a product of two finite sets. Thus $xy^{-1} \in F$ and F is a subgroup. By Lemma A5.15, $\nu(F) > 0$. Then by Lemma A5.14(iii),(iv) and the kind of subgroups respected by ν, we conclude that F is an open subgroup. □

If G acts automorphically on a compact group X, we let $\pi: G \to \mathrm{Aut}\ X$ be the representation given by $\pi(g)(x) = g{\cdot}x$. Let id_X denote the identity function of X. Then the fixed point set X_g is the equalizer of the morphisms $\pi(g)$ and id_G and is therefore a closed subgroup of X. Let $I \subseteq G$ denote the set of all $g \in G$ for which X_g has inner points.

Lemma A5.17. *Assume that G and X are compact groups and assume the following hypotheses:*

(a) *μ and ν are the Haar measures on G and X, respectively.*

(b) *G acts automorphically on X.*

(c) *G is finite.*

Then $P(E) = \frac{1}{|G|}\cdot\sum_{g\in I} |X/X_g|^{-1}$. In particular, $P(E)$ is a rational number.

Proof. By (∗∗) preceding Definition A5.13 and the fact that Haar measure on a finite group G is counting measure with $\mu(\{g\}) = |G|^{-1}$, we have $P(E) = \frac{1}{|G|}\cdot\sum_{g\in G}\nu(X_g)$. If a closed subgroup Y of the compact group X has no inner points, its Haar measure $\nu(Y)$ is zero. If it has inner points, it is open and its measure $\nu(Y)$ is the reciprocal of its index, that is $\nu(Y) = |X/Y|^{-1}$. Hence $P(E) = \frac{1}{|G|}\cdot\sum_{g\in I} |X/X_g|^{-1}$ and the assertion follows. □

Our conclusions sum up to the following result:

Proposition A5.18. *Let G and X be compact groups and assume that G acts automorphically on X. Let μ and ν be normalized positive Borel measures on G and X, respectively. Define*

$$E = \{(g, x) \in G \times X : g{\cdot}x = x\} \subseteq G \times X,$$

whence $(\mu \times \nu)(E) = \int_{x\in F} \mu(G_x)d\nu(x)$. Assume that μ respects closed subgroups and that at least one of the following conditions is satisfied

(α) *ν respects Borel subgroups of X,*

(β) *X is a Lie group and ν respects closed subgroups of X,*

then the following statements are equivalent:

(1) $(\mu \times \nu)(E) > 0$.

(2) *The subgroup $F \leq X$ of all elements with finite G-orbits is open and thus has finite index in X.*

Proof. The proof follows from the preceding discourse. □

The main application of this general situation will be the case of a compact group G and the automorphic action of G on $X = G$ via inner automorphisms:

$$(g, x) \mapsto g{\cdot}x = gxg^{-1} : G \times X \to X.$$

The orbit $G{\cdot}x$ of x is the conjugacy class $C(x)$ of x, and the isotropy group G_x of the action at x is the centralizer $Z(x, G) = \{g \in G : gx = xg\}$ of x in G. The set E is the set $D = \{(x, y) \in G \times G : [x, y] = 1\}$, and F is the union of all finite conjugacy classes. In particular, F is a characteristic F_σ subgroup of G whose elements have finite conjugacy classes, that is, the FC-center of G. Recall that a group agreeing with its FC-center is called an FC-group .

Corollary A5.19. *Let G be a compact group and let μ and ν be Borel probability measures on G and assume that μ respects closed subgroups and that ν respects Borel subgroups or, if G is a Lie group, that ν respects closed subgroups.*

Let F be the FC*-center of G. Then F is an F_σ and we define*

$$D = \{(g, x) \in G \times G : [g, x] = 1\} \subseteq G \times G.$$

Then

$$P(D) = \int_{x \in F} \mu(Z(x, G)) d\nu(x),$$

and the following statements are equivalent:

(1) $(\mu \times \nu)(D) > 0$.

(2) *F is open in G and thus has finite index in G.*

Proof. This is a consequence of Proposition A5.18. □

Moreover, under these conditions, $Z(F, G)$ contains the identity component G_0, and the profinite group $G/Z(F, G)$ is acting effectively on F with orbits being exactly the finite conjugacy classes of G.

In the Structure Theorem 9.102 on the probability that two elements commute in a compact group, we show that the center $Z(F)$ is an open subgroup, whence $Z(F, G)$, containing $Z(F)$, is an open subgroup of G. Therefore $G/Z(F, G)$ in fact turns out to be finite.

Nonmeasurable Subgroups of Compact Groups

Hewitt and Ross provided in 1963 an instructive and far-reaching discussion of the topic of subsets of a (locally) compact group which are not measurable with respect to Haar measure in [147], pp. 226ff. However, more than twenty years later, in 1985, S. Saeki and K. Stromberg published the following question in [310]:

Question 1. *Does every infinite compact group necessarily have a subgroup which is not Haar measurable?.*

For compact abelian groups the answer is affirmative, as Comfort et al have shown in [74]. Yet in general this question is still not completely answered and continues to interest authors in the field after an article by Hernández and the authors [145] in 2016. It looks like another case for which the structure theory of compact groups reaches into the domain of set theory and logic. We saw such situations in the contexts of the undecidability of the Torus Proposition (see Theorem 8.48), or in the environment of the question whether the arc component of the identity in a compact abelian group is a Borel set (see Theorem 8.94 ff. and Theorem 9.60 ff.).

In any compact totally disconnected, that is, profinite, group any open, hence closed, subgroup has finite index. In Corollary 1.3 of [144] we show that every infinite power $G = K^X$ for a profinite group K has nonclosed subgroups of finite index.

In the following Definition we single out a rather small class of compact groups:

Definition A5,20. Let G be a topological group and G' its (algebraic) commutator subgroup. Then G is called an HHM-*group* if it satisfies the following conditions:

(a) *G is profinite.*
(b) *The only subgroups of G of finite index are open.*
(c) *G' has finite index in G.*

A result of M. G. Smith and J. S. Wilson [336] says the following:

Proposition A5.21. *A profinite group satisfies* (b) *if and only if there are only countably many finite index subgroups.* ☐

Since in any profinite group the open subgroups form a basis of the filter of identity neighborhoods, this implies, in particular, that

(d) *any* HHM-*group is metric.*

Profinite groups satifying (b) are also called *strongly complete* (see [298], Section 4.2, pp. 124ff.) Nikolov and Segal [280] showed that all topologically finitely generated metric profinite groups are strongly complete, as had been conjectured by Serre.

Typical examples in this class of groups are countable products of pairwise nonisomorphic simple finite groups. A result of Saxl and Wilson [316] says:

Proposition A5.22. *Let $\{G_n : n \in \mathbb{N}\}$ be a sequence of finite simple nonabelian groups and $G = \prod_{n\in\mathbb{N}} G_n$. Then condition (b) holds if and only if there are not infinitely many of the G_n which are isomorphic.* ☐

A partial but not fully satisfactory answer to Question 1 now is the following:

Theorem A5.23. [145] *Any compact group in which all subgroups are measurable is an* HHM-*group.*

Therefore Question 1 may be rephrased as follows:

Question 2. *Does every* HHM-*group contain a nonmeasurable subgroup?*

In the contemplation of the quest for nonmeasurable subgroups of a compact group, the following is an instructive exercise:

Exercise EA5.2. *Let G be a compact group and H a subgroup.*

(a) *If H has countably infinite index, then H is nonmeasurable. In particular, H is not a Borel subset.*

(b) *If H is measurable, then either it has measure* 0*, or it has positive measure in which case it is open (thus having finite index).*

(c) *If H has finite index in G and is not closed, then H is nonmeasurable. In particular, a countable index subgroup H of G is either closed with finite index or is nonmeasurable.*

(d) *If H is nonmeasurable in G, then $\overline{H}$ is an open (and therefore finite index) subgroup of G.*

(e) *If H is a finite index subgroup, $G = H \cup g_1H \cup \cdots \cup g_nH$, then the largest normal subgroup $N = H \cap g_1Hg_1^{-1} \cap \cdots \cap g_nHg_n^{-1}$ has finite index in G.*

(f) *Assume that H is nonmeasurable and that N is the largest normal subgroup contained in the open subgroup $\overline{H}$. Then N is open and $H \cap N$ is dense in N and nonmeasurable in N.*

(g) *Assume that $f: G \to G_1$ is a surjective morphism of compact groups and that $H_1 \subseteq G_1$ is a nonmeasurable subgroup of countable index. Then $H \stackrel{\text{def}}{=} f^{-1}(H_1)$ is a nonmeasurable subgroup of G.*

[See [145], Proposition 1.1.]

Since Haar measure is a Borel measure, every Borel subset of G is measurable. In the present context it is natural that we should calculate some cardinal invariants which we had not computed in Chapter 12. So let $\mathcal{S}(G)$ denote the set of (not necessarily closed!) subgroups of G. We let again $\mathfrak{c} = 2^{\aleph_0}$ denote the cardinality of the continuum and let $\mathcal{B}(G)$ be the set of all Borel subsets of G.

Proposition A5.24. If X is an infinite second countable metric space, then

$$\operatorname{card}(\mathcal{B}(X)) \leq \mathfrak{c}.$$

Proof. In [34], Exercise 4 c), §6, Chap. 9 it is established that the cardinality of the set of Borel subsets of a metric second countable space is $\leq \mathfrak{c}$. □

Remark A5.25. For Haar measure μ on a compact metric group G, a subset X is measurable iff there are sets $B_1, B_2 \in \mathcal{B}(G)$ such that $B_1 \subseteq X \subseteq B_2$ such that $\mu(B_2 \setminus X) = 0 = \mu(X \setminus B_1)$. □

(See e.g. [307], 10.10; the argument given there is quite general.)

We now observe that an infinite compact metric group has as many subgroups as it has subsets.

Theorem A5.26. *Let G be an infinite metric compact group. Then*

$$\mathrm{card}(\mathcal{S}(G)) = 2^{\mathfrak{c}}.$$

Proof. By Zelmanov's Theorem 9.91a, G contains an infinite abelian subgroup A which we may assume to be closed. Then A is a compact metric abelian group. Therefore, if the assertion of the Theorem is true for abelian groups, then it is true in general. Thus the claim follows from Corollary 1.2 of [22]. □

Corollary A5.27. *Every infinite compact group has a subgroup which is not a Borel subset.*

Proof. By Proposition A5.24 and Theorem A5.26, every infinite compact *metric group* has more subgroups than it has Borel subgroups. □

By Theorem A5.23, every compact group which fails to be a metric profinite group has a subgroup which is nonmeasurable for Haar measure. Since all Borel subgroups are Haar measurable, none of these is a Borel subgroup.

Brian and Mislove show 2016 in [50] that

The assertion that every compact group has a nonmeasurable subgroup is consistent with ZFC.

Przezdzieck, Szewczak, and Tsaban, use in 2018 a consequence of the Continuum Hypothesis to derive a positive answer to Question 1, see [297].

Postscript

While we did not wish to disrupt the train of thought in the very early Chapters 1 and 2 by detouring into the technical details of Radon measure theory, it is neverthess true that Haaar measure on a compact group is basic for the structure theory of compact groups as an essential tool. Numerous proofs of the existence and uniqueness of Haar measure of compact groups (and indeed of locally compact group) are sprinkled all over the literature. The proof, of which we give a self-contained presentation here has the unique charm of being based on the topological algebra of compact monoids and gives additional insights into a compact affine monoid coming along with each compact group. It is, however, not suitable for dealing with locally compact groups by this very feature, as noncompact locally compact groups lack this accompaniment. Our presentation parallels that of [177].

The second part of this appendix provides some measure theoretical technicalities which in the text are used for a surprising structure theorem on compact groups, in which the probability that two randomly picked elements commute is nonzero (Theorem 9.102, due to Hofmann and Russo [202]).

The third part deals with a problem on compact groups and their Haar measure of 1985 [310], which was revitalized through a paper by Hernández and the authors [145] and which remains without a definitive solution. The challenge is the simply formulated question whether an infinite compact group necessarily contains a subgroup which is nonmeasurable with respect to the normalized Haar measure of the group whose existence was discussed in the first part of this Appendix. A definitive result describes a small class of metric profinite groups for which the question remains open (Theorem A5.25); for all other compact groups the existence of nonmeasurable subgroup is assured. If A_n denote the finite simple group of even permutations of $n \geq 5$ elements, the group

$$G = A_5 \times A_6 \times A_7 \times \cdots$$

is a typical member of the group for which a definitive answer to the question is not available. Beyond that situation authors have established results depending on the set theory one is allowed to use to assert an affirmative answer to the question [50, 297].

References for this Appendix—Additional Reading

[37], [38], [50], [147], [177], [202], [203], [280], [297].

Appendix 6

Well-Ordered Projective Limits, Supercompactness, and Compact Homeomorphism Groups

In this appendix we indicate an alternative projective limit approach to the structure theory of compact groups first proposed by L. S. Pontryagin in his book [295], Definition 42ff. We explain how it can be used to prove that every compact group is supercompact. Finally we record that homeomorphism groups can be compact groups only if they are profinite.

Well-ordered Lie chains

A total order is called a *well-order* if every nonempty subset has a minimum. Let G be a compact group and let $\mathfrak{N}(G)$ denote the complete lattice of all closed normal subgroups. Then $\mathfrak{N}(G)$ contains the filter basis $\mathcal{N}(G)$ of normal subgroups N such that G/N is a Lie group.

Definition A6.1. Let us say that a $\supseteq$-well-ordered chain $\mathfrak{C} \subseteq \mathfrak{N}(G)$ is a *Lie-chain* if the following conditions are satisfied:

(1) $\max \mathfrak{C} = G$,
(2) If $N \in \mathfrak{C}$ has a predecessor $M \in \mathfrak{C}$, then M/N is a Lie group,
(3) If $N \in \mathfrak{C}$ has no predecessor, then $N = \bigcap\{M \in \mathfrak{C} : N \subseteq M\}$,
(4) $\bigcap \mathfrak{C} = \{1\}$.

We say that $\mathfrak{C}$ is a *standard Lie chain* if, in addition, the following condition is satisfied where $w(G)$ denotes the weight of G:

(5) If G is not a Lie group, then the ordinal number of $\mathfrak{C}$ is $w(G)$. □

We recall that the class of cardinals is considered as a subclass of the class of ordinals and that ordinals are viewed as representing the set of preceding ordinals. Thus $\aleph_0$, in this spirit, is considered as the set of natural numbers (it is often called ω). The next ordinal $\omega + 1$ is not a cardinal. There is a first subsequent cardinal $\aleph_1$. The Continuum Hypothesis expresses the assumption that $\aleph_1 = 2^{\aleph_0}$, the cardinality of the continuum.

Let us say that a well-ordering of a set is *canonical* if its ordinal number is a cardinal. By the Well-Ordering Principle every set I can be well-ordered. The well-order can be chosen so that its ordinal number is $\operatorname{card} I$ and that all proper initial subintervals have smaller cardinality than $\operatorname{card} I$.

Example A6.2. Let $\{G_j : j \in J\}$ be an infinite family of nondegenerate compact Lie groups and set $G = \prod_{j\in J} G_j$. Let $\leq$ be any canonical well-ordering of I. For each $j \in I$ we set $N_k = \prod_{k\leq j} G_j$ where a product over a subset of I is identified with a partial product of G. For $j \in I$ let $f_j\colon N_j \to N_{j+1}$ for the successor $j+1$ of j be the projection onto the partial product with kernel $\ker f_j = G_j$ considered as a subgroup of N_j in the obvious way. Then $\mathfrak{M} \stackrel{\text{def}}{=} \{N_k : k \in I\}$ is a standard Lie chain. □

Exercise EA6.1. Verify the claim of Example A6.2.

[Hint. Observe Exercise EA4.3.] □

We shall call $\mathfrak{M}$ in Example A6.2 a *Lie chain associated with the product G and a canonical well-ordering $\leq$ of its index set I.*

Notice that if $\mathfrak{M}$ is a Lie chain in a compact group G and $M \supseteq M^*$ is a successive pair of of members of $\mathfrak{M}$, then there is a canonical morphism $p_M\colon G/M^* \to G/M$, $p_M(gM^*) = gM$ whose kernel is the Lie group M/M^*. Moreover, if $N \in \mathfrak{M}$ has no predecessor, then $G/N = \lim_{M\in\mathfrak{M},\, M\supset N} G/M$. In particular, for a standard Lie chain, $\mathfrak{C}$, we have.

$$G = \lim_{M\in\mathfrak{C}} G/M.$$

If $\mathfrak{M} = \{N_k : k \in I\}$ is the Lie chain associated with a product $G = \prod_{j\in I} G_j$ and a canonical well-ordering of I, then $G/N_k \cong \prod_{j<k} G_j$ and $\ker f_{j+1} = G_j$.

With the aid of the Axiom of Choice one obtains Pontryagin's theorem:

Proposition A6.3. *Every compact group has a standard Lie chain.*

Proof. Exercise AE6.2. □

Exercise AE6.2. Prove Proposition A6.3.

[Hint. See [295], Theorem 68. Here Pontryagin considers a basis $\mathcal{U}$ of neighborhoods of the identity of a given compact group G, where the cardinality of $\mathcal{U}$ is the weight $w(G)$ of G considered as an ordinal, selects a well-ordering $(U_\alpha)_{\alpha<w(G)}$, $\alpha \leq 0$ of $\mathcal{U}$, and picks in each $U_\alpha \in \mathcal{U}$ a normal subgroup M_α such that $M_\alpha \subseteq U_\alpha$ and that G/M_α is a Lie group. Then the chain of normal subgroups $N_\alpha = \bigcap_{\beta<\alpha} M_\beta$, $\alpha \geq 1$ is a canonical Lie chain.] □

Corollary A6.4. *Let G be a compact group and Z a central closed subgroup. Then G has a Lie chain $\mathfrak{C}$ containing Z. Moreover, unless G is a Lie group,* $\operatorname{card}\mathfrak{C} = \max\{w(G/Z), w(Z)\}$.

Proof. By Proposition A6.3 applied to G/Z, there is a Lie chain $\mathfrak{C}_1$ of G with $Z = \bigcap \mathfrak{C}_1$. Applying A6.3 again with Z in place of G we obtain a Lie chain $\mathfrak{C}_2$ inside Z with maximal element Z and $\bigcap \mathfrak{C}_2 = \{1\}$. Since Z is central, all subgroups of Z are normal in G. Then $\mathfrak{M} \stackrel{\rm def}{=} \mathfrak{C}_1 \cup \mathfrak{C}_2$ is the required Lie chain of G containing N and having $\{1\}$ as minimal element. We have $w(G) = \max\{w(G/Z), w(Z)\}$ and $\mathrm{card}(\mathfrak{C}_1) = w(G/Z)$, $\mathrm{card}(\mathfrak{C}_2) = w(Z)$. □

In the case of *connected* compact groups one can obtain more specific results.

Definition A6.5. A morphism $f: G \to H$ is a *projection morphism* if there is a morphism $g: H \to G$ such that $fg = \mathrm{id}_H$ and $\mathrm{im}\, g$ is normal. □

If we denote $\ker f = M$ and $\mathrm{im}\, g = N$ we have an isomorphism $\mu: M \times N \to G$, $\mu(m,n) = mn$ and a commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\mu} & G \\ {\scriptstyle \mathrm{pr}_N}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ N & \xrightarrow[f|N]{} & H \end{array}$$

in which the horizontal maps are isomorphisms.

For a Lie chain $\mathfrak{C}$ of a compact group and for $N \in \mathfrak{C}$ we let again $N^* \subseteq N$ denote the successor of N. The morphisms $G/N^* \to G/N$ for a Lie chain associated with a product and a well-ordering of its index set are all projection morphisms.

Definition A6.6. The Lie chain $\mathfrak{C}$ is called *special* if for each $N \in \mathfrak{C}$ the morphism $f_N: G/N^* \to G/N$ is a projection morphism whose kernel is either a centerfree simple Lie group, or a circle group, or a finite group. □

Example A6.7. Let $\{G_j : j \in I\}$ be a family of compact Lie groups such that each G_j is a centerfree simple compact Lie group or a circle group or a finite group. Then the Lie chain associated with the product $G = \prod_{j\in I} G_j$ and any canonical well-ordering of I is a standard special Lie chain. □

Exercise EA6.3. Verify the claim in Example A6.7. □

Recall that every strictly reductive group G according to Definition 9.88(ii) is of the type of Example A6.7.

Example A6.8. Let G be a profinite group. Then any standard Lie chain of G is special. □

This is trivial since the kernels of the connecting morphisms $f_N: G/N^* \to G/N$ for N in a Lie chain $\mathfrak{C}$ of G are finite.

Furthermore, recalling that a product of circle groups is called a torus, according to Exercise E9.16 following Corollary 9.79 we have

- *every connected compact group G has a totally disconnected central subgroup D such that G/D is a product of simple centerfree compact Lie groups and a torus, that is, a product of circles.*

Moreover, Proposition 12.10(iii) shows $w(G) = w(G/D)$. As was observed in the paragraph following the Definition A4.7 of the weight of a space, $w(D) \leq w(G)$.

With the aid of Corollary A6.4, the preceding statements yield without undue difficulties the following result.

Proposition A6.9. *Every compact connected group has a special Lie chain* $\mathfrak{M}$ *and* $\operatorname{card} \mathfrak{M} = w(G)$. □

In light of the Countable Layer Theorem 9.91 it is of interest to note in passing that every strictly reductive group has a special Lie chain as well.

Notice that the Lie chain whose existence is asserted in Proposition A6.9 will not be a standard Lie chain unless $D = \{1\}$.

Supercompactness

Definition A6.10. A Hausdorff space is called *supercompact* if it possesses a subbasis for the open sets such that every cover of the topological space from elements of the subbasis has a subcover with at most two elements. □

This concept was introduced by J. de Groot in 1967 [81]. Not every compact space is supercompact [20], but every compact metric space is supercompact [260]. Results of the kind of Proposition A6.9 were used in [234] to prove the following theorem, first announced by Mills [261] and belonging to the ambience of this book.

Theorem A6.11. (Supercompactness Theorem for Compact Groups) (Mills, Kubiś and Turek) *Every compact group is supercompact.*

To see this, note that the Dyadicity Theorem 10.40 says that

- *a compact group is homeomorphic to a product of a compact connected group and a Cantor Cube.*

Since products of supercompact spaces are supercompact, and a Cantor cube is supercompact, the Supercompactness Theorem follows if it is proved for compact connected groups.

Kubiś and Turek proved the following result:

Supercompactness Lemma. *Let X be the projective limit of a well-ordered inverse system*

$$\{f_{jk} : X_k \to X_k | (j,k) \in I \times I, j \leq k\}$$

of compact spaces and assume that

(i) X_0 *is supercompact,*

(ii) $(\forall j \in I)\, f_{j,j+1}\colon X_{j+1} \to X_j$ *is either a local homeomorphism, or else is equivalent to a product projection* $X_j \times S_j \to X_j$ *for some supercompact space* S_j*,*

(iii) *for all limit elements* $k \in (I, \leq)$ *we have* $X_k = \lim_{j<k} X_j$*.*

Then X is supercompact.

The Supercompactness Lemma and Proposition A6.9 yield the Supercompactness Theorem for Compact Groups.

Compact Homeomorphism Groups

For the remainder of this appendix, let $\mathcal{I}$ denote the group of all self-homeomorphisms of $\mathbb{I} = [0, 1]$ fixing the endpoints, endowed with the compact-open topology. One knows that this group is homeomorphic to $\ell^2(\mathbb{N})$, that is, separable Hilbert space. (For further references see e.g. [191].)

Recall that a *Tychonoff space* is a completely regular Hausdorff space.

Lemma A6.12. (Main Lemma) *Let G be a compact group acting on a Tychonoff space X. If there is at least one orbit which has nondegenerate connected subspaces, then the homeomorphism group $\mathcal{H}(X)$ has a closed subgroup G allowing a continuous morphism $p: G \to \mathcal{I}$ with a continuous cross section.* □

The proof of this principal lemma in [191] is technical and involved. We note that, under these circumstances, G is homeomorphic to $\ker p \times \mathcal{I}$. One proves easily the following:

Lemma A6.13. *A topological group cannot be locally compact if it has a closed subgroup G such that for a closed normal subgroup $N \subseteq G$ the quotient group G/N is isomorphic to $\mathcal{I}$.* □

Lemma A6.14. *If a G-space X has at least one orbit failing to be totally disconnected, then $\mathcal{H}(X)$ is not locally compact.*

Proof. This is a consequence of the preceding Lemmas A6.12 and Lemma A6.13. □

Now we obtain quickly

Theorem A6.15. *A compact homeomorphism group $\mathcal{H}(X)$ of a Tychonoff space X is profinite.*

Proof. Assume that $G \stackrel{\text{def}}{=} \mathcal{H}(X)$ is compact. Then X is a G-space to which Lemma A6.14 applies, and it follows that all G-orbits on X are totally disconnected. Since the functions $f \mapsto f(x) : G \to G{\cdot}x$, $x \in X$ separate the points, the group G itself is a totally disconnected compact group, that is, a profinite group. □

Postscript

It was Pontryagin's idea to represent a compact group in terms of a projective limit of a well-ordered inverse system of compact groups in which each successive member $G_{\alpha+1}$ arises as an extension of a Lie group L_α by the predecessor G_α in the form of $G_\alpha \cong G_{\alpha+1}/L_\alpha$ and in which G_α for a limit ordinal α is the projective limit of the system of all of its predecessors. Whenever a property of G is pulled up by transfinite induction across such a well-ordered system, such a presentation can be very useful as a sequence of recent applications shows ([5], [11], [120], [234]). Pontryagin used this idea to prove theorems of the type of Corollary 10.75.

In general, however, the structure theory of a compact group G has traditionally used projective limits over inverse systems which arise naturally from given data, such as the filter base $\mathcal{N}$ of normal subgroups N for which G/N is a Lie group. These have well-ordered bases only in the case of metric groups. This use of projective limits is reflected in this book from Chapter 1, Defininition 1.25 on.

The last portion of this appendix shows that a representation theory of compact groups in terms of groups of homeomorphisms on completely regular Hausdorff spaces is somewhat limited: A surjective representation $\pi\colon G \to \mathcal{H}(X)$ of a compact group G must have G_0 in its kernel. In [119], Gartside and Glyn show that every *metric* profinite group can indeed be represented as a homeomorphism group. Whether this remains true for arbitrary profinite groups is an open question.

References for this Appendix—Additional Reading

[11], [120], [20], [119], [81], [191], [234], [261], [295], [347].

Appendix 7

Weakly Complete Topological Vector Spaces

In this book, duality is certainly an important topic. Chapter 7 is devoted to Pontryagin duality of locally compact abelian groups and the structure theory of those groups. Chapter 3, Part 3 has a proof of Tannaka duality for compact groups. In this appendix, we record some very useful material on an elementary duality of vector spaces which will, however, be of considerable importance to us, especially in Chapter 3, Part 3.

If E is a real vector space, then E is isomorphic as a vector space to the restricted direct sum $\mathbb{R}^{(J)}$ of $\mathrm{card}(J)$ copies of $\mathbb{R}$. Now E has a natural topology, $\mathcal{O}(E)$, which is the finest topology such that $(E, \mathcal{O}(E))$ is a locally convex space. If we consider the topological dual E' of the topological vector space $(E, \mathcal{O}(E))$, then E' is isomorphic as a topological vector space to the space $\mathbb{R}^J$ with the Tychonoff product topology. Topological vector spaces of the type $\mathbb{R}^J$ are called weakly complete. We shall see that there is a duality between real vector spaces and weakly complete locally convex spaces.

The situation for complex vector spaces will be seen to be slightly different. While continuous real linear functionals correspond precisely to characters (continuous homomorphisms into the circle group $\mathbb{T}$), the same is not true for continuous complex valued linear functionals. We shall show that there are natural tensor products for vector spaces and for weakly complete locally convex spaces and the duality mentioned above extends to tensor products.

Also in this Appendix we shall introduce weakly complete $\mathbb{K}$-algebras and their duals which are coassociative $\mathbb{K}$-coalgebras. This allows us to state Cartier's Fundamental Theorem and to derive from it the significant portion of pro-Lie-group theory for weakly complete unital algebras.

Character Groups of Topological Vector Spaces

For topological vector spaces the study of vector space duals turned out to be eminently fruitful. We want to make the connection between character theory and vector space duality. A first step is the following observation:

Proposition A7.1. (i) *Assume that E_1 and E_2 are $\mathbb{R}$-vector spaces such that the underlying additive groups are topological groups and that for each $v \in E_j$, $j = 1, 2$, the function $r \mapsto r{\cdot}v: \mathbb{R} \to E_j$ is continuous. Then every morphism $f: E_1 \to E_2$ of abelian topological groups is linear.*

(ii) *Let E be a real topological vector space and $E' = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ the space of all continuous linear forms $E \to \mathbb{R}$ endowed with the compact open topology. Then $E' = \mathrm{Hom}(E, \mathbb{R})$* (*in the sense of topological* Hom-*groups*)*, and if $q: \mathbb{R} \to \mathbb{T}$ is the* *quotient morphism, then* $\mathrm{Hom}(E,q)\colon E' = \mathrm{Hom}(E,\mathbb{R}) \to \mathrm{Hom}(E,\mathbb{T}) = \widehat{E}$ *is an isomorphism of topological vector spaces.*

Proof. (i) Let $f\colon E_1 \to E_2$ be additive. If $m \in \mathbb{N}$, $n \in \mathbb{Z}$, then $m{\cdot}f(\frac{n}{m}{\cdot}v) = f(n{\cdot}v) = n{\cdot}f(v)$, whence $f(\frac{n}{m}{\cdot}v) = \frac{n}{m}{\cdot}f(v)$. Thus f is $\mathbb{Q}$-linear, i.e. $r{\cdot}f(v) = f(r{\cdot}v)$ for $r \in \mathbb{Q}$. By the continuity of all $r \mapsto r{\cdot}v$ and the continuity of f we get the desired $\mathbb{R}$-linearity.

(ii) Each continuous linear form $E \to \mathbb{R}$ is trivially a member of $\mathrm{Hom}(E,\mathbb{R})$. Conversely, every member f of $\mathrm{Hom}(E,\mathbb{R})$ is $\mathbb{R}$-linear by (i). It follows that $E' = \mathrm{Hom}(E,\mathbb{R})$. Now $\mathrm{Hom}(E,q)\colon E' \to \widehat{E}$ is a morphism of topological groups as is readily checked.

The additive topological group of E as that of a real topological vector space is simply connected (see for instance Definition A2.6, Proposition A2.9, Proposition A2.10(i)). Hence every character $\chi\colon E \to \mathbb{T}$ has a unique lifting $\widetilde{\chi}\colon E \to \mathbb{R}$ such that

$$\mathrm{Hom}(E,q)(\widetilde{\chi}) = q \circ \widetilde{\chi} = \chi$$

(see for instance Appendix 2, A2.32). Thus $\chi \mapsto \widetilde{\chi}\colon \widehat{E} \to E'$ is an inverse of $\mathrm{Hom}(E,q)$. It remains to be verified that it is continuous. We set $\mathbb{D} = \{z \in \mathbb{R} : |z| \leq 1\}$. Let C be a compact subset of E and $U = \,]-\varepsilon,\varepsilon[\, \subseteq \mathbb{R}$ with $0 < \varepsilon \leq \frac{1}{4}$. Now $\mathbb{D}{\cdot}C$ is compact connected and contains C. Consider $\chi \in V_E\big(\mathbb{D}{\cdot}C, q(U)\big)$. Then $\widetilde{\chi}(\mathbb{D}{\cdot}C)$ is a connected subset of $q^{-1}(q(U))$ containing 0. The component of 0 in $q^{-1}\big(q(U)\big) = U + \mathbb{Z}$ is U. Hence $\widetilde{\chi}(\mathbb{D}{\cdot}C) \subseteq U$. Thus $\widetilde{\chi} \in V_E(\mathbb{D}{\cdot}C, U) \subseteq V_{\mathbb{R}}(C,U)$, proving the continuity of $\chi \mapsto \widetilde{\chi}\colon \widehat{E} \to E'$. This completes the proof. □

One is wondering to what extent the ground field $\mathbb{R}$ is unique with respect to the conclusions of Proposition A7.1. The following exercise shows that it is and to what extent statements similar to A7.1(ii) may be formulated for complex vector spaces. For this purpose we write $\mathbb{C}^\times \stackrel{\text{def}}{=} (\mathbb{C} \setminus \{0\}, .) \cong \mathbb{R} \times \mathbb{T}$ and note that the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, $\exp z = e^{2\pi i z}$ is a universal covering homomorphism of $\mathbb{C}^\times$ in the sense of Definitions A2.19. Thus every morphism of abelian topological groups $f\colon E \to \mathbb{C}^\times$ defined on a topological $\mathbb{R}$-vector space E lifts uniquely to a morphism $\widetilde{f}\colon E \to \mathbb{C}$ such that $\exp \circ \widetilde{f} = f$ by Proposition A2.32 again. For a complex topological vector space let $E' = \mathrm{Hom}(E,\mathbb{C})$ be the complex dual of E endowed with the compact open topology.

Exercise AE7.1. Let E be a complex topological vector space and $E' = \mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$ the space of all continuous linear forms $E \to \mathbb{C}$ endowed with the compact open topology.

$$f \mapsto \exp \circ f : E' \to \mathrm{Hom}(E,\mathbb{C}^\times)$$

is an isomorphism of topological groups.

[Hint. One has noted that $E' \to \mathrm{Hom}(E,\mathbb{C}^\times)$ is an isomorphism of abelian groups. Its continuity and openness is proved as in the proof of A7.1(ii).]

It should be observed that in the complex case, the abelian topological group $\mathrm{Hom}(E,\mathbb{C}^\times)$ takes the role of the character group of E in the real case whereas here

$$\mathrm{Hom}(E,\mathbb{C}^\times) \cong \mathrm{Hom}(E,\mathbb{R}) \times \mathrm{Hom}(E,\mathbb{T}) = (E_\mathbb{R})' \times \widehat{E}$$

where $E_\mathbb{R}$ is the underlying real topological vector space and $\widehat{E}$ the character group of the underlying abelian topological group E.

Let us recall some basic facts on topological vector spaces in an exercise. In this Appendix we shall denote the real or complex ground field by $\mathbb{K}$. We write $\mathbb{D} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{K} : |x| \leq 1\}$. A subset U in a $\mathbb{K}$-vector space is called *balanced* if $\mathbb{D}{\cdot}U = U$. It is called *absorbing* if

$$(\forall v \in E)(\exists r > 0)(\forall t \in \mathbb{K}) \quad (|t| > r) \Rightarrow (v \in t{\cdot}U).$$

A balanced set is absorbing if every vector is contained in a multiple of the set.

Finite dimensional topological vector spaces

Recall that a *topological vector space over* $\mathbb{K}$ (or over any topological field $\mathbb{K}$, for that matter) is an abelian topological group E with a continuous scalar multiplication $(t,v) \mapsto t{\cdot}v \colon \mathbb{K} \times E \to E$.

Exercise EA7.2. Show that the filter of zero neighborhoods $\mathcal{U}$ in a Hausdorff topological vector space satisfies

(0) $\bigcap \mathcal{U} = \{0\}$.
(i) $(\forall U \in \mathcal{U})(\exists V \in \mathcal{U}) \quad V - V \subseteq U$.
(ii) $(\forall U \in \mathcal{U})(\exists V \in \mathcal{U}) \quad \mathbb{D}{\cdot}V \subseteq U$.
(iii) Every $U \in \mathcal{U}$ is absorbing.

Conversely show that, if a filter $\mathcal{U}$ satisfies (i), (ii), (iii), then the set $\mathcal{O}$ of all subsets U of E such that for $v \in U$ there is a $W \in \mathcal{U}$ with $v + W \subseteq U$ is a vector space topology whose filter of identity neighborhoods is $\mathcal{U}$. If it also satisfies (0), then $\mathcal{O}$ is a Hausdorff topology. □

We shall always assume that our topological vector spaces are Hausdorff.

A subset P of a topological group is called *precompact*, if for every nonempty open subset U there is a finite subset F such that $P \subseteq FU$. We call an abelian topological group G *locally precompact* if it has a precompact identity neighborhood.

For topological vector spaces over the reals (or indeed any locally compact field), the finite-dimensional ones form a topologically distinguished class.

Proposition A7.2. (i) *On a one-dimensional* $\mathbb{K}$*-vector space* E *there is only one vector space topology. For each* $0 \neq v \in E$ *the map* $r \mapsto r{\cdot}v \colon \mathbb{K} \to E$ *is an isomorphism of topological vector spaces.*

(ii) *A locally compact subgroup H of a Hausdorff topological group G is a closed subset.*

(iii) *A finite-dimensional $\mathbb{K}$-vector space admits one and only one vector space topology. If E is a $\mathbb{K}$-vector space with $\dim E = n$ and $E \to \mathbb{K}^n$ is an isomorphism then it is an isomorphism of topological vector spaces where $\mathbb{K}^n$ has the product topology.*

(iv) *A locally precompact topological vector space over $\mathbb{K}$ is finite-dimensional.*

Proof. (i) Let $0 \neq e \in \mathbb{K}$, set $f{:}\,\mathbb{K} \to E$, $f(t) = t{\cdot}e$. Let $\mathcal{V} \stackrel{\text{def}}{=} f^{-1}\big(\mathcal{U}(E)\big)$ be the inverse image of the filter of zero neighborhoods of E. Since f is linear, $\mathcal{V}$ has a basis of balanced and absorbing sets. In $\mathbb{K}$ a set V is balanced iff it is of the form $V = \{r \in \mathbb{K} : |r| < \varepsilon\}$ or of the form $V = \{r \in \mathbb{K} : |r| \leq \varepsilon\}$. The only one among these which is not absorbing is $\{0\}$. Note that $\mathcal{V}$ intersects in $\{0\}$ and conclude that $\mathcal{V}$ is the neighborhood filter of 0 in $\mathbb{K}$.

(ii) It is no loss of generality to assume $G = \overline{H}$; show $H = G$. Let K be a compact identity neighborhood of H and let U be an open identity neighborhood of G with $U \cap H \subseteq K$. Consider $g \in G$; we must show $g \in H$. Since H is dense in G there is an $h \in H \cap U^{-1}g$, say $h = u^{-1}g$. Since U is open, $U \cap H$ is dense in U. Thus $U \cap K$ is dense in U, i.e. $U \subseteq \overline{K} = K$ (as K is compact in a Hausdorff space). Thus $u \in U \subseteq K \subseteq H$, whence $g = uh \in HH \subseteq H$.

(iii) Let $e_1, \dots, e_n$ be a basis of E and set $f{:}\,\mathbb{K}^n \to E$, $f(x_1, \dots, x_n) = x_1{\cdot}e_1 + \cdots + x_n{\cdot}e_n$. We prove by induction on n that f is an isomorphism of topological vector spaces. In (i) one dealt with $n = 1$. We consider vector subspaces $N \stackrel{\text{def}}{=} \mathbb{K}{\cdot}e_1 \oplus \cdots \oplus \mathbb{K}{\cdot}e_n$ and $H \stackrel{\text{def}}{=} \mathbb{K}{\cdot}e_{n+1}$. We show that $(n, h) \mapsto n+h{:}\, N \times H \to E$ is an isomorphism of topological vector spaces. This map is a continuous algebraic isomorphism. We must show that its inverse is continuous. By the induction hypothesis $N \cong \mathbb{K}^n$, $H \cong \mathbb{K}$, and so both vector spaces are locally compact hence closed by (ii). Then E/N is Hausdorff, and hence isomorphic to $\mathbb{K}$ by (i). Thus $h \mapsto h+N{:}\, H \to E/N$ is an isomorphism of topological vector spaces by (i). Hence the projection of $p_H{:}\, E \to H$, $p_H(x_1{\cdot}e_1 + \cdots + x_{n+1}{\cdot}e_{n+1}) = x_{n+1}{\cdot}e_{n+1}$, factoring through the quotient $E \to E/N$ and the isomorphism $E/N \to H$ is continuous. Hence the projection $p_N = \mathrm{id} - p_H$ onto N is continuous. So $(n, h) \mapsto n + h$ has a continuous inverse.

(iv) Let U be a balanced zero neighborhood of E such that $U+U$ is precompact. The sets $U + u$, $u \in U$ form an open cover of $U + U$ by translates of an identity neighborhood. By precompactness there is a finite subset $F \subseteq U$ such that $U+U \subseteq U + F$. Let $E_1 = \operatorname{span}_K F$. Then $U + U \subseteq U + E_1$. Since the vector space E_1 is finite-dimensional and therefore locally compact by (iii), it is closed by (ii). Hence E/E_1 is Hausdorff. Set $V = (U + E_1)/E_1$, then V is a balanced 0-neighborhood of E/E_1 satisfying $V + V \subseteq V = -V$. Then V is a vector space. However, V is precompact since U is precompact. Claim: V is a singleton. Suppose it is not a singleton; then by (i) the $\mathbb{K}$-vector space contains a vector subspace isomorphic to $\mathbb{K}$. Hence $\mathbb{K}$ would have to be precompact. But then the subspace $\mathbb{N} \subseteq \mathbb{K}$ would have to be precompact, but it is not because it cannot be covered by a finite number of translates of a disc of radius $\frac{1}{2}$. Thus V is singleton and $E = E_1$. □

Duals of vector spaces

A topological vector space E over $\mathbb{K}$ is called *locally convex* if every zero neighborhood contains a convex one. Now let E be any real vector space and let $\mathcal{B}(E)$ denote the set of all balanced, absorbing and *convex* subsets of E. Let us observe that there are plenty of those, in fact enough to allow only $\{0\}$ to be their intersection. Let F be a basis of E and $\rho: F \to]0, \infty[$ any function. Then the set

$$U(F;\rho) = \Big\{ \sum_{e\in F} r_e{\cdot}e : |r_e| < \rho(e) \Big\}$$

is balanced, absorbing and convex. We call it a *box neighborhood* with respect to F. The box neighborhoods with respect to a single basis already intersect in $\{0\}$. Thus the filter of all supersets of sets from $\mathcal{B}(E)$ satisfies (0), (i), (ii), and (iii) of EA7.2. If we set

$$\mathcal{O}(E) = \{W \subseteq E \mid (\forall w \in W)\big(\exists U \in \mathcal{B}(E)\big) \quad w + U \subseteq W\},$$

then $\mathcal{O}(E)$ is a locally convex vector space topology. From its definition it is immediate that it contains every other locally convex vector space topology. It is clearly an algebraic invariant in so far as it depends only on the vector space structure of E. A convex subset U of E belongs to $\mathcal{O}(E)$ if and only if for every $u \in U$ and every $x \in E$ the set $\{r \in \mathbb{R} \mid u + r{\cdot}x \in U\}$ is an open interval of $\mathbb{R}$ containing 0. It follows that a convex subset U of E belongs to $\mathcal{O}(E)$ if and only if for each finite-dimensional vector subspace F and each $v \in E$ the intersection $F \cap (U - v)$ is open in F (in the unique vector space topology of F).

Let us record some of the basic properties of $\mathcal{O}(E)$. We shall see that $\mathcal{O}(E)$ is not only an isomorphism invariant, but is, in a variety of aspects, a purely algebraic entity attached to the real vector space E.

Proposition A7.3. *Let E be an arbitrary vector space over $\mathbb{K}$.*

(i) *If E_1 and E_2 are vector spaces, $T: E_1 \to E_2$ is a linear map, and E_2 is a locally convex topological vector space, then T is continuous for the topology $\mathcal{O}(E_1)$.*

In particular, every algebraic linear form $E \to \mathbb{K}$ is $\mathcal{O}(E)$-continuous; i.e. the algebraic dual $E^* = \mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$ is the underlying vector space of the topological dual $E' = \mathrm{Hom}(E, \mathbb{K})$ (which is considered to carry the compact open topology and which is isomorphic to $\widehat{E}$ if $\mathbb{K} = \mathbb{R}$).

(ii) *Every vector subspace of E is $\mathcal{O}(E)$-closed and is a direct summand algebraically and topologically. Moreover, the topology induced on each vector subspace is its finest locally convex topology.*

(iii) *Let F be a linearly independent subset of E. Then there is a zero neighborhood $U \in \mathcal{O}(E)$ such that $\{v + U \mid v \in F\}$ is a disjoint open cover of F. In particular, any linearly independent subset of $\big(E, \mathcal{O}(E)\big)$ is discrete.*

(iv) *If C is an $\mathcal{O}(E)$-precompact subset, then $\mathrm{span}_{\mathbb{K}}(C)$ is finite-dimensional.*

Proof. (i) If U is any balanced and convex zero neighborhood of E_2 then $T^{-1}(U)$ is balanced, absorbing and convex and thus belongs to $\mathcal{B}(E_1)$. This shows the continuity of T with respect to $\mathcal{O}(E_1)$. The remainder of (i) then follows at once.

(ii) Let E_1 be an arbitrary vector subspace of E and let E_2 be a vector space complement; i.e. $E = E_1 \oplus E_2$. The function $x \mapsto (\mathrm{pr}_1(x), \mathrm{pr}_2(x)): E \to E_1 \times E_2$ is a vector space isomorphism and then is continuous by (i). The function $\alpha: E_1 \times E_2 \to E$, $\alpha(x, y) = x+y$, is its inverse. Since it is the restriction of the continuous addition $(x, y) \mapsto x + y: E \times E \to E$ to the subspace $E_1 \times E_2$ it is continuous. Hence α is an isomorphism of topological vector spaces. This proves assertion (ii).

(iii) Let F be a linearly independent subset. Since by the Axiom of Choice, F can be supplemented to a basis $\{e_j \mid j \in J\}$ we may just as well assume that F is this basis. Let $\sigma: F \to]0, \infty[$ be the function with the constant value $\frac{1}{2}$. We claim that for different elements $e \in F$ we have

$$\big(e + U(F;\sigma)\big) \cap \Big(\bigcup_{f\in F,\, f\neq e} f + U(F;\sigma)\Big) = \emptyset.$$

Indeed if $\omega: E \to \mathbb{K}$ is the linear functional defined by $\omega(f) = 0$ for $e \neq f \in F$ and by $\omega(e) = 1$, then

$$\Big|\omega\Big(\bigcup_{f\in F,\, f\neq e} f + U(F;\sigma)\Big)\Big| \subseteq [0, \tfrac{1}{2}[,$$

and

$$|\omega(e + U(F;\sigma))| \subseteq]\tfrac{1}{2}, 1[.$$

This proves (iii).

(iv) Let K be a precompact subset of $\big(E, \mathcal{O}(E)\big)$. We want to show that $\dim \mathrm{span}_{\mathbb{K}} K < \infty$. It is no loss of generality to assume that $E = \mathrm{span}_{\mathbb{K}} K$. Then we can select a basis $B = \{e_j \mid j \in J\}$ of elements $e_j \in K$. Then B is precompact. By (iv) we find an open zero neighborhood U such that $e_j + U$ is a disjoint cover of B by translates of U. Since B is precompact this implies that J is finite. Hence $\dim E = \mathrm{card}\, J < \infty$. □

The topology $\mathcal{O}(E)$ is called the *finest locally convex topology on* E.

For a vector space E over $\mathbb{K}$ we shall denote the set of all finite-dimensional vector subspaces by $\mathrm{Fin}(E)$. For a topological vector space E over $\mathbb{K}$ we denote the set of cofinite-dimensional *closed* vector subspaces (i.e. closed vector subspaces M with $\dim E/M < \infty$) by $\mathrm{Cofin}(E)$.

Exercise A7.2a. Use Proposition A7.3(i) to prove the following observation:

Corollary A7.3a. *The assignment $E \mapsto (E, \mathcal{O}(E))$ is a functor from the category of real vector spaces to the category of locally convex vector spaces and continuous linear maps which is left adjoint in the sense of Definition A3.31 (and Proposition A3.33) to the grounding functor assigning to a locally convex real topological vector space the underlying real vector space.* □

According to Proposition A3.21 for each set X there is a free real vector space FX over X, and so $(FX, \mathcal{O}(FX))$ is the *free locally convex vector space* on the set X.

Lemma A7.4. *Let E be a $\mathbb{K}$-vector space endowed with its finest locally convex vector space topology $\mathcal{O}(E)$.*

(i) *Then the compact open topology on E' is the weak*-topology, i.e. the topology of pointwise convergence.*

(ii) *Every continuous linear functional $\Omega: E' \to \mathbb{K}$ is of the form $\omega \mapsto \omega(x) : E' \to \mathbb{K}$ for a unique $x \in E$.*

(iii) $F \mapsto F^\perp: \mathrm{Fin}(E) \to \mathrm{Cofin}(E')$ *is an order reversing bijection.*

Proof. (i) The compact open topology on E' is generated by the set of basic zero-neighborhoods $V_E(C, U) = \{f \in E' : f(C) \subseteq U\}$ for a compact subset C of E and some zero neighborhood U of $\mathbb{R}$. This topology is always equal to or finer than the weak*-topology. It is no loss of generality to assume that C is convex balanced, since C is contained in a finite-dimensional subspace by A7.3(iv), where the closed convex circled hull C^* of a compact set C is compact and then $V_E(C^*, U) \subseteq V_E(C, U)$. Assume C is convex balanced now. For any $\varepsilon > 0$ we write $D_\varepsilon \stackrel{\text{def}}{=} \{t \in \mathbb{K} : |t| < \varepsilon\}$. For a zero neighborhood U in $\mathbb{K}$ there is always an $\varepsilon > 0$ such that $D_\varepsilon \subseteq U$. Then

$$V_E(\tfrac{1}{\varepsilon}C, D_1) = V_E(C, D_\varepsilon) \subseteq V_E(C, U).$$

Thus we may assume that the filter of zero-neighborhoods of E' for the compact open topology is generated by basic zero neighborhoods of the form $V_E(C, D_1)$ as C ranges through the compact convex and balanced subsets of E. Now $\operatorname{span} C$ is finite-dimensional; then there is a basis $e_1, \ldots, e_n$ of $\operatorname{span} C$ such that $C \subseteq K \stackrel{\text{def}}{=} \{\sum_{j=1}^n r_j{\cdot}e_j \mid |r_j| \le 1,\, j = 1, \ldots, n\}$, where K is the convex balanced hull of $\{e_1, \ldots, r_n\}$. Then

$$V_E(\{e_1, \ldots, e_n\}, D_1) = V_E(K, D_1) \subseteq V_E(C, D_1).$$

Hence the topology of E' and the weak*-topology agree.

(ii) Let $\Omega: E' \to \mathbb{K}$ be a continuous linear functional. By (i), its continuity implies the existence of a finite set $F \subseteq E$ of vectors in E such that

$$\Omega(V_E(F, D_1)) \subseteq D_1.$$

Since $F^\perp \subseteq E'$ is contained in $V_E(F, D_1)$ we have $\Omega(F^\perp) \subseteq D_1$, and since $\Omega(F^\perp)$ is a vector space, we conclude $F^\perp \subseteq \ker(\Omega)$. Therefore we have a linear functional $\Omega': E'/F^\perp \to \mathbb{R}$ such that $\Omega = \Omega' \circ q$ for the quotient map $q: E' \to E'/F^\perp$. If $x \in E$, then $x^\perp = \{\omega \in E' : \omega(x) = 0\}$ is a closed hyperplane in E', and thus $F^\perp = \bigcap_{x \in F} x^\perp$ is a finite intersection of hyperplanes. Thus $E'/F^\perp$ is a finite dimensional vector space. Its dual may be identified with $\operatorname{span} F$ in the sense that every linear functional of $E'/F^\perp$ is of the form $\omega + F^\perp \mapsto \omega(x)$ for an $x \in \operatorname{span} F$. We apply this to Ω' and find some vector $x \in \operatorname{span} F$ such that $\Omega'(\omega) = \omega(x)$. Then $\Omega(\omega) = \omega(x)$ for all $\omega \in E'$, and this is what we had to show.

(iii) If $F \in \mathrm{Fin}(E)$, then $F^\perp \in \mathrm{Cofin}(E')$ as we saw in the proof of (i). Conversely let $M \in \mathrm{Cofin}(E')$. Then M is the intersection of finitely many closed hyperplanes. Each one of these is the kernel of a continuous functional Ω; we saw in (ii) that each one of these is of the form $\omega \mapsto \omega(x)$. Hence M is the annihilator $M = F^\perp$ of some $F \in \mathrm{Fin}(E)$. Hence $F \mapsto F^\perp : \mathrm{Fin}(E) \to \mathrm{Cofin}(E')$ is a containment reversing bijection. □

Let E be a locally convex topological vector space over $\mathbb{K}$ and E' its topological dual. If $\eta_E : E \to E''$, $\eta_E(x)(\omega) = \omega(x)$, denotes the *evaluation morphism*, then for each subset $H \subset E$ we set $H^\circ \stackrel{\text{def}}{=} \{\omega \in E' : |\omega(H)| \subseteq [0,1]\} = \bigcap_{h\in H} \eta_E(h)^{-1}B_1$ with $B_1 = \{r \in \mathbb{K} : |r| \leq 1\}$ and call this set the *polar* of H in E'. Similarly for a subset $\Omega \subseteq E'$ we define the polar of Ω in E to be $\Omega^\circ \stackrel{\text{def}}{=} \{x \in E : |\Omega(x)| \subseteq [0,1]\} = \bigcap_{\omega\in\Omega} \omega^{-1}(B_1)$. Again as in the case of annihilators of subsets of abelian topological groups one must specify where the polars are taken. Polars are always closed.

For the following, recall that E is called *semireflexive* if the morphism $\eta_E : E \to E''$, $\eta_E(x)(\omega) = \omega(x)$ is bijective. It is called *reflexive* if η_E is an isomorphism of topological groups.

Lemma A7.5. (The Bipolar Lemma). Let E be a locally convex vector space and U be a convex balanced subset of E. Let Ω be a convex balanced subset of E'. Then

(i) $U^{\circ\circ} = \overline{U}$, and

(ii) if E is semireflexive, $\Omega^{\circ\circ} = \overline{\Omega}$.

Proof. (i) The taking of polars is containment reversing. Hence $U \subseteq U^{\circ\circ}$ and since polars are closed we have $\overline{U} \subseteq U^{\circ\circ}$. In order to prove the converse containment let $x \in U^{\circ\circ}$. This means that $|U^\circ(x)| \subseteq [0,1]$. We claim that this implies $x \in \overline{U}$. Suppose it does not. Then by the Theorem of Hahn and Banach (see e.g. [40]), there is a real linear functional $\rho: E \to \mathbb{R}$ with $\rho(x) > 1$ and $\rho(U) \subseteq [-1,1]$. If $\mathbb{K} = \mathbb{R}$, then this says that $\rho \in U^\circ$, contradicting $|U^\circ(x)| \subseteq [0,1]$. If $\mathbb{K} = \mathbb{C}$, then $\omega(y) = \rho(x) - i\rho(i\cdot x)$ defines a complex linear functional such that $|\omega(x)| \geq \mathrm{Re}\,|(\omega(x))| = \rho(x) > 1$, while, on the other hand, for $0 \neq u \in U$ we define z with $|z| = 1$ by $z\omega(u) = |\omega(u)|$. Then $zu \in U$ and $|\omega(u)| = z\omega(u)\omega(zu) = \mathrm{Re}\,\omega(zu) = \rho(zu) = |\rho(zu)| \leq 1$. Thus $\omega \in U^\circ$. This proves the claim.

(ii) If E is semireflexive, then E may be identified with the vector space of all continuous linear functionals of E'. Since E' is locally convex, the proof of part (i) applies here and proves the assertion. □

Weakly complete topological vector spaces

Our main interest will be with vector spaces dual to those we just discussed. Their topology was determined by the finite-dimensional vector subspaces. Dually we may consider vector space topologies which are determined by the cofinite-dimensional closed vector subspaces.

Proposition A7.6. *Let E be a topological $\mathbb{K}$-vector space. Then for $M, N \in \operatorname{Cofin}(E)$ with $N \subseteq M$, there is a canonical quotient map $q_{MN}: E/N \to E/M$. Since $\operatorname{Cofin}(E)$ is a filter basis, there is an inverse system and, in the category of topological vector spaces, there is a projective limit $E_{\operatorname{Cofin}(E)} = \lim_{M\in\operatorname{Cofin}(E)} E/M$, the vector subspace of all*

$$(v_M + M)_{M\in\operatorname{Cofin}(E)} \in \prod_{M\in\operatorname{Cofin}(E)} E/M$$

such that $N \subseteq M$ implies $v_N - v_M \in M$. The function

$$\gamma_E: E \to E_{\operatorname{Cofin}(E)}, \qquad \gamma_E(v) = (v+M)_{M\in\operatorname{Cofin}(E)}$$

is a morphism of topological vector spaces which is injective if and only if $\bigcap \operatorname{Cofin}(E) = \{0\}$.

Proof. This is straightforward. □

Lemma A7.7. *For a topological $\mathbb{K}$-vector space E, the following statements are equivalent:*

(1) *There is a set J and an isomorphism of topological vector spaces $E \to \mathbb{K}^J$.*
(2) *There exists a $\mathbb{K}$-vector space P such that $E = P^* = \operatorname{Hom}(P, \mathbb{K}) \subseteq \mathbb{K}^{\mathbb{P}}$ with the topology of pointwise convergence on P^*.*
(3) *The evaluation map $\operatorname{ev}: E \to E'^*$, $\operatorname{ev}(v)(f) = f(v)$ is an isomorphism of topological vector spaces.*
(4) *The function $\gamma_E: E \to E_{\operatorname{Cofin}(E)}$ is an isomorphism of topological vector spaces.*
(5) *E is isomorphic to a closed vector subspace of $\mathbb{K}^X$ for some set X.*

Proof. (1) ⇒(2): If $E = \mathbb{K}^J$, let $P = \mathbb{K}^{(J)} = \bigoplus_{j\in J} \mathbb{K}_j$, $\mathbb{K}_j = \mathbb{K}$. Then $P^* \cong P^J$ under the pairing $\langle\cdot,\cdot\rangle : P^J \times P^{(J)} \to \mathbb{K}$ given by $\langle (x_j)_{j\in J}, (y_j)_{j\in J}\rangle = \sum_{j\in J} x_j y_j$, where we note that the sum is well-defined since all but finitely many of the y_j are 0.

(2) ⇒(3): Assume $E = P^*$. By the Axiom of Choice, P has a basis $\{e_j : j \in J\}$. Hence P may be identified with $\mathbb{K}^{(J)}$. 'Now $E = (\mathbb{K}^{(J)})'$ may be identified with $\mathbb{K}^J$. Then $E' = \operatorname{Hom}(\mathbb{K}^J, \mathbb{K})$ in the category of topological $\mathbb{K}$-vector spaces, and this vector space has as a basis the projections $\operatorname{pr}_m: \mathbb{K}^J \to \mathbb{K}$, $\operatorname{pr}((x_j)_{j\in J}) = x_m$, $m \in J$. Thus E' may be identified with $\mathbb{K}^{(J)}$ via the dual pairing $\langle\cdot,\cdot\rangle: \mathbb{K}^{(J)}, \mathbb{K}^J \to \mathbb{K}$. Assertion (3) follows from this fact.

(3) ⇒(4): By Lemma A7.4(iii) upon interchanging the roles of E' and E we observe that $M \mapsto M^\perp : \operatorname{Cofin}(E) \to \operatorname{Fin}(E')$ is an order reversing bijection. The limit $\lim_{M\in\operatorname{Cofin} E} E/M$ is dual to the colimit $\bigcup_{F\in\operatorname{Fin}(E')} F$ in the category of $\mathbb{K}$-vector spaces. The obvious colimit morphism $\tau_{E'} : \bigcup_{F\in\operatorname{Fin}(E')} F \to E'$ is dual to γ_E. Since $\tau_{E'}$ is an isomorphism, so is γ_E

(4) ⇒(5): By (4), E is isomorphic to a closed vector subspace of $\prod_{M\in\operatorname{Cofin}(E)} E/M$ where each E/M is ismorphic to a finite product of copies

of $\mathbb{K}$. Hence E is isomorphic to a closed topological $\mathbb{K}$-vector subspace of $\mathbb{K}^X$ for some set X.

(5) $\Rightarrow$(1): The topological dual $(\mathbb{K}^X)' = \mathrm{Hom}(\mathbb{K}^X, \mathbb{K})$ may be identified with the (abstract) $\mathbb{K}$-vector space $\mathbb{K}^{(X)}$, and $\mathbb{K}^X$ may be considered as the dual $\mathrm{Hom}(\mathbb{K}^{(X)}, \mathbb{K})$ with the topology of pointwise convergence. In view of the Axiom of Choice, The annihilator $E^\perp$ of E in $\mathbb{K}^{(X)}$ has a basis $e_j : j \in J_0$ for some set J_0 which may be extended to a basis $\{e_x : x \in X\}$, $X = J_0 \dot{\cup} J$ so that $E' \cong \mathbb{K}^{(X)}/\mathbb{K}^{(J_0)} \cong \mathbb{K}^{(J)}$. So $E \cong \mathbb{K}^J$ follows. □

The Definition of weakly complete topological vector spaces

Definition A7.8. A topological vector space E is called *weakly complete* if it satisfies any of the equivalent conditions (1)–(5) of Lemma A7.7. □

The *weak topology* on a $\mathbb{K}$-vector space is the smallest topology making all linear functionals $f: E \to \mathbb{K}$ continuous. All finite-dimensional vector spaces are weakly complete. On a weakly complete topological vector space, the continuous functionals separate points. Our definition establishes completeness in the weak topology as the name of a *"weakly complete topological vector space"* suggests.

Recall from A7.3(i) that the algebraic dual E^* of a $\mathbb{K}$-vector space E is at the same time the topological dual $(E, \mathcal{O}(E))'$, consisting of all continuous linear functionals on E when E is endowed with the finest locally convex topology $\mathcal{O}(E)$. On the basis of bare linear algebra one always has the weak $*$-topology on E^*, that is, the topology of pointwise convergence induced by the natural inclusion $E^* \to \mathbb{K}^E$. The first item in the following lemma will show that this topology agrees with the topology of uniform convergence on compact sets which is the topology we consider in order to have the isomorphism $E' \cong \widehat{E}$ for $\mathbb{K} = \mathbb{R}$ according to A7.1.

Let us denote the category of $\mathbb{K}$-vector spaces, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and linear maps by $\mathcal{V}_\mathbb{K}$ and the category of weakly complete $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps by $\mathcal{W}_\mathbb{K}$. For a $\mathbb{K}$-vector space V we write V^* for its algebraic dual $\mathcal{V}_\mathbb{K}(V, \mathbb{K}) \subseteq \mathbb{K}^V$ equipped with the topology induced from $\mathbb{K}^V$, i.e., the topology of pointwise convergence, and for any weakly complete $\mathbb{K}$-vector space W, we write W' for its topological dual $\mathcal{W}_K(W, \mathbb{K})$ as an abstract $\mathbb{K}$-vector space. Clearly $V \mapsto V^* : \mathcal{V}_\mathbb{K} \to \mathcal{W}_\mathbb{K}$ and $W \mapsto W' : \mathcal{W}_\mathbb{K} \to \mathcal{V}_K$ are contravariant functors.

The Duality of $\mathcal{V}_K$ and $\mathcal{W}_K$

Theorem A7.9. *The natural evaluation morphisms*

$$(A) \qquad \mathrm{ev}_V : V \to V^{*\prime}, \quad \mathrm{ev}_V(v)(f) = f(v),$$

and

$$(B) \qquad \mathrm{ev}_W : W \to W'^{*}, \quad \mathrm{ev}_W(w)(f) = f(w),$$

are isomorphisms. The categories $\mathcal{V}_\mathbb{K}$ and $\mathcal{W}_\mathbb{K}$ are dual to each other.

Proof. The naturality of the evaluation morphisms ev_* is straightforward. By the basis theorem, every object V of $\mathcal{V}_{\mathbb{K}}$ is $\mathcal{V}_K$-isomorphic to $\mathbb{K}^{(J)}$ for some set J, and $(\mathbb{K}^{(J)})^* \cong \mathbb{K}^J$, naturally. Also by Definition A7.8. and Lemma A7.7 every $\mathcal{W}_K$ object is $\mathcal{W}_K$-isomorphic to $\mathbb{K}^J$ for some set J. The fact that $\mathrm{ev}_{\mathbb{K}^{(J)}} : \mathbb{K}^{(J)} \to (\mathbb{K}^{(J)})^{*\prime} \cong (\mathbb{K}^J)'$ is an isomorphism was established in Lemma A7.4(ii), and the fact that $\mathrm{ev}_{\mathbb{K}^J} : \mathbb{K}^J \to (\mathbb{K}^J)'^* \cong (\mathbb{K}^J)'$ is an isomorphism is clear from the fact that any continuous linear functional $f\colon \mathbb{K}^J \to \mathbb{K}$ must vanish on some vector subspace $K^I \subseteq \mathbb{K}^J$ such that $F \stackrel{\text{def}}{=} J \setminus I$ is finite, plus the elementary linear algebra information that $\mathrm{ev}_{\mathbb{K}^F}$ is an isomorphism for finite F. □

If $W = \mathbb{R}^J$, then the cardinal $\operatorname{card} J$ is called the *topological dimension* of W. (See [185].) Thus the topological dimension of a weakly complete topological vector space is the linear dimension of its dual.

In this spirit, the weakly complete topological vector spaces are generalisations of the familiar euclidean vector spaces $\mathbb{R}^n$, and we provide enough evidence in this book that they are the *correct* generalisation.

What may be considered as lacking in this duality theory is any information how it fits into Pontryagin Duality of abelian topological groups as it was expounded in Chapter 7. We have secured the background material to provide this information now for the real ground field $\mathbb{K} = \mathbb{R}$. We utilize this aspect in Chapter 7 in the section entitled "Character Groups of Topological Vector Spaces" (headline preceding Exercise E7.12).

We shall stay in the category $\mathbb{TAB}$ of all abelian topological groups, and we shall denote the full subcategory of all reflexive abelian topological groups by $\mathbb{ABD}$, that is all abelian topological groups G such that $\eta_G\colon G \to \widehat{\widehat{G}}$, $\eta_G(x)(\chi) = \chi(x)$, is an isomorphism of abelian topological groups.

Duality of Real Vector Spaces

Theorem A7.10. *Let V be a real vector space and endow it with its finest locally convex vector space topology $\mathcal{O}(V)$, and let W be a weakly complete real topological vector space. Then*

(i) *V is reflexive; that is, $\eta_V\colon V \to \widehat{\widehat{V}}$ is an isomorphism of topological vector spaces.*

(ii) *W is reflexive; that is, $\eta_W\colon W \to \widehat{\widehat{W}}$ is an isomorphism of topological vector spaces.*

(iii) *The contravariant functor $\widehat{\cdot}\colon \mathbb{ABD} \to \mathbb{ABD}$ exchanges the full subcategory of real vector spaces (given the finest locally convex topology) and the full subcategory of weakly complete topological vector spaces.*

Proof. (i) The topological dual V' and the character group $\widehat{V}$ are isomorphic by 7.5(iii). Since the linear functionals separate the points, η_V is injective. By A7.4(ii), η_V is surjective. As vector spaces, therefore, V, V'', and $\widehat{\widehat{V}}$ may be identified.

A subset K in $V' \cong \widehat{V}$ is compact if and only if it is closed in V' and for each compact subset C of V and each $\varepsilon > 0$ there is an $M \in \mathcal{B}(V)$ such that $|K(C \cap M)| \subseteq [0, \varepsilon[$ (see for instance Proposition 7.6). In view of the fact that V^* is weakly complete and that $V' = \widehat{V}$ by A7.4(i), we may express this as follows:

$$(*) \qquad (\forall F \in \mathrm{Fin}(V),\, \varepsilon > 0)(\exists M \in \mathcal{B}(V)) \quad K(F \cap M) \subseteq [0, \varepsilon].$$

We shall now show that $\mathcal{B}(V)$ has a basis of zero neighborhoods $V_{V'}(K, B_\varepsilon) = \{x \in V : |K(x)| \subseteq [0, \varepsilon]\}$ for the compact open topology on V when V is identified with $V'' \cong \widehat{\widehat{V}}$. For a proof of the claim let $U \in \mathcal{B}(V)$. Set $K \stackrel{\mathrm{def}}{=} \{\omega \in V' : |\omega(U)| \subseteq [0, 1]\} = U^\circ$. We claim that K is compact. Clearly, K is closed in V' since V' has the topology of pointwise convergence by (i) above. Let $F \in \mathrm{Fin}(V)$ and $\varepsilon > 0$. Then $|K(F \cap \varepsilon{\cdot}U)| \subseteq \varepsilon{\cdot}[0, 1]$ by the definition of K. Hence $(*)$ is satisfied and K is compact as asserted. Now we observe that $V_{V'}(K, B_1) = U^{\circ\circ}$. By the Bipolar Lemma A7.5 we have $V_{V'}(K, B_1) = \overline{U}$. Since the filter basis $\mathcal{B}(V)$ has a basis of closed sets, we have shown that it has a basis of sets $V_{V'}(K, B_1)$. Thus the compact open topology on V is finer than or equal to the given topology of V which is the finest locally convex topology. But since the sets $V_{V'}(K, B_1)$ are convex, the two topologies agree and thus V is the dual of V' and V is reflexive.

(ii) Set $V = W'$. Then each nonzero $\omega \in V$ has a closed hyperplane as kernel, and each closed hyperplane is the kernel of such an ω. It follows that $M \mapsto M^\perp : \mathrm{Cofin}(W) \to \mathrm{Fin}(V)$ is an order reversing bijection. Now let $\Theta \in V''$ and consider $F \in \mathrm{Fin}(V)$. Set $M = F^\perp$. Now F may be canonically identified with the dual of W/M so that $\langle \omega, v + M \rangle = \omega(v)$ for $\omega \in F = M^\perp$, $v \in W$ (see the Annihilator Mechanism Lemma 7.17(i)). But F and W/M are finite-dimensional vector spaces which are reflexive. Hence there is a unique element $\Theta_M \in w/M$ such that $\langle \omega, \Theta|F \rangle = \langle \omega, \Theta_M \rangle$ for all $\omega \in F$. Moreover, if $F_1 \subseteq F_2$ then $M_2 = F_2^\perp \subseteq F_1^\perp = M_1$ and since $(\Theta|F_2)|F_1 = \Theta|F_1$ the quotient map $q_{M_1 M_2} : W/M_2 \to W/M_1$ maps Θ_{M_2} to Θ_{M_1}. Hence $(\Theta_M)_{M \in \mathrm{Cofin}(W)} \in \prod_{M \in \mathrm{Cofin}(W)} W/M$ is contained in $\lim_{M \in \mathrm{Cofin}(W)} W/M$. By hypothesis, W is weakly complete; hence there is an element $v \in W$ such that $v + M = \Theta_M$. Let $\omega \in V$. Then $F \stackrel{\mathrm{def}}{=} \mathbb{K}{\cdot}\omega \in \mathrm{Fin}(V)$. Thus, letting $M = F^\perp$ we get $\Theta(\omega) = \langle \omega, \Theta|F \rangle = \langle \omega, \Theta_M \rangle = \langle \omega, v + M \rangle = \omega(v)$. Hence $\Theta = \eta_W(v)$. This suffices to show that W is semireflexive.

Now we investigate the compact open topology of V and show that it agrees with the finest locally convex topology. Let U be in the set $\mathcal{B}(V)$ of all balanced, absorbing, and convex subsets of V. Set $U^\circ \stackrel{\mathrm{def}}{=} \{x \in W : |U(x)| \subseteq [0, 1]\}$ and consider $0 \neq \omega \in V$. Set $M = \omega^{-1}(0)$. Since U is absorbing, $B_\varepsilon{\cdot}\omega \subseteq U$ for some $\varepsilon > 0$, $B_\varepsilon = \{r \in \mathbb{K} : |r| \leq \varepsilon\}$. Thus $U^\circ \subseteq (B_\varepsilon{\cdot}\omega)^\circ \stackrel{\mathrm{def}}{=} \{x \in W : |(B_\varepsilon{\cdot}\omega)(x)| \subseteq [0, 1]\} = \{x \in W : |\omega(x)| \leq \frac{1}{\varepsilon}\} = \omega^{-1} B_{1/\varepsilon}$. Since $W = \lim_{M \in \mathrm{Cofin}(W)} W/M$, the sets $\omega^{-1}(B_r)$, $\omega \in V$ and $r > 0$ are subbasic zero neighborhoods of W (meaning that the collection of all finite intersections of these form a basis of the filter of zero neighborhoods), and the topology of W is the smallest making all $\omega \in V$ continuous. Thus $x \mapsto (\omega(x))_{\omega \in W \setminus \{0\}} : W \to \mathbb{R}^{V \setminus \{0\}}$ is an embedding and U° is mapped onto a closed subset of $\prod_{\omega \in W \setminus \{0\}} B_{r(\omega)}$ for a family of positive numbers $r(\omega)$. Hence U° is compact in W. Now $U^{\circ\circ} = \{\omega \in V : \omega(U^\circ) \subseteq B_1\}$ is a zero neighborhood for the compact open topology. Since W is semireflexive, by the Bipolar Lemma A7.5(ii), $U^{\circ\circ} = \overline{U}$. Thus the closed sets which are members of $\mathcal{B}(V)$ are zero neighborhoods of the compact open topology. This says that the compact open topology of V and the finest locally convex topology of V agree.

Now V' carries the topology of pointwise convergence by A7.4(i); by the characterisation of the topology of W just derived we know that $\eta_W : W \to W'' = V'$ is an algebraic and topological embedding. Since we have seen η_W to be bijective, it is an isomorphism of topological vector spaces.

(iii) is a consequence of (i) and (ii). □

Duality at Work for Weakly Complete Topological Vector Spaces

Duality: Vector Subspaces

Theorem A7.11. *Let V be a weakly complete real topological vector space. Then the following statements hold:*

(i) *Every closed vector subspace V_1 of V is algebraically and topologically a direct summand; that is there is a closed vector subspace V_2 of V such that $(x, y) \mapsto x+y : V_1 \times V_2 \to V$ is an isomorphism of topological vector spaces.*

(ii) *Let E in $\mathcal{V}_K$ be V'. We may identify E^* and V by duality. For every closed vector subspace H of E, the relation $H^{\perp\perp} = H \cong (V/H^\perp)'$ holds and $V/H^\perp$ is isomorphic to $\widehat{H}$.*

(iii) *The map $F \mapsto F^\perp$ is an antiisomorphism of the complete lattice of vector subspaces of E onto the lattice of closed vector subspaces of V.*

$$\begin{array}{ll} E & \\ \big| & \Big\} = E/H \\ H & \\ \big| & \Big\} \cong (V/H^\perp)^* \\ \{0\} & \end{array} \qquad\qquad (E/H)' = \left\{ \begin{array}{c} \{0\} \\ \big| \\ H^\perp \\ \big| \\ V. \end{array} \right.$$

Proof. (i) Let $E = V'$. Then V' is a vector space in $\mathcal{V}_{\mathbb{K}}$. Let $E_1 = V_1^\perp$. Then there is a vector subspace E_2 such that $E = E_1 \oplus E_2$ in $\mathcal{V}_{\mathbb{K}}$. Set $V_2 = E_2^\perp$ and conclude $V = V_2 \oplus V_1$ with $V_1 \cong (E_2)^*$ and $V_2 \cong (E_1)^*$.

(ii) We consider a vector subspace H of E and recall that $E = V'$ and that the linear functionals of E/H separate the points. Then the Annihilator Mechanism applies (see for instance Lemma 7.17(iii)) and shows $H = H^{\perp\perp}$. Since $E = V'$ and the continuous functionals of E/H separate points, we know that $(V/H^\perp)'$ and H are isomorphic vector spaces. (Cf. 7.17(v).) There is a vector subspace K of E such that $E = H \oplus K$, and we obtain $E' = K^\perp \oplus H^\perp$ with a closed and hence weakly complete topological vector subspace $K^\perp \cong \widehat{H}$ of V. Then $V/H^\perp \cong K^\perp$

is a weakly complete topological vector space and thus is isomorphic to its bidual. This implies that $V/H^\perp \cong H^*$ (cf. for instance Lemma 7.17(vi)).

(iii) is a consequence of these facts. □

Duality: Morphisms

Theorem A7.12. (a) *Let $f\colon V \to W$ be a morphism of weakly complete topological vector spaces. Assume that f has a dense image. Then $f\colon V \to W$ splits; that is, there is a morphism $\sigma\colon W \to V$ such that $f \circ \sigma = \mathrm{id}_W$.*

(b) *Let $f\colon V \to W$ be a morphism of weakly complete topological vector spaces. Then $f(V)$ is a closed vector subspace of W, and the natural bijection $V/\ker f \to f(V)$ is an isomorphism of topological vector spaces.*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle j} \\ V/\ker f & \xrightarrow[f']{} & f(V) \end{array}$$

(c) (The Second Isomorphism Theorem) *If V and W are closed vector subspaces of a weakly complete topological vector space U, then $V + W$ is closed, and the function $f\colon V/(V \cap W) \to (V + W)/W$, $f\big(v + (V \cap W)\big) = v + W$ is an isomorphism of topological vector spaces.*

Proof. (a) Let $f\colon V \to W$ be a morphism of weakly complete topological vector spaces with dense image. Then the morphism f' is an epimorphism in the category of topological vector spaces since it has a zero cokernel. Then $f'\colon W' \to V'$ is a monomorphism of real vector spaces. Thus f' has a zero kernel and so is injective. Now the linear algebra of vector spaces provides, with the use of the Axiom of Choice, a linear function $\tau\colon V' \to W'$ such that $\tau \circ f' = \mathrm{id}_{W'}$. the map τ is continuous and is thus a morphism of topological vector spaces. From (iii) we know that ${f'}^* \circ \tau^*$ is the identity map of ${W'}^*$. We get a morphism $\sigma\colon W \to V$ such that ${\sigma'}^* = \tau^*$ and that $f \circ \sigma = \mathrm{id}_W$.

(b) In the category of weakly complete topological vector spaces, we have a canonical decomposition

$$(*)\qquad \begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle j} \\ V/\ker f & \xrightarrow[\varphi]{} & \overline{f(V)} \end{array}$$

where $q(v) = v + \ker f$, $j(w) = w$, $\varphi(v + \ker f) = f(v)$. After replacing f by φ we may assume without loss of generality that f is injective and has a dense image. Then f is both a monic and an epic in the category of weakly complete topological vector spaces. By (a) it is also a retraction, and a monic retraction is an isomorphism. In particular, it is surjective and thus (b) is proved.

(c) We define a morphism of weakly complete topological vector spaces

$$F\colon V/(V \cap W) \to U/W \text{ by } F\big(v + (V \cap W)\big) = v + W.$$

By Part (a), the vector subspace $(V+W)/V = F(V)$ is closed in U, and thus $V+W$, the full inverse image under the quotient morphism $U \to U/W$ is closed in U. Moreover, the corestriction of the injective morphism F to the image. namely, $f\colon V/(V\cap W) \to (V+W)/W$, is an isomorphism of weakly complete topological vector spaces. □

Filter Bases of Closed Linear Subspaces

Theorem A7.13. *Let W be a weakly complete topological vector space and $\mathcal{F}$ a filter basis of closed linear subspaces.*

(a) *Assume that F is a closed vector subspace of W. Then*

$$(*) \qquad F + \bigcap \mathcal{F} = \bigcap_{H\in\mathcal{F}} (F+H).$$

(b) *The following conditions are equivalent*

(i) $\lim \mathcal{F} = 0$.

(ii) $\bigcap \mathcal{F} = \{0\}$.

(c) *Assume that $\mathcal{F} \subseteq \mathrm{Cofin}(W)$. Then the following conditions are equivalent:*

(i) $\lim \mathcal{F} = 0$.

(iii) *$\mathcal{F}$ is a basis of* $\mathrm{Cofin}(W)$.

Proof. (a) By duality of weakly complete topological vector spaces according to Theorems A7.9(iii) and A7.11.(ii) by passing to annihilators, the assertion is equivalent to the following assertion in a vector space E with a vector subspace S and a directed (ascending) set $\mathcal{D}$ of vector subspaces

$$S \cap \bigcup \mathcal{D} = \bigcup_{U\in\mathcal{D}} S\cap U.$$

This relation holds for elementary set-theoretic reasons.

(b) Let $\mathcal{U}$ denote the filter of zero neighborhoods of W. Then by definition, (i) is equivalent to

(i)′ The filter $\langle\mathcal{F}\rangle$ generated by $\mathcal{F}$ contains $\mathcal{U}$.

This in turn is equivalent to

(i)″ $(\forall U\in\mathcal{U})(\exists V\in\mathcal{F})\, V\subseteq U$.

Now (i)″⇒(ii) is clear since $\bigcap\mathcal{F} \subseteq \bigcap\mathcal{U} = \{0\}$.

We prove (ii)⇒(i)″. Let U be a zero neighborhood. Since W is weakly complete the filter basis $\mathcal{V}$ of cofinite dimensional vector spaces V converges to 0. Hence we may assume that there is a cofinite-dimensional vector space V contained in U, and we may even assume that V is the only vector space containing V and being contained in U. By (a) we have $\bigcap_{V'\in\mathcal{F}}(V+V') = V + \bigcap\mathcal{F} = V+\{0\} = V$. Now $\{(V+V')/V : V'\in\mathcal{F}\}$ is a filter basis of vector subspaces of the finite-dimensional vector space W/V intersecting in $\{0\}$, and thus there is a $V'\in\mathcal{F}$ such that $(V+V')/V$ is zero, that is, $V'\subseteq V\subseteq U$. This proves (i)″.

(c) Trivially, (ii) implies (i) because $\lim \mathrm{Cofin}(W) = 0$. For a proof of (i) implies (ii), let $V\in\mathrm{Cofin}(W)$. Then W/V is finite-dimensional, and by the continuity of the quotient map $W \to W/V$ the filter basis $\{B+V/V : B \in \mathcal{B}\}$ of vector subspaces of W/V converges to zero. Since W/V has no small subgroups, there is a $B \in \mathcal{B}$ such that $(B+V)/V = V/V$, that is $B+V = V$ or, equivalently, $B \subseteq V$. □

In lattice theoretic terminology conclusion (a) reads:
The lattice of closed vector subspaces of V is meet continuous.

For the following, recall that an *affine subspace* A of a vector space W is a subset of the form $A = g + V$ for some vector subspace V. The affine subspace is *linear* iff $g \in V$.

Filter Bases of Affine Subspaces

Theorem A7.14. *Let W be a weakly complete topological vector space and $\mathcal{F}$ a filter basis of closed affine subspaces. Then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. (a) We write the affine members of $\mathcal{F}$ in the form $g_j + V_j$ with closed vector subspaces V_j and elements g_j, $j \in J$. We claim that the set $\{V_j : j \in J\}$ is a filter basis. Indeed let $i, j \in J$, then there is a $k \in J$ such that $g_k + V_k \subseteq (g_i + V_i) \cap (g_j + V_j)$, since $\mathcal{F}$ is a filter basis. Therefore $g_i + V_i = g_k + V_i$ and $g_j + V_j = g_k + V_j$. Now $g_k + V_k \subseteq (g_k + V_i) \cap (g_k + V_j)$, and hence $V_k \subseteq V_i \cap V_j$. Let $V = \bigcap_{j \in J} V_j$. Then W/V is a weakly complete topological vector space and $\mathcal{F}/V = \{(g_j + V) + V_j/V : j \in J\}$ is a filter basis of closed affine subsets. It clearly suffices to show that $\mathcal{F}/V$ has a nonempty intersection. Thus we assume from here on that $V = \{0\}$, that is the filter basis $\mathcal{V} \stackrel{\text{def}}{=} \{V_j : j \in J\}$ has the intersection $\{0\}$. But then $\lim \mathcal{V} = 0$ in W by (b) above. This implies that $\mathcal{F}$ is a Cauchy filter: Let U be an identity neighborhood; then there is a $j \in J$ such that $V_j \subseteq U$. Then $(g_j + V_j) - (g_j + V_j) = V_j \subseteq U$. Since W is a complete topological vector space, every Cauchy filter basis converges. Let $g = \lim \mathcal{F}$. Since all $g_j + V_j$ are closed, we have $g \in g_j + V_j$ for all $j \in J$ and this completes the proof of the Lemma. □

In terms of a terminology that has been used for situations resembling the one we have in the previous theorem, this result can be expressed in the following form:
Weakly complete topological vector spaces are linearly compact.

Topological Properties of Weakly Complete Topological Vector Spaces

A topological group G is *topologically compactly generated* if there is a compact subset $C \subseteq G$ such that $G = \overline{\langle C \rangle}$.

It is *compactly generated* if there is a compact subset $C \subseteq G$ such that $G = \langle C \rangle$.

We begin by observing that the idea of a weakly complete topological vector space being a *topologically compactly generated* pro-Lie group is not very restrictive.

Remark A7.15. Any weakly complete topological vector group is topologically compactly generated.

Proof. For the purposes of the proof we may and will assume that $W = \mathbb{R}^J$ for some set J. For any subset I of J we identify $\mathbb{R}^I$ naturally with a subgroup of $\mathbb{R}^J$. The dual $E \stackrel{\text{def}}{=} \widehat{W}$ may and will be identified with $\mathbb{R}^{(J)}$, the set of all $f\colon J \to \mathbb{R}$ with finite support, in such a fashion that $f \in E$ and $g \in W$ gives us $\langle f, g\rangle = \sum_{j\in J} f(j)g(j)$.

Let $K = \{\delta_j \in \mathbb{R}^J : j \in J\} \cup \{0\}$. Let V be a cofinite-dimensional vector subspace of W. Then $V^\perp$ is a finite-dimensional vector subspace of the dual $E \stackrel{\text{def}}{=} \widehat{W}$. Let $\operatorname{Fin}(J)$ denote the set of finite subsets of J. Since $E = \bigcup_{I\in\operatorname{Fin}(J)} \mathbb{R}^{(I)}$ and since $V^\perp$ is finite-dimensional, there is an $I \in \operatorname{Fin}(J)$ such that $V^\perp \subseteq \mathbb{R}^{(I)}$ and thus $V \subseteq \left(\mathbb{R}^{(I)}\right)^\perp = \mathbb{R}^{J\setminus I}$. Hence $K \setminus V = \{\delta_i : i \in I\}$ is finite. Therefore K is compact. On the other hand, $W = \overline{\mathbb{R}^{(J)}} = \overline{\langle [0,1]\cdot K\rangle}$ and $[0,1]\cdot K$ is a compact subset of $\mathbb{R}^{(J)}$. Hence W is topologically compactly generated.

Since $\delta_j \in \mathbb{Z}^J \subseteq \mathbb{R}^J$, the assertion on $\mathbb{Z}^J$ follows analogously, as $\mathbb{Z}^J = \overline{\mathbb{Z}^{(J)}} = \overline{\langle K\rangle}$. □

In Chapter 5 we specified the following concepts: A topological space is called a *Polish space* if it is completely metrizable and second countable. It is said to be *σ-compact*, if it is a countable union of compact subspaces. It is said to be *separable* if it has a dense countable subset. Countable products of Polish spaces are Polish. For instance, $\mathbb{R}^\mathbb{N}$ is Polish.

Remark A7.16. (i) Every almost connected locally compact group is compactly generated.

(ii) Every compactly generated topological group is σ-compact.

(iii) A topological group whose underlying space is a Baire space and which is σ-compact is a locally compact topological group.

(iv) A σ-compact Polish group is locally compact.

(v) A compactly generated Baire group is locally compact.

Proof. (i) Let K be a compact neighborhood of the identity. Then $\langle K\rangle$ is an open subgroup which has finite index in G. Let F be any finite set which meets each coset modulo $\langle K\rangle$. Then $K \cup F$ is a compact generating set of G.

(ii) If K is a compact generating set of G, then $C \stackrel{\text{def}}{=} KK^{-1}$ is a compact generating set satisfying $C^{-1} = C$; then $G = \langle C\rangle = \bigcup_{n=1}^\infty C^n$.

(iii) A Baire space cannot be the union of a countable set of nowhere dense closed subsets. A topological group containing a compact set with nonempty interior is locally compact.

(iv) By the Baire Category Theorem (see [34], Chapter 9, §5, n° 3, Théorème 1.), every Polish space is a Baire space.

(v) is clear from the preceding. □

Proposition A7.17. *For a weakly complete topological vector space W, the following statements are equivalent:*
(A) *W is σ-compact.*
(B) *W is locally compact.*
(C) *W is finite-dimensional.*
(D) *W is compactly generated.*

Proof. The equivalence of (B) and (C) was shown in A7.2(iii) and (iv). Locally compact connected groups are compactly generated by A7.16(i) and so (B) implies (D); and (D) implies (A) by A7.16(ii).

In order to prove that (A) implies (C), let W be a weakly complete σ-compact topological vector space. Its dual is a vector space E and W is finite-dimensional iff E is finite-dimensional. Suppose that E is infinite-dimensional. Selecting from a basis an infinite countable subset we get a vector subspace F with a countable basis. Then $W/F^\perp$ is isomorphic to the dual of $F \cong \mathbb{R}^{(\mathbb{N})}$ and therefore $W/F^\perp$ is a homomorphic image of W which is isomorphic to $\mathbb{R}^{\mathbb{N}}$ and therefore is a Polish topological vector space. Since it is also σ-compact as a homomorphic image of a σ-compact group, it is locally compact by A7.16. But then it is finite-dimensional, a contradiction. □

Proposition A7.18. *For a weakly complete topological vector space W, the following statements are equivalent:*
(i) $W \cong \mathbb{R}^J$ *with* $\operatorname{card} J \leq \aleph_0$.
(ii) *W is locally compact or is isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(iii) *W is finite-dimensional or is isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(iv) *W is second countable.*
(v) *W is first countable.*
(vi) *W is Polish.*

Proof. By the remarks preceding the proposition, for each cardinal $\aleph$, there is, up to isomorphy of topological vector spaces and of topological groups one and only one weakly complete topological vector space of topological dimension $\aleph$, namely, $\mathbb{R}^{\aleph}$. Conditions (i), (ii), (iii) are ostensibly all equivalent to saying that $\aleph$ is countable. The weight $w(W)$, that is the smallest cardinal representing the cardinality of a basis for the topology of $W \cong \mathbb{R}^{\aleph}$ is $\aleph_0$ if $\aleph$ is countable, and is $\aleph$ if $\aleph$ is uncountable (see e.g. Exercise EA4.3 following Proposition A7.8), so (iv) is likewise equivalent to (ii), and implies (v). Then W is metrizable (see e.g. Theorem A4.16) and thus (vi) follows by the completeness of W. Since trivially (vi) implies (iv), the equivalence of (iv), (v), and (vi) follows, and the proof is complete. □

Proposition A7.19. *For a weakly complete topological vector space W, the following statements are equivalent:*
(a) *W is separable.*
(b) *W contains a dense vector subspace of countable linear dimensions over $\mathbb{R}$.*
(c) *W is isomorphic as a topological vector space to $\mathbb{R}^J$ with* $\operatorname{card} J \leq 2^{\aleph_0}$.

These conditions are implied by the equivalent statements of Proposition A7.17.

Proof. A second countable space is always separable: It suffices to pick a point in every set in a countable basis for the topology: this yields a countable dense set. What remains therefore is to see the equivalence of (a), (b), and (c). We may safely assume that W is infinite dimensional, since the finite-dimensional case is clear.

(a)⇒(b): Let C be a countable dense subset of $\mathbb{R}^J$. Then the real linear span of C is dense vector subspace of $\mathbb{R}^J$ whose linear dimension is countable.

(b) ⇒(c): Assume that $\iota\colon \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}^J$ is a linear map between vector spaces such that $\overline{\mathrm{im}(\iota)} = \mathbb{R}^J$. We give $\mathbb{R}^{(\mathbb{N})}$ the finest locally convex topology. The vector space dual of $\mathbb{R}^{(\mathbb{N})}$ may be identified with $\mathbb{R}^{\mathbb{N}}$, and that of $\mathbb{R}^J$ with $\mathbb{R}^{(J)}$. The morphism ι is both an epic (and a monic) in the category of (Hausdorff) topological vector spaces. Its adjoint morphism $\iota'\colon \mathbb{R}^{(J)} \to \mathbb{R}^{\mathbb{N}}$ is a monic (and epic) and is therefore an injection (with dense image). Thus $\mathrm{card}(J) \leq \dim_{\mathbb{R}} \mathbb{R}^{\mathbb{N}} = 2^{\aleph_0}$.

(c)⇒(a): Let $W = \mathbb{R}^J$ with $\mathrm{card}(J) = 2^{\aleph_0}$. We shall show that W is separable; since $\mathbb{R}^I$ with $\mathrm{card}(I) \leq \mathrm{card}(J)$ is a homomorphic image of $\mathbb{R}^J$, this will yield the implication. The topological vector space dual of $\mathbb{R}^J$ may be identified with $\mathbb{R}^{(J)}$ and then there is a linear bijection $\beta\colon \mathbb{R}^{(J)} \to \mathbb{R}^{\mathbb{N}}$. If we give $\mathbb{R}^{(J)}$ the finest locally convex topology and $\mathbb{R}^{\mathbb{N}}$ the product topology, then β is an epic (and a monic) in the category of topological vector spaces and thus its adjoint $\beta'\colon \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}^J$ has a dense image (and is injective). Even in the finest locally convex topology, $\mathbb{Q}^{(\mathbb{N})}$ is dense in $\mathbb{R}^{(\mathbb{N})}$, and $\mathbb{Q}^{(\mathbb{N})}$ is countable. Hence $\mathbb{R}^J$ is separable as asserted. □

Tensor Products

We need to endow each of the categories $\mathcal{V}$ of $\mathbb{K}$-vector spaces and $\mathcal{W}$ of weakly complete vector spaces with tensor products so that each may be considered as a *commutative monoidal category* according to Definition A3.62, also called *symmetric monoidal cagegory* $\mathcal{A}$ which supports a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. The precise definitions were collected in Appendix 3, see notably Definition A3.62. What is relevant here is that not only does the category $\mathcal{V}$ of $\mathbb{K}$-vector spaces have the familiar tensor product $\otimes_{\mathcal{V}}$ but that the category $\mathcal{W}$ of weakly complete vector spaces has a tensor product as well. It was first introduced by R. Dahmen in [76] and was used readily in [78]. The essence of this tensor product is that for two weakly complete vector spaces $W_1 = \mathbb{K}^X$ and $W_2 = \mathbb{K}^Y$ we have $W_1 \otimes W_2 \cong \mathbb{K}^{X \times Y}$ and that for a natual topological embedding $W_1 \times W_2 \to W_1 \otimes_{\mathcal{W}} \mathcal{W}_2$ denoted $(w_1, w_2) \mapsto w_1 \otimes w_2 : W_1 \times W_2 \to W_1 \otimes W_2$ we have the following universal property: For any continuous bilinear function $b\colon W_1 \times W_2 \to W_3$ with a weakly complete object W_3 of $\mathcal{W}$ there is a unique continuous linear map $b'\colon W_1 \otimes W_2 \to W_3$ so that $b(w_1, w_2) = b'(w_1 \otimes_{\mathcal{W}} w_2)$ for all $w_1 \in W_1$, $w_2 \in W_2$.

Proposition A7.20. *The category $\mathcal{W}$ together with its tensor product $\otimes_{\mathcal{W}}$ is a commutative monoidal category such that for any two $\mathbb{K}$-vector spaces V_1 and V_2*

and two weakly complete $\mathbb{K}$*-vector spaces* W_1 *and* W_2 *we have natural isomorphisms*

$$(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^* \quad \textit{and} \quad (W_1 \otimes_{\mathcal{W}} W_2)' \cong W_1' \otimes_{\mathcal{V}} W_2'.$$

Proof. The verification is left as an exercise. □

Exercise EA7.3. Fill in the details of the proof of Proposition A7.20. (Cf. [76]). □

Corollary A7.21. *The symmetric monoidal categories* $(\mathcal{V}, \otimes_{\mathcal{V}})$ *and* $(W, \otimes_{\mathcal{W}})$ *are naturally dual.*

Proof. This is a reformulation of Theorem A7.21 and Proposition A7.20. □

Pro-Lie Groups

In one small section here it is impossible to do the topic "Pro-Lie Groups" justice. For a complete treatment see [188] and for a subsequent survey see [192].

Definition A7.22. A topological group G is called a *pro-Lie group* if is is complete and if every identity neighborhood of G contains a normal subgroup N such that G/N is a Lie group. □

The topological groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{R}^m$, for every cardinal number m, every finite-dimensional Lie group, every compact group, and every locally compact abelian group is a pro-Lie group.

Proposition A7.23. (Yamabe's Theorem) *Every almost connected locally compact group is a pro-Lie group.*

To each topological group G one can easily associate a topological space $\mathfrak{L}(G)$, namely, the space $\mathrm{Hom}(\mathbb{R}, G)$ of all continuous group homomorphisms from the additive topological group $\mathbb{R}$ of real numbers to the topological group G, endowed with the topology of uniform convergence on compact sets. We also have a continuous function $\exp\colon \mathfrak{L}(G) \to G$ given by $\exp X = X(1)$ and a "scalar multiplication" $(r, X) \mapsto r{\cdot}X : \mathbb{R} \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ given by $(r{\cdot}X)(s) = X(sr)$. If G is a pro-Lie group, then $\mathfrak{L}(G)$ is equippend with the sructure of a weakly complete $\mathbb{R}$-vector space and a topological Lie algebra.

Proposition A7.24. *Every pro-Lie group* G *has a Lie algebra* $\mathfrak{L}(G)$ *and the image* $\exp \mathfrak{L}(G)$ *of the exponential function algebraically generates a subgroup which is dense in the connected component* G_0 *of the identity.*

For the present record, let us recall what we did in Definition 1.25ff. when we introduced projective limits. So, a projective system of topological groups is a family of topological groups $(C_j)_{j\in J}$ indexed by a directed set J and a family of morphisms $\{f_{jk}\colon C_k \to C_j \mid (j,k) \in J \times J,\ j \le k\}$, such that f_{jj} is always the identity morphism and $i \le j \le k$ in J implies $f_{ik} = f_{ij} \circ f_{jk}$. Then the *projective* *limit of the system* $\lim_{j\in J} C_j$ is the subgroup of $\prod_{j\in J} C_j$ consisting of all J-tuples $(x_j)_{j\in J}$ for which the equation $x_j = f_{jk}(x_k)$ holds for all $j, k \in J$ such that $j \leq k$.

Theorem A7.25. *Every projective limit of pro-Lie groups is a pro-Lie group. Every closed subgroup of a pro-Lie group is a pro-Lie group. A topological group is a pro-Lie group if and only if it is isomorphic to a closed subgroup of a product of Lie groups.*

Theorem A7.26. *The category of pro-Lie groups and continuous homomorphisms is closed in the category of topological groups and continuous homomorphisms under the formation of all limits and is therefore complete. It is the smallest full subcategory of the category of all topological groups and continuous homomorphisms that contains all finite dimensional Lie groups and is closed under the formation of all limits.*

We consider a topological Lie algebra $\mathfrak{g}$ and on it the filterbasis of closed ideals $\mathfrak{j}$ such that $\dim \mathfrak{g}/\mathfrak{j} < \infty$; we shall denote it by $\mathcal{I}(\mathfrak{g})$.

Definition A7.27. A topological Lie algebra $\mathfrak{g}$ is called a *pro-Lie algebra* (short for *profinite dimensional Lie algebra*) if $\mathcal{I}(\mathfrak{g})$ converges to 0 and if $\mathfrak{g}$ is a complete topological vector space. □

Under these circumstances, $\mathfrak{g} \cong \lim_{\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{j}$, and the underlying topological vector space is a weakly complete topological vector space.

Theorem A7.28. *Every pro-Lie group G has a pro-Lie algebra $\mathfrak{g}$ as Lie-algebra, and the assignment $\mathfrak{L}$ which associates with a pro-Lie group G its pro-Lie algebra is a limit preserving functor.*

Theorem A7.29. *The Lie algebra functor from the category of pro-Lie groups to the category of pro-Lie algebras has a left adjoint Γ. It associates with every pro-Lie algebra $\mathfrak{g}$ a unique simply connected pro-Lie group $\Gamma(\mathfrak{g})$ and a natural isomorphism $\eta_\mathfrak{g}\colon \mathfrak{g} \to \mathfrak{L}\big(\Gamma(\mathfrak{g})\big)$ such that for every morphism $\varphi\colon \mathfrak{g} \to \mathfrak{L}(G)$ for some pro-Lie group G, there is a unique morphism $\varphi'\colon \Gamma(\mathfrak{g}) \to G$ of pro-Lie groups such that $\varphi = \mathfrak{L}(\varphi') \circ \eta_\mathfrak{g}$.*

The abelian pro-Lie groups we know best are the compact abelian groups and the weakly complete vector groups. So it is very pleasing that we can state in the end of our review of some basic pro-Lie Theory the fact that our Vector Group Splitting Theorem 7.57 for locally compact abelian groups largely generalizes to abelian pro-Lie groups. Indeed, every abelian pro-Lie G group has a weakly complete vector subgroup V such that G is isomorphic to the direct product $V \times (G/V)$ where the factor G/V has no nontrivial vector subgroup. We call any such subgroup V a *vector group complement.* Recall from Definition 7.44 that for a topological group G we let $\mathrm{comp}(G)$ denote the set of all elements which are contained in a compact subgroup. A topological group is called *prodiscrete* if it is complete and every identity neighborhood contains an open normal subgroup.

It may be helpful for understanding the following theorem to consult and keeping in mind the diagram concluding the statement of the Vector Group Splitting Theorem 7.57 for locally compact abelian groups.

Theorem A7.31. (Vector Group Splitting Theorem for Abelian Pro-Lie Groups) *Let G be an abelian pro-Lie group and V a vector group complement. Then there is a closed subgroup H such that*
(i) $(v,h)\mapsto v+h : V\times H\to G$ *is an isomorphism of topological groups,*
(ii) H_0 *is compact and equals* $\operatorname{comp} G_0$ *and* $\operatorname{comp}(H)=\operatorname{comp}(G)$*; in particular,* $\operatorname{comp}(G)\subseteq H$.
(iii) $H/H_0\cong G/G_0$*, and this group is prodiscrete.*
(iv) $G/\operatorname{comp}(G)\cong V\times S$ *for some prodiscrete abelian group S without nontrivial compact subgroups.*
(v) *G has a characteristic closed subgroup $G_1=G_0\operatorname{comp}(G)$ which is isomorphic to $V\times\operatorname{comp}(H)$ such that G/G_1 is prodiscrete without nontrivial compact subgroups.*

Weakly Complete Unital Algebras

In any abstract or topological category, an algebra V with a multiplication poses the problem that multiplication
$$(x,y)\mapsto xy : V\times V\to V$$
is not a morphism because it is bilinear rather than linear. The presence of a tensor product "⊗" that transforms bilinearity into linearity is therefore an ideal tool to deal with algebras in a systematic way.

An algebra is called *unital* if it has an identity.

Definition A7.32. A *weakly complete unital algebra* is an associative algebra A over $\mathbb{K}$ with identity, whose underlying vector space is weakly complete, and whose multiplication $(a,b)\mapsto ab : A\times A\to A$ is continuous. □

A product of any family of finite dimensional associative $\mathbb{K}$-algebras is obviously a weakly complete unital algebra, as is any closed $\mathbb{K}$-subalgebra containing the identity element.

After what was said before, the multiplication can be written as a $\mathcal{W}$-morphism $m: A\otimes_{\mathcal{W}} A\to A$ making $(A,\otimes_{\mathcal{W}})$ into a monoid in the symmetric monoidal category $(\mathcal{W},\otimes_{\mathcal{W}})$ in the sense of Appendix 3, Definition A3.63a. (Cf. also [78], Definition 1.6.)

A $\mathbb{K}$-vector space C together with a linear map $c: C\to C\otimes_V C$ of $\mathbb{K}$-vector spaces and a linear map $k: A\to\mathbb{K}$ making C into a *comonoid* in the symmetric monoidal category $(\mathcal{V},\otimes_{\mathcal{V}})$ in the sense of Appendix 3, Definition A3.63a (or [78], Definition 2.5) is called a (coassociative and counital) *coalgebra* (over $\mathbb{K}$) .

There is a fundamental theorem attributed to Cartier on these purely algebraic objects for which we refer the reader to the informative handbook essay by W. Michaelis [258].

For us, the following version is relevant. It should be clear that a vector subspace S of a coalgebra C is a subcoalgebra if $c_C(S)\subseteq S\otimes S$ and $k_C(S)=\mathbb{R}$.

Theorem A7.33. (Fundamental Theorem of Coalgebras) *Every coalgebra C is the directed union of the set of its finite dimensional subcoalgebras.*

Proof. See [258], Theorem 4.12, p. 742. □

This is sometimes formulated as follows: *Every coalgebra is the injective limit of its finite dimensional subcoalgebras.*

Now if we take Theorems A7.21 and A7.33 together, we arrive at the following theorem [25]. Its consequences are surprising. In this book we see projective limits everywhere, and in the sense of category theory they are discussed in Definition A3.41ff., and in the concrete case of limits and notably projective limits of topological groups we also refer to [188], pp. 63ff., respectively, pp. 77ff.)

A projective limit of topological groups is *strict* if all bonding morphisms and all limit morphisms are surjective (see Definition 1.32 in this book or Definition 1.24 in [188]). We shall call a projective limit of topological groups *a strict projective limit of quotients* if all bonding maps and all limit morphisms are surjective *and open*; that is, are quotient morphisms.

Theorem A7.34. (The Fundamental Theorem of Weakly Complete Topological Algebras) *Every weakly complete unital topological $\mathbb{K}$-algebra is the strict projective limit of a projective system of quotient morphisms between its finite dimensional unital quotient algebras.*

Proof. Theorem A7.12 implies that for any injective morphism $f\colon E_1 \to E_2$ in the category $\mathcal{V}$ of vector spaces, the dual morphism

$$f^*\colon E_2^* \to E_1^*$$

in the category $\mathcal{W}$ of weakly complete vector spaces is automatically surjective and open. Therefore, the present theorem is a direct consequence of the Fundamental Theorem of Coalgebras A7.33. □

The literature on locally compact groups shows considerable attention to structural results derived from the information that a group, say, is a projective limit of Lie groups. It is therefore remarkable that a result concluding the presence of a projective limit of additive Lie groups emerges out of the vector space duality between $\mathcal{V}$ and $\mathcal{W}$ and the Fundamental Theorem on Coalgebras.

For a weakly complete unital algebra A let $\mathbb{I}(A)$ denote the filter basis of closed two sided proper ideals $I \subset A$ such that A/I is a finite dimensional.

Corollary A7.35. *In a weakly complete topological unital algebra A each neighborhood of* 0 *contains an ideal $I \in \mathbb{I}(A)$. That is, the filter basis $\mathbb{I}(A)$ converges to* 0*. In short,* $\lim \mathbb{I}(A) = 0$ *and*

$$(*) \qquad A \cong \lim_{I \in \mathbb{I}(A)} A/I.$$

Proof. The assertion is a reformulation of Theorem A7.34. (Cf. [78], Corollary 3.3.) □

An element a in an algebra A is called a *unit* if it has a multiplicative inverse, that is, there exists an element $a' \in A$ such that $aa' = a'a = 1$. The set A^{-1} of units of an algebra is a group with respect to multiplication.

Lemma A7.36. *The group of units A^{-1} of a weakly complete unital algebra A is a topological group.*

Proof. We must show that the function $a \mapsto a^{-1} : A^{-1} \to A^{-1}$ is continuous. In every finite dimensional real or complex unital algebra, the group of units is a topologial group. This applies to each factor algebra A/I, for $I \in \mathbb{I}(A)$. Then $a \mapsto a^{-1}I : A^{-1} \to (A/I)^{-1}$ is continuous for all $I \in \mathbb{I}(A)$. Since the isomorphism $A \cong \lim_{I \in \mathbb{I}(A)} A/I$ holds also in the category of topological spaces, the continuity of $a \mapsto a^{-1}$ follows by the universal property of the limit (see Definition A3.41). □

Let us denote the category of weakly complete unital K-algebras by $\mathcal{A}$ and the category of (Hausdorff) topological groups by $\mathcal{T}$. Then $A \mapsto A^{-1} : \mathcal{A} \to \mathcal{T}$ is readily seen to be a functor preserving products and intersections. Hence by Proposition A3.51 it preserves arbitrary limits.

Proposition A7.37. *The group A^{-1} of units of a weakly complete unital $\mathbb{K}$-algebra A is a pro-Lie group and*

$$(**) \qquad A^{-1} \cong \lim_{I \in \mathbb{I}(A)} (A/I)^{-1}$$

where $(A/I)^{-1}$ is a linear Lie group for each $I \in \mathbb{I}(A)$.

Proof. Since $A \mapsto A^{-1}$ is a limit preserving functor, relation (**) follows from relation (*). Since A/I is a finite dimensional $\mathbb{K}$-algebra for each $I \in \mathbb{I}(A)$, the group $(A/I)^{-1}$ of its units is a linear Lie group according to Definition 5.32. □

We observe that we do not have at this time any obvious conclusion on the nature of the bonding and limit maps in (∗∗). We shall obtain the final piece of information in Corollary A7.43.

In the category of (abstract) $\mathbb{K}$-algebras, the polynomial algebra $\mathbb{K}[X]$ is the free object in one generator. If by a slight extension of notation we let $\mathbb{I}(\mathbb{K}[X])$ denote the set of all ideals of $\mathbb{K}[X]$ such that $\mathbb{K}[X]/I$ is finite dimensional, we obtain

$$\mathbb{K}\langle X\rangle \stackrel{\text{def}}{=} \lim_{I \in \mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/I$$

as the free weakly complete unital $\mathbb{K}$-algebra in one generator.

Let $\mathbb{P}$ denote the set of all irreducible polynomials p with leading coefficient 1. Then we have the following:

Lemma A7.38. *There is an isomorphism of weakly complete* $\mathbb{K}$*-algebras*

$$\mathbb{K}\langle X\rangle \cong \prod_{p\in\mathbb{P}} \mathbb{K}_p\langle x\rangle, \ \ \textit{where } \mathbb{K}_p\langle X\rangle = \lim_{k\in\mathbb{N}} \frac{\mathbb{K}[X]}{(p^k)}.$$

Proof. Since $\mathbb{K}[X]$ is a principal ideal domain, every ideal $J \in \mathbb{I}(\mathbb{K}[X])$ is generated by a nonzero polynomial $f = f(X)$, that is, $J = (f)$. Furthermore, each polynomial f admits a unique decomposition into irreducible factors:

$$\mathbb{I}(\mathbb{K}[X]) = \left\{ \left(\prod_{p\in\mathbb{P}} p^{k_p} \right) : (k_p)_{p\in\mathbb{P}} \in (\mathbb{N}_0)^{(\mathbb{P})} \right\}.$$

Here, $(\mathbb{N}_0)^{(\mathbb{P})}$ denotes the set of all families of nonnegative integers where all but finitely many indices are zero. For each $f = \prod_{p\in\mathbb{P}} p^{k_p}$ we have

$$\mathbb{K}[X]/(f) \cong \prod_{p\in\mathbb{P}} \mathbb{K}[X]/(p^{k_p})$$

by the Chinese Remainder Theorem.

This enables us to rewrite the projective limit in the definition of $\mathbb{K}\langle x\rangle$ as

$$(\#) \qquad \lim_{J\in\mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/J \to \prod_{p\in\mathbb{P}} \left(\frac{\lim_{k\in\mathbb{N}} \mathbb{K}[X]}{(p^k)} \right). \qquad \square$$

We remark that if $p \in \mathbb{P}$ is of degree 1, the algebra $\mathbb{K}_p\langle X\rangle$ is isomorphic to $\mathbb{K}[[X]]$, the power series algebra in one variable.

Since for $\mathbb{K} = \mathbb{C}$, all $p \in \mathbb{P}$ are of degree 1, it follows that the algebra $\mathbb{C}\langle X\rangle$ is isomorphic to $\mathbb{C}[[X]]^{\mathbb{C}}$.

Recall that a commutative unital ring R is called a *local ring* if it has a unique maximal ideal M in which case R/M is a field.

Lemma A7.39. (Density Lemma) *For each irreducible polynomial* p *over* $\mathbb{K}$ *with leading coefficient* 1*, the weakly complete algebra* $A \overset{\text{def}}{=} \mathbb{K}_p\langle X\rangle$ *is a local ring and its group* A^{-1} *of units is open and dense in* A.

Proof. Let $\pi\colon A \to \mathbb{K}[X]/(p)$ denote the bonding morphism for $k = 1$ in (#) and let $J \overset{\text{def}}{=} \ker\pi$. For every $f \in J$, the series $\sum_{m=0}^{\infty} f^m$ converges in A to $(1 - f)^{-1}$. So

$$(*) \qquad 1 - J \subseteq A^{-1}.$$

Now let $f \in A \setminus J$. Since $F \overset{\text{def}}{=} \mathbb{K}[X]/(p)$ is a field, $\pi(f)$ has an inverse in F. Thus there is an element $g \in A$ with $h \overset{\text{def}}{=} fg \in 1 - J$. By $(*)$, h^{-1} exists and $fgh^{-1} = 1$. Hence f is invertible. This shows that

$$(**) \qquad A \setminus J \subseteq A^{-1}.$$

Trivially $A^{-1} \cap J = \emptyset$ and so equality holds in $(**)$.

This shows that the closed ideal J is maximal and thus A is a local ring. Moreover, $A^{-1} = A \setminus J = \pi^{-1}(F \setminus \{0\})$ is open and dense as the inverse of a dense set under an open surjective map. □

Theorem A7.40. (The Density Theorem) *For any weakly complete unital $\mathbb{K}$-algebra A, the group A^{-1} of units is dense in A.*

Proof. Let $0 \neq a \in A$ and let V denote an open neighborhood of a of A. According to the universal property of $\mathbb{K}\langle X\rangle$ there is a morphism $\varphi\colon \mathbb{K}\langle X\rangle \to A$ with $\varphi(X) = a$. Then $U \stackrel{\text{def}}{=} \varphi^{-1}(V)$ is an open neighborhood of X in $\mathbb{K}\langle X\rangle$. If we find a unit $u \in \mathbb{K}\langle X\rangle^{-1}$ in U, then $\varphi(u) \in V \cap A^{-1}$ is a unit, and this will prove the density of A^{-1} in A. By Lemma A7.28 we have $\mathbb{K}\langle X\rangle \cong \prod_{p\in\mathbb{P}} \mathbb{K}_p\langle X\rangle$, and so the problem reduces to finding a unit near X in $\mathbb{K}_p\langle X\rangle$ for each $p \in \mathbb{P}$. The preceding Density Lemma A7.39 says that this is possible. □

The Exponential Function

Every finite dimensional unital $\mathbb{K}$-algebra is, in particular, a unital Banach algebra over $\mathbb{K}$ with respect to a suitable norm. By Proposition 1.4, in any unital Banach algebra A over $\mathbb{K}$ the group A^{-1} of units is an open subgroup of the monoid $(A, \cdot)$, and it is a (real) linear Lie group with Lie algebra $\mathfrak{L}(A) = A_{\text{Lie}}$, the real vector space underlying A with the Lie bracket given by $[x, y] = xy - yx$, with the exponential function $\exp\colon \mathfrak{L}(A^{-1}) \to A^{-1}$ given by the everywhere absolutely convergent power series $\exp x = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot x^n$. (For $\mathbb{K} = \mathbb{R}$ this is discussed extensively in Chapter 5, notably Definition 5.32.)

Now let A be a weakly complete unital $\mathbb{K}$-algebra. Every closed (2-sided) ideal J of A is a closed Lie algebra ideal of A_{Lie}. We apply the Theorem A7.34 and note that the Lie algebra A_{Lie} is (up to natural isomorphism of topological Lie algebras) the strict projective limit of quotients

$$\lim_{J\in\mathbb{I}(A)} \left(\frac{A}{J}\right)_{\text{Lie}} \subseteq \prod_{J\in\mathbb{I}(A)} \left(\frac{A}{J}\right)_{\text{Lie}}$$

of its finite dimensional quotient algebras Each of these quotient Lie algebras is the domain of an exponential function

$$\exp_{A/J}\colon A_{\text{Lie}}/J \to (A/J)^{-1} \subseteq A/J, \quad (\forall a_J \in A/J)\ \exp_{A/J} a_J = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot a_J^n.$$

This yields a componentwise exponential function on $\prod_{J\in\mathbb{I}(A)} A/J$ which respects the bonding morphisms of the subalgebra $\lim_{J\in\mathbb{I}(A)} A/J$. Thus we obtain the following basic result which one finds in [25, 192].

Theorem A7.41. *If A is a weakly complete unital $\mathbb{K}$-algebra, then the exponential series $1 + a + \frac{1}{2!}a^2 + \cdots$ converges on all of A and defines the exponential* *function*

$$\exp_A : A_{\rm Lie} \to A^{-1}, \quad \exp_A a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$$

of the pro-Lie group A^{-1}. The Lie algebra $\mathfrak{L}(A^{-1})$ of the pro-Lie group A^{-1} may be identified with the topological Lie algebra $A_{\rm Lie}$, whose underlying weakly complete vector space is the underlying weakly complete vector space of A. □

Corollary A7.42. *Let $f: A \to B$ be a surjective morphism of weakly complete unital $\mathbb{K}$-algebras. Then the induced morphism of topological groups $F: A^{-1} \to B^{-1}$ is a quotient morphism of almost connected pro-Lie groups.*

Proof. By the Density Theorem A7.30, A^{-1} is dense in A. So $F(A^{-1})$ is dense in B^{-1}. Abbreviate A^{-1} by G and B^{-1} by H. Then $A_{\rm Lie} = L(G)$ and $B_{\rm Lie} = \mathfrak{L}(H)$. Accordingly, $\mathfrak{L}(F): \mathfrak{L}(G) \to \mathfrak{L}(H)$ agrees with f and therefore is surjective. The image $\exp_G(\mathfrak{L}(G)) = \exp_A(A)$ is in G and so $\exp_H \mathfrak{L}(H) = \exp_H \mathfrak{L}(F)(\mathfrak{L}(G)) = F(\exp_G(\mathfrak{L}(G))) \subseteq F(G)$. But from Proposition A7.24 we know that for any pro-Lie group H with exponential function $\exp_H: \mathfrak{L}(H) \to H$ the closure of the subgroup $\langle \exp_H(\mathfrak{L}(H)) \rangle$ generated by the image of the exponential function is precisely the connected component H_0 of the identity. Therefore $H_0 \subseteq F(G)$, and so $F(G)/H_0$ is dense in H/H_0 since $F(G)$ is dense in H. Since $F(G_0) \subseteq H_0$, the morphism $g \mapsto F(g)H_0 : G \to H/H_0$ induces a morphism $\varphi: G/G_0 \to H/H_0$ with dense image.

It was shown in [78], Theorem 4.1, that G/G_0 is compact. Therefore φ is surjective, which shows that $F(G)H_0 = H$ and so $F(G) = H$, since $H_0 \subseteq F(A)$. The Open Mapping Theorem for pro-Lie Groups (see Theorem 9.60 in [188]) shows that the surjective morphism $F: A^{-1} \to B^{-1}$ is open and thus is a quotient morphism. □

We warn the reader that the references which we cited in the proof from [78] and [188] are not trivial. This indicatees that the Corollary we proved is not superficial.

Proposition A7.37 now has a significantly sharper corollary:

Corollary A7.43. *Let A be any weakly complete unital algebra. Then the projective limit representation of the pro-Lie group A^{-1} of units of A in the form*

$$A^{-1} \cong \lim_{I \in \mathbb{I}(A)} \frac{A^{-1}}{(A^{-1} \cap (1+I))}$$

is a strict projective limit of quotient limit maps.

Proof. This is an immediate consequence of the preceding corollary. □

Postscript

In this Appendix we introduce weakly complete topological vector spaces over the field $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We denote the category of $\mathbb{K}$-vector spaces and linear maps by $\mathcal{V}_\mathbb{K}$ and the category of weakly complete $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps by $\mathcal{W}_\mathbb{K}$. For a $\mathbb{K}$-vector space V we write V^* for its algebraic dual $\mathcal{V}_\mathbb{K}(V,\mathbb{K}) \subseteq \mathbb{K}^V$ equipped with the topology induced from $\mathbb{K}^V$, i.e., the topology of pointwise convergence, and for any weakly complete $\mathbb{K}$-vector space W, we write W' for its topological dual $\mathcal{W}_\mathbb{K}(W,\mathbb{K})$ as an abstract $\mathbb{K}$-vector space. We prove two duality theorems, one which holds for vector spaces over the field $\mathbb{K}$, where $\mathbb{K}$ equals $\mathbb{R}$ or $\mathbb{C}$, and one which holds only over $\mathbb{R}$. The first one is in the spirit of linear algebra and says that the natural evaluation morphisms

$$(A) \qquad \mathrm{ev}_V\colon V \to V^{*\prime}, \quad \mathrm{ev}_V(v)(f) = f(v),$$

and

$$(B) \qquad \mathrm{ev}_W\colon W \to W'^{*}, \quad \mathrm{ev}_W(w)(f) = f(w),$$

are isomorphisms. This means that the categories $\mathcal{V}_\mathbb{K}$ and $\mathcal{W}_\mathbb{K}$ are dual to each other.

The other duality is between two full subcategories of the category of all locally convex Hausdorff topological vector spaces. The first one has as objects those real locally convex vector spaces whose topology is maximal among all its locally convex topologies and the second one has all those complete vector spaces with the smallest locally convex Hausdorff topology. These two categories are duals of each other, and the first category is equivalent to the category of all real vector spaces (without topologies) and the second is equivalent to the category of all weakly complete locally convex vector topological vector spaces.

We let V be a real vector space and endow it with its finest locally convex vector space topology $\mathcal{O}(V)$, and we let W be a weakly complete real topological vector space. Then

(i) V is reflexive; that is, $\eta_V\colon V \to \widehat{\widehat{V}}$ is an isomorphism of topological vector spaces (where $\widehat{V}$ is the Pontryagin dual of the abelian topological group V);
(ii) W is reflexive; that is, $\eta_W\colon W \to \widehat{\widehat{W}}$ is an isomorphism of topological vector spaces;
(iii) The contravariant functor $\widehat{\cdot}\colon \mathbb{ABD} \to \mathbb{ABD}$ exchanges the full subcategory of real vector spaces (given the finest locally convex topology) and the full subcategory of weakly complete topological vector spaces.

We endow each of the categories $\mathcal{V}$ of $\mathbb{K}$-vector spaces and $\mathcal{W}$ of weakly complete vector spaces with tensor products so that each may be considered as a commutative monoidal category $\mathcal{A}$ which supports a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. What is relevant here is that not only does the category $\mathcal{V}$ of $\mathbb{K}$-vector spaces have the familiar tensor product $\otimes_\mathcal{V}$ but that the category $\mathcal{W}$ of weakly complete vector spaces has a tensor product as well. The essence of this tensor product is that for two weakly complete vector spaces $W_1 = \mathbb{K}^X$ and $W_2 = \mathbb{K}^Y$ we have $W_1 \otimes W_2 \cong \mathbb{K}^{X\times Y}$

and that for a natural topological embedding $W_1 \times W_2 \to W_1 \otimes_{\mathcal{W}} \mathcal{W}_2$ denoted $(w_1, w_2) \mapsto w_1 \otimes w_2 : W_1 \times W_2 \to W_1 \otimes W_2$

We see that the category $\mathcal{W}$ together with its tensor product $\otimes_{\mathcal{W}}$ is a commutative monoidal category such that for two $\mathbb{K}$-vector spaces V_1 and V_2 and two weakly complete $\mathbb{K}$-vector spaces W_1 and W_2 we have natural isomorphisms

$$(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^* \quad \text{and} \quad (W_1 \otimes_{\mathcal{W}} W_2)' \cong W_1' \otimes_{\mathcal{V}} W_2'.$$

The symmetric monoidal categories $(\mathcal{V}, \otimes_{\mathcal{V}})$ and $(W, \otimes_{\mathcal{W}})$ are naturally dual.

Before moving on to weakly complete unital algebras we remind the reader about the theory of pro-Lie groups as appeared in [188] and we mentioned the subsequent survey [192]. We noted that the category of pro-Lie groups and continuous homomorphisms is the smallest full subcategory of the category of all topological groups and continuous homomorphisms that contains all finite dimensional Lie groups and is closed under the formation of all limits. Further, every pro-Lie group G has a pro-Lie algebra $\mathfrak{g}$ as Lie-algebra, and the assignment $\mathfrak{L}$ which associates with a pro-Lie group G its pro-Lie algebra is a limit preserving functor.

Finally in this Appendix we discuss weakly complete unital algebras. We note that in any abstract or topological category, an algebra V with a multiplication poses the problem that multiplication $(x, y) \mapsto xy : V \times V \to V$ is not a morphism because it is bilinear rather than linear. The presence of a tensor product "$\otimes$" that transforms bilinearity into linearity is therefore an ideal tool to deal with algebras in a systematic way.

A weakly complete unital algebra is then an associative algebra A over $\mathbb{K}$ with identity, whose underlying vector space is weakly complete, and whose multiplication $(a, b) \mapsto ab : A \times A \to A$ is continuous. A product of any family of finite dimensional associative $\mathbb{K}$-algebras is a weakly complete unital algebra, as is any closed $\mathbb{K}$-subalgebra containing the identity element. The multiplication can be written as a $\mathcal{W}$-morphism $m: A \otimes_{\mathcal{W}} A \to A$ making $(A, \otimes_{\mathcal{W}})$ into a monoid in the symmetric monoidal category $(\mathcal{W}, \otimes_{\mathcal{W}})$. A $\mathbb{K}$-vector space C together with a linear map $c: C \to C \otimes_V C$ of $\mathbb{K}$-vector spaces and a linear map $k: A \to \mathbb{K}$ making C into a comonoid in the symmetric monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}})$ is called a (coassociative and counital) coalgebra (over $\mathbb{K}$).

There is a fundamental theorem attributed to Cartier on these purely algebraic objects. It says that every coalgebra C is the directed union of the set of its finite dimensional subcoalgebras. This can be expressed by saying every coalgebra is the injective limit of its finite dimensional subcoalgebras.

The Fundamental Theorem of Weakly Complete Topological Algebras says that every weakly complete unital topological $\mathbb{K}$-algebra A is the strict projective limit of a projective system of quotient morphisms between its finite dimensional unital quotient-algebras. Further each neighborhood of 0 in A contains an ideal $I \in \mathbb{I}(A)$. So $\lim \mathbb{I}(A) = 0$ and $A \cong \lim_{I \in \mathbb{I}(A)} A/I$. We show that the group of units A^{-1} of a weakly complete unital algebra A is a topological group.

We denote the category of weakly complete unital K-algebras by $\mathcal{A}$ and the category of (Hausdorff) topological groups by $\mathcal{T}$. Then $A \mapsto A^{-1} : \mathcal{A} \to \mathcal{T}$ is readily seen to be a functor preserving arbitrary limits.

We prove that the group A^{-1} of units of a weakly complete unital $\mathbb{K}$-algebra A is a pro-Lie group and $A^{-1} \cong \lim_{I \in \mathbb{I}(A)} (A/I)^{-1}$, where $(A/I)^{-1}$ is a linear Lie group for each $I \in \mathbb{I}(A)$. We also prove the Density Theorem which says that for any weakly complete unital algebra $\mathbb{K}$-algebra A, the group A^{-1} of units is dense in A.

Finally we show that if $f: A \to B$ is a surjective morphism of weakly complete unital $\mathbb{K}$-algebras, then the induced morphism of topological groups $A^{-1} \to B^{-1}$ is a quotient morphism of almost connected pro-Lie groups.

The results in this Appendix which may seem at this point to be curious and only mildly interesting. However they in fact turn out to be absolutely crucial for Part 3 of Chapter 3, which culminates in a new proof of, and approach to, the Tannaka-Hochschild Duality Theorem.

References for this Appendix—Additional Reading

[75], [76], [171].

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