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RECENT DEVELOPMENTS IN STURM-LIOUVILLE THEORY

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Introduction

We discuss recent results in the one- and in two-interval theory of Sturm–Liouville problems. The one-interval theory is covered in Chapters 1–7, the two-interval theory in Chapters 8–10. The extension of the 2-interval theory to finitely many intervals is routine. A list of notations is given in Appendix A, and open problems are given in Appendix B.

The Prũfer transformation is a powerful tool, theoretically and computationally, for studying the eigenvalues and eigenfunctions of self-adjoint Sturm–Liouville problems with *separated boundary conditions*. In 2012, Bailey and Zettl [13] developed an algorithm based on the Prũfer transformation, which can be used to compute the eigenvalues of self-adjoint Sturm–Liouville problems with *coupled* boundary conditions using "families" of problems with separated conditions. This is discussed at the end of Chapter 1.

Problems with periodic coefficients are discussed in Chapter 2. This chapter was motivated, to some extend, by numerous discussions with Shang Yuan Ren, the author of the book *Electronic States in Crystals of Finite Size*, Quantum Confinement of Bloch Waves, Springer Tracts in Modern Physics, 2005, second edition, volume 270, 2017.

It also uses some of the methods used by M. S. P. Eastham in his well-known book *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh and London, 1973, but with the following major differences:

(1) We use quasi-derivatives (py') instead of the classical derivative y'; in particular, for the periodic boundary conditions, we have

$$y(a) = y(b), \quad (py')(a) = (py')(b)$$

instead of

$$y(a) = y(b), \quad y'(a) = y'(b).$$

(2) We do not assume that p is differentiable nor that q and w are piecewise continuous and that w is bounded away from 0.
(3) We do assume that p is positive. This seems to be an oversight by Eastham. If p has positive and negative values, each on a set of positive Lebesgue measure (such as a subinterval, but it need not be a subinterval), then the eigenvalues are unbounded above and below. So there is no unique ordering of the eigenvalues, and consequently λ_n is not well defined.
(4) The quasi-derivative (py') is continuous on the interval $[a,b]$, whereas the classical derivative $y'(t)$ may not exist for all t in $[a,b]$.
(5) We use the interval $(0,\pi)$, instead of $(0,1)$, to parameterize the complex self-adjoint boundary conditions. This provides a simple visualization of the "movement" of the eigenvalues $\lambda_n(\gamma)$ on the unit circle of the complex plane relative to the points 0 and π, which correspond to the periodic and semiperiodic eigenvalues.

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(6) We use a notation that makes it easier to "keep track" of the dependence of the eigenvalues on the many parameters a, b, k, n, π, θ, etc. of the problem. This dependence sometimes requires a very delicate analysis.

It is well known that for h-periodic coefficients, the eigenvalues of complex self-adjoint boundary conditions on the base interval $(a, a+h)$ are related to the periodic eigenvalues on the larger intervals $(a, a+hk)$, $k = 1, 2, 3, 4, \ldots$ For fixed a, h, let $\lambda_n(\gamma)$, $\lambda_n^P(k)$, and $\lambda_n^S(k)$ denote the complex, periodic, and semiperiodic eigenvalues on the intervals $(a, a+hk)$, $k = 1, 2, 3, 4, \ldots$. In 2017, 2018 Yuan, Sun, and Zettl [107, 108] found a similar relationship for the semiperiodic boundary conditions and – for both cases – found a one-to-one correspondence between the eigenvalues of the complex self-adjoint boundary conditions on the base interval $J = (a, a+h)$ and the periodic and semiperiodic boundary conditions on the larger intervals. The complex boundary conditions on J can be parameterized by $\gamma \in (0, \pi)$.

Let $P(k) = \bigcup_{n=0}^{\infty} \lambda_n^P(k), S(k) = \bigcup_{n=0}^{\infty} \lambda_n^S(k)$, and $\Gamma(\gamma) = \bigcup_{n=0}^{\infty} \lambda_n(\gamma)$ denote the periodic, semiperiodic, and complex eigenvalues. The sets of periodic and semiperiodic eigenvalues $P(k)$ and $S(k)$ are countable; the set of complex eigenvalues $\Gamma(\gamma)$, $\gamma \in (0, \pi)$, is not countable. Given $\lambda_n^P(k)$ for any $n = 1, 2, 3, \ldots$ for which γ and which m is $\lambda_n^P(k) = \lambda_m(\gamma)$? This question and a similar question for semiperiodic eigenvalues is answered in Chapter 2.

Chapters 3 and 4 discuss various extensions of the classical Sturm–Liouville theory, including the Atkinson extension. In his classical book, Atkinson [4] hints at the existence of self-adjoint regular Sturm–Liouville problems with finite spectrum. This was confirmed in 2001 by Kong, Wu, and Zettl [65]. These authors later showed that Sturm–Liouville problems of "Atkinson type" are equivalent to matrix problems.

Chapter 5 discusses the inverse theory for problems with finite spectrum developed by Kong and Zettl [69] in 2012. This finite spectrum inverse theory applies to problems with both the leading coefficient p and the potential function q and weight function w in contrast to the infinite spectrum inverse theory, where p and w are assumed to be the constant function 1.

The eigenvalues below the essential spectrum of singular problems developed by Zhang, Sun, and Zettl [114] are discussed in Chapter 6. For operators that are bounded below we can now claim to understand the continuous dependence of the eigenvalues of self-adjoint Sturm–Liouville problems on the boundary conditions. Chapter 7 discusses results on lambda-dependent boundary conditions, also found by these authors.

Recently, there has been a lot of interest in the literature of self-adjoint Sturm–Liouville problems with discontinuous boundary conditions specified at regular interior points of the underlying interval. Such conditions are known by various names including transmission conditions [1, 2, 82, 87, 88, 98], interface conditions [61, 76, 92, 109], discontinuous conditions [51, 91, 81], multipoint conditions [55, 76, 36, 112], point interactions (in the physics literature) [42, 21, 23, 35], conditions on trees, graphs, or networks [90, 87, 88], etc. For an informative survey of such problems arising in applications, including an extensive bibliography and historical notes, see Pokornyi and-Borovskikh [87] and Prokornyi and Pryadiev [88]. These problems are not covered by the classical Sturm–Liouville theory since, in this theory, solutions and their quasi-derivatives are continuous at all interior points of the underlying interval $J = (a, b)$. In particular, this applies to all eigenfunctions. These two-interval problems are introduced in Chapter 8. Chapter 9 develops the Neuberger construction of the two-interval Green's function.

Chapter 10 is based on the 2011 paper by Littlejohn and Zettl [74]. It discusses the Legendre equation and its self-adjoint operators on the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$, and $(-\infty, \infty)$ in detail. Most of the results discussed here can be inferred from known results scattered widely in the literature; others require some additional work. Some are new in this paper, for example, the construction of a regular Legendre equation on the interval $(-1, 1)$, which is equivalent to the classical singular Legendre equation on the same interval. It is remarkable that we can find some new results about this well-studied classical equation and its associated operators.

Appendix A is a list of notations used.

Appendix B discusses some open problems. These problems are "open" as far as the author knows at the time of this writing and are stated in random order. Some may be intractable, some accessible but challenging, and others routine.

> The world of Mathematics is full of wonders and of mysteries, at least as much so as the physical world. Without Mathematics (M) there would be no Science (S), without Science there would be no Engineering (E), and without Science and Engineering there would be no modern Technology (T). STEM should be spelled MSET.
>
> Mathematics exists in all Galaxies and in all Universes.

Contents

Part I: **One-interval problems**

1 Classical regular self-adjoint problems

1.1 Introduction

In this chapter, we discuss properties of the eigenvalues of classical regular self-adjoint Sturm–Liouville problems. Such a problem consists of the equation

$$My = -(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty \le a < b \le \infty, \tag{1.1}$$

with coefficients satisfying

$$1/p, q, w \in L^1(J,\mathbb{R}), \quad p > 0, \quad w > 0 \quad \text{a. e. on } J, \tag{1.2}$$

and boundary conditions

$$AY(a) + BY(b) = 0, \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix} \tag{1.3}$$

satisfying

$$A, B \in M_2(\mathbb{C}), \quad AEA^* = BEB^*, \quad \operatorname{rank}(A : B) = 2, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.4}$$

Here $M_2(\mathbb{C})$ denotes the 2×2 matrices with complex entries. Recall the system formulation of equation (1.1):

$$Y' = (P - \lambda W)Y \tag{1.5}$$

with

$$P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}. \tag{1.6}$$

From (1.2) it follows that

$$Y(a) = \begin{bmatrix} y(a) \\ (py')(a) \end{bmatrix}, \quad Y(b) = \begin{bmatrix} y(b) \\ (py')(b) \end{bmatrix}$$

exist as finite limits so that the boundary conditions (1.3) are well defined.

Definition 1.1.1. Let $\Phi(t,u,P,w,\lambda)$ be the primary fundamental matrix of (1.5) and recall that

$$\Phi'(t) = [P(t) - \lambda W(t)]\Phi(t), \quad \Phi(u,u,\lambda) = I, \quad a \le u, t \le b, \quad \lambda \in \mathbb{C}. \tag{1.7}$$

Define the characteristic function δ by

$$\delta(\lambda) = \delta(a,b,A,B,P,w,\lambda) = \det[A + B\Phi(b,a,P,w,\lambda)], \quad \lambda \in \mathbb{C}. \tag{1.8}$$

Definition 1.1.2. This function δ is the characteristic function of problem (1.1)–(1.4).

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Lemma 1.1.1. *Let* (1.1) *to* (1.4) *hold, and let* $\delta(\lambda)$ *be defined by* (1.8). *Then* $\delta(\lambda)$ *is an entire function of* $\lambda \in \mathbb{C}$, *and its zeros are precisely the eigenvalues of problem* (1.1)–(1.4).

Proof. The fact that $\delta(\lambda)$ is an entire function of λ is well known. A direct computation shows that its zeros are precisely the eigenvalues of the problem. □

It is convenient to classify the self-adjoint boundary conditions into two mutually exclusive classes, separated and coupled.

Lemma 1.1.2 (Separated boundary conditions). *Let* (1.1)–(1.4) *hold. Fix* P, W, J *and assume that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \end{bmatrix}.$$

Then $\delta(\lambda) = -A_2B_1\phi_{11}(b,a,\lambda) - A_2B_2\phi_{21}(b,a,\lambda) + A_1B_1\phi_{12}(b,a,\lambda) + A_1B_2\phi_{22}(b,a,\lambda)$ *for* $\lambda \in \mathbb{C}$.

Proof. This follows from the definition of δ and a direct computation. □

The characterization of the eigenvalues as zeros of $\delta(\lambda)$ reduces to a simpler and more informative form when the boundary conditions are coupled. This reduction is given by the next lemma.

Lemma 1.1.3 (Coupled self-adjoint boundary conditions). *Let* (1.1)–(1.4) *hold. Let* $\Phi = (\phi_{ij})$ *be the primary fundamental matrix of system* (1.5). *Fix* P, W, J *and assume that*

$$B = -I, \quad A = e^{i\gamma}K, \quad -\pi < \gamma \leq \pi, \quad K \in \mathrm{SL}_2(\mathbb{R}), \tag{1.9}$$

that is, K *is a real* 2×2 *matrix with determinant* 1. *Let* $K = (k_{ij})$ *and define*

$$D(\lambda, K) = k_{11}\phi_{22}(b,a,\lambda) - k_{12}\phi_{21}(b,a,\lambda) - k_{21}\phi_{12}(b,a,\lambda) + k_{22}\phi_{11}(b,a,\lambda) \tag{1.10}$$

for $\lambda \in \mathbb{C}$. *Note that* $D(\lambda, K)$ *does not depend on* γ. *Then*

(1) *The real number* λ *is an eigenvalue of* (1.1)–(1.4) *if and only if*

$$D(\lambda, K) = 2\cos\gamma, \quad -\pi < \gamma \leq \pi. \tag{1.11}$$

(2) *If* λ *is an eigenvalue for* $A = e^{i\gamma}K, B = -I$, $0 < \gamma < \pi$, *with eigenfunction* u, *then* λ *is also an eigenvalue for* $A = e^{-i\gamma}K, B = -I$, *but with eigenfunction* $\overline{u}$.

Proof. Since Φ is a primary fundamental matrix, we have $\det \Phi(b,a,\lambda) = 1$. We abbreviate $(\phi_{ij}(b,a,\lambda))$ to ϕ_{ij}. Noting that $\det K = 1$, we get

$$\begin{aligned}\delta(\lambda) = \det(e^{i\gamma}K - \Phi) &= \begin{vmatrix} e^{i\gamma}k_{11} - \phi_{11} & e^{i\gamma}k_{12} - \phi_{12} \\ e^{i\gamma}k_{21} - \phi_{21} & e^{i\gamma}k_{22} - \phi_{22} \end{vmatrix} \\ &= 1 + e^{2i\gamma} - e^{i\gamma}D(\lambda).\end{aligned}$$

Hence $\delta(\lambda) = 0$ if and only if (1.11) holds. Part (2) follows from (1.11) and by taking conjugates of equation (1.1). □

Remark 1.1.1. Throughout Chapters 1 and 2, we assume that (1.1)–(1.4) hold and use the notations (1.5)–(1.11).

1.2 Self-adjoint operators in Hilbert space

In this section, we survey self-adjoint operator realizations of equation (1.1) in the Hilbert space $L^2(J, w)$ determined by two-point boundary conditions and their spectrum $\{\lambda_n : n \in \mathbb{N}_0\}$. The dependence of the eigenvalues of regular self-adjoint Sturm–Liouville problems (SLP) on all parameters of the problem, the coefficients, the endpoints of the domain interval J, and the boundary conditions, is now well understood due to some surprisingly recent results given the long history and voluminous literature of Sturm–Liouville problems dating back at least to the seminal 1836 paper of Sturm and Liouville.

Notation 1.2.1. *$M_2(\mathbb{C})$ denotes the 2×2 matrices over the complex numbers $\mathbb{C}$, and $L^1(J, \mathbb{R})$ denotes the real-valued Lebesgue-integrable functions on the entire interval J. We also use the notation $M_2(\mathbb{R})$ for the real 2×2 matrices and $L_{\mathrm{loc}}(J, \mathbb{R})$ for the real-valued functions integrable on all compact subintervals of J. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.*

Lemma 1.2.1. *Let $\lambda \in \mathbb{C}$. Let $1/p, q, w \in L^1(J, \mathbb{R})$. Then the appropriate one-sided limits*

$$\lim_{t\to a} y(t), \quad \lim_{t\to a}(py')(t), \quad \lim_{t\to b} y(t), \quad \lim_{t\to b}(py')(t), \tag{1.12}$$

exist and are finite for each solution y and its quasi-derivative (py'). Furthermore, every initial value problem has a unique solution y defined on the entire interval J, and both y and (py') are continuous at each t, $a \le t \le b$.

Proof. See Everitt and Race [34] and [113]. □

Remark 1.2.1. Note that each endpoint of J can be finite or infinite. In much of the literature an infinite endpoint is automatically classified as singular in contrast to here. From the basic existence–uniqueness theorem (see Theorem 1.2.1 in [113]) it follows that each solution y and its quasi-derivative (py') are continuous for all $t \in J$ and, by Lemma 1.2.1, can be continuously extended to the (finite or infinite) endpoints a, b. Also note that, under condition (1.13), $y'(t)$ may not exist for some $t \in J$. This is the main reason for using the quasi-derivative (py') as one function. Note the parentheses around py' since $(py')(t)$ not always can be separated into $p(t)y'(t)$. The existence of the limits (1.12) shows that the boundary condition (1.3) is well defined. Although (1.3) and (1.4) consist of two independent conditions, we refer to the pair (1.3)–(1.4) as one self-adjoint boundary condition.

Remark 1.2.2. Why only "two-point" boundary conditions of the form (1.3)? Rather than three-point conditions or integral conditions or others? The answers can be seen from the next theorem.

Definition 1.2.1. Let $H = L^2(J, w)$ and define

$$D_{\max} = \left\{ f \in H : \frac{1}{w} Mf \in H \right\},$$
$$S_{\max} f = Mf \quad (f \in D_{\max}). \tag{1.13}$$

Theorem 1.2.1. *Let $D_{\max}$ be defined by* (1.13). *Then $D_{\max}$ is dense in H. Let $S_{\min} = S^*_{\max}$ and denote its domain by $D_{\min}$. Then $S_{\min} \subset S_{\max}$. Define the operator S from H to H by $Sy = \frac{1}{w} My$ for all $y \in H$ satisfying the two-point boundary conditions* (1.3)–(1.4). *Then S satisfies*

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{1.14}$$

Furthermore, if S satisfies (1.14) *and is generated by two-point boundary conditions* (1.3), *these conditions satisfy* (1.4).

Proof. This is well known; see [113].

It is clear from Theorem 1.2.1 that – for a fixed equation (1.1) – the operators S satisfying (1.14) differ from each other only by their domains. In the rest of this chapter, we survey how the eigenvalues of each operator S change when the coefficients and the boundary conditions, including the endpoints, change. This is now, due to some surprisingly recent results, well understood. In Section 1.7, we survey inequalities among eigenvalues of different boundary conditions. □

1.3 Canonical forms of self-adjoint boundary conditions

The self-adjoint boundary condition (1.3)–(1.4) is homogeneous and thus clearly invariant under left multiplication by a nonsingular matrix. This is a serious obstacle to studying the continuous dependence of the eigenvalues on the boundary condition and for their numerical computation. In preparation for the investigation of how eigenvalues change when the boundary condition is changed, in this section, we discuss canonical forms of self-adjoint boundary conditions.

At first glance, it may seem that the self-adjoint boundary conditions (1.3)–(1.4) always connect the endpoints a, b with each other. This is not the case: they can be divided into three mutually exclusive classes: separated, real coupled, and complex coupled. The three classes are:

(1) **Separated self-adjoint BCs.** These are

$$A_1 y(a) + A_2 (py')(a) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0), \tag{1.15}$$
$$B_1 y(b) + B_2 (py')(b) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0,0). \tag{1.16}$$

These separated conditions can be parameterized as follows:

$$\cos\alpha y(a) - \sin\alpha(py')(a) = 0, \quad 0 \le \alpha < \pi, \tag{1.17}$$
$$\cos\beta y(b) - \sin\beta(py')(b) = 0, \quad 0 < \beta \le \pi, \tag{1.18}$$

by choosing $\alpha \in [0,\pi)$ such that

$$\tan\alpha = \frac{-A_2}{A_1} \quad \text{if } A_1 \ne 0, \quad \text{and} \quad \alpha = \pi/2 \quad \text{if } A_1 = 0, \tag{1.19}$$

and similarly, by choosing $\beta \in (0,\pi]$ such that

$$\tan\beta = \frac{-B_2}{B_1}, \quad \text{if } B_1 \ne 0, \quad \text{and} \quad \beta = \pi/2 \quad \text{if } B_1 = 0. \tag{1.20}$$

Note the different normalization in (1.20) for β from that used for α in (1.19). This is for convenience in using the Prüfer transformation, which is widely used for the theoretical studies of eigenvalues and their eigenfunctions and for the numerical computation of these. For example, the FORTRAN code SLEIGN2 [10, 13, 12, 9] uses this normalization. This code can be downloaded free from the internet and comes with a user-friendly interface.

(2) **All *real* coupled self-adjoint BCs.** These can be formulated as follows:

$$Y(b) = KY(a), \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \tag{1.21}$$

where $K \in SL_2(\mathbb{R})$, that is, K satisfies

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad k_{ij} \in \mathbb{R}, \quad \det K = 1. \tag{1.22}$$

(3) **All *complex* coupled self-adjoint BCs.** These are:

$$Y(b) = e^{i\gamma}KY(a), \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \tag{1.23}$$

where K satisfies (1.22), and $-\pi < \gamma < 0$ or $0 < \gamma < \pi$.

Lemma 1.3.1. *Given a boundary condition* (1.3)–(1.4)*, it is equivalent to exactly one of the separated, real coupled, or complex coupled boundary conditions defined above, and each of these conditions can be written in the form* (1.3)–(1.4).

Proof. See [113]. □

Notation 1.3.1. *Given the canonical forms of the boundary conditions, we use the following notation for the eigenvalues:*

$$\lambda_n(a,b,\alpha,\beta,p,q,w),\ \lambda_n(a,b,K,p,q,w),\ \lambda_n(a,b,\gamma,K,p,q,w), \quad n \in \mathbb{N}_0. \tag{1.24}$$

To study the dependence of the eigenvalues for a fixed equation, we abbreviate this notation to $\lambda_n(\alpha,\beta)$, $\lambda_n(K), \lambda_n(\gamma, K)$*; to study the eigenvalues for a fixed boundary condition, the notation is abbreviated to* $\lambda_n(p,q,w)$. *Since the eigenvalues depend not on p, but rather on* $1/p$*, we should use* $1/p$ *in* (1.24)*, but since the use of p is so well established in the literature, we continue to use* (1.24). *Note that this notation covers all self-adjoint boundary conditions. Since each of these has a unique representation as a separated, real coupled, or complex coupled condition, we can study how the eigenvalues change when the boundary condition changes. The existence of eigenvalues is discussed in the next section.*

Remark 1.3.1. This unique representation of the boundary conditions as separated, real coupled, or complex coupled is of fundamental importance for the study of the theoretical properties of the eigenvalues as functions of the boundary conditions (e. g., continuity) as well as for their numerical computation. Although the characterization (1.3)–(1.4) of the self-adjoint boundary conditions extends naturally to equations of general order $n > 2$,

$$AY(a) + BY(b) = 0$$

with solution vector Y, where A, B satisfy

$$A, B \in M_n(\mathbb{C}), \quad AEA^* = BEB^*, \quad \operatorname{rank}(A : B) = n,$$

and all entries of E are zeros except those on the counter diagonal that alternate between -1 and $+1$, there is no *comparable* canonical representation of the self-adjoint boundary conditions for $n > 2$. Recently, for $n = 4$, Wang, Sun, and Zettl [101] have shown that there are three classes of self-adjoint boundary conditions, separated, coupled, and mixed, and found canonical forms for each class. The separated and coupled canonical forms for $n = 4$ are more complicated than the corresponding ones for $n = 2$.

Remark 1.3.2. In this remark, we comment on what happens when the normalization conditions $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$ are violated. For fixed $\beta \in (0,\pi]$, as $\alpha \to 0^-$, $\lambda_n(\alpha,\beta)$ has an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n \in \mathbb{N}$. Similarly, for fixed $\alpha \in [0,\pi)$, as $\beta \to \pi^+$, $\lambda_n(\alpha,\beta)$ has an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n \in \mathbb{N}$. In each case where $\lambda_n(\alpha,\beta)$ has a jump discontinuity the eigenvalue can be embedded in a "continuous eigenvalue branch", which is defined by two indices n and $n + 1$; in other words, the eigencurves from the left and right of the point where the jump occurs "match up" continuously when one of the indices n is changed to $n + 1$. Furthermore, the resulting matched eigencurve determined by two consecutive indices is not only continuous but also differentiable everywhere including at the matched point.

1.4 Existence of eigenvalues

Given a self-adjoint realization S of equation (1.12) in H, what is its spectrum $\sigma(S)$? This is the question we discuss in this section.

Theorem 1.4.1. *Let S satisfy* (1.14). *Then the spectrum of S is bounded below and discrete. Furthermore:*

(1) *There are an infinite but countable number of eigenvalues with no finite accumulation point.*

(2) *The eigenvalues can be ordered to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots; \quad \lambda_n \to +\infty \quad \text{as } n \to \infty. \tag{1.25}$$

Each eigenvalue may be simple or double, but there cannot be two consecutive equalities in (1.25) *since, for any value of λ, equation* (1.12) *has exactly two linearly independent solutions. Note that λ_n is well defined for each $n \in \mathbb{N}_0$ but there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue since every nontrivial solution of the equation for such an eigenvalue is an eigenfunction. Let $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$, where the eigenvalues are ordered to satisfy* (1.25).

(3) *If the boundary condition is separated, then strict inequality holds everywhere in* (1.25). *Furthermore, if u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples and has exactly n zeros in the open interval $J = (a, b)$ for each $n \in \mathbb{N}_0$.*

(4) *Let S be determined by a real coupled boundary condition matrix K, and let u_n be a real-valued eigenfunction of $\lambda_n(K)$. Then the number of zeros of u_n in the open interval J is* 0 *or* 1 *if $n = 0$ and $n-1$, n, or $n+1$ if $n \ge 1$.*

(5) *Let S be determined by a complex coupled boundary condition (K, γ), and let $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$. Then all eigenvalues are simple, and strict inequality holds everywhere in* (1.25). *Moreover, if u_n is an eigenfunction of λ_n, then the number of zeros of* $\operatorname{Re} u_n$ *on $[a, b)$ is* 0 *or* 1 *if $n = 0$ and $n-1$, n, or $n+1$ if $n \ge 1$. The same conclusion holds for* $\operatorname{Im} u_n$. *Moreover, u_n has no zero in $[a, b]$, $n \in \mathbb{N}_0$.*

(6) *For any self-adjoint boundary condition, separated, real coupled, or complex coupled, we have the following asymptotic formula:*

$$\frac{\lambda_n}{n^2} \to c = \pi^2 \left(\int_a^b \sqrt{\frac{w}{p}} \right)^{-2} \quad \text{as } n \to \infty. \tag{1.26}$$

Proof. See [113]. □

Remark 1.4.1. Note that Theorem 1.4.1 justifies notation (1.24). Thus for each S satisfying (1.14), we have that the spectrum $\sigma(S)$ of S is given by

(1) $\sigma(S) = \{\lambda_n(\alpha, \beta), n \in \mathbb{N}_0\}$ if the boundary condition of S is separated and determined by the parameters α, β;

(2) $\sigma(S) = \{\lambda_n(K), n \in \mathbb{N}_0\}$ if the boundary condition of S is real coupled with coupling constant K;
(3) $\sigma(S) = \{\lambda_n(\gamma, K), n \in \mathbb{N}_0\}$ if the boundary condition of S is complex coupled with coupling constants K, γ.

Remark 1.4.2. Canonical forms of the boundary conditions make it possible to introduce the notation of Remark 1.4.1. This notation identifies λ_n uniquely and makes it possible to study the dependence of the eigenvalues on the boundary conditions and on the coefficients as well as inequalities among eigenvalues of different boundary conditions. As mentioned before, no comparable canonical representation of all self-adjoint boundary conditions is known for higher-order ordinary differential equations. There are some recent results by Hao et al. [48, 46], but these are much more complicated and thus more difficult to use for the study of the dependence of the eigenvalues on the problem. This is a major open problem for $n > 2$.

The next result shows what happens to the eigenvalues when the interval shrinks to one endpoint. This study was motivated by a problem in fuel cell dynamics [8].

Theorem 1.4.2. *Let $c \in (a, b)$, and let $\{\lambda_n(c, \alpha, \beta); n \in \mathbb{N}_0\}$, $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$, denote the eigenvalues on the interval (a, c) with all other parameters fixed. Then:*
(1)

$$\lambda_n(c, \alpha, \beta) \to +\infty \quad \text{as } c \to a^+, \quad n \in \mathbb{N}. \tag{1.27}$$

(2) *If $\alpha < \beta$, then*

$$\lambda_0(c, \alpha, \beta) \to +\infty \quad \text{as } c \to a^+. \tag{1.28}$$

(3) *If $\alpha > \beta$, then*

$$\lambda_0(c, \alpha, \beta) \to +\infty \quad \text{as } c \to a^+. \tag{1.29}$$

(4) *$\lambda_n(c, \alpha, \beta)$ may have a finite limit as $c \to a^+$ if and only if $\alpha = \beta$ and $n = 0$.*
(5) *Similar results hold at the endpoint b.*

Proof. See the paper by Kong, Wu, and Zettl [66]. □

Remark 1.4.3. Kong, Wu, and Zettl [66], under certain conditions on the coefficients, found the finite limits of $\lambda_0(c, \alpha, \alpha)$ as $c \to a^+$ and showed that these finite limits do not always exist. They may be $\pm\infty$, or they may not exist.

1.5 Continuity of eigenvalues

In this section, we survey the continuity of the eigenvalues as functions of each parameter of the problem. Recall notation (1.24) for the eigenvalues:

$$\lambda_n(a,b,\alpha,\beta,p,q,w),\ \lambda_n(a,b,K,p,q,w),\ \lambda_n(a,b,\gamma,K,p,q,w), \quad n \in \mathbb{N}_0.$$

When we study the dependence on one parameter x with the others fixed, we abbreviate the notation to $\lambda_n(x)$; thus $\lambda_n(q)$ indicates that we are studying λ_n as a function of $q \in L^1(J,\mathbb{R})$ with all other parameters of the problem fixed, $\lambda_n(a)$ indicates that we are studying λ_n as a function of the left endpoint with all other parameters fixed, and so on.

The eigenvalues are continuous functions of each of $\frac{1}{p}, q, w, a, b$;, in general, they are not continuous functions of the boundary conditions. The continuity on the coefficients $\frac{1}{p}, q, w$ is with respect to the $L^1(J,\mathbb{R})$ norm, the continuity on K is with respect to any matrix norm, and the continuity with respect to $a, b, \alpha, \beta, \gamma$ is in the reals $\mathbb{R}$. It is shown in [60] (see also [113]) that even though, in general, λ_n is not a continuous function of the boundary conditions for fixed n, it can always be embedded in a "continuous branch" of eigenvalues by varying the index n. For separated boundary conditions, there is a jump discontinuity when either $y(a) = 0$ or $y(b) = 0$. The coupled boundary conditions at which the eigenvalues are not continuous are characterized in [113], and it is shown that all discontinuities are finite or infinite jumps. We call the set of boundary conditions at which the eigenvalues have discontinuities "the jump set" since all discontinuities are of the jump type.

We start with the continuous dependence on the coefficients and endpoints.

Theorem 1.5.1 (Kong, Wu, and Zettl). *Let $n \in \mathbb{N}_0$. Then:*
(1) $\lambda_n(1/p)$ *is a continuous function of* $1/p \in L^1(J,\mathbb{R})$*;*
(2) $\lambda_n(q)$ *is a continuous function of* $q \in L^1(J,\mathbb{R})$*;*
(3) $\lambda_n(w)$ *is a continuous function of* $w \in L^1(J,\mathbb{R})$*;*
(4) $\lambda_n(a)$ *is a continuous function of* a*; and*
(5) $\lambda_n(b)$ *is a continuous function of* b*.*

Proof. See Section 2 of Kong, Wu, and Zettl [60]. □

Next, we characterize the boundary conditions at which λ_n is not continuous, and we call this set the "jump" set since all discontinuities are of jump type.

Definition 1.5.1 (Jump set of boundary conditions). The "jump set of boundary conditions" $\mathbb{J}$ is the union of
(1) the (real and complex) coupled conditions

$$Y(b) = e^{i\gamma} K Y(a), \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \quad -\pi < \gamma \le \pi, \tag{1.30}$$

where the 2×2 matrix $K = (k_{ij}) \in \mathrm{SL}(2,\mathbb{R})$ satisfies $k_{12} = 0$, and

(2) the separated boundary conditions

$$A_1y(a) + A_2(py')(a) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0), \tag{1.31}$$
$$B_1y(b) + B_2(py')(b) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0,0),$$

satisfying $A_2B_2 = 0$. Note that these are precisely the conditions where either $\alpha = 0$ or $\beta = \pi$ or both $\alpha = 0$ and $\beta = \pi$.

Theorem 1.5.2. *Let $n \in \mathbb{N}_0$. Let $\mathbb{J}$ be given by Definition* 1.5.1. *Then:*
(1) *If the boundary condition is not on the jump set $\mathbb{J}$, then λ_n is a continuous function of the boundary condition.*
(2) *If $n \in \mathbb{N}$, $k_{12} = 0$, and $\lambda_n = \lambda_{n-1}$, then λ_n is continuous at K.*
(3) *The lowest eigenvalue λ_0 has an infinite jump discontinuity at each separated or (real or complex) coupled boundary condition in $\mathbb{J}$.*
(4) *Let $n \in \mathbb{N}$. If the boundary condition is in $\mathbb{J}$ and λ_n is simple, then λ_n has a finite jump discontinuity at this boundary condition.*

Proof. See Section 3 in [60]. □

For the important particular case of separated boundary conditions in canonical form, there is a stronger result:

Lemma 1.5.1. *For any $n \in \mathbb{N}_0$, $\lambda_n(\alpha, \beta)$ is jointly continuous on $[0, \alpha) \times (0, \pi]$ and strictly decreasing in α for each fixed β and strictly increasing in β for each fixed α.*

Proof. See [60]. □

The next theorem gives more detailed information about separated boundary conditions not in canonical form, in particular, for the separated jump boundary conditions.

Theorem 1.5.3 (Everitt, Möller, and Zettl). *Fix a, b, p, q, w and consider conditions* (1.31).
– *Fix B_1, B_2 and let $A_1 = 1$. Consider $\lambda_n = \lambda_n(A_2)$ as a function of $A_2 \in \mathbb{R}$. Then for each $n \in \mathbb{N}_0, \lambda_n(A_2)$ is continuous at A_2 for $A_2 > 0$ and $A_2 < 0$ but has a jump discontinuity at $A_2 = 0$. More precisely, we have:*
  (1) $\lambda_n(A_2) \to \lambda_n(0)$ *as* $A_2 \to 0^-$, $n \in \mathbb{N}_0$.
  (2) $\lambda_0(A_2) \to -\infty$ *as* $A_2 \to 0^+$.
  (3) $\lambda_{n+1}(A_2) \to \lambda_n(0)$ *as* $A_2 \to 0^+$.
– *Fix A_1, A_2 and let $B_1 = 1$. Consider $\lambda_n = \lambda_n(B_2)$ as a function of $B_2 \in \mathbb{R}$. Then for each $n \in \mathbb{N}_0, \lambda_n(B_2)$ is continuous at B_2 for $B_2 > 0$ and $B_2 < 0$ but has a jump discontinuity at $B_2 = 0$. More precisely, we have:*
  (1) $\lambda_n(B_2) \to \lambda_n(0)$ *as* $B_2 \to 0^+$, $n \in \mathbb{N}_0$.
  (2) $\lambda_0(B_2) \to -\infty$ *as* $B_2 \to 0^-$.
  (3) $\lambda_{n+1}(B_2) \to \lambda_n(0)$ *as* $B_2 \to 0^-$.

Proof. See Everitt, Möller, and Zettl [32, 33]. □

Remark 1.5.1. Note that $\lambda_0(A_2)$ has an infinite jump discontinuity at $A_2 = 0$, but for all $n \geq 1$, $\lambda_n(A_2)$ has a finite jump discontinuity at $A_2 = 0$, and $\lambda_n(A_2)$ is left but not right continuous at 0. Similarly, $\lambda_0(B_2)$ has an infinite jump discontinuity at $B_2 = 0$, but for all $n \geq 1$, $\lambda_n(B_2)$ has a finite jump discontinuity at $B_2 = 0$, and $\lambda_n(B_2)$ is right but not left continuous at 0. In all cases, $\lambda_n(0)$ is embedded in a continuous branch of eigenvalues as A_2 or B_2 passes through zero, but this branch is not given by a fixed index n; to preserve the continuity, the index "jumps" from n to $n+1$ as A_2 or B_2 passes through zero from the appropriate direction.

Remark 1.5.2. This forced "index jumping" to stay on a continuous branch of eigenvalues plays an important role in some of the algorithms and their numerical implementations used in the code SLEIGN2 [10] for the numerical approximation of the spectrum of regular and singular Sturm–Liouville problems.

Remark 1.5.3. This "index jumping" phenomenon to stay on a "continuous eigenvalue branch" is quite general: It applies to all simple eigenvalues for all boundary conditions on the jump set $\mathbb{J}$, separated, real coupled, or complex coupled. For details, the reader is referred to [113], Theorems 3.39, 3.73, and 3.76, and Propositions 3.71 and 3.72 in [60].

Remark 1.5.4. Kong and Zettl [68] have shown that each continuous eigenvalue branch is in fact differentiable everywhere including the point $A_2 = 0$ (or $B_2 = 0$) where the index jumps. This also follows from Möller and Zettl [79].

Remark 1.5.5. It is remarkable that if the boundary condition is in $\mathbb{J}$ and λ_n is simple, then it can be embedded in a continuous eigenvalue branch, and this branch is differentiable. Möller and Zettl [79] extended this result to abstract operators in a Banach space.

1.6 Differentiability of eigenvalues

Now that as the continuities of λ_n have been characterized, it is natural to investigate the differentiability of λ_n as a function of the parameters of the problem. We embark upon this next. Here for each $n \in \mathbb{N}_0$, u_n denotes a normalized eigenfunction of $\lambda_{n.}$ For all cases except where $\gamma \neq 0$, we choose u_n to be real valued.

Theorem 1.6.1 (Kong and Zettl). *Let* (1.1)–(1.4) *hold, and let* $n \in \mathbb{N}_0$.

(1) *Assume that p, q, w are continuous at a and $p(a) \neq 0$. Then $\lambda_n(a)$ is differentiable at a, and*

$$\lambda_n'(a) = \frac{1}{p(a)}\left|pu_n'\right|^2(a) - |u_n|^2(a)\left[q(a) - \lambda_n(a)w(a)\right]. \tag{1.32}$$

(2) *Assume that* p, q, w *are continuous at* b *and* $p(b) \neq 0$. *Then* $\lambda_n(b)$ *is differentiable at* b, *and*

$$\lambda_n'(b) = -\frac{1}{p(b)}|pu_n'|^2(b) + |u_n|^2(b)[q(b) - \lambda_n(b)w(b)]. \tag{1.33}$$

(3) *Let* $-\pi < \gamma < 0$ *or* $0 < \gamma < \pi$. *Then* $\lambda_n(\gamma)$ *is differentiable at* γ, *and*

$$\lambda_n'(\gamma) = -2\,\mathrm{Im}[u_n(b)(pu_n')(b)], \tag{1.34}$$

where $\mathrm{Im}[z]$ *denotes the imaginary part of* z.

(4) *Let* $\alpha \in (0, \pi)$. *Then* $\lambda_n(\alpha)$ *is differentiable, and its derivative is given by*

$$\lambda_n'(\alpha) = -u^2(a) - (pu')^2(a). \tag{1.35}$$

(5) *Let* $\beta \in (0, \pi)$. *Then* $\lambda_n(\beta)$ *is differentiable, and its derivative is given by*

$$\lambda_n'(\beta) = u^2(b) + (pu')^2(b). \tag{1.36}$$

Proof. See [68]. □

Next, we survey the differentiability of the eigenvalues with respect to the remaining parameters: $\frac{1}{p}, q, w$, and K.

Theorem 1.6.2 (Kong and Zettl). *Let* (1.1)–(1.4) *hold, and let* $n \in \mathbb{N}_0$.

(1) *Assume that* $\lambda_n(q)$ *is a simple eigenvalue with real-valued normalized eigenfunction* $u_n(\cdot, q)$. *Then* $\lambda_n(\cdot, q)$ *is differentiable in* $L^1(J, \mathbb{R})$, *and its Fréchet derivative is given by*

$$\lambda_n'(q)h = \int_a^b |u_n(\cdot, q)|^2 h, \quad h \in L^1(J, \mathbb{R}). \tag{1.37}$$

(2) *Assume that* $\lambda_n(1/p)$ *is a simple eigenvalue with real-valued normalized eigenfunction* $u_n(\cdot, \frac{1}{p})$. *Then* $\lambda_n(\cdot, 1/p)$ *is differentiable in* $L^1(J, \mathbb{R})$, *and its Fréchet derivative is given by*

$$\lambda_n'(1/p)h = -\int_a^b |u_n^{[1]}(\cdot, 1/p)|^2 h, \quad h \in L^1(J, \mathbb{R}). \tag{1.38}$$

(3) *Assume that* $\lambda_n(w)$ *is a simple eigenvalue with real-valued normalized eigenfunction* $u_n(\cdot, w)$. *Then* $\lambda_n(\cdot, w)$ *is differentiable in* $L^1(J, \mathbb{R})$, *and its Fréchet derivative is given by*

$$\lambda_n'(w)h = -\lambda_n(w)\int_a^b |u_n(\cdot, w)|^2 h, \quad h \in L^1(J, \mathbb{R}). \tag{1.39}$$

(4) *Assume that* $\lambda_n(K)$ *is a simple eigenvalue with real-valued normalized eigenfunction* $u_n(\cdot, K)$. *Then* $\lambda_n(\cdot, K)$ *is differentiable, and its Fréchet derivative is given by the bounded linear transformation defined by*

$$\lambda_n'(K)H = [p\overline{u_n}'(b), -\overline{u}_n(b)]HK^{-1}\begin{bmatrix} u_n(b) \\ (pu_n')(b) \end{bmatrix}, \quad H \in M_{2,2}(\mathbb{C}). \tag{1.40}$$

Proof. See [68]; also, see [79] for (4). □

1.7 Eigenvalue inequalities

In this section, we describe how, for a fixed equation, the eigenvalues change when the boundary conditions change. Since the Dirichlet and Neumann boundary conditions play a special role, we introduce the notation

$$\lambda_n^D = \lambda_n(0, \pi), \quad \lambda_n^N = \lambda_n(\pi/2, \pi/2), \quad n \in \mathbb{N}_0. \tag{1.41}$$

Theorem 1.7.1. *Let* λ_n^D *be defined by* (1.41). *Then for all* (A, B) *satisfying the self-adjoint boundary conditions* (1.4), *we have:*

(1)

$$\lambda_n(A, B) \le \lambda_n^D, \quad n \in \mathbb{N}_0. \tag{1.42}$$

Equality can hold in (1.42) *for non-Dirichlet eigenvalues; see Theorem* 1.7.2 *and Remark* 1.7.1.

(2) *For all* (A, B) *satisfying* (1.4), *we have*

$$\lambda_n^D \le \lambda_{n+2}(A, B), \quad n \in \mathbb{N}_0. \tag{1.43}$$

(3) *The range of* $\lambda_0(A, B)$ *is* $(-\infty, \lambda_0^D]$.

(4) *The range of* $\lambda_1(A, B)$ *is* $(-\infty, \lambda_0^D]$.

(5) *The range of* $\lambda_n(A, B)$ *is* $(\lambda_{n-2}^D, \lambda_n^D]$ *for* $n \ge 2$.
Moreover, (3), (4), and (5) still hold when A, B *are restricted to be real.*

Proof. See [113]. □

Equality can occur in (1.42). The next result characterizes all such cases of equality for $n = 0$.

Theorem 1.7.2. *Let* (1.1)–(1.4) *hold, let* λ_n^D *be defined by* (1.41), *and let* $\Phi(t, \lambda) = (\phi_{ij}(t, \lambda))$ *be the primary fundamental matrix of the system representation of equation* (1.12). *Then*

$$\lambda_0(A, B) = \lambda_0^D$$

if and only if the boundary condition is the Dirichlet condition or the boundary condition matrices A, B are given by

$$A = \begin{bmatrix} \phi_{11}(b,\lambda_0^D) & 0 \\ \phi_{21}(b,\lambda_0^D) & \phi_{22}(b,\lambda_0^D) \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -d & -1 \end{bmatrix} \quad \text{with } d \le 0. \tag{1.44}$$

In canonical form, these conditions are given by the coupling matrix

$$K = \begin{bmatrix} \phi_{11}(b,\lambda_0^D) & 0 \\ d\phi_{11}(b,\lambda_0^D) + \phi_{21}(b,\lambda_0^D) & \phi_{22}(b,\lambda_0^D) \end{bmatrix}, \quad d \le 0. \tag{1.45}$$

Proof. See Corollary 4.5 in Haertzen, Kong, Wu, and Zettl [45]. □

Remark 1.7.1. We make a number of observations about Theorem 1.7.1.

(1) $\lambda_0(\alpha,\beta) = \lambda_0^D$ if and only if $\alpha = 0$ and $\beta = \pi$. In other words, for no *separated* boundary condition other than the Dirichlet condition does equality hold in (1.42) when $n = 0$.

(2) For no complex coupled boundary condition does equality hold in (1.42) when $n = 0$.

(3) For $n = 0$, equality holds in (1.42) for some coupled real boundary conditions. All these are characterized in Theorem 1.7.1, and all these lie on the jump set $\mathbb{J}$. (Recall that this is the set of boundary conditions on which all eigenvalues λ_n have jump discontinuities as functions of the boundary conditions.)

(4) Friedrichs Extension. Among all self-adjoint realizations of the Sturm–Liouville equation (1.12) with $p > 0$, $w > 0$, that is, among all operators S satisfying (1.14), there is a special one ("eine ausgezeichnete") often singled out in applied mathematics and mathematical physics, which is called the *Friedrichs extension* in honor of K. O. Friedrichs, who constructed it without any direct reference to a boundary condition. One of its basic properties is that it preserves the lower bound of the minimal operator $S_{\min}$ associated with equation (1.12) in the Hilbert space $L^2(J,w)$; however, it is not characterized by this property, that is, there may be other self-adjoint extensions of the minimal operator that preserve its lower bound. This lower bound is λ_0^D. Thus Theorem 1.7.1 gives examples of operators S that have the same lower bound as $S_{\min}$ and are not the Friedrichs extensions of $S_{\min}$ and characterizes all these.

(5) It is interesting to note that all operators S that have the same lower bound as $S_{\min}$ are determined by boundary conditions that lie on the jump set $\mathbb{J}$, and all, except for the Dirichlet condition, are determined by real coupled boundary conditions.

Next, we investigate more closely how the eigenvalues change when the boundary conditions change.

According to a well-known classical result (see [27] and [24] for the case of smooth coefficients and [105] for the general case), we have the following inequalities for $K = I$, the identity matrix:

$$\begin{aligned}
\lambda_0^N &\le \lambda_0(I) < \lambda_0(e^{i\gamma}I) < \lambda_0(-I) \le \{\lambda_0^D, \lambda_1^N\} \\
&\le \lambda_1(-I) < \lambda_1(e^{i\gamma}I) < \lambda_1(I) \le \{\lambda_1^D, \lambda_2^N\} \\
&\le \lambda_2(I) < \lambda_2(e^{i\gamma}I) < \lambda_2(-I) \le \{\lambda_2^D, \lambda_3^N\} \\
&\le \lambda_3(-I) < \lambda_3(e^{i\gamma}I) < \lambda_3(I) \le \{\lambda_3^D, \lambda_4^N\} \le \cdots,
\end{aligned} \tag{1.46}$$

where $\gamma \in (-\pi, \pi)$ and $\gamma \ne 0$. In (1.46) the notation $\{\lambda_n^D, \lambda_{n+1}^N\}$ means either of λ_n^D and λ_{n+1}^N, and there is no comparison made between these two. These inequalities are well known in Flochet theory.

Eastham, Kong, Wu, and Zettl [26] extended these inequalities to general $K \in \mathrm{SL}(2,\mathbb{R})$. A key feature of this extension is the identification of separated boundary conditions, which play the role of the Dirichlet and Neumann conditions. These are given next.

For $K \in \mathrm{SL}_2(\mathbb{R})$, $K = \left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$, denote by $\mu_n = \mu_n(K)$ and $\nu_n = \nu_n(K)$, $n \in \mathbb{N}_0$, the eigenvalues of the separated boundary conditions

$$y(a) = 0, \quad k_{22}y(b) - k_{12}y^{[1]}(b) = 0; \tag{1.47}$$

$$y^{[1]}(a) = 0, \quad k_{21}y(b) - k_{11}y^{[1]}(b) = 0; \tag{1.48}$$

respectively. For convenience, we let $y^{[1]} = (py')$, the quasi-derivative of y. Note that $(k_{22}, k_{12}) \ne (0,0) \ne (k_{21}, k_{11})$ since $\det K = 1$. Therefore each of these is a self-adjoint separated boundary condition with a countably infinite number of only real eigenvalues.

Theorem 1.7.3. *Let* (1.1)–(1.4) *hold. Let* μ_n *and* ν_n, $n \in \mathbb{N}_0$, *be the eigenvalues for* (1.47) *and* (1.48), *respectively. Then we have:*

- Suppose that $k_{12} < 0$ and $k_{11} \le 0$. Then
  (1) $\lambda_0(K)$ is simple;
  (2) $\lambda_0(K) < \lambda_0(-K)$; and
  (3) the following inequalities hold for $-\pi < \gamma < 0$ and $0 < \gamma < \pi$:

$$\begin{aligned}
-\infty &< \lambda_0(K) < \lambda_0(\gamma, K) < \lambda_0(-K) \le \{\mu_0, \nu_0\} \\
&\le \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \le \{\mu_1, \nu_1\} \\
&\le \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \le \{\mu_2, \nu_2\} \\
&\le \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(-K) \le \{\mu_3, \nu_3\} \le \cdots.
\end{aligned} \tag{1.49}$$

– Suppose that $k_{12} \le 0$ and $k_{11} > 0$. Then
  (1) $\lambda_0(K)$ is simple;
  (2) $\lambda_0(K) < \lambda_0(-K)$; and
  (3) the following inequalities hold for $-\pi < \gamma < 0$ and $0 < \gamma < \pi$:

$$\begin{aligned} \nu_0 &\le \lambda_0(K) < \lambda_0(\gamma, K) < \lambda_0(-K) \le \{\mu_0, \nu_1\} \\ &< \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \le \{\mu_1, \nu_2\} \\ &\le \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \le \{\mu_2, \nu_3\} \\ &\le \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(K) \le \{\mu_3, \nu_4\} \le \cdots . \end{aligned} \tag{1.50}$$

– Furthermore, for $0 < \alpha < \beta < \pi$, we have

$$\begin{aligned} \lambda_0(\beta, K) &< \lambda_0(\alpha, K) < \lambda_1(\alpha, K) < \lambda_1(\beta, K) < \lambda_2(\beta, K) < \lambda_2(\alpha, K) \\ &< \lambda_3(\alpha, K) < \lambda_3(\beta, K) < \cdots . \end{aligned}$$

– If neither of the above cases holds for K, then one of them must hold for $-K$. The notation $\{\mu_n, \nu_m\}$ is used to indicate either υ_n or ν_m, but no comparison is made between μ_n and ν_m.

Proof. For a diagonal matrix K, these inequalities were established by Weidmann [105]. The general result is proven by Eastham, Kong, Wu, and Zettl [26]. □

Next, we mention some interesting consequences of Theorem 1.7.3.

Remark 1.7.2. For separated boundary conditions, the Prüfer transformation is a powerful tool for proving the existence of eigenvalues, studying their properties and computing them numerically. There is no comparable tool for coupled conditions. For coupled conditions, the standard existence proof for the eigenvalues is based on operator theory in a Hilbert space; the Green's function is constructed and used as a kernel in the definition of an integral operator whose eigenvalues are those of the problem or their reciprocals; see Coddington and Levinson [24] or Weidmann [105].

A proof based on Theorem 1.7.3 is given in [26] and goes as follows: Starting with the eigenvalues μ_n and ν_n, $n \in \mathbb{N}_0$, of the separated boundary conditions (1.47)–(1.48), the proof of Theorem 4.8.1 in [26] (although this is not explicitly pointed out there) in fact shows that there is exactly one eigenvalue of the coupled condition determined by K in the interval $(-\infty, \mu_0]$, and it is $\lambda_0(\gamma, K)$; there is exactly one eigenvalue in the interval $[\mu_0, \mu_1]$, and it is $\lambda_1(\gamma, K)$; there is exactly one eigenvalue in the interval $[\mu_n, \mu_{n+1}]$, and it is $\lambda_{n+1}(\gamma, K)$ for $n \in \mathbb{N}_0$. This not only proves the existence of the eigenvalues of K but can be used to construct an algorithm to compute them. Such an algorithm is used by SLEIGN2; see [10]; see also [11, 12]. This seems to be the first existence proof for coupled eigenvalues that does not use self-adjoint operators in a Hilbert space and thus can be considered as the first "elementary" existence proof.

Remark 1.7.3. By Theorem 1.7.3, for *any* $K \in \mathrm{SL}(2,\mathbb{R})$, either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple. This extends the classical result that the lowest periodic eigenvalue is simple to the general case of arbitrary coupled self-adjoint boundary conditions. Here "simple" refers to both the algebraic and geometric multiplicities, since these are equal.

Theorem 1.7.4. *Let* (1.1)–(1.4) *hold. Let* μ_n *and* ν_n, $n \in \mathbb{N}_0$, *be the eigenvalues for* (1.47) *and* (1.48), *respectively.*
(1) *An eigenvalue* $\lambda_n(K)$ *is double if and only if there exist* $k, m \in \mathbb{N}_0$ *such that*

$$\lambda_n(K) = \mu_k = \nu_m.$$

(2) *Given eigenvalues* $\lambda_n(K)$ *and* $\lambda_{n+1}(K)$ *of* K, *distinct or not, there exist eigenvalues* υ_k, ν_m *of the separated boundary conditions* (1.47) *and* (1.48) *such that*

$$\lambda_n(K) \le \{\mu_k, \nu_m\} \le \lambda_{n+1}(K).$$

Proof. See Theorem 4.3 and Corollary 4.2 in Kong, Wu, and Zettl [60]. □

1.8 Monotonicity and multiplicity of eigenvalues

In this section, we fix a boundary condition and study how the eigenvalues change when a coefficient changes monotonically and discuss their multiplicity.

Theorem 1.8.1. *Let* (1.1)–(1.4) *hold, and let* $n \in \mathbb{N}_0$. *Assume that* $-\infty < a < b < \infty$.
(1) *Fix* p, w. *Suppose* $Q \in L^1((a,b),\mathbb{R})$ *and assume that* $Q \ge q$ *a. e. on* $[a,b]$.
Then $\lambda_n(Q) \ge \lambda_n(q)$. *If* $Q > q$ *on a subset of* $[a,b]$ *having positive Lebesgue measure, then* $\lambda_n(Q) > \lambda_n(q)$.
(2) *Fix* q, w. *Suppose* $1/P \in L^1((a,b),\mathbb{R})$ *and* $0 < P \le p$ *a. e. on* $[a,b]$.
Then $\lambda_n(1/P) \ge \lambda_n(1/p)$; *if* $1/P < 1/p$ *on a subset of* $[a,b]$ *having positive Lebesgue measure, then* $\lambda_n(1/P) < \lambda_n(1/p)$.
(3) *Fix* p, q. *Suppose* $W \in L^1((a,b),\mathbb{R})$ *and* $W \ge w > 0$ *a. e. on* $[a,b]$.
Then $\lambda_n(W) \ge \lambda_n(w)$ *if* $\lambda_n(W) < 0$ *and* $\lambda_n(w) < 0$; *but* $\lambda_n(W) \le \lambda_n(w)$ *if* $\lambda_n(W) > 0$ *and* $\lambda_n(w) > 0$. *Furthermore, if strict inequality holds in the hypothesis on a set of positive Lebesgue measure, then strict inequality holds in the conclusion.*

Proof. We give the proof for (1); the proofs of (2) and (3) are similar. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \lambda_n(s(t)), \quad s(t) = q + t(Q - q), \quad t \in [0,1].$$

Then $s(t) \in L^1((a,b),\mathbb{R})$ for each $t \in [0,1]$. From the chain rule in a Banach space and formula (1.37) for $\lambda_n'(q)$ we have

$$f'(t) = \lambda_n'((s(t))s'(t) = \int_a^b |u^2(r,s(t))|(Q(r) - q(r))\,dr \geq 0, \quad t \in [0,1].$$

Hence f is nondecreasing on $[0,1]$, and $f(1) = \lambda_n(Q) \geq \lambda_n(q) = f(0)$. The strict inequality part of the theorem also follows from this argument. □

The next theorem shows that the algebraic and geometric multiplicities of the eigenvalues of classical regular self-adjoint SLP are the same. Recall that the geometric multiplicity of an eigenvalue is the dimension of its eigenspace, that is, the number of linearly independent eigenfunctions of this eigenvalue. The algebraic multiplicity of an eigenvalue is the order of its zero as a root of the characteristic function

$$\delta(\lambda) = \delta(a,b,A,B,P,w,\lambda) = \det[A + B\Phi(b,a,P,w,\lambda)], \quad \lambda \in \mathbb{C}.$$

Theorem 1.8.2. *The algebraic and geometric multiplicities of the eigenvalues of regular self-adjoint Sturm–Liouville problems* (1.1)–(1.4) *are the same.*

Proof. For coupled boundary conditions, this is given in [26]. The separated case is proven in Theorem 4.12 of [60]. □

From here on we speak only of the multiplicity of an eigenvalue.

Theorem 1.8.3. *Let* (1.1)–(1.4) *hold. Fix all components except q, and fix $n \in \mathbb{N}_0$. Let*

$$S_1 = \{q \in L^1(J,\mathbb{R}) : \lambda_n(q) \text{ is simple}\};$$
$$S_2 = \{q \in L^1(J,\mathbb{R}) : \lambda_n(q) \text{ is double}\}.$$

Then S_1 is an open set in $L^1(J,\mathbb{R})$, and S_2 is closed and nowhere dense in $L^1(J,\mathbb{R})$. The same results hold when q is replaced by either $1/p$ or w.

Proof. This follows from Theorem 4.3 of [68] and the continuous dependence of λ_n on $1/p, q$, and w established in [60]. □

1.9 The Prüfer transformation and separated boundary conditions

In this section, we briefly describe the well-known Prüfer transformation and its relationship to separated boundary conditions. An elementary proof of the existence of eigenvalues and their theoretical and numerical properties can be based on this transformation.

To discuss the relation between the Sturm–Liouville equation and the equations arising from the Prüfer transformation, we consider the equations

$$-(py')' + qy = \lambda wy \quad \text{on } J, \tag{1.51}$$

$$\theta' = p^{-1}\cos^2\theta + (\lambda w - q)\sin^2\theta \quad \text{on } J, \tag{1.52}$$
$$\rho' = [(p^{-1} + q - \lambda w)\sin\theta\cos\theta]\rho \quad \text{on } J, \tag{1.53}$$

where

$$1/p, q, w \in L^1(J, \mathbb{R}), \quad \lambda \in \mathbb{R}, \quad p > 0 \quad \text{a.e. on } J = [a,b], \quad -\infty < a < b < \infty. \tag{1.54}$$

Theorem 1.9.1. *Let* (1.51)–(1.54) *hold.*

(1) *Then every initial value problem for equation* (1.52) *has a unique real-valued solution, and this solution is defined on* J.
(2) *Suppose* θ *and* ρ *be solutions of* (1.52) *and* (1.53), *respectively. Then* $y = \rho\sin\theta$ *is a solution of* (1.51) *on* J, *and* $(py') = \rho\cos\theta$.
(3) *Suppose* y *is a nontrivial solution of* (1.51). *Then there exist a solution* θ *of* (1.52) *and a solution* ρ *of* (1.51) *satisfying* $\rho(t) \neq 0$ *for* $t \in J$ *such that* $y = \rho\sin\theta$ *and* $(py') = \rho\cos\theta$.

Proof. This is well known and classical. □

Consider the SLP consisting of the equation

$$-(py')' + qy = \lambda wy, \quad \text{on } (a,b), \quad -\infty < a < b < \infty, \tag{1.55}$$

together with separated boundary conditions

$$A_1 y(a) + A_2(py')(a) = 0, \quad (A_1, A_2) \neq (0,0), \quad A_1, A_2 \in \mathbb{R}, \tag{1.56}$$
$$B_1 y(b) + B_2(py')(b) = 0, \quad (B_1, B_2) \neq (0,0), \quad B_1, B_2 \in \mathbb{R}, \tag{1.57}$$

and coefficients satisfying

$$p, q, w : (a,b) \to \mathbb{R}, \quad 1/p, q, w \in L(a,b), \quad p > 0, \quad w > 0 \quad \text{a.e. on } (a,b). \tag{1.58}$$

Theorem 1.9.2. *Let* (1.55)–(1.58) *hold. Then:*

(1) *All eigenvalues are real and simple.*
(2) *There are an infinite but countable number of eigenvalues* $\{\lambda_n : n \in \mathbb{N}_0\}$; *they are bounded below and can be ordered to satisfy the inequalities*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

and $\lambda_n \to \infty$ *as* $n \to \infty$.
(3) *If* $u_n = u_n(\cdot, \lambda_n)$ *is an eigenfunction of* λ_n, *then* u_n *has exactly* n *zeros in the open interval* (a,b).
(4) *Choose* $\alpha \in [0, \pi)$ *such that*

$$\tan\alpha = \frac{-A_2}{A_1} \quad \text{if } A_1 \neq 0, \quad \text{and} \quad \alpha = \pi/2 \quad \text{if } A_1 = 0;$$

similarly, choose $\beta \in (0, \pi]$ *such that*

$$\tan\beta = \frac{-B_2}{B_1}, \quad \textit{if } B_1 \neq 0, \quad \textit{and} \quad \beta = \pi/2 \quad \textit{if } B_1 = 0.$$

Then each eigenvalue λ_n *is the unique solution* $\lambda = \lambda_n$ *of the equation*

$$\theta(b, \lambda) = \beta + n\pi, \quad n \in \mathbb{N}_0, \tag{1.59}$$

where θ *is the solution of* (1.52) *determined by the initial condition* $\theta(a, \lambda) = \alpha$ *for each* $\lambda \in \mathbb{R}$.

(5) *The sequence of eigenfunctions* $\{u_n = u_n(\cdot, \lambda_n) : n \in \mathbb{N}_0\}$ *can be normalized to be an orthonormal sequence in the Hilbert space* $H = L^2(J, w)$*, that is,*

$$\int_a^b u_n \overline{u}_m w = \begin{cases} 0 & \textit{if } n \neq m, \\ 1 & \textit{if } n = m. \end{cases}$$

Furthermore, the orthonormal sequence $\{u_n = u_n(\cdot, \lambda_n) : n \in \mathbb{N}_0\}$ *is complete in* H*, that is, for any* $f \in H$*, we have*

$$f = \sum_0^\infty c_n u_n, \quad c_n = \int_a^b f u_n w.$$

Here the left equality means that the partial sums of the series on the right side of the equation converge to f *in the norm of* H.

Proof. This is well known. Although the proof given by Coddington and Levinson [24] has stronger hypotheses, it can readily be adapted to the given hypotheses. □

The characterization (1.59) of Theorem 1.9.2 is established under the hypotheses $p > 0$ and $w > 0$. These assumptions on p and w guarantee that the spectrum is bounded below and (1.59) holds for each λ_n, $n \in \mathbb{N}_0$. This characterization of λ_n is interesting from theoretical and numerical perspectives. It can be used to numerically compute each eigenvalue independently of all other eigenvalues; this is done in the code SLEIGN2. Theoretically, it can be used to study the dependence of λ_n on the problem. Also, it follows directly from (1.52) that each eigenfunction of λ_n has exactly n zeros in the open interval J. When p changes sign, the spectrum is unbounded above and below. Does (1.59) hold in this case for all positive and negative eigenvalues? The next theorem gives an affirmative answer to this question. We state this theorem in full even though part of it is repetitive.

Theorem 1.9.3 (Binding and Volkmer). *Consider the SLP consisting of the equation*

$$-(py')' + qy = \lambda wy \quad \textit{on } J = (a, b),$$

together with separated boundary conditions

$$A_1 y(a) + A_2(py')(a) = 0, \quad (A_1, A_2) \neq (0,0), \quad A_1, A_2 \in \mathbb{R},$$
$$B_1 y(b) + B_2(py')(b) = 0, \quad (B_1, B_2) \neq (0,0), \quad B_1, B_2 \in \mathbb{R},$$

and coefficients satisfying

$$1/p, q, w \in L(J, \mathbb{R}), \quad w > 0 \quad a.e. \text{ on } J = (a,b), \quad -\infty < a < b < \infty.$$

Assume that p changes sign on J. Then this SLP has only real and simple eigenvalues, there are an infinite but countable number of them, and they are unbounded below and above and can be indexed and ordered to satisfy

$$\cdots < \lambda_{-3} < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots. \tag{1.60}$$

Let θ be defined as before. Then for each integer $n \in \mathbb{Z} = (\dots, -2, -1, 0, 1, 2, 3, \dots\}$, there is exactly one eigenvalue λ_n, which is the unique solution of the equation

$$\theta(b, \lambda_n) = n\pi + \beta. \tag{1.61}$$

There are no other eigenvalues. Here β is defined as in equation (1.59).

Proof. See Binding and Volkmer [15] and the next remark. □

Remark 1.9.1. The fact that all eigenvalues are real and that there are an infinite but countable number of them follows from the "standard" Hilbert space proof – using the Hilbert space $L^2(J, w)$ – and the self-adjoint operator realization of this SLP; see [24]. Möller [78] showed that these eigenvalues are unbounded above and below. By the characterization of the eigenvalues as zeros of the characteristic function (see [113]) the eigenvalues are isolated with no finite accumulation point. The simplicity of the eigenvalues is clear from the separated boundary conditions. The indexing of the eigenvalues so that (1.60) holds is not unique, in fact, rather arbitrary. It can be made more definite as follows: If $\lambda = 0$ is an eigenvalue, then denote it by λ_0 and let λ_1 denote the smallest positive eigenvalue. The latter exists: Let

$$\lambda_1 = \inf\{\lambda_n : \lambda_n > 0\},$$

then λ_1 is an eigenvalue by the continuity of the characteristic function $\delta(\lambda)$ and is positive since $\lambda = 0$ is isolated. Similarly, $\lambda_2 = \inf\{\lambda_n > \lambda_1\}$ is an eigenvalue greater than λ_1, and so on. The same argument can be used when $\lambda = 0$ is not an eigenvalue. This is the indexing scheme used by the code SLEIGN2 for the numerical computation of the eigenvalues in the singular limit-circle oscillatory case. As already mentioned, it is rather arbitrary: we can replace $\lambda = 0$ in this scheme by any real λ and use it for a "pivot".

In sharp contrast the characterization of the eigenvalues in terms of the Prüfer angle θ given by equation (1.61) is definite and explicit. It is interesting from theoretical and numerical perspectives. For instance, if we compute an eigenvalue as a root of the characteristic equation $\delta(\lambda) = 0$, then the following question arises: Which eigenvalue is it? Characterization (1.61) gives a definite answer to this question for this class of SLPs. In general, indexing the eigenvalues in some definite and explicit manner is a difficult open problem for nonclassical problems, even in the case where the coefficients are real-valued and the boundary conditions are self-adjoint, for example, when both p and w change sign. The Binding–Volkmer characterization (1.61) is an important result for problems where p changes sign.

1.10 A Prüfer characterization for real coupled conditions

In the previous section, we saw that the Prüfer transformation is a powerful tool for studying properties of eigenvalues and eigenfunctions; for example, to prove that the nth eigenfunction has exactly n zeros in the open domain interval (a, b). In 1966, Bailey [7] showed that the Prüfer transformation can be used to compute the eigenvalues for separated conditions very effectively and efficiently. The nth eigenvalue can be computed without any prior knowledge of the previous or subsequent eigenvalues for any n.

The code SLEIGN2 [10] computes eigenvalues for separated and real or complex coupled self-adjoint boundary conditions. The algorithm used by SLEIGN2 is based on the inequalities discussed in Section 1.7. These inequalities locate the coupled eigenvalues uniquely between two separated ones. The Prüfer transformation is then used to compute the separated eigenvalues followed by a search mechanism to compute the coupled eigenvalue within the bounds given by the separated ones.

Bailey and Zettl [13] developed an algorithm to characterize and compute the eigenvalues of general real coupled boundary conditions

$$Y(b) = KY(a), \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix},$$

where $K \in SL_2(\mathbb{R})$, that is, K is real, and $\det K = 1$. They constructed a one-parameter family of separated conditions and proved that the extrema of this family were eigenvalues for K or $-K$ and all eigenvalues for K and $-K$ can be obtained in this way. Given the index n, for any eigenvalue of K, they determined an appropriate separated boundary condition and determined which eigenvalue of this separated condition was equal to the coupled eigenvalue with this index n.

Thus the Prüfer characterization for separated boundary conditions discussed in the previous section can be used to study the eigenvalues of any real coupled condition. If λ_n is a simple eigenvalue for K, then the number of its zeros in the open domain interval is determined exactly by this characterization. *Furthermore, this characterization can be used to compute the eigenvalues for any* $K \in SL_2(\mathbb{R})$ *using any code that works for separated conditions.*

In stark contrast to this close relation between the eigenvalues $\lambda_n(K)$ and $\lambda_n(\alpha,\beta)$, no eigenvalue of the constructed family of separated conditions determined by K or $-K$ is an eigenvalue $\lambda_n(K,\gamma)$ for any $\gamma \neq 0$.

As mentioned before, for each $K \in SL_2(\mathbb{R})$, all eigenvalues for K and $-K$ can be found from the eigenvalues of a related family of separated conditions constructed from K. Next, we define this separated family and present an algorithm.

Definition 1.10.1 (α-family of K). Let $K = (k_{ij}) \in SL_2(\mathbb{R})$. For each $\alpha \in [0,\pi)$, consider the separated boundary conditions

$$\begin{aligned} y(a)\cos\alpha - (py')(a)\sin\alpha &= 0, \\ y(b)(k_{21}\sin\alpha + k_{22}\cos\alpha) - (py')(b)(k_{11}\sin\alpha + k_{12}\cos\alpha) &= 0. \end{aligned} \tag{1.62}$$

Define $\alpha^* \in [0,\pi)$ by

$$\alpha^* = \begin{cases} \tan(-k_{12}/k_{11}) & \text{if } k_{11} \neq 0, \\ \pi/2 & \text{if } k_{11} = 0. \end{cases} \tag{1.63}$$

Note that $\alpha^* = 0$ when $k_{12} = 0$ since $k_{11} \neq 0$ in this case.

Remark 1.10.1. Note that condition (1.62) for $-K$ is equivalent to (1.62) for K. Since $K \in SL_2(\mathbb{R})$, $(k_{i1}, k_{i2}) \neq (0,0) \neq (k_{1i}, k_{2i})$, $i = 1,2$, and (1.62) is a self-adjoint boundary condition for each $\alpha \in [0,\pi)$. When $\alpha = 0$, (1.62) reduces to $y(a) = 0 = y(b)k_{22} - (py')(b)k_{12}$; when $\alpha = \pi/2$, (1.62) is equivalent with $(py')(a) = 0 = y(b)k_{21} - (py')(b)k_{11}$. When $\alpha = \pi/2$ and $k_{11} = 0$, (1.62) becomes $(py')(a) = 0 = y(b)$.

Definition 1.10.2. Let $K = (k_{ij}) \in SL_2(\mathbb{R})$. For each $\alpha \in [0,\pi)$, let

$$\{\mu_n(\alpha) : n \in \mathbb{N}_0\} \tag{1.64}$$

denote the eigenvalues of (1.62).

These eigenvalues $\{\mu_n(\alpha) : n \in \mathbb{N}_0\}$ determine all eigenvalues for K and for $-K$ for each $K \in SL_2(\mathbb{R})$. The next theorem defines the continuous eigenvalue curves whose maxima and minima are the eigenvalues for K and $-K$.

Theorem 1.10.1. *Let* $K = (k_{ij}) \in SL_2(\mathbb{R})$.

(1) *Suppose* $k_{12} \neq 0$ *and* α^* *is defined by* (1.63). *Then* $\alpha^* \in (0,\pi)$. *For each* $n \in \mathbb{N}_0$, *define the eigencurves* R_n *and* L_n *as follows:*

$$R_n(\alpha) = \mu_n(\alpha), \quad \alpha^* \leq \alpha < \pi; \tag{1.65}$$

$$L_n(\alpha) = \mu_n(\alpha), \quad 0 \leq \alpha < \alpha^*. \tag{1.66}$$

Then $R_n(\alpha)$ *is continuous on* $[\alpha^*,\pi)$, *and* $L_n(\alpha)$ *is continuous on* $[0,\alpha^*)$.

(2) *Suppose* $k_{12} = 0$. *Define* R_n *by*

$$R_n(\alpha) = \mu_n(\alpha), \quad 0 \le \alpha < \pi. \tag{1.67}$$

Then $R_n(\alpha)$ *is continuous on* $[0, \pi)$. *(There is no* L_n *in this case.)*

Proof. This follows from Lemma 1.5.1. □

The selection process for the eigenvalues for K and $-K$ is given by the following algorithm.

Algorithm 1. Let $K \in \mathrm{SL}_2(\mathbb{R})$.

– If $\lambda_n(K) = \lambda_{n+1}(K)$ for some $n \in \mathbb{N}$, then $\mu_n(\alpha) = \lambda_n(K)$ for all $\alpha \in [0, \pi)$.

(1) Assume $k_{12} = 0$ and $k_{11} > 0$. Then $\lambda_0(K)$ is simple, and

a:

$$\lambda_0(K) = \max R_0(\alpha), \quad 0 \le \alpha < \pi. \tag{1.68}$$

b: If n is even, then

$$\lambda_n(K) = \max R_n(\alpha), \quad 0 \le \alpha < \pi. \tag{1.69}$$

c: If n is odd, then

$$\lambda_n(K) = \min R_{n+1}(\alpha), \quad 0 \le \alpha < \pi. \tag{1.70}$$

(2) Assume $k_{12} = 0$ and $k_{11} < 0$. (Note that if $k_{12} = 0$, then $k_{11} \neq 0$ since $\det(K) = 1$.) Then

a:

$$\lambda_0(K) = \min R_1(\alpha), \quad 0 \le \alpha < \pi.$$

b: If n is even, then

$$\lambda_n(K) = \min R_{n+1}(\alpha), \quad 0 \le \alpha < \pi.$$

c: If n is odd, then

$$\lambda_n(K) = \max R_n(\alpha), \quad 0 \le \alpha < \pi.$$

(3) Assume $k_{12} < 0$. Then

a:

$$\lambda_0(K) = \max R_0(\alpha), \quad \alpha^* \le \alpha < \pi.$$

b: If n is even, then

$$\lambda_n(K) = \max R_n(\alpha), \quad \alpha^* \le \alpha < \pi.$$

c: If n is odd, then

$$\lambda_n(K) = \min L_n(\alpha), \quad 0 \le \alpha < \alpha^*.$$

(4) Assume $k_{12} > 0$. Then

a:

$$\lambda_0(K) = \min L_0(\alpha), \quad 0 \le \alpha < \alpha^*.$$

b: If n is even, then

$$\lambda_n(K) = \min L_n(\alpha), \quad 0 \le \alpha < \alpha^*.$$

c: If n is odd, then

$$\lambda_n(K) = \max R_n(\alpha), \quad \alpha^* \le \alpha < \pi.$$

Proof. See below. □

Except for the case where $k_{12} < 0$ and $n = 0$, the inequalities given in Section 1.7 locate each eigenvalue $\lambda_n(K)$, $n \in \mathbb{N}_0$, uniquely between two consecutive eigenvalues of the separated boundary conditions. The next corollary fills this gap.

Corollary 1.10.1. *Assume that $k_{12} > 0$. Then for any $\alpha \in [\alpha^*, \pi)$, we have that*

$$\lambda_0(K) \le \mu_0(\alpha).$$

In particular, for any $\varepsilon > 0$, $\mu_0(\alpha) - \varepsilon$ is a lower bound of $\lambda_0(K)$.

Proof. This follows from part 3(a) of the Algorithm. □

1.11 Another family of separated boundary conditions

In this section we study another family of separated boundary conditions generated by a coupling matrix K. This family and its relation to the *α-family* constructed in the previous section is used in the proof of Algorithm 1 and, we believe, is of independent interest. But we first recall the characterization of the eigenvalues by means of the characteristic function.

For any $\lambda \in \mathbb{C}$, define two linearly independent solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the differential equation (1.1) by the initial conditions

$$\begin{aligned} \varphi(a,\lambda) &= 0, \quad (p\varphi')(a,\lambda) = 1, \\ \psi(a,\lambda) &= 1, \quad (p\psi')(a,\lambda) = 0. \end{aligned} \tag{1.71}$$

Then any solution $y(x,\lambda)$ of equation (1.1) can be expressed in the form

$$\begin{aligned} y(x,\lambda) &= y(a,\lambda)\psi(x,\lambda) + (py')(a,\lambda)\varphi(x,\lambda), \\ (py')(x,\lambda) &= y(a,\lambda)(p\psi')(x,\lambda) + (py')(a,\lambda)(p\varphi')(x,\lambda) \end{aligned} \tag{1.72}$$

for all $x \in [a,b]$ and $\lambda \in \mathbb{C}$. In particular, we have

$$y(b,\lambda) = y(a,\lambda)\psi(b,\lambda) + (py')(a,\lambda)\varphi(b,\lambda), \qquad (1.73)$$
$$(py')(b,\lambda) = y(a,\lambda)(p\psi')(b,\lambda) + (py')(a,\lambda)(p\varphi')(b,\lambda),$$

and two basic results follow.

Theorem 1.11.1. *Let $\lambda \in \mathbb{C}$. The differential equation* (1.1) *has a nontrivial solution satisfying the separated boundary conditions* (1.15)–(1.16) *if and only if*

$$\delta(\lambda) := -A_1B_2(p\varphi')(b,\lambda) + A_2B_1\psi(b,\lambda) + A_2B_2(p\psi')(b,\lambda) - A_1B_1\varphi(b,\lambda) = 0. \qquad (1.74)$$

Proof. Simply substitute (1.73) into (1.1) to get two equations in $y(a,\lambda)$ and $(py')(a,\lambda)$, which must be consistent. This condition is (1.74). □

Theorem 1.11.2. *Let $K \in SL_2(\mathbb{R})$, $\lambda \in \mathbb{C}$, $-\pi < \gamma \le \pi$. The differential equation* (1.1) *has a nontrivial solution satisfying the coupled boundary conditions* (1.9) *if and only if*

$$D(\lambda) = k_{11}(p\varphi')(b,\lambda) - k_{21}\varphi(b,\lambda) + k_{22}\psi(b,\lambda) - k_{12}(p\psi')(b,\lambda) = 2\cos\gamma. \qquad (1.75)$$

Proof. Proceed as in the proof of Theorem 1.11.1 using the coupled boundary conditions (1.9); see [113] for details. □

Now we define a family of separated boundary conditions in terms of $r \in \mathbb{R} \cup \{\pm\infty\}$ as follows: Given $K \in SL_2(\mathbb{R})$, for each $r \in \mathbb{R}$, consider the boundary conditions

$$y(a) - r(py')(a) = 0, \qquad (1.76)$$
$$(k_{21}r + k_{22})y(b) - (k_{11}r + k_{12})(py')(b) = 0,$$

and also the condition

$$(py')(a) = 0 = k_{21}y(b) - k_{11}(py')(b) = 0. \qquad (1.77)$$

Condition (1.77) corresponds to $r = \pm\infty$; its eigenvalues are denoted by $\nu_n = \lambda_n(\pm\infty)$, $n \in \mathbb{N}_0$. Here it is important to keep in mind that conditions (1.76) for all $r \in \mathbb{R}$ and condition (1.77) *together* form one family of separated conditions generated by K; we refer to this family as the *r-family* of K. Next, we study this family.

Notation 1.11.1. *Let $\sigma(r) = \{\lambda_n(r), n \in \mathbb{N}_0\}$ denote the eigenvalues of the r-family with $\nu_n = \lambda_n(\pm\infty)$ corresponding to $r \pm \infty$.*

The next lemma discusses the continuity properties of the eigenvalues of the r-family.

Lemma 1.11.1. *For a fixed $n \in \mathbb{N}_0$, the eigenvalue function $\lambda_n(r)$ is a continuous function of $r \in \mathbb{R}$ except in the following three cases:*

(1) *as* $r \to 0^-$.
(2) *when* $k_{12} = 0$ *and* $r = 0$ *(note that* $r_{22} \neq 0$ *in this case).*
(3) *when* $k_{11} \neq 0$ *and* $r = -k_{12}/k_{11}$.

Proof. This follows from the continuity theorem. □

Next, for each $K = (k_{ij}) \in \mathrm{SL}_2(\mathbb{R})$, we construct a family of separated boundary conditions associated with K. Let $r \in \mathbb{R} \cup \{\pm\infty\}$, let

$$R = \frac{rk_{11} + k_{12}}{rk_{21} + k_{22}}, \tag{1.78}$$

and note that:
(1) Not both of k_{11}, k_{12} or k_{21}, k_{22} can be 0 since $\det(K) = 1$.
(2) If $k_{12} = 0$, then $k_{11} \neq 0$ and $R = 0$ if and only if $r = 0$.
(3) If $k_{21} \neq 0$ and $r = -k_{22}/k_{21}$, then R is undefined.
(4) If $k_{21} = 0$, then $k_{22} \neq 0$, and R is well defined for all $r \in \mathbb{R}$.

The next theorem relates the eigenvalues of K and $-K$ with those of the r-family of K.

Theorem 1.11.3. *Let* $K \in \mathrm{SL}_2(\mathbb{R})$*, and let* $D(\lambda)$ *be defined by* (1.75)*. If* λ *is an eigenvalue of any member of the r-family of* K*, then* $D^2(\lambda) - 4 \geq 0$*, that is,* $D(\lambda) \geq 2$ *or* $D(\lambda) \leq -2$.

Proof. We first prove the case where $r \in \mathbb{R}$. If λ is such an eigenvalue, then its boundary conditions are of the form (1.15)–(1.16) with

$$A_1 = 1, \quad A_2 = -r, \quad B_1 = 1, \quad B_2 = -R. \tag{1.79}$$

Substituting into (1.78) gives a quadratic equation in r,

$$Ar^2 + Br + C = 0, \tag{1.80}$$

where

$$\begin{aligned} A &= k_{21}\psi(b,\lambda) - k_{11}(p\psi')(b,\lambda), \qquad (1.81)\\ B &= k_{21}\varphi(b,\lambda) + k_{22}\psi(b,\lambda) - k_{11}(p\psi')(b,\lambda) - k_{12}(p\varphi')(b,\lambda),\\ C &= k_{22}\varphi(b,\lambda) - k_{12}(p\varphi')(b,\lambda). \end{aligned}$$

Since λ is an eigenvalue for some fixed number r, the left-hand side of (1.80) must vanish. Hence

$$4A^2\left\{\left(r + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A^2}\right\} = 0,$$

or

$$4A^2\left(r + \frac{B}{2A}\right)^2 = B^2 - 4AC.$$

A direct computation shows that $B^2 - 4AC = D^2(\lambda) - 4$. Therefore

$$4A^2\left(r + \frac{B}{2A}\right)^2 = D^2(\lambda) - 4.$$

Since r, A, B, C are real, it follows that

$$D^2(\lambda) \geq 4 \quad \text{and} \quad D(\lambda) \geq 2 \quad \text{or} \quad D(\lambda) \leq -2. \tag{1.82}$$

This concludes the proof for $r \in \mathbb{R}$.

For $r = \pm\infty$, the member of the r-family is (1.77), whose eigenvalues are ν_n, $n \in \mathbb{N}_0$. The conclusion (1.82) for ν_n, $n \in \mathbb{N}_0$, was established in [113] pp. 80–84. This concludes the proof. □

Although the next result is a corollary of Theorem 1.11.2 and other known results, we state it here as a theorem because we think it is surprising and provides a stark contrast with Algorithm 1.

Theorem 1.11.4. *Let $K \in \mathrm{SL}_2(\mathbb{R})$. Let $\sigma(r)$ for $r \in \mathbb{R} \cup \{\pm\infty\}$, and let $\sigma(K, \gamma)$ for $\gamma \in (-\pi, 0) \cup (0, \pi)$ be defined as before. Then no eigenvalue of any member of any r-family is an eigenvalue in $\sigma(K, \gamma)$ for any $\gamma \in (-\pi, 0) \cup (0, \pi)$. More explicitly,*

$$\sigma(r) \cap \sigma(K, \gamma) = \emptyset \tag{1.83}$$

for all $r \in \mathbb{R} \cup \{\pm\infty\}$ and $\gamma \in (-\pi, 0) \cup (0, \pi)$.

Proof. If λ is an eigenvalue in $\sigma(K, \gamma)$ for any $\gamma \in (-\pi, 0) \cup (0, \pi)$, then $|D(\lambda)| < 2$ by (1.82), and the conclusion follows from Theorem 1.11.2. □

Theorem 1.11.5. *Let $K \in \mathrm{SL}_2(\mathbb{R})$. If $\lambda_n'(r_0) = 0$ for some $n \in \mathbb{N}_0$ and some $r_0 \in \mathbb{R} \cup \{\pm\infty\}$, then $\lambda_n(r_0)$ is an eigenvalue of either K or $-K$.*

Proof. Suppose $\lambda = \lambda_n(r)$ satisfies equation (1.75). Each term of this equation is a function of r, so we differentiate the left-hand side of the equation with respect to r and obtain

$$2Ar + B + \left\{r^2 \frac{\partial A}{\partial \lambda} + r \frac{\partial B}{\partial \lambda} + C \frac{\partial C}{\partial \lambda}\right\} \frac{d\lambda}{dr} = 0 \quad \text{at } \lambda = \lambda_n(r). \tag{1.84}$$

By assumption, $\lambda_n'(r_0) = 0$. Hence (1.84) reduces to

$$2Ar_0 + B = 0 \quad \text{at } \lambda_n(r_0). \tag{1.85}$$

This equation yields $B^2 - 4AC = 0$, and this implies that $D^2(\lambda) - 4 = 0$, which means that λ is an eigenvalue for either K or $-K$. □

Remark 1.11.1. In other words, Theorem 1.11.3 says that the eigenvalues for K and $-K$ are the extrema of the continuous eigencurves $L_n(\alpha)$ and $R_n(\alpha)$ defined before. For any K and n, the algorithm explicitly states which extremum is equal to $\lambda_n(K)$.

Next, we study the relation of the α- and r-families with each other.

Let $r \in \mathbb{R} \cup \{\pm\infty\}$ be determined by

$$\tan(\alpha) = r, \quad \alpha \in [0, \pi), \tag{1.86}$$

where $r = \pm\infty$ when $\alpha = \pi/2$.

Let R be given by (1.86). Now define $\beta = \beta(\alpha) = \beta(\alpha(r))$ by

$$\tan(\beta) = R, \quad \beta \in (0, \pi] \tag{1.87}$$

and observe that

(1) $\beta = \pi/2$ when $k_{21} \neq 0$ and $r = -k_{22}/k_{12}$;
(2) $\beta = \pi$ if and only if $k_{12} = 0$ and $r = 0$, and
(3) by (1.87) $\beta = \pi/2$ corresponds to $R = \pm\infty$.

By definition, $\alpha = \pi/2$ corresponds to $r = \pm\infty$.

In each of these three cases, there is an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n > 0$.

1.12 Proof of the algorithm

The proof is basically obtained by combining Theorems 1.11.3 and 1.11.4 with the known inequalities given in Section 1.7.

Proof of Algorithm 1. First, consider the particular case $k_{12} = 0$. Then $\alpha^* = 0$. If $k_{11} > 0$, then by the inequalities theorem, the interval $[\lambda_{n-1}(K), \lambda_n(K)]$ for even n contains $\nu_n = \upsilon_n(\pi/2)$, and the function $\upsilon_n(\alpha)$ is continuous at $\alpha = \pi/2$. By Theorem 1.4.2, $\upsilon_n(\alpha)$ cannot move outside the interval $[\lambda_{n-1}(K), \lambda_n(K)]$ as α varies continuously away from $\alpha = \pi/2$ since $\upsilon_n(\alpha) > \lambda_n(K)$ or $\upsilon_n(\alpha) < \lambda_{n-1}(K)$ would contradict $D^2(\upsilon_n(\alpha)) - 4 \geq 0$. Therefore $\lambda_{n-1}(K) \leq \upsilon_n(\alpha) \leq \lambda_n(K)$ for all $\alpha \in [0, \pi)$. If $\lambda_{n-1}(K) = \lambda_n(K)$, then $\upsilon_n(\alpha) = \lambda_n(K)$ for all $\alpha \in [0, \pi)$.

If $\lambda_{n-1}(K) < \lambda_n(K)$, then the continuous function $\upsilon_n(\alpha)$ has a maximum and a minimum in the compact interval $[\lambda_{n-1}(K), \lambda_n(K)]$ as α varies in $[0, \pi)$. If the maximum is not $\lambda_n(K)$, then it occurs in the interior of this interval, and by Theorem 1.4.2 $\lambda_n(K) = \max\{\upsilon_n(\alpha) : \alpha \in [0, \pi)\}$. Similarly, $\lambda_{n-1}(K) = \min\{\upsilon_n(\alpha) : \alpha \in [0, \pi)\}$. The proof for $k_{12} = 0$ and $k_{11} < 0$ is similar.

If $k_{12} \neq 0$, then $0 < \alpha^*$. In this case the proof is also similar to the above proof but with one important difference: The interval $[0, \pi)$ in the above argument is replaced by two intervals $[0, \alpha^*)$ and $[\alpha^*, \pi)$. This is due to the fact that the function $\upsilon_n(\alpha)$ has a jump discontinuity at α^* – see part (1) of Theorem 1.2.1 for a discussion of jump discontinuities of eigencurves for separated boundary conditions. This discontinuity is due to the fact that $\tan(\beta) = 0$ when $\alpha = \alpha^*$, and hence (recall the normalization for β in (1.5)) $\beta = \pi$ when $\alpha = \alpha^*$. See part (1) of Theorem 1.2.1 for a discussion of jump discontinuities. Although this discussion is for the case where β is fixed as α varies, it extends readily to our situation where β is a continuous function of α. In fact, the results mentioned in part (1) of Theorem 1.2.1 have far reaching extensions; see Sections 3.4, 3.5, and 3.6 in [113]. □

Remark 1.12.1. Note that the intersections of the function $D(\lambda)$ with the horizontal lines at +2 and −2, which are the eigenvalues for K and $-K$, correspond precisely with the local extrema of the continuous eigencurves $R_n(\alpha)$ and $L_n(\alpha)$ of the related α-family in an appropriate α interval. See the graph of a typical characteristic function $D(\lambda)$ on page 92 in [113].

The eigenvalues corresponding to each index n lie along two distinct continuous curves, one on $[0, \alpha^*)$ and the other on $[\alpha^*, \pi)$.

When the parameter $\gamma = 0$ (or π), then $D(\lambda) = 2$ if and only if λ is an eigenvalue for the problem with coupled boundary conditions defined by a matrix K; $D(\lambda) = -2$ when λ is an eigenvalue of the problems defined by $-K$. Thus to compute eigenvalues of such coupled boundary condition problems, we simply compute the values of functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ at $x = b$, evaluate $D(\lambda)$, and search for values of λ for which $D(\lambda) = 2$. But then we have to determine the index n. For this, the upper and lower bounds obtained from Theorem 1.2.1 (see Remark 1.2.1) can be used. See [13] for examples of computed eigenvalues of coupled boundary conditions computed with SLEIGN2 and this algorithm.

1.13 Comments

Most of the material covered in Sections 1–9 can be found in the book [113]. Sections 10, 11, and 12 describe an algorithm developed by Bailey and Zettl [13] in 2012, which can be used with SLEIGN2 to compute the eigenvalues of regular and singular Sturm–Liouville problems with real *coupled* self-adjoint boundary conditions. This algorithm is based on using the Prüfer transformation on families of separated boundary conditions.

In 1978, Bailey introduced the code SLEIGN to compute the eigenvalues of Sturm–Liouville problems with regular self-adjoint *separated* boundary conditions and for a singular problem selected by the code. This singular problem is usually, but not always, the Friedrichs extension. In 1991, Bailey, Everitt, and Zettl [6] introduced the code SLEIGN2 based on new algorithms. SLEIGN2, when used together with appropriate theoretical results, can also provide some information about the spectrum of some singular problems, for example, the starting point of the essential spectrum, the number and numerical value of eigenvalues below the essential spectrum, and an approximation of the first few spectral bands and gaps; see the paper [10] of these authors for examples and additional information.

The SLEIGN2 “package” and a number of related papers can be downloaded from the Web at

http://www.math.niu.edu/~zettl/SL2

This package contains a user-friendly interface consisting of six FORTRAN files, two tex files, and three pdf files. All these files can be downloaded by clicking on the given links.

Also see [10] for a comparison of SLEIGN2, which uses the Prüfer transformation, and the Fulton–Pruess code SLEDGE [40], which is based on approximating the coefficients. Both codes are used in [10] on some examples to compute eigenvalues, and the results are compared.

2 Periodic coefficients

2.1 Eigenvalues of periodic, semiperiodic, and complex boundary conditions

Consider the equation

$$-(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \tag{2.1}$$

with coefficients satisfying

$$1/p, q, w \in L_{\text{loc}}(\mathbb{R}, \mathbb{R}), \quad p > 0, \quad w > 0 \quad \text{a. e. on } \mathbb{R}. \tag{2.2}$$

The coefficients are h-periodic if for some $h \in \mathbb{R}$, $0 < h < \infty$,

$$p(t+h) = p(t), \quad q(t+h) = q(t), \quad w(t+h) = w(t), \quad t \in \mathbb{R}; \tag{2.3}$$

the complex boundary conditions on the interval $[a, a+h]$ for any given $a \in \mathbb{R}$ are defined by

$$\begin{aligned} y(a+h) &= e^{i\gamma} y(a), \\ (py')(a+h) &= e^{i\gamma}(py')(a), \quad 0 < \gamma < \pi; \end{aligned} \tag{2.4}$$

and the periodic and semiperiodic boundary conditions on each of the k intervals $[a, a+kh]$, $k \in \mathbb{N}$, are defined by

$$\begin{aligned} y(a+kh) &= y(a), \\ (py')(a+kh) &= (py')(a), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} y(a+kh) &= -y(a), \\ (py')(a+kh) &= -(py')(a). \end{aligned} \tag{2.6}$$

Remark 2.1.1. Note that condition (2.2) implies that the coefficients satisfy

$$1/p, q, w \in L^1(J, \mathbb{R}), \quad p > 0, \quad w > 0 \quad \text{a. e. on } J \tag{2.7}$$

for any interval $J = [a, a+kh]$, $a \in \mathbb{R}$, $k \in \mathbb{N}$. Thus all these problems are regular classical self-adjoint boundary value problems. So the results of Chapter 1 apply to each of these problems.

Remark 2.1.2. Also note that we do not assume that h is the smallest positive number for which (2.3) holds.

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Each of these boundary value problems has a discrete spectrum consisting of a countable number of real eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$, which are bounded below and unbounded above and can be ordered to satisfy

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots. \tag{2.8}$$

Furthermore, for the complex boundary condition (2.4), all inequalities are strict, that is, each eigenvalue has multiplicity 1; for the periodic and semiperiodic conditions (2.5) and (2.6) – with $k = 1$ – the eigenvalues may have multiplicity 1 or 2 with the exception of the lowest periodic eigenvalue, which has multiplicity 1. Here multiplicity can be interpreted as either the geometric or algebraic multiplicity since these are the same.

For these and other basic results, definitions, and notation about Sturm–Liouville problems used further, we refer the reader to Chapter 1 or [113].

Notation 2.1.1. *We denote the complex, periodic, and semiperiodic eigenvalues by $\lambda_n(\gamma)$, $\lambda_n^P(k)$, and $\lambda_n^S(k)$, respectively, for $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $\gamma \in (0,\pi)$. Note that each eigenvalue λ_n is uniquely defined, although there may be some ambiguity for its eigenfunctions in case of multiplicity 2 for the periodic and semiperiodic cases. We also use the notations $\lambda_n^P(1) = \lambda_n^P = \lambda_n^P(0)$, $\lambda_n^S(1) = \lambda_n^S = \lambda_n^S(\pi)$, $n \in \mathbb{N}_0$, since the periodic eigenvalues correspond to the endpoint 0, and the semiperiodic eigenvalues correspond to the endpoint π of the interval $(0,\pi)$ in a natural sense as we will see further. Also, $\lambda_n^D(k)$ and $\lambda_n^N(k)$ denote the Dirichlet and Neumann eigenvalues; these play a special role for eigenvalue inequalities for different boundary conditions.*

For each $k > 2$, we identify which values of γ generate the periodic eigenvalues $\lambda_n^P(k)$ and which ones generate the semiperiodic eigenvalues $\lambda_n^S(k)$. The case where $k = 2$ is special in the sense that no value of γ generates these eigenvalues.

Remark 2.1.3. For smooth coefficients, periodic boundary conditions of the form

$$y(a) = y(b), \quad y'(a) = y'(b)$$

and semiperiodic conditions of the form

$$y(a) = -y(b), \quad y'(a) = -y'(b),$$

as well as different parameterizations of γ and relation between the eigenvalues $\lambda_n(\gamma)$ and $\lambda_n^P(k)$ are investigated in the well-known book by Eastham [27]. Although we are influenced by some of the methods used in [27], there are a number of significant differences in our approach:

(1) We assume neither that p is differentiable nor that q and w are piecewise continuous and that w is bounded away from 0.

(2) We do assume that p is positive. This seems to be an oversight in [27]. If p has positive and negative values, both on sets of positive Lebesgue measure (such as subintervals), then the eigenvalues are unbounded above and below [78]. So there is no unique ordering (2.8) of the eigenvalues. This ordering is critical for the results below and in [27].

(3) We believe that definitions (2.5) and (2.6) for periodic and semiperiodic boundary conditions are more natural than those used in [27]. Under condition (2.2), the quasi-derivative $y^{[1]} = (py')$ is continuous on $[a,b]$, whereas the classical derivative y' may not be continuous on $[a,b]$ and may not exist at all points of this interval. See [113] for further elaborations of this point.

(4) We use the interval $(0,\pi)$ to parameterize the complex boundary conditions (2.4), which generate the eigenvalues $\{\lambda_n^P(k), \lambda_n^S(k) : n \in \mathbb{N}_0, k > 2\}$. We believe that this is more natural, simpler, and more transparent than the interval $[-1,1]$ used in [27]. In particular, it provides a simple visualization of the points on the unit circle in the complex plane that generate these eigenvalues and naturally associates the "boundary" point 0 and π with the periodic and semiperiodic eigenvalues, respectively.

2.2 General eigenvalue inequalities

For convenience of the reader, we review eigenvalue inequalities for equation (2.1) with coefficients satisfying

$$1/p, q, w \in L^1(J,\mathbb{R}), \quad p > 0, \quad w > 0 \quad \text{a. e. on } J, \tag{2.1}$$

on an interval $J = [a,b]$, $-\infty < a < b < \infty$.

Remark 2.2.1. Note that in this section we do not assume that the coefficients are periodic. We will apply the results from this section to the intervals $[a, a+kh]$ for $k \in \mathbb{N}$ in the next sections.

Definition 2.2.1. For $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, determine solutions $u = u(\cdot, a, \lambda)$, $v = v(a, \cdot, \lambda)$ of equation (2.1) with initial conditions

$$u(a,\lambda) = 1 = v^{[1]}(a,\lambda), \quad v(a,\lambda) = 0 = u^{[1]}(a,\lambda), \tag{2.2}$$

and let

$$D(\lambda) = u(b,\lambda) + v^{[1]}(b,\lambda), \quad \lambda \in \mathbb{R}. \tag{2.3}$$

Note that u and v are linearly independent solutions on J. Since a is fixed, we may abbreviate the notation to $u(t,\lambda)$, $v(t,\lambda)$ and occasionally to just $u(t)$, $v(t)$ or even u, v when λ is fixed. The function $D(\lambda)$ is known as the discriminant or the characteristic function. It plays a major role in the study of the eigenvalues of the boundary value problems studied further.

Theorem 2.2.1. *Let* (2.1) *hold. Then*

(1) *The number $\lambda = \lambda_n(\gamma)$ for some $n \in \mathbb{N}_0$ and some $\gamma \in (0, \pi)$ if and only if*

$$D(\lambda) = 2\cos\gamma, \quad -\pi < \gamma < \pi. \tag{2.4}$$

In this case,

$$-2 < D(\lambda) < 2.$$

(2) *Let $0 < \gamma < \pi$. Then $\lambda_n(\gamma)$ is simple, and $\lambda_n(\gamma) = \lambda_n(-\gamma)$, $n \in \mathbb{N}_0$. If u_n is an eigenfunction of $\lambda_n(\gamma)$, then it is unique up to constant multiples, and its complex conjugate $\overline{u}_n$ is an eigenfunction of $\lambda_n(-\gamma)$, $n \in \mathbb{N}_0$.*

(3) *$\lambda = \lambda_n^P$ for some $n \in \mathbb{N}_0$ if and only if*

$$D(\lambda) = 2.$$

(4) *$\lambda = \lambda_n^S$ for some $n \in \mathbb{N}_0$ if and only if*

$$D(\lambda) = -2.$$

(5) *We have the following inequalities for $0 < \gamma < \pi$:*

$$\begin{aligned} \lambda_0^N \le \lambda_0^P < \lambda_0(\gamma) < \lambda_0^S &\le \{\lambda_0^D, \lambda_1^N\} \\ \le \lambda_1^S < \lambda_1(\gamma) < \lambda_1^P &\le \{\lambda_1^D, \lambda_2^N\} \\ \le \lambda_2^P < \lambda_2(\gamma) < \lambda_2^S &\le \{\lambda_2^D, \lambda_3^N\} \\ \le \lambda_3^S < \lambda_3(\gamma) < \lambda_3^P &\le \{\lambda_3^D, \lambda_4^N\} \le \cdots. \end{aligned}$$

Here the notation $\{\lambda_n^D, \lambda_{n+1}^N\}$ means either of λ_n^D and λ_{n+1}^N, and there is no comparison made between these two.

(6) *$\lambda_n \le \lambda_n^D \le \lambda_{n+2}$, $n \in \mathbb{N}_0$, where λ_n is the nth eigenvalue for any self-adjoint boundary condition on $[a, b]$; there is no lower bound for λ_0 and λ_1 as functions of the self-adjoint boundary conditions.*

(7) *λ_0^P is simple, and each of the other eigenvalues λ_n^P, $n \in \mathbb{N}$, and λ_n^S, $n \in \mathbb{N}_0$, may be simple or double. If λ_{2n+1}^P is simple, then λ_{2n+2}^P is also simple. If there is a double periodic eigenvalue, then the first double periodic eigenvalue is preceded by an odd number of simple periodic eigenvalues.*

(8) *For $\lambda = \lambda_n^D$ or $\lambda = \lambda_n^N$, $n \in \mathbb{N}_0$, we have*

$$D^2(\lambda) \ge 4.$$

(9) *For $0 < \alpha < \beta < \pi$, we have*

$$\begin{aligned} \lambda_0(\beta) < \lambda_0(\alpha) < \lambda_1(\alpha) < \lambda_1(\beta) < \lambda_2(\beta) < \lambda_2(\alpha) \\ < \lambda_3(\alpha) < \lambda_3(\beta) < \cdots. \end{aligned}$$

(10) *Suppose* $\lambda \in \{\lambda_n^P : n \in \mathbb{N}_0\} \cup \{\lambda_n^S : n \in \mathbb{N}_0\}$. *Then* λ *is a double eigenvalue if and only if* $\lambda = \lambda_n^D = \lambda_m^N$ *for some n and m.*
(11) $D(\lambda)$ *is strictly decreasing in the intervals* $(\lambda_{2n}^P, \lambda_{2n}^S)$, $n \in \mathbb{N}_0$, *and strictly increasing in the intervals* $(\lambda_{2n+1}^S, \lambda_{2n+1}^P)$, $n \in \mathbb{N}$.
(12) $D'(\lambda) \neq 0$ *for* $\lambda \in (0, \pi)$.

Proof. This is a particular case of the eigenvalue inequalities for general regular self-adjoint boundary conditions; see 4.8, pp. 91–95 in [113]. □

2.3 Structure of solutions

In this section, we assume that the coefficients of equation (2.1) are periodic, that is, (2.1), (2.2), and (2.3) hold. Solutions of equations with periodic coefficients have special properties. These are studied and summarized here.

It is well known [34] that the local integrability condition (2.2) is necessary and sufficient for each initial value problem of equation (2.1) to have a unique solution y defined on $\mathbb{R}$, and y and $y^{[1]}$ are continuous on $\mathbb{R}$. (Recall that, in general, y' is not continuous on $\mathbb{R}$.)

Recall that the scalar equation (2.1) is equivalent to the first-order system

$$Y' = PY, \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \tag{2.1}$$

where

$$P = \begin{bmatrix} 0 & 1/p \\ q - \lambda w & 0 \end{bmatrix}. \tag{2.2}$$

Let

$$Y(t,\lambda) = \begin{bmatrix} u(t,\lambda) & v(t,\lambda) \\ u^{[1]}(t,\lambda) & v^{[1]}(t,\lambda) \end{bmatrix}, \quad t \in \mathbb{R}, \quad \lambda \in \mathbb{C}. \tag{2.3}$$

The next theorem shows that both $Y(t,\lambda)$ and $Y(t+h,\lambda)$ are fundamental matrix solutions of (2.2), determines how they are related, and establishes the corresponding properties of the solutions of the scalar equation (2.1).

Theorem 2.3.1. *Fix* $a \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$. *Let* $Y(a) = I$, *the identity matrix. Then:*
(1) $Y(t,\lambda)$ *is a fundamental matrix solution of* (2.1).
(2) $Y(t+h,\lambda)$ *is also a fundamental matrix solution of* (2.1).
(3) $Y(t+h,\lambda) = Y(t,\lambda)A(\lambda)$, $t \in \mathbb{R}$, $\lambda \in \mathbb{C}$,
where

$$A(\lambda) = \begin{bmatrix} u(a+h,\lambda) & v(a+h),\lambda) \\ u^{[1]}(a+h,\lambda) & v^{[1]}(a+h),\lambda) \end{bmatrix} \tag{2.4}$$

and $\det A(\lambda) = 1$.

(4) *For some complex number* $\rho \neq 0$, *there exists a nontrivial solution* y *of* (2.1) *such that*

$$\begin{aligned} y(t+h,\lambda) &= \rho(\lambda)y(t,\lambda), \\ y^{[1]}(t+h,\lambda) &= \rho(\lambda)y^{[1]}(t,\lambda), \quad t \in \mathbb{R}, \end{aligned} \tag{2.5}$$

if and only if ρ *is an eigenvalue of* $A(\lambda)$.

(5) *The eigenvalues* $\rho_1(\lambda)$ *and* $\rho_2(\lambda)$ *of* $A(\lambda)$ *are roots of the quadratic equation*

$$\rho(\lambda)^2 - \operatorname{trace} A(\lambda)\rho(\lambda) + 1 = 0 \tag{2.6}$$

and satisfy

$$\rho_1(\lambda)\rho_2(\lambda) = 1, \quad \lambda \in \mathbb{C}. \tag{2.7}$$

(6) *Each* ρ_j *has a representation*

$$\rho_j = e^{m_j h} \tag{2.8}$$

for some $m_j \in \mathbb{C}$, $j = 1, 2$. *The real part of* m_j *is unique, and the imaginary part of* m_j *is not unique.*

Proof. Let $Z(t,\lambda) = Y(t+h,\lambda)$ and note that Z is a matrix solution of (2.1):

$$Z'(t,\lambda) = Y'(t+h,\lambda) = P(t+h,\lambda)Y(t+h,\lambda) = P(t,\lambda)Z(t,\lambda).$$

Since $\operatorname{trace} P = 0$, it follows from Abel's theorem that

$$\det Y(t,\lambda) = \det Y(a,\lambda) = 1 = \det Y(t+h,\lambda) = \det A(y).$$

Hence both $Y(t+h,\lambda)$ and $Y(t,\lambda)$ are fundamental matrix solutions of (2.1), proving (1) and (2). Therefore by a basic result of linear ordinary differential equations we have

$$Y(t+h,\lambda) = Y(t,\lambda)A(\lambda), \quad t \in \mathbb{R}, \tag{2.9}$$

where $A(\lambda)$ is a nonsingular matrix independent of t, and (2.4) follows by evaluating (2.9) at a, establishing (3). To prove (4), note that $\det A(\lambda) = 1$ by (1) and (2). Hence, if ρ is an eigenvalue of $A(\lambda)$, then $\rho \neq 0$, and there is a nonzero constant vector C such that

$$[A - \rho I]C = 0;$$

therefore from (2.9) we get

$$Y(t+h,\lambda)[A - \rho I]C = Y(t,\lambda)[A - \rho I]C,$$

and (2.5) follows, proving (4). (Here we omit λ in the notation for simplicity of exposition, but ρ depends on λ.) From (2.4) we get

$$\det(A - \rho I) = \rho^2 - (\operatorname{trace} A)\rho + \det A = \rho^2 - (\operatorname{trace} A)\rho + 1 = 0.$$

Now (2.7) follows from the fact that the product of the roots of a quadratic equation of the form $x^2 + bx + c = 0$ is equal to c, completing the proof of (5).

To get the representation (2.8) in part (6), we recall that $\rho \neq 0$. Let $\rho = re^{i\theta}$ with $r > 0$, and let $r = e^x$. Then take $mh = x + i\theta$. Note that the real part of m is unique but the imaginary part of m is not unique. For $\rho > 0$, take $\theta = 0$ and note that the representation $\rho = e^{mh}$ is unique. For $\rho < 0$, take $\rho = e^{mh+i\pi} = e^{(m+i\pi/h)h}$. This completes the proof of Theorem 2.3.1. □

The eigenvalues $\rho_1(\lambda)$ and $\rho_2(\lambda)$ of the matrix $A(\lambda)$ strongly influence the structure of the solutions of equation (2.1) on $\mathbb{R}$. We study this influence next.

Lemma 2.3.1. *Assume that y is a nontrivial solution of equation* (2.1) *satisfying* (2.5) *with $\rho = e^{mh}$, $m \in \mathbb{C}$. Then there exists an h-periodic function g of* (2.1) *such that*

$$y(t) = e^{mt} g(t), \quad t \in \mathbb{R}.$$

Proof. Define $g(t) = e^{-mt} y(t)$. Then

$$g(t+h) = e^{-m(t+h)} y(t+h) = e^{-mt} e^{-mh} \rho y(t) = e^{-mt} e^{-mh} e^{mh} y(t) = e^{-mt} y(t) = g(t),$$

and therefore $y(t) = e^{mt} g(t)$, $t \in \mathbb{R}$. □

Theorem 2.3.2. *Let $\rho_1(\lambda)$ and $\rho_2(\lambda)$ be the eigenvalues of the matrix $A(\lambda)$ with representations given by $\rho_j = e^{m_j h}$, $j = 1, 2$, where m_1 and m_2 are complex constants, not necessarily distinct. Then:*

a: *If $\rho_1(\lambda) \neq \rho_2(\lambda)$, then there are two linearly independent solutions y_1 and y_2 of equation* (2.1) *such that*

$$y_1(t) = e^{m_1 t} g_1(t), \quad y_2(t) = e^{m_2 t} g_2(t), \quad t \in \mathbb{R},$$

where g_1 and g_2 are h-periodic.

b: *If $\rho_1(\lambda) = \rho_2(\lambda) = \rho = e^{mh}$, then there is a solution y_1 of equation* (2.1) *such that*

$$y_1(t) = e^{mt} g_1(t),$$

where g_1 is h-periodic.

Let y_2 be a solution such that y_1 and y_2 are linearly independent, and let $y(t) = cy_1(t) + dy_2(t)$. Then there are two subcases:

(1) $c = 0$. *In this case,*

$$y_2(t) = e^{mt} g_2(t),$$

where g_2 is also h-periodic.

(2) $c \neq 0$. *There are two linearly independent solutions* y_1 *and* y_2 *of equation* (2.1) *such that*

$$y_1(t) = e^{mt} g_1(t), \quad y_2(t) = e^{mt}[t \quad g_1(t) + g_2(t)], \quad t \in \mathbb{R},$$

where g_1 *and* g_2 *are h-periodic functions.*

Proof. Case (a) and case (b), part (1), are immediate corollaries of Lemma 2.3.1. For case (b), part (2), recall that the Wronskian $W(y_1, y_2)$ of two linearly independent solutions is a nonzero constant and observe that by a direct computation we have

$$W(y_1, y_2)(t+h) = \rho d W(y_1, y_2)(t),$$

and therefore $d = \rho$ by (2.7). Hence $y(t) = cy_1(t) + \rho y_2(t)$.

Let

$$k_1(t) = e^{-mt} y_1(t), \quad k_2(t) = e^{-mt} y_2(t) - \frac{c}{\rho h} t \quad g_1(t).$$

Then g_1 is h-periodic by Theorem 3.2.1, and so is g_2:

$$\begin{aligned} g_2(t+h) &= e^{-m(t+h)} y_2(t+h) - \frac{c}{\rho h}(t+h) g_1(t) \\ &= e^{-mt} e^{-mh} [cy_1(t) + \rho y_2(t)] - \frac{c}{\rho h}(t+h) g_1(t) \\ &= e^{-mt} e^{-mh} cy_1(t) + e^{-mt} y_2(t) - \frac{c}{\rho h}(t+h) g_1(t) \\ &= e^{-mt} y_2(t) - \frac{c}{\rho h} t g_1(t) = g_2(t). \end{aligned}$$

Now part (2) follows, and the proof is complete. □

2.4 Eigenvalues on one interval

In this section, we assume that (2.1), (2.2), and (2.3) hold. Let

$$D(\lambda) = u(b, \lambda) + v^{[1]}(b, \lambda), \quad \lambda \in \mathbb{R}, \quad b = a + h, \tag{2.1}$$

where u, v are defined by (2.2), and note that $D(\lambda) = \operatorname{trace} A(\lambda)$ for $A(\lambda)$ given by (2.4).

The next theorem is basically a corollary of Theorems 2.3.1 and 2.3.2 and Lemma 2.3.1. It will be used further to get information about the eigenvalues for the periodic, semiperiodic, and complex boundary conditions on the interval $[a, a+h]$.

Theorem 2.4.1. *We have*

A: *Suppose* $D(\lambda) > 2$. *By Theorem* (2.2) *the eigenvalues* ρ_1 *and* ρ_2 *of* $A(\lambda)$ *are distinct and positive but not equal to* 1. *Let m be the unique positive number such that*

$$\rho_1 = e^{mh}, \quad \rho_2 = e^{-mh}.$$

Then two linearly independent solutions of (2.1) *have the form*

$$y_1(t) = e^{mt}g_1(t), \quad y_2(t) = e^{-mt}g_2(t), \quad t \in \mathbb{R},$$

where $g_1(t)$ *and* $g_2(t)$ *are h-periodic.*

B: *Suppose* $D(\lambda) < -2$. *This is similar to case A except that* $\rho_1(\lambda)$ *and* $\rho_2(\lambda)$ *are now negative but not equal to* -1. *Let m be the unique positive number such that*

$$\rho_1 = e^{(m+i\pi/h)h}, \quad \rho_2 = e^{-(m+i\pi/h)h}.$$

Then two linearly independent solutions of (2.1) *have the form*

$$y_1(t) = e^{(m+i\pi/h)t}g_1(t), \quad y_2(t) = e^{-(m+i\pi/h)t}g_2(t), \quad t \in \mathbb{R},$$

where $m > 0$, *and* $g_1(t)$ *and* $g_2(t)$ *are h-periodic.*

C: *Suppose* $-2 < D(\lambda) < 2$. *Then* $\rho_1(\lambda)$ *and* $\rho_2(\lambda)$ *are nonreal and distinct. By* (2.8) *there is a real number* γ *with* $0 < \gamma < \pi$ *such that* $e^{i\gamma} = \rho_1$ *and* $e^{-i\gamma} = \rho_2$. *Choose m such that* $\gamma = mh$. *Then we have*

$$\rho_1 = e^{imh}, \quad \rho_2 = e^{-imh}, \quad 0 < \gamma = mh < \pi,$$

and from Lemma 2.3.1 *it follows that*

$$y_1(t) = e^{imt}g_1(t), \quad y_2(t) = e^{-imt}g_2(t),$$

where $g_1(t)$ *and* $g_2(t)$ *are h-periodic linearly independent solutions of* (2.1) *on* $\mathbb{R}$.

D: *Suppose* $D(\lambda) = 2$. *Then* $\rho_1(\lambda) = \rho_2(\lambda) = 1$, *and* λ *is a periodic eigenvalue on the interval* $[a, a+h]$. *Conversely, if* λ *is a periodic eigenvalue on* $[a, a+h]$, *then* $D(\lambda) = 2$. *The eigenvalue* λ *may have multiplicity* 1 *or* 2.

E: *Suppose* $D(\lambda) = -2$. *In this case,* $\rho_1(\lambda) = \rho_2(\lambda) = -1$, *and* λ *is a semiperiodic eigenvalue on the interval* $[a, a+h]$, *and, conversely, if* λ *is a semiperiodic eigenvalue on* $[a+h]$, *then* $D(\lambda) = -2$. *The eigenvalue* λ *may have multiplicity* 1 *or* 2.

Proof. All theses cases follow from Theorems 2.3.1 and 2.3.2, Lemma 2.3.1, and the quadratic equation (2.6). D and E follow from general Theorem 2.2.1. □

Corollary 2.4.1. *Since* $\rho(\lambda)$ *is an eigenvalue of* $A(\lambda)$, *the rank of* $A(\lambda) - \rho I$ *is either* 1 *or* 0.

(1) *If* $\operatorname{rank}[A(\lambda) - \rho(\lambda)I] = 1$, *then there is exactly one linearly independent solution y of* (2.1) *satisfying* (2.5).
(2) *If* $\operatorname{rank}[A(\lambda) - \rho(\lambda)I] = 0$, *then there are exactly two linearly independent solution y of* (2.1) *satisfying* (2.5).
(3) *Assume that* $\rho_1(\lambda) = \rho_2(\lambda) = 1$. *Then* λ *is real and is an eigenvalue on the interval* $[a, a+h]$ *of the periodic boundary condition*

$$y(a+h) = y(a),$$
$$y^{[1]}(a+h) = y^{[1]}(a).$$

It has multiplicity 2 *if and only if* $\operatorname{rank}[A(\lambda) - I] = 0$. *Since* $\operatorname{rank}[A^t(\lambda) - I] = \operatorname{rank}[A(\lambda) - I]$ *and this holds if and only if*

$$\begin{bmatrix} u(a+h,\lambda) & v(a+h),\lambda) \\ u^{[1]}(a+h,\lambda) & v^{[1]}(a+h),\lambda) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

in this case, both u and v are eigenfunctions of λ *on the interval* $[a, a+h]$.

(4) *Assume that* $\rho_1(\lambda) = \rho_2(\lambda) = -1$. *Then* λ *is real and is an eigenvalue on the interval* $[a, a+h]$ *of the semiperiodic boundary condition*

$$y(a+h) = -y(a),$$
$$y^{[1]}(a+h) = -y^{[1]}(a).$$

It has multiplicity 2 *if and only if* $\operatorname{rank}[A(\lambda) - I] = 0$. *Since* $\operatorname{rank}[A^t(\lambda) - I] = \operatorname{rank}[A(\lambda) - I]$ *and this holds if and only if*

$$\begin{bmatrix} u(a+h,\lambda) & v(a+h),\lambda)) \\ u^{[1]}(a+h,\lambda) & v^{[1]}(a+h),\lambda) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

in this case, both u and v are eigenfunctions of λ *on the interval* $[a, a+h]$.

Proof. This follows from Theorem 2.3.1 and its proof. □

2.5 Eigenvalues on different intervals

In this section, for each $k > 2$, we find explicitly a value of $\gamma \in (0, \pi)$ such that the eigenvalues $\lambda_n(\gamma)$ from the interval $[a, a+h]$ are also periodic eigenvalues on the interval $[a, a+kh]$ and a corresponding result for the semiperiodic eigenvalues. For $k = 2$, every periodic eigenvalue on $[a, a+2h]$ is either a periodic or semiperiodic eigenvalue on $[a, a+h]$. Recall the notation $\lambda_n(\gamma)$, $\lambda_n^P(k)$, and $\lambda_n^S(k)$ for the eigenvalues of the complex, periodic, and semiperiodic, boundary conditions.

Key to this analysis is a simple but important observation, which we make with the next remark.

Remark 2.5.1. Hypothesis (2.2) that the coefficients p, q, w are h-periodic does not assume that h is the smallest positive number for which p, q, w are h-periodic. Also note that if p, q, w are h-periodic, then they are also hk-periodic for each $k \in \mathbb{N}$. Therefore the results of Sections 2.3 and 2.4 hold when the interval $[a, a+h]$ is replaced by $[a, a+kh]$ for any $k \in \mathbb{N}$ and, correspondingly, h by kh. Note that if y is an eigenfunction of an eigenvalue for a periodic boundary condition on any interval $[a, a+kh]$, then y can be extended to a periodic solution of equation (2.1) on $\mathbb{R}$, and, conversely, if y is a kh-periodic solution on $\mathbb{R}$, then y is a periodic eigenvalue on $[a, a+kh]$. We will use the same notation y for an eigenfunction on $[a, a + kh]$ of a periodic boundary condition and for its periodic extension to $\mathbb{R}$. This should not be confusing since it is clear from the context.

To illustrate how the results of Sections 2.3 and 2.4 apply to the kh-periodic case, we state the kh version of Theorem 3.2.1.

Lemma 2.5.1. *Let* (2.1), (2.2), *and* (2.3) *hold, and let* $k \in \mathbb{N}$. *Then* p, q, w *are* kh*-periodic. Define*

$$D_k(\lambda) = u(a + kh, \lambda) + v^{[1]}(a + kh, \lambda), \quad \lambda \in \mathbb{R}. \tag{2.1}$$

Assume that y *is a nontrivial solution of equation* (2.1) *satisfying* (2.5) *with* $\rho = e^{mkh}$, $m \in \mathbb{C}$. *Then there exists a* kh*-periodic function* g *of* (2.1) *such that*

$$y(t) = e^{mt} g(t), \quad t \in \mathbb{R}.$$

Proof. Replace h by kh in the proof of Lemma 2.3.1. □

Note that when the underlying interval is $[a, a + kh]$, $k \in \mathbb{N}$, then to restrict γ to the interval $(0, \pi)$, we must have $0 < \gamma = mkh < \pi$ rather than $0 < \gamma = mh < \pi$.

Next, to find the values of $\gamma \in (0, \pi)$ such that every eigenvalue $\lambda_n(\gamma)$ is a periodic eigenvalue on the interval $[a, a + kh]$, we use Theorem 2.4.1 to construct the periodic eigenfunctions for this interval, and similarly for the semiperiodic case. This is done by (i) eliminating cases A and B and then by using C to construct the appropriate solutions and D and E to show that the constructed solutions are periodic or semiperiodic and that all periodic and semiperiodic solutions can be obtained this way.

We start with the elimination of cases A and B.

Lemma 2.5.2. *Let* $k \in \mathbb{N}$, *and let* (2.2) *hold. If* $D_k(\lambda) > 2$ *or* $D_k(\lambda) < -2$, *then all solutions of equation* (2.1) *are unbounded on* $\mathbb{R}$ *and therefore cannot be eigenfunctions for the periodic or semiperiodic boundary conditions.*

Proof. Any periodic or semiperiodic eigenfunction can be extended to a periodic or semiperiodic function on $\mathbb{R}$ and is therefore bounded. If $D_k(\lambda) > 2$ or $D_k(\lambda) < -2$, then the solutions of (2.1) are not bounded by A and B of Theorem 2.4.1. □

In the construction of periodic and semiperiodic solutions for the intervals $[a, a + kh]$ the roots of unity play an important role. There are many different representations of these, so next we present the representations used in the proofs below.

Lemma 2.5.3. *For each integer* $k \in \mathbb{N}$, *there are exactly* k *complex roots of the number* 1*: For* $k = 1$, *there is only one root, namely,* 1. *For* $k = 2$, *there are two roots,* -1 *and* 1. *For* $k > 2$, *the* k *roots*

$$e^{i2l\pi/k}, \quad l = 0, 1, \ldots, k - 1,$$

can be represented by

(1) $k = 2s$, $s > 1$,

$$1, -1, e^{i2\pi l/k}, e^{-i2\pi l/k}, \quad l = 1, \ldots, s-1,$$

(2) $k = 2s+1$, $s > 0$,

$$e^{i2\pi l/k}, e^{-i2\pi l/k}, \quad l = 1, \ldots, s.$$

Similarly, for each integer $k \in \mathbb{N}$, there are exactly k complex roots of the number -1:

$$e^{i(2l+1)\pi)/k}, \quad l = 0, 1, \ldots, k-1,$$

which can be represented by

(1) $k = 2s$, $s \geq 1$,

$$i, -i, e^{i(2l+1)\pi/k}, e^{-i(2l+1)\pi/k}, \quad l = 0, \ldots, s-1,$$

(2) $k = 2s+1$, $s \geq 1$,

$$e^{i(2l+1)\pi/k}, e^{-i(2l+1)\pi/k}, \quad l = 0, \ldots, s-1.$$

Proof. This is well known. These representations can be derived from the standard one using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ and properties of the sin and cos functions. □

For $\gamma \in (0, \pi)$, since $\lambda_n(\gamma) = \lambda_n(-\gamma)$, the eigenvalues $\lambda_n(\gamma)$ can be visualized as being determined by γ in the upper open half-circle in the complex plane and also in the lower open half-circle with the exception of the two "boundary points" $\gamma = 0$ and $\gamma = \pi$. These two points correspond to the periodic and semiperiodic eigenvalues, respectively. The next theorem makes this statement precise.

Theorem 2.5.1. *Let* (2.2) *hold, let* $J = [a, a+h]$, *and let* $D(\lambda)$ *be defined by* (2.1) *with* $k = 1$. *Then for* $\gamma \in (0, \pi)$,

$$\lim_{\gamma \to 0} \lambda_{2n}(\gamma) = \lambda_{2n}^{P} \quad \textit{and} \quad \lim_{\gamma \to \pi} \lambda_{2n+1}(\gamma) = \lambda_{2n+1}^{S}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

The limits are the appropriate one-sided limits given the restriction $\gamma \in (0, \pi)$.

Proof. This is well known; see [113] p. 92. □

Theorem 2.5.1 helps us to visualize the "movements" of the eigenvalues $\lambda_n(\gamma) = \lambda_n(-\gamma)$ toward the limits in (2.2); see the graph on p. 92 of [113].

The next theorem determines exactly which of the eigenvalues $\lambda_n(\gamma)$ from the interval $[a, a+h]$ are also periodic eigenvalues on the intervals $[a, a+kh]$ for $k \in \mathbb{N}$.

Theorem 2.5.2. *Let* (2.2) *and* (2.3) *hold, let* $k \in \mathbb{N}$*, and let* $D(\lambda)$ *be defined by* (2.1) *with* $b = a + h$*. Then equation* (2.1) *has a nontrivial solution with period* kh *on* $\mathbb{R}$ *if and only if*

$$D(\lambda) = 2\cos(2l\pi/k)$$

for some integer $l \in \mathbb{Z}$*.*

Furthermore,

(1) *If* $k = 2s$, $s > 1$, $\gamma = 2l\pi/2s$, $l = 1, \dots, s-1$, *and* $\lambda = \lambda_n(\gamma)$ *for some* $n \in \mathbb{N}_0$*, then* λ *is a periodic eigenvalue with multiplicity* 2 *on the interval* $[a, a+kh]$*, and every eigenfunction of* λ *can be extended to a periodic solution on* $\mathbb{R}$*.*

(2) *If* $k = 2s+1$, $s \geq 1$, $\gamma = \frac{2l\pi}{2s+1}$, $l = 1, \dots, s-1$, *and* $\lambda = \lambda_n(\gamma)$ *for some* $n \in \mathbb{N}_0$*, then* λ *is a periodic eigenvalue with multiplicity* 2 *on the interval* $[a, a+kh]$*, and every eigenfunction of* λ *can be extended to a periodic solution on* $\mathbb{R}$*.*

The word "multiplicity" here can be taken as either the geometric or the algebraic multiplicity, where the algebraic multiplicity is defined as the order of λ *as a zero of the function* $D(\lambda)$*.*

Proof. Note that any periodic eigenfunction on $[a, a+kh]$ for any $k \in \mathbb{N}$ when extended to $\mathbb{R}$ is bounded. Hence parts A and B of Theorem 2.4.1 do not apply, and we need to consider only $-2 \leq D(\lambda) = 2\cos\gamma \leq 2$.

Suppose $-2 < D(\lambda) = 2\cos\gamma < 2$. Then case C of Theorem 2.4.1 applies: $\rho_1(\lambda)$ and $\rho_2(\lambda)$ are nonreal and distinct, and there is a real number γ satisfying $0 < \gamma < \pi$ such that $e^{i\gamma} = \rho_1(\lambda)$, $e^{-i\gamma} = \rho_2(\lambda)$. Choose m such that $\gamma = mh$ with m real and not 0. Then we have

$$\rho_1 = e^{imh}, \rho_2 = e^{-imh}, \quad 0 < \gamma = mh < \pi,$$

and

$$y_1(t) = e^{imt} g_1(t), \quad y_2(t) = e^{-imt} g_2(t),$$

where $g_1(t)$ and $g_2(t)$ are linearly independent h-periodic solutions of (2.1) on $\mathbb{R}$. Note that for any $k \in \mathbb{N}$, $g_1(t)$ and $g_2(t)$ are hk-periodic, and for $0 < \gamma = mhk < \pi y = cy_1 + dy_2$ with constants c, d, not both 0, we have

$$\begin{aligned} & y(t+kh) - y(t) \\ & \quad = ce^{imk(t+h)} g_1(t+h) + de^{-imkt+h)} g_2(t+h) - ce^{imkt} g_1(t) - de^{-imkt} g_2(t) \\ & \quad = c[e^{imkh} - 1] e^{imtk} g_1(t) + d[e^{-imkh} - 1] e^{-imkt} g_2(t), \quad t \in \mathbb{R}. \end{aligned}$$

Note that $e^{imkh} = 1$ if and only if $e^{-imkh} = 1$. Hence y_1 and y_2 and therefore y are kh-periodic if and only if e^{imh} is a kth root of 1, and this is true if and only if

$$mkh = l(2\pi)$$

for some $l \in \mathbb{Z}$.

For $k = 1$, e^{ix} has one and only one root when $x = l(2\pi)$. In particular, this holds with $l = 1$ (or $l = 0$).

For $k = 2$, there are exactly two roots of 1: e^{ix} when $x = 0$ and $x = \pi$. Note that neither root is in the range $(0, \pi)$ but both are boundary points of this open interval. Therefore none of λ satisfying $-2 < D(\lambda) < 2$ is a periodic eigenvalue on the interval $[a, a + 2h]$, and case C is eliminated. Therefore only cases D and E remain, and λ is either a periodic eigenvalue on $[a, a + h]$ or a semiperiodic eigenvalue on $[a, a + h]$.

These two cases $k = 1$ and $k = 2$ are "special" in the sense that for neither case, there is a value of $\lambda \in \mathbb{R}$ with $-2 < D(\lambda) < 2$ such that λ is also a periodic eigenvalue on $[a, a + h]$ or on $[a, a + 2h]$.

Next, we show that for $k > 2$, there is at least one λ in the range $(0, \pi)$ such that λ is a periodic eigenvalue on the interval $[a, a + kh]$.

Choose m and l such that

$$mkh = 2l\pi \quad \text{and} \quad 0 < \frac{2l}{k} < 1.$$

Then $0 < \gamma = mh < \pi$. Such a choice for m and l is always possible when $k > 2$.

It follows that:

(1) If $k = 2s$, $s > 1$, then $\lambda \in \{\lambda_n(\gamma) : n \in \mathbb{N}_0, \gamma \in (0, \pi)\}$ is a periodic eigenvalue on $[a, a + kh]$ if and only if $\gamma = 2l\pi/2s$, $l = 1, \ldots, s - 1$, and it has multiplicity 2.

(2) If $k = 2s + 1$, $s \geq 1$, then $\lambda \in \{\lambda_n(\gamma) : n \in \mathbb{N}_0, \gamma \in (0, \pi)\}$ is a periodic eigenvalue on $[a, a + kh]$ if and only if $\gamma = \frac{2\pi l}{2s+1}$, $l = 1, \ldots, s$, and it has multiplicity 2.

(3) Furthermore, for both cases, each λ has multiplicity 2 since both eigenfunctions g_1 and g_2 are kh-periodic.

If λ is a periodic eigenvalue on $[a, a + kh]$ for some $k \in \mathbb{N}$, then its extension to $\mathbb{R}$ is a kh-periodic solution of equation (2.1). Conversely, if λ is a kh-periodic solution of equation (2.1), then λ is an eigenfunction on the interval $[a, a + kh]$ for any $a \in \mathbb{R}$. This completes the proof. □

Now we list some examples to illustrate Theorem 2.5.1.

Example 2.5.1. For each $k \in \mathbb{N}$, there are exactly kth roots of unity, that is, complex roots of the number 1. These have different representations as shown by Lemma 2.5.1. For $k = 1$, there is one and only one root, namely, 1; for $k = 2$, there are exactly two roots, 1 and -1. These can also be represented by $e^{i2\pi} = 1$ and $e^{i\pi} = -1$. For $k > 2$, it is convenient to use the representation given in the previous lemma:

$k = 3$: 1, $e^{i2\pi/3}, e^{-i\pi/3}$ are representations of the three distinct roots of 1. Note that $\gamma = 2\pi/3$ satisfies $0 < \gamma < \pi$ and $\gamma = -2\pi/3$ satisfies $-\pi < \gamma < 0$. Recall that $D(\lambda) = 2\cos(\gamma) = 2\cos(-\gamma)$, so λ is an eigenvalue for γ and (the same λ) is also an eigenvalue for $-\gamma$, but the eigenfunctions are complex conjugates of each other; see part C.

$k = 4$**:** $1, -1, e^{i2\pi/4}, e^{-i2\pi/4}$ are the four distinct roots of 1. Note that $\gamma = 2\pi/4 \in (0, \pi)$ and $\gamma = -2\pi/4 \in (-\pi, 0)$.

$k = 5$**:** $1, e^{i2\pi/5}, e^{i4\pi/5}, e^{-i2\pi/5}, e^{-i4\pi/5}$ are the five distinct roots of 1. Note that $\gamma = 2\pi/5$ and $\gamma = 4\pi/5$ are both in $(0, \pi)$ and $\gamma = -2\pi/5$ and $\gamma = -4\pi/5$ are both in $(-\pi, 0)$.

$k = 6$**:** $1, -1, e^{i2\pi/6}$, $e^{i4\pi/6}$, $e^{-i2\pi/6}$, $e^{-i4\pi/6}$ are the six distinct roots of 1. Note that $2\pi/6$, $4\pi/6 \in (0, \pi)$ and $-2\pi/6$, $-4\pi/6 \in (0, \pi)$.

$k = 7$**:** $1, e^{i2\pi/7}, e^{i4\pi/7}, e^{i6\pi/7}, e^{-i2\pi/7}, e^{-i4\pi/7}, e^{-i6\pi/7}$ are the seven distinct roots of 1. Note that $\gamma = 2\pi/7$, $\gamma = 4\pi/7$, and $\gamma = 6\pi/7$ are in $(0, \pi)$ and their negatives are in $(-\pi, 0)$.

$k = 8$**:** $1, -1$, $e^{i2\pi/8}$, $e^{i4\pi/8}, e^{i6\pi/8}, e^{-i2\pi/8}, e^{-i4\pi/8}, e^{-i6\pi/8}$ are the 8 distinct roots of 1. Note that $2\pi/8$, $4\pi/8, 4\pi/8 \in (0, \pi)$ and $-2\pi/8$, $-4\pi/8, -6\pi/8 \in (-\pi, 0)$.

Remark 2.5.2. As mentioned in the proof of Theorem 2.5.2, the cases $k = 1$ and $k = 2$ are "special". Here λ_0^P on the interval $[a, a + h]$ is simple, and each of the other eigenvalues for both intervals $[a, a+h]$ and $[a, a+2h]$ may be simple or double. Determining explicitly which eigenvalue is simple on $[a, a + h]$ and which one is double is an open problem, not only for periodic coefficients, but also for general self-adjoint Sturm–Liouville problems. See Corollary 2.4.1 for necessary and sufficient conditions in terms of $u^{[1]}$ and v at the endpoint $a + h$. (See [113] for a more detailed statement of this and many other open problems for the Sturm–Liouville equation (2.1).

Next, we investigate the relation between the eigenvalues of the complex boundary conditions parameterized by $\gamma \in (0, \pi)$ and the semiperiodic eigenvalues on the intervals $[a, a + kh]$, $k \in \mathbb{N}$. This is similar to the case of periodic eigenvalues studied above, but there is a difference here for k even or odd, and the special case $k = 2$ in the periodic case is different in the semiperiodic case.

Theorem 2.5.3. *Let* (2.2) *and* (2.3)*, hold, let* $k \in \mathbb{N} = \{1, 2, 3, \dots\}$*, and let* $D(\lambda)$ *be defined by* (2.1) *with* $b = a + h$*. Then equation* (2.1) *has a nontrivial solution with semiperiod* kh *on* $\mathbb{R}$ *if and only if*

$$D(\lambda) = 2\cos\big((2l + 1)\pi/k\big)$$

for some integer $l \in \mathbb{Z}$*.*

Furthermore,

(1) *If* $k = 2s$*,* $s \geq 1$*, then* $\lambda \in \{\lambda_n(\gamma) : n \in \mathbb{N}_0, \gamma \in (0, \pi)\}$ *is a semiperiodic eigenvalue on* $[a, a + kh]$ *if and only if* $\gamma = (2l + 1)\pi/2s$*,* $l = 1, \dots, s$*, and* λ *has multiplicity* 2*.*

(2) *If* $k = 2s + 1$*,* $s \geq 1$*, then* $\lambda \in \{\lambda_n(\gamma) : n \in \mathbb{N}_0, \gamma \in (0, \pi)\}$ *is a semiperiodic eigenvalue on* $[a, a + kh]$ *if and only if* $\gamma = \frac{(2l+1)\pi}{2s+1}$*,* $l = 1, \dots, s$*, and* λ *has multiplicity* 2 *if* $s > 1$*.*

Proof. As in the proof of the periodic case, we only need to consider the case $-2 < D(\lambda) < 2$. Here $\rho_1(\lambda)$ and $\rho_2(\lambda)$ are nonreal and distinct, and there is a real number γ

satisfying $0 < \gamma < \pi$ such that $e^{i\gamma} = \rho_1(\lambda)$ and $e^{-i\gamma} = \rho_2(\lambda)$. Choose m such that $\gamma = mh$ with m real and not 0. Then we have

$$\rho_1 = e^{imh}, \quad \rho_2 = e^{-imh}, \quad 0 < \gamma = mh < \pi,$$

and

$$y_1(t) = e^{imt} g_1(t), \quad y_2(t) = e^{-imt} g_2(t),$$

where $g_1(t)$ and $g_2(t)$ are h-periodic and linearly independent solutions of (2.1) on $\mathbb{R}$. Let $\lambda = \lambda_n(\gamma)$ and $0 < \gamma = mh < \pi$. Let $y = cy_1 + dy_2$ for constants c and d, not both 0. Proceeding as in Theorem 2.5.2, we obtain

$$\begin{aligned} &y(t+kh) + y(t) \\ &\quad = ce^{im(t+kh)} g_1(t+kh) + de^{-im(t+kh)} g_2(t+kh) + ce^{imt} g_1(t) + de^{-imt} g_2(t) \\ &\quad = c[e^{imkh} + 1] e^{imt} g_1(t) + d[e^{-imkh} + 1] e^{-imt} g_2(t), \quad t \in \mathbb{R}. \end{aligned}$$

Note that $e^{imkh} = -1$ if and only if $e^{-imkh} = -1$. Hence y_1 and y_2 are kh-semi-periodic, and y is kh-semiperiodic, if and only if e^{imh} is a k-th root of -1 and this is true if and only if

$$mkh = (2l + 1)\pi$$

for some $l \in \mathbb{Z}$.

For $k = 1$, -1 has one and only one root, which can be taken as π since $e^{i\pi} = -1$. For $k = 2$, the two distinct roots can be taken as π ($l = 1$) and $-\pi$ ($l = -1$). Note that neither one is in $(0, \pi)$, and therefore for none of $\gamma \in (0, \pi)$ is a semi-periodic eigenvalue on the interval $[a, a + 2h]$.

For $k = 3$, the three roots of $e^{i3x} = -1$ can be represented by $x = \frac{\pi}{3}$, $x = \pi$, and $x = -\pi$. Note that only one of these roots is in $(0, \pi)$. For $l = 1$ (when $\gamma = \pi$), λ is a semiperiodic eigenvalue on $[a, a + 3h]$. Hence if λ is a semiperiodic eigenvalue on $[a, a + 3h]$, then λ is a semiperiodic eigenvalue on $[a, a + h]$.

Next, we show that for any $k > 2$, there is at least one λ in the range $(0, \pi)$ that is a semiperiodic eigenvalue on the interval $[a, a + kh]$.

Choose m and l such that

$$mkh = (2l + 1)\pi \quad \text{and} \quad 0 < \frac{2l + 1}{k} < 1.$$

Then $0 < \gamma = mh < \pi$. Such a choice for m and l is always possible when $k > 1$.

Proceeding as in Theorem 2.5.2, we find that for both $k = 2s$, $s > 1$, and $k = 2s + 1$, $s \geq 1$, s of the k roots of -1,

$$e^{i(2(l+1)\pi)/k}, \quad l = 0, 1, \ldots, k - 1,$$

have the representation

$$e^{i(2l+1)\pi/k}, \quad l = 0, 1, 2, \ldots, s-1.$$

Since $\gamma = mh$, it follows that $e^{imkh} = e^{ik\gamma} = -1$ if and only if

$$\gamma = (2l+1)\pi/k, \quad l = 0, 1, 2, \ldots, s-1.$$

This completes the proof. □

2.6 Eigenvalues of periodic, semiperiodic, and complex boundary conditions

For $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\gamma \in (0, \pi)$, and any fixed $a \in \mathbb{R}$, we introduce the notations

$$P(k) = \bigcup_{n=0}^{\infty} \lambda_n^P(k), \quad S(k) = \bigcup_{n=0}^{\infty} \lambda_n^S(k), \quad \Gamma(\gamma) = \bigcup_{n=0}^{\infty} \lambda_n(\gamma), \tag{2.1}$$

where $\lambda_n^P(k)$ and $\lambda_n^S(k)$ are the periodic and semiperiodic eigenvalues on the interval $[a, a+kh]$, respectively, and $\lambda_n(\gamma)$ are the γ eigenvalues on $[a, a+h]$.

From Section 2.5, if the coefficients of equation (2.1) are periodic with period h, $0 < h < \infty$, then we have that

$$P(k) \cup S(k) \subset \Gamma(\gamma). \tag{2.2}$$

In other words, every periodic and semiperiodic eigenvalue from every k interval for $k = 1, 2, 3, \ldots$ is also an eigenvalue of a complex self-adjoint boundary condition on the interval $[a, a+h]$ determined by some $\gamma \in (0, \pi)$.

Given $\lambda_n^P(k), n \in \mathbb{N}_0$, $k \in \mathbb{N}$, for which m and which γ is

$$\lambda_n^P(k) = \lambda_m(\gamma)?$$

And, given $\lambda_n^S(k)$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, for which m and which γ is

$$\lambda_n^S(k) = \lambda_m(\gamma)?$$

We answer each of these two questions in two steps. First, we find γ, and then we construct an algorithm to determine m. This algorithm constructs a one-to-one correspondence between the eigenvalues of the countable set $P(k) \cup S(k)$ (since the union of countable sets is countable) and the subset of the uncountable set $\Gamma(\gamma)$ consisting of the eigenvalues that correspond to the set $P(k) \cup S(k)$.

Next, we summarize and restate the parts of the theorems of Section 2.5, which are further used for simplicity of exposition. First, for the periodic case given by Theorem 2.5.2, we have the following:

Theorem 2.6.1. *For $k = 2s$, $s \geq 1$, and for $k = 2s + 1$, $s \geq 0$, we have*

$$P(k) = \bigcup_{l=0}^{s} \Gamma\left(\frac{2l\pi}{k}\right). \tag{2.3}$$

Furthermore, if $k > 2$, then every eigenvalue in $S(k)$ has multiplicity 2.
For $k = 1$, we have $P(1) = \Gamma(0) = \{\lambda_n^P(1) = \lambda_n^P : n \in \mathbb{N}_0\}$.

Proof. This follows from Theorem 2.5.2. □

The case $k = 2$ is "special" in the sense that there is no γ in the open interval $(0, \pi)$ that generates a periodic eigenvalue in the $k = 2$ interval. For every $k > 2$, there is at least one such γ. It is clear that if λ is a periodic eigenvalue for $k = 1$, then it is also a periodic eigenvalue for $k = 2$. Also, if λ is a semiperiodic eigenvalue for $k = 1$, then λ is a periodic eigenvalue for $k = 2$. The next corollary shows that the converse is true: If λ is a periodic eigenvalue for $k = 2$, then it is either a periodic or semiperiodic eigenvalue for $k = 1$.

Corollary 2.6.1. *Let the hypotheses and notation of Theorem 2.5.2 hold. Then*

$$P(2) = \Gamma(0) \cup \Gamma(\pi) = P(1) \cup S(1).$$

Proof. This follows directly from Theorem 2.5.2. □

The next theorem reviews the semiperiodic case given by Theorem 2.5.3.

Theorem 2.6.2. *For $k = 2s$, $s \geq 1$, and for $k = 2s + 1$, $s \geq 0$, we have*

$$S(k) = \bigcup_{l=0}^{s} \Gamma\left(\frac{(2l+1)\pi}{k}\right). \tag{2.4}$$

Furthermore, if $k > 2$, then every eigenvalue in $P(k)$ has multiplicity 2.
In particular, for $k = 1$, we have $S(1) = \Gamma(\pi) = \{\lambda_n^S(1) = \lambda_n^S : n \in \mathbb{N}_0\}$.

Proof. This follows directly from Theorem 2.5.3. □

2.7 Eigenvalue equalities from different intervals

It is clear that $\lambda_0^P(k)$ stays constant as k changes, but how do the other eigenvalues change? More specifically:

Given an eigenvalue λ in $P(k)$ for some $k > 1$, by Theorem 2.5.2 λ is also an eigenvalue in $\Gamma(\gamma)$. Which one?

Given an eigenvalue λ in $S(k)$ for some $k > 1$, by Theorem 2.5.3 λ is also an eigenvalue in $\Gamma(\gamma)$. Which one?

These questions are answered in this and the next section.

For each $k = 1, 2, 3, \ldots$, Theorem 2.6.1 determines the values of $\gamma \in (0, \pi)$ that determine all the periodic eigenvalues on the k interval $[a, a + kh]$. Similarly, Theorem 2.6.2 determines all semiperiodic eigenvalues on these intervals. The set $\bigcup_{k=1}^{\infty} P(k)$ is a countable union of countable sets and is therefore countable, whereas the set $\Gamma(\gamma) = \{\bigcup_{n=0}^{\infty} \lambda_n(\gamma) : \gamma \in (0, \pi)\}$ is not countable, so there cannot be one-to-one correspondence between these two sets, and similarly for $\bigcup_{k=1}^{\infty} S(k)$ and $\Gamma(\gamma)$.

Recall that the eigenvalues of these sets can be ordered to satisfy

$$-\infty < \lambda_0^P(k) \le \lambda_1^P(k) \le \lambda_2^P(k) \le \lambda_3^P(k) \le \cdots \tag{2.1}$$

$$-\infty < \lambda_0^S(k) \le \lambda_1^S(k) \le \lambda_2^S(k) \le \lambda_3^S(k) \le \cdots \tag{2.2}$$

$$-\infty < \lambda_0(\gamma) < \lambda_1(\gamma) < \lambda_2(\gamma) < \lambda_3(\gamma) < \cdots, \tag{2.3}$$

and this ordering identifies each eigenvalue $\lambda_n^P(k), \lambda_n^S(k), \lambda_n(\gamma)$ uniquely. (Although the eigenfunctions of the periodic and semiperiodic eigenvalues are not unique if their multiplicity is 2.)

This is the "natural" ordering that defines λ_n uniquely when the eigenvalues are bounded below. In [27] the assumption that p is positive seems to be omitted. Möller [78] has shown that if p is positive and negative, each on a set of positive Lebesgue measure, then the eigenvalues are unbounded above and below. In this case, λ_n is not well defined.

Using Theorem 2.5.2 and the general Theorem 2.2.1, we will find a different ordering and construct a one-to-one correspondence between these two orderings. We will illustrate this new correspondence with some examples for both periodic and semiperiodic cases.

We start with a remark.

Remark 2.7.1. Although we have defined $\Gamma(\gamma)$ only for γ in the open interval $(0, \pi)$, Theorem 2.5.2 shows that the "boundary sets" $\Gamma(0)$ and $\Gamma(\pi)$ represent the periodic eigenvalues and semiperiodic eigenvalues on the interval $[a, a+h]$, respectively. However, it is important to keep in mind that the eigenvalues when $\gamma \in (0, \pi)$ are all simple but the eigenvalues in $\Gamma(0)$ and $\Gamma(\pi)$ may be simple or double, except for λ_0^P, which is always simple. It follows from Theorem 2.5.2 that $\Gamma(0) = \Gamma(2l\pi)$ and that $\Gamma(\pi) = \Gamma((2l + 1)\pi)$ for any $l \in \mathbb{Z}$.

In the next two theorems, we review the known inequalities between the eigenvalues of

$$P(k) = \bigcup_{n=0}^{\infty} \lambda_n^P(k), \quad S(k) = \bigcup_{n=0}^{\infty} \lambda_n^S(k), \quad \text{and} \quad \Gamma(\gamma) = \cup_{n=0}^{\infty} \lambda_n(\gamma).$$

Theorem 2.7.1. *Let* (2.1)–(2.5) *hold. Fix* $k > 2$. *Let* $P(k), S(k), \Gamma(\gamma)$ *be defined as before, and let*

$$P(1) = \{\lambda_n^P(1) : n \in \mathbb{N}_0\} = \Gamma(0) = \{\lambda_n(0) : n \in \mathbb{N}_0\},$$

$$S(1) = \{\lambda_n^S(1) : n \in \mathbb{N}_0\} = \Gamma(\pi) = \{\lambda_n(\pi) : n \in \mathbb{N}_0\}.$$

Then:

(1) *If* $k = 2s$, $s > 1$, *we have*

$$\begin{aligned}\lambda_0^P(0) &= \lambda_0(0) < \lambda_0(2\pi/k) < \lambda_0(4\pi/k) < \cdots < \lambda_0\big(2(s-1)\pi)/k\big) < \lambda_0(\pi)\\ &\le \lambda_1(\pi) < \lambda_1\big(2(s-1)\pi/k\big) < \lambda_1\big(2(s-2)\pi/k\big) < \cdots < \lambda_1(2\pi/k) < \lambda_1(0)\\ &\le \lambda_2(0) < \lambda_2(2\pi/k) < \lambda_2(4\pi/k) < \cdots < \lambda_2\big(2(s-1)\pi/k\big) < \lambda_2(\pi)\\ &\le \lambda_3(\pi) < \lambda_3\big(2(s-1)\pi/k\big) < \lambda_3\big(2(s-2)\pi/k\big) \cdots < \lambda_3(2\pi/k) < \lambda_3(0)\\ &\le \lambda_4(0) < \lambda_4(2\pi/k) < \cdots.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_0^P(k) &= \lambda_0^P,\\ \lambda_s^P(k) &= \lambda_0(2s\pi/k) = \lambda_0^S,\\ \lambda_{s+1}^P(k) &= \lambda_1(2s\pi/k) = \lambda_1^S,\\ \lambda_{s+2}^P(k) &= \lambda_1\big((2s-2)\pi/k\big),\\ &\cdots\end{aligned}$$

(2) *If* $k = 2s + 1$, $s > 1$, *we have*

$$\begin{aligned}\lambda_0^P &= \lambda_0(0) < \lambda_0(2\pi/k) < \lambda_0(4\pi/k) < \lambda_0(6\pi/k) \cdots < \lambda_0(2s\pi/k)\\ &< \lambda_1(2s\pi/k) < \lambda_1\big(2(s-1)\pi/k\big) < \cdots < \lambda_1(2\pi/k) < \lambda_1(0)\\ &\le \lambda_2(0) < \lambda_2((2\pi/k) < \lambda_2(4\pi/k) < \cdots < \lambda_2(2s\pi/k)\\ &< \lambda_3(2s\pi/k) < \lambda_3\big(2(s-1)\pi/k\big) < \cdots < \lambda_3(2\pi/k) < \lambda_3(0)\\ &\le \lambda_4(0) < \lambda_4(2\pi/k) \ldots.\end{aligned} \tag{2.4}$$

Therefore

$$\begin{aligned}\lambda_0^P(k) &= \lambda_0^P,\\ \lambda_s^P(k) &= \lambda_0(2s\pi/k),\\ \lambda_{s+1}^P(k) &= \lambda_1(2s\pi/k),\\ \lambda_{s+2}^P(k) &= \lambda_1\big((2s-2)\pi/k\big),\\ &\cdots\end{aligned} \tag{2.5}$$

Proof. These inequalities follow from Theorems 2.2.1, 2.5.1, 2.5.2, and 2.5.3. The fact that $\lambda_0(\gamma)$ is decreasing, $\lambda_1(\gamma)$ is increasing, $\lambda_2(\gamma)$ decreasing, $\lambda_3(\gamma)$ increasing, … for $\gamma \in (0, \pi)$ is reflected in the pattern for the alternating rows in (2.4) and (2.5). This pattern is clearly seen in the examples below. See the papers by Yuan, Sun, and Zettl [107, 108] for more detail. □

Theorem 2.7.2. *Let* (2.1)–(2.5) *hold. Then:*

(1) *If* $k = 2s$, $s > 1$, *we have*

$$\begin{aligned}\lambda_0(\pi/k) &< \lambda_0(3\pi/k) < \cdots < \lambda_0((2s-1)\pi/k)\\ &< \lambda_1((2s-1)\pi/k) < \lambda_1((2s-3)\pi/k) < \cdots < \lambda_1(\pi/k)\\ &< \lambda_2(\pi/k) < \cdots < \lambda_2(3\pi/k) < \cdots < \lambda_3((2s-1)\pi/k)\\ &< \lambda_3((2s-1)\pi/k) < \lambda_3((2s-3)\pi/k) < \cdots < \lambda_3(\pi/k)\\ &< \lambda_4(\pi/k) < \cdots < \lambda_4(3\pi/k) < \cdots < \lambda_4((2s-1)\pi/k)\ldots.\end{aligned} \tag{2.6}$$

Therefore

$$\begin{aligned}\lambda_0^S(k) &= \lambda_0(\pi/k),\\ \lambda_{s-1}^S(k) &= \lambda_0((2s-1)\pi/k),\\ \lambda_s(k) &= \lambda_1((2s-1)\pi/k),\\ \lambda_{s+1}^S(k) &= \lambda_1((2s-3)\pi/k),\\ &\ldots\end{aligned} \tag{2.7}$$

(2) *If* $k = 2s+1$, $s > 1$, *then*

$$\begin{aligned}\lambda_0(\pi/k) &< \lambda_0(3\pi/k) < \cdots < \lambda_0((2s+1)\pi/k) = \lambda_0^S\\ &\le \lambda_1^S = \lambda_1(\pi) < \lambda_1((2s-1)\pi/k) < \cdots < \lambda_1(\pi/k)\\ &< \lambda_2(\pi/k) < \lambda_2(3\pi/k) < \cdots < \lambda_2((2s+1)\pi/k) = \lambda_2^S\\ &\le \lambda_3^S = \lambda_3(\pi) < \lambda_3((2s-1)\pi/k) < \cdots < \lambda_3(\pi/k)\\ &< \cdots.\end{aligned} \tag{2.8}$$

Therefore

$$\begin{aligned}\lambda_0^S(k) &= \lambda_0(\pi/k),\\ \lambda_s^S(k) &= \lambda_0^S,\\ \lambda_{s+1}^S(k) &= \lambda_1^S,\\ \lambda_{s+2}^S(k) &= \lambda_1((2s-1)\pi/k),\\ &\ldots\end{aligned} \tag{2.9}$$

Proof. These inequalities follow from Theorems 2.2.1, 2.5.1, 2.5.2, and 2.5.3. The fact $\lambda_0(\gamma)$ is decreasing, $\lambda_1(\gamma)$ is increasing, $\lambda_2(\gamma)$ decreasing, $\lambda_3(\gamma)$ increasing, … for $\gamma \in (0,\pi)$ is reflected in the pattern for the alternating rows in (2.4) and (2.5). This pattern is clearly seen in the examples below. See the papers by Yuan, Sun, and Zettl [107, 108] for more detail. □

Now we list some examples to illustrate Theorem 2.3.1 and clarify its proof. We start with the periodic case for $k = 2$. This case is special and does not illustrate the general pattern because it does not involve γ.

As k gets large, the eigenvalues $\lambda_n^P(k)$ and $\lambda_n^S(k)$ approach $\lambda_0^P(1) = \lambda_0^P$ from the right. More precisely, we have the following:

Theorem 2.7.3. *Let* (2.1)–(2.5) *hold. For any* $n \in \mathbb{N}$, *we have*

$$\lim_{k\to\infty} \lambda_n^P(k) = \lambda_0^P \quad \textit{and} \quad \lim_{k\to\infty} \lambda_n^S(k) = \lambda_0^P. \tag{2.10}$$

Proof. Let $n \in \mathbb{N}$. For $k = 2(n+1) = 2s$, from Theorem 2.5.2 we get that $\lambda_n^P(k) = \lambda_0(2s\pi/k)$, and therefore

$$\lim_{k\to\infty} \lambda_n^P(k) = \lambda_0^P.$$

For $k = 2n + 1 = 2s + 1$, we get from Theorem (2.5.2) that $\lambda_n^P(k) = \lambda_0(2s\pi/k)$, and (2.10) follows. Since $\lambda_n^P(k) > \lambda_0^P$ for k even or odd, the limit in (2.10) is from the right.

The proof of $\lim_{k\to\infty} \lambda_n^S(k) = \lambda_0^P$ is similar, and the limit is also from the right. □

It is well known that equation (2.1) is oscillatory on $\mathbb{R}$ when $\lambda > \lambda_0^P$ and nonoscillatory when $\lambda \le \lambda_0^P$. In the next theorem, we give an elementary proof of this under our general hypotheses (2.2).

Theorem 2.7.4. *Let* (2.1)–(2.5) *hold. Then equation* (2.1) *is oscillatory on* $\mathbb{R}$ *when* $\lambda > \lambda_0^P$ *and nonoscillatory when* $\lambda \le \lambda_0^P$.

Proof. Suppose that $\lambda = \lambda_0^P$ and u is an eigenfunction of λ. Then by Theorem 8 in [107] u has no zero in the closed interval $[a, a+h]$. Hence the extension of u to $\mathbb{R}$ has no zero on $\mathbb{R}$. By the Sturm comparison theorem equation (2.1) is nonoscillatory for $\lambda \le \lambda_0^P$. Let $\lambda > \lambda_0^P$. By Theorems 2.7.1 and 2.7.3 $\lambda_0^P < \lambda_n^P(k) < \lambda$ for all sufficiently large n and k. Since $\lambda_n^P(k)$ has zeros in the interval $[a, kh]$, its extension to $\mathbb{R}$ has infinitely many zeros, that is, it is oscillatory. □

2.8 Construction of the one-to-one correspondence

The next two theorems give an explicit one-to-one correspondence between the periodic and semiperiodic eigenvalues on the k interval, $k > 1$, and the corresponding γ eigenvalues from the interval $k = 1$.

Theorem 2.8.1. *Let* (2.1)–(2.5) *hold, and let the eigenvalues* $\lambda_n^P(k)$ *be ordered according to* (2.1). *Then:*

– *If* $k = 2s$, $s \in \mathbb{N}$, *then*
  (1) *for even* m, *we have*

$$\lambda_{ms+n}^P(k) = \lambda(2(n-m)\pi/k), \quad n = m, m+1, \dots, m+s.$$

(2) *for odd m, we have*

$$\lambda^P_{ms+n}(k) = \lambda\big(2(m+s-n)\pi/k\big), \quad n = m, m+1, \dots, m+s.$$

– *If* $k = 2s+1$, $s > 0$, *then*
 (1) *for even m, we have*

$$\lambda^P_{ms+n}(k) = \lambda\big(2(n-m)\pi/k\big), \quad n = m, m+1, \dots, m+s.$$

 (2) *for odd m, we have*

$$\lambda^P_{ms+n}(k) = \lambda\big(2(m+s-n)\pi/k\big), \quad n = m, m+1, \dots, m+s.$$

Proof. Suppose $k = 2s$, $s \in \mathbb{N}$. From (2.6) and the natural ordering it follows that

$$\begin{aligned}
&\lambda^P_0 = \lambda^P_0, \quad \lambda^P_1(k) = \lambda_0(2\pi/k), \dots, \lambda^P_{s-1}(k) = \lambda_0\big(2(s-1)\pi\big)/k), \quad \lambda^P_s(k) = \lambda^S_0,\\
&\lambda^P_{s+1}(k) = \lambda^S_1, \quad \lambda^P_{s+2}(k) = \lambda_1(2(s-1)\pi/k), \dots, \lambda^P_{2s}(k) = \lambda_1(2\pi/k), \quad \lambda^P_{2s+1}(k) = \lambda^P_1,\\
&\lambda^P_{2s+2}(k) = \lambda^P_2, \quad \lambda^P_{2s+3}(k) = \lambda_2(2\pi/k), \dots, \lambda^P_{3s+1}(k) = \lambda_2\big(2(s-1)\pi/k\big), \quad \lambda^P_{3s+2}(k) = \lambda^S_2,\\
&\lambda^P_{3s+3}(k) = \lambda^S_3, \quad \lambda^P_{s+4}(k) = \lambda_3\big(2(s-1)\pi/k\big), \dots, \lambda^P_{4s+2}(k) = \lambda_3(2\pi/k), \quad \lambda^P_{4s+3}(k) = \lambda^P_3,
\end{aligned}$$

and so on.

Note that for $\lambda^P_{ms+n}(k)$, the values of γ increase (0, $2\pi/k, \dots, 2(s-1)\pi/k$, $2s\pi/k = \pi$) as the index n goes from m to $m+s$ when m is even and decrease ($2s\pi/k = \pi$, $2(s-1)\pi/k, \dots, 2\pi/k$, 0) when m is odd.

Suppose $k = 2s+1$, $s > 0$. From (2.7) and the natural ordering it follows that

$$\begin{aligned}
&\lambda^P_0(k) = \lambda^P_0, \quad \lambda^P_1(k) = \lambda_0(2\pi/k) \dots \lambda^P_{s-1}(k) = \lambda_0\big(2(s-1)\pi/k\big), \quad \lambda^P_s(k) = \lambda_0(2s\pi/k),\\
&\lambda^P_{s+1}(k) = \lambda_1(2s\pi/k), \quad \lambda^P_{s+2}(k) = \lambda_1\big(2(s-1)\pi/k\big) \dots \lambda^P_{2s}(k) = \lambda_1(2\pi/k), \quad \lambda^P_{2s+1}(k) = \lambda^P_1,\\
&\lambda^P_{2s+2}(k) = \lambda^P_2, \quad \lambda^P_{2s+3}(k) = \lambda_2(2\pi/k) \dots \lambda^P_{3s+1}(k) = \lambda_2\big(2(s-1)\pi/k\big), \quad \lambda^P_{3s+2}(k) = \lambda_2(2s\pi/k),\\
&\lambda^P_{3s+3}(k) = \lambda_3(2s\pi/k), \quad \lambda^P_{3s+4}(k) = \lambda_3\big(2(s-1)\pi/k\big) \dots \lambda^P_{4s+2}(k) = \lambda_3(2\pi/k), \quad \lambda^P_{4s+3}(k) = \lambda^P_3,
\end{aligned}$$

and so on.

Note that for $\lambda^P_{ms+n}(k)$, the values of γ increase (0, $2\pi/k, \dots, 2(s-1)\pi/k$, $2s\pi/k = \pi$) as the index n goes from m to $m+s$ when m is even and decrease ($2s\pi/k = \pi$, $2(s-1)\pi/k, \dots, 2\pi/k$, 0) when m is odd. □

Theorem 2.8.2. *Let* (2.1)–(2.5) *hold, and let the eigenvalues* $\lambda^S_n(k)$ *be ordered according to* (2.2). *Then:*
– *If* $k = 2s$, $s > 1$, *then*
 (1) *for even m, we have*

$$\lambda^S_{ms+n}(k) = \lambda\big((2n+1)\pi/k\big), \quad n = 0, 1, \dots, s-1.$$

(2) *for odd m, we have*

$$\lambda^S_{ms+n}(k) = \lambda\big(2(s-1-n)\pi/k\big), \quad n = 0,1,\dots,s-1.$$

– *If $k = 2s+1$, $s > 0$, then*
  (1) *for even m and $n \in [m, m+s]$, we have*

$$\lambda^S_{ms+n}(k) = \lambda\big(2(n-m)\pi + 1)/k\big), \quad n = m, m+1,\dots,m+s.$$

  (2) *for odd m and $n \in [m, m+s]$, we have*

$$\lambda^S_{ms+n}(k) = \lambda\big((2(m+s-n)+1)\pi/k\big), \quad n = m, m+1,\dots,m+s.$$

Proof. For clarity, we use the notation discussed before. Suppose $k = 2s$, $s \in \mathbb{N}$. From the ordering of $\lambda^S_n(k)$ and the natural ordering of $\lambda_n(\gamma)$ it follows that

$$\begin{aligned}
&\lambda^S_0(k) = \lambda_0(\pi/k), \dots \lambda^S_{s-2}(k) = \lambda_0\big((2s-3)\pi/k\big), \quad \lambda^S_{s-1}(k) = \lambda_0\big((2s-1)\pi/k\big),\\
&\lambda^S_s(k) = \lambda_1\big((2s-1)\pi/k\big), \dots \lambda^S_{2s-2}(k) = \lambda_1(3\pi/k), \quad \lambda^S_{2s-1}(k) = \lambda_1(\pi/k),\\
&\lambda^S_{2s}(k) = \lambda_2(\pi/k) \dots \lambda^S_{3s-2}(k) = \lambda_2\big((2s-3)\pi/k\big), \quad \lambda^S_{3s-1}(k) = \lambda_2\big((2s-1)\pi/k\big),\\
&\lambda^S_{3s}(k) = \lambda_3\big((2s-1)\pi/k\big), \dots \lambda^S_{4s-2}(k) = \lambda_3(3\pi/k), \quad \lambda^S_{4s-1}(k) = \lambda_3(\pi/k),
\end{aligned}$$

and so on.

Note that for $\lambda^S_{ms+n}(k)$, the values of γ increase $(\pi/k,\dots,(2s-1)\pi/k)$ as the index n goes from 0 to $s-1$ when m is even and decrease $((2s-1)\pi/k,\dots,\pi/k)$ when m is odd.

Suppose $k = 2s+1$, $s > 0$. From the ordering of $\lambda^S_n(k)$ and the natural ordering of $\lambda_n(\gamma)$ it follows that

$$\begin{aligned}
&\lambda^S_0(k) = \lambda_0(\pi/k), \dots \lambda^S_{s-1}(k) = \lambda_0\big((2s-1)\pi/k\big), \quad \lambda^S_s(k) = \lambda_0\big((2s+1)\pi/k\big) = \lambda^S_0,\\
&\lambda^S_{s+1}(k) = \lambda^S_1, \quad \lambda^S_{s+2}(k) = \lambda_1\big((2s-1)\pi/k\big) \dots \lambda^S_{2s}(k) = \lambda_1(3\pi/k), \quad \lambda^S_{2s+1}(k) = \lambda_1(\pi/k),\\
&\lambda^S_{2s+2}(k) = \lambda_2(\pi/k), \dots \lambda^S_{3s+1}(k) = \lambda_2\big((2s-1)\pi/k\big), \quad \lambda^S_{3s+2}(k) = \lambda_2\big((2s+1)\pi/k\big) = \lambda^S_2,\\
&\lambda^S_{3s+3}(k) = \lambda^S_3, \lambda^S_{3s+4}(k) = \lambda_3\big((2s-1)\pi/k\big) \dots \lambda^S_{4s+2}(k) = \lambda_3(3\pi/k), \quad \lambda^S_{4s+3}(k) = \lambda_3(\pi/k),
\end{aligned}$$

and so on.

Note that for $\lambda^S_{ms+n}(k)$, the values of γ increase $(\pi/k,\dots,(2s-1)\pi/k)$ as the index n goes from m to $m+s$ when m is even and decrease $((2s+1)\pi/k = \pi,\dots,\pi/k)$ when m is odd. □

Next, we show that the set of all periodic eigenvalues from all the intervals $[a, a+k]$, $k = 1,2,3,\dots$, is dense in the uncountable set of all eigenvalues from interval $k = 1$ and $\gamma \in (0,\pi]$.

Definition 2.8.1. Define the sets E and Γ as follows:

$$\begin{aligned}
E &= \{\lambda^P_n(k) : n \in \mathbb{N}_0,\ k \in \mathbb{N}\},\\
\Gamma &= \{\lambda_n(\gamma) : n \in \mathbb{N}_0,\ \gamma \in (0,\pi]\}.
\end{aligned}$$

Remark 2.8.1. Note that the set E is countable since it is the countable union of countable sets; and the set Γ is not countable since the interval $(0,\pi]$ is not countable.

Theorem 2.8.3. *The closure of the set E is Γ, that is,*

$$\overline{E} = \Gamma.$$

Proof. If $e^{i\gamma}$ is a kth root of 1, $k = 1,2,3,\dots$, then $\lambda_n(\gamma, I)$ is a periodic eigenvalue for the interval $[a, a+kh]$. It is well known that the set of all kth roots of 1 for all $k \in \mathbb{N}$ that lie in the interval $(0,\pi)$ is dense in this interval. The conclusion follows from the characterization of the eigenvalues

$$D(\lambda) = u(a+h,\lambda) + v^{[1]}(a+h,\lambda) = 2\cos\gamma,$$

the continuity of $D(\lambda)$ as a function of $\lambda \in \mathbb{R}$, and the continuous dependence of each $\lambda_n(\gamma, I)$ as a function of $\gamma \in (0,\pi]$ by Theorem 2.5.1. (Since $\lambda_n(-\gamma, I) = \lambda_n(\gamma, I)$ we do not need to consider the interval $(-\pi, 0)$.) □

2.9 Examples of the one-to-one correspondence

In this section, we give some examples. First, for the cases $k = 2,3,4$ and then for some higher-order cases. There are some key differences between even and odd k. For the periodic even-order case, any periodic eigenvalue for $k = 1$ is also a periodic eigenvalue for $k > 1$. Also, a semiperiodic eigenvalue for $k = 1$ is a periodic eigenvalue for even k. A more subtle difference is the effect of the inequalities of Theorem 2.7.1 on the one-to-one correspondence. This has to do with the alternating increasing and decreasing values of γ for the even- and odd-order cases. These will be illustrated in our examples.

Example 2.9.1. $k = 2$. As mentioned before, the case $k = 2$ is special. By Corollary 2.6.1 $P(2) = P(1) \cup S(1) = \Gamma(0) \cup \Gamma(\pi)$. From this and from (9) of Theorem 2.2.1 we get

$$\lambda_0^P < \lambda_0^S \le \lambda_1^S < \lambda_1^P \le \lambda_2^P < \lambda_2^S \le \lambda_3^S < \lambda_3^P \le \lambda_4^P < \cdots.$$

Hence the one-to-one correspondence is

$$\lambda_0^P(2) = \lambda_0^P(1) = \lambda_0^P, \quad \lambda_1^P(2) = \lambda_0^S, \quad \lambda_2^P(2) = \lambda_1^S, \quad \lambda_3^P(2) = \lambda_1^P, \quad \lambda_4^P(2) = \lambda_2^P, \quad \dots.$$

Example 2.9.2. $k = 3$. This case is similar to Example 2.9.1. In this case, $\gamma = 2\pi/3$ generates the additional eigenvalues rather than the semiperiodic ones, which can be identified with $\gamma = \pi$. Thus we have

$$\lambda_0^P < \lambda_0(2\pi/3) < \lambda_1^P \le \lambda_2^P < \lambda_2(2\pi/3) < \lambda_3^P \le \lambda_4^P < \lambda_4(2\pi/3) < \cdots.$$

Hence the one-to-one correspondence is

$$\lambda_0^P(2) = \lambda_0^P(1) = \lambda_0^P, \quad \lambda_1^P(2) = \lambda_0(2\pi/3), \quad \lambda_2^P(2) = \lambda_2^P, \quad \lambda_3^P(2) = \lambda_3(2\pi/3),$$
$$\lambda_4^P(2) = \lambda_4^P, \quad \ldots.$$

Example 2.9.3. $k = 2s$, $s = 4$. This and the next example illustrate the fact that the values of γ increase $(\pi/k, \ldots, (2s-1)\pi/k)$ as the index n goes from m to $m+s$ when m is even and decrease $((2s+1)\pi)/k = \pi, \ldots, \pi/k)$ when m is odd. By Theorem 2.6.1 we have

$$\begin{aligned}\lambda_0^P(0) = \lambda_0(0) &< \lambda_0(2\pi/8) < \lambda_0(4\pi/8) < \lambda_0(6\pi/8) < \lambda_0(\pi)\\ &\le \lambda_1(\pi) < \lambda_1(6\pi/8) < \lambda_1(4\pi/8) < \lambda_1(2\pi/8) < \lambda_1(0)\\ &\le \lambda_2(0) < \lambda_2(2\pi/8) < \lambda_2(4\pi/8) < \lambda_2(6\pi/8) < \lambda_2(\pi)\\ &\le \lambda_3(\pi) < \lambda_3(6\pi/8) < \lambda_3(4\pi/8) < \lambda_3(2\pi/8) < \lambda_3(0)\\ &\le \lambda_4(0) < \lambda_4(2\pi/8) < \cdots.\end{aligned}$$

Thus we have:

(1) $m = 0$:

$$\lambda_0^P(8) = \lambda_0^P, \quad \lambda_1^P(8) = \lambda_0(2\pi/8), \quad \lambda_2^P(8) = \lambda_0(4\pi/8), \quad \lambda_3^P(8) = \lambda_0(6\pi/8),$$
$$\lambda_4^P(8) = \lambda_0(8\pi/8) = \lambda_0^S;$$

(2) $m = 1$:

$$\lambda_5^P(8) = \lambda_1^S, \quad \lambda_6^P(8) = \lambda_1(6\pi/8), \quad \lambda_7^P(8) = \lambda_1(4\pi/k), \quad \lambda_8^P(8) = \lambda_1(2\pi/8),$$
$$\lambda_9^P(8) = \lambda_1(0) = \lambda_1^P;$$

(3) $m = 2$:

$$\lambda_{10}^P(8) = \lambda_2^P, \quad \lambda_{11}^P(8) = \lambda_2(2\pi/8), \quad \lambda_{12}^P(8) = \lambda_2(4\pi/8), \quad \lambda_{13}^P(8) = \lambda_2(6\pi/8),$$
$$\lambda_{14}^P(8) = \lambda_2(\pi) = \lambda_2^S;$$

(4) $m = 3$:

$$\lambda_{15}^P(8) = \lambda_3^S, \quad \lambda_{16}^P(8) = \lambda_3(6\pi/8), \quad \lambda_{17}^P(8) = \lambda_3(4\pi/8), \quad \lambda_{18}^P(8) = \lambda_3(2\pi/8),$$
$$\lambda_{19}^P(8) = \lambda_3^P.$$

Example 2.9.4. $k = 2s + 1$, $s = 4$. By Theorem 2.6.1 we have

$$\begin{aligned}\lambda_0^P &< \lambda_0(2\pi/9) < \lambda_0(4\pi/9) < \lambda_0(6\pi/9) < \lambda_0(8\pi/9)\\ &< \lambda_1^S = \lambda_1(\pi) < \lambda_1\big((2s-1)\pi/9\big) < \cdots < \lambda_1(\pi/9)\\ &< \lambda_2(\pi/9) < \lambda_2(3\pi/9) < \cdots < \lambda_2\big((2s+1)\pi/9\big) = \lambda_2^S\end{aligned}$$

$$\le \lambda_3^S = \lambda_3(\pi) < \lambda_3\big((2s-1)\pi/9\big) < \cdots < \lambda_3(\pi/9)$$
$$< \cdots.$$

Thus we have:

(1) $m = 0$:

$$\lambda_0^P(9) = \lambda_0^P, \quad \lambda_1^P(9) = \lambda_0(\pi/9), \quad \lambda_2^P(9) = \lambda_0(3\pi/9), \quad \lambda_3^P(9) = \lambda_0(5\pi/9),$$
$$\lambda_4^P(9) = \lambda_0(7\pi/9);$$

(2) $m = 1$:

$$\lambda_5^P(9) = \lambda_1(7\pi/9), \quad \lambda_6^P(9) = \lambda_1(5\pi/9), \quad \lambda_7^P(9) = \lambda_1(3\pi/9),$$
$$\lambda_8^P(9) = \lambda_1(\pi/9) < \lambda_9^P(9) = \lambda_1^P;$$

(3) $m = 2$:

$$\lambda_{10}^P(9) = \lambda_2^P, \quad \lambda_{11}^P(9) = \lambda_2(\pi/9), \quad \lambda_{12}^P(9) = \lambda_2(3\pi/9), \quad \lambda_{13}^P(9) = \lambda_2(5\pi/9),$$
$$\lambda_{14}^P(9) = \lambda_2(7\pi/9);$$

(4) $m = 3$:

$$\lambda_{15}^P(9) = \lambda_3(7\pi/9), \quad \lambda_{16}^P(9) = \lambda_3(5\pi/9), \quad \lambda_{17}^P(9) = \lambda_3(3\pi/9),$$
$$\lambda_{18}^P(9) = \lambda_3(\pi/9) < \lambda_{19}^P(9) = \lambda_3^P.$$

The next examples illustrate the semiperiodic case. For $S(2) = \Gamma(\frac{\pi}{2})$, the one-to-one correspondence is just the identity, so we start with $S(3)$.

Example 2.9.5. $k = 3$. For $S(3) = S(1) \cup \Gamma(\frac{\pi}{3}) = \Gamma(\pi) \cup \Gamma(\frac{\pi}{3})$, from Theorem 2.6.2 we get the inequalities:

$$\lambda_0(\pi/3) < \lambda_0(\pi) = \lambda_0^S \le \lambda_1^S = \lambda_1(\pi) < \lambda_1(\pi/3) < \lambda_2(\pi/3) < \lambda_2(\pi) = \lambda_2^S$$
$$\le \lambda_3^S = \lambda_3(\pi) < \lambda_3(\pi/3) < \lambda_4(\pi/3) < \lambda_4(\pi) = \lambda_4^S \le \lambda_5^S < \cdots.$$

Hence $\lambda_0^S(3) = \lambda_0(\pi/3), \lambda_1^S(3) = \lambda_0^S, \lambda_2^S(3) = \lambda_1^S, \lambda_4^S(3) = \lambda_2(\pi/3), \ldots$.

Example 2.9.6. $k = 2s$, $s = 4$. By Theorem 2.6.2 we know that

$$S(8) = \Gamma(\pi/8) \cup \Gamma(3\pi/8) \cup \Gamma(5\pi/8) \cup \Gamma(7\pi/8),$$

and we have the inequalities

$$\lambda_0(\pi/8) < \lambda_0(3\pi/8) < \lambda_0(5\pi/8) < \lambda_0(7\pi/8)$$
$$< \lambda_1(7\pi/8) < \lambda_1(5\pi/8) < \lambda_1(3\pi/8) < \lambda_1(\pi/8)$$

$$
\begin{aligned}
&< \lambda_2(\pi/8) < \lambda_2(3\pi/8) < \lambda_2(5\pi/8) < \lambda_2(7\pi/8)\\
&< \lambda_3(7\pi/8) < \lambda_3(5\pi/8) < \lambda_3(3\pi/8) < \lambda_3(\pi/8)\\
&< \lambda_4(\pi/8) < \lambda_4(3\pi/8) < \lambda_4(5\pi/8) < \lambda_4(7\pi/8) < \cdots.
\end{aligned}
$$

From these inequalities and Theorem 2.7.1 we have:

(1) $m = 0$:

$$\lambda_0^S(k) = \lambda_0(\pi/k), \quad \lambda_1^S(k) = \lambda_0(3\pi/k), \quad \lambda_2^S(k) = \lambda_0(5\pi/k), \quad \lambda_3^S(k) = \lambda_0(7\pi/k);$$

(2) $m = 1$:

$$\lambda_4^S(k) = \lambda_1(7\pi/k), \quad \lambda_5^S(k) = \lambda_1(5\pi/k), \quad \lambda_6^S(k) = \lambda_1(3\pi/k), \quad \lambda_7^S(k) = \lambda_0(\pi/k);$$

(3) $m = 2$:

$$\lambda_8^S(k) = \lambda_2(\pi/k), \quad \lambda_9^S(k) = \lambda_2(3\pi/k), \quad \lambda_{10}^S(k) = \lambda_2(5\pi/k), \quad \lambda_{11}^S(k) = \lambda_2(7\pi/k);$$

(4) $m = 3$:

$$\lambda_{12}^S(k) = \lambda_3(7\pi/k), \quad \lambda_{13}^S(k) = \lambda_3(5\pi/k), \quad \lambda_{14}^S(k) = \lambda_3(3\pi/k), \quad \lambda_{15}^S(k) = \lambda_3(\pi/k).$$

Example 2.9.7. $k = 2s + 1$, $s = 4$. From Theorem 2.6.2 we have

$$S(9) = S(1)\cup\Gamma\left(\frac{\pi}{9}\right)\cup\Gamma\left(\frac{3\pi}{9}\right)\cup\Gamma\left(\frac{5\pi}{9}\right)\cup\Gamma\left(\frac{7\pi}{9}\right) = \Gamma(\pi)\cup\Gamma\left(\frac{\pi}{9}\right)\cup\Gamma\left(\frac{3\pi}{9}\right)\cup\Gamma\left(\frac{5\pi}{9}\right)\cup\Gamma\left(\frac{7\pi}{9}\right).$$

This yields the inequalities

$$
\begin{aligned}
\lambda_0(\pi/9) &< \lambda_0(3\pi/9) < \lambda_0(5\pi/9) < \lambda_0(7\pi/9) < \lambda_0(9\pi/9) = \lambda_0(\pi) \le \lambda_1(\pi)\\
&< \lambda_1(7\pi/9) < \lambda_1(5\pi/9) < \lambda_1(3\pi/9) < \lambda_1(1\pi/9)\\
&< \lambda_2(1\pi/9) < \lambda_2(3\pi/9) < \lambda_2(5\pi/9) < \lambda_2(7\pi/9) < \lambda_2(\pi) \le \lambda_3(\pi)\\
&< \lambda_3(7\pi/9) < \lambda_3(5\pi/9) < \lambda_3(3\pi/9) < \lambda_3(1\pi/9)\\
&< \lambda_4(1\pi/9) < \lambda_4(3\pi/9) < \lambda_4(5\pi/9) < \lambda_4(7\pi/9) < \lambda_4(\pi) \le \lambda_5(\pi) < \cdots.
\end{aligned}
$$

From these inequalities and Theorem 2.7.2 we have:

(1) $m = 0$:

$$\lambda_0^S(9) = \lambda_0(\pi/9), \quad \lambda_1^S(9) = \lambda_0(3\pi/9), \quad \lambda_2^S(9) = \lambda_0(5\pi/9), \quad \lambda_3^S(9) = \lambda_0(7\pi/9);$$
$$\lambda_4^S(9) = \lambda_0^S;$$

(2) $m = 1$:

$$\lambda_5^S(9) = \lambda_1^S, \quad \lambda_6^S(9) = \lambda_1(7\pi/9), \quad \lambda_7^S(9) = \lambda_1(5\pi/9), \quad \lambda_8^S(9) = \lambda_1(3\pi/9),$$
$$\lambda_9^S(9) = \lambda_1(\pi/9);$$

(3) $m = 2$:

$$\lambda_{10}^S(9) = \lambda_2(\pi/9), \quad \lambda_{11}^S(9) = \lambda_2(3\pi/9), \quad \lambda_{12}^S(9) = \lambda_2(5\pi/9), \quad \lambda_{13}^S(9) = \lambda_2(7\pi/9),$$
$$\lambda_{14}^S(9) = \lambda_2^S;$$

(4) $m = 3$:

$$\lambda_{15}^S(9) = \lambda_3^S, \quad \lambda_{16}^S(9) = \lambda_3(7\pi/9), \quad \lambda_{17}^S(9) = \lambda_3(5\pi/9), \quad \lambda_{18}^S(9) = \lambda_3(3\pi/9),$$
$$\lambda_{19}^S(9) = \lambda_3(\pi/9).$$

2.10 Spectrum of the minimal operator

In this section, we comment on the spectrum of the half-line minimal operator $S_{\min}(0,\infty)$ with 0 a regular endpoint and the boundary condition $y(0) = 0$ and on the whole-line operator $S_{\min}(-\infty,\infty)$, which is self-adjoint (without any boundary condition) and has no proper self-adjoint extension. From part (4) of Theorem 2.3.1 it follows that both operators have no eigenvalues.

Hence their spectrum consists entirely [113] of the essential spectrum, which is the same for both operators:

$$\sigma_e(S_{\min}(0,\infty)) = \sigma_e(S_{\min}(-\infty,\infty)) = \bigcup_{n=0}^{\infty} J_n,$$

where

$$J_0 = [\lambda_0^P, \lambda_0^S], \quad J_1 = [\lambda_1^S, \lambda_1^P], \quad J_2 = [\lambda_2^P, \lambda_2^S], \quad J_3 = [\lambda_3^S, \lambda_3^P], \quad J_4 = [\lambda_4^P, \lambda_4^S], \dots$$

with λ_n^P and λ_n^S denoting the periodic and semiperiodic eigenvalues on the interval $J = [0,h]$, respectively.

In particular, the starting point σ_0 of the essential spectrum is given by $\sigma_0 = \inf \sigma_e(S_{\min}) = \lambda_0^P$. The gaps of the spectrum consist of the open intervals $(\lambda_0^S, \lambda_1^S)$, $(\lambda_1^P, \lambda_2^P)$, $(\lambda_2^S, \lambda_3^S)$, $(\lambda_3^P, \lambda_4^P)$, ... If λ_0^S is a double eigenvalue, then the “first gap is missing”; if λ_1^P is a double eigenvalue, then the “second gap is missing”; and so on. Recall that λ_0^P is always simple. (The open interval $(-\infty, \lambda_0^P)$ is also considered a gap by some authors.) If all gaps are missing, then $\sigma_e(S) = [\lambda_0^P, \infty)$. There may be no gaps, a finite number of gaps, or an infinite number of gaps. The compact intervals J_n are called the spectral bands.

There is a large literature on L^1 perturbations of the coefficients and the eigenvalues they generate. These may be below or above σ_0. Each gap may contain a finite or infinite number of eigenvalues of S. If there are infinitely many eigenvalues in a gap, then they can converge only at an endpoint of the gap.

Brown, McCormack, and Zettl [20, 19] introduce a new method for proving the existence of eigenvalues below and above the essential spectrum of L^1 perturbations of periodic Sturm–Liouville problems and illustrate this method for the following class of problems:

$$-y'' + qy = \lambda y, \quad q(t) = \sin t, \quad t \in J = (0, \infty), \quad y(0) = 0,$$

with $L^1(J, \mathbb{R})$ perturbations given by

$$q(t) = c \sin\left(t + \frac{1}{1 + t^2}\right), \quad t \in J, \quad c > 0.$$

They find such eigenvalues for a number of values of the positive constant c. These problems were motivated by applications.

This method has three primary ingredients: functional analysis, interval analysis, and interval arithmetic. The method not only establishes the existence of these eigenvalues but also computes provably correct bounds for their values.

In Chapter 6, we study properties of eigenvalues below σ_0 for general Sturm–Liouville problems, that is, without assuming that the coefficients are periodic. The eigenfunctions of eigenvalues above σ_0 are oscillatory. We do not study boundary conditions that generate eigenvalues above λ_0^P; little seems to be known about such boundary conditions other than their existence.

Remark 2.10.1. The eigenvalues $\lambda_n^P(k)$ and $\lambda_n^S(k)$, which determine the spectral bands and gaps, can be computed with the Bailey–Everitt–Zettl Fortran code SLEIGN2 [10], which can be downloaded for free and comes with a user-friendly interface. Using the one-to-one correspondence given by Theorems 2.5.2 and 2.5.3, any periodic and semiperiodic eigenvalues $\lambda_n^P(k)$ and $\lambda_n^S(k)$ for any $k > 2$ can be computed by computing the corresponding eigenvalue $\lambda_m(\gamma)$.

2.11 Comments

As mentioned in Introduction of the book, this chapter was motivated by numerous discussions with Shang Yuan Ren, the author of the book *Electronic States in Crystals of Finite Size*, Quantum Confinement of Bloch Waves, Springer Tracts in Modern Physics, 2005, second edition, volume 270, 2017.

I am indebted to Shang Yuan Ren for sharing his insight in the application of Sturm–Liouville theory with periodic coefficients to the study of the theory of the electronic states of crystals.

Also, this chapter was influenced by some of the methods used by M. S. P. Eastham in his book *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh and London, 1973, but with the following major differences:

(1) We use quasi-derivatives (py') instead of the classical derivative y'; in particular, for the periodic boundary conditions, we have

$$y(a) = y(b), \quad (py')(a) = (py')(b)$$

instead of

$$y(a) = y(b), \quad y'(a) = y'(b).$$

(2) We do not assume that p is differentiable nor that q and w are piecewise continuous and that w is bounded away from 0.
(3) We do assume that p is positive. This seems to be an oversight by Eastham. If p has positive and negative values, each on a set of positive Lebesgue measure (such as a subinterval), then the eigenvalues are unbounded above and below. So there is no unique ordering of the eigenvalues, and λ_n is not well defined.
(4) The quasi-derivative (py') is continuous on the interval $[a, b]$, whereas the classical derivative $y'(t)$ may not exist for all t in $[a, b]$.
(5) We use the interval $(0, \pi)$, instead of $(0, 1)$, to parameterize the complex self-adjoint boundary conditions. This provides a simple visualization of the "movement" of the eigenvalues $\lambda_n(\gamma)$ on the unit circle of the complex plane relative to the points 0 and π, which correspond to the periodic and semiperiodic eigenvalues.
(6) We use a notation that makes it easier to "keep track" of the dependence of the eigenvalues on the many parameters a, b, k, n, π, θ, and so on of the problem. This dependence sometimes requires a very delicate analysis.

3 Extensions of the classical problem

3.1 Introduction

In this chapter, we study the equation

$$My = -(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty \leq a < b \leq \infty, \tag{3.1}$$

with self-adjoint boundary conditions

$$AY(a) + BY(b) = 0, \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \quad AEA^* = BEB^*, \quad \operatorname{rank}(A : B) = 2, \tag{3.2}$$

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.3}$$

but with coefficients satisfying only the condition

$$r = 1/p, q, w \in L^1(J, \mathbb{R}). \tag{3.4}$$

We will introduce additional conditions on r and w when needed.

In his well-known book, Atkinson [4] weakened the conditions $r = \frac{1}{p} > 0, w > 0$ on J to $r \geq 0, w \geq 0$. At first glance, this may seem to be a minor extension, but, as we will see, it leads to some surprising results including Sturm–Liouville problems, which are equivalent to finite-dimensional matrix problems. This establishes a new and surprising connection between Sturm–Liouville and matrix theories. Each of these fields is well established with a voluminous literature dating back at least to the early 1800s; moreover, with this new connection, some results from each field seem to yield new results in the other field. See [59, 65, 70, 97] for some examples.

Other extensions are obtained by allowing p and w to change sign. The sign changes of p have to be "mild" because of the assumption $r = 1/p \in L^1(J, \mathbb{R})$, but for w, it can be quite general. (There is no sign restriction on q in (1.2).)

Remark 3.1.1. It is remarkable that conditions (1.2) on the coefficients can be extended significantly while retaining most of the major results discussed before, but, as far as the author knows, there are no significant – or even minor – extensions of the self-adjoint boundary conditions (1.3), (1.4) that preserve the above properties of the eigenvalues. More specifically, given matrices (A, B) satisfying (3.3), which is the same as (1.4), we do not know any theorem that guarantees many of the discussed properties of the eigenvalues when one entry of either A or B is changed by a small amount such that the changed matrices do not satisfy (1.4).

Remark 3.1.2. Note that there is no sign restriction on any of the coefficients r, q, w in (3.4). Also, each of r, q, w is allowed to be identically zero in subintervals of J. (Also in the entire interval J, but this is a pathological case, which we do not discuss here.) If

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r is identically zero on a subinterval I of J, then there exist solutions y that are zero on I but whose quasi-derivatives (py') are nonzero constants on I. A maximal such interval of zeros is counted as a single zero in the results on the number of zeros of eigenfunctions.

Definition 3.1.1. By a trivial solution of equation (3.1) on a subinterval I of J we mean a solution y that is identically zero on I and whose quasi-derivative (py') is also identically zero on I.

Definition 3.1.2. A real-valued function f on an interval I is said to change sign on I if it assumes positive values on a set of positive Lebesgue measure and assumes negative values on a set of positive Lebesgue measure.

Remark 3.1.3. By (3.3) the boundary condition (3.4) is well defined for bounded and unbounded intervals J.

3.2 The leading coefficient changes sign

For convenience of the reader, in the next theorem, we review the basic properties of eigenvalues.

Theorem 3.2.1. *Let* (3.1), (3.3), *and* (3.4) *hold and assume that*

$$w > 0 \quad a.e.\ on\ J. \tag{3.5}$$

Then:

(1) *All eigenvalues are real and isolated with no finite accumulation point, and their number infinite but countable.*
(2) *If p changes sign on J, then the eigenvalues are unbounded below and above and can be ordered to satisfy*

$$\cdots \le \lambda_{-2} \le \lambda_{-1} \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \tag{3.6}$$

with $\lambda_n \to +\infty$ and $\lambda_{-n} \to -\infty$ as $n \to \infty$. Each eigenvalue may be geometrically simple or double, but there cannot be two consecutive equalities in (3.5), *since for each λ, equation* (3.1) *has exactly two linearly independent solutions. Note that λ_0 can be any of the eigenvalues in this indexing scheme.*
(3) *If p changes sign and the boundary conditions are separated, then there is strict inequality everywhere in* (3.6).
(4) *If p changes sign on J and $p^+(t) = \max(p(t), 0)$, $p^-(t) = \max(-p(t), 0)$, then the asymptotic form of the eigenvalues is given by*

$$\frac{\lambda_n}{n^2} \to c = \pi^2\left(\int_a^b \sqrt{\frac{w}{p^+}}\right)^{-2} \quad as\ n \to \infty. \quad \frac{\lambda_n}{n^2} \to c = \pi^2\left(\int_a^b \sqrt{\frac{w}{p^-}}\right)^{-2} \quad as\ n \to -\infty. \tag{3.7}$$

Proof. For part (1), we note that the "standard" Hilbert space proof as given, for example, in Coddington and Levinson [24] does not use the positivity assumption on p. Although stronger assumptions of the coefficients are used in [24], the proof given there extends readily to accommodate our assumptions. The positivity of w is used to apply the Hilbert space method in $L^2(J,w)$. This proof also applies to unbounded intervals. When p is positive, part (2) follows from the characterization of the eigenvalues as zeros of an entire characteristic function; see [113]. Although the existence proof for the eigenvalues "works" when $p > 0$, the eigenvalues are not bounded below when $p > 0$ and $w > 0$, and the beautiful oscillation properties of the eigenfunctions in general do not hold when p changes sign.

Möller [78] proved that the spectrum is not bounded below when p is negative on a set of positive Lebesgue measure even when this set contains no interval. Similarly, the spectrum is not bounded above when p is positive on a set of positive Lebesgue measure even when this set contains no interval.

The asymptotic formulas (3.7) are due to Atkinson and Mingarelli [5]. □

3.3 Complex coefficients

In this section, we discuss regular Sturm–Liouville problems (SLPs) with complex coefficients and general, not necessarily self-adjoint, two-point boundary conditions. The main tool for this study is the theory of analytic functions of a complex variable.

A regular two-point SLP consists of the equation

$$-(py')' + qy = \lambda wy \quad \text{on } J = (a,b), \quad -\infty \le a < b \le \infty, \tag{3.8}$$

where

$$r = 1/p, q, w \in L(J,\mathbb{C}), \quad \lambda \in \mathbb{C}, \tag{3.9}$$

together with boundary conditions

$$AY(a) + BY(b) = 0, \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \quad A, B \in M_2(\mathbb{C}). \tag{3.10}$$

Here $M_2(\mathbb{C})$ denotes the 2×2 matrices with complex entries. From Section 1.6 we know that $Y(a)$ and $Y(b)$ exist as finite limits (for finite or infinite a, b), so that (3.10) is well defined. Let

$$P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}.$$

Then, as shown in Chapter 1, the scalar equation (3.5) is equivalent to the first-order system

$$Y' = (P - \lambda W)Y = \begin{bmatrix} 0 & 1/p \\ q - \lambda w & 0 \end{bmatrix} Y, \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix}.$$

Let $\Phi(\cdot, u, P, w, \lambda)$ be the primary fundamental matrix of (3.8) and recall that

$$\Phi' = (P - \lambda W)\Phi \quad \text{on } J, \quad \Phi(u, u, \lambda) = I, \quad a \leq u \leq b, \quad \lambda \in \mathbb{C}. \tag{3.11}$$

Define the characteristic function Δ by

$$\Delta(\lambda) = \Delta(a, b, A, B, P, w, \lambda) = \det[A + B\Phi(b, a, P, w, \lambda)], \quad \lambda \in \mathbb{C}. \tag{3.12}$$

We will further show that its zeros are precisely the eigenvalues of the problem.

Definition 3.3.1. By a trivial solution of equation (3.8) on some interval I we mean a solution y that is identically zero on I *and whose quasi-derivative* $z = (py')$ *is also identically zero on* I. (I may be a subinterval of J, or it may be the whole interval J.) Note that, under general hypotheses (3.9), a solution y may be identically zero on I but its quasi-derivative (py') is not necessarily zero on I.

Definition 3.3.2. Let (3.9) hold. A complex number λ is called an eigenvalue of the boundary value problem consisting of (3.8), (3.9) if equation (3.8) has a *nontrivial* solution on J satisfying boundary conditions (3.9). Such a solution is called an eigenfunction of λ. The theorems of this section give far reaching extensions of Theorem 1.25 for classical problems.

Definition 3.3.3. Any multiple of an eigenfunction is also an eigenfunction. If there are two linearly independent eigenfunctions for the same λ, then we say that λ has geometric multiplicity two. If there is only one linearly independent eigenfunction of λ, then we say that λ is a simple eigenvalue or that λ has geometric multiplicity one. Since for each $\lambda \in \mathbb{C}$ equation (3.8) has exactly two linearly independent solutions, each eigenvalue λ has geometric multiplicity either one or two. The theorems of this section give far reaching extensions of Theorem 1.25 for classical problems.

Remark 3.3.1. Condition (3.9) does not restrict the coefficients to be real valued, and if they are real valued, then it does not restrict the sign of any of the coefficients r, q, w. Also, each of r, q, w is allowed to be identically zero on one or more subintervals of J. If r is identically zero on a subinterval I, then all solutions y are constant on I. Note that if this constant is zero for some solution y, then its quasi-derivative $z = py'$ may be a nonzero constant on I. Similarly, if both q and w are identically zero on a subinterval I, then py' is constant on I for any solution y. This constant may be nonzero even when y is identically zero on I. These statements can be clearly seen and are best interpreted from the system formulation of equation (3.8):

$$y' = rz, \quad z' = (q - \lambda w)y \quad \text{on } J, \quad z = (py'), \quad r = \frac{1}{p}. \tag{3.13}$$

An interval of zeros of a nontrivial solution y is counted as a single zero in the results on the numbers of zeros of solutions, in particular, of eigenfunctions.

Remark 3.3.2. Recall from Section 2.3 that condition (3.9) implies that y and (py') exist as finite limits at each (finite or infinite) endpoint a, b. Hence the boundary condition (3.10) is well defined.

Lemma 3.3.1. *Let* (3.8), (3.9), *and* (3.10) *hold. Then the characteristic function* Δ *is well defined and is an entire function of* λ *for fixed* (a, b, A, B, P, w).

Proof. It follows from Theorem 1.5.2 that for fixed λ, P, w, the primary fundamental matrix $\Phi(b, a, \lambda, P, w)$ exists and is continuous at a and b. The entire dependence on λ follows from Theorem 2.5.3. □

Lemma 3.3.2. *Let* (3.9) *hold. Then:*
(1) *A complex number* λ *is an eigenvalue of the BVP* (3.8), (3.10) *if and only if* $\Delta(\lambda) = 0$.
(2) *The geometric multiplicity of an eigenvalue* λ *is equal to the number of linearly independent vector solutions* $C = Y(a)$ *of the linear algebra system*

$$[A + B\Phi(b, a, \lambda)]C = 0. \tag{3.14}$$

Proof. Suppose $\Delta(\lambda) = 0$. Then (3.14) has a nontrivial vector solution for C. Solve the IVP

$$Y' = (P - \lambda W)Y \quad \text{on } J, \quad Y(a) = C.$$

Then

$$Y(b) = \Phi(b, a, \lambda)Y(a) \quad \text{and} \quad [A + B\Phi(b, a, \lambda)]Y(a) = 0.$$

From this it follows that the top component of Y, say, y, is an eigenfunction of the BVP (3.8), (3.10); this means λ is an eigenvalue of this BVP. Conversely, if λ is an eigenvalue and y is an eigenvector of λ, then $Y = \left[\begin{smallmatrix} y \\ (py') \end{smallmatrix}\right]$ satisfies $Y(b) = \Phi(b, a, \lambda)Y(a)$, and consequently $[A + B\Phi(b, a, \lambda)]Y(a) = 0$. Since $Y(a) = 0$ would imply that y is the trivial solution in contradiction to it being an eigenfunction, we have that $\det[A + B\Phi(b, a, \lambda)] = 0$. If the algebraic equation has two linearly independent solutions for C, say C_1, C_2, then solve the IVP with initial conditions $Y(a) = C_1, Y(a) = C_2$ to obtain solutions Y_1, Y_2. Then Y_1, Y_2 are linearly independent vector solutions of the differential system, and their top components y_1, y_2 are linearly independent solutions of the scalar equation. Conversely, if y_1, y_2 are linearly dependent solutions of the scalar differential equation, we can reverse the previous steps to obtain two linearly independent vector solutions of the algebraic system. □

The next result shows that any given complex number is an eigenvalue of geometric multiplicity two for precisely one boundary condition.

Lemma 3.3.3. *Let* (3.8)–(3.10) *hold with* $B = -I$. *A number* $\lambda \in \mathbb{C}$ *is an eigenvalue of geometric multiplicity two if and only if*

$$A = \Phi(b, a, \lambda).$$

Proof. This follows from Lemma 3.3.2 and its proof. □

Lemma 3.3.4. *For the boundary problems* (3.8)–(3.10)*, exactly one of the following four cases holds:*
(1) *There are no eigenvalues in* $\mathbb{C}$.
(2) *Every complex number is an eigenvalue.*
(3) *There are exactly* n *eigenvalues in* $\mathbb{C}$ *for some* $n \in \mathbb{N}$.
(4) *There are an infinite but countable number of eigenvalues in* $\mathbb{C}$*, and these have no finite accumulation point in* $\mathbb{C}$.

Proof. This follows directly from Lemmas 3.3.1 and 3.3.2 and the well-known fact that the zeros of an entire function are isolated and therefore have no accumulation point in the finite complex plane $\mathbb{C}$. □

Remark 3.3.3. Every self-adjoint regular SLP (see Chapter 4 for a definition of self-adjoint regular problems) with positive weight function w falls into category 4 of Lemma 3.3.4. Section 3.3 contains simple examples illustrating cases 1 and 2. Are there examples for case 3? Such examples are constructed in Chapter 4.

It is convenient to classify the self-adjoint boundary conditions into two mutually exclusive classes, separated and coupled. Note that since the boundary conditions are homogeneous, multiplication by a nonzero constant or left multiplication by a nonsingular matrix leads to equivalent boundary conditions.

Lemma 3.3.5 (Separated boundary conditions). *Let* (3.8)–(3.10) *hold. Fix* P, W, J *and assume that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \end{bmatrix}.$$

Then

$$\Delta(\lambda) = -A_2B_1\phi_{11}(b,a,\lambda) - A_2B_2\phi_{21}(b,a,\lambda) + A_1B_1\phi_{12}(b,a,\lambda) + A_1B_2\phi_{22}(b,a,\lambda)$$

for $\lambda \in \mathbb{C}$.

Proof. This follows from the definition of Δ and a direct computation. □

The characterization of the eigenvalues as zeros of an entire function reduces to a simpler and more informative form when the boundary conditions are self-adjoint and coupled. This reduction is given by the next lemma.

Lemma 3.3.6 (Coupled self-adjoint boundary conditions). *Let* (3.8)–(3.10) *hold, and let* $\Phi = (\phi_{ij})$ *be the primary fundamental matrix of the corresponding system. Fix* P, W, J *and assume that*

$$B = -I, \quad A = e^{i\gamma}K, \quad -\pi < \gamma \le \pi, \quad K \in \mathrm{SL}_2(\mathbb{R}),$$

that is, K is a real 2×2 matrix with determinant 1. Let $K = (k_{ij})$ and define

$$D(\lambda, K) = k_{11}\phi_{22}(b,a,\lambda) - k_{12}\phi_{21}(b,a,\lambda) - k_{21}\phi_{12}(b,a,\lambda) + k_{22}\phi_{11}(b,a,\lambda)$$

for $\lambda \in \mathbb{C}$. Note that $D(\lambda, K)$ does not depend on γ. Then

(1) *The complex number λ is an eigenvalue of BVP (3.8)–(3.10) if and only if*

$$D(\lambda, K) = 2\cos\gamma, \quad -\pi < \gamma \le \pi.$$

(2) *If p, q, w are real valued and λ is an eigenvalue for $A = e^{i\gamma}K, B = -I$, $0 < \gamma < \pi$, with eigenfunction u, then λ is also an eigenvalue for $A = e^{-i\gamma}K, B = -I$, but with eigenfunction $\overline{u}$.*

Proof. Note that $\det \Phi(b,a,\lambda) = 1$. We abbreviate $\phi_{ij}(b,a,\lambda)$ to ϕ_{ij}. From the definition of $\Delta(\lambda)$ and $D(\lambda)$, noting that $\det K = 1$, we get

$$\begin{aligned}\Delta(\lambda) = \det(e^{i\gamma}K - \Phi) &= \begin{vmatrix} e^{i\gamma}k_{11} - \phi_{11} & e^{i\gamma}k_{12} - \phi_{12} \\ e^{i\gamma}k_{21} - \phi_{21} & e^{i\gamma}k_{22} - \phi_{22} \end{vmatrix} \\ &= 1 + e^{2i\gamma} - e^{i\gamma}D(\lambda).\end{aligned}$$

Hence $\Delta(\lambda) = 0$ if and only if $D(\lambda, K) = 2\cos\gamma$. Part (2) follows similarly by reversing the steps and taking conjugates of equation (3.5). □

Remark 3.3.4. Although the matrices A, B determine self-adjoint boundary conditions (these are the canonical form of all coupled self-adjoint BC; see Chapters 4 and 10), no conditions other than (3.9) are assumed on p, q, w in Lemma 3.3.6, part (1). In particular, no symmetry (formal self-adjointness) or definiteness assumption is made on equation (3.8). Thus the characterization of the eigenvalues applies not only to so-called left-definite, right-definite, and indefinite Sturm–Liouville problems, but the coefficients p, q and the weight function w can be complex valued. Furthermore, one or more of $1/p$, q, w can be identically zero on one or more subintervals of J.

3.4 The weight function changes sign

When the weight function w changes sign, there may be nonreal eigenvalues. In this section, we give a brief overview of the so-called left-definite problems. These are problems with a weight function that changes sign but with all real eigenvalues. Here we consider the same equation (3.1) with the same boundary conditions (3.3) but with coefficients satisfying

$$r = 1/p, q, w \in L^1(J, \mathbb{R}), \quad p > 0, \quad |w| > 0 \quad \text{on } J, \tag{3.15}$$

and w changing sign on J. Here we use the index set

$$\{\ldots, -3, -2, -1, -0, 0, 1, 2, 3, \ldots\} \tag{3.16}$$

to order the eigenvalues and the equation

$$My = -(py')' + qy = \lambda|w|y, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty \le a < b \le \infty, \tag{3.17}$$

as a comparison equation.

Theorem 3.4.1. *Let* (3.1)–(3.4) *hold. Assume that w changes sign on J and that $\lambda_0(|w|) > 0$. Then all eigenvalues are real, and there are a countably infinite number of negative eigenvalues and a countably infinite number of positive eigenvalues. The eigenvalues are not bounded below and are not bounded above, and they have no finite accumulation point. Also:*

(1) *$\lambda = 0$ is not an eigenvalue.*

(2) *If the boundary conditions are separated, then the eigenvalues can be ordered to satisfy*

$$\cdots < \lambda_{-3} < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \tag{3.18}$$

and every λ_n has exactly $|n|$ zeros in the open interval J.

(3) *If the boundary conditions are real coupled, then the eigenvalues can be ordered to satisfy*

$$\cdots < \lambda_{-3} \le \lambda_{-2} \le \lambda_{-1} \le \lambda_{-0} < 0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 < \cdots.$$

(4) *If the boundary condition is complex coupled, then the eigenvalues can be ordered to satisfy* (3.18).

(5) *In all three cases the following asymptotic formula holds:*

$$\frac{\lambda_{\pm n}}{n^2} \to c = \pm\pi^2 \left(\int_a^b \sqrt{\frac{w_\pm}{p}} \right)^{-2} \quad \text{as } n \to \infty, \tag{3.19}$$

where w_+ and w_- denote the positive and negative parts of w.

3.5 Nonnegative leading coefficient and weight function

To avoid needless repetitions, since we are dealing with Lebesgue-integrable functions, we just write $f > 0$ on J instead of $f > 0$ a. e. on J; similarly, for $f \ge 0$.

The next three theorems generalize Theorem 1.25 to nonclassical problems.

Theorem 3.5.1 (Everitt, Kwong, and Zettl). *Assume that*

$$r = 1/p, q, w \in L^1(J, \mathbb{R}), \quad J = (a,b), \quad -\infty \le a < b \le \infty,$$
$$r > 0, \quad w \ge 0 \quad \text{on } J, \quad \int_a^b w > 0. \tag{3.20}$$

Then the boundary value problem (3.20) *with separated self-adjoint boundary conditions has only real and simple eigenvalues, there are an infinite but countable number of them, and they are bounded below and can be ordered to satisfy*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \quad \textit{and} \quad \lambda_n \to +\infty \quad \textit{as } n \to \infty. \tag{3.21}$$

If u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples. Let z_n denote the number of zeros of u_n in the open interval (a,b) (with a maximal subinterval of zeros counted as one zero), $n \in \mathbb{N}_0$. Then

$$z_{n+1} = z_n + 1, \quad n \in \mathbb{N}_0. \tag{3.22}$$

For any integer $m \geq 0$, there exists an SLP with separated boundary conditions such that $z_0 = m$. A sufficient but not necessary condition that $z_0 = 0$ is that $w > 0$ a. e. on J.

Proof. See [29]. □

Remark 3.5.1. This theorem seems to be the most general result available to establish the existence of an infinite number of isolated real eigenvalues for the case where $r > 0$, and $w \geq 0$ on J. The integral condition on w eliminates the case where w is identically zero on J, which would mean that λ has no effect on the boundary conditions. In the next theorem the hypothesis $r > 0$ on J is weakened to $r \geq 0$ on J, but at the expense of considerably more restrictions on w. Note the subtle but important difference between the hypotheses $r > 0$ on J and $r \geq 0$ on J.

Theorem 3.5.2 (Atkinson). *Assume that*

$$\begin{gathered} r = 1/p, q, w \in L^1(J, \mathbb{R}), \quad J = (a,b), \quad -\infty \leq a < b \leq \infty, \\ r \geq 0, w \geq 0, \ \textit{on } J, \quad \int_a^t w > 0, \ \int_t^b w > 0, \ \int_a^b r > 0 \quad \textit{for all } t \in J, \\ \textit{and} \quad w = 0 \quad \textit{on } (c,d) \subset J \textit{ implies } q = 0 \textit{ on } (c,d). \end{gathered} \tag{3.23}$$

Then equation (3.1) *with separated self-adjoint boundary conditions has only real and simple eigenvalues, which are bounded below and can be ordered as a* finite or infinite sequence *satisfying*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$

If u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples and has exactly n zeros in the open interval J. (Recall that under these conditions a "zero" may be a whole subinterval.)

Proof. See Theorem 8.4.5 in [4]. □

Remark 3.5.2. The phrase a "*finite or infinite sequence*" is a quote from Atkinson's book [4]. It suggests to us that Atkinson was aware that under the conditions of this theorem there may only be a finite number of eigenvalues. But he gave no example and made no such conjecture. Such examples were constructed by Kong, Wu, and Zettl [65], who showed that for any positive integer n, there exist regular self-adjoint S-L problems with exactly n eigenvalues. These seem to be the first examples of regular self-adjoint Sturm–Liouville problems with only a finite number of eigenvalues. This theorem and the next one are stated by Atkinson [4] only for bounded intervals J. However, Atkinson mentioned that this is only for "convenience"; in other words, the results and their proofs extend readily to unbounded intervals J. Further comments are given in the next remark.

Theorem 3.5.3 (Atkinson). *Let the hypotheses and notation of Theorem* 2.5.1 *hold, and, in addition, suppose that there exists an infinite increasing sequence* $\{c_i : i \in \mathbb{N}\}$ *of points in* J *such that*

$$\int_{c_{2i}}^{c_{2i+1}} w > 0, \qquad \int_{c_{2i+1}}^{c_{2i+2}} r > 0. \tag{3.24}$$

Then equation (3.1) *with separated self-adjoint boundary conditions has only real and simple eigenvalues. There are an infinite but countable number of them. They are bounded below and can be ordered to satisfy*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \text{and} \quad \lambda_n \to +\infty \quad \text{as } n \to \infty. \tag{3.25}$$

If u_n *is an eigenfunction of* λ_n*, then* u_n *is unique up to constant multiples and has exactly* n *zeros in the open interval* J*. (Recall that under these conditions a "zero" may be a whole subinterval.)*

Proof. See Theorem 8.4.6 in [4]. □

Corollary 3.5.1. *Let the hypotheses and notation of Theorem* 3.5.3 *hold. If* $r = 1/p$ *and* w *are both positive on a common subinterval of* J*, then the conclusions hold; in particular, there are an infinite number of eigenvalues.*

Proof. Just choose distinct points c_i in the common subinterval where r and w are positive and (3.24) holds. □

Remark 3.5.3. By Corollary 3.5.1, if w and r are positive on a common subinterval of J, then there are an infinite number of eigenvalues. This raises the question: Can there be only a finite number of eigenvalues if w and r are not positive on any common subinterval of J? Kong, Wu, and Zettl [63] showed that the answer is yes, and for each positive integer n, there exists an SLP with exactly n eigenvalues. In Chapter 4, we will find a matrix representation of SLP with finite spectrum and, conversely, show that certain matrix problems can be represented as Sturm–Liouville problems. These matrix problems have no boundary conditions since the boundary conditions are built into the matrix. In this construction, the interval J is partitioned into subintervals such that w and r are, alternatively, identically zero on adjacent subintervals. In Chapter 4, we will also discuss analogues of the classical results mentioned including eigenvalue inequalities.

Remark 3.5.4. When $p > 0$, $w > 0$ on J, and the boundary condition is separated, each eigenfunction u_n of λ_n has exactly n zeros in the open interval J. But when p changes sign, there is no analogous result in spite of the beautiful characterization of the eigenvalues λ_n by equation (1.61). Binding and Volkmer show by theorems and examples that the zero properties of eigenfunctions under the conditions of the previous section can be quite "strange"; for instance, eigenfunctions can have an infinite number of zeros in the interval J.

We end this section with two examples illustrating the delicate dependence of the spectrum on the coefficients.

Example 3.5.1 (Binding and Volkmer). Let $J = (0, 1)$, $w = 1$, $q \in L^1(J, \mathbb{R})$, and define p by

$$\frac{1}{p(t)} = 2t\cos(1/t) + \sin(1/t), \quad 0 < t < 1.$$

Consider the SLP

$$-(py')' + qy = \lambda y \quad \text{on } J$$

with boundary condition

$$y(0) = 0, \quad B_1 y(1) + B_2(py')(1) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0, 0).$$

Every eigenfunction has an infinite number of zeros accumulated at the left endpoint 0.

Note that this is a regular problem on J; in particular, 0 is a regular endpoint.

Hinton and Lewis [49] introduced property BD of the spectrum: the spectrum is bounded below and discrete. The next example shows that property BD does not depend continuously on the coefficient $1/p$. This is just one of many illustrations of the delicate dependence of the spectrum on the problem.

Example 3.5.2. Consider the boundary value problem with Dirichlet boundary conditions and the equation

$$-(p_\varepsilon y')' = \lambda y \quad \text{on } (0, 1),$$

where $\varepsilon \in [0, 1]$, and

$$p_{\varepsilon\text{Binding}}(t) = \begin{cases} -1 & \text{if } 0 \le t \le \varepsilon, \\ 1 & \text{if } \varepsilon < t \le 1. \end{cases}$$

Then for $\varepsilon = 0$, the spectrum is bounded below, but for each $\varepsilon > 0$, the spectrum is unbounded below. Note that $1/p_\varepsilon \to 1/p_0$ in $L^1((0,1),\mathbb{R})$.

3.6 Comments

Many people have found extensions of the classical theory discussed in Chapter 1. In this chapter, we focused on the extensions of Atkinson, Binding, Everitt, Hinton, Kong, Kwong, Lewis, Möller, Volkmer, Wu, and Zettl.

In the next chapter, we will see that some of these extensions lead to the surprising result that there is a class of Sturm–Liouville boundary value problems that are equivalent to matrix eigenvalue problems. These matrix problems have the boundary conditions built into the matrix.

4 Finite spectrum

4.1 Introduction

In this chapter, we explore the relation between regular self-adjoint Sturm–Liouville problems of Atkinson type (see Section 4.2 for a definition) and matrix eigenvalue problems of the form

$$DX = \lambda WX, \tag{4.1}$$

where D and W are real matrices, and W is diagonal. We use the notation $D = P + Q$ to remind us that the matrix D depends on the leading coefficient p and on the potential function q. All three matrices P, Q, W also depend on the parameters α, β when the boundary conditions are separated and on the coupling function K when the boundary conditions are real coupled. Thus the Sturm–Liouville equations of Atkinson type have equivalent matrix representations with boundary conditions "built into" the matrices for both separated and real coupled boundary conditions.

Consider the equation

$$-(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty < a < b < \infty, \tag{4.2}$$

with coefficients satisfying

$$r = 1/p, q, w \in L^1(J, \mathbb{R}) \tag{4.3}$$

and boundary conditions that are either separated

$$\begin{aligned} \cos\alpha y(a) - \sin\alpha(py')(a) &= 0, \quad 0 \le \alpha < \pi, \\ \cos\beta y(b) - \sin\beta(py')(b) &= 0, \quad 0 < \beta \le \pi, \end{aligned} \tag{4.4}$$

or real coupled

$$Y(b) = KY(a), \quad K \in \mathrm{SL}_2(\mathbb{R}), \quad K = (k_{ij}), \quad \det K = 1. \tag{4.5}$$

We do not discuss coupled complex self-adjoint boundary conditions in this chapter.

Here it is convenient to use the system representation of equation (4.2): with $u = y$ and $v = (py')$, we have

$$u' = rv, \quad v' = (q - \lambda w)u \quad \text{on } J. \tag{4.6}$$

4.2 Matrix representations of Sturm–Liouville problems

Recall from Chapter 3 that condition (4.3) allows r, q, w to be identically zero on subintervals of J.

Following Volkmer and Zettl [96], we associate a special class of Sturm–Liouville problems with the name of Atkinson.

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Definition 4.2.1. A Sturm–Liouville equation (4.1) is said to be of Atkinson type if, for some positive integer $n > 2$, there exists a partition of the interval J

$$a = a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n = b \tag{4.7}$$

such that

$$r = 0 \quad \text{on } [a_k, b_k], \quad k = 0, \ldots, n, \quad \int_{b_{k-1}}^{a_k} r > 0, \quad k = 1, 2, \ldots, n, \tag{4.8}$$

and

$$q = 0 = w \quad \text{on } [b_{k-1}, a_k], \quad k = 1, \ldots, n, \quad \int_{a_k}^{b_k} w > 0, \quad k = 0, 1, \ldots n. \tag{4.9}$$

Let

$$p_k = \left(\int_{b_{k-1}}^{a_k} r \right)^{-1}, \quad k = 1, 2, \ldots n; \quad q_k = \int_{a_k}^{b_k} q, \quad w_k = \int_{a_k}^{b_k} w, \quad k = 0, 1, \ldots n. \tag{4.10}$$

Definition 4.2.2. A Sturm–Liouville problem is of Atkinson type if the equation is of Atkinson type and the self-adjoint boundary condition is separated or real coupled, that is, either (4.4) or (4.5) holds.

In this section, we construct matrix eigenvalue problems that have exactly the same eigenvalues as the corresponding Sturm–Liouville problems of Atkinson type.

Remark 4.2.1. As mentioned before, in 1964, Atkinson, in his classical book [4, Chapter 8], hinted the existence of self-adjoint regular S-L problems whose spectra consist entirely of finitely many eigenvalues. This was confirmed in 2001 by Kong, Wu, and Zettl [65], who showed that for every positive integer $n > 2$, there exist S-L problems whose spectrum consists of exactly n eigenvalues. Surprisingly, these problems of Atkinson type include the "generalized" Sturm–Liouville problems of Feller and Krein [71]; see [96] and references therein.

From (4.8) and (4.9) we see that for any solution u, v of (4.6), u is constant on the intervals $[a_k, b_k]$, $k = 0, \ldots, n$, and v is constant on $[b_{k-1}, a_k]$, $k = 1, \ldots, n$. Let

$$u_k = u(t), \quad t \in [a_k, b_k], \quad k = 0, 1, \ldots, n; \quad v_k = v(t), \quad t \in [b_{k-1}, a_k], \quad k = 1, \ldots, n; \tag{4.11}$$

and set

$$v_0 = v(a_0) = v(a), \quad v_{n+1} = v(b_n) = v(b). \tag{4.12}$$

Lemma 4.2.1. *Assume that equation* (4.2) *is of Atkinson type with partition* (4.7)–(4.11). *Then for any solution* u, v *of equation* (4.6), *we have*

$$p_k(u_k - u_{k-1}) = v_k, \quad k = 1, 2, \ldots, n, \tag{4.13}$$

and

$$v_{k+1} - v_k = u_k(q_k - \lambda w_k), \quad k = 0, 1, \ldots, n. \tag{4.14}$$

Conversely, for any solution $u_k, k = 0, 1, \ldots, n$, *and* $v_k, k = 0, 1, \ldots, n+1$, *of system* (4.13)–(4.14), *there is a unique solution* $u(t)$ *and* $v(t)$ *of equation* (4.7) *satisfying* (4.11) *and* (4.12).

Proof. From the first equation of (4.6), for $k = 1, \ldots, n$, we have

$$u_k - u_{k-1} = u(a_k) - u(a_{k-1}) = \int_{a_{k-1}}^{a_k} u' = \int_{a_{k-1}}^{a_k} rv = v_k \int_{b_{k-1}}^{a_k} r = v_k / p_k.$$

This establishes (4.13), and (4.14) follows similarly from the second equation of (4.6).

On the other hand, if u_k, v_k satisfy (4.13) and (4.14), then define $u(t)$ and $v(t)$ according to (4.11) and (4.12) and extend them continuously to the whole interval J as a solution of the equation by integrating over subintervals. □

Theorem 4.2.1. *Assume that equation* (4.2)–(4.3) *is of Atkinson type with partition* (4.7)–(4.11). *Let* $\alpha \in [0, \pi)$ *and* $\beta \in (0, \pi]$. *Define the* $(n+1) \times (n+1)$ *tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 \sin\alpha + \cos\alpha & -p_1 \sin\alpha & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & & -p_n \sin\beta & p_n \sin\beta - \cos\beta \end{bmatrix} \tag{4.15}$$

and diagonal matrices

$$Q_{\alpha\beta} = \operatorname{diag}(q_0 \sin\alpha, q_1, \ldots, q_{n-1}, q_n \sin\beta), \quad W_{\alpha\beta} = \operatorname{diag}(w_0 \sin\alpha, w_1, \ldots, w_{n-1}, w_n \sin\beta). \tag{4.16}$$

Then the SLP consisting of equation (4.2)–(4.3) *with boundary condition* (4.4) *is equivalent to the matrix eigenvalue problem*

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U, \tag{4.17}$$

where $U = [u_0, u_1, \ldots, u_n]^T$. *Moreover, all eigenvalues are geometrically simple, and the eigenfunction* $u(t)$ *of the SLP and the corresponding eigenvector* U *of the matrix eigenvalue problem associated with the same eigenvalue are related by* $u(t) = u_k$, $t \in [a_k, b_k]$, $k = 0, 1, \ldots, n$.

Proof. There is a one-to-one correspondence between the solutions of system (4.13)–(4.14) and the solutions of the following system:

$$p_1(u_1 - u_0) - v_0 = u_0(q_0 - \lambda w_0), \tag{4.18}$$

$$p_{k+1}(u_{k+1} - u_k) - p_k(u_k - u_{k-1}) = u_k(q_k - \lambda w_k), \quad k = 1, 2, \dots, n-1, \tag{4.19}$$

$$v_{n+1} - p_n(u_n - u_{n-1}) = u_n(q_n - \lambda w_n). \tag{4.20}$$

In fact, assume that $u_k, k = 0, 1, \dots, n$, and $v_k, k = 0, 1, \dots, n+1$, is a solution of system (4.13)–(4.14). Then (4.18)–(4.20) follow from (4.13) and (4.14). On the other hand, let $u_k, k = 0, 1, \dots, n$, be a solution of system (4.18)–(4.20). Then v_0 and v_{n+1} are determined by (4.18) and (4.20), respectively. Let $v_k, k = 1, 2, \dots, n$, be defined by (4.13). Then using (4.18) and induction on (4.19), we obtain (4.14).

Therefore by Lemma 4.2.1 any solution of equation (4.13), and hence of (4.11), is uniquely determined by a solution of system (4.18)–(4.20). Note that from (4.11) we have

$$u_0 \cos\alpha = v_0 \sin\alpha, \quad u_n \cos\beta = v_{n+1} \sin\beta.$$

The equivalence follows from (4.18)–(4.20). □

Corollary 4.2.1.

(i) *Let* $\alpha, \beta \in (0, \pi)$. *Define the* $(n+1) \times (n+1)$ *symmetric tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & & -p_n & p_n - \cot\beta \end{bmatrix} \tag{4.21}$$

and diagonal matrices

$$Q_{\alpha\beta} = \operatorname{diag}(q_0, q_1, \dots, q_{n-1}, q_n), \quad W_{\alpha\beta} = \operatorname{diag}(w_0, w_1, \dots, w_{n-1}, w_n). \tag{4.22}$$

Then SLP (4.7), (4.11) *is equivalent to the matrix eigenvalue problem*

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U, \tag{4.23}$$

where $U = [u_0, u_1, \dots, u_{n-1}, u_n]^T$.

(ii) *If* $\alpha = 0$ *and/or* $\beta = \pi$, *then a similar statement holds with matrices* P, Q, W *obtained from the matrices* (4.22)–(4.23) *by deleting their first row and column if* $\alpha = 0$ *and/or the last row and column if* $\beta = \pi$.

Proof. (i) In this case, we divide the first and last rows of system (4.17) by $\sin\alpha$ and $\sin\beta$, respectively, to obtain (4.23).

(ii) If $\alpha = 0$, then $u_0 = 0$, so the first row and column of the matrices P, Q, W can be deleted. Similarly, if $\beta = \pi$, then $u_n = 0$, so the last row and column can be deleted. □

Theorem 4.2.1 and its corollary show that every SLP of Atkinson type with a self-adjoint separated boundary condition has a representation as a tridiagonal matrix eigenvalue problem.

Next, we show that all SLPs of Atkinson type with a coupled real self-adjoint BC also have matrix representations. In this case, the matrix P is symmetric and "almost tridiagonal" in the sense that the entries in the upper right and lower left corners are nonzero.

Theorem 4.2.2. *Consider the boundary condition* (4.5) *with* $k_{12} = 0$. *Define the* $n \times n$ *symmetric matrix, which is tridiagonal except for the* $(1, n)$ *and* $(n, 1)$ *entries,*

$$P_0 = \begin{bmatrix} -k_{11}k_{21} + p_1 + k_{11}^2 p_n & -p_1 & & & & -k_{11}p_n \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-2} & p_{n-2} + p_{n-1} & -p_{n-1} \\ -k_{11}p_n & & & & -p_{n-1} & p_{n-1} + p_n \end{bmatrix} \tag{4.24}$$

and diagonal matrices

$$Q_0 = \operatorname{diag}(q_0 + k_{11}^2 q_n, q_1, \ldots, q_{n-1}), \quad W_0 = \operatorname{diag}(w_0 + k_{11}^2 w_n, w_1, \ldots, w_{n-1}). \tag{4.25}$$

Then SLP (4.2), (4.5) *is equivalent to the matrix eigenvalue problem*

$$(P_0 + Q_0)U = \lambda W_0 U, \tag{4.26}$$

where $U = [u_0, u_1, \ldots, u_{n-1}]^T$, *admitting n eigenvalues according to multiplicity, provided that* $w_0 + k_{11}^2 w_n \neq 0$. *Moreover, eigenfunctions* $u(t)$ *of SLP* (4.2), (4.5) *and eigenvectors* U *of the matrix eigenvalue problem* (4.26) *associated with the same eigenvalue are related by* $u(t) = u_k$, $t \in [a_k, b_k]$, $k = 0, 1, \ldots, n-1$, *and* $u(t) = k_{11}u_0$ *when* $t \in [a_n, b_n]$.

Proof. Since $k_{12} = 0$, boundary condition (4.5) is the same as

$$u_n = k_{11}u_0, \quad v_{n+1} = k_{21}u_0 + k_{22}v_0, \tag{4.27}$$

where $k_{11}k_{22} = 1$. We claim that there is a one-to-one correspondence between the solutions of system (4.13)–(4.14) with boundary condition (4.27) and the solutions of the following system:

$$[-k_{11}k_{21} + (p_1 + q_0 - \lambda w_0) + k_{11}^2(p_n + q_n - \lambda w_n)]u_0 - p_1 u_1 - k_{11}p_n u_{n-1} = 0, \tag{4.28}$$

$$p_{k+1}(u_{k+1} - u_k) - p_k(u_k - u_{k-1}) = u_k(q_k - \lambda w_k), \quad k = 1, 2, \ldots, n-1. \tag{4.29}$$

In fact, assume that $u_k, k = 0, 1, \ldots, n$, and $v_k, k = 0, 1, \ldots, n+1$, is a solution of system (4.13)–(4.14), (4.27). Then (4.29) easily follows from (4.13) and (4.14). From (4.13) with $k = 1$ and (4.14) with $k = 0$ we have

$$v_0 = p_1(u_1 - u_0) - u_0(q_0 - \lambda w_0). \tag{4.30}$$

From (4.14) and (4.15) with $k = n$ we have

$$v_{n+1} = p_n(u_n - u_{n-1}) + u_n(q_n - \lambda w_n). \tag{4.31}$$

Combining (4.11), (4.12), and (4.13), we obtain that

$$p_n(k_{11}u_0 - u_{n-1}) + k_{11}u_0(q_n - \lambda w_n) = k_{21}u_0 + k_{22}[p_1(u_1 - u_0) - u_0(q_0 - \lambda w_0)]. \tag{4.32}$$

Note that $k_{11}k_{22} = 1$. Then (4.32) becomes (4.28).

On the other hand, assume that $u_k, k = 0, 1, \dots, n$, is a solution of system (4.28)–(4.29). Then u_n, v_0, and v_n are determined by (4.27), (4.30), and (4.31), respectively. Let $v_k, k = 1, 2, \dots, n$, be defined by (4.13). Then using (4.12), by induction on (4.13) we obtain (4.14). From (4.30)–(4.32) we see that $v_{n+1} = k_{21}u_0 + k_{22}v_0$. Hence the boundary condition (4.27) is satisfied.

Therefore by Lemma 4.2.1 any solution of SLP (4.6), (4.5), and hence of SLP (4.2), (4.5), is uniquely determined by a solution of system (4.28)–(4.29). □

Remark 4.2.2. The particular case $k_{12} = 0$ considered in Theorem 4.2.2 includes the "generalized" periodic-type boundary conditions

$$k_{12} = k_{21} = 0, \quad k_{11} = c, \quad \text{and} \quad k_{22} = 1/c, \quad c \neq 0.$$

This is a periodic condition when $c = 1$ and a semiperiodic condition when $c = -1$.

Theorem 4.2.3. *Consider the boundary condition* (4.5) *with* $k_{12} \neq 0$. *Define the* $(n+1) \times (n+1)$ *symmetric matrix, which is tridiagonal except for the* $(1, n)$ *and* $(n, 1)$ *entries,*

$$P_1 = \begin{bmatrix} p_1 - k_{11}/k_{12} & -p_1 & & & & 1/k_{12} \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ 1/k_{12} & & & & -p_n & p_n - k_{22}/k_{12} \end{bmatrix} \tag{4.33}$$

and diagonal matrices

$$Q_1 = \operatorname{diag}(q_0, q_1, \dots, q_{n-1}, q_n), \quad W_1 = \operatorname{diag}(w_0, w_1, \dots, w_{n-1}, w_n). \tag{4.34}$$

Then SLP (4.2), (4.5) *is equivalent to the matrix eigenvalue problem*

$$(P_1 + Q_1)U = \lambda W_1 U, \tag{4.35}$$

where $U = [u_0, u_1, \dots, u_n]^T$. *Moreover, eigenfunctions* $u(t)$ *of SLP* (4.2), (4.5) *and the corresponding eigenvectors* U *of the matrix eigenvalue problem* (4.35) *associated with the same eigenvalue are related by* $u(t) = u_k$, $t \in [a_k, b_k]$, $k = 0, 1, \dots, n$.

Proof. The boundary condition (4.5) is the same as

$$u_n = k_{11}u_0 + k_{12}v_0, \quad v_{n+1} = k_{21}u_0 + k_{22}v_0.$$

Since $k_{11}k_{22} - k_{12}k_{21} = 1$, this can be written as

$$v_0 = -\frac{k_{11}}{k_{12}}u_0 + \frac{1}{k_{12}}u_n, \quad v_{n+1} = -\frac{1}{k_{12}}u_0 + \frac{k_{22}}{k_{12}}u_n. \tag{4.36}$$

We claim that there is a one-to-one correspondence between the solutions of system (4.13), (4.14) with boundary condition (4.18) and the solutions of the following system:

$$\left(p_1 - \frac{k_{11}}{k_{22}} + q_0 - \lambda w_0\right)u_0 - p_1u_1 + \frac{1}{k_{12}}u_n = 0, \tag{4.37}$$

$$p_{k+1}(u_{k+1} - u_k) - p_k(u_k - u_{k-1}) = u_k(q_k - \lambda w_k), \quad k = 1, 2, \ldots, n-1, \tag{4.38}$$

$$\frac{1}{k_{12}}u_0 - p_nu_{n-1} + \left(p_n - \frac{k_{22}}{k_{22}} + q_n - \lambda w_n\right)u_n = 0. \tag{4.39}$$

In fact, assume that $u_k, k = 0, 1, \ldots, n$, and $v_k, k = 1, 2, \ldots, n$, is a solution of system (4.13)–(4.14) with boundary condition (4.36). Then (4.38) follows from (4.13)–(4.14) easily. Reasoning as in the proof of Theorem 4.2.2, v_0 and v_{n+1} also satisfy (4.30) and (4.31), respectively. Hence (4.37) and (4.39) are consequences of (4.30), (4.31), and (4.36).

On the other hand, if $u_k, k = 0, 1, \ldots, n$, is a solution of system (4.37)–(4.39), then v_0 and v_n are determined by (4.30) and (4.31), respectively. Let $v_k, k = 1, 2, \ldots, n$, be defined by (4.32). Then using (4.30) and (4.31), by induction on (4.38) we obtain (4.14). Also, from (4.30), (4.31), (4.37), and (4.39) we see that boundary condition (4.36) is satisfied.

Therefore by Lemma 4.2.1 any solution of SLP (4.6), (4.5), and hence of SLP (4.2), (4.5), is uniquely determined by a solution of system (4.37)–(4.39). □

Remark 4.2.3. When comparing Theorems 4.2.2 and 4.2.3, we note that the dimension of the matrix system for the former is n and for the latter is $n + 1$. The reason for this is that the condition $u_n = k_{11}u_0$ is used to express u_n in terms of u_0, thus eliminating the need for u_n in (4.35). Thus there are exactly n eigenvalues, counting multiplicity, in Theorem 4.2.2 and exactly $n + 1$ in Theorem 4.2.3.

Theorems 4.2.2, 4.2.3, and 4.2.4 prove that for any fixed separated or real coupled self-adjoint boundary condition, every Sturm–Liouville problem of Atkinson type has a matrix representation with the same eigenvalues. The next result highlights the fact that every such matrix representation is equivalent to not just one but to a whole family of Sturm–Liouville problems of Atkinson type and one member of this family has piecewise constant coefficients.

Theorem 4.2.4. *Let equation* (4.2) *be of Atkinson type, and let* $p_k, k = 1, 2, \ldots, n$*, and* $q_k, w_k, k = 0, 1, \ldots, n$*, be given by* (4.10)*. Define the piecewise constant functions* $\bar{p}, \bar{q}$*, and* $\bar{w}$ *on* J *by*

$$\begin{aligned}
&\bar{p}(t) = p_k(a_k - b_{k-1}), \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \ldots, n,\\
&\bar{p}(t) = \infty, \quad t \in [a_k, b_k], \quad k = 0, 1, \ldots, n;\\
&\bar{q}(t) = q_k/(b_k - a_k), \quad t \in [a_k, b_k], \quad k = 0, 1, \ldots, n,\\
&\bar{q}(t) = 0, \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \ldots, n;\\
&\bar{w}(t) = w_k/(b_k - a_k), \quad t \in [a_k, b_k], \quad k = 0, 1, \ldots, n,\\
&\bar{w}(t) = 0, \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \ldots, n.
\end{aligned} \tag{4.40}$$

Here $\bar{p}(t) = \infty$ *means that* $\bar{r} = 1/\bar{p} = 0$*. Suppose that the self-adjoint boundary condition* (4.4) *is either separated or real coupled. Then SLP* (4.2), (4.4) *has exactly the same eigenvalues as the SLP consisting of the equation with piecewise constant coefficients*

$$-(\bar{p}y')' + \bar{q}y = \lambda \bar{w}y \quad \text{on } J \tag{4.41}$$

and the same boundary condition (4.4).

Proof. Observe that both SLPs (4.2), (4.4) and (4.41), (4.4) determine the same

$$p_k, k = 1, 2, \ldots, n, \quad \text{and} \quad q_k, w_k, k = 0, 1, \ldots, n.$$

Thus by one of Theorems 4.2.1–4.2.3, depending on which boundary condition (4.4) is involved, they are equivalent to the same matrix eigenvalue problem, and hence they have the same eigenvalues. □

By Theorem 4.2.4 we see that for a fixed boundary condition (4.4) on a given interval J, there is a family of SLPs of Atkinson type that have exactly the same eigenvalues as SLP (4.41), (4.4). Such a family is called the *equivalent family* of SLPs (4.41), (4.4).

Remark 4.2.4. Note that all eigenvalues of the matrix problems (4.17), (4.26), and (4.35) are real. Moreover, under separated boundary conditions, there are exactly $n+1$ eigenvalues if $\alpha, \beta \in (0, \pi)$, exactly n eigenvalues if $\alpha = 0$ or $\beta = \pi$, and exactly $n - 1$ eigenvalues if $\alpha = 0$ and $\beta = \pi$.

4.3 Sturm–Liouville representations of matrix eigenvalue problems

In this section, we show that matrix eigenvalue problems of the form

$$DX = \lambda BX, \tag{4.42}$$

where $D = (d_{ij})$ is an $m \times m$ real symmetric matrix with $d_{i,i+1} \neq 0$, $i = 1, \ldots, m-1$, which is either tridiagonal or "almost tridiagonal", and $B = \operatorname{diag}(b_{11}, \ldots, b_{mm})$ with $b_{kk} \neq 0$, $k = 1, \ldots, m$, have representations as SLPs of Atkinson type. Such representations are not unique as shown by Theorem 4.2.4. Here we characterize all Sturm–Liouville representations of the matrix problem (4.42) using SLPs (4.21), (4.9), and their equivalent families.

First, we consider the case of separated boundary conditions (4.4) and find a kind of converse to Theorem 4.2.1.

Theorem 4.3.1. *Let $m > 2$, let D be an $m \times m$ symmetric tridiagonal matrix*

$$D = \begin{bmatrix} d_{11} & d_{12} & & & & \\ d_{12} & d_{22} & d_{23} & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{m-1,m-2} & d_{m-1,m-1} & d_{m-1,m} \\ & & & & d_{m-1,m} & d_{mm} \end{bmatrix} \tag{4.43}$$

where $d_{ij} \in \mathbb{R}$, $1 \leq i, j \leq m$, $d_{j,j+1} \neq 0$, $j = 1, \ldots, m-1$, and let

$$B = \operatorname{diag}(b_{11}, \ldots, b_{mm}), \quad 0 \neq b_{jj} \in \mathbb{R}, \quad 1 \leq j \leq m. \tag{4.44}$$

Then, given any separated self-adjoint BC (4.4)*, the matrix eigenvalue problem* (4.42) *has representations as SLPs of Atkinson type in the form of SLP* (4.2), (4.4)*. Moreover, given a fixed partition* (4.7) *of J, it has a unique representation in the form of SLP* (4.41), (4.4) *provided that, with the notation in* (4.10)*, one of the following holds:*

(1) $\alpha, \beta \in (0, \pi)$;

(2) $\alpha = 0$, $\beta \in (0, \pi)$, *and p_n, q_n, w_n are fixed;*

(3) $\alpha \in (0, \pi)$, $\beta = \pi$, *and p_1, q_0, w_0 and p_n, q_n, w_n are fixed.*

In each of these cases, all Sturm–Liouville representations of the matrix problem (4.42) *are given by the corresponding equivalent families* (4.41), (4.4) *with all possible choices of the parameters; for example, with all possible choices of p_1, q_0, w_0 in case $\alpha = 0$ and $\beta \in (0, \pi)$.*

Proof. First, consider the case where $a, \beta \in (0, \pi)$. Let $n = m - 1$ and $J = (a, b)$, $-\infty < a < b < \infty$. Define a partition of (a, b) by (4.7). We construct piecewise constant functions $\bar{p}, \bar{q}, \bar{w}$ on $[a, b]$ satisfying (4.3) and (4.8)–(4.9). We need to define the values of these functions on those subintervals of $[a, b]$ where they are not defined as zero in (4.8)–(4.9). To do this, we let

$$p_k = -d_{k,k+1}, \quad k = 1, 2, \ldots, n; \quad w_k = b_{k+1,k+1}, \quad k = 0, 1, \ldots, n;$$

and

$$q_0 = d_{11} - p_1 - \cot \alpha, \quad q_n = d_{n+1,n+1} - p_n + \cot \beta,$$

$$q_k = d_{k+1,k+1} - p_k - p_{k+1}, \quad k = 1, 2, \dots, n-1.$$

Then define $\bar{p}(t)$, $\bar{q}(t)$, and $\bar{w}(t)$ by (4.40). Such $\bar{p}, \bar{q}, \bar{w}$ are piecewise constant functions on J satisfying (4.8), (4.9), and (4.3). Equation (4.41) is of Atkinson type, and (4.10) is satisfied with p, q, w replaced by $\bar{p}, \bar{q}, \bar{w}$, respectively. It is easy to see that problem (4.7) is of the same form as problem (4.31). Therefore by Corollary 4.2.1 problem (4.7) is equivalent to the SLP (4.7), (4.11). The last part follows from Theorem (4.9). The cases $\alpha = 0$ and/or $\beta = \pi$ can be proven similarly. □

Next, we consider the coupled boundary condition (4.5) and find a kind of converse to Theorems 4.2.2 and 4.2.3. Note that in both of these two theorems the matrix corresponding to D in (4.42) is not tridiagonal but "almost tridiagonal".

Theorem 4.3.2. *Let $m > 2$, and let D be a symmetric matrix, which is tridiagonal except for nonzero entries $d_{1m} = d_{m1}$,*

$$D = \begin{bmatrix} d_{11} & d_{12} & & & & d_{1m} \\ d_{12} & d_{22} & d_{23} & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{m-1,m-2} & d_{m-1,m-1} & d_{m-1,m} \\ d_{1m} & & & & d_{m-1,m} & d_{mm} \end{bmatrix}, \tag{4.45}$$

where

$$d_{ij} \in \mathbb{R}, \quad b_{jj} \in \mathbb{R}, 1 \le i, \quad j \le m, \quad d_{j,j+1} \ne 0, \quad j = 1, \dots, m-1, \quad d_{1m} \ne 0; \tag{4.46}$$

and let

$$B = \operatorname{diag}(b_{11}, \dots, b_{mm}), \quad 0 \ne b_{jj} \in \mathbb{R}, \quad 1 \le j \le m. \tag{4.47}$$

Then, given any real coupled self-adjoint boundary condition (4.5)*, the matrix eigenvalue problem* (4.42) *has representations as SLPs of Atkinson type in the form of SLP* (4.2), (4.4). *Moreover, given a fixed partition* (4.7) *of J, it has a unique representation in the form of SLP* (4.41), (4.4)*, provided that, with the notation in* (4.10)*, one of the following holds:*

(1) $k_{12} \ne 0$*;*
(2) $k_{12} = 0$*, and q_0 and w_0 are fixed.*

In each of these cases, all Sturm–Liouville representations of the matrix problem (4.42) *are given by the corresponding equivalent families* (4.41), (4.4) *with all possible choices of the parameters.*

Proof. First, we consider the case $k_{12} \ne 0$. Note that we can normalize the matrices D and B such that $d_{1m} = 1/k_{12}$ by multiplying equation (4.42) by $(k_{12} d_{1m})^{-1}$. This operation does not change the eigenvalues of problem (4.42). Choose $n = m - 1$ and let $a < b$. Define the partition of $[a, b]$ by (4.7). Let

$$p_k = -d_{k,k+1}, \quad k = 1, 2, \dots, n; \quad w_k = b_{k+1,k+1}, \quad k = 0, 1, \dots, n;$$

and

$$q_0 = d_{11} - p_1 + k_{11}/k_{12}, \quad q_n = d_{n+1,n+1} - p_n + k_{22}/k_{12},$$
$$q_k = d_{k+1,k+1} - p_k - p_{k+1}, \quad k = 1, 2, \dots, n-1.$$

Then define $\bar{p}(t)$, $\bar{q}(t)$, and $\bar{w}(t)$ by (4.40). Similarly to the proof of Theorem 4.3.1, we see that problem (4.35) is the same as problem (4.40). Therefore by Theorem 4.2.3 the matrix problem (4.42) is equivalent to SLP (4.2), (4.4).

For the case $k_{12} = 0$, we choose $n = m$ and fix q_0 and w_0, and then proceed similarly. □

4.4 The study of Jacobi and cyclic Jacobi matrix eigenvalue problems using Sturm–Liouville theory

We study the eigenvalues of matrix problems involving Jacobi and cyclic Jacobi matrices as functions of certain entries. In Sections 4.2 and 4.3, we saw that Sturm–Liouville problems of Atkinson type have only a finite number of eigenvalues and are equivalent to matrix eigenvalue problems. Due to results by numerous researchers, many of which are of surprisingly recent origin given the long history of these problems, the dependence of eigenvalues of self-adjoint regular SLPs on the problem is now well understood. In this chapter, we apply some of these SLP results to matrix problems. Although some of our results, in particular, the behavior of the eigenvalues as certain entries of the matrices approach plus or minus infinity, seem to be new, we emphasize the approach, which we believe has not been used before: obtaining results about matrix eigenvalues using methods from Sturm–Liouville theory.

For $n \geq 3$ [41], let $\mathbb{M}_n$ be the set of $n \times n$ matrices over the reals. For any $C \in \mathbb{M}_n$, we denote by $\sigma(C)$ the set of eigenvalues of C. Furthermore, when $n \geq 3$, we denote by $C^{[1]}$ the submatrix of C obtained by removing the first row and column, by $C^{[n]}$ the submatrix obtained by removing the last row and column, and by $C^{[1,n]}$ the submatrix obtained by removing both the first and last rows and columns.

For any $C, D \in \mathbb{M}_n$, we say that λ is an eigenvalue of the matrix pair (C, D) if there exists a nontrivial vector $u \in \mathbb{R}^n$ such that $(C - \lambda D)u = 0$. We denote by $\sigma(C, D)$ the set of eigenvalues of (C, D). Clearly, $\lambda \in \sigma(C)$ if and only if $\lambda \in \sigma(C, I_n)$, where I_n is the identity matrix in $\mathbb{M}_n$.

Throughout this book, for any $C, D \in \mathbb{M}_n$, we use the following notation:

$$\begin{aligned}
\sigma(C, D) &= \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \\
\sigma(C^{[1]}, D^{[1]}) &= \{\lambda_1^{[1]}, \lambda_2^{[1]}, \ldots, \lambda_{n-1}^{[1]}\}, \\
\sigma(C^{[n]}, D^{[n]}) &= \{\lambda_1^{[n]}, \lambda_2^{[n]}, \ldots, \lambda_{n-1}^{[n]}\}, \\
\sigma(C^{[1,n]}, D^{[1,n]}) &= \{\lambda_1^{[1,n]}, \lambda_2^{[1,n]}, \ldots, \lambda_{n-2}^{[1,n]}\}.
\end{aligned} \tag{4.48}$$

When C is symmetric and D is positive definite, all these eigenvalues are real. In this case, each of the above sets of eigenvalues is arranged in nondecreasing order.

For $n \geq 2$, we study the spectrum $\sigma(P + Q, W)$ for the following two classes of symmetric matrices:

(I) $P, Q, W \in \mathbb{M}_{n+1}$ are such that P is a Jacobi matrix of the form

$$P = \begin{bmatrix} p_1 & -p_1 & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & & -p_n & p_n \end{bmatrix}$$

and

$$Q = \operatorname{diag}(q_0, q_1, \ldots, q_{n-1}, q_n), \quad W = \operatorname{diag}(w_0, w_1, \ldots, w_{n-1}, w_n),$$

where $p_i, w_i > 0$ and $q_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$;

(II) $P, Q, W \in \mathbb{M}_{n+1}$ are such that P is a cyclic Jacobi matrix of the form

$$P = \begin{bmatrix} p_1 & -p_1 & & & & p_{n+1} \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ p_{n+1} & & & & -p_n & p_n \end{bmatrix}$$

and

$$Q = \operatorname{diag}(q_0, q_1, \ldots, q_{n-1}, q_n), \quad W = \operatorname{diag}(w_0, w_1, \ldots, w_{n-1}, w_n),$$

where $p_i > 0$ for $i = 1, \ldots, n$, $p_{n+1} \neq 0$, $w_i > 0$, and $q_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$.

It is well known [41] that all eigenvalues in $\sigma(P+Q, W)$ are simple when $(P+Q, W)$ is in class (I) and simple or double when $(P+Q, W)$ is in class (II).

4.4.1 Main results

In this section, using notation (4.1) with $C = P + Q$ and $D = W$ for $(P + Q, W)$, we state our results for classes (I) and (II). Proofs are given in the next section.

We first consider class (I).

Theorem 4.4.1. *Let $(P+Q,W)$ be in class (I). For fixed $p_i > 0$ $(i=1,\dots,n)$, $q_i \in \mathbb{R}$ $(i=1,\dots,n-1)$, and $w_i > 0$ $(i=0,\dots,n)$, consider $\lambda_i = \lambda_i(q_0,q_n)$ as a function of q_0 and q_n. Then for $i=1,\dots,n+1$, $\lambda_i(q_0,q_n)$ is strictly increasing in both q_0 and q_n. Furthermore, we have:*

(a) *For each $q_n \in \mathbb{R}$,*

$$\lim_{q_0\to-\infty}\lambda_1(q_0,q_n)=-\infty,\quad \lim_{q_0\to-\infty}\lambda_i(q_0,q_n)=\lambda_{i-1}^{[1]}(q_n)\quad \text{for } i=2,\dots,n+1;$$
$$\lim_{q_0\to\infty}\lambda_i(q_0,q_n)=\lambda_i^{[1]}(q_n)\quad \text{for } i=1,\dots,n,\quad \lim_{q_0\to\infty}\lambda_{n+1}(q_0,q_n)=\infty.$$

(b) *For each $q_0 \in \mathbb{R}$,*

$$\lim_{q_n\to-\infty}\lambda_1(q_0,q_n)=-\infty,\quad \lim_{q_n\to-\infty}\lambda_i(q_0,q_n)=\lambda_{i-1}^{[n+1]}(q_0)\quad \text{for } i=2,\dots,n+1;$$
$$\lim_{q_n\to\infty}\lambda_i(q_0,q_n)=\lambda_i^{[n+1]}(q_0)\quad \text{for } i=1,\dots,n,\quad \lim_{q_0\to\infty}\lambda_{n+1}(q_0,q_n)=\infty.$$

(c) *In general,*

$$\lim_{q_0\to-\infty,q_n\to-\infty}\lambda_i(q_0,q_n)\to-\infty\quad \text{for } i=1,2;$$
$$\lim_{q_0\to-\infty,q_n\to-\infty}\lambda_i(q_0,q_n)\to\lambda_{i-2}^{[1,n+1]}\quad \text{for } i=3,\dots,n+1;$$
$$\lim_{q_0\to\infty,q_n\to-\infty}\lambda_1(q_0,q_n)\to-\infty;$$
$$\lim_{q_0\to\infty,q_n\to-\infty}\lambda_i(q_0,q_n)\to\lambda_{i-1}^{[1,n+1]}\quad \text{for } i=2,\dots,n;$$
$$\lim_{q_0\to\infty,q_n\to-\infty}\lambda_{n+1}=\infty;$$
$$\lim_{q_0\to-\infty,q_n\to\infty}\lambda_1(q_0,q_n)\to-\infty;$$
$$\lim_{q_0\to-\infty,q_n\to\infty}\lambda_i(q_0,q_n)\to\lambda_{i-1}^{[1,n+1]}\quad \text{for } i=2,\dots,n;$$
$$\lim_{q_0\to-\infty,q_n\to\infty}\lambda_{n+1}=\infty;$$
$$\lim_{q_0\to\infty,q_n\to\infty}\lambda_i(q_0,q_n)\to\lambda_{i-1}^{[1,n+1]}\quad \text{for } i=1,\dots,n-1;$$
$$\lim_{q_0\to\infty,q_n\to\infty}\lambda_i(q_0,q_n)\to\infty\quad \text{for } i=n,n+1.$$

Theorem 4.4.2. *For fixed $p_i > 0$ $(i=1,\dots,n)$, $q_i \in \mathbb{R}$ $(i=1,\dots,n-1)$, and $w_i > 0$ $(i=1,\dots,n)$, consider $\lambda_i^{[1]} = \lambda_i^{[1]}(q_n)$ as a function of q_n. Then for $i=1,\dots,n$, $\lambda_i^{[1]}(q_n)$ is strictly increasing in q_n. Furthermore,*

$$\lim_{q_n\to-\infty}\lambda_1^{[1]}(q_n)=-\infty,\quad \lim_{q_n\to-\infty}\lambda_i^{[1]}(q_n)=\lambda_{i-1}^{[1,n+1]}\quad \text{for } i=2,\dots,n;$$
$$\lim_{q_n\to\infty}\lambda_i^{[1]}(q_n)=\lambda_i^{[1,n+1]}\quad \text{for } i=1,\dots,n-1,\quad \lim_{q_n\to\infty}\lambda_n^{[1]}(q_n)=\infty.$$

Theorem 4.4.3. *For fixed $p_i > 0$ ($i = 1, \ldots, n$), $q_i \in \mathbb{R}$ ($i = 1, \ldots, n-1$), and $w_i > 0$ ($i = 0, \ldots, n-1$), consider $\lambda_i^{[n+1]} = \lambda_i^{[n+1]}(q_0)$ as a function of q_0. Then for $i = 1, \ldots, n$, $\lambda_i^{[n+1]}(q_0)$ is strictly increasing in q_0. Furthermore, we have:*

$$\lim_{q_0\to-\infty} \lambda_1^{[n+1]}(q_0) = -\infty, \quad \lim_{q_0\to-\infty} \lambda_i^{[n+1]}(q_0) = \lambda_{i-1}^{[1,n+1]} \quad \text{for } i = 2, \ldots, n;$$
$$\lim_{q_0\to\infty} \lambda_i^{[n+1]}(q_0) = \lambda_i^{[1,n+1]} \quad \text{for } i = 1, \ldots, n-1, \quad \lim_{q_0\to\infty} \lambda_n^{[n+1]}(q_0) = \infty.$$

The following corollaries are immediate consequences of Theorems 4.4.1, 4.4.2, and 4.4.3 and the well-known continuous dependence of eigenvalues on the matrix entries.

Corollary 4.4.1. *In addition to the notation in Theorem 4.4.1, we let $\mathcal{R}(\lambda_i(q_0, q_n))$ be the range of λ_i as a function of q_0 and q_n, $i = 1, \ldots, n+1$. Then we have:*

(a) *for each $q_n \in \mathbb{R}$,*

$$\begin{aligned}
&\mathcal{R}(\lambda_1(q_0, q_n)) = (-\infty, \lambda_1^{[1]}(q_n)),\\
&\mathcal{R}(\lambda_i(q_0, q_n)) = (\lambda_{i-1}^{[1]}(q_n), \lambda_i^{[1]}(q_n)) \quad \text{for } i = 2, \ldots, n,\\
&\mathcal{R}(\lambda_{n+1}(q_0, q_n)) = (\lambda_n^{[1]}(q_n), \infty);
\end{aligned}$$

(b) *for each $q_0 \in \mathbb{R}$,*

$$\begin{aligned}
&\mathcal{R}(\lambda_1(q_0, q_n)) = (-\infty, \lambda_1^{[n+1]}(q_n)),\\
&\mathcal{R}(\lambda_i(q_0, q_n)) = (\lambda_{i-1}^{[n+1]}(q_n), \lambda_i^{[n+1]}(q_n)) \quad \text{for } i = 2, \ldots, n,\\
&\mathcal{R}(\lambda_{n+1}(q_0, q_n)) = (\lambda_n^{[n+1]}(q_n), \infty);
\end{aligned}$$

(c) *in general,*

$$\begin{aligned}
&\mathcal{R}(\lambda_i(q_0, q_n)) = (-\infty, \lambda_i^{[1,n+1]}) \quad \text{for } i = 1, 2,\\
&\mathcal{R}(\lambda_i(q_0, q_n)) = (\lambda_{i-2}^{[1,n+1]}, \lambda_i^{[1,n+1]}) \quad \text{for } i = 3, \ldots, n-1,\\
&\mathcal{R}(\lambda_i(q_0, q_n)) = (\lambda_{i-2}^{[1,n+1]}, \infty) \quad \text{for } i = n, n+1.
\end{aligned}$$

Corollary 4.4.2. *In addition to the notation in Theorem 4.4.1, let $\mathcal{R}(\lambda_i^{[1]}(q_n))$ be the range of $\lambda_i^{[1]}$ as a function of q_n, $i = 1, \ldots, n$. Then we have:*

$$\begin{aligned}
&\mathcal{R}(\lambda_1^{[1]}(q_n)) = (-\infty, \lambda_1^{[1,n+1]}),\\
&\mathcal{R}(\lambda_i^{[1]}(q_n)) = (\lambda_{i-1}^{[1,n+1]}, \lambda_i^{[1,n+1]}) \quad \text{for } i = 2, \ldots, n-1,\\
&\mathcal{R}(\lambda_n^{[1]}(q_n)) = (\lambda_{n-1}^{[1,n+1]}, \infty).
\end{aligned}$$

Corollary 4.4.3. *In addition to the notation in Theorem 4.4.3, let $\mathcal{R}(\lambda_i^{[n+1]}(q_0))$ be the range of $\lambda_i^{[n+1]}$ as a function of q_0, $i = 1, \ldots, n$. Then we have:*

$$\mathcal{R}(\lambda_1^{[n+1]}(q_0)) = (-\infty, \lambda_1^{[1,n+1]}),$$

$$\mathcal{R}(\lambda_i^{[n+1]}(q_0)) = (\lambda_{i-1}^{[1,n+1]}, \lambda_i^{[1,n+1]}) \quad \textit{for } i = 2, \ldots, n-1,$$
$$\mathcal{R}(\lambda_n^{[n+1]}(q_0)) = (\lambda_{n-1}^{[1,n+1]}, \infty).$$

Next, we consider class (II).

The next theorem is a minor extension of Theorem 4.3.8 in the book by Horn and Johnson [50] with $W = I$. However, our proof is entirely different from the algebraic approach used there.

Theorem 4.4.4. *Let $(P + Q, W)$ be in class (II). Then we have the following inequalities:*

$$\lambda_1 \le \lambda_1^{[1]} \le \lambda_2 \le \lambda_2^{[1]} \le \cdots \le \lambda_n \le \lambda_n^{[1]} \le \lambda_{n+1}.$$

Moreover, there are no adjacent equalities in these inequalities.

The following corollaries are immediate consequences of Theorem 4.4.2 and Corollary 4.4.2.

Corollary 4.4.4. *For fixed $p_i > 0$ $(i = 1, \ldots, n)$, $p_{n+1} \neq 0$, $q_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n-1)$, and $w_i > 0$ $(i = 0, 1, \ldots, n-1)$, consider $\lambda_i = \lambda_i(q_n)$ as a function of q_n. Then we have:*

$$\lim_{q_n \to -\infty} \lambda_1(q_n) = -\infty \quad \textit{and} \quad \lim_{q_n \to \infty} \lambda_n(q_n) = \infty.$$

Corollary 4.4.5. *Let $\mathcal{R}(\lambda_i(q_n))$ be the range of λ_i as a function of q_n, $i = 1, \ldots, n+1$. Then we have:*

$$\mathcal{R}(\lambda_i(q_n)) \subset (-\infty, \lambda_i^{[1,n+1]}) \quad \textit{for } i = 1, 2,$$
$$\mathcal{R}(\lambda_i(q_n)) \subset (\lambda_{i-2}^{[1,n+1]}, \lambda_i^{[1,n+1]}) \quad \textit{for } i = 3, \ldots, n,$$
$$\mathcal{R}(\lambda_{n+1}(q_n)) \subset (\lambda_{n-1}^{[1,n+1]}, \infty).$$

Remark 4.4.1. In the above theorems and corollaries, all entries of P, Q, and W remain fixed except q_0 and q_n, and the eigenvalues are studied as functions of q_0 and q_n. We do not expect parallel results for the other entries. This is because q_0 and q_n play special roles in the corresponding Sturm–Liouville problems in the sense that they depend on the boundary conditions. Mpreover, as mentioned in the introduction to this chapter, due to some surprisingly recent results, the dependence of the eigenvalues of regular self-adjoint Sturm–Liouville problems on the boundary conditions is now well understood; see [113].

4.5 Comments

Kong, Möller, Wu, Volkmer, and Zettl in papers [65, 59, 96, 58] seem to be the first authors to have found relations between self-adjoint Sturm–Liouville problems with finite spectrum and the corresponding matrix theory.

5 Inverse Sturm–Liouville problems with finite spectrum

5.1 Introduction

In this chapter, we study the inverse problem. We show that given two sets of interlacing real numbers, there exists a Sturm–Liouville equation of Atkinson type with two separated boundary conditions such that the given numbers are the eigenvalues of these two problems. Parallel results are also obtained for some, but not all, real coupled boundary conditions.

Consider the equation

$$-(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty < a < b < \infty, \tag{5.1}$$

with coefficients satisfying

$$r = 1/p, q, w \in L^1(J, \mathbb{R}), \tag{5.2}$$

and self-adjoint boundary conditions, which are either separated conditions

$$\begin{aligned} &\cos\alpha y(a) - \sin\alpha (py')(a) = 0, \quad 0 \le \alpha < \pi, \\ &\cos\beta y(b) - \sin\beta (py')(b) = 0, \quad 0 < \beta \le \pi, \end{aligned} \tag{5.3}$$

or real coupled conditions

$$Y(b) = KY(a), \quad K \in \mathrm{SL}_2(\mathbb{R}), \quad K = (k_{ij}), \quad \det K = 1. \tag{5.4}$$

We do not consider complex self-adjoint coupled boundary conditions in this chapter.

For some positive integer $n > 2$, consider a partition of the interval J

$$a = a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n = b \tag{5.5}$$

such that

$$r = 0 \quad \text{on } [a_k, b_k], \quad k = 0, \ldots, n, \quad \int_{b_{k-1}}^{a_k} r > 0, \quad k = 1, 2, \ldots, n, \tag{5.6}$$

and

$$q = 0 = w \quad \text{on } [b_{k-1}, a_k], \quad k = 1, \ldots, n, \quad \int_{a_k}^{b_k} w > 0, \quad k = 0, 1, \ldots n. \tag{5.7}$$

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Let

$$p_k = \left(\int_{b_{k-1}}^{a_k} r \right)^{-1}, \quad k = 1, 2, \dots n; \quad q_k = \int_{a_k}^{b_k} q, \quad w_k = \int_{a_k}^{b_k} w, \quad k = 0, 1, \dots n. \tag{5.8}$$

Define piecewise constant functions $\bar{p}, \bar{q}$, and $\bar{w}$ on J by

$$\begin{aligned}
&\bar{p}(t) = p_k(a_k - b_{k-1}), \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \dots, n, \\
&\bar{p}(t) = \infty, \quad t \in [a_k, b_k], \quad k = 0, 1, \dots, n; \\
&\bar{q}(t) = q_k/(b_k - a_k), \quad t \in [a_k, b_k], \quad k = 0, 1, \dots, n, \\
&\bar{q}(t) = 0, \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \dots, n; \\
&\bar{w}(t) = w_k/(b_k - a_k), \quad t \in [a_k, b_k], \quad k = 0, 1, \dots, n, \\
&\bar{w}(t) = 0, \quad t \in [b_{k-1}, a_k], \quad k = 1, 2, \dots, n.
\end{aligned} \tag{5.9}$$

Here $\bar{p}(t) = \infty$ means that $\bar{r} = 1/\bar{p} = 0$.

Then the Sturm–Liouville problem consisting of equation (5.1) with either the separated boundary conditions (5.3) or the coupled conditions (5.4) has exactly the same eigenvalues as the SLP consisting of the equation with piecewise constant coefficients

$$-(\bar{p}y')' + \bar{q}y = \lambda \bar{w}y \quad \text{on } J \tag{5.10}$$

and the same boundary condition.

Observe that for both boundary conditions, we have the same piecewise constant coefficients:

$$p_k, \quad k = 1, 2, \dots, n, \quad \text{and} \quad q_k, w_k, \quad k = 0, 1, \dots, n.$$

Thus by Theorem 4.2.1 and its corollary and Theorems 4.2.2 and 4.2.3, depending on which boundary condition is involved, each of these problems is equivalent to the same matrix eigenvalue problem and hence has the same eigenvalues as the matrix problem. In other words, the two Sturm–Liouville problems with coefficients p, q, w and $\bar{p}, \bar{q}, \bar{w}$ and the same boundary condition have the same eigenvalues because both are equivalent to the same matrix eigenvalue problem.

Thus for a fixed boundary condition (5.3) or (5.4) on a given interval J, there is a family of SLPs of Atkinson type that have exactly the same eigenvalues. Such a family is called an *equivalent family* of SLPs of Atkinson type.

5.2 Main results

Definition 5.2.1. For a given equation of Atkinson type with coefficients satisfying (5.7)–(5.9), $\sigma(\alpha, \beta)$ denotes the spectrum of the separated boundary condition (5.3), and $\sigma(K)$ denotes the spectrum of the real coupled condition (5.4).

Next, we state our two theorems for separated boundary conditions; proofs will be given in Section 5.3.

Theorem 5.2.1. *Suppose* (5.1)–(5.9) *hold. Let* $\alpha, \beta \in (0, \pi)$. *Suppose that* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *are two sets of real numbers satisfying the strict interlacing relation*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k. \tag{5.11}$$

Let $n = k - 1$. *Then for any interval* $J = (a, b)$, $-\infty < a < b < \infty$, *any partition* (5.5), *and any* $w \in L(a, b)$ *satisfying* (5.7), *we have:*

a: *There exist coefficients* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family* (5.10) *has the same spectrum*

$$\sigma(\alpha, \beta) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \beta) = \{\mu_i : i = 1, \ldots, k-1\}.$$

b: *There exist coefficients* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family* (5.10) *has the same spectrum*

$$\sigma(\alpha, \beta) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(\alpha, \pi) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Furthermore, the equivalent families in (a) and (b) are uniquely determined.

Theorem 5.2.2. *Suppose* (5.1)–(5.9) *hold. Let* $\alpha, \beta \in (0, \pi)$, *and let* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *be two sets of real numbers satisfying the strict interlacing relation*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k.$$

Let $n = k$. *Then for any interval* $J = (a, b)$, $-\infty < a < b < \infty$, *any partition* (5.5), *and any* $w \in L(a, b)$ *satisfying* (5.7), *we have:*

a: *There exist coefficients* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family* (5.10) *has the spectrum*

$$\sigma(0, \beta) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \pi) = \{\mu_i : i = 1, \ldots, k-1\}.$$

b: *There exist coefficients* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family* (5.10) *has the spectrum*

$$\sigma(\alpha, \pi) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \pi) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Furthermore, the equivalent families in (a) and (b) are uniquely determined.

Remark 5.2.1. From these theorems it follows that given any finite set of distinct real numbers $\{\lambda_i : i = 1, \ldots, k\}$, $k > 3$, there exists a Sturm–Liouville equation of Atkinson type with self-adjoint separated boundary conditions whose spectrum is the given set. In particular, we this holds:

(1) For any finite set of prime numbers.
(2) For any finite set of twin primes. At the time of this writing, it is not known if there are infinitely many pairs of twin primes. From the perspective of Sturm–Liouville problems of Atkinson type there seems to be no obstacle to the existence of an infinite number of pairs of twin primes. As of 28 May 2018 the largest known twin primes are:

$$2996863034895 \cdot 2^{1290000} \pm 1.$$

(3) The Sturm–Liouville theory, which constructs a boundary value problem whose eigenvalues are a given set of primes or twin primes, also generates eigenfunctions whose zeros can be characterized by the Prüfer transformation. Does this “extra” information give any clues about the location of the “large” primes or twin primes?

The next two theorems are for some real coupled boundary conditions with coupling matrix K. Recall that for coupled boundary conditions, some eigenvalues may have multiplicity two.

Theorem 5.2.3. *Let* $\{\lambda_i : i = 1, \dots, k\}$ *and* $\{\mu_i : i = 1, \dots, k-1\}$ *be two sets of real numbers satisfying the following three conditions:*
(1)

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \lambda_{k-1} \le \mu_{k-1} \le \lambda_k,$$

(2) $\mu_i \ne \mu_j$ *if* $i \ne j$,
(3) *there exists a number* $d > 0$ *such that for all* $j = 1, \dots, k-1$,

$$\prod_{i=1}^{k} |\mu_i - \lambda_i| \ge 2d\big[1 + (-1)^{k+1-j}\big].$$

Let

$$n = k - 1.$$

Then for any interval $J = (a, b)$, $-\infty < a < b < \infty$, *any partition* (5.5) *of the interval* (a, b), *and any* $w \in L(a, b)$ *satisfying* (5.9), *we have:*
(a) *For any* $\beta \in (0, \pi)$, *there exist* $K = (k_{ij}) \in \mathrm{SL}_2(\mathbb{R})$ *satisfying* $k_{12} < 0$ *and* $\cot\beta = k_{22}/k_{12}$ *and* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family of SLPs* (5.10) *has the spectrum*

$$\sigma(K) = \{\lambda_i : i = 1, \dots, k\} \quad \text{and} \quad \sigma(0, \beta) = \{\mu_i : i = 1, \dots, k-1\}.$$

(b) *For any* $\alpha \in (0, \pi)$, *there exist* $K = (k_{ij}) \in \mathrm{SL}_2(\mathbb{R})$ *satisfying* $k_{12} < 0$ *and* $\cot\alpha = -k_{11}/k_{12}$ *and* $r, q \in L(J)$ *satisfying* (5.6) *and* (5.7) *such that the associated equivalent family of SLPs* (5.10) *has the spectrum*

$$\sigma(K) = \{\lambda_i : i = 1, \dots, k\} \quad \text{and} \quad \sigma(\alpha, \pi) = \{\mu_i : i = 1, \dots, k-1\}.$$

Remark 5.2.2. Note that conditions (1)–(3) of this theorem imply that
(1) The multiplicities of the eigenvalues $\lambda_i, i = 1, \ldots, k$, can be 1 or 2.
(2) The eigenvalues $\mu_i, i = 1, \ldots, k-1$, are distinct.
(3) For all $j = 1, 2, \ldots, k-1$ if $\mu_j = \lambda_i$ for some $i \in \{1, \ldots, k\}$, then j must be even when k is even, and j must be odd when k is odd.

Theorem 5.2.4. *Let $K = (k_{ij}) \in \mathrm{SL}_2(\mathbb{R})$ with $k_{12} = 0$ and $k_{11} > 0$, let $\{\lambda_i : i = 1, \ldots, k\}$ and $\{\mu_i : i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying conditions (1)–(3), and let $n = k$. Then for any interval $J = (a, b)$, $-\infty < a < b < \infty$ and $w \in L(a, b)$ satisfying* (5.9), *there exist $r, q \in L(a, b)$ satisfying* (5.6) *and* (5.7) *such that the associated equivalent family of SLPs* (5.10) *has the spectrum*

$$\sigma(K) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \pi) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Remark 5.2.3. In the conclusions of the theorems for coupled boundary conditions the existence of inverse problems is guaranteed for all matrices K with $k_{12} = 0$ and $k_{11} > 0$, but only for some matrices K with $k_{12} \neq 0$. The case where $k_{12} = 0$ and $k_{11} < 0$ remains unsolved. In particular, the semiperiodic case $k_{11} = -1 = k_{22}$ and $k_{12} = 0 = k_{21}$ is open.

Remark 5.2.4. In all four theorems, $\{\mu_i : i = 1, \ldots, k-1\}$ are eigenvalues for a Dirichlet boundary condition either at a or b. This shows that Dirichlet boundary conditions play a special role in the inverse spectral theory of Sturm–Liouville problems of Atkinson type.

Our proofs of these theorems use inverse matrix theory for equations of the type

$$DX = \lambda BX,$$

where D is a Jacobi or a cyclic Jacobi matrix, and B is a diagonal matrix. This theory is given in the book by Xu [106, Chapter 2] for the case where B is the identity matrix. We extend these theorems in [106, Chapter 2] to diagonal matrices B and prove this extension in Section 5.3 since we do not know a reference for it.

5.3 Inverse matrix eigenvalue problems with a weight function

We develop the inverse matrix eigenvalue problems for Jacobi and cyclic Jacobi matrices with a diagonal weight matrix.

Let $\mathbb{M}_k$ be the set of $k \times k$ matrices over the reals. For any $C \in \mathbb{M}_k$, we denote by $\sigma(C)$ the set of eigenvalues of C. Let C_1 be the principal submatrix obtained from C by removing its first row and column, and let C^1 be the submatrix obtained from C by removing the kth row and column.

For any $C, D \in \mathbb{M}_k$, we say that λ^* is an eigenvalue of the matrix pair (C, D) if there exists a nontrivial vector $u \in \mathbb{R}^k$ such that $(C - \lambda^* D)u = 0$. We denote by $\sigma(C, D)$ the set of eigenvalues of (C, D). Clearly, $\lambda^* \in \sigma(C)$ if and only if $\lambda^* \in \sigma(C, I_k)$, where I_k is the identity matrix in $\mathbb{M}_k$.

Consider symmetric matrices in $\mathbb{M}_k$ of the form

$$\begin{bmatrix} c_1 & d_1 & & & & \\ d_1 & c_2 & d_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{k-2} & c_{k-1} & d_{k-1} \\ & & & & d_{k-1} & c_k \end{bmatrix}. \tag{5.12}$$

Definition 5.3.1. A matrix $J \in \mathbb{M}_k$ of the form (5.12) is called a positive Jacobi matrix if $d_i > 0$ for all $i = 1, 2, \ldots, k$, and it is called a negative Jacobi matrix if $d_i < 0$ for all $i = 1, 2, \ldots, k$. We say that J is a Jacobi matrix if it is either a positive or negative Jacobi matrix.

Now we state a lemma from the book by Xu [106, Theorem 2.3.3] on the inverse eigenvalue problem for positive Jacobi matrices.

Lemma 5.3.1. *Let* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *be two sets of real numbers satisfying the strict interlacing relation*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k. \tag{5.13}$$

Then there exists a unique positive Jacobi matrix $J \in \mathbb{M}_k$ *such that*

$$\sigma(J) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(J_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

The next two theorems are extensions of Lemma 5.3.1.

Theorem 5.3.1. *Let* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *be two sets of real numbers satisfying the strict interlacing relation* (5.13). *Let* $W = \operatorname{diag}(w_1, \ldots, w_k)$ *be a diagonal matrix with* $w_i > 0$ *for* $i = 1, \ldots, k$. *Then there exists a unique positive Jacobi matrix* $M \in \mathbb{M}_k$ *such that*

$$\sigma(M, W) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(M_1, W_1) = \{\mu_i : i = 1, \ldots, k-1\}. \tag{5.14}$$

Proof. By Lemma 5.3.1 there exists a unique positive Jacobi matrix $J \in \mathbb{M}_k$ such that

$$\sigma(J) = \{\lambda_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(J_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Hence for each $\lambda = \lambda_i$, $i = 1, \ldots, k$, there exists a nontrivial $u \in \mathbb{R}^k$ such that $(J - \lambda I_k)u = 0$. Let $R = \sqrt{W} := \operatorname{diag}(\sqrt{w_1}, \ldots, \sqrt{w_k})$, and let $u = R\tilde{u}$. Multiplying the above equation by R we get

$$(RJR - \lambda R^2)\tilde{u} = 0, \quad \text{that is,} \; (M - \lambda W)\tilde{u} = 0,$$

where $M = RJR$. Clearly, $\lambda \in \sigma(M, W)$, and M is also a positive Jacobi matrix. Similarly, for each $\mu = \mu_i$, $i = 1, \ldots, k-1$, there exists a nontrivial $v \in \mathbb{R}^{k-1}$ such that $(J_1 - \lambda I_{k-1})v = 0$. We let $v = R_1\tilde{v}$. Then multiplying the above equation by R_1, we obtain $(R_1J_1R_1 - \mu R_1^2)\tilde{v} = 0$. We note that $M_1 = R_1J_1R_1$ and $W_1 = R_1^2$. This shows that $\mu \in \sigma(M_1, W_1)$. Thus

$$\sigma(J) \subset \sigma(M, W) \quad \text{and} \quad \sigma(J_1) \subset \sigma(M_1, W_1).$$

By reversing the steps in this argument we see that

$$\sigma(J) \supset \sigma(M, W) \quad \text{and} \quad \sigma(J_1) \supset \sigma(M_1, W_1).$$

Therefore

$$\sigma(J) = \sigma(M, W) \quad \text{and} \quad \sigma(J_1) = \sigma(M_1, W_1).$$

To show the uniqueness, let M be any positive Jacobi matrix satisfying (5.14). Then

$$\sigma(R^{-1}MR^{-1}) = \{\lambda_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(R_1^{-1}M_1R_1^{-1}) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Note that $R^{-1}MR^{-1}$ is a positive Jacobi matrix and $(R^{-1}MR^{-1})_1 = R_1^{-1}M_1R_1^{-1}$. By Lemma 5.3.1 $R^{-1}MR^{-1}$, and hence M, is uniquely determined. This completes the proof. □

Theorem 5.3.2. *Let $\{\lambda_i : i = 1, \ldots, k\}$ and $\{\mu_i : i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the strict interlacing property* (5.13). *Let $W = \operatorname{diag}(w_1, \ldots, w_k)$ be a diagonal matrix with $w_i > 0$ for $i = 1, \ldots, k$. Then there exists a unique negative Jacobi matrix $M \in \mathbb{M}_k$ such that*

$$\sigma(M, W) = \{\lambda_i : i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(M_1, W_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Proof. Let $\xi_i = -\lambda_{k+1-i}$, $i = 1, \ldots, k+1$, and $v_i = -\mu_{k-i}$, $i = 1, \ldots, k-1$. Then

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_{k-1} < \eta_{k-1} < \xi_k.$$

By Theorem 5.3.1 there exists a unique positive Jacobi matrix $M \in \mathbb{M}_k$ such that

$$\sigma(M, W) = \{\xi_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(M_1, W_1) = \{\eta_i : i = 1, \ldots, k-1\}.$$

It follows that

$$\sigma(-M, W) = \{-\xi_i : i = 1, \ldots, k\} = \{\lambda_i : i = 1, \ldots, k\}$$

and

$$\sigma(-M_1, W_1) = \{-\eta_i : i = 1, \ldots, k-1\} = \{\mu_i : i = 1, \ldots, k-1\}.$$

Note that M is a negative Jacobi matrix and $(-M)_1 = -M_1$. The proof is complete. □

Corollary 5.3.1. *Theorems* 5.3.1 *and* 5.3.2 *hold when* M_1 *and* W_1 *are replaced by* M^1 *and* W^1*, respectively.*

Proof. Let $\widetilde{M} = GMG$ and $\widetilde{W} = GWG$ with $G = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$. Then the ith row of $\widetilde{M}$ is the same as the $(k-i)$th row of M, and the same holds for the columns. Hence $\widetilde{M}_1 = M^1$. Similarly, $\widetilde{W}_1 = W^1$. Therefore the conclusion follows from Theorems 5.3.1 and 5.3.2 with M and W replaced by $\widetilde{M}$ and $\widetilde{W}$, respectively. □

Corollary 5.3.2. *Theorems* 5.3.1 *and* 5.3.2 *hold when* $w_i > 0$ *is replaced by* $w_i < 0$ *for* $i = 1, \dots, k$.

Proof. Let $\{\lambda_i : i = 1, \dots, k\}$ and $\{\mu_i : i = 1, \dots, k-1\}$ be two sets of real numbers satisfying the strict interlacing relation (5.13). By Theorem 5.3.1 there exists a unique positive Jacobi matrix $M \in \mathbb{M}_k$ such that

$$\sigma(M, -W) = \{-\lambda_i : i = 1, \dots, k\} \quad \text{and} \quad \sigma(M_1, W_1) = \{-\mu_i : i = 1, \dots, k-1\}.$$

Hence

$$\sigma(M, W) = \{\lambda_i : i = 1, \dots, k\} \quad \text{and} \quad \sigma(M_1, W_1) = \{\mu_i : i = 1, \dots, k-1\}.$$

This shows that Theorem 5.3.1 holds when $w_i > 0$ is replaced by $w_i < 0$ for $i = 1, \dots, k$. The same argument applies to Theorem 5.3.2. □

Next, consider symmetric matrices in $\mathbb{M}_k$ of the form

$$\begin{bmatrix} c_1 & d_1 & & & & d_k \\ d_1 & c_2 & d_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{k-2} & c_{k-1} & d_{k-1} \\ d_k & & & & d_{k-1} & c_k \end{bmatrix}. \tag{5.15}$$

Definition 5.3.2. A matrix $J \in \mathbb{M}_k$ of the form of (5.15) is called a positive cyclic Jacobi matrix if $d_i > 0$ for all $i = 1, 2, \dots, k-1$, and it is called a negative cyclic Jacobi matrix if $d_i < 0$ for all $i = 1, 2, \dots, k-1$. We say that J is a cyclic Jacobi matrix if it is either a positive or negative cyclic Jacobi matrix. (In the literature, cyclic Jacobi matrices are sometimes called periodic Jacobi matrices.)

Now we state another lemma from Xu [106, Theorem 2.8.3] on the inverse eigenvalue problem for positive cyclic Jacobi matrices. Note that the uniqueness is not guaranteed by this lemma; see [106, p. 78].

Lemma 5.3.2. *Let* $\{\lambda_i : i = 1, \dots, k\}$ *and* $\{\mu_i : i = 1, \dots, k-1\}$ *be two sets of real numbers satisfying the conditions*

(1) $\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \lambda_{k-1} \le \mu_{k-1} \le \lambda_k$;
(2) $\mu_i \ne \mu_j$ *if* $i \ne j$, *and*

(3) *there exists a* $d > 0$ *such that for all* $j = 1, \ldots, k-1$,

$$\prod_{i=1}^{k} |\mu_i - \lambda_i| \geq 2d\big[1 + (-1)^{k+1-j}\big].$$

Then there exists a positive cyclic Jacobi matrix J *of the form of* (5.15) *such that* $d = \prod_{i=1}^{k} d_i > 0$ *and*

$$\sigma(J) = \{\lambda_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(J_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

The following theorems are extensions of Lemma 5.3.2. Since the proofs are similar to those of Theorems 5.2.1 and 5.2.2; we omit the details.

Theorem 5.3.3. *Let* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *be two sets of real numbers satisfying conditions (1)–(3) of Lemma* 5.3.2. *Let* $W = \operatorname{diag}(w_1, \ldots, w_k)$ *be a diagonal matrix with* $w_i > 0$ *for* $i = 1, \ldots, k$. *Then for any* $d > 0$ *satisfying (3) of Lemma* 5.3.2, *there exists a positive cyclic Jacobi matrix* N *of the form* (5.15) *such that* $\prod_{i=1}^{k} d_i = d$ *and*

$$\sigma(N, W) = \{\lambda_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(N_1, W_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

Theorem 5.3.4. *Let* $\{\lambda_i : i = 1, \ldots, k\}$ *and* $\{\mu_i : i = 1, \ldots, k-1\}$ *be two sets of real numbers satisfying conditions (1)–(3) of Lemma* 5.3.2. *Let* $W = \operatorname{diag}(w_1, \ldots, w_k)$ *be a diagonal matrix with* $w_i > 0$ *for* $i = 1, \ldots, k$. *Then for any* $d > 0$ *satisfying (3) of Lemma* 5.3.2, *there exists a negative cyclic Jacobi matrix* N *of the form* (5.15) *such that* $\prod_{i=1}^{k} d_i = d$ *and*

$$\sigma(N, W) = \{\lambda_i : i = 1, \ldots, k\} \quad \text{and} \quad \sigma(N_1, W_1) = \{\mu_i : i = 1, \ldots, k-1\}.$$

With the same arguments as in the previous corollaries, we have the following:

Corollary 5.3.3.

a: *The conclusions of Theorems* 5.3.1 *and* 5.3.2 *hold when* M_1 *and* W_1 *are replaced by* M^1 *and* W^1, *respectively.*

b: *The conclusions of Theorems* 5.3.4 *and* 5.3.4 *hold when* $w_i > 0$ *is replaced by* $w_i < 0$ *for* $i = 1, \ldots, k$.

5.4 Proofs of the main results

To prove Theorems 5.2.1 and 5.2.2, we use Theorems 5.3.1 and 5.3.2 for the extended inverse Jacobi matrix problems and Theorem 4.2.1 with its corollary. This theorem and its corollary show that every Sturm–Liouville problem of Atkinson type with separated self-adjoint boundary conditions has a representation as a matrix eigenvalue problem. For clarity of exposition, we state the four matrix representations corresponding to $\alpha, \beta \in (0, \pi)$; $\alpha = 0$, $\beta \in (0, \pi)$; $\alpha \in (0, \pi)$, $\beta = 0$; and $\alpha = 0$, $\beta = \pi$ as Propositions 1–4 using the hypotheses and notation of Section 5.1. Recall that $\alpha = 0$ determines the Dirichlet boundary condition at the endpoint a and $\beta = \pi$ determines the Dirichlet boundary condition at the endpoint b. In the inverse spectral theory of Sturm–Liouville problems of Atkinson type the Dirichlet boundary conditions at one or both endpoints play a special role. Each Dirichlet condition at an endpoint reduces the number of eigenvalues by one.

Proposition 5.4.1. *Let $\alpha, \beta \in (0, \pi)$. Define the $(n+1) \times (n+1)$ Jacobi matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & & -p_n & p_n - \cot\beta \end{bmatrix} \tag{5.16}$$

and diagonal matrices

$$Q_{\alpha\beta} = \operatorname{diag}(q_0, q_1, \dots, q_{n-1}, q_n), \quad W_{\alpha\beta} = \operatorname{diag}(w_0, w_1, \dots, w_{n-1}, w_n).$$

Then the spectrum $\sigma(\alpha, \beta)$ and the spectrum $\sigma(P_{\alpha\beta} + Q_{\alpha\beta}, W_{\alpha\beta})$ of the matrix pair $(P_{\alpha\beta} + Q_{\alpha\beta}, W_{\alpha\beta})$ are the same.

Proposition 5.4.2. *Let $\alpha = 0$ and $\beta \in (0, \pi)$. Define the $n \times n$ Jacobi matrix*

$$P_{0\beta} = \begin{bmatrix} p_1 + p_2 & -p_2 & & & & \\ -p_2 & p_2 + p_3 & -p_3 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & & -p_n & p_n - \cot\beta \end{bmatrix} \tag{5.17}$$

and diagonal matrices

$$Q_{0\beta} = \operatorname{diag}(q_1, \dots, q_{n-1}, q_n), \quad W_{0\beta} = \operatorname{diag}(w_1, \dots, w_{n-1}, w_n).$$

Then the spectrum $\sigma(0, \beta)$ and the spectrum $\sigma(P_{0\beta} + Q_{0\beta}, W_{0\beta})$ of the matrix pair $(P_{0\beta} + Q_{0\beta}, W_{0\beta})$ are the same.

Proposition 5.4.3. *Let $\alpha \in (0, \pi)$ and $\beta = \pi$. Define the $n \times n$ Jacobi matrix*

$$P_{\alpha\pi} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-2} & p_{n-2} + p_{n-1} & -p_{n-1} \\ & & & & -p_{n-1} & p_{n-1} + p_n \end{bmatrix} \tag{5.18}$$

and diagonal matrices

$$Q_{\alpha\pi} = \operatorname{diag}(q_0, q_1, \dots, q_{n-1}), \quad W_{\alpha\pi} = \operatorname{diag}(w_0, w_1, \dots, w_{n-1}).$$

Then the spectrum $\sigma(\alpha, \pi)$ and the spectrum $\sigma(P_{\alpha\pi} + Q_{\alpha\pi}, W_{\alpha\pi})$ of the matrix pair $(P_{\alpha\pi} + Q_{\alpha\pi}, W_{\alpha\pi})$ are the same.

Proposition 5.4.4. *Let $\alpha = 0$ and $\beta = \pi$. Define the $(n-1)\times(n-1)$ Jacobi matrix*

$$P_{0\pi} = \begin{bmatrix} p_1+p_2 & -p_2 & & & \\ -p_2 & p_2+p_3 & -p_3 & & \\ & \cdots & \cdots & \cdots & \\ & & -p_{n-2} & p_{n-2}+p_{n-1} & -p_{n-1} \\ & & & -p_{n-1} & p_{n-1}+p_n \end{bmatrix} \tag{5.19}$$

and diagonal matrices

$$Q_{0\pi} = \operatorname{diag}(q_1, q_{2,} \cdots, q_{n-1}), \quad W_{0\pi} = \operatorname{diag}(w_1, w_2, \ldots, w_{n-1}).$$

Then the spectrum $\sigma(0,\pi)$ and the spectrum $\sigma(P_{0\pi}+Q_{0\pi}, W_{0\pi})$ of the matrix pair $(P_{0\pi}+Q_{0\pi}, W_{0\pi})$ are the same.

Remark 5.4.1. Note that for these four propositions, we have

$$(P_{\alpha\beta}+Q_{\alpha\beta})_1 = P_{0\beta}+Q_{0\beta}, \quad (P_{\alpha\beta}+Q_{\alpha\beta})^1 = P_{\alpha\pi}+Q_{\alpha\pi}$$

and

$$(P_{0\beta}+Q_{0\beta})^1 = P_{0\pi}+Q_{0\pi}, \quad (P_{\alpha\pi}+Q_{\alpha\pi})_1 = P_{0\pi}+Q_{0\pi}.$$

Proof of Theorem 5.2.1. (a) For a given partition (5.5) of (a,b), define

$$w_i = \int_{a_i}^{b_i} w, \quad i = 0,1,\ldots n, \quad \text{and} \quad W_{\alpha\beta} = \operatorname{diag}(w_0, w_1, \ldots, w_n).$$

By (5.7) $w_i > 0$, $i = 0,1,\ldots,n$. Since $k = n+1$, by Theorem 5.3.2 there exists a unique negative Jacobi matrix $M \in \mathbb{M}_{n+1}$ of the form (5.12) such that

$$\sigma(M,W) = \{\lambda_i : i = 1,\ldots,n+1\} \quad \text{and} \quad \sigma(M_1,W_1) = \{\mu_i : i = 1,\ldots,n\}.$$

Let

$$p_i = d_i, \quad i = 1,\ldots,n; \quad q_i = c_{i+1} - p_i - p_{i+1}, \quad i = 1,\ldots,n-1,$$
$$q_0 = c_1 - p_1 - \cot\alpha, \quad q_n = c_{n+1} - p_n - \cot\beta,$$

and define $P_{\alpha\beta}, Q_{\alpha\beta}, P_{0\beta}$, and $Q_{0\beta}$ as before. Note that $p_i > 0$, $i = 1,\ldots,n$. It is easy to see that $M = P_{\alpha\beta}+Q_{\alpha\beta}$ and $M_1 = P_{0\beta}+Q_{0\beta}$. With the above notation, we have $(W_{\alpha\beta})_1 = W_{0\beta}$. Therefore

$$\sigma(P_{\alpha\beta}+Q_{\alpha\beta}, W_{\alpha\beta}) = \{\lambda_i : i = 1,\ldots,n+1\}$$

and

$$\sigma(P_{0\beta}+Q_{0\beta}, W_{0\beta}) = \{\mu_i : i = 1,\ldots,n\}.$$

From Propositions 5.4.1 and 5.4.2 it follows that

$$\sigma(\alpha,\beta) = \{\lambda_i : i = 1,\dots,n+1\} \quad \text{and} \quad \sigma(0,\beta) = \{\mu_i : i = 1,\dots,n\}.$$

Observe that the choice of $p_i, i = 1,\dots,n$, and $q_i, i = 0,\dots,n$, is unique and all $r, q \in L(a,b)$ by this choice form an equivalent family of SLPs. This completes the proof.

(b) The proof is similar using Corollary 5.3.1 and Propositions 5.4.1 and 5.4.3. We omit the details. □

Also, the proof of Theorem 5.2.2 is similar to that of Theorem 5.2.1 using Theorem 5.3.2, Corollary 5.3.1, and Propositions 5.4.2, 5.4.3, and 5.4.3. We omit the details.

To prove Propositions 5.4.3 and 5.4.4, we use, in addition to Theorems 5.3.3 and 5.3.4 for the inverse Jacobi matrix problems, Theorems 4.2.3 and 4.2.4 on the equivalence between Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems. For clarity of exposition, we state these propositions here.

Proposition 5.4.5. *Consider the real coupled boundary condition* (5.4) *with* $k_{12} \neq 0$. *Define the* $(n+1)\times(n+1)$ *cyclic Jacobi matrix*

$$P_I = \begin{bmatrix} p_1 - k_{11}/k_{12} & -p_1 & & & & 1/k_{12} \\ -p_1 & p_1+p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1}+p_n & -p_n \\ 1/k_{12} & & & & -p_n & p_n - k_{22}/k_{12} \end{bmatrix} \tag{5.20}$$

and diagonal matrices

$$Q_I = \operatorname{diag}(q_0, q_1, \dots, q_{n-1}, q_n), \quad W_I = \operatorname{diag}(w_0, w_1, \dots, w_{n-1}, w_n).$$

Then the spectrum $\sigma(K)$ *and the spectrum* $\sigma(P_I + Q_I, W_I)$ *of the matrix pair* $(P_I + Q_I, W_I)$ *are the same.*

Proposition 5.4.6. *Consider the real coupled boundary condition* (5.4) *with* $k_{12} = 0$. *Define the* $n \times n$ *cyclic Jacobi matrix*

$$P_\Theta = \begin{bmatrix} -k_{11}k_{21} + p_1 + k_{11}^2 p_n & -p_1 & & & & -k_{11}p_n \\ -p_1 & p_1+p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-2} & p_{n-2}+p_{n-1} & -p_{n-1} \\ -k_{11}p_n & & & & -p_{n-1} & p_{n-1}+p_n \end{bmatrix} \tag{5.21}$$

and diagonal matrices

$$Q_\Theta = \operatorname{diag}(q_0 + k_{11}^2 q_n, q_1, \dots, q_{n-1}), \quad W_\Theta = \operatorname{diag}(w_0 + k_{11}^2 w_n, w_1, \dots, w_{n-1}).$$

Then the spectrum $\sigma(K)$ *and the spectrum* $\sigma(P_\Theta + Q_\Theta, W_\Theta)$ *of the matrix pair* $(P_\Theta + Q_\Theta, W_\Theta)$ *are the same.*

Proof of Theorem 4.2.3. (a) For a given partition (5.5) of (a, b), define

$$w_i = \int_{a_i}^{b_i} w, \quad i = 0, 1, \dots n, \quad \text{and} \quad W_I = \operatorname{diag}(w_0, w_1, \dots, w_n).$$

By (5.7), $w_i > 0$, $i = 0, 1, \dots, n$. Since $k = n + 1$, by Theorem 4.2.4 there exists a negative cyclic Jacobi matrix $N \in \mathbb{M}_{n+1}$ in the form (4.44) such that

$$\sigma(N, W) = \{\lambda_i : i = 1, \dots, n + 1\} \quad \text{and} \quad \sigma(N_1, W_1) = \{\mu_i : i = 1, \dots, n\}.$$

Let $p_i = -d_i$, $i = 1, \dots, n$. Then $p_i > 0$, $i = 1, \dots, n$. Let $k_{12} = 1/d_{n+1}$. Then $k_{12} < 0$. For $\beta \in (0, \pi)$, choose $K \in \mathrm{SL}_2(\mathbb{R})$ such that $\cot \beta = k_{22}/k_{12}$ $k_{12} = 1/d_{n+1}$. Thus K defines a coupled boundary condition (5.4). Let

$$q_i = c_{i+1} - p_i - p_{i+1}, \quad i = 1, \dots, n - 1,$$
$$q_0 = c_1 - p_1 + k_{11}/k_{12}, \quad q_n = c_{n+1} - p_n + k_{22}/k_{12}.$$

Define $P_I, Q_I, P_{0\beta}$, and $Q_{0\beta}$ as before and note that $N = P_I + Q_I$ and $N_1 = P_{0\beta} + Q_{0\beta}$. With the above notation, we also have $(W_I)_1 = W_{0\beta}$. Therefore

$$\sigma(P_I + Q_I, W_I) = \{\lambda_i : i = 1, \dots, n + 1\}$$

and

$$\sigma(P_{0\beta} + Q_{0\beta}, W_{0\beta}) = \{\mu_i : i = 1, \dots, n\}.$$

By Propositions 5.4.5 and 5.4.2 we have that

$$\sigma(K) = \{\lambda_i : i = 1, \dots, n + 1\} \quad \text{and} \quad \sigma(0, \beta) = \{\mu_i : i = 1, \dots, n\}.$$

This completes the proof.

(b) The proof is similar using Corollary 5.3.3(a) and Propositions 5.4.6 and 5.4.3. We omit the details. □

The proof of Theorem 4.2.4 is similar to that of Theorem 4.2.3 using Theorem 5.3.4, Corollary 5.3.1, and Propositions 5.4.6 and 5.4.4. We only need to note that the condition $k_{11} > 0$ is needed to guarantee that $p_n > 0$ in the matrix P_Θ. We omit the details.

5.5 Comments on the inverse theories for finite and infinite spectra

This chapter is based on the paper by Kong and Zettl [69]; see this paper for additional details.

Sturm–Liouville problems of Atkinson type are clearly a very special subclass of all regular self-adjoint Sturm–Liouville problems. However, Volkmer [97] has shown, using the Radon–Nikodym theorem, that problems of Atkinson type include those studied by Feller [38] and Krein [72] in connection with their works on frequencies of vibrating strings and diffusion operators.

Most of the literature on inverse problems is restricted to the case where both the leading coefficient p and the weight function w are identically 1 on the whole interval (a, b) and the boundary conditions are separated. It is clear that the fact that $1/p, q, w$ are identically zero on certain subintervals of (a, b) plays a fundamental role for all our theorems in Chapter 5.

Many authors prove a result for the case where $p = 1 = w$ and claim their result holds when all three coefficients p, q, w with $p > 0$ and $w > 0$ are present because this more general equation can be transformed to an equation with $p = 1 = w$. This claim can be highly misleading as is illustrated in [113] with the well-known Molchanov criterion for the discreteness of the spectrum; see pages 213–214. How do the conditions on the potential function q transform back to the original equation with all three coefficients p, q, w present? The Molchanov criterion has been extended by Müller and Pfeiffer [80] and Kwong and Zettl [73]; these extensions are far from trivial.

One of the celebrated papers on inverse Sturm–Liouville problems is the paper of Borg [17], who showed that when p and w are identically 1, the spectra of two given separated boundary conditions determine the potential q uniquely.

Next, we comment on some differences between the results on inverse theory for finite spectrum in Chapter 5 and the classical inverse theory using Borg's theorem as an illustration.

(1) In Borg's theorem [17], there is an a priori assumption that the two given sets of infinite numbers are spectra. Theorems 5.2.1 and 5.2.2 are for two arbitrarily given finite sets of real numbers that satisfy the interlacing property.
(2) Borg's theorem guarantees that the Sturm–Liouville equation is uniquely determined by the two preassigned spectra, whereas Theorems 5.2.1 and 5.2.2 determine a unique family of equations. This family consists of an uncountable number of equations, but each member of this family can be represented by the same matrix eigenvalue problem.
(3) Borg's theorem requires that the two spectra are from two prescribed boundary conditions, whereas for Theorems 5.2.1 and 5.2.2, one boundary condition is arbitrarily given, the other is related to the given one, and there is an arbitrarily chosen weight function w. Then the equivalent family of Sturm–Liouville equations is determined by these.
(4) Theorem 5.2.3 is for some real coupled boundary conditions. Not all real coupled boundary conditions are covered by this theorem. The periodic conditions are covered, but the semiperiodic case is open. All complex self-adjoint boundary conditions are open.

6 Eigenvalues below the essential spectrum

6.1 Introduction

In this chapter, we study the existence and properties of eigenvalues of singular problems with one limit endpoint. The case where neither endpoint is LP has been extensively studied and is now well understood for both regular and singular endpoints due to some surprisingly recent results [114, 111, 103, 10, 20, 19, 10, 62, 64, 67, 57, 26, 8, 56, 74, 100, 66]. See [113] for a more comprehensive list of references for basic definitions, notations, and general information.

Here our approach is based on the paper by Zhang et al. [114]. It uses the spectral theorem for self-adjoint operators in a Hilbert space, regular approximations of singular problems, and the relation between the number of linearly independent square-integrable solutions for real values of the spectral parameter and the spectrum.

We start with a brief review of properties of self-adjoint operators in the weighted Hilbert space $H = L^2(J, w)$ where $w > 0$ a. e. on $J = (a, b)$, $-\infty \leq a < b \leq \infty$.

6.2 The Lagrange form and maximal and minimal domains

In this section, we start by reviewing the Lagrange form and maximal and minimal operators. Consider the differential expression M defined by

$$My = -(py')' + qy, \quad \text{with } r = 1/p, q \in L_{\text{loc}}(J, \mathbb{R}), \quad J = (a, b). \tag{6.1}$$

The expression My is defined a. e. for functions y such that y and (py') are in $AC_{\text{loc}}(J)$; we refer to this as the expression domain of M. The maximal domain $D_{\max} = D_{\max}(M, w, J)$ of M on J with weight function $w \in L_{\text{loc}}(J, \mathbb{R})$, $w > 0$ a. e., is defined by

$$D_{\max} = \{y : J \to \mathbb{C} : y, (py') \in AC_{\text{loc}}(J), y, w^{-1} M y \in L^2(J, w)\}. \tag{6.2}$$

For y and z in the expression domain of M, the Lagrange sesquilinear form $[\cdot, \cdot]$ is given by

$$[y, z] = y(p\bar{z}') - \bar{z}(py'). \tag{6.3}$$

Lemma 6.2.1. *For any y and z in the expression domain of M, we have*

$$\overline{z}My - y\overline{Mz} = [y, z]'. \tag{6.4}$$

Proof. This can be verified by a direct computation. □

Lemma 6.2.2. *For any y, z in the expression domain of M and $\alpha, \beta \in J$, $\alpha < \beta$, we have*

$$\int_\alpha^\beta \{\overline{z}My - y\overline{Mz}\} = [y, z](\beta) - [y, z](\alpha). \tag{6.5}$$

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Proof. This is obtained by integrating (6.4). □

The Lagrange form also exists at the endpoints of the underlying interval J, and the values of this form at the endpoints are of critical importance in the characterization of self-adjoint realizations of the Sturm–Liouville equation (6.4) at a singular endpoint.

Lemma 6.2.3. *For any y, z in $D_{\max}$, both limits*

$$[y,z](b) = \lim_{t\to b^-}[y,z](t), \quad [y,z](a) = \lim_{t\to a^+}[y,z](t) \tag{6.6}$$

exist and are finite.

Proof. Fix $\alpha \in J$ and let $\beta \to b^-$ in (6.5). It follows from the definition of $D_{\max}$ that both integrals on the left of (6.5) have a finite limit at b. Hence the first limit in (6.6) exists and is finite. Letting $\alpha \to a$, we see that the second limit of (6.6) exists and is finite. □

Definition 6.2.1 (The maximal and minimal operators). Let the maximal domain $D_{\max}$ and the expression M be defined by (6.1) and (6.2), and let

$$w \in L_{\mathrm{loc}}(J, \mathbb{R}), \quad w > 0.$$

Define

$$S_{\max} f = w^{-1} M f \quad \text{for } f \in D_{\max}.$$
$$S'_{\min} f = Mf, \quad f \in D_{\max}, \quad f \text{ has compact support in } J.$$

Then $S_{\max}$ is called the maximal operator of (M, w) on J, and $S'_{\min}$ is called the preminimal operator. The minimal operator $S_{\min}$ of (M, w) on J is defined as the closure of $S'_{\min}$. The preminimal operator is closable, and so $S_{\min}$ is well defined as given in the next lemma.

Lemma 6.2.4. *The maximal and minimal domains are dense in the Hilbert space*

$$H = L^2(J, w) = \left\{ f : J \to \mathbb{C}, \int_J |f|^2 w < \infty \right\}.$$

The preminimal operator is closable so that the minimal operator $S_{\min}$ is a closed symmetric densely defined operator, and the operators $S_{\min}$ and $S_{\max}$ are an adjoint pair in the sense that

$$S^*_{\min} = S_{\max} \quad \text{and} \quad S^*_{\max} = S_{\min}. \tag{6.7}$$

Hence any self-adjoint extension of $S_{\min}$ is also a self-adjoint restriction of $S_{\max}$, and conversely.

Proof. See [84, 105]. □

From (6.7) it is clear that any self-adjoint extension S of the minimal operator $S_{\min}$ satisfies

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{6.8}$$

Any operator S satisfying (6.8) can be determined by two-point boundary conditions specified at the endpoints a, b of the interval J. These, however, are vacuous at an LP endpoint. To describe these conditions, it is convenient to take cases depending on the LP/LC classification of the endpoints. Here LC/LP will mean that the left endpoint a is LC and the right endpoint b is LP.

An operator S satisfying (6.8) is called a self-adjoint extension of $S_{\min}$ on J, or a self-adjoint restriction of $S_{\max}$ on J, or simply a self-adjoint realization of the equation $My = \lambda wy$ on J, or a self-adjoint realization of (M, w) on J.

We now review the definition of the spectrum of closed densely defined linear operators in Hilbert spaces; see the book by Weidmann [104] for more detail. Let T be a closed (not necessarily self-adjoint) linear operator with dense domain $D(T)$ on the Hilbert space H. Let I denote the identity operator on H. A number λ in $\mathbb{C}$ is an eigenvalue of T if there exists $u \in H$, $u \neq 0$, such that $Tu = \lambda u$. In this case, $T - \lambda I$ is not one-to-one, and the null space of $T - \lambda I$ is not empty; its dimension is the geometric multiplicity of the eigenvalue λ. Each nonzero element of the null space of $T - \lambda I$ is called an eigenfunction of λ.

If $\lambda \in \mathbb{C}$ is not an eigenvalue of T, then

$$R(T, \lambda) = (T - \lambda I)^{-1}$$

is well defined and is a closed linear operator on H, but its domain may not be all of H. Let

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is 1-1 and onto}\}.$$

The set $\rho(T)$ is called the resolvent set of T. The spectrum $\sigma(T)$ of T is defined by

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

In other words, the spectrum of T consists of all complex numbers that are not in the resolvent set $\rho(T)$. Various parts of the spectrum, absolutely continuous, continuous, discrete, essential, point, and singular continuous, are studied in [104]. In this monograph, we consider only the discrete and essential spectra. The discrete spectrum consists of all isolated eigenvalues of finite geometric multiplicity and is denoted by $\sigma_d(T)$. The rest of the spectrum is called the essential spectrum and is denoted by $\sigma_e(T)$. Since the eigenvalues of the differential operators are isolated, we have that for operators T discussed here,

$$\sigma_e(T) = \sigma(T) \setminus \sigma_d(T).$$

6.3 Summary of spectral properties

In this section, we summarize spectral properties of self-adjoint regular and singular Sturm–Liouville problems with positive weight function. We consider self-adjoint realizations of the equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J \tag{6.1}$$

with the following conditions on the coefficients:

$$J = (a, b), \quad -\infty \le a < b \le \infty, \quad 1/p, q, w \in L_{\text{loc}}(J, \mathbb{R}), \quad w > 0, \tag{6.2}$$

together with self-adjoint boundary conditions.

An eigenvalue is simple if it has exactly one linearly independent eigenfunction; otherwise, it is double. The geometric multiplicity of an eigenvalue is the number of its linearly independent eigenfunctions; the algebraic multiplicity of an eigenvalue is the order of its root as a zero of the characteristic function defined in [113], Section 10.4. The algebraic and geometric multiplicity of the eigenvalues of the operators S satisfying (6.8) are equal. So here we will just speak of the multiplicity of an eigenvalue.

Theorem 6.3.1. *Let* (6.1) *and* (6.2) *hold. (Note that* (6.2) *does not assume that* $p > 0$ *on* J*.) Assume that* $S_{\min} \subset S = S^* \subset S_{\max}$*. Then the spectrum of* S *is real.*

(1) *If* p *changes sign on* J*, then the spectrum of* S *is unbounded above and below. (This holds even if there is no subinterval of* J *on which* p *is negative.)*

(2) *If neither endpoint is LP, then the spectrum of* S *is discrete. It may be bounded below or above, but not both.*

(3) *If* $p > 0$ *on* J *and each endpoint is either regular or LCNO (limit-circle nonoscillatory; see Section* 6.4*), then the spectrum of* S *is discrete and bounded below. Let* $\sigma = \sigma(S)$ *denote its spectrum. Then* $\sigma = \{\lambda_n : n \in \mathbb{N}_0\}$ *can be ordered and indexed to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \tag{6.3}$$

and $\lambda_n \to \infty$ *as* $n \to \infty$*. Equality cannot hold for two consecutive terms since the equation has exactly two linearly independent solutions for each* λ*. The algebraic multiplicity of each eigenvalue is the same as its geometric multiplicity.*

(4) *If* $p > 0$ *on* J*, each endpoint is either regular or LCNO, and the boundary conditions are separated, then all eigenvalues are simple, and strict inequality holds throughout* (6.3)*. Furthermore, if* u_n *is an eigenfunction of* λ_n*, then* u_n *has exactly* n *zeros in the open interval* J *for any* $n \in \mathbb{N}_0$*.*

(5) *If* $p > 0$ *on* J*, each endpoint is either regular or LCNO, and the boundary conditions are nonreal coupled, then all eigenvalues are simple, and strict inequality holds throughout* (6.3)*. If* u_n *is an eigenfunction of* λ_n*, then* u_n *is complex valued and has no zero in the closed interval* $[a, b]$*; the number of zeros of* $\text{Re}(u_n)$ *on the half open*

interval $[a, b)$ *is* 0 *or* 1 *if* $n = 0$ *and* $n-1$, n, *or* $n+1$ *if* $n \geq 1$*; the number of zeros of* $\mathrm{Im}(u_n)$ *on the half open interval* $[a, b)$ *is* 0 *or* 1 *if* $n = 0$ *and* $n-1$, n, *or* $n+1$ *if* $n \geq 1$.

(6) *If* $p > 0$ *on* J, *each endpoint is either regular or LCNO, and the boundary conditions are real coupled, then each eigenvalue may be simple or double. (Note that the eigenvalues are uniquely determined by* (6.3), *but there is some ambiguity in the meaning of an eigenfunction* u_n *if* λ_n *is a double eigenvalue. Since linearly independent solutions of* (6.1) *may not have the same number of zeros in* J, *the exact number of zeros of an eigenfunction* u_n *of a double eigenvalue* λ_n *cannot be determined.) If* u_n *is a real-valued eigenfunction of* λ_n, *then the number of zeros of* u_n *on the open interval* J *is* 0 *or* 1 *if* $n = 0$ *and* $n-1$, n, *or* $n+1$ *if* $n \geq 1$.

(7) *If* $p > 0$ *on* J *and (at least) one endpoint is LCO (limit-circle oscillatory), then the spectrum* S *is unbounded above and below. If* λ *is an eigenvalue and* u *is an eigenfunction of* λ, *then* u *has an infinite number of zeros in* J.

(8) *If* $p > 0$ *on* J *and (at least) one endpoint is LP, then* $\sigma_e(S) = \sigma_e(S_{\min})$ *for every self-adjoint realization* S *of equation* (6.1). *In particular, the essential spectrum does not depend on the boundary conditions and therefore depends only on the coefficients* p, q, w *(more precisely, on* $1/p, q, w$*). The discrete spectrum* $\sigma_d(S)$ *depends on the boundary conditions. Either one of* $\sigma_e(S)$, $\sigma_d(S)$, *but not both, may be empty. Let* $\sigma_0 = \inf \sigma_e(S_{\min})$. *There are three possibilities for* σ_0*:*

  (i) $\sigma_0 = -\infty$. *In this case,* σ_e *may be the whole line, or it may consist of disjoint closed intervals separated by "gaps". The number of gaps may be finite or infinite. If there is an eigenvalue, then its eigenfunctions have an infinite number of zeros in* J.

  (ii) $\sigma_0 = +\infty$. *In this case the spectrum of every self-adjoint realization of* (6.1) *is discrete and unbounded above. Either the spectrum of every self-adjoint realization is bounded below (but there is no uniform lower bound for all self-adjoint realizations), or the spectrum of no self-adjoint realization is bounded below. If the spectrum is unbounded below, then every eigenfunction has an infinite number of zeros in* J. *If* S *is a self-adjoint realization and its spectrum* $\sigma(S)$ *is bounded below, then* $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$*; the eigenvalues* λ_n *are all simple and can be ordered to satisfy*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \tag{6.4}$$

  and $\lambda_n \to \infty$ *as* $n \to \infty$. *In this case, if* u_n *is an eigenfunction of* λ_n, *then* u_n *has exactly* n *zeros in* J.

  (iii) $-\infty < \sigma_0 < \infty$. *In this case,* σ_0 *is also the oscillation number of equation* (6.1)*: For* $\lambda < \sigma_0$, *each nontrivial solution of* (6.1) *has either no zero or a finite number of zeros in* J. *If* $\lambda > \sigma_0$, *then every solution of* (6.1) *has an infinite number of zeros in* J. *If* $\lambda = \sigma_0$, *then both cases can occur in general:* (i) *every solution has an infinite number of zeros, or* (ii) *no nontrivial solution has an infinite number of zeros; but if the coefficients are h-periodic, then* $\sigma_0 = \lambda_0^P$, *the first* *periodic eigenvalue on the interval* $[0, h]$, λ_0^P *is simple, and its eigenfunction is oscillatory. There may be no eigenvalues below* σ_0, *a finite number of such eigenvalues, or an infinite number. All eigenvalues below* σ_0 *are simple. If there are an infinite number of eigenvalues below* σ_0, *they must accumulate at* σ_0. *If* λ_n *are the finite or infinite number of eigenvalues below* σ_0, *ordered as in* (6.4), *and* u_n *is an eigenfunction of* λ_n, *then* u_n *has exactly n zeros in* J.

(9) *Let* $J = (-\infty, \infty)$ *and* $p > 0$ *on* J. *Assume that each of* p, q, w *is h-periodic (*$0 < h < \infty$*, and h is the fundamental period). Then there is a unique self-adjoint realization S of* (6.1), $S = S_{\min} = S_{\max}$; *S has no eigenvalues, and*

$$\sigma(S) = \sigma_e(S) = \bigcup_{n=0}^{\infty} J_n,$$
$$J_0 = [\lambda_0^P, \lambda_0^S], \quad J_1 = [\lambda_1^S, \lambda_1^P], \quad J_2 = [\lambda_2^P, \lambda_2^S], \quad J_3 = [\lambda_3^S, \lambda_3^P], \quad J_4 = [\lambda_4^P, \lambda_4^S], \quad \ldots .$$

Here λ_n^P *and* λ_n^S *denote the periodic and semiperiodic eigenvalues of* (6.1) *on the interval* $J = [0, h]$, *respectively. In particular,* λ_0^P *is simple, and* $\sigma_0 = \inf \sigma_e(S_{\min}) = \lambda_0^P$. *The gaps of the spectrum consist of the open intervals* $(\lambda_0^S, \lambda_1^S)$, $(\lambda_1^P, \lambda_2^P)$, $(\lambda_2^S, \lambda_3^S)$, $(\lambda_3^P, \lambda_4^P)$, ... *If* λ_0^S *is a double eigenvalue, then the "first gap is missing"; if* λ_1^P *is a double eigenvalue, then the "second gap is missing"; and so on. (By Theorem* 2.2.1 λ_0^P *is simple.) (The open interval* $(-\infty, \lambda_0^P)$ *is also considered a gap by some authors.) If all gaps are missing, then* $\sigma_e(S) = [\lambda_0^P, \infty)$. *There may be no gaps, a finite number of gaps or an infinite number of gaps. The closed intervals* J_n *are called the spectral bands of S.*

(10) *Suppose* $J = (a, \infty)$, $-\infty < a < \infty$, *and* $p > 0$ *on* J. *Assume that each of* p, q, w *is h-periodic with fundamental interval* $[a, a + h]$. *Then the endpoint a is regular, and* ∞ *is LP. In this case, there are an infinite number of self-adjoint realization of equation* (6.1), *each determined by a boundary condition at a:*

$$\cos(\alpha)y(a) + \sin(\alpha)(py')(a) = 0, \quad \alpha \in [0, \pi);$$

for any $\alpha \in [0, \pi)$, *let* $S(\alpha)$ *be this operator. Then*

$$\sigma(S(\alpha)) = \sigma_e(S(\alpha)) = \bigcup_{n=0}^{\infty} J_n,$$
$$J_0 = [\lambda_0^P, \lambda_0^S], \quad J_1 = [\lambda_1^S, \lambda_1^P], \quad J_2 = [\lambda_2^P, \lambda_2^S], \quad J_3 = [\lambda_3^S, \lambda_3^P], \quad J_4 = [\lambda_4^P, \lambda_4^S], \quad \ldots,$$

where λ_n^P *and* λ_n^S *denote the periodic and semiperiodic eigenvalues of* (6.1) *on the interval* $J = [a, a + h]$, *respectively.*

Proof. See [113] Section 2.10 and below. □

6.4 The LPNO case

For the rest of this chapter, we assume that

$$My = -(py')' + qy = \lambda wy \quad \text{on } J \tag{6.1}$$

with the following conditions on the coefficients:

$$J = (a,b), \quad -\infty \le a < b \le \infty, \quad 1/p, q, w \in L_{\text{loc}}(J, \mathbb{R}), \quad w > 0. \tag{6.2}$$

In this section, we study the existence and continuity of the eigenvalues for the case where one endpoint is regular or LCNO and the other is LPNO. Although the LCNO case reduces to the regular case in a natural way, we nevertheless consider the regular case separately in view of the wide interest in it.

Recall the following basic definitions for (6.1) with coefficients satisfying (6.2).

Definition 6.4.1. The endpoint a is:
(1) regular if, in addition to (6.2), $\frac{1}{p}, q, w \in L(a,c)$ for some (and hence any) $c \in (a,b)$;
(2) singular if it is not regular;
(3) limit-circle (LC) if it is singular and all solutions of equation (6.1) are in $L^2((a,c),w)$ for some $c \in (a,b)$;
(4) limit-point (LP) if it is singular and not LC;
(5) oscillatory (O) if there is a nontrivial real-valued solution of equation (6.1) with an infinite number of zeros in any right neighborhood of a.
(6) nonoscillatory (NO) if it is not oscillatory (this depends on λ);
(7) limit-circle nonoscillatory (LCNO) if it is both LC and NO;
(8) limit-point Nonoscillatory (LPNO) if it is LP and NO for every $\lambda \in \mathbb{R}$.
Similar definitions are made at b.

All these classifications are independent of $c \in (a,b)$. The LC and LP classifications are independent of $\lambda \in \mathbb{R}$. If an endpoint is LC, then the O and NO classifications at that endpoint are also independent of $\lambda \in \mathbb{R}$, but if an endpoint is LP, then, in general, the O and NO classifications depend on λ as the Fourier equation $-y'' = \lambda y$ on $(0,\infty)$ illustrates. Thus the assumption "for every $\lambda \in \mathbb{R}$" in definition (8) is important. This dependence will play an important role here. Throughout this section, we assume that the endpoint a is regular or LCNO and the endpoint b is LP. When the endpoint b is regular or LCNO and a is LP, similar results can be obtained.

Consider the boundary conditions

$$\cos\alpha y(a) - \sin\alpha (py')(a) = 0, \quad \alpha \in [0,\pi). \tag{6.3}$$

It is well known [113] that if b is LP and a is regular, then the boundary conditions (6.3) generate all self-adjoint realizations of equation (6.1) in the Hilbert space $H = L^2(J,w)$. The spectrum of any such realization S consists, in general, of eigenvalues and essential spectrum: $\sigma(S) = \sigma_p(S) \cup \sigma_e(S)$, where $\sigma_p(S)$ is the set of eigenvalues, and $\sigma_e(S)$ is the essential spectrum. Either one, but not both, of these two sets may be empty. The essential spectrum of any self-adjoint realization does not depend on the boundary condition, that is, on α, but the eigenvalues depend on α, and it is this dependence we study in the next section. All eigenvalues have multiplicity 1 or 2.

Our starting point is the following:

Lemma 6.4.1. *Let* (6.1) *and* (6.2) *hold. Assume that a is regular and b is LP. Then the spectrum of every self-adjoint realization S is discrete and bounded below if and only if the endpoint b is LPNO.*

Proof. See Hinton and Lewis [49]. □

We can now state one of our main results.

Theorem 6.4.1. *Let* (6.1) *and* (6.2) *hold and assume that the endpoint a is regular and the endpoint b is LPNO. Then:*

(1) *The spectrum $\sigma(S(\alpha))$ of any self-adjoint realization $S(\alpha)$ is discrete, bounded below, and not bounded above. Let*

$$\sigma(S(\alpha)) = \{\lambda_n(\alpha) : n \in \mathbb{N}_0\}.$$

(2) *The eigenvalues are all simple and can be ordered to satisfy*

$$\lambda_0(\alpha) < \lambda_1(\alpha) < \lambda_2(\alpha) < \lambda_3(\alpha) < \cdots$$

(3) *For any $n \in \mathbb{N}_0$ and $\alpha \in (0, \pi)$, we have $\lambda_n(\alpha) < \lambda_n(0)$.*
(4) *For any $n \in \mathbb{N}_0$ and $0 < \alpha_1 < \alpha_2 < \pi$, we have $\lambda_n(\alpha_2) < \lambda_n(\alpha_1)$.*
(5) *For any $n \in \mathbb{N}_0$ and $\alpha, \beta \in [0, \pi)$, we have $\lambda_n(\alpha) < \lambda_{n+1}(\beta)$.*
(6) *For any $n \in \mathbb{N}_0$, $\lambda_n(\alpha)$ is a continuous function of $\alpha \in [0, \pi)$.*
(7) *We have $\lambda_0(\alpha) \to -\infty$ as $\alpha \to \pi^-$.*
(8) *For any $n \in \mathbb{N}$, $\lambda_n(\alpha) \to \lambda_{n-1}(0)$ as $\alpha \to \pi^-$.*

Proof. This will be given in Section 6.6. □

We illustrate an application of Theorem 6.4.1 with the following:

Example 6.4.1. Let $S(\alpha)$ denote the self-adjoint realization of the problem

$$\begin{aligned} &-y'' = \lambda w y \quad \text{on } (0, \infty), \quad w \in L_{\mathrm{loc}}([0, \infty), \mathbb{R}), \quad w > 0, \\ &\cos \alpha y(a) - \sin \alpha y'(a) = 0, \quad \alpha \in [0, \pi). \end{aligned}$$

Assume that w is monotone and satisfies

$$\int_0^\infty \sqrt{w}\, dt < \infty, \quad \lim_{t\to\infty} t \int_t^\infty w(s)\, ds = 0.$$

Then the spectrum $\sigma(S(\alpha)) = \{\lambda_n(\alpha) : n \in \mathbb{N}_0\}$ is discrete and bounded below and satisfies

$$\lim_{n\to\infty} \frac{n}{\lambda_n(\alpha)^{\frac{1}{2}}} = \frac{1}{\pi} \int_0^\infty \sqrt{w}\, dt.$$

Proof. The fact that the spectrum is discrete and bounded below was established by Glazman [43]. For the case $\alpha = 0$, this asymptotic formula was established by Birman and Borzov [16] and Naimark and Solomyak [83]. Then the general case for $\alpha \in [0, \pi)$ follows from Theorem 6.4.1. □

Next, we give a result for normalized eigenfunctions. Here by a normalized eigenfunction we mean a real normalized eigenfunction. In the case of limit-point nonoscillation, all the eigenvalues are simple, and the real normalized eigenfunctions are unique up to sign. We have the following:

Theorem 6.4.2. *Let the hypotheses and notation of Theorem* 6.4.1 *hold. Let* $n \in \mathbb{N}_0$ *and* $\alpha_0 \in [0, \pi)$, *and let* $u_n(\cdot, \alpha_0)$ *denote a normalized eigenfunction of the eigenvalue* $\lambda_n(\alpha_0)$. *Then there exists a normalized eigenfunction* $u_n(\cdot, \alpha)$ *of* $\lambda_n(\alpha)$ *such that*

$$u_n(\cdot, \alpha) \to u_n(\cdot, \alpha_0), \quad (pu_n')(\cdot, \alpha) \to (pu_n')(\cdot, \alpha_0), \quad \alpha \to \alpha_0, \tag{6.4}$$

both uniformly on any compact subinterval of $[a, b)$.

Proof. This proof is similar to that of Theorem 3.2(i) in [67]. □

Next, we show that the eigenvalues $\lambda_n(\alpha)$ are differentiable functions of α and find a formula for the derivative.

Theorem 6.4.3. *Let the hypotheses and notation of Theorem* 6.4.1 *hold. Let* $n \in \mathbb{N}_0$, *and let* $u_n(\cdot, \alpha)$ *be a normalized eigenfunction* $u_n(\cdot, \alpha)$ *of* $\lambda_n(\alpha)$. *Then* λ_n *is continuously differentiable with respect to α, and the derivative is given by*

$$\lambda_n'(\alpha) = -u_n^2(a, \alpha) - (pu_n')^2(a, \alpha), \quad 0 \le \alpha < \pi. \tag{6.5}$$

Proof. See Section 6.6. □

Next, we study the case where the endpoint a is singular and LCNO and b is LPNO. In this case the boundary condition (6.3) does not make sense because, in general, $y(a)$ and $(py')(a)$ are not defined for solutions y of equation (6.1).

To construct all self-adjoint boundary conditions for this case, we use the Lagrange bracket $[\cdot, \cdot]$ defined by (6.3)

$$[y, z] = y(p\bar{z}') - z(p\bar{y}'), \quad y, z \in D_{\max}(a, b),$$

where

$$D_{\max}(a,b) = \left\{ f \in L^2(J,w) : f, (pf') \in AC_{\text{loc}}(a,b), \frac{1}{w}[-(pf')' + qf] \in L^2(J,w) \right\}, \quad (6.6)$$

and note that $[y,z](a)$ exists as a finite limit for all $y, z \in D_{\max}(a,b)$ even though the individual terms $y, z, (p\overline{y}'), (p\overline{z}')$ may blow up or oscillate wildly at a.

Definition 6.4.2. Let u, v be real solutions of (6.1). Then u is called a principal solution at the endpoint a if

(1) $u(x) \neq 0$ for all $x \in (a,c]$ and some $c \in (a,b)$;

(2) every solution y of (6.1) that is not a multiple of u satisfies $\frac{u(x)}{y(x)} \to 0$ as $x \to a^+$;
v is called a nonprincipal solution at the endpoint a if $v(x) \neq 0$ for al $x \in (a,c]$ and some $c \in (a,b)$ and $v(x)$ is not a principal solution.

Definition 6.4.3. Two functions u, v are called a boundary condition basis at a if they satisfy the following five conditions:

(1) $u, v \in D_{\max}(a,b)$ and are real-valued on (a,b).

(2) u is a principal solution at a for some fixed $\lambda = r_0 \in \mathbb{R}$.

(3) v is a nonprincipal solution at a for $\lambda = r_0 \in \mathbb{R}$.

(4) $[u,v](a) = 1$.

(5) $v > 0$ on (a,b).

Remark 6.4.1. Such functions u, v in these two definitions exist [86]; see also [113], Theorem 6.2.1. Note that u, v are solutions in a right neighborhood of a but need not be solutions on the entire interval (a,b). The principal solution u is unique up to constant multiples; the nonprincipal solution v is not unique.

We now define the self-adjoint boundary conditions for the LCNO/LPNO case. Let u, v be a boundary condition basis at a and consider the boundary conditions:

$$\cos\alpha[y,u](a) - \sin\alpha[y,v](a) = 0, \quad \alpha \in [0,\pi). \quad (6.7)$$

In the LCNO/LPNO case, it is known [113] that the boundary conditions (6.7) determine all self-adjoint realizations S of equation (6.1) in the space $L^2(J,w)$ that satisfy $S_{\min} \subset S = S^* \subset S_{\max}$.

Theorem 6.4.4. *Assume that the endpoint a is LCNO and the endpoint b is LPNO. Let the hypotheses and notation of Theorem* 6.4.1 *hold. Then all eight parts of the conclusion of Theorem* 6.4.1 *hold.*

Proof. This will be given in Section 6.6. □

As in the case where a is regular, the eigenvalues are differentiable functions of α. This is the next result.

Theorem 6.4.5. *Let the hypotheses and notation of Lemma* 6.2.4 *hold, and let* $y = y_n(\cdot, \alpha)$ *be a normalized eigenfunction of* $\lambda_n(\alpha)$. *Then* $\lambda_n'(\alpha)$ *exists and is given by*

$$\lambda_n'(\alpha) = -[y, u]^2(a) - [y, v]^2(a), \quad 0 \le \alpha < \pi. \tag{6.8}$$

Proof. To be given in Section 6.6. □

6.5 The general LP case

In this section, we investigate the case where the endpoint b is LP but not necessarily LPNO. In this case the spectrum may be very complicated. In particular, it may be discrete, consist entirely of essential spectrum, or contain both eigenvalues and essential spectrum. Let

$$\sigma_0 = \inf \sigma_e.$$

There are three possibilities for σ_0: (i) $\sigma_0 = -\infty$; this means that if there are any eigenvalues, they are either in gaps of the essential spectrum σ_e or embedded in it; (ii) $\sigma_0 = +\infty$; this means that there is no essential spectrum, so the spectrum is discrete, but it may or may not be bounded below; if it is bounded below, then we are back to the case discussed in Section 6.4; (iii)

$$-\infty < \sigma_0 < \infty. \tag{6.1}$$

Case (iii) is our main focus in this section. We search for answers to the questions: Are there eigenvalues below σ_0? If so, how many? There may be no eigenvalues below σ_0, a finite number of them, or an infinite number of them. See Chapter 14 in [113] for examples of all three types.

Some results from Section 6.4 can be extended to eigenvalues – if they exist – below σ_0. We will see that the existence and number of eigenvalues below σ_0 depends on α.

Theorem 6.5.1. *Let the hypotheses and notation of Theorem* 6.4.1 *hold. Assume that the endpoint a is regular, the endpoint b is LP, the spectrum is bounded below, and* (6.1) *holds. Denote the eigenvalues below* σ_0, *if they exist, by* $\lambda_k(\alpha)$, $k = 0, 1, 2, 3, \ldots$, *and index them in increasing order:*

$$\lambda_0(\alpha) < \lambda_1(\alpha) < \lambda_2(\alpha) < \lambda_3(\alpha) < \cdots.$$

Then we have:
1. *There exist* $k \in \mathbb{N}_0$ *and* $\alpha_0 \in [0, \pi)$ *such that* $\lambda_k(\alpha_0) < \sigma_0$.
2. *If* $\lambda_k(\alpha_0)$ *exists for some* $\alpha_0 \in [0, \pi)$ *and* $k > 0$, *then* $\lambda_k(\alpha)$ *exists for any* $\alpha \in [\alpha_0, \pi)$.
3. *If* $\lambda_k(\alpha_0)$ *exists for some* $\alpha_0 \in [0, \pi)$ *and* $k > 0$, *then* $\lambda_0(\alpha), \ldots, \lambda_{k-1}(\alpha)$ *exist for all* $\alpha \in [0, \pi)$.

(4) *If there exist a finite number of eigenvalues below* σ_0 *for some* $\alpha \in [0, \pi)$*, then there exist* $k \in \mathbb{N}_0$ *and* $\alpha_0 \in [0, \pi)$ *such that there are exactly* $k + 1$ *eigenvalues below* σ_0 *for any* $\alpha \in (\alpha_0, \pi)$ *and there are exactly k eigenvalues below* σ_0 *for any* $\alpha \in [0, \alpha_0]$*.*
(5) *If there exist an infinite number of eigenvalues below* σ_0 *for some* $\alpha \in [0, \pi)$*, then* $\lambda_n(\alpha)$ *exists for any* $n \in \mathbb{N}_0$ *and* $\alpha \in [0, \pi)$*.*

Proof. To be given in Section 6.7. □

Next, we give the result for the LCNO/LP case. Just as in the LCNO/LPNO case discussed before, we will see that also in this case, the results for *a* regular and *a* LCNO are the same provided, of course, that the regular boundary condition is replaced by the corresponding singular self-adjoint condition.

Theorem 6.5.2. *Let* (6.1), (6.2)*, and* (6.7) *hold. Assume that the endpoint a is LCNO and the endpoint b is LP, the spectrum is bounded below, and* (6.1) *holds. Index the eigenvalues as in Theorem* 6.5.1*. Then all five parts of Theorem* 6.5.1 *hold.*

Proof. This follows from Theorem 6.5.1 and the "regularization" of LCNO endpoints used by Niessen and Zettl in [86]; see also the proof of Theorem 6.4.4. □

Next, we consider some further analogues of the LPNO case.

Theorem 6.5.3. *Let* (6.1) *and* (6.2) *hold. Suppose that either* (i) *the endpoint a is regular, b is limit point, and the boundary condition is given by* (6.3) *or* (ii) *the endpoint a is LCNO, b is limit point, and the boundary condition is given by* (6.7)*.*

Assume that $-\infty < \sigma_0 < \infty$*, the spectrum is bounded below, and there exist a finite number of eigenvalues* $\lambda_i(\alpha) : i = 0, \dots, k$*, below* σ_0 *for some* $\alpha \in [0, \pi)$*. Then:*
(1) *There exist* $k \in \mathbb{N}_0$ *and* $\alpha_0 \in [0, \pi)$ *such that there are exactly* $k + 1$ *eigenvalues* $\{\lambda_i(\alpha) : i = 0, \dots, k,$ *for* $\alpha \in [\alpha_0, \pi)\}$ *below* σ_0 *and there are exactly k eigenvalues* $\{\lambda_i(\alpha) : i = 0, \dots, k - 1,$ *for* $\alpha \in [0, \alpha_0)\}$ *below* σ_0*.*
(2) *For any* $n \in \{0, \dots, k - 1\}$ *and* $\alpha \in (0, \pi)$*, we have* $\lambda_n(\alpha) < \lambda_n(0)$*.*
(3) *For any* $n \in \{0, \dots, k - 1\}$ *and for* $0 < \alpha_1 < \alpha_2 < \pi$*, we have* $\lambda_n(\alpha_2) < \lambda_n(\alpha_1)$*. For* $n = k$ *and* $\alpha_0 < \alpha_1 < \alpha_2 < \pi$*, we have* $\lambda_n(\alpha_2) < \lambda_n(\alpha_1)$*.*
(4) *For any* $n \in \{0, \dots, k-2\}$ *and* $\alpha, \beta \in [0, \pi)$*, we have that* $\lambda_n(\alpha) < \lambda_{n+1}(\beta)$*. For* $n = k-1$ *and any* $\alpha, \beta \in (\alpha_0, \pi)$*, we have* $\lambda_n(\alpha) < \lambda_{n+1}(\beta)$*.*
(5) *For any* $n \in \{0, \dots, k - 1\}$*,* $\lambda_n(\alpha)$ *is continuous on* $\alpha \in [0, \pi)$*. For* $n = k$*,* $\lambda_n(\alpha)$ *is continuous on* $\alpha \in (\alpha_0, \pi)$*.*
(6) *We have* $\lambda_0(\alpha) \to -\infty$ *as* $\alpha \to \pi^-$*.*
(7) *For any* $n \in \{1, 2, \dots, k\}$*,* $\lambda_n(\alpha) \to \lambda_{n-1}(0)$ *as* $\alpha \to \pi^-$*.*
(8) *Let* $n \in \{0, \dots, k - 1\}$*. If a is regular and* $u_n(\cdot, \alpha)$ *is a normalized eigenfunction of* $\lambda_n(\alpha)$*, then* λ_n *is continuously differentiable with respect to* α*, and the derivative is given by*

$$\lambda_n'(\alpha) = -u_n^2(a, \alpha) - (pu_n')^2(a, \alpha), \quad 0 \le \alpha < \pi.$$

(9) *Let $n \in \{0, \dots, k-1\}$. If a is LCNO and $y_n(\cdot, \alpha)$ is a normalized eigenfunction of $\lambda_n(\alpha)$, then λ_n is continuously differentiable with respect to α, and the derivative is given by*

$$\lambda_n'(\alpha) = -[y_n, u]^2(a) - [y_n, v]^2(a).$$

Assume that there exist an infinite number of eigenvalues below σ_0 for some $\alpha \in [0, \pi)$ that are bounded below. (They may converge to σ_0 but not to $-\infty$.) Then the results of Theorems 6.4.1, 6.4.3, 6.4.4, *and* 6.4.5 *hold except for part (1) of Theorems* 6.4.1 *and* 6.4.4.

Proof. This proof is similar to that of Theorems 6.4.1, 6.4.3, 6.4.4, and 6.4.5 and hence omitted. □

The Fourier equation illustrates some of the results of Theorems 6.4.1, 6.4.3, and 6.4.4.

Example 6.5.1. Consider the boundary value problem

$$-y'' = \lambda y \quad \text{on } (0, \infty),$$
$$\cos \alpha y(0) - \sin \alpha y'(0) = 0, \quad \alpha \in [0, \pi).$$

Let $S(\alpha)$ denote the self-adjoint realizations in $L^2((0, \infty), 1)$. Then it is well known that $\sigma_e(S(\alpha)) = [0, \infty)$ and the spectrum is bounded below for each $\alpha \in [0, \pi)$.

Direct computations show that:

(1) If $\alpha \in [0, \frac{\pi}{2}]$, then there are no eigenvalues below $\sigma_0 = 0$.

(2) If $\alpha \in (\frac{\pi}{2}, \pi)$, then there exists one and only one eigenvalue

$$\lambda_0(\alpha) = -\cot^2 \alpha,$$

and the normalized eigenfunction of the eigenvalue $\lambda_0(\alpha)$ is

$$y(x) = \sqrt{-2 \cot \alpha} e^{\cot \alpha x}.$$

Note that $\lambda_0(\alpha)$ is continuously differentiable and decreasing for $\alpha \in (\frac{\pi}{2}, \pi)$.

6.6 Proofs of theorems in Section 6.4

In this section, we give the proofs of Theorems 6.4.1, 6.4.3, 6.4.4, and 6.4.5. For this, we need some definitions and lemmas.

Recall that the deficiency index d is the number of linearly independent solutions of (6.1) with $\lambda = i$ that lie in $H = L^2((a, b), w)$. It is well known that d is independent of λ for all $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$ and if the endpoint a is regular or LC, then $d = 1$ or $d = 2$, and both values are realized. The minimal deficiency case $d = 1$ is called the limit-point (LP) case, and the maximal deficiency case $d = 2$ is called the limit-circle

(LC) case. For real λ, let $r(\lambda)$ denote the number of linearly independent solutions of (6.1) that lie in $H = L^2((a,b),w)$. The following lemma describes a relation between $r(\lambda)$ and the spectrum.

Lemma 6.6.1. *Suppose that* (6.1) *and* (6.2) *hold and assume that the endpoint a is regular. Let d denote the deficiency index of the minimal operator $S_{\min}$ generated by* (6.1), *and let $r(\lambda)$ be defined as before. Then:*
(1) *For every $\lambda \in \mathbb{R}$, we have $r(\lambda) \le d$.*
(2) *If $r(\lambda) < d$, then λ is in the essential spectrum of every self-adjoint realization.*
(3) *If $r(\lambda) = d$ for some $\lambda \in \mathbb{R}$, then λ is an eigenvalue of geometric multiplicity d for some self-adjoint realization S of* (6.1).

Proof. This can be found in [47, 93]. □

A singular Sturm–Liouville problem with a regular endpoint a can be approximated by a sequence of regular SLPs on truncated intervals (a,b_r), where

$$a < b_r < b,$$

and the sequence $\{b_r : r \in \mathbb{N}\}$ converges increasingly to b. By S and S_r we denote self-adjoint realizations on the interval (a,b) and (a,b_r), respectively. To relate the operators S_r to S, we construct "induced restriction operators" $\{S_r\}$ [9], which are determined by (6.3) and

$$f(b_r) = 0.$$

With each of these operators $\{S_r\}$ in the Hilbert space $L_r^2((a,b_r),w)$, we associate an operator $\{S_r'\}$ in the Hilbert space $H = L^2((a,b),w)$ as follows:

$$S_r' = S_r \dotplus \Theta_r,$$

where Θ_r is the zero operator in the space

$$H_r^\perp = L^2((b_r,b),w),$$

and S_r' is defined by

$$D(S_r') = D(S_r) \dotplus H_r^\perp.$$

Thus S_r and S_r' are self-adjoint operators in the Hilbert spaces $H_r = L^2((a,b_r),w)$ and $H = L^2((a,b),w)$, respectively. Denote the spectrum of the operators S, S_r, and S_r' by $\sigma(S)$, $\sigma(S_r)$, $\sigma(S_r')$ and their eigenvalues by $\lambda_n(\alpha)$, $\lambda_n^r(\alpha)$, and $\lambda_n^{r'}(\alpha)$ for $n \in \mathbb{N}_0$. On the convergence of the induced operators $\{S_r\}$ and of the spectrum, we have the following:

Lemma 6.6.2. *Suppose that* (6.1) *and* (6.2) *hold and the endpoint a is regular. Let $\{S_r\}$ and $\{S_r'\}$ be defined above. In addition, assume that the operator S has spectrum bounded below with $-\infty < \sigma_0 \le +\infty$.*

(1) *If the operator $S(\alpha)$ has exactly k eigenvalues $\lambda_n^r(\alpha)$ below σ_0, then*

$$\lim_{r\to\infty} \lambda_n^r(\alpha) = \lambda_n(\alpha), \quad n = 0, \dots, k-1.$$

(2) *If the operator $S(\alpha)$ has an infinite number of eigenvalues $\lambda_n^r(\alpha)$ below σ_0, then*

$$\lim_{r\to\infty} \lambda_n^r(\alpha) = \lambda_n(\alpha), \quad n \in \mathbb{N}_0.$$

In particular, if $\sigma_0 = +\infty$, then the spectrum is discrete. In this case, the above limits hold for each $n \in \mathbb{N}_0$.

Proof. This is given in [9] and [31]. □

Remark 6.6.1. It follows from Lemmas 6.6.1 and 6.6.2 that the nth eigenvalue of problem (6.1)–(6.3) with b in the LPNO case can be approximated by the nth eigenvalue of the inherited operators $\{S_r\}$.

For regular Sturm–Liouville operators $\{S_r\}$, we have the following lemma, which is important in our proof.

Lemma 6.6.3. *For any $n \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $\alpha \in (0, \pi)$, we have*
(1) $\lambda_n^r(\alpha) < \lambda_n^r(0)$;
(2) $\lambda_0^r(\alpha) \to -\infty$ *as* $\alpha \to \pi^-$;
(3) $\lambda_{n+1}^r(\alpha) \to \lambda_n^r(0)$ *as* $\alpha \to \pi^-$;
(4) *if* $0 < \alpha_1 < \alpha_2 < \pi$, *then* $\lambda_n^r(\alpha_2) < \lambda_n^r(\alpha_1)$.

Proof. See Lemma 3.32 in [60]. □

Next, we prove Theorem 6.4.1.

Proof. Parts (1) and (2) are direct consequences of Lemma 6.4.1. For the regular operator S_r, by Lemma 6.6.3 we have that

$$\lambda_n^r(\alpha) < \lambda_n^r(0), \quad \alpha \in (0, \pi).$$

It follows from Lemma 6.6.2(2) that

$$\lambda_n(\alpha) \leq \lambda_n(0).$$

Since all the eigenvalues are simple, (3) follows. The argument for (4) is similar, since $\lambda_n^r(\alpha)$ is a decreasing function of $\alpha \in [0, \pi)$ by Lemma 6.6.3(4). It follows from Lemma 6.6.3(3) that

$$\lambda_{n+1}^r(\alpha) \to \lambda_n^r(0) \quad \text{as } \alpha \to \pi^-.$$

Again by Lemma 6.6.3(4), we have $\lambda_n^r(\alpha) < \lambda_{n+1}^r(\beta)$ for any $\alpha, \beta \in [0, \pi)$. Similarly to the proof of (3), part (5) is verified.

It follows from Lemma 6.6.1 and 6.6.2(2) that LPNO implies that $r(\lambda) = 1$ for any real number $\lambda \in \mathbb{R}$. From Lemma 6.6.3(3) it follows that for any $\lambda \in \mathbb{R}$, λ is an eigenvalue of geometric multiplicity 1 for some self-adjoint realization S of (6.1). Thus for any real number $\lambda \in \mathbb{R}$, there must exist a unique $\alpha \in [0, \pi)$, that is, a unique self-adjoint boundary condition such that λ is an eigenvalue of this self-adjoint Sturm–Liouville problem. Conversely, for any boundary condition parameter $\alpha \in [0, \pi)$, there must exist an infinite but countable number of eigenvalues for SLP (6.1)–(6.3) with LPNO endpoint b. It follows from the monotonicity of the eigenvalues $\lambda_n(\alpha)$ on $\alpha_0 \in (0, \pi)$ that there exist the limits $\lim_{\alpha \to \alpha_0^-} \lambda_n(\alpha)$ and $\lim_{\alpha \to \alpha_0^+} \lambda_n(\alpha)$; moreover,

$$\lim_{\alpha \to \alpha_0^-} \lambda_n(\alpha) \geq \lambda_n(\alpha_0) \geq \lim_{\alpha \to \alpha_0^+} \lambda_n(\alpha).$$

Assume that

$$M = \lim_{\alpha \to \alpha_0^-} \lambda_n(\alpha) > \lambda_n(\alpha_0)$$

and $\tilde{\lambda} \in (\lambda_n(\alpha_0), M)$. Then $\tilde{\lambda}$ cannot be an eigenvalue of some self-adjoint problem by conclusions (3), (4), and (5), which is a contradiction to Lemma 6.4.1(3). Therefore

$$\lim_{\alpha \to \alpha_0^-} \lambda_n(\alpha) = \lambda_n(\alpha_0).$$

Similarly,

$$\lim_{\alpha \to \alpha_0^+} \lambda_n(\alpha) = \lambda_n(\alpha_0).$$

The continuity of $\lambda_n(\alpha)$ on $\alpha \in (0, \pi)$ is proved. The rest of (6) and the argument for (7) and (8) are similar to this. □

Next, we review a technical lemma to be used further.

Lemma 6.6.4.
(1) $D_{\min} = \{y \in D_{\max} : [y, z](b) - [y, z](a) = 0$ *for all* $z \in D_{\max}\}$.
(2) *The endpoint b is LP if and only if* $[y, z](b) = 0$ *for all* $y, z \in D_{\max}$.

Proof. See [84]. □

Next, we give a proof of Theorem 6.4.3.

Proof. Let $u = u_n(\cdot, \alpha)$ be normalized eigenfunctions of $\lambda_n(\alpha)$. Then it follows from Theorem 6.4.2 that there exist normalized eigenfunctions $v = u_n(\cdot, \alpha + h)$ of $\lambda_n(\alpha + h)$ such that $v(\cdot) \to u(\cdot)$, $(pv')(\cdot) \to (pu')(\cdot)$, $h \to 0$, uniformly on any compact subinterval of $[a, b)$. Since b is LP, it follows from Lemma 6.6.4(2) that $[u, v](b) = 0$.

If $\alpha \neq \frac{\pi}{2}$, then we have

$$\begin{aligned}[\lambda_n(\alpha+h)-\lambda_n(\alpha)]\int_a^b u\bar{v}w &= [u,v](a)-[u,v](b)=[u,v](a)\\ &= [u(p\bar{v}')-v(p\bar{u}')](a)\\ &= (\tan\alpha-\tan(\alpha+h))(p\bar{u}')(a)(pv')(a).\end{aligned}$$

Dividing both sides by h and taking the limit as $h \to 0$, we obtain

$$\lambda_n'(\alpha)\int_a^b u\bar{u}w = -\sec^2\alpha(pu')^2(a) = -(1+\tan^2\alpha)(pu')^2(a).$$

From the boundary conditions (6.3) we have $\tan\alpha(pu')(a) = u(a)$. Thus the conclusion is obtained.

If $\alpha = \frac{\pi}{2}$, then $(pu')(a) = 0$. Thus, similarly to the above computation, we have

$$\begin{aligned}[\lambda_n(\alpha+h)-\lambda_n(\alpha)]\int_a^b u\bar{v}w &= [u,v](a)-[u,v](b)=[u,v](a)\\ &= [u(p\bar{v}')-v(p\bar{u}')](a)\\ &= u(a)(p\bar{v}')(a)\\ &= u(a)\cot(\alpha+h)v(a).\end{aligned}$$

Dividing both sides by h and taking the limit as $h \to 0$, we obtain

$$\lambda_n'(\alpha)\int_a^b u\bar{u}w = -u^2(a) = -u^2(a)-(pu')^2(a).$$

This completes the proof. □

Next, we prove Theorem 6.4.4.

Proof. We use the regularization method for singular SLP; see [86, 113] for more detail. Define

$$P = v^2p, \quad Q = v[-(pv')'^2w$$

and consider the equation

$$-(Pz')' + Qz = \lambda Wz. \tag{6.1}$$

If z is a solution, then $y = zv$ is a solution of (6.1). Since the endpoint a is LCNO and b is LPNO, this equation is regular at a and is LPNO at b. Moreover, the boundary condition (6.3) can be transformed into the following:

$$\cos\alpha z(a) - \sin\alpha(Pz')(a) = 0. \tag{6.2}$$

Thus this boundary value problem and problem (6.1), (6.7) have the same eigenvalues. The conclusion follows from Theorem 6.4.2. □

Lemma 6.6.5. *Let* $y, z, u, v \in D_{\max}$. *For any* $c \in [a, b]$, *we have*

$$[y, z](c)[v, u](c) = [y, v](c)[\bar{z}, \bar{u}](c) - [y, \bar{u}](c)[\bar{z}, v](c).$$

Proof. See [113]. □

In the following, we give the proof of Theorem 6.4.5.

Proof. Let $y = y_n(\cdot, \alpha)$ denote a normalized eigenfunction of the eigenvalue $\lambda_n(\alpha)$. Singular initial value problems at the LC endpoint a for equation (6.1) and the initial conditions

$$[y, u](a) = c_1, \quad [y, v](a) = c_2$$

with constants c_1, c_2 can be found in [113], Chapter 8. Similarly to Theorem 6.4.2, there exists a normalized eigenfunction $\tilde{y} = y_n(\cdot, \alpha + h)$ of $\lambda_n(\alpha + h)$ such that

$$\tilde{y}(\cdot) \to y(\cdot), \quad (p\tilde{y}')(\cdot) \to (py)'(\cdot), \quad h \to 0,$$

and

$$[\tilde{y}, u](\cdot) \to [y, u](\cdot), \quad [\tilde{y}, v](\cdot) \to [y, v](\cdot), \quad h \to 0,$$

uniformly on any compact subinterval of $[a, b)$.

By a direct computation we obtain that if $\alpha \neq \frac{\pi}{2}$, then

$$\begin{aligned}
[\lambda_n(\alpha + h) - \lambda_n(\alpha)] \int_a^b y\bar{\tilde{y}}w &= [y, \tilde{y}](a) - [y, \tilde{y}](b) = [y, \tilde{y}](a) \\
&= [y, \tilde{y}](a)[u, v](a) \\
&= [y, u](a)[\tilde{y}, v](a) - [y, v](a)[\tilde{y}, u](a) \\
&= \big(\tan\alpha - \tan(\alpha + h)\big)[y, v](a)[\tilde{y}, v](a).
\end{aligned}$$

Similarly to the proof of Theorem 6.4.3, the conclusion follows. The proof for $\alpha = \frac{\pi}{2}$ is similar. □

6.7 Proofs of theorems in Section 6.5

In this section, we prove the theorems of Section 6.5. First, we introduce a technical lemma.

Lemma 6.7.1. *Assume that* (6.1) *and* (6.2) *hold. Then:*

(1) $\dim E(\lambda^-) = N(\lambda)$, *where* $N(\lambda)$ *is the number of zeros of the solution of equation* (6.1) *on* (a, b) *satisfying the following initial value condition*

$$y(a) = \sin\alpha, \quad (py')(a) = \cos\alpha,$$

where $E(\lambda)$ *is the spectral measure of the self-adjoint Sturm–Liouville operator* S;

(2) *Let S be bounded below with separated boundary conditions and $\sigma_e \subset [\sigma_0, \infty)$. Then for the eigenvalues*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots < \sigma_0,$$

the corresponding eigenfunction y_n has exactly n zeros on the interval (a, b);

(3) *Let $M(\lambda)$ be the minimal number of zeros of a solution of* (6.1) *in (a, b). Then*

$$\dim E(\lambda^-) - 1 \le M(\lambda) \le \dim E(\lambda^-).$$

Proof. Parts (1) and (3) are direct consequences of Theorems 14.2(2) and 14.8(2) in Weidmann [105], respectively, and (2) follows from Theorem 14.10 in [105]. □

This lemma gives a relation between the oscillation of solutions of equation (6.1), the spectral measure, and the essential spectrum.

Next, we prove Theorem 6.5.1.

Proof. (1) Assume that $-\infty < \sigma_0 < \infty$. It follows from Lemma 6.4.1 that for any $\lambda < \sigma_0$, equation (6.1) has exactly one real linearly independent square-integrable solution $y(x)$. Therefore there exists a unique $\alpha \in [0, \pi)$ such that $y(x)$ satisfies the self-adjoint boundary condition (6.3) and λ is an eigenvalue of (6.1)–(6.3). Thus there exist α_0 and k and an eigenvalue $\lambda_k(\alpha_0)$ such that $\lambda_k(\alpha_0) < \sigma_0$.

(2) Let $\alpha \in (\alpha_0, \pi)$. If there exists $\lambda_k(\alpha_0)$ for some $k \in \mathbb{N}_0$ and $\alpha_0 \in [0, \pi)$, then the solution $y(x, \alpha_0, \lambda_{n_0}(\alpha_0))$ of the initial value problem

$$\begin{aligned} -(py')' + qy &= \lambda_k(\alpha_0) wy, \\ y(a) &= \sin\alpha_0, \\ (py')(a) &= \cos\alpha_0 \end{aligned}$$

has exactly k zeros in the interval (a, b) by Lemma 7 in [114]. Note that

$$\dim E(\lambda_k(\alpha_0), \alpha_0) = k + 1.$$

Then by Lemma 6.7.1(1), for any sufficiently small $\varepsilon > 0$, the solution $y(x, \alpha_0, \lambda_k(\alpha_0) + \varepsilon)$ of the initial value problem

$$\begin{aligned} -(py')' + qy &= (\lambda_k(\alpha_0) + \varepsilon) wy, \\ y(a) &= \sin\alpha_0, \\ (py')(a) &= \cos\alpha_0 \end{aligned}$$

has at least $k + 1$ zeros in (a, b). Let x_{k+1} be the $(k + 1)$th zero of $y(x, \alpha_0, \lambda_k(\alpha_0) + \varepsilon)$. Let $y(x, \alpha, \lambda_k(\alpha_0) + \varepsilon)$ be the solution of the following problem:

$$-(py')' + qy = (\lambda_k(\alpha_0) + \varepsilon) wy,$$

$$y(a) = \sin\alpha,$$
$$(py')(a) = \cos\alpha.$$

From the Prüfer transformation

$$y(x,\alpha,\lambda_k(\alpha_0)+\varepsilon) = \rho(x)\sin\theta(x,\alpha),$$
$$(py')(x,\alpha,\lambda_k(\alpha_0)+\varepsilon) = \rho(x)\cos\theta(x,\alpha),$$

and

$$0 \le \theta(a,\alpha_0) = \alpha_0 < \theta(a,\alpha) = \alpha < \pi$$

it follows that

$$\theta(x_{k+1},\alpha_0) = (k+1)\pi < \theta(x_{k+1},\alpha).$$

Therefore $y(x,\alpha,\lambda_k(\alpha_0)+\varepsilon)$ has at least $k+1$ zeros in (a,x_{k+1}) for any $\pi > \alpha > \alpha_0$. It follows from Lemma 6.7.1(1) that

$$\dim E(\lambda_k(\alpha_0)+\varepsilon,\alpha) \ge k+1.$$

Since the spectral measure $E(\lambda)$ is right continuous and ε is an arbitrary small positive number, we have

$$\dim E(\lambda_k(\alpha_0),\alpha) \ge k+1.$$

Thus there must exist the kth eigenvalue $\lambda_k(\alpha)$ for any $\alpha \in (\alpha_0,\pi)$.

(3) For sufficiently small $\varepsilon > 0$, we have that

$$\dim E(\lambda_k(\alpha_0)+\varepsilon^-,\alpha_0) = k+1$$

by the proof of (2). It follows from Lemma 6.7.1(3) that

$$M(\lambda_k(\alpha_0)+\varepsilon) \ge k.$$

Hence $y(x,\alpha,\lambda_k(\alpha_0)+\varepsilon)$ has at least k zeros at (a,b) for any $\alpha \in [0,\pi)$, where $y(x,\alpha,\lambda_k(\alpha_0)+\varepsilon)$ is the solution of the problem

$$-(py')' + qy = (\lambda_k(\alpha_0)+\varepsilon)wy,$$
$$y(a) = \sin\alpha,$$
$$(py')(a) = \cos\alpha.$$

Thus $\dim E(\lambda_k(\alpha_0)+\varepsilon^-,\alpha) \ge k$ by Lemma 6.7.1(1), $\dim E(\lambda_k(\alpha_0),\alpha) \ge k$, and (3) is verified.

(4) It follows from (1), (2), and (3) that there must exist $k \in \mathbb{N}_0$ and α_0 such that there are exactly $k+1$ eigenvalues below σ_0 for $\alpha \in (\alpha_0,\pi)$ and k eigenvalues below σ_0 for $\alpha \in [0,\alpha_0]$. Part (5) follows from (2) and (3) of this theorem. □

6.8 Comments

This chapter is based on the paper by Zhang et al. [114].

The definition of regular endpoint follows that given in [113] and does not require the endpoint to be finite in contrast to much of the older literature. A primary motivation for this definition is that condition (6.2) on the coefficients implies that at such an endpoint every solution y and its quasi-derivative (py') have finite limits; see [113] for details.

Remark 6.8.1. Assume that neither endpoint is LCO or LPO. (If one endpoint is LCO or LPO, then the spectrum is unbounded above and below.) With the results given in [113] and this chapter, it is now fair to say that the dependence of the eigenvalues on the boundary conditions of self-adjoint Sturm–Liouville problems on any interval $J = (a, b), \leq a < b \leq \infty$, with endpoints a, b that are regular or singular is now well understood. This includes the eigenvalues below the essential spectrum.

For all classifications of the endpoints, R, LP, LCNO, or LCO, the Bailey–Everitt–Zettl FORTRAN code SLEIGN2 can be used to numerically compute the eigenvalues; see [10]. Furthermore, this code, combined with some theoretical results, can be used to get information about the essential spectrum, for example, its starting point σ_0, some spectral bands and gaps, the number of eigenvalues below the starting point of the essential spectrum σ_0, and so on. See [110, 20, 19] for illustrations.

Also see [10] for a comparison of SLEIGN2, which uses the Prúfer transformation, and the Fulton–Pruess [40] code SLEDGE, which is based on approximating the coefficients. As mentioned in Introduction of the book, both codes are used by Bailey, Everitt, and Zettl [10] to compute eigenvalues of some examples, and the results are compared.

7 Spectral parameter in the boundary conditions

7.1 Introduction

In this chapter, we discuss the spectrum of singular problems with eigendependent boundary conditions and its approximation with eigenvalues from a sequence of regular problems.

Our approach is in the spirit of Bailey, Everitt, Weidmann, and Zettl [9] using approximations based on truncating the interval, "inherited" boundary conditions, and "reduced restriction" operators to study the approximations of the spectrum of singular problems using eigenvalues from a sequence of regular problems. This allows us to use methods from functional analysis for both theoretical and numerical studies of these problems, in particular, strong resolvent and norm resolvent convergence of operators in a Hilbert space.

Bailey, Everitt, Weidmann, and Zettl [9] and Everitt, Marletta, and Zettl [31] studied the approximation of the spectrum of singular Sturm–Liouville problems (SLPs) with eigenvalues of regular problems. This played a critical role in the development of the code SLEIGN2 by Bailey, Everitt, and Zettl [10]. This code can be used for the computation of the eigenvalues of regular or singular SLPs with either separated or coupled self-adjoint boundary conditions. In the singular case, each endpoint may be a limit point (LP) or a limit circle (LC), and in the LP case, SLEIGN2 can also be used to detect and approximate parts of the essential (sometimes called continuous) spectrum [110], especially its starting point.

7.2 Construction of operators

In this section, we construct operators in a direct sum Hilbert space:

$$H_1 = L^2(a,b,w) \quad \text{and} \quad H = H_1 \dotplus \mathbb{C},$$

where $\mathbb{C}$ is the space of complex numbers with the usual inner product for H_1,

$$(f,g) = \int_J f(t)\overline{g(t)}w(t)\,dt,$$

and the inner product $\langle \cdot, \cdot \rangle$ for H given by

$$\big\langle (f,g),(u,v) \big\rangle = (f,g) + \rho u\bar{v}, \quad \rho > 0.$$

Note that the space H_1 can be identified with the closed subspace $\{(f,0) : f \in H_1\}$ of H and that this subspace has codimension 1 in H. Below, for convenience, we will use the notation (f,u) for the elements of H.

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Let

$$J = (a, b), \quad -\infty < a < b \le +\infty$$

and consider

$$My = \frac{1}{w}[-(py')' + qy] = \lambda y \quad \text{on } J, \tag{7.1}$$

where

$$p, q, w : J \to \mathbb{R}, \quad \frac{1}{p}, q, w \in L_{\mathrm{loc}}(J), \quad p > 0, \quad w > 0 \quad \text{a. e. on } J. \tag{7.2}$$

For the rest of this chapter, we assume that the endpoint a is regular.

The endpoint b is arbitrary, but we state and prove our results for singular b since the case where b is regular then follows as a particular case. No restrictions are placed on b, which can be finite or infinite, oscillatory or nonoscillatory, a limit circle (LC) or a limit point (LP) in H_1.

For $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and the spectral parameter λ, we consider boundary conditions at the regular endpoint a:

$$y(a)(\alpha_1\lambda + \beta_1) = (py')(a)(\alpha_2\lambda + \beta_2), \quad \eta = \beta_1\alpha_2 - \alpha_1\beta_2 > 0, \quad \rho = 1/\eta. \tag{7.3}$$

7.2.1 The classical minimal and maximal operators in H_1

For clarity of exposition, we briefly recall some basic facts about self-adjoint operators S in H_1. These satisfy

$$S_{\min} \subset S = S^* \subset S_{\max},$$

where $S_{\min}$ and $S_{\max}$ are the classical minimal and maximal operators (7.1) in H_1. These can be defined by

$$\begin{aligned} D(S_{\max}) &= \{f \in H_1 : f, (pf') \in AC_{\mathrm{loc}}(J), Mf \in H_1\}, \\ S_{\max} f &= Mf, \quad f \in D(S_{\max}), \\ S_{\min} &= S_{\max}^*. \end{aligned}$$

Let

$$D_{\max} = D(S_{\max}), \quad D_{\min} = D(S_{\min}).$$

The domain $D_{\min}$ is dense in H_1, and $S_{\min}$ is a closed symmetric operator in H_1 with equal deficiency indices $d = d(S_{\min})$; since a is regular, $d = 1, 2$, depending on the LC/LP classification of b.

Recall the Lagrange form $[\cdot,\cdot]$ defined by

$$[f,g] = f\overline{(pg')} - g\overline{(pf')}, \quad f,g \in D_{\max},$$

and that

$$[f,g](b) = \lim_{t\to b^-}[f,g](t)$$

exists as a finite limit.

Next, we summarize some well-known properties of the operators $S_{\min}$, $S_{\max}$ and their domains.

Proposition 7.2.1. *Let* (7.1) *and* (7.2) *hold, and let* $S_{\min}$, $S_{\max}, D_{\min}, D_{\max}$ *be defined as before.*

(1) *The (one-sided)* $[f,g](b)$ *exists and is finite for all* $f,g \in D_{\max}$.
(2) $D_{\min} = \{f \in D_{\max} : [f,g](b) - [f,g](a) = 0$ *for all* $g \in D_{\max}\}$.
(3) *The endpoint b is LP if and only if* $[f,g](b) = 0$ *for all* $f,g \in D_{\max}$.
(4) *For any* $c,d \in J$ *and* $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, *there exists* $g \in D_{\max}$ *such that*

$$g(c) = \alpha, \quad (pg')(c) = \beta, \quad g(d) = \gamma, \quad (pg')(d) = \delta.$$

Also, for any $\alpha,\beta \in \mathbb{C}$, *there is* $g \in D_{\max}$ *such that*

$$g(a) = \alpha, \quad (pg')(a) = \beta.$$

(5) *For any* $g \in D_{\min}$, *we have*

$$g(a) = 0, \quad (pg')(a) = 0.$$

(6) $S_{\min}^* = S_{\max}$ *and* $S_{\min} = S_{\max}^*$.

7.2.2 Construction of operators in *H*

In this subsection, we construct two operators T_1 and T_0 in H generated by (7.1)–(7.3); these can be thought of playing the roles of $S_{\max}$ and $S_{\min}$ in H_1. Define the operators T_1 and T_0 from H to H by

$$\begin{aligned}
D(T_1) &= \{(f,f_1) \in H : f \in D_{\max}, f_1 = \alpha_1 f(a) - \alpha_2(pf')(a)\},\\
T_1(f,f_1) &= (S_{\max}f, f_2) : f_2 = -\beta_1 f(a) + \beta_2(pf')(a),\\
D(T_0) &= \{(f,f_1) \in D(T_1) : [f,g](b) = 0 \text{ for all } (g,g_1) \in D(T_1)\},\\
T_0(f,f_1) &= T_1(f,f_1) : (f,f_1) \in D(T_0).
\end{aligned}$$

Theorem 7.2.1. *Let* T_0, T_1 *be defined as before. Then the operator* T_0 *is closed, densely defined, and symmetric, and* $T_0^* = T_1$ *in* H.

Proof. Since $S_{\min}$ is closed, the closedness of T_0 follows from the fact that $D(T_0) = D(S_{\min}) \dotplus N$, where N is a finite-dimensional space.

Assume that $U = (u, u_1) \in H$ and $U \perp D(T_0)$. Let $f \in D(S_{\min})$. Then $f_1 = 0$ by part (5) of Proposition 7.2.1 and $F = (f, f_1) \in D(T_0)$. It follows from

$$\langle F, U\rangle = \int_a^b f(t)\overline{u(t)}w(t)\,dt = 0$$

that $u(t) = 0$ almost everywhere on (a, b) and thus $U = (0, u_1)$. Assume that $G = (g, g_1) \in D(T_0)$. Then $G \perp U$, that is,

$$\langle G, U\rangle = \rho g_1\overline{u_1} = 0, \quad \rho = \frac{1}{\eta}.$$

Since $g_1 = \alpha_1 g(a) - \alpha_2(pg')(a)$ can be chosen arbitrarily, it follows that $u_1 = 0$. Hence the operator T_0 is densely defined.

Next, we show that $T_0 \subset T_0^*$, and therefore T_0 is symmetric. For $(f, f_1) \in D(T_0)$ and $(g, g_1) \in D(T_1)$, we have

$$\begin{aligned}
&\big\langle T_0(f, f_1), (g, g_1)\big\rangle - \big\langle (f, f_1), T_1(g, g_1)\big\rangle \\
&\quad = \big\langle (S_{\max} f, f_2), (g, g_1)\big\rangle - \big\langle (f, f_1), (S_{\max} g, g_2)\big\rangle \\
&\quad = \int_J (Mf)\overline{g}w + \rho f_2 g_1 - \left\{\int_J \overline{(Mg)} f w + \rho f_1 g_2\right\} \\
&\quad = [f, g](b) - [f, g](a) + \rho\{f_2 g_1 - f_1 g_2\}.
\end{aligned} \tag{7.4}$$

From the definition of f_1, g_1, f_2, g_2 we have

$$\begin{aligned}
&f_2 g_1 - f_1 g_2 \\
&\quad = [-\beta_1 f(a) + \beta_2(pf')(a)][\alpha_1\overline{g}(a) - \alpha_2(p\overline{g}')(a)] \\
&\qquad - [-\beta_1\overline{g}(a) + \beta_2(p\overline{g}')(a)][\alpha_1 f(a) - \alpha_2(pf')(a)] \\
&\quad = -\beta_1 f(a)\alpha_1\overline{g}(a) + \beta_1 f(a)\alpha_2(p\overline{g}')(a) + \beta_2(pf')(a)\alpha_1\overline{g}(a) \\
&\qquad - \beta_2(pf')(a)\alpha_2(p\overline{g}')(a) - \{[-\beta_1\overline{g}(a)\alpha_1 f(a) + \beta_1\overline{g}(a)\alpha_2(pf')(a) \\
&\qquad + \beta_2(p\overline{g}')(a)\alpha_1 f(a) - \beta_2(p\overline{g}')(a)\alpha_2(pf')(a)\} \\
&\quad = (\alpha_2\beta_1 - \beta_2\alpha_1)[(f(a)(p\overline{g}')(a) - (pf')(a)\overline{g}(a)] \\
&\quad = (\alpha_2\beta_1 - \beta_2\alpha_1)[f, g](a) = \eta[f, g](a).
\end{aligned} \tag{7.5}$$

Using $\rho = 1/\eta$ and substituting (7.5) into (7.4), we obtain

$$\big\langle T_0(f, f_1), (g, g_1)\big\rangle - \big\langle (f, f_1), T_0(g, g_1)\big\rangle = [f, g](b) - [f, g](a) + [f, g](a) = 0. \tag{7.6}$$

In the last step, we used the fact that $[f, g](b) = 0$ since $(f, f_1) \in D(T_0)$. From (7.6) we conclude that $T_0 \subset T_0^*$ and, in particular, T_0 is symmetric.

For any $F \in D(T_0)$ and $G \in D(T_1)$, from (7.6) we have that

$$\langle T_0 F, G\rangle - \langle F, T_1 G\rangle = 0.$$

Hence $G \in D(T_0^*)$ and $T_1 \subset T_0^*$. Thus to show that $T_1 = T_0^*$, it suffices to show that $F \in D(T_1)$ for any $F = (f, \tilde{f}_1) \in D(T_0^*)$. Let $U = (u, u_1) \in D(T_0)$. Then we have

$$\left\langle T_0^* F, U\right\rangle = \int_a^b (T_0^* f)\overline{u(x)}w(x)\,dx + \frac{1}{\eta}\tilde{f}_1\big(\overline{\alpha_1 u(a) - \alpha_2 (pu')(a)}\big). \tag{7.7}$$

On the other hand,

$$\left\langle T_0^* F, U\right\rangle = \langle F, T_0 U\rangle = \int_a^b f\overline{Mu(x)}w(x)\,dx + \frac{1}{\eta}\tilde{f}_1\big(\overline{-\beta_1 u(a) + \beta_2 (pu')(a)}\big). \tag{7.8}$$

Let $u(a) = (pu')(a) = 0$. Then (7.8) implies $f \in D_{\max}$ and

$$(S_{\max} f, u) = (f, S_{\min} u).$$

From this it follows that

$$f, pf' \in AC_{\mathrm{loc}}(J), \quad Mf \in H_1. \tag{7.9}$$

Now (7.7) minus (7.8), together with (7.9), yield

$$\begin{aligned}\int_a^b [(-pf')' + qf]\bar{u}\,dx - \int_a^b \overline{[(-pu')' + qu]}f\,dx \\ + \frac{1}{\eta}\tilde{f}_1\big(\overline{\alpha_1 u(a)\alpha_2 (pu')(a)}\big) \\ - \frac{1}{\eta}\tilde{f}_1\big(\overline{-\beta_1 u(a) + \beta_2 (pu')(a)}\big) = 0.\end{aligned}$$

Since $U \in D(T_0)$, we have $[f, u](b) = 0$. Integrating by parts, we obtain

$$\frac{1}{\eta}\tilde{f}_1\overline{\alpha_1 u(a) - \alpha_2 (pu')(a)} - \frac{1}{\eta}\tilde{f}_1\big(\overline{-\beta_1 u(a) + \beta_2 (pu')(a)}\big) - f(a)\overline{(pu')(a)} + (pf')(a)\overline{u(a)} = 0. \tag{7.10}$$

First, assume that $\alpha_1 + \beta_1 \neq 0$ and $\alpha_2 + \beta_2 \neq 0$. Let $(pu')(a) = 0$, $u(a) = 1$. Then

$$\tilde{f}_1 = \frac{-\eta}{\alpha_1 + \beta_1}(pf')(a). \tag{7.11}$$

Let $(pu')(a) = 1$ and $u(a) = 0$. Then

$$\tilde{f}_1 = \frac{-\eta}{\alpha_2 + \beta_2} f(a). \tag{7.12}$$

It follows from (7.11) and (7.12) that

$$\frac{(pf')(a)}{\alpha_1+\beta_1} = \frac{f(a)}{\alpha_2+\beta_2}. \tag{7.13}$$

Now (7.11), (7.12), and (7.13) yield

$$\tilde{f}_1 = \frac{-\eta}{\alpha_1+\beta_1}(pf')(a) = -\frac{\beta_1\alpha_2-\alpha_1\beta_2}{\alpha_1+\beta_1}(pf')(a) = \alpha_1 f(a) - \alpha_2(pf')(a).$$

If $\alpha_1+\beta_1 = 0$, then by (7.10) we have $(pf')(a) = 0$. Again by (7.13)

$$\tilde{f}_1 = \frac{-\eta}{\alpha_2+\beta_2} f(a) = \alpha_1 f(a) - \alpha_2(pf')(a).$$

Similarly, if $\alpha_2+\beta_2 = 0$, then this equality still holds. Thus $F \in D(T_1)$. □

Since T_0 is a symmetric restriction of T_1, we now study its deficiency index and self-adjoint extensions in H. Since $T_0^* = T_1$, any self-adjoint extension T of T_0 satisfies

$$T_0 \subset T \subset T_1 = T_0^*$$

and thus is a self-adjoint restriction of T_1.

Recall that if all the solutions of the differential equation (7.1) are in $L^2((c,b),w)$ for some $c \in (a,b)$, then the endpoint b is a limit circle (LC). Otherwise, the endpoint b is called a limit point (LP). This LP/LC classification is independent of $c \in (a,b)$ and $\lambda \in \mathbb{C}$. The next theorem relates the deficiency index of T_0 in H to the deficiency index of $S_{\min}$ in H_1. Since H_1 can be identified with a closed subspace of H having codimension 1 in H, it is not surprising that the deficiency indices $d(S_{\min})$ and $d(T_0)$ are related by $d(T_0) = d(S_{\min}) - 1$. The next theorem confirms this.

Theorem 7.2.2. *The deficiency index d of T_0 in H is either* 0 *or* 1. *If the endpoint b is LC in H_1, then the deficiency index of T_0 is* 1 *in H. If the endpoint b is LP in H_1, then the deficiency index d of T_0 is* 0 *in H.*

Proof. Let

$$d_+ = \dim\ker(T_0^* + iI) = \dim\ker(T_1 + iI)$$

and

$$d_- = \dim\ker(T_0^* - iI) = \dim\ker(T_1 - iI)$$

be the positive and negative deficiency indices of T_0, respectively. Then we have $d_+ = d_-$, and thus $d = d_+ = d_- = d(T_0)$.

Consider the equation $T_1(f,f_1) = i(f,f_1)$, that is,

$$(Mf, -\beta_1 f(a) + \beta_2(pf')(a)) = (if, i(\alpha_1 f(a) - \alpha_2(pf')(a)). \tag{7.14}$$

If the endpoint b is LC, then equation (7.1) has two linearly independent solutions in H_1 for each $\lambda \in \mathbb{C}$. Therefore the number of linearly independent solutions satisfying equation (7.14) in the space H is 1. Thus the deficiency index of T_0 is 1.

If the endpoint b is LP, then equation (7.1) has exactly one linearly independent solution f in H_1 for each nonreal value of λ. Therefore the number of linearly independent solutions satisfying equation (7.14) in the space H is at most 1. A limit point at the endpoint b implies that the solution f satisfies

$$[f,g](b) = 0, \quad \text{for any } g \in D_{\max}$$

by Proposition 7.2.1. Thus $f \in D(T_0)$. In view of the decomposition theorem of von Neumann, the deficiency index d of T_0 is 0. □

The following theorem characterizes all self-adjoint extensions of T_0.

Theorem 7.2.3.

(1) *Assume that b is LP. Then $d = 0$, and T_0 is self-adjoint in H and has no proper self-adjoint extension in H. Thus no boundary conditions are required or allowed at b to determine self-adjoint extensions of T_0 in the Hilbert space H.*

(2) *Assume that b is LC. Then $d = 1$. In this case, there exist $\lambda_0 \in \mathbb{R}$ and two linearly independent solutions u, v of* (7.1) *with $\lambda = \lambda_0$ such that*

$$D(T) = \{(f, f_1) \in D(T_1) : \cos\alpha[f,u](b) - \sin\alpha[f,v](b) = 0\}, \alpha \in (0,\pi] \tag{7.15}$$

is the domain of a self-adjoint operator T in H. Conversely, given any self-adjoint extension T satisfying $T_0 \subset T = T^* \subset T_1$, there exists (for given u, v) $\alpha \in (0,\pi]$ such that the domain of T is given by (7.15).

Proof. Part (1) is obtained from the classical deficiency index theory and the theory of self-adjoint extension of symmetric operators in a Hilbert space; see [22]. The argument that $D(T)$ is the domain of a self-adjoint operator T in H can be seen from [28]. Note that the self-adjoint extension theory of general ordinary differential operators is given by the well-known Glazman–Krein–Naimark theorem [36]; see Section 9; also see [84]. The converse part of (2) can be obtained from the Calkin description of self-adjoint extensions of abstract symmetric operators in a Hilbert space [22]; see also [75]. □

7.2.3 Inherited boundary conditions and induced restriction operators

In this subsection, we define inherited boundary conditions and construct the induced restriction operators, which will be used further to approximate the spectra of singular operators with eigenvalues from sequences of these regular induced restriction operators.

Let

$$I_r = (a, b_r), \quad a < b_r < b, \quad r \in \mathbb{N}, \quad \lim_{r\to\infty} b_r = b,$$

where b_r is an increasing sequence of $r \in \mathbb{N}$. We are interested in approximating a given self-adjoint realization T in the Hilbert space H with regular operators $\{T_r : r \in \mathbb{N}\}$ acting in the Hilbert spaces

$$H_r = L^2\big((a, b_r), w\big) \dotplus \mathbb{C}.$$

Definition 7.2.1. Let T_1 and T_0 be defined as before. Let $S_{r\max}$ and $S_{r\min}$ be the maximal and minimal operators with domains $D_{r\max}$ and $D_{r\min}$, respectively, in the space $H_{r1} = L^2((a, b_r), w)$. Define the operators T_{r1} and T_{r0} in H_{r1} as follows:

$$\begin{aligned} D(T_{r1}) &= \{(f, f_1) \in H_r : f \in D_{r\max}, f_1 = \alpha_1 f(a) - \alpha_2 (pf')(a)\}, \\ T_{r1}(f, f_1) &= (S_{r\max} f, f_2) : f_2 = -\beta_1 f(a) + \beta_2 (pf')(a), \\ D(T_{r0}) &= \{(f, f_1) \in D(T_{r1}) : [f, g](b_r) = 0 \text{ for any } (g, g_1) \in D(T_{r1})\}, \\ T_{r0}(f, f_1) &= T_{r1}(f, f_1) : (f, f_1) \in D(T_{r0}). \end{aligned}$$

We define the induced restriction operators $\{T_r : r \in \mathbb{N}\}$ in H_r as follows:

(1) $d = 0$. Choose any ψ_r in $D(T_{r1})$ not in $D(T_{r0})$ satisfying

$$[\psi_r, \psi_r](b_r) = 0.$$

Define

$$\begin{aligned} D(T_r) &= \{F = (f, f_1) \in D(T_{r1}) : [f, \psi_r](b_r) = 0\}, \\ T_r(f, f_1) &= T_{r1}(f, f_1), \quad (f, f_1) \in D(T_r). \end{aligned}$$

(2) $d = 1$. Let T_r be defined by

$$\begin{aligned} D(T_r) &= \{F = (f, f_1) \in D(T_{r1}) : \cos\alpha [f, u](b_r) - \sin\alpha [f, v](b_r) = 0\}, \quad \alpha \in (0, \pi], \\ T(f, f_1) &= T_{r1}(f, f_1), \quad (f, f_1) \in D(T_r), \end{aligned}$$

where u, v are defined as before, and $[f, g]$ denotes the Lagrange bracket.

Remark 7.2.1. If $d = 0$, then for any real λ, choose any nontrivial real-valued solution ψ of (7.1) and let ψ_r be its restriction to $[a, b_r)$. In this case the boundary condition $[f, \psi_r](b_r) = 0$ can be reduced to the form

$$f(b_r)\cos\beta - (pf')(b_r)\sin\beta = 0$$

for some β, which may vary with r.

Remark 7.2.2. Suppose that the endpoint b is LP and $\sigma_e(S) \neq (-\infty,\infty)$ and $\lambda \notin \sigma_e(S)$ for some S, $S_{\min} \subset S \subset S_{\max}$. (Recall that $\sigma_e(S)$ is independent of S.) Choose any nontrivial real-valued solution ψ of (7.1) for this λ, let ψ_r be its restriction to $[a, b_r)$, and note that $[\psi_r, \psi_r](b_r) = 0$; see also [99]. In fact, for the operator T, we can also choose any nontrivial real-valued solution ψ of (7.1) for some $\lambda \notin \sigma_e(T)$; note that $[\psi_r, \psi_r](b_r) = 0$.

Theorem 7.2.4. *The induced restriction operators* $\{T_r\}$ *of* T *are self-adjoint operators in the Hilbert space* H_r.

Proof. This can be found in [22, 39]. □

With each one of these operators T_r, we associate an operator T_r' in the Hilbert space H as follows:

$$T_r' = T_r \dotplus O_r = T_r P_r, \quad D(T_r') = D(T_r) \dotplus H_r^{\perp},$$

where P_r is the orthogonal projection of H onto H_r, and O_r is the zero operator in the space $H_r^{\perp} = L^2((b_r, b), w)$. It is clear that $\{T_r'\}$ are self-adjoint operators in the space H with dense domains.

7.3 Spectral properties

In this section, we study spectral properties of the self-adjoint operators in H constructed in Theorem 7.2.3 for both cases where b is LC in H_1 and b is LP in H_1.

7.3.1 Assume that *b* is LC in H_1

Let T denote a self-adjoint realization in H as constructed before.

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ be two linearly independent solutions of equation (7.1) with $\lambda \in \mathbb{C}$ satisfying the initial conditions

$$\varphi(a,\lambda) = 1, \quad (p\varphi')(a) = 0; \quad \psi(a) = 0, (p\psi')(a) = 1.$$

Let

$$\Phi(x,\lambda) = \begin{bmatrix} [\varphi, u](x) & [\psi, u](x) \\ [\varphi, v](x) & [\psi, v](x) \end{bmatrix}, \quad x \in J.$$

Then we have the following:

Theorem 7.3.1. *The complex number* λ *is an eigenvalue of SLP* (7.1), (7.3), (7.15) *if and only if* λ *satisfies the equation*

$$\varpi(\lambda) = \det\big(A + B\Phi(b,\lambda)\big) = 0, \tag{7.16}$$

where

$$A = \begin{bmatrix} \alpha_1\lambda + \beta_1 & -(\alpha_2\lambda + \beta_2) \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \cos\alpha & -\sin\alpha \end{bmatrix}.$$

Proof. Assume that λ is an eigenvalue and

$$y(x,\lambda) = c_1\varphi(x,\lambda) + c_2\psi(x,\lambda)$$

is an eigenfunction of λ. Then $y(x,\lambda)$ satisfies the two boundary conditions (7.3) and (7.15), that is, the linear equations

$$(\alpha_1\lambda + \beta_1)c_1 - (\alpha_2\lambda + \beta_2)c_2 = 0$$

and

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \end{bmatrix} \begin{bmatrix} [\varphi,u](b) & [\psi,u](b) \\ [\varphi,v](b) & [\psi,v](b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

have nonzero solutions $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$. Therefore (7.16) holds. Conversely, if (7.16) holds, then these linear equations have a nontrivial solution $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$. Let

$$y(x,\lambda) = c_1\varphi(x,\lambda) + c_2\psi(x,\lambda).$$

Then $y(x,\lambda)$ satisfies the boundary conditions and the differential equation, and therefore λ is an eigenvalue. □

Note that the eigenvalues of SLP (7.1), (7.3), (7.15) coincide with the eigenvalues of the self-adjoint operator T constructed before. In fact, the spectrum of the operator T is discrete:

Theorem 7.3.2. *Assume that T is defined by case (2) of Theorem* 7.3.1. *Then the operator T has only a point spectrum, that is, $\sigma(T) = \sigma_p(T)$.*

Proof. Assume that $\lambda \in \mathbb{R}$ is not an eigenvalue of T. It suffices to show that the equation

$$TY = \lambda Y + G$$

has a solution $Y \in D(T)$, where $G = (g, g_1) \in H$, that is, the equations

$$\begin{cases} My = \lambda y + g, \\ -\beta_1 y(a) + \beta_2(py')(a) = \lambda(\alpha_1 y(a) - \alpha_2(py')(a)) + g_1, \\ \cos\alpha[y,u](b) - \sin\alpha[y,v](b) = 0 \end{cases} \tag{7.17}$$

have a unique solution $y(x,\lambda) \in H_1$. For the differential equation in (7.17), by the variation-of-parameters formula we have that

$$y(x,\lambda) = c_1\varphi(x,\lambda) + c_2\psi(x,\lambda) + \int_a^b K(x,\xi,\lambda)g(\xi)\,d\xi \tag{7.18}$$

is a solution of this equation, where c_1, c_2 are constants, and

$$K(x,\xi,\lambda) = \begin{cases} \varphi(x,\lambda)\psi(\xi,\lambda) - \psi(x,\lambda)\varphi(\xi,\lambda), & a \le \xi \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Since $y(x,\lambda)$ needs to satisfy the last two equalities of (7.17), by direct computation we have

$$c_1 = \frac{\alpha_2\lambda + \beta_2}{\varpi(\lambda)}L, \quad c_2 = \frac{\alpha_1\lambda + \beta_1}{\varpi(\lambda)}L,$$

where

$$\begin{aligned} L = \{\cos\alpha[u,\varphi](b) - \sin\alpha[v,\varphi](b)\} \int_a^b \psi(\xi,\lambda)g(\xi)\,d\xi \\ - \{\cos\alpha[u,\psi](b) - \sin\alpha[v,\psi](b)\} \int_a^b \varphi(\xi,\lambda)g(\xi)\,d\xi. \end{aligned}$$

It follows from $\varphi(x,\lambda), \psi(x,\lambda) \in L^2(a,b,w)$ that (7.17) has a unique solution $y(x,\lambda) \in H_1$. □

7.3.2 Assume that *b* is LP in H_1

In this case, in general, the spectrum consists of eigenvalues of essential spectrum. In the classical case, i. e., when the boundary conditions do not depend on the eigenparameter, it is well known that the essential spectrum depends only on the coefficients, not on the boundary conditions. The next theorem shows that this, not surprisingly, also holds for eigenparameter-dependent boundary conditions and furthermore that, in this case, the essential spectrum of T in H is the same as the essential spectrum of the corresponding self-adjoint realization of (7.1) in the space H_1.

Theorem 7.3.3. *Let the operator T be defined as in case 1 of Theorem 7.2.3. Let the operator S be given by*

$$\begin{aligned} Sy = \frac{1}{w}[-(py')' + qy], \quad y \in D(S), \\ D(S) = \{f : f, (pf') \in AC_{\text{loc}}(J), \quad Sf \in H_1 = L^2(a,b,w), f(a) = 0\}. \end{aligned}$$

Denote the essential spectra of S and T by $\sigma_e(S)$ and $\sigma_e(T)$, respectively. Then

$$\sigma_e(S) = \sigma_e(T).$$

Proof. For $c \in (a,b)$, consider the operators S_1 and S_2:

$$\begin{aligned} D(S_1) &= \{f : f, (pf') \in AC_{\mathrm{loc}}(a,c), S_1 f \in L^2(a,c,w), f(c) = 0\}, \\ S_1 f &= Sf, \quad f \in D(S_1); \\ D(S_2) &= \{f : f, (pf') \in AC_{\mathrm{loc}}(c,b), S_2 f \in L^2(c,b,w), f(c) = 0\}, \\ S_2 f &= Sf, \quad f \in D(S_2). \end{aligned}$$

Then the operators S_1 and S_2 are self-adjoint operators in the spaces $L^2(a,c,w)$ and $L^2(c,b,w)$, respectively. Let S_0 be the orthogonal sum of the operators S_1 and S_2. Then we have

$$\sigma_e(S_0) = \sigma_e(S_1) \cup \sigma_e(S_2) = \sigma_e(S_2).$$

Note that the self-adjoint domain of the operator S is a finite-dimensional extension of the domain of the operator S_0. It follows from $\sigma_e(S) = \sigma_e(S_0)$ that

$$\sigma_e(S) = \sigma_e(S_0) = \sigma_e(S_2).$$

Construct the following three operators $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_0$:

$$\begin{aligned} D(\tilde{T}_1) &= \{(f,f_1) : f, (pf') \in AC_{\mathrm{loc}}(a,c), Mf \in L^2(a,c,w), \\ &\qquad f_1 = \alpha_1 f(a) - \alpha_2 (pf')(a), f(c) = 0\}, \\ \tilde{T}_1(f,f_1) &= (Mf, f_2), f_2 = -\beta_1 f(a) + \beta_2 (pf')(a); \\ D(\tilde{T}_2) &= \{(f,0) : f, (pf') \in AC_{\mathrm{loc}}(c,b), Mf \in L^2(c,b,w), f(c) = 0\}, \\ \tilde{T}_2(f,0) &= (Mf, 0); \end{aligned}$$

and

$$\tilde{T}_0 = \tilde{T}_1 \dotplus \tilde{T}_2.$$

By [22, 39] it follows that $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_0$ are all self-adjoint. Therefore from the theory of the direct sum of the operators [28] we have

$$\sigma_e(T) = \sigma_e(\tilde{T}_0) = \sigma_e(\tilde{T}_1) \cup \sigma_e(\tilde{T}_2) = \sigma_e(\tilde{T}_2).$$

From $\sigma_e(S_2) = \sigma_e(\tilde{T}_2)$ it follows that $\sigma_e(S) = \sigma_e(T)$. □

7.4 Approximation of eigenvalues

In this section, we introduce some definitions about strong resolvent convergence and norm resolvent convergence of operators. These play important roles in the study of the approximation of the spectrum of singular operators with eigenvalues from a sequence of approximating regular operators. These definitions and the following lemmas can be found in [9, 89, 104].

Definition 7.4.1. Let $\{T_r : r \in \mathbb{N}\}$ and T be self-adjoint operators in the Hilbert space H. Then T_r is said to converge to T in the strong resolvent sense (SRC) if

$$(T_r - z)^{-1} f \to (T - z)^{-1} f, \quad r \to \infty,$$

for all $f \in H$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$; T_r is said to converge to T in the norm resolvent sense (NRC) if

$$\|(T_r - z)^{-1} - (T - z)^{-1}\| \to 0, \quad r \to \infty.$$

In the following lemma we give a sufficient condition of SRC.

Lemma 7.4.1. *Let H be a Hilbert space, and let $\{T_r : r \in \mathbb{N}\}$ and T be given as before. Suppose that there is a core $C(T)$ of T such that for any $f \in C(T)$, there exists $r_0 \in \mathbb{N}$ such that $f \in D(T_r)$ for any $r > r_0$ and $T_r f \to Tf$ as $r \to \infty$ for all $f \in C(T)$. Then $\{T_r : r \in \mathbb{N}\}$ is SRC to T in H.*

On the convergence of spectrum, we have the following definitions.

Definition 7.4.2.
(1) The sequence $\{T_r : r \in \mathbb{N}\}$ is spectral included for T if for any $\lambda \in \sigma(T)$, there exists a sequence $\{\lambda_r : r \in \mathbb{N}\}$ with $\lambda_r \in \sigma(T_r)$ such that $\lim_{r\to\infty} \lambda_r = \lambda$.
(2) The sequence $\{T_r : r \in \mathbb{N}\}$ is spectral exact for T if it is spectral included for T and if any limit point of a sequence $\{\lambda_r : r \in \mathbb{N}\}$ with $\lambda_r \in \sigma(T_r)$ belongs to $\sigma(T)$.

The next lemma describes the relation between the convergence of operators and of their spectrum.

Lemma 7.4.2.
(1) *Suppose that T is a self-adjoint operator on a Hilbert space H and $\{T_r\}$ is a sequence of self-adjoint operators on H which is SRC to T. Then $\{T_r\}$ is spectral included for T. Let $\{E(T_r, \lambda); \lambda \in \mathbb{R}\}$ and $\{E(T, \lambda); \lambda \in \mathbb{R}\}$ with λ not an eigenvalue of T denote the spectral projections of T_r and T, respectively. Then for all $f \in H$,*

$$\|E(T_r, \lambda) f - E(T, \lambda) f\| \to 0, \quad r \to \infty.$$

(2) *Suppose that T is a self-adjoint operator on a Hilbert space H and $\{T_r\}$ is a sequence of self-adjoint operators on H that is NRC to T. Then $\{T_r\}$ is spectral exact for T. Let $\{E(T_r, \lambda); \lambda \in \mathbb{R}\}$ and $\{E(T, \lambda); \lambda \in \mathbb{R}\}$ with λ not an eigenvalue of T denote the spectral projection of T_r and T, respectively. Then*

$$\|E(T_r, \lambda) - E(T, \lambda)\| \to 0, \quad r \to \infty.$$

In Section 7.2, we constructed the self-adjoint operator T with the induced restriction operators $\{T_r : r \in \mathbb{N}\}$ and the corresponding induced operators $\{T_r'\}$. We have the following:

Theorem 7.4.1. *Let T, T_r, T_r' be defined as in Section 7.2 and suppose $\{T_r'\}$ is SRC to T. Then for any $\lambda \in \mathbb{R}$ that is not an eigenvalue of T, we have that $E(T_r, \lambda)P_r$ is strongly convergent to $E(T, \lambda)$ and the sequence $\{T_r\}$ is spectral included for T.*

Proof. It is similar to the proof of Theorem 3.6 in [9]. □

7.4.1 The case where *b* is limit circle

First, we study the limit circle case at the endpoint b, which implies that the deficiency index of T_0 in H is $d = 1$. In this case the spectrum is discrete, and the approximation of the eigenvalues of singular problems is relatively simple.

Theorem 7.4.2. *Let T, T_r, T_r' be defined as above and assume that the endpoint b is limit circle. Then:*
(1) *the sequence $\{T_r'\}$ is SRC to T;*
(2) *for any $z \in \mathbb{C}/\mathbb{R}$, the sequence $\{(T_r - z)^{-1}P_r : r \in \mathbb{N}\}$ converges to $\{(T - z)^{-1}\}$ in the Hilbert–Schmidt norm;*
(3) *the sequence $\{T_r : r \in \mathbb{N}\}$ is spectral exact for T;*
(4) *for any $\lambda \in \mathbb{R}$ not an eigenvalue of T, the sequence $E(T_r, \lambda)P_r$ converges to $E(T, \lambda)$ not only strongly but also in norm.*

Proof. (1) For fixed α, define

$$C(T) = \{F \in D(T) : f = c(u\cos\alpha - v\sin\alpha)\},$$

where $f = c(u\cos\alpha - v\sin\alpha)$ in $[b', b)$ for some constant c and some b' in (a, b). Then $C(T)$ is a core of T. This is due to the fact that

$$\dim(D(T)/D(T_0)) = 1, \quad C(T) = D'(T_0) \dotplus N,$$

where

$$D'(T_0) = \{(f, f_1) \in D(T_1) : f \text{ has compact support in } [a, b)\},$$
$$\dim N = 1, \quad N \cap D'(T_0) = \{0\}.$$

Given $F \in C(T)$, we have that $F \in D(T_r')$ for all sufficiently large r and

$$T_r'F \to TF \quad \text{in } H, \quad r \to \infty.$$

Therefore $\{T_r'\}$ is SRC to T in H by Lemma 7.4.1.

(2) For any $z \in \mathbb{C}/\mathbb{R}$, let $\varphi(x, z)$, $\psi(x, z)$ be a fundamental set of solutions of (7.1) with $\lambda = z$. By the proof of Theorem 7.3.1 we have that the resolvents $\{(T - z)^{-1}\}$ and $\{(T_r - z)^{-1}\}$ have the form

$$(T - z)^{-1}G = (y(x, z), \alpha_1 y(a) - \alpha_2(py')(a)),$$

$$(T_r - z)^{-1}G = (y_r(x,z), \alpha_1 y_r(a) - \alpha_2(py_r')(a)),$$

where

$$y(x,z) = c_1\varphi(x,z) + c_2\psi(x,z) + \int_a^b K(x,\xi,z)g(\xi)\,d\xi,$$

$$y_r(x,z) = c_{r,1}\varphi(x,z) + c_{r,2}\psi(x,z) + \int_a^b K_r(x,\xi,z)g(\xi)\,d\xi,$$

and $K, K_r, c_1, c_2, c_{r,1}, c_{r,2}$ are defined as in the proof of Theorem 7.4.2. It is clear that

$$K_r \to K, \quad c_{r,1} \to c_1, \quad c_{r,2} \to c_2, \quad y_r(a) \to y(a), \quad (py_r')(a) \to (py')(a).$$

The Hilbert–Schmidt convergence follows. This easily implies the convergence of the spectrum and the desired norm convergence of the spectral projections from Lemma 7.4.2. □

Recall that if b is oscillatory, then the spectrum of any self-adjoint realization S in H_1 and therefore also of any self-adjoint realization T in H is unbounded both above and below. On the other hand, if b is nonoscillatory, then both S and T are bounded below. The convergence properties of the eigenvalues for these two cases is very different as shown by the next theorem.

Theorem 7.4.3. *Suppose that T, T_r, T_r' are defined as in case* (2) *of Definition* 7.4.1.

(1) *Assume that the spectrum of the operator T is bounded below and denote the nth eigenvalue of the operators T_r and T by $\lambda_n(T_r)$ and $\lambda_n(T)$, respectively, for all $n \in \mathbb{N}$. Then*

$$\lambda_n(T_r) \to \lambda_n(T), \quad r \to \infty.$$

(2) *If the spectrum of the operator T is unbounded below and $\lambda_n(T_r)$ ($n \in \mathbb{N}$) is the nth eigenvalue of T_r, then*

$$\lambda_n(T_r) \to -\infty, \quad r \to \infty.$$

Proof. The argument for (1) is directly obtained from Theorem 7.4.1. We now verify case (2). We associate a right-hand Dirichlet problem by appending the boundary condition $y(b_r) = 0$. The corresponding eigenvalues are denoted by $\lambda_n^D(S_r)$, $n \in \mathbb{N}$. Then by Corollary 2.3 in [14] we have

$$\lambda_n(T_r) \le \lambda_n^D(S_r) < \lambda_{n+1}(T_r), \quad n = 0, 1, 2, \dots. \tag{7.19}$$

Thus it suffices to prove that $\lambda_n^D(S_r) \to -\infty$, $r \to \infty$, $n = 0, 1, 2, \dots$.

Let the sets C_r be defined as follows:

$$C_r = \{F \in D(T) : f \text{ has compact support in } [a, b_r)\}.$$

Then by the variational characterization of the eigenvalues we have

$$\lambda_n^D(S_r) = \inf_{\dim V = n, V \subset C_r} \left\{ \sup_{F \in V} \frac{(S_r F, F)}{(F, F)} \right\}.$$

Since b_r is increasing, it is clear that $H_r \subset H_{r+1}$ for all r. Thus $\lambda_n^D(S_{r+1}) \le \lambda_n^D(S_r)$, and $\lambda_n^D(S_r)$ is a decreasing function of r. Therefore the limit $\lim_{r\to\infty} \lambda_n^D(S_r)$ exists or equals $-\infty$. Suppose

$$\lim_{r\to\infty} \lambda_n^D(S_r) = c > -\infty.$$

Choose $\mu_k \in \sigma(T)$ such that

$$\mu_n < \mu_{n-1} < \cdots < \mu_1 < c.$$

By Theorem 7.4.1(3) there exists an index sequence $n(r, \mu_k)$ such that $\lambda_{n(r,\mu_k)}(T_r) \to \mu_k$. From (7.20) we have $n(r, \mu_1) < n$ for sufficiently large r. Similarly, $n(r, \mu_{n-1}) < \cdots < n(r, \mu_1) < n$. Thus $n(r, \mu_{n-1})$ is at most 1. Therefore there does not exist a sequence $n(r, \mu_n)$ such that $\lambda_{n(r,\mu_n)}(T_r) \to \mu_n$, which is a contradiction. □

7.4.2 The case where *b* is limit point

Next, we consider the LP case for b, that is, $d = 0$. In this case the approximation of the spectrum of the singular operator T by eigenvalues of the regular operators T_r is much more complicated.

Theorem 7.4.4. *Let T, T_r, T_r' be defined as in case* (1) *in Definition* 7.4.1. *Then the sequence $\{T_r'\}$ is SRC to T in H, and $\{T_r\}$ is spectral included for T.*

Proof. Define

$$C(T) = \{F \in D(T) : f \text{ has compact support in } [a, b)\}.$$

Then $C(T)$ is a core of T. (See [77, 104] for a definition of a core of T.) By Lemma 7.4.1 the sequence $\{T_r'\}$ is SRC to T in H. The spectral inclusion follows from Theorem 7.4.1. □

When T_0 is bounded below, there is a simpler approximation of the singular eigenvalues. The following lemma plays an important role in the proof.

Lemma 7.4.3. *Let $\{P_r : r \in \mathbb{N}\}$ be a sequence of self-adjoint projections in a Hilbert space H, and let P be a bounded self-adjoint projection in H such that P_r is strongly convergent to P. Assume that*

$$\dim P_n \le \dim P < \infty$$

for all n. Then $\|P_r - P\| \to 0$, $r \to \infty$.

Proof. This is a direct consequence of Lemmas 1.23 and 1.24 in [53], Chapter 8. □

Theorem 7.4.5. *Let T, T_r, T_r' be defined as in case* (1) *of Definition* 7.4.1. *In addition, assume that*

(1) *the operator T_0 is bounded below in H;*
(2) *the induced restrictions $\{T_r : r \in \mathbb{N}\}$ are determined by the boundary conditions* (7.3) *at the regular endpoint a and*

$$f(b_r) = 0. \tag{7.20}$$

Then:

(1) *$\{T_r : r \in \mathbb{N}\}$ is spectral exact for T below the essential spectrum $\sigma_e(T)$ of T, that is, if there exist exactly k eigenvalues $\lambda_0(T), \lambda_1(T), \ldots, \lambda_{k-1}(T)$, then*

$$\lambda_n(T_r) \to \lambda_n(T), \quad r \to \infty, \quad n = 0, 1, 2, \ldots, k-1.$$

If there exist infinitely many eigenvalues below the essential spectrum $\sigma_e(T)$ of T, then we have

$$\lambda_n(T_r) \to \lambda_n(T), \quad r \to \infty, \quad n \in \mathbb{N}.$$

In particular, in the case where the spectrum is discrete, $\{T_r : r \in \mathbb{N}\}$ is spectral exact for T, and the above equality still holds.

(2) *For any λ below the essential spectrum $\sigma_e(T)$ with λ not an eigenvalue of T, $E(T_r, \lambda)P_r$ converges to $E(T, \lambda)$ not only strongly but also in norm.*
(3) *Assume that there exist exactly k eigenvalues $\lambda_0(T), \lambda_1(T), \ldots, \lambda_{k-1}(T)$ below σ_0. Then*

$$\lambda_n(T_r) \to \sigma_0, \quad r \to \infty, \quad n = k, k+1, k+2, \ldots,$$

where $\sigma_0 = \inf \sigma_e(T)$.

Proof. The strong convergence of $E(T_r, \lambda)P_r$ to $E(T, \lambda)$ follows from Theorems 7.4.1 and 7.4.2. Let Q_r and Q be the quadratic forms corresponding to the operators T_r and T, respectively. Let $D(Q_r)$ and $D(Q)$ denote the form domains of T_r and T, respectively. Since

$$D(Q_r) = \{F \in D(T_{r1}) : f(b_r) = 0\}, \quad D(Q) = D(T_0),$$

we have $Q_r \subset Q$. Thus the range of $E(T_r, \lambda)$ is a maximal subspace of $D(Q_r)$ consisting of functions F satisfying $Q_r(F, F) \le \lambda(F, F)$. Similarly, for Q, we can obtain that

$$\dim E(T_r, \lambda) \le \dim E(T, \lambda) < \infty$$

for all λ lying below $\sigma_e(T)$. By Lemma 7.4.2 we have that $E(T_r, \lambda)P_r$ converges to $E(T, \lambda)$ in norm, and thus the argument for (2) holds. By Lemma 7.4.2(2), (1) is verified.

Since the essential spectrum is closed, there exists a strictly decreasing sequence $\{\mu_k \in \sigma(T)\}$ converging to σ_0. Similarly to the proof of Theorem 7.4.2, we have that $\lambda_n(T_r)$ is a decreasing function of $r \in (0, \infty)$. Thus the limit $\lim_{r\to\infty} \lambda_n(T_r)$ $(n = k, k+1, k+2, \dots)$ exists. Again by (1), the above limit is equal to or greater than σ_0. Assume that the limit

$$\lim_{r\to\infty} \lambda_{k+1}(T_r) = c > \sigma_0.$$

For those spectral points between c and σ_0, it is not possible to find a sequence $\lambda_{n(r)}(T_r)$ that is convergent to these points. This is a contradiction, and the proof of (3) is completed. □

7.5 Examples

In this section, we illustrate some results from Section 7.4 using the Fourier equation. Consider

$$-y'' = \lambda y \quad \text{on } [0, \infty) \tag{7.21}$$

with boundary condition

$$y'(0) = -\lambda y(0) - 6y(0). \tag{7.22}$$

Note that equation (7.21) is regular at 0 and LP at ∞. Let the operator T in the space H and the operator S in the space H_1 be constructed as before. It is well known that $\sigma_e(S) = [0, \infty)$, and therefore $\sigma_e(T) = [0, \infty)$ by Theorem 7.3.3.

A direct computation shows that $\lambda = -4$ is the only eigenvalue of T below the essential spectrum $\sigma_e(T)$.

In this case, for $0 < b_r < \infty$, the induced restriction operator T_r is generated by the equation

$$-y'' = \lambda y \quad \text{on } [0, b_r)$$

with boundary conditions

$$y'(0) = -\lambda y(0) - 6y(0), \quad y(b_r) = 0.$$

A direct computation shows that the first eigenvalue $\lambda_0^r = \lambda_0(T_r)$ of this problem satisfies the equality

$$t = \frac{1}{2}\left[\sqrt{\left(\frac{-e^{-2tb_r} - 1}{-e^{-2tb_r} + 1}\right)^2 + 24} + \frac{-e^{-2tb_r} - 1}{-e^{-2tb_r} + 1}\right], \tag{7.23}$$

where $\lambda_0^r = -t^2$. Note that when $\lim_{r\to\infty} b_r = \infty$, then the limit $\lim_{r\to\infty} t$ exists. Taking the limits as $r \to \infty$ of the two sides of equality (7.23), we obtain

$$\lim_{r\to\infty} t = 2.$$

Thus

$$\lim_{r\to\infty} \lambda_0^r = -4.$$

This illustrates the result of Theorem 7.4.2.

7.6 Comments

This chapter is based on the papers [115, 77] by Zhang, Sun, and Zettl.

Part II: **Two-interval problems**

8 Discontinuous boundary conditions

8.1 Introduction

As mentioned in the introduction of the book, recently there has been a lot of interest in the literature of self-adjoint Sturm–Liouville problems with discontinuous boundary conditions specified at *regular interior points* of the underlying interval. Such conditions are known by various names including transmission conditions [1, 2, 82, 87, 88, 98], interface conditions [61, 76, 92, 109], discontinuous conditions [51, 91, 81], multipoint conditions [55, 76, 36, 112], point interactions (in the physics literature) [42, 21, 23, 35], conditions on trees, graphs, or networks [90, 87, 88], and so on. For an informative survey of such problems arising in applications including an extensive bibliography and historical notes, see Pokornyi and Borovskikh [87] and Prokornyi and Pryadiev [88].

In this chapter, we study these problems for both *regular and singular interior points*.

These problems are not covered by the classical Sturm–Liouville theory since, in this theory, solutions and their quasi-derivatives are continuous at all interior points of the underlying interval J. In particular, this applies to all eigenfunctions. Below we will call this theory and its extensions discussed in Part I (The one-interval theory).

Motivated by applications, in particular, the paper of Boyd [18] and its references, in 1986, Everitt and Zettl [37] introduced a framework for the rigorous study of Sturm–Liouville problems that have a singularity in the interior of the domain interval since the existing theory did not cover such problems. The Boyd paper, which was based on several previous papers by atmospheric scientists, studied eddies in the atmosphere using a mathematical model based on the SL problem

$$-y'' + \frac{1}{x}y = \lambda y, \quad y(-1) = 0 = y(1), \quad -1 < x < 1. \tag{8.1}$$

Note that 0 is a singular point in the interior of the underlying interval $(-1, 1)$ and condition (1.2) of the one-interval theory does not hold.

The framework introduced in [37] is the direct sum of Hilbert spaces, one for each interval $(-1, 0)$ and $(0, 1)$. The primary goal of this study is a characterization of all self-adjoint realizations from the two intervals. A simple way of getting self-adjoint operators in the direct sum space is taking the direct sum of operators from the separate spaces. However, there are many self-adjoint operators in the direct sum space that are not obtained this way. These “new” operators involve interactions between the two intervals; see Chapter 13 of [113].

Mukhtarov and Yakubov [82] observed that the set of self-adjoint operator realizations developed in [37] and discussed in [113] could be further enlarged by using different multiples of the usual inner products associated with each of the intervals. Sun, Wang, and Zettl [101, 94] used the Mukhtarov and Yakubov modification of the

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Everitt–Zettl theory in [37] to obtain more general self-adjoint two interval boundary conditions. In particular, it was shown in [101, 94] that for coupled self-adjoint boundary conditions determined by a coupling matrix K, the condition $\det(K) = 1$ required in the one-interval case can be replaced by $\det(K) > 1$. Wang and Zettl [103] extended the Mukhtarov and Yakubov result further to $\det(K) \neq 0$.

We briefly review the one-interval theory in Section 8.2 and the general two-interval theory in Section 8.3. In Section 8.4, we apply the two-interval theory to regular and singular problems with discontinuous boundary conditions and give a number of illustrative examples. Further examples will be given in Chapter 10 for the Legendre equation. These Legendre examples give explicit illustrations of singular self-adjoint discontinuous jump boundary conditions.

8.2 The one-interval theory

Spectral properties of the classical SL problems are studied by considering the symmetric SL equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J = (a,b), \quad \lambda \in \mathbb{C}, \quad -\infty \leq a < b \leq \infty, \tag{8.2}$$

with coefficients p, q and weight function w satisfying

$$p^{-1}, q, w \in L_{loc}(J, \mathbb{R}), \quad w > 0 \quad \text{a.e. on } J, \tag{8.3}$$

where $L_{\text{loc}}(J, \mathbb{R})$ denotes the real-valued functions Lebesgue integrable on all compact subintervals of J. For later reference, we note that there is no sign restriction on q or p in (8.3).

For convenience, we let $y^{[1]} = (py')$; this is the quasi-derivative of y. Under conditions (8.3), every solution y of (8.2) and its quasi-derivative $y^{[1]}$ are defined and continuous on J (but $y'(t)$ may not exist for some t in J).

Equation (8.2) generates minimal and maximal operators $S_{\min}$ and $S_{\max}$ in the Hilbert space $L^2(J, w)$ and self-adjoint operators S in this space. Each of the mentioned classical problems has such an operator realization S. These operators S satisfy

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{8.4}$$

From (8.4) it is clear that these operators S are distinguished from each other only by their domains. These domains can be determined by boundary conditions specified only at the endpoints a, b of the interval J.

Note that (8.3) holds when (a, b) is replaced by (a, c) or (c, b) for any $c \in (a, b)$. We further use the notation $D_{\max}(a, c)$ and so on to indicate the dependence on the interval (a, c).

To characterize the domains $D(S)$ of the operators S satisfying (8.4), we start with some definitions.

Definition 8.2.1. The endpoint a is regular if $p^{-1}, q, w \in L(a,c)$ for some (and hence any) $c \in (a,b)$. Similarly, b is regular if $p^{-1}, q, w \in L(c,b)$ for $a < c < b$. If an endpoint is not regular, then it is called singular. If a is singular, it is said to be in the limit-circle (LC) case if all solutions of (8.2) are in the Hilbert space $H = L^2((a,c), w)$. This is known to hold for some $\lambda \in \mathbb{C}$ if and only if it holds for all $\lambda \in \mathbb{C}$. If a is singular and not LC, then it is said to be in the limit-point (LP) case, and similarly for the endpoint b. We say that an operator S in the Hilbert space $H = L^2(J, w)$ is a self-adjoint realization of equation (8.2) if and only if (8.4) holds.

Definition 8.2.2. Recall that the Lagrange form $[\cdot,\cdot]$ is defined for all $y, z \in D_{\max}$ by

$$[y,z] = y(p\bar{z}') - \bar{z}(py'). \tag{8.5}$$

Definition 8.2.3. Assume that the endpoint a is either regular or LC. A real-valued function pair $(u,v) \in D_{\max}(a,c)$ is said to be a boundary condition basis at a if there exists a point $c \in (a,b)$ such that each of u, v is linearly independent modulo $D_{\min}(a,c)$ and normalized to satisfy $[u,v](a) = 1$. A similar definition is made for the endpoint b. A simple way to get such (u,v) at a is taking linearly independent real-valued solutions for any real λ in some interval (a,c) and normalizing them as indicated, and similarly for b.

The number of boundary conditions needed to characterize the operators S satisfying (8.4) depends on the deficiency index d of $S_{\min}$, which depends on the classification of the endpoints a, b as regular, LC, or LP. This classification depends on the coefficients p, q, w, and this dependence is implicit and complicated. There is a vast literature on this dependence, and much is known, but there still exist equations (8.2) for which the LC/LP classification is not known; see [54]. The number d is given in terms of the endpoint classifications by the next proposition.

Proposition 8.2.1. *The deficiency index d of $S_{\min}$ in $L^2(J,w)$ satisfies $0 \le d \le 2$, and all three values are realized. Furthermore:*
(1) *If $d = 0$, then $S_{\min}$ is self-adjoint and has no proper self-adjoint extension.*
(2) *$d = 1$ if and only if one endpoint is LP and the other regular or LC.*
(3) *$d = 2$ if and only if each endpoint is either regular or LC.*

Proof. See [113, 84, 104] for proofs. □

We can now state the characterization of all operators S that satisfy (8.4).

Theorem 8.2.1. *Let* (8.2) *and* (8.3) *hold. Then:*
(1) *If both endpoints are LP, then $S_{\min}$ is self-adjoint with no proper self-adjoint extension.*
(2) *Suppose that a is LP and b is LC. Assume that (u,v) is a boundary condition basis at b. If $c, d \in \mathbb{R}$, $(c,d) \neq (0,0)$, and*

$$D(S) = \{y \in D_{\max} : c[y,u](b) + d[y,v](b) = 0\}, \tag{8.6}$$

then the operator S with domain $D(S)$ *satisfies* (8.4).
If a is LC, b is LP, and (u, v) *is a boundary condition basis at a, then replace b by a in* (8.6).
If a is LP and b is regular, then (8.6) *reduces to (but not necessarily with the same c, d)*

$$D(S) = \{y \in D_{\max} : cy(b) + d(py')(b) = 0\}. \tag{8.7}$$

If a is regular and b is LP, then (8.6) *reduces to*

$$D(S) = \{y \in D_{\max} : cy(a) + d(py')(a) = 0\}. \tag{8.8}$$

(3) *Assume each of a and b are, independently, regular or LC, and let* (u_a, v_a) *and* (u_b, v_b) *be boundary condition bases at a and b, respectively. Suppose* $A, B \in M_2(\mathbb{C})$ *satisfy*

$$\operatorname{rank}(A : B) = 2 \quad \textit{and} \quad AEA^* = BEB^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.9}$$

If

$$D(S) = \left\{ y \in D_{\max} : A \begin{bmatrix} [y, u_a](a) \\ [y, v_a](a) \end{bmatrix} + B \begin{bmatrix} [y, u_b](b) \\ [y, v_b](b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \tag{8.10}$$

then $D(S)$ *is the domain of a self-adjoint extension S satisfying* (8.4). *Moreover, for fixed* (u_a, v_a) *and* (u_b, v_b), *all operators S satisfying* (8.4) *are generated this way. Furthermore, if a is regular, then the term multiplied by A can be replaced by*

$$\begin{bmatrix} y(a) \\ (py')(a) \end{bmatrix}. \tag{8.11}$$

Similarly, if b is regular, then the term multiplied by B can be replaced by

$$\begin{bmatrix} y(b) \\ (py')(b) \end{bmatrix}. \tag{8.12}$$

Thus if both a and b are regular, then (8.10) *can be reduced to the more familiar regular self-adjoint boundary conditions*

$$A \begin{bmatrix} y(a) \\ (py')(a) \end{bmatrix} + B \begin{bmatrix} y(b) \\ (py')(b) \end{bmatrix} = 0. \tag{8.13}$$

It is well known that the boundary conditions (8.13) *can be categorized into two mutually exclusive classes, separated and coupled. The separated conditions have the form* (8.7) *when b is regular and the same form with b is replaced by a, but not* *necessarily with the same c, d when a is regular, and these separated conditions have the familiar canonical form*

$$\cos\beta y(b) - \sin\beta(py')(b) = 0, \quad 0 < \beta \leq \pi, \tag{8.14}$$

when b is regular and the canonical form

$$\cos\alpha y(a) - \sin\alpha(py')(a) = 0, \quad 0 \leq \alpha < \pi, \tag{8.15}$$

when a is regular. (The different parameterizations for α and β are customary and used for convenience in using the Prüfer transformation and stating results, but these different parameterizations play no role in this book.)
The coupled singular conditions (8.10) *have the canonical form*

$$Y(b) = e^{i\gamma}KY(a), \quad -\pi < \gamma \leq \pi, \quad i = \sqrt{-1}, \tag{8.16}$$

where $K \in M_2(\mathbb{R})$ satisfies $\det(K) = 1$*, and*

$$Y(a) = \begin{bmatrix} [y, u_a](a) \\ [y, v_a](a) \end{bmatrix}, \quad Y(b) = \begin{bmatrix} [y, u_b](b) \\ [y, v_b](b) \end{bmatrix}. \tag{8.17}$$

When b is regular, $Y(b)$ can be replaced by (8.12)*, and $Y(a)$ can be replaced by* (8.11) *when a is regular. So when both a and b are regular,* (8.16) *can be reduced to*

$$Y(b) = \begin{bmatrix} y(b) \\ (py')(b) \end{bmatrix} = e^{i\gamma}K \begin{bmatrix} y(a) \\ (py')(a) \end{bmatrix}, \quad -\pi < \gamma \leq \pi, \quad i = \sqrt{-1}, \tag{8.18}$$

with $K \in M_2(\mathbb{R})$, $\det(K) = 1$.

Proof. See [113]. □

Remark 8.2.1. See Chapter 14 in [113] for explicit boundary conditions that determine the problems generating the special functions associated with the names of Bessel, Chebychev, Fourier, Jacobi, Legendre, Morse, and so on.

Remark 8.2.2. For reference below, we note that the characterization of the self-adjoint operators S satisfying (8.4) given by Theorem 8.2.1 is unchanged if the usual inner product

$$(f, g) = \int_J f\overline{g}w \tag{8.19}$$

in $H = L^2(J, w)$ is replaced by

$$(f, g) = h\int_J f\overline{g}w \tag{8.20}$$

for any $h > 0$.

Remark 8.2.3. From another perspective we can say that the characterization of the self-adjoint operators S satisfying (8.4) given by Theorem 8.2.1 is unchanged if the weight function w is replaced by hw where h is any positive constant. The positivity of h is important for (8.20) to be an inner product and for the weight function to satisfy (8.3). However, we make the following interesting observation: The characterization given by Theorem 8.2.1 remains valid and unchanged if the weight function w is replaced by hw where h is any positive or **negative** constant. In the negative case, write the right-hand side of equation (8.2) as $(-\lambda)(-hw)y$ and use the previous observation for the positive weight function $(-hw)$. This leaves the characterization of the self-adjoint operators given by Theorem 8.2.1 unchanged. However, the spectrum of the operator changes, in particular, if λ is changed to $-\lambda$, and then the spectrum is "flipped" accordingly.

Remark 8.2.4. The simple observations of the last two remarks take on added significance in the two-interval theory as we will see further. In particular, they are used to extend the self-adjointness condition $\det(K) = 1$ to $\det(K) > 0$ using Remark 8.2.3 as observed by Mukhtarov and Yakubov [82]. Using Remark 8.2.4, Wang and Zettl [103] observed the further extension to nonsingular K.

Remark 8.2.5. From the perspective of the modern classical one-interval theory we can say that Theorem 8.2.1 characterizes the self-adjoint two point boundary conditions that determine self-adjoint SL operators in the Hilbert space $L^2(J, w)$. It follows from (8.3) that all solutions of equation (8.2) and their quasi-derivatives are continuous on the interval $J = (a, b)$. In particular, the eigenfunctions of every self-adjoint operator S satisfying (8.4) are continuous on J. But see the next remark.

Remark 8.2.6. However, we will see in Section 8.3 that the two-interval theory, when specialized to adjacent intervals, produces more self-adjoint operators in $L^2(J, w)$, and in Section 8.4, we will see that the additional self-adjoint operators generated by the two-interval theory when the intervals have a common endpoint generate all regular and singular self-adjoint operators determined by discontinuous boundary conditions. All these operators S satisfy (8.4), but in general their eigenfunctions are not continuous on the interval $J = (a, b)$.

8.3 The two-interval theory

Let

$$J_1 = (a, b), \quad -\infty \le a < b \le \infty, \quad J_2 = (c, d), \quad -\infty \le c < d \le \infty, \tag{8.21}$$

and assume the coefficients and weight functions satisfy

$$p_r^{-1}, q_r, w_r \in L_{loc}(J_r, \mathbb{R}), \quad w_r > 0 \quad \text{a. e. on } J_r, \quad r = 1, 2. \tag{8.22}$$

Note that the intervals J_1 and J_2 are independent; they may be disjoint, have a common endpoint, overlap, or be identical.

Define the differential expressions M_r by

$$M_r y = -(p_r y')' + q_r y \quad \text{on } J_r, \quad r = 1, 2, \tag{8.23}$$

and consider the equations

$$M_r y = \lambda w_r y \quad \text{on } J_r, \quad r = 1, 2. \tag{8.24}$$

Let

$$H_r = L^2(J_r, w_r), \quad r = 1, 2. \tag{8.25}$$

A simple way of getting self-adjoint operators S in the direct sum space

$$H_u = H_1 \dotplus H_2, \quad \text{where } H_r = L^2(J_r, w_r), \quad r = 1, 2, \tag{8.26}$$

is to take the direct sum of self-adjoint operators from H_1 and H_2. If these were all the self-adjoint operator realizations from the two intervals, there would be no need for a "2-interval" theory. As noted in [37], there are many self-adjoint operators in H_u that are not merely the sum of self-adjoint operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals.

We further use the notation with a subscript r to denote the rth interval. The subscript r is sometimes omitted when it is clear from the context.

Elements of $H_u = H_1 + H_2$ will be denoted in boldface: $\mathbf{f} = \{f_1, f_2\}$ with $f_1 \in H_1, f_2 \in H_2$. The usual inner product in H_u is given by

$$(\mathbf{f}, \mathbf{g}) = (f_1, g_1)_1 + (f_2, g_2)_2, \tag{8.27}$$

where $(\cdot, \cdot)_r$ is the usual inner product in H_r:

$$(f_r, g_r)_r = \int_{J_r} f_r \overline{g_r} w_r, \quad r = 1, 2. \tag{8.28}$$

Mukhtarov and Yakubov [82] observed that the set of self-adjoint operator realizations developed in [37] can be further enlarged by using a different Hilbert space

$$H = \left(L^2(J_1, w_1) \dotplus L^2(J_2, w_2), \langle \cdot, \cdot \rangle\right) \tag{8.29}$$

with inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = h(f_1, g_1)_1 + k(f_2, g_2)_2, \quad h > 0, \quad k > 0. \tag{8.30}$$

Remark 8.3.1. Note that (8.30) is an inner product in H for any positive numbers h and k. The elements of the Hilbert spaces H and H_u are the same; thus these spaces differ from each other only by their inner products. As we will further see, the parameters h, k influence the boundary conditions that yield self-adjoint realizations in the two-interval case. From another perspective, the Hilbert space $(H, \langle,\rangle)$ can be viewed as a "usual" direct sum space H_u with summands $H_r = L^2(J_r, w_r)$ but with w_1 replaced by hw_1 and w_2 replaced by kw_2.

We will further see that for coupled boundary conditions, the self-adjoint operator realizations S, and therefore their eigenvalues, depend on h and k. In the one-interval theory of Part I, we studied the dependence of the eigenvalues on the boundary conditions and observed that this dependence is invariant with respect to the inner product. In the Mukhtarov–Yakubov Hilbert space $(H, \langle \mathbf{f}, \mathbf{g}\rangle)$ the eigenvalues for coupled boundary conditions depend on the conditions and on the space.

The Mukhtarov–Yakubov theory was applied to get very general self-adjoint regular and singular boundary conditions by Sun, Wang, and Zettl, [94]. In particular, they showed, as mentioned before, that the condition $\det(K) = 1$ required in the one-interval theory can be extended to $\det(K) > 1$. This was further extended to $\det(K) \neq 0$ by Wang and Zettl [102]; this means that the Mukhtarov–Yakubov theory applies to all coupled self-adjoint boundary conditions.

Remark 8.3.2. Note that $w > 0$ ensures that $L^2(J, w)$ is a Hilbert space. However, if $w < 0$ on J, then we can multiply the equation by -1 to obtain

$$-(-py')' + (-q)y = \lambda(-w)y \quad \text{on } J$$

and observe that the one-interval theory applies to this equation since there is no sign restriction on either p or q and $-w > 0$. Also, the boundary conditions are homogeneous and thus invariant with respect to multiplication by -1. We will further apply these observations to one or both equations (8.23) to extend the restriction $\det(K) > 0$ to $\det(K) \neq 0$. The assumption $p > 0$ is commonly used in the literature and in books, but it is not needed for the characterization of the self-adjoint operators characterized by the equation. This fact allows us to extend the Mukhtarov–Yakubov restriction $h > 0$, $k > 0$ to any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. (However, note that, in general, the spectral properties, the oscillatory behavior of eigenfunctions, and so on of a given self-adjoint operator S are different when p is not positive.)

As in the one-interval case, the Lagrange sesquilinear form $[\cdot,\cdot]$ is fundamental to the study of boundary value problems. It is defined, for $\mathbf{y} = \{y_1, y_2\}$, $\mathbf{z} = \{z_1, z_2\}$, $y_1, z_1 \in D_{\max}(J_1)$, and $y_2, z_2 \in D_{\max}(J_2)$, by

$$[\mathbf{y}, \mathbf{z}] = h[y_1, z_1]_1(b) - h[y_1, z_1]_1(a) + k[y_2, z_2]_2(d) - k[y_2, z_2]_2(c), \tag{8.31}$$

where

$$[y_r, z_r]_r = y_r(p_r\overline{z_r'}) - \overline{z_r}(p_r y_r') = Z_r^* E Y_r \tag{8.32}$$

and

$$Y_r = \begin{bmatrix} y_r \\ y_r^{[1]} \end{bmatrix}, \quad Z_r = \begin{bmatrix} z_r \\ z_r^{[1]} \end{bmatrix}, \quad y_r^{[1]} = (p_r y_r'), \quad r = 1,2; \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.33}$$

Note that the two-interval Lagrange form $[\mathbf{y}, \mathbf{z}]$ "connects" all four endpoints with each other and depends on h and k.

The two-interval maximal and minimal domains and operators are simply the direct sums of the corresponding one-interval domains and operators:

$$D_{\max}(J_1, J_2) = D_{\max}(J_1) \dotplus D_{\max}(J_2), \quad D_{\min}(J_1, J_2) = D_{\min}(J_1) \dotplus D_{\min}(J_2), \tag{8.34}$$

$$S_{\max}(J_1, J_2) = S_{\max}(J_1) \dotplus S_{\max}(J_2), \quad S_{\min}(J_1, J_2) = S_{\min}(J_1) \dotplus S_{\min}(J_2). \tag{8.35}$$

Note that the maximal and minimal domains and operators do not depend on h and k.

Definition 8.3.1. Let the above hypotheses and notation hold. By a self-adjoint realization of the two equations

$$-(p_r y')' + q_r y = \lambda w_r y \quad \text{on } J_r, \quad r = 1,2,$$

in the space $H = (L^2(J_1, w_1) \dotplus L^2(J_2, w_2), \langle \cdot, \cdot \rangle)$ we mean an operator S from H into H that satisfies

$$S_{\min}(J_1, J_2) \subset S = S^* \subset S_{\max}(J_1, J_2). \tag{8.36}$$

From (8.36) it is clear that the two-interval self-adjoint realizations are distinguished from each other only by their domains. A characterization of *these* domains in terms of boundary conditions is a major goal of the two-interval theory. Each operator S satisfying (8.36) can be considered an extension of the minimal operator $S_{\min}(J_1, J_2)$ or, equivalently, a restriction of the maximal operator $S_{\max}(J_1, J_2)$. Let d_1 and d_2 be the deficiency indices on J_1 and J_2, respectively.

Our starting point for the two-interval theory is the following:

Lemma 8.3.1. *We have*

(1) $S^*_{\min}(J_1 J_2) = S^*_{\min}(J_1) \dotplus S^*_{\min}(J_2) = S_{\max}(J_1) \dotplus S_{\max}(J_2) = S_{\max}(J_1, J_2)$ *and* $S^*_{\max}(J_1, J_2) = S^*_{\max}(J_1) \dotplus S^*_{\max}(J_2) = S_{\min}(J_1) \dotplus S_{\min}(J_2) = S_{\min}(J_1, J_2)$.
In particular, $D_{\max}(J_1, J_2) = D(S_{\max}(J_1, J_2)) = D(S_{\max}(J_1)) \dotplus D(S_{\max}(J_2))$ *and* $D_{\min}(J_1, J_2) = D(S_{\min}(J_1, J_2)) = D_{\min}(J_1) \dotplus D_{\min}(J_2)$.

(2) *The minimal operator* $S_{\min}(J_1, J_2)$ *is a closed symmetric densely defined operator in the Hilbert space* H *with deficiency index* $d = d_1 + d_2$.

Proof. See Lemma 13.3.1 in [113]. Since the coefficients and weight functions are all real, the upper and lower deficiency indices are equal, the common value is denoted by d in the two-interval case, and by d_1 and d_2 for one and two intervals. □

We state the next theorem for endpoints that are either LP or LC but indicate at the end of the theorem how the characterizations can be simplified at each regular endpoint.

Theorem 8.3.1. *Let the two-interval minimal and maximal domains be* $D_{\min} = D_{\min}(J_1, J_2)$ *and* $D_{\max} = D_{\max}(J_1, J_2)$, *and let the operators* $S_{\min} = S_{\min}(J_1, J_2)$ *and* $S_{\max} = S_{\max}(J_1, J_2)$ *be defined as before. Let* d *denote the deficiency index of* $S_{\min}$ *in* H. *Then* $0 \leq d \leq 4$, *and all values in this range are realized. Let the Lagrange form* $[\cdot, \cdot]$ *be given by* (8.31). *Then all self-adjoint operators* S *satisfying* (8.36) *can be characterized as follows.*

Case 1. $d = 0$. *This case occurs if and only if all four endpoints are LP. In this case,* $S_{\min} = S_{\max}$, *which is a self-adjoint operator in* H *with no proper self-adjoint extension. Thus in this case, there are no boundary conditions required or allowed. Also note that for all* $\mathbf{f} = \{f_1, f_2\}$, $\mathbf{g} = \{g_1, g_2\} \in D_{\max}$, *we have* $[f_1, g_1]_1(b) = 0$, $[f_1, g_1]_1(a) = 0$, $[f_2, g_2]_2(d) = 0$, $[f_2, g_2]_2(c) = 0$, *and therefore*

$$[\mathbf{f}, \mathbf{g}] = h[f_1, g_1]_1(b) - h[f_1, g_1]_1(a) + k[f_2, g_2]_2(d) - k[f_2, g_2]_2(c) = 0.$$

Case 2. $d = 1$. *This case occurs if and only if exactly three endpoints are LP; the other, say* $s \in \{a, b, c, d\}$, *is regular or LC. Let* (u, v) *be a boundary condition basis at* s. *Then*

$$D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max} : c_{11}[y, u](s) + c_{12}[y, v](s) = 0, \\ c_{11}, c_{12} \in \mathbb{R},\ (c_{11}, c_{12}) \neq (0, 0)\},$$

is a self-adjoint domain. Conversely, if $D(S)$ *is a self-adjoint domain, then there exist* $c_{11}, c_{12} \in \mathbb{R}$ *with* $(c_{11}, c_{12}) \neq (0, 0)$ *such that* (8.36) *holds.*

To summarize this case, we say that all self-adjoint extensions of the minimal operator are determined by separated boundary conditions of the form (8.36) *at the non-LP endpoint* s.

Case 3. $d = 2$. *This case occurs if and only if exactly two of the four endpoints are LP. There are two subcases.*

(i) *Assume that* a, b *are the two non-LP endpoints.*

In this case, all the self-adjoint extensions S *in* H *are given by* $S = S(J_1) \dotplus S_{\min}(J_2)$, *where* $S(J_1)$ *is an arbitrary self-adjoint extension in* H_1 *obtained from the one-interval theory on* J_1. *Note that* $S_{\min}(J_2)$ *is self-adjoint by the one-interval theory since both endpoints* c, d *are LP and all* $S(J_1)$ *are obtained from the one-interval theory discussed in Part* I. *There is a similar result when* a, b *are both LP and* c, d *are non-LP.*

To summarize this case, we can say that all self-adjoint operators in H *are obtained simply as direct sums of the minimal operator from the interval with the two LP endpoints together with all the self-adjoint operators from the other interval, and these are characterized by the one-interval theory of Part* I.

(ii) *The two non-LP endpoints are from different intervals. In this subcase, there are nontrivial interactions between the two intervals that are not directly obtainable from the one-interval theory. These depend on h and k when the boundary conditions are coupled.*

Assume that b and c are the two non-LP endpoints. Let (u_1, v_1) *be a boundary basis at b, and let* (u_2, v_2) *be a boundary basis at c. Suppose the matrices* $B, C \in M_2(\mathbb{C})$ *satisfy the following two conditions:*

(1) *The matrix* $(B : C)$ *has full rank.*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kBEB^* - hCEC^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max} : B\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) = 0\},$$

where

$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \quad \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}$$

is the domain of a self-adjoint operator S in H that satisfies (8.36)*, and every operator S in H satisfying* (8.36) *is obtained this way.*

Assume that a and d are the two non-LP endpoints. Let (u_1, v_1) *be a boundary basis at a, and let* (u_2, v_2) *a boundary basis at d. Suppose* $A, D \in M_2(\mathbb{C})$ *satisfy the following two conditions:*

(1) *The matrix* $(A : D)$ *has full rank.*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kAEA^* - hDED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max} : A\mathbf{Y}_1(a) + D\mathbf{Y}_2(d) = 0\},$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y_1, u_1]_1(a) \\ [y_1, v_1]_1(a) \end{bmatrix}, \quad \mathbf{Y}_2(d) = \begin{bmatrix} [y_2, u_2]_2(d) \\ [y_2, v_2]_2(d) \end{bmatrix}$$

is the domain of a self-adjoint operator S in H satisfying (8.36)*, and every operator S in H satisfying* (8.36) *is obtained this way.*

Assume that a and c are the two non-LP endpoints. Let (u_1, v_1) *be a boundary basis at a, and let* (u_2, v_2) *be a boundary basis at c. Suppose* $A, C \in M_2(\mathbb{C})$ *satisfy the following two conditions:*

(1) *The matrix* $(A : C)$ *has full rank.*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kAEA^* + hCEC^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max} : A\mathbf{Y}_1(a) + C\mathbf{Y}_2(c) = 0\},$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y_1, u_1]_1(a) \\ [y_1, v_1]_1(a) \end{bmatrix}, \quad \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}$$

is the domain of a self-adjoint operator S in H satisfying (8.36)*, and every operator S in H satisfying* (8.36) *is obtained this way.*

Assume that b and d are the two non-LP endpoints. Let (u_1, v_1) *be a boundary basis at b, and let* (u_2, v_2) *be a boundary basis at d. Suppose* $B, D \in M_2(\mathbb{C})$ *satisfy the following two conditions:*
(1) *The matrix* $(B : D)$ *has full rank.*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kBEB^* + hDED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max} : B\mathbf{Y}_1(b) + D\mathbf{Y}_2(d) = 0\},$$

where

$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \quad \mathbf{Y}_2(d) = \begin{bmatrix} [y_2, u_2]_2(d) \\ [y_2, v_2]_2(d) \end{bmatrix}$$

is the domain of a self-adjoint operator S in H satisfying (8.36)*, and every operator S in H satisfying* (8.36) *is obtained this way.*

Case 4. $d = 3$. *In this case, there is exactly one LP endpoint. Assume that a is LP. Let* (u_1, v_1) *be a boundary basis at b, let* (u_2, v_2) *be a boundary basis at c, and let* (u_3, v_3) *be a boundary basis at d. Suppose* $B = (b_{ij})$, $C = (c_{ij})$, *and* $D = (d_{ij})$ *are* 3×2 *matrices with complex entries satisfying the following two conditions:*
(1) *The matrix* (B, C, D) *has full rank,*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kBEB^* - hCEC^* + hDED^* = 0.$$

Then

$$D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max} : B\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) + D\mathbf{Y}_3(d) = 0\},$$

where

$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \quad \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}, \quad \mathbf{Y}_3(d) = \begin{bmatrix} [y_2, u_3]_2(d) \\ [y_2, v_3]_2(d) \end{bmatrix}$$

is the domain of a self-adjoint operator S in H satisfying (8.36)*, and every operator S in H satisfying* (8.36) *is obtained this way.*

The cases where exactly one of b, c, d is LP are similar.

Case 5. $d = 4$. *This is the case where there is no LP endpoint, that is, each endpoint is either regular or LC. Let (u_1, v_1) be a boundary basis at a, let (u_2, v_2) be a boundary basis at b, let (u_3, v_3) be a boundary basis at c, and let (u_4, v_4) be a boundary basis at d. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ satisfying* (8.36) *if there exist* 4×2 *matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$ with complex entries such that the* 4×8 *matrix (A, B, C, D) whose first two columns are those of A, the second two columns are those of B, and so on satisfies the following two conditions:*

(1) *The matrix (A, B, C, D) has full rank.*
(2) *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$kAEA^* - kBEB^* + hCEC^* - hDED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and $D(S)$ is the set of $\mathbf{y} = \{y_1, y_2\} \in D_{\max}$ satisfying

$$A\mathbf{Y}_1(a) + B\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) + D\mathbf{Y}_2(d) = 0, \quad \mathbf{Y}_i = \begin{bmatrix} [y_i, u_i]_i \\ [y_i, v_i]_i \end{bmatrix}, \quad i = 1, 2.$$

Furthermore, every operator S satisfying (8.36) *is obtained this way.*

In each of these cases, if $t \in \{a, b, c, d\}$ is a regular endpoint, then $Y_r(t)$ can be replaced by

$$\begin{bmatrix} y_r(t) \\ y_r^{[1]}(t) \end{bmatrix}, \quad r = 1, 2.$$

Proof. See [94] for a proof for positive h and k and Remark 8.3.2 for the case where one or both of h and k are negative. □

Next, we give some illustrative examples for both regular and singular problems.

8.3.1 Regular endpoints

Although, as stated in Theorem 8.3.1, the conditions at a regular endpoint can be obtained from the LC conditions at that point, here we give some examples to illustrate this in view of the wide interest in regular problems.

Example 8.3.1. Separated boundary conditions at all four regular endpoints:

$$\begin{aligned} A_1y(a) + A_2y^{[1]}(a) &= 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0); \\ B_1y(b) + B_2y^{[1]}(b) &= 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0,0); \\ C_1y(c) + C_2y^{[1]}(c) &= 0, \quad C_1, C_2 \in \mathbb{R}, \quad (C_1, C_2) \neq (0,0); \\ D_1y(d) + D_2y^{[1]}(d) &= 0, \quad D_1, D_2 \in \mathbb{R}, \quad (D_1, D_2) \neq (0,0). \end{aligned}$$

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case the 4×8 matrix (A, B, C, D) has full rank, and

$$0 = AEA^* = BEB^* = CEC^* = DED^*.$$

Note that this case is independent of h, k.

Example 8.3.2. Separated boundary conditions at a and d and coupled conditions at b, c:

$$\begin{aligned} A_1y(a) + A_2(py')(a) &= 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0); \\ D_1y(d) + D_2(py')(d) &= 0, \quad D_1, D_2 \in \mathbb{R}, \quad (D_1, D_2) \neq (0,0) \end{aligned}$$

and

$$Y(c) = e^{i\gamma}KY(b), \quad Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix},$$

$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad 1 \leq i, j \leq 2, \quad \det K \neq 0, \quad -\pi < \gamma \leq \pi.$$

Let A, D be as in Example 8.3.1, then $\operatorname{rank}(A, D) = 2$ and $kAEA^* - hDED^* = 0$ for any h, k since $0 = AEA^* = DED^*$. Let

$$C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = e^{i\gamma} \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad -\pi < \gamma \leq \pi.$$

Then a straightforward computation shows that

$$hCEC^* = kBEB^*$$

is equivalent with

$$hE = k(\det K)E,$$

which is equivalent with

$$h = k \det K.$$

Since this holds for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$, it follows from Theorem 8.3.1 that the boundary conditions of this example are self-adjoint for any $K \in M_2(\mathbb{R})$ with $\det(K) \neq 0$.

Remark 8.3.3. In the one-interval theory, $\det K = 1$ is required for self-adjointness. We find it remarkable that the one-interval condition $\det K = 1$ extends to $\det(K) \neq 0$ in the two-interval theory and that this generalization follows from two simple observations: (i) The Mukhtarov–Yakubov [82] observation that for $h > 0$ and $k > 0$, using inner product multiples produces an interaction between the two intervals yielding $\det(K) > 0$, and (ii) the Wang–Zettl [102] observation that the boundary value problem is invariant under multiplication by -1 yields the further extension $\det(K) \neq 0$. This is optimal in the sense that when K is singular, the boundary condition is separated, not coupled. Let $h < 0$ and $k > 0$. Now apply Theorem 8.3.1 to the equations

$$M_1 y = -(-p_1 y')' + (-q_1)y = \lambda(-hw_1)y \quad \text{on } J_1$$

and

$$M_2 y = -(p_2 y')' + q_2 y = \lambda(kw_2)y \quad \text{on } J_2$$

to obtain the result for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. Note that both the equation $r = 1$ and its boundary conditions are invariant under multiplication by -1. If $h > 0$ and $k < 0$, then the proof is the same with the roles of equations $r = 1$ and $r = 2$ interchanged.

If the boundary conditions are coupled for the endpoint pair a, d and the pair b, c, then the parameters h, k play a role in both sets of coupled boundary conditions. The next example illustrates this point.

Example 8.3.3. Two pairs of coupled conditions, with $-\pi < \gamma_1, \gamma_2 \leq \pi$,

$$Y(d) = e^{i\gamma_1} G Y(a), \quad G = (g_{ij}), \quad g_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det G \neq 0,$$
$$Y(c) = e^{i\gamma_2} K Y(b), \quad K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det K \neq 0, \quad Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}.$$

Proceeding as in the previous example, we obtain the equivalence of the conditions for self-adjointness:

$$kGEG^* = hE \quad \text{and} \quad kKEK^* = hE,$$
$$k\det G = h \quad \text{and} \quad k\det K = h,$$

that is,

$$\det G = \det K = \frac{h}{k}.$$

This shows that these boundary conditions are self-adjoint for any positive or negative h, k.

See the next section for examples with discontinuous boundary conditions.

8.3.2 Singular endpoints

Here we illustrate the self-adjoint boundary conditions given by Theorem 8.3.1 when at least one endpoint is singular. The conditions when $d = 0$ or 1 are the same as in the one-interval case and are independent of h and k. In these cases the self-adjoint extensions in the Hilbert space H are the same as those of the usual direct sum Hilbert space H_u. So we give examples here only for $d = 2$, $d = 3$, and $d = 4$.

Notation 8.3.1. *In the following examples in this section and the next, (u_1, v_1) denotes a boundary condition basis at a, (u_2, v_2) a boundary condition basis at b, (u_3, v_3) a boundary condition basis at c, and (u_4, v_4) a boundary condition basis at d. Also, we use $[y, u_r]$ as an abbreviation for $[y_r, u_r]$ and $[y, v_r]$ as an abbreviation for $[y_r, v_r]$, $r = 1, 2, 3, 4$.*

Example 8.3.4. Let $d = 2$. Let a and d be the two non-LP endpoints. Suppose that the boundary conditions at a and d are coupled:

$$\begin{bmatrix} [y, u_1](a) \\ [y, v_1](a) \end{bmatrix} = K \begin{bmatrix} [y, u_3](d) \\ [y, v_3](d) \end{bmatrix},$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det K \neq 0.$$

Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D = K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

Then $\operatorname{rank}(A, D) = 2$. From a straightforward computation it follows that

$$kAEA^* = hDED^*$$

is equivalent with

$$k = h \det K.$$

By Theorem 8.3.1 we have that if $h = 1$ and $k > 0$ satisfies $\det K = k$, then these boundary conditions are self-adjoint.

Using Remark 8.3.3, this result extends to any positive or negative h, k as in the previous examples.

Example 8.3.5. Let $d = 2$. Let a and c be the two non-LP endpoints. Let the boundary conditions at a, c be given by

$$\begin{bmatrix} [y, u_3](c) \\ [y, v_3](c) \end{bmatrix} = K \begin{bmatrix} [y, u_1](a) \\ [y, v_1](a) \end{bmatrix},$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det K \neq 0.$$

Let

$$A = K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $\operatorname{rank}(A, C) = 2$. By a straightforward computation we see that

$$kAEA^* + hCEC^* = 0$$

is equivalent with

$$k \det K = -h.$$

Therefore, if $k = 1$, $h > 0$, and $\det K = -h$, then these boundary conditions are self-adjoint. This extends to $\det K = +h$ as in the previous examples.

Remark 8.3.4. By changing the weight function w_1 to hw_1 we can generate self-adjoint operators for any real coupling matrix K satisfying $\det K \neq 0$.

Example 8.3.6. Let $d = 3$. Let b, c, d be regular or LC endpoints. Consider separated boundary conditions at d and coupled conditions at b, c:

$$A_1[y, u_4](d) + A_2[y_2, v_4](d) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0, 0);$$
$$\begin{bmatrix} [y_2, u_3](c) \\ [y_2, v_3](c) \end{bmatrix} = K \begin{bmatrix} [y_1, u_2](b) \\ [y_1, v_2](b) \end{bmatrix},$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det K \neq 0.$$

Let

$$B = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ A_1 & A_2 \end{bmatrix}.$$

In this case, $\operatorname{rank}(B, C, D) = 3$, and $DED^* = 0$. Then, in terms of Theorem 8.3.1, we obtain the equivalence of the conditions for self-adjointness:

$$h = k \det K.$$

Thus, if $k = 1$ and $h > 0$ satisfies $\det K = h$, then these boundary conditions are self-adjoint, and this extends to $\det K = -h$ as before.

In the following example, we still let three endpoints b, c, and d be regular or LC, but let boundary conditions at c be separated, and let the boundary conditions at b, d be coupled.

Example 8.3.7. Let $d = 3$. Let

$$C_1[y, u_3](c) + C_2[y, v_3](c) = 0, \quad C_1, C_2 \in \mathbb{R}, \quad (C_1, C_2) \neq (0, 0);$$
$$\begin{bmatrix} [y, u_4](d) \\ [y, v_4](d) \end{bmatrix} = K \begin{bmatrix} [y, u_2](b) \\ [y, v_2](b) \end{bmatrix},$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det K \neq 0.$$

Let

$$B = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Then $\operatorname{rank}(B, C, D) = 3$, and $hCEC^* = 0$ for any h since $CEC^* = 0$. Proceeding as in the previous example, we obtain the equivalence of the conditions for self-adjointness:

$$k \det K + h = 0.$$

This shows that these are self-adjoint boundary conditions when $k = 1$, $h > 0$, and $\det K = -h$, and this extends to $\det K = +h$ as before.

Example 8.3.8. Let $d = 3$. Separated boundary conditions at b and coupled conditions at c, d:

$$B_1[y, u_2](b) + B_2[y, v_2](b) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0, 0);$$
$$C \begin{bmatrix} [y, u_3](c) \\ [y, v_3](c) \end{bmatrix} + D \begin{bmatrix} [y, u_4](d) \\ [y, v_4](d) \end{bmatrix} = 0.$$

Then $kBEB^* = 0$ for any k since $BEB^* = 0$. In terms of Theorem 8.3.1, these boundary conditions are self-adjoint if and only if $\operatorname{rank}(C, D) = 2$ and

$$CEC^* - DED^* = 0.$$

Note that these conditions are independent of h and k and are simply the one-interval self-adjointness conditions for each of the two intervals separately. Thus this example just gives the two-interval self-adjointness conditions that are generated by the direct sum of self-adjoint operators from each of the two intervals separately.

Example 8.3.9. Let $d = 4$, that is, each endpoint is either regular or LC. Separated boundary conditions at b and at d and coupled conditions at a, c:

$$B_1[y,u_2](b) + B_2[y,v_2](b) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0,0);$$
$$D_1[y,u_4](d) + D_2[y,v_4](d) = 0, \quad D_1, D_2 \in \mathbb{R}, \quad (D_1, D_2) \neq (0,0);$$
$$\begin{bmatrix} [y,u_3](c) \\ [y,v_3](c) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} [y,u_1](a) \\ [y,v_1](a) \end{bmatrix},$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i,j = 1,2, \quad \det K \neq 0.$$

Let

$$A = \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case, $\operatorname{rank}(A,B,C,D) = 4$, and $kBEB^* + hDED^* = 0$ for any h, k since $0 = BEB^* = DED^*$. Therefore these boundary conditions are self-adjoint if

$$k \det K + h = 0.$$

If we choose $k = 1$ and $h > 0$ such that $\det K = -h < 0$, then the boundary conditions are self-adjoint. As before, this extends to $\det K \neq 0$.

For $d = 4$, see [113, 109, 94], and the next section for more examples.

8.4 Transmission and interface conditions

In this section we show that the particular case of the one-interval theory,

$$-\infty \leq a < b = c < d \leq +\infty, \quad J_1 = (a,b), \quad J_2 = (c,d), \quad J = (a,d), \tag{8.37}$$

produces the regular transmission and interface conditions used in the cited references and more general regular conditions and singular analogues of all these regular conditions.

Remark 8.4.1. When comparing the results with those of the one-interval theory from Part I, it is important to note that the interval (a,d) in (8.37) plays the role of the interval (a,b) in Part I.

We start with two simple but important observations, which help illustrate how the special case (8.37) of the two-interval theory produces regular and singular transmission and interface conditions.

Remark 8.4.2. To connect the two-interval theory discussed in Section 8.3 to the transmission and interface conditions mentioned in the Introduction, a key observation is that the direct sum Hilbert space $L^2(J_1, w_1) \dotplus L^2(J_2, w_2)$ can be identified with the space $L^2(J, w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2. Note that even though $b = c$, there are still four endpoint classifications since the endpoint c may have LC and LP classifications on (a, c) different from those on (c, d). To emphasize this point and to relate to the notation commonly used for regular transmission and interface conditions, we use the notation c^+ when c is a right endpoint, that is, for the interval (a, c), and c^- for c as an endpoint of the interval (c, d).

Remark 8.4.3. In this section, we show that the self-adjointness conditions for the "interval" (c^+, c^-), which produce the transmission and interface conditions, are, surprisingly, more general than the corresponding one-interval conditions of Part I. This is due to the influence of the parameters h, k and the observation that the one-interval boundary conditions are invariant with respect to multiplication by -1.

Throughout this section, we assume that (8.21) holds and note that this implies that (8.22) holds when c is a regular endpoint for both intervals (a, c) and (c, b). In this case the one-interval theory of Part I can be applied to the interval $J = (a, d)$.

Remark 8.4.4. Note that c is the right endpoint of the interval J_1 and the left endpoint of the interval J_2. When c is a regular endpoint for both intervals (a, c) and (c, b), the one-interval theory can be applied to the interval $J = (a, d)$, but this theory does not produce any self-adjoint operator in $L^2(J, w)$ with a condition that requires a jump discontinuity at c, since condition (8.22) implies that all functions in the maximal domain $D_{\max}(a, d)$ and thus all solutions of (8.21) and their quasi-derivatives are continuous at c. But, as we will see further, the two-interval theory generates self-adjoint operators in $L^2(J, w)$ with boundary conditions that specify jump discontinuities at regular interior points and self-adjoint boundary conditions at interior singular points, which in general have infinite jumps. Such boundary conditions may be separated or coupled; in the separated case, they are generally called "transmission" conditions in the literature; in the coupled case, they are often referred to as "interface" conditions. But both "transmission" and "interface" conditions transmit conditions from one interval to the other.

We start with the case where both outer endpoints are LP and the interior point c is regular from both sides because this case highlights the jump discontinuities at c.

Corollary 8.4.1. *Let* (8.3) *hold and assume that c is a regular endpoint for both intervals* (a, c) *and* (c, d) *and that both outer endpoints a and d are LP. Let* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. *Suppose the matrices* $C, D \in M_2(\mathbb{C})$ *satisfy*

$$\operatorname{rank}(C, D) = 2, \quad kCEC^* = hDED^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.38}$$

Let

$$D(S)=\left\{y\in D_{\max}(J_1,J_2):CY(c^+)+DY(c^-)=0,\,Y=\begin{bmatrix} y\\ (py')\end{bmatrix}\right\}. \tag{8.39}$$

Define the operator S in $L^2(J,w)$ *by* $S(y)=S_{\max}(J_1,J_2)y$ *for* $y\in D(S)$. *Then S is self-adjoint in* $L^2(J,w)$.

In (8.39),

$$Y=\begin{bmatrix} y\\ (py')\end{bmatrix}, \tag{8.40}$$

and $y(c^+)$, $(py')(c^+)$, $y(c^-)$, $(py')(c^-)$ denote the appropriate one-sided limits. Note that these limits exist and are finite.

As in Theorem 8.3.1, the boundary conditions (8.39) can be categorized into two mutually exclusive classes, separated and coupled. The separated conditions have the general form

$$h_1y(c^-)+k_1(py')(c^-)=0,\quad h_1,k_1\in\mathbb{R},\quad (h_1,k_1)\neq(0,0), \tag{8.41}$$

$$h_2y(c^+)+k_2\quad(py')(c^+)=0,\quad h_2,k_2\in\mathbb{R},\quad (h_2,k_2)\neq(0,0), \tag{8.42}$$

and these have the canonical forms

$$\cos\alpha y(c^-)-\sin\alpha(py')(c^-)=0,\quad \alpha\in[0,\pi), \tag{8.43}$$

$$\cos\beta y(c^+)-\sin\beta(py')(c^+)=0,\quad \beta\in(0,\pi]. \tag{8.44}$$

Note that these separated conditions do not depend on h and k. (We follow the customary parameterizations for α and β even though these play no role in this book.)

The coupled conditions have the canonical form

$$\begin{bmatrix} y(c^+)\\ (py')(c^+)\end{bmatrix}=e^{i\gamma}K\begin{bmatrix} y(c^-)\\ (py')(c^-)\end{bmatrix},\quad -\pi<\gamma\leq\pi,\quad i=\sqrt{-1}, \tag{8.45}$$

with $K\in M_2(\mathbb{R})$ and $\det(K)\neq 0$.

Proof. Note that in this case, we can identify the direct sum space $L^2(J_1,w_1)\dotplus L^2(J_2,w_2)$ with the space $L^2(J,w)$ where $w=w_1$ on J_1 and $w=w_2$ on J_2. For $y=(y_1,y_2)\in D_{\max}(J_1,J_2)$, let $y=y_1$ on J_1 and y_2 on J_2. Then define $y(c)$ using (8.40) and note that $y\in D_{\max}(J)$. The conclusion then follows from case 3, part (ii), of Theorem 8.3.1 and Remark 8.3.2. □

Remark 8.4.5. We comment on Corollary 8.4.1. Note the similarity between the conditions of Corollary 8.4.1 and the regular one-interval self-adjoint boundary conditions of Theorem 1.15. These are similar to conditions (1.15)–(1.18) and (1.23). Comparing

Corollary 8.4.1 with the regular one-interval theory, we see that the regular separated self-adjointness conditions on the nondegenerate interval (a, d) are the same as the separated jump conditions on the "interval" (c^+, c^-). However, remarkably, the coupled condition on the "interval" (c^+, c^-) only requires $\det K \neq 0$, in contrast with the requirement that $\det K = 1$ in the one-interval theory in Chapter 1. This is due to the influence of the inner product parameters h, k and the observation that on each interval the boundary value problem is invariant under multiplication by −1. Although we have conditions on the narrow "interval" (c^+, c^-) rather than on nondegenerate interval (a, d), the influence of k is felt by c^- and the influence of h by c^+. Note that when $K = I$, the identity matrix, and $\gamma = 0$, condition (8.45) is just the continuity condition for y and (py') at c, and therefore this case generates the one-interval minimal operator $S_{\min}(a, d)$, and this is the only self-adjoint operator in this case since a, d are both LP.

Remark 8.4.6. In much of the literature the separated jump conditions (8.41)–(8.42) and (8.43)–(8.44) are called "transmission" conditions, whereas special cases of the coupled conditions (8.45) are called "interface" conditions. The separated conditions are real, but note that the coupled jump conditions are nonreal when $\gamma \neq 0$ and $\gamma \neq \pi$.

Next, we give an analogue of Corollary 8.4.1 when c is LC.

Corollary 8.4.2. *Assume that c is LC for both intervals (a, c) and (c, d) and that both outer endpoints a and d are LP. Let (u_1, v_1) be a boundary condition bases at c for the interval (a, c), and let (u_2, v_2) be a boundary condition bases at c for the interval (c, d).*

Suppose the matrices $C, D \in M_2(\mathbb{C})$ satisfy (8.38).

Let

$$D(S) = \left\{ \mathbf{y} \in D_{\max}(J_1, J_2) : C \begin{bmatrix} [y, u_1](c^+) \\ [y, v_1](c^+) \end{bmatrix} + D \begin{bmatrix} [y, u_2](c^-) \\ [y, v_2](c^-) \end{bmatrix} = 0 \right\}. \tag{8.46}$$

Define the operator S in $L^2(J, w)$ by $S(\mathbf{y}) = M_{\max}(J_1, J_2)\mathbf{y}$ for $\mathbf{y} \in D(S)$. Then S is self-adjoint in $L^2(J, w)$.

Here

$$[y, u_r](c^+),\ [y, v_r](c^+),\ [y, u_r](c^-),\ [y, v_r](c^-), \quad r = 1, 2, \tag{8.47}$$

exist as finite limits.

As in Theorem 8.3.1, *the boundary conditions* (8.39)–(8.42) *can be categorized into two mutually exclusive classes, separated and coupled. The separated conditions have the general form*

$$h_1[y, u_1](c^+) + k_1[y, v_1](c^+) = 0, \quad h_1, k_1 \in \mathbb{R}, \quad (h_1, k_1) \neq (0, 0), \tag{8.48}$$

$$h_2[y, u_2](c^-) + k_2[y, v_2](c^-) = 0, \quad h_2, k_2 \in \mathbb{R}, \quad (h_2, k_2) \neq (0, 0), \tag{8.49}$$

and these have the canonical form

$$\cos\alpha[y,u_1](c^+) - \sin\alpha[y,v_1](c^+) = 0, \quad \alpha \in [0,\pi), \tag{8.50}$$

$$\cos\beta[y,u_2](c^-) - \sin\beta[y,v_2](c^-) = 0, \quad \beta \in (0,\pi]. \tag{8.51}$$

The coupled conditions have the canonical form

$$\begin{bmatrix} [y,u_2](c^-) \\ [y,v_2](c^+) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} [y,u_1](c^+) \\ [y,v_1](c^+) \end{bmatrix}, \quad -\pi < \gamma \le \pi, \quad i = \sqrt{-1}, \tag{8.52}$$

with $K \in M_2(\mathbb{R})$ *and* $\det(K) \neq 0$.

Proof. As in Corollary 8.4.1, we note that we can identify the direct sum space $L^2(J_1, w_1) \dotplus L^2(J_2, w_2)$ with the space $L^2(J,w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2 and then use Theorem 8.3.1. □

Remark 8.4.7. Note that Corollary 8.4.2 parallels Corollary 8.4.1 with the jump conditions on the Lagrange forms rather than on y and $y^{[1]}$. Such a parallel result holds generally when the assumption that an endpoint is regular is replaced by the assumption that this endpoint is LC. At an LP endpoint, there is no boundary condition. Conditions (8.48)–(8.49) and (8.50)–(8.51) are the singular analogues of the regular separated jump conditions; condition (8.52) is the singular analogue of the regular jump condition (8.45). Thus (8.41)–(8.42) and (8.43)–(8.44) could be called singular transmission conditions and (8.52) singular coupled interface conditions, but we have not seen any of these singular jump conditions studied in the literature before the publication of [103]. Note that whereas the Lagrange brackets $[y,u_1]$, $[y,v_1]$, $[y,u_2](c^+)$, $[y,v_2](c^+)$ exist and are finite at c^+ and c^-, the solutions y and their quasi-derivatives (py') in general are not continuous at c; they may blow up, that is, be infinite at c^+ or c^-, or they may oscillate wildly at c^+ or c^-.

The next corollary shows how case 5 of Theorem 8.3.1 can be used to get self-adjoint jump conditions at an interior point c when c is LC for both intervals (a,c) and (c,d) and each of a,d is LC. The cases where one or more of these four endpoints are regular then follows as before in Corollary 8.4.1.

Corollary 8.4.3. *Assume that c is LC for both intervals* (a,c) *and* (c,d) *and that both outer endpoints a and d are LC. Let* (u_1,v_1) *be a boundary condition basis at a,* (u_2,v_2) *be a boundary condition basis at* c^-, (u_3,v_3) *be a boundary condition bases at* c^+, *and* (u_4,v_4) *be a boundary condition basis at d, respectively.*

Suppose that for some $h,k \in \mathbb{R}$, $h \neq 0 \neq k$, *the matrices* $A,B,C,D \in M_2(\mathbb{C})$ *satisfy*

$$\operatorname{rank}(A,D) = 2, \quad kAEA^* = hDED^*, \quad \operatorname{rank}(B,C) = 2, \quad hBEB^* = kCEC^*, \tag{8.53}$$

where $E = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. *With* $Y_r = \left[\begin{smallmatrix} [y,u_r] \\ [y,v_r] \end{smallmatrix}\right]$, $r = 1,2,3,4$, *define* $D(S)$ *to be the set of all* $\mathbf{y} \in D_{\max}(J_1,J_2)$ *satisfying*

$$AY_1(a) + DY_4(d) = 0, \quad CY_3(c^+) + BY_2(c^-) = 0, \tag{8.54}$$

and define the operator S in $L^2(J,w)$ *by* $S(\mathbf{y}) = M_{\max}(J_1,J_2)\mathbf{y}$ *for* $\mathbf{y} \in D(S)$, $\mathbf{y} = \{y_1,y_2\}$. *Then S is self-adjoint in* $L^2(J,w)$.

Here

$$[y,u_r](t),\ [y,v_r](t), \quad \text{for } t = a,c^-,c^+,d; \quad r = 1,2,3,4, \tag{8.55}$$

exist as finite limits.

As before, each of the two boundary conditions (8.54) *can be categorized into two mutually exclusive classes, separated and coupled, and these have the canonical forms given there.*

Proof. As before, we identify the direct sum space $L^2(J_1,w_1) \dotplus L^2(J_2,w_2)$ with the space $L^2(J,w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2. For $y = (y_1,y_2) \in D_{\max}(J_1,J_2)$, let $y = y_1$ on J_1 and y_2 on J_2. Then define $y(c)$ using (8.40) and note that $y \in D_{\max}(J)$. Now we apply case 5 of Theorem 8.3.1. Let

$$A_0 = \begin{bmatrix} A \\ O \end{bmatrix}, \quad B_0 = \begin{bmatrix} O \\ B \end{bmatrix}, \quad C_0 = \begin{bmatrix} O \\ C \end{bmatrix}, \quad D_0 = \begin{bmatrix} D \\ O \end{bmatrix}, \tag{8.56}$$

where O denotes the 2×2 zero matrix. Note that these 4×2 matrices satisfy the conditions of case 5 of Theorem 8.3.1; the matrix (A_0,B_0,C_0,D_0) has full rank, the self-adjointness conditions hold, and the matrices A_0,B_0,C_0,D_0 and boundary condition reduce to (8.54). □

As before, at each regular endpoint t, $Y_r(t) = \begin{bmatrix} [y,u_r](t) \\ [y,v_r](t) \end{bmatrix}$ in (8.54) can be replaced by $\begin{bmatrix} y(t) \\ (py')(t) \end{bmatrix}$, and the boundary conditions can be given in canonical form. We do this in the next corollary for the case where c is regular for both intervals (a,c) and (c,d). But we emphasize that this can be done for each combination of endpoints, that is, for each Y_r independent of the other three Y_r.

Corollary 8.4.4. *Assume that c is regular for both intervals* (a,c) *and* (c,d) *and that both outer endpoints a and d are LC. Let* (u_1,v_1) *be a boundary condition bases at a, and let* (u_4,v_4) *be a boundary condition bases at d. Suppose the matrices* $A,B,\ C,D \in M_2(\mathbb{C})$ *satisfy* (8.53).

Define $D(S)$ *to be the set of all* $y \in D_{\max}(J_1,J_2)$ *satisfying*

$$AY_1(a) + DY_4(d) = 0, \quad CY_3(c^+) + BY_2(c^-) = 0, \tag{8.57}$$

where

$$Y_1(a) = \begin{bmatrix} [y,u_1](a) \\ [y,v_1](a) \end{bmatrix}, \quad Y_2(c^-) = \begin{bmatrix} y(c^-) \\ (py')(c^-) \end{bmatrix},$$
$$Y_3(c^+) = \begin{bmatrix} y(c^+) \\ (py')(c^+) \end{bmatrix}, \quad Y_4(d) = \begin{bmatrix} [y,u_4](d) \\ [y,v_4](d) \end{bmatrix},$$

and define the operator S in $L^2(J, w)$ *by*

$$S(y) = S_{\max}(J_1, J_2)y, \quad y \in D(S). \tag{8.58}$$

Then S is self-adjoint in $L^2(J, w)$.

Recall that each of the two boundary conditions in (8.57) *consists of separated and coupled conditions and these have the following canonical forms:*

$$\cos\alpha[y, u_1](a) - \sin\alpha[y, v_1](a) = 0, \quad 0 \le \alpha < \pi,$$
$$\cos\beta[y, u_4](d) - \sin\beta[y, v_4](d) = 0, \quad 0 < \beta \le \pi;$$
$$Y(d) = e^{i\gamma}KY(a), \quad -\pi < \gamma \le \pi, \tag{8.59}$$

where $Y(a)$*,* $Y(d)$ *are given in* (8.58)*, and*

$$\cos\alpha \quad y(c^-) - \sin\alpha(py')(c^-) = 0, \quad \alpha \in [0, \pi),$$
$$\cos\beta \quad y(c^+) - \sin\beta(py')(c^+) = 0, \quad \beta \in (0, \pi];$$
$$\begin{bmatrix} y(c^+) \\ (py')(c^+) \end{bmatrix} = e^{i\gamma}K \begin{bmatrix} y(c^-) \\ (py')(c^-) \end{bmatrix}, \quad -\pi < \gamma \le \pi, \tag{8.60}$$

with $K \in M_2(\mathbb{R})$ *and* $\det(K) \ne 0$.

Proof. This follows from Theorem 8.3.1 and Corollaries 8.4.1, 8.4.2, and 8.4.3. □

Example 8.4.1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ m & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{8.61}$$

It is easy to check that if $h = k > 0$, then the self-adjointness conditions of Theorem 8.3.1 are satisfied for any $m \in \mathbb{R}$. These four matrices yield the following boundary conditions:

$$y(a) = 0 = y(d), \quad y(b) = y(c), \quad (py')(b) - (py')(c) = -my(c). \tag{8.62}$$

Thus, if $b = c$, conditions (8.61) require y to be continuous at $b = c$ but allow the quasi-derivative to have a jump discontinuity at c. If this jump is proportional to the value of y at c with real proportionality constant $-m$ ($m = 0$ is allowed and reduces to the continuous case), then the jump is self-adjoint. Note that the conditions at a, d are independent of those at c, b and the conditions at a, d can be replaced by any self-adjoint conditions at these two endpoints, that is, by

$$A_1EA_1^* = D_1ED_1^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{rank}(A_1, D_1) = 2,$$

where A_1, D_1 are 2×2 matrices, and A, D are the 4×2 matrices respectively obtained by inserting two rows of zeros between the two rows of A_1 and between the two rows of D_1.

Example 8.4.2. Replacing the matrix C in the previous example by

$$C = \begin{bmatrix} 0 & 0 \\ -1 & m \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

we get a self-adjoint problem for any real m by choosing $h = k > 0$. When $b = c$, the quasi-derivatives are continuous at b, but the solutions are discontinuous when $m \neq 0$. In this case the self-adjoint boundary conditions are:

$$y(a) = 0 = y(d), \quad (py')(c^+) = (py')(c^-), \quad y(c^+) - y(c^-) = -m(py')(c).$$

These two examples can be found in [109], where they were established by a completely different method using Green's functions.

Remark 8.4.8. We remark that the above corollaries are only a few of the particular cases of case 5 of Theorem 8.3.1 when the endpoints satisfy (8.37). There are many more. Others can be obtained with other choices of the matrices A_0, B_0, C_0, D_0 and with other endpoint classifications. Also, corollaries of cases 3 and 4 of Theorem 8.3.1 can be similarly obtained.

Remark 8.4.9. From the perspective of the two-interval theory and the corollaries discussed in this section, there are many self-adjoint S-L operators in Hilbert spaces $L^2(J, w)$ and direct sums of such spaces, which are generated by discontinuous regular and singular, separated and coupled, boundary conditions.

Remark 8.4.10. The extension of the two-interval theory discussed before to any finite number of intervals is routine when the inner product constants corresponding to h and k are all equal to 1; see Everitt and Zettl [36], where this is done for SL and higher-order problems. For unequal h' and k', the multiinterval theory has to take into account the interactions between these. An infinite interval theory is also developed in [36], but the extension to an infinite number of intervals is not routine and requires additional technical considerations.

Remark 8.4.11. See Everitt, Shubin, Stolz, and Zettl [35] for a discussion of self-adjoint Sturm–Liouville problems with an infinite number of interior singularities including an extension of the Titchmarsh–Weyl dichotomy for square-integrable solutions and the corresponding m-coefficient. Applications to the one-dimensional Schrödinger equation extend the earlier work of Gesztesy and Kirsch [42].

Remark 8.4.12. From the perspective of the finite and infinite interval theories we see that there are many self-adjoint S-L operators in the spaces $L^2(J, w)$ and the direct sums of these spaces with modified inner products, which are not covered by the classical modern one-interval theory. Which of these will become celebrated classical operators in applied mathematics corresponding to the Bessel, Legendre, Laguerre, Jacobi, and other operators?

8.5 Comments

In this chapter, the two-interval theory is applied to get discontinuous boundary conditions at an interior point of the underlying interval. But the general two-interval theory of this chapter can also be applied to disjoint intervals. In that case the "discontinuous" boundary conditions are "jump conditions" from one interval to another. These jumps are finite if they occur at a point that is regular from both sides and may be infinite otherwise. The general two-interval theory can also be applied to overlapping intervals.

9 The Green's and characteristic functions

9.1 Introduction

We construct the Green's function for two-interval regular self-adjoint and nonself-adjoint Sturm–Liouville problems. The two intervals may be disjoint, overlap, or be identical.

As mentioned in the Introduction to Chapter 8, Sturm–Liouville problems with boundary conditions requiring discontinuous eigenfunctions or discontinuous derivatives of eigenfunctions have been studied recently by many authors. As a particular case, our construction of the Green's and characteristic functions applies to such problems. The Green's function construction is modeled on a construction of Neuberger [85] for the one-interval case. This construction differs from the usual one found in textbooks and in most of the literature, in that the discontinuity of the derivative of the Green's function along the diagonal occurs naturally, in contrast to the usual construction as found, for example, in Coddington and Levinson [24], where this discontinuity is assigned a priori as part of the construction.

9.2 The characteristic function

First, we study the two-interval characteristic function for general, not necessarily self-adjoint, boundary conditions. Let

$$J_r = (a_r, b_r), \quad -\infty < a_r < b_r < \infty, \quad r = 1, 2,$$

and assume that the coefficients and weight functions satisfy

$$\frac{1}{p_r}, q_r, w_r \in L(J_r, \mathbb{C}), \quad r = 1, 2. \tag{9.1}$$

Define the differential expressions M_r by

$$M_r y = -(p_r y')' + q_r y \quad \text{on } J_r, \quad r = 1, 2. \tag{9.2}$$

We further use the notation with a subscript r denoting the rth interval. The subscript r is sometimes omitted when it is clear from the context. We consider the second-order scalar differential equations

$$-(p_r y')' + q_r y = \lambda w_r y \quad \text{on } J_r, \quad r = 1, 2, \quad \lambda \in \mathbb{C}, \tag{9.3}$$

together with boundary conditions

$$A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0, \quad Y_r = \begin{bmatrix} y_r \\ (p_r y_r') \end{bmatrix}, \quad r = 1, 2. \tag{9.4}$$

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Here $A_r, B_r \in M_{4\times 2}(\mathbb{C})$, $r = 1, 2$. By (9.1) and the basic theory of linear ordinary differential equations the boundary condition (9.4) is well defined.

Next, we construct the characteristic function whose zeros are precisely the eigenvalues of the two-interval Sturm–Liouville problem. Let

$$P_r = \begin{bmatrix} 0 & \frac{1}{p_r} \\ q_r & 0 \end{bmatrix}, \quad W_r = \begin{bmatrix} 0 & 0 \\ w_r & 0 \end{bmatrix}. \tag{9.5}$$

Then the scalar equation (9.2) is equivalent to the first-order system

$$Y' = (P_r - \lambda W_r)Y = \begin{bmatrix} 0 & \frac{1}{p_r} \\ q_r - \lambda w_r & 0 \end{bmatrix} Y, \quad Y = \begin{bmatrix} y \\ p_r y' \end{bmatrix}. \tag{9.6}$$

Note that given any scalar solution y_r of $-(p_r y')' + q_r y = \lambda w_r y$ on J_r, the vector Y_r defined by (9.6) is a solution of the system $Y' = (P_r - \lambda W_r)Y$ on J_r. Conversely, given any vector solution Y_r of system $Y' = (P_r - \lambda W_r)Y$ its top component y_r is a solution of $-(p_r y')' + q_r y = \lambda w_r y$.

Let $\Phi_r(\cdot, u_r, P_r, w_r, \lambda)$ be the primary fundamental matrix of (9.6). We have

$$\Phi_r' = (P_r - \lambda W_r)\Phi_r \quad \text{on } J_r, \quad \Phi_r(u_r, u_r, \lambda) = I, \quad a_r \le u_r \le b_r, \quad \lambda \in \mathbb{C}, \tag{9.7}$$

where I denotes the 2×2 identity matrix. Here we use the notation $\Phi_r = \Phi_r(\cdot, u_r, P_r, w_r, \lambda)$ to indicate the dependence of the primary fundamental matrix on these quantities. Since here P_r, w_r are fixed, we simplify the notation to $\Phi_r(\cdot, u_r, \lambda)$. By (9.1) $\Phi(b_r, a_r, \lambda)$ exists.

Define the characteristic function Δ by

$$\begin{aligned} \Delta(\lambda) &= \Delta(a_1, b_1, a_2, b_2, A_1, B_1, A_2, B_2, P_1, P_2, w_1, w_2, \lambda) \\ &= \det[(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))], \quad \lambda \in \mathbb{C}, \end{aligned} \tag{9.8}$$

where $(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda))$ denotes the 4×4 complex matrix whose first two columns are those of $A_1 + B_1\Phi_1(b_1, a_1, \lambda)$, and the second two columns are those of $A_2 + B_2\Phi_2(b_2, a_2, \lambda)$.

Definition 9.2.1. By a trivial solution of equation $M_r y = \lambda w_r y$ on some interval I_r we mean a solution y_r that is identical zero on I_r and whose quasi-derivative $(p_r y_r')$ is also identically zero on I_r. (I_r may be a subinterval of J_r, or it may be the whole interval J_r.) Note that under assumptions (9.1), a solution y_r might be identically zero on I_r, but its quasi-derivative $(p_r y_r')$ might not be identically zero on I_r.

Definition 9.2.2. By a trivial solution of the two-interval Sturm–Liouville equations (9.1) we mean a solution $\mathbf{y} = \{y_1, y_2\}$ each of whose components y_r is a trivial solution of equation $M_r y = \lambda w_r y$ on J_r, $r = 1, 2$, that is, y_r and $(p_r y_r')$ both are identically zero on J_r, $r = 1, 2$.

Definition 9.2.3. A complex number λ is called an eigenvalue of the two-interval SL boundary value problems (BVP) if the two-interval SL equations (9.1) have a nontrivial solution $\mathbf{y}$ satisfying the boundary conditions (9.4). Such a solution $\mathbf{y}$ is called an eigenfunction of λ. Any multiple of an eigenfunction is also an eigenfunction.

The next theorem characterizes the eigenvalues of boundary value problems as zeros of the characteristic function.

Theorem 9.2.1. *A complex number λ is an eigenvalue of the boundary value problems* (9.3)–(9.4) *if and only if* $\Delta(\lambda) = 0$.

Proof. If λ is an eigenvalue and $\mathbf{y} = \{y_1, y_2\}$ is an eigenfunction of λ, then there exist $C_r \in M_{2\times 1}(\mathbb{C})$, $r = 1, 2$, and at least one of the vectors C_1 and C_2 is nonzero, so that

$$Y_r(t) = \Phi_r(t, a_r, \lambda)C_r. \tag{9.9}$$

Note that

$$\Phi_r(a_r, a_r, \lambda) = I, \quad r = 1, 2.$$

Substituting (9.9) into the boundary conditions, we obtain

$$A_1C_1 + B_1\Phi_1(b_1, a_1, \lambda)C_1 + A_2C_2 + B_2\Phi_2(b_2, a_2, \lambda)C_2 = 0. \tag{9.10}$$

Set $C = \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right]$. Then (9.8) can be written as

$$\big(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda)\big)C = 0. \tag{9.11}$$

Since $C \neq 0$ and λ is an eigenvalue of BVP by assumption, it follows that

$$\det\big[\big(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda)\big)\big] = 0,$$

that is, $\Delta(\lambda) = 0$.

Conversely, suppose $\Delta(\lambda) = 0$. Then (9.11) has a nontrivial vector solution $C \in M_{4\times 1}(\mathbb{C})$. We use the notation $C_1 \in M_{2\times 1}(\mathbb{C})$ to denote the vector whose rows are the first two rows of C, and $C_2 \in M_{2\times 1}(\mathbb{C})$ denotes the vector whose rows are the last two rows of C. At least one of the vectors C_1 and C_2 is nontrivial. Solve the initial value problems

$$Y' = (P_r - \lambda W_r)Y \quad \text{on } J_r, \quad Y_r(a_r) = C_r, \quad r = 1, 2.$$

Then

$$Y_r(b_r) = \Phi_r(b_r, a_r, \lambda)Y_r(a_r)$$

and

$$\big(A_1 + B_1\Phi_1(b_1, a_1, \lambda)\big)Y_1(a_1) + \big(A_2 + B_2\Phi_2(b_2, a_2, \lambda)\big)Y_2(a_2) = 0.$$

Therefore we have that $\mathbf{y} = \{y_1, y_2\}$ is an eigenfunction of the BVP (9.3)–(9.4), where y_r is the top component of Y_r, $r = 1, 2$. This shows that λ is an eigenvalue of this BVP. □

9.3 The Green's function

Since, as mentioned before, our method of constructing the Green's function – even in the one-interval case – is not the standard one generally found in the literature and in textbooks. We make it self-contained by presenting the basic theory used in the construction for convenience of the reader.

Let p_r^{-1}, q_r, w_r satisfy (9.1), and let $f_r \in L(J_r, \mathbb{C})$. We consider the two-interval inhomogeneous boundary value problem

$$-\left(p_r y'\right)' + q_r y = \lambda w_r y + f_r \quad \text{on } J_r = (a_r, b_r), \quad r = 1,2, \quad \lambda \in \mathbb{C}, \tag{9.12}$$

$$A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0, \quad Y_r = \begin{bmatrix} y_r \\ (p_r y_r') \end{bmatrix}, \quad r = 1,2. \tag{9.13}$$

The boundary value problem (9.12)–(9.13) is equivalent to the system boundary value problem

$$Y' = (P_r - \lambda W_r)Y + F_r, \quad A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0, \tag{9.14}$$

where P_r, W_r are defined by (9.5), and

$$F_r = \begin{bmatrix} 0 \\ -f_r \end{bmatrix}. \tag{9.15}$$

Let $\Phi_r = \Phi_r(\cdot,\cdot,\lambda)$ be the primary fundamental matrix of the homogeneous system

$$Y' = (P_r - \lambda W_r)Y. \tag{9.16}$$

Note that

$$\Phi_r(t, u_r, \lambda) = \Phi_r(t, a_r, \lambda)\Phi_r(a_r, u_r, \lambda) \tag{9.17}$$

for $a_r \le t, u_r \le b_r$.

The next theorem is a particular case of the well-known Fredholm alternative.

Theorem 9.3.1. *The following statements are equivalent:*

(1) *when* $\mathbf{f} = \{f_1, f_2\} = 0$*, that is,* $f_r = 0$ *on* J_r*,* $r = 1,2$*, the two-interval boundary value problem* (9.12)–(9.13) *and consequently also* (9.16) *has only the trivial solution.*
(2) *The matrix* $[A_1 + B_1\Phi_1(b_1, a_1, \lambda) | A_2 + B_2\Phi_2(b_2, a_2, \lambda)]$ *has an inverse.*
(3) *For every* $\mathbf{f} = \{f_1, f_2\}$*,* $f_r \in L(J_r, \mathbb{C})$*,* $r = 1,2$*, each of the problems* (9.12), (9.13)*, and* (9.16) *has only the trivial solution.*

Proof. We know that Y_r is a solution of

$$Y' = (P_r - \lambda W_r)Y + F_r \quad \text{on } J_r \tag{9.18}$$

if and only if y_r is a solution of

$$-\left(p_r y'\right)' + q_r y = \lambda w_r y + f_r \quad \text{on } J_r, \tag{9.19}$$

where

$$Y_r = \begin{bmatrix} y_r \\ (p_r y_r') \end{bmatrix}.$$

For $C_r = \left[\begin{smallmatrix} c_{r1} \\ c_{r2} \end{smallmatrix}\right]$, $c_{r1}, c_{r2} \in \mathbb{C}$, $r = 1, 2$, determine a solution Y_r of (9.16) on J_r by the initial condition

$$Y_r(a_r, \lambda) = C_r.$$

Then y_r is a solution of (9.12) determined by the initial conditions $y_r(a_r, \lambda) = c_{r1}$ and $(p_r y_r')(a_r, \lambda) = c_{r2}$.

By the variation-of-parameters formula we have

$$Y_r(t, \lambda) = \Phi_r(t, a_r, \lambda) C_r + \int_{a_r}^{t} \Phi_r(t, s, \lambda) F_r(s)\, ds, \quad a_r \le t \le b_r. \tag{9.20}$$

In particular,

$$Y_r(b_r, \lambda) = \Phi_r(b_r, a_r, \lambda) C_r + \int_{a_r}^{b_r} \Phi_r(b_r, s, \lambda) F_r(s)\, ds.$$

Let $D(\lambda) = (A_1 + B_1 \Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2 \Phi_2(b_2, a_2, \lambda))$ and $C = \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right]$. Then

$$\begin{aligned} &A_1 Y_1(a_1, \lambda) + B_1 Y_1(b_1, \lambda) + A_2 Y_2(a_2, \lambda) + B_2 Y_2(b_2, \lambda) \\ &\quad = D(\lambda) C + B_1 \int_{a_1}^{b_1} \Phi_1(b_1, s, \lambda) F_1(s)\, ds + B_2 \int_{a_2}^{b_2} \Phi_2(b_2, s, \lambda) F_2(s)\, ds. \end{aligned} \tag{9.21}$$

When $f_r = 0$ on J_r $(r = 1, 2)$, $\mathbf{Y} = \{Y_1, Y_2\}$ and $\mathbf{y} = \{y_1, y_2\}$ are nontrivial solutions if and only if C is not the zero vector. By (9.21) we have that when $f_r = 0$ on $J_r (r = 1, 2)$, there is a nontrivial solution $\{Y_1, Y_2\}$ (and a nontrivial solution $\{y_1, y_2\}$) of (9.12) satisfying the boundary conditions

$$A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0$$

if and only if $D(\lambda)$ is singular. It also follows from (9.21) that there is a unique solution $\{Y_1, Y_2\}$ satisfying the boundary conditions (9.13) for every $f_r \in L(J_r, \mathbb{C})$, $r = 1, 2$, if and only if $D(\lambda)$ is nonsingular. Similarly, there is a unique solution $\mathbf{y} = \{y_1, y_2\}$ satisfying the boundary conditions (9.13) for every $\mathbf{f} = \{f_1, f_2\}$, $f_r \in L(J_r, \mathbb{C})$, $r = 1, 2$, if and only if $D(\lambda)$ is nonsingular. □

Next, we construct the Green's function for two-interval boundary value problems. Assume that

$$D(\lambda) = \big(A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda)\big)$$

is nonsingular. We use the notation $D_1(\lambda)$ to denote the 2×4 matrix whose rows are the first two rows of $D^{-1}(\lambda)$, and $D_2(\lambda)$ denotes the 2×4 matrix whose rows are the last two rows of $D^{-1}(\lambda)$. Let

$$G_1(t,s,\lambda) = \begin{cases} -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_1\Phi_1(b_1,s,\lambda), & a_1 \le t < s \le b_1, \\ -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_1\Phi_1(b_1,s,\lambda) + \Phi_1(t,s,\lambda), & a_1 \le s \le t \le b_1, \end{cases}$$

$$\widetilde{G}_1(t,s,\lambda) = -\Phi_1(t,a_1,\lambda)D_1(\lambda)B_2\Phi_2(b_2,s,\lambda), \quad a_1 \le t \le b_1, \quad a_2 \le s \le b_2,$$

$$G_2(t,s,\lambda) = -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_1\Phi_1(b_1,s,\lambda), \quad a_2 \le t \le b_2, a_1 \le s \le b_1,$$

$$\widetilde{G}_2(t,s,\lambda) = \begin{cases} -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_2\Phi_2(b_2,s,\lambda), & a_2 \le t < s \le b_2, \\ -\Phi_2(t,a_2,\lambda)D_2(\lambda)B_2\Phi_2(b_2,s,\lambda) + \Phi_2(t,s,\lambda), & a_2 \le s \le t \le b_2. \end{cases}$$

Theorem 9.3.2. *Assume $D(\lambda)$ is nonsingular, that is, $[A_1 + B_1\Phi_1(b_1, a_1, \lambda) \mid A_2 + B_2\Phi_2(b_2, a_2, \lambda)]^{-1}$ exists. Then for any $\mathbf{f} = \{f_1, f_2\}$, $f_r \in L(J, \mathbb{C})$, $r = 1, 2$, the unique solution $\mathbf{y} = \{y_1, y_2\}$ of* (9.12)–(9.13) *and the unique solution $\mathbf{Y} = \{Y_1, Y_2\}$ of* (9.18)*, respectively, are given by*

$$y_1(t) = -\int_{a_1}^{b_1} G_{1,(12)}(t,s,\lambda)f_1(s)ds - \int_{a_2}^{b_2} \widetilde{G}_{1,(12)}(t,s,\lambda)f_2(s)ds, \quad a_1 \le t \le b_1, \tag{9.22}$$

$$y_2(t) = -\int_{a_1}^{b_1} G_{2,(12)}(t,s,\lambda)f_1(s)ds - \int_{a_2}^{b_2} \widetilde{G}_{2,(12)}(t,s,\lambda)f_2(s)ds, \quad a_2 \le t \le b_2, \tag{9.23}$$

$$Y_1(t) = \int_{a_1}^{b_1} G_1(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_1(t,s,\lambda)F_2(s)\,ds, \quad a_1 \le t \le b_1, \tag{9.24}$$

$$Y_2(t) = \int_{a_1}^{b_1} G_2(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_2(t,s,\lambda)F_2(s)\,ds, \quad a_2 \le t \le b_2. \tag{9.25}$$

Set $\mathbf{K}(t,s,\lambda) = \{K_1(t,s,\lambda), K_2(t,s,\lambda)\}$, where

$$K_1(t,s,\lambda) = \begin{cases} G_1(t,s,\lambda), & a_1 \le s \le b_1, \\ \widetilde{G}_1(t,s,\lambda), & a_2 \le s \le b_2, \end{cases} \quad a_1 \le t \le b_1,$$

$$K_2(t,s,\lambda) = \begin{cases} G_2(t,s,\lambda), & a_1 \le s \le b_1, \\ \widetilde{G}_2(t,s,\lambda), & a_2 \le s \le b_2, \end{cases} \quad a_2 \le t \le b_2.$$

We call $\mathbf{K}(t,s,\lambda) = \mathbf{K}(t,s,\lambda,P_1,P_2,W_1,W_2,A_1,A_2,B_1,B_2)$ (here we use the complete notation to highlight the dependence of $\mathbf{K}$ on these quantities), the Green's matrix of the *regular boundary value problem*

$$Y_2(t) = \int_{a_1}^{b_1} G_2(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_2(t,s,\lambda)F_2(s)\,ds, \quad a_2 \le t \le b_2,$$

(9.6), *and* (9.4). *And we call* $\mathbf{K}_{12} = \{K_{1,(12)}, K_{2,(12)}\}$ *the Green's function of two-interval boundary value problem*

$$Y_2(t) = \int_{a_1}^{b_1} G_2(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_2(t,s,\lambda)F_2(s)\,ds, \quad a_2 \le t \le b_2,$$

(9.3), *and* (9.4).

Proof. Let

$$C = D^{-1}(\lambda)\left(-B_1\int_{a_1}^{b_1} \Phi_1(b_1,s,\lambda)F_1(s)\,ds - B_2\int_{a_2}^{b_2} \Phi_2(b_2,s,\lambda)F_2(s)\,ds\right).$$

By (9.21) we have

$$A_1Y_1(a_1) + B_1Y_1(b_1) + A_2Y_2(a_2) + B_2Y_2(b_2) = 0.$$

Recalling the notation $D_1(\lambda)$ and $D_2(\lambda)$, we have

$$C_1 = D_1(\lambda)\left(-B_1\int_{a_1}^{b_1} \Phi_1(b_1,s,\lambda)F_1(s)\,ds - B_2\int_{a_2}^{b_2} \Phi_2(b_2,s,\lambda)F_2(s)\,ds\right),$$
$$C_2 = D_2(\lambda)\left(-B_1\int_{a_1}^{b_1} \Phi_1(b_1,s,\lambda)F_1(s)\,ds - B_2\int_{a_2}^{b_2} \Phi_2(b_2,s,\lambda)F_2(s)\,ds\right).$$

From (9.13) we obtain that

$$\begin{aligned} Y_1(t) &= \Phi_1(t,a_1,\lambda)D_1(\lambda)\left(-B_1\int_{a_1}^{b_1} \Phi_1(b_1,s,\lambda)F_1(s)\,ds - B_2\int_{a_2}^{b_2} \Phi_2(b_2,s,\lambda)F_2(s)\,ds\right) \\ &\quad + \int_{a_1}^{t} \Phi_1(t,s,\lambda)F_1(s)\,ds \\ &= \int_{a_1}^{b_1} [\Phi_1(t,a_1,\lambda)D_1(\lambda)(-B_1\Phi_1(b_1,s,\lambda)F_1(s))]\,ds + \int_{a_1}^{t} \Phi_1(t,s,\lambda)F_1(s)\,ds \\ &\quad + \int_{a_2}^{b_2} [\Phi_1(t,a_1,\lambda)D_1(\lambda)(-B_2\Phi_2(b_2,s,\lambda)F_2(s))]\,ds \end{aligned}$$

$$= \int_{a_1}^{b_1} G_1(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_1(t,s,\lambda)F_2(s)\,ds, \quad a_1 \le t \le b_1.$$

$$\begin{aligned} Y_2(t) &= \Phi_2(t,a_2,\lambda)D_2(\lambda)\left(-B_1\int_{a_1}^{b_1}\Phi_1(b_1,s,\lambda)F_1(s)\,ds - B_2\int_{a_2}^{b_2}\Phi_2(b_2,s,\lambda)F_2(s)\,ds\right) \\ &\quad + \int_{a_2}^{t}\Phi_2(t,s,\lambda)F_2(s)\,ds \\ &= \int_{a_1}^{b_1}\big[\Phi_2(t,a_2,\lambda)D_2(\lambda)\big(-B_1\Phi_1(b_1,s,\lambda)F_1(s)\big)\big]\,ds + \int_{a_2}^{t}\Phi_2(t,s,\lambda)F_2(s)\,ds \\ &\quad + \int_{a_2}^{b_2}\big[\Phi_2(t,a_2,\lambda)D_2(\lambda)\big(-B_2\Phi_2(b_2,s,\lambda)F_2(s)\big)\big]\,ds \\ &= \int_{a_1}^{b_1} G_2(t,s,\lambda)F_1(s)\,ds + \int_{a_2}^{b_2} \widetilde{G}_2(t,s,\lambda)F_2(s)\,ds, \quad a_2 \le t \le b_2. \end{aligned}$$

Note that (9.22) and (9.23) follow directly from these formulas for $Y_1(t)$ and $Y_2(t)$, respectively, by taking the upper right component, that is,

$$y_1(t) = -\int_{a_1}^{b_1} G_{1,(12)}(t,s,\lambda)f_1(s)\,ds - \int_{a_2}^{b_2} \widetilde{G}_{1,(12)}(t,s,\lambda)f_2(s)\,ds, \quad a_1 \le t \le b_1,$$

$$y_2(t) = -\int_{a_1}^{b_1} G_{2,(12)}(t,s,\lambda)f_1(s)\,ds - \int_{a_2}^{b_2} \widetilde{G}_{2,(12)}(t,s,\lambda)f_2(s)\,ds, \quad a_2 \le t \le b_2. \qquad \square$$

Remark 9.3.1. Note that the above construction of the Green's function and the characteristic function does not assume any symmetry or self-adjointness of the problem. The coefficients p_r, q_r, w_r may be complex valued, and the boundary conditions need not be self-adjoint. If w_r is identically zero on the whole interval J_r, then there is no λ dependence, and the problem becomes degenerate. Similarly, in the case where $1/p_r$ is identically zero on J_r, the problem can be considered degenerate.

9.4 Examples

In this section, we give examples to illustrate that the construction of the two-interval Green's function can be applied to problems with transmission and interface conditions as mentioned in the Introduction.

To avoid unnecessary subscripts, we let

$$J_1 = (a,b), \quad J_2 = (c,d), \quad b = c \tag{9.26}$$

and use $c^+ = b$ for the right endpoint of J_1 and $c^- = c$ for the left endpoint of J_2. Also, we let $A = A_1$, $B = B_1$, $C = A_2$, and $D = B_2$ in (9.13).

Using this notation, we make the following simple but key observation.

Remark 9.4.1. Recall the simple but important observation: when $b = c$, the direct sum of the Hilbert spaces from the two intervals (a,b) and (c,d) can be identified with the Hilbert space of the "outer" interval (a,d) :

$$L^2((a,b),w_1) \dotplus L^2((c,d),w_2) = L^2((a,d),w), \tag{9.27}$$

where w_1 is the restriction of w to J_1, and w_2 is the restriction of w to J_2. In each example below the given boundary conditions generate a self-adjoint operator in the Hilbert space $L^2((a,d),w)$.

The first example has separated boundary conditions: these are often called "transmission conditions" in the literature.

Example 9.4.1 (Transmission conditions). Separated boundary conditions:

$$\begin{aligned} A_1y(a) + A_2y^{[1]}(a) &= 0, \quad A_1,A_2 \in \mathbb{R}, \quad (A_1,A_2) \neq (0,0); \\ B_1y(b) + B_2y^{[1]}(b) &= 0, \quad B_1,B_2 \in \mathbb{R}, \quad (B_1,B_2) \neq (0,0); \\ C_1y(c) + C_2y^{[1]}(c) &= 0, \quad C_1,C_2 \in \mathbb{R}, \quad (C_1,C_2) \neq (0,0); \\ D_1y(d) + D_2y^{[1]}(d) &= 0, \quad D_1,D_2 \in \mathbb{R}, \quad (D_1,D_2) \neq (0,0). \end{aligned} \tag{9.28}$$

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case the 4×8 matrix (A,B,C,D) has full rank, and

$$0 = AEA^* = BEB^* = CEC^* = DED^*.$$

Considering $(a,c] \cup [c,d)$ as one interval (a,d), the next example has transmission conditions at the outer endpoint a,d and interface conditions at c. This example is chosen to highlight the (discontinuous) interface conditions at an interior point c. The roles of the endpoints a, c^+, c^-, d can be interchanged in this example (but care must be taken regarding the signs of the matrices A,B,C,D; see [102]).

Example 9.4.2. Let $h,k \in \mathbb{R}$, $h \neq 0 \neq k$. Separated boundary conditions at a and at d and coupled jump conditions at c:

$$\begin{aligned} A_1y(a) + A_2(py')(a) &= 0, \quad A_1,A_2 \in \mathbb{R}, \quad (A_1,A_2) \neq (0,0); \\ D_1y(d) + D_2(py')(d) &= 0, \quad D_1,D_2 \in \mathbb{R}, \quad (D_1,D_2) \neq (0,0); \end{aligned}$$

and

$$Y(c) = e^{i\gamma} K Y(b), \quad Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}, \quad K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad 1 \le i, j \le 2,$$

$$\det K \neq 0, \quad -\pi < \gamma \le \pi.$$

Let A, D be as in Example 9.4.1. Then $\operatorname{rank}(A, D) = 2$ and $kAEA^* - hDED^* = 0$ for any h, k since $0 = AEA^* = DED^*$. Let

$$C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = e^{i\gamma} \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad -\pi < \gamma \le \pi.$$

Then a straightforward computation shows that

$$hCEC^* = kBEB^*$$

is equivalent with

$$hE = k(\det K)E,$$

which is equivalent with

$$h = k \det K.$$

Since this holds for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$, it follows from Theorem 2 in [102] that the boundary conditions of this example are self-adjoint for any $K \in M_2(\mathbb{R})$ with $\det(K) \neq 0$.

The next remark highlights a remarkable comparison with the well-known classical one-interval self-adjoint boundary conditions; see [113].

Remark 9.4.2. It is well known that in the one-interval theory, $\det K = 1$ is required for self-adjointness of the coupled boundary conditions. We find it remarkable that the one-interval condition $\det K = 1$ extends to $\det(K) \neq 0$ in the two-interval theory and that this generalization follows from two simple observations: (i) The Mukhtarov–Yakubov [82] observation that for $h > 0$ and $k > 0$, using inner product multiples produces an interaction between the two intervals yielding $\det(K) > 0$ and (ii) the Wang–Zettl observation that the boundary value problem is invariant under multiplication by -1, and this yields the further extension $\det(K) \neq 0$. Note that the parameters h, k play no role in Example 9.4.1 when the boundary conditions are separated.

The next example illustrates the situation where there are two sets of coupled, that is, "jump" boundary conditions. In one case the jumps are between the outer endpoints a, d and the other between the inner "endpoints", $b = c^+$ and $c = c^-$.

Example 9.4.3. Two pairs of coupled conditions with $-\pi < \gamma_1, \gamma_2 \leq \pi$:

$$Y(d) = e^{i\gamma_1} GY(a), \quad G = (g_{ij}), \quad g_{ij} \in \mathbb{R}, \quad i,j = 1,2, \quad \det G \neq 0,$$
$$Y(c) = e^{i\gamma_2} KY(b), \quad K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, \quad i,j = 1,2, \quad \det K \neq 0, \quad Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}. \tag{9.29}$$

Proceeding as in the previous example, we obtain the equivalence of the conditions for self-adjointness:

$$kGEG^* = hE \quad \text{and} \quad kKEK^* = hE;$$
$$k \det G = h \quad \text{and} \quad k \det K = h;$$

that is,

$$\det G = \det K = \frac{h}{k}.$$

This shows that (9.29) are self-adjoint boundary conditions for any positive or negative h, k.

9.5 Comments

This chapter is based on the paper by Wang and Zettl [102]. As mentioned in the Introduction of this chapter, the Green's function construction used here is modeled by a construction of Neuberger [85] for the one-interval case. This one-interval construction differs from the usual one found in textbooks and in most of the literature in that the discontinuity of the derivative of the Green's function along the diagonal occurs naturally, in contrast to the usual construction as found, for example, in Coddington and Levinson [24], where this discontinuity is assigned a priori as part of the construction. This one-interval construction is extended here to two-interval self-adjoint and nonself-adjoint problems.

The next remark is from J. W. Neuberger and published here with his permission. It is of interest not only because we refer to "a construction of Neuberger" in the Introduction of this chapter but also for pedagogical reasons.

Remark 9.5.1 (J. W. Neuberger). In the spring of 1958, I taught my first graduate course. It was an introduction to functional analysis by means of Sturm–Liouville problems. As was, and still is, my custom, I didn't lecture, but rather I broke up material for the class into a sequence of problems. The night before I was concerned with finding problems that gave a good introduction to Green's functions to the class. The standard "recipe" with its prescribed discontinuity, seemed contrived. I managed to come up with the algebraic method discussed here. Problems for some simple examples quickly led to the general case, again algebraically. To me this remains an example of how "teaching" and "research" can impact one another, particularly in a nonlecture situation. If I had been lecturing, I would have given the standard approach, the only one I knew the day before. The algebraic approach to Green's functions might have never seen the light of day, and some nice mathematics would have been missed.

10 The Legendre equation and its operators

10.1 Introduction

The Legendre equation

$$-\left(py'\right)' = \lambda y, \quad p(t) = 1 - t^2, \quad \lambda \in \mathbb{C}, \tag{10.1}$$

is one of the simplest singular Sturm–Liouville differential equations. Its potential function q is zero, its weight function w is the constant 1, and its leading coefficient p is a simple quadratic.

Equation (10.1) and its associated self-adjoint operators exhibit a surprisingly wide variety of interesting phenomena. We survey some of these. Of course, one of the main reasons this equation is important in many areas of pure and applied mathematics stems from the fact that it has interesting solutions. Indeed, the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ form a complete orthogonal set of functions in $L^2(0,\infty)$, and for $n \in \mathbb{N}_0$, $y = P_n(t)$ is a solution of (10.1) when $\lambda = \lambda_n = n(n+1)$. Properties of the Legendre polynomials can be found in several textbooks including the remarkable book of Szego [95]. Most of the results discussed here can be inferred from known results scattered widely in the literature; others require some additional work. It is remarkable that we can find some new results about this simple and well-studied equation and the operators it generates.

In this chapter, equation (10.1) and its associated self-adjoint operators are studied on each of the three intervals

$$J_1 = (-\infty,-1), \quad J_2 = (-1,1), \quad J_3 = (1,\infty). \tag{10.2}$$

For each interval, the corresponding operator setting is the Hilbert space $H_i = L^2(J_i)$, $i = 1,2,3$, consisting of complex-valued functions $f \in AC_{\text{loc}}(J_i)$ such that

$$\int_{J_i} |f|^2 < \infty. \tag{10.3}$$

Since $p(t)$ is negative when $|t| > 1$, we let

$$r(t) = t^2 - 1. \tag{10.4}$$

Then (10.4) is equivalent to

$$-\left(ry'\right)' = \xi y, \quad \xi = -\lambda. \tag{10.5}$$

Note that $r(t) > 0$ for $t \in J_1 \cup J_3$, so that equation (10.5) has the usual Sturm–Liouville form with positive leading coefficient r.

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We also discuss Legendre operators on the whole real line $\mathbb{R}$ in the Hilbert space $L^2(\mathbb{R})$, which we identify with the direct sum

$$H_5 = L^2(\mathbb{R}) = L^2(-\infty,-1) \dotplus L^2(-1,1) \dotplus L^2(1,\infty). \tag{10.6}$$

This is a three-interval problem, but the modifications needed to apply the two-interval results from Chapter 8 to three intervals are straightforward. For this three-interval theory, we take the Mukhtarov–Yakubov constants h, k to be $h = 1 = k$.

To illustrate the two-interval theory, we also discuss operators on the intervals $(-1,1)$ and $(1,\infty)$ by identifying the space $L^2(-1,\infty)$ with

$$H_4 = L^2(-1,1) \dotplus L^2(1,\infty), \tag{10.7}$$

but here we use the general Mukhtarov–Yakubov [82] constants h, k and give examples.

In addition, we construct regular problems that are equivalent to the singular Legendre problem, and we construct the Green's function.

10.2 General properties

Equation (10.1) has singularities at the points ± 1 and at $\pm\infty$. The singularities at ± 1 are due to the fact that $1/p$ is not Lebesgue integrable in left and right neighborhoods of these points; the singularities at $-\infty$ and at $+\infty$ are due to the fact that the weight function $w(t) = 1$ is not integrable in a neighborhood of these infinite endpoints.

Before proceeding to the details of the study of the Legendre equation on each of the three intervals J_i, $i = 1, 2, 3$, using the one-interval theory on the whole line $\mathbb{R}$, using the 3-interval theory, and on the two intervals $(-1,1)$ and $(1,\infty)$, using the two-interval theory with modified inner products, we make some general observations. This can also be considered one interval $(-1,\infty)$ with a singularity at the interior point 1.

For $\lambda = \xi = 0$, two linearly independent solutions are given by

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{1-t}{t+1}\right|\right). \tag{10.8}$$

Since these two functions u, v play an important role further, we make some observations about them.

Observe that for all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$(pv')(t) = +1. \tag{10.9}$$

Thus the quasi-derivative (pv') can be continuously extended so that it is well defined and continuous on the whole real line $\mathbb{R}$ including two singular points -1 and $+1$. It is interesting to observe that u, (pu'), and (the extended) (pv') can be defined to be continuous on $\mathbb{R}$, and only v blows up at the singular points -1 and $+1$.

These simple observations about solutions of (10.1) when $\lambda = 0$ extend in a natural way to solutions for all $\lambda \in \mathbb{C}$ and are given in the next theorem whose proof may be of more interest than the theorem. It is based on a "system regularization" of (10.1) using the above functions u, v.

The standard system formulation of the scalar equation (10.1) has the form

$$Y' = (P - \lambda W)Y \quad \text{on } (-1,1), \tag{10.10}$$

where

$$Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1/p \\ 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{10.11}$$

For u and v given by (10.8), let

$$U = \begin{bmatrix} u & v \\ (pu') & (pv') \end{bmatrix} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}. \tag{10.12}$$

Note that $\det U(t) = 1$ for $t \in J_2 = (-1,1)$ and set

$$Z = U^{-1}Y. \tag{10.13}$$

Then

$$\begin{aligned} Z' &= (U^{-1})'Y + U^{-1}Y' = -U^{-1}U'U^{-1}Y + (U^{-1})(P - \lambda V)Y \\ &= -U^{-1}U'Z + (U^{-1})(P - \lambda W)UZ \\ &= -U^{-1}(PU)Z + U^{-1}(DU)Z - \lambda(U^{-1}WU)Z = -\lambda(U^{-1}WU)Z. \end{aligned}$$

Letting $G = (U^{-1}WU)$, we may conclude that

$$Z' = -\lambda GZ. \tag{10.14}$$

Observe that

$$G = U^{-1}WU = \begin{bmatrix} -v & -v^2 \\ 1 & v \end{bmatrix}. \tag{10.15}$$

Definition 10.2.1. We call (10.14) a "regularized" Legendre system.

This definition is justified by the next theorem.

Theorem 10.2.1. *Let $\lambda \in \mathbb{C}$, and let G be given by* (10.15).

(1) *Every component of G is in $L^1(-1,1)$, and therefore* (10.14) *is a regular system.*

(2) *For any $c_1, c_2 \in \mathbb{C}$, the initial value problem*

$$Z' = -\lambda GZ, \quad Z(-1) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{10.16}$$

has a unique solution Z defined on the closed interval $[-1,1]$.

(3) *If* $Y = \left[\begin{smallmatrix} y(t,\lambda) \\ (py')(t,\lambda) \end{smallmatrix}\right]$ *is a solution of* (10.10) *and* $Z = U^{-1}Y = \left[\begin{smallmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{smallmatrix}\right]$, *then* Z *is a solution of* (10.14), *and for all* $t \in (-1,1)$, *we have*

$$y(t,\lambda) = uz_1(t,\lambda) + v(t)z_2(t,\lambda) = z_1(t,\lambda) + v(t)z_2(t,\lambda), \tag{10.17}$$
$$(py')(t,\lambda) = (pu')z_1(t,\lambda) + (pv')(t)z_2(t,\lambda) = -z_2(t,\lambda). \tag{10.18}$$

(4) *For every solution* $y(t,\lambda)$ *of the singular scalar Legendre equation* (10.8), *the quasi-derivative* $(py')(t,\lambda)$ *is continuous on the compact interval* $[-1,1]$. *More specifically, we have*

$$\lim_{t\to -1^+}(py')(t,\lambda) = -z_2(-1,\lambda), \quad \lim_{t\to 1^-}(py')(t,\lambda) = -z_2(1,\lambda). \tag{10.19}$$

Thus the quasi-derivative is a continuous function on the closed interval $[-1,1]$ *for every* $\lambda \in \mathbb{C}$.

(5) *Let* $y(t,\lambda)$ *be given by* (10.17). *If* $z_2(1,\lambda) \neq 0$, *then* $y(t,\lambda)$ *is unbounded at* 1; *if* $z_2(-1,\lambda) \neq 0$, *then* $y(t,\lambda)$ *is unbounded at* -1.

(6) *Fix* $t \in [-1,1]$. *Let* $c_1, c_2 \in \mathbb{C}$. *If* $Z = \left[\begin{smallmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{smallmatrix}\right]$ *is the solution of* (10.16) *determined by the initial conditions* $z_1(-1,\lambda) = c_1$, $z_2(-1,\lambda) = c_2$, *then* $z_i(t,\lambda)$ *is an entire function of* λ, $i = 1,2$. *Similarly for the initial condition* $z_1(1,\lambda) = c_1$, $z_2(1,\lambda) = c_2$.

(7) *For each* $\lambda \in \mathbb{C}$, *there is a nontrivial solution that is bounded in a (two-sided) neighborhood of* 1, *and there is a (generally different) nontrivial solution that is bounded in a (two-sided) neighborhood of* -1.

(8) *A nontrivial solution* $y(t,\lambda)$ *of the singular scalar Legendre equation* (10.1) *is bounded at* 1 *if and only if* $z_2(1,\lambda) = 0$; *a nontrivial solution* $y(t,\lambda)$ *of the singular scalar Legendre equation* (10.1) *is bounded at* -1 *if and only if* $z_2(-1,\lambda) = 0$.

Proof. Part (1) follows from (10.15); (2) is a direct consequence of (1) and the theory of regular systems; $Y = UZ$ implies (3) $\Longrightarrow$ (4) and (5); (6) follows from (2) and the basic theory of regular systems. For (7), determine solutions $y_1(t,\lambda), y_{-1}(t,\lambda)$ by applying the Frobenius method to obtain power series solutions of (10.1) in the form (see [3], page 5 with different notations):

$$y_1(t,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda)(t-1)^n, \quad |t-1| < 2;$$
$$y_{-1}(t,\lambda) = 1 + \sum_{n=1}^{\infty} b_n(\lambda)(t+1)^n, \quad |t+1| < 2. \tag{10.20}$$

To prove (8), it follows from (10.17) that if $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is not bounded at 1. Suppose $z_2(1,\lambda) = 0$. If the corresponding $y(t,\lambda)$ is not bounded at 1, then there are two linearly unbounded solutions at 1, and hence all nontrivial solutions are unbounded at 1. This contradiction establishes (8) and completes the proof of the theorem. □

Remark 10.2.1. From Theorem 10.2.1 we see that, *for every* $\lambda \in \mathbb{C}$, the Legendre equation (10.1) has a solution y_1 that is bounded at 1 and has a solution y_{-1} that is bounded at -1.

It is well known that for $\lambda_n = n(n+1)$, $n \in \mathbb{N}_0$, the Legendre polynomials P_n (see the formula below) are solutions on $(-1,1)$ and hence are bounded at -1 and at $+1$.

For later reference, we introduce the primary fundamental matrix of system (10.11).

Definition 10.2.2. Fix $\lambda \in \mathbb{C}$. Let $\Phi(\cdot,\cdot,\lambda)$ be the primary fundamental matrix of (10.11), that is, for each $s \in [-1,1]$, $\Phi(\cdot,s,\lambda)$ is the unique matrix solution of the initial value problem

$$\Phi(s,s,\lambda) = I, \tag{10.21}$$

where I is the 2×2 identity matrix. Since (10.11) is regular, $\Phi(t,s,\lambda)$ is defined for all $t,s \in [-1,1]$, and, for any fixed t,s, $\Phi(t,s,\lambda)$ is an entire function of λ.

We now consider two-point boundary conditions for (10.16); later, we will relate these to singular boundary conditions for (10.1).

Let $A,B \in M_2(\mathbb{C})$, the set of 2×2 complex matrices, and consider the boundary value problem

$$Z' = -\lambda GZ, \quad AZ(-1) + BZ(1) = 0. \tag{10.22}$$

Lemma 10.2.1. *A complex number* $-\lambda$ *is an eigenvalue of* (10.22) *if and only if*

$$\Delta(\lambda) = \det[A + B\Phi(1,-1,-\lambda)] = 0. \tag{10.23}$$

Furthermore, a complex number $-\lambda$ *is an eigenvalue of geometric multiplicity two if and only if*

$$A + B\Phi(1,-1,-\lambda) = 0. \tag{10.24}$$

Proof. Note that a solution for the initial condition $Z(-1) = C$ is given by

$$Z(t) = \Phi(t,-1,-\lambda)C, \quad t \in [-1,1]. \tag{10.25}$$

The boundary value problem (10.22) has a nontrivial solution for Z if and only if the algebraic system

$$[A + B\Phi(1,-1,-\lambda)]Z(-1) = 0 \tag{10.26}$$

has a nontrivial solution for $Z(-1)$.

To prove the furthermore part, observe that two linearly independent solutions of the algebraic system (10.24) for $Z(-1)$ yield two linearly independent solutions $Z(t)$ of the differential system, and conversely. □

Given any $\lambda \in \mathbb{R}$ and any solutions y, z of (10.8), applying the Lagrange form (6.3), we have

$$[u,v](t) = +1, \quad [v,u](t) = -1, \quad [y,u](t) = -(py')(t), \quad t \in \mathbb{R},$$
$$[y,v](t) = y(t) - v(t)(py')(t), \quad t \in \mathbb{R}, \quad t \neq \pm 1.$$

We will further see that although v blows up at ± 1, the form $[y,v](t)$ is well defined at -1 and $+1$ since the limits

$$\lim_{t \to -1} [y,v](t), \quad \lim_{t \to +1} [y,v](t)$$

exist and are finite from both sides. This holds for any solution y of equation (10.8) for any $\lambda \in \mathbb{R}$. Note that since v blows up at 1, this means that y must blow up at 1 except, possibly, when $(py')(1) = 0$. We will expand on this observation further in the section on "Regular Legendre" equations.

Now we make the following additional observations. For definitions of the technical terms used here, see [113].

Proposition 10.2.1.

(1) *Both equations* (10.1) *and* (10.5) *are singular at* $-\infty$, $+\infty$ *and at* -1, $+1$ *from both sides.*

(2) *In the* L^2 *theory the endpoints* $-\infty$ *and* $+\infty$ *are in the limit-point (LP) case, whereas* -1^-, -1^+, 1^-, *and* 1^+ *are all in the limit-circle (LC) case. In particular, both solutions* u, v *are in* $L^2(-1,1)$. *Here we use the notation* -1^- *to indicate that the equation is studied on an interval that has* -1 *as its right endpoint. Similarly for* -1^+, 1^-, *and* 1^+.

(3) *For every* $\lambda \in \mathbb{R}$, *equation* (10.1) *has a solution that is bounded at* -1 *and another solution that blows up logarithmically at* -1. *Similarly for* $+1$.

(4) *When* $\lambda = 0$, *the constant function u is a principal solution at each of the endpoints* -1^-, -1^+, 1^-, *and* 1^+, *but u is a nonprincipal solution at both endpoints* $-\infty$ *and* $+\infty$. *On the other hand, v is a nonprincipal solution at* -1^-, -1^+, 1^-, *and* 1^+, *but it is the principal solution at* $-\infty$ *and* $+\infty$. *Recall that at each endpoint, the principal solution is unique up to constant multiples, but a nonprincipal solution is never unique since the sum of principal and nonprincipal solutions is nonprincipal.*

(5) *On the interval* $J_2 = (-1,1)$, *equation* (10.1) *is nonoscillatory at* -1^-, -1^+, 1^-, *and* 1^+ *for every real* λ.

(6) *On the interval* $J_3 = (1,\infty)$, *equation* (10.5) *is oscillatory at* ∞ *for every* $\lambda > -1/4$ *and nonoscillatory at* ∞ *for every* $\lambda < -1/4$.

(7) *On the interval* $J_3 = (1,\infty)$, *equation* (10.1) *is nonoscillatory at* ∞ *for every* $\lambda < 1/4$ *and oscillatory at* ∞ *for every* $\lambda > 1/4$.

(8) *On the interval* $J_1 = (-\infty,-1)$, *equation* (10.1) *is nonoscillatory at* $-\infty$ *for every* $\lambda < +1/4$ *and oscillatory at* $-\infty$ *for every* $\lambda > +1/4$.

(9) *On the interval* $J_1 = (-\infty,-1)$, *equation* (10.5) *is oscillatory at* $-\infty$ *for every* $\lambda > -1/4$ *and nonoscillatory at* $-\infty$ *for every* $\lambda < -1/4$.

(10) *The spectrum of the classical Sturm–Liouville problem (SLP) consisting of equation* (10.1) *on* $(-1,1)$ *with the boundary condition*

$$(py')(-1) = 0 = (py')(+1)$$

is discrete and is given by

$$\sigma(S_F) = \{n(n+1) : n \in \mathbb{N}_0\}.$$

Here S_F *denotes the classical Legendre operator, that is, the self-adjoint operator in the Hilbert space* $L^2(-1,1)$ *that represents the Sturm–Liouville problem (SLP)* (10.1) *with boundary condition* $(py')(-1) = 0 = (py')(+1)$. *The notation* S_F *is used to indicate that this is the celebrated Friedrichs extension. Its orthonormal eigenfunctions are the Legendre polynomials* $\{P_n : n \in \mathbb{N}_0\}$ *given by*

$$P_n(t) = \sqrt{\frac{2n+1}{2}} \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{2^n j!(n-j)!(n-2j)!} t^{n-2j} \quad (n \in \mathbb{N}_0),$$

where $[n/2]$ *denotes the greatest integer* $\leq n/2$.
The special (ausgezeichnete) operator S_F *is one of an uncountable number of self-adjoint realizations of the equation on* $(-1,1)$ *in the Hilbert space* $H = L^2(-1,1)$. *The singular boundary conditions determining the other self-adjoint realizations will be given explicitly below.*

(11) *The essential spectrum of every self-adjoint realization in the Hilbert spaces* $L^2(1,\infty)$ *and* $L^2(-\infty,-1)$ *is given by*

$$\sigma_e = (-\infty, -1/4].$$

For each interval, every self-adjoint realization is bounded above and has at most two eigenvalues. Each eigenvalue is $\geq -1/4$. *The existence of* 0, 1, *or* 2 *eigenvalues and their locations depends on the boundary condition. There is no uniform bound for all self-adjoint realizations. For more information, see* [74].

(12) *The essential spectrum of every self-adjoint realization of the Legendre equation in the Hilbert spaces* $L^2(1,\infty)$ *and* $L^2(-\infty,-1)$ *is given by*

$$\sigma_e = [1/4, \infty).$$

For each interval, every self-adjoint realization is bounded below and has at most two eigenvalues. Each eigenvalue is $\geq -1/4$. *There is no uniform bound for all self-adjoint realizations. The existence of* 0, 1, *or* 2 *eigenvalues and their locations depends on the boundary condition.*

Proof. Parts (1), (2), and (4) are basic results in Sturm–Liouville theory [113]. The proof of (3) will be given further in the section on regular Legendre equations. For these and other basic facts mentioned below, the reader is referred to [113]. Part (10) is the well-known celebrated classical theory of the Legendre polynomials; see [86] for a characterization of the Friedrichs extension. In the other parts, the statements about oscillation, nonoscillation, and the essential spectrum σ_e follow from the well-known general fact that when the leading coefficient is positive, the equation is oscillatory for all $\lambda > \inf \sigma_e$ and nonoscillatory for all $\lambda < \inf \sigma_e$. Thus $\inf \sigma_e$ is called the oscillation number of the equation. It is well known that the oscillation number of equation (10.5) on $(1, \infty)$ is $-1/4$. Since (10.5) is nonoscillatory at 1^+ for all $\lambda \in \mathbb{R}$, oscillation can occur only at ∞. The transformation $t \to -1$ shows that the same results hold for (10.5) on $(-\infty, -1)$. Since $\xi = -\lambda$, the above-mentioned results hold for the standard Legendre equation (10.1) but with the sign reversed. To compute the essential spectrum on $(1, \infty)$, we first note that the endpoint 1 makes no contribution to the essential spectrum since it is limit circle nonoscillatory. Note that $\int_2^\infty \frac{1}{\sqrt{r}} = \infty$ and

$$\lim_{t\to\infty} \frac{1}{4}\left(r''(t) - \frac{1}{4}\frac{[r'(t)]^2}{r(t)}\right) = \lim_{t\to\infty} \frac{1}{4}\left(2 - \frac{1}{4}\frac{4t^2}{t^2-1}\right) = \frac{1}{4}.$$

From this and Theorem XIII.7.66 in Dunford and Schwartz [25], part (12) follows, and part (11) follows from (12). Parts (6)–(10) follow from the fact that the starting point of the essential spectrum is the oscillation point of the equation; that is, the equation is oscillatory for all λ above the starting point and nonoscillatory for all λ below the starting point. (Note that there is a sign change correction needed in the statement of Theorem XIII.7.66 since $1 - t^2$ is negative when $t > 1$, and this theorem applies to a positive leading coefficient.) □

10.3 Regular Legendre equations

In this section, we construct *regular* Sturm–Liouville equations that are equivalent to the classical *singular Legendre equation* (10.1). This construction is based on a transformation used by Niessen and Zettl [86]. We apply this construction to the classical Legendre problem on the interval $(-1, 1)$:

$$My = -(py') = \lambda y \quad \text{on } J_2 = (-1,1), \quad p(t) = 1 - t^2, \quad -1 < t < 1. \tag{10.27}$$

This transformation depends on a modification of the function v given by (10.8). Note that v changes sign in $(-1, 1)$ at 0 and we need a function that is positive on the entire interval $(-1, 1)$ and is a nonprincipal solution at both endpoints.

This modification consists of using a multiple of v that is positive near each endpoint and changing the function v in the middle of J_2:

$$v_m(t) = \begin{cases} \frac{-1}{2}\ln(\frac{1-t}{1+t}), & \frac{1}{2} \le t < 1, \\ m(t) & \frac{-1}{2} \le t \le \frac{1}{2}, \\ \frac{1}{2}\ln(\frac{1-t}{1+t}), & -1 \le t \le \frac{-1}{2}, \end{cases} \tag{10.28}$$

where the “middle function” m is chosen so that the modified function v_m defined on $(-1,1)$ satisfies the following two properties:
(1) $v_m(t) > 0$, $-1 < t < 1$.
(2) $v_m, (pv_m') \in AC_{\text{loc}}(-1,1)$, $v_m, (pv_m') \in L^2(-1,1)$.
(3) v_m is a nonprincipal solution at both endpoints.

For later reference, we note that

$$\begin{aligned}
&(pv_m')(t) = +1, \quad \frac{1}{2} \le t < 1,\\
&(pv_m')(t) = -1, \quad -1 < t < \frac{-1}{2},\\
&[u,v_m](t) = u(t)(pv_m')(t) - v(t)(pu')(t) = (pv_m')(t) = 1, \quad \frac{1}{2} \le t < 1,\\
&[u,v_m](t) = u(t)(pv_m')(t) - v(t)(pu')(t) = (pv_m')(t) = -1, \quad -1 < t < -\frac{1}{2}.
\end{aligned} \tag{10.29}$$

Niessen and Zettl [86, Lemmas 2.3 and 3.6] showed that such choices for m are possible in general. Although in the Legendre case studied here, an explicit such m can be constructed, we do not do so here since our focus is on boundary conditions at the endpoints that are independent of the choice of m.

Definition 10.3.1. Let M be given by (10.27). Define

$$P = v_m^2 p, \quad Q = v_m M v_m, \quad W = v_m^2 \quad \text{on } J_2 = (-1,1). \tag{10.30}$$

Consider the equation

$$Nz = -(Pz')' + Qz = \lambda W z \quad \text{on } J_2 = (-1,1). \tag{10.31}$$

In (10.30), P denotes a scalar function; this notation should not be confused with P defined above where P denotes a matrix.

Lemma 10.3.1. *Equation* (10.31) *is regular with* $P > 0$ *on* J_2 *and* $W > 0$ *on* J_2.

Proof. The positivity of P and W is clear. To prove that equation (10.31) is regular on $(-1,1)$, we have to show that

$$\int_{-1}^{1} \frac{1}{P} < \infty, \quad \int_{-1}^{1} Q < \infty, \quad \int_{-1}^{1} W < \infty. \tag{10.32}$$

The third integral is finite since $v \in L^2(-1,1)$.

Since v_m is a nonprincipal solution at both endpoints, it follows from SL theory [113] that

$$\int_{-1}^{c} \frac{1}{pv_m^2} < \infty, \quad \int_{d}^{1} \frac{1}{pv_m^2} < \infty$$

for some c, d, $-1 < c < d < 1$. By (10.10) $1/v_m^2$ is bounded on $[c, d]$, and therefore

$$\int_c^d \frac{1}{p} < \infty,$$

and so we can conclude that the first integral (10.32) is finite. The middle integral is finite since Mv_m is identically zero near each endpoint and $v_m, (pv_m') \in AC_{\text{loc}}(-1, 1)$. □

Corollary 10.3.1. *Let* $\lambda \in \mathbb{C}$. *For every solution* z *of* (10.31), *the limits*

$$\begin{aligned} z(-1) &= \lim_{t\to -1^+} z(t), \quad z(1) = \lim_{t\to 1^-} z(t), \\ (Pz')(-1) &= \lim_{t\to -1^+} (Pz')(t), \quad (Pz')(1) = \lim_{t\to 1^-} (Pz')(t) \end{aligned} \tag{10.33}$$

exist and are finite.

Proof. This follows directly from SL theory [113]; every solution and its quasi-derivative have finite limits at each regular endpoint. □

We call equation (10.31) a “regularized Legendre equation”. It depends on the function v that depends on m. The key property of v is that it is a positive nonprincipal solution at each endpoint. Note that v_m in (10.28) is “patched together” from two different nonprincipal solutions, one from each endpoint, and the “patching” function m plays no significant role in this book.

Note that (10.31) is also defined on $(-1, 1)$ but can be considered on the compact interval $[-1, 1]$, in contrast to the singular Legendre equation (10.1). A significant consequence of this is that, for each $\lambda \in \mathbb{C}$, every solution z of (10.31) and its quasi-derivative (Pz') can be continuously extended to the endpoints ± 1. We use the notation (Pz') to remind the reader that the product (Pz') has to be considered as one function when evaluated at ± 1 since P is not defined at -1 and at 1.

Remark 10.3.1. Note that we use the theory of *quasi-differential* equations. Conditions (10.32) show that equation (10.31) is a regular quasi-differential equation. We take full advantage of this fact in this chapter.

Let $S_{\min}(N)$ and $S_{\max}(N)$ denote the minimal and maximal operators associated with (10.31), and denote their domains by $D_{\min}(N)$ and $D_{\max}(N)$, respectively. Note that these are operators in the weighted Hilbert space with weight function v_m^2, which we denote by $L^2(v_m) = L^2(J_2, v_m^2)$. A self-adjoint realization $S(N)$ of (10.31) is an operator in $L^2(v_m)$, which satisfies

$$S_{\min}(N) \subset S(N) = S^*(N) \subset S_{\max}(N). \tag{10.34}$$

Applying the theory of self-adjoint regular Sturm–Liouville problems to the regularized Legendre equation (10.12), we obtain the following:

Theorem 10.3.1. *Let A and B be 2×2 complex matrices satisfying the following two conditions:*

$$\operatorname{rank}(A:B) = 2, \tag{10.35}$$

$$AEA^* = BEB^*. \tag{10.36}$$

Then the set of all $z \in D_{\max}(N)$ satisfying

$$A\begin{bmatrix} z(-1) \\ (Pz')(-1) \end{bmatrix} + B\begin{bmatrix} z(1) \\ (Pz')(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10.37}$$

is a self-adjoint domain. Conversely, given any self-adjoint realization of (10.31) *in the space $L^2(v)$, that is, any operator $S(N)$ satisfying* (10.34)*, there exist 2×2 complex matrices A and B satisfying* (10.35) *and* (10.36) *such that the domain of $S(N)$ is the set of all $z \in D_{\max}(N)$ satisfying* (10.37)*. Here (A, B) is the 2×4 matrix whose first two columns are the columns of A and last two columns are those of B.*

For a proof of the theorem, see Chapter 1.

It is convenient to divide the self-adjoint boundary conditions (10.37) into two disjoint mutually exclusive classes, the separated and coupled conditions. The former have the well-known canonical representation

$$\begin{aligned} \cos(\alpha)z(-1) + \sin(\alpha)(Pz')(-1) &= 0, \quad 0 \le \alpha < \pi, \\ \cos(\beta)z(1) + \sin(\beta)(Pz')(1) &= 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{10.38}$$

The latter have the canonical representation

$$\begin{bmatrix} z(1) \\ (Pz')(1) \end{bmatrix} = e^{i\gamma}K\begin{bmatrix} z(-1) \\ (Pz')(-1) \end{bmatrix}, \quad -\pi < \gamma \le \pi. \tag{10.39}$$

Examples of separated conditions are the well-known Dirichlet condition

$$z(-1) = 0 = z(1) \tag{10.40}$$

and the Neumann condition

$$(Pz')(-1) = 0 = (Pz')(1). \tag{10.41}$$

Examples of coupled conditions are the periodic conditions

$$\begin{aligned} z(-1) &= z(1), \\ (Pz')(-1) &= (Pz')(1) \end{aligned} \tag{10.42}$$

and the semiperiodic (also called antiperiodic) conditions

$$\begin{aligned} z(-1) &= -z(1), \\ (Pz')(-1) &= -(Pz')(1). \end{aligned} \tag{10.43}$$

Note that when $\gamma \neq 0$, we have complex matrices A and B defining regular self-adjoint operators. Next, we explore the relation between solutions y of the singular Legendre equation (10.1) and solutions z of the regularized Legendre equation (10.1). Note that both equations are on the same interval $(-1, 1)$.

Lemma 10.3.2. *For any $\lambda \in \mathbb{C}$, the solutions $y(\cdot, \lambda)$ of the singular equation* (10.1) *and the solutions $z(\cdot, \lambda)$ of the regular equation* (10.31) *are related by*

$$\frac{y(t,\lambda)}{v_m(t)} = z(t,\lambda), \quad -1 < t < 1, \quad \lambda \in \mathbb{C}, \tag{10.44}$$

and the correspondence $y(\cdot,\lambda) \to z(\cdot,\lambda)$ is one-to-one onto. Note that λ is the same on both sides.

Proof. Fix $\lambda \in \mathbb{C}$ and simplify the notation for this proof so that $v = v_m$ and let

$$z = \frac{y}{v} \quad \text{on } (-1, 1).$$

Then $z' = \frac{vy' - yv'}{v^2}$, and

$$\begin{aligned}((pv^2)z')' &= (v(py') - y(pv'))' = v(py')' + v'py' - y'pv' - y(pv')' \\ &= v(-\lambda y) + y(Mv) = -\lambda v^2 \frac{y}{v} + \frac{y}{v} vMv = -\lambda v^2 z + Qz,\end{aligned}$$

from which (10.31) follows. Reversing the steps shows that the correspondence is one-to-one. □

Remark 10.3.2. We comment on the relation between the classical singular Legendre equation (10.1) and its regularizations (10.31); this remark will be amplified further after we discuss the self-adjoint operators generated by the singular Legendre equation (10.1). In particular, we will see that the operator $S(N)$ determined by the Dirichlet condition (10.41), which we denote by $S_F(N)$, is a regular representation of the celebrated classical singular Friedrichs operator, denoted by S_F, whose eigenvalues are $\{n(n+1) : n \in \mathbb{N}_0\}$ and whose eigenfunctions are the classical Legendre polynomials P_n. Note that the solutions $y(t,\lambda)$ and $z(t,\lambda)$ have exactly the same zeros in the open interval $(-1, 1)$ but not in the closed interval $[-1, 1]$ since z may be zero at the endpoints and y may not be defined there.

Remark 10.3.3. Each solution z and its quasi-derivative (Pz') is continuous on the compact interval $[-1, 1]$. Note that $v(t)$ does not depend on λ. Therefore the singularity of every solution $y(t,\lambda)$ for all $\lambda \in \mathbb{C}$ is contained in v; in other words, the nature of the singularities of the solutions $y(t,\lambda)$ are invariant with respect to λ. Although $v(t)$ does not exist for $t = -1$ and $t = 1$ and $y(t)$ also may not exist for $t = -1$ and $t = 1$, the limits

$$\lim_{t\to -1^+} \frac{y(t,\lambda)}{v(t)} = z(-1,\lambda), \quad \lim_{t\to 1^-} \frac{y(t,\lambda)}{v(t)} = z(1,\lambda) \tag{10.45}$$

exist for all solutions $y(t,\lambda)$ of the Legendre equation (10.1). If $z(1,\lambda) \neq 0$, then $y(t,\lambda)$ blows up logarithmically as $t \to 1$, and similarly at -1.

Remark 10.3.4. Applying the correspondence (10.44) to the Legendre polynomials, we obtain a factorization of these polynomials:

$$P_n(t) = v(t)z_n(t), \quad -1 < t < 1, \quad n \in \mathbb{N}_0. \tag{10.46}$$

Since P_n is continuous at -1 and at 1 and v blows up at these points, it follows that $z_n(-1) = 0 = z_n(1)$, $n \in \mathbb{N}_0$. Note that z_n has exactly the same zeros as P_n in the open interval $(-1,1)$. However, also note that this is not the case for the closed interval $[-1,1]$ since $z_n(-1) = 0 = z_n(1)$ but $P_n(1) \neq 0 \neq P_n(-1)$ for each $n \in \mathbb{N}_0$.

Remark 10.3.5. Following the characterization of the self-adjoint Legendre realizations S of the singular Legendre equation (10.1) using singular SL theory, we will further specify a one-to-one correspondence between the self-adjoint realizations $S(N)$ of the regularized Legendre equation (10.31) and the self-adjoint operators of the singular classical Legendre equation (10.1). In particular, we will see that the operator $S_D(N)$ determined by the regular *Dirichlet boundary condition*

$$z(-1) = 0 = z(1) \tag{10.47}$$

corresponds to the celebrated classical Friedrichs Legendre operator S_F determined by the singular boundary condition

$$(py')(-1) = 0 = (py')(1),$$

whose eigenvalues are $\{n(n+1), n \in \mathbb{N}_0\}$ and whose eigenfunctions are the classical Legendre polynomials P_n. The Dirichlet operator $S_D(N)$ has the same eigenvalues as S_F, but its eigenfunctions are given by

$$z_n = \frac{P_n}{v^2}, \quad n \in \mathbb{N}_0.$$

Note that each z_n has exactly the same zeros in the open interval $(-1,1)$ but not in the closed interval $[-1,1]$ since $z_n(-1) = 0 = z_n(1)$. Also note that S_F is a self-adjoint operator in the space $L^2(-1,1)$ and $S_D(N)$ is a self-adjoint operator in the weighted Hilbert space $L^2((-1,1),v^2)$. Thus all the Legendre polynomial P_n can be factored as

$$P_n = v^2 z_n, \quad n \in \mathbb{N}_0,$$

with the same factor v^2 for all n.

10.4 Self-adjoint operators in $L^2(-1,1)$

By a self-adjoint operator associated with equation (10.1) in $H_2 = L^2(-1,1)$ we mean a self-adjoint operator S satisfying (10.52). Let

$$D_{\max} = \{f : (-1,1) \to \mathbb{C} : f, pf' \in AC_{\text{loc}}(-1,1); f, pf' \in H_2\}, \tag{10.48}$$

$$S_{\max} f = -(pf')', \quad f \in D_{\max}. \tag{10.49}$$

Note that all bounded continuous functions on $(-1,1)$ are in $D_{\max}$; in particular, all polynomials are in $D_{\max}$. (More precisely, the restriction of every polynomial to $(-1,1)$ is in $D_{\max}$.) However, $D_{\max}$ also contains functions that are not bounded on $(-1,1)$, for example, $f(t) = \ln(1-t)$.

Lemma 10.4.1. *The operator $S_{\max}$ is densely defined in H_2 and therefore has a unique adjoint in H_2 denoted by $S_{\min}$:*

$$S^*_{\max} = S_{\min}. \tag{10.50}$$

Furthermore, the minimal operator $S_{\min}$ in H_2 is symmetric, closed, and densely defined, and

$$S^*_{\min} = S_{\max}. \tag{10.51}$$

Moreover, if S is a self-adjoint extension of $S_{\min}$, then S is also a self-adjoint restriction of $S_{\max}$, and conversely. Thus we have

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{10.52}$$

It is clear from (10.52) that each self-adjoint operator S is determined by its domain $D(S)$.

Next, we characterize these self-adjoint domains. It is remarkable that all self-adjoint Legendre operators can be described explicitly in terms of two-point singular boundary conditions. For this, the functions u, v from Section 10.2 play an important role. We will use them to describe all self-adjoint boundary singular boundary conditions explicitly.

Let

$$My = -(py')'. \tag{10.53}$$

Of critical importance in the characterization of all self-adjoint boundary conditions is the Lagrange sesquilinear form $[\cdot,\cdot]$ defined for all maximal domain functions,

$$[f,g] = fp(\overline{g}') - gp(\overline{f}') \quad (f,g \in D_{\max}), \tag{10.54}$$

and the associated Green's formula

$$\int_a^b \{\overline{g}Mf - \overline{f}Mg\} = [f,g](b) - [f,g](a), \quad f,g \in D_{\max}, \quad -1 < a < b < 1. \tag{10.55}$$

From this equality it follows that the limits

$$\lim_{a\to -1^+} [f,g](t), \quad \lim_{b\to +1^-} [f,g](t) \tag{10.56}$$

exist and are finite.

The next theorem characterizes all self-adjoint Legendre operators S in $L^2(-1,1)$.

Theorem 10.4.1. *Let the functions u, v be given by* (10.8)*, that is,*

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{1-t}{t+1}\right|\right), \quad -1 < t < 1.$$

Let A and B be 2×2 complex matrices satisfying the following two conditions:

$$\operatorname{rank}(A : B) = 2, \tag{10.57}$$

$$AEA^* = BEB^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10.58}$$

Define $D(S) = \{y \in D_{\max}\}$ such that

$$A\begin{bmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{bmatrix} + B\begin{bmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10.59}$$

Then $D(S)$ is a self-adjoint domain. Furthermore, all self-adjoint domains are generated this way. Here $(A : B)$ denotes the 2×4 matrix whose first two columns are those of A and last two columns are the columns of B.

Proof. See Section 10.4 in [113] for a proof of the general characterization and use the functions u, v to obtain (10.59). □

Remark 10.4.1. We comment on some aspects of this remarkable characterization of all self-adjoint Legendre operators in $L^2(-1,1)$.

(1) Just as in the regular case, the singular self-adjoint boundary conditions (10.59) are explicit since u and v are given explicitly.

(2) Note that $[y,u] = -(py')$ and $[y,v] = y(pv') - v(py')$. Hence $-(py')$ and $(ypv' - v(py'))$ exist as finite limits at -1 and 1 for all maximal domain functions y. In particular, these limits exist and are finite for all solutions y of equation (10.1) for any λ. Thus a number λ is an eigenvalue of the singular boundary value problem (10.57)–(10.58) if and only if equation (10.1) has a nontrivial solution y satisfying (10.59). Note that the separate terms $y(pv')$ and $v(py')$ may not exist at -1 or at $+1$, they may blow up or oscillate wildly at these points, but the Lagrange bracket $[y,v]$ has finite limits at -1 and $+1$ for any maximal domain functions y, v.

(3) Choose

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then (10.57)–(10.58) hold, and the singular boundary condition (10.59) reduces to

$$(py')(-1) = 0 = (py')(1). \tag{10.60}$$

This is the boundary condition that determines, among the uncountable number of self-adjoint conditions, the special ("ausgezeichnete") Friedrichs extension S_F. It is interesting to observe that even though (10.60) has the appearance of a regular Neumann condition, in fact, it is the singular analogue of the regular Dirichlet condition. It is well known [113] that, in general, the Dirichlet boundary condition determines the Friedrichs extension S_F of regular SLP and that for singular nonoscillatory limit-circle problems, in general, the Friedrichs extension S_F is determined by the condition

$$[y, u_a](a) = 0 = [y, u_b](b),$$

where u_a is the principal solution at the left endpoint a, and u_b is the principal solution at the right endpoint b. (The principal solution is unique up to constant multiples at each endpoint.) Since the constant function $u = 1$ is the principal solution *at both endpoints* -1 and 1 in the Legendre case, we have $[y, u] = -(py')$, and (10.60) follows.

(4) Condition (10.59) includes separated and coupled conditions. We will further give a canonical form for these two classes of conditions, which is analogous to the regular case. We will also see that (10.59) includes *complex* boundary conditions. These are coupled; it is known that all separated self-adjoint conditions can be taken as real, that is, each complex separated condition (10.59) is equivalent to a real such condition.

(5) Since each endpoint is LCNO (limit-circle nonoscillatory), it is well known that the spectrum σ of every self-adjoint extension S, $\sigma(S)$, is discrete, bounded below, and unbounded above with no finite cluster point. For S_F, we have the celebrated result that

$$\sigma(S_F) = n(n+1) \quad (n \in \mathbb{N}_0),$$

and the corresponding orthonormal eigenfunctions are the polynomials P_n. For other self-adjoint Legendre operators S, the eigenvalues and eigenfunctions are not known in closed form. However, they can be computed numerically with the FORTRAN code SLEIGN2, developed by Bailey, Everitt, and Zettl [10]; this code, and a number explanatory files related to it, can be downloaded free from the internet. It comes with a user-friendly interface.

(6) It is known from general Sturm–Liouville theory that the eigenfunctions of every self-adjoint Legendre realization S are dense in $L^2(-1, 1)$. In particular, the Legendre polynomials P_n are dense in $L^2(-1, 1)$.

(7) If S is generated by a separated boundary condition, then the nth eigenfunction of S has exactly n zeros in the open interval $(-1, 1)$ for each $n \in \mathbb{N}_0$. In particular, this is true for the Legendre polynomials P_n.

(8) The self-adjoint boundary conditions (10.59) depend on the function v given by (10.8). Note that only the values of v near the endpoints play a role in (10.59), and

therefore v can be replaced by any function that is asymptotically equivalent to it; in particular, v can be replaced by any function that has the same values as v in a neighborhood of -1 and of 1.

Now that we have determined all self-adjoint singular Legendre operators with Theorem 10.3.1, and we compare these with the self-adjoint operators determined by the regularized Legendre equation given by Theorem 10.4.1. In making this comparison, it is important to keep in mind that these operators act in different Hilbert spaces, $L^2(-1,1)$ for the singular classical case and $L^2(v^2) = L^2((-1,1),v^2)$ for the regularized case.

But first we show that the correspondence

$$\frac{y}{v} = z, \quad y = vz \tag{10.61}$$

extends from solutions to functions in the domains of the operator realizations of the classical Legendre equation and its regularization. Since we now compare operator realizations of the singular equation (10.1) and its regularization (10.31) with each other, we use the notation $S(M)$ for operators associated with the former and $S(N)$ for those of the latter.

We denote the Lagrange forms associated with these equations by

$$[y,f]_M = y(p\overline{f'}) - \overline{f}(py'), \quad y,f \in D_{\max}(M) \tag{10.62}$$

and by

$$[z,g]_N = z(P\overline{g}') - \overline{g}(Pz'), \quad z,g \in D_{\max}(N), \quad P = v^2p, \tag{10.63}$$

respectively.

Notation 10.4.1. *We say that $D(N)$ is a self-adjoint domain for* (10.31) *if the operator with this domain is a self-adjoint realization of* (10.31) *in the Hilbert space $L(v^2)$. Similarly, $D(M)$ is a self-adjoint domain for* (10.1) *if the operator with this domain is a self-adjoint realization of* (10.1) *in the Hilbert space $L^2(-1,1)$.*

The next theorem compares the singular self-adjoint Legendre operators with the self-adjoint regularized Legendre operators.

Theorem 10.4.2. *Let* (10.1) *and* (10.31) *hold; let v be given by* (10.8).
(1) *A function $z \in D_{\max}(N)$ if and only if $vz \in D_{\max}(M)$.*
(2) *$D(N)$ is a self-adjoint domain for* (10.31) *if and only if $D(M) = \{y = vz : z \in D(M)\}$.*
(3) *In particular, we have a new characterization of the Friedrichs domain for* (10.1)*:*

$$D(S_F(M)) = \{vz : z \in D_{\max}(N) : z(-1) = 0 = z(1)\}.$$

Proof. Let $y, f \in D_{\max}(M)$, and let $z = \frac{y}{v}$, $g = \frac{f}{v}$. Then we have

$$\begin{aligned}
[z,g]_N = \left[\frac{y}{v},\frac{f}{v}\right]_N &= \frac{y}{v}P\left(\frac{\overline{f}}{v}\right)' - \frac{\overline{f}}{v}P\left(\frac{y}{v}\right)' \\
&= \frac{y}{v}pv^2\frac{v\overline{f'} - \overline{f}v'}{v_2} - \frac{\overline{f}}{v}pv^2\frac{vy' - yv'}{v_2} \\
&= yp\overline{f'} - \frac{y}{v}p\overline{f}v' - \overline{f}py' + \frac{y}{v}p\overline{f}v' \\
&= yp\overline{f'} - \overline{f}py' = [y,f]_M. \qquad (10.64)
\end{aligned}$$

Part (2) follows from (1) and (10.43). To prove (1), assume that

$$y \in D_{\max}(M) = \{y \in L^2(J_2) : py' \in AC_{\mathrm{loc}}(J_2),\ My = (py')' \in L^2(J_2)\}.$$

We must show that

$$z \in D_{\max}(N) = \{z \in L^2(v^2), Pz' \in AC_{\mathrm{loc}}(J_2), Nz = (Pz')' \in L^2(v^2)\}.$$

Note that

$$\begin{aligned}
&\int_{-1}^{1} |z^2|v^2 = \int_{-1}^{1} |y^2| < \infty, \\
&Pz' = v(py') - y(pv') = v(py') - y \in AC_{\mathrm{loc}}(J_2),
\end{aligned}$$

and

$$(Pz')' = v'py' + v(py')' - y'(pv') - y(pv)' = vMy \in L^2(v^2).$$

The converse follows similarly by reversing the steps in this argument. □

10.4.1 Eigenvalue properties

In this subsection, we study the variation of the eigenvalues as functions of the boundary conditions for the Legendre problem consisting of the equation

$$My = -(py')' = \lambda y \quad \text{on } J_2 = (-1,1), \quad p(t) = 1 - t^2, \quad -1 < t < 1, \qquad (10.65)$$

together with the boundary conditions

$$A\begin{bmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{bmatrix} + B\begin{bmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (10.66)$$

Here v is given by (10.8) near the endpoints, and the matrices A and B satisfy the self-adjointness condition.

Since these homogeneous boundary conditions are invariant under multiplication by a nonsingular matrix, to study the dependence of the eigenvalues on the boundary conditions, it is very useful to have their canonical representation. For such a representation, it is convenient to classify the boundary conditions into two mutually exclusive classes, separated and coupled. The separated conditions have the form [113]

$$\begin{aligned} \cos(\alpha)[y,u](-1) + \sin(\alpha)[y,v](-1) = 0, \quad 0 \le \alpha < \pi, \\ \cos(\beta)[y,u](1) + \sin(\beta)[y,v](1) = 0, \quad 0 < \beta \le \pi, \end{aligned} \tag{10.67}$$

whereas the coupled conditions have the following canonical representation [113]:

$$Y(1) = e^{i\gamma} K Y(-1), \tag{10.68}$$

where

$$Y = \begin{bmatrix} [y,u] \\ [y,v] \end{bmatrix}, \quad -\pi < \gamma \le \pi, \quad K \in SL_2(\mathbb{R}), \tag{10.69}$$

that is, $K = (k_{ij})$, $k_{ij} \in \mathbb{R}$, and $\det(K) = 1$.

Definition 10.4.1. The boundary conditions (10.67) are called *separated*, and (10.68) are *coupled*; if $\gamma = 0$, then we say they are *real coupled*, and with $\gamma \neq 0$, they are *complex coupled*.

We have the following theorem.

Theorem 10.4.3. *Let S be a self-adjoint Legendre operator in* $L^2(-1,1)$ *according to Theorem* 10.4.1 *and denote its spectrum by* $\sigma(S)$.

(1) *Then the boundary conditions determining S are either given by* (10.59) *and each such boundary condition determines a self-adjoint Legendre operator in* $L^2(-1,1)$.
(2) *The spectrum* $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$ *is real and discrete, and can be ordered to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots, \tag{10.70}$$

Here equality cannot hold for two consecutive terms.
(3) *If the boundary conditions are separated, then strict inequality holds everywhere in* (10.70), *and if* u_n *is an eigenfunction of* λ_n, *then* u_n *is unique up to constant multiples and has exactly n zeros in the open interval* $(-1,1)$ *for each* $n = 0, 1, 2, 3, \ldots$.
(4) *If the boundary conditions are coupled and real* ($\gamma = 0$) *and* u_n *is a real eigenfunction of* λ_n, *then the number of zeros of* u_n *in the open interval* $(-1,1)$ *is* 0 *or* 1 *if* $n = 0$ *and* $n-1$, *n, or* $n+1$ *if* $n \ge 1$. *(Note that although there may be eigenvalues of multiplicity* 2, *the indexing of the eigenvalues* λ_n *is uniquely determined, but there may be some ambiguity about the indexing of the eigenfunctions* u_n.*)*

(5) *If the boundary conditions are coupled and complex* ($\gamma \neq 0$)*, then all eigenvalues are simple, and strict inequality holds in* (10.70)*. If* u_n *is an eigenfunction of* λ_n*, then the complex eigenfunction* u_n *has no zero in the closed interval* $[-1, 1]$*. The number of zeros of both real part* $\mathrm{Re}(u_n)$ *and imaginary part* $\mathrm{Im}(u_n)$ *in the half-open interval* $[-1, 1)$ *is* 0 *or* 1 *if* $n = 0$ *and is* $n - 1$*,* n*, or* $n + 1$ *if* $n \geq 1$*.*

(6) *If the boundary condition is the classical condition*

$$(py')(-1) = 0 = (py')(1),$$

then the eigenvalues are given by

$$\lambda_n = n(n+1), \quad n \in \mathbb{N}_0,$$

and the normalized eigenfunctions are the classical Legendre polynomials P_n*.*

(7) *For any boundary conditions, separated, real coupled, or complex coupled, we have*

$$\lambda_n \leq n(n+1), \quad n \in \mathbb{N}_0. \tag{10.71}$$

In other words, the eigenvalues of the self-adjoint Legendre operator determined by the classical boundary conditions maximize the eigenvalues of all other self-adjoint Legendre operators.

(8) *For any self-adjoint boundary conditions, separated, real coupled, or complex coupled, we have*

$$n(n+1) \leq \lambda_{n+2}, \quad n \in \mathbb{N}_0. \tag{10.72}$$

In other words, the nth eigenvalue of the self-adjoint Legendre operator determined by the classical boundary conditions is a lower bound of λ_{n+2} *for all other self-adjoint Legendre operators. These bounds are precise:*

(9) *The range of* $\lambda_0(S) = (-\infty, 0]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1, 1)$*.*

(10) *The range of* $\lambda_1(S) = (-\infty, 0]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1, 1)$*.*

(11) *The range of* $\lambda_n(S) = ((n-2)(n-1), n(n+1)]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1, 1)$*.*

(12) *The last three statements about the range of the eigenvalues are still valid if the operators* S *are restricted to those determined by real boundary condition only.*

(13) *Let* S *be any self-adjoint Legendre operator in* $L^2(-1, 1)$ *determined by separated, real coupled, or complex coupled boundary conditions, and let* $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$ *denote its spectrum. Then*

$$\frac{\lambda_n}{n^2} \to 1 \quad \text{as } n \to \infty.$$

Proof. Part (6) is the well-known classical result about the Legendre equation and its polynomial solutions. All the other parts follow from applying the known corresponding results for regular problems; see [113, Chapter 4] for the above regularization of the singular Legendre equation. □

10.5 The maximal and Friedrichs domains

In this section, we develop properties of the maximal and Friedrichs domains including various their characterizations. Recall that the maximal domain $D_{\max}$ is defined as follows. Let $H = L^2(-1,1)$ and

$$D_{\max} = \{y \in H : (py') \in AC_{\mathrm{loc}}(-1,1), \quad (py')' \in H.$$

The next lemma describes maximal domain functions and their quasi-derivatives.

Lemma 10.5.1. *Let v be given by* (10.8). *For every $y \in D_{\max}$, there exist two constants $c, d \in \mathbb{C}$ and a function $g \in H$ such that*

$$y(t) = c + dv(t) + \int_{-1}^{t} [v(t) - v(s)]g(s)\,ds, \quad -1 < t < 1,$$

$$(py')(t) = d + \int_{-1}^{t} g(s)\,ds, \quad -1 < t < 1. \tag{10.73}$$

Conversely, for every $c, d \in \mathbb{C}$ and $g \in H$, the function y defined by (10.73) *is in $D_{\max}$.*

Proof. Suppose $y \in D_{\max}$. Then $(py')' \in H$. Let $(py')' = g$. Since u, v are linearly independent solutions of $(py')' = 0$, (10.73) follows directly from the variation-of-parameters formula. (The integrals exist since $v \in H$ and $v \in L^1(-1,1)$.) Differentiating (10.73) yields, for almost all $t \in (-1,1)$,

$$y'(t) = dv'(t) + v'(t) \int_{-1}^{t} g(s)\,ds.$$

Multiplying by $p(t)$ and noting that $(pv')(t) = 1$ yield the first part of (10.73). To prove the converse statement, note that y is in H since each its term is in $L^2(-1,1)$. Clearly, $(py') \in AC_{\mathrm{loc}}(-1,1)$, and $(py')' = g \in H$. □

Corollary 10.5.1. *The quasi-derivative (py') of every maximal domain function y can be continuously extended to the compact interval $[-1,1]$ and is therefore continuous and bounded on $[-1,1]$.*

Proof. This follows from Corollary 10.5.1. □

Lemma 10.5.2. *Let v be given by* (10.8). *For every $y \in D_{\max}$, we have:*

(1) *Both limits*

$$\lim_{t\to -1^+} \frac{y(t)}{v(t)} \quad \textit{and} \quad \lim_{t\to 1^-} \frac{y(t)}{v(t)} \tag{10.74}$$

exist and are finite.

(2) *For any c, d such that $-1 < c < 0 < d < 1$,*

$$(\sqrt{p})v\left(\frac{y}{v'}\right) \in L^2(-1, c), \quad (\sqrt{p})v\left(\frac{y}{v'}\right) \in L^2(d, 1). \tag{10.75}$$

Proof. In Section 10.3, we showed that $z = y/v_m$ is a solution of the regular Legendre equation (10.1). Therefore z can be continuously extended to both endpoints. Since v_m agrees with v near both endpoints, (10.74) follows. For part (2), see Niessen and Zettl [86, Theorem 4.2, p. 558]. □

Recall the definition of the Friedrichs domain D_F:

$$D_F = \{y \in D_{\max} : (py')(-1) = 0 = (py')(1)\}. \tag{10.76}$$

The next theorem gives a number of equivalent characterizations of the Friedrichs domain; see also [30] and [52].

Theorem 10.5.1. *Let v be given by* (10.8). *For any $y \in D_{\max}$, the following statements are equivalent:*

(i) *In* (10.73) *of Lemma* (10.5.1) *the constant $d = 0$.*

(ii) *y is bounded on* $(-1, 1)$.

(iii) *The limits*

$$\frac{y(t)}{v(t)} \to 0 \quad \textit{as } t \to -1^+ \quad \textit{and as } t \to +1$$

exist and are finite.

(iv)

$$\lim_{t\to -1^+} (py')(t) = 0 = \lim_{t\to 1^-} (py')(t).$$

(v) *The limits*

$$\lim_{t\to -1^+} y(t), \quad \lim_{t\to 1^-} y(t)$$

exist and are finite.

(vi) $y \in AC[-1, 1]$.

(vii) $y' \in L^2(-1, 1)$. *Furthermore, this result is best possible in that there exists $g \in D(S_F)$ such that $g' \notin L^q(-1, 1)$ for any $q > 2$, where g is independent of q.*

(viii) $p^{1/2}y' \in L^2(-1,1)$.
(ix) *For any* $-1 < c < 0 < d < 1$, *we have*

$$\frac{y}{(\sqrt{p})v} \in L^2(-1,c) \quad \textit{and} \quad \frac{y}{(\sqrt{p})v} \in L^2(d,1), \quad -1 < c < 0 < d < 1.$$

(x) $y, y' \in AC_{\text{loc}}(-1,1)$ *and* $py'' \in L^2(-1,1)$. *Furthermore, this result is best possible in the sense that there exists* $g \in D(S_F)$ *such that* $pg'' \notin L^q(-1,1)$ *for any* $q > 2$, *where* g *is independent of* q.

Proof. The equivalence of (i), (ii), (iii), (v), and (vi) is clear from (10.73) of Lemma (10.5.1) and the definition of $v(t)$ in (10.8). We now prove the equivalence of (ii) and (iv) by using the method used to construct regular Legendre equations. In particular, we use the "regularizing" function v_m and other notation from Section 10.2. Recall that v_m agrees with v near both endpoints and is positive on $(-1,1)$. As in Section 10.2, $[\cdot,\cdot]_M$ and $[\cdot,\cdot]_N$ denote the Lagrange brackets of M and N, respectively. Let $z = y/v$ and $x = u/v$. Then

$$\begin{aligned} -(py')(1) &= \left[\frac{y}{v_m}, \frac{u}{v_m}\right]_M (1) = [z,x]_N(1) \\ &= \lim_{t\to1} z(t) \lim_{t\to1}(Px')(1) - \lim_{t\to1} x(t) \lim_{t\to1}(Pz')(1) = \lim_{t\to1} z(t) \lim_{t\to1}(Px')(1) = 0. \end{aligned}$$

All these limits exist and are finite since N is a regular problem. Since u is a principal solution and v is a nonprincipal solution, it follows that $\lim_{t\to1} x(t) = 0$. The proof for the endpoint -1 is entirely similar. Thus we have shown that (ii) implies (iv). The converse is obtained by reversing the steps. Thus we conclude that (i)–(vi) are equivalent. Proofs of (vii), (viii), (ix), and (x) can be found in [3]. □

10.6 The Legendre Green's function

In this subsection, we construct the Legendre Green's function. Let

$$-(py')' = \lambda y, \quad p(t) = 1 - t^2 \quad \text{on } J = (-1,1). \tag{10.77}$$

This construction is a five-step procedure:
(1) Formulate the singular second-order scalar equation as a first-order singular system.
(2) "Regularize" this singular system by constructing regular systems equivalent to it.
(3) Construct the Green's matrix for boundary value problems of the regular system.
(4) Construct the singular Green's matrix for the equivalent singular system from the regular one.

(5) Extract the upper right corner element from the singular Green's matrix. This is the Green's function for singular scalar boundary value problems for equation (10.1).

For convenience of the reader, we present these five steps here even though some of them were given above.

For $\lambda = 0$, recall the two linearly independent solutions u, v of (10.1) given by

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{1-t}{t+1}\right|\right). \tag{10.78}$$

The standard system formulation of (10.77) has the form

$$Y' = (P - \lambda W)Y \quad \text{on } (-1, 1), \tag{10.79}$$

where

$$Y = \begin{bmatrix} y \\ (py') \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1/p \\ 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{10.80}$$

Let

$$U = \begin{bmatrix} u & v \\ (pu') & (pv') \end{bmatrix} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}. \tag{10.81}$$

Note that $\det U(t) = 1$ for $t \in J = (-1, 1)$ and set

$$Z = U^{-1}Y. \tag{10.82}$$

Then

$$\begin{aligned} Z' &= (U^{-1})'Y + U^{-1}Y' = -U^{-1}U'U^{-1}Y + (U^{-1})(P - \lambda W)Y \\ &= -U^{-1}U'Z + (U^{-1})(P - \lambda W)UZ \\ &= -U^{-1}(PU)Z + U^{-1}(PU)Z - \lambda(U^{-1}WU)Z = -\lambda(U^{-1}WU)Z. \end{aligned}$$

Letting $G = (U^{-1}WU)$, we can conclude that

$$Z' = -\lambda GZ, \tag{10.83}$$

where

$$G = U^{-1}WU = \begin{bmatrix} -v & -v^2 \\ 1 & v \end{bmatrix}. \tag{10.84}$$

Note that (10.83) is the regularized Legendre system discussed in Section 10.3.

The next theorem summarizes the properties of system (10.83) and its relation to (10.77).

Theorem 10.6.1. *Let $\lambda \in \mathbb{C}$ and let G be given by* (10.84).

(1) *Every component of G is in $L^1(-1,1)$ and therefore* (10.84) *is a regular system.*

(2) *For any $c_1, c_2 \in \mathbb{C}$, the initial value problem*

$$Z' = -\lambda G Z, \quad Z(-1) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{10.85}$$

has a unique solution Z defined and continuous on the closed interval $[-1,1]$.

(3) *If $Y = \begin{bmatrix} y(t,\lambda) \\ (py')(t,\lambda) \end{bmatrix}$ is a solution of* (10.79) *and $Z = U^{-1}Y = \begin{bmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{bmatrix}$, then Z is a solution of* (10.83)*, and for all $t \in (-1,1)$, we have*

$$y(t,\lambda) = uz_1(t,\lambda) + v(t)z_2(t,\lambda) = z_1(t,\lambda) + v(t)z_2(t,\lambda), \tag{10.86}$$

$$(py')(t,\lambda) = (pu')z_1(t,\lambda) + (pv')(t)z_2(t,\lambda) = z_2(t,\lambda). \tag{10.87}$$

(4) *For every solution $y(t,\lambda)$ of the singular scalar Legendre equation* (10.77)*, the quasi-derivative $(py')(t,\lambda)$ is continuous on the compact interval $[-1,1]$. More specifically, we have*

$$\lim_{t\to -1^+} (py')(t,\lambda) = z_2(-1,\lambda), \quad \lim_{t\to 1^-} (py')(t,\lambda) = z_2(1,\lambda). \tag{10.88}$$

Thus the quasi-derivative $(py')(t,\lambda)$ is a continuous function on the closed interval $[-1,1]$ for every $\lambda \in \mathbb{C}$.

(5) *Let $y(t,\lambda)$ be given by* (10.86)*. If $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is unbounded at 1; if $z_2(-1,\lambda) \neq 0$, then $y(t,\lambda)$ is unbounded at -1.*

(6) *Fix $t \in [-1,1]$. Let $c_1, c_2 \in \mathbb{C}$. If $Z = \begin{bmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{bmatrix}$ is the solution of* (10.85) *determined by the initial conditions $z_1(-1,\lambda) = c_1$ and $z_2(-1,\lambda) = c_2$, then $z_i(t,\lambda)$ is an entire function of λ, $i = 1,2$, and similarly for the initial conditions $z_1(1,\lambda) = c_1$ and $z_2(1,\lambda) = c_2$.*

(7) *For each $\lambda \in \mathbb{C}$, there is a nontrivial solution that is bounded in a (two-sided) neighborhood of 1; and there is a (generally different) nontrivial solution that is bounded in a (two-sided) neighborhood of -1.*

(8) *A nontrivial solution $y(t,\lambda)$ of the singular scalar Legendre equation* (10.77) *is bounded at 1 if and only if $z_2(1,\lambda) = 0$. A nontrivial solution $y(t,\lambda)$ of the singular scalar Legendre equation* (10.77) *is bounded at -1 if and only if $z_2(-1,\lambda) = 0$.*

Proof. Part (1) follows from (10.84); (2) is a direct consequence of (1) and the theory of regular systems; $Y = UZ$ implies (3) $\Longrightarrow$ (4) and (5); (6) follows from (2) and the basic theory of regular systems. For (7), determine the solutions $y_1(t,\lambda)$ and $y_{-1}(t,\lambda)$ by applying the Frobenius method to obtain power series solutions of the following form (see [30], p. 5 with different notations):

$$y_1(t,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda)(t-1)^n, \quad |t-1| < 2;$$

$$y_{-1}(t,\lambda) = 1 + \sum_{n=1}^{\infty} b_n(\lambda)(t+1)^n, \quad |t+1| < 2. \tag{10.89}$$

To prove (8), it follows from (10.86) that if $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is not bounded at 1. Suppose $z_2(1,\lambda) = 0$. If the corresponding $y(t,\lambda)$ is not bounded at 1, then there are two linearly unbounded solutions at 1, and hence all nontrivial solutions are unbounded at 1. This contradiction establishes (8) and completes the proof of the theorem. □

Remark 10.6.1. From Theorem 10.6.1 we see that *for every* $\lambda \in \mathbb{C}$, equation (10.77) has a solution y_1 that is bounded at 1 and has a solution y_{-1} that is bounded at -1. It is well known that for $\lambda_n = n(n+1)$, $n \in \mathbb{N}_0$, the Legendre polynomials P_n are solutions on $(-1,1)$ and hence are bounded at -1 and $+1$.

We now consider two-point boundary conditions for (10.85); later, we will relate these to singular boundary conditions for (10.77).

Let $A, B \in M_2(\mathbb{C})$, the set of 2×2 complex matrices, and consider the boundary value problem

$$Z' = -\lambda GZ, \quad AZ(-1) + BZ(1) = 0. \tag{10.90}$$

Recall that $\Phi(t,s,-\lambda)$ is the primary fundamental matrix of the system $Z' = -\lambda GZ$.

Lemma 10.6.1. *A complex number $-\lambda$ is an eigenvalue of* (10.90) *if and only if*

$$\Delta(\lambda) = \det(A + B\Phi(1,-1,-\lambda) = 0. \tag{10.91}$$

Furthermore, a complex number $-\lambda$ is an eigenvalue of geometric multiplicity two if and only if

$$A + B\Phi(1,-1,-\lambda) = 0. \tag{10.92}$$

Proof. Note that a solution for the initial condition $Z(-1) = C$ is given by

$$Z(t) = \Phi(t,-1,-\lambda)C, \quad t \in [-1,1]. \tag{10.93}$$

The boundary value problem (10.90) has a nontrivial solution for Z if and only if the algebraic system

$$[A + B\Phi(1,-1,-\lambda)]Z(-1) = 0 \tag{10.94}$$

has a nontrivial solution for $Z(-1)$.

To prove the furthermore part, observe that two linearly independent solutions of the algebraic system (10.93) for $Z(-1)$ yield two linearly independent solutions $Z(t)$ of the differential system, and conversely. □

Given any $\lambda \in \mathbb{R}$ and solutions y, z of (10.1), the Lagrange form $[y,z](t)$ is defined by

$$[y,z](t) = y(t)(p\overline{z'})(t) - \overline{z}(t)(py')(t).$$

So, in particular, we have

$$[u,v](t) = +1, \quad [v,u](t) = -1, \quad [y,u](t) = -(py')(t), \quad t \in \mathbb{R},$$
$$[y,v](t) = y(t) - v(t)(py')(t), \quad t \in \mathbb{R}, \quad t \neq \pm 1.$$

We will further see that although v blows up at ± 1, the form $[y,v](t)$ is well defined at -1 and $+1$ since the limits

$$\lim_{t\to -1}[y,v](t), \quad \lim_{t\to +1}[y,v](t)$$

exist and are finite from both sides for any solution y of equation (10.1) and any $\lambda \in \mathbb{R}$. Note that since v blows up at 1, this means that y must blow up at 1, except, possibly, when $(py')(1) = 0$.

We are now ready to construct the Green's function of the singular scalar Legendre problem consisting of the equation

$$My = -(py')' = \lambda y + h \quad \text{on } J = (-1,1), \quad p(t) = 1 - t^2, \quad -1 < t < 1, \tag{10.95}$$

together with two-point boundary conditions

$$A\begin{bmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{bmatrix} + B\begin{bmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{10.96}$$

where u, v are given by (10.78), and A, B are 2×2 complex matrices. This construction is based on the system regularization discussed before, and we will use the notation from above. Consider the regular nonhomogeneous system

$$Z' = -\lambda GZ + F, \quad AZ(-1) + BZ(1) = 0, \tag{10.97}$$

where

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad f_j \in L^1(J, \mathbb{C}), \quad j = 1,2. \tag{10.98}$$

Theorem 10.6.2. *Let* $-\lambda \in \mathbb{C}$*, and let* $\Delta(-\lambda) = [A + B\Phi(1,-1,-\lambda)]$. *Then the following statements are equivalent:*
(1) *For* $F = 0$ *on* $J = (-1,1)$*, the homogeneous problem has only the trivial solution.*
(2) $\Delta(-\lambda)$ *is nonsingular.*
(3) *For every* $F \in L^1(-1,1)$*, the nonhomogeneous problem* (10.78) *has a unique solution* Z*, and this solution is given by*

$$Z(t,-\lambda) = \int_{-1}^{1} K(t,s,-\lambda)F(s)\,ds, \quad -1 \le t \le 1, \tag{10.99}$$

where

$$K(t,s,-\lambda) = \begin{cases} \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) \\ \quad \text{if } -1 \le t < s \le 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \phi(t,s-\lambda) \\ \quad \text{if } -1 \le s < t \le 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \frac{1}{2}\phi(t,s-\lambda) \\ \quad \text{if } -1 \le s = t \le 1. \end{cases}$$

The proof is a minor modification of the Neuberger construction given in [85]; see also [113].

From the regular Green's matrix we now construct the singular Green's matrix and, from the latter, the singular scalar Legendre Green's function.

Definition 10.6.1. Let

$$L(t,s,\lambda) = U(t)K(t,s,-\lambda)U^{-1}(s), \quad -1 \le t,s \le 1. \tag{10.100}$$

The next theorem shows that L_{12}, the upper right component of L, is the Green's function of the singular scalar Legendre problem (10.95)–(10.96).

Theorem 10.6.3. *Assume that* $[A + B\Phi(1,-1,-\lambda)]$ *is nonsingular. Then for every function* h *satisfying*

$$h, vh \in L^1(J,\mathbb{C}), \tag{10.101}$$

the singular scalar Legendre problem (10.95)–(10.96) *has a unique solution* $y(\cdot,\lambda)$ *given by*

$$y(t,\lambda) = \int_{-1}^{1} L_{12}(t,s)h(s)\,ds, \quad -1 < t < 1. \tag{10.102}$$

Proof. Let

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = U^{-1}H, \quad H = \begin{bmatrix} 0 \\ -h \end{bmatrix}. \tag{10.103}$$

Then $f_j \in L^1(J_2,\mathbb{C})$, $j = 1,2$. Since $Y(t,\lambda) = U(t)Z(t,-\lambda)$, from (10.99) we get

$$\begin{aligned} Y(t,\lambda) &= U(t)Z(t,-\lambda) = U(t)\int_{-1}^{1} K(t,s,-\lambda)F(s)\,ds \\ &= \int_{-1}^{1} U(t)K(t,s,-\lambda)U^{-1}(s)H(s)\,ds = \int_{-1}^{1} L(t,s,\lambda)H(s)\,ds, \quad -1 < t < 1. \end{aligned} \tag{10.104}$$

Therefore

$$y(t, \lambda) = -\int_{-1}^{1} L_{12}(t, s, \lambda) h(s)\, ds, \quad -1 < t < 1. \tag{10.105}$$

□

An important property of the Friedrichs extension S_F is the well-known fact that it has the same lower bound as the minimal operator $S_{\min}$. But this fact does not characterize the Friedrichs extension of $S_{\min}$. Haertzen, Kong, Wu, and Zettl [44] characterized all self-adjoint regular Sturm–Liouville operators that preserve the lower bound of the minimal operator; see also Proposition 4.8.1 in [113]. The next theorem uniquely characterizes the Legendre Friedrichs extension S_F.

Theorem 10.6.4 (Everitt, Littlejohn, and Marić). *Let $S \neq S_F$ be a self-adjoint Legendre operator in $L^2(-1, 1)$.Then there exists $f \in D(S)$ such that*

$$pf'' \notin L^2(-1, 1) \textit{ and } f' \notin L^2(-1, 1).$$

Proof. See [30]. □

10.7 Operators on the interval $(1, +\infty)$

In this section, we discuss self-adjoint operators for the interval $J_3 = (1, \infty)$. A similar discussion for $(-\infty, -1)$ can be obtained from the change of variable $t \to -t$. Consider

$$My = -(py')' = \lambda y \quad \text{on } J_3 = (1, \infty), \quad p(t) = 1 - t^2. \tag{10.106}$$

Note that $p(t) < 0$ for $t > 1$. So to conform to the standard notation for Sturm-Liouville problems, we study the equivalent equation

$$Ny = -(ry')' = \xi y \quad \text{on } J_3 = (1, \infty), \quad r(t) = t^2 - 1 > 0, \quad \xi = -\lambda. \tag{10.107}$$

This N should not be confused with the notation N used for the regularized equation in Section 10.3.

Recall that for $\lambda = \xi = 0$, two linearly independent solutions are given by

$$u(t) = 1, \quad v(t) = \frac{1}{2} \ln\left(\left|\frac{t-1}{t+1}\right|\right). \tag{10.108}$$

Although we focus on the interval $(1, \infty)$ in this section, we make the following general observations. For all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$(pv')(t) = -1, \tag{10.109}$$

so for any $\lambda \in \mathbb{R}$ and any solution y of the Legendre equation, we have the following Lagrange forms:

$$[y,u] = -py', \quad [y,v] = -y - v(py'), \quad [u,v] = -1, \quad [v,u] = 1. \tag{10.110}$$

As before for the interval $(-1,1)$, these play an important role in the study of self-adjoint operators in $L^2(1,\infty)$. Recall that although v blows up at -1 and $+1$ from both sides, it turns out that these forms are defined and finite at all points of $\mathbb{R}$ including -1 and $+1$, provided thet we define the appropriate one-sided limits

$$[y,u](1^+) = \lim_{t\to 1^+}[y,u](t), \quad [y,u](1^+) = \lim_{t\to -1^-}[y,u](t) \tag{10.111}$$

for all $y \in D_{\max}(J_3)$. Since $u \in L^2(1,2)$ and $v \in L^2(1,2)$, it follows from general Sturm–Liouville theory that 1, the left endpoint of J_3, is limit-circle nonoscillatory (LCNO). In particular, all solutions of equations (10.106) and (10.107) are in $L^2(1,2)$ for each $\lambda \in \mathbb{C}$.

In the mathematics and physics literature, when a singular Sturm–Liouville problem is studied on a half-line (a,∞), it is generally assumed that the endpoint a us regular. Here the left endpoint $a = 1$ is singular. Therefore regular conditions such as $y(a) = 0$ or, more generally,

$$A_1y(a) + A_2(py')(a) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0)$$

do not make sense. It is interesting that, as pointed out before, in the Legendre case studied here, although the Dirichlet condition

$$y(1) = 0$$

does not make sense, the Neumann condition

$$(py')(1) = 0 \tag{10.112}$$

does in fact determine a self-adjoint Legendre operator in $L^2(1,\infty)$, the Friedrichs extension! So although (10.112) has the appearance of a regular Neumann condition, in the Legendre case, it is actually an analogue of the Dirichlet condition!

By a self-adjoint operator associated with equation (10.78) in $H_3 = L^2(1,\infty)$, that is, a self-adjoint realization of equation (10.78) in H_3, we mean a self-adjoint restriction of the maximal operator $S_{\max}$ associated with (10.78). This is defined as follows:

$$D_{\max} = \{f : (-1,1) \to \mathbb{C} \mid f,\ pf' \in AC_{\text{loc}}(-1,1);\ f, pf' \in H_3\}, \tag{10.113}$$

$$S_{\max}f = -(rf')', f \in D_{\max}. \tag{10.114}$$

Note that, in contrast to the $(-1,1)$ case, the Legendre polynomials neither are in $D_{\max}$ nor are solutions of (10.78) in general. As in the case for $(-1,1)$, we have the following basic lemma.

Lemma 10.7.1. *The operator S_{max} is densely defined in H_3 and therefore has a unique adjoint in H_3, denoted by S_{min}:*

$$S^*_{max} = S_{min}.$$

The minimal operator S_{min} in H_3 is symmetric, closed, and densely defined and satisfies

$$S^*_{min} = S_{max}.$$

Its deficiency index $d = d(S_{min}) = 1$. If S is a self-adjoint extension of S_{min}, then S is also a self-adjoint restriction of S_{max}, and conversely. Thus we have

$$S_{min} \subset S = S^* \subset S_{max}.$$

Proof. See [113]. □

It is clear from Lemma 10.6.1 that each self-adjoint operator S is determined by its domain. Next, we describe these self-adjoint domains. As before for the interval $(-1, 1)$, the functions u, v play an important role.

The Legendre operator theory for the interval $(1, \infty)$ is similar to that on $(-1, 1)$, except for the fact that the endpoint ∞ is in the limit-point case and therefore there are no boundary conditions required or allowed at ∞. Thus all self-adjoint Legendre operators in $H_3 = L^2(1, \infty)$ are generated by separated singular self-adjoint boundary conditions at 1. These have the form

$$A_1[y, u](1) + A_2[y, v](1) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0, 0). \tag{10.115}$$

Theorem 10.7.1. *Let $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0, 0)$, and define the linear manifold $D(S)$ consisting of all $y \in D_{max}$ satisfying* (10.115)*. Then the operator S with domain $D(S)$ is self-adjoint in $L^2(1, \infty)$. Moreover, given any operator S satisfying $S_{min} \subset S = S^* \subset S_{max}$, there exist $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0, 0)$, such that $D(S)$, the domain of S, is determined by* (10.115).

The proof of this theorem is based on the next three lemmas.

Lemma 10.7.2. *Suppose $S_{min} \subset S = S^* \subset S_{max}$. Then there exists a function $g \in D(S) \subset D_{max}$ satisfying*

(1) *g is not in D_{min}, and*

(2)
$$[g, g](1) = 0,$$

so that $D(S)$ consists of all $y \in D_{max}$ satisfying

(3)
$$[y, g](1) = 0. \tag{10.116}$$

Conversely, given $g \in D_{max}$ that satisfies conditions (1) and (2), the set $D(S) \subset D_{max}$ consisting of all y satisfying (3) is a self-adjoint extension of S_{min}.

Proof. The proof of the lemma follows from the GKN theory [113] applied to (10.78). □

The next lemma also plays an important role and is called the "bracket decomposition lemma" in [113].

Lemma 10.7.3 (Bracket decomposition lemma). *For any $y, z \in D_{\max}$, we have*

$$[y,z](1) = [y,v](1)[\overline{z},u](1) - [y,u](1)[\overline{z},v](1). \tag{10.117}$$

Proof. See [113, pp. 175–176]. □

Lemma 10.7.4. *For any $\alpha, \beta \in \mathbb{C}$, there exists a function $g \in D_{\max}(J_3)$ such that*

$$[g,u](1^+) = \alpha, \quad [g,v](1^+) = \beta. \tag{10.118}$$

Proof. See [113, pp. 175–176]. □

Armed with these lemmas, we can now proceed to the proof of Theorem 10.7.1.

Proof. Let $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0,0)$. By Lemma 10.7.3 there exists $g \in D_{\max}(J_3)$ such that

$$[g,u](1^+) = A_2, \quad [g,v](1^+) = -A_1. \tag{10.119}$$

From (10.116) we get that for any $y \in D_{\max}$,

$$[y,g](1) = [y,v](1)[g,u](1) - [y,u](1)[g,v](1) = A_1[y,u](1) + A_2[y,v](1).$$

Now consider the boundary condition

$$A_1[y,u](1) + A_2[y,v](1) = 0. \tag{10.120}$$

If (10.120) holds for all $y \in D_{\max}$, then it follows from Lemma 10.4.1, p. 175 of [113], that $g \in D_{\min}$. But this implies, also by Lemma 10.4.1, that $(A_1, A_2) \neq (0,0)$, which is a contradiction. From this it follows that

$$\begin{aligned}[g,g](1) &= [g,v](1)[g,u](1) - [g,u](1)[g,v](1)\\ &= A_1[g,u](1) + A_2[g,v](1) = A_1A_2 - A_2A_1 = 0.\end{aligned}$$

Therefore g satisfies conditions (1) and (2) of Lemma 10.7.2, and, consequently,

$$[y,g](1) = A_1[y,u](1) + A_2[y,v](1) = 0 \tag{10.121}$$

is a self-adjoint boundary condition.

To prove the converse, reverse the steps in this argument. □

It is clear from Theorem 10.7.1 that there are an uncountable number of self-adjoint Legendre operators in $L^2(1,\infty)$. It is also clear that the Legendre polynomials P_n are not eigenfunctions of any such operator since they are not in the maximal domain and therefore not in the domain of any self-adjoint restriction S of $D_{\max}$.

Next, we discuss the spectrum of the self-adjoint Legendre operators in $H_3 = L^2(1,\infty)$. Recall that the essential spectrum of all self-adjoint extensions of $S_{\min}$ is the same.

Lemma 10.7.5. *Let $S_{\min} \subset S = S^* \subset S_{\max}$, where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators in $L^2(1,\infty)$ associated with equation* (10.106). *Then:*

Lemma 10.7.6. *S has no discrete spectrum.*

The essential spectrum $\sigma_e(S)$ is given by

$$\sigma_e(S) = \left(-\infty, -\frac{1}{4}\right].$$

The proof of this lemma is given in Proposition 10.2.1. The next theorem gives the version of this lemma for the Legendre equation in the more commonly used form (10.1).

Theorem 10.7.2. *Let $S_{\min} \subset S = S^* \subset S_{\max}$, where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators in $L^2(1,\infty)$ associated with equation* (10.1). *Then:*
- *S has no discrete spectrum.*
- *The essential spectrum $\sigma_e(S)$ is given by*

$$\sigma_e(S) = \left[\frac{1}{4}, \infty\right).$$

Proof. This follows from the preceding lemma by simply changing the sign. □

10.8 The Legendre operators on the whole line

In this section, we study the self-adjoint operators generated by the Legendre equation (10.1) on the whole real line $\mathbb{R} = (-\infty,\infty)$. Our approach is using the direct sum method developed by Everitt and Zettl [37]. To apply this method, we identify the Hilbert space $L^2(\mathbb{R})$ with the direct sum

$$L^2(\mathbb{R}) = L^2(-\infty,-1) \dotplus L^2(-1,1) \dotplus L^2(1,\infty). \tag{10.122}$$

The Legendre equation (10.1) has singular points at $-\infty$ and $+\infty$ and at the interior points -1 and $+1$. So we study the three-interval problem for the three intervals

$$J_1 = (-\infty,-1), \quad J_2 = (-1,1), \quad J_3 = (1,\infty). \tag{10.123}$$

We use the notations -1^-, -1^+, $+1^-$, and $+1^+$ to indicate that -1 is a right endpoint for J_1 and a left endpoint for J_2; $+1$ is a left endpoint for J_2 and a left endpoint for J_3.

Although only two-interval problems are discussed in Chapter 8, their extension to three intervals is straightforward, especially, since in this section, we use only the usual inner product on each interval, not the Mukhtarov–Yakubov [82] modification.

Let $S_{\min}(J_i)$ and $S_{\max}(J_i)$ denote the minimal and maximal operators in $L^2(J_i)$, $i = 1,2,3$, with domains $D_{\min}(J_i)$ and $D_{\max}(J_i)$, respectively.

Definition 10.8.1. The minimal and maximal Legendre operators $S_{\min}$ and $S_{\max}$ in $L^2(\mathbb{R})$ and their domains $D_{\min}$, $D_{\max}$ are defined as follows:

$$\begin{aligned} D_{\min} &= D_{\min}(J_1) \dotplus D_{\min}(J_2) \dotplus D_{\min}(J_3), \\ D_{\max} &= D_{\max}(J_1) \dotplus D_{\max}(J_2) \dotplus D_{\max}(J_3), \\ S_{\min} &= S_{\min}(J_1) \dotplus S_{\min}(J_2) \dotplus S_{\min}(J_3), \\ S_{\min} &= S_{\max}(J_1) \dotplus S_{\max}(J_2) \dotplus S_{\max}(J_3). \end{aligned}$$

Lemma 10.8.1. *The minimal operator $S_{\min}$ is a closed densely defined symmetric operator in $L^2(\mathbb{R})$ satisfying*

$$S_{\min}^* = S_{\max}, \quad S_{\max}^* = S.$$

Its deficiency index $d = d(S_{\min}) = 4$. Each self-adjoint extension S of $S_{\min}$ is a restriction of $S_{\max}$, that is, we have

$$S_{\min} \subset S = S^* \subset S_{\max}.$$

Proof. The adjoint properties follow from the corresponding properties of the component operators, and it follows that

$$\begin{aligned} \operatorname{def}(S_{\min}) &= \operatorname{def}(S_{\min}(J_1)) + \operatorname{def}(S_{\min}(J_2)) + \operatorname{def}(S_{\min}(J_3)) \\ &= 1 + 2 + 1 = 4, \end{aligned}$$

since $-\infty$ and $+\infty$ are LP and -1^-, -1^+, $+1^-$, $+1^+$ are all LC. □

Remark 10.8.1. Although the minimal and maximal operators $S_{\min}$, $S_{\max}$ are the direct sums of the corresponding operators on each of the three intervals, we will further see that there are many self-adjoint extensions S of $S_{\min}$ that are not simply direct sums of operators from these three intervals.

For $y, z \in D_{\max}$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$, we define the “three-interval” or “whole-line” Lagrange sesquilinear form $[\cdot,\cdot]$ as follows:

$$\begin{aligned} [y,z] &= [y_1,z_1]_1(-1^-) - [y_1,z_1]_1(-\infty) + [y_2,z_2]_2(+1^-) - [y_2,z_2]_2(-1^+) \\ &\quad + [y_3,z_3]_3(+\infty) - [y_3,z_3]_3(+1^+) \\ &= [y_1,z_1]_1(-1^-) + [y_2,z_2]_2(+1^-) - [y_2,z_2]_2(-1^+) - [y_3,z_3]_3(+1^+). \end{aligned} \tag{10.124}$$

Here $[y_i, z_i]_i$ denotes the Lagrange form on the interval J_i, $i = 1, 2, 3$. In the last step, we noted that the Lagrange forms evaluated at $-\infty$ and at $+\infty$ are zeros because these are LP endpoints. The fact that each of these one-sided limits exists and is finite follows from the one-interval theory.

As noted before for $\lambda = 0$, the Legendre equation

$$My = -(py')' = \lambda y \tag{10.125}$$

has two linearly independent solutions

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{t-1}{t+1}\right|\right).$$

Observe that u is defined on the whole $\mathbb{R}$, but v blows up logarithmically at the two interior singular points from both sides. Observe that

$$[u, v](t) = u(t)(pv')(t) - v(t)(pu')(t) = 1, \quad -\infty < t < \infty, \tag{10.126}$$

where we have taken appropriate one-sided limits at ± 1, and for all $y \in D$, we have

$$[y, u] = -py', \quad [y, v] = y - v(py'), \tag{10.127}$$

and again by taking appropriate one-sided limits, if necessary, $[y, u](t)$ is defined (finitely) for all $t \in \mathbb{R}$. Similarly, the vector

$$Y = \begin{bmatrix} [y, u] \\ [y, v] \end{bmatrix} = \begin{bmatrix} -py' \\ y - v(py') \end{bmatrix} \tag{10.128}$$

is well defined. In particular,

$$Y(-1^-),\ Y(-1^+),\ Y(1^-),\ Y(1^+) \tag{10.129}$$

are all well defined and finite. Note also that $Y(-\infty)$ and $Y(-\infty)$ are well defined and

$$Y(-\infty) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Y(\infty). \tag{10.130}$$

Remark 10.8.2. For any $y \in D_{\max}$, the one-sided limits of (py') and of $y - v(py')$ exist and are finite at -1 and at 1. Hence if (py') has a nonzero finite limit, then y must blow up logarithmically.

Now we can state the theorem giving the characterization of all self-adjoint extensions S of the minimal operator $S_{\min}$; recall that these are all operators S satisfying $S_{\min} \subset S = S^* \subset S_{\max}$ in the Hilbert space $L^2(\mathbb{R})$, which we identify with the direct sum space $L^2(-\infty, -1) \dotplus L^2(-1, 1) \dotplus L^2(1, \infty)$.

Theorem 10.8.1. *Suppose $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$ are 4×2 complex matrices satisfying the following two conditions:*
(1)

$$\operatorname{rank}(A, B, C, D) = 4, \tag{10.131}$$

(2)

$$AEA^* - BEB^* + CEC^* - DED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10.132}$$

Componentwise, conditions (10.132) *are written for $j, k = 1, 2, 3, 4$ as*

$$(a_{j1}\overline{a}_{k2} - a_{j2}\overline{a}_{k1}) - (b_{j1}\overline{b}_{k2} - b_{j2}\overline{b}_{k1}) + (c_{j1}\overline{c}_{k2} - c_{j2}\overline{c}_{k1}) - (d_{j1}\overline{d}_{k2} - d_{j2}\overline{d}_{k1}) = 0.$$

Define D to be the set of all $y \in D_{\max}$ satisfying

$$AY(-1^-) + BY(-1^+) + CY(1^-) + DY(1^+) = 0, \tag{10.133}$$

where

$$Y = \begin{bmatrix} -(py') \\ y - v(py') \end{bmatrix}.$$

Then D is the domain of a self-adjoint extension S of the three-interval minimal operator $S_{\min}$.

Conversely, given any self-adjoint operator S satisfying $S_{\min} \subset S = S^* \subset S_{\max}$ with domain $D = D(S)$, there exist 2×4 complex matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$ satisfying conditions (1) and (2) such that $D(S)$ is given by (10.133).

Proof. The proof of the theorem is based on three lemmas, which we establish next. Also see Example 13.3.4, pp. 273–275, in [113]. □

Remark 10.8.3. The boundary conditions are given by (10.133); (10.131) determines the *number* of independent conditions, and (10.132) specifies the conditions on the boundary conditions needed for self-adjointness.

Using the three-interval Lagrange form, the next lemma gives an extension of the GKN characterization for the whole-line Legendre problem.

Lemma 10.8.2. *Suppose $S_{\min} \subset S = S^* \subset S_{\max}$. Then there exist $v_1, v_2, v_3, v_4 \in D(S) \subset D_{\max}$ satisfying the following conditions:*

(1) *v_1, v_2, v_3, v_4 are linear independent modulo $D_{\min}$, that is, no nontrivial linear combination is in $D_{\min}$;*

(2)

$$[v_i, v_j] = 0, \quad i, j = 1, 2, 3, 4, \tag{10.134}$$

so that $D(S)$ consists of all $y \in D_{\max}$ satisfying

(3)

$$[y, v_j] = 0, \quad j = 1, 2, 3, 4. \tag{10.135}$$

Conversely, given $v_1, v_2, v_3, v_4 \in D_{\max}$ satisfying conditions (1) and (2), the set $D(S) \subset D_{\max}$ consisting of all y satisfying (3) is a self-adjoint extension of $S_{\min}$.

The lemma follows from [37, Theorem 3.3], extended to three intervals and applied to the Legendre equation. The next lemma is called the "bracket decomposition" lemma in [113]. It applies to each of the intervals J_i, $i = 1, 2, 3$, but for simplicity of notation, we omit the subscripts.

Lemma 10.8.3 (Bracket decomposition lemma). *Let $J_i = (a, b)$, let $y, z, u, v \in D_{\max} = D_{\max}(J_i)$, $J_i = (a, b)m$ and assume that $[v, u](c) = 1$ for some c, $a \le c \le b$. Then*

$$[y, z](c) = [y, v](c)[\overline{z}, \overline{u}](c) - [y, \overline{u}](c)[\overline{z}, v](c). \tag{10.136}$$

For a proof of the lemma, see [113, pp. 175–176]. The next lemma applies a one-interval result to three intervals J_i, $i = 1, 2, 3$.

For this lemma, we extend the definitions of the functions u, v, but we will continue using the same notation:

$$u(t) = \begin{cases} 1, & -1 < t < 1,\ -2 < t < -1,\ 1 < t < 2, \\ 0, & |t| > 3m \end{cases} \tag{10.137}$$

$$v(t) = \begin{cases} -\frac{1}{2}\ln(\frac{t-1}{t+1}), & -1 < t < 1,\ -2 < t < -1,\ 1 < t < 2, \\ 0, & |t| > 3, \end{cases} \tag{10.138}$$

and define both functions on the intervals $[-3, -2]$ and $[2, 3]$ so that they are continuously differentiable on these intervals.

Lemma 10.8.4. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.*

- *There exists $g \in D_{\max}(J_2)$ that is not in $D_{\min}(J_2)$ such that*

$$[g, u](-1^+) = \alpha, \quad [g, v](-1^+) = \beta, \quad [g, u](1^+) = \gamma, \quad [g, v](1^+) = \delta. \tag{10.139}$$

- *There exists $g \in D_{\max}(J_1)$ that is not in $D_{\min}(J_1)$ such that*

$$[g, u](-1^-) = \alpha, \quad [g, v](-1^-) = \beta. \tag{10.140}$$

- *There exists $g \in D_{\max}(J_3)$ that is not in $D_{\min}(J_3)$ such that*

$$[g, u](1^+) = \gamma, \quad [g, v](1^+) = \delta. \tag{10.141}$$

Proof of Theorem 10.8.1. The method of proof is the same as that used in the proof of Theorem 10.7.1, but the computations are longer. It consists in showing that each part of Theorem 10.7.1 is equivalent to the corresponding part of Lemma 10.7.1. □

10.8.1 A self-adjoint Legendre operator on the whole real line

The boundary condition

$$(py')(-1^-) = (py')(-1^+) = (py')(1^-) = (py')(1^+) = 0 \tag{10.142}$$

satisfies the conditions of Theorem 10.8.1 and therefore determines a self-adjoint operator S_L in $L^2(\mathbb{R})$. Let S_1 in $L^2(-\infty,-1)$ be determined by $(py')(-1^-) = 0$, $S_2 = S_F$ in $(-1,1)$ by $(py')(-1^+) = (py')(1^-) = 0$, and S_3 by $(py')(1^+) = 0$. Then each S_i is self-adjoint, and the direct sum

$$S = S_1 \dotplus S_2 \dotplus S_3 \tag{10.143}$$

is a self-adjoint operator in $L^2(-\infty,\infty)$. It is well known that the essential spectrum of a direct sum of operators is the union of the essential spectra of these operators. From this, the above proposition, and the fact that the spectrum of S_2 is discrete we have

$$\sigma_e(S) = (-\infty,-1/4].$$

Note that the Legendre polynomials satisfy all four conditions of (10.142). Therefore the triple

$$P_L = (0, P_n, 0) \quad (n \in \mathbb{N}_0) \tag{10.144}$$

are eigenfunctions of S_L with eigenvalues

$$\lambda_n = n(n+1) \quad (n \in \mathbb{N}_0). \tag{10.145}$$

Thus we may conclude that

$$(-\infty,-1/4] \cup \{\lambda_n = n(n+1), n \in \mathbb{N}_0\} \subset \sigma(S). \tag{10.146}$$

We conjecture that

$$(-\infty,-1/4] \cup \{\lambda_n = n(n+1) : n \in \mathbb{N}_0\} = \sigma(S). \tag{10.147}$$

By using equation (10.1) on the interval $(-1,1)$ and equation (10.5) on the intervals $(-\infty,-1)$ and $(1,\infty)$, in other words, by using $p(t) = 1-t^2$ for $-1 < t < 1$ and $p(t) = t^2-1$

for $-\infty < t < -1$ and for $1 < t < \infty$ and applying the three-interval theory as in Example 1, we obtain an operator whose essential spectrum is $[1/4, \infty)$ and whose discrete spectrum contains the classical Legendre eigenvalues

$$\{\lambda_n = n(n+1) : n \in \mathbb{N}_0\}.$$

Note that $\lambda_0 = 0$ is below the essential spectrum and all other eigenvalues λ_n for $n > 0$ are embedded in the essential spectrum. Each triple

$$(0, P_n, 0) \quad \text{when } n \in \mathbb{N}_0$$

is an eigenfunction with eigenvalue λ_n for $n \in \mathbb{N}_0$.

10.9 Singular transmission and interface conditions for the Legendre equation

In this section, we illustrate the two-interval characterization of self-adjoint domains given by case 4, $d = 3$, of Theorem 8.3.1 and the application of this theorem discussed in Section 8.4 by describing the two-interval self-adjoint realizations of the Legendre equation

$$-\left(py'\right)' = \lambda y, \quad p(t) = 1 - t^2 \tag{10.148}$$

in the Hilbert space $H = L^2(-1, \infty)$. For this, we identify the direct sum space $L^2(-1,1) \dotplus L^2(1,\infty)$ with this Hilbert space:

$$H = L^2(-1,\infty) = L^2(-1,1) \dotplus L^2(1,\infty). \tag{10.149}$$

For clarity of exposition, we state case 4 of Theorem 8.3.1 as the next corollary.

Corollary 10.9.1. *Consider the two-interval problem consisting of the Legendre equation* (10.148) *on the intervals* $J_1 = (-1,1)$, $J_2 = (1,\infty)$ *with endpoints* $a = -1^-$, $b = 1^+$, $c = 1^-$ *and* $d = \infty$. *Let* (u_1, v_1) *be a boundary basis at* a, *let* (u_2, v_2) *be a boundary basis at* b, *and let* (u_3, v_3) *be a boundary basis at* c. *Suppose* $A = (a_{ij})$, $B = (b_{ij})$, *and* $C = (c_{ij})$ *are* 3×2 *matrices with complex entries satisfying the following two conditions:*

(1) *The matrix* (A, B, C) *has full rank,*

(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kAEA^* - kBEB^* + hCEC^* = 0.$$

Then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$ *such that*

$$A\mathbf{Y}_1(a) + B\mathbf{Y}_2(b) + C\mathbf{Y}_3(c) = 0\},$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y,u_1]_1(a) \\ [y,v_1]_1(a) \end{bmatrix}, \mathbf{Y}_2(b) = \begin{bmatrix} [y,u_2]_2(b) \\ [y,v_2]_2(b) \end{bmatrix}, \quad \mathbf{Y}_3(c) = \begin{bmatrix} [y,u_3]_3(c) \\ [y,v_3]_3(c) \end{bmatrix},$$

is the domain of a self-adjoint operator S in $H = L^2(-1,\infty)$ satisfying

$$S_{\min}(J_1,J_2) \subset S = S^* \subset S_{\max}(J_1,J_2), \tag{10.150}$$

and every operator S in H satisfying (10.150) *is obtained this way.*

Proof. This is case 4 of Theorem 8.3.1 where the right endpoint of the second interval J_2 is LP and each of the other endpoints a, b, c is singular LC. In this case the deficiency index is 3. □

Next, we recall some observations about the Legendre equation and use these to give explicit singular self-adjoint transmission and interface conditions at the interior singular point 1.

For $\lambda = 0$, two linearly independent solutions of (10.148) are given by

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{1-t}{t+1}\right|\right).$$

Since these two functions u, v further play an important role, we list some their properties.

Observe that for all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$v^{[1]}(t) = (pv')(t) = +1.$$

Thus the quasi-derivative (pv') can be continuously extended so that it is well defined and continuous on the whole real line $\mathbb{R}$ including the two singular points -1 and $+1$. It is interesting to observe that u, (pu'), and the extended (pv') can be defined to be continuous on $\mathbb{R}$ and only v blows up at the singular points -1 and $+1$, and this blowup is logarithmic.

Note that

$$[u,v](t) = u(t)(pv')(t) - v(t)(pu')(t) = 1, \quad -\infty < t < \infty,$$

where we have taken appropriate one-sided limits at ± 1.

For all $y = (y_1, y_2) \in D_{\max}(J_1, J_2)$, we have

$$[y,u] = -(py'), \quad [y,v] = y - v(py'),$$

and again by taking appropriate one-sided limits, if necessary, we see that $[y,u](t)$ is defined and finite for all $t \in \mathbb{R}$. Thus the vector

$$Y(t) = \begin{bmatrix} [y,u](t) \\ [y,v](t) \end{bmatrix} = \begin{bmatrix} -(py')(t) \\ (y - v(py'))(t) \end{bmatrix}$$

is well defined for all $t \in \mathbb{R}$. In particular,

$$Y(-1^-),\ Y(1^-),\ Y(1^+)$$

are well defined and finite. Note also that $Y(\infty)$ is well defined and

$$Y(\infty) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since ∞ is LP.

Using these observations, we can make Corollary 10.9.1 more explicit.

Corollary 10.9.2. *Consider the two-interval problem consisting of the Legendre equation* (10.148) *on the intervals* $(-1,1)$ *and* $(1,\infty)$ *with endpoints* $a = -1^-$, $b = 1^+$, $c = 1^-$ *and* $d = \infty$ *and note that the deficiency index is* 3. *Suppose* $A = (a_{ij})$, $B = (b_{ij})$, *and* $C = (c_{ij})$ *are* 3×2 *matrices with complex entries satisfying the following two conditions:*
(1) *The matrix* (A, B, C) *has full rank,*
(2) *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$kAEA^* - kBEB^* + hCEC^* = 0.$$

Then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}(J_1, J_2)$ *such that*

$$A\mathbf{Y}_1(-1^-) + B\mathbf{Y}_1(1^+) + C\mathbf{Y}_2(1^-) = 0\},$$

where

$$\mathbf{Y}_1(-1^-) = \begin{bmatrix} -(py')(-1^-) \\ (y - v(py'))(-1^-) \end{bmatrix}, \quad \mathbf{Y}_1(1^+) = \begin{bmatrix} -(py')(1^+) \\ (y - v(py'))(1^+) \end{bmatrix},$$
$$\mathbf{Y}_2(1^-) = \begin{bmatrix} -(py')(1^-) \\ (y - v(py'))(1^-) \end{bmatrix},$$

is the domain of a self-adjoint operator S in $H = L^2(-1,\infty)$ *satisfying* (10.150), *and every operator S in H satisfying* (10.150) *is obtained this way.*

Proof. Note that (u_1, v_1) where $u_1 = u$ on J_1 and $v_1 = v$ on J_1 are a boundary condition basis at both endpoints -1^- and 1^+. Also, $u_3 = u$ and $v_3 = v$ are a boundary condition basis for 1^-. The explicit form of the singular boundary conditions then follows from the observations above. □

Remark 10.9.1. We comment on Corollary 10.9.2. Let $\mathbf{y} = \{y_1, y_2\} \in D_{\max}(J_1, J_2)$. Since $(py_r')(t)$ and $y_r - v(py_r')$ are finite for all $t \in \mathbb{R}$ and v blows up at $a = -1^-$, $b = 1^+$, $c = 1^-$, it follows that y_r, $r = 1, 2$, also blow up at these points. In particular, this holds for any solution of equation (10.148) on J_1 and J_2 for any $\lambda \in \mathbb{R}$.

Next, we give some examples.

Example 10.9.1. Let $a_{1j}, b_{1j}, c_{1j} \in \mathbb{R}$, $j = 1, 2$, with at least one member of each pair not 0.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ b_{11} & b_{12} \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ c_{11} & c_{12} \end{bmatrix},$$

$$\begin{aligned} 0 &= -a_{11}(py')(-1^-) + a_{12}(y - v(py'))(-1^-) \\ &= -b_{11}(py')(1^+) + b_{12}(y - v(py'))(1^+) \\ &= -c_{11}(py')(1^-) + c_{12}(y - v(py'))(1^-). \end{aligned}$$

Example 10.9.2. Note that in the previous example, these singular transmission conditions are independent of the parameters h, k. We mention a couple of particular cases:

–

$$0 = (py')(-1^-) = (py')(1^+) = (py')(1^-).$$

It is interesting to note that each of these conditions looks like a regular Neumann conditions but is actually a singular analogue of the regular Dirichlet condition [113]. The singular analogues of the regular Neumann conditions are given by

–

$$0 = (y - v(py'))(-1^-) = (y - v(py'))(1^+).$$

These depend on the function v.

The next example illustrates singular self-adjoint interface conditions. These are "jump" conditions involving a solution y that blows up, that is, has an infinite jump at the singular interior point 1, where the condition is specified.

Example 10.9.3. In this example, we have a separated condition at -1^- and a coupled condition "coupling" the endpoints 1^+ and 1^-. For $a_{11}, a_{12} \in \mathbb{R}$, $(a_{11}, a_{12}) \neq (0, 0)$,

$$0 = -a_{11}(py')(-1^-) + a_{12}(y - v(py'))(-1^-),$$

and

$$Y_2(1^-) = e^{i\gamma} K Y_1(1^+), \quad -\pi < \gamma \leq \pi, \quad i = \sqrt{-1},$$

where $Y_1(1^+)$ and $Y_2(1^-)$ are given previously, and K is a real 2×2 nonsingular matrix.

Remark 10.9.2. The particular case $K = I$ and $\gamma = 0$ is a singular analogue of the regular periodic boundary condition. Similarly, the case where $K = -I$ and $\gamma = 0$ is a singular analogue of the regular semiperiodic (antiperiodic) boundary condition. However, in both cases, these conditions depend on the function v.

Remark 10.9.3. Consider $y = 0$ and

$$K = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

Then the self-adjoint boundary condition reduces to

$$(py')(1^-) = (py')(1^+),$$

that is, the quasi-derivative (py') is continuous at the singular interior point 1, and

$$(y - v(py'))(1^-) - (y - v(py'))(1^+) = r(-(py')(1)).$$

Note that the right-hand side is finite. On the left-hand side, as remarked before, y must blow up asymptotically like v so that each of $(y - v(py'))(1^-)$ and $(y - v(py'))(1^+)$ is finite and the self-adjointness condition is that the difference between these two finite numbers is equal to the right-hand side.

10.10 Comments

This chapter is based largely on papers by Littlejohn and Zettl [74], Niessen and Zettl [86], Arvesu, Littlejohn, and Marcellan [3], Everitt and Zettl [37], Mukhtarov and Yakubov [82], Neuberger [85], and Wang and Zettl [103].

A Notation

A good notation has a subtlety and suggestiveness which at times make it almost seem like a live teacher.

Russell, Bertrand (1872–1970). In J. R. Newman (ed.) The World of Mathematics, New York: Simon and Schuster, 1956.

In this Appendix, we provide a brief list of the mathematical notation used.

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

$\mathbb{N} = \{1, 2, 3, \ldots\}$.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{R}$ is the set of real numbers.

$\mathbb{C}$ is the set of complex numbers.

(a, b) denotes the open interval with finite or infinite endpoints, $-\infty \leq a < b \leq \infty$.

$[a, b]$ denotes the closed interval with finite endpoints.

$[a, b)$ includes a but not b; similarly for $(a, b]$.

$L(J, \mathbb{C})$ is the set of Lebesgue-integrable complex-valued functions defined almost everywhere on J.

$L(J, \mathbb{R})$ is the set of real-valued Lebesgue-integrable functions on J.

$L_{\mathrm{loc}}(J, \mathbb{R})$ is the set of functions y satisfying $y \in L([a, b], \mathbb{R})$ for every compact subinterval $[a, b]$ of J.

$L_{\mathrm{loc}}(J, \mathbb{C})$ is the set of functions y satisfying $y \in L([a, b], \mathbb{C})$ for every compact subinterval $[a, b]$ of J.

$AC_{\mathrm{loc}}(J)$ is the set of complex-valued functions that are absolutely continuous on all compact subintervals of J. (Note: $AC_{\mathrm{loc}}(J) = AC_{\mathrm{loc}}(J, \mathbb{C})$, but we do not need this notation since we have no need for just the real-valued absolutely continuous functions on compact subintervals.)

$M_{n,m}(S)$ is the set of $n \times m$ matrices with entries from S; if $n = m$, we abbreviate this to $M_n(S)$. Also, if $m = 1$, then we sometimes write S^n for $M_{n,1}(S)$.

$|P|$ denotes the absolute value of P if P is a real or complex number or a function. If P is a real or complex matrix constant or function, then $|P|$ denotes the matrix norm. Since all matrix norms are topologically equivalent (in a finite-dimensional vector space), this matrix norm can be taken as the 1-norm:

$$|P| = \sum |p_{ij}|.$$

$\|Y\|$ denotes a norm in a vector space; this space is either specified or is clear from the context.

$\{X_n : n \in N\}$ denotes the sequence $X_1, X_2, X_3, \ldots$.

$L^2(J, w) = \{f : J \to \mathbb{C}, \int_J |f|^2 w < \infty\}$ is the Hilbert space of square-integrable functions with weight $w > 0$ a. e. on J. Since we have no need for the Hilbert space of real-valued square-integrable functions, we do not use or need the notations $L^2(J, \mathbb{C}, w)$ and $L^2(J, \mathbb{R}, w)$.

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$L^2(J,w) = \{f : J \to \mathbb{C}, \int_J |f|^2 w < \infty\}$ is the Krein space of square-integrable functions with weight w changing sign on J. Since we have no need for the Krein space of real-valued square-integrable functions, we do not use or need the notations $L^2(J,\mathbb{C},w)$ and $L^2(J,\mathbb{R},w)$.

$\Phi(t,u,)$ or $\Phi(t,u,P)$ is the "primary fundamental matrix" of the system $Y' = PY$ on J. The primary fundamental matrix Φ is a matrix solution satisfying $\Phi(u,u) = I$ for all $u \in J$. Here I is the identity matrix.

$T'(x)$ is the Fréchet derivative in Banach spaces; see Section 1.7.

$o(|h|)$ See Section 1.7.

$\sigma(C)$ is the set of eigenvalues of a matrix $C \in \mathbb{M}_n$.

$\sigma(C,D)$ is the set of eigenvalues of the matrix pair (C,D), $(C - \lambda D)u = 0$.

$C^{[1]}$ is the submatrix of C obtained by removing the first row and column.

$C^{[n]}$ is the submatrix obtained by removing the last row and column.

$C^{[1,n]}$ is the submatrix obtained by removing both the first and last rows and columns.

B Open problems

> Unfortunately, what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.
>
> Galois, Évariste. In N. Rose (ed.) Mathematical Maxims and Minims, Raleigh NC: Rome Press Inc., 1988.

These problems are "open" as far as the author knows at the time of this writing and are stated in random order. Some may be intractable, some accessible but challenging, and others routine. I have made no effort to grade these problems by their difficulty. Some I feel can be done with a moderate amount of effort, others with considerable effort, and some I have no idea how to do.

1. Chapter 10 "The Legendre Equation and its Operators" contains an extensive discussion of the Legendre equation. Chapter 14 in [113] has a list of other equations, which are of considerable interest in mathematics, physics, and other fields, including those named after Bessel, Laplace, Laguerre, Fichera, Latzko, etc.
   Problem: Write Chapter 10 for one of these.
2. In Chapter 5, it is shown that given any finite set of primes, there is a Sturm–Liouville boundary value problem whose spectrum is this set. Show that there is a Sturm–Liouville boundary value problem whose spectrum is the set of all primes. This may be a one-interval problem or a multiinterval problem with a finite or infinite number of intervals.
   Note that Example 22 in Chapter 14 of [113] gives a class of Laguerre problems all having spectrum given by

$$\sigma = \{\lambda_n = n : n \in \mathbb{N}_0\}.$$

3. Does there exist a regular classical self-adjoint SLP with an infinite set of eigenvalues all of which are simple except for arbitrary three $\lambda_{n_1}, \lambda_{n_2}, \lambda_{n_3}$?
4. Consider the equation

$$-(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \quad \text{on } J = (a,b), \quad -\infty < a < b < \infty,$$

   with coefficients satisfying

$$r = 1/p, q, w \in L^1(J, \mathbb{R}), \quad p > 0, \quad w > 0 \text{ a. e.}$$

   and separated self-adjoint boundary conditions

$$\cos\alpha y(a) - \sin\alpha(py')(a) = 0, \quad 0 \le \alpha < \pi,$$
$$\cos\beta y(b) - \sin\beta(py')(b) = 0, \quad 0 < \beta \le \pi.$$

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Given real numbers $\{\lambda_i, \mu_i : i \in \mathbb{N} = 1, 2, 3, \dots\}$ satisfying the strict interlacing relation

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k$$

and a weight function w, do there exist coefficients p, q such that for some $\alpha_1 \in [0, \pi)$ and $\beta_1 \in (0, \pi]$, we have

$$\sigma(\alpha_1, \beta_1) = \{\lambda_i, i \in \mathbb{N}\}$$

and

$$\sigma(\alpha_2, \beta_2) = \{\mu_i : i \in \mathbb{N}\}?$$

5. Extend the inverse theorems of Chapter 5 to regular self-adjoint classical problems for any coupling matrix K.
6. Extend the inverse theorems of Chapter 5 to regular complex self-adjoint boundary conditions, that is, to $\gamma \neq 0$ in

$$Y(b) = e^{i\gamma} KY(a).$$

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