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Giovanni Molica Bisci, Patrizia Pucci

NONLINEAR PROBLEMS WITH LACK OF COMPACTNESS

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Authors
Prof. Giovanni Molica Bisci
Dipartimento di Scienze Pure e Applicate
(DiSPeA)
Università degli Studi di Urbino Carlo Bo
Piazza della Repubblica, 13
61029 Urbino
Italy
giovanni.molicabisci@uniurb.it

Prof. Patrizia Pucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Via Vanvitelli, 1
06123 Perugia
Italy
patrizia.pucci@unipg.it

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To our respective parents
for the greatest present
we ever got: our sisters,
Antonella and Maria Paola

Preface

The book is dedicated to the study of elliptic problems when a lack of compactness occurs. This research area has been intensively developed in recent years also in connection with nonlinear phenomena that naturally arise in General Topology, Geometric Analysis, Functional and Convex Analysis, Game Theory, Mathematical Economics, and other branches of pure and applied sciences. Fundamental works in the field are due, among others, to T. Aubin [18], H. Brézis [47], L. Nirenberg [197], J. Serrin [230], and N. Trudinger [241].

The main mathematical interest lies in the fact that some classical results of Functional Analysis, mainly based on variational arguments, cannot be directly used for problems with a lack of compactness, and new techniques have to be produced. For instance, following the seminal ideas due to H. Brézis and L. Nirenberg [50], P.-L. Lions [160], and R. Palais [202], the presence of symmetries allows us to obtain the existence of solutions preserving the geometrical nature.

The aim of the monograph is to present some of these techniques, together with their applications to elliptic problems with a variational structure. The current literature on these abstract tools and on their applications is therefore very interesting and quite large. We refer to the recent outstanding monograph of A. Ambrosetti and A. Malchiodi [11], as well as to the references therein. The book is addressed to researchers and postgraduate, as well as graduate students, for a comprehensive introduction to the existence theory for elliptic partial differential equations with a lack of compactness, and can serve as a textbook. The extensive reference list and index make it as a reference book.

This monograph would never have been written without the encouragements of V. D. Rădulescu, but with great pleasure we thank also some other dear friends and colleagues as R. Aftabizadeh, A. Ambrosetti, M. F. Bidaut-Véron, L. Boccardo, M. Chipot, J. I. Díaz, A. Farina, G. Fusco, N. Fusco, N. Garofalo, F. Gazzola, E. Lanconelli, A. Malchiodi, P. Marcellini, J. Mawhin, G. Mingione, E. Mitidieri, F. Pacella, P. Rabinowitz, D. D. Repovš, G. Restuccia, B. Ricceri, M. Rigoli, S. Salsa, C. Sbordone, X. Tang, G. Tarantello, S. Terracini, L. Verón, R. Xu, F. Zanolin, and B. Zhang.

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July, 2020 Giovanni Molica Bisci and Patrizia Pucci

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Part III: **Variational Principles in Geometric Analysis**

Introduction

The book tries to be an up-to-date unified exposition for a series of nonlinear problems with a lack of compactness via critical point theory, obtained by ourselves or by direct collaboration with other coauthors. For an extensive bibliography of the pioneering papers on variational Dirichlet second order elliptic problems with critical exponents and on the Yamabe equation, we refer to the seminal review article [47] due to H. Brézis.

Much of the impressive advance has been recently performed in this field, though many problems still remain open. The close relationship between analysis and geometry allows the use of methods which simplify many arguments and proofs. The main theorems are entirely self-contained and given in detail, since we desire to make the book accessible to a large audience, including graduate and postgraduate students, and researchers in the field of partial differential equations.

We assume the reader to have a standard background in nonlinear analysis, including Sobolev spaces and a first course of functional analysis. A useful assortment of classical results and techniques can be found in [119, 151, 214].

The monograph is divided into three parts. In the first part of the book, the existence of solutions for elliptic equations in $\mathbb{R}^N$ with nonstandard growth is studied.

Chapter 1 concerns the existence theorems for a quasilinear elliptic equation in $\mathbb{R}^N$, involving general operators with nonstandard growth, as well as critical nonlinearities. Problems with nonstandard growth have been largely studied in the literature. Existence and qualitative properties of solutions already appear in the famous well-known papers of the theory. For instance, in the setting of the Calculus of Variations, an extensive treatment has been provided starting by the seminal papers of P. Marcellini [168–170], and V. V. Zhikov [259], and more recently by G. Mingione and his collaborators; see, among others, the papers [28–30, 66, 67, 73, 74, 176] and the references therein.

The quasilinear elliptic equations in $\mathbb{R}^N$ considered along Chapter 1 are of the form

$$-\mathrm{div}\big(A(|\nabla u|)\nabla u\big) + B(|u|)u = \lambda f(u) + |u|^{q^*-2}u, \tag{I.1}$$

where q^* is the critical exponent related to q, with $1 < q < N$, while λ is a real positive parameter. Furthermore, A and B are strictly positive and continuous in $\mathbb{R}^+$, while $t \mapsto tA(t)$, $t \mapsto tB(t)$ approach 0 as $t \to 0^+$ and are of class $C^1(\mathbb{R}^+)$. For further natural technical assumptions, we refer to Section 1.1. General elliptic operators of type A take inspiration from [96, 100] and from Chapter 5 of the monograph [214] due to P. Pucci and J. Serrin.

The presence of the critical nonlinearity, as well as the fact that (I.1) is set in the whole $\mathbb{R}^N$, produces new interesting nonlinear phenomena. On the other hand, equations of type (I.1) arise in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics, and

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game theory, as they are the typical outcome of the stochastic stabilization of Lévy processes.

The main existence result given in Theorem 1.1.1 applies well to problems in $\mathbb{R}^N$ like the following:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \Delta_4 u + u + u^3 = \lambda f(u) + |u|^{4^*-2}u,$$

involving the Minkowski mean curvature operator; see W. M. Ni and J. Serrin [194–196], as well as L. A. Peletier and J. Serrin [204]. The extension of Theorem 1.1.1 in several directions and into the vectorial case has been given in [100]. However, even Theorem 1.1.1 extends and complements the results of [31, 33, 54, 55, 158], and of the references therein.

In Section 1.2 it will be shown that the energy functional I associated to (I.1) has the geometric features to get the existence result via the famous mountain pass theorem due to A. Ambrosetti and P. Rabinowitz [12]. On the other hand, Section 1.2 contains also preliminary results of independent theoretical interest, for example, as in Lemma 1.2.1.

Section 1.3 is devoted to the proof of the existence Theorem 1.1.1 for (I.1), which relies on a direct intriguing alternative of Lions type presented in Proposition 1.3.1.

In Chapter 2 the following equation in $\mathbb{R}^N$ is considered:

$$-\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u - \sigma\frac{|u|^{p-2}u}{|x|^p} = \lambda h(x)u_+^{q-1} + \gamma g(x,u) \tag{I.2}$$

where $1 < p < N$, $N \geq 2$, $1 < q < N$, $u_+ = \max\{u, 0\}$, and h is a positive function of class $L^\theta(\mathbb{R}^N)$, with $\theta = N/(N-q)$, while $\lambda > 0$, $\gamma > 0$, and σ is a real parameter. The function g is of exponential type and is assumed to satisfy certain natural structural properties; see Section 2.1.

Equations in the whole $\mathbb{R}^N$, involving elliptic operators with standard N-growth, as well as critical Trudinger–Moser nonlinearities, have been studied in the literature. Existence and multiplicity results are obtained by using different methods and techniques; see, among others, the papers [5–7, 82], as well as [3, 4, 75, 83] and the references therein.

In Section 2.2 we give a brief and self-contained introduction to the variational setting for equation (I.2) and to some technical lemmas that are crucial in order to get solutions on the Sobolev space

$$W = W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,N}},$$

where $\|u\|_{W^{1,\wp}} = \left(\|u\|_\wp^\wp + \|\nabla u\|_\wp^\wp\right)^{1/\wp}$ for every $u \in W^{1,\wp}(\mathbb{R}^N)$ and $\|\cdot\|_\wp$ denotes the canonical $L^\wp(\mathbb{R}^N)$ norm for any $\wp > 1$; see Lemmas 2.2.1 and 2.2.2.

On account of the preliminary results recalled above, Section 2.3 is completely devoted to the proof of the main Theorem 2.1.1. The aforementioned proof combines new and classical tools in Nonlinear Analysis, such as a Brézis–Lieb type lemma for exponential nonlinearities, a Trudinger–Moser inequality, and the Ekeland variational principle; see Lemmas 2.2.3 and 2.3.3.

Chapter 3 continues the study of quasilinear elliptic equations in $\mathbb{R}^N$ with nonstandard growth. More precisely, Chapter 3 deals with the existence of nontrivial solutions for Kirchhoff equations in $\mathbb{R}^N$ whose form is given by

$$M(\|u\|^p_{W^{1,p}})(-\Delta_p u + |u|^{p-2}u) + M(\|u\|^q_{W^{1,q}})(-\Delta_q u + |u|^{q-2}u) \\ = \lambda f(x,u) + |u|^{q^*-2}u, \tag{I.3}$$

where $2 \le p < q < N$ and $q^* = Nq/(N-q)$. The parameter λ in $(\mathcal{E}_M)$ is strictly positive and the Kirchhoff term $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, as well as the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, verify some natural and mild hypotheses; see Section 3.1.

The prototype for the function M proposed by *Kirchhoff* in 1883, namely

$$M(t) = a + b\theta t^{\theta-1}, \quad a,\, b \ge 0, \quad a+b > 0, \;\; \theta \ge 1, \tag{I.4}$$

is clearly monotone. Along this direction, several authors studied the existence of solutions of Kirchhoff equations assuming that M is nondecreasing in $\mathbb{R}^+_0$. However, from a mathematical point of view, it is interesting to treat cases in which this monotonicity condition is relaxed. Hence, the main assumption $(\mathscr{M})$ in Chapter 3 does not force the Kirchhoff term M to be monotone; see condition $(\mathscr{M})$ in Section 3.1. Let us refer to other two main contributions [175, 253], besides [71], in which M is not monotone, and to the references therein.

The so-called degenerate Kirchhoff problems, that is, when the continuous function M is zero in 0 and positive in $\mathbb{R}^+$, are extremely interesting and delicate. Recently, in [21, 63] the degenerate case was covered, as well as in [23, 26, 254], but the involved Kirchhoff term M was assumed to be nonnegative and nondecreasing as in (I.4).

The main Theorems 3.1.1 and 3.1.2 generalize, in different and nontrivial ways, to the Kirchhoff setting the results contained in [1, 14, 15], while extend and complete the existence result given in [38, Theorem 1.1]. Moreover, Theorem 3.1.1 deals with both the degenerate and nondegenerate Kirchhoff equations, while Theorem 3.1.2 treats only the nondegenerate case not covered in Theorem 3.1.1; see Sections 3.2 and 3.3.

The proof techniques should therefore overcome the nonlocal structure of problem (I.3) due to the presence of the Kirchhoff term M, as well as the intrinsic lack of compactness that the domain $\mathbb{R}^N$ naturally produces. It is worthy to emphasize the great interest in stationary Kirchhoff problems in closed Riemannian manifolds; see

[124, 126] and the references therein. In Chapters 6 and 9 Kirchhoff problems on Riemannian manifolds are considered. Some special cases of the above results are worth specific note.

In the second part of the book, the existence of multiple solutions has been treated via a group-theoretical invariance in the Hilbertian framework for different problems, in which the settings are responsible for the loss of compactness.

Chapter 4 deals with the one-parameter critical elliptic equation in $\mathbb{R}^N$ given by

$$-\Delta u + u = \lambda w(x)|u|^{m-2}u - h(x)|u|^{2^*-2}u, \tag{I.5}$$

where $\lambda > 0$ is a real parameter, $1 < m < 2^*$, and the main coefficients h and w combine each other and verify suitable summability conditions in order to overcome the loss of compactness. Equations of this type in bounded domains have been largely studied in the literature, and we refer for historical comments, as well as for preliminary results on weighted Lebesgue spaces, to Section 4.1.

Section 4.2 is devoted to the proof of Theorem 4.1.1 in the difficult case $1 < m < 2^*$ using a strategy which first appears in [217]. When $1 < m < 2$, multiplicity is obtained for (I.5) via the genus theory.

Finally, in Section 4.3 at the core of the new approach based on a symmetric group theory, first developed in the pioneering paper [36] due to T. Bartsch and M. Willem, we prove Theorem 4.3.1; see also [148]. In recent years, these techniques have been successfully applied to several elliptic problems set in the Hilbertian space $H^1(\mathbb{R}^N)$. Regrettably, the extension to equations driven by the general p-Laplacian operator, $p > 1$, seems not yet completely understood. Among others, we cite the papers [146, 172].

Chapter 5 is concerned with solutions of a scalar field equation settled on a strip-like domain of the Euclidean space $\mathbb{R}^N$. More precisely, the general form of the main problem is given by

$$\begin{cases} -\Delta u = \lambda f(x, y, u) & \text{in } \mathcal{O} \times \mathbb{R}^{N-m}, \\ u = 0 & \text{on } \partial\mathcal{O} \times \mathbb{R}^{N-m}, \end{cases} \tag{I.6}$$

where λ is a positive real parameter, $f : \mathcal{O} \times \mathbb{R}^{N-m} \times \mathbb{R} \to \mathbb{R}$ is a suitable continuous nonlinear term, and $\mathcal{O} \times \mathbb{R}^{N-m}$ is a strip-like domain in $\mathbb{R}^N$, in which $\mathcal{O}$ is a bounded open set in $\mathbb{R}^m$, with $m \geq 1$ and $N \geq m + 2$.

The existence and multiplicity theorems proved in Chapter 5 represent a more precise form of some results that have already appeared in the recent literature; see, among others, the papers [140, 141, 148, 151]. To overcome the lack of compactness in order to prove the existence of solutions, we make use of a sort of flower-shape geometry for symmetric subspaces of the Sobolev space $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ constructed in [79] and mainly inspired by the results contained in [141] and [144].

From a purely mathematical point of view, Theorem 5.2.1 furnishes an accurate description of the number of block-symmetric solutions for (D_λ) that are not cylindrically symmetric when the dimension N is either $N = m + 4$ or $N \geq m + 6$. Recently, a hemivariational inequality problem, via nonsmooth analysis arguments, was treated in [141]; see also [90] for related topics. When the nonlinearity of [141] is regular, the problem reduces to (I.6). However, Theorem 3.1 in [141] gives the existence of only cylindrically symmetric solutions. It is worth noting that the main conclusion of Theorem 5.2.1 remains valid for the hemivariational problem treated in [141] by using the W. Krawcewicz and W. Marzantowicz principle for locally Lipschitz functionals established in [139]. Moreover, to the contrary of Theorem 3.1 in [141], Theorem 5.2.1 has been achieved thanks to Propositions 5.1.3 and 5.1.4, which are derived from a careful analysis of the classical compactness argument due to P.-L. Lions in [160, Théorème III.2]; see Section 5.1 for a detailed discussion on this topic.

Subsequently, in Section 5.3, the classical fountain theorem provides not only a finite number of infinitely many cylindrically symmetric solutions, but also cylindrically nonsymmetric solutions in certain dimensions. The main Theorem 5.3.1 can be viewed as a refined version of a classical existence result proved by T. Bartsch and M. Willem in [36] for Schrödinger equations. Again, the existence of infinitely many cylindrically symmetric solutions for hemivariational inequalities has been proved in Theorem 3.2 of [141], by using a suitable version of the fountain theorem valid for nonsmooth functionals. Inspired by [140, 141], in Theorem 5.3.1 the existence of a precise number of sequences of symmetric solutions with no cylindrical structure has been proved, completing somehow the picture.

In both Theorems 5.2.1 and 5.3.1, a crucial role in our approach is played by a careful algebraic analysis of some symmetric structures defined by the natural action of the group $\widehat{O}(N-m) = \{\mathbb{I}_m\} \times O(N-m)$, where $\mathbb{I}_m$ is the identity matrix of order m, over the Sobolev space $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$; see Section 5.1.

Chapter 6 deals with elliptic equations on the unit sphere $\mathbb{S}^N \hookrightarrow \mathbb{R}^{N+1}$, with $N \geq 2$, endowed by the induced Riemannian metric and involving a possibly critical nonlinear term. In this chapter we first consider the polyharmonic problem on the unit sphere

$$\begin{cases} \mathfrak{D}^m v = |u|^{2^*_m-2}u & \text{in } \mathbb{S}^N, \\ u \in H^m(\mathbb{S}^N), & N > 2m, \end{cases} \tag{I.7}$$

where m and N are two positive integers, with $m > 2$, $2^*_m = 2N/(N-2m)$, $\mathfrak{D}^m$ is the polyharmonic operator on the sphere defined by

$$\mathfrak{D}^m = \prod_{k=1}^{m}\left(-\Delta_h + \frac{1}{4}(N-2k)(N+2k-2)\mathrm{id}_{L^2(\mathbb{S}^N)}\right),$$

and Δ_h denotes the usual Laplace–Beltrami operator on $\mathbb{S}^N$.

In Section 6.2 the existence of sequences of sign-changing solutions, which are mutually symmetrically distinct, is attained, and a lower estimate of the number of these sequences is also given; see Theorem 6.2.1. To this aim, some fine properties of certain symmetric subspaces of the Sobolev space $H^m(\mathbb{S}^N)$ are studied in Section 6.1

via a group-theoretical analysis of the natural action of the orthogonal group $O(N+1)$ on the sphere $\mathbb{S}^N$. The main result given in Theorem 6.2.1 is crucially based on the abstract tools developed in [144, Proposition 3.2]. Theorem 6.2.1 ensures that the critical polyharmonic equation (I.7) admits al least

$$s_N = [N/2] + (-1)^{N+1} - 1$$

sequences of infinitely many finite energy nodal solutions, which are unbounded in $H^m(\mathbb{S}^N)$ and mutually symmetrically distinct.

Critical polyharmonic problems have been intensively studied in the mathematical literature also in connection with the famous results obtained by H. Brézis and L. Nirenberg in [50] for the semilinear critical Dirichlet eigenvalue problem. Among others, we mention here the paper [213] due to P. Pucci and J. Serrin dedicated to critical polyharmonic Dirichlet eigenvalue problems in the Euclidean ball. Paper [213] has served as inspiration for subsequent research in different directions; see the monograph [115] as a general reference on this subject.

Along this direction, inspired by Y. Ding [80], T. Bartsch, M. Schneider, and T. Weth in [35] show for the critical polyharmonic equation

$$\begin{cases} (-\Delta)^m u = |u|^{2_m^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{m,2}(\mathbb{R}^N), \quad N > 2m, \end{cases} \tag{I.8}$$

the existence of a sequence of infinitely many finite energy nodal solutions which are unbounded in the Beppo Levi space $\mathcal{D}^{m,2}(\mathbb{R}^N)$. A more precise version of Theorem 6.2.1, obtained in [185], ensures that the critical polyharmonic equation (I.8) admits at least an asymptotically exponential number of sequences of infinitely many finite energy nodal solutions which are unbounded in $\mathcal{D}^{m,2}(\mathbb{R}^N)$. A general result valid for a wide class of $O(N+1)$-invariant variational problems that correctly encode also the critical polyharmonic equation (I.8) has been recently proved by W. Marzantowicz in [174], via the intrinsic linking between orthogonal Borel subgroups in $O(N+1)$ with partial and orthogonal flags in $\mathbb{R}^{N+1}$. Theorem 6.2.1, on the contrary of the Marzantowicz result, is based on some explicit symmetric structures defined through the action of the orthogonal group on the unit sphere $\mathbb{S}^N$. Indeed, this action naturally arises in the theory of Lie groups of transformations.

Actually, elliptic equations on the unit sphere are relevant to the theoretical point of view also in connection with the study of the following parametrized Emden–Fowler (or, Lane–Emden) equation:

$$-\Delta u = \lambda |x|^{s-2} w(x/|x|) f(|x|^{-s}u), \quad x \in \mathbb{R}^{N+1} \setminus \{0\}, \tag{I.9}$$

where $s \in \mathbb{R}$, with $1 - N < s < 0$, and

$$w \in \Lambda_+(\mathbb{S}^N) = \left\{ w \in L^\infty(\mathbb{S}^N) \,:\, \operatorname*{ess\,inf}_{\mathbb{S}^N} w > 0 \right\}.$$

Existence results for (I.9) have been established recently in [42, 150, 151] via variational methods. The key transformation of M. F. Bidaut-Véron and L. Véron in [39] reduces (I.9) to

$$-\Delta_h v + \alpha v = \lambda w(\sigma) f(v), \quad \sigma \in \mathbb{S}^N,\ \alpha = s(1-s-N) > 0. \tag{I.10}$$

Equations of type (I.10) have been largely studied, and we refer to the pioneering papers [68] of A. Cotsiolis and D. Iliopoulos and [244] by J. L. Vázquez and L. Véron. See also [151, Chapters 9 and 10] for an intensive treatment of this argument. Along this direction, in Theorems 6.2.4 and 6.2.5, we establish existence of infinitely many arbitrarily small solutions of (I.9) via (I.10). Furthermore, the main variational idea is based on the general approach proposed by B. Ricceri in [226]. This method was first applied in problems similar to (I.9) by J. Saint Raymond in [228].

A persisting assumption on the current literature dedicated to the existence of infinitely many solutions for a large class of problems driven by a second order elliptic operator is expressed by

$$-\infty < \liminf_{t\to L} \frac{F(t)}{t^2} \le \limsup_{t\to L} \frac{F(t)}{t^2} = \infty, \tag{I.11}$$

where either $L = 0^+$ or $L = \infty$; see, among others, the papers of F. I. Njoku, P. Omari, and F. Zanolin [198], F. Obersnel and P. Omari [199], as well as P. Omari and F. Zanolin [200, 201]. On the contrary of the above results, in Theorems 6.2.4 and 6.2.5, condition (I.11) is not required any longer, and the primitive F is supposed to have a more general oscillating behavior near the origin or at infinity, including the case

$$\limsup_{t\to L} \frac{F(t)}{t^2} < \infty.$$

Weaker forms of (I.11) are analyzed in Chapters 8 and 9, in which the existence of infinitely many solutions for a wide class of elliptic problems on homogeneous Hadamard manifolds is presented. The last Section 6.3 of Chapter 6 is dedicated to elliptic problems on the unit sphere involving a critical nonlinear term. It is worth mentioning that in [126] E. Hebey investigates existence and compactness properties of stationary Kirchhoff equations settled on general compact manifolds.

The main Theorem 6.3.3 is peculiar of stationary nondegenerate Kirchhoff equations on the sphere $\mathbb{S}^N$ when $N > 4$. For instance, an analogous result cannot be achieved for Dirichlet problems on bounded Euclidean domains. In the spirit of the conclusions contained in [126], a direct and meaningful consequence of Theorem 6.3.3 ensures that the critical Kirchhoff equation

$$(a + b\|u\|^2)(-\Delta_h u + u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{S}^N,$$

in which $q \in (2, 2^*)$ and

$$\|u\| = \left(\int_{\mathbb{S}^N} |\nabla_h u|^2 d\sigma_h + \int_{\mathbb{S}^N} |u|^2 d\sigma_h \right)^{1/2},$$

has at least $3s_N$ solutions, provided that $N > 4$,

$$a^{\frac{N-4}{2}} b > \frac{2(N-4)^{\frac{N-4}{2}}}{(N-2)^{\frac{N-2}{2}} S^{\frac{N}{2}}}, \quad \text{where} \quad S = \inf_{u \in H^1(\mathbb{S}^N)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2^*}^2}, \tag{I.12}$$

and $\lambda > 0$; see Corollary 6.3.4.

The variational analysis we use to prove Theorem 6.3.3 and its consequences is based on some fine topological results obtained in [225, 226] and makes use of an interesting technical approach developed in [89]. More precisely, a condition like (I.12) has been introduced in [89] in order to recover the weak lower semicontinuity property of the energy functional associated to a stationary Kirchhoff problem defined on a bounded Euclidean domain in the presence of a critical nonlinear term. The restriction on the dimension required in Theorem 6.3.3 is sharp. Indeed, it cannot be improved for symmetry reasons and coerciveness arguments.

The theorems of Sections 6.2 and 6.3 are obtained by the preliminary abstract results contained in Section 6.1 and partly inspired by [144] and the monograph [151, Chapter 10].

The third part of the book is dedicated to non compact problems arising from geometry.

Chapter 7 deals with subelliptic problems on Carnot groups. In sub-Riemannian structures, when a lack of compactness occurs, the problems are fairly intriguing and have been intensively studied in recent years by many authors. For instance, in the case of subelliptic problems defined on stratified Lie groups, we refer to the papers of L. D'Ambrosio and E. Mitidieri [70], N. Garofalo and E. Lanconelli [113], S. Maad [163, 164], I. Schindler and K. Tintarev [229], K. Tintarev [240], and to the references therein.

In particular, when a domain Ω of a Carnot group $\mathbb{G}$ is not bounded, the Folland–Stein space $HW_0^{1,2}(\Omega)$ fails to be compactly embedded into suitable Lebesgue spaces. This lack of compactness produces several difficulties to apply variational methods. In order to recover compactness in the unbounded case, a persisting assumption in the above cited papers is the *strong asymptotical contractiveness* condition on Ω, introduced in [163]. In the Euclidean framework, we refer to the pioneering paper [77] due to M. A. Del Pino and P. L. Felmer. However, a strongly asymptotically contractive domain Ω is geometrically thin at infinity.

In the presence of symmetries, subelliptic problems, in which the domains are possibly large at infinity, can be successfully treated under the more general geometrical requirement $(\mathscr{H})$, introduced recently in [27]; see Lemma 7.1.1. More precisely, in Section 7.2 we work with a topological group T, acting continuously on $HW_0^{1,2}(\Omega)$, such that the T-invariant closed subspace $HW_{0,T}^{1,2}(\Omega)$ can be compactly embedded in suitable Lebesgue spaces.

Assuming the left invariance of the standard Haar measure μ of the Carnot group $\mathbb{G}$, with respect to the action of the group $* : T \times HW_0^{1,2}(\Omega) \to HW_0^{1,2}(\Omega)$, as discussed in Chapter III § 2 No 4 of N. Bourbaki [46] and Chapter 7 § 1 No 1 of N. Bourbaki [45], variational methods can be applied to the energy Euler–Lagrange functional associated to the main problem of Chapter 7; see Lemma 7.2.1, as well as Theorems 7.2.2, 7.2.3, and 7.2.6.

The main existence and multiplicity results cover the case of Dirichlet problems driven by a nonlinear subcritical continuous term f that is superlinear at zero and either superlinear or sublinear at infinity; see respectively Theorems 7.2.2 and 7.2.6. Similar variational approaches have been extensively used in several contexts, in order to prove multiplicity results, such as elliptic problems on either bounded or unbounded domains of Euclidean spaces, elliptic equations involving the Laplace–Beltrami operator on compact Riemannian manifolds without boundary, and, more recently, elliptic equations on the ball endowed with Funk–type metrics; see [142, 152, 153, 155], [150] and [154], respectively. Moreover, Theorem 7.2.3 inspired by [222] emphasizes the role of the celebrated Ambrosetti–Rabinowitz condition in order to obtain direct multiplicity results. Elliptic problems on unbounded domains appear fairly involved, since the Palais–Smale condition of the associated Euler–Lagrange functionals does not hold at any level, but just below suitable thresholds; see, among others, the papers [44, 162, 185, 189, 207]. In Section 7.3 we overcome these difficulties adopting several strategies already used in different contexts.

The theoretical arguments presented above are successful for proving existence of weak solutions for subelliptic problems defined on a special class of (unbounded) domains of the Heisenberg group $\mathbb{H}^N = \mathbb{C}^N \times \mathbb{R}$, $N \geq 1$. More precisely, let us consider $\psi_1, \psi_2 : \mathbb{R}_0^+ \to \mathbb{R}$ that are bounded on bounded sets, with $\psi_1(t) < \psi_2(t)$ for every $t \in \mathbb{R}_0^+$. Define

$$\Omega_\psi = \{\sigma \in \mathbb{H}^N : \sigma = (z,t),\ \text{with}\ \psi_1(|z|) < t < \psi_2(|z|)\}, \tag{I.13}$$

where $|z| = \sqrt{\sum_{i=1}^N |z_i|^2}$.

If the functions ψ_1 and ψ_2 are bounded, the domain Ω_ψ is strongly asymptotically contractive, and the whole space $HW_0^{1,2}(\Omega_\psi)$ is compactly embedded in $L^\wp(\Omega_\psi)$ for every $\wp \in (2, 2^*)$; see [27, 169], as well as [113, 229]. Otherwise, thanks to the Rubik-cube geometry, Lemma 7.1.1 recovers the compact embedding above.

The attempt to develop a Rubik-cube technique on the Heisenberg group is very recent; the first result in this sense has been proved in [27]. Following [27], a detailed description of some symmetric subgroups of $\mathbb{U}(N)$ and variational arguments allow us to obtain further multiplicity results; see, for instance, Corollary 7.2.5. In this spirit, Section 7.3 is devoted to an application of an abstract critical point theorem due to P. Rabier in [219] in problems settled on strip-like domains $\Omega_\psi \subset \mathbb{H}^N$; see Theorem 7.3.1. Multiplicity results can be directly derived for a wide class of nonlinear problems from Theorem 7.3.1.

Good references on the arguments treated in Chapter 7 are the monographs [43, 151] and the papers [70, 245], as well as their bibliographies.

Chapter 8 treats elliptic problems on homogeneous Hadamard manifolds, i. e., Riemannian manifolds which are complete, simply connected, with everywhere nonpositive sectional curvature, and with a transitive group of isometries. The existence or nonexistence of solutions for elliptic problems defined on an Hadamard manifold $\mathcal{M} = (\mathcal{M}, g)$ is a topic in Differential Geometry that dates back to the 1970s. In the last years, several questions have been studied in this setting in connection to sharp isometric inequalities; see [122, 123].

Among these intriguing geometric implications, conditions on the sectional curvature produce meaningful compact embeddings of certain Sobolev spaces associated to $\mathcal{M}$ into Lebesgue spaces; see L. Skrzypczak and C. Tintarev [232, Theorem 1.3 and Proposition 3.1]. The compactness properties are essential to apply the critical point theory to the energy functionals associated to the problem on $\mathcal{M}$ in question.

In Section 8.1 we recall some well known concepts in Riemannian geometry. In the presentation we are as concise as possible, in order to correctly introduce the main problem. We refer, for example, to [18–20, 122, 123] for a detailed derivation of the geometric quantities, their motivations, and applications.

The heart of Chapter 8 is the problem

$$\begin{cases} -\Delta_g u + u = w(\sigma)[f(u) + \lambda \mathrm{f}(u)] & \text{in } \mathcal{M}, \\ u \geq 0 \text{ in } \mathcal{M}, \quad u \in H^1_g(\mathcal{M}), \end{cases} \tag{I.14}$$

treated in Section 8.3. In (I.14) the symbol Δ_g denotes the classical Laplace–Beltrami operator on an N-dimensional homogeneous Hadamard manifold $\mathcal{M}$, with $N \geq 3$, λ is a real parameter, $w : \mathcal{M} \to \mathbb{R}$ is a suitable symmetric positive potential, $f : \mathbb{R}^+_0 \to \mathbb{R}$ is a continuous function oscillating near the origin or at infinity, and $\mathrm{f} : \mathbb{R}^+_0 \to \mathbb{R}$ is any continuous function, with $\mathrm{f}(0) = 0$.

The existence part of the main results given in Theorems 8.3.1 and 8.3.2 is mainly based on minimization techniques on a truncated problem and on a nonsmooth version of the Palais principle due to J. Kobayashi and M. Ôtani in [138], which is valid for Szulkin-type functionals defined on reflexive Banach spaces; see Theorem 8.2.1 of Section 8.2 and Theorem A.2.2 in the Appendix. Even if the variational methods for proving Theorems 8.3.1 and 8.3.2 are classical, the interest of such a more general approach goes much beyond their proofs because it seems to be flexible and useful for other purposes.

For instance, in connection with the main theorems, some results have been proved for elliptic problems on Cartan–Hadamard manifolds with poles in [88, Theorem 4.1] and, in presence of symmetries, for Schrödinger–Maxwell systems on Hadamard manifolds in [91, Theorem 1.3]. It is easily seen that similar versions of Theorems 8.3.1 and 8.3.2 can be proved for Schrödinger–Maxwell systems of the form

$$\begin{cases} -\Delta_g v + v + ev\phi = \lambda w(\sigma)[f(v) + \lambda \mathrm{f}(v)] & \text{in } \mathcal{M}, \\ -\Delta_g \phi + \phi = qv^2 & \text{in } \mathcal{M}, \end{cases}$$

where $\mathcal{M}$ is a homogeneous Hadamard manifold of dimension N, with $3 \leq N \leq 5$, f, w, f are as before, and $e, q > 0$ are positive constants; see [91, Remark 1.5].

A crucial tool used along the proof of Theorems 8.3.1 and 8.3.2 is the existence of a suitable topological group $\mathscr{G}$ acting on the Sobolev space $H^1_g(\mathcal{M})$ and such that the $\mathscr{G}$-invariant closed subspace $H^1_{\mathscr{G},g}(\mathcal{M})$ is compactly embedded in suitable Lebesgue spaces; see Proposition 8.1.1 in Section 8.1.

As it is well known that the Poincaré ball model is a significative example of a homogeneous Hadamard manifold, which is noncompact and of infinite Riemannian measure. In the last Chapter 9, in connection with the results proved in Chapter 8 and taking the advantage of the intrinsic nature of the hyperbolic geometry, elliptic problems on the Poincaré ball model are studied; see [111, 167, 182], as well as [218, 233, 234] for related topics and methods.

More precisely, we investigate the existence of multiple solutions for Kirchhoff problems whose simple prototype is given by

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)\Delta_H u = \lambda w(\sigma) f(u) & \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N), \end{cases} \tag{I.15}$$

settled on the Poincaré ball model $\mathbb{B}^N$, with dimension $N \geq 3$, and positive real parameters λ, a and b. Here, Δ_H denotes the Laplace–Beltrami operator on $\mathbb{B}^N$, the potential $w \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is a nontrivial nonnegative radially symmetric function, and f is a continuous function.

To the best of our knowledge, no results comparable to the theorems of Chapter 9 are available in the literature concerning Kirchhoff problems on curved structures. The proof of Theorem 9.2.1 is based on critical point arguments similar to that carried out by several authors in different contexts; see, among others the papers [16, 17, 184] and references therein. However, the noncompact hyperbolic setting presents additional difficulties with respect to the aforementioned cases, and suitable geometrical and algebraic tools need to be adopted in order to get existence.

For instance, a crucial step in the main approach is the continuity of the superposition operator due to M. Marcus and V. Mizel [171, Theorem 1, p. 219] given in the hyperbolic context instead of the classical Euclidean setting; for additional comments and remarks in the Riemannian framework, see also [123, Proposition 2.5, p. 24]. The continuity of the superposition operator replaces the nonsmooth analysis method used in Chapter 8. Indeed, direct minimization gives the existence of constrained local minima of the associated energy functional J_λ on appropriate weakly closed subsets $(C_k^{\mathscr{G}})_k \subset H^1_{\mathscr{G}}(\mathbb{B}^N)$, which are actually local minima of J_λ in the entire symmetric Sobolev space $H^1_{\mathscr{G}}(\mathbb{B}^N)$ thanks to the Marcus–Mizel property, and so solutions of (I.15) by the symmetric criticality principle of R. Palais.

Theorem 9.2.1 covers nonlinearity models f for which the potential F satisfies

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = -\infty,$$

i. e., condition (I.11) is violated, and so the results contained in Chapter 8 cannot be applied; see also Theorem 9.2.5 in which

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} \in \mathbb{R}^+ \cup \{\infty\}.$$

Theorem 9.2.4 gives a suitable hyperbolic version of Theorem 7.2.3 of Chapter 7 and shows how assumption (I.11) implies the existence of a nontrivial solution of (I.15); see also Theorem 9.3.3 for a general version of Theorem 9.2.4 valid for Kirchhoff problems defined on homogeneous Hadamard manifolds. Finally, the existence established in Theorem 9.3.2 continues to hold also in the presence of small subcritical perturbations as shown in Theorem 9.3.4. Comments and open problems are presented at the end of Chapter 9.

Appendix A is dedicated to the celebrated principle of symmetric criticality of R. Palais intensively used along Parts II and III of the book. The origin of the principle is rather unclear, and its first implicit use seems to be due to H. Weyl [246] around 1950, and later in 1975 by S. Coleman [65] in a more explicit form. In 1979, for smooth symmetric functionals, the general criterion was rigorously formulated by R. Palais in his celebrated paper [202]. A simple version of the Palais principle reads as follows: *Let $\mathscr{G}$ be a compact Lie group which acts linearly on a real Banach space X and let I be a $\mathscr{G}$-invariant functional on X, then*

$$\begin{cases} I'(u) = 0 \ \textit{in} \ \mathrm{Fix}_{\mathscr{G}}(X) \\ u \in \mathrm{Fix}_{\mathscr{G}}(X) \end{cases} \Rightarrow I'(u) = 0 \ \textit{in}\ X,$$

where $\mathrm{Fix}_{\mathscr{G}}(X)$ *is the closed subspace of $\mathscr{G}$-symmetric points of X defined by*

$$\mathrm{Fix}_{\mathscr{G}}(X) = \{u \in X : gu = u \ \textit{for every}\ g \in \mathscr{G}\}.$$

Theorem A.1.1 is the general version of the Palais result due to J. Kobayashi and M. Ôtani in [138]. Successively, Theorems A.1.2 and A.1.5 treat two special meaningful cases of Theorem A.1.1. The first, whose original version is due to R. Palais himself in [202, Proposition 4.2], deals with the so-called compact case, while the latter concerns the isometric case.

Roughly speaking, if I is a C^1 functional on a Banach space X, then a good strategy to search its critical points is to find a suitable topological group $\mathscr{G}$, acting either continuously or isometrically on X, and such that the functional I is invariant with respect to the action of $\mathscr{G}$ on X. Hence, if I restricted to $\mathrm{Fix}_{\mathscr{G}}(X)$ admits a critical point u, then u is also a critical point of I in the entire space X thanks to the symmetric criticality principle.

The appendix ends with Theorem A.2.2, which is a significant version of Theorem A.1.2 and is applicable to the so-called Szulkin functionals that are sums of a C^1 functional and of a proper convex lower semicontinuous functional. For further details, we refer to the celebrated paper [239] of A. Szulkin.

The techniques which we discuss and describe in this book go far beyond all the equations we study, and the methods used here can be applied to other classes of elliptic equations, Hamiltonian systems, as well as hemivariational inequalities. Many of the proofs and derivations are new and, though difficult, make the subject available to a general reader.

The conclusions also raise, and leave open, a number of other intriguing questions. Some of them are briefly presented at the end of every chapter. Finally, the bibliography is far from being complete, and we just listed papers we closely use. Naturally, we apologize for possible omissions.

Part I: **Elliptic equations in $\mathbb{R}^N$ with nonstandard growth**

1 Critical quasilinear equations of Marcellini's type

Lessi così di tutto un po', disordinatamente;
ma libri, in ispecie, di filosofia. Pesano tanto:
eppure, chi se ne ciba e se li mette in corpo,
vive tra le nuvole.

Luigi Pirandello
from *Il fu Mattia Pascal*

This chapter deals with existence of nontrivial solutions for critical quasilinear equations driven by general (p,q) elliptic operators of Marcellini's type.

The importance of studying problems involving (p,q) operators, or operators with nonstandard growth conditions, begins with the pioneering papers of P. Marcellini [168], see also [169, 170], and V. V. Zhikov [259]. Since then the subject has been drawing increasing attention to the existence, regularity, and qualitative properties of solutions of different problems.

We refer to [66, 67, 176] for historical details and a wide list of recent contributions along with [110].

1.1 The quasilinear equation $(\mathcal{E})$

In this chapter, we study the existence of solutions for a quasilinear elliptic equation, involving general (p,q) elliptic operators, as well as critical nonlinearities. Since the study of the scalar equation is fairly involved, we refer the interested reader to the proofs of the vectorial case contained in the original paper [100], where these operators and problems were introduced. More precisely, we consider the equation in $\mathbb{R}^N$ written as

$$-\operatorname{div}(A(|\nabla u|)\nabla u) + B(|u|)u = \lambda f(u) + |u|^{q^*-2}u, \qquad (\mathcal{E})$$

where q^* is the critical exponent related to q, with $1 < q < N$, while $\lambda > 0$ is a real parameter. We require the following condition:

(C_1) *A and B are strictly positive and continuous in $\mathbb{R}^+$, with $tA(t) \to 0$ and $tB(t) \to 0$ as $t \to 0^+$.*

Let us introduce for simplicity the functions $\mathcal{A}$ and $\mathcal{B}$ as the potentials, which are 0 at 0 and obtained by integration from

$$\mathcal{A}'(t) = tA(t), \quad \mathcal{B}'(t) = tB(t) \quad \text{for all } t \in \mathbb{R}^+_0,$$

where $tA(t)$ and $tB(t)$ are defined to be 0 at 0 thanks to (C_1). Taking inspiration from [96, 100, 214], we furthermore require

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(C_2) *There exist constants* a_0, $\mathfrak{a}_0$, b_0, $\mathfrak{b}_0$, a_1, $\mathfrak{a}_1$, b_1, $\mathfrak{b}_1$ *all strictly positive, and there are exponents p and q, with* $1 < p < q < N$, *such that for all* $t \in \mathbb{R}_0^+$,

$$a_0 t^{p-1} + a_1 t^{q-1} \le \mathcal{A}'(t) \le \mathfrak{a}_0 t^{p-1} + \mathfrak{a}_1 t^{q-1},$$
$$b_0 t^{p-1} + b_1 t^{q-1} \le \mathcal{B}'(t) \le \mathfrak{b}_0 t^{p-1} + \mathfrak{b}_1 t^{q-1}.$$

Moreover, we assume that

(C_3) *There exist constants* θ *and* ϑ, *with* $q \le \min\{\theta, \vartheta\} < q^*$, *such that*

$$\theta\mathcal{A}(t) \ge t\mathcal{A}'(t), \quad \vartheta\mathcal{B}(t) \ge t\mathcal{B}'(t) \quad \text{for all } t \in \mathbb{R}_0^+$$

holds. Moreover, there exists a constant $c_q > 0$ *such that for all* $\xi, \eta \in \mathbb{R}^N$,

$$\big(A(|\xi|)\xi - A(|\eta|)\eta\big) \cdot (\xi - \eta) \ge c_q|\xi - \eta|^q. \tag{1.1}$$

Note that (1.1) holds whenever $\mathcal{A}$ is of the form $\mathcal{A}(t) = \mathcal{A}_1(t) + t^q/q$, $t \in \mathbb{R}_0^+$, where $q \ge 2$, $\mathcal{A}_1$ is convex and of class $C^1(\mathbb{R}_0^+)$, with $\mathcal{A}_1(0) = 0$. Indeed, (1.1) follows at once by convexity and by the famous Simon inequality, see Lemma 2.1 of [231]. For more general considerations we refer to [100].

Condition (C_1), together with (C_3), yields that

$$\mathcal{A}(t) \le t\mathcal{A}'(t) \le \theta\mathcal{A}(t), \quad \mathcal{B}(t) \le t\mathcal{B}'(t) \le \vartheta\mathcal{B}(t)$$

for all $t \in \mathbb{R}_0^+$.

We just present few examples which illustrate the general equations covered under the assumptions (C_1)–(C_3) and refer to [100] for general systems. In the examples we tacitly suppose that $1 < p < q$ and $2 \le q < N$, without mentioning.

Similarly, for $\mathcal{A}(t) = \mathcal{B}(t) = t^p/p + t^q/q$, $t \in \mathbb{R}_0^+$, with $1 < p < 2 \le q < N$, one has $a_0 = \mathfrak{a}_0 = a_1 = \mathfrak{a}_1 = b_0 = \mathfrak{b}_0 = b_1 = \mathfrak{b}_1 = 1$, $\theta = \vartheta = q$, and $c_q > 0$ comes from convexity and the famous Simon inequality, as noted above. Hence ($\mathcal{E}$) becomes

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda f(u) + |u|^{q^*-2}u,$$

where from here on $\Delta_\wp u = \operatorname{div}(|\nabla u|^{\wp-2}\nabla u)$ for any $\wp > 1$.

If $\mathcal{A}(t) = \sqrt{1+t^2} - 1 + t^4/4$ and $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, with $2 = p < q = 4$, then $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 1/2$, $\theta = \vartheta = 4$, $q^* = 4^*$, $c_4 > 0$ in (1.1). Now ($\mathcal{E}$) reads as

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \Delta_4 u + u + u^3 = \lambda f(u) + |u|^{4^*-2}u.$$

Taking $\mathcal{A}(t) = t\arctan t - \log\sqrt{1+t^2} + t^4/4$ and again $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, we get $2 = p < q = 4$, $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 2/3$, $\theta = \vartheta = 4$, $q^* = 4^*$, $c_4 > 0$ is the constant above, and ($\mathcal{E}$) reduces to

$$-\operatorname{div}\left(\frac{\arctan|\nabla u|}{|\nabla u|}\nabla u\right) - \Delta_4 u + u + u^3 = \lambda f(u) + |u|^{4^*-2}u.$$

For other examples, we refer to [100].

The parameter λ in $(\mathcal{E})$ is strictly positive and *f is a continuous function while $F(t) = \int_0^t f(\tau)d\tau$ satisfies the subcritical growth conditions*

(F) *$F \geq 0$ in $\mathbb{R}$, $f(t) = 0$ for all $t \leq 0$. Furthermore, there exist m and ν such that $1 < p < q < m < q^*$, $\max\{\theta, \vartheta\} < \nu < q^*$, and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequalities*

$$\left|f(t)\right| \leq q\varepsilon|t|^{q-1} + mC_\varepsilon|t|^{m-1} \quad \textit{for any } t \in \mathbb{R} \tag{1.2}$$

and

$$0 < \nu F(t) \leq tf(t) \quad \textit{for all } t \in \mathbb{R}^+$$

hold, where θ, ϑ are given in (C_3).

The symbol $D^{1,q}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm $\|\nabla u\|_q = (\int_{\mathbb{R}^N} |\nabla u|^q dx)^{1/q}$. Moreover, $c_{q^*} > 0$ is the best Sobolev constant, for which

$$\|u\|_{q^*} \leq c_{q^*}\|\nabla u\|_q \quad \text{for all } u \in D^{1,q}(\mathbb{R}^N). \tag{1.3}$$

The natural space for finding solutions of $(\mathcal{E})$ is

$$W = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,q}},$$

where $\|u\|_{W^{1,\wp}} = \|u\|_\wp + \|\nabla u\|_\wp$ for all $u \in W^{1,\wp}(\mathbb{R}^N)$, and $\|\cdot\|_\wp$ denotes the canonical $L^\wp(\mathbb{R}^N)$ norm for any $\wp > 1$.

Theorem 1.1.1. *Suppose that A and B satisfy (C_1)–(C_3) and (F) holds. Then there exists $\lambda^* > 0$ such that equation $(\mathcal{E})$ admits at least one nontrivial solution u_λ in W for all $\lambda \geq \lambda^*$. Moreover,*

$$\lim_{\lambda\to\infty} \|u_\lambda\| = 0 \tag{1.4}$$

holds.

It is interesting to point out that for the basic equation $(\mathcal{E})$ the proof follows a direct argument presented in Proposition 1.3.1. Moreover, the threshold λ^* is just obtained in the key Lemma 1.2.4, on which Theorem 1.1.1 is strongly based. Thus, we thought to present the model equation $(\mathcal{E})$, since it appears still interesting in applications and, moreover, since the existence argument is fairly elegant. Indeed, the proof relies on the alternative Proposition 1.3.1 of Lions type. However, even Theorem 1.1.1 and its extension to the vectorial case given in the original paper [100] continue to improve or complement previous results for the quasilinear (p,q) scalar or vectorial problems; cf., e. g., [31, 33, 54, 55, 158] and the references therein.

Indeed, Theorem 1.1.1 extends previous results in several directions also for the mild growth conditions on the main elliptic operator A, which exhibits a (p,q) growth by (C_2). This is more evident from the fact that the solution space W has a strong dependence on (p,q), since we consider existence of entire solutions. Usually (p,q) problems are settled in bounded domains Ω of $\mathbb{R}^N$, so that the natural solution space is $W = W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$. In this chapter the situation is much more delicate.

Equation $(\mathcal{E})$ has a relevant physical interpretation in applied sciences, as well as a mathematical challenge in overcoming the new difficulties intrinsic to $(\mathcal{E})$. For additional physical motivations, we mention [58]. The presence of the critical nonlinearity, as well as the fact that $(\mathcal{E})$ is studied in the entire space $\mathbb{R}^N$, cause, roughly speaking, a double loss of compactness, which produces new challenging complications. The interest in equation $(\mathcal{E})$ is twofold. On the one hand, $(\mathcal{E})$ is quite challenging from an analytical point of view since the (p,q) operator is not homogeneous and, because of the lack of compactness, several technical difficulties arise when applying the usual methods of the theory of elliptic equations. On the other hand, $(\mathcal{E})$ has a relevant physical interpretation in applied sciences. In other words, if u denotes a concentration of a chemical substance, $(\mathcal{E})$ derives from a general reaction–diffusion equation

$$u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\big) + g(u), \tag{1.5}$$

which arises not only in physics, but also in biophysics, in plasma physics, in chemical reaction design, and in models of elementary particles. In most cases, g is of polynomial type, but for the von Karman model this nonlinear term has critical growth at infinity.

1.2 Existence of weak solutions for $(\mathcal{E})$

In this section, for simplicity, we assume, without further mentioning, that the structural assumptions required in Theorem 1.1.1 hold.

We say that a function $u \in W$ is a (weak) *solution* of equation $(\mathcal{E})$ if

$$\int_{\mathbb{R}^N} A(|\nabla u|)\nabla u \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} B(|u|)u\varphi dx = \lambda \int_{\mathbb{R}^N} f(u)\varphi dx + \frac{1}{q^*}\int_{\mathbb{R}^N} |u|^{q^*-2}u\varphi dx$$

for any $\varphi \in W$.

Clearly, the (weak) solutions of $(\mathcal{E})$ are exactly the critical points of the Euler–Lagrange functional $I = I_\lambda$ associated with $(\mathcal{E})$, where $I : W \to \mathbb{R}$ is given for all $u \in W$ by

$$I(u) = \int_{\mathbb{R}^N} \mathcal{A}(|\nabla u|)dx + \int_{\mathbb{R}^N} \mathcal{B}(|u|)dx - \lambda \int_{\mathbb{R}^N} F(u)dx - \frac{1}{q^*}\int_{\mathbb{R}^N} |u|^{q^*} dx,$$

which is well defined and of class $C^1(W)$ by (C_1) and (F).

It will be shown here that the functional I has the geometric features to get the existence of a Palais–Smale sequence at special levels via the mountain pass theorem of *Ambrosetti* and *Rabinowitz*. Taking inspiration from [100], we prove some preliminary properties. Combining the classical results in Sobolev space theory, we easily get the next lemma.

Lemma 1.2.1. *The embedding $W \hookrightarrow L^{\wp}(\mathbb{R}^N)$ is continuous for all $\wp \in [p, q^*]$, and*

$$\|u\|_{\wp} \le c_{\wp}\|u\| \quad \textit{for all } u \in W, \tag{1.6}$$

where $c_{\wp}$ depends on $\wp$, N, p, and q.

If $\wp \in [1, q^*)$, then the embedding $W \hookrightarrow\hookrightarrow L^{\wp}_{\mathrm{loc}}(\mathbb{R}^N)$ is compact.

Before showing the geometrical mountain pass structure of I, let us note that for any $\varepsilon > 0$ condition (F) gives the existence of $C_\varepsilon > 0$ such that

$$|F(t)| \le \varepsilon|t|^q + C_\varepsilon|t|^m \quad \text{for all } t \in \mathbb{R} \tag{1.7}$$

holds. For simplicity in notation, in what follows we put

$$a = \min\{a_0,\ b_0,\ a_1,\ b_1\}, \quad \tilde{a} = \min\{a_0,\ a_1\}, \tag{1.8}$$

while $\mathfrak{a} = \max\{\mathfrak{a}_0,\ \mathfrak{b}_0,\ \mathfrak{a}_1,\ \mathfrak{b}_1\}$. Clearly, $0 < a \le \tilde{a}$ by (C_2).

Lemma 1.2.2. *Fix any $\lambda > 0$. Then there exists a nonnegative function $e \in C_c^\infty(\mathbb{R}^N)$, independent of λ, such that $I(e) < 0$, $\|e\| \ge 2$ and $\|e\|_{q^*} > 0$.*

Furthermore, there exist numbers $\jmath = \jmath(\lambda) > 0$ and $\rho = \rho(\lambda) \in (0, 1]$ such that $I(u) \ge \jmath$ for any $u \in W$, with $\|u\| = \rho$.

Proof. Fix $\lambda > 0$. Let $u \in C_c^\infty(\mathbb{R}^N)$ be such that $u \ge 0$ in $\mathbb{R}^N$, $\|u\| = 1$, and $\|u\|_{q^*} > 0$. Therefore, by (C_2), (F), and the definition of $\mathfrak{a}$ given in Section 1.1, we have as $t \to \infty$,

$$I(tu) \le \frac{\mathfrak{a}}{p}t^q - \frac{t^{q^*}}{q^*}\|u\|_{q^*}^{q^*} \to -\infty,$$

since $1 < p < q < q^*$, as assumed in (C_3). Hence, taking $e = \tau_0 u$, with $\tau_0 > 0$ sufficiently large, we obtain at once that $e \ge 0$ in $\mathbb{R}^N$, $\|e\| \ge 2$, $I(e) < 0$, and $\|e\|_{q^*} > 0$, as stated.

Moreover, for the second part of the lemma, we note that for all $u \in W$,

$$\|u\|^q \le 2^{q-1}\{\|u\|^q_{W^{1,p}} + \|u\|^q_{W^{1,q}}\}.$$

Hence, (C_2), (1.8), and (1.6) imply that for all $u \in W$, with $\|u\| \le 1$,

$$\begin{aligned} I(u) &\ge \frac{a}{p}\|u\|^p_{W^{1,p}} + \frac{a}{q}\|u\|^q_{W^{1,q}} - \lambda \int_{\mathbb{R}^N} \varepsilon |u|^q dx - \lambda \int_{\mathbb{R}^N} C_\varepsilon |u|^m dx - \frac{1}{q^*}\|u\|^{q^*}_{q^*} \\ &\ge \frac{a}{q}\{\|u\|^q_{W^{1,p}} + \|u\|^q_{W^{1,q}}\} - \lambda\varepsilon c_q^q \|u\|^q - \lambda C_\varepsilon c_m^m \|u\|^m - \frac{c_{q^*}^{q^*}}{q^*}\|u\|^{q^*} \\ &\ge \frac{a}{q} 2^{1-q}\|u\|^q - \lambda\varepsilon c_q^q \|u\|^q - \lambda C_\varepsilon c_m^m \|u\|^m - \frac{c_{q^*}^{q^*}}{q^*}\|u\|^{q^*}, \end{aligned}$$

since $1 < p < q < m < q^*$. Therefore, we are able to fix $\varepsilon > 0$ so small that

$$\kappa = 2^{1-q}\frac{a}{q} - \lambda\varepsilon c_q^q > 0.$$

Hence, there exists $\rho \in (0, 1]$ such that

$$\max_{t\in[0,1]} y(t) = y(\rho) > 0, \quad \text{where } y(t) = \kappa\, t^q - \lambda C_\varepsilon c_m^m t^m - \frac{c_{q^*}^{q^*}}{q^*} t^{q^*},$$

since $q < m < q^*$ and due to the choice of ε. Consequently, $I(u) \ge y(\rho) = \jmath$ for all $u \in W$, with $\|u\| = \rho$, as desired. This concludes the proof. □

From the proof of Lemma 1.2.2 it is evident that $e = \tau_0 u$ is selected at some $\lambda_0 > 0$, that is, $\tau_0 = \tau_0(\lambda_0)$, then $I(e) < 0$ for all $\lambda \ge \lambda_0$. Moreover, $\|e\| \ge 2 > \rho$ for all $\lambda \ge \lambda_0$, since $\rho = \rho(\lambda) \in (0, 1]$.

We recall in passing that, if X is a real Banach space, a $C^1(X)$ functional J satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ if any Palais–Smale sequence $(u_k)_k$ at level c, briefly $(PS)_c$ sequence, such that

$$J(u_k) \to c \quad \text{and} \quad J'(u_k) \to 0 \quad \text{in } X' \text{ as } k \to \infty, \tag{1.9}$$

admits a convergent subsequence in X.

Now, for fixed $\lambda > 0$, thanks to the geometry given in Lemma 1.2.2, we introduce the special levels of I by

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{1.10}$$

where $\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}$. Obviously, $c_\lambda > 0$ thanks to Lemma 1.2.2, since in particular $\|e\| > \rho$. We introduce an asymptotic condition for the level c_λ. This result was already observed in the scalar case in [96], cf. Lemma 2.2 and Remark 2.3, and the vectorial case in [100], and will be crucial to overcome the lack of compactness due to the presence of the critical nonlinearities.

Lemma 1.2.3. *One has*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof. Fix $\lambda > 0$. Let e be the nonnegative function determined in Lemma 1.2.2. Since the functional I satisfies the mountain pass geometry at 0 and e, there exists $t_\lambda > 0$ verifying $I(t_\lambda e) = \max_{t\geq 0} I(te)$. Therefore, $\langle I'(t_\lambda e), e\rangle = 0$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^N} A(t_\lambda|\nabla e|)t_\lambda|\nabla e|^2 dx + \int_{\mathbb{R}^N} B(t_\lambda|e|)t_\lambda|e|^2 dx \\ &= \lambda \int_{\mathbb{R}^N} f(t_\lambda e)e dx + t_\lambda^{q^*-1}\|e\|_{q^*}^{q^*} \\ &\geq t_\lambda^{q^*-1}\|e\|_{q^*}^{q^*},\end{aligned} \tag{1.11}$$

by (F), since $\lambda > 0$.

We claim that $\{t_\lambda\}_{\lambda>0}$ is bounded in $\mathbb{R}$. Indeed, from (C_2), putting $\Lambda = \{\lambda > 0 : t_\lambda\|e\| \geq 1\}$, we derive that

$$\int_{\mathbb{R}^N} A(t_\lambda|\nabla e|)t_\lambda^2|\nabla e|^2 dx + \int_{\mathbb{R}^N} B(t_\lambda|e|)t_\lambda^2|e|^2 dx \leq \mathfrak{a} t_\lambda^q\|e\|^q \tag{1.12}$$

for any $\lambda \in \Lambda$, since $1 < p < q$. Therefore, (1.11) and (1.12) imply that

$$\mathfrak{a}\|e\|^q \geq t_\lambda^{q^*-q}\|e\|_{q^*}^{q^*} \quad \text{for any } \lambda \in \Lambda,$$

which yields that $\{t_\lambda\}_{\lambda\in\Lambda}$ is bounded, since $\|e\|_{q^*} > 0$ by Lemma 1.2.2. It follows at once that $\{t_\lambda\}_{\lambda>0}$ is bounded. This proves the claim.

Fix now a sequence $(\lambda_k)_k \subset \mathbb{R}^+$ such that $\lambda_k \to \infty$ as $k \to \infty$. Obviously, $(t_{\lambda_k})_k$ is bounded. Thus, there exist a $t_0 \geq 0$ and a subsequence of $(\lambda_k)_k$, still denoted by $(\lambda_k)_k$, such that $t_{\lambda_k} \to t_0$ as $k \to \infty$. By the continuity of $\mathcal{A}'$ and $\mathcal{B}'$ in $\mathbb{R}_0^+$, as stated in (C_1), combined with (1.11), there exists $C > 0$ such that, for any $k \in \mathbb{N}$,

$$\lambda_k \int_{\mathbb{R}^N} f(t_\lambda e)e\, dx + t_{\lambda_k}^{q^*-1}\|e\|_{q^*}^{q^*} \leq C. \tag{1.13}$$

We assert that $t_0 = 0$. Otherwise, (F) and the dominated convergence theorem yield, as $k \to \infty$,

$$\int_{\mathbb{R}^N} f(t_{\lambda_k} e)e\, dx \to \int_{\mathbb{R}^N} f(t_0 e)e\, dx > 0,$$

by (F) and the fact that e is nonnegative, with $\|e\|_{q^*} > 0$, as constructed in Lemma 1.2.2. Therefore, recalling that $\lambda_k \to \infty$, we get at once that

$$\lim_{k\to\infty}\left(\lambda_k \int_{\mathbb{R}^N} f(t_{\lambda_k} e)e\, dx + t_{\lambda_k}^{q^*-1}\|e\|_{q^*}\right) = \infty,$$

which contradicts (1.13). Thus $t_0 = 0$ and $t_\lambda \to 0$ as $\lambda \to \infty$, since the sequence $(\lambda_k)_k$ is arbitrary.

Now the path $\gamma(t) = t\,e$, $t \in [0,1]$, belongs to Γ, so that Lemma 1.2.2 and (C_2) give

$$0 < c_\lambda \le \max_{t\ge 0} I\big(\gamma(t)\big) \le I(t_\lambda e) \le \frac{\mathfrak{a}}{p}\|e\|^q t_\lambda^p \to 0$$

as $\lambda \to \infty$, since e does not depend on λ. This completes the proof of the lemma. □

Lemma 1.2.2 and the mountain pass lemma yield that the set of Palais–Smale sequences of I at the level c_λ given in (1.10) is nonempty. Now we are ready to prove crucial properties of the Palais–Smale sequences of I at the special level c_λ.

Lemma 1.2.4. *Let $(u_k)_k \subset W$ be a $(PS)_{c_\lambda}$ sequence of I for all $\lambda > 0$. Then,*
(*i*) *up to a subsequence, $u_k \rightharpoonup u_\lambda$ in W as $k \to \infty$;*
(*ii*) *there exists $\lambda^* > 0$ such that the weak limit u_λ is a solution of $(\mathcal{E})$ for all $\lambda \ge \lambda^*$;*
(*iii*) *the set $\{u_\lambda\}_{\lambda\ge\lambda^*}$ has the asymptotic property* (1.4).

Proof. Let $(u_k)_k \subset W$ be a $(PS)_{c_\lambda}$ sequence of I for any $\lambda > 0$ such that

$$I(u_k) \to c_\lambda \quad \text{and} \quad I'(u_k) \to 0 \quad \text{in } W' \text{ as } k \to \infty. \tag{1.14}$$

By (C_3) and (F), we get

$$\begin{aligned} I(u_k) - \frac{1}{\nu}\langle I'(u_k), u_k\rangle &= \int_{\mathbb{R}^N} \mathcal{A}(|\nabla u_k|)dx - \frac{1}{\nu}\int_{\mathbb{R}^N} A(|\nabla u_k|)|\nabla u_k|^2 dx \\ &+ \int_{\mathbb{R}^N} \mathcal{B}(|u_k|)dx - \frac{1}{\nu}\int_{\mathbb{R}^N} B(|u_k|)|u_k|^2 dx \\ &- \lambda \int_{\mathbb{R}^N}\Big(F(u_k) - \frac{1}{\nu}f(u_k)u_k\Big)dx + \Big(\frac{1}{\nu} - \frac{1}{q^*}\Big)\|u_k\|_{q^*}^{q^*} \\ &\ge \Big(\frac{1}{\theta} - \frac{1}{\nu}\Big)\int_{\mathbb{R}^N} \mathcal{A}'(|\nabla u_k|)|\nabla u_k|dx + \Big(\frac{1}{\vartheta} - \frac{1}{\nu}\Big)\int_{\mathbb{R}^N} \mathcal{B}'(|u_k|)|u_k|dx, \end{aligned} \tag{1.15}$$

since $q \le \max\{\theta, \vartheta\} < \nu < q^*$ by (F) and (C_3). Then, thanks to (1.14) and (C_2), there exists $d_\lambda > 0$ such that, as $k \to \infty$,

$$c_\lambda + d_\lambda\|u_k\| + o(1) \ge \ell\big(\|u_k\|_{W^{1,p}}^p + \|u_k\|_{W^{1,q}}^q\big), \tag{1.16}$$

where

$$\ell = a\Big(\frac{1}{\max\{\theta,\vartheta\}} - \frac{1}{\nu}\Big), \tag{1.17}$$

and $\ell > 0$ by (1.8), (C_2), and (F). We claim that $(u_k)_k$ is bounded in W.

Assume for a contradiction that $\|u_k\| \to \infty$ as $k \to \infty$. Then, passing if necessary to a subsequence, still labeled by $(u_k)_k$, it has norm diverging as n diverges. Then, we could have either exactly one Sobolev norm diverging, say $\|\cdot\|_{W^{1,p}}$, or both norms diverging. In the first case, as $k \to \infty$,

$$\begin{aligned} 0 < \ell &\le d_\lambda \frac{\|u_k\|_{W^{1,p}} + \|u_k\|_{W^{1,q}}}{\|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}}} + o(1) \\ &\le d_\lambda \left(\frac{\|u_k\|_{W^{1,p}}}{\|u_k\|^p_{W^{1,p}}} + \frac{\|u_k\|_{W^{1,q}}}{\|u_k\|^p_{W^{1,p}}} \right) + o(1) \\ &\le d_\lambda \|u_k\|^{1-p}_{W^{1,p}} + o(1) = o(1). \end{aligned}$$

Again this gives the required contradiction and proves the claim.

Finally, in the second case, as $k \to \infty$,

$$\begin{aligned} 0 < \ell &\le d_\lambda \frac{\|u_k\|_{W^{1,p}} + \|u_k\|_{W^{1,q}}}{\|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}}} + o(1) \\ &\le d_\lambda \left(\frac{\|u_k\|_{W^{1,p}}}{\|u_k\|^p_{W^{1,p}}} + \frac{\|u_k\|_{W^{1,q}}}{\|u_k\|^q_{W^{1,q}}} \right) + o(1) \\ &\le d_\lambda \left(\|u_k\|^{1-p}_{W^{1,p}} + \|u_k\|^{1-q}_{W^{1,q}} \right) + o(1) = o(1). \end{aligned}$$

Consequently, the claim is proved in all the possible cases.

Thus, since $(u_k)_k$ is bounded in the reflexive Banach space W, there exist $u_\lambda \in W$, nonnegative numbers $\imath_\lambda$, δ_λ, and bounded nonnegative Radon measures μ and ω on $\mathbb{R}^N$, by virtue of Proposition 1.202 of [107], such that, up to a subsequence still denoted by $(u_k)_k$, we have

$$\begin{gathered} u_k \rightharpoonup u_\lambda \text{ in } W, \quad \|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}} \to \imath_\lambda, \\ u_k \to u_\lambda \text{ in } L^{\wp}_{\mathrm{loc}}(\mathbb{R}^N), \quad u_k \to u_\lambda \text{ a. e. in } \mathbb{R}^N, \\ |u_k| \le g_R \text{ a. e. in } \mathbb{R}^N, \text{ for some } g_R \in L^q(B_R) \text{ and all } R > 0, \\ \|u_k - u_\lambda\|^{q^*}_{q^*} \to \delta_\lambda, \\ |u_k|^{q^*-2}u_k \rightharpoonup |u_\lambda|^{q^*-2}u_\lambda \text{ in } L^{q^*/(q^*-1)}(\mathbb{R}^N), \\ \tilde{a}|\nabla u_k|^q dx \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbb{R}^N), \quad |u_k|^{q^*} dx \overset{*}{\rightharpoonup} \omega \text{ in } \mathcal{M}(\mathbb{R}^N), \end{gathered} \tag{1.18}$$

with $\wp \in [1, q^*)$, by (1.6) and Lemma 1.2.1. Let us recall that $\tilde{a}$ is the key positive number introduced in (1.8). This completes the proof of (i).

Theorem 2 of [134] can be applied thanks to (1.18). Therefore, there exist two nonnegative numbers μ_0 and ω_0, at most countable set J, three families of points $\{x_j\}_{j\in J}$

and of nonnegative numbers $\{\mu_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that

$$\omega = |u_\lambda|^{q^*}dx + \omega_0\delta_0 + \sum_{j\in J}\omega_j\delta_{x_j},$$
$$\mu \geq \widetilde{a}|\nabla u_\lambda|^q dx + \mu_0\delta_0 + \sum_{j\in J}\mu_j\delta_{x_j}, \quad \omega_0^{q/q*} \leq \frac{\mu_0}{S}, \tag{1.19}$$
$$\omega_j^{q/q*} \leq \frac{\mu_j}{S} \text{ for all } j \in J,$$

where δ_0 and δ_{x_j} are the Dirac functions at the points 0 and x_j of $\mathbb{R}^N$, and

$$S = \inf_{\substack{u\in D^{1,q}(\mathbb{R}^N)\\ u\neq 0}} \frac{\widetilde{a}\|\nabla u\|_q^q}{\|u\|_{q^*}^q}. \tag{1.20}$$

From (1.15)–(1.18), we derive the main formula

$$c_\lambda + o(1) \geq \ell\{\|u_k\|_{W^{1,p}}^p + \|u_k\|_{W^{1,q}}^q\} + \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\|u\|_{q^*}^{q^*} \tag{1.21}$$

as $k \to \infty$.

First we assert that

$$\lim_{\lambda\to\infty} \imath_\lambda = 0. \tag{1.22}$$

Otherwise, $\limsup_{\lambda\to\infty} \imath_\lambda = \imath > 0$. Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\imath_{\lambda_j} \to \imath$ as $j \to \infty$. Then, letting $j \to \infty$, we get from (1.21) and Lemma 1.2.3 that

$$0 \geq \ell\,\imath > 0.$$

This contradiction proves the assertion (1.22). Moreover,

$$\|u_\lambda\|_{W^{1,p}}^p + \|u_\lambda\|_{W^{1,q}}^q \leq \imath_\lambda,$$

since $u_k \rightharpoonup u_\lambda$ in W, so that (1.6) and (1.22) imply that

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{q^*} = \lim_{\lambda\to\infty} \|u_\lambda\| = 0. \tag{1.23}$$

Fix a test function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in the closed ball B_1 of center 0 and radius 1, while $\varphi \equiv 0$ in B_2^c, where B_2 is the closed ball of center 0 and radius 2, and $\|\nabla\varphi\|_\infty \leq 2$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(x) = \varphi((x-x_j)/\varepsilon)$, $x \in \mathbb{R}^N$, for any fixed $j \in J$, where $\{x_j\}_{j\in J}$ is introduced in (1.19), and $\varphi_{\varepsilon,0}(x) = \varphi(x/\varepsilon)$, $x \in \mathbb{R}^N$. Fix $j \in J \cup \{0\}$. Then $\varphi_{\varepsilon,j}u_k \in W$ and so $\langle I'(u_k), \varphi_{\varepsilon,j}u_k\rangle = o(1)$ as $k \to \infty$. Therefore, as $k \to \infty$,

$$o(1) = \int_{\mathbb{R}^N} A(|\nabla u_k|)u_k\nabla u_k \cdot \nabla\varphi_{\varepsilon,j}dx$$

$$+\int_{\mathbb{R}^N}\varphi_{\varepsilon,j}\{A(|\nabla u_k|)|\nabla u_k|^2+B(|u_k|)|u_k|^2\}dx \tag{1.24}$$
$$-\lambda\int_{\mathbb{R}^N}\varphi_{\varepsilon,j}f(u_k)u_k dx-\int_{\mathbb{R}^N}\varphi_{\varepsilon,j}|u_k|^{q^*}dx.$$

Thus, by (C_2), the Hölder inequality, and a change of variable,

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int_{\mathbb{R}^N}A(|\nabla u_k|)u_k\nabla u_k\cdot\nabla\varphi_{\varepsilon,j}dx\right|\\
&\le\limsup_{k\to\infty}\left\{\int_{B(x_j,2\varepsilon)}\{\mathfrak{a}_0|\nabla u_k|^{p-1}|u_k|\cdot|\nabla\varphi_{\varepsilon,j}|+\mathfrak{a}_1|\nabla u_k|^{q-1}|u_k|\cdot|\nabla\varphi_{\varepsilon,j}|\}dx\right\}\\
&\le\limsup_{k\to\infty}\left\{\mathfrak{a}_0\|\nabla u_k\|_p^{p-1}\left(\int_{B(x_j,2\varepsilon)}\left|u_k\nabla\varphi_{\varepsilon,j}(x)\right|^p dx\right)^{1/p}\right.\\
&\qquad\left.+\mathfrak{a}_1\|\nabla u_k\|_q^{q-1}\left(\int_{B(x_j,2\varepsilon)}\left|u_k\nabla\varphi_{\varepsilon,j}(x)\right|^q dx\right)^{1/q}\right\}\\
&\le c_0\mathfrak{a}_0\left(\int_{B(x_j,2\varepsilon)}\left|u_\lambda\nabla\varphi_{\varepsilon,j}(x)\right|^p dx\right)^{1/p}+c_1\mathfrak{a}_1\left(\int_{B(x_j,2\varepsilon)}\left|u_\lambda\nabla\varphi_{\varepsilon,j}(x)\right|^q dx\right)^{1/q}\\
&\le c_\varphi\left\{c_0\mathfrak{a}_0\left(\int_{B(x_j,2\varepsilon)}|u_\lambda|^{p^*}dx\right)^{1/p^*}+c_1\mathfrak{a}_1\left(\int_{B(x_j,2\varepsilon)}|u_\lambda|^{q^*}dx\right)^{1/q^*}\right\},
\end{aligned}$$

where $c_0=\sup_{k\in\mathbb{N}}\|\nabla u_k\|_p^{p-1}$, $c_1=\sup_{k\in\mathbb{N}}\|\nabla u_k\|_q^{q-1}$, and $c_\varphi=(\int_{B_2}|\nabla\varphi(y)|^N dy)^{1/N}$. Consequently,

$$\lim_{\varepsilon\to0^+}\limsup_{k\to\infty}\left|\int_{\mathbb{R}^N}A(|\nabla u_k|)u_k\nabla u_k\cdot\nabla\varphi_{\varepsilon,j}dx\right|=0. \tag{1.25}$$

Clearly, by (C_2), the properties of φ, and (1.18), as $k\to\infty$,

$$\begin{aligned}
0&\le\int_{\mathbb{R}^N}\varphi_{\varepsilon,j}B(|u_k|)|u_k|^2dx\le\int_{B(x_j,2\varepsilon)}(\mathfrak{b}_0|u_k|^p+\mathfrak{b}_1|u_k|^q)dx\\
&\to\int_{B(x_j,2\varepsilon)}(\mathfrak{b}_0|u_\lambda|^p+\mathfrak{b}_1|u_\lambda|^q)dx,
\end{aligned}$$

since $1<p<q<q^*$. In conclusion,

$$\lim_{\varepsilon\to0^+}\lim_{k\to\infty}\int_{\mathbb{R}^N}\varphi_{\varepsilon,j}B(|u_k|)|u_k|^2dx=0. \tag{1.26}$$

Likewise, by (F) and (1.18), as $k \to \infty$,

$$0 \le \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} f(u_k) u_k dx \le \int_{B(x_j,2\varepsilon)} (q|u_k|^q + m\,C_1|u_k|^m)dx$$
$$\to \int_{B(x_j,2\varepsilon)} (q|u_\lambda|^q + m\,C_1|u_\lambda|^m)dx,$$

since $1 < p < q < m < q^*$. Therefore,

$$\lim_{\varepsilon\to0^+} \lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} f(u_k) u_k dx = 0. \tag{1.27}$$

In conclusion, (C_2), (1.8), (1.24)–(1.27) give the crucial formula for all $j \in J \cup \{0\}$, namely

$$\int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\mu + o(1) \le \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\omega \tag{1.28}$$

as $\varepsilon \to 0^+$.

By Lemma 1.2.3, there exists $\lambda^* = \lambda^*(N, q, \omega) > 0$ such that

$$c_\lambda < \left(\frac{1}{\nu} - \frac{1}{q^*}\right) S^{N/q} \quad \text{for all } \lambda \ge \lambda^*. \tag{1.29}$$

We divide the proof in two parts. First, (1.19) and (1.28) yield $S\,\omega_j^{q/q^*} \le \mu_j \le \omega_j$ for all $j \in J$. Assume by contradiction that $\omega_j > 0$ for some $j \in J$. Then, $\omega_j \ge S^{N/q}$, and so (1.21) implies

$$c_\lambda + o(1) \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) \|u_k\|_{q^*}^{q^*} \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\omega$$

as $k \to \infty$. On the other hand, as $k \to \infty$ and $\varepsilon \to 0^+$, we have

$$c_\lambda \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) \omega_j \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) S^{N/q} > 0,$$

which is an obvious contradiction by (1.29). Hence, $\omega_j = 0$ for all $j \in J$ and for all $\lambda \ge \lambda^*$.

Similarly, when the center of the ball is 0, then (1.19) and (1.28) give $S\omega_0^{q/q^*} \le \mu_0 \le \omega_0$. Assume by contradiction that $\omega_0 \ne 0$. Then, $\omega_0 \ge S^{N/q}$. As above, by (1.21) we obtain, as $k \to \infty$ and $\varepsilon \to 0^+$,

$$c_\lambda \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) \omega_0 \ge \left(\frac{1}{\nu} - \frac{1}{q^*}\right) S^{N/q} > 0,$$

which is again a contradiction by (1.29). Therefore, $\omega_0 = 0$ and so $\mu_0 = 0$ for all $\lambda \ge \lambda^*$.

In summary, we have shown that there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$,

$$|u_k|^{q^*} dx \overset{*}{\rightharpoonup} \omega = |u_\lambda|^{q^*} dx \quad \text{in } \mathcal{M}(\mathbb{R}^N)$$

as $k \to \infty$, by (1.18) and (1.19). In particular, for all $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \phi |u_k|^{q^*} dx = \int_{\mathbb{R}^N} \phi |u_\lambda|^{q^*} dx. \tag{1.30}$$

Take $R > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi \equiv 1$ in B_R, $\varphi \equiv 0$ in B_{2R}^c and $\|\nabla\varphi\|_\infty \leq 2$. Now by (1.1) of (C_3) we have

$$\begin{aligned} c_q \int_{B_R} |\nabla u_k - \nabla u_\lambda|^q dx &\leq \int_{B_R} \big(A(|\nabla u_k|)\nabla u_k - A(|\nabla u_\lambda|)\nabla u_\lambda\big) \cdot (\nabla u_k - \nabla u_\lambda) dx \\ &\leq \int_{\mathbb{R}^N} \big(A(|\nabla u_k|)\nabla u_k - A(|\nabla u_\lambda|)\nabla u_\lambda\big) \cdot (\nabla u_k - \nabla u_\lambda)\varphi\, dx \\ &= \int_{\mathbb{R}^N} \varphi A(|\nabla u_k|)|\nabla u_k|^2 dx - \int_{\mathbb{R}^N} \varphi A(|\nabla u_k|)\nabla u_k \cdot \nabla u_\lambda dx + o(1) \end{aligned} \tag{1.31}$$

as $k \to \infty$ by (1.18). Clearly,

$$\langle I'(u_k), \varphi u_k\rangle - \langle I'(u_k), \varphi u_\lambda\rangle = o(1) \quad \text{as } k \to \infty$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi A(|\nabla u_k|)\{|\nabla u_k|^2 - \nabla u_k \cdot \nabla u_\lambda\} dx &= \langle I'(u_k), \varphi u_k\rangle - \langle I'(u_k), \varphi u_\lambda\rangle \\ &\quad - \int_{\mathbb{R}^N} A(|\nabla u_k|)(u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx - \int_{\mathbb{R}^N} \varphi B(|u_k|)u_k(u_k - u_\lambda) dx \\ &\quad + \lambda \int_{\mathbb{R}^N} \varphi f(u_k)(u_k - u_\lambda) dx \\ &\quad + \int_{\mathbb{R}^N} \varphi |u_k|^{q^*} dx - \int_{\mathbb{R}^N} \varphi |u_k|^{q^*-2} u_k u_\lambda dx. \end{aligned}$$

By (C_2) and the Hölder inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} A(|\nabla u_k|)(u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx \right| &\leq 2\Bigg\{ \mathfrak{a}_0 \|\nabla u_k\|_p^{p-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^p dx \right)^{1/p} \\ &\quad + \mathfrak{a}_1 \|\nabla u_k\|_q^{q-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^q dx \right)^{1/q} \Bigg\}, \end{aligned}$$

which yields by (1.18) that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} A(|\nabla u_k|)(u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx = 0. \tag{1.32}$$

Similarly, again by (C_2) and the Hölder inequality,

$$\left|\int_{\mathbb{R}^N} \varphi B(|u_k|)u_k(u_k - u_\lambda)dx\right| \le \left\{\mathfrak{b}_0\|u_k\|_p^{p-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^p dx\right)^{1/p} + \mathfrak{b}_1\|u_k\|_q^{q-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^q dx\right)^{1/q}\right\},$$

which yields by (1.18) that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} \varphi B(|u_k|)u_k(u_k - u_\lambda)dx = 0. \tag{1.33}$$

Likewise, by (F), the Hölder inequality, and (1.18), as $k \to \infty$,

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} \varphi f(u_k)(u_k - u_\lambda)dx \\ &\le \int_{B_{2R}} \varphi(q|u_k|^{q-1}|u_k - u_\lambda| + m\,C_1|u_k|^{m-1}|u_k - u_\lambda|)dx \\ &\le C\left\{\left(\int_{B_{2R}} |u_k - u_\lambda|^q dx\right)^{1/q} + \left(\int_{B_{2R}} |u_k - u_\lambda|^m dx\right)^{1/m}\right\} \to 0, \end{aligned} \tag{1.34}$$

where $C = q\sup_{k\in\mathbb{N}}\|u_k\|^{q-1} + m\,C_1\sup_{k\in\mathbb{N}}\|u_k\|^{m-1} < \infty$, since $1 < p < q < m < q^*$. Finally, as $k \to \infty$,

$$\int_{\mathbb{R}^N} \varphi|u_k|^{q^*} dx - \int_{\mathbb{R}^N} \varphi|u_k|^{q^*-2}u_k u_\lambda dx \to 0, \tag{1.35}$$

by (1.18) and (1.30). Thus, combining (1.31)–(1.35), we have

$$c_q\int_{B_R} |\nabla u_k - \nabla u_\lambda|^q dx \le o(1) \quad \text{as } k \to \infty.$$

Consequently, $\nabla u_k \to \nabla u_\lambda$ in $[L^q(B_R)]^N$ for all $R > 0$, since $c_q > 0$ by (C_3). Therefore, up to subsequence, not relabeled, we get that

$$\nabla u_k \to \nabla u_\lambda \quad \text{a. e. in } \mathbb{R}^N, \tag{1.36}$$

and for all $R > 0$ there exists a function $h_R \in L^q(B_R)$ such that $|\nabla u_k| \le h_R$ a. e. in B_R and for all $k \in \mathbb{N}$.

Fix ϕ in $C_c^\infty(\mathbb{R}^N)$ and let $R > 0$ be so large that $\operatorname{supp}\phi \subset B_R$. By the above construction and (C_2), we have a. e. in B_R that

$$\begin{aligned}\big|A(|\nabla u_k|)\nabla u_k \cdot \nabla\phi\big| &\le \big(\mathfrak{a}_0|\nabla u_k|^{p-1} + \mathfrak{a}_1|\nabla u_k|^{q-1}\big)|\nabla\phi| \\ &\le \big(\mathfrak{a}_0 h_R^{p-1} + \mathfrak{a}_1 h_R^{q-1}\big)|\nabla\phi| = \mathfrak{h},\end{aligned}$$

where $\mathfrak{h} \in L^1(B_R)$. Therefore, the dominated convergence theorem gives at once as $k \to \infty$ that

$$\begin{aligned}\int_{\mathbb{R}^N} A(|\nabla u_k|)\nabla u_k \cdot \nabla\phi dx &= \int_{B_R} A(|\nabla u_k|)\nabla u_k \cdot \nabla\phi dx \\ &\to \int_{\mathbb{R}^N} A(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla\phi dx.\end{aligned}$$

Similarly, using again (C_2) and (1.18), we have a. e. in B_R that

$$\big|B(|u_k|)u_k\phi\big| \le \big(\mathfrak{b}_0 g_R^{p-1} + \mathfrak{b}_1 g_R^{q-1}\big)|\phi| = \mathfrak{g} \in L^1(B_R),$$

and so the dominated convergence theorem gives, as $k \to \infty$,

$$\int_{\mathbb{R}^N} B(|u_k|)u_k\phi dx \to \int_{\mathbb{R}^N} B(|u_\lambda|)u_\lambda\phi dx,$$

while by (F),

$$\big|f(u_k)\phi\big| \le \big(q|u_k|^{q-1}\phi + m\, C_1|u_k|^{m-1}\big)|\phi| \le \mathfrak{G} \in L^1(B_R),$$

so again by the dominated convergence theorem, as $k \to \infty$,

$$\int_{\mathbb{R}^N} f(u_k)\phi dx \to \int_{\mathbb{R}^N} f(u_\lambda)\phi dx.$$

Finally, since $\langle I'(u_k), \phi\rangle = o(1)$ as $k \to \infty$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} A(|\nabla u_k|)\nabla u_k \cdot \nabla\phi dx + \int_{\mathbb{R}^N} B(|u_k|)u_k\phi dx \\ &\qquad = \lambda \int_{\mathbb{R}^N} f(u_k)\phi dx + \frac{1}{q^*}\int_{\mathbb{R}^N} |u_k|^{q^*-2}u_k\phi dx + o(1).\end{aligned}$$

Thus, letting $k \to \infty$, using the above arguments and (1.18), we get at once that

$$\begin{aligned}&\int_{\mathbb{R}^N} A(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla\phi dx + \int_{\mathbb{R}^N} B(|u_\lambda|)u_\lambda\phi dx \\ &\qquad = \lambda \int_{\mathbb{R}^N} f(u_\lambda)\phi dx + \frac{1}{q^*}\int_{\mathbb{R}^N} |u_\lambda|^{q^*-2}u_\lambda\phi dx\end{aligned} \tag{1.37}$$

for all ϕ in $C_c^\infty(\mathbb{R}^N)$.

Now fix φ in W. Then the sequence $(\phi_k)_k$ in $C_c^\infty(\mathbb{R}^N)$, defined by $\phi_k = \zeta_k(\rho_k * \varphi)$, where $(\rho_k)_k$ is a sequence of mollifiers and $(\zeta_k)_k$ is a sequence of cut-off functions, has the properties that $\phi_k \to \varphi$ in $W = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and $\phi_k \to \varphi$, $\nabla\phi_k \to \nabla\varphi$ a. e. in $\mathbb{R}^N$ as $k \to \infty$. Of course, (1.37) holds along $(\phi_k)_k$ for all k. Passing to the limit as $k \to \infty$ under the sign of integrals, by the dominated convergence theorem, we obtain the validity of (1.37) for all $\varphi \in W$. In conclusion,

$$\langle I'(u_\lambda), \varphi\rangle = 0 \quad \text{for all } \varphi \in W, \tag{1.38}$$

that is, u_λ is a solution of $(\mathcal{E})$ for all $\lambda \geq \lambda^*$. This completes the proof of (ii).

Finally, (1.4), and so (iii), is a direct consequence of (1.23) and (ii). □

1.3 Proof of Theorem 1.1.1

Let us finish the chapter with a result of independent interest, which implies useful consequences.

Proposition 1.3.1. *For any $\lambda > 0$ let $(u_k)_k \subset W$ be a $(PS)_{c_\lambda}$ sequence of I such that $u_k \rightharpoonup 0$ in W as $k \to \infty$. Then, either*
(i) *$u_k \to 0$ in W, or*
(ii) *there exists $R > 0$ and a sequence $(y_k)_k \subset \mathbb{R}^N$ such that*

$$\limsup_{k\to\infty} \int_{B_R(y_k)} |u_k|^p dx > 0.$$

Moreover, $(y_k)_k$ is not bounded in $\mathbb{R}^N$.

Proof. If (ii) does not occur, then for all $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_k|^p dx = 0.$$

Therefore, $u_k \to 0$ in $L^\wp(\mathbb{R}^N)$ as $k \to \infty$ for all $\wp \in (p, q^*)$ by Lemma I.1 of [161], since $(u_k)_k$ is bounded in $L^p(\mathbb{R}^N)$, while $(\nabla u_k)_k$ is bounded in $[L^q(\mathbb{R}^N)]^N$. Consequently, by (F) and (1.2), as $k \to \infty$,

$$0 \leq \int_{\mathbb{R}^N} f(u_k)u_k dx \leq \int_{\mathbb{R}^N} (q|u_k|^q + m\,C_1|u_k|^m) dx \to 0,$$

since $1 < p < q < m < q^*$. Now, $(u_k)_k \subset W$ is a $(PS)_{c_\lambda}$ sequence of I, so that, arguing as in the proof of Lemma 1.3.1, passing to subsequences if necessary, and using the notation in (1.18), we find as $k \to \infty$ that

$$\int_{\mathbb{R}^N} A(|\nabla u_k|)|\nabla u_k|^2 dx + \int_{\mathbb{R}^N} B(|u_k|)|u_k|^2 dx = \|u_k\|^{q^*} + o(1) = \delta_\lambda + o(1).$$

Therefore, by (C_2), (1.8), and (1.23), as $k \to \infty$ and $\lambda \to \infty$,

$$a\big(\|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}}\big) \le \int_{\mathbb{R}^N} A(|\nabla u_k|)|\nabla u_k|^2 dx + \int_{\mathbb{R}^N} B(|u_k|)|u_k|^2 dx + o_k(1) = o_{k,\lambda}(1).$$

Thus, $\|u_k\| \to 0$ as $k \to \infty$, as required. In conclusion, (i) holds.

Assume now that (ii) is verified and suppose by contradiction that $(y_k)_k$ is bounded in $\mathbb{R}^N$. Consequently, there exists $M > 0$ so large that $B_R(y_k) \subset B_M$ for all k. Now, $u_k \to 0$ in $L^{\wp}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $\wp \in [1, q^*)$. Therefore,

$$0 = \lim_{k\to\infty} \int_{B_M} |u_k|^p dx \ge \limsup_{k\to\infty} \int_{B_R(y_k)} |u_k|^p dx > 0,$$

which gives the required contradiction. In conclusion, $(y_k)_k$ is not bounded in $\mathbb{R}^N$. □

All the results proved up to now in the chapter continue to be valid when f is a Carathéodory function and satisfies (F) when we request (1.2) in the form

$$\big|f(x,t)\big| \le q\varepsilon|t|^{q-1} + mC_\varepsilon|t|^{m-1} \quad \text{for a. e. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R},$$

with $q < m < q^*$. In the last part of Section 1.3 we need that f does not depend on x. Let us now conclude the chapter with the proof of Theorem 1.1.1 based on Proposition 1.3.1.

Proof of Theorem 1.1.1. By Lemmas 1.2.2–1.2.4, for any $\lambda > 0$ the functional I has the geometry of the mountain pass lemma, so that it admits a $(PS)_{c_\lambda}$ sequence $(u_k)_k$ of I, which, up to a subsequence, weakly converges to the limit $u_\lambda \in W$. The weak limit $u_\lambda \in W$ is a critical point of I for all $\lambda \ge \lambda^*$, with $\lambda^* > 0$, as asserted in Lemma 1.2.4(ii).

Assume by contradiction that $u_\lambda = 0$. Of course, $(u_k)_k$ cannot converge strongly to 0 in W, since otherwise $I'(u_\lambda) = 0$ and $0 = I(u_\lambda) = c_\lambda > 0$ by Lemma 1.2.2. Therefore, Proposition 1.3.1 implies that there exist $R > 0$ and a sequence $(y_k)_k \subset \mathbb{R}^N$ such that

$$\limsup_{k\to\infty} \int_{B_R(y_k)} |u_k|^p dx > 0. \tag{1.39}$$

Now, the new sequence $(\tilde{u}_k)_k$, with $\tilde{u}_k = u_k(\cdot + y_k)$, is again a $(PS)_{c_\lambda}$ sequence of I, since $I(\tilde{u}_k) = I(u_k)$ and, moreover, $I'(\tilde{u}_k) \to 0$ as $k \to \infty$ in W'. Indeed, for all $\varphi \in W$, with

$\|\varphi\| = 1$, putting $\varphi(z - y_k) = \varphi_k(z)$, $z \in \mathbb{R}^N$, we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} A(|\nabla \tilde{u}_k|)\nabla \tilde{u}_k \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} B(|\tilde{u}_k|)\tilde{u}_k \varphi dx - \lambda \int_{\mathbb{R}^N} f(\tilde{u}_k)\varphi dx \\
&\qquad - \frac{1}{q^*}\int_{\mathbb{R}^N} |\tilde{u}_k|^{q^*-2}\tilde{u}_k \varphi dx \Big| \\
&= \Big| \int_{\mathbb{R}^N} A(|\nabla u_k|)\nabla u_k \cdot \nabla \varphi_k dz + \int_{\mathbb{R}^N} B(|u_k|)u_k \varphi_k dz - \lambda \int_{\mathbb{R}^N} f(u_k)\varphi_k dz \\
&\qquad - \frac{1}{q^*}\int_{\mathbb{R}^N} |u_k|^{q^*-2}u_k \varphi_k dz \Big| \\
&= \big|\langle I'(u_k), \varphi_k\rangle\big| \le \|I'(u_k)\|_{W'}\|\varphi_k\| = \|I'(u_k)\|_{W'},
\end{aligned}$$

since $1 = \|\varphi\| = \|\varphi_k\|$. Therefore, as $k \to \infty$,

$$\|I'(\tilde{u}_k)\|_{W'} = \sup_{\substack{\varphi\in W \\ \|\varphi\|=1}} \big|\langle I'(\tilde{u}_k), \varphi\rangle\big| \le \|I'(u_k)\|_{W'} = o(1).$$

Consequently, $(\tilde{u}_k)_k$ weakly converges to some $\tilde{u}_\lambda$ in W by Lemma 1.2.4. Furthermore, by (1.39),

$$0 < \limsup_{k\to\infty} \int_{B_R(y_k)} |u_k|^p dx = \lim_{k\to\infty} \int_{B_R} |\tilde{u}_k|^p dz = \int_{B_R} |\tilde{u}_\lambda|^p dz.$$

Hence, $\tilde{u}_\lambda \neq 0$. Finally, (1.4) follows straight from Lemma 1.2.4(*iii*). This completes the proof. □

Comments on Chapter 1

For the sake of completeness, we point out that the results presented in this chapter could be also investigated for a larger class of elliptic equations where the leading term is governed by some differential operators such as those considered in [28–30]. For instance, in [28] a class of nonautonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been considered. The results contained in [28] are the borderline counterpart of the classical cases valid for functionals with (p, q) growth. However, in order to get Theorem 1.1.1 for this wide class of equations, some different technical approaches need to be developed in suitable Musielak–Orlicz spaces in which a Lions type result as Proposition 1.3.1 can be recovered; see, for instance, the recent paper of F. Colasuonno and M. Squassina [64] in bounded domains. A noteworthy difference with respect to the classical elliptic case given in [134, Theorem 2] is that, in order to handle this kind of problem, some new concentration compactness arguments in weighted Orlicz spaces seem to be essential. This analysis allows proving a variety of existence results which are outside the scope of the book.

2 On (p, N) Laplacian equations in $\mathbb{R}^N$ with exponential nonlinearities

M'illumino
d'immenso

Giuseppe Ungaretti
Mattina

This chapter deals with the existence of nontrivial solutions for (p, N) equations in $\mathbb{R}^N$ with critical exponential growth. The main features and novelty of the chapter are the (p, N) growth of the elliptic operator, combined with the triple lack of compactness, as we shall note below. As explained in Chapter 1, also $(\mathcal{E}_N)$ has a relevant physical interpretation in applied sciences, as well as a mathematical challenge in overcoming the new difficulties intrinsic to $(\mathcal{E}_N)$. Furthermore, equations with Hardy potentials arise from many physical contexts, such as molecular physics, quantum cosmology, and linearization of combustion models. But from the mathematical point of view, the main reason of interest in Hardy potentials lies in their criticality. In other words, the noncompactness of the embedding $D^{1,\wp}(\mathbb{R}^N) \hookrightarrow L^{\wp}(\mathbb{R}^N, |x|^{-\wp}dx)$, $\wp > 1$, even locally in any neighborhood of zero, leads to other difficulties and, more importantly, to a new phenomenon concerning the possibility of a blow-up. Finally, the presence of the Hardy terms and critical nonlinearities, as well as the fact that $(\mathcal{E}_N)$ is studied in the entire space $\mathbb{R}^N$, cause, roughly speaking, a triple loss of compactness which produces new interesting complications. In particular, let u denote the concentration of a chemical substance in (1.5). Then, even if usually the right-hand side of (1.5) has polynomial growth with variable coefficients, in the Liouville–Bratu–Gelfand and Frank–Kamenetsky models the right-hand side of (1.5) has exponential growth at infinity.

2.1 The environment and existence results

In this chapter we study the following equation in $\mathbb{R}^N$:

$$-\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u - \sigma\frac{|u|^{p-2}u}{|x|^p} = \lambda h(x)u_+^{q-1} + \gamma g(x, u), \qquad (\mathcal{E}_N)$$

where $1 < p < N < \infty$, $N \geq 2$, $1 < q < p$, $u_+ = \max\{u, 0\}$, and h is a positive function of class $L^\theta(\mathbb{R}^N)$, with $\theta = N/(N-q)$, while $\lambda > 0$, $\gamma > 0$, and σ is a real parameter. The function g is of exponential type and is assumed to satisfy

(H_1) *g admits partial derivative in u and $\partial_t g$ is a Carathéodory function, with $\partial_t g(\cdot, t) = 0$ for all $t \leq 0$, and such that there exists $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$\partial_t g(x, t)t \leq \varepsilon u^{N-1} + \kappa_\varepsilon(e^{\alpha_0 t^{N'}} - S_{N-2}(\alpha_0, t))$$

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for a. e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0,\infty)$,

$$N' = \frac{N}{N-1} \quad \text{and} \quad S_{N-2}(\alpha_0, t) = \sum_{j=0}^{N-2} \frac{\alpha_0^j t^{jN'}}{j!};$$

(H_2) *There exists a number $\nu > N$ such that $0 < \nu G(x,t) \le tg(x,t)$ for a.e $x \in \mathbb{R}^N$ and any $t \in \mathbb{R}^+$, where $G(x,t) = \int_0^t g(x,\tau)d\tau$ for a.e $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$.*

A canonical prototype of a function verifying assumptions (H_1)–(H_2) is given by $g(t) = t_+(e^{t^2} - 1)$, $t \in \mathbb{R}$, in the simplest case $N = 2$. Indeed, the associated partial derivative $\partial_t g$ of g verifies (H_1) with $\alpha_0 > 1$. Moreover, its primitive $G(t) = (e^{t^2} - 1 - t^2)/2$, $t \in \mathbb{R}_0^+$, and $G(t) = 0$, $u \in \mathbb{R}_0^-$, satisfies (H_2), with $\nu = 4 > 2 = N$. Similarly, in the general case $N > 2$, the example becomes $g(t) = t_+^{N-1}(e^{t_+^{N'}} - S_{N-2}(1,t_+))$. Again $\partial_t g$ verifies (H_1) with $\alpha_0 > 1$, and the primitive G of g satisfies (H_2), with $\nu = 2N$.

Of course, any function $g(x,t) = a(x)\phi(t)$, where ϕ is of the exponential type presented above and $a \in L^\infty(\mathbb{R}^N)$, with $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} a(x) > 0$, continues to satisfy (H_1)–(H_2).

Note that (H_1) implies a similar exponential growth condition on g. Indeed, fix $\varepsilon > 0$. Then by (H_1) there exists $\kappa_\varepsilon > 0$ such that for a. e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}_0^+$,

$$\begin{aligned} g(x,t) &= \int_0^1 \frac{d}{d\tau} g(x,\tau t)d\tau = \int_0^1 \frac{1}{\tau}\partial_t g(x,\tau t)\tau t d\tau \\ &\le \varepsilon u^{N-1} + \kappa_\varepsilon(e^{\alpha_0 t^{N'}} - S_{N-2}(\alpha_0, t)), \end{aligned} \tag{2.1}$$

as stated.

The natural space for finding solutions of $(\mathcal{E}_N)$ is

$$W = W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,N}},$$

where $\|u\|_{W^{1,\wp}} = (\|u\|_\wp^\wp + \|\nabla u\|_\wp^\wp)^{1/\wp}$ for all $u \in W^{1,\wp}(\mathbb{R}^N)$ and $\|\cdot\|_\wp$ denotes the canonical $L^\wp(\mathbb{R}^N)$ norm for any $\wp > 1$. Furthermore, we also set

$$\|u\|_{q,h}^q = \int_{\mathbb{R}^N} h(x)|u|^q dx \quad \text{for all } u \in L^N(\mathbb{R}^N).$$

A crucial role is also played by the best Hardy constant $\mathcal{H}_p = [(N-p)/p]^p$ in $W^{1,p}(\mathbb{R}^N)$, given by

$$\mathcal{H}_p = \inf_{\substack{u\in W^{1,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{\|\nabla u\|_p^p}{\|u\|_{\mathcal{H}_p}^p}, \quad \|u\|_{\mathcal{H}_p} = \left(\int_{\mathbb{R}^N} |u(x)|^p \frac{dx}{|x|^p}\right)^{1/p}.$$

For a proof, we refer to Lemma 2.1 of [112].

In the literature, there are very few contributions devoted to the study of exponential nonlinear problems driven by operators with nonstandard growth conditions. A (p, N) equation similar to $(\mathcal{E}_N)$ first appeared in [255], but set on a bounded domain Ω and with g exactly equal to an exponential function. The authors of [255] were able to get an existence result via a suitable minimax argument, which strongly relies on the requirement that Ω is bounded. More recently, the existence of one solution for critical exponential problems, set on bounded domains Ω and driven by a general (p, N) operator, was given in [97], via the Nehari manifold approach. Finally, in [102] the existence and multiplicity results for equation $(\mathcal{E}_N)$ were proved in the case $\sigma = 0$.

For equations in the entire space, involving elliptic operators with standard N-growth, as well as critical Trudinger–Moser nonlinearities, we refer to [5–7, 82] for the existence results, to [3, 4, 75, 83] for the multiplicity results, and to the references therein. In the nonlocal fractional framework, we refer instead to the very recent papers [178, 251]. Singular equations have been also studied in [116, 186] and the references therein.

Let us first prove the existence of a nontrivial nonnegative solution for $(\mathcal{E}_N)$.

Theorem 2.1.1. *Let $1 < p < N < \infty$ and $1 < q < N$. Let h be a positive function in $L^\theta(\mathbb{R}^N)$, with $\theta = N/(N-q)$. Suppose that g verifies (H_1)–(H_2). Then, for any $\sigma \in (-\infty, \mathcal{H}_p)$ there exists $\widetilde{\lambda} = \widetilde{\lambda}(\sigma) > 0$, independent of $\gamma \in (0,1]$, such that for all $\lambda \in (0, \widetilde{\lambda})$ there exists $\gamma^* = \gamma^*(\sigma, \lambda) \in (0,1]$ with the property that $(\mathcal{E}_N)$ admits at least one nontrivial nonnegative solution $u_{\sigma,\lambda,\gamma}$ in W for all $\gamma \in (0, \gamma^*)$. Moreover,*

$$\lim_{\lambda \to 0^+} \|u_{\sigma,\lambda,\gamma}\| = 0 \tag{2.2}$$

holds true.

The proof of Theorem 2.1.1 is based on an application of the Ekeland variational principle. Theorem 2.1.1 somehow extends in several directions Theorem 1.1 of [5], Theorem 1.4 of [72], Theorem 1 of [82], Theorem 1.2 of [97], Theorem 1.2 of [255] and Theorem 1.1 of [102]. For the multiplicity results when $\sigma = 0$, we refer to [102].

2.2 Sobolev space framework and preliminary lemmas

In this section we briefly recall the variational setting for equation $(\mathcal{E}_N)$ and the technical lemmas for the separable reflexive real Banach space W, which we use throughout the chapter.

We say that $u \in W$ is a (weak) *solution* of problem $(\mathcal{E}_N)$ if

$$\begin{aligned}\int_{\mathbb{R}^N} \{(|\nabla u|^{p-2} + |\nabla u|^{N-2})\nabla u \cdot \nabla\varphi + (|u|^{p-2} + |u|^{N-2})u\varphi\}dx - \sigma \int_{\mathbb{R}^N} |u|^{p-2}u\varphi \frac{dx}{|x|^p} \\ = \lambda \int_{\mathbb{R}^N} h(x)u_+^{q-1}\varphi dx + \gamma \int_{\mathbb{R}^N} g(x,u)\varphi dx\end{aligned}$$

for any $\varphi \in W$.

Clearly, the (weak) solutions of $(\mathcal{E}_N)$ are exactly the critical points of the Euler–Lagrange functional $I = I_{\sigma,\lambda}$ associated to $(\mathcal{E}_N)$, where $I : W \to \mathbb{R}$ is given by

$$I(u) = \frac{1}{p}\|u\|^p_{W^{1,p}} + \frac{1}{N}\|u\|^N_{W^{1,N}} - \frac{\sigma}{p}\|u\|^p_{\mathcal{H}_p} - \lambda\|u_+\|^q_{q,h} - \gamma\int_{\mathbb{R}^N} G(x,u)dx.$$

The functional I is well defined and of class $C^1(W)$ by the structural assumption (H_1).

Indeed, the model function $g(t) = t_+^{N-1}(e^{t_+^{N'}} - S_{N-2}(1,t_+))$ clearly satisfies (H_1), with $\alpha_0 > 1$, and so (2.1). In order to show the validity of (H_2), let us consider the function

$$\Phi(t) = G(t) - \frac{1}{2N}tg(t), \quad t \in \mathbb{R}_0^+,$$

where, by integration by parts, the primitive G of the model function is given by

$$G(t) = \frac{t^{2N}}{2N}\sum_{k=N-1}\frac{t^{kN'-N}}{k!} - \int_0^t \frac{v^{2N}}{2N}\bigg(\sum_{k=N-1}\frac{v^{kN'-N}}{k!}\bigg)' dv.$$

Thus, by direct derivation in $\mathbb{R}^+$,

$$\begin{aligned}\Phi'(t) = t^{2N-1}\sum_{k=N-1}\frac{t^{kN'-N}}{k!} + \frac{t^{2N}}{2N}\bigg(\sum_{k=N-1}\frac{t^{kN'-N}}{k!}\bigg)' - \frac{t^{2N}}{2N}\bigg(\sum_{k=N-1}\frac{t^{kN'-N}}{k!}\bigg)'\\ - t^{2N-1}\sum_{k=N-1}\frac{t^{kN'-N}}{k!} - \frac{t^{2N}}{2N}\bigg(\sum_{k=N-1}\frac{t^{kN'-N}}{k!}\bigg)'\\ = -\frac{t^{2N}}{2N}\bigg(\sum_{k=N-1}\frac{t^{kN'-N}}{k!}\bigg)' < 0,\end{aligned}$$

which implies that $\Phi(t) \le 0$ in $\mathbb{R}_0^+$, since $\Phi(0) = 0$. This proves (H_2), with $\nu = 2N$.

By the classical results in Sobolev space theory, we have the following first embedding.

Lemma 2.2.1. *The embedding $W \hookrightarrow L^{\wp}(\mathbb{R}^N)$ is continuous for all $\wp \in [p,p^*] \cup [N,\infty)$, and*

$$\|u\|_{\wp} \le c_{\wp}\|u\| \quad \textit{for all } u \in W,$$

where $c_{\wp}$ depends on $\wp$, p, and N.

Clearly, $p^* > N$, whenever $(N/2) < p < N$. By Proposition A.6 of [24], we know that the Banach space $L^q(\mathbb{R}^N,h) = (L^q(\mathbb{R}^N,h), \|\cdot\|_{q,h})$ is uniformly convex. Furthermore, combining some ideas of Lemma 2.3 of [24], Lemma 2.2 of [25], Theorem 2.1 of [250], and Lemma 2.1 of [52], in Lemma 2.2 of [102] the next technical result was proved. For the sake of completeness, we report the proof.

Lemma 2.2.2. *The embedding $L^N(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, h)$ is continuous and*

$$\|u\|_{q,h} \leq \|h\|_\theta^{1/q} \|u\|_N \quad \textit{for all } u \in L^N(\mathbb{R}^N).$$

Furthermore, the embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, h)$ is compact.

Proof. By the Hölder inequality, for all $u \in L^N(\mathbb{R}^N)$,

$$\|u\|_{q,h} \leq \Big(\int_{\mathbb{R}^N} h^\theta(x)dx \Big)^{1/\theta q} \cdot \Big(\int_{\mathbb{R}^N} |u|^N dx \Big)^{1/N},$$

that is, the first embedding holds.

To prove the second part of the lemma, we need to show that if $u_k \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$, then $\|u_k - u\|_{q,h} \to 0$ as $k \to \infty$. Thanks to the Hölder inequality,

$$\int_{\mathbb{R}^N \setminus B_R} h(x)|u_k - u|^q dx \leq M \Big(\int_{\mathbb{R}^N \setminus B_R} h^\theta(x)dx \Big)^{1/\theta} = o(1)$$

as $R \to \infty$, where $h \in L^\theta(\mathbb{R}^N)$ by assumption and $\|u_k - u\|_N^q = M < \infty$ for all $k \in \mathbb{N}$. Hence, for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so large that $\int_{\mathbb{R}^N \setminus B_R} h(x)|u_k - u|^q dx < \varepsilon/2$.

Fix $\varepsilon > 0$ and a subsequence $(u_{k_n})_n \subseteq (u_k)_k$. Since $u_{k_n} \to u$ in $L^\nu(B_{R_\varepsilon})$ for all $\nu \in [1, N)$, we can assume, up to a further subsequence, that $u_{k_n} \to u$ a. e. in B_{R_ε}. Thus $h(x)|u_k - u|^q \to 0$ a. e. in B_{R_ε}. Furthermore, for each measurable subset $E \subseteq B_{R_\varepsilon}$, by the Hölder inequality we have

$$\int_E h(x)|u_{k_n} - u|^q dx \leq M \Big(\int_E h^\theta(x)dx \Big)^{1/\theta}.$$

Hence, we obtain that $(h(x)|u_{k_n} - u|^q)_n$ is equiintegrable and uniformly bounded in $L^1(B_{R_\varepsilon})$, since $h \in L^\theta(\mathbb{R}^N)$. Then, the Vitali convergence theorem implies

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon}} h(x)|u_{k_n} - u|^q dx = 0,$$

and so $u_k \to u$ in $L^q(B_{R_\varepsilon}, h)$, since the sequence $(u_{k_n})_n$ is arbitrary. Consequently, there exists $k_0 \in \mathbb{N}$ such that $\int_{B_{R_\varepsilon}} h(x)|u_k - u|^q dx < \varepsilon/2$ for all $k \geq k_0$. In conclusion, for all $k \geq k_0$,

$$\|u_k - u\|_{q,h}^q = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} h(x)|u_k - u|^q dx + \int_{B_{R_\varepsilon}} h(x)|u_k - u|^q dx < \varepsilon,$$

as required. □

We end the section by stating Lemma 2.4 of [3] in the form we use several times in what follows.

Lemma 2.2.3. *Let $(u_k)_k$ be a sequence in $W^{1,N}(\mathbb{R}^N)$ such that*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^{N'} < \frac{\alpha_N}{2\alpha_0},$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the $(N-1)$-dimensional measure of the unit sphere $\mathbb{S}^{N-1}$ of $\mathbb{R}^N$. Then for all $m \in (\alpha_N/2\alpha_0, \alpha_N/\alpha_0)$ and all $\alpha > \alpha_0$ and $l > 1$ so small that $l\alpha m < \alpha_N$ it results that

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^N} \left(e^{\alpha|u_k|^{N'}} - S_{N-2}(\alpha, u_k)\right)^l dx = C_m,$$

where $C_m = C_m(m, \alpha, l)$ is a nonnegative number.

2.3 Proof of Theorem 2.1.1

In this section, for simplicity we assume, without further mentioning, that the structural assumptions required in Theorem 2.1.1 hold.

We start proving the geometric properties of the functional I, necessary to apply both a minimization argument and the mountain pass lemma.

Lemma 2.3.1. *Any solution $u \in W$ of $(\mathcal{E}_N)$ is nonnegative in $\mathbb{R}^N$ for all $\sigma < \mathcal{H}_p$, all $\lambda > 0$ and $\gamma > 0$.*

For any fixed $\sigma < \mathcal{H}_p$ there exists $\rho \in (0, 1]$ and two positive numbers λ_* and $\jmath$, depending on ρ, such that $I(u) \geq \jmath$ for all $u \in W$, with $\|u\| = \rho$, and for all $\lambda \in (0, \lambda_*]$ and $\gamma > 0$.

Furthermore, for all $\sigma < \mathcal{H}_p$, $\lambda \in (0, \lambda_*]$ and all $\gamma \in (0, 1]$, there exist in $\overline{B}_\rho$, where $B_\rho = \{u \in W : \|u\| < \rho\}$, a sequence $(u_k)_k$ of nonnegative functions and some nonnegative function $u_{\sigma,\lambda,\gamma}$ such that for all $k \in \mathbb{N}$,

$$\begin{gathered} \|u_k\| < \rho, \qquad m_{\sigma,\lambda,\gamma} \leq I(u_k) \leq m_{\sigma,\lambda,\gamma} + \frac{1}{k}, \\ u_k \rightharpoonup u_{\sigma,\lambda,\gamma} \text{ in } W, \quad u_k \to u_{\sigma,\lambda,\gamma} \text{ a.e. in } \mathbb{R}^N \quad \text{and} \quad I'(u_k) \to 0 \end{gathered} \tag{2.3}$$

as $k \to \infty$, where

$$m_{\sigma,\lambda,\gamma} = \inf\{I(u) : u \in \overline{B}_\rho\} < 0.$$

Proof. Let $\sigma < \mathcal{H}_p$, $\lambda > 0$ and $\gamma > 0$ be fixed and let u be any solution of $(\mathcal{E}_N)$ in W. Putting $u = u_+ - u_-$, we have that both u_+ and u_- are in W and that

$$\begin{aligned} \int_{\mathbb{R}^N} \{(|\nabla u|^{p-2} + |\nabla u|^{N-2})\nabla u \cdot \nabla u_- + (|u|^{p-2} + |u|^{N-2})uu_-\}dx \\ = -\|u_-\|_{W^{1,p}}^p - \|u_-\|_{W^{1,N}}^N. \end{aligned}$$

Thus, by the definition of solution for $(\mathcal{E}_N)$ and (H_1), we get

$$-\|u_-\|^p_{W^{1,p}} - \|u_-\|^N_{W^{1,N}} = -\sigma\|u_-\|^p_{\mathcal{H}_p} \geq -\frac{\sigma_+}{\mathcal{H}_p}\|u_-\|^p_{W^{1,p}} \geq -\|u_-\|^p_{W^{1,p}},$$

taking as a test function $\varphi = u_- \in W$. Hence, $u_- = 0$ a. e. in $\mathbb{R}^N$. Thus u is nonnegative in $\mathbb{R}^N$, as stated.

Fix $\varepsilon > 0$. Then there exists $\kappa_\varepsilon > 0$ by (2.1) such that as $t \to 0^+$,

$$t^{1-N} g(x,t) \leq \varepsilon + \kappa_\varepsilon \frac{\alpha_0^{N-1}}{(N-1)!} t + o(t).$$

In other words, $\limsup_{t\to 0^+} t^{1-N} g(x,t) \leq \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, this implies at once that

$$\lim_{t\to 0^+} t^{1-N} g(x,t) = 0 \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{2.4}$$

For the second part of the lemma, fix $\varepsilon > 0$. Thus by (2.4), there exists $\delta = \delta(\varepsilon) > 0$ such that

$$G(x,t) \leq \frac{\varepsilon}{N} t^N \quad \text{for a. e. } x \in \mathbb{R}^N \text{ and all } t \in [0,\delta]. \tag{2.5}$$

Take now $s \geq 1$ and $\alpha > \alpha_0$. Then by (2.1), there exists $\tilde{\kappa}_\varepsilon = \tilde{\kappa}(\alpha_0, s, \varepsilon) > 0$ such that for a. e. $x \in \mathbb{R}^N$ and all $t \in [\delta, \infty)$,

$$G(x,t) \leq \tilde{\kappa}_\varepsilon t^s (e^{\alpha t^{N'}} - S_{N-2}(\alpha, t)). \tag{2.6}$$

In conclusion, (2.5) and (2.6) yield

$$G(x,t) \leq \frac{\varepsilon}{N} t^N + \tilde{\kappa}_\varepsilon t^s (e^{\alpha t^{N'}} - S_{N-2}(\alpha, t)) \tag{2.7}$$

for a. e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}_0^+$ Furthermore, for any $a < \alpha_N$, where α_N is defined in Lemma 2.2.3, then Lemma 1 of [82] gives the existence of a constant $C_a = C_a(a, N, 1)$ such that

$$\int_{\mathbb{R}^N} (e^{a|v|^{N'}} - S_{N-2}(a, v))dx \leq C_a \quad \text{for all } v \in W, \text{ with } \|v\| \leq 1. \tag{2.8}$$

Following somehow [75], for $\wp > 1$, we apply the Hölder inequality and find

$$\begin{aligned}\mathcal{I}_\alpha &= \int_{\mathbb{R}^N} |u|^s (e^{\alpha|u|^{N'}} - S_{N-2}(\alpha, u))dx \\ &\leq \|u\|^s_{s\wp} \Big(\int_{\mathbb{R}^N} (e^{\alpha|u|^{N'}} - S_{N-2}(\alpha, u))^{\wp'} dx \Big)^{1/\wp'}.\end{aligned}$$

For $\beta > \wp'$, Lemma 2.3 of [4] yields for all $u \in W$, with $0 < \|u\| \le \delta$, that

$$\begin{aligned}\mathcal{I} = \int_{\mathbb{R}^N} (e^{\alpha|u|^{N'}} - S_{N-2}(\alpha,u))^{\wp'} dx &\le \int_{\mathbb{R}^N} (e^{\beta\alpha|u|^{N'}} - S_{N-2}(\beta\alpha,u))dx \\ &\le \int_{\mathbb{R}^N} (e^{\beta\alpha r^{N'}(|u|/\|u\|)^{N'}} - S_{N-2}(\beta\alpha r^{N'}, |u|/\|u\|))dx.\end{aligned}$$

We choose $\delta \in (0,1]$ so small that $\beta\alpha\delta^{N'} < \alpha_N$ and, applying (2.8) with $a = \beta\alpha\delta^{N'}$, we get

$$\mathcal{I}_\alpha \le \|u\|^s_{s\wp} \mathcal{I}^{1/\wp'} \le C_a^{1/\wp'} \|u\|^s_{s\wp} \quad \text{for all } u \in W, \text{ with } \|u\| \le \delta. \tag{2.9}$$

Fix $\sigma < \mathcal{H}_p$, $\lambda > 0$, and $\gamma \in (0,1]$. Taking $s > N$ for later purposes, by Lemma 2.2.1, (2.7), and (2.9), we obtain

$$\begin{aligned}I(u) &\ge \frac{1}{p}\left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right)\|u\|^p_{W^{1,p}} + \frac{1}{N}\|u\|^N_{W^{1,N}} - \frac{\lambda}{q}\|h\|_\theta \|u_+\|^q_N \\ &\qquad - \gamma\frac{\varepsilon}{N}\|u\|^N_N - \gamma\tilde{\kappa}_\varepsilon C_a^{1/\wp'}\|u\|^s_{s\wp} \\ &\ge \frac{1}{N}\left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right)(\|u\|^N_{W^{1,p}} + \|u\|^N_{W^{1,N}}) - \frac{\lambda}{q}\|h\|_\theta\|u_+\|^q_N \\ &\qquad - \frac{\varepsilon}{N}\|u\|^N_N - \tilde{\kappa}_\varepsilon C_a^{1/\wp'}\|u\|^s_{s\wp} \\ &\ge \frac{1}{N}\left[2^{1-N}\left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right) - \varepsilon\right]\|u\|^N - \frac{\lambda}{q}\|h\|_\theta\|u\|^q - C_\wp\|u\|^s,\end{aligned}$$

for all $u \in W$, with $\|u\| \le \delta$, where $C_\wp = \tilde{\kappa}_\varepsilon C_a^{1/\wp'} c^s_{s\wp}$ and $c_{s\wp}$ is given in Lemma 2.2.1, when $\mathfrak{p} = s\wp$. Choose $\varepsilon = 2^{-N}(1 - \sigma_+/\mathcal{H}_p)$ and consider the function

$$\psi(\tau) = \frac{1 - \sigma_+/\mathcal{H}_p}{2^{1+N}N}\tau^N - C_\wp\tau^s, \quad \tau \in [0,\delta].$$

Then ψ admits a positive maximum $\jmath$ in $[0,\delta]$ at a point $\rho \in (0,\delta]$, since $s > N$ and $\delta \le 1$. Consequently, for all $u \in W$, with $\|u\| = \rho$, we obtain

$$I(u) \ge \frac{1 - \sigma_+/\mathcal{H}_p}{2^N N}\rho^N - \frac{\lambda}{q}\|h\|_\theta\rho^q - C_\gamma\rho^s \ge \psi(\rho) = \jmath > 0,$$
$$\text{for all } \lambda \in (0,\lambda_*], \text{ with } \lambda_* = \frac{q(1 - \sigma_+/\mathcal{H}_p)}{2^{1+N}N\|h\|_\theta}\rho^{N-q},$$

as stated.

Fix $\lambda \in (0,\lambda_*]$ and a nonnegative function $u \in C_0^\infty(\mathbb{R}^N)$, with $\|u\| = 1$. Thus

$$I(\tau u) \le \frac{1}{p}\tau^N + \frac{\sigma_-}{p}\|u\|^p_{\mathcal{H}_p}\tau^p - \frac{\lambda}{q}\tau^q\|u\|^q_{q,h} < 0$$

for all $\tau \in (0,1]$ sufficiently small, since $1 < q < p < N$. Hence,

$$m_{\sigma,\lambda,\gamma} = \inf\{I(u) : u \in \overline{B}_\rho\} < 0.$$

Then, by the Ekeland variational principle in $\overline{B}_\rho$ and the first part of the lemma, there exists a sequence $(u_k)_k \subset B_\rho$ such that

$$m_{\sigma,\lambda,\gamma} \le I(u_k) \le m_{\sigma,\lambda,\gamma} + \frac{1}{k} \quad \text{and} \quad I(u) \ge I(u_k) - \frac{1}{k}\|u - u_k\| \tag{2.10}$$

for all $k \in \mathbb{N}$ and for any $u \in \overline{B}_\rho$. Fixing $k \in \mathbb{N}$, for all $w \in S_W$, where $S_W = \{u \in W : \|u\| = 1\}$, and for all $\tau > 0$ so small that $u_k + \tau w \in \overline{B}_\rho$, we have

$$I(u_k + \tau w) - I(u_k) \ge -\frac{\tau}{k}$$

by (2.10). Since I is Gâteaux differentiable in W, we get

$$\langle I'(u_k), w\rangle = \lim_{\tau\to 0} \frac{I(u_k + \tau w) - I(u_k)}{\tau} \ge -\frac{1}{k}$$

for all $w \in S_W$. Consequently, $|\langle I'(u_k), w\rangle| \le 1/k$, since $w \in S_W$ is arbitrary. Therefore, $I'(u_k) \to 0$ in W' as $k \to \infty$ and, clearly, up to a subsequence, the bounded sequence $(u_k)_k$ weakly converges to some $u_{\sigma,\lambda,\gamma} \in \overline{B}_\rho$ and $u_k \to u_{\sigma,\lambda,\gamma}$ a. e. in $\mathbb{R}^N$. Furthermore, we assume w. l. o. g. that $(u_{k,-})_k$ weakly converges to $u_{\sigma,\lambda,-} \in \overline{B}_\rho$ in W and $u_{k,-} \to u_{\sigma,\lambda,-}$ a. e. in $\mathbb{R}^N$, since $u_k \to u_{\sigma,\lambda,\gamma}$ a. e. in $\mathbb{R}^N$ implies at once that $u_{k,+} \to u_{\lambda,\gamma,+}$ and $u_{k,-} \to u_{\lambda,\gamma,-}$ a. e. in $\mathbb{R}^N$. Moreover, as $k \to \infty$,

$$\begin{aligned} o(1) = -\langle I'(u_k), u_{k,-}\rangle = -\int_{\mathbb{R}^N} \{&(|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})\nabla u_k \cdot \nabla u_{k,-} \\ &+ (|u_k|^{p-2} + |u_k|^{N-2})u_k u_{k,-}\}dx - \sigma\|u_{k,-}\|^p_{\mathcal{H}_p} \\ \ge \left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right)&\|u_{k,-}\|^p_{W^{1,p}} + \|u_{k,-}\|^N_{W^{1,N}}. \end{aligned}$$

Therefore, $(u_{k,-})_k$ strongly converges to 0 in W and $u_{k,-} \to 0$ a. e. in $\mathbb{R}^N$. Thus $u_{\sigma,\lambda,-} = 0$ a. e. in $\mathbb{R}^N$. In particular, $u_{\sigma,\lambda,\gamma} \ge 0$ in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume that $u_k = u_{k,+}$, since $u_{k,-} \to 0$ in W. This completes the proof of (2.3). □

Lemma 2.3.2. *The weak limit $u = u_{\sigma,\lambda,\gamma}$ of the sequence constructed in Lemma* 2.3.1 *is a solution of $(\mathcal{E}_N)$ provided that $\sigma < \mathcal{H}_p$, $\gamma \in (0,1]$, and $\lambda \in (0,\widetilde{\lambda})$, where $\widetilde{\lambda} = \min\{\lambda_*, \lambda_0\}$, λ_* is given in Lemma* 2.3.1, *and λ_0 is well defined by*

$$\lambda_0 = \frac{(\nu - N)q}{N\|h\|_\theta(\nu - q)}\left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{(N-q)/N'} > 0, \tag{2.11}$$

where $1 < q < N < \nu$ by (H_2).

Proof. Fix $\sigma < \mathcal{H}_p$, $\gamma \in (0,1]$, and $\lambda \in (0, \widetilde{\lambda})$, as in the statement.

Lemma 2.3.1 gives the existence of the sequence $(u_k)_k$ of nonnegative functions in $\overline{B}_\rho$ and of a function $u_{\sigma,\lambda,\gamma}$, which for brevity will be denoted simply by u, unless otherwise specified, satisfying (2.3). Consequently,

$$\begin{gathered} u_k \rightharpoonup u \text{ in } L^p(\mathbb{R}^N, |x|^{-p}), \\ u_k \to u \text{ in } L^{\wp}_{\text{loc}}(\mathbb{R}^N), \quad \wp \in [1,\infty), \\ u_k \le g_R \text{ a. e. in } \mathbb{R}^N, \text{ for some } g_R \in L^N(B_R) \text{ and all } R > 0, \end{gathered} \tag{2.12}$$

hold. Take $R > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^N$, $\varphi \equiv 1$ in B_R and $\varphi \equiv 0$ in B_{2R}^c. By convexity, we have

$$\left(\left|\nabla u_k(x)\right|^{p-2}\nabla u_k(x) - \left|\nabla u(x)\right|^{p-2}\nabla u(x)\right) \cdot \left(\nabla u_k(x) - \nabla u(x)\right) \ge 0 \quad \text{a. e. in } \mathbb{R}^N$$

for any $k \in \mathbb{N}$. Thus, the well known Simon inequality, see Lemma 2.1 of [231], with $N \ge 2$, yields the existence of $c_N > 0$ such that

$$\begin{aligned} c_N \int_{B_R} |\nabla u_k - \nabla u|^N dx &\le \int_{B_R} \left(|\nabla u_k|^{N-2}\nabla u_k - |\nabla u|^{N-2}\nabla u\right) \cdot (\nabla u_k - \nabla u) dx \\ &\le \int_{B_R} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right) \cdot (\nabla u_k - \nabla u) dx \\ &\quad + \int_{B_R} \left(|\nabla u_k|^{N-2}\nabla u_k - |\nabla u|^{N-2}\nabla u\right) \cdot (\nabla u_k - \nabla u) dx \\ &\le \int_{\mathbb{R}^N} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right) \cdot (\nabla u_k - \nabla u)\varphi dx \\ &\quad + \int_{\mathbb{R}^N} \left(|\nabla u_k|^{N-2}\nabla u_k - |\nabla u|^{N-2}\nabla u\right) \cdot (\nabla u_k - \nabla u)\varphi dx. \end{aligned}$$

Therefore, as $k \to \infty$,

$$\begin{aligned} c_N \int_{B_R} |\nabla u_k - \nabla u|^N dx \le & \int_{\mathbb{R}^N} \varphi\left(|\nabla u_k|^p + |\nabla u_k|^N\right) dx \\ & - \int_{\mathbb{R}^N} \varphi\left(|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2}\right)\nabla u_k \cdot \nabla u dx + o(1), \end{aligned} \tag{2.13}$$

since $u_k \rightharpoonup u$ in W. Clearly, (2.3) gives

$$\langle I'(u_k), \varphi u_k \rangle - \langle I'(u_k), \varphi u \rangle = o(1) \quad \text{as } k \to \infty,$$

where

$$\int_{\mathbb{R}^N} \varphi\left(|\nabla u_k|^p + |\nabla u_k|^N\right) dx - \int_{\mathbb{R}^N} \varphi\left(|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2}\right)\nabla u_k \cdot \nabla u \, dx$$

$$
\begin{aligned}
&= \langle I'(u_k), \varphi u_k\rangle - \langle I'(u_k), \varphi u\rangle \\
&\quad - \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})(u_k - u)\nabla u_k \cdot \nabla\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \varphi(u_k^{p-1} + u_k^{N-1})(u_k - u)dx + \sigma \int_{\mathbb{R}^N} \varphi u_k^{p-1}(u_k - u)\frac{dx}{|x|^p} \\
&\quad + \lambda \int_{\mathbb{R}^N} \varphi h(x) u_k^{q-1}(u_k - u)dx + \gamma \int_{\mathbb{R}^N} \varphi g(x, u_k)(u_k - u)dx.
\end{aligned} \tag{2.14}
$$

By the Hölder inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})(u_k - u)\nabla u_k \cdot \nabla\varphi dx\right| \\
&\qquad \le \|\nabla\varphi\|_\infty \left\{\|\nabla u_k\|_p^{p-1}\left(\int_{B_{2R}} |u_k - u|^p dx\right)^{1/p}\right. \\
&\qquad\qquad \left. + \|\nabla u_k\|_N^{N-1}\left(\int_{B_{2R}} |u_k - u|^N dx\right)^{1/N}\right\},
\end{aligned}
$$

which yields by (2.12) that

$$
\lim_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})(u_k - u)\nabla u_k \cdot \nabla\varphi dx = 0. \tag{2.15}
$$

Similarly, again by the Hölder inequality,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N} \varphi(u_k^{p-1} + u_k^{N-1})(u_k - u)dx\right| &\le \|u_k\|_p^{p-1}\left(\int_{B_{2R}} |u_k - u|^p dx\right)^{1/p} \\
&\quad + \|u_k\|_N^{N-1}\left(\int_{B_{2R}} |u_k - u|^N dx\right)^{1/N},
\end{aligned}
$$

which implies that

$$
\lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi(u_k^{p-1} + u_k^{N-1})(u_k - u)dx = 0. \tag{2.16}
$$

Furthermore, taking $\wp$, with $1 < p < \wp < N$, and applying the Hölder inequality, with p', $\wp$ and $\mathfrak{q} = p\wp/(\wp - p)$, we find

$$
\left|\int_{\mathbb{R}^N} \varphi u_k^{p-1}(u_k - u)\frac{dx}{|x|^p}\right| \le \big\|1/|x|\big\|_{L^\wp(B_{2R})} \|u_k\|_H^{p-1}\left(\int_{B_{2R}} |u_k - u|^{\mathfrak{q}} dx\right)^{1/\mathfrak{q}}.
$$

Thus, (2.12) gives

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi u_k^{p-1}(u_k - u)\frac{dx}{|x|^p} = 0. \tag{2.17}$$

Likewise, by the Hölder inequality, as $k \to \infty$,

$$\begin{aligned} \left|\int_{\mathbb{R}^N} \varphi h(x) u_k^{q-1}(u_k - u)dx\right| &\le \|u_k\|_{q,h}^{q-1}\Bigg(\int_{B_{2R}} h(x)|u_k - u|^q dx\Bigg)^{1/q} \\ &\le C_q\|h\|_\theta^{1/q}\Bigg(\int_{B_{2R}} |u_k - u|^N dx\Bigg)^{1/N} \\ &\to 0, \end{aligned} \tag{2.18}$$

where $C_q = \sup_{k\in\mathbb{N}} \|u_k\|_{q,h}^{q-1} < \infty$ by Lemma 2.2.2, since $(u_k)_k$ is bounded in W.

Using the notation of Lemma 2.3.1, thanks to (2.3), (H_2), the fact that $\gamma \in (0, 1]$, and Lemma 2.2.2, we get, as $k \to \infty$,

$$\begin{aligned} 0 > m_{\sigma,\lambda,\gamma} &= I(u_k) - \frac{1}{\nu}\langle I'(u_k), u_k\rangle + o(1) \\ &\ge \left(\frac{1}{p} - \frac{1}{\nu}\right)\left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right)\|u_k\|_{W^{1,p}}^p + \left(\frac{1}{N} - \frac{1}{\nu}\right)\|u_k\|_{W^{1,N}}^N \\ &\qquad - \lambda\left(\frac{1}{q} - \frac{1}{\nu}\right)\|h\|_\theta\|u_k\|_{W^{1,N}}^q + o(1). \end{aligned}$$

Consequently, since $\sigma < \mathcal{H}_p$, as $k \to \infty$,

$$\left(\frac{1}{N} - \frac{1}{\nu}\right)\|u_k\|_{W^{1,N}}^N - \lambda\left(\frac{1}{q} - \frac{1}{\nu}\right)\|h\|_\theta\|u_k\|_{W^{1,N}}^q + o(1) < 0,$$

so that

$$\limsup_{k\to\infty} \|u_k\|_{W^{1,N}}^N \le \left[\frac{\lambda N\|h\|_\theta(\nu - q)}{(\nu - N)q}\right]^{N/(N-q)} < \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1},$$

since $\lambda < \lambda_0$, with λ_0 given in (2.11). Therefore, passing to a subsequence, if necessary, which is still labeled $(u_k)_k$ for simplicity,

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^N < \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1}. \tag{2.19}$$

By (2.19), we fix $m \in (\alpha_N/2\alpha_0, \alpha_N/\alpha_0)$, $\alpha > \alpha_0$, and $l > 1$ close to 1, with $l\alpha m < \alpha_N$. Now, by (2.1) there exists $\tilde\kappa = \tilde\kappa(\alpha_0, 1) > 0$ such that for a. e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}_0^+$,

$$g(x,t) \le t^{N-1} + \tilde\kappa(e^{\alpha t^{N'}} - S_{N-2}(\alpha, t)). \tag{2.20}$$

Hence, the Hölder inequality and (2.20) yield

$$\left|\int\limits_{\mathbb{R}^N} \varphi g(x,u_k)(u_k-u)dx\right| \le \int\limits_{B_{2R}} [u_k^{N-1} + \tilde{\kappa}(e^{\alpha u_k^{N'}} - S_{N-2}(\alpha,u_k))](u_k-u)dx$$
$$\le \|u_k\|_N^{N-1}\Big(\int\limits_{B_{2R}} |u_k-u|^N dx\Big)^{1/N}$$
$$+ \tilde{\kappa}\Big(\int\limits_{\mathbb{R}^N} (e^{\alpha u_k^{N'}} - S_{N-2}(\alpha,u_k))^l dx\Big)^{1/l}\Big(\int\limits_{B_{2R}} |u_k-u|^{l'} dx\Big)^{1/l'}.$$

Thus, for all $k \in \mathbb{N}$,

$$\left|\int\limits_{\mathbb{R}^N} \varphi g(x,u_k)(u_k-u)dx\right| \le d_1\Big(\int\limits_{B_{2R}} |u_k-u|^N dx\Big)^{1/N} + d_2\Big(\int\limits_{B_{2R}} |u_k-u|^{l'} dx\Big)^{1/l'}, \tag{2.21}$$

with $d_1 = \sup_{k\in\mathbb{N}} \|u_k\|_N^{N-1} < \infty$, since $(u_k)_k$ is bounded in W, while

$$d_2 = \tilde{\kappa} \sup_{k\in\mathbb{N}}\Big(\int\limits_{\mathbb{R}^N} (e^{\alpha u_k^{N'}} - S_{N-2}(\alpha,u_k))^l dx\Big)^{1/l} < \infty$$

by Lemma 2.2.3, thanks to the choices of the exponents α and l.

Thus, combining (2.13)–(2.18) and (2.21), we obtain

$$c_N \int\limits_{B_R} |\nabla u_k - \nabla u|^N dx \le o(1) \quad \text{as } k \to \infty.$$

Consequently, $\nabla u_k \to \nabla u$ in $[L^N(B_R)]^N$ for all $R > 0$. Therefore, up to subsequence, not relabeled, we get that

$$\nabla u_k \to \nabla u \quad \text{a. e. in } \mathbb{R}^N, \tag{2.22}$$

and for all $R > 0$ there exists a function $h_R \in L^N(B_R)$ such that $|\nabla u_k| \le h_R$ a. e. in B_R and for all $k \in \mathbb{N}$.

Fix ϕ in $C_c^\infty(\mathbb{R}^N)$ and let $R > 0$ be so large that $\operatorname{supp}\phi \subset B_R$. By the above construction, we have a. e. in B_R that

$$\big|(|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})\nabla u_k \cdot \nabla\phi\big| \le (h_R^{p-1} + h_R^{N-1})|\nabla\phi| = \mathfrak{h},$$

where $\mathfrak{h} \in L^1(B_R)$. Therefore, as $k \to \infty$, the dominated convergence theorem gives at once

$$\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})\nabla u_k \cdot \nabla\phi dx = \int_{B_R} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})\nabla u_k \cdot \nabla\phi dx$$
$$\to \int_{\mathbb{R}^N} (|\nabla u|^{p-2} + |\nabla u|^{N-2})\nabla u \cdot \nabla\phi dx.$$

Similarly, using (2.12), we have a. e. in B_R that

$$\left|(u_k^{p-1} + u_k^{N-1})\phi\right| \leq (g_R^{p-1} + g_R^{N-1})|\phi| = \mathfrak{g} \in L^1(B_R),$$

and so the dominated convergence theorem gives, as $k \to \infty$,

$$\int_{\mathbb{R}^N} (u_k^{p-1} + u_k^{N-1})\phi dx \to \int_{\mathbb{R}^N} (u^{p-1} + u^{N-1})\phi dx.$$

Using (2.3), we have a. e. in B_R that

$$\left|\frac{u_k^{p-1}}{|x|^p}\phi\right| \leq g_R^{p-1}\left|\frac{\phi}{|x|^p}\right| = \mathfrak{f} \in L^1(B_R).$$

Therefore, again the dominated convergence theorem yields

$$\int_{\mathbb{R}^N} \frac{u_k^{p-1}}{|x|^p}\phi dx \to \int_{\mathbb{R}^N} \frac{u^{p-1}}{|x|^p}\phi dx,$$

while, since $h \in L^\theta(\mathbb{R}^N)$ and again by (2.3),

$$\left|h(x)\, u_k^{q-1}\phi\right| \leq \mathfrak{G} \in L^1(B_R),$$

so by again the dominated convergence theorem, as $k \to \infty$,

$$\int_{\mathbb{R}^N} h(x)\, u_k^{q-1}\phi dx \to \int_{\mathbb{R}^N} h(x)\, u^{q-1}\phi dx.$$

In the same way, by Lemma 2.2.3, (2.19), (2.20), and (2.3),

$$\left|g(x, u_k)\phi\right| \leq \mathfrak{H} \in L^1(B_R).$$

Thus, the dominated convergence theorem applies and gives, as $k \to \infty$,

$$\int_{\mathbb{R}^N} g(x, u_k)\phi dx \to \int_{\mathbb{R}^N} g(x, u)\phi dx.$$

Finally, since $\langle I'(u_k), \phi\rangle = o(1)$ as $k \to \infty$ by (2.3), we have

$$\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} + |\nabla u_k|^{N-2})\nabla u_k \cdot \nabla \phi dx + \int_{\mathbb{R}^N} (u_k^{p-1} + u_k^{N-1})\phi dx - \sigma \int_{\mathbb{R}^N} \frac{u_k^{p-1}}{|x|^p}\phi dx$$
$$= \lambda \int_{\mathbb{R}^N} h(x)\, u_k^{q-1}\phi dx + \gamma \int_{\mathbb{R}^N} g(x, u_k)\phi dx + o(1).$$

Thus, letting $k \to \infty$, using the above arguments, (2.3), and (2.12), we get at once that

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u|^{p-2} + |\nabla u|^{N-2})\nabla u \cdot \nabla \phi dx \\ &\quad + \int_{\mathbb{R}^N} (u^{p-1} + u^{N-1})\phi dx - \sigma \int_{\mathbb{R}^N} \frac{u^{p-1}}{|x|^p}\phi dx \\ &\quad = \lambda \int_{\mathbb{R}^N} h(x)\, u^{q-1}\phi dx + \gamma \int_{\mathbb{R}^N} g(x, u)\phi dx. \end{aligned} \tag{2.23}$$

for all ϕ in $C_c^\infty(\mathbb{R}^N)$.

Fix φ in W. The sequence $(\phi_k)_k$ in $C_c^\infty(\mathbb{R}^N)$, defined by $\phi_k = \zeta_k(\rho_k * \varphi)$, where $(\rho_k)_k$ is a sequence of mollifiers and $(\zeta_k)_k$ is a sequence of cut-off functions, has the properties that $\phi_k \to \varphi$ in $W = W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and, up to subsequences, $\phi_k \to \varphi$, $\nabla\phi_k \to \nabla\varphi$ a. e. in $\mathbb{R}^N$ as $k \to \infty$, and there exist functions $\psi \in L^p(\mathbb{R}^N)$ and $\tilde{\psi} \in L^N(\mathbb{R}^N)$ such that $|\phi_k| \le \psi$, $|\nabla\phi_k| \le \psi$, and $|\phi_k| \le \tilde{\psi}$, $|\nabla\phi_k| \le \tilde{\psi}$ a. e. in $\mathbb{R}^N$ and for all k. Of course, (2.23) holds along $(\phi_k)_k$ for all $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$ under the sign of integrals, by the dominated convergence theorem, we obtain the validity of (2.23) for all $\varphi \in W$. In conclusion,

$$\langle I'(u), \varphi\rangle = 0 \quad \text{for all } \varphi \in W. \tag{2.24}$$

Hence u is a solution of $(\mathcal{E}_N)$. □

Before completing the proof of Theorem 2.1.1, let us present a lemma of Brézis and Lieb type for exponential nonlinearities, as given for the first time in the original paper [102]. Here, we use assumption (H_1) in its full strength for the first time.

Lemma 2.3.3. *Let $(u_k)_k$ be a sequence in W and let u be in W such that $u_k \rightharpoonup u$ in W, $\|u_k\|_{W^{1,N}} \to \ell_N$, $u_k \to u$ a. e. in $\mathbb{R}^N$, $\nabla u_k \to \nabla u$ a. e. in $\mathbb{R}^N$, and*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^{N'} < \frac{\alpha_N}{2^{N'+1}\alpha_0} \tag{2.25}$$

hold true. Then,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \big|g(x, u_k)u_k - g(x, u - u_k)(u - u_k) - g(x, u)u\big| dx = 0.$$

Proof. Arguing similarly as for (2.7) and using (2.4), for $\varepsilon = 1$, $\alpha > \alpha_0$, and any $s \geq 1$, there exists $\tilde{\kappa} = \tilde{\kappa}(s, \alpha_0, 1) > 0$ such that

$$\begin{aligned} g(x,t) &\leq |t|^{N-1} + \tilde{\kappa}\,|t|^{s-1}(e^{\alpha|t|^{N'}} - S_{N-2}(\alpha, |t|)),\\ \partial_t g(x,t)t &\leq |t|^{N-1} + \tilde{\kappa}\,|t|^{s-1}(e^{\alpha|t|^{N'}} - S_{N-2}(\alpha, |t|)) \end{aligned} \tag{2.26}$$

for a. e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}_0^+$. The validity of (2.26) in the entire $\mathbb{R}$ holds with obvious changes, since $\partial_u g(x,\cdot) = 0$ in $\mathbb{R}_0^-$ for a. e. in $\mathbb{R}^N$. From these inequalities, with $s = 1$, for any a, $b \in \mathbb{R}$,

$$\begin{aligned} |g(x,a+b)(a+b) - g(x,a)a| &= \left|\int_0^1 \frac{d}{d\tau}[g(x,a+\tau b)(a+\tau b)]d\tau\right|\\ &= \left|\int_0^1 [\partial_u g(x,a+\tau b)b(a+\tau b) + g(x,a+\tau b)]d\tau\right|\\ &\leq 2\int_0^1 [|a+\tau b|^{N-1}|b|\\ &\quad + \tilde{\kappa}|b|(e^{\alpha|a+\tau b|^{N'}} - S_{N-2}(\alpha, |a+\tau b|))]d\tau\\ &\leq 2\int_0^1 [2^{N-2}(|a|^{N-1}|b| + \tau^{N-1}|b|^N)\\ &\quad + \tilde{\kappa}|b|(e^{\alpha(|a|+\tau|b|)^{N'}} - S_{N-2}(\alpha, |a| + \tau|b|))]d\tau. \end{aligned}$$

Hence, for any a, $b \in \mathbb{R}$,

$$\begin{aligned} |g(x,a+b)(a+b) - g(x,a)a| &\leq 2^{N-1}(|a|^{N-1}|b| + |b|^N)\\ &\quad + 2\tilde{\kappa}|b|(e^{\alpha(|a|+|b|)^{N'}} - S_{N-2}(\alpha, |a| + |b|)). \end{aligned} \tag{2.27}$$

From the last step, using the Young inequality, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |g(x,a+b)(a+b) - g(x,a)a| &\leq 2^{N-1}(\varepsilon|a|^N + C_\varepsilon|b|^N)\\ &\quad + 2\tilde{\kappa}|b|(e^{\alpha(|a|+|b|)^{N'}} - S_{N-2}(\alpha, |a| + |b|)). \end{aligned}$$

By (2.27), with $a = u_k - u$ and $b = u$, putting $v_k = u_k - u$, we get

$$\begin{aligned} |g(x,v_k+u)(v_k+u) - g(x,v_k)v_k| &\leq 2^{N-1}(|v_k|^{N-1}|u| + |u|^N)\\ &\quad + 2\tilde{\kappa}|u|(e^{\alpha(|v_k|+|u|)^{N'}} - S_{N-2}(\alpha, |v_k| + |u|)). \end{aligned} \tag{2.28}$$

From this fact, setting $\mathfrak{f}_k(x) = |g(x, v_k + u)(v_k + u) - g(x, v_k)v_k - g(x, u)u|$, we easily obtain

$$\mathfrak{f}_k(x) \le 2^{N-1}|v_k|^{N-1}|u| + (2^{N-1} + 1)|u|^N + 2\tilde{\kappa}|u|Q_k + 2\tilde{\kappa}|u|Q, \tag{2.29}$$

where $Q_k = e^{\alpha(|v_k|+|u|)^{N'}} - S_{N-2}(\alpha, |v_k| + |u|)$ and $Q = e^{\alpha|u|^{N'}} - S_{N-2}(\alpha, |u|)$. Of course, $Q_k \to Q$ a. e. in $\mathbb{R}^N$, since $v_k \to 0$ a. e. in $\mathbb{R}^N$.

Now, by the structural assumptions on $(u_k)_k$ and the Brézis and Lieb lemma, we have

$$\|v_k\|^N_{W^{1,N}} = \|u_k\|^N_{W^{1,N}} - \|u\|^N_{W^{1,N}} + o(1) \le \|u_k\|^N_{W^{1,N}} + o(1)$$

as $k \to \infty$. Thus,

$$\limsup_{k\to\infty} \|v_k\|^N_{W^{1,N}} = \lim_{k\to\infty} \|v_k\|^N_{W^{1,N}} \le \ell_N^N \le \sup_{k\in\mathbb{N}} \|u_k\|^N_{W^{1,N}} < \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1}$$

by (2.25). Hence there exists J such that

$$\sup_{k\ge J} \|v_k\|^N_{W^{1,N}} < \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1}.$$

Of course, (2.25) implies at once that

$$\|u\|^N_{W^{1,N}} \le \ell_N^N \le \sup_{k\in\mathbb{N}} \|u_k\|^N_{W^{1,N}} < \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1}.$$

Hence, we have

$$\sup_{k\ge J} \big\||v_k| + |u|\big\|^N_{W^{1,N}} < 2^N \left(\frac{\alpha_N}{2^{N'+1}\alpha_0}\right)^{N-1} \le \left(\frac{\alpha_N}{2\alpha_0}\right)^{N-1}. \tag{2.30}$$

Thanks to (2.30), we can apply Lemma 2.2.3 to the sequence $(|v_k| + |u|)_{k\ge J}$, with fixed $m \in (\alpha_N/2\alpha_0, \alpha_N/\alpha_0)$, $\alpha > \alpha_0$, and l, where $1 < l \le N'$ is so close to 1 that $l\alpha m < \alpha_N$. Fix any measurable set E in $\mathbb{R}^N$. Then, by (2.29) and the Hölder inequality, we have for all $k \ge J$ that

$$\begin{aligned}\int_E \mathfrak{f}_k(x)dx &\le 2^{N-1}\|v_k\|_N^{N-1}\Big(\int_E |u|^N dx\Big)^{1/N} + (2^{N-1}+1)\int_E |u|^N dx \\ &\quad + 2\tilde{\kappa}(\|Q_k\|_l + \|Q\|_l)\Big(\int_E |u|^{l'} dx\Big)^{1/l'} \\ &\le C_Q\Big\{\Big(\int_E |u|^N dx\Big)^{1/N} + \int_E |u|^N dx + \Big(\int_E |u|^{l'} dx\Big)^{1/l'}\Big\},\end{aligned} \tag{2.31}$$

where $C_Q = 2^{N-1}\sup_{k\in\mathbb{N}}\|v_k\|_N^{N-1} + 2^{N-1} + 1 + 2\tilde{\kappa}(\sup_{k\geq J}\|Q_k\|_l + \|Q\|_l) < \infty$, since $(u_k)_k$ is bounded in W and $(Q_k)_{k\geq J}$ is bounded in $L^l(\mathbb{R}^N)$ by Lemma 2.2.3 and the choices of the parameters taken above. Hence $(\mathfrak{f}_k)_{k\geq J}$ is bounded in $L^1(\mathbb{R}^N)$ by Lemma 2.2.2, since $u \in W$ and $l' \geq N$.

Clearly, (2.31) implies at once that the sequence $(\mathfrak{f}_k)_{k\geq J}$ of $L^1(\mathbb{R}^N)$ verifies the two properties of Vitali. Indeed, fixing $\varepsilon > 0$, it is enough to choose $\delta = \delta(\varepsilon) > 0$ so small and $R = R(\varepsilon) > 0$ so large that for all measurable sets U, with $|U| < \delta$,

$$C_Q\left\{\left(\int_U |u|^N dx\right)^{1/N} + \int_U |u|^N dx + \left(\int_U |u|^{l'} dx\right)^{1/l'}\right\} < \varepsilon,$$

$$C_Q\left\{\left(\int_{\mathbb{R}^N\setminus B_R} |u|^N dx\right)^{1/N} + \int_{\mathbb{R}^N\setminus B_R} |u|^N dx + \left(\int_{\mathbb{R}^N\setminus B_R} |u|^{l'} dx\right)^{1/l'}\right\} < \varepsilon.$$

Finally, $v_k \to 0$ a. e. in $\mathbb{R}^N$ and so $\mathfrak{f}_k \to 0$ a. e. in $\mathbb{R}^N$ by (H_1). An application of the Vitali criterion, Corollary 4.5.5 of [40], gives the assertion. □

Proof of Theorem 2.1.1. Fix $\sigma < \mathcal{H}_p$, $\gamma \in (0,1]$, and $\lambda \in (0,\widetilde{\lambda})$, as in the statement of Lemma 2.3.2. Let (u_k) be the sequence constructed in Lemma 2.3.1 and $u = u_{\sigma,\lambda,\gamma}$ its weak limit in W. In particular, by (2.3), (2.12), (2.22), and (2.24), up to a subsequence, there exist nonnegative numbers ℓ_p, ℓ_N, ℓ_H, and δ such that

$$\begin{gathered}
u_k \rightharpoonup u \text{ in } W, \quad u_k \rightharpoonup u \text{ in } L^p(\mathbb{R}^N, |x|^{-p}), \quad I'(u) = 0 \text{ in } W', \\
\nabla u_k \to \nabla u \text{ and } u_k \to u \text{ a. e. in } \mathbb{R}^N, \\
|\nabla u_k|^{p-2}\nabla u_k \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } [L^{p'}(\mathbb{R}^N)]^N, \\
|\nabla u_k|^{N-2}\nabla u_k \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \text{in } [L^{N'}(\mathbb{R}^N)]^N, \\
\|u_k\|_{W^{1,p}} \to \ell_p, \quad \|u_k\|_{W^{1,N}} \to \ell_N, \quad \|u_k\|_{\mathcal{H}_p} \to \ell_H, \\
\int_{\mathbb{R}^N} g(x,u_k)u_k dx \to \delta, \quad u_k \to u \text{ in } L^{\wp}_{\text{loc}}(\mathbb{R}^N), \quad \wp \in [1,\infty), \\
u_k \leq \psi_R \text{ a. e. in } \mathbb{R}^N \text{ for some } \psi_R \in L^N(B_R) \text{ and all } R > 0.
\end{gathered} \tag{2.32}$$

Moreover, (2.1), implied by (H_1), (2.32), and the Fatou lemma yield

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} g(x,u_k)u dx \geq \int_{\mathbb{R}^N} g(x,u)u dx. \tag{2.33}$$

Thus, Lemma 2.2.2, (2.32), and (2.33) give, as $k \to \infty$,

$$o(1) = \langle I'(u_k), u_k - u\rangle = \|u_k\|^p_{W^{1,p}} - \int_{\mathbb{R}^N} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla u dx - \int_{\mathbb{R}^N} u_k^{p-1} u dx$$

$$
\begin{aligned}
&+\|u_k\|_{W^{1,N}}^N - \int_{\mathbb{R}^N} |\nabla u_k|^{N-2}\nabla u_k\cdot\nabla u dx - \int_{\mathbb{R}^N} u_k^{N-1} u dx\\
&-\sigma\Big[\|u_k\|_{H_p}^p - \int_{\mathbb{R}^N}\frac{u_k^{p-1}}{|x|^p}udx\Big]\\
&-\lambda\int_{\mathbb{R}^N} h(x)u_k^{q-1}(u_k-u)dx - \gamma\int_{\mathbb{R}^N} g(x,u_k)(u_k-u)dx\\
\geq{}& \ell_p^p - \|u\|_{W^{1,p}}^p + \ell_N^N - \|u\|_{W^{1,N}}^N - \sigma(\ell_H^p - \|u\|_{\mathcal{H}_p}^p)\\
&-\gamma\int_{\mathbb{R}^N}[g(x,u_k)u_k - g(x,u)u]dx + o(1).
\end{aligned}
$$

Now (2.32), Lemma 2.3.3, and the Brézis and Lieb lemma give

$$
\begin{aligned}
\|u_k\|_{W^{1,p}}^p - \|u_k-u\|_{W^{1,p}}^p &= \|u\|_{W^{1,p}}^p + o(1),\\
\|u_k\|_{W^{1,N}}^N - \|u_k-u\|_{W^{1,N}}^N &= \|u\|_{W^{1,N}}^N + o(1),\\
\|u_k\|_{H_p}^p - \|u_k-u\|_{H_p}^p &= \|u\|_{H_p}^p + o(1),\\
\int_{\mathbb{R}^N}[g(x,u_k)u_k - g(x,u)u]dx &= \int_{\mathbb{R}^N} g(x,u_k-u)(u_k-u)dx + o(1)
\end{aligned}
$$

as $k\to\infty$. From this, the argument above yields the main formula

$$
\begin{aligned}
\gamma\lim_{k\to\infty}\int_{\mathbb{R}^N} g(x,u_k-u)(u_k-u)dx \geq{}& \lim_{k\to\infty}\|u_k-u\|_{W^{1,p}}^p\\
&+\lim_{k\to\infty}\|u_k-u\|_{W^{1,N}}^N - \sigma\lim_{k\to\infty}\|u_k-u\|_{H_p}^p.
\end{aligned}
\tag{2.34}
$$

Arguing as in (2.7), we obtain that for any $m\in(\alpha_N/2\alpha_0,\alpha_N/\alpha_0)$, $\alpha>\alpha_0$, with α_0 given in (H_1), $s>N$, and $l>1$ sufficiently small such that $l\alpha m<\alpha_N$, there exists a positive constant $C^* = C^*(m,\alpha,s,l)$ such that

$$
\begin{aligned}
\sup_{k\geq J}\int_{\mathbb{R}^N}|u_k-u|^s\big(e^{\alpha|u_k-u|^{N'}} - S_{N-2}(\alpha,u_k-u)\big)dx&\\
\leq C^*\sup_{k\in\mathbb{N}}\|u_k-u\|_{sl'}^s<\infty&
\end{aligned}
\tag{2.35}
$$

is satisfied, as long as $sl'>N$. Finally, by Lemma 2.3.3 and (2.32),

$$
\sup_{k\geq J}\int_{\mathbb{R}^N} g(x,u_k-u)(u_k-u)dx<\infty. \tag{2.36}
$$

Then, by the Hölder inequality, we obtain for all $k\geq J$ that

$$
\int_{\mathbb{R}^N}|u_k-u|^s\big(e^{\alpha|u_k-u|^{N'}} - S_{N-2}(\alpha,u_k-u)\big)dx
$$

$$\leq \|u_k - u\|_{sl'}^s \left(\int\limits_{\mathbb{R}^N} (e^{\alpha|u_k - u|^{N'}} - S_{N-2}(\alpha, u_k - u))^l dx \right)^{1/l}$$
$$\leq C^* \|u_k - u\|_{sl'}^s.$$

From this last fact and (2.34), we get, as $k \to \infty$,

$$C^* \gamma \tilde{\kappa} \|u_k - u\|_{sl'}^s + o(1) \geq \left(1 - \frac{\sigma_+}{\mathcal{H}_p}\right) \|u_k - u\|_{W^{1,p}}^p + \frac{1}{2} \|u_k - u\|_{W^{1,N}}^N. \tag{2.37}$$

The continuity of the embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^{sl'}(\mathbb{R}^N)$ gives

$$C^* \gamma \tilde{\kappa} \|u_k - u\|_{sl'}^s + o(1) \geq \frac{1}{2c_{sl'}^N} \|u_k - u\|_{sl'}^N.$$

Passing eventually to a further subsequence, we assume that there exists $\ell_\gamma \geq 0$ such that

$$\lim_{k \to \infty} \|u_k - u\|_{sl'} = \ell_\gamma.$$

We assert that $\ell_\gamma = 0$. Otherwise,

$$\gamma \ell_\gamma^{s-N} \geq \frac{1}{2c_{sl'}^N C^* \tilde{\kappa}} = \mathfrak{c}. \tag{2.38}$$

Let us define

$$\gamma^* = \begin{cases} \inf\{\gamma \in (0,1] \ : \ \ell_\gamma > 0\}, & \text{if there exists } \gamma \in (0,1] \text{ such that } \ell_\gamma > 0, \\ 1, & \text{if } \ell_\gamma = 0 \text{ for all } \gamma \in (0,1]. \end{cases}$$

We claim that $\gamma^* > 0$ if there exists $\gamma \in (0,1]$ such that $\ell_\gamma > 0$. Otherwise, there exists a sequence $(\gamma_k)_k$, with $\ell_{\gamma_k} > 0$, such that $\gamma_k \to 0$ as $k \to \infty$. Thus, (2.38) implies that

$$\gamma_k \ell_{\gamma_k}^{s-N} \geq \mathfrak{c} > 0.$$

This is an obvious contradiction, since $\{\ell_\gamma\}_{\gamma \in (0,1]}$ is uniformly bounded above by the embedding theorem. Indeed, $(u_k)_k \subset B_\rho$, $u \in \overline{B}_\rho$ and ρ, given in Lemma 2.3.1, is independent of γ.

Hence, $\ell_\gamma = 0$ for any $\gamma \in (0, \gamma^*)$. Therefore, for all $\gamma \in (0, \gamma^*)$,

$$\lim_{k \to \infty} \|u_k - u\|_{sl'} = 0,$$

which, together with (2.37), gives at once that $u_k \to u$ as $k \to \infty$ in W for all $\gamma \in (0, \gamma^*)$.

In conclusion, for fixed $\sigma \in (-\infty, \mathcal{H}_p)$ and $\lambda \in (0, \widetilde{\lambda})$, and for any $\gamma \in (0, 1]$, Lemma 2.3.1 and the Ekeland variational principle give the existence of a $(PS)_{m_{\sigma,\lambda,\gamma}}$ sequence $(u_k)_k$ in W of I. Moreover, the argument above shows the existence of $\gamma^* =$

$\gamma^*(\sigma,\lambda) > 0$ such that, up to a subsequence, $(u_k)_k$ strongly converges to $u = u_{\sigma,\lambda,\gamma}$ in W, with $m_{\sigma,\lambda,\gamma} = I(u_{\sigma,\lambda,\gamma}) < 0 < \jmath \le I(v)$ for all $v \in \partial B_\rho$. Consequently, $u_{\sigma,\lambda,\gamma} \in B_\rho$, so that $I'(u_{\sigma,\lambda,\gamma}) = 0$. In other words, $u_{\sigma,\lambda,\gamma}$ is a nontrivial nonnegative solution of $(\mathcal{E}_N)$ for any $\gamma \in (0,\gamma^*)$.

It remains to show (2.2). First, we recall that the nonnegative solution $u_{\sigma,\lambda,\gamma} \in B_\rho$, with $\rho > 0$ independent of λ as asserted in Lemma 2.3.1. Hence, $\{u_{\sigma,\lambda,\gamma}\}_{\lambda\in(0,\widetilde{\lambda})}$ is uniformly bounded in W. Thus, by (2.3) and (H_2), we have

$$\begin{aligned} m_{\sigma,\lambda,\gamma} \ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\left(1-\frac{\sigma_+}{\mathcal{H}_p}\right)\|u_{\sigma,\lambda,\gamma}\|^p_{W^{1,p}} + \left(\frac{1}{N}-\frac{1}{\nu}\right)\|u_{\sigma,\lambda,\gamma}\|^N_{W^{1,N}} \\ -\lambda\left(\frac{1}{q}-\frac{1}{\nu}\right)\|h\|_\theta\|u_{\sigma,\lambda,\gamma}\|^q_{W^{1,N}} + o(1) \\ \ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\left(1-\frac{\sigma_+}{\mathcal{H}_p}\right)\|u_{\sigma,\lambda,\gamma}\|^p_{W^{1,p}} + \left(\frac{1}{N}-\frac{1}{\nu}\right)\|u_{\sigma,\lambda,\gamma}\|^N_{W^{1,N}} - \lambda C_h, \end{aligned}$$

where

$$C_h = \left(\frac{1}{q}-\frac{1}{\nu}\right)\|h\|_\theta \sup_{\lambda\in(0,\widetilde{\lambda})} \|u_{\sigma,\lambda,\gamma}\| < \infty.$$

We first assert that

$$\lim_{\lambda\to 0} \|u_{\sigma,\lambda,\gamma}\|_{W^{1,p}} = 0. \tag{2.39}$$

Otherwise, $\limsup_{\lambda\to 0} \|u_{\sigma,\lambda,\gamma}\|_{W^{1,p}} = \upsilon_p > 0$. Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\|u_{\sigma,\lambda_j,\gamma}\|_{W^{1,p}} \to \ell_p$ as $j \to \infty$. Then, letting $j \to \infty$, we get from (2.3) and Lemma 2.2.2 that

$$0 \ge \limsup_{\lambda\to 0} m_{\sigma,\lambda,\gamma} \ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\left(1-\frac{\sigma_+}{\mathcal{H}_p}\right)\ell_p^p > 0,$$

which is the desired contradiction, proving (2.39). Similarly,

$$\lim_{\lambda\to 0} \|u_{\sigma,\lambda,\gamma}\|_{W^{1,N}} = 0. \tag{2.40}$$

Otherwise, $\limsup_{\lambda\to 0} \|u_{\sigma,\lambda,\gamma}\|_{W^{1,N}} = \ell_N > 0$. Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\|u_{\sigma,\lambda_j,\gamma}\|_{W^{1,N}} \to \ell_N$ as $j \to \infty$. Then, letting $j \to \infty$, we get from (2.3) and Lemma 2.2.2 that

$$0 \ge \limsup_{\lambda\to 0} m_{\sigma,\lambda,\gamma} \ge \left(\frac{1}{N}-\frac{1}{\nu}\right)\ell_N^N > 0,$$

which is the desired contradiction, proving (2.40). Of course, (2.39) and (2.40) imply at once the validity of (2.2). □

Comments on Chapter 2

A substantial progress for Moser–Trudinger inequalities on Riemannian manifolds has been achieved in the last years. For instance, in the compact case, the study of these inequalities started with the pioneering works due to T. Aubin [18], P. Cherrier [59], and L. Fontana [108]. In the presence of lack of compactness, Sobolev inequalities are more delicate and the different geometric notions of curvature play a crucial role in this case. For instance, inspired by the above cited paper [156] and by using some fine estimates on the density function of the volume form, Q. Yang, D. Su, and Y. Kong in [256] proved that on a complete, simply connected N-dimensional Riemannian manifold with negative sectional curvature there exists a constant $k_N(\mathcal{M}) > 0$ such that

$$\sup_{\substack{u \in W^{1,N}(\mathcal{M}) \\ \|u\|_{W^{1,N}} \le 1}} \int_{\mathcal{M}} (e^{\beta |u|^{N'}} - S_{N-2}(\beta, |u|)) d\sigma_g = k_N(\mathcal{M})$$

for every $\beta \in [0, N\omega_{N-1}^{1/(N-1)}]$. A challenging problem is to extend Theorem 2.1.1 when $(\mathcal{E}_N)$ is set on a noncompact Riemannian manifold $\mathcal{M}$; see Chapter 8 for related results.

3 Critical Hardy–Kirchhoff equations in $\mathbb{R}^N$

Ognuno sta solo sul cuor della terra
trafitto da un raggio di sole:
ed è subito sera.

Salvatore Quasimodo
Ed è subito sera

This chapter deals with the existence of nontrivial solutions for stationary critical, possibly degenerate, Kirchhoff (p, q) equations in $\mathbb{R}^N$. For clarity, the results are presented in the scalar case, and we refer to [101] for the extension into the vectorial as well as fractional framework. The main difficulties arise because of the (p, q)-Laplacian operator, the double lack of compactness, as well as the fact that the Kirchhoff equation can be degenerate, that is, $M(0) = 0$.

Lately, great attention has been drawn to the study of nonlocal elliptic problems that lack compactness. These models arise in a quite natural way in many different applications, and we refer to [210] for details. In the fractional setting we cite the recent monograph [188], the extensive paper [81], and the references cited there for further comments. More recently, the study has been extended to problems involving fractional (p, q) elliptic operators, see [1, 14, 15, 38, 101].

3.1 The stationary Kirchhoff framework

In this chapter, we study the existence of nontrivial solutions for possibly degenerate Kirchhoff equations involving the (p, q)-Laplacian as well as critical nonlinearities. For the sake of clarity, we present the results in the scalar case. More precisely, we consider the following equation in $\mathbb{R}^N$:

$$M(\|u\|^p_{W^{1,p}})(-\Delta_p u + |u|^{p-2}u) + M(\|u\|^q_{W^{1,q}})(-\Delta_q u + |u|^{q-2}u) \\ = \lambda f(x, u) + |u|^{q^*-2}u, \qquad (\mathcal{E}_M)$$

where $2 \le p < q < N$ and $q^* = Nq/(N - q)$.

The natural solution space is $W = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,q}}.$$

The space W is a separable reflexive Banach space.

Throughout the chapter, we suppose for the Kirchhoff coefficient that

($\mathscr{M}$) $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ *is a continuous function such that*
 (M_1) *there exists* $\theta \in [1, p^*/p)$, *yielding* $M(t)t \le \theta\mathscr{M}(t)$ *for all* $t \in \mathbb{R}^+_0$, *where* $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$,
 and either

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($\widetilde{M}_2$) $\inf_{t\in\mathbb{R}_0^+} M(t) = a > 0$,
or $M(0) = 0$ *and* M *verifies both properties*:
(M_2) *for any* $\tau > 0$ *there exists* $m = m(\tau) > 0$ *such that* $M(t) \geq m$ *for all* $t \geq \tau$,
(M_3) *there exists a positive number* $c > 0$ *such that* $M(t) \geq c\,t^{\theta-1}$ *for all* $t \in [0, 1]$.

Usually, the existence of solutions of Kirchhoff problems is obtained when M is also nondecreasing in $\mathbb{R}_0^+$. For more comments, we refer, e. g., to [105, 211, 216]. However, the entire condition ($\mathscr{M}$) does not force M to be monotone as the example of $M(t) = (1+t)^k + (1+t)^{-1}$ for $t \in \mathbb{R}_0^+$, with $0 < k < 1$, shows; see Figure 3.1 below. For details, we suggest looking at [22, 215].

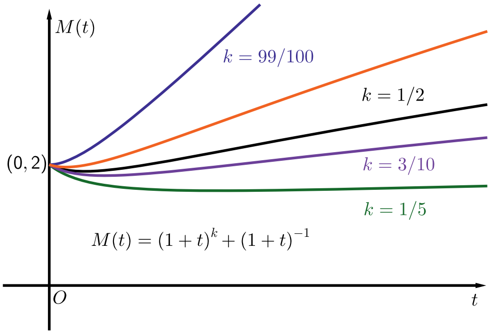

Figure 3.1: The behavior of $M(t)$ for different values of k.

Let $D^{1,p}(\mathbb{R}^N)$ be the Banach space defined in Section 1.2 so that (1.3) continues to hold when $\wp = p$.

The parameter λ in ($\mathcal{E}_M$) is strictly positive and the perturbed subcritical term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ verifies

($\mathcal{F}$) f *is a Carathéodory function. For a. e.* $x \in \mathbb{R}^N$, $F(x, \cdot) > 0$ *in* $\mathbb{R}^+$, *where* $F(x,t) = \int_0^t f(x,s)ds$. *Furthermore, there exist* r *and* γ *such that* $\theta q < r < q^*$, $\theta q < \gamma < q^*$, *and for every* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *for which the inequalities*

$$|f(x,t)| \leq \varepsilon\theta q|t|^{\theta q-1} + rC_\varepsilon|t|^{r-1} \quad \text{for any } t \in \mathbb{R}$$

and

$$0 \leq \nu F(x,t) \leq f(x,t)t \ \text{ for all } t \in \mathbb{R}$$

hold for a. e. $x \in \mathbb{R}^N$.

The definition ($\mathcal{F}$) makes sense since $\theta < p^*/p < q^*/q$.

For critical equations in $\mathbb{R}^N$, driven by the fractional p-Laplacian, we refer the reader to [25, 32, 52, 81, 98, 99, 101, 211, 216] and the references therein for the study of equations with critical nonlinearities. In the vectorial case very few contributions are in the case of $\mathbb{R}^N$, e. g., [13, 100, 101, 103, 104, 159, 177], while in bounded domains we mention [57, 92, 93, 120, 121, 173, 179, 180, 249] and the references therein.

Let us recall that Kirchhoff problems, with Kirchhoff function M, are said to be *nondegenerate* if $(\widetilde{M_2})$ holds, and *degenerate* if $M(0) = 0$ and 0 is the unique zero of M. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero and M measures the change of the tension on the string caused by the change of its length during the vibration. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. For example, the existence of solutions for nondegenerate fractional Kirchhoff stationary problems is treated in [104, 105, 159], and for degenerate problems this problem is considered in [22, 52, 98, 99, 216, 249] and the references therein.

The main novelty of $(\mathcal{E}_M)$ is that it involves elliptic operators with (p, q) growth, as well as critical nonlinearities. In this last direction, we recall the recent works [1, 14, 15, 38], devoted to the study of critical (p, q)-fractional problems with $M \equiv 1$, namely without the Kirchhoff coefficient. To overcome the lack of compactness, the authors in [1, 14, 15, 38] exploited suitable concentration compactness arguments which seem not to work in the presence of a general Kirchhoff coefficient $M \not\equiv 1$. For this reason, following [101], we tried to adopt the method which was introduced in [22] and was further improved successfully in [52, 98, 99, 104, 211] for different contexts. Here we use a tricky step analysis which allows us to handle the nonlocal nature and the double loss of compactness of $(\mathcal{E}_M)$. Actually, this approach has been useful also to provide the existence of solutions for critical fractional Kirchhoff problems, as in [22, 101, 211]. However, it should be noted that in [52, 98, 99, 104] the presence of fractional Hardy potentials does not allow us to use the strategy in the degenerate Kirchhoff setting. In the critical (p, q)-fractional Kirchhoff equation $(\mathcal{E}_M)$, the application of the tricky step analysis is fairly delicate because of the double structure of the norm $\|\cdot\|$ in W. Indeed, we have to split the study on the behavior of the Palais–Smale sequences in different cases, examining all the possible situations as the norms $\|\cdot\|_{W^{1,p}}$ and $\|\cdot\|_{W^{1,q}}$ approach zero, due to the degenerate nature of $(\mathcal{E}_M)$. However, the scheme provides us with a positive answer to the question of solution existence for $(\mathcal{E}_M)$.

Theorem 3.1.1. *Suppose that M verifies $(\mathscr{M})$ and f fulfills $(\mathcal{F})$. Then there exists a threshold $\lambda^* > 0$ such that $(\mathcal{E}_M)$ admits at least one nontrivial solution u_λ in W for all $\lambda \geq \lambda^*$. Moreover,*

$$\lim_{\lambda \to \infty} \|u_\lambda\| = 0 \tag{3.1}$$

holds.

We now study $(\mathcal{E}_M)$ under the sole assumption $(\widetilde{M_2})$ on the positive continuous Kirchhoff function M, which is an addition to Theorem 3.1.1, when $(\mathcal{E}_M)$ is nondegenerate.

Theorem 3.1.2. *Suppose that the positive continuous Kirchhoff function M verifies* $(\widetilde{M_2})$ *and f fulfills* $(\mathcal{F})$, *with* $\theta = 1$ *and the exponents ν and r satisfying*

$$q < \nu < q^*, \quad q < r < q^*, \quad q\,M(0) < \nu\,a. \tag{3.2}$$

Then for any $\epsilon \in (M(0), a\nu/q)$ *there exists* $\lambda^* = \lambda^*(\epsilon) > 0$ *such that* $(\mathcal{E}_M)$ *admits at least one nontrivial solution* u_λ *in W for all* $\lambda \geq \lambda^*$. *Furthermore,* (3.1) *continues to hold.*

Clearly, the requirement $(3.2)_3$ is automatically satisfied whenever $M(0) = a$, due to $\nu > q$ by $(3.2)_1$. The assumption $M(0) = a$, together with standard nondecreasing monotonicity of M, was assumed in [105, 211], as well as in numerous papers. A very interesting open problem is to construct a nontrivial solution u_λ of $(\mathcal{E}_M)$ when $\nu\,a \leq q\,M(0)$ and the growth condition on M stated in (M_1) does not hold; in other words, when both Theorems 3.1.1 and 3.1.2 cannot be applied. For the already mentioned Kirchhoff function

$$M(t) = (1+t)^k + (1+t)^{-1}, \quad t \in \mathbb{R}_0^+, \quad k \in (0,1),$$

we have $M(0) = 2$ and $a = k^{-k/(k+1)}(1+k) < 2$. Furthermore, if k is so small that $k+1 < \nu/q$, then M verifies all the assumptions of Theorem 3.1.1, with $\theta = k+1$. While if $k \in (0,1)$ is sufficiently large, then $q\,M(0) = 2q < \nu\,a$, since $q < \nu$ by (3.2), and M satisfies all the hypotheses of Theorem 3.1.2. It is therefore evident that Theorem 3.1.2 is applicable even when neither M is increasing in $\mathbb{R}_0^+$, nor $M(0) = a$.

The two Theorems 3.1.1 and 3.1.2 extend, in different and nontrivial ways, to the Kirchhoff setting described in Theorem 1.2 of [1], Theorem 1.1 of [14], Theorem 1.1 of [15], while extending and completing the existence result given in Theorem 1.1 of [38].

For the generalization of the previous results to a general framework as well as to the vectorial case, we refer the interested reader to the original paper [101].

3.2 Proof of Theorem 3.1.1

For the relevant definitions and notations related to the separable reflexive real Banach space W, we refer to [52, 56, 100, 101, 103, 215, 216]. In this section, we first assume, without further mentioning, that the hypotheses required in Theorem 3.1.1 are satisfied.

Lemma 3.2.1. *Let* $(u_k)_k$ *and u be in W and such that* $u_k \rightharpoonup u$ *weakly in W, and* $u_k \to u$ *a. e. in* $\mathbb{R}^N$. *Then,*

$$\int_{\mathbb{R}^N} f(x,u_k)(u_k - u)dx \to 0 \tag{3.3}$$

as $k \to \infty$.

Proof. By $(\mathcal{F})$, with $\varepsilon = 1$, the Hölder inequality gives, for a suitable constant $C > 0$,

$$\left|\int_{\mathbb{R}^N} f(x,u_k)(u_k - u)dx\right| \le \int_{\mathbb{R}^N} \{\theta q|u_k|^{\theta q-1}|u_k - u| + r\,C_1|u_k|^{r-1}|u_k - u|\}dx$$
$$\le C(\|u_k - u\|_{\theta q} + \|u_k - u\|_r) \to 0$$

as $k \to \infty$, by Theorem 2.7 of [100], since $2 \le p \le \theta p < \theta q < r < q^*$ by (M_1) and $(\mathcal{F})$. This proves (3.3). □

We say that $u \in W$ is a (weak) *solution* of $(\mathcal{E}_M)$ if

$$M(\|u\|^p_{W^{1,p}})\langle u,\varphi\rangle_{W^{1,p}} + M(\|u\|^q_{W^{1,q}})\langle u,\varphi\rangle_{W^{1,q}}$$
$$= \int_{\mathbb{R}^N} f(x,u)\varphi dx + \frac{1}{q^*}\int_{\mathbb{R}^N} |u|^{q^*-2}u\varphi dx$$

for any $\varphi \in W$, where

$$\langle u,\varphi\rangle_{W^{1,p}} = \langle \nabla u,\nabla\varphi\rangle_p + \langle u,\varphi\rangle_p,\ \ \langle u,\varphi\rangle_{W^{1,q}} = \langle \nabla u,\nabla\varphi\rangle_q + \langle u,\varphi\rangle_q,$$
$$\langle \nabla u,\nabla\varphi\rangle_p = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx, \quad \langle u,\varphi\rangle_q = \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\cdot\nabla\varphi dx,$$
$$\langle u,\varphi\rangle_p = \int_{\mathbb{R}^N} |u|^{p-2}u\varphi dx, \quad \langle u,\varphi\rangle_q = \int_{\mathbb{R}^N} |u|^{q-2}u\varphi dx.$$

Indeed, the simplified notation is reasonable, since $\langle u,\cdot\rangle_{W^{1,p}}$, $\langle u,\cdot\rangle_{W^{1,q}}$ are linear bounded functionals on W for all $u \in W$.

Clearly, the entire (weak) solutions of $(\mathcal{E}_M)$ are exactly the critical points of the Euler–Lagrange functional $I : W \to \mathbb{R}$, $I = I_\lambda$, associated to $(\mathcal{E}_M)$, given for all $u \in W$ by

$$I(u) = \frac{1}{p}\mathscr{M}(\|u\|^p_{W^{1,p}}) + \frac{1}{q}\mathscr{M}(\|u\|^q_{W^{1,q}}) - \lambda\int_{\mathbb{R}^N} F(x,u)dx - \frac{1}{q_s^*}\|u\|^{q^*}_{q^*},$$

which is well defined and of class $C^1(W)$ by $(\mathcal{F})$ and the continuity of M.

We start by showing that the functional I has the mountain pass geometric features to guarantee the existence of the Palais–Smale sequence at special levels. The proof of this behavior is quite standard, but we give it for completeness.

Lemma 3.2.2. *There exists $e \in C_c^\infty(\mathbb{R}^N)$ such that $I(e) < 0$, $\|e\|_{W_{W^{1,p}}} \ge 1$, and $\|e\|_{W_{W^{1,q}}} \ge 1$ for all $\lambda > 0$.*

Furthermore, for all $\lambda > 0$ there exist $\jmath = \jmath(\lambda) > 0$ and $\rho = \rho(\lambda) \in (0,1]$ such that $I(u) \ge \jmath$ for any $u \in W$, with $\|u\| = \rho$.

Proof. Now, let $u \in C_c^\infty(\mathbb{R}^N)$ be such that $\|u\| = 1$. The assumption (M_1) implies that

$$\mathscr{M}(t) \le \mathscr{M}(1)t^\theta \quad \text{for all } t \ge 1 \quad \text{and} \quad \mathscr{M}(t) \ge \mathscr{M}(1)t^\theta \quad \text{for all } t \in [0,1]. \tag{3.4}$$

Thus, by $(\mathcal{F})$ and (3.4), we have for all $\lambda > 0$,

$$\begin{aligned} I(tu) &= \frac{1}{p}\mathscr{M}\big(\|tu\|^p_{W^{1,p}}\big) + \frac{1}{q}\mathscr{M}\big(\|tu\|^q_{W^{1,q}}\big) - \lambda \int_{\mathbb{R}^N} F(x,tu)dx - \frac{t^{q^*}}{q^*}\|u\|^{q^*}_{q^*} \\ &\le \mathscr{M}(1)\left(\frac{t^{\theta p}}{p} + \frac{t^{\theta q}}{q}\right) - \frac{t^{q^*}}{q^*}\|u\|^{q^*}_{q^*} \to -\infty, \end{aligned} \tag{3.5}$$

as $t \to -\infty$, since $\theta p < p^* < q^*$ and $\theta q < q^*$, since $(p^*/p) < (q^*/q)$ by (M_1). Hence, taking $e = \tau_0 u$ with $\tau_0 > 0$ sufficiently large, we obtain at once that $\|e\|_{W_{W^{1,p}}} \ge 1$, $\|e\|_{W_{W^{1,q}}} \ge 1$, and $I(e) < 0$ for all $\lambda > 0$, as stated.

For the second part, note that $(\mathcal{F})$ gives for any $\varepsilon > 0$ the existence of $C_\varepsilon > 0$ such that

$$0 < F(x,t) \le \varepsilon t^{\theta q} + C_\varepsilon t^r \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \tag{3.6}$$

holds. Hence, fixing $\lambda > 0$, (1.3), (1.6), and (3.4) imply that for all $u \in W$ with $\|u\| \le 1$,

$$\begin{aligned} I(u) &\ge \frac{\mathscr{M}(1)}{p}\|u\|^{\theta p}_{W^{1,p}} + \frac{\mathscr{M}(1)}{q}\|u\|^{\theta q}_{W^{1,q}} - \lambda\varepsilon\|u\|^{\theta q}_{\theta q} - \lambda C_\varepsilon\|u\|^r_r - \frac{1}{q^*}\|u\|^{q^*}_{q^*} \\ &\ge \left(\frac{\mathscr{M}(1)}{q2^{\theta q-1}} - \lambda\varepsilon c^{\theta q}_{\theta q}\right)\|u\|^{\theta q} - \lambda C_\varepsilon c^r_r\|u\|^r - \frac{c^{q^*}_{q^*}}{q^*}\|u\|^{q^*}. \end{aligned}$$

From this, we choose $\varepsilon > 0$ sufficiently small so that

$$\mathfrak{m}_\lambda = \frac{\mathscr{M}(1)}{q2^{\theta q-1}} - \lambda\varepsilon c^{\theta q}_{\theta q} > 0.$$

Clearly, there exists $\rho \in (0,1]$ such that

$$\max_{t\in[0,1]} y(t) = y(\rho) > 0, \text{ where } y(t) = \mathfrak{m}_\lambda t^{\theta q} - \lambda C_\varepsilon c^r_r t^r - \frac{c^{q^*}_{q^*}}{q^*}t^{q^*},$$

since $\theta q < r < q^*$. Consequently, $I(u) \ge y(\rho) = \jmath$ for all $u \in W$, with $\|u\| = \rho$, as desired. This concludes the proof. □

Now we discuss the compactness property for the functional I, given by the Palais–Smale condition at a suitable level. For this, we fix $\lambda > 0$ and set

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\big(\gamma(t)\big), \quad \Gamma = \{\gamma \in C\big([0,1];W\big) : \gamma(0) = 0,\ I\big(\gamma(1)\big) < 0\}. \tag{3.7}$$

Obviously, $c_\lambda > 0$ thanks to Lemma 3.2.2, since in particular $\|e\| > \rho$. Before proving that I satisfies the Palais–Smale condition at level c_λ, we introduce an asymptotic condition for the level c_λ. This result will be crucial to overcome the lack of compactness and was first used in [96] for the (p,q)-Laplacian critical equations.

Lemma 3.2.3. *One has*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof. Fix $\lambda > 0$. Let e be the function determined in Lemma 3.2.2, which is independent of $\lambda > 0$. Since I satisfies the mountain pass geometry at 0 and e, there exists $t_\lambda > 0$ verifying $I(t_\lambda e) = \max_{t\geq 0} I(te)$ for all $\lambda > 0$. Therefore, $\langle I'(t_\lambda e), e\rangle = 0$. Thus,

$$\begin{aligned} t_\lambda^{p-1} M(\|t_\lambda e\|^p_{W^{1,p}})\|e\|^p_{W^{1,p}} + t_\lambda^{q-1} M(\|t_\lambda e\|^q_{W^{1,q}})\|e\|^q_{W^{1,q}} \\ = \lambda \int_{\mathbb{R}^N} f(x, t_\lambda e) e dx + t_\lambda^{q^*-1}\|e\|^{q^*}_{q^*} \geq t_\lambda^{q^*-1}\|e\|^{q^*}_{q^*}, \end{aligned} \tag{3.8}$$

by $(\mathcal{F})$, since $\lambda > 0$.

We claim that $\{t_\lambda\}_{\lambda>0}$ is bounded in $\mathbb{R}$. Put $\Lambda = \{\lambda > 0 : t_\lambda \geq 1\}$. If $\Lambda = \emptyset$, we are done. So consider the case $\Lambda \neq \emptyset$. Now, from (M_1), (3.4), the fact that $\|e\|_{W_{W^{1,p}}} \geq 1$ and also $\|e\|_{W_{W^{1,p}}} \geq 1$, we derive that

$$\begin{aligned} t_\lambda^p M(\|t_\lambda e\|^p_{W^{1,p}})\|e\|^p_{W^{1,p}} + t_\lambda^q M(\|t_\lambda e\|^q_{W^{1,q}})\|e\|^q_{W^{1,q}} \\ \leq \theta\{\mathscr{M}(\|t_\lambda e\|^p_{W^{1,p}}) + \mathscr{M}(\|t_\lambda e\|^q_{W^{1,q}})\} \\ \leq \theta \mathscr{M}(1) t_\lambda^{\theta q}\|e\|^{\theta q} \end{aligned} \tag{3.9}$$

for any $\lambda \in \Lambda$, since $1 < p < q$ and $\theta \geq 1$. Therefore, (3.8) and (3.9) imply that

$$\theta \mathscr{M}(1)\|e\|^{\theta q} \geq t_\lambda^{q^*-\theta q}\|e\|^{q^*}_{q^*} \quad \text{for any } \lambda \in \Lambda,$$

which yields that $\{t_\lambda\}_{\lambda\in\Lambda}$ is bounded since $\theta q < q^*$, as already noted above. It follows at once that $\{t_\lambda\}_{\lambda>0}$ is bounded. This proves the claim.

Fix now a sequence $(\lambda_k)_k \subseteq \mathbb{R}^+$ such that $\lambda_k \to \infty$ as $k \to \infty$. Obviously, $\{t_{\lambda_k}\}_k$ is bounded. Thus, there exist a $t_0 \geq 0$ and subsequence of $(\lambda_k)_k$, still denoted by $(\lambda_k)_k$, such that $t_{\lambda_k} \to t_0$. By the continuity of M, also $\{M(t^p_{\lambda_k}\|e\|^p_{W^{1,p}})\}_k$ and $\{M(t^q_{\lambda_k}\|e\|^q_{W^{1,q}})\}_k$ are bounded, and so by (3.8) there exists $C > 0$ such that, for any $k \in \mathbb{N}$,

$$\lambda_k \int_{\mathbb{R}^N} f(x, t_{\lambda_k} e) e\, dx + t_{\lambda_k}^{q^*-1}\|e\|^{q^*}_{q^*} \leq C. \tag{3.10}$$

We assert that $t_0 = 0$. Otherwise, $(\mathcal{F})$ and the dominated convergence theorem yield

$$\int_{\mathbb{R}^N} f(x, t_{\lambda_k} e) e\, dx \to \int_{\mathbb{R}^N} f(x, t_0 e) e\, dx > 0 \tag{3.11}$$

as $k \to \infty$. Recalling that $\lambda_k \to \infty$, we get

$$\lim_{k\to\infty}\left[\lambda_k \int_{\mathbb{R}^N} f(x, t_{\lambda_k} e) e dx + t_{\lambda_k}^{q^*-1}\|e\|^{q^*}_{q^*}\right] = \infty,$$

which contradicts (3.10). Thus $t_0 = 0$ and $t_\lambda \to 0$ as $\lambda \to \infty$, since the sequence $(\lambda_k)_k$ is arbitrary.

Now the path $\gamma(t) = te$, $t \in [0,1]$, belongs to Γ, so that Lemma 3.2.2 gives

$$0 < c_\lambda \le \max_{t\ge 0} I(\gamma(t)) \le I(t_\lambda e) \le \frac{1}{p}\mathscr{M}(\|t_\lambda e\|^p_{W^{1,p}}) + \frac{1}{q}\mathscr{M}(\|t_\lambda e\|^q_{W^{1,q}}).$$

Moreover, $\mathscr{M}(\|t_\lambda e\|^p_{W^{1,p}}) \to 0$ and $\mathscr{M}(\|t_\lambda e\|^q_{W^{1,q}}) \to 0$ as $\lambda \to \infty$, by the continuity of $\mathscr{M}$ and the fact that e does not depend on $\lambda > 0$. This completes the proof of the lemma. □

Now we are ready to prove the compactness property of I at the special level (3.7) for λ sufficiently large. To this aim, we use somehow an argument which first appeared in [22]. This method has been adopted during the years, after several improvements and refinements, in different problems where the lack of global compactness was present; see [52, 98, 99, 104, 211].

Lemma 3.2.4. *Any $(PS)_{c_\lambda}$ sequence $(u_k)_k$ of I is bounded in W for all $\lambda > 0$. Moreover, there exist $u_\lambda \in W$, nonnegative numbers ℓ_p, ℓ_q, δ_λ such that, up to a subsequence, still denoted by $(u_k)_k$,*

$$\begin{gathered} u_k \rightharpoonup u_\lambda \text{ in } W, \quad \|u_k\|^p_{W^{1,p}} \to \ell_p, \quad \|u_k\|^q_{W^{1,q}} \to \ell_q, \\ u_k \to u_\lambda \text{ in } L^{\wp}_{\mathrm{loc}}(\mathbb{R}^N), \quad u_k \to u_\lambda \text{ a.e. in } \mathbb{R}^N, \\ |u_k| \le g_R \text{ a.e. in } \mathbb{R}^N, \text{ for some } g_R \in L^{\wp}(B_R) \text{ and all } R > 0, \\ u_k \rightharpoonup u_\lambda \text{ in } L^{q^*}(\mathbb{R}^N), \quad \|u_k\|^{q^*}_{q^*} \to \delta_\lambda, \\ |u_k|^{q^*-2}u_k \rightharpoonup |u_\lambda|^{q^*-2}u_\lambda \text{ in } L^{q^*/(q^*-1)}(\mathbb{R}^N), \end{gathered} \tag{3.12}$$

hold for all $\wp \in [1, q^*)$. Finally, up to further subsequence, if necessary, not relabeled,

$$\nabla u_k \to \nabla u_\lambda \quad \text{a.e. in } \mathbb{R}^N \tag{3.13}$$

is valid.

Proof. Fix $\lambda > 0$ and let $(u_k)_k \subset W$ be a $(PS)_{c_\lambda}$ sequence of I. Assume for contradiction that $(u_k)_k$ is not bounded in W. Then, going to a subsequence, still called $(u_k)_k$ for simplicity, either

$$\begin{gathered} \lim_{k\to\infty} \|u_k\|_{W^{1,p}} = \infty, \quad \lim_{k\to\infty} \|u_k\|_{W^{1,q}} = \infty, \\ \|u_k\|_{W^{1,p}} \ge 1, \quad \|u_k\|_{W^{1,q}} \ge 1 \quad \text{for all } k, \end{gathered} \tag{3.14}$$

or

$$\lim_{k\to\infty} \|u_k\|_{W^{1,p}} = \infty, \quad \sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}} < \infty, \quad \|u_k\|_{W^{1,p}} \ge 1 \quad \text{for all } k \tag{3.15}$$

is valid.

Assume first that (3.14) holds. If $M(0) = 0$, then by (M_2), with $\tau = 1$, there exists $m > 0$ such that

$$M(\|u_k\|^p_{W^{1,p}}) \geq m \quad \text{and} \quad M(\|u_k\|^q_{W^{1,q}}) \geq m \quad \text{for all } k. \tag{3.16}$$

Clearly, inequalities (3.16) continue to hold also in the nondegenerate case, that is, when $(\widetilde{M_2})$ is true, by taking $m = a$. Furthermore, from (M_1) and $(\mathcal{F})$ in both cases it follows that

$$\begin{aligned} I(u_k) - \frac{1}{\nu}\langle I'(u_k), u_k\rangle &\geq \frac{1}{p}\mathscr{M}(\|u_k\|^p_{W^{1,p}}) - \frac{1}{\nu}M(\|u_k\|^p_{W^{1,p}})\|u_k\|^p_{W^{1,p}} \\ &\quad + \frac{1}{q}\mathscr{M}(\|u_k\|^q_{W^{1,q}}) - \frac{1}{\nu}M(\|u_k\|^q_{W^{1,q}})\|u_k\|^q_{W^{1,q}} \\ &\quad + \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\|u_k\|^{q^*}_{q^*} \\ &\geq \left(\frac{1}{p\theta} - \frac{1}{\nu}\right)M(\|u_k\|^p_{W^{1,p}})\|u_k\|^p_{W^{1,p}} \\ &\quad + \left(\frac{1}{q\theta} - \frac{1}{\nu}\right)M(\|u_k\|^q_{W^{1,q}})\|u_k\|^q_{W^{1,q}}, \end{aligned}$$

since $p\theta < q\theta < \nu < q^*$. Hence, by (1.9) there exists β_λ such that as $k \to \infty$,

$$\begin{aligned} &c_\lambda + \beta_\lambda\|u_k\| + o(1) \geq \alpha\{\|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}}\}, \quad \alpha > 0, \\ &\alpha = \left(\frac{1}{q\theta} - \frac{1}{\nu}\right) \times \begin{cases} m, & \text{in the degenerate case,} \\ a, & \text{in the nondegenerate case,} \end{cases} \end{aligned} \tag{3.17}$$

where m is given in (3.16), while a is the positive constant in $(\widetilde{M_2})$. Therefore,

$$0 < \alpha \leq \beta_\lambda \frac{\|u_k\|_{W^{1,p}} + \|u_k\|_{W^{1,q}}}{\|u_k\|^p_{W^{1,p}} + \|u_k\|^q_{W^{1,q}}} + o(1) \leq \beta_\lambda(\|u_k\|^{1-p}_{W^{1,p}} + \|u_k\|^{1-q}_{W^{1,q}}) + o(1)$$

as $k \to \infty$. This is impossible by (3.14), since $\alpha > 0$ is a fixed number.

It remains to consider the case when (3.15) holds. Arguing as above, we now get

$$0 < \alpha \leq \beta_\lambda \frac{\|u_k\|_{W^{1,p}} + \|u_k\|_{W^{1,q}}}{\|u_k\|^p_{W^{1,p}}} + o(1) = \beta_\lambda \|u_k\|^{1-p}_{W^{1,p}} + o(1)$$

as $k \to \infty$. Again this cannot occur by (3.15). The claim is now completely proved, since the other alternative can be handled in the same way.

Clearly, (3.12) follows at once by the fact that W is a reflexive Banach space, by (1.6) and by Lemma 1.2.1.

It remains to prove (3.13). To this aim, we have to distinguish three cases for the possibly degenerate nature of $(\mathcal{E}_M)$. The case $\ell_p = \ell_q = 0$ cannot occur, otherwise $u_k \to 0 = u_\lambda$ in W and so $0 = I(u_\lambda) = c_\lambda > 0$ gives the required contradiction. Hence either $\ell_p > 0$ or $\ell_q > 0$.

We mimic the argument given in the proof of Lemma 2.3.2. Take $R > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^N$, $\varphi \equiv 1$ in B_R and $\varphi \equiv 0$ in B_{2R}^c. By convexity, we get

$$\left(\left|\nabla u_k(x)\right|^{\wp-2}\nabla u_k(x) - \left|\nabla u_\lambda(x)\right|^{\wp-2}\nabla u(x)_\lambda\right)\cdot\left(\nabla u_k(x) - \nabla u_\lambda(x)\right) \ge 0 \quad \text{a.e. in } \mathbb{R}^N$$

for any $k \in \mathbb{N}$ and for $\wp \in \{p, q\}$.

Let us first suppose that $\ell_p > 0$. Thus, the well known Simon inequality, see Lemma 2.1 of [231], with $p \ge 2$, yields the existence of $c_p > 0$ such that

$$\begin{aligned}
& c_p M(\|u_k\|_{W^{1,p}}^p)\int_{B_R}|\nabla u_k - \nabla u_\lambda|^p dx \\
&\quad \le M(\|u_k\|_{W^{1,p}}^p)\int_{B_R}(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx \\
&\quad \le M(\|u_k\|_{W^{1,p}}^p)\int_{\mathbb{R}^N}\varphi(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx \\
&\quad + M(\|u_k\|_{W^{1,q}}^q)\int_{\mathbb{R}^N}\varphi(|\nabla u_k|^{q-2}\nabla u_k - |\nabla u_\lambda|^{q-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx.
\end{aligned}$$

Therefore, as $k \to \infty$,

$$\begin{aligned}
& c_p M(\|u_k\|_{W^{1,p}}^p)\int_{B_R}|\nabla u_k - \nabla u_\lambda|^p dx \\
&\quad \le \int_{\mathbb{R}^N}\varphi\{M(\|u_k\|_{W^{1,p}}^p)|\nabla u_k|^p + M(\|u_k\|_{W^{1,q}}^q)|\nabla u_k|^q\}dx \\
&\quad - \int_{\mathbb{R}^N}\varphi\{M(\|u_k\|_{W^{1,p}}^p)|\nabla u_k|^{p-2} \\
&\qquad + M(\|u_k\|_{W^{1,q}}^q)|\nabla u_k|^{q-2}\}\nabla u_k\cdot\nabla u_\lambda dx + o(1).
\end{aligned} \tag{3.18}$$

Clearly, (1.9) and (3.12) give

$$\langle I'(u_k), \varphi u_k\rangle - \langle I'(u_k), \varphi u\rangle = o(1) \quad \text{as } k \to \infty,$$

that is,

$$\begin{aligned}
& \int_{\mathbb{R}^N}\varphi\{M(\|u_k\|_{W^{1,p}}^p)|\nabla u_k|^p + M(\|u_k\|_{W^{1,q}}^q)|\nabla u_k|^q\}dx \\
& - \int_{\mathbb{R}^N}\varphi\{M(\|u_k\|_{W^{1,p}}^p)|\nabla u_k|^{p-2} + M(\|u_k\|_{W^{1,q}}^q)|\nabla u_k|^{q-2}\}\nabla u_k\cdot\nabla u_\lambda dx \\
& = \langle I'(u_k), \varphi u_k\rangle - \langle I'(u_k), \varphi u_\lambda\rangle - \int_{\mathbb{R}^N}\{M(\|u_k\|_{W^{1,p}}^p)|\nabla u_k|^{p-2}
\end{aligned}$$

$$
\begin{aligned}
&+ M(\|u_k\|^q_{W^{1,q}})|\nabla u_k|^{q-2}\}(u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx \\
&- \int_{\mathbb{R}^N} \varphi\{M(\|u_k\|^p_{W^{1,p}})|u_k|^{p-2}u_k \\
&+ M(\|u_k\|^q_{W^{1,q}})|u_k|^{q-2}u_k\}(u_k - u_\lambda)dx \\
&+ \lambda \int_{\mathbb{R}^N} \varphi f(x,u_k)(u_k - u_\lambda)dx + \int_{\mathbb{R}^N} \varphi|u_k|^{q^*}dx - \int_{\mathbb{R}^N} \varphi|u_k|^{q^*-2}u_k u_\lambda dx.
\end{aligned}
\tag{3.19}
$$

By the Hölder inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} \{M(\|u_k\|^p_{W^{1,p}})|\nabla u_k|^{p-2} + M(\|u_k\|^q_{W^{1,q}})|\nabla u_k|^{q-2}\}(u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx\right| \\
&\quad \leq \|\nabla\varphi\|_\infty \mathfrak{m}_{p,q}\left\{\|\nabla u_k\|_p^{p-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^p dx\right)^{1/p}\right. \\
&\qquad \left. + \|\nabla u_k\|_q^{q-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^q dx\right)^{1/q}\right\},
\end{aligned}
$$

where $\mathfrak{m}_{p,q} = \max\{\sup_k M(\|u_k\|^p_{W^{1,p}}),\ \sup_k M(\|u_k\|^q_{W^{1,q}})\}$. This yields by (3.12) that

$$
\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^N} &\{M(\|u_k\|^p_{W^{1,p}})|\nabla u_k|^{p-2} + M(\|u_k\|^q_{W^{1,q}})|\nabla u_k|^{q-2})\} \\
&\times (u_k - u_\lambda)\nabla u_k \cdot \nabla\varphi dx = 0.
\end{aligned}
\tag{3.20}
$$

Similarly, again by the Hölder inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} \varphi\{M(\|u_k\|^p_{W^{1,p}})|u_k|^{p-2} + M(\|u_k\|^q_{W^{1,q}})u_k^{q-2}\}u_k(u_k - u_\lambda)dx\right| \\
&\quad \leq \mathfrak{m}_{p,q}\left\{\|u_k\|_p^{p-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^p dx\right)^{1/p} + \|u_k\|_q^{q-1}\left(\int_{B_{2R}} |u_k - u_\lambda|^q dx\right)^{1/q}\right\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^N} &\varphi(M(\|u_k\|^p_{W^{1,p}})|u_k|^{p-2} \\
&+ M(\|u_k\|^q_{W^{1,q}})u_k^{q-2})u_k(u_k - u_\lambda)dx = 0.
\end{aligned}
\tag{3.21}
$$

Likewise, by $(\mathcal{F})$ and the Hölder inequality, as $k \to \infty$,

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} \varphi f(x,u_k)(u_k - u_\lambda)dx \\ &\le \int_{B_{2R}} \varphi(\theta q|u_k|^{\theta q-1}|u_k - u_\lambda| + r\,C_1|u_k|^{r-1}|u_k - u_\lambda|)dx \\ &\le C\left\{\left(\int_{B_{2R}} |u_k - u_\lambda|^{\theta q}dx\right)^{1/\theta q} + \left(\int_{B_{2R}} |u_k - u_\lambda|^r dx\right)^{1/r}\right\} \to 0, \end{aligned} \tag{3.22}$$

where $C = \theta q \sup_k \|u_k\|_{\theta q}^{\theta q-1} + r\,C_1 \sup_k \|u_k\|_r^{r-1} < \infty$ by Lemma 2.2.2, since $(u_k)_k$ is bounded in W. Finally, as $k \to \infty$,

$$\int_{\mathbb{R}^N} \varphi|u_k|^{q^*}dx - \int_{\mathbb{R}^N} \varphi|u_k|^{q^*-2}u_k u_\lambda dx \to 0, \tag{3.23}$$

by (3.12). Thus, combining (3.18)–(3.23), we obtain

$$c_p M(\|u_k\|_{W^{1,p}}^p)\int_{B_R} |\nabla u_k - \nabla u_\lambda|^p dx \le o(1) \quad \text{as } k \to \infty,$$

which implies at once that

$$\nabla u_k \to \nabla u_\lambda \quad \text{in } [L^p(B_R)]^N \text{ for all } R > 0,$$

since $M(\|u_k\|_{W^{1,p}}^p) \to M(\ell_p^p) > 0$ by (M_2) when $M(0) = 0$ and by $(\widetilde{M_2})$ in the nondegenerate case. Therefore, up to subsequence, not relabeled, we get (3.13).

It remains finally to consider the case in which $\ell_q > 0$. Again, the well known Simon inequality, with $q > 2$, yields the existence of $c_q > 0$ such that

$$\begin{aligned} c_q M(\|u_k\|_{W^{1,q}}^q)&\int_{B_R} |\nabla u_k - \nabla u_\lambda|^q dx \\ &\le M(\|u_k\|_{W^{1,q}}^q)\int_{B_R} (|\nabla u_k|^{q-2}\nabla u_k - |\nabla u_\lambda|^{q-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx \\ &\le M(\|u_k\|_{W^{1,p}}^p)\int_{\mathbb{R}^N} \varphi(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx \\ &+ M(\|u_k\|_{W^{1,q}}^q)\int_{\mathbb{R}^N} \varphi(|\nabla u_k|^{q-2}\nabla u_k - |\nabla u_\lambda|^{q-2}\nabla u_\lambda)\cdot(\nabla u_k - \nabla u_\lambda)dx. \end{aligned}$$

Therefore, as $k \to \infty$,

$$\begin{aligned} c_q M(\|u_k\|^q_{W^{1,q}}) \int_{B_R} |\nabla u_k - \nabla u_\lambda|^q dx \\ \leq \int_{\mathbb{R}^N} \varphi\{M(\|u_k\|^p_{W^{1,p}})|\nabla u_k|^p + M(\|u_k\|^q_{W^{1,q}})|\nabla u_k|^q\} dx \\ - \int_{\mathbb{R}^N} \varphi\{M(\|u_k\|^p_{W^{1,p}})|\nabla u_k|^{p-2} \\ + M(\|u_k\|^q_{W^{1,q}})|\nabla u_k|^{q-2}\}\nabla u_k \cdot \nabla u_\lambda dx + o(1). \end{aligned} \tag{3.24}$$

Again, (1.9) and (3.12) give (3.19)–(3.23). Thus, combining (3.19)–(3.23) with now (3.24), we get

$$c_q M(\|u_k\|^q_{W^{1,q}}) \int_{B_R} |\nabla u_k - \nabla u_\lambda|^q dx \leq o(1) \quad \text{as } k \to \infty,$$

which implies at once that

$$\nabla u_k \to \nabla u_\lambda \quad \text{in } [L^q(B_R)]^N \text{ for all } R > 0,$$

since $M(\|u_k\|^q_{W^{1,q}}) \to M(\ell_q^q) > 0$ by (M_2) when $M(0) = 0$ and by $(\widetilde{M_2})$ in the nondegenerate case. Therefore, up to subsequence, not relabeled, we get also in this case (3.13). □

Lemma 3.2.5. *There exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ the functional I satisfies the Palais–Smale condition at level c_λ.*

Proof. Fix $\lambda > 0$ and let $(u_k)_k \subset W$ be a $(PS)_{c_\lambda}$ sequence of I. Thus, up to a subsequence, still called $(u_k)_k$, there exists $u_\lambda \in W$ such that (3.12) and (3.13) hold thanks to Lemma 3.2.4. By (M_1), $(\mathcal{F})$, (1.9), and (3.12), we also have

$$c_\lambda + o(1) \geq \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\|u_k\|^{q^*}_{q^*} \tag{3.25}$$

as $k \to \infty$. In particular, by (3.12), (3.25) and the Brézis and Lieb lemma in [49], letting $k \to \infty$, we get

$$c_\lambda \geq \left(\frac{1}{\nu} - \frac{1}{q^*}\right)(\ell_\lambda^{q^*} + \|u_\lambda\|^{q^*}_{q^*}) \geq \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\ell_\lambda^{q^*}. \tag{3.26}$$

On the other hand, (1.9), (3.3), (3.12), and (3.13) imply that, as $k \to \infty$,

$$\begin{aligned} o(1) = \langle I'(u_k), u_k - u_\lambda\rangle = M(\|u_k\|^p_{W^{1,p}})\|u_k\|^p_{W^{1,p}} - M(\|u_k\|^p_{W^{1,p}})\langle u_k, u_\lambda\rangle_{W^{1,p}} \\ + M(\|u_k\|^q_{W^{1,q}})\|u_k\|^q_{W^{1,q}} - M(\|u_k\|^q_{W^{1,q}})\langle u_k, u_\lambda\rangle_{W^{1,q}} \end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{\mathbb{R}^N} (|u_k|^{q^*-2} u_k (u_k - u_\lambda) dx - \lambda \int\limits_{\mathbb{R}^N} f(x, u_k)(u_k - u_\lambda) dx \\
&= M(\ell_p^p)(\ell_p^p - \|u_\lambda\|_{W^{1,p}}^p) + M(\ell_q^q)(\ell_q^q - \|u_\lambda\|_{W^{1,q}}^q) \\
&\quad - \|u_k\|_{q^*}^{q^*} + \|u_\lambda\|_{q^*}^{q^*} + o(1).
\end{aligned}
$$

Thus, (3.12), (3.13), Theorem 2.7 of [100], and the Brézis and Lieb lemma yield the crucial formula

$$
\begin{aligned}
M(\ell_p^p) \lim_{k\to\infty} \|u_k - u_\lambda\|_{W^{1,p}}^p + M(\ell_q^q) \lim_{k\to\infty} \|u_k - u_\lambda\|_{W^{1,q}}^q \\
= \lim_{k\to\infty} \|u_k - u_\lambda\|_{q^*}^{q^*} = \delta_\lambda^{q^*}.
\end{aligned}
\tag{3.27}
$$

On the other hand, by Lemma 3.2.3, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$,

$$
c_\lambda < \left(\frac{1}{\nu} - \frac{1}{q^*}\right) \begin{cases} \min\{(m\,S)^{q^*/(q^*-q)}, (c\,S^\theta)^{q^*/(q^*-\theta q)}\}, \\ \qquad\qquad \text{in the degenerate case,} \\ (a\,S)^{q^*/(q^*-q)} \quad \text{in the nondegenerate case,} \end{cases}
\tag{3.28}
$$

where m is the number depending on $\tau = 1$ selected as in (M_2), c is the positive constant in assumption (M_3) in the degenerate case, while a is given in $(\widetilde{M_2})$ in the nondegenerate case. Finally, S is the best constant of the Sobolev embedding for $W^{1,q}(\mathbb{R}^N)$, given by

$$
S = \inf_{\substack{u\in W^{1,q}(\mathbb{R}^N)\\ u\neq 0}} \frac{\|u\|_{W^{1,q}}^q}{\|u\|_{q^*}^q} > 0.
\tag{3.29}
$$

Since $(\mathcal{E}_M)$ could be degenerate, that is, the Kirchhoff function M could be possibly 0 at 0, we split the proof in two steps.

Step 1. The Kirchhoff function M verifies $M(0) = 0$, (M_1), (M_2), and (M_3).

Due to the degenerate nature of $(\mathcal{E}_M)$, several situations must be considered, and we divide the proof of the current step into further three cases; we shall show that the first two cannot occur.

Case 1. *$\ell_p = 0$, but $\ell_q > 0$ for a fixed $\lambda \geq \lambda^*$.*

Of course, $u_k \to 0$ strongly in $W^{1,p}(\mathbb{R}^N)$. Thus (3.12) implies that $u_\lambda = 0$ in W. Therefore, the crucial formula (3.27) becomes

$$
M(\ell_q^q)\ell_q^q = \delta_\lambda^{q^*}.
\tag{3.30}
$$

We claim that $\delta_\lambda = 0$. Otherwise, $\delta_\lambda > 0$ and (3.29) and (3.30) imply

$$
\delta_\lambda^{q^*-q} \geq S\,M(\ell_q^q).
\tag{3.31}
$$

Since we do not know the exact behavior of M, we distinguish two situations.

If $\ell_q \geq 1$, then (M_2) gives $m > 0$, corresponding to $\tau = 1$, such that $M(\ell_q^q) \geq m$. Hence, (3.31) yields

$$\delta_\lambda^{q^*-q} \geq m\,S. \tag{3.32}$$

While if $\ell_q \in (0,1)$, then $\delta_\lambda^{q^*-q} \geq c\,S\ell_q^{q(\theta-1)} \geq c\,S^\theta \delta_\lambda^{q(\theta-1)}$ by (3.31), (M_3), and (3.29). This gives

$$\delta_\lambda^{q^*-\theta q} \geq c\,S^\theta, \tag{3.33}$$

since $\delta_\lambda > 0$ by contradiction. By combining (3.26), (3.32), and (3.33), we have

$$\begin{aligned} c_\lambda &\geq \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\ell_\lambda^{q^*} \\ &\geq \left(\frac{1}{\nu} - \frac{1}{q^*}\right)\min\left\{(m\,S)^{q^*/(q^*-q)}, \left(c\,S^\theta\right)^{q^*/(q^*-\theta q)}\right\}, \end{aligned} \tag{3.34}$$

which is impossible by (3.28). Thus $\delta_\lambda = 0$, as claimed.

But $\delta_\lambda = 0$ denies the validity of (3.30), since $\ell_q > 0$ implies $M(\ell_q^q) > 0$ by (M_2). Therefore, *Case* 1 cannot occur, as stated.

Case 2. $\ell_p > 0$ *but* $\ell_q = 0$ *for some fixed* $\lambda > 0$.

It is obvious that $u_\lambda = 0$ in W and that $u_k \to 0$ strongly in $L^{q^*}(\mathbb{R}^N)$. Therefore, (3.27) yields at once the required contradiction

$$M(\ell_p^p)\ell_p^p = 0,$$

since $\ell_p > 0$ and so $M(\ell_p^p) > 0$ by (M_2).

In conclusion, we have to consider only

Case 3. $\ell_p > 0$ *and* $\ell_q > 0$ *for some fixed* $\lambda \geq \lambda^*$.

Let us prove that $(u_k)_k$, up to a possibly further subsequence, converges strongly to u_λ in W. Arguing as above, we assert that $\delta_\lambda = 0$. Otherwise, $\delta_\lambda > 0$ so that (3.27) and (3.29) give (3.31) again, and we can proceed as in *Case* 1. Let us distinguish two situations. If $\ell_q \geq 1$, then (M_2) gives $m > 0$, corresponding to $\tau = 1$, such that $M(\ell_q^q) \geq m$. Hence (3.31) yields again (3.32). While if $\ell_q \in (0,1)$, then (3.12), (3.29), (3.31), (M_3), and Theorem 2.7 of [100] imply

$$\begin{aligned} \delta_\lambda^{q^*} &\geq S\,M(\ell_q^q)\delta_\lambda^q \geq c\,S\,\delta_\lambda^q \ell_q^{q(\theta-1)} \\ &= c\,S\,\delta_\lambda^q\left(\|u_k - u_\lambda\|_{W^{1,q}}^q + \|u_\lambda\|_{W^{1,q}}^q\right)^{\theta-1} + o(1) \\ &\geq c\,S\,\delta_\lambda^q\|u_k - u_\lambda\|_{W^{1,q}}^{q(\theta-1)} + o(1) \geq c\,S^\theta\delta_\lambda^{\theta q} + o(1) \end{aligned} \tag{3.35}$$

as $k \to \infty$. Consequently, (3.33) is valid again. Therefore, (3.26), (3.32), and (3.33) give once more (3.34), which contradicts (3.28). Thus $\delta_\lambda = 0$, as claimed. Hence,

$$\lim_{k\to\infty} \|u_k - u_\lambda\|_{q^*} = \delta_\lambda = 0.$$

Clearly, (3.27) yields that $u_k \to u_\lambda$ in W as $k \to \infty$, due to $M(\ell_p^p) > 0$ and $M(\ell_q^q) > 0$ by (M_2) and the fact that $\ell_p^p > 0$ and $\ell_q^q > 0$. This completes the proof of *Step* 1.

Step 2. The Kirchhoff function M satisfies (M_1) and $(\widetilde{M_2})$.

In this case, the proof of Step 1 simplifies further. Indeed, the argument produces the main formula (3.27), and so (3.29) gives at once

$$\delta_\lambda^{q^*} \ge S\, M(\ell_q^q)\delta_\lambda^q \ge a\, S\, \delta_\lambda^q$$

by $(\widetilde{M_2})$. Hence $\delta_\lambda = 0$ by (3.26) and (3.28). Hence, we may finish the proof of Step 2 proceeding as at the end of *Case* 3. This completes the proof. □

Proof of Theorem 3.1.1. Lemmas 3.2.2 and 3.2.5 guarantee that there exists $\lambda^* > 0$ such that for any $\lambda \ge \lambda^*$, the functional I satisfies all the assumptions of the mountain pass theorem at the level c_λ. Hence, there exists a critical point $u_\lambda \in W$ of I at level c_λ. Clearly, $u_\lambda \ne 0$, since $I(u_\lambda) = c_\lambda > 0 = I(0)$.

Furthermore, from (M_1) and $(\mathcal{F})$,

$$\begin{aligned} c_\lambda \ge \left(\frac{1}{p\theta} - \frac{1}{\nu}\right) & M(\|u_\lambda\|_{W^{1,p}}^p)\|u_\lambda\|_{W^{1,p}}^p \\ & + \left(\frac{1}{q\theta} - \frac{1}{\nu}\right) M(\|u_\lambda\|_{W^{1,q}}^q)\|u_\lambda\|_{W^{1,q}}^q, \end{aligned} \tag{3.36}$$

which yields (3.1) in the nondegenerate case by virtue of Lemma 3.2.3 and $(\widetilde{M_2})$.

In the degenerate case, we argue as follows. Suppose first that

$$\limsup_{\lambda\to\infty} \|u_\lambda\|_{W^{1,p}} = \omega_p > 0.$$

Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\|u_{\lambda_j}\|_{W^{1,p}} \to \omega_p$ as $j \to \infty$. Then (3.36) gives

$$c_{\lambda_j} \ge \left(\frac{1}{p\theta} - \frac{1}{\nu}\right) M(\|u_{\lambda_j}\|_{W^{1,p}}^p)\|u_{\lambda_j}\|_{W^{1,p}}^p.$$

Lemma 3.2.3 as $j \to \infty$ yields

$$0 \ge \left(\frac{1}{p\theta} - \frac{1}{\nu}\right) M(\omega_p^p)\omega_p^p > 0$$

by (M_2). This contradiction proves that

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{W^{1,p}} = 0. \tag{3.37}$$

We assert that

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{W^{1,q}} = 0. \tag{3.38}$$

Otherwise, $\limsup_{\lambda\to\infty} \|u_\lambda\|_{W^{1,q}} = \omega_q > 0$. Hence there is a sequence $j \mapsto \mu_j \uparrow \infty$ such that $\|u_{\mu_j}\|_{W^{1,q}} \to \omega_q$ as $j \to \infty$. Then, (3.36) gives

$$c_{\mu_j} \geq \left(\frac{1}{q\theta} - \frac{1}{\nu}\right) M(\|u_{\mu_j}\|^q_{W^{1,q}})\|u_{\mu_j}\|^q_{W^{1,q}}.$$

Lemma 3.2.3 as $j \to \infty$ yields

$$0 \geq \left(\frac{1}{q\theta} - \frac{1}{\nu}\right) M(\omega_q^q)\omega_q^q > 0,$$

again by (M_2). This contradiction proves (3.38). Of course, (3.37) and (3.38) imply at once the validity of (3.1). □

3.3 Proof of Theorem 3.1.2

We conclude this chapter proving the result stated in Theorem 3.1.2 for the nondegenerate case. For this, we need a truncation argument, as in [98, 99, 104, 105] and the references therein, in order to control the growth of the elliptic part of $(\mathcal{E}_M)$. From now until the end of the section, we require all the assumptions of Theorem 3.1.2.

Proof of Theorem 3.1.2. Take $\varepsilon \in \mathbb{R}^+$, with $0 < a \leq M(0) < \varepsilon < a\nu/q$, which is possible by (3.2). Put for all $t \in \mathbb{R}^+_0$,

$$\begin{gathered} M_\varepsilon(t) = \begin{cases} M(t), & \text{if } M(t) \leq \varepsilon, \\ \varepsilon, & \text{if } M(t) > \varepsilon, \end{cases} \quad \text{so that} \\ M_\varepsilon(0) = M(0), \quad \min_{t\in\mathbb{R}^+_0} M_\varepsilon(t) = a, \end{gathered} \tag{3.39}$$

and let $\mathscr{M}_\varepsilon(t) = \int_0^t M_\varepsilon(\tau)d\tau$. Consider the auxiliary equation in $\mathbb{R}^N$ such that

$$\begin{aligned} M_\varepsilon(\|u\|^p_{W^{1,p}})(-\Delta_p u + |u|^{p-2}u) + M_\varepsilon(\|u\|^q_{W^{1,q}})(-\Delta_q u + |u|^{q-2}u) \\ = \lambda f(x,u) + |u|^{q^*-2}u. \end{aligned} \tag{3.40}$$

We are going to solve (3.40), using the proof of Step 2 of Theorem 3.1.1, but replacing the Kirchhoff function M with M_ε.

Clearly, (3.40) can be thought as the Euler–Lagrange equation of the C^1 functional

$$I_\varepsilon(u) = \frac{1}{p}\mathscr{M}_\varepsilon(\|u\|^p_{W^{1,p}}) + \frac{1}{q}\mathscr{M}_\varepsilon(\|u\|^q_{W^{1,q}}) - \lambda \int_{\mathbb{R}^N} F(x,u)dx - \frac{1}{q^*}\|u\|^{q^*}_{q^*}$$

for all $u \in W$. For the functional I_ε, Lemmas 3.2.2 and 3.2.3 continue to hold. Indeed, for Lemma 3.2.2 it is enough to observe that (3.5) is now replaced by

$$I_\varepsilon(tu) \leq \varepsilon\left(\frac{t^p}{p} + \frac{t^q}{q}\right) - \frac{t^{q^*}}{q^*}\|u\|_{q^*}^{q^*} \to -\infty$$

as $t \to \infty$, since $p < q < q^*$. Similarly, also Lemma 3.2.3 can be proved in a simpler way. Indeed, $t_\lambda > 0$, so that (3.9) becomes

$$\begin{aligned}\varepsilon\big(t_\lambda^p\|e\|_{W^{1,p}}^p + t_\lambda^q\|e\|_{W^{1,q}}^q\big) \geq\; & t_\lambda^p M_\varepsilon\big(\|t_\lambda e\|_{W^{1,p}}^p\big)\|e\|_{W^{1,p}}^p \\ & + t_\lambda^q M_\varepsilon\big(\|t_\lambda e\|_{W^{1,q}}^q\big)\|e\|_{W^{1,q}}^q\end{aligned}$$

for any $\lambda > 0$. This and (3.8) imply at once that $\{t_\lambda\}_{\lambda>0}$ is bounded in $\mathbb{R}$. The rest of the proof is unchanged. Hence Lemmas 3.2.2–3.2.4 are valid, and it remains to prove the main Lemma 3.2.5 for I_ε.

Proceeding as in the proof of *Claim* 1 of Lemma 3.2.5, by $(\widetilde{M_2})$ now (3.17) becomes, as $n \to \infty$,

$$c_\lambda + \beta_\lambda\|u_\lambda\| + o(1) \geq \alpha\big\{\|u_n\|_{W^{1,p}}^p + \|u_n\|_{W^{1,q}}^q\big\}, \quad \alpha = \frac{a}{q} - \frac{\varepsilon}{\nu} > 0, \tag{3.41}$$

since $\varepsilon < a\nu/q$, while the other key formulas hold true with no relevant modifications. Thus, arguing as before in Step 2 of Lemma 3.2.5, we find that for all $\varepsilon \in (M(0), a\nu/q)$ there exists a suitable $\widetilde{\lambda} = \widetilde{\lambda}(\varepsilon) > 0$ such that (3.40) admits a nontrivial solution $u_\lambda \in W$, with $I_\varepsilon(u_\lambda) = c_\lambda$. Hence, (3.41) implies that for all $\lambda \geq \widetilde{\lambda}$,

$$c_\lambda \geq \alpha\big\{\|u_\lambda\|_{W^{1,p}}^p + \|u_\lambda\|_{W^{1,q}}^q\big\},$$

so that (3.1) follows at once by Lemma 3.2.3.

Fix $\varepsilon \in (M(0), a\nu/q)$. By (3.1),

$$a \leq M(0) = M_\varepsilon(0) = \lim_{\lambda\to\infty} \max\big\{M_\varepsilon\big(\|u_\lambda\|_{W^{1,p}}\big), M_\varepsilon\big(\|u_\lambda\|_{W^{1,q}}\big)\big\}.$$

Therefore, there exists $\lambda^* = \lambda^*(\varepsilon) \geq \widetilde{\lambda}$ such that

$$a \leq \max\big\{M_\varepsilon\big(\|u_\lambda\|_{W^{1,p}}\big), M_\varepsilon\big(\|u_\lambda\|_{W^{1,q}}\big)\big\} < \varepsilon \quad \text{for all } \lambda \geq \lambda^*.$$

In conclusion, for all $\varepsilon \in (M(0), a\nu/q)$ there exists a threshold $\lambda^* = \lambda^*(\varepsilon) > 0$ such that for all $\lambda \geq \lambda^*$ the solution u_λ of (3.40) is also a solution of $(\mathcal{E}_M)$. □

Comments on Chapter 3

Equation $(\mathcal{E}_M)$ possesses some interesting features: it is governed by two operators and contains a critical term. Problems in which both the p and q Laplacians appear, set also in unbounded domains, have been recently considered in the literature but, as far as we know, no problem like $(\mathcal{E}_M)$ with either p or q variable has been examined. A very interesting area of nonlinear analysis lies in the study of elliptic equations involving anisotropic elliptic operators. Recently, great attention has been focused on these problems; see, among others, the paper [8] and the references therein. From the variational viewpoint, several intriguing difficulties naturally arise in this new setting. For instance, a precise formulation of the concentration compactness principle in spaces with variable exponent need to be investigated; see [109]. We also refer to the monographs [221] for a theoretical account on these spaces. The extension to this new setting will be an object of future studies.

Part II: Existence of multiple solutions via group-theoretical invariance in the Hilbertian setting

4 Multiple solutions for critical equations in $\mathbb{R}^N$

Se non dovessi tornare,
sappiate che non sono mai
partito.
Il mio viaggiare
è stato tutto un restare
qua, dove non fui mai.

Giorgio Caproni
Biglietto lasciato prima di non andar via

In [9] A. Ambrosetti, H. Brézis, and G. Cerami studied the existence and multiplicity of solutions for semilinear elliptic Dirichlet problems in bounded domains, analyzing the combined effects of concave and convex nonlinearities with respect to a real parameter λ. Later, S. Alama and G. Tarantello in [2] dealt with a related semilinear Dirichlet problem in a bounded domain, with weighted nonlinear terms. The competing nonlinear terms combine each other, with the first being subcritical and the latter critical or supercritical. Here, following [24, 217], we prove the existence of a critical value $\lambda^* > 0$ with the property that $(\mathcal{E}_\lambda)$ admits nontrivial nonnegative entire solutions if and only if $\lambda \geq \lambda^*$. Furthermore, if $(\mathcal{E}_\lambda)$ possesses a nontrivial nonnegative entire solution for some $\lambda > 0$, then $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions. In the last section a multiplicity result has been proved in the presence of symmetries.

4.1 Nonnegative entire solutions

In this chapter we study the following one-parameter critical elliptic equation in $\mathbb{R}^N$:

$$-\Delta u + u = \lambda w(x)|u|^{m-2}u - h(x)|u|^{2^*-2}u, \quad N \geq 3, \quad 2^* = \frac{2N}{N-2}, \tag{$\mathcal{E}_\lambda$}$$

where $\lambda \in \mathbb{R}$ and the main coefficients verify

$(\mathcal{H}_1)$ (i) *The exponent m is such that* $1 < m < 2^*$;
(ii) $0 < h \in L^\infty(\mathbb{R}^N)$, $0 \leq w \in L^{2^*/(2^*-m)}(\mathbb{R}^N)$, *and* $w \not\equiv 0$.

$(\mathcal{H}_2)$ *The coefficients h and w are related by the condition that*

$$\begin{aligned} w(w/h)^{(m-1)/(2^*-m)} &\in L^{2^{*\prime}}(\mathbb{R}^N), \\ \|w(w/h)^{(m-1)/(2^*-m)}\|_{2^{*\prime}} &= \mathscr{W} \in \mathbb{R}^+ \end{aligned} \tag{4.1}$$

hold, where $2^{*\prime}$ *is the Hölder conjugate of* 2^*.

Note that the requirement that $h \in L^\infty(\mathbb{R}^N)$ stated in $(\mathcal{H}_1)$-(ii) implies that the potential $w \in L^{2^*/(2^*-m)}(\mathbb{R}^N)$ under condition $(\mathcal{H}_2)$.

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Equation $(\mathcal{E}_\lambda)$ is a semilinear elliptic problem involving a Sobolev critical nonlinearity with combined effects. For quasilinear problems of this kind, we just quote, for example, [24, 53, 209, 217] and the comments and references therein.

Clearly, $(\mathcal{E}_\lambda)$ has a variational nature and the underlying functional $I = I_\lambda$ is well defined in $H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$, which is the solution space of $(\mathcal{E}_\lambda)$ and is given by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{m}\|u\|_{m,w}^m + \frac{1}{2^*}\|u\|_{2^*,h}^{2^*}, \tag{4.2}$$

where $\|\cdot\| = \|\cdot\|_{H^1}$. We combine the main results in the following statement.

Theorem 4.1.1. *Let* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold.*

(i) *There exists* $\lambda_* \geq 0$ *such that* $(\mathcal{E}_\lambda)$ *has at least one nontrivial nonnegative entire solution for all* $\lambda > \lambda_*$ *and has no nontrivial nonnegative entire solutions for* $\lambda < \lambda_*$;

(ii) *If* $2 < m < 2^*$, *then there exists* $\lambda_* > 0$ *such that* $(\mathcal{E}_\lambda)$ *admits at least one nontrivial nonnegative entire solution if and only if* $\lambda \geq \lambda_*$;

(iii) *Furthermore, if* $(\mathcal{E}_\lambda)$ *has a nontrivial nonnegative entire solution for some* $\lambda > 0$, *then* $(\mathcal{E}_\lambda)$ *admits at least two nontrivial nonnegative entire solutions*;

(iv) *Finally, for all* $\lambda > 0$ *equation* $(\mathcal{E}_\lambda)$ *has at least two nontrivial nonnegative entire solutions, and if* $1 < m < 2$ *then* $(\mathcal{E}_\lambda)$ *possesses infinitely many solutions* $(u_k)_k$, *whose negative critical values* $c_k = I(u_k)$ *tend to* 0 *as* $k \to \infty$, *where* I *is the underlying functional of* $(\mathcal{E}_\lambda)$, *given in* (4.2).

It is worthwhile to see that

$$\int_{\mathbb{R}^N} \frac{w^{2^*/[2^*-m]}}{h^{m/[2^*-m]}}\,dx = \mathscr{H} \in \mathbb{R}^+ \tag{4.3}$$

implies that (4.1) holds. Indeed, the Hölder inequality, with $(2^*-1)m/(2^*-m)$ and $m'/2^{*\prime}$, where $2^{*\prime}$ and m' are the Hölder conjugates of 2^* and m, respectively, gives

$$\begin{aligned}\int_{\mathbb{R}^N}\left[w\left(\frac{w}{h}\right)^{(m-1)/(2^*-m)}\right]^{2^{*\prime}} dx &= \int_{\mathbb{R}^N} w^{2^{*\prime}/m}\left[\left(\frac{w^{2^*}}{h^m}\right)^{1/(2^*-m)}\right]^{2^{*\prime}/m'} dx \\ &\leq \|w\|_{2^*/(2^*-m)}^{2^{*\prime}/m}\mathscr{H}^{2^{*\prime}/m'}.\end{aligned}$$

Therefore property (4.1) holds by virtue of assumption (4.3), since $w \in L^{2^*/(2^*-m)}(\mathbb{R}^N)$ and $1 < m < 2^*$ by $(\mathcal{H}_1)$-(i).

Throughout the chapter *we require condition* $(\mathcal{H}_1)$, *without further mentioning*. Since we are interested in weighted Lebesgue spaces, denoting by ω a generic weight on $\mathbb{R}^N$ of class $L^1_{\text{loc}}(\mathbb{R}^N)$, we put for any $\wp$, with $1 < \wp < \infty$,

$$L^\wp(\mathbb{R}^N, \omega) = \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \omega^{1/\wp}|u| \in L^\wp(\mathbb{R}^N)\},$$

endowed with the norm $\|u\|_{\wp,\omega} = \|\omega^{1/\wp}u\|_{\wp}$. In particular, the next result summarizes the main properties of the weighted spaces $L^m(\mathbb{R}^N, w)$ and $L^{2^*}(\mathbb{R}^N, h)$ we are interested in. By Proposition A.6 in [24], we have

Lemma 4.1.2. *Let the weights w, h be of class $L^1_{\text{loc}}(\mathbb{R}^N)$, and let m be a finite Lebesgue exponent strictly greater than 1. Then $L^m(\mathbb{R}^N, w)$ and $L^{2^*}(\mathbb{R}^N, h)$ are separable uniformly convex Banach spaces.*

By standard Sobolev theory we have

Lemma 4.1.3. *The embeddings $H^1(\mathbb{R}^N) \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are continuous, with $\|\nabla u\|_2 \le \|u\|$ for all $u \in H^1(\mathbb{R}^N)$,*

$$\|u\|_{2^*} \le c_{2^*}\|\nabla u\|_2 \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N). \tag{4.4}$$

For any $R > 0$ and $x_0 \in \mathbb{R}^N$, the embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{\wp}(B_R(x_0))$ is compact for all $\wp$, with $1 \le \wp < 2^*$.

Lemma 4.1.4. *If either $(\mathcal{H}_2)$ or $2 < m < 2^*$ is satisfied, then the embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^m(\mathbb{R}^N, w)$ is compact.*

Proof. We divide the proof into two parts.

(*i*) *Assume that $(\mathcal{H}_2)$ is satisfied.* Again $w \in L^{\frac{2^*}{2^*-m}}(\mathbb{R}^N)$ by $(\mathcal{H}_1)$-(*ii*). Let us recall the following elementary inequality: *for every $k_1 \ge 0$, $k_2 > 0$ and α, β, with $0 < \alpha < \beta$, there exists $C_{\alpha\beta} = \frac{\beta-\alpha}{\beta}(\frac{\alpha}{\beta})^{\alpha/(\beta-\alpha)} \in (0,1)$ such that*

$$k_1|t|^\alpha - k_2|t|^\beta \le C_{\alpha\beta}k_1\left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \le k_1\left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \tag{4.5}$$

for all $t \in \mathbb{R}$. Hence, taking $k_1 = w$, $k_2 = 1/2$, $\alpha = m-1$, $\beta = 2^*-1$ in (4.5), we get, for all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\|u\|^m_{m,w} &= \int_{\mathbb{R}^N}\left(w|u|^{m-1} - \frac{1}{2}|u|^{2^*-1}\right)|u|dx + \frac{1}{2}\int_{\mathbb{R}^N}|u|^{2^*}dx \\ &\le C_1\int_{\mathbb{R}^N} w^{(2^*-1)/(2^*-m)}|u|dx + \frac{1}{2}\|u\|^{2^*}_{2^*} \\ &\le C_1\|w\|_{2^*/(2^*-m)}\|u\|_{2^*} + \frac{1}{2}\|u\|^{2^*}_{2^*} \\ &\le C_1c_{2^*}\mathscr{W}\|\nabla u\|_2 + \frac{1}{2}\|u\|^{2^*}_{2^*}\end{aligned} \tag{4.6}$$

by (4.4), where $C_1 = 2^{(m-1)/(2^*-m)}$. Therefore, for any $v \in H^1(\mathbb{R}^N)$, $v \ne 0$, putting $u = v/\|v\|$ and $C_{\mathscr{W}} = C_1c_{2^*}\mathscr{W}$, we have

$$\int_{\mathbb{R}^N} w\left|\frac{v}{\|v\|}\right|^m dx \le C_{\mathscr{W}}\left\|\nabla\frac{v}{\|v\|}\right\|_2 + \frac{1}{2}\int_{\mathbb{R}^N}\left|\frac{v}{\|v\|}\right|^{2^*}dx$$

$$\le C_{\mathscr{W}} \left\| \nabla \frac{v}{\|v\|} \right\|_2 + \frac{1}{2} \int_{\mathbb{R}^N} \left| \frac{v}{\|v\|_{2^*}} \right|^{2^*} dx$$
$$\le C_{\mathscr{W}} + 1.$$

Hence,

$$1 = \int_{\mathbb{R}^N} w \left| \frac{v}{\|v\|_{m,w}} \right|^m dx = \left(\frac{\|v\|}{\|v\|_{m,w}} \right)^m \int_{\mathbb{R}^N} w \left| \frac{v}{\|v\|} \right|^m dx$$
$$\le \left(\frac{\|v\|}{\|v\|_{m,w}} \right)^m (C_{\mathscr{W}} + 1).$$

Thus $\|v\|_{m,w} \le (C_{\mathscr{W}}+1)^{1/m}\|v\|$, and so $H^1(\mathbb{R}^N)$ is continuously embedded into $L^m(\mathbb{R}^N, w)$.

It remains to prove that the embedding is actually compact. To this aim, let $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Again, up to a subsequence, still denoted by $(u_k)_k$, we have $u_k \to u$ a. e. in $\mathbb{R}^N$ by Lemma A.10 of [24]. We claim that for all $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$\int_{\mathbb{R}^N} w|u_k - u|^m dx < \varepsilon \quad \text{for all } k \ge K. \tag{4.7}$$

Since $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, one has $\|u_k - u\|_{2^*}^{2^*} \le M$ for all $k = 1, 2, \dots$ and a suitable positive constant M. Fix $\varepsilon > 0$. By (4.5), as in the proof of (4.6), taking $k_1 = w$, $k_2 = \varepsilon/2M$, $\alpha = m - 1$, $\beta = 2^* - 1$, we get, for all $k = 1, 2, \dots,$

$$\begin{aligned} \int_{\mathbb{R}^N} w|u_k - u|^m dx &\le \int_{\mathbb{R}^N} C_\varepsilon w^{(2^*-1)/(2^*-m)}|u_k - u| dx \\ &\quad + \frac{\varepsilon}{2M} \int_{\mathbb{R}^N} |u_k - u|^{2^*} dx \\ &\le \int_{\mathbb{R}^N} C_\varepsilon w^{(2^*-1)/(2^*-m)}|u_k - u| dx + \frac{\varepsilon}{2}, \end{aligned} \tag{4.8}$$

where $C_\varepsilon = (2M/\varepsilon)^{(m-1)/(2^*-m)}$. We assert that $(w^{(2^*-1)/(2^*-m)}|u_k - u|)_k$ is uniformly integrable in $\mathbb{R}^N$. Indeed, for any measurable subset $U \subset \mathbb{R}^N$, we have

$$\int_U w^{(2^*-1)/(2^*-m)}|u_k - u| dx \le \left(\int_U w^{2^*/(2^*-m)} dx \right)^{1-1/2^*} \|u_k - u\|_{2^*}.$$

From the last inequality we get at once the assertion. Hence the Vitali convergence theorem yields

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} C_\varepsilon w(w/h)^{(m-1)/(2^*-m)}|u_k - u| dx$$

$$= \int_{\mathbb{R}^N} \lim_{k\to\infty} C_\varepsilon w(w/h)^{(m-1)/(2^*-m)}|u_k - u|dx = 0.$$

Thus there exists $K = K(\varepsilon)$ such that

$$\int_{\mathbb{R}^N} C_\varepsilon w(w/h)^{(m-1)/(2^*-m)}|u_k - u|dx < \frac{\varepsilon}{2}$$

for all $k \geq K$. In conclusion, the claim (4.7) is valid by (4.8). Therefore, the embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^m(\mathbb{R}^N, w)$ is compact.

(*iii*) *Assume* $2 < m < 2^*$. Similarly, $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^m(\mathbb{R}^N, w)$ is compact. It is simply enough to replace in the main argument $w^{(2^*-1)/(2^*-m)}$ by w. Indeed, for all $u \in H^1(\mathbb{R}^N)$, the Hölder inequality and (4.4) give

$$\begin{aligned}\int_{\mathbb{R}^N} w|u|^m dx &\leq \|w\|_{2^*/(2^*-m)}\|u\|_{2^*}^m \leq c_{2^*}^m \|w\|_{2^*/(2^*-m)}\|\nabla u\|_2^m \\ &\leq c_{2^*}^m \|w\|_{2^*/(2^*-m)}\|u\|^m.\end{aligned}$$

Thus, $\|u\|_{m,w} \leq c_{2^*}\|w\|_{2^*/(2^*-m)}^{1/m}\|u\|$ for all $u \in H^1(\mathbb{R}^N)$, that is, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N, w)$ is continuous. Let us prove that it is actually compact. Fix $(u_k)_k$, $u \in H^1(\mathbb{R}^N)$ and assume that $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then, up to a subsequence, still denoted by $(u_k)_k$, we get that $u_k \to u$ a. e. in $\mathbb{R}^N$ as $k \to \infty$. Moreover, there exists a constant $M > 0$ such that $\|u_k\| + \|u\| \leq M$. We claim that $u_k \to u$ in $L^m(\mathbb{R}^N, w)$. Fix $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon) > 0$ and $r = r(\varepsilon) > 0$ such that for any measurable subset $U \subset \mathbb{R}^N$, with $|U| < \delta$,

$$\begin{aligned}\int_U w^{2^*/(2^*-m)}dx &< \left[\frac{\varepsilon}{(Mc_{2^*})^m}\right]^{2^*/(2^*-m)}, \\ \int_{\mathbb{R}^N\setminus B_R} w^{2^*/(2^*-m)}dx &< \left[\frac{\varepsilon}{(Mc_{2^*})^m}\right]^{2^*/(2^*-m)} \quad \text{for all } R \geq r,\end{aligned}$$

since $w \in L^{2^*/(2^*-m)}(\mathbb{R}^N)$. Consequently, we have for any measurable subset $U \subset \mathbb{R}^N$, with $|U| < \delta$,

$$\begin{aligned}\int_U w|u_k - u|^m dx &\leq \left(\int_U w^{2^*/(2^*-m)}dx\right)^{(2^*-m)/2^*} \|u_k - u\|_{2^*}^m < \varepsilon, \\ \int_{\mathbb{R}^N\setminus B_R} w|u_k - u|^m dx &\leq \left(\int_{\mathbb{R}^N\setminus B_R} w^{2^*/(2^*-m)}dx\right)^{(2^*-m)/2^*} \|u_k - u\|_{2^*}^m < \varepsilon\end{aligned}$$

for all $R \geq r$. Therefore, the fact that $w|u_k - u| \to 0$ a. e. in $\mathbb{R}^N$ and the Vitali convergence theorem yield

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} w|u_k - u|^m dx = 0,$$

that is, $u_k \to u$ in $L^m(\mathbb{R}^N, w)$, as claimed. This completes the proof. □

Lemma 4.1.5. *If $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\lambda \in \mathbb{R}$ satisfy*

$$\|u\|^2 + \|u\|_{2^*,h}^{2^*} = \lambda\|u\|_{m,w}^m, \tag{4.9}$$

then $\lambda > 0$. Moreover,
(i) if $(\mathcal{H}_2)$ is valid, then $\|u\|_{m,w} \leq \kappa_1\lambda^{(m-2+2^*)/(2^*-m)m}$;
(ii) if $2 < m < 2^*$ is satisfied, then $\kappa_2\lambda^{1/(2-m)} \leq \|u\|_{m,w}$,
and the positive constants κ_1 and κ_2 are independent of u.

Proof. Let $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\lambda \in \mathbb{R}$ satisfy (4.9), then $\lambda > 0$, since $u \neq 0$.
(*i*) *Assume that* $(\mathcal{H}_2)$ *is verified*. By (4.5) and (4.9), we have

$$\begin{aligned}
\|u\|^2 + \frac{1}{2}\|u\|_{2^*,h}^{2^*} &= \int_{\mathbb{R}^N} \left[\lambda w|u|^{m-1} - \frac{1}{2}h|u|^{2^*-1}\right]|u|dx \\
&\leq C_1 \int_{\mathbb{R}^N} \lambda w(\lambda w/h)^{(m-1)/(2^*-m)}|u|dx \\
&\leq C_1\lambda^{(2^*-1)/(2^*-m)}\|w(w/h)^{(m-1)/(2^*-m)}\|_{2^{*\prime}}\|u\|_{2^*} \\
&\leq C_2\lambda^{(2^*-1)/(2^*-m)}\|\nabla u\|_2 \leq C\lambda^{(2^*-1)/(2^*-m)}\|u\|,
\end{aligned} \tag{4.10}$$

where $C = C_1 c_{2^*}\mathscr{W}$, $C_1 = 2^{(m-1)/(2^*-m)}$, as in (4.6). Thus

$$\|u\| \leq C\lambda^{(2^*-1)/(2^*-m)}. \tag{4.11}$$

By (4.9), since $\lambda > 0$, we finally get, thanks to (4.10) and (4.11),

$$\begin{aligned}
\|u\|_{m,w} &\leq \left(\frac{2}{\lambda}\left[\|u\|^2 + \frac{1}{2}\|u\|_{2^*,h}^{2^*}\right]\right)^{1/m} \\
&\leq (2C^2\lambda^{2(2^*-1)/(2^*-m)-1})^{1/m} \\
&= \kappa_1\lambda^{(m-2+2^*)/(2^*-m)m},
\end{aligned}$$

where $\kappa_1 = (2C^2)^{1/m}$.
(*ii*) *Assume that* $2 < m < 2^*$. Then Lemma 4.1.4 and (4.9) imply that

$$\|u\|_{m,w} \leq C_m\|u\| \leq C_m\lambda^{1/2}\|u\|_{m,w}^{m/2},$$

where $C_m > 0$ is the constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N, w)$. In conclusion, since $u \neq 0$ and $2 < m < 2^*$, we get $\|u\|_{m,w} \geq \kappa_2 \lambda^{1/(2-m)}$, where $\kappa_2 = C_m^{2/(2-m)}$. This completes the proof. □

We say that $u \in H^1(\mathbb{R}^N)$ is a (weak) *entire solution of* $(\mathcal{E}_\lambda)$ if

$$\langle u, v\rangle = \lambda \int_{\mathbb{R}^N} w|u|^{m-2}uvdx - \int_{\mathbb{R}^N} h|u|^{2^*-2}uvdx$$

for all $v \in H^1(\mathbb{R}^N)$. Hence the entire solutions of $(\mathcal{E}_\lambda)$ correspond to the critical points of the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{m}\|u\|_{m,w}^m + \frac{1}{2^*}\|u\|_{2^*,h}^{2^*}$$

for all $u \in H^1(\mathbb{R}^N)$.

If $(\mathcal{E}_\lambda)$ *admits a nontrivial entire solution* $u \in H^1(\mathbb{R}^N)$, *then* (4.9) *holds and so* $\lambda > 0$ *by Lemma* 4.1.5. *From now on we consider only the case* $\lambda > 0$.

Lemma 4.1.6. *Assume that* $(\mathcal{H}_2)$ *holds.*

(i) *The functional I is coercive in* $H^1(\mathbb{R}^N)$ *and any sequence* $(u_k)_k$ *in* $H^1(\mathbb{R}^N)$, *with* $(I(u_k))_k$ *bounded, admits a weakly convergent subsequence in* $H^1(\mathbb{R}^N)$;

(ii) *For a fixed* $\lambda > 0$, *all the critical points of I are uniformly bounded in* $H^1(\mathbb{R}^N)$.

Proof. (i) By (4.5), arguing as in (4.6) and using the same notations, we get, for all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \left[\frac{\lambda}{m}\|u\|_{m,w}^m - \frac{1}{2\cdot 2^*}\|u\|_{2^*,h}^{2^*}\right] + \frac{1}{2\cdot 2^*}\|u\|_{2^*,h}^{2^*} \\ &\geq \frac{1}{2}\|u\|^2 + \frac{1}{2\cdot 2^*}\|u\|_{2^*,h}^{2^*} - \int_{\mathbb{R}^N}\left[\lambda\frac{w}{m}|u|^{m-1} - \frac{h}{2\cdot 2^*}|u|^{2^*-1}\right]\cdot|u|dx \\ &\geq \frac{1}{2}\|u\|^2 + \frac{1}{2\cdot 2^*}\|u\|_{2^*,h}^{2^*} - C_1\int_{\mathbb{R}^N}\lambda w(\lambda w/h)^{(m-1)/(2^*-m)}|u|dx \\ &\geq \frac{1}{2}\|u\|^2 + \frac{1}{2\cdot 2^*}\|u\|_{2^*,h}^{2^*} - C_{\mathscr{W}}\|\nabla u\|_2 \geq \frac{1}{2}\|u\|^2 - C_{\mathscr{W}}\|u\|, \end{aligned}$$

with $C_{\mathscr{W}} = C_1 c_{2^*}\mathscr{W}$ by Lemma 4.1.3. Thus I is coercive in $H^1(\mathbb{R}^N)$.

(ii) Fix $\lambda > 0$ and let $\mathcal{S}_\lambda = \{u \in H^1(\mathbb{R}^N) : u \text{ is a critical point of } I\}$. Clearly, every $u \in \mathcal{S}_\lambda$ is a solution of $(\mathcal{E}_\lambda)$ and so satisfies (4.9). Hence Lemma 4.1.5(i) is valid. In conclusion, $\mathcal{S}_\lambda$ is bounded in $L^m(\mathbb{R}^N, w)$, and so in $H^1(\mathbb{R}^N)$ and also in $L^{2^*}(\mathbb{R}^N, h)$ by (4.9). This completes the proof. □

Lemma 4.1.7. *Assume* $(\mathcal{H}_2)$ *holds. Then I is of class* $C^1(H^1(\mathbb{R}^N))$ *and sequentially weakly lower semicontinuous in* $H^1(\mathbb{R}^N)$. *Hence, if* $u_k \rightharpoonup u$ *in* $H^1(\mathbb{R}^N)$, *then*

$$I(u) \leq \liminf_{k\to\infty} I(u_k).$$

Proof. Clearly, I is of class $C^1(H^1(\mathbb{R}^N))$. Hence it remains to show that I is sequentially weakly lower semicontinuous in $H^1(\mathbb{R}^N)$. To this aim, fix $(u_k)_k$, u in $H^1(\mathbb{R}^N)$, with $u_k \rightharpoonup u$ as $k \to \infty$.

Since $(\mathcal{H}_2)$ is satisfied, and due to the fact that $\lambda > 0$,

$$\begin{aligned}\liminf_{k\to\infty} I(u_k) &\geq \liminf_{k\to\infty}\left(\frac{1}{2}\|u_k\|^2 + \frac{1}{2^*}\|u_k\|_{2^*,h}^{2^*}\right) - \frac{\lambda}{m}\limsup_{k\to\infty}\|u_k\|_{m,w}^m \\ &\geq \frac{1}{2}\|u\|^2 + \frac{1}{2^*}\|u\|_{2^*,h}^{2^*} - \frac{\lambda}{m}\|u\|_{m,w}^m = I(u),\end{aligned}$$

by Lemma 4.1.4. Therefore I is sequentially weakly lower semicontinuous in $H^1(\mathbb{R}^N)$. □

Let $\mathscr{I} : H^1(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\mathscr{I}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2^*}\|u\|_{2^*,h}^{2^*}$$

for all $u \in H^1(\mathbb{R}^N)$ and let $\mathscr{I}' : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ be its derivative operator, where $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$. Moreover, $\mathscr{I}'$ can be represented as

$$\left\langle \mathscr{I}'(u), v\right\rangle = \langle u, v\rangle + \int_{\mathbb{R}^N} h|u|^{2^*-2}uv\,dx$$

for all $u, v \in H^1(\mathbb{R}^N)$.

Lemma 4.1.8.
(i) $\mathscr{I}' : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ *is a continuous, bounded, strictly monotone operator*;
(ii) $\mathscr{I}$ *is a mapping of type* (S_+), *i. e., if* $u_k \rightharpoonup u$ *in* $H^1(\mathbb{R}^N)$ *and*

$$\limsup_{k\to\infty}\left\langle \mathscr{I}'(u_k) - \mathscr{I}'(u), u_k - u\right\rangle \leq 0,$$

then $u_k \to u$ *in* $H^1(\mathbb{R}^N)$;
(iii) $\mathscr{I}' : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ *is a homeomorphism.*

Proof. (i) This is an obvious statement thanks to the representation of $\mathscr{I}'$.

(ii) Property (S_+) for $\mathscr{I}'$ is a direct consequence of convexity and the fact that the setting is Hilbertian.

(iii) The strict monotonicity of $\mathscr{I}'$ implies that $\mathscr{I}'$ is an injection operator. Clearly,

$$\lim_{\|u\|\to\infty}\frac{\langle \mathscr{I}'(u), u\rangle}{\|u\|} = \lim_{\|u\|\to\infty}\frac{\|u\|^2 + \|u\|_{2^*,h}^{2^*}}{\|u\|} = \infty,$$

so that $\mathscr{I}'$ is coercive in $H^1(\mathbb{R}^N)$. Hence $\mathscr{I}'$ is a surjection in view of the Minty–Browder theorem, see Theorem 26A of [258]. Thus $\mathscr{I}'$ has an inverse operator $(\mathscr{I}')^{-1}$:

$H^{-1}(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ and the continuity of $(\mathscr{I}')^{-1}$ is sufficient to ensure that $\mathscr{I}'$ is a homeomorphism.

To this aim, fix $(f_k)_k, f \in H^{-1}(\mathbb{R}^N)$, with $f_k \to f$ in $H^{-1}(\mathbb{R}^N)$. Put $u_k = (\mathscr{I}')^{-1}(f_k)$ for all k and $u = (\mathscr{I}')^{-1}(f)$. Then $\mathscr{I}'(u_k) = f_k$, $\mathscr{I}'(u) = f$, and $(u_k)_k$ is bounded in $H^1(\mathbb{R}^N)$, since $\mathscr{I}'$ is coercive in $H^1(\mathbb{R}^N)$. Without loss of generality, we assume that $u_k \rightharpoonup u_\infty$ in $H^1(\mathbb{R}^N)$ for some $u_\infty \in H^1(\mathbb{R}^N)$. Consequently, as $k \to \infty$,

$$\begin{aligned}\langle \mathscr{I}'(u_k) - \mathscr{I}'(u_\infty), u_k - u_\infty \rangle &= \langle \mathscr{I}'(u_k) - \mathscr{I}'(u), u_k - u_\infty \rangle \\ &\quad + \langle \mathscr{I}'(u) - \mathscr{I}'(u_\infty), u_k - u_\infty \rangle \\ &= \langle f_k - f, u_k - u_\infty \rangle + \langle \mathscr{I}'(u) - \mathscr{I}'(u_\infty), u_k - u_\infty \rangle \\ &= o(1),\end{aligned}$$

since $f_k \to f$ in $H^{-1}(\mathbb{R}^N)$ and $u_k \rightharpoonup u_\infty$ in $H^1(\mathbb{R}^N)$. Therefore, $u_k \to u_\infty$ as $k \to \infty$ by the (S_+) property of $\mathscr{I}'$. But $f_k \to f$ in $H^{-1}(\mathbb{R}^N)$ and $\mathscr{I}'$ is continuous in $H^1(\mathbb{R}^N)$, so that $\mathscr{I}'(u_\infty) = \lim_{k\to\infty} \mathscr{I}'(u_k) = \lim_{k\to\infty} f_k = f = \mathscr{I}'(u)$. Since $\mathscr{I}'$ is bijective, we conclude that $u_\infty = u$. Hence $(\mathscr{I}')^{-1}$ is continuous, and this completes the proof. □

Lemma 4.1.9. *If $(\mathcal{H}_2)$ holds, then I satisfies the (PS) condition, namely, $(u_k)_k \subset H^1(\mathbb{R}^N)$, with $I(u_k) \to c$ and $I'(u_k) \to 0$ in $H^{-1}(\mathbb{R}^N)$, admits a convergent subsequence in X.*

Proof. Fix $c \in \mathbb{R}$ and $(u_k)_k \subset H^1(\mathbb{R}^N)$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Then $(u_k)_k$ is bounded in $H^1(\mathbb{R}^N)$, due to I being coercive in $H^1(\mathbb{R}^N)$ by Lemma 4.1.6 and $(\mathcal{H}_2)$. Thus $(u_k)_k$ has a weakly convergent subsequence, still denoted by $(u_k)_k$, to some $u \in H^1(\mathbb{R}^N)$.

By Lemma 4.1.4 and $(\mathcal{H}_2)$, the embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^m(\mathbb{R}^N, w)$ is compact, so that $I_w'(u_k) \to I_w'(u)$ in $H^{-1}(\mathbb{R}^N)$, where $I_w(u) = \|u\|_{m,w}^m/m$. Consequently, since $(\mathscr{I}')^{-1}$ is continuous from $H^{-1}(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$ by Lemma 4.1.8, it follows that $u_k \to (\mathscr{I}')^{-1} \circ I_w'(u)$ in $H^1(\mathbb{R}^N)$, and we are done. □

For all $\lambda > 0$ and $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, put

$$g_\lambda(x, u) = -u + \lambda w|u|^{m-2}u - h|u|^{2^*-2}u.$$

Lemma 4.1.10. *Assume that $(\mathcal{H}_2)$ holds. Then $g_\lambda(\cdot, u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ along any $u \in H^1(\mathbb{R}^N)$.*

Proof. Fix $\lambda > 0$, $u \in H^1(\mathbb{R}^N)$, and $R > 0$. Clearly,

$$\int_{B_R} |u| dx = c_R < \infty,$$

$$\int_{B_R} h|u|^{2^*-1} dx \le \|h\|_\infty \int_{B_R} (|u|^{2^*} + 1) dx \le \|h\|_\infty (\|u\|_{2^*}^{2^*} + |B_R|).$$

By Lemma 4.1.4 and $(\mathcal{H}_2)$,

$$\lambda \int_{B_R} w|u|^{m-1}dx \le \lambda \int_{B_R} w(|u|^m + 1)dx \le \lambda\Big(\|u\|_{m,w}^m + \int_{B_R} wdx\Big)$$
$$\le \lambda C\Big(\|u\|^m + \int_{B_R} wdx\Big),$$

where $C = \max\{1, C_m\}$. Hence, summarizing the above inequalities, we obtain

$$\int_{B_R} |g_\lambda(x,u)|dx < \infty.$$

Thus $g_\lambda(\cdot, u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for $u \in H^1(\mathbb{R}^N)$. □

From now on we assume also condition $(\mathcal{H}_2)$, without further mentioning. Let us introduce the *crucial* value

$$\overline{\lambda} = \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ I_w(u)=1}} \Big\{\frac{1}{2}\|u\|^2 + \frac{1}{2^*}\|u\|_{2^*,h}^{2^*}\Big\} = \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ I_w(u)=1}} \mathscr{I}(u).$$

We claim that $\overline{\lambda} > 0$. Indeed, for any $u \in H^1(\mathbb{R}^N)$, with $I_w(u) = 1$, one has $\|u\|_{m,w} = m^{1/m} > 1$. By Lemma 4.1.4 and $(\mathcal{H}_2)$, there exists a constant $C_m > 0$ such that $\|u\|_{m,w} \le C_m\|u\|$ for any $u \in H^1(\mathbb{R}^N)$. Thus $\|u\| \ge 1/C_m$ for any $u \in H^1(\mathbb{R}^N)$, with $I_w(u) = 1$. Thus $\mathscr{I}(u) \ge C > 0$ for all $u \in H^1(\mathbb{R}^N)$, with $I_w(u) = 1$. Thus $\overline{\lambda} \ge C > 0$, and the claim is proved.

Lemma 4.1.11. *For all $\lambda > \overline{\lambda}$, there exists a global nontrivial nonnegative minimizer $e \in H^1(\mathbb{R}^N)$ of I, with $I(e) < 0$.*

Proof. By Lemma 4.1.6 and $(\mathcal{H}_2)$, the functional I is coercive in $H^1(\mathbb{R}^N)$, and Lemma 4.1.7 gives that I is sequentially weakly lower semicontinuous in $H^1(\mathbb{R}^N)$. Hence for all $\lambda > 0$ there exists a global minimizer $e \in H^1(\mathbb{R}^N)$ of I, that is,

$$I(e) = \inf_{v\in H^1(\mathbb{R}^N)} I(v).$$

Clearly, e is a solution of $(\mathcal{E}_\lambda)$. The definition of $\overline{\lambda}$ yields that $\inf_{v\in H^1(\mathbb{R}^N)} I(v) < 0$ for all $\lambda > \overline{\lambda}$. Thus $e \ne 0$. In conclusion, for any $\lambda > \overline{\lambda}$, equation $(\mathcal{E}_\lambda)$ has a nontrivial solution $e \in H^1(\mathbb{R}^N)$ such that $I(e) < 0$. Finally, we may assume $e \ge 0$ a. e. in $\mathbb{R}^N$, since $|e| \in H^1(\mathbb{R}^N)$ and $I(e) = I(|e|)$. □

Put $\mathcal{E} = \{\lambda \in \mathbb{R} : (\mathcal{E}_\lambda)$ admits a nontrivial nonnegative entire solution$\}$. Lemma 4.1.11 assures that $\mathcal{E}$ is nonempty. Set

$$\lambda_* = \sup\{\lambda : (\mathcal{E}_\mu) \text{ admits only the trivial solution for all } \mu < \lambda\},$$

$$\lambda_{**} = \inf\{\lambda : \lambda \in \mathcal{E}\}.$$

Clearly, $\lambda_* \geq 0$ and $\lambda_{**} \geq 0$ by Lemma 4.1.5.

Theorem 4.1.12. *For all $\lambda > \lambda_{**}$, equation $(\mathcal{E}_\lambda)$ admits a nontrivial nonnegative entire solution $u_\lambda \in H^1(\mathbb{R}^N)$. Moreover, $\lambda_* = \lambda_{**}$.*

Proof. Fix $\lambda > \lambda_{**}$. By definition of λ_{**}, there exists $\mu \in (\lambda_{**}, \lambda)$ such that I_μ has a nontrivial critical point $u_\mu \in H^1(\mathbb{R}^N)$. We assume, without loss of generality, that $u_\mu \geq 0$ a. e. in $\mathbb{R}^N$, since $|u_\mu|$ is also a solution of $(\mathcal{E})_\mu$. Of course, u_μ is a subsolution for $(\mathcal{E})_\lambda$. Only in this proof we denote explicitly the dependence of I on the parameter λ. Consider the following minimization problem:

$$\inf_{v\in\mathcal{M}} I_\lambda(v), \quad \mathcal{M} = \{v \in H^1(\mathbb{R}^N) : v \geq u_\mu\}.$$

First note that $\mathcal{M}$ is closed and convex, and in turn also weakly closed. Moreover, I_λ is coercive in $\mathcal{M}$, being coercive in $H^1(\mathbb{R}^N)$ by Lemma 4.1.6. Finally, I_λ is sequentially weakly lower semicontinuous in $H^1(\mathbb{R}^N)$ and so in $\mathcal{M}$ by Lemma 4.1.7. Hence, Corollary 3.23 of [48] assures that I_λ is bounded from below in $\mathcal{M}$ and attains its infimum in $\mathcal{M}$, i. e., there exists $u_\lambda \geq u_\mu$ such that $I_\lambda(u_\lambda) = \inf_{v\in\mathcal{M}} I_\lambda(v)$.

We claim that u_λ is a solution of $(\mathcal{E})_\lambda$. Indeed, take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Put

$$\varphi_\varepsilon = \max\{0, u_\mu - u_\lambda - \varepsilon\varphi\} \geq 0 \quad \text{and} \quad v_\varepsilon = u_\lambda + \varepsilon\varphi + \varphi_\varepsilon,$$

so that $v_\varepsilon \in \mathcal{M}$. Of course,

$$0 \leq \langle I'_\lambda(u_\lambda), v_\varepsilon - u_\lambda\rangle = \varepsilon\langle I'_\lambda(u_\lambda), \varphi\rangle + \langle I'_\lambda(u_\lambda), \varphi_\varepsilon\rangle,$$

and in turn

$$\langle I'_\lambda(u_\lambda), \varphi\rangle \geq -\frac{1}{\varepsilon}\,\langle I'_\lambda(u_\lambda), \varphi_\varepsilon\rangle. \tag{4.12}$$

Define

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N : u_\lambda(x) + \varepsilon\varphi(x) \leq u_\mu(x) < u_\lambda(x)\}.$$

Clearly, $\Omega_\varepsilon \subset \operatorname{supp}\varphi$. Since u_μ is a subsolution of $(\mathcal{E})_\lambda$ and $\varphi_\varepsilon \geq 0$, it turns out that $\langle I'_\lambda(u_\mu), \varphi_\varepsilon\rangle \leq 0$. Hence, we have

$$\begin{aligned}\langle I'_\lambda(u_\lambda), \varphi_\varepsilon\rangle &= \langle I'_\lambda(u_\mu), \varphi_\varepsilon\rangle + \langle I'_\lambda(u_\lambda) - I'_\lambda(u_\mu), \varphi_\varepsilon\rangle \\ &\leq \int_{\Omega_\varepsilon} (\nabla u_\lambda - \nabla u_\mu)\cdot\nabla(u_\mu - u_\lambda - \varepsilon\varphi)dx \\ &\quad + \int_{\Omega_\varepsilon} (u_\lambda - u_\mu)(u_\mu - u_\lambda - \varepsilon\varphi)dx \\ &\quad - \int_{\Omega_\varepsilon} (f(x,u_\lambda) - f(x,u_\mu))(u_\mu - u_\lambda - \varepsilon\varphi)dx,\end{aligned} \tag{4.13}$$

where $f(x,v) = \lambda w|v|^{m-2}v - h|v|^{2^*-2}v$. Clearly,

$$\int_{\Omega_\varepsilon} (\nabla u_\lambda - \nabla u_\mu)\cdot(\nabla u_\mu - \nabla u_\lambda)dx = -\int_{\Omega_\varepsilon} |\nabla u_\lambda - \nabla u_\mu|^2 dx \le 0,$$

while, since $0 \le u_\mu - u_\lambda - \varepsilon\varphi = u_\mu - u_\lambda + \varepsilon|\varphi| < \varepsilon|\varphi|$ in Ω_ε, we get

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} (u_\lambda - u_\mu)(u_\mu - u_\lambda - \varepsilon\varphi)dx\right| &\le \int_{\Omega_\varepsilon} |u_\lambda - u_\mu|(u_\mu - u_\lambda - \varepsilon\varphi)dx \\ &\le \varepsilon \int_{\Omega_\varepsilon} |u_\lambda - u_\mu|\cdot|\varphi|dx,\end{aligned}$$

and similarly,

$$\left|\int_{\Omega_\varepsilon} (f(x,u_\lambda) - f(x,u_\mu))(u_\mu - u_\lambda - \varepsilon\varphi)dx\right| \le \varepsilon \int_{\Omega_\varepsilon} \left|f(x,u_\lambda) - f(x,u_\mu)\right|\cdot|\varphi|dx.$$

Therefore, (4.13) yields

$$\langle I'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \le \varepsilon \int_{\Omega_\varepsilon} \psi(x)dx,$$

where $\psi = (\nabla u_\mu - \nabla u_\lambda)\cdot\nabla\varphi + (|u_\lambda - u_\mu| + |f(x,u_\lambda) - f(x,u_\mu)|)|\varphi|$. We claim that $\psi \in L^1(\operatorname{supp}\varphi)$. Indeed, ∇u_μ and ∇u_λ are in $[L^2(\mathbb{R}^N)]^N$, while u_λ and u_μ are in $L^1_{\text{loc}}(\mathbb{R}^N)$. Finally, also $|f(x,u_\lambda) - f(x,u_\mu)|$ is in $L^1_{\text{loc}}(\mathbb{R}^N)$, since

$$\left|f(x,u_\lambda) - f(x,u_\mu)\right| \le \lambda w(x)\big(|u_\lambda|^{m-1} + |u_\mu|^{m-1}\big) + h(x)\big(|u_\lambda|^{2^*-1} + |u_\mu|^{2^*-1}\big).$$

Thus,

$$\lim_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon} \psi(x)dx = 0,$$

since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$. In conclusion, $\langle I'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \le o(\varepsilon)$ as $\varepsilon \to 0^+$, so that by (4.12) it follows that $\langle I'_\lambda(u_\lambda), \varphi\rangle \ge o(1)$ as $\varepsilon \to 0^+$. Therefore, $\langle I'_\lambda(u_\lambda), \varphi\rangle \ge 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, that is, $\langle I'_\lambda(u_\lambda), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $H^1(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|}$, we obtain that u_λ is a solution of $(\mathcal{E})_\lambda$. Finally, u_λ is nontrivial and nonnegative, since $u_\lambda \ge u_\mu$.

The first part of the statement shows that $\lambda_{**} \ge \lambda_*$ Suppose by contradiction that $\lambda_{**} > \lambda_*$. Then $(\mathcal{E}_\lambda)$ cannot admit a nontrivial solution $u \in H^1(\mathbb{R}^N)$ if $\lambda < \lambda_{**}$, since this would contradict the minimality of λ_{**}. Hence, for all $\lambda \in [\lambda_*, \lambda_{**})$ the unique solution of $(\mathcal{E}_\lambda)$ is $u \equiv 0$. But this is again impossible since it would contradict the maximality of λ_*. Hence $\lambda^{**} = \lambda^*$. □

Theorem 4.1.13. *Assume* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *and let* $2 < m < 2^*$. *Then* $(\mathcal{E}_{\lambda_*})$ *admits a nontrivial nonnegative entire solution* $u \in H^1(\mathbb{R}^N)$ *and so* $\lambda_* > 0$.

Proof. Let $(\lambda_k)_k$ be a strictly decreasing sequence converging to λ_* and let u_k in $H^1(\mathbb{R}^N)$ be a nontrivial nonnegative entire solution of $(\mathcal{E}_{\lambda_k})$. Then for all $v \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla u_k, \nabla v)dx = \int_{\mathbb{R}^N} g_k v dx, \tag{4.14}$$

where $g_k = -u_k + \lambda_k w u_k^{m-1} - h u_k^{2^*-1}$ for all k.

By (4.9) and the monotonicity of $(\lambda_k)_k$, we obtain

$$\|u_k\|^2 + \|u_k\|_{2^*,h}^{2^*} = \lambda_k \|u_k\|_{m,w}^m \le C,$$

where $C = \kappa_1^m \lambda_1^{2(2^*-1)/(2^*-m)}$, thanks to Lemma 4.1.5-$(i)$ by (4.1) in $(\mathcal{H}_2)$. Therefore, the sequences $(\|u_k\|)_k$ and $(\|u_k\|_{2^*,h})_k$ are bounded. Hence, $(g_k)_k$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$, since also $(\lambda_k)_k$ is bounded. Moreover, by Lemma 4.1.4, it is possible to extract a subsequence, still denoted $(u_k)_k$, satisfying

$$\begin{gathered} u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^N), \quad u_k \to u \text{ in } L^m(\mathbb{R}^N, w), \\ u_k \rightharpoonup u \text{ in } L^{2^*}(\mathbb{R}^N, h), \quad u_k \to u \text{ a. e. in } \mathbb{R}^N, \\ \nabla u_k \rightharpoonup \nabla u \text{ in } [L^2(\mathbb{R}^N)]^N, \end{gathered} \tag{4.15}$$

for some $u \in H^1(\mathbb{R}^N)$. We claim that u, which is clearly nonnegative in $\mathbb{R}^N$ by (4.1), is the solution we are looking for.

Indeed, for all $v \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla u_k, \nabla v)dx \to \int_{\mathbb{R}^N} (\nabla u, \nabla v)dx, \quad \int_{\mathbb{R}^N} u_k v dx \to \int_{\mathbb{R}^N} u v dx \tag{4.16}$$

as $k \to \infty$ by (4.1). Again, for all $v \in H^1(\mathbb{R}^N)$,

$$\begin{gathered} \int_{\mathbb{R}^N} w|u_k|^{m-2} u_k v dx \to \int_{\mathbb{R}^N} w|u|^{m-2} u v dx, \\ \int_{\mathbb{R}^N} h|u_k|^{2^*-2} u_k v dx \to \int_{\mathbb{R}^N} h|u|^{2^*-2} u v dx, \end{gathered} \tag{4.17}$$

as $k \to \infty$ by (4.1) and Lemma 4.1.4. In conclusion, passing to the limit as $k \to \infty$ in (4.14), we get by (4.16)–(4.17) that u is a nonnegative entire solution of $(\mathcal{E}_{\lambda_*})$.

We claim that $u \not\equiv 0$. Indeed, since $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ by (4.1), Lemma 4.1.4 yields in particular that $\|u\|_{m,w} = \lim_{k\to\infty} \|u_k\|_{m,w}$. Moreover, Lemma 4.1.5$(ii)$ applied to each $u_k \not\equiv 0$ implies that $\|u_k\|_{m,w} \ge \kappa_2 \lambda_k^{1/(2-m)} \ge \kappa_2 \lambda_1^{1/(2-m)}$, since $\lambda_k \searrow \lambda_*$ and $2 < m$. Consequently, $\|u\|_{m,w} \ge \kappa_2 \lambda_1^{1/(2-m)} > 0$. Hence u is nontrivial and nonnegative by (4.1). Lemma 4.1.5 yields now that $\lambda_* > 0$. ☐

4.2 Proof of Theorem 4.1.1

Thanks to the preliminary key results of the previous section, we are now able to prove Theorem 4.1.1. However, for the sake of clarity, we shall divide the proof in two parts. The first contains (*i*)–(*iii*), and the latter shows (*iv*).

First part of the proof of Theorem 4.1.1. (*i*) Theorem 4.1.12 says that there exists $\lambda_* \geq 0$ such that $(\mathcal{E}_\lambda)$ has at least a nontrivial nonnegative entire solution for $\lambda > \lambda_*$ and $(\mathcal{E}_\lambda)$ has no nonnegative entire solution for $\lambda < \lambda_*$ by definition of λ_*.

(*ii*) Lemmas 4.1.5 and 4.1.11 as well as Theorems 4.1.12 and 4.1.13 show the existence of $\lambda_* > 0$ such that $(\mathcal{E}_\lambda)$ admits at least a nontrivial nonnegative entire solution if and only if $\lambda \geq \lambda_*$.

(*iii*) Let $(\mathcal{E}_\lambda)$ possess a nontrivial nonnegative entire solution $u_1 \in H^1(\mathbb{R}^N)$. We claim that $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions.

To this aim let us consider

$$-\Delta u + u + h|u|^{2^*-2}u = \lambda w|u|^{m-2}u^+, \tag{4.18}$$

where $u^+ = \max\{u, 0\}$. The embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact by Lemma 4.1.4 and $(\mathcal{H}_2)$. Thus, recalling that $I_w(u) = \|u\|^m_{m,w}/m$, we have that $I'_w : H^1(\mathbb{R}^N) \to (L^m(\mathbb{R}^N, w))'$ is compact, i. e., if $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ then $I'_w(u_k) \to I'_w(u)$ in $(L^m(\mathbb{H}^n, w))'$.

Now $\mathscr{I}' : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is a homeomorphism by Lemma 4.1.8(*iii*). Hence u is a solution of (4.18) if and only if u is a solution of the operator equation $u = (\mathscr{I}')^{-1} \circ I'_w(u^+)$. Let $\delta \in [0, 1]$ and consider

$$u = (\mathscr{I}')^{-1} \circ I'_{\delta w}(u^+). \tag{4.19}$$

Define $G : [0, 1] \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ by $G(\delta, u) = (\mathscr{I}')^{-1} \circ I'_{\delta w}(u^+)$ for all $(\delta, u) \in [0, 1] \times H^1(\mathbb{R}^N)$. Thus G is continuous and compact. Lemma 4.1.6(*ii*) yields that all the solutions of (4.19) are uniformly bounded in $H^1(\mathbb{R}^N)$, so that there exists $R > 0$ sufficiently large such that (4.19) has no solutions on $\partial B_R \subset H^1(\mathbb{R}^N)$. Therefore,

$$\deg_{LS}(I - G(1, \cdot), B_R, 0) = \deg_{LS}(I - G(0, \cdot), B_R, 0) = 1.$$

Since (4.18) has the trivial solution zero and the nontrivial nonnegative entire solution u_1, then (4.18) has another nontrivial entire solution $u_2 \in H^1(\mathbb{R}^N)$.

We claim that u_2 is nonnegative. Suppose the contrary. Put $u_2^- = \max\{-u_2, 0\}$. Then $u_2^- \in H^1(\mathbb{R}^N)$ and take u_2^- as a test function. Therefore by (4.18),

$$0 = \lambda \int_{\mathbb{R}^N} w|u_2|^{m-2}u_2^+ u_2^- dx = -\|u_2^-\|^2 - \|u^-\|^{2^*}_{2^*,h} \leq 0.$$

In conclusion, $\|u_2^-\| = 0$, that is, $u_2^- = 0$, as required. Thus u_2 is nonnegative. Finally, u_2 is a nontrivial nonnegative solution of $(\mathcal{E}_\lambda)$. □

It remains to prove (*iv*) of Theorem 4.1.1. Before doing it, let us present some introductory properties. Since $H^1(\mathbb{R}^N)$ is separable and clearly reflexive, there exist two sequences $(e_j)_j \subset H^1(\mathbb{R}^N)$ and $(e_j^*)_j \subset H^{-1}(\mathbb{R}^N)$ such that

$$H^1(\mathbb{R}^N) = \overline{\operatorname{span}}\,\{e_j, j = 1, 2, \dots\}, \quad H^{-1}(\mathbb{R}^N) = \overline{\operatorname{span}}^{w^*}\{e_j^*, j = 1, 2, \dots\},$$

and $< e_i^*, e_j >= \delta_{ij}$, $i, j = 1, 2, \dots$, where $< \cdot, \cdot >$ is the dual pairing between $H^1(\mathbb{R}^N)$ and its dual space $H^{-1}(\mathbb{R}^N)$, while δ_{ij} denotes the Kronecker symbol and $\overline{}^{w^*}$ is the closure of a subset of $H^{-1}(\mathbb{R}^N)$ with respect to the weak-star topology on $H^{-1}(\mathbb{R}^N)$. For brevity, we put

$$X_j = \operatorname{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{4.20}$$

Let us state for completeness a useful corollary of the general Lemma 5.1 proved in [217], which we state in our context.

Lemma 4.2.1. *Let $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ be sequentially weakly continuous in $H^1(\mathbb{R}^N)$, with $\Phi(0) = 0$. Fix $R > 0$ and put*

$$\beta_k = \sup\{\Phi(u) \,:\, \|u\| \le R,\ u \in Z_k\}$$

for all k. Then $\beta_k \to 0$ as $k \to \infty$.

Last part of the proof of Theorem 4.1.1. (*iv*) Fix $\lambda > 0$ and recall that by assumption

$$1 < m < 2 \tag{4.21}$$

holds.

The functional I is weakly lower semicontinuous in $H^1(\mathbb{R}^N)$ by Lemma 4.1.7 and also coercive in $H^1(\mathbb{R}^N)$ by Lemma 4.1.6(*i*). Hence, by Lemma 4.1.11, the functional I attains its infimum at some nontrivial nonnegative function $e \in H^1(\mathbb{R}^N)$ and, clearly, e is a solution of $(\mathcal{E}_\lambda)$, with $I(e) < 0$.

By Theorem 4.1.1(*iii*), equation $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions in $H^1(\mathbb{R}^N)$.

Thanks to the fact that in this case $1 < m < 2$, we claim that $(\mathcal{E}_\lambda)$ has a sequence of solutions $(\pm u_k)_k$ such that $I(\pm u_k) < 0$ and $I(\pm u_k) \to 0$ as $k \to \infty$.

The functional I is even in $H^1(\mathbb{R}^N)$. Moreover, I is coercive in $H^1(\mathbb{R}^N)$ by Lemma 4.1.6(*i*) and I satisfies the (*PS*) condition in $H^1(\mathbb{R}^N)$ by Lemma 4.1.9. Using Definition 5.1 on page 94 of [238], we denote by $\gamma(B)$ the genus of $B \in \mathscr{C}$, where

$$\mathscr{C} = \{B \subset H^1(\mathbb{R}^N)\setminus\{0\} \,:\, B \text{ is compact and } B = -B\},$$
$$\mathscr{C}_k = \{B \in \mathscr{C} \,:\, \gamma(B) \ge k\}, \quad c_k = \inf_{B \in \mathscr{C}_k} \sup_{u \in B} I(u), \quad k = 1, 2, \dots$$

Thus

$$-\infty < c_1 \le c_2 \le \cdots \le c_k \le c_{k+1} \le \cdots.$$

We assert that $c_k < 0$ for every k.

Fix $k \in \mathbb{N}$ and choose a k-dimensional linear subspace F_k of $C_c^\infty(\mathbb{R}^N)$. Since all the norms on F_k are equivalent, there exists $\rho_k \in (0,1)$ such that $\varphi \in F_k$ and $\|\varphi\| \le \rho_k$ implies that $\|\varphi\|_\infty \le \delta < 1$. Put

$$S_{\rho_k}^{(k)} = \{u \in F_k \,:\, \|u\| = \rho_k\}.$$

From the compactness of $S_{\rho_k}^{(k)}$ and the fact that $w > 0$ in Ω, for all k there exist constants $\theta_k, \eta_k > 0$ such that for all $\varphi \in S_{\rho_k}^{(k)}$,

$$I_w(\varphi) = \frac{1}{m}\int_{\mathbb{R}^N} w|\varphi|^m dq \ge \frac{1}{m}\int_\Omega w(q)|\varphi|^m dq \ge \theta_k \quad \text{and} \quad \mathscr{I}(\varphi) \le \eta_k.$$

Therefore, for $\varphi \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$,

$$I(t\varphi) = \mathscr{I}(t\varphi) - \lambda I_w(t\varphi) \le \eta_k(t^2 + t^{2^*}) - \lambda\,\theta_k t^m.$$

Since $1 < m < 2$ by (4.21), for all k there exist $t_k \in (0,1)$ and $\varepsilon_k > 0$ so small that for all $\varphi \in S_{\rho_k}^{(k)}$,

$$I(t_k\varphi) \le -\varepsilon_k < 0, \quad \text{that is, } I(u) \le -\varepsilon_k < 0$$

for all $u \in S_{t_k\rho_k}^{(k)}$. Finally, $\gamma(S_{t_k\rho_k}^{(k)}) = k$, so that $c_k \le -\varepsilon_k < 0$ for all k and the assertion is proved.

By the genus theory, see, for instance, Theorem 4.2. and the remark on page 97 of [238], each c_k is a critical value of I. Hence there is a sequence of solutions $(\pm u_k)_k$ such that $I(\pm u_k) < 0$. It only remains to show that $c_k \to 0$ as $k \to \infty$.

Since I is coercive in $H^1(\mathbb{R}^N)$ by Lemma 4.1.6(i), there exists a constant $R > 0$ such that $I(u) > 0$ for all u, with $\|u\| \ge R$. Fix k and let Y_k, Z_k be as in (4.20). Take $B \in \mathscr{C}_k$, so that $\gamma(B) \ge k$. Therefore, according to the properties of genus, $B \cap Z_k \ne \emptyset$. Put

$$\beta_k = \sup\{\lambda\, I_w(u) \,:\, u \in Z_k,\ \|u\| \le R\}.$$

Thus $\beta_k \to 0$ as $k \to \infty$ by Lemma 4.2.1, since I_w is sequentially weakly continuous in $H^1(\mathbb{R}^N)$ by Lemma 4.1.4. If $u \in Z_k$ and $\|u\| \le R$, then

$$I(u) = \mathscr{I}(u) - \lambda I_w(u) \ge -\lambda I_w(u) \ge -\beta_k.$$

Hence $\sup_{u\in B} I(u) \ge -\beta_k$, and so $0 > c_k \ge -\beta_k$. This implies at once that $c_k \to 0$ as $k \to \infty$. □

4.3 Sign-changing multiple solutions

We study now the existence of multiple solutions (radial and nonradial) for $(\mathcal{E}_\lambda)$ assuming that the weights h and w are radial in $\mathbb{R}^N$ and possibly higher dimensions. More precisely, we are going to prove

Theorem 4.3.1. *Let $N \geq 3$. Suppose that $(\mathcal{H}_2)$ holds and that w and h are radial in $\mathbb{R}^N$. Then, there exists $\overline{\lambda} > 0$ such that for all $\lambda > \overline{\lambda}$ equation $(\mathcal{E}_\lambda)$ admits a nontrivial nonnegative radial solution and ζ_N sign-changing solutions, with mutually symmetric different structures, where*

$$\zeta_N = \begin{cases} 0, & \text{if } N = 3, \\ (-1)^N + [\frac{N-3}{2}], & \text{if } N \geq 4. \end{cases} \tag{4.22}$$

All the ζ_N sign-changing solutions have negative energy.

Clearly, $\zeta_5 = 0$, as we shall see in the construction of the symmetries and $[\cdot]$ in (4.22) denotes the integer part of a real number.

We make use the group-theoretical construction given in [148, Section 2.2]. More precisely, let either $N = 4$ or $N \geq 6$ and consider the subgroup $H_{N,i} \subset O(N)$ defined by

$$H_{N,i} = \begin{cases} O(N/2) \times O(N/2) & \text{if } i = \frac{N-2}{2}, \\ O(i+1) \times O(N-2i-2) \times O(i+1) & \text{if } i \neq \frac{N-2}{2}, \end{cases}$$

for every $i \in J_N = \{1, \ldots, \zeta_N\}$, where ζ_N is introduced in (4.22). Let us introduce the involution $\eta_{H_{N,i}} : \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$\eta_{H_{N,i}}(x) = \begin{cases} (x_3, x_1) & \text{if } i = \frac{N-2}{2},\ x = (x_1, x_3) \in \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}, \\ (x_3, x_2, x_1) & \text{if } i \neq \frac{N-2}{2}, x = (x_1, x_2, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{N-2i-2} \times \mathbb{R}^{i+1}, \end{cases}$$

for every $i \in J_N$. By definition, $\eta_{H_{N,i}} \notin H_{N,i}$, as well as

$$\eta_{H_{N,i}} H_{N,i} \eta_{H_{N,i}}^{-1} = H_{N,i}, \quad \text{and} \quad \eta_{H_{N,i}}^2 = \mathrm{id}_{\mathbb{R}^N},$$

for every $i \in J_N$. Moreover, for every $i \in J_N$, let us consider the compact group

$$H_{N,\eta_i} = \langle H_{N,i}, \eta_{H_{N,i}} \rangle, \quad \text{that is, } H_{N,\eta_i} = H_{N,i} \cup (\eta_{H_{N,i}} H_{N,i}),$$

and the action $\circledast_i : H_{N,\eta_i} \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ of H_{N,η_i} on $H^1(\mathbb{R}^N)$, given by

$$g \circledast_i u(x) = \begin{cases} u(g^{-1}x) & \text{if } g \in H_{N,i}, \\ -u(\tau^{-1}\eta_{H_{N,i}}^{-1} x) & \text{if } g = \eta_{H_{N,i}} \tau \in H_{N,\eta_i} \setminus H_{N,i}, \tau \in H_{N,i}, \end{cases} \tag{4.23}$$

for a. e. $x \in \mathbb{R}^N$.

Note that $\circledast_i$ is defined for every element of H_{N,η_i}. Indeed, if $g \in H_{N,\eta_i}$, then either $g \in H_{N,i}$ or $g = \eta_{H_{N,i}}\tau \in H_{N,\eta_i} \setminus H_{N,i}$, with $\tau \in H_{N,i}$. Moreover, set $E_i = \mathrm{Fix}_{H_{N,\eta_i}}(H^1(\mathbb{R}^N))$, that is,

$$E_i = \{u \in H^1(\mathbb{R}^N) \,:\, g \circledast_i u = u \text{ for all } g \in H_{N,\eta_i}\}$$

for every $i \in J_N$.

Fix $i \in J_N$. Every nonzero element of the space E_i changes sign in $\mathbb{R}^N$. Indeed, if $u \in E_i \setminus \{0\}$, then $u(x) = -u(g_i^{-1}x)$ for every $x \in \mathbb{R}^N$ thanks to the H_{N,η_i}-invariance of u and to (4.23). Consequently, u should change sign in $\mathbb{R}^N$, since $u \neq 0$.

Finally, for all $u \in H^1(\mathbb{R}^N)$ and for a. e. $x \in \mathbb{R}^N$,

$$g \circledast_0 u(x) = u(g^{-1}x) \quad \text{for all } g \in O(N). \tag{4.24}$$

Then, putting $E_0 = \{u \in H^1(\mathbb{R}^N) \,:\, g \circledast_0 u = u \text{ for all } g \in O(N)\}$, we get that $E_0 = H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and the following facts:

if $N = 4$ *or* $N \geq 6$, then

$$E_i \cap E_0 = \{0\} \quad \text{for every } i \in J_N; \tag{4.25}$$

if $N = 6$ *or* $N \geq 8$, then

$$E_i \cap E_j = \{0\} \quad \text{for every } i, j \in J_N, \text{ with } i \neq j. \tag{4.26}$$

We refer the interested reader to [148, Theorem 2.2] for details.

Naturally, F_0 is well defined also when $N = 3$ and $N = 5$. In conclusion, the number ζ_N in (4.22) is well defined only when either $N = 4$ or $N \geq 6$, but we extended the definition of ζ_N for all $N \geq 3$, putting $\zeta_N = 0$ and $J_N = \emptyset$, when $N = 3$ and $N = 5$. Finally, set $\tilde{J}_N = J_N \cup \{0\}$ and $G_i = H_{N,\eta_i}$ if $i \in J_N$, while $G_0 = O(N)$.

Let us introduce for all $i \in \tilde{J}_N$ the new *crucial* values

$$\overline{\lambda}_i = \inf_{\substack{u \in E_i \\ I_w(u)=1}} \left\{ \frac{1}{2}\|u\|^2 + \frac{1}{2^*}\|u\|^{2^*}_{2^*,h} \right\} = \inf_{\substack{u \in E_i \\ I_w(u)=1}} \mathscr{I}(u).$$

We claim that $\overline{\lambda}_i > 0$. Indeed, for any $u \in E_i$, with $I_w(u) = 1$, then $\|u\|_{m,w} = m^{1/m} > 1$. By Lemma 4.1.4 and $(\mathcal{H}_2)$, there exists a constant $C_m > 0$ such that $\|u\|_{m,w} \leq C_m\|u\|$ for any $u \in E_i$. Thus $\|u\| \geq 1/C_m$ for any $u \in E_i$, with $I_w(u) = 1$. Therefore, $\mathscr{I}(u) \geq C > 0$ for all $u \in E_i$, with $I_w(u) = 1$. Thus $\overline{\lambda}_i \geq C > 0$, and the claim is proved.

Proof of Theorem 4.3.1. Suppose $N \geq 3$ and let us first claim that for all $\lambda > \overline{\lambda}$, where

$$\overline{\lambda} = \max\{\overline{\lambda}_i \,:\, i \in \tilde{J}_N\},$$

there exist a radial nonnegative minimizer $e_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ of I in $E_0 = H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and ζ_N sign changing minimizers $e_i \in E_i$ of I in E_i for all $i \in J_N$, with $I(e_i) < 0$ for all $i \in \tilde{J}_N$.

Indeed, by Lemma 4.1.6 and $(\mathcal{H}_2)$, the functional I is coercive in E_i, and Lemma 4.1.7 gives that I is sequentially weakly lower semicontinuous in $H^1(\mathbb{R}^N)$, and so in E_i. Hence for all $\lambda > 0$ there exists a global minimizer $e_i \in E_i$ of I in E_i, that is,

$$I(e_i) = \inf_{v \in E_i} I(v).$$

Clearly, e_i is a critical point of $I|_{E_i}$ in E_i for all $i \in \tilde{J}_N$. The definition of $\bar{\lambda}$ yields that $\inf_{v \in E_i} I(v) < 0$ for all $\lambda > \bar{\lambda}$. Thus $e_i \neq 0$ for all $i \in \tilde{J}_N$. The radial case $j = 0$ can be treated exactly as in the proof of Lemma 4.1.11, since $e_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ gives at once that $|e_0|$ is still in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Finally, the constructed $1 + \zeta_N$ minimizers are distinct by (4.25) and (4.26).

The functional I is even in $H^1(\mathbb{R}^N)$, so that (4.23) and (4.24), as well as the fact that h and w are radial in $\mathbb{R}^N$, imply that $I(g \circledast_i u) = I(u)$ for every $g \in G_i$, each $u \in E_i$, and all $i \in \tilde{J}_N$. Therefore, I is G_i-invariant on $H^1(\mathbb{R}^N)$. Indeed, G_i acts isometrically on $H^1(\mathbb{R}^N)$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(\frac{\lambda}{m} w(x)|g \circledast_i u(x)|^m - \frac{h(x)}{2^*}|g \circledast_i u(x)|^{2^*}\right)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\lambda}{m} w(x)|u(g^{-1}x)|^m - \frac{h(x)}{2^*}|u(g^{-1}x)|^{2^*}\right)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\lambda}{m} w(y)|u(y)|^m - \frac{h(y)}{2^*}|u(y)|^{2^*}\right)dy\end{aligned}$$

if either $g \in G_i = H_{N,\eta_i}$ for $i \in J_N$ or $g \in G_0 = O(N)$ for $N \in \{3, 5\}$ thanks to (4.23) and (4.24), while

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(\frac{\lambda}{m} w(x)|g \circledast_i u(x)|^m - \frac{h(x)}{2^*}|g \circledast_i u(x)|^{2^*}\right)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\lambda}{m} w(x)\left|u(\tau^{-1}\eta^{-1}_{H_{N,i}}x)\right)\right|^m - \frac{h(x)}{2^*}|u(\tau^{-1}\eta^{-1}_{H_{N,i}}x)|^{2^*})dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\lambda}{m} w(y)|u(y)|^m - \frac{h(y)}{2^*}|u(y)|^{2^*}\right)dy\end{aligned}$$

if $g = \eta_{H_{N,i}}\tau \in H_{N,\eta_i} \setminus H_{N,i}$ for $i \in J_N$ thanks to (4.23).

By the principle of symmetric criticality due to R. Palais in [202], see also Theorem A.1.2, the critical points e_i, $i \in \tilde{J}_N$, of $I|_{E_i}$ in E_i are also critical points of I in $H^1(\mathbb{R}^N)$. In summary, we have shown the needed multiplicity result. □

Comments on Chapter 4

Elliptic problems in bounded domains involving concave and convex terms have been studied extensively since the work [9]; see, among others, the paper [248] and the references therein. The combined effect of concave and convex nonlinearities on the number of positive solutions for semilinear elliptic equations in the entire space $\mathbb{R}^N$ and involving sign-changing weight functions is an argument of genuine mathematical interest. Some results in this direction are obtained in Theorems 1.1 and 1.2 of [248] by using the Nehari manifold method and some classical theorems due to B. Gidas, W. M. Ni, and L. Nirenberg [117]. An interplay between symmetries results and classical straightforward minimization arguments seems to be fruitful in order to get new multiplicity results for elliptic problems governed by sign-changing weight functions.

5 Weak solutions of a scalar field equation

Lo spirito ha bisogno del finito
per incarnare slanci d'infinito.

Maria Luisa Spaziani
from *Lo spirito ha bisogno del finito*

In this chapter we study the multiplicity of solutions for a class of Dirichlet eigenvalue problems defined on strip-like domains of the Euclidean space $\mathbb{R}^N$, with $N \geq 3$. The first main result presented in Section 5.2 is based on an abstract critical point theorem for smooth functionals and establishes the existence of multiple solutions which are not cylindrically symmetric for certain eigenvalues. Afterwards, in Section 5.3, the classical fountain theorem provides not only a finite number of infinitely many cylindrically symmetric solutions but also cylindrically nonsymmetric solutions for special dimensions.

In both cases a crucial role in our approach is played by the principle of symmetric criticality for smooth functionals and by a group-theoretical approach on certain subgroups of the orthogonal group $O(N)$ in $\mathbb{R}^N$ developed in Section 5.1.

The theorems presented here represent a more precise form of some results already known on the same subject contained in [140, 141, 148, 151].

5.1 Sobolev spaces with symmetry on strip-like domains

In this section we give some preliminaries and introduce suitable group-theoretical arguments that will be fundamental in the approach used along this chapter. In the sequel $\mathcal{O}$ is a bounded open set in $\mathbb{R}^m$, $m \geq 1$, with a smooth boundary $\partial\mathcal{O}$, and finally $\mathcal{O} \times \mathbb{R}^{N-m}$ is the main strip-like domain in $\mathbb{R}^N$, with $N \geq m+2$. A point $(x, y) \in \mathcal{O} \times \mathbb{R}^{N-m}$ is of the form $(x_1, \ldots, x_m, y_1, \ldots, y_{N-m})$; see Figure 5.1 below.

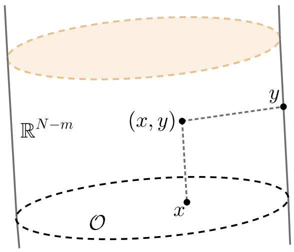

Figure 5.1: A strip-like domain $\mathcal{O} \times \mathbb{R}^{N-m}$ in $\mathbb{R}^N$.

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Furthermore, $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ denotes the usual Hilbert space endowed with the inner product

$$\langle u, v \rangle = \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \nabla u \cdot \nabla v \, dxdy$$

and the induced norm

$$\|u\| = \Bigg(\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla u|^2 dxdy \Bigg)^{1/2},$$

while $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$, with $\wp \in [1, \infty]$, is the classical Lebesgue space, having the norm defined by

$$\|u\|_{\wp} = \begin{cases} (\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |u|^{\wp} \, dxdy)^{1/\wp}, & \text{if } \wp \in [1, \infty), \\ \inf\{c \geq 0 : |u| \leq c \text{ a. e. in } \mathcal{O} \times \mathbb{R}^{N-m}\}, & \text{if } \wp = \infty. \end{cases}$$

Since the embedding $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m}) \hookrightarrow L^{\wp}(\mathcal{O}\times\mathbb{R}^{N-m})$ is continuous for any $\wp \in [2, 2^*]$, there exists $c_{\wp} > 0$ such that

$$\|u\|_{\wp} \leq c_{\wp}\|u\| \quad \text{for any } u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}). \tag{5.1}$$

Let $(O(N-m), \cdot)$ be the orthogonal group in $\mathbb{R}^{N-m}$ and consider the group

$$\widehat{O}(N-m) = \{\mathbb{I}_m\} \times O(N-m),$$

where $\mathbb{I}_m$ is the identity matrix of order m. The natural multiplication in $\widehat{O}(N-m)$ maps any pair $(\widehat{g}_1, \widehat{g}_2)$ into

$$\widehat{g}_1 \cdot \widehat{g}_2 = \mathbb{I}_m \times (g_1 g_2) \quad \text{for any } \widehat{g}_1 = \mathbb{I}_m \times g_1, \quad \widehat{g}_2 = \mathbb{I}_m \times g_2 \in \widehat{O}(N-m).$$

Clearly, every $\widehat{g} \in \widehat{O}(N-m)$ can be identified canonically with the element of $O(N)$ given by

$$\widehat{g} = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & g \end{pmatrix},$$

for some $g \in O(N-m)$. Taking into account the above remark, it is easily seen that every group $\widehat{O}(N-m)$ is isomorphic to a subgroup of $O(N)$.

Let $+ : (\mathbb{R}^m \times \mathbb{R}^{N-m}) \times (\mathbb{R}^m \times \mathbb{R}^{N-m}) \to \mathbb{R}^m \times \mathbb{R}^{N-m}$ be the natural addition law given by

$$(x, y) + (x', y') = (x + y, x' + y') \quad \text{for all } (x, y), (x', y') \in \mathbb{R}^m \times \mathbb{R}^{N-m}.$$

The next lemma states that the group $\widehat{O}(N-m)$ acts continuously and left-distributively on $(\mathbb{R}^m \times \mathbb{R}^{N-m}, +)$ by the map

$$* : \widehat{O}(N-m) \times \mathbb{R}^m \times \mathbb{R}^{N-m} \to \mathbb{R}^m \times \mathbb{R}^{N-m},$$

defined by

$$(\widehat{g}, (x,y)) \mapsto \widehat{g} * (x,y) = (x, gy)$$

for every $\widehat{g} \in \widehat{O}(N-m)$ and $(x,y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$.

Lemma 5.1.1. *The group $\widehat{O}(N-m)$ acts continuously $\mathbb{R}^m \times \mathbb{R}^{N-m}$ by $*$, i. e., the following conditions hold*:

(g_1) $(\mathbb{I}_m \times \mathbb{I}_{N-m}) * (x,y) = (x,y)$ *for every* $(x,y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$;

(g_2) $(\widehat{g}_1\widehat{g}_2) * (x,y) = \widehat{g}_1 * (\widehat{g}_2 * (x,y))$ *for every pair of elements* $\widehat{g}_1, \widehat{g}_2 \in \widehat{O}(N-m)$ *and for all* $(x,y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$.

Furthermore the action $*$ is left-distributive, that is,

(g_3) $\widehat{g} * ((x,y) + (x',y')) = \widehat{g} * (x,y) + \widehat{g} * (x',y')$ *for every* $\widehat{g} \in \widehat{O}(N-m)$ *and all* $(x,y), (x',y') \in \mathbb{R}^m \times \mathbb{R}^{N-m}$.

Proof. Conditions (g_1) and (g_2) come directly from the definition of $*$. On the other hand, the natural action law of the group $O(N-m)$ on $(\mathbb{R}^{N-m}, +)$ is left-distributive, so that

$$\begin{aligned}\widehat{g} * ((x,y) + (x',y')) &= \widehat{g} * (x + x', y + y') \\ &= (x + x', g(y + y')) = (x + x', gy + gy') \\ &= (x, gy) + (x', gy') = \widehat{g} * (x,y) + \widehat{g} * (x',y')\end{aligned}$$

for every $\widehat{g} \in \widehat{O}(N-m)$ and $(x,y), (x',y') \in \mathbb{R}^m \times \mathbb{R}^{N-m}$. Hence, also condition (g_3) is easily verified. □

A set $\Omega \subseteq \mathbb{R}^m \times \mathbb{R}^{N-m}$ is said to be $\widehat{O}(N-m)$-*invariant* if $\widehat{O}(N-m) * \Omega = \Omega$, i. e., $\widehat{g} * (x,y) \in \Omega$ for every $\widehat{g} \in \widehat{O}(N-m)$. Moreover, if Ω is $\widehat{O}(N-m)$-invariant, then $\widehat{g} * \Omega = \Omega$ for every $\widehat{g} \in \widehat{O}(N-m)$ by (g_1) and (g_2). Of course, a strip-like domain $\mathcal{O} \times \mathbb{R}^{N-m}$ is $\widehat{O}(N-m)$-invariant.

The natural induced action

$$\sharp : \widehat{O}(N-m) \times H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) \to H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$$

of the group $\widehat{O}(N-m)$ on $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ maps any pair $(\widehat{g}, u)$ into the function $\widehat{g}\sharp u$ defined pointwise by setting for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$,

$$\widehat{g}\sharp u(x,y) = u(x, g^{-1}y) \quad \text{if } \widehat{g} = \mathbb{I}_m \times g,\ g \in O(N-m), \tag{5.2}$$

i. e., in a more direct form,

$$\widehat{g}\sharp u(x,y) = u(\widehat{g}^{-1} * (x,y))$$

for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$.

Lemma 5.1.2. *The group $\widehat{O}(N-m) = \{\mathbb{I}_m\} \times O(N-m)$ acts isometrically on the Sobolev space $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ by* (5.2).

Proof. Fix $u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$. It is enough to show that

$$\|\widehat{g}\sharp u\| = \|u\| \quad \text{for all } \widehat{g} = \mathbb{I}_m \times g,\ g \in O(N-m), \tag{5.3}$$

where the operation $\sharp : \widehat{O}(N-m) \times H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) \to H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ is given in (5.2). In order to prove (5.3), it is enough to check that

$$\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla v|^2 dxdy = \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla u|^2 dxdy, \tag{5.4}$$

where $v(x,y) = u(x,g^{-1}y)$ for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$. Since

$$\nabla v(x,y) = (\widehat{g}^{-1})^T \nabla u(x,g^{-1}y),$$

where $(\widehat{g}^{-1})^T$ denotes the transpose of $\widehat{g}^{-1}$, relation (5.4) becomes

$$\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} (\widehat{g}^{-1})^T \nabla u(x,g^{-1}y) \cdot (\widehat{g}^{-1})^T \nabla u(x,g^{-1}y) dxdy = \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla u|^2 dxdy,$$

that is,

$$\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \widehat{g}^{-1}(\widehat{g}^{-1})^T \nabla u(x,g^{-1}y) \cdot \nabla u(x,g^{-1}y) dxdy = \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla u|^2 dxdy. \tag{5.5}$$

Since $g \in O(N-m)$, taking into account that

$$\widehat{g}^{-1} = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & g^{-1} \end{pmatrix} \quad \text{and} \quad (\widehat{g}^{-1})^T = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & (g^{-1})^T \end{pmatrix}, \tag{5.6}$$

we have $\widehat{g}^{-1}(\widehat{g}^{-1})^T = \mathbb{I}_N$. Then, by (5.5) and (5.6), claim (5.3) is proved. □

From now on the symbol $E_0 = \mathrm{Fix}_{\widehat{O}(N-m)}(H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}))$ denotes the elements of $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ which are fixed with respect to the action $\sharp$ of the group $\widehat{O}(N-m)$ on the space Sobolev space $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$, i. e.,

$$E_0 = \{u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) : \widehat{g}\sharp u = u,\ \widehat{g} \in \widehat{O}(N-m)\}. \tag{5.7}$$

Clearly, $E_0 = \mathrm{Fix}_{\widehat{O}(N-m)}(H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}))$ is a closed subspace of the main space $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ and it is exactly the space $H^1_{0,\mathrm{cyl}}(\mathcal{O}\times\mathbb{R}^{N-m})$ of cylindrically symmetric functions defined by

$$H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m}) = \{u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) \;:\; u(x,y) = u(x,y') \text{ for a. e. } (x,y),(x,y') \in \mathcal{O} \times \mathbb{R}^{N-m} \text{ such that } |y| = |y'|\}. \tag{5.8}$$

Of course, by (5.1), the Sobolev embedding

$$H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m}) \hookrightarrow L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m}) \tag{5.9}$$

is continuous for any $\wp \in [2, 2^*]$. Moreover, by the celebrated paper of M. Esteban and P.-L. Lions [85], the Sobolev embedding

$$H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m}) \hookrightarrow\hookrightarrow L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m}) \tag{5.10}$$

is compact for any $\wp \in (2, 2^*)$.

Let either $N = m + 4$ or $N \geq m + 6$. Put $J_{N,m} = \{1, \ldots, \zeta_{N,m}\} \subset \mathbb{N}$, where

$$\zeta_{N,m} = (-1)^{N-m} + \left[\frac{N-m-3}{2}\right]. \tag{5.11}$$

Clearly, the set of indices $J_{N,m}$ is nonempty, since either $N = m + 4$ or $N \geq m + 6$.

By *grouping together* the $N - m$ variables of the unbounded part of the strip in blocks of at least two variables, we easily see that there are $\zeta_{N,m} \geq 1$ subgroups of $O(N - m)$ given for every $i \in J_{N,m}$ by

$$H_{N,m,i} = \begin{cases} O((N-m)/2) \times O((N-m)/2), & \text{if } i = \frac{N-m-2}{2}, \\ O(i+1) \times O(N-m-2i-2) \times O(i+1), & \text{if } i \neq \frac{N-m-2}{2}. \end{cases}$$

Hence subgroups $H_{N,m,i}$ define the subgroups

$$\widehat{H}_{N,m,i} = \{\mathbb{I}_m\} \times H_{N,m,i} \subset \widehat{O}(N-m)$$

for every $i \in J_{N,m}$. On account of the group isomorphism

$$\widehat{H}_{N,m,i} \ni \widehat{g} = \mathbb{I}_m \times g \mapsto \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & g \end{pmatrix} \in O(N),$$

every group $\widehat{H}_{N,m,i}$ can be identified as a subgroup of $O(N)$.

Note that the sets $\mathscr{E}_i = \mathrm{Fix}_{\widehat{H}_{N,m,i}}(H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}))$, where

$$\mathscr{E}_i = \{u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) \;:\; \widehat{g}\sharp u = u \text{ for all } \widehat{g} \in \widehat{H}_{N,m,i}\},$$

are the subspaces of block-radial functions and each $\mathscr{E}_i$ is compactly embedded into $L^{\wp}(\mathcal{O}\times\mathbb{R}^{N-m})$ for every $\wp\in(2,2^*)$, see [160, Théorème III.2]. Regrettably, this is not enough to get the multiplicity result, since the $\zeta_{N,m}$ subspaces $\mathscr{E}_i$ are not *mutually disjoint*. Thus, for any $i\in J_{N,m}$, we define the involution $\eta_{N,m,i}:\mathbb{R}^{N-m}\to\mathbb{R}^{N-m}$ as follows:

$$\eta_{N,m,i}(y)=\begin{cases}(y_3,y_1), & \text{if } i=\frac{N-m-2}{2} \text{ and}\\ & y=(y_1,y_3)\in\mathbb{R}^{(N-m)/2}\times\mathbb{R}^{(N-m)/2},\\ (y_3,y_2,y_1), & \text{if } i\neq\frac{N-m-2}{2} \text{ and}\\ & y=(y_1,y_2,y_3)\in\mathbb{R}^{i+1}\times\mathbb{R}^{N-m-2i-2}\times\mathbb{R}^{i+1},\end{cases}$$

and we set

$$\eta_{\widehat{H}_{N,m,i}}=\mathbb{I}_m\times\eta_{N,m,i}.$$

It is easily seen that for any $i\in J_{N,m}$,

$$\eta_{\widehat{H}_{N,m,i}}\in\widehat{O}(N-m),\quad \eta^2_{\widehat{H}_{N,m,i}}=\mathrm{id}_{\mathcal{O}\times\mathbb{R}^{N-m}},$$
$$\eta_{\widehat{H}_{N,m,i}}\notin\widehat{H}_{N,m,i},\quad \eta_{\widehat{H}_{N,m,i}}\widehat{H}_{N,m,i}\,\eta^{-1}_{\widehat{H}_{N,m,i}}=\widehat{H}_{N,m,i},$$

where

$$\eta_{\widehat{H}_{N,m,i}}\widehat{H}_{N,m,i}\,\eta^{-1}_{\widehat{H}_{N,m,i}}=\{\eta_{\widehat{H}_{N,m,i}}\widehat{g}\,\eta^{-1}_{\widehat{H}_{N,m,i}}\ :\ \widehat{g}\in\widehat{H}_{N,m,i}\}.$$

Finally, for every $i\in J_{N,m}$, we consider the compact subgroup of $\widehat{O}(N-m)$

$$\widehat{H}_{N,m,\eta_i}=\langle\widehat{H}_{N,m,i},\eta_{\widehat{H}_{N,m,i}}\rangle,$$

where $\langle\widehat{H}_{N,m,i},\eta_{\widehat{H}_{N,m,i}}\rangle$ denotes the subgroup generated by the subgroup $\widehat{H}_{N,m,i}$ and the element $\eta_{\widehat{H}_{N,m,i}}\in\widehat{O}(N-m)\setminus\widehat{H}_{N,m,i}$, that is,

$$\widehat{H}_{N,m,\eta_i}=\widehat{H}_{N,m,i}\cup(\eta_{\widehat{H}_{N,m,i}}\widehat{H}_{N,m,i}),$$

and the action

$$\circledast_i:\widehat{H}_{N,m,\eta_i}\times H^1_0(\mathcal{O}\times\mathbb{R}^{N-m})\to H^1_0(\mathcal{O}\times\mathbb{R}^{N-m})$$

of $\widehat{H}_{N,m,\eta_i}$ on $H^1_0(\mathcal{O}\times\mathbb{R}^{N-m})$ is given by setting

$$\widehat{g}\circledast_i u(x,y)=\begin{cases}u(x,g^{-1}y) & \text{if } \widehat{g}=\mathbb{I}_m\times g \text{ in } \widehat{H}_{N,m,i},\\ -u(x,\tau^{-1}\eta^{-1}_{N,m,i}y) & \text{if } \widehat{g}=\mathbb{I}_m\times\eta_{N,m,i}\tau \text{ in}\\ & \widehat{H}_{N,m,\eta_i}\setminus\widehat{H}_{N,m,i},\ \tau\in H_{N,m,i},\end{cases}\tag{5.12}$$

for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$.

Bearing in mind (5.2) and fixing $i \in J_{N,m}$, the action $\circledast_i$ can be written as follows:

$$\widehat{g} \circledast_i u(x,y) = \begin{cases} \widehat{g} \sharp u(x,y) & \text{if } \widehat{g} \in \widehat{H}_{N,m,i}, \\ -(\eta_{\widehat{H}_{N,m,i}} \widehat{\tau}) \sharp u(x,y) & \text{if } \widehat{g} = \eta_{\widehat{H}_{N,m,i}} \widehat{\tau} \in \widehat{H}_{N,m,\eta_i} \setminus \widehat{H}_{N,m,i} \\ & \text{and } \tau \in H_{N,m,i}, \end{cases}$$

for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$.

Let us note that $\circledast_i$ is defined for every element of $\widehat{H}_{N,m,\eta_i}$. Indeed, if $\widehat{g} \in \widehat{H}_{N,m,\eta_i}$, then either $\widehat{g} \in \widehat{H}_{N,m,i}$ or $\widehat{g} = \mathbb{I}_m \times \eta_{N,m,i}\tau \in \widehat{H}_{N,m,\eta_i} \setminus \widehat{H}_{N,m,i}$, with $\tau \in H_{N,m,i}$.

We are now ready to introduce $E_i = \mathrm{Fix}_{\widehat{H}_{N,m,\eta_i}}(H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}))$ for any $i \in J_{N,m}$, which is the set of the functions of $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ fixed with respect to the action $\circledast_i$ of the group $\widehat{H}_{N,m,\eta_i}$, that is,

$$E_i = \{u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) : \widehat{g} \circledast_i u = u \text{ for any } \widehat{g} \in \widehat{H}_{N,m,\eta_i}\}. \tag{5.13}$$

It easily seen that each set E_i is a nontrivial closed subspace of $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$. We first prove that E_i for each $i \in J_{N,m}$ is compactly embedded in the Lebesgue space $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ for any $\wp \in (2, 2^*)$ and then show some geometrical properties of E_i.

Actually, as we shall see below, the compactness result (5.10) continues to hold if $H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m})$ is replaced by E_i. Let us recall that throughout the chapter, in all the properties involving $J_{N,m}$, we tacitly handle with the case $m \geq 1$, and either $N = m+4$ or $N \geq m+6$. Therefore, the action $\circledast_i$ and the subspace E_i are defined in (5.12) and (5.13), respectively.

Proposition 5.1.3. *For each $i \in J_{N,m}$, the embedding $E_i \hookrightarrow L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ is continuous for any $\wp \in [2, 2^*]$ and compact for any $\wp \in (2, 2^*)$.*

Proof. Fix $i \in J_{N,m}$. The space $\mathscr{E}_i$ is continuously embedded in $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ for any $\wp \in [2, 2^*]$ and is compactly embedded in $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ for any $\wp \in (2, 2^*)$ by Théorème III.2 of [160]. Hence, the embedding $E_i \hookrightarrow L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ is also continuous for any $\wp \in [2, 2^*]$ and compact for any $\wp \in (2, 2^*)$, since $\widehat{H}_{N,m,i} \subset \widehat{H}_{N,m,\eta_i}$, the first relation of (5.12) that defines the action $\circledast_i$ implies that E_i is a subspace of the space of block-radial functions $\mathscr{E}_i$. This completes the proof. □

Now, we prove a sort of *flower-shape geometry* for the configuration of the subspaces E_i, as stated below.

Proposition 5.1.4. *The following statements hold true*:

(i) *If either $N = m+4$ or $N \geq m+6$, then $E_i \cap H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m}) = \{0\}$ for any $i \in J_{N,m}$*;

(ii) *If either $N = m+6$ or $N \geq m+8$, then $E_i \cap E_j = \{0\}$ for any $i, j \in J_{N,m}$, with $i \neq j$.*

Proof. (*i*) Let either $N = m + 4$ or $N \geq m + 6$. Fix $i \in J_{N,m}$ and let u be in $E_i \cap H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m})$. Since u is $\widehat{H}_{N,m,\eta_i}$-invariant, taking into account (5.12), we have

$$u(x,y) = -u(x, \eta^{-1}_{N,m,i} y) \tag{5.14}$$

for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$.

Moreover, since u is radial in the second component, i. e., $u(x,y) = u(x,y')$ if $|y| = |y'|$, and $|y| = |\eta^{-1}_{N,m,i} y|$ for every $y \in \mathbb{R}^{N-m}$, then (5.14) yields that $u(x,y) = -u(x,y)$ for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$ and so u must be identically zero in $\mathcal{O} \times \mathbb{R}^{N-m}$.

(*ii*) Let either $N = m+6$ or $N \geq m+8$ so that $\zeta_{N,m} \geq 2$. Then, fix $i, j \in J_{N,m}$, with $i < j$, and $u \in E_i \cap E_j$. It can be easily seen that the function u is both $\widehat{H}_{N,m,i}$- and $\widehat{H}_{N,m,j}$-invariant. Therefore, u is also $\langle \widehat{H}_{N,m,i}, \widehat{H}_{N,m,j} \rangle$-invariant, where $\langle \widehat{H}_{N,m,i}, \widehat{H}_{N,m,j} \rangle$ denotes the subgroup of $\widehat{O}(N-m)$ generated by $\widehat{H}_{N,m,i}$ and $\widehat{H}_{N,m,j}$, that is,

$$u(x,y) = u(x, g^{-1}_{ij} y)$$

for every $g_{ij} \in \langle H_{N,m,i}, H_{N,m,j} \rangle$ and for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$. Now, as proved in Theorem 2.2(*ii*) of [148], the group $\langle H_{N,m,i}, H_{N,m,j} \rangle$ acts transitively on the sphere $\mathbb{S}^{N-m-1}$. Hence, for any $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$,

$$\langle \widehat{H}_{N,m,i}, \widehat{H}_{N,m,j} \rangle (x,y) = \{x\} \times |y|\, \mathbb{S}^{N-m-1}.$$

Hence, the function u is cylindrically symmetric, and we can apply (*i*), obtaining that u is identically zero in $\mathcal{O} \times \mathbb{R}^{N-m}$. This concludes the proof. □

For additional comments and remarks concerning the theoretical methods recalled in this section, we refer to [79] and references therein.

In the next part of the chapter, we shall present two cases which are complementary and describe the existence of solutions with block-cylindrical symmetries. Namely, the first treats nonlinear terms f which are sublinear at infinity, and the latter handles the case when the nonlinearities are superlinear at infinity.

5.2 Finitely many solutions

The main purpose of this section is to use the group-theoretical properties presented in Section 5.1 to treat eigenvalue problems on strip-like domains. The main approach is based on a critical point result proved in [223], which will be combined with the principle of symmetric criticality for smooth functionals, establishing the existence of at least three distinct not cylindrically symmetric solutions for an eigenvalue Dirichlet problem, lacking compactness.

Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^m$ with a smooth boundary $\partial\mathcal{O}$ and let $\mathcal{O} \times \mathbb{R}^{N-m}$ be a strip-like domain in $\mathbb{R}^N$, with $m \geq 1$, and either $N = m + 4$ or $N \geq m + 6$.

We shall deal with a multiplicity result for the elliptic Dirichlet problem given by

$$\begin{cases} -\Delta u = \lambda f(x,y,u) & \text{in } \mathcal{O} \times \mathbb{R}^{N-m}, \\ u = 0 & \text{on } \partial\mathcal{O} \times \mathbb{R}^{N-m}, \end{cases} \tag{D_λ}$$

where λ is a positive real parameter and f is a nonlinear term. Assume that

(k_1) *$f : \mathcal{O} \times \mathbb{R}^{N-m} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is $p \in (2, 2^*)$, with the property that for every $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$\left|f(x,y,t)\right| \leq \varepsilon|t| + \kappa_\varepsilon |t|^{p-1} \text{ for every } (x,y,t) \in \mathcal{O} \times \mathbb{R}^{N-m} \times \mathbb{R};$$

(k_2) *$f(\cdot,\cdot,t)$ is cylindrically symmetric for all $t \in \mathbb{R}$, that is, $f(x,y,t) = f(x,|y|,t)$ for every $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$;*

(k_3) *$f(x,y,\cdot)$ is even in $\mathbb{R}$ for every $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$;*

(k_4) *there exist positive numbers $q \in (0,2)$, $\mathfrak{q} \in [2, 2^*]$ and measurable functions $\alpha \in L^{\mathfrak{q}/(\mathfrak{q}-q)}(\mathcal{O} \times \mathbb{R}^{N-m})$ and $\beta \in L^1(\mathcal{O} \times \mathbb{R}^{N-m})$ such that*

$$F(x,y,t) \leq \alpha(x,y)|t|^q + \beta(x,y), \quad \text{where } F(x,y,t) = \int_0^t f(x,y,s)ds$$

for every $(x,y,t) \in \mathcal{O} \times \mathbb{R}^{N-m} \times \mathbb{R}$;

(k_5) *there are positive numbers r and t_0 such that $F(x,y,t) \geq 0$ for all (x,y,t) in $\mathcal{O} \times B(0,r) \times [0,t_0]$, and $F(x,y,t_0) > 0$ for every $(x,y) \in \mathcal{O} \times B(0,r)$.*

The function

$$f(x,y,t) = \frac{|t|^{p-2}t}{(1+|y|^{N-m})^2} \cos |t|^p,$$

and its potential

$$F(x,y,t) = \frac{\sin |t|^p}{p(1+|y|^{N-m})^2}$$

satisfy conditions (k_1)–(k_5) in $\mathcal{O} \times \mathbb{R}^{N-m} \times \mathbb{R}$, with $m \geq 1$, either $N = m+4$ or $N \geq m+6$, $q = 1$, $\alpha = 0$, and $\beta(x,y) = (1+|y|^{N-m})^{-2}$. Of course, a subcase is when $f(x,y,t) = f(t) = |t|^{p-2}t \cos |t|^p$. A simple prototype is given in Figure 5.2.

The main result of this section reads as follows.

Theorem 5.2.1. *Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^m$ with a smooth boundary $\partial\mathcal{O}$ and let $\mathcal{O} \times \mathbb{R}^{N-m}$ be a strip-like domain in $\mathbb{R}^N$, with $m \geq 1$, and either $N = m+4$ or $N \geq m+6$. Let f satisfy (k_1)–(k_5). Then, for every $i \in J_{N,m}$ there exist an open interval $\Lambda_i \subset \mathbb{R}^+$ and a number $\sigma_i > 0$ such that for each $\lambda \in \Lambda_i$ there are at least three distinct not cylindrically symmetric solutions of (D_λ), whose norms are strictly less than σ_i.*

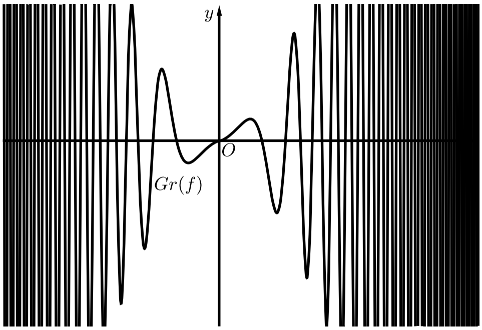

Figure 5.2: The function $f(t) = \sqrt{|t|}t\cos\sqrt{|t|^5}$ with $p = 5/2$.

Theorem 5.2.1 furnishes a precise information on the number of solutions of problem (D_λ) with different symmetry structure; see the above special example of f as well as Example 3.1 in [141].

The main abstract tool in order to prove Theorem 5.2.1 is the next critical point result valid for smooth functionals defined on separable reflexive Banach spaces; see [223] for a detailed proof.

Theorem 5.2.2. *Let $X = (X, \|\cdot\|)$ be a separable reflexive Banach space, let Q, I be functionals belonging to $C^1(X)$, and let $\Sigma \subseteq \mathbb{R}$ be a real interval. Suppose that*

(*i*) *Q is weakly sequentially lower semicontinuous in X and I is weakly sequentially continuous in X;*

(*ii*) *for every $\lambda \in \Sigma$ the functional $\mathcal{I}_\lambda = Q + \lambda I$ satisfies the (PS) condition and is coercive, that is,*

$$\lim_{\|u\|\to\infty} \mathcal{I}_\lambda(u) = \infty;$$

(*iii*) *there exists a continuous concave function $\ell : \Sigma \to \mathbb{R}$ such that*

$$\sup_{\lambda\in\Sigma}\inf_{u\in X}(\mathcal{I}_\lambda(u) + \ell(\lambda)) < \inf_{u\in X}\sup_{\lambda\in\Sigma}(\mathcal{I}_\lambda(u) + \ell(\lambda)).$$

Then there are an open interval $\Lambda \subseteq \Sigma$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$ the functional $\mathcal{I}_\lambda$ has at least three distinct critical points in X with norms strictly less than σ.

The existence of cylindrically symmetric solutions for hemivariational inequality problems has been studied in Theorem 3.1 of [141] by using an abstract critical point theorem similar to Theorem 5.2.2 but for nonsmooth functionals. In the case of strip-like domains, the space of cylindrically symmetric functions has been the main tool in the investigation, due to the presence of compact embeddings into classical Lebesgue spaces. Inspired by the analysis given in [140, 141, 148, 151] and using the compact embedding result proved in Proposition 5.1.3 of Section 5.1, in Theorem 5.2.1 we are interested in the existence of multiple not cylindrically symmetric solutions for problem (D_λ). We emphasize that the main conclusion in Theorem 5.2.1 remains valid for the hemivariational problem treated in [141], using the Krawcewicz and Marzantowicz principle for locally Lipschitz functionals, established in [139]. More general versions of Theorem 5.2.1 in the nonsmooth case can be obtained by using the results contained in [90].

Fix $\lambda > 0$ and consider the energy functional $I_\lambda : H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}) \to \mathbb{R}$ given by

$$I_\lambda(u) = \Phi(u) + \lambda\Psi(u), \tag{5.15}$$

where

$$\Phi(u) = \frac{1}{2}\|u\|^2 = \frac{1}{2}\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} |\nabla u|^2 dxdy$$

and

$$\Psi(u) = -\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u)dxdy,$$

for every $u \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$.

Now, let $i \in J_{N,m}$. Theorem 5.2.2 can be applied to the energy functional $\mathcal{I}_{\lambda,i} : E_i \to \mathbb{R}$ defined by

$$\mathcal{I}_{\lambda,i}(u) = \Phi|_{E_i}(u) + \lambda\Psi|_{E_i}(u)$$

for every $u \in E_i$, by choosing

$$X = E_i, \quad Q = \Phi|_{E_i}, \quad I = \Psi|_{E_i}, \quad \Sigma = \mathbb{R}_0^+.$$

From now on, for the sake of simplicity, we fix $i \in J_{N,m}$ and write $\mathcal{I}_\lambda$ instead of $\mathcal{I}_{\lambda,i}$ and X in place of E_i. In order to prove Theorem 5.2.1, we show here the following semicontinuity property.

Lemma 5.2.3. *Thanks to (k_1), the functional*

$$u \mapsto I(u) \quad \textit{for all } u \in X$$

is sequentially weakly continuous on X. In particular, if $u_k \rightharpoonup u_\infty$ in X then

$$\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} f(x,y,u_k)(u_k - u_\infty)dxdy \to 0 \tag{5.16}$$

as $k \to \infty$.

Proof. Fix $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$ and $t_1, t_2 \in \mathbb{R}$. By the Lagrange mean value theorem, there exists $\theta \in (0,1)$ such that

$$F(x,y,t_1) - F(x,y,t_2) = f\big(x,y,\theta t_1 + (1-\theta)t_2\big)(t_1 - t_2).$$

Hence, by (k_1), for every $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that

$$\begin{aligned} \big|F(x,y,t_1) - F(x,y,t_2)\big| \le \big[&\varepsilon\big(|t_1| + |t_2|\big) \\ &+ 2^{p-2}\kappa_\varepsilon\big(|t_1|^{p-1} + |t_2|^{p-1}\big)\big]|t_2 - t_1|, \end{aligned} \tag{5.17}$$

Let $(u_k)_k \subset X$ converge weakly to an element $u_\infty \in X$. By (5.17) and the Hölder inequality, it follows that

$$\begin{aligned} \big|I(u_k) - I(u_\infty)\big| &\le \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \big|F(u_k) - F(u_\infty)\big| dxdy \\ &\le \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \big(\varepsilon\big(|u_k| + |u_\infty|\big) + C_\varepsilon\big(|u_k|^{p-1} + |u|^{p-1}\big)\big)|u_k - u_\infty| dxdy \\ &\le \varepsilon\big(\|u_k\|_2 + \|u_\infty\|_2\big)\|u_k - u_\infty\|_2 \\ &\qquad + C_\varepsilon\big(\|u_k\|_p^{p-1} + \|u_\infty\|_p^{p-1}\big)\|u_k - u_\infty\|_p, \end{aligned}$$

where $C_\varepsilon = 2^{p-2}\kappa_\varepsilon$.

Now, X is compactly embedded in $L^p(\mathcal{O} \times \mathbb{R}^{N-m})$ by Proposition 5.1.3 since $p \in (2, 2^*)$. Thus $\|u_k - u_\infty\|_p \to 0$ as $k \to \infty$, since $u_k \rightharpoonup u_\infty$ in X. Consequently,

$$\limsup_{k\to\infty} \big|I(u_k) - I(u_\infty)\big| \le \varepsilon C, \tag{5.18}$$

where $C = \sup_k(\|u_k\|_2 + \|u_\infty\|_2)\|u_k - u_\infty\|_2 < \infty$, since $(u_k)_k$ is bounded in X. Thus, since ε is arbitrary, (5.18) gives that $I(u_k) \to I(u_\infty)$ as $k \to \infty$. This shows the first part of the lemma.

To show (5.16), fix $(u_k)_k \subset X$ weakly convergent to an element $u_\infty \in X$. Repeating again the argument above, (k_1) implies

$$\begin{aligned} \Big|\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} f(x,y,u_k)(u_k - u_\infty) dxdy\Big| &\le \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \big(\varepsilon|u_k| + \kappa_\varepsilon|u_k|^{p-1}\big)|u_k - u_\infty| dxdy \\ &\le \varepsilon\|u_k\|_2\|u_k - u_\infty\|_2 \\ &\qquad + \kappa_\varepsilon\|u_k\|_p^{p-1}\|u_k - u_\infty\|_p, \end{aligned}$$

that is, by Proposition 5.1.3,

$$\limsup_{k\to\infty} \Big|\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} f(x,y,u_k)(u_k - u_\infty) dxdy\Big| \le \varepsilon C.$$

Thus, since ε is arbitrary, the above inequality gives at once (5.16). □

We notice that the proof of Lemma 5.2.3 for functions with cylindrical symmetries directly follows by classical arguments due to M. Esteban and P.-L. Lions. Indeed, by (k_1), it follows that $F(x,y,t) = o(t^2)$ as $t \to 0$ and $F(x,y,t) = o(t^{2^*})$ as $t \to \infty$ for a. e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$. Arguing as in the proof of Corollary 3 of [85], it easily seen that $I(u_k) \to I(u_\infty)$; see also [84, Lemma 4, p. 368].

Proof of Theorem 5.2.1. Fix $i \in J_{N,m}$ and let us divide the argument into four steps.

Step 1. *Q is weakly sequentially lower semicontinuous and I is weakly sequentially continuous in X.*

The statement follows from Lemma 5.2.3 and a standard reasoning applied to the functional Q.

Step 2. *For every $\lambda \in \Lambda$, the functional $\mathcal{I}_\lambda = Q + \lambda I$ fulfils the (PS) condition and*

$$\lim_{\|u\|\to\infty} \mathcal{I}_\lambda(u) = \infty.$$

Fix $\lambda \in \Lambda$. First, (k_4), the Hölder inequality, and (5.1) give

$$\begin{aligned}\mathcal{I}_\lambda(u) &= Q(u) + \lambda I(u)\\ &\geq \frac{1}{2}\|u\|^2 - \lambda \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \alpha(x,y)|u|^q dxdy - \lambda \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \beta(x,y)dxdy\\ &\geq \frac{1}{2}\|u\|^2 - \lambda c_{\mathrm{q}}^q \|\alpha\|_{\mathrm{q}/(\mathrm{q}-q)}\|u\|^q - \lambda\|\beta\|_1,\end{aligned}$$

for every $u \in X$. Since $q \in (0,2)$, one gets $\mathcal{I}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, that is, $\mathcal{I}_\lambda$ is coercive.

Moreover, $\mathcal{I}_\lambda$ satisfies the (PS) condition at level $c \in \mathbb{R}$, that is, for any sequence $(u_k)_k$ in X such that

$$\mathcal{I}_\lambda(u_k) \to c \quad \text{and} \quad \mathcal{I}'_\lambda(u_k) \to 0 \quad \text{in } X' \tag{5.19}$$

as $k \to \infty$, there exists $u_\infty \in X$ such that, up to a subsequence,

$$\|u_k - u_\infty\| \to 0 \tag{5.20}$$

as $k \to \infty$.

First of all, we notice that the coerciveness of the functional $\mathcal{I}_\lambda$ implies that the sequence $(u_k)_k$ is bounded in X, and consequently in $L^p(\mathcal{O} \times \mathbb{R}^{N-m})$. Since X is a reflexive space, we also get that, up to a subsequence, still denoted by $(u_k)_k$, there exists $u_\infty \in X$ such that, thanks also to Proposition 5.1.3,

$$\begin{gathered} u_k \to u_\infty \quad \text{weakly in } X, \quad u_k \to u_\infty \quad \text{a. e. in } \mathcal{O}\times\mathbb{R}^{N-m},\\ u_k \to u_\infty \quad \text{in } L^{\wp}(\mathcal{O}\times\mathbb{R}^{N-m}),\ \wp \in (2,2^*), \end{gathered} \tag{5.21}$$

as $k \to \infty$. Now

$$\langle \Phi'(u_k), u_k - u_\infty \rangle = \langle \mathcal{I}'_\lambda(u_k), u_k - u_\infty \rangle + \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} f(x,y,u_k)(u_k - u_\infty)dxdy \tag{5.22}$$

for every $k \in \mathbb{N}$.

Of course, taking into account that $(u_k)_k$ is bounded in X, by (5.19) it follows that

$$\left|\langle \mathcal{I}'_\lambda(u_k), u_k - u_\infty \rangle\right| \le \left\|\mathcal{I}'_\lambda(u_k)\right\|_{X'} \|u_k - u_\infty\| \to 0 \tag{5.23}$$

as $k \to \infty$. In conclusion, (5.16), (5.22), and (5.23) yield, as $k \to \infty$,

$$\begin{aligned}\|u_k - u_\infty\|^2 &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} \nabla u_k \cdot \nabla(u_k - u_\infty)dxdy + o(1)\\ &= \langle \Phi'(u_k), u_k - u_\infty \rangle \to 0,\end{aligned}$$

which is (5.20).

Step 3. *There exists a continuous concave function* $\ell : \Sigma \to \mathbb{R}$ *satisfying*

$$\sup_{\lambda\in\Sigma} \inf_{u\in X} (\mathcal{I}_\lambda(u) + \ell(\lambda)) < \inf_{u\in X} \sup_{\lambda\in\Sigma} (\mathcal{I}_\lambda(u) + \ell(\lambda)). \tag{5.24}$$

First of all, we prove that the structural assumptions on the nonlinear term f give the existence of a parameter $\varrho \in (0,1]$ and of a nontrivial function $u_{\varrho,i}$ in X such that

$$I(u_{\varrho,i}) = \Psi|_X(u_{\varrho,i}) = -\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u_{\varrho,i})dxdy < 0. \tag{5.25}$$

Inspired by the construction given in [148], let $r > 0$ be as in (k_5) and fix r_1 and r_2, with $0 < (5 + 4\sqrt{2})r_1 < r_2 < r$.

For any $\varrho \in (0,1]$, set

$$S_{\varrho,i} = \{y \in \mathbb{R}^{N-m} \,:\, y \text{ satisfies } (5.27)\}, \tag{5.26}$$

where

$$\begin{cases} \left(|y_1| - \dfrac{r_2+3r_1}{4}\right)^2 + |y_3|^2 \le \varrho^2\left(\dfrac{r_2-r_1}{4}\right)^2 & \text{if } i = \dfrac{N-m-2}{2},\\ \left(|y_3| - \dfrac{r_2+3r_1}{4}\right)^2 + |y_1|^2 \le \varrho^2\left(\dfrac{r_2-r_1}{4}\right)^2 & \\ \text{and } |y_2| \le \varrho\dfrac{r_2-r_1}{4}, & \text{if } i \ne \dfrac{N-m-2}{2}, \end{cases} \tag{5.27}$$

with

$$y = \begin{cases} (y_1, y_3) \in \mathbb{R}^{(N-m)/2} \times \mathbb{R}^{(N-m)/2} & \text{if } i = \frac{N-m-2}{2},\\ (y_1, y_2, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{N-m-2i-2} \times \mathbb{R}^{i+1} & \text{if } i \ne \frac{N-m-2}{2}. \end{cases} \tag{5.28}$$

Now, let $t_0 > 0$ be as in (k_5) and define the function $v_{\varrho,i} : \mathcal{O} \times \mathbb{R}^{N-m} \to \mathbb{R}$ as follows:

$$\begin{aligned} v_{\varrho,i}(y) = \Bigg[& \left(\frac{r_2 - r_1}{4} - \max\left\{ \sqrt{\left(|y_1| - \frac{r_2 + 3r_1}{4} \right)^2 + |y_3|^2}, \varrho \frac{r_2 - r_1}{4} \right\} \right)^+ \\ & - \left(\frac{r_2 - r_1}{4} - \max\left\{ \sqrt{\left(|y_3| - \frac{r_2 + 3r_1}{4} \right)^2 + |y_1|^2}, \varrho \frac{r_2 - r_1}{4} \right\} \right)^+ \Bigg] \cdot \\ & \times \left(\frac{r_2 - r_1}{4} - \max\left\{ |y_2|, \varrho \frac{r_2 - r_1}{4} \right\} \right)^+ \frac{16 t_0}{(r_2 - r_1)^2 (1 - \varrho)^2}, \end{aligned}$$

where y is as in (5.28).

Let ω and Ω be open sets in $\mathbb{R}^m$, with Lebesgue measure $|\omega| > 0$ and $\omega \subset\subset \Omega \subset\subset \mathcal{O}$. Then, let us fix a nonnegative function $\varphi \in C_0^\infty(\mathcal{O})$, with $\operatorname{supp} \varphi \subset \Omega$, $\varphi \equiv 1$ on ω and $\|\varphi\|_\infty = 1$. Define

$$u_{\varrho,i}(x, y) = \varphi(x) v_{\varrho,i}(y) \quad \text{for } (x, y) \in \mathcal{O} \times \mathbb{R}^{N-m}. \tag{5.29}$$

Direct computation ensures that $u_{\varrho,i} \in H^1(\mathcal{O} \times \mathbb{R}^{N-m})$. Moreover, since $\operatorname{supp} u_{\varrho,i}$ is a compact subset of $\mathcal{O} \times \mathbb{R}^{N-m}$, one gets $u_{\varrho,i} \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ by [48, Lemma 9.5]. Actually, $u_{\varrho,i} \in X$, since $u_{\varrho,i} \in H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$, and $u_{\varrho,i}$ is $\circledast_i$-invariant by (5.12), arguing as in the proof of Theorem 1.1(*iii*) of [148]. More precisely, we have

$$\begin{aligned} \operatorname{supp} u_{\varrho,i} \subset \mathcal{O} \times S_{1,i} \subseteq \{(x, y) \in \mathcal{O} \times \mathbb{R}^{N-m} : r_1 \le |y| \le r_2\} \\ |u_{\varrho,i}(x, y)| = t_0 \ \text{ for } (x, y) \in \omega \times S_{\varrho,i}, \quad \|u_{\varrho,i}\|_\infty \le t_0. \end{aligned} \tag{5.30}$$

Since $|u_{\varrho,i}(x, y)| = |\varphi(x) v_{\varrho,i}(y)| \le |v_{\varrho,i}(y)| \in [0, t_0]$ for $(x, y) \in \mathcal{O} \times \mathbb{R}^{N-m}$ and in force of (k_3) and (k_5), we have

$$F(x, y, u_{\varrho,i}(x, y)) \ge 0 \quad \text{for a. e. } (x, y) \in \mathcal{O} \times S_{1,i}. \tag{5.31}$$

Finally, $0 < r_1 < r_2 < r$ so that, by (5.29), (5.30), and (5.31), we get

$$\begin{aligned} I(u_{\varrho,i}) &= - \iint\limits_{\mathcal{O} \times \mathbb{R}^{N-m}} F(x, y, u_{\varrho,i}) dx dy = - \iint\limits_{\mathcal{O} \times (S_{\varrho,i} \cup (S_{1,i} \setminus S_{\varrho,i}))} F(x, y, u_{\varrho,i}) dx dy \\ &= - \iint\limits_{\mathcal{O} \times S_{\varrho,i}} F(x, y, u_{\varrho,i}) dx dy - \iint\limits_{\mathcal{O} \times (S_{1,i} \setminus S_{\varrho,i})} F(x, y, u_{\varrho,i}) dx dy \\ &\le - \iint\limits_{\mathcal{O} \times S_{\varrho,i}} F(x, y, \varphi t_0) dx dy - \iint\limits_{\mathcal{O} \times (S_{1,i} \setminus S_{\varrho,i})} F(x, y, u_{\varrho,i}) dx dy \\ &\le - \iint\limits_{\mathcal{O} \times S_{\varrho,i}} F(x, y, \varphi t_0) dx dy \le - \iint\limits_{\omega \times S_{\varrho,i}} F(x, y, t_0) dx dy < 0, \end{aligned}$$

i. e., inequality (5.25) is verified.

Now, let us define the function $\chi : \mathbb{R}_0^+ \to \mathbb{R}$ by

$$\chi(s) = \sup_{\substack{Q(u)\leq s\\ u\in X}} \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u)dxdy,$$

for every $s \in \mathbb{R}_0^+$. By (k_1) and direct integration, we get that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that $|F(x,y,t)| \leq \varepsilon t^2 + \kappa_\varepsilon |t|^p$ for every $(x,y,t) \in \mathcal{O}\times\mathbb{R}^{N-m}\times\mathbb{R}$. Consequently,

$$0 \leq \chi(s) \leq 2\varepsilon c_2^2 s + 2^p \kappa_\varepsilon c_p^p s^{\frac{p}{2}},$$

for every $s \in \mathbb{R}_0^+$. Hence as $s \to 0^+$,

$$0 \leq \frac{\chi(s)}{s} < 2\varepsilon c_2^2 + 2^p \kappa(\varepsilon) c_p^p s^{\frac{p}{2}-1} \sim 2\varepsilon c_2^2,$$

since $p > 2$. Therefore, since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{s\to 0^+} \frac{\chi(s)}{s} = 0. \tag{5.32}$$

Inequality (5.25) gives at once that $u_{\varrho,i} \neq 0$ and there exists a number η such that

$$0 < \eta < \frac{2}{\|u_{\varrho,i}\|^2} \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u_{\varrho,i})dxdy.$$

On the other hand, by (5.32), there exist numbers $s_0 \in (0, \|u_{\varrho,i}\|^2/2)$ and $\rho_0 > 0$ such that

$$\chi(s_0) < \rho_0 < \frac{2s_0}{\|u_{\varrho,i}\|^2} \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u_{\varrho,i})dxdy = -\frac{s_0 I(u_{\varrho,i})}{Q(u_{\varrho,i})}. \tag{5.33}$$

Hence, the choice of s_0 yields

$$\rho_0 < \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u_{\varrho,i})dxdy, \quad \text{that is, } \rho_0 + I(u_{\varrho,i}) < 0. \tag{5.34}$$

Define $\Sigma = \mathbb{R}_0^+$ and $\ell : \Sigma \to \mathbb{R}$ by $\ell(\lambda) = \rho_0\lambda$. We claim that such ℓ satisfies (5.24). The real function

$$\Sigma \ni \lambda \mapsto \inf_{u\in X}(\mathcal{I}_\lambda(u) + \rho_0\lambda), \quad \mathcal{I}_\lambda(u) = Q(u) + \lambda I(u),$$

is upper semicontinuous on Σ. Relation (5.34) implies that

$$\lim_{\lambda\to\infty} \inf_{u\in X}(\mathcal{I}_\lambda(u) + \rho_0\lambda) \leq \lim_{\lambda\to\infty} \{Q(u_{\varrho,i}) + \lambda(\rho_0 + I(u_{\varrho,i}))\} = -\infty.$$

Thus, the Tonelli–Weierstrass Theorem 1.2 of [238] guarantees the existence of an element $\overline{\lambda} \in \Sigma$ such that

$$\sup_{\lambda\in\Sigma}\inf_{u\in X}\{\mathcal{I}_\lambda(u) + \rho_0\lambda\} = \inf_{u\in X}\{Q(u) + \overline{\lambda}(\rho_0 + I(u))\}. \tag{5.35}$$

Moreover, $I(u) > -\rho_0$ for every $u \in Q^{-1}((-\infty, s_0])$, since $\chi(s_0) < \rho_0$. Hence,

$$s_0 \leq \inf_{I(u)\geq-\rho_0} Q(u). \tag{5.36}$$

On the other hand,

$$\begin{aligned}\inf_{u\in X}\sup_{\lambda\in\Sigma}(\mathcal{I}_\lambda(u) + \rho_0\lambda) &= \inf_{u\in X}\Big\{Q(u) + \sup_{\lambda\in\Sigma}\lambda(\rho_0 + I(u))\Big\}\\ &= \inf_{I(u)\geq-\rho_0} Q(u).\end{aligned}$$

Therefore, relation (5.36) can be written as

$$s_0 \leq \inf_{u\in X}\sup_{\lambda\in\Sigma}(\mathcal{I}_\lambda(u) + \rho_0\lambda). \tag{5.37}$$

There are two cases to be considered in (5.35).

Case 1. If $\overline{\lambda} \in [0, s_0/\rho_0)$, then, recalling that $0 \in X$ and (5.37), we get

$$\begin{aligned}\sup_{\lambda\in\Sigma}\inf_{u\in X}(\mathcal{I}_\lambda(u) + \rho_0\lambda) &= \inf_{u\in X}(Q(u) + \overline{\lambda}(\rho_0 + I(u)))\\ &\leq \overline{\lambda}\rho_0 < s_0 \leq \inf_{u\in X}\sup_{\lambda\in\Sigma}(\mathcal{I}_\lambda(u) + \rho_0\lambda),\end{aligned}$$

that is, we get the validity of (5.24).

Case 2. If $\overline{\lambda} \leq [s_0/\rho_0, \infty)$, then by (5.33),

$$\begin{aligned}\sup_{\lambda\in\Sigma}\inf_{u\in X}(\mathcal{I}_\lambda(u) + \rho_0\lambda) &= \inf_{u\in X}(Q(u) + \overline{\lambda}(\rho_0 + I(u)))\\ &\leq Q(u_{\varrho,i}) + \overline{\lambda}(\rho_0 + I(u_{\varrho,i}))\\ &\leq Q(u_{\varrho,i}) + \frac{s_0}{\rho_0}(\rho_0 + I(u_{\varrho,i}))\\ &= Q(u_{\varrho,i}) + \frac{s_0}{\rho_0}I(u_{\varrho,i}) + s_0\\ &< s_0 \leq \inf_{u\in X}\sup_{\lambda\in\Sigma}(\mathcal{I}_\lambda(u) + \rho_0\lambda),\end{aligned}$$

that is, (5.24) holds. This completes the proof of Step 3.

Step 4. Every critical point of $\mathcal{I}_\lambda$ weakly solves (D_λ) in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$.

Clearly, I_λ is even in $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$ by (k_3). Furthermore, $I_\lambda(\widehat{g}\circledast_i u) = I_\lambda(u)$ for every $\widehat{g} \in \widehat{H}_{N,m,\eta_i}$ and $u \in H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$, where $\circledast_i$ is defined in (5.12), i. e., the functional I_λ is $\widehat{H}_{N,m,\eta_i}$-invariant on $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$. Indeed, $\widehat{H}_{N,m,\eta_i}$ acts isometrically on $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$ by (5.12), taking into account that $\widehat{O}(N-m)$ acts isometrically on $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$ by Lemma 5.1.2. Moreover, (k_2), (k_3), and the fact that the strip-like domain $\mathcal{O}\times\mathbb{R}^{N-m}$ is $\widehat{O}(N-m)$-invariant give

$$\begin{aligned}\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,\widehat{g}\circledast_i u)dxdy &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,u(x,g^{-1}y))dxdy\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,gz,u(x,z))dxdz\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,z,u(x,z))dxdz\end{aligned}$$

if $\widehat{g} = \mathbb{I}_m \times g \in \widehat{H}_{N,m,i}$, and

$$\begin{aligned}\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,\widehat{g}\circledast_i u)dxdy &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,y,-u(x,\tau^{-1}\eta_{H_{N,m,i}}^{-1}y))dxdy\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,\eta_{N,m,i}\tau z,u(x,z))dxdz\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} F(x,z,u(x,z))dxdz,\end{aligned}$$

if $\widehat{g} = \mathbb{I}_m \times \eta_{N,m,i}\tau \in \widehat{H}_{N,m,\eta_i} \setminus \widehat{H}_{N,m,i}$ and $\tau \in H_{N,m,i}$.

By the principle of symmetric criticality, Theorem A.1.5, the critical points of the restriction $\mathcal{I}_\lambda$ in X are also critical points of the energy functional I_λ in $H_0^1(\mathcal{O}\times\mathbb{R}^{N-m})$. Then, Theorem 5.2.2 guarantees that for every $i \in J_{N,m}$ there are an open interval $\Lambda_i \subset \Sigma = \mathbb{R}_0^+$ and a number $\sigma_i > 0$ such that for each $\lambda \in \Lambda_i$ there are at least three distinct not cylindrically symmetric solutions of problem (D_λ), whose norms are strictly less than σ_i. This completes the proof of Theorem 5.2.1. □

5.3 Infinitely many solutions

In this section we study, as an application of Proposition 5.1.4, the existence of a finite number of infinitely many solutions without cylindrical symmetry for a Dirichlet problem defined on a strip-like domain $\mathcal{O}\times\mathbb{R}^{N-m}$, with $m \geq 1$, and either $N = m+4$ or $N \geq m+6$, provided that the nonlinear term verifies suitable hypotheses. Indeed, we treat nonlinearities which satisfy special forms of (k_1)–(k_3) and (k_5), but which are superlinear at infinity. More precisely, we shall deal with infinitely many solutions of the semilinear elliptic Dirichlet problem

$$\begin{cases} -\Delta u = w(x,y)f(u) & \text{in } \mathcal{O}\times\mathbb{R}^{N-m}, \\ u = 0 & \text{on } \partial\mathcal{O}\times\mathbb{R}^{N-m}, \end{cases} \tag{D_w}$$

under the following assumptions on the weight w and reaction term f:

(h_1) *$f:\mathbb{R}\to\mathbb{R}$ is a continuous function and there is an exponent $p\in(2,2^*)$, with the property that for every $\varepsilon>0$ there exists $\kappa_\varepsilon>0$ such that*

$$|f(t)|\le\varepsilon|t|+\kappa_\varepsilon|t|^{p-1}$$

for every $t\in\mathbb{R}$;

(h_2) *$w\in L^1(\mathcal{O}\times\mathbb{R}^{N-m})\cap L^\infty(\mathcal{O}\times\mathbb{R}^{N-m})$ is cylindrically symmetric, that is, $w(x,y)=w(x,|y|)$ for a. e. $(x,y)\in\mathcal{O}\times\mathbb{R}^{N-m}$ and*

$$w(x,y)\ge w_0>0$$

for a. e. $(x,y)\in\mathcal{O}\times\mathbb{R}^{N-m}$;

(h_3) *f is odd in $\mathbb{R}$;*

(h_4) *there exist $\nu>2$ and $t_0>0$ such that*

$$0<\nu F(t)\le f(t)t \quad \text{for any } t, \text{ with } |t|\ge t_0, \text{ where } F(t)=\int_0^t f(s)ds.$$

A model function verifying (h_1)–(h_4) is $w(x,y)(|u|^{\wp-2}u+|u|^{p-2}u)$, where $2<\wp<p<2^*$, $\nu=\wp$, t_0 is any positive number, and $w\in L^1(\mathcal{O}\times\mathbb{R}^{N-m})\cap L^\infty(\mathcal{O}\times\mathbb{R}^{N-m})$ is a positive cylindrically symmetric function in y satisfying (h_2); see the Figure 5.3.

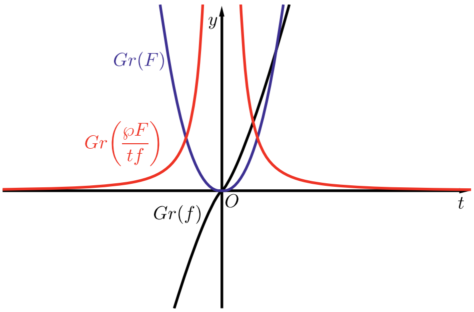

Figure 5.3: The case $f(t)=|t|^{1/5}u+|u|^{2/5}u$, with $\wp=11/5$.

Problem (D_w) has a variational nature and its Euler–Lagrange functional I is given by

$$I(u)=\frac{1}{2}\|u\|^2-\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(u)dxdy, \quad u\in H_0^1(\mathcal{O}\times\mathbb{R}^{N-m}). \tag{5.38}$$

Clearly, under the main assumptions on the nonlinear term f, the functional I is well defined in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ and is of class $C^1(H_0^1(\mathcal{O} \times \mathbb{R}^{N-m}))$. Furthermore, the critical points of I are exactly the solutions of problem (D_w) in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$.

As in Section 5.2, the difficulty in working with the functional I is the lack of compactness. Using the compactness result given in Proposition 5.1.4, it is possible to find suitable subspaces of $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$, compactly embedded into $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ for all $\wp \in (2, 2^*)$.

As done in Sections 4.3 and 5.2, let us put $J_{N,m} = \{1, \dots, \zeta_{N,m}\}$, where

$$\zeta_{N,m} = (-1)^{N-m} + \left[\frac{N-m-3}{2}\right], \tag{5.39}$$

when either $N - m = 4$ or $N - m \geq 6$. We extend the definition of $\zeta_{N,m}$ and of $J_{N,m}$ for all $N - m \geq 3$, putting $\zeta_{N,m} = 0$ and $J_{N,m} = \emptyset$, when $N - m = 3$ and $N - m = 5$.

With the above notations, we can now state the main result assuming, without further mentioning, that conditions (h_1)–(h_4) hold.

Theorem 5.3.1. *Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^m$ with a smooth boundary $\partial\mathcal{O}$ and let $\mathcal{O} \times \mathbb{R}^{N-m}$ be a strip-like domain in $\mathbb{R}^N$, with $m \geq 1$ and $N - m \geq 3$. Then, problem (D_w) admits at least one cylindrically symmetric unbounded sequence $(u_k^{(0)})_k$ of solutions in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ and $\zeta_{N,m}$ unbounded sequences $(u_k^{(i)})_k$, $i \in J_{N,m}$, of solutions in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$, with symmetric mutually different structure and being not cylindrically symmetric.*

The main variational tool to prove Theorem 5.3.1 is the classical critical point result, the fountain theorem for even smooth functionals, established originally by T. Bartsch in Theorem 2.5 of [34].

Theorem 5.3.2. *Let H be a Hilbert space and $(e_j)_j \subset H$ be an orthonormal sequence. Set*

$$H_n = \bigoplus_{j=1}^{n} \mathbb{R}e_j \quad \text{and} \quad H_n^{\perp} = \overline{\bigoplus_{j=n}^{\infty} \mathbb{R}e_j},$$

for every $n \geq 1$. Consider a C^1-functional $\mathcal{I} : H \to \mathbb{R}$ which satisfies the following properties:

(j_1) *$\mathcal{I}$ is even in H;*

(j_2) $b_{k-1} = \sup_{\rho \geq 0} \inf_{\substack{u \in H_k^{\perp} \\ \|u\| = \rho}} \mathcal{I}(u) \to \infty$ *as* $k \to \infty$;

(j_3) $\inf_{r>0} \sup_{\substack{u \in H_k \\ \|u\| \geq r}} \mathcal{I}(u) < 0$ *for every* $k \in \mathbb{N}$;

(j_4) *the $(PS)_c$ condition holds for all $c > 0$.*

Then the functional $\mathcal{I}$ possesses an unbounded sequence of critical values $(c_k)_k$. In fact, for each $k \geq 1$, with $b_k > 0$, there exists a critical value $c_k \geq b_k$, which can be characterized as

$$c_k = \inf_{\gamma \in \Gamma_k} \sup_{u \in B_k} \mathcal{I}(\gamma(u)),$$

where

$$B_k = \{u \in H_k \,:\, \|u\| \le r_k\},$$

with r_k *large enough, so that* $\mathcal{I}(u) < 0$ *for every* $u \in H_k$, $\|u\| \ge r_k$, *and*

$$\Gamma = \{\gamma : B_k \to H \,:\, \gamma \text{ is odd}, \gamma(u) = u \text{ if } \|u\| = r_k\}.$$

Theorem 5.3.2 is even available when (j_1) is replaced by the requirement that $\mathcal{I}$ is G-invariant with respect to a compact Lie group G, which acts transitively on H, as shown in Chapter 3 of [247].

As mentioned above, the functional I defined by (5.38) does not satisfy the $(PS)_c$ condition (j_4) if the subspaces are not chosen carefully. Thus, we have to search for suitable subspaces of $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$ such that the restrictions I on them satisfy (j_4). The next steps will give us a guess on how to construct such subspaces.

We notice that the existence of infinitely many axially symmetric solutions for hemivariational inequalities has been proved in Theorem 3.2 of [141], by using a nonsmooth version of the fountain theorem recalled above. By exploiting some ideas contained in [140, 141], in Theorem 5.3.1 we are able to prove the existence of a precise number of sequences of solutions with different symmetries. The main result remains valid for the hemivariational problem studied in [141] and it can be viewed as a more precise form of the existence theorem proved in the classical paper [36].

Proof of Theorem 5.3.1. Let $m \ge 1$ and $N - m \ge 3$. First of all, we shall apply Theorem 5.3.2 to the functional $\mathcal{I} : H \to \mathbb{R}$, $H = H^1_{0,\text{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m})$, $I|_{H^1_{0,\text{cyl}}(\mathcal{O}\times\mathbb{R}^{N-m})} = \mathcal{I}$. Of course, $\mathcal{I}$ satisfies (j_1) since the potential F is even as a consequence of (h_3).

Now, let $(e_k)_k \subset H$ be an orthonormal basis of H and, using the notation of Theorem 5.3.2, the next preparatory lemma follows directly by the compact embedding in (5.10).

Lemma 5.3.3. *For any* $\wp \in (2, 2^*)$,

$$\mu_k = \sup_{\substack{u \in H^\perp_{k-1} \\ u \neq 0}} \frac{\|u\|_\wp}{\|u\|} \to \infty$$

as $k \to \infty$.

Proof. First of all, by construction

$$0 < \mu_{k+1} = \sup_{\substack{u \in H^\perp_{k} \\ u \neq 0}} \frac{\|u\|_\wp}{\|u\|} \le \mu_k = \sup_{\substack{u \in H^\perp_{k-1} \\ u \neq 0}} \frac{\|u\|_\wp}{\|u\|}, \quad k \ge 2,$$

i. e., the real sequence $(\mu_k)_k$ given in the statement is positive and nonincreasing. Suppose, arguing by contradiction, that $\mu_k \to \mu_\infty > 0$ as $k \to \infty$. Then, there exists a sequence $(u_k)_k \subset H$, such that $u_k \in H_{k-1}^{\perp}$, $\|u_k\| = 1$ and $\|u_k\|_{\wp} \geq \mu_\infty/2$ for all k. By definition of $H_{k-1}^{\perp}$, this implies that $u_k \rightharpoonup 0$ in H along a subsequence. Indeed, the sequence $(u_k)_k$ is bounded in H, thus, by reflexivity, there exists $u_\infty \in H$ such that, up to a subsequence, still denoted by $(u_k)_k$, $u_k \rightharpoonup u_\infty$ in H. Any $\varphi \in H$ can be represented by its Fourier series, that is, $\varphi = \sum_{j=1}^{\infty} c_j e_j$. Hence, for every $k \geq 2$ we have

$$\langle u_k, \varphi \rangle = \sum_{j=k}^{\infty} c_j \langle u_k, e_j \rangle \to 0$$

as $k \to \infty$. Thus $u_k \rightharpoonup 0$ in H as claimed. Now, as proved by M. Esteban and P.-L. Lions in [85], the separable Hilbert Sobolev space $H = H_{0,\mathrm{cyl}}^1(\mathcal{O} \times \mathbb{R}^{N-m})$ is compactly embedded in $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ for all $\wp \in (2, 2^*)$. In conclusion, $u_k \to 0$ in $L^{\wp}(\mathcal{O} \times \mathbb{R}^{N-m})$ as $k \to \infty$. This is impossible and completes the proof. □

With the above notations and assumptions, we prove the following facts:

Claim 1. The functional $\mathcal{I}$ satisfies condition (j_2).

By (h_1), for every $\varepsilon > 0$ there exists a constant $\kappa_\varepsilon > 0$ such that

$$\left|F(t)\right| \leq \varepsilon |t|^2 + \kappa_\varepsilon |t|^p \tag{5.40}$$

for every $t \in \mathbb{R}$. Thus, for a fixed $k \geq 2$ and $u \in H_{k-1}^{\perp}$, it follows that

$$\begin{aligned}
\mathcal{I}(u) &= \frac{1}{2}\|u\|^2 - \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(u)\,dxdy \\
&\geq \frac{1}{2}\|u\|^2 - \|w\|_\infty \left(\varepsilon \|u\|_2^2 + \kappa_\varepsilon \|u\|_p^p\right) \\
&\geq \|u\|^2 \left(\frac{1}{2} - \varepsilon \|w\|_\infty\right) - \kappa_\varepsilon \mu_k^p \|u\|^p .
\end{aligned}$$

Now, we choose $\varepsilon = (p-2)/4p\|w\|_\infty$, and so

$$\rho_k = \frac{1}{(p\kappa_\varepsilon \mu_k^p)^{p-2}} \to \infty$$

as $k \to \infty$ by Lemma 2.3.2. Hence, for all $u \in H_{k-1}^{\perp}$, with $\|u\| = \rho_k$, we have

$$\mathcal{I}(u) \geq \left(\frac{1}{4} - \frac{1}{2p}\right)\frac{1}{\rho_k^2}.$$

Therefore,

$$b_k \geq \inf_{\substack{u \in H_{k-1}^{\perp} \\ \|u\| = \rho_k}} \mathcal{I}(u) \to \infty \quad \text{as} \quad k \to \infty,$$

i. e., condition (j_2) holds.

Claim 2. The functional $\mathcal{I}$ satisfies condition (j_3).

To prove Claim 2, we first show the next property.

Lemma 5.3.4. *The primitive F of the nonlinear term f satisfies the inequality*

$$w(x,y)F(t) \geq c_0|t|^\nu - w(x,y)F_\infty,$$

for a.e. $(x,y) \in \mathcal{O} \times \mathbb{R}^{N-m}$ *and any* $t \in \mathbb{R}$, *where* $c_0 = w_0 t_0^{-\nu} F(t_0) > 0$ *and* $F_\infty = \max_{|t|\leq t_0} |F(t)|$.

Proof. Let $t_0 > 0$ be as in (h_4). A direct computation yields at once that

$$F(t) \geq \frac{F(t_0)}{t_0^\nu}\, t^\nu \quad \text{for any } t \in \mathbb{R}, \text{ with } t \geq t_0$$

and $F(t_0) > 0$. Hence, since F is even in $\mathbb{R}$ by (h_3), we get $F(t) \geq \tilde{c}_0|t|^\nu$ for any $t \in \mathbb{R}$, with $|t| \geq t_0 > 0$, where $\tilde{c}_0 = t_0^{-\nu}F(t_0) > 0$. Clearly, by continuity, $|F(t)| \leq F_\infty$ for any $t \in \mathbb{R}$, with $|t| \leq t_0$. Hence, the stated estimate holds at once by (h_2). □

Since $w \in L^1(\mathcal{O} \times \mathbb{R}^{N-m})$ by (h_2), thanks to Lemma 5.3.4, we have

$$\mathcal{I}(u) \leq \frac{1}{2}\|u\|^2 - c_0\|u\|_\nu^\nu - F_\infty\|w\|_1 \tag{5.41}$$

for every $u \in H$. Now, taking into account that H_k is a finite-dimensional space, all the norms are equivalent on it. Therefore, $\nu > 2$ implies

$$\lim_{r\to\infty} \sup_{\substack{u\in H_k \\ \|u\|\geq r}} \mathcal{I}(u) \to -\infty.$$

This proves the claim.

Claim 3. The functional $\mathcal{I}$ satisfies condition (j_4).

To this aim, for any $c > 0$ fix a $(PS)_c$ sequence $(u_k)_k$ in H of $\mathcal{I}$. Let us first show that $(u_k)_k$ is bounded in H.

For any $k \in \mathbb{N}$ there exists $\kappa > 0$ such that

$$|\mathcal{I}(u_k)| \leq \kappa \quad \text{and} \quad \|\mathcal{I}'(u_k)\|_{H'} \leq \kappa. \tag{5.42}$$

Moreover, by (h_1) and so (5.40), applied with $\varepsilon = 1$, we have

$$\left| \iint_{\mathcal{O}\times\mathbb{R}^{N-m}\cap\{|u_k|\leq t_0\}} w(x,y)\left(F(u_k) - \frac{1}{\nu} f(u_k)\, u_k\right) dxdy \right| \leq \left(t_0^2 + \kappa_1 t_0^p + \frac{t_0^2}{\nu} + \frac{\kappa_1}{\nu}\, t_0^p\right)\|w\|_1 = \tilde{\kappa}.$$

Hence, thanks to (h_4), we get

$$\mathcal{I}(u_k) - \frac{1}{\nu}\langle \mathcal{I}'(u_k), u_k\rangle \geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_k\|^2 - \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}\cap\{|u_k|\leq t_0\}} w(x,y)\left(F(u_k) - \frac{1}{\nu} f(u_k)\, u_k\right) dxdy.$$

Therefore, we obtain

$$\mathcal{I}(u_k) - \frac{1}{\nu}\langle \mathcal{I}'(u_k), u_k\rangle \geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_k\|^2 - \tilde{\kappa}. \tag{5.43}$$

As a consequence of (5.42), we also have

$$\mathcal{I}(u_k) - \frac{1}{\nu}\langle \mathcal{I}'(u_k), u_k\rangle \leq \kappa(1 + \|u_k\|),$$

so that, using (5.43), we find a suitable constant $c > 0$ such that

$$\|u_k\|^2 \leq c(1 + \|u_k\|)$$

for any $k \in \mathbb{N}$. Hence, $(u_k)_k$ is bounded in the Hilbert space H and so there exists a subsequence, still denoted by $(u_k)_k$, weakly converging to some $u_\infty \in H$.

Now, let $\varepsilon > 0$ be fixed. Then, by (h_1) there exists a corresponding $\kappa_\varepsilon > 0$ such that the Hölder inequality, as well as the fact that $\|u_k\|_p + \|u_\infty\|_p$ is uniformly bounded in k, gives

$$\begin{aligned}
&\left| \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)(f(u_k) - f(u_\infty))(u_k - u_\infty) dxdy \right| \\
&\quad \leq \varepsilon\|w\|_\infty \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} (|u_k| + |u_\infty|)^2 dxdy \\
&\quad\quad + \kappa_\varepsilon\|w\|_\infty \iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} (|u_k|^{p-1} + |u_\infty|^{p-1})|u_k - u_\infty| dxdy \\
&\quad \leq \|w\|_\infty [2\varepsilon(\|u_k\|_2^2 + \|u_\infty\|_2^2) + \kappa_\varepsilon(\|u_k\|_p^{p-1} + \|u\|_p^{p-1})]\|u_k - u_\infty\|_p \\
&\quad \leq C(2\varepsilon + \kappa_\varepsilon\|u_k - u_\infty\|_p),
\end{aligned}$$

and the right-hand side approaches $2C\varepsilon$ as $k \to \infty$ by (5.10). This gives at once

$$\iint\limits_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)(f(u_k) - f(u_\infty))(u_k - u_\infty) dxdy \to 0$$

as $k \to \infty$, since $\varepsilon > 0$ is arbitrary. Hence, as $k \to \infty$,

$$\|u_k - u_\infty\|^2 = \langle \mathcal{I}'(u_k) - \mathcal{I}'(u_\infty), u_k - u_\infty\rangle$$

$$+ \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)\big(f(u_k) - f(u_\infty)\big)(u_k - u_\infty)\,dxdy$$
$$= o(1),$$

by (5.42) and the fact that $u_k \rightharpoonup u_\infty$ in H. The claim is proved.

In conclusion, $\mathcal{I}$ satisfies all the assumptions of Theorem 5.3.2 and so possesses an unbounded sequence of critical points in $H = H^1_{0,\mathrm{cyl}}(\mathcal{O} \times \mathbb{R}^{N-m})$.

Clearly, (5.2) gives that $I(\widehat{g}\sharp u) = I(u)$ for every $\widehat{g} \in \widehat{O}(N-m)$ and any $u \in H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$, i. e., the functional $I : H^1_0(\mathcal{O}\times\mathbb{R}^{N-m}) \to \mathbb{R}$ is $\widehat{O}(N-m)$-invariant. Indeed, $\widehat{O}(N-m)$ acts isometrically on $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$ by (5.2), as proved in Lemma 5.1.2. Moreover, $\mathcal{O} \times \mathbb{R}^{N-m}$ is $\widehat{O}(N-m)$-invariant, as a strip-like domain. By (h_2) and (h_3),

$$\begin{aligned}\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(\widehat{g}\sharp u)\,dxdy &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F\big(u(x,g^{-1}y)\big)\,dxdy\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,gz)F\big(u(x,z)\big)\,dxdz\\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,z)F\big(u(x,z)\big)\,dxdz,\end{aligned}$$

if $\widehat{g} = \mathbb{I}_m \times g \in \widehat{O}(N-m)$, $g \in O(N-m)$.

By the principle of symmetric criticality, Theorem A.1.5, the critical points of the restriction $\mathcal{I}$ are also critical points of the energy functional I. Then, by virtue of Theorem 5.3.2, problem (D_w) admits at least one unbounded sequence $(u_k)_k$ of solutions in $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$ with cylindrical symmetry.

Now, assume that either $N = m + 4$ or $N \geq m + 6$. We apply Theorem 5.3.2 to each E_i and to the corresponding functional $\mathcal{I}_i$ on E_i, given by

$$\mathcal{I}_i(u) = \frac{1}{2}\|u\|^2 - \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(u)\,dxdy, \quad u \in E_i,$$

for every $i \in J_{N,m}$.

Repeating a similar argument as in the cylindrical case, we state that for every $i \in J_{N,m}$, the functional $\mathcal{I}_i$ admits at least one unbounded sequence $(u_k^{(i)})_k$ of critical points in E_i. Since f is an odd function by (h_3), the main energy functional I, defined in (5.38), is even. Thus, I is $\widehat{H}_{N,m,\eta_i}$-invariant, when the action of $\widehat{H}_{N,m,\eta_i}$ on $H^1_0(\mathcal{O}\times\mathbb{R}^{N-m})$ is given by (5.12). Indeed, $\widehat{H}_{N,m,\eta_i}$ acts isometrically on $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$ by (5.12), since $\widehat{O}(N-m)$ acts isometrically on $H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$ by Lemma 5.1.2. Moreover, (h_2), (h_3), and the fact that $\mathcal{O} \times \mathbb{R}^{N-m}$ is $\widehat{O}(N-m)$-invariant yield

$$\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(\widehat{g} \circledast_i u)\,dxdy = \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F\big(u(x,g^{-1}y)\big)\,dxdy$$

$$= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,gz)F(u(x,z))dxdz$$
$$= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,z)F(u(x,z))dxdz$$

if $\widehat{g} = \mathbb{I}_m \times g \in \widehat{H}_{N,m,i}$, $g \in H_{N,m,i}$, and

$$\begin{aligned}\iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(\widehat{g} \circledast_i u)dxdy &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,y)F(-u(x,\tau^{-1}\eta^{-1}_{H_{N,m,i}}y))dxdy \\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,\eta_{N,m,i}\tau z)F(u(x,z))dxdz \\ &= \iint_{\mathcal{O}\times\mathbb{R}^{N-m}} w(x,z)F(u(x,z))dxdz,\end{aligned}$$

if $\widehat{g} = \mathbb{I}_m \times \eta_{N,m,i}\tau \in \widehat{H}_{N,m,\eta_i} \setminus \widehat{H}_{N,m,i}$, and $\tau \in H_{N,m,i}$.

The principle of symmetric criticality, Theorem A.1.5, implies that the critical points of $\mathcal{I}_i$ are also critical points for the functional I, therefore, solutions of problem (D_w).

Summing up the above facts, on the basis of Proposition 5.1.4, problem (D_w) admits at least one cylindrically symmetric unbounded sequence $(u_k^{(0)})_k$ of solutions in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$ and $\zeta_{N,m}$ unbounded sequences $(u_k^{(i)})_k$, $i \in J_{N,m}$, of solutions in $H_0^1(\mathcal{O} \times \mathbb{R}^{N-m})$, with symmetric mutually different structure and being not cylindrically symmetric. This concludes the proof of Theorem 5.3.1. □

Comments on Chapter 5

In the last years many papers deals with different aspects of problems similar to (D_w). Special kinds of oscillations at infinity produce infinitely many solutions for a wide class of elliptic problems in the Euclidean setting as proved in [200, 201]. These results suggest to study elliptic or semilinear elliptic equations on strip-like domains under an appropriate assumption on f at infinity. As we shall see in Chapter 9, in the multiplicity Theorem 9.2.1 there are two key tools,

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} > -\infty$$

and Theorem 1 of [171], that is, the continuity of the superposition operator defined in (9.14). An interesting open question is the study of (D_w) when

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = -\infty.$$

However, to get a multiplicity result under this new assumption, the continuity of the corresponding superposition operator remains essential.

6 Elliptic equations on the sphere

Sento i miei passi passare…
In questo infinito oggi
sono l'oggi che va
senza andare.

Antonella Coletti
Attimi

The chapter deals with the existence of infinitely many sign-changing solutions of higher order elliptic problems settled on the unit sphere $\mathbb{S}^N \hookrightarrow \mathbb{R}^{N+1}$, $N \geq 2$, and involving a possibly critical nonlinear term. Here $\mathbb{S}^N$ is endowed with the induced Riemannian metric h. To overcome the lack of compactness, symmetry properties on the Sobolev space $H^m(\mathbb{S}^N)$ are carefully studied in Section 6.1 via a group-theoretical argument. Thus the existence of sequences of sign-changing solutions, which are mutually symmetrically distinct, is attained, and a lower estimate of the number of those sequences is also given; see Theorem 6.2.1 in Section 6.2.

Then, in Theorems 6.2.4 and 6.2.5, we use the reduction method to the unit sphere in order to prove the existence of infinitely many solutions for some parameterized Emden–Fowler equations that naturally arise in astrophysics, conformal Riemannian geometry, and in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion.

In the last Section 6.3, the existence of multiple symmetric solutions for a critical stationary nondegenerate Kirchhoff problem on the unit sphere is proved under minimal assumptions on the forcing nonlinear term, provided that a combination of the Kirchhoff coefficients a and b is sufficiently large with respect to the critical Sobolev embedding constant; see Theorem 6.3.3. This result is peculiar of Kirchhoff problems on the Euclidean sphere.

The abstract approach we use to prove Theorem 6.3.3 is inspired by [226] and by the recent paper [89]. For instance, some fine topological properties of the energy functional associated to the main problem are obtained using abstract tools introduced in [225, 226] and recalled in Theorems 6.3.1 and 6.3.2. Finally, a key role along the proof is played by a compact argument given in Theorem 6.1.2 and by Proposition 6.1.3. See [144] and the monograph [151, Chapter 10] for a nice detailed discussion on the subject and on related topics.

6.1 Group actions on Sobolev spaces

Let m be an integer and $N > 2m$. In what follows, $H^m(\mathbb{S}^N)$ denotes the classical Hilbert Sobolev space consisting of functions on $\mathbb{S}^N$ with weak derivatives of order D^α, $|\alpha| \leq m$, in $L^2(\mathbb{S}^N)$. Due to the usual role of the critical exponent, the Sobolev space $H^m(\mathbb{S}^N)$

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cannot be compactly embedded into the Lebesgue space $L^{2^*_m}(\mathbb{S}^N)$, where as usual $2^*_m = 2N/(N-2m)$. In order to prove the main existence results of the chapter, we recover compactness on suitable symmetric subspaces of $H^m(\mathbb{S}^N)$, which are compactly embedded into $L^q(\mathbb{S}^N)$, even when q is supercritical. Such properties have been observed in specific contexts by several authors, see [129, 144, 185] and the references therein for related topics. This approach is fruitful in the study of a wide class of variational elliptic problems in the presence of a suitable group action on the Sobolev space, thanks to the principle of symmetric criticality given in the Appendix.

In this section we describe in detail the construction of s_N subspaces $H^m_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ of the Sobolev space $H^m(\mathbb{S}^N)$ related to certain subgroups $G^{\tau_i}_{N,i}$ of the orthogonal group $O(N+1)$. The main useful tool of the chapter, and in particular to prove Theorem 6.2.1, is the geometrical profile of the subspaces $H^m_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ defined in Proposition 6.1.3.

In this last part of the section, we take $s_N = [N/2] + (-1)^{N+1} - 1$, which is well defined if $N > 4$. In this case, for every $i \in J_N = \{1, \dots, s_N\}$,

$$G_{N,i} = \begin{cases} O(i+1) \times O(N-2i-1) \times O(i+1), & \text{if } i \neq \frac{N-1}{2}, \\ O(i+1) \times O(i+1), & \text{if } i = \frac{N-1}{2}. \end{cases}$$

Furthermore, $G^N_{i,j}$ denotes the group generated by $G_{N,i}$ and $G_{N,j}$ whenever $i, j \in J_N$ and $i \neq j$. The following result, proved in Proposition 3.2 of [144], will be crucial in the sequel.

Proposition 6.1.1. *Let $N > 4$. For every $i, j \in J_N$, with $i \neq j$, the group $G^N_{i,j}$ acts transitively on $\mathbb{S}^N$, i. e., there exists $\sigma_0 \in \mathbb{S}^N$ such that $G^N_{i,j}\sigma_0 = \mathbb{S}^N$.*

Fix $N > 4$ and $i \in J_N$. Let $\tau_i : \mathbb{S}^N \to \mathbb{S}^N$ be the involution function associated to $G_{N,i}$ and defined for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^N$ by

$$\tau_i(\sigma) = \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if } i \neq \frac{N-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1},\ \sigma_2 \in \mathbb{R}^{N-2i-1}, \\ (\sigma_3, \sigma_1), & \text{if } i = \frac{N-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}. \end{cases}$$

By construction,

$$\tau_i \notin G_{N,i}, \quad \tau_i G_{N,i} \tau_i^{-1} = G_{N,i} \quad \text{and} \quad \tau_i^2 = \mathrm{id}_{\mathbb{S}^N}.$$

Let us present explicit forms of some groups $G_{N,i}$ and of related functions τ_i, as summarized in Chapter 10 of [151]. For instance, if $N = 11$, then $s_{11} = 5$ and the groups and the involution functions are:

(*a*) $G_{11,1} = O(2) \times O(8) \times O(2)$, $\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$ for $\sigma_1, \sigma_3 \in \mathbb{R}^2$ and $\sigma_2 \in \mathbb{R}^8$, when $i = 1$;

(*b*) $G_{11,2} = O(3) \times O(6) \times O(3)$, $\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$ for $\sigma_1, \sigma_3 \in \mathbb{R}^3$ and $\sigma_2 \in \mathbb{R}^6$, when $i = 2$;

(*c*) $G_{11,3} = O(4) \times O(4) \times O(4)$, $\tau_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_2, \sigma_1, \sigma_3)$ for $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^4$, when $i = 3$;
(*d*) $G_{11,4} = O(5) \times O(2) \times O(5)$, $\tau_4(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$ for $\sigma_1, \sigma_3 \in \mathbb{R}^5$ and $\sigma_2 \in \mathbb{R}^2$, when $i = 4$;
(*e*) $G_{11,5} = O(6) \times O(6)$, $\tau_5(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$ for $\sigma_1, \sigma_2 \in \mathbb{R}^6$, when $i = 5$.

As in [144], we fix $N > \max\{4, 2m\}$ from now on, if not otherwise stated. For all $i \in J_N$ let $\widehat{\circledast}_i$ be an action of the compact group

$$G_{N,i}^{\tau_i} = \langle G_{N,i}, \tau_i \rangle \subset O(N+1) \tag{6.1}$$

on the Sobolev space $H^m(\mathbb{S}^N)$.

More precisely, we consider the action $\widehat{\circledast}_i : G_{N,i}^{\tau_i} \times H^m(\mathbb{S}^N) \to H^m(\mathbb{S}^N)$, $(\tilde{g}, u) \mapsto g\widehat{\circledast}_i u$, which is defined pointwise for a. e. $\sigma \in \mathbb{S}^N$ by

$$(g\widehat{\circledast}_i u)(\sigma) = \begin{cases} u(g^{-1}\sigma), & \text{if } g \in G_{N,i}, \\ -u(g^{-1}\tau_i^{-1}\sigma), & \text{if } g = \tau_i \tilde{g} \in G_{N,i}^{\tau_i} \setminus G_{N,i},\ \tilde{g} \in G_{N,i}. \end{cases} \tag{6.2}$$

This can be done by the properties of τ_i. Therefore, $\widehat{\circledast}_i$ is well defined, linear, and continuous.

Let us consider for every $i \in J_N$ the subspace $H^m_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$ of $H^m(\mathbb{S}^N)$ given by

$$E_i = H^m_{G_{N,i}^{\tau_i}}(\mathbb{S}^N) = \{u \in H^m(\mathbb{S}^N) : g\widehat{\circledast}_i u = u \text{ for all } g \in G_{N,i}^{\tau_i}\}.$$

Clearly, $E_i = H^m_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$ contains all the functions $u \in H^m(\mathbb{S}^N)$ which are symmetric with respect to the action $\widehat{\circledast}_i$ of the compact group $G_{N,i}^{\tau_i}$. Moreover, for every $i \in J_N$ we also introduce

$$\mathscr{E}_i = H^m_{G_{N,i}}(\mathbb{S}^N) = \{u \in H^m(\mathbb{S}^N) : g \circledast_i u = u \text{ for all } g \in G_{N,i}\},$$

where the action $\circledast_i : G_{N,i} \times H^m(\mathbb{S}^N) \to H^m(\mathbb{S}^N)$ of the compact group $G_{N,i}$ on $H^m(\mathbb{S}^N)$, $(g, u) \mapsto g\circledast_i u$, is defined pointwise for a. e. $\sigma \in \mathbb{S}^N$ by

$$(g\circledast_i u)(\sigma) = u(g^{-1}\sigma). \tag{6.3}$$

Note that every $u \in E_i \setminus \{0\}$ has no constant sign. Indeed, $u(\sigma) = -u(\tau_i^{-1}\sigma)$ for every $\sigma \in \mathbb{S}^N$, since u is $G_{N,i}^{\tau_i}$-invariant by (6.3). The conclusion then follows immediately from the fact that u is not zero.

For the sake of clarity, let us recall Lemma 3.2 of [35] in the form we shall use later.

Proposition 6.1.2. *Let G be a closed topological subgroup of the orthogonal group $O(N+1)$ and let $\cdot : G \times H^m(\mathbb{S}^N) \to H^m(\mathbb{S}^N)$ be the natural action of the topological group G on the Hilbert Sobolev space $H^m(\mathbb{S}^N)$. Set*

$$H^m_G(\mathbb{S}^N) = \{u \in H^m(\mathbb{S}^N) : gu = u \text{ for all } g \in G\}.$$

Let

$$N_G = \min_{\sigma \in \mathbb{S}^N} \dim(G\sigma)$$

be the minimal dimension of the orbits in $\mathbb{S}^N$*, where the orbit* $G\sigma$ *of an element* $\sigma \in \mathbb{S}^N$ *is given by*

$$G\sigma = \{g\sigma \ : \ \textit{for all}\ g \in G\},$$

and $g\sigma$ *is the natural multiplicative action. Then the Sobolev embedding*

$$H^m_G(\mathbb{S}^N) \hookrightarrow L^q(\mathbb{S}^N)$$

is compact for every $q \in [1, q_G)$*, where*

$$q_G = \begin{cases} \frac{m(N-N_G)}{N-N_G-2m} & \textit{if}\ N > 2m + N_G, \\ \infty & \textit{if}\ N \le 2m + N_G. \end{cases}$$

If $N > 2m + N_G$*, then the space* $H^m_G(\mathbb{S}^N)$ *is continuously embedded in* $L^{q_G}(\mathbb{S}^N)$*.*

If G is a connected algebraic group, which acts on a variety Y (not necessarily affine), then for each $y \in Y$ the orbit Gy is an irreducible variety, that is, Gy is open in its closure. Moreover, its boundary, $\partial Gy = \overline{Gy} \setminus Gy$, is the union of orbits of strictly smaller dimension. Finally, in this case orbits of minimal dimension are closed.

By Proposition 6.1.1, arguing as in the proof of Theorem 3.1 of [144], the next result holds.

Proposition 6.1.3. *Let* $N > 2m$*, with* $m \ge 1$*. Then the following statements hold for any fixed* $i \in J_N$ *:*

(i) *the Hilbert Sobolev space* $\mathscr{E}_i = H^m_{G_{N,i}}(\mathbb{S}^N)$ *is compactly embedded into* $L^q(\mathbb{S}^N)$*, whenever* $q \in [1, q_i^\star)$*, where*

$$q_i^\star = \begin{cases} \frac{2(N-1)}{N-2m-1}, & \textit{if}\ N > 2m + 1, \\ \infty, & \textit{if}\ N = 2m + 1; \end{cases}$$

(ii) $\mathscr{E}_i \cap \mathscr{E}_j = \{\text{constant functions on } \mathbb{S}^N\}$ *for every* $j \in J_N$*, with* $j \neq i$*;*

(iii) $E_i \cap E_j = \{0\}$ *for every* $j \in J_N$*, with* $j \neq i$*.*

Proof. Part (i). A careful analysis of the definition of $G_{N,i}$ shows that the $G_{N,i}$-orbit of every point $\sigma \in \mathbb{S}^N$ has at least dimension 1, i. e., $\dim(G_{N,i}\sigma) \ge 1$ for every $\sigma \in \mathbb{S}^N$, and

$$N_{G_{N,i}} = \min_{\sigma \in \mathbb{S}^N} \dim(G_{N,i}\sigma) \ge 1.$$

Hence, by Proposition 6.1.2, the space $\mathscr{E}_i$ is compactly embedded into $L^q(\mathbb{S}^N)$ for every $q \in [1, q_i^\star)$. Since $N > 2m$, one has

$$q_i^\star > 2_m^* = \frac{2N}{N-2m}$$

and $\mathscr{E}_i \subset E_i$, so that the embedding

$$\mathscr{E}_i \hookrightarrow\hookrightarrow L^{2_m^*}(\mathbb{S}^N)$$

is compact for every $i \in J_N$.

Part (*ii*). Fix $j \in J_N$, with $j \neq i$, and $u \in \mathscr{E}_i \cap \mathscr{E}_j$. Since u is both $G_{N,i}$- and $G_{N,j}$-invariant, then u is also $G_{i,j}^N$-invariant, i. e., $u(g\sigma) = u(\sigma)$ for every $g \in G_{i,j}^N$ and $\sigma \in \mathbb{S}^N$. According to Proposition 6.1.1, the group $G_{i,j}^N$ acts transitively on the sphere $\mathbb{S}^N$, i. e., $G_{i,j}^N \sigma = \mathbb{S}^N$ for each $\sigma \in \mathbb{S}^N$. Thus, u is a constant function.

Part (*iii*). Fix $j \in J_N$, with $j \neq i$, and $u \in E_i \cap E_j$. The second relation of (6.2) shows that $u(\sigma) = -u(\tau_i^{-1}\sigma) = -u(\tau_j^{-1}\sigma)$ for every $\sigma \in \mathbb{S}^N$. But, Part (*ii*) shows that u is constant. Thus, u must be identically zero in $\mathbb{S}^N$. □

Finally, following [151], we construct explicit functions belonging to E_i that will be useful in the sequel. To this aim, we say that a set $D \subset \mathbb{S}^N$ is $G_{N,i}^{\tau_i}$-invariant if $gD \subseteq D$ for every $g \in G_{N,i}^{\tau_i}$.

Proposition 6.1.4. *Let $N > 2m$ and $m \geq 1$. Let $i \in J_N$ be fixed. Then there exist a number $C_i > 0$ and a $G_{N,i}^{\tau_i}$-invariant set $D_i \subset \mathbb{S}^N$, with $\mathrm{Vol}_h(D_i) > 0$, and a function $v \in E_i$ such that*
(*i*) $\|v\|_\infty \leq 1$;
(*ii*) $|\nabla_h v|_h \leq C_i$ *a. e. in* $\mathbb{S}^N$;
(*iii*) $|v| = 1$ *in* D_i.

An explicit function $v : \mathbb{S}^N \to \mathbb{R}$ fulfilling all the requirements of Proposition 6.1.4 is given by

$$v(\sigma) = \frac{8}{R-r}\ \mathrm{sgn}(|\sigma_1| - |\sigma_3|)\max\{0, \mathfrak{m}(\sigma_1, \sigma_3)\}, \quad \text{with}$$
$$\mathfrak{m}(\sigma_1, \sigma_3) = \min\left\{\frac{R-r}{8}, \frac{R-r}{4} - \mathfrak{M}(\sigma_1, \sigma_3)\right\},$$
$$\mathfrak{M}(\sigma_1, \sigma_3) = \max\left\{\left||\sigma_1| + |\sigma_3| - \frac{R+3r}{4}\right|, \left|||\sigma_1| - |\sigma_3|| - \frac{R+3r}{4}\right|\right\},$$

where $R > r$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^N$, with $\sigma_1, \sigma_3 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{N-2i-1}$, whenever $i \neq (N-1)/2$, and $\sigma = (\sigma_1, \sigma_3) \in \mathbb{S}^N$, with $\sigma_1, \sigma_3 \in \mathbb{R}^{(N+1)/2}$, whenever $i = (N-1)/2$. The $G_{N,i}^{\tau_i}$-invariant set $D_i \subset \mathbb{S}^N$ can be defined as

$$D_i = \left\{\sigma \in \mathbb{S}^N : \mathfrak{M}(\sigma_1, \sigma_3) \leq \frac{R-r}{8}\right\}.$$

The above construction can be found in [144]. See also [151, Chapter 10] for additional remarks.

6.2 Geometrically distinct sequences of solutions

In this section we first consider the polyharmonic problem on the sphere

$$\begin{cases} \mathfrak{D}^m v = |u|^{2_m^*-2}u & \text{in } \mathbb{S}^N, \\ u \in H^m(\mathbb{S}^N), & N > 2m, \end{cases} \tag{6.4}$$

where $\mathfrak{D}^m$ is the polyharmonic operator given by

$$\mathfrak{D}^m = \prod_{k=1}^{m}\left(-\Delta_h + \frac{1}{4}(N-2k)(N+2k-2)\mathrm{id}_{L^2(\mathbb{S}^N)}\right),$$

and Δ_h denotes the usual Laplace–Beltrami operator on $\mathbb{S}^N$.

We present here for simplicity a meaningful consequence of Theorem 1.1 in [185].

Theorem 6.2.1. *Let m and N be two positive integers, with $N > 2m \geq 4$. Set*

$$s_N = [N/2] + (-1)^{N+1} - 1.$$

Then, the critical polyharmonic equation (6.4) *admits at least s_N sequences of infinitely many finite energy nodal solutions, which are unbounded in $H^m(\mathbb{S}^N)$ and mutually symmetrically distinct.*

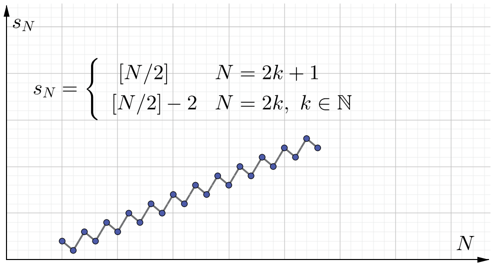

Figure 6.1: Number s_N of sequences of solutions ($5 \leq N \leq 28$).

Figure 6.1 above shows the behavior of the number s_N of sequences of solutions when the dimension N is small. We study (6.4) from the point of view of the $O(N+1)$ symmetry theory, as in [35, 144, 174]. This approach presents new and challenging features in the higher order case. For instance, A. Maalaoui and V. Martino in [165] and A. Maalaoui, V. Martino, and G. Tralli in [166], motivated again by the original paper of

W. Y. Ding, establish the existence of sign-changing solutions for the Yamabe problem on the Heisenberg group $\mathbb{H}^N = \mathbb{C}^N \times \mathbb{R}$.

Finally, for the sake of completeness, we cite the paper [145], in which A. Kristály proves a more general multiplicity existence theorem of sign-changing solutions for the fractional Yamabe problem on the Heisenberg group $\mathbb{H}^N$ via a nonlocal version of a compactness result due to W. Y. Ding, E. Hebey, and M. Vaugon on the Cauchy–Riemann unit sphere $\mathbb{S}^{2N+1}$ and an algebraic-theoretical approach on suitable subgroups of the unitary group $U(N+1)$.

For the sake of clarity, we present a consequence of Theorem 6.2.1, when $m = 2$, that is, when the biharmonic operator in equation (6.4) reduces to the celebrated Paneitz operator, introduced by S. Paneitz himself in [203] for smooth Riemannian manifolds. For further details in this context, we refer to [127], as well as to the monograph [115] and to the references therein. More precisely, in the case $m = 2$, the operator $\mathfrak{D}^2$ has the form

$$\Delta_h^2 - \alpha\,\Delta_h + a\,\mathrm{id}_{L^2(\mathbb{S}^N)},$$

where $\alpha = (N^2 - 2N - 4)/2$ and $a = N(N^2-4)(N-4)/16$. Thus, $\alpha > 0$ and $a > 0$ for all $N > 4$.

Corollary 6.2.2. *Let $N > 4$. Then, the critical Paneitz equation*

$$\Delta_h^2 u - \alpha\,\Delta_h u + a\,u = |u|^{8/(N-4)}u \quad \text{in } \mathbb{S}^N,$$

admits at least s_N sequences of infinitely many finite energy nodal weak solutions $u_k^{(i)} \in H^2(\mathbb{S}^N)$, $i = 1,\dots,s_N$, which are unbounded in $H^2(\mathbb{S}^N)$ and mutually symmetrically distinct.

More precisely, for each i the unbounded sequence $(u_k^{(i)})_k$ lies in the subspace $H^2_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ of the $G^{\tau_i}_{N,i}$-invariant functions of $H^2(\mathbb{S}^N)$, with respect to the action

$$\widehat{\circledast}_i : G^{\tau_i}_{N,i} \times H^2(\mathbb{S}^N) \to H^2(\mathbb{S}^N), \quad (g,v) \mapsto g\widehat{\circledast}_i u,$$

defined pointwise by

$$(g\widehat{\circledast}_i u)(\sigma) = \begin{cases} u(g^{-1}\sigma), & \text{if } g \in G_{N,i}, \\ -u(g^{-1}\tau_i^{-1}\sigma), & \text{if } g = \tau_i\tilde{g} \in G^{\tau_i}_{N,i} \setminus G_{N,i},\ \tilde{g} \in G_{N,i}, \end{cases}$$

where $G^{\tau_i}_{N,i}$ is the compact group of $O(N+1)$ generated by the compact subgroup

$$G_{N,i} = \begin{cases} O(i+1) \times O(N-2i-1) \times O(i+1), & \text{if } i \neq \frac{N-1}{2}, \\ O(i+1) \times O(i+1), & \text{if } i = \frac{N-1}{2}, \end{cases}$$

of $O(N+1)$ and by an involution $\tau_i : \mathbb{S}^N \to \mathbb{S}^N$, with the properties that

$$\tau_i \notin G_{N,i}, \quad \tau_i G_{N,i} \tau_i^{-1} = G_{N,i} \quad \text{and} \quad \tau_i^2 = \mathrm{id}_{\mathbb{S}^N}$$

for every $i = 1, \ldots, s_N$. *The mutual symmetry difference comes from the fact that* $H^2_{G^{\tau_i}_{N,i}}(\mathbb{S}^N) \cap H^2_{G^{\tau_j}_{N,j}}(\mathbb{S}^N) = \{0\}$ *for all* $i, j \in J_N$, *with* $i \neq j$.

The Hilbertian structure of $H^m(\mathbb{S}^N)$ is given by the scalar product

$$\langle u, v\rangle_{H^m(\mathbb{S}^N)} = \begin{cases} \int_{\mathbb{S}^N} (\Delta_h^k u\, \Delta_h^k v + uv) d\sigma_h, & \text{if } m = 2k, \\ \int_{\mathbb{S}^N} (\nabla_h \Delta_h^k u \cdot \nabla_h \Delta_h^k v + uv) d\sigma_h, & \text{if } m = 2k+1, \end{cases} \tag{6.5}$$

for every $u, v \in H^m(\mathbb{S}^N)$. We denote by $\|\cdot\|_{H^m(\mathbb{S}^N)}$ the norm induced by the scalar product in (6.5).

In order to handle the variational formulation of problem (6.4), we introduce a different Hilbertian norm $\|\cdot\|_*$ on the Sobolev space $H^m(\mathbb{S}^N)$, which is equivalent to the norm $\|\cdot\|_{H^m(\mathbb{S}^N)}$. This equivalence will be more readable if we express (6.5) in a convenient form given in terms of the Fourier coefficients of the functions u and v. To this aim, let $L^2(\mathbb{S}^N)$ be the standard Lebesgue space of square summable functions on $\mathbb{S}^N$ endowed by the natural inner product

$$\langle u, v\rangle_{L^2(\mathbb{S}^N)} = \int_{\mathbb{S}^N} u\, v d\sigma_h \quad \text{for every } u, v \in L^2(\mathbb{S}^N).$$

Clearly, $L^2(\mathbb{S}^N)$ can be decomposed as a direct sum of the orthogonal eigenspaces connected with the eigenfunctions of $-\Delta_h$ on $H^1(\mathbb{S}^N)$, that is,

$$L^2(\mathbb{S}^N) = \bigoplus_{\ell=0}^{\infty} K_\ell, \tag{6.6}$$

where for every $\ell \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the ℓth eigenspace $K_\ell = \operatorname{Ker}(-\Delta_h - \lambda_\ell \mathrm{id}_{L^2(\mathbb{S}^N)})$ is generated by the ℓth degree orthonormal (real valued) spherical harmonics Y_ℓ^j, with $j = 1, \ldots, c_\ell$ and

$$c_\ell = \binom{\ell + N}{N} - \binom{\ell + N - 2}{N}.$$

More precisely, the ℓth graded component of $L^2(\mathbb{S}^N)$ is generated by harmonic polynomial maps $P : \mathbb{R}^{N+1} \to \mathbb{R}$ restricted to $\mathbb{S}^N$ that are homogeneous of degree ℓ. Moreover, the representation of the orthogonal group $O(N+1)$ on the linear space K_ℓ is irreducible, in the sense of the representation theory, see Chapter IV of the celebrated monograph [235] due to E. M. Stein and G. Weiss.

By (6.6), every function $u \in L^2(\mathbb{S}^N)$ admits a unique Fourier decomposition

$$u = \sum_{\ell=0}^{\infty} \sum_{j=1}^{c_\ell} \widehat{u}(\ell, j) Y_\ell^j, \tag{6.7}$$

where $\widehat{u}(\ell,j)$ denotes the Fourier coefficient of u given by

$$\widehat{u}(\ell,j) = \langle u, Y_\ell^j\rangle_{L^2(\mathbb{S}^N)}$$

for every $\ell \in \mathbb{N}_0$ and $j = 1,\dots,c_\ell$. In other words, (6.7) has the expected expression

$$u = \sum_{\ell=0}^{\infty}\sum_{j=1}^{c_\ell}\langle u, Y_\ell^j\rangle_{L^2(\mathbb{S}^N)} Y_\ell^j$$

for every $u \in L^2(\mathbb{S}^N)$. Accordingly to (6.7), we can rewrite the inner product given in (6.5) as

$$\langle u,v\rangle_{H^m(\mathbb{S}^N)} = \sum_{\ell=0}^{\infty}(b_\ell^m + 1)\sum_{j=1}^{c_\ell}\widehat{u}(\ell,j)\widehat{v}(\ell,j) \quad \text{for every } u,v \in H^m(\mathbb{S}^N), \tag{6.8}$$

where $b_\ell = \ell(\ell + N - 1)$ denotes the ℓth eigenvalue of $-\Delta_h$ in $H^1(\mathbb{S}^N)$, that is,

$$-\Delta_h Y_\ell^j = b_\ell Y_\ell^j \quad \text{in } \mathbb{S}^N \tag{6.9}$$

for all $j = 1,\dots,c_\ell$. Moreover, as it is well known, by (6.8) the inner product on $H^m(\mathbb{S}^N)$, defined for every u, $v \in H^m(\mathbb{S}^N)$ by

$$\langle u,v\rangle_* = \sum_{\ell=0}^{\infty}\gamma_\ell(N,m)\sum_{j=1}^{c_\ell}\widehat{u}(\ell,j)\widehat{v}(\ell,j), \quad \gamma_\ell(N,m) = \frac{\Gamma(\frac{N}{2} + m + \ell)}{\Gamma(\frac{N}{2} - m + \ell)}, \tag{6.10}$$

induces the norm

$$\|u\|_* = \left(\sum_{\ell=0}^{\infty}\gamma_\ell(N,m)\sum_{j=1}^{c_\ell}\left|\widehat{u}(\ell,j)\right|^2\right)^{1/2} \quad \text{for every } u \in H^m(\mathbb{S}^N),$$

which is equivalent to $\|\cdot\|_{H^m(\mathbb{S}^N)}$.

Now, we claim that

$$\mathfrak{D}^m Y_\ell^j = \gamma_\ell(N,m)Y_\ell^j \tag{6.11}$$

for every $\ell \in \mathbb{N}_0$ and $j = 1,\dots,c_\ell$. Fix $\ell \in \mathbb{N}_0$ and $j = 1,\dots,c_\ell$. Then, by (6.9) and (6.10),

$$\begin{aligned}
\mathfrak{D}^m Y_\ell^j &= \prod_{k=1}^{m}\left(-\Delta_h + \frac{1}{4}(N-2k)(N+2k-2)\mathrm{id}_{L^2(\mathbb{S}^N)}\right)Y_\ell^j \\
&= \prod_{k=1}^{m}\left(b_\ell + \frac{1}{4}(N-2k)(N+2k-2)\right)Y_\ell^j \\
&= \gamma_\ell(N,m)Y_\ell^j,
\end{aligned}$$

as claimed.

Let us now prove that

$$\langle u, v\rangle_* = \int_{\mathbb{S}^N} (\mathfrak{D}^m u) v d\sigma_h \tag{6.12}$$

for every $u, v \in H^m(\mathbb{S}^N)$. To see this, fix $u, v \in H^m(\mathbb{S}^N)$. By (6.7), clearly,

$$\int_{\mathbb{S}^N} (\mathfrak{D}^m u) v d\sigma_h = \sum_{\ell=0}^{\infty} \sum_{j=1}^{c_\ell} \hat{u}(\ell, j) \int_{\mathbb{S}^N} (\mathfrak{D}^m Y_\ell^j) v d\sigma_h.$$

On the other hand, (6.11) yields

$$\int_{\mathbb{S}^N} (\mathfrak{D}^m Y_\ell^j) v d\sigma_h = \gamma_\ell(N, m) \int_{\mathbb{S}^N} Y_\ell^j v d\sigma_h.$$

Thus

$$\int_{\mathbb{S}^N} (\mathfrak{D}^m u) v d\sigma_h = \sum_{\ell=0}^{\infty} \gamma_\ell(N, m) \sum_{j=1}^{c_\ell} \hat{u}(\ell, j) \int_{\mathbb{S}^N} Y_\ell^j v d\sigma_h. \tag{6.13}$$

By (6.6), it follows that

$$\begin{aligned} \int_{\mathbb{S}^N} Y_\ell^j v d\sigma_h &= \sum_{\tilde{\ell}=0}^{\infty} \sum_{\tilde{j}=1}^{c_\imath} \hat{v}(\tilde{\ell}, \tilde{j}) \int_{\mathbb{S}^N} Y_\ell^j Y_{\tilde{\ell}}^{\tilde{j}} d\sigma_h \\ &= \sum_{\tilde{\ell}=0}^{\infty} \sum_{\tilde{j}=1}^{c_\imath} \hat{v}(\tilde{\ell}, \tilde{j}) \delta_{\tilde{\ell},\ell} \delta_{\tilde{j},j} = \hat{v}(\ell, j). \end{aligned} \tag{6.14}$$

Then, (6.13) and (6.14) give

$$\int_{\mathbb{S}^N} (\mathfrak{D}^m u) v d\sigma_h = \sum_{\ell=0}^{\infty} \gamma_\ell(N, m) \sum_{j=1}^{c_\ell} \hat{u}(\ell, j) \hat{v}(\ell, j),$$

i. e., (6.12) is verified.

In conclusion, we have shown that problem (6.4) has a variational nature. Consequently, we say that a function $u \in H^m(\mathbb{S}^N)$ is a solution of (6.4) if

$$\langle u, \varphi\rangle_* = \int_{\mathbb{S}^N} |u|^{2_m^* - 2} u \varphi d\sigma_h$$

for every $\varphi \in H^m(\mathbb{S}^N)$.

Proof of Theorem 6.2.1. As already noted, (6.4) has a variational nature and its Euler–Lagrange functional $\mathcal{J}$ is given by

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_*^2 - \int_{\mathbb{S}^N} |u|^{2_m^*} d\sigma_h, \quad u \in H^m(\mathbb{S}^N). \tag{6.15}$$

The functional $\mathcal{J}$ is well defined in $H^m(\mathbb{S}^N)$ and is of class $C^1(H^m(\mathbb{S}^N))$. Moreover, for each $u \in H^m(\mathbb{S}^N)$,

$$\langle \mathcal{J}(u), \varphi \rangle = \langle u, \varphi \rangle_* - \int_{\mathbb{S}^N} |u|^{2_m^*-2} u\varphi d\sigma_h \tag{6.16}$$

for every $\varphi \in H^m(\mathbb{S}^N)$. Hence, the critical points of $\mathcal{J}$ in $H^m(\mathbb{S}^N)$ are exactly the solutions of (6.4).

Let G be a topological group. We say that $u \in H_G^m(\mathbb{S}^N)$ is a *solution of* (6.4) *only in the* $H_G^m(\mathbb{S}^N)$ *sense* if

$$\langle \mathcal{J}(u), \varphi \rangle = \langle u, \varphi \rangle_* - \int_{\mathbb{S}^N} |u|^{2_m^*-2} u\varphi d\sigma_h$$

for any $\varphi \in H_G^m(\mathbb{S}^N)$. Then, $u \in H_G^m(\mathbb{S}^N)$ is a solution of (6.4) in the whole space $H^m(\mathbb{S}^N)$, that is, in sense of definition (6.16), if the symmetric criticality Theorem A.1.5 of Palais holds. For details and comments, we refer to Section 5 of [52].

We emphasize that the invariance of $\mathcal{J}$, with respect to translations and dilations, implies that the functional $\mathcal{J}$ does not satisfy the Palais–Smale condition. However, as observed in Proposition 3.1 of [35], the symmetric mountain pass theorem in addition to the principle of symmetric criticality of Palais yield the following critical point result.

Theorem 6.2.3. *Let G be a compact topological group. Let*

$$\diamond : G \times H^m(\mathbb{S}^N) \to H^m(\mathbb{S}^N), \quad (g, u) \mapsto g \diamond u,$$

be a linear and isometric action of G on $H^m(\mathbb{S}^N)$ *and denote by*

$$H_G^m(\mathbb{S}^N) = \{u \in H^m(\mathbb{S}^N) : g \diamond u = u \text{ for all } g \in G\}$$

the subspace of $H^m(\mathbb{S}^N)$ *containing all the symmetric functions with respect to the group G. Let* $\mathcal{J}$ *be the energy functional associated to* (6.4) *and assume that*

(*i*) $\mathcal{J}$ *is G-invariant*;
(*ii*) *the embedding* $H_G^m(\mathbb{S}^N) \hookrightarrow L^{2_m^*}(\mathbb{S}^N)$ *is compact*;
(*iii*) $H_G^m(\mathbb{S}^N)$ *has infinite dimension.*

Then, the functional $\mathcal{J}$ *admits a sequence of critical points* $(u_k)_k \subset H^m_G(\mathbb{S}^N)$ *such that*

$$\int_{\mathbb{S}^N} |u_k|^{2^*_m} d\sigma_h \to \infty$$

as $k \to \infty$.

Let $\mathcal{J}$ be the energy functional associated to (6.4) and given in (6.15). Fix $i \in J_N$ and consider the compact group

$$G^{\tau_i}_{N,i} \subset O(N+1),$$

given in (6.1), and let $\widehat{\circledast}_i : G^{\tau_i}_{N,i} \times H^m(\mathbb{S}^N) \to H^m(\mathbb{S}^N)$ be the action defined in (6.2). Thanks to the definition of $\widehat{\circledast}_i$, the functional $\mathcal{J}$ is $G^{\tau_i}_{N,i}$-invariant, that is,

$$\mathcal{J}(g\widehat{\circledast}_i u) = \mathcal{J}(u)$$

for every $(g,u) \in G^{\tau_i}_{N,i} \times H^m(\mathbb{S}^N)$. Then, the subspace

$$E_i = \{u \in H^m(\mathbb{S}^N) : g\widehat{\circledast}_i u = u \text{ for all } g \in G^{\tau_i}_{N,i}\}$$

of $H^m(\mathbb{S}^N)$, which consists of $G^{\tau_i}_{N,i}$-invariant functions, has infinite dimension.

Since $\dim(G^{\tau_i}_{N,i}\sigma) \geq 1$ for every $\sigma \in \mathbb{S}^N$, one gets $q^\star_i > 2^*_m$. Thus, Proposition 6.1.2 ensures that the embedding

$$E_i \hookrightarrow\hookrightarrow L^{2^*_m}(\mathbb{S}^N)$$

is compact.

Hence, by Theorem 6.2.3, the functional $\mathcal{J}$ admits a sequence of critical points $(u^{(i)}_k)_k$ in $E_i = H^m_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ such that

$$\int_{\mathbb{S}^N} |u^{(i)}_k|^{2^*_m} d\sigma_h \to \infty, \quad \text{as } k \to \infty.$$

The symmetric criticality Theorem A.1.5 implies that (6.4) admits a sequence $(u^{(i)}_k)_k \subset H^m(\mathbb{S}^N)$ of solutions satisfying

$$\|u^{(i)}_k\|_* \to \infty \quad \text{as } k \to \infty. \tag{6.17}$$

Consequently, Proposition 6.1.3(*ii*) gives that (6.4) admits at least

$$s_N = [N/2] + (-1)^{N+1} - 1$$

sequences $(u^{(i)}_k)_k \subset H^m(\mathbb{S}^N)$ of solutions satisfying (6.17). The remarks on the structure of the symmetric Sobolev spaces E_i yield that the solutions $u^{(i)}_k$ for every $k \in \mathbb{N}$ and $i \in J_N$ are sign-changing. This completes the proof of Theorem 6.2.1. □

Since the appearance of the celebrated paper of W. Y. Ding [80] on the conformally invariant scalar field equation in $\mathbb{R}^N$, concerning the existence of infinitely many conformally inequivalent sign-changing solutions, with finite energy, the method of pulling back the problem into the unit sphere $\mathbb{S}^N$ of $\mathbb{R}^{N+1}$ by means of a stereographic projection (see Figure 6.2) and then into its variational formulation has seen extensive use in the literature for different problems involving critical nonlinearities in the sense of Sobolev.

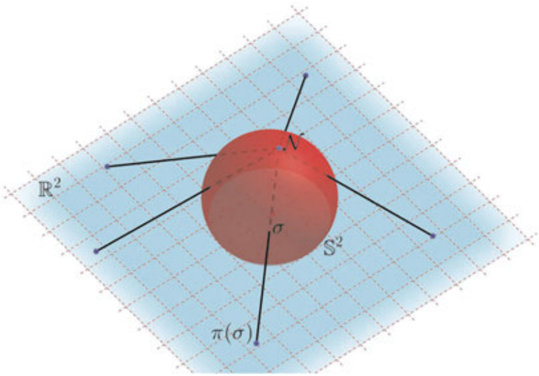

Figure 6.2: The stereographic projection $\pi : \mathbb{S}^2 \setminus \{\mathcal{N}\} \to \mathbb{R}^2$.

In other words, the case of $N > 2 = 2m$, that is, $m = 1$, reduces equation (6.4) into

$$-\Delta_h u + \frac{N(N-2)}{2} u = |u|^{4/(N-2)} u \quad \text{in } \mathbb{S}^N.$$

The search of positive solutions is the well known Yamabe problem, which arises from the conformal geometry. For details we refer to Chapter 7 of the monograph [10].

For instance, inspired by [80], T. Bartsch, M. Schneider, and T. Weth in [35] showed for the critical polyharmonic equation

$$\begin{cases} (-\Delta)^m u = |u|^{2_m^*-2} u \quad \text{in } \mathbb{R}^N, \quad u \in \mathcal{D}^{m,2}(\mathbb{R}^N), \\ N > 2m, \quad 2_m^* = \dfrac{2N}{N-2m}, \end{cases} \tag{6.18}$$

the existence of a sequence of infinitely many finite energy nodal solutions which are unbounded in the Sobolev space $\mathcal{D}^{m,2}(\mathbb{R}^N)$.

The polyharmonic operator $(-\Delta)^m$ that appears in (6.18) is the most popular prototype of an elliptic operator of order $2m$, formally given by

$$(-\Delta)^m = (-1)^m \sum_{j_1+\cdots+j_N=m} \frac{m!}{j_1! j_2! \cdots j_N!} \frac{\partial^{2m}}{\partial x_1^{2j_1} \cdots \partial x_N^{2j_N}}.$$

For $m \geq 2$, polyharmonic functions have interesting applications in physics. Airy functions, which appear in optics, quantum mechanics, electromagnetics and radiative transfer, are biharmonic functions.

In this spirit, starting from the pioneering paper [35] and encouraged by a wide interest in the current literature on polyharmonic problems, in [185] the existence of at least a finite number of sequences of infinitely many finite energy nodal solutions which are unbounded in the Beppo Levi space $\mathcal{D}^{m,2}(\mathbb{R}^N)$ has been proved. Finally, we recall that every nontrivial nonnegative solution $u \in \mathcal{D}^{m,2}(\mathbb{R}^N)$ of (6.18) is positive in $\mathbb{R}^N$ and has the form

$$u_{\varepsilon,\xi}(x) = \varepsilon^{-\frac{N-2m}{2}} U\big((x-\xi)/\varepsilon\big), \quad \text{where } U(x) = P_{m,N}^{\frac{N-2m}{4m}} \big(1+|x|^2\big)^{-\frac{N-2m}{2}},$$

$\varepsilon > 0$, $\xi \in \mathbb{R}^N$ and $P_{m,N} = \prod_{k=-m}^{m}(N+2k)$.

See the quoted paper [35] for additional remarks and comments. For historical details and a wide list of recent contributions on semilinear problems involving the biharmonic or polyharmonic operator as the principal part, we refer to the modern excellent monograph [115] and the references therein.

More recently, in [174] the author describes a group-theoretical scheme, which arises in previous papers on $O(N+1)$-invariant variational problems, as a method to show the existence of several geometrically different sequences of solutions, distinguished by their symmetry properties. The special topological compact groups $G_i \subset O(N+1)$, $i = 1, \dots, \kappa_N$, constructed in [174] via an abstract approach, can be applied to $H^m(\mathbb{S}^N)$ in order to find a finite family $\{H^m_{G_i}(\mathbb{S}^N)\}_{i=1}^{\kappa_N}$ of subspaces $H^m_{G_i}(\mathbb{S}^N) \subset H^m(\mathbb{S}^N)$ such that $H^m_{G_i}(\mathbb{S}^N) \cap H^m_{G_j}(\mathbb{S}^N) = \{0\}$ and $(O(N+1)u) \cap H^m_{G_i}(\mathbb{S}^N) = \emptyset$ for every $u \in H^m_{G_j}(\mathbb{S}^N) \setminus \{0\}$ and $i \neq j$. The theoretical procedure of [174], the Palais symmetry Theorem A.1.5, and Proposition 6.1.3 ensure that equation (6.18) admits at least κ_N geometrically different sequences of solutions distinguished by their symmetry properties. This result is summarized in Theorem 4.8 of [174] in a more general form, in which W. Marzantowicz studies the intrinsic linking between orthogonal Borel subgroups in $O(N+1)$ with partial and orthogonal flags in $\mathbb{R}^{N+1}$. The key tool is the use of the number of the unrestricted partitions of the Euclidean dimension $N+1$. A consequence of the Marzantowicz approach is given in Theorem 1.1 of [185], to which we refer the interested reader.

Let us now prove an existence result for special Emden–Fowler problems by using the reduction to the unit sphere. Let s be a fixed constant, with $1 - N < s < 0$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function, or more generally, locally Hölder continuous, and w is a smooth and positive function on the unit sphere $\mathbb{S}^N$. Consider the parameterized Emden–Fowler equation

$$-\Delta u = \lambda |x|^{s-2} w\big(x/|x|\big) f\big(|x|^{-s} u\big), \quad x \in \mathbb{R}^{N+1} \setminus \{0\}. \tag{6.19}$$

Existence results for (6.19) has been established recently in [42, 150, 151], via variational methods. Using the key transformation as M. F. Bidaut–Véron and L. Véron in [39], we shall reduce (6.19) to

$$-\Delta_h v + \alpha v = \lambda w(\sigma) f(v), \quad \sigma \in \mathbb{S}^N, \; \alpha = s(1-s-N) > 0. \tag{6.20}$$

Equations as in (6.20) have been largely studied, and we refer to the pioneering papers [68] of A. Cotsiolis and D. Iliopoulos and [244] by J. L. Vázquez and L. Véron. Moreover, (6.20) perfectly fits the scope of the chapter.

For the main results of (6.19), let us introduce

$$\Lambda_+(\mathbb{S}^N) = \{w \in L^\infty(\mathbb{S}^N) \; : \; \operatorname*{ess\,inf}_{\mathbb{S}^N} w > 0\}$$

and on the Hilbert Sobolev space $H^1(\mathbb{S}^N)$ the norm

$$\|v\| = \left(\int_{\mathbb{S}^N} |\nabla_h v|^2 d\sigma_h + \alpha \int_{\mathbb{S}^N} |v|^2 d\sigma_h \right)^{1/2},$$

which is equivalent to $\| \cdot \|_{H^1(\mathbb{S}^N)}$ given in (6.5).

Theorem 6.2.4. *Let $N \geq 3$ and let $s \in \mathbb{R}$, with $1-N < s < 0$. Let $w \in \Lambda_+(\mathbb{S}^N)$ and $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, with $f(0) \geq 0$, and such that for some $q \in (1, 2^*)$, $2^* = 2N/(N-2)$,*

$$\sup_{t \in \mathbb{R}} \frac{|f(t)|}{1+|t|^{q-1}} < \infty.$$

Assume that

(i_1) There are two real sequences $(\xi_k)_k$ and $(\zeta_k)_k$, with $0 \leq \xi_k < \zeta_k$, and such that

$$F(\xi_k) = \sup_{t \in [\xi_k, \zeta_k]} F(t), \quad \lim_{k \to \infty} \xi_k = \infty,$$

(i_2) $F_\infty = \limsup_{t \to \infty} \frac{F(t)}{t^2} \in \mathbb{R}^+ \cup \{\infty\}$.

Then, for every

$$\lambda > \lambda_*, \quad \lambda_* = \begin{cases} \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2F_\infty \|w\|_{L^1(\mathbb{S}^N)}} & \text{if } F_\infty \in \mathbb{R}^+, \\ 0 & \text{if } F_\infty = \infty, \end{cases}$$

equation (6.19) *admits a sequence $(u_k)_k$ of nonnegative classical solutions such that the function $|\cdot|^{-s} u_k \in H^1(\mathbb{S}^N)$ for every $k \in \mathbb{N}$ and*

$$\lim_{k \to \infty} \int_{\mathbb{S}^N} \left(\left|\nabla_h(|x|^{-s} u_k)\right|^2 + \left||x|^{-s} u_k\right|^2\right) d\sigma_h = \infty.$$

Proof. The solutions of (6.19) are being sought in the particular form

$$u(x) = r^s v(\sigma), \tag{6.21}$$

where $(r,\sigma) = (|x|, x/|x|) \in \mathbb{R}^+ \times \mathbb{S}^N$ are the spherical coordinates in the space $\mathbb{R}^{N+1} \setminus \{0\}$ and v is a smooth function defined on $\mathbb{S}^N$. This type of transformation comes from [39], where the asymptotics of a special form of (6.19) has been studied. Thanks to (6.21) and taking into account that

$$\begin{aligned}\Delta u &= r^{-N}\frac{\partial}{\partial r}\left(r^N \frac{\partial}{\partial r}(r^s v)\right) + r^{s-2}\Delta_h v \\ &= (-\alpha + \Delta_h v)r^{s-2}, \quad \alpha = s(1-s-N) > 0,\end{aligned}$$

equation (6.19) reduces to (6.20). Set for every $\sigma \in \mathbb{S}^N$,

$$\mathsf{f}(\sigma,t) = \begin{cases} w(\sigma)f(t) & \text{if } t \geq 0, \\ w(\sigma)f(0) & \text{if } t < 0, \end{cases}$$

and consider the equation

$$-\Delta_h v + \alpha v = \lambda \mathsf{f}(\sigma, v), \quad \sigma \in \mathbb{S}^N, \; v \in H^1(\mathbb{S}^N). \tag{6.22}$$

Put for every $v \in H^1(\mathbb{S}^N)$,

$$\Phi(v) = \frac{1}{2}\|v\|^2 \quad \text{and} \quad \Psi(v) = \int_{\mathbb{S}^N}\left(\int_0^{v(\sigma)} \mathsf{f}(\sigma,t)dt\right)d\sigma_h.$$

Standard arguments ensure that the solutions of (6.22) are the critical points of the energy functional

$$J_\lambda(v) = \frac{1}{\lambda}\Phi(v) - \Psi(v) \quad \text{for all } v \in H^1(\mathbb{S}^N).$$

Owing to the compact embedding of $H^1(\mathbb{S}^N)$ into the Lebesgue spaces $L^{\wp}(\mathbb{S}^N)$, with $\wp \in [1, 2^*)$, the functional J_λ is well defined and sequentially weakly lower semicontinuous and continuously Gâteaux differentiable in $H^1(\mathbb{S}^N)$. For every $k \in \mathbb{N}$ define

$$\mathbb{E}_k = \{v \in H^1(\mathbb{S}^N) \,:\, 0 \leq v \leq \zeta_k \text{ a.e. in } \mathbb{S}^N\}.$$

Following the arguments used in [181, Theorem 3.1] it is possible to prove that there exists $v_k \in \mathbb{E}_k$ such that $J_\lambda(v_k) = \inf_{v\in\mathbb{E}_k} J_\lambda(v) = m_k$ for every $k \in \mathbb{N}$.

Now, let us prove that $\liminf_{k\to\infty} m_k = -\infty$. Assume first that $F_\infty < \infty$ in (i_2). Since

$$\lambda > \lambda_* = \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2F_\infty \|w\|_{L^1(\mathbb{S}^N)}},$$

clearly,

$$\frac{1}{2\lambda} < \frac{F_\infty \|w\|_{L^1(\mathbb{S}^N)}}{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}.$$

Let $L \in \mathbb{R}^+$ be such that

$$\frac{1}{2\lambda} < L < \frac{F_\infty \|w\|_{L^1(\mathbb{S}^N)}}{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}.$$

Then, there exists a sequence $(\eta_k)_k \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} \eta_k = \infty \quad \text{and} \quad \frac{F(\eta_k)}{\eta_k^2} > \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{\|w\|_{L^1(\mathbb{S}^N)}} \tag{6.23}$$

for every $k \in \mathbb{N}$ by (i_2). The sequence $(\eta_k)_k$ with the property (6.23), clearly, exists also in the case $F_\infty = \infty$ and $\lambda_* = 0$ as a direct consequence of (i_2).

Let us choose a subsequence $(\zeta_{k_j})_j$ of $(\zeta_k)_k$ such that $\eta_j < \zeta_{k_j}$ for every $j \in \mathbb{N}$. Thus, the function $v_j = \eta_j$ in $\mathbb{S}^N$ belongs to $\mathbb{E}_{k_j}$. This implies that for every $j \in \mathbb{N}$,

$$\begin{aligned} J_\lambda(v_j) &= \frac{1}{\lambda}\Phi(v_j) - F(v_j) \\ &\le \frac{\eta_j^2}{2\lambda}\alpha \mathrm{Vol}_h(\mathbb{S}^N) - \int_{\mathbb{S}^N} w(\sigma)\left(\int_0^{v_j(\sigma)} f(t)dt\right)d\sigma_h \\ &< \frac{\eta_j^2}{2}\alpha \mathrm{Vol}_h(\mathbb{S}^N)\left(\frac{1}{2\lambda} - L\right) < 0. \end{aligned}$$

Thus, $\lim_{j\to\infty} J_\lambda(v_j) = -\infty$. Moreover, since

$$m_{k_j} = \inf_{v\in\mathbb{E}_{k_j}} J_\lambda(v) \le J_\lambda(v_j),$$

the previous inequality implies that $\lim_{j\to\infty} m_{k_j} = -\infty$.

Let us prove that the sequence of local minima $(v_{k_j})_j$ must be unbounded in $H^1(\mathbb{S}^N)$. Otherwise, there would be a subsequence, still denoted by $(v_{k_j})_j$, weakly convergent to some function $v_\infty \in H^1(\mathbb{S}^N)$. Then,

$$J_\lambda(v_\infty) \le \liminf_{j\to\infty} J_\lambda(v_{k_j}) = -\infty,$$

which is the desired contradiction. The assertion is proved.

Finally, the solutions of (6.22) are classical since the nonlinear term f is a locally Lipschitz continuous function. □

Put

$$a_k = \frac{2k!(k+2)! - 1}{4(k+1)!} \quad \text{and} \quad b_k = \frac{2k!(k+2)! + 1}{4(k+1)!},$$

for every $k \geq 1$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} ((k+1)!^2 - k!^2)\frac{g_k(t)}{\int_{a_k}^{b_k} g_k(\tau)d\tau} & \text{if } t \in \bigcup_{k\geq 1}[a_k, b_k], \\ 0 & \text{otherwise,} \end{cases}$$

where $g_k : [a_k, b_k] \to \mathbb{R}$ is given by

$$g_k(t) = \sqrt{\frac{1}{16(k+1)!} - \left(t - \frac{k!(k+2)}{2}\right)^2}, \quad t \in [a_k, b_k]$$

for every $k \in \mathbb{N}$; see Figure 6.3.

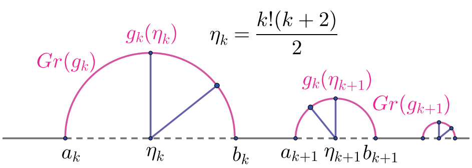

Figure 6.3: The sequence $(g_k)_k$.

More precisely, $(g_k)_k$ is a sequence of semicircles with decreasing radii and supported on the intervals $[a_k, b_k]$ for every $k \in \mathbb{N}$.

As noted in [41, Example 4.1], a direct computation ensures that

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} = F_\infty = 4.$$

Moreover,

$$F(b_k) = \sup_{t\in[b_k, a_{k+1}]} F(t) \quad \text{for every } k \in \mathbb{N}.$$

Then, Theorem 6.2.4 asserts that

$$-\Delta u = \lambda |x|^{s-2} w(x/|x|) f(|x|^{-s}u), \quad x \in \mathbb{R}^{N+1} \setminus \{0\},$$

admits for every $\lambda > \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{8\|w\|_{L^1(\mathbb{S}^N)}}$ a sequence $(u_k)_k$ of nonnegative classical solutions such that $|\cdot|^{-s}u_k \in H^1(\mathbb{S}^N)$ for every $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} \left(\left|\nabla_h(|x|^{-s}u_k)\right|^2 + \left||x|^{-s}u_k\right|^2\right) d\sigma_h = \infty.$$

We end the section proving the existence of infinitely many arbitrarily small solutions of problem (6.19). In this case, global growth conditions on the nonlinear term f are not required any longer, but the potential F is supposed to have an oscillating behavior near the origin expressed by condition (j_2) below. The statement of the result is as follows.

Theorem 6.2.5. *Let $N \geq 3$ and let $s \in \mathbb{R}$ with $1 - N < s < 0$. Let $w \in \Lambda_+(\mathbb{S}^N)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, with $f(0) = 0$. Assume that*

(j_1) *There are two real sequences $(\xi_k)_k$ and $(\zeta_k)_k$, with $0 \leq \xi_k < \zeta_k$ for every $k \in \mathbb{N}$ and such that*

$$F(\xi_k) = \sup_{t\in[\xi_k,\zeta_k]} F(t), \qquad \lim_{k\to\infty} \zeta_k = 0;$$

(j_2) $F_0 = \limsup_{t\to 0^+} \frac{F(t)}{t^2} \in \mathbb{R}^+ \cup \{\infty\}$.

Then, for every

$$\lambda > \lambda^*, \qquad \lambda^* = \begin{cases} \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2F_0\|w\|_{L^1(\mathbb{S}^N)}} & \text{if } F_0 \in \mathbb{R}^+, \\ 0 & \text{if } F_0 = \infty, \end{cases}$$

equation (6.19) *admits a sequence $(u_k)_k$ of nonnegative classical solutions such that the function $|\cdot|^{-s}u_k \in H^1(\mathbb{S}^N) \cap L^\infty(\mathbb{S}^N)$ for every $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \||\cdot|^{-s}u_k\|_\infty = \lim_{k\to\infty} \int_{\mathbb{S}^N} \left(\left|\nabla(|x|^{-s}u_k)\right|^2 + \left||x|^{-s}u_k\right|^2\right) d\sigma_h = 0. \tag{6.24}$$

Proof. Since the term f is continuous, fixing $t_0 > 0$, there exists $\kappa > 0$ such that

$$\left|w(\sigma)f(t)\right| \leq \kappa \quad \text{for all } (\sigma, t) \in \mathbb{S}^N \times [0, t_0].$$

Without loss of generality, we suppose that $\zeta_k \leq t_0$ for every $k \in \mathbb{N}$.

Fix $\lambda > \lambda^*$ and define $\mathtt{f} : \mathbb{R} \to \mathbb{R}$ by

$$\mathtt{f}(t) = \begin{cases} f(t_0) & \text{if } t > t_0, \\ f(t) & \text{if } 0 \leq t \leq t_0, \\ 0 & \text{if } t < 0. \end{cases}$$

Whence, for a. e. $\sigma \in \mathbb{S}^N$ and $t \in \mathbb{R}$, it turns out that

$$\left|w(\sigma)\mathtt{f}(t)\right| \leq \kappa. \tag{6.25}$$

Now, consider the equation

$$-\Delta_h v + \alpha v = \lambda w(\sigma)\mathtt{f}(v), \quad \sigma \in \mathbb{S}^N, \; v \in H^1(\mathbb{S}^N), \tag{6.26}$$

and set

$$J_\lambda(v) = \frac{1}{\lambda}\Phi(v) - \Psi(v) \quad \text{for all } v \in H^1(\mathbb{S}^N),$$

where

$$\Phi(v) = \frac{1}{2}\|v\|^2 \quad \text{and} \quad \Psi(v) = \int_{\mathbb{S}^N} w(\sigma)\left(\int_0^{v(\sigma)} \mathrm{f}(t)dt\right)d\sigma_h,$$

for every $v \in H^1(\mathbb{S}^N)$. Clearly, the solutions of (6.26) are the critical points of the functional J_λ. Owing to (6.25) and the compact embedding of $H^1(\mathbb{S}^N)$ into $L^\wp(\mathbb{S}^N)$, with $\wp \in [1, 2^*)$, the functional J_λ is well defined and sequentially weakly lower semicontinuous and continuously Gâteaux differentiable in $H^1(\mathbb{S}^N)$. Moreover, taking into account (6.25) and (j_1), by using direct minimization arguments, J_λ admits a local minimum v_k that belongs to the set

$$\mathbb{E}_k = \{v \in H^1(\mathbb{S}^N) \, : \, 0 \le v \le \zeta_k \text{ a.e. in } \mathbb{S}^N\}$$

for every $k \in \mathbb{N}$. More precisely, every v_k assumes its values in the interval $[0, \xi_k]$ except for a null measure subset of $\mathbb{S}^N$. In fact, fix $k \in \mathbb{N}$, define the function $\varrho_k : \mathbb{R} \to \mathbb{R}$ by

$$\varrho_k(t) = \begin{cases} \xi_k & \text{if } t > \xi_k, \\ t & \text{if } 0 \le t \le \xi_k, \\ 0 & \text{if } t < 0, \end{cases}$$

and consider the superposition operator $T_k : H^1(\mathbb{S}^N) \to H^1(\mathbb{S}^N)$ such that $v \mapsto T_k v$, where

$$T_k v(\sigma) = \varrho_k(v(\sigma)) \quad \text{a.e. in } \mathbb{S}^N$$

for every $v \in H^1(\mathbb{S}^N)$; see Figure 6.4.

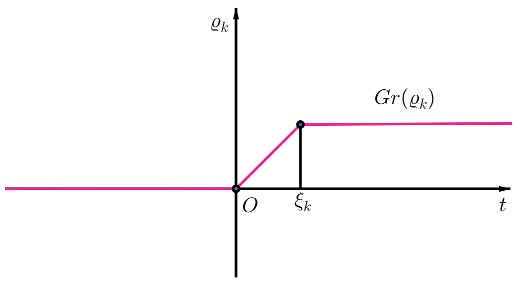

Figure 6.4: The graph $\mathrm{Gr}(\varrho_k)$ of the function ϱ_k.

Moreover, $T_k v \in H^1(\mathbb{S}^N)$ for every $v \in H^1(\mathbb{S}^N)$. Indeed, since ϱ_k is Lipshitz continuous, with $\varrho_k(0) = 0$, one gets $T_k v \in H^1(\mathbb{S}^N)$ for every $v \in H^1(\mathbb{S}^N)$. More precisely, $T_k v \in \mathbb{E}_k$ for every $k \in \mathbb{N}$. Now, set $\mathsf{v}_k = T_k v_k$ and let

$$X_k = \{\sigma \in \mathbb{S}^N \;:\; v_k(\sigma) \notin [0, \xi_k]\}.$$

If the Riemann measure $\mathrm{Vol}_h(X_k) = 0$ our conclusion is achieved. Otherwise, suppose that $\mathrm{Vol}_h(X_k) > 0$. Then, $\xi_k < v_k(\sigma) \leq \zeta_k$, as well as

$$\mathsf{v}_k(\sigma) = T_k v_k(\sigma) = \xi_k$$

for a. e. $\sigma \in X_k$. Now, (j_1) gives

$$\int_0^{v_k(\sigma)} \mathsf{f}(t)dt \leq \sup_{t\in[\xi_k,\zeta_k]} \int_0^t \mathsf{f}(\xi)d\xi = \int_0^{\xi_k} \mathsf{f}(t)dt = \int_0^{\mathsf{v}_k(\sigma)} \mathsf{f}(t)dt$$

for a. e. $\sigma \in X_k$. Moreover,

$$\begin{aligned}
\|\mathsf{v}_k\|^2 - \|v_k\|^2 &= \alpha \int_{\mathbb{S}^N} (|\mathsf{v}_k|^2 - |v_k|^2)d\sigma_g + \int_{\mathbb{S}^N} (|\nabla_h \mathsf{v}_k|^2 - |\nabla_h v_k|^2)d\sigma_g \\
&= \alpha \int_{X_k} (\xi_k^2 - v_k^2)d\sigma_g - \int_{X_k} |\nabla_h v_k|^2 d\sigma_g \\
&\leq -\alpha \int_{X_k} |v_k - \xi_k|^2 d\sigma_g - \int_{X_k} |\nabla_h \mathsf{v}_k - \nabla_h v_k|^2 d\sigma_g \\
&= -\alpha \int_{\mathbb{S}^N} |\,\mathsf{v}_k - v_k|^2 d\sigma_g - \int_{\mathbb{S}^N} |\nabla_h \mathsf{v}_k - \nabla_h v_k|^2 d\sigma_g \\
&= -\|\mathsf{v}_k - v_k\|^2.
\end{aligned}$$

The above inequalities ensure that

$$\begin{aligned}
J_\lambda(\mathsf{v}_k) - J_\lambda(v_k) &= \frac{1}{2\lambda}(\|\mathsf{v}_k\|^2 - \|v_k\|^2) - \int_{\mathbb{S}^N} w(\sigma)\left(\int_{v_k}^{\mathsf{v}_k} \mathsf{f}(t)dt\right)d\sigma_g \\
&\leq -\frac{1}{2\lambda}\|\mathsf{v}_k - v_k\|^2 - \int_{X_k} w(\sigma)\left(\int_{v_k}^{\mathsf{v}_k} \mathsf{f}(t)dt\right)d\sigma_g \\
&\leq -\frac{1}{2\lambda}\|\mathsf{v}_k - v_k\|^2.
\end{aligned}$$

Since $\mathsf{v}_k \in \mathbb{E}_k$, it follows that $J_\lambda(\mathsf{v}_k) \geq J_\lambda(v_k)$. Then

$$\|\mathsf{v}_k - v_k\|^2 = 0 \quad \text{for all } k,$$

that is,

$$\|\mathrm{v}_k - v_k\|^2 = \int_{X_k} \left|\nabla(\mathrm{v}_k - v_k)\right|^2 d\sigma_g + \int_{X_k} \alpha|\mathrm{v}_k - v_k|^2 d\sigma_g = 0.$$

Since $\mathrm{Vol}_h(X_k) > 0$, one gets $v_k = \mathrm{v}_k$ a. e. in $\mathbb{S}^N$. Hence $v_k \in [0, \xi_k]$ a. e. in $\mathbb{S}^N$, as claimed. Now, following the arguments used in [181, Theorem 3.2], the function v_k is a local minimum point of functional J_λ in the Sobolev space $H^1(\mathbb{S}^N)$ for every $k \in \mathbb{N}$. Set $m_k = \inf_{v \in \mathbb{E}_k} J_\lambda(v) = J_\lambda(v_k)$. By (6.25), we get for every $v \in \mathbb{E}_k$,

$$J_\lambda(v) = \frac{1}{\lambda}\Phi(v) - \Psi(v) \geq -\int_{\mathbb{S}^N} w(\sigma)\left(\int_0^{v(\sigma)} \mathrm{f}(t)dt\right)d\sigma_h \geq -\kappa \mathrm{Vol}_h(\mathbb{S}^N)\zeta_k.$$

Then, since $-\kappa \mathrm{Vol}_h(\mathbb{S}^N)\zeta_k \leq m_k \leq 0$, it follows that

$$\lim_{k\to\infty} m_k = \lim_{k\to\infty} \inf_{v\in\mathbb{E}_k} J_\lambda(v) = 0.$$

Moreover,

$$\begin{aligned}\frac{1}{\lambda}\Phi(v_k) = \Psi(v_k) + J_\lambda(v_k) &\leq \int_{\mathbb{S}^N} w(\sigma)\left(\int_0^{v_k(\sigma)} \mathrm{f}(t)dt\right)d\sigma_h + m_k \\ &\leq \kappa \mathrm{Vol}_h(\mathbb{S}^N)\zeta_k + m_k.\end{aligned}$$

Hence, the last inequality yields

$$\lim_{k\to\infty} \|v_k\| = 0. \tag{6.27}$$

To obtain the conclusion, it is enough to prove that such local minima are pairwise distinct. Assume first that $F_0 < \infty$ in (j_2). Since $\lambda > \lambda^*$, we get

$$F_0 > \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2\lambda\|w\|_{L^1(\mathbb{S}^N)}}.$$

Hence, there exists a sequence $(\eta_k)_k \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} \eta_k = 0 \quad \text{and} \quad \frac{F(\eta_k)}{\eta_k^2} > \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2\lambda\|w\|_{L^1(\mathbb{S}^N)}} \tag{6.28}$$

for every $k \in \mathbb{N}$. The sequence $(\eta_k)_k$ with the property (6.28) exists also in the easier case $F_0 = \infty$ and $\lambda^* = 0$ as a direct consequence of (j_2). Let $k_0 \in \mathbb{N}$ be so large that $\eta_{k_0} < \zeta_k$. Thus, the constant function η_{k_0} belongs to $H^1(\mathbb{S}^N)$ and this implies that

$$J_\lambda(v_k) \leq J_\lambda(\eta_{k_0}) \quad \text{for every } k \in \mathbb{N}.$$

Moreover,

$$\frac{F(\eta_{k_0})}{\eta_{k_0}^2} > \frac{\alpha \mathrm{Vol}_h(\mathbb{S}^N)}{2\lambda\|w\|_{L^1(\mathbb{S}^N)}}.$$

Consequently, $J_\lambda(v_k) < 0$ for every $k \in \mathbb{N}$. Then, the sequence $(v_k)_k$ has a subsequence, still denoted by $(v_k)_k$, of pairwise distinct elements that solve (6.26). On the other hand, $\|v_k\|_\infty \leq t_0$ for every $k \in \mathbb{N}$. Thus, $(v_k)_k$ is a sequence of solutions of (6.26) and, thanks to (6.21), as in the proof of Theorem 6.2.4, also solutions of (6.19). The proof is complete on account of (6.21) and (6.27). □

We observe that a similar variational approach with respect to that used here in order to prove Theorems 6.2.4 and 6.2.5 has been used in [17], where the existence of infinitely many solutions for elliptic Neumann problems on bounded domains was studied.

Put

$$a_k = \frac{1}{k!k} \quad \text{and} \quad b_k = \frac{1}{k!},$$

for every $k \geq 2$, and let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 4(b_k^2 - b_{k+1}^2)\frac{t-b_{k+1}}{(a_k-b_{k+1})^2} & \text{if } b_{k+1} \leq t \leq \frac{a_k+b_{k+1}}{2}, \\ 4(b_k^2 - b_{k+1}^2)\frac{a_k-t}{(a_k-b_{k+1})^2} & \text{if } \frac{a_k+b_{k+1}}{2} < t \leq a_k, \\ 0 & \text{otherwise.} \end{cases}$$

An easy description of the behavior of f is given below; see Figure 6.5. More precisely, the graph of the function f is given by a sequence of triangles with decreasing height and supported on the intervals $[b_{k+1}, a_k]$, for every $k \geq 2$.

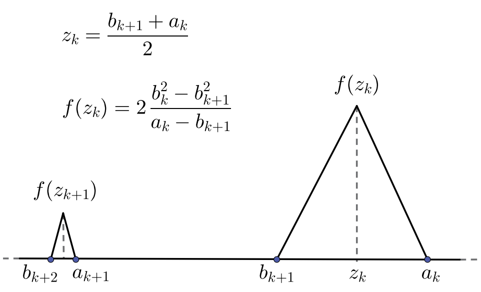

Figure 6.5: The structure of the function f.

As noted in [86, Example 2.4], a direct computation ensures that

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = 1 \quad \text{and} \quad \limsup_{t\to 0^+} \frac{F(t)}{t^2} = F_0 = \infty.$$

Moreover,

$$F(a_{k+1}) = \sup_{t\in[a_{k+1},b_{k+1}]} F(t) \quad \text{for every } k \geq 2.$$

Then, Theorem 6.2.5 asserts that

$$-\Delta u = \lambda|x|^{s-2}w(x/|x|)f\big(|x|^{-s}u\big), \quad x \in \mathbb{R}^{N+1} \setminus \{0\},$$

admits for every $\lambda > 0$ a sequence $(u_k)_k$ of nonnegative classical solutions such that $|\cdot|^{-s}u_k \in H^1(\mathbb{S}^N) \cap L^\infty(\mathbb{S}^N)$ for every $k \in \mathbb{N}$ and (6.24) holds.

6.3 Stationary Kirchhoff critical equations on the sphere

Motivated by a wide interest in Kirchhoff equations on manifolds, in this section we apply Theorem 6.1.2 and Proposition 6.1.3 to get multiple symmetric solutions for stationary Kirchhoff critical problems on the unit sphere $\mathbb{S}^N$.

Existence of a smooth positive solution of nonlinear critical problems on the unit sphere are related to the celebrated Yamabe and Nirenberg problems; see [252], as well as [197]. Celebrated results for nonlinear critical problems on the unit sphere, as those obtained by T. Aubin in [20], A. Cotsiolis and D. Iliopoulos in [68], E. Hebey in [122], J. L. Kazdan and F. W. Warner in [136], J. L. Vázquez and L. Véron in [244], are contained in the remarkable survey [157] due to J. M. Lee and T. H. Parker.

To prove the main theorem of the section, we recall two abstract results. The first is settled on Banach spaces. Let X be a real Banach space. We indicate by $\mathcal{W}_X$ the class of all functionals $I : X \to \mathbb{R}$ with the property that if $u_k \rightharpoonup u$ in X and

$$\liminf_{k\to\infty} I(u_k) \leq I(u),$$

then $u_k \to u$ up to a subsequence. With the above notation, the following result holds.

Theorem 6.3.1. *Let X be a separable reflexive real Banach space. Assume that $Q, P : X \to \mathbb{R}$ are two sequentially weakly lower semicontinuous functionals and that $Q \in \mathcal{W}_X$, with*

$$\lim_{\|u\|\to\infty} \big(Q(u) + P(u)\big) = \infty.$$

Then, any strict local minimum of the functional $Q + P$ in the strong topology is such also in the weak topology of X.

For a simple detailed proof, we refer to [226, Theorem C]. The second tool is given in the framework of topological spaces. More precisely, the next existence result is proved in [225, Theorem 4], in which $\Sigma_\zeta = (-\infty, \zeta)$ for any real number ζ.

Theorem 6.3.2. *Let $X = (X, \tau_X)$ be a Hausdorff topological space and let $Q, P : X \to \mathbb{R}$ be two sequentially lower semicontinuous functionals. Assume that there is $\zeta > \inf_{u\in X} Q(u)$ such that the set $\overline{Q^{-1}(\Sigma_\zeta)}$ is compact and first countable. Finally, if $u_0 \in X$ is a strict local minimum of Q such that*

$$\inf_{u\in X} Q(u) < Q(u_0) < \zeta,$$

then there exists $\delta > 0$ such that for each $\vartheta \in [0,\delta]$ the functional $Q + \vartheta P$ admits at least two τ_Q-local minima lying in $Q^{-1}(\Sigma_\zeta)$, where τ_Q is the smallest topology on X which contains the topology τ_X and the family of sets $\{Q^{-1}(\Sigma_\xi)\}_{\xi\in\mathbb{R}}$.

Put $\mathscr{X} = \{f \in C(\mathbb{R}) : f$ is odd and $\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|^{q-1}} < \infty$ for some $q \in (2, 2^*)\}$. Let S be the positive constant given by

$$S = \inf_{\substack{u\in H^1(\mathbb{S}^N)\\ u\neq 0}} \frac{\|u\|^2_{H^1(\mathbb{S}^N)}}{\|u\|^2_{2^*}}.$$

From here on, with abuse of notation, but for simplicity, we denote $\|\cdot\|_{H^1(\mathbb{S}^N)}$ by $\|\cdot\|$. The main result involves the key number

$$s_N = [N/2] + (-1)^{N+1} - 1$$

introduced in Theorem 6.2.1.

Theorem 6.3.3. *Let $N > 4$ and let w, $\mathrm{w} \in L^\infty(\mathbb{S}^N)$ be two radially symmetric weights, with* $\operatorname{ess\,inf}_{\mathbb{S}^N} w > 0$*. Let a, b be two positive real numbers, with*

$$a^{\frac{N-4}{2}} b > \frac{2(N-4)^{\frac{N-4}{2}}}{(N-2)^{\frac{N-2}{2}} S^{\frac{N}{2}}}. \tag{6.29}$$

Suppose that $f \in \mathscr{X}$ satisfies

(k_1) $\lim_{t\to 0^+} \frac{F(t)}{t^2} \leq 0$;

(k_2) $F(t_0) > 0$ *for some* $t_0 > 0$.

Then, there exists $\lambda^* > 0$ such that for each compact interval $[\alpha, \beta] \subset (\lambda^*, \infty)$ there is $r > 0$ with the property that for every $\lambda \in [\alpha, \beta]$ and for every $\mathrm{f} \in \mathscr{X}$ there is $\vartheta^* > 0$ such that the Kirchhoff critical equation

$$(a + b\|u\|^2)(-\Delta_h u + u) = |u|^{2^*-2}u + \lambda w(\sigma) f(u) + \vartheta \mathrm{w}(\sigma)\mathrm{f}(u) \quad \text{in } \mathbb{S}^N \tag{6.30}$$

has at least $3s_N$ solutions whose norms are strictly less that r for every $\vartheta \in [0, \vartheta^*]$.

Proof. Let us fix $i \in J_N$ and consider the energy functional $\Psi : H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N) \to \mathbb{R}$ given by

$$\Psi(u) = \int_{\mathbb{S}^N} w(\sigma)F(u)\,d\sigma_h.$$

By Proposition 6.1.3, the Sobolev space $H^1_{G_{N,i}}(\mathbb{S}^N)$ is compactly embedded into $L^{\wp}(\mathbb{S}^N)$, whenever $\wp \in [1, q_i^*)$, $q_i^* = (2N-2)/(N-3)$. Of course, $2^* < q_i^*$, so that from $E_i = H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N) \subset \mathscr{E}_i = H^1_{G_{N,i}}(\mathbb{S}^N) \hookrightarrow L^{2^*}(\mathbb{S}^N)$ it follows that the embedding $E_i \hookrightarrow L^{2^*}(\mathbb{S}^N)$ is compact. Clearly, the above compactness property ensures the sequentially weak continuity of the smooth functional Ψ. Set

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \quad \text{for every } u \in E_i = H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N).$$

Let us consider the real function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$ defined by

$$\varphi(t) = \frac{a}{2} + \frac{b}{4}t^2 - \frac{S^{-\frac{2^*}{2}}}{2^*}t^{2^*-2} \quad \text{for every } t \in \mathbb{R}_0^+,$$

see Figure 6.6. The minimum of φ is attained at $m_\varphi > 0$, where

$$m_\varphi = \left(\frac{2^* b}{2(2^*-2)} S^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-4}},$$

and $2^* < 4$ since $N > 4$. By (6.29), it follows that

$$\varphi(m_\varphi) = \frac{1}{2}\left(a - 2^{\frac{4}{N-4}} b^{-\frac{2}{N-4}} \frac{N-4}{N^{\frac{N-2}{N-4}} S^{\frac{N}{N-4}}}\right) > 0.$$

Thus φ is positive in $\mathbb{R}_0^+$ and

$$\Phi(u) \geq \varphi(m_\varphi)\|u\|^2 \quad \text{for every } u \in H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N). \tag{6.31}$$

Since $f \in \mathscr{X}$, by (k_1) it follows that

$$\limsup_{u \to 0} \frac{\Psi(u)}{\Phi(u)} \leq 0. \tag{6.32}$$

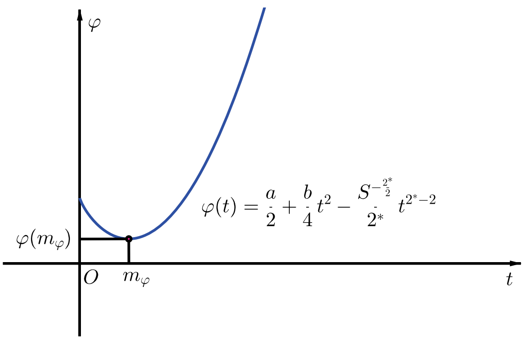

Figure 6.6: The graph of the function φ.

To prove (6.32), fix $\varepsilon > 0$. Then, since F is odd, (k_1) yields the existence of a $\delta_\varepsilon > 0$ such that

$$F(t) \le \frac{\varepsilon}{\|w\|_\infty} t^2 \quad \text{for every } t, \text{ with } |t| \le \delta_\varepsilon. \tag{6.33}$$

Let us divide the argument in two cases.

Since $q \in (2, 2^*)$, one gets that $t \mapsto \frac{f(t)}{|t|^{q-1}}$ is bounded for every $t \in \mathbb{R}$ with $|t| > \delta_\varepsilon$. Therefore, there is $m_\varepsilon > 0$ such that

$$F(t) \le \frac{\varepsilon}{\|w\|_\infty} t^2 + \frac{m_\varepsilon}{q} |t|^q \;\; \text{for every } t \in \mathbb{R}. \tag{6.34}$$

Thus, for $u \in H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$,

$$\begin{aligned}
\Psi(u) &\le \int_{\mathbb{S}^N} w(\sigma)\left[\frac{\varepsilon}{\|w\|_\infty} u^2 + \frac{m_\varepsilon}{q}|u|^q\right] d\sigma_h \\
&\le \int_{\mathbb{S}^N} \left[\varepsilon u^2 + \frac{m_\varepsilon}{q} w(\sigma)|u|^q\right] d\sigma_h \\
&\le \varepsilon \|u\|^2 + m_\varepsilon c^q_{i,q} \|w\|_\infty \|u\|^q,
\end{aligned} \tag{6.35}$$

where $c_{i,q} > 0$ is the best embedding constant in $E_i = H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N) \hookrightarrow L^q(\mathbb{S}^N)$, thanks to Proposition 6.1.3. Thus, (6.31) implies that for every $u \in E_i \setminus \{0\}$, $E_i = H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$,

$$\begin{aligned}
\frac{\Psi(u)}{\Phi(u)} &\le \frac{\varepsilon \|u\|^2 + m_\varepsilon c^q_{i,q} \|w\|_\infty \|u\|^q}{\Phi(u)} \\
&\le \varphi(m_\varphi)^{-1} \big(\varepsilon + m_\varepsilon c^q_{i,q} \|w\|_\infty \|u\|^{q-2}\big).
\end{aligned}$$

Consequently, since $q > 2$,

$$\limsup_{u \to 0} \frac{\Psi(u)}{\Phi(u)} \le \frac{\varepsilon}{\varphi(m_\varphi)}.$$

Since $\varepsilon > 0$ is arbitrary, the above relation immediately gives (6.32).

By (6.32), the energy functional $\mathcal{J}_{i,\lambda} : H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N) \to \mathbb{R}$, defined by

$$\mathcal{J}_{i,\lambda}(u) = \Phi(u) - \lambda \Psi(u) \quad \text{for every } u \in H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N),$$

has a strong local minimum at zero for every $\lambda > 0$. Moreover, the function $\mathcal{J}_{i,\lambda}$ is bounded from below and coercive on $H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ since $N > 4$. Indeed, by (6.35), it follows that

$$\mathcal{J}_{i,\lambda}(u) \ge \left(\frac{a}{2} - \varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|^{2^*}_{2^*} - m_\varepsilon c^q_{i,q} \|w\|_\infty \|u\|^q$$

for every $u \in H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$. Hence, since $q \in (2, 2^*)$ and $N > 4$, the above inequality ensures that

$$\lim_{\|u\|\to\infty} \mathcal{J}_{i,\lambda}(u) = \infty,$$

as claimed.

Now, we claim that the global minimum of $\mathcal{J}_{i,\lambda}$ in $H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ is nonzero. Indeed, by (k_1) there is a strict decreasing sequence $(t_k)_k \subset (0, 1)$ such that

$$\lim_{k\to\infty} \frac{F(t_k)}{t_k^2} = F_0 \le 0 \quad \text{and} \quad \lim_{k\to\infty} t_k = 0.$$

Moreover, $F(t_k) = F(-t_k)$ for every $k \in \mathbb{N}$, since $f \in \mathscr{Z}$. Let $D_i \subset \mathbb{S}^N$ be $G^{\tau_i}_{N,i}$-invariant and let $C_i > 0$ be as in the statement of Proposition 6.1.4. Fix

$$\ell > -F_0 \max\left\{1, \frac{\mathrm{Vol}_h(D_i)}{\mathrm{Vol}_h(\mathbb{S}^N)}\right\} \in \mathbb{R}^+_0. \tag{6.36}$$

Since F is even, then (k_1) implies that there exists $\varrho > 0$ such that

$$F(t) \ge -\ell t^2 \quad \text{for every } t \in (-\varrho, \varrho). \tag{6.37}$$

Let $v_k = t_k v \in H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ be the function from Proposition 6.1.4 corresponding to the value $t_k > 0$. Then

$$\mathcal{J}_{i,\lambda}(v_k) \le \kappa_i t_k^2 - \lambda \int_{D_i} F(v_k)d\sigma_h - \lambda \int_{\mathbb{S}^N\setminus D_i} F(v_k)d\sigma_h,$$

with

$$\kappa_i = \frac{1}{2}(C_i^2 + 1)\mathrm{Vol}_h(\mathbb{S}^N)\left(a + \frac{b}{2}(C_i^2 + 1)\mathrm{Vol}_h(\mathbb{S}^N)\right),$$

where the constant C_i is given in Proposition 6.1.4(ii). Moreover, Proposition 6.1.4(iii) yields

$$\int_{D_i} F(v_k)d\sigma_h = F(t_k)\mathrm{Vol}_h(D_i).$$

On the other hand, due to relation (6.37) and Proposition 6.1.4(i), we have

$$\int_{\mathbb{S}^N\setminus D_i} F(v_k)d\sigma_h \ge -\ell \int_{\mathbb{S}^N\setminus D_i} v_k^2(\sigma)d\sigma_h > -\ell\mathrm{Vol}_h(\mathbb{S}^N)t_k^2.$$

Hence

$$\mathcal{J}_{i,\lambda}(v_k) \le t_k^2\left[\kappa_i - \lambda\left(\frac{F(t_k)}{t_k^2}\mathrm{Vol}_h(D_i) + \ell\mathrm{Vol}_h(\mathbb{S}^N)\right)\right].$$

By (6.36) and taking $\lambda > \lambda_i^*$, where

$$\lambda_i^* = \frac{\kappa_i}{F_0\mathrm{Vol}_h(D_i) + \ell\mathrm{Vol}_h(\mathbb{S}^N)},$$

there exists $\bar{k}$ such that

$$\mathcal{J}_{i,\lambda}(v_{\bar{k}}) \le t_{\bar{k}}^2\left[\kappa_i - \lambda\left(\frac{F(t_{\bar{k}})}{t_{\bar{k}}^2}\mathrm{Vol}_h(D_i) + \ell\mathrm{Vol}_h(\mathbb{S}^N)\right)\right] < 0.$$

Consequently,

$$\inf_{u\in H^1_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)} \mathcal{J}_{i,\lambda}(u) \le \mathcal{J}_{i,\lambda}(v_{\bar{k}}) < 0,$$

which proves the claim.

On account of Theorem 6.3.1, the trivial function $v_0 = 0$ turns out to be a local minimizer of $\mathcal{J}_{i,\lambda}$ in the weak topology of $H^1_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$.

Now, let us fix $[\alpha,\beta] \subset (\lambda_i^*,\infty)$ and $\zeta > 0$. Since $\mathcal{J}_{i,\lambda}$ is coercive on $H^1_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$, it follows that the weak closure of the sublevel $\mathcal{J}_{i,\lambda}^{-1}(\Sigma_\zeta)$, namely $\overline{\mathcal{J}_{i,\lambda}^{-1}(\Sigma_\zeta)}^{w}$, is compact and metrizable with respect to the weak topology. Moreover, let $\eta > 0$ be such that

$$\bigcup_{\lambda\in[\alpha,\beta]} \mathcal{J}_{i,\lambda}^{-1}(\Sigma_\zeta) \subseteq B_\eta,$$

where

$$B_{i,\eta} = \{u \in H^1_{G_{N,i}^{\tau_i}}(\mathbb{S}^N) : \|u\| < \eta\}.$$

Let $r > \eta$ be such that

$$\bigcup_{\lambda\in[\alpha,\beta]} \mathcal{J}_{i,\lambda}^{-1}((-\infty, c^\star + 2]) \subseteq B_r, \tag{6.38}$$

where $c^\star = \sup_{u\in B_\eta}\Psi(u) + \beta\sup_{u\in B_\eta}|\Psi(u)|$.

Now, let the functional $\Upsilon : H^1_{G_{N,i}^{\tau_i}}(\mathbb{S}^N) \to \mathbb{R}$ be defined by

$$\Upsilon(u) = \int_{\mathbb{S}^N} \mathtt{w}(\sigma)\,\mathtt{F}(u)d\sigma_h, \text{ where } \mathtt{F}(t) = \int_0^t \mathtt{f}(s)ds.$$

Since $\mathrm{f} \in \mathscr{X}$, it follows that $\Upsilon \in C^1(H^1_{G^{\tau_i}_{N,i}}(\mathbb{S}^N))$. Furthermore, the derivative Υ' of Υ is a compact operator since the Sobolev space $H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ is compactly embedded into $L^{\wp}(\mathbb{S}^N)$ for every $\wp \in [1, 2^*]$ by Proposition 6.1.3. Let $\phi \in C^1(\mathbb{R})$ be a bounded function such that

$$\phi(t) = t \quad \text{for every} \quad t \in \Big[-\sup_{u\in B_r}|\Upsilon(u)|, \sup_{u\in B_r}|\Upsilon(u)|\Big].$$

Define $\widetilde{\Upsilon} = \phi \circ \Upsilon$. Clearly, $\widetilde{\Upsilon}$ is a C^1 functional with compact derivative and such that $\widetilde{\Upsilon} = \Upsilon$ in B_r.

Denote by $\tau_{\mathcal{J}_{i,\lambda}}$ the smallest topology on $H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ containing the weak topology and the family $\{\mathcal{J}^{-1}_{i,\lambda}(\Sigma_\xi)\}_{\xi\in\mathbb{R}}$.

An application of Theorem 6.3.2 to the functionals $Q = \mathcal{J}_{i,\lambda}$, $P = -\widetilde{\Upsilon}$, and $X = H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$, endowed with the weak topology, gives the existence of some $\delta_i > 0$ such that for every $\vartheta \in [0, \delta_i]$ the functional $\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon}$ has two local minimizers u_1, $u_2 \in H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ in the $\tau_{\mathcal{J}_{i,\lambda}}$ topology, with

$$u_1, u_2 \in \mathcal{J}^{-1}_{i,\lambda}(\Sigma_\zeta) \subseteq B_\eta \subset B_r.$$

Since the topology $\tau_{\mathcal{J}_{i,\lambda}}$ is weaker than the strong topology, the functions u_1, u_2 are local minimizers of $\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon}$.

Let us denote by $\vartheta^*_i = \min\{\delta_i, \frac{1}{\sup_{t\in\mathbb{R}}\phi(t)}\}$. Due to the compactness of embedding of $H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ into $L^{\wp}(\mathbb{S}^N)$ for every $\wp \in [1, 2^*]$, it is easily seen that the functional $\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon}$ verifies the (PS) condition thanks to the compactness arguments given in [89]; see also [257, Example 38.25] for classical results.

The Pucci and Serrin Theorem 1 in [212] ensures that the functional $\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon}$ admits a critical point $u_3 \in H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)$ such that

$$(\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon})(u_3) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} (\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon})(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_{G^{\tau_i}_{N,i}}(\mathbb{S}^N)) : \gamma(0) = u_1 \text{ and } \gamma(1) = u_2\}$. Now, if $\tilde{\gamma}(t) = tu_1 + (1-t)u_2$, with $t \in [0,1]$, then $\tilde{\gamma} \in \Gamma$ and

$$\tilde{\gamma}(t) \in B_\eta \quad \text{for every } t \in [0,1].$$

Thus, it follows that

$$(\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon})(u_3) \le \sup_{t\in[0,1]} (\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon})(\tilde{\gamma}(t)) \le c^* + \vartheta^* \sup_{t\in\mathbb{R}} \phi(t) \le c^* + 1.$$

Since

$$(\mathcal{J}_{i,\lambda} - \vartheta\widetilde{\Upsilon})(u_3) \le c^* + 1 + \vartheta^* \sup_{t\in\mathbb{R}} \phi(t) \le c^* + 2,$$

by (6.38), we have $u_3 \in B_r$. Hence $\widetilde{\Upsilon}(u_i) = \Upsilon(u_i)$, with $i \in \{1, 2, 3\}$.

Consequently, the functions $u_1, u_2, u_3 \in H_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$ are critical points of the functional $\mathcal{J}_{i,\lambda} - \vartheta\Upsilon$ in $H_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$.

Since f and g are odd functions, the energy functional $\mathcal{I}_\lambda$,

$$\mathcal{I}_\lambda(u) = \Phi(u) - \lambda \int_{\mathbb{S}^N} w(\sigma)F(u)d\sigma_h - \vartheta \int_{\mathbb{S}^N} \mathtt{w}(\sigma)\mathtt{F}(u)d\sigma_h,$$

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \quad \text{for every } u \in H^1(\mathbb{S}^N),$$

is even. Thus, for every $i \in J_N$ the functional $\mathcal{I}_\lambda$ is $G_{N,i}^{\tau_i}$-invariant, since $G_{N,i}^{\tau_i}$ acts isometrically on $H^1(\mathbb{S}^N)$, by virtue of (6.2).

Moreover, thanks to the symmetry assumptions on f, $\mathtt{f}$, w, and $\mathtt{w}$, it is easily seen that the functional $N_{\lambda,\vartheta} : H^1(\mathbb{S}^N) \to \mathbb{R}$, given by

$$N_{\lambda,\vartheta}(u) = \lambda \int_{\mathbb{S}^N} w(\sigma)F(u)d\sigma_h + \vartheta \int_{\mathbb{S}^N} \mathtt{w}(\sigma)\mathtt{F}(u)d\sigma_h,$$

is $G_{N,i}^{\tau_i}$-invariant.

Hence, the symmetric criticality Theorem A.1.5 yields that the critical points of $\mathcal{J}_{i,\lambda} - \vartheta\Upsilon$ are also critical points for the functional $\mathcal{I}_\lambda$, and so solutions of equation (6.30).

Thus, for every $i \in J_N$ the functional $\mathcal{J}_{i,\lambda} - \vartheta\Upsilon$ admits at least three distinct solutions $(u_k^{(i)})_{k=1}^3$ in $E_i = H_{G_{N,i}^{\tau_i}}(\mathbb{S}^N)$, provided that $\lambda > \lambda_i^*$.

Now, it remains to count the number of distinct solutions of the above type. More precisely, on the basis of Proposition 6.1.3, there are at least s_N subspaces $H_{G_{N,i}^{\tau_i}}(\mathbb{S}^N) \subset H^1(\mathbb{S}^N)$ whose mutual intersections contain only the zero function. Put

$$\lambda^* = \max\{\lambda_i^* : i \in J_N\} \quad \text{and} \quad \vartheta^* = \min\{\vartheta_i^* : i \in J_N\}.$$

Each $H_{G_{N,i}^{\tau_i}}(\mathbb{S}^N) \subset H^1(\mathbb{S}^N)$ contains three distinct pairs of nonzero solutions of (6.30), whenever

$$\lambda > \lambda^* \quad \text{and} \quad \vartheta < \vartheta^*.$$

This concludes the proof. □

A meaningful consequence of Theorem 6.3.3 is the next multiplicity result.

Corollary 6.3.4. *Let $N > 4$ and let $a, b \in \mathbb{R}^+$ satisfy* (6.29). *Furthermore, let $q \in (2, 2^*)$. Then, for every $\lambda > 0$ the stationary Kirchhoff critical equation*

$$(a + b\|u\|^2)(-\Delta_h u + u) = \lambda|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{S}^N$$

has at least $3s_N$ solutions.

In Corollary 6.3.4 the threshold λ^* in Theorem 6.3.3 is zero. Clearly, in this case $F(u) = |u|^q/q$ with $q \in (2, 2^*)$. Consequently, $F_0 = \lim_{t\to 0^+} \frac{F(t)}{t^2} = 0$. Fix $\lambda_0 > 0$ and take

$$\ell > \frac{\max\{\kappa_i : i \in J_N\}}{\lambda_0 \mathrm{Vol}_h(\mathbb{S}^N)}.$$

Put

$$\lambda_i^* = \frac{\kappa_i}{\ell \mathrm{Vol}_h(\mathbb{S}^N)} \quad \text{for every } i \in J_N.$$

Thus, (6.36) holds and the conclusion is achieved.

Comments on Chapter 6

The method introduced in [172] allowed W. Marzantowicz to obtain also nice information on the number of the solutions of critical problems on the sphere via suitable algebraic methods mainly based on the classification of the Borel subgroups of the orthogonal group. Now, very little it is known about the existence, multiplicity and blow-up of nodal solutions for the supercritical problem on the Euclidean sphere. For instance, some new symmetry results for positive solutions of elliptic problems in the whole space $\mathbb{R}^{N+1}$ and on the sphere $\mathbb{S}^N$ have been studied in [51]. The proofs of the results contained in [51] are proved through the stereographic projection and they are essentially based on the moving plane method and rearrangement techniques. The methods in [172] and the above theoretical results seem to be useful to obtain the existence of an infinite number of nodal solutions of elliptic equations involving critical exponents; see, among others, the paper [95] and the references therein. Furthermore, Ding-type results and the Hebey–Vaugon compactness properties can be used in the study of equations similar to (6.18) in a fractional Heisenberg setting. We refer to [27] and to Chapter 9 for related equations in the Heisenberg group. Moreover, in the framework of complete Riemannian manifolds, some reduction methods also apply and have been combined with the Lyapunov–Schmidt arguments in order to obtain sequences of positive and sign-changing solutions of supercritical equations; see, among others, the paper [94]. The above remarks motivate the idea that the methods developed in this chapter can be successfully applied to different classes of problems having Riemannian and subelliptic structures.

Part III: **Variational Principles in Geometric Analysis**

7 Subelliptic problems on Carnot groups

Bacio che sopporti il peso
della mia anima breve
in te il mondo del mio discorso
diventa suono e paura.

Alda Merini
Bacio

The chapter deals with the existence of solutions for a wide class of eigenvalue subelliptic critical problems in possibly unbounded domains Ω of a Carnot group $\mathbb{G}$ via the symmetric criticality principle of Palais, together with variational arguments based on certain recent compactness results due to Z. Balogh and A. Kristály in [27]. In this way, we do not have to require any longer the strong asymptotical contractiveness condition on the domain Ω, which is a persisting assumption in the current literature.

More precisely, we study semilinear equations, when either the Ambrosetti–Rabinowitz geometrical condition is satisfied or the nonlinear terms are monotone so that the trick due to P. Rabier can be used. In the first case, the main variational tools are the mountain pass theorem of A. Ambrosetti and P. Rabinowitz, the principle of symmetric criticality of Palais, and a suitable flower-shape geometry on the horizontal Sobolev space associated to the domain Ω of $\mathbb{G}$. In the latter case, following [219], the existence of bounded Palais–Smale sequences for the energy functional associated to the equation is obtained via a generalization of the quite classical approach due to M. Struwe and L. Jeanjean, combined with a rescaling argument.

At the end of each section of the chapter, we present applications of the main results on the meaningful subcase in which the Kohn–Laplace subelliptic problems are settled on unbounded domains of the Heisenberg group $\mathbb{H}^N$. Indeed, in this special framework, the geometrical group arguments are particularly expressive.

7.1 Basic theory on stratified Lie groups

In this section we briefly recall some basic facts on Carnot groups and the functional Folland–Stein space $HW_0^{1,2}(\Omega)$. A Carnot group $\mathbb{G} = (\mathbb{G}, \circ)$ is a connected, simply connected, nilpotent Lie group, whose Lie algebra $\mathfrak{G}$ admits a stratification, i. e.,

$$\mathfrak{G} = \bigoplus_{k=1}^{r} \mathfrak{G}_k,$$

where the integer r is called the *step* of $\mathbb{G}$, while $\mathfrak{G}_k$ is the linear subspace of finite dimension d_k of $\mathfrak{G}$ for every $k \in \{1, \dots, r\}$, and

$$[\mathfrak{G}_1, \mathfrak{G}_k] = \mathfrak{G}_{k+1} \quad \text{for all } k, \text{ with } 1 \leq k < r-1, \quad \text{and} \quad [\mathfrak{G}_1, \mathfrak{G}_r] = \{O\}.$$

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In this context the symbol $[\mathfrak{G}_1, \mathfrak{G}_k]$ denotes the subalgebra of $\mathfrak{G}$ generated by the commutators $[X, Y]$, where $X \in \mathfrak{G}_1$ and $Y \in \mathfrak{G}_k$, and where the last bracket denotes the Lie bracket of vector fields, that is, $[X, Y] = XY - YX$.

The left translation by $\sigma_1 \in \mathbb{G}$ on $\mathbb{G}$ is given by $\ell_{\sigma_1}(\sigma) = \sigma_1 \circ \sigma$ for every $\sigma \in \mathbb{G}$. Let $\Gamma(T\mathbb{G})$ be the space of global sections of the tangent bundle $T\mathbb{G}$ on $\mathbb{G}$. A vector field $X \in \Gamma(T\mathbb{G})$ is left invariant if

$$X(\varphi \circ \ell_\sigma) = (X\varphi) \circ \ell_\sigma$$

for any $\sigma \in \mathbb{G}$ and $\varphi \in C^\infty(\mathbb{G})$.

Moreover, the Lie algebra $\mathfrak{G}$ associated to $\mathbb{G}$ consists of left invariant vector fields X on $\mathfrak{G}$ and $\mathfrak{G}$ is canonically isomorphic to the tangent space $T_e\mathbb{G}$. Let

$$d = \sum_{k=1}^{r} d_k$$

be the *topological dimension* of the Carnot group $\mathbb{G}$.

The exponential map $\exp_{\mathbb{G}} : \mathfrak{G} \to \mathbb{G}$ is given by $\exp_{\mathbb{G}}(X) = \gamma_X(1)$, where γ_X is the unique integral curve associated to the left invariant vector field X such that $\gamma_X(0) = e$, where e is the neutral element of $\mathbb{G}$. In other words, the curve γ_X is the unique solution of the Cauchy problem

$$\dot{\gamma}_X(t) = X(\gamma_X(t)), \quad \gamma_X(0) = e.$$

The curve γ_X is defined for any $t \in \mathbb{R}$, since from the identity $\gamma_X(t + s) = \gamma_X(s)\,\gamma_X(t)$ for all s, $t \in \mathbb{R}$, it is clear that γ_X can be extended in the entire $\mathbb{R}$.

Since $\mathbb{G}$ is nilpotent, connected, and simply connected Lie group, the exponential map $\exp_{\mathbb{G}}$ is a smooth diffeomorphism from $\mathfrak{G}$ onto $\mathbb{G}$.

Let $\langle\cdot,\cdot\rangle_0$ be a fixed inner product on the first graduated component $\mathfrak{G}_1$ of $\mathfrak{G}$, with associated orthonormal basis $\mathcal{B} = \{X_1, X_2, \ldots, X_{d_1}\}$. From now on, we consider the extension of the inner product $\langle\cdot,\cdot\rangle_0$ to the whole tangent bundle $T\mathbb{G}$ by group translation. The corresponding norm is denoted by $\|\cdot\|_0$. A left invariant vector field $X \in \mathfrak{G}$ is said to be *horizontal* if

$$X(\sigma) \in \operatorname{span}\{X_1(\sigma), \ldots, X_{d_1}(\sigma)\}$$

for every $\sigma \in \mathbb{G}$. Indeed, $\mathfrak{G}_1$ is considered to be the *horizontal direction*, while the remaining layers $\mathfrak{G}_2, \ldots, \mathfrak{G}_r$ are viewed as the *vertical directions*. In particular, the last layer $\mathfrak{G}_r$ is the center of the Lie algebra, and the horizontal direction G_1 generates in the sense of Lie algebras the whole $\mathfrak{G}$. More precisely,

$$\mathfrak{G}_k = \underbrace{[\mathfrak{G}_1, [\mathfrak{G}_1, [\mathfrak{G}_1, \ldots [\mathfrak{G}_1, \mathfrak{G}_1] \cdots]]]}_{k \text{ times}}$$

for all $k = 2, \ldots, r$.

Since the map $\exp_{\mathbb{G}}$ is bijective, for every element $\sigma \in \mathbb{G}$ there exists a unique vector field $X = \sum_{k=1}^{d_1} x_k X_k + \sum_{k=d_1+1}^{d} x_k X'_k \in \mathfrak{G}$ such that

$$\sigma = \exp_{\mathbb{G}}(X) = \exp_{\mathbb{G}}\left(\sum_{k=1}^{d_1} x_k X_k + \sum_{k=d_1+1}^{d} x_k X'_k \right),$$

where $\{X_{d_1+1}, \dots, X_d\}$ are vertical vector fields that extend $\mathcal{B}$ to an orthonormal basis $\mathcal{B}^*$ of $\mathfrak{G}$.

Now, observe that $\mathfrak{G} \cong \mathbb{R}^d$. Thus, we often identify every element $\sigma \in \mathbb{G}$ with its *exponential coordinates* $(x_1, \dots, x_{d_1}, x_{d_1+1}, \dots, x_d) \in \mathbb{R}^d$ in connection to the basis $\mathcal{B}^*$ of $\mathfrak{G}$.

More precisely, it is possible to identify the Carnot group $(\mathbb{G}, \circ)$ with $(\mathbb{R}^d, \star)$, where the expression of the group operation $\star$ is given by

$$x \star y = \varrho^{-1}\big(\varrho(x) \circ \varrho(y)\big) \quad \text{for all } x, y \in \mathbb{R}^d$$

and is explicitly determined by the Baker–Campbell–Hausdorff formula.

Whenever we are in the presence of a stratification, it is possible to define a one-parameter group $\{\Delta_\eta\}_{\eta>0}$ of dilatations of the algebra. More precisely, for a fixed real number $\eta > 0$ and all $X \in \mathfrak{G}_k$, we set $\Delta_\eta(X) = \eta^k X$ and extend the map Δ_η to the whole $\mathfrak{G}$ by linearity.

Furthermore, the family $\{\Delta_\eta\}_{\eta>0}$ induces a family $\{\delta_\eta\}_{\eta>0}$ of the group automorphisms on $\mathbb{G}$ by the exponential map such that the diagram in Figure 7.1 is commutative.

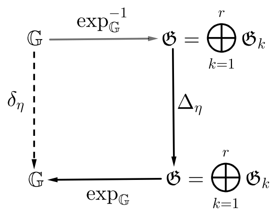

Figure 7.1: The automorphism δ_η.

The *homogeneous dimension* Q of $\mathbb{G}$, attached to the automorphisms $\{\delta_\eta\}_{\eta>0}$, is defined by

$$Q = \sum_{k=1}^{r} k \, \dim \mathfrak{G}_k = d_1 + 2d_2 + \cdots + r d_r.$$

In particular, the above definition of Q and the fact that $\{\delta_\eta\}_{\eta>0}$ is a family of automorphisms on $\mathbb{G}$ imply that the Jacobian determinant of the dilation δ_η is constant in σ and given by η^Q.

Moreover, let μ denote the push-forward of the d-dimensional Lebesgue measure λ_d on $\mathfrak{G}$ via the exponential map. Then, $d\mu$ defines a biinvariant Haar measure on $\mathbb{G}$ and

$$d\mu(\sigma \circ \delta_\eta) = \eta^Q d\mu(\sigma).$$

Since $\mathbb{G}$ can be identified with $(\mathbb{R}^d, \star)$ by using the exponential map, if $E \subseteq \mathbb{G}$ is a measurable subset, its Haar measure μ is explicitly given by $\mu(E) = \lambda_d(\rho^{-1}(E))$. Therefore, the same notation will be used for both measures.

Take $\sigma_1, \sigma_2 \in \mathbb{G}$ and let $H\Gamma_{\sigma_1,\sigma_2}(\mathbb{G})$ be the set of piecewise smooth curves γ, such that $\gamma : [0,1] \to \mathbb{G}$, $\dot{\gamma}(t) \in \mathfrak{G}_1$ a. e. $t \in [0,1]$, $(\gamma(0), \gamma(1)) = (\sigma_1, \sigma_2)$ and

$$\int_0^1 \|\dot{\gamma}(t)\|_0 dt < \infty.$$

Since $H\Gamma_{\sigma_1,\sigma_2}(\mathbb{G}) \neq \emptyset$ by the celebrated Chow–Rashevskiĭ theorem in [60], it is possible to define the *Carnot–Carathéodory distance* on $\mathbb{G}$ as follows:

$$d_{CC}(\sigma_1, \sigma_2) = \inf_{\gamma \in H\Gamma_{\sigma_1,\sigma_2}(\mathbb{G})} \int_0^1 \|\dot{\gamma}(t)\|_0 dt.$$

The metric d_{CC} is left invariant on $\mathbb{G}$ and for every $\eta > 0$,

$$d_{CC}(\delta_\eta(\sigma_1), \delta_\eta(\sigma_2)) = \eta\, d_{CC}(\sigma_1, \sigma_2)$$

for every $\sigma_1, \sigma_2 \in \mathbb{G}$.

The Euclidean norm $|\cdot|$ induces two homogeneous pseudonorms $|\cdot|_{\mathfrak{G}}$ on $\mathfrak{G}$ and $|\cdot|_{\mathbb{G}}$ on the group $\mathbb{G}$ via the exponential map. Indeed, for $X \in \mathfrak{G}$, with $X = \sum_{k=1}^r X_k$, where $X_k \in \mathfrak{G}_k$, define a pseudonorm on $\mathfrak{G}$ as follows:

$$|X|_{\mathfrak{G}} = \left(\sum_{k=1}^r |X_k|^{2r!/k} \right)^{2r!}.$$

The induced pseudonorm on $\mathbb{G}$ has the form

$$|\sigma|_{\mathbb{G}} = \left|\exp_{\mathbb{G}}^{-1}(\sigma)\right|_{\mathfrak{G}} \quad \text{for all } \sigma \in \mathbb{G}$$

and is usually known as the *nonisotropic gauge*. It defines a pseudodistance on $\mathbb{G}$ given by

$$d(\sigma_1, \sigma_2) = \left|\sigma_2^{-1} \circ \sigma_1\right|_{\mathbb{G}} \quad \text{for all } \sigma_1, \sigma_2 \in \mathbb{G},$$

which is equivalent to the *Carnot–Carathéodory distance* d_{CC} on $\mathbb{G}$.

Thus, Carnot groups are endowed with the intrinsic Carnot–Carathéodory geometry. The adjective "intrinsic" is meant to emphasize a privileged role played by the horizontal layer and by group translations and dilations. It is worth stressing that the Carnot–Carathéodory geometry is not Riemannian at any scale. In fact, Carnot groups can be seen as a particular case of more general structures, the so called *sub-Riemannian spaces*.

The most basic second-order partial differential operator in a Carnot group $\mathbb{G}$ is the *sub-Laplacian*, or equivalently, the *horizontal Laplacian in* $\mathbb{G}$, given by

$$\Delta_{\mathbb{G}} = \sum_{k=1}^{d_1} X_k^2.$$

We shall denote by $D_{\mathbb{G}} = (X_1, \dots, X_{d_1})$ the related *horizontal gradient* and set $\|D_{\mathbb{G}} u\|_0 = (\sum_{k=1}^{d_1} (X_k u)^2)^{1/2}$.

Obviously, Euclidean spaces are commutative Carnot groups, and, more precisely, the only commutative Carnot groups. The simplest example of a Carnot group of step two is provided by the Heisenberg group $\mathbb{H}^N$ of topological dimension $d = 2N + 1$ and homogeneous dimension $Q = 2N + 2$, that is, the Lie group whose underlying manifold is $\mathbb{R}^{2N+1}$, endowed with the non-Abelian group law

$$\sigma_1 \circ \sigma_2 = \left(z + z', t + t' + 2 \sum_{i=1}^{N} (y_i x_i' - x_i y_i') \right)$$

for all $\sigma_1, \sigma_2 \in \mathbb{H}^N$, with $\sigma_1 = (z, t) = (x_1, \dots, x_N, y_1, \dots, y_N, t)$, $\sigma_2 = (z', t') = (x_1', \dots, x_N', y_1', \dots, y_N', t')$. The vector fields, for $j = 1, \dots, N$,

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t},$$

constitute a basis $\mathcal{B}^*$ for the real Lie algebra $\mathfrak{H} = \mathfrak{G}$ of left invariant vector fields on $\mathbb{H}^N$. The basis $\mathcal{B}^*$ satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero.

If $u \in C^2(\mathbb{H}^N)$, then the *horizontal Laplacian in* $\mathbb{H}^N$ of u, called the *Kohn–Spencer Laplacian*, is defined as follows:

$$\begin{aligned}\Delta_{\mathbb{H}^N} u &= \sum_{j=1}^{N} (X_j^2 + Y_j^2) u \\ &= \sum_{j=1}^{N} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},\end{aligned}$$

and $\Delta_{\mathbb{H}^N}$ is *hypoelliptic* according to the celebrated Theorem 1.1 due to L. Hörmander in [131].

In order to study the variational problems considered in the next sections, we need to introduce suitable solution spaces. To this goal, let Ω be a nonempty open subset of $\mathbb{G}$. The Folland–Stein horizontal Sobolev space $HW_0^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Hilbertian norm

$$\begin{aligned} \|u\| &= \left(\int_\Omega \|D_\mathbb{G} u\|_0^2 d\mu + \int_\Omega |u|^2 d\mu\right)^{1/2}, \\ \langle u, \varphi\rangle &= \int_\Omega \langle D_\mathbb{G} u, D_\mathbb{G}\varphi\rangle_0\, d\mu + \int_\Omega u\varphi d\mu. \end{aligned} \tag{7.1}$$

Of course, if $\Omega = \mathbb{G}$, then $HW^{1,2}(\mathbb{G}) = HW_0^{1,2}(\mathbb{G})$, where $HW^{1,2}(\mathbb{G})$ denotes the horizontal Sobolev space of the functions $u \in L^2(\mathbb{G})$ such that $D_\mathbb{G} u$ exists in the sense of distributions, and $\|D_\mathbb{G} u\|_0$ is in $L^2(\mathbb{G})$ endowed with the Hilbertian norm (7.1).

The embedding

$$HW_0^{1,2}(\Omega) \hookrightarrow L^\wp(\Omega)$$

is continuous for any $\wp \in [2, 2^*]$; see G. B. Folland and E. M. Stein [106]. Furthermore, by [114, 132, 243] we know that, if Ω is a bounded open set of $\mathbb{G}$, the embedding

$$HW_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^\wp(\Omega)$$

is compact for all $\wp$, with $1 \le \wp < 2^*$.

Let $(\mathbb{G}, \circ)$ be a Carnot group, and $(T, \cdot)$ be a closed topological group, with neutral element j_T. The group T is said to *act continuously* on $\mathbb{G}$, if there exists a map $* : T \times \mathbb{G} \to \mathbb{G}$ such that the following conditions:

(T_1) $j_T * \sigma = \sigma$ *for every* $\sigma \in \mathbb{G}$;

(T_2) $\tau_1 * (\tau_2 * \sigma) = (\tau_1 \cdot \tau_2) * \sigma$ *for every* $\tau_1, \tau_2 \in T$ *and* $\sigma \in \mathbb{G}$,

hold. In addition, the action $*$ is said to be *left distributed* if

(T_3) $\tau * (\sigma_1 \circ \sigma_2) = (\tau * \sigma_1) \circ (\tau * \sigma_2)$ *for every* $\tau \in T$ *and* $\sigma_1, \sigma_2 \in \mathbb{G}$

is satisfied. A set $\Omega \subseteq \mathbb{G}$ is T-*invariant*, with respect to $*$, if $T * \Omega = \Omega$.

We assume that T induces an action $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$, defined for every $(\tau, u) \in T \times HW_0^{1,2}(\mathbb{G})$ by

$$(\tau \sharp u)(\sigma) = u(\tau^{-1} * \sigma) \quad \text{for all } \sigma \in \mathbb{G}. \tag{7.2}$$

The group T acts *isometrically* on $HW_0^{1,2}(\Omega)$ if

$$\|\tau \sharp u\| = \|u\| \quad \text{for all } (\tau, u) \in T \times HW_0^{1,2}(\mathbb{G}).$$

Let

$$HW_{0,T}^{1,2}(\Omega) = \{u \in HW_0^{1,2}(\Omega) : \tau \sharp u = u \text{ for all } \tau \in T\}$$

be the T-invariant subspace of $HW_0^1(\Omega)$. Clearly, $HW_{0,T}^{1,2}(\Omega)$ is closed, since the action $\sharp$ of T on $HW_0^{1,2}(\Omega)$ is continuous.

Hereafter we shall frequently use on $\mathbb{G}$ the Carnot–Carathéodory distance d_{CC} : $\mathbb{G} \times \mathbb{G} \to \mathbb{R}_0^+$ and the natural Haar measure μ on $\mathbb{G}$. In the next definition "lim inf" is the Kuratowski lower limit of sets.

Let Ω be a nonempty open T-invariant subset of $\mathbb{G}$, with a nontrivial boundary $\partial\Omega$, and assume that

($\mathscr{H}$) *For every* $(\sigma_k)_k \subset \mathbb{G}$ *such that*

$$\lim_{k\to\infty} d_{CC}(e,\sigma_k) = \infty \quad \text{and} \quad \mu\Big(\liminf_{k\to\infty}(\sigma_k \circ \Omega)\Big) > 0,$$

where $\sigma_k \circ \Omega = \{\sigma_k \circ \sigma : \sigma \in \Omega\}$, *then there exist a subsequence* $(\sigma_{k_j})_j$ *of* $(\sigma_k)_k$ *and a sequence of subgroups* $(T_{\sigma_{k_j}})_j$ *of* T, *with cardinality* $\operatorname{card}(T_{\sigma_{k_j}}) = \infty$, *having the property that for all* $\tau_1, \tau_2 \in T_{\sigma_{k_j}}$, *with* $\tau_1 \neq \tau_2$, *it results*

$$\lim_{j\to\infty} \inf_{\sigma\in\mathbb{G}} d_{CC}\big((\tau_1 * \sigma_{k_j}) \circ \sigma, (\tau_2 * \sigma_{k_j}) \circ \sigma\big) = \infty.$$

A domain Ω of $\mathbb{G}$, for which condition ($\mathscr{H}$) holds, is simply called an $\mathcal{H}$ *domain*.

The following compactness result is due to Z. Balog and A. Kristály and is given in [27, Theorem 3.1].

Lemma 7.1.1. *Let* $\mathbb{G} = (\mathbb{G}, \circ)$ *be a Carnot group of step* r *and homogeneous dimension* $Q > 2$, *with neutral element denoted by* e. *Let* $T = (T, \cdot)$ *be a closed infinite topological group acting continuously and left distributively on* $\mathbb{G}$ *by the map* $* : T \times \mathbb{G} \to \mathbb{G}$. *Assume furthermore that* T *acts isometrically on* $HW_0^{1,2}(\mathbb{G})$, *where the action* $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$ *is defined in* (7.2). *Let* Ω *be a nonempty* T*-invariant open subset of* $\mathbb{G}$, *satisfying condition* ($\mathscr{H}$). *Then the embedding*

$$HW_{0,T}^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^{\wp}(\Omega)$$

is compact for every $\wp \in (2, 2^*)$.

A direct application of Lemma 7.1.1 gives a compactness result for suitable Sobolev spaces associated to a class of unbounded domains of the Heisenberg group $\mathbb{H}^N = \mathbb{C}^N \times \mathbb{R}$, $N \geq 1$.

More precisely, let $\psi_1, \psi_2 : \mathbb{R}_0^+ \to \mathbb{R}$ be two functions that are bounded on bounded sets, with $\psi_1(t) < \psi_2(t)$ for every $t \in \mathbb{R}_0^+$. Define

$$\Omega_\psi = \{\sigma \in \mathbb{H}^N : \sigma = (z,t) \text{ with } \psi_1(|z|) < t < \psi_2(|z|)\}, \tag{7.3}$$

where $|z| = \sqrt{\sum_{i=1}^N |z_i|^2}$; see Figure 7.2 below.

Let $\mathbb{U}(N) = U(N) \times \{1\}$, where

$$U(N) = \{\tau \in \mathrm{GL}(N;\mathbb{C}) : \langle \tau z, \tau z' \rangle_{\mathbb{C}^N} = \langle z, z' \rangle_{\mathbb{C}^N} \text{ for all } z, z' \in \mathbb{C}^N\},$$

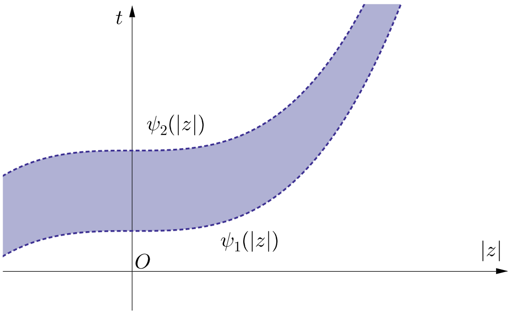

Figure 7.2: A strip-like domain Ω_ψ.

that is, $U(N)$ is the usual unitary group. Here $\langle\cdot,\cdot\rangle_{\mathbb{C}^N}$ denotes the standard Hermitian product on $\mathbb{C}^N$, in other words, $\langle z,z'\rangle_{\mathbb{C}^N} = \sum_{k=1}^N z_k \cdot \overline{z'_k}$.

Hence, $\mathbb{U}(N)$ is the unitary group endowed with the natural multiplication law $\cdot : \mathbb{U}(N)\times\mathbb{U}(N) \to \mathbb{U}(N)$, which acts continuously and left distributively on $\mathbb{H}^N$ by the map $* : \mathbb{U}(N) \times \mathbb{H}^N \to \mathbb{H}^N$, defined by

$$\hat{\tau} * \sigma = (\tau z, t) \quad \text{for all } \hat{\tau} = (\tau, 1) \in \mathbb{U}(N) \text{ and all } \sigma = (z,t) \in \mathbb{H}^N,$$

thanks to [27, Lemma 3.1]. Taking $T = \mathbb{U}(N)$, we get that Ω_ψ is $\mathbb{U}(N)$-invariant and an $\mathcal{H}$ domain, as shown in the proof of Theorem 1.1 of [27]. Moreover,

$$HW^{1,2}_{0,\mathbb{U}(N)}(\Omega_\psi) = \{u \in HW^{1,2}_0(\Omega_\psi) : u(z,t) = u(|z|,t) \text{ for all } (z,t) \in \Omega_\psi\},$$

that is, $HW^{1,2}_{0,\mathbb{U}(N)}(\Omega_\psi) = HW^{1,2}_{0,\mathrm{cyl}}(\Omega_\psi)$ is the space of cylindrically symmetric functions of $HW^{1,2}_0(\Omega_\psi)$.

Finally, $\mathbb{U}(N)$ acts isometrically on the horizontal Folland–Stein space $HW^{1,2}_0(\mathbb{H}^N)$, where the action $\sharp : \mathbb{U}(N) \times HW^{1,2}_0(\mathbb{H}^N) \to HW^{1,2}_0(\mathbb{H}^N)$ is defined for every $(\hat{\tau}, u)$ in $\mathbb{U}(N) \times HW^{1,2}_0(\mathbb{H}^N)$ by

$$(\hat{\tau}\sharp u)(\sigma) = u(\tau^{-1}z, t) \quad \text{for all } \sigma = (z,t) \in \mathbb{H}^N, \tag{7.4}$$

in view of [27, Lemma 3.2].

Thus, let $T \subseteq \mathbb{U}(N)$ be the subgroup of the form

$$T = U(N_1) \times \cdots \times U(N_\ell) \times \{1\}, \quad N = \sum_{k=1}^{\ell} N_k, \quad \text{with} \quad N_k \geq 1 \text{ and } \ell \geq 1,$$

and consider the Sobolev space

$$HW^{1,2}_{0,T}(\Omega_\psi) = \{u \in HW^{1,2}_0(\Omega_\psi) : u(z,t) = u(|z_1|, |z_2|, \ldots, |z_\ell|, t), z_k \in \mathbb{C}^{N_k}\}.$$

On account of the above results, since Ω_ψ is a T-invariant open subset of $\mathbb{H}^N$, Theorem 1.1 in [27] ensures that the next special case of Lemma 7.1.1 holds.

Lemma 7.1.2. *Let $N \geq 1$ and let Ω_ψ be a strip-like domain of $\mathbb{H}^N$, as given in* (7.3). *Then the following embedding:*

$$HW^{1,2}_{0,T}(\Omega_\psi) \hookrightarrow\hookrightarrow L^{\wp}(\Omega_\psi)$$

is compact for any $\wp \in (2, 2^*)$.

Inspired by Section 4 of [27], we introduce the main definitions and notations necessary to state the key compactness Proposition 4.1 of [27] and stated here in Lemma 7.1.3 and Proposition 7.1.4. From here until the end of the section, we assume that $N \geq 2$, unless state differently, and put $J_N = \{1, \ldots, [N/2]\}$.

For every $j \in J_N$, consider the subgroup $T_{N,j} \subset U(N)$, defined by

$$T_{N,j} = \begin{cases} U(N/2) \times U(N/2), & \text{if } j = N/2, \\ U(j) \times U(N-2j) \times U(j), & \text{if } j \neq N/2, \end{cases}$$

and the matrix

$$\omega_j = \begin{cases} \begin{pmatrix} 0 & \mathbb{I}_{\mathbb{C}^{N/2}} \\ \mathbb{I}_{\mathbb{C}^{N/2}} & 0 \end{pmatrix}, & \text{if } j = N/2, \\ \begin{pmatrix} 0 & 0 & \mathbb{I}_{\mathbb{C}^j} \\ 0 & \mathbb{I}_{\mathbb{C}^{N-2j}} & 0 \\ \mathbb{I}_{\mathbb{C}^j} & 0 & 0 \end{pmatrix}, & \text{if } j \neq N/2. \end{cases}$$

By definition, $\omega_j \in U(N) \setminus T_{N,j}$, as well as

$$\omega_j T_{N,j} \omega_j^{-1} = T_{N,j} \quad \text{and} \quad \omega_j^2 = \mathrm{id}_{\mathbb{C}^N}$$

for every $j \in J_N$.

Let $\widehat{T}^{\omega_j}_{N,j}$ be the subgroup of $\mathbb{U}(N)$ generated by ω_j and $\widehat{T}_{N,j} = T_{N,j} \times \{1\}$, that is,

$$\widehat{T}^{\omega_j}_{N,j} = \langle \widehat{T}_{N,j}, \omega_j \rangle = \widehat{T}_{N,j} \cup \widehat{\omega}_j \widehat{T}_{N,j}$$

for every $j \in J_N$.

Define the action $\widehat{T}^{\omega_j}_{N,j} \natural HW^{1,2}_0(\Omega_\psi) \to HW^{1,2}_0(\Omega_\psi)$ of $\widehat{T}^{\omega_j}_{N,j}$ on $HW^{1,2}_0(\Omega_\psi)$ given by

$$(\widetilde{\tau} \natural u)(\sigma) = \begin{cases} (\widehat{\tau} \sharp u)(\sigma), & \text{if } \widetilde{\tau} = \widehat{\tau} \in \widehat{T}^{\omega_j}_{N,j}, \\ -((\widehat{\omega}_j \widehat{\tau}) \sharp u)(\sigma), & \text{if } \widetilde{\tau} = \widehat{\omega}_j \widehat{\tau} \in \widehat{T}^{\omega_j}_{N,j} \setminus \widehat{T}_{N,j}, \text{ with } \widehat{\tau} \in \widehat{T}_{N,j}, \end{cases} \tag{7.5}$$

for every $\sigma \in \Omega_\psi$.

The action $\natural$ is defined on the whole subgroup $\widehat{T}^{\omega_j}_{N,j}$. Indeed, if $\widetilde{\tau} \in \widehat{T}^{\omega_j}_{N,j}$, then either $\widetilde{\tau} = \widehat{\tau} \in \widehat{T}^{\omega_j}_{N,j}$ or $\widetilde{\tau} = \widehat{\omega}_j \widehat{\tau} \in \widehat{T}^{\omega_j}_{N,j} \setminus \widehat{T}_{N,j}$, with $\widehat{\tau} \in \widehat{T}_{N,j}$. Moreover, set

$$E_j = HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi) = \{u \in HW^{1,2}_0(\Omega_\psi) \;:\; \widetilde{\tau} \natural u = u \text{ for all } \widetilde{\tau} \in \widehat{T}^{\omega_j}_{N,j}\}$$

for every $j \in J_N$.

By Lemma 7.1.2, since $HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi) \subset HW^{1,2}_{0,\widehat{T}_{N,j}}(\Omega_\psi)$, the following compactness result holds.

Lemma 7.1.3. *The embedding*

$$HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi) \hookrightarrow\hookrightarrow L^{\wp}(\Omega_\psi)$$

is compact for any $\wp \in (2, 2^*)$ *and every* $j \in J_N$.

The next result provides precise information on the mutually symmetric differences for the spaces of $\widehat{T}^{\omega_j}_{N,j}$-invariant functions in $HW^{1,2}_0(\Omega_\psi)$.

Proposition 7.1.4. *The following statements hold true*:

(*i*) $E_j \cap HW^{1,2}_{0,\mathrm{cyl}}(\Omega_\psi) = \{0\}$ *for every* $j \in J_N$;

(*ii*) *If* $N \geq 4$, *then*

$$E_j \cap E_k = \{0\}$$

for every $j, k \in J_N$, *with* $j \neq k$.

The geometrical meaning of Proposition 7.1.4 is clearly expressed in Figure 7.3.

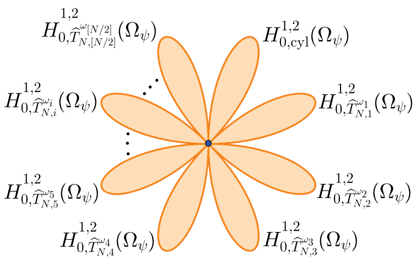

Figure 7.3: The flower-shape geometry given by Proposition 7.1.4.

From now on in this chapter, unless otherwise specified, we assume that:

- ➢ $\mathbb{G} = (\mathbb{G}, \circ)$ *is a Carnot group of step* r *and homogeneous dimension* $Q > 2$, *with neutral element denoted by* e;
- ➢ $T = (T, \cdot)$ *is a closed infinite topological group acting continuously and left distributively on* $\mathbb{G}$ *by the map* $* : T \times \mathbb{G} \to \mathbb{G}$;
- ➢ $T = (T, \cdot)$ *acts isometrically on the Hilbert Sobolev space* $HW^{1,2}_0(\mathbb{G})$ *by the action* $\sharp$, *where* $\sharp : T \times HW^{1,2}_0(\mathbb{G}) \to HW^{1,2}_0(\mathbb{G})$ *is defined in* (7.2).

7.2 Semilinear problems on unbounded domains of Carnot groups

We are ready now to study the existence of solutions of the nonlinear eigenvalue subelliptic problem

$$\begin{cases} -\Delta_{\mathbb{G}} u + u = \lambda w(\sigma) f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.6}$$

where Ω is a $\mathcal{H}$ domain of a Carnot group $\mathbb{G}$ with boundary $\partial\Omega$. For our purpose, we assume that the right-hand side of equation (7.6) is a function f verifying the conditions:

(f_1) $f \in C(\mathbb{R})$ *and for some* $q \in (2, 2_Q^*)$,

$$\sup_{t\in\mathbb{R}\setminus\{0\}} \frac{|f(t)|}{|t| + |t|^{q-1}} < \infty,$$

where 2_Q^* *is the critical Sobolev exponent given by* $2_Q^* = 2Q/(Q-2)$;

(f_2) $f(t) = o(|t|)$ *as* $|t| \to 0$;

(f_3) *There exists* $\nu > 2$ *such that* $0 < \nu F(t) \leq t f(t)$ *for any* $t \in \mathbb{R} \setminus \{0\}$, *where* $F(t) = \int_0^t f(s) ds$.

Moreover, we require

(w) $w \in L^1(\Omega) \cap L^\infty(\Omega)$ *with* $\operatorname{ess\,inf}_\Omega w > 0$;

(C_Ψ) *The functional* $\Psi : HW_0^{1,2}(\Omega) \to \mathbb{R}$ *given by*

$$\Psi(u) = \int_\Omega w(\sigma) F(u) d\mu \quad \textit{for every } u \in HW_0^{1,2}(\Omega)$$

is T*-invariant, that is,* $\Psi(\tau \sharp u) = \Psi(u)$ *for every* $(\tau, u) \in T \times HW_0^{1,2}(\Omega)$.

The next lemma is based on well-known tools in abstract group measure theory. The result shows when the key assumption (C_Ψ) is satisfied, imposing some additional conditions on the weight w and on the Haar measure on $\mathbb{G}$.

Lemma 7.2.1. *Suppose that the action* $*$ *of the group* T *on the Carnot group* $\mathbb{G}$ *satisfies conditions* (T_1)–(T_3). *Assume furthermore that the natural Haar measure* μ, *defined on* $\mathbb{G}$, *is left* $*$ *invariant, that is, for all measurable subset* E *of* $\mathbb{G}$ *and for all* $\tau \in T$,

$$\mu(\tau * E) = \mu(E),$$

where $\tau * E = \{\tau * \sigma : \sigma \in E\}$. *If* $f \in C(\mathbb{R})$ *is a subcritical function and* $w \in L^1(\Omega) \cap L^\infty(\Omega)$ *is* T*-invariant, that is,* $w(\tau * \sigma) = w(\sigma)$ *for every* $\tau \in T$ *and* $\sigma \in \mathbb{G}$, *then the functional* Ψ *satisfies* (C_Ψ).

Proof. Fix $\tau \in T$ and $u \in HW_0^{1,2}(\Omega)$. Then, putting $\tau^{-1} * \sigma = \tilde{\sigma}$, we get by (T_1)–(T_3) that

$$\begin{aligned}\Psi(\tau\sharp u) &= \int_\Omega w(\sigma)F\big((\tau\sharp u)(\sigma)\big)d\mu(\sigma) = \int_\Omega w(\sigma)F\big(u(\tau^{-1} * \sigma)\big)d\mu(\sigma)\\ &= \int_{\tau*\Omega} w(\tau * \tilde{\sigma})F\big(u(\tilde{\sigma})\big)d\mu(\tau * \tilde{\sigma})\\ &= \int_\Omega w(\tilde{\sigma})F\big(u(\tilde{\sigma})\big)d\mu(\tilde{\sigma}) = \Psi(u),\end{aligned}$$

since Ω and w are T-invariant by assumption, and the left $*$ invariance of the measure μ implies that

$$d\mu(\tau * \tilde{\sigma}) = d\mu(\tilde{\sigma}) \quad \text{for all } \tilde{\sigma} \in \mathbb{G},$$

which is exactly [45, formula (10)]. This shows that Ψ is T-invariant, that is, Ψ satisfies (C_Ψ), concluding the proof. □

In the first result of this section, we consider the semilinear equation (7.6) when f satisfies the superlinear condition given in (f_3). The main tools are given by the mountain pass theorem of [12], and the algebraic tools developed in Section 7.1.

Theorem 7.2.2. *Let Ω be a nonempty T-invariant open subset of $\mathbb{G}$, satisfying condition $(\mathscr{H})$, and let λ be a positive parameter. Assume that (f_1)–(f_3), (w), and (C_Ψ) hold. Then, for any $\lambda > 0$ problem (7.6) admits a nontrivial T-invariant solution $u_\lambda \in HW_0^{1,2}(\Omega)$. In addition, if f is odd, then (7.6) admits a sequence of nontrivial T-invariant solutions for any $\lambda > 0$.*

Proof. Problem (7.6) has a clear variational structure. Indeed, its solutions can be found as critical points of the underlying energy functional defined for all $u \in HW_0^{1,2}(\Omega)$ by

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_\Omega w(\sigma)F(u)d\mu,$$

where $\|\cdot\|$ is the standard norm on $HW_0^{1,2}(\Omega)$ introduced in (7.1).

Since the problem is settled on the domain Ω, possibly unbounded, no compact embeddings can be used for the whole Folland–Stein horizontal Sobolev space $HW_0^{1,2}(\Omega)$. In order to find a solution of (7.6), we shall work with $HW_{0,T}^{1,2}(\Omega)$, where T is defined above, in order to recover compactness. It is clear from Lemma 7.1.1 that the crucial role is played by $\mathcal{J}_\lambda$, which is the restriction of $\mathcal{I}_\lambda$ to the space $HW_{0,T}^{1,2}(\Omega)$, i. e.,

$$\mathcal{J}_\lambda(u) = \mathcal{I}_\lambda|_{HW_{0,T}^{1,2}(\Omega)}(u), \quad u \in HW_{0,T}^{1,2}(\Omega). \tag{7.7}$$

First of all, let us show that $\mathcal{J}_\lambda$ possesses the geometric mountain pass structure. By continuity and conditions (f_1) and (f_2), it is clear that for any $\varepsilon > 0$ there exists $M = M_\varepsilon > 0$ such that for any $t \in \mathbb{R}$,

$$|f(t)| \leq \varepsilon|t| + M_\varepsilon |t|^{q-1}$$

and, consequently,

$$|F(t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{M_\varepsilon}{q}|t|^q . \tag{7.8}$$

Now, let us proceed by steps.

Step 1. *There exist $\rho > 0$ and $\jmath_\rho > 0$ such that $\mathcal{J}_\lambda(u) \geq \jmath_\rho$ for any $u \in HW^{1,2}_{0,T}(\Omega)$, with $\|u\| = \rho$.*

Let u be a function in $HW^{1,2}_{0,T}(\Omega)$. On account of (7.8) and the positivity of λ, we easily get that for any $\varepsilon > 0$,

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon\lambda}{2}\|w\|_\infty \|u\|_2^2 - \frac{M_\varepsilon \lambda}{q}\|w\|_\infty \|u\|_q^q \\ &\geq \frac{1}{2}(1 - \varepsilon\lambda c_2^2 \|\alpha\|_\infty)\|u\|^2 - \frac{M_\varepsilon \lambda c_q^q}{q}\|w\|_\infty \|u\|^q \\ &= \|u\|^2 \left[\frac{1}{2}(1 - \varepsilon\lambda c_2^2\|w\|_\infty) - \frac{M_\varepsilon \lambda c_q^q}{q}\|w\|_\infty \|u\|^{q-2}\right],\end{aligned}$$

where $c_\wp$, with $\wp \in \{2, q\}$, is the Sobolev constant of the continuous embedding $HW^{1,2}_{0,T}(\Omega) \hookrightarrow L^\wp(\Omega)$.

By choosing $\varepsilon > 0$ small enough to have $\varepsilon\lambda c_2^2 \|w\|_\infty < 1$, we get that there exist suitable positive constants $\bar{\kappa}$ and $\tilde{\kappa}$ such that

$$\inf_{\substack{u \in HW^{1,2}_{0,T}(\Omega) \\ \|u\| = \rho}} \mathcal{J}_\lambda(u) \geq \rho^2(\bar{\kappa} - \tilde{\kappa}\rho^{q-2}) = \jmath_\rho > 0,$$

provided ρ is sufficiently small.

Step 2. *There exists a function $\overline{u} \in HW^{1,2}_{0,T}(\Omega)$ such that $\|\overline{u}\| > \rho$ and $\mathcal{J}_\lambda(\overline{u}) < \jmath_\rho$.*

First of all, note that as a consequence of (f_1) and (f_4), we easily have that there exist two positive constants a_1 and a_2 such that

$$F(t) \geq a_1|t|^\nu - a_2 \quad \text{for any } t \in \mathbb{R}.$$

Let $u \in HW^{1,2}_{0,T}(\Omega)$ be such that $\|u\| = 1$ and let $s > 0$. Moreover, let $w_0 > 0$ be such that $\operatorname{ess\,inf}_\Omega w \geq w_0$. Bearing in mind that $\lambda > 0$ and $w \in L^1(\Omega) \cap L^\infty(\Omega)$, the above superquadratic condition immediately yields

$$\begin{aligned}\mathcal{J}_\lambda(su) &= \frac{s^2}{2}\|u\|^2 - \lambda \int_\Omega w(\sigma)F(su)\,d\mu \\ &\leq \frac{s^2}{2} - \lambda a_1 w_0 s^\nu \|u\|_\nu^\nu + \lambda a_2 \|w\|_1 .\end{aligned}$$

Since $\nu > 2$, passing to the limit as $s \to \infty$, we get that $\mathcal{J}_\lambda(su) \to -\infty$, so that the claim follows taking $\overline{u} = \bar{s}u$, with $\bar{s}$ sufficiently large.

Thanks to Lemma 7.1.1, arguing as in the proof of Claim 3 in Chapter 5, Section 5.3, it is possible to show that the functional $\mathcal{J}_\lambda$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$. Hence, by Steps 1 and 2, the mountain pass theorem ensures the existence of a nontrivial solution $u_\lambda \in HW^{1,2}_{0,T}(\Omega)$ which is a critical point for the energy functional $\mathcal{J}_\lambda$. Thus, u_λ is a constrained critical point of $\mathcal{I}_\lambda$ on $HW^{1,2}_{0,T}(\Omega)$. It remains to prove that $HW^{1,2}_{0,T}(\Omega)$ is a natural constraint for $\mathcal{I}_\lambda$. To achieve this goal, we notice that T acts isometrically on $HW^{1,2}_0(\mathbb{G})$, where the action $\sharp : T \times HW^{1,2}_0(\mathbb{G}) \to HW^{1,2}_0(\mathbb{G})$ is defined in (7.2). Thus, thanks to assumption (C_Ψ), the functional $\mathcal{I}_\lambda$ is T-invariant, that is, $\mathcal{I}_\lambda(\tau\sharp u) = \mathcal{I}_\lambda(u)$ for every $u \in HW^{1,2}_0(\Omega)$. The principle of symmetric criticality, Theorem A.1.5, ensures that u_λ is a critical point of $\mathcal{I}_\lambda$, i. e., $u_\lambda \in HW^{1,2}_0(\Omega)$ is a solution of problem (7.6).

Finally, in the case of f being odd, as usual when dealing with even functionals, we apply the classical symmetric version of the mountain pass theorem to the energy functional $\mathcal{J}_\lambda$. Similar geometrical arguments as in Steps 1 and 2 and Theorem A.1.5 give that for any $\lambda > 0$ problem (7.6) admits a sequence of nontrivial T-invariant solutions in $HW^{1,2}_0(\Omega)$. The proof is now complete. □

Now, we prove that, for small values of the parameter, problem (7.6) admits at least two solutions requiring that the continuous term f *satisfies only* (f_3) *and the new condition*

(f_1') *For some* $q \in (2, 2^*_Q)$,

$$C_f = \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1 + |t|^{q-1}} < \infty.$$

An application of the principle of symmetric criticality and suitable variational methods give the following result.

Theorem 7.2.3. *Let Ω be a nonempty T-invariant open subset of $\mathbb{G}$, satisfying condition $(\mathscr{H})$, and let λ be a positive parameter. Assume that $w \in L^1(\Omega) \cap L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega w > 0$, and let $f \in C(\mathbb{R})$ verify assumptions (f_1') and (f_3). Finally, suppose that (C_Ψ) holds. Then, there exists $\lambda_\star > 0$ such that, for every $\lambda \in (0, \lambda_\star)$, problem* (7.6) *admits at least two solutions in $HW^{1,2}_{0,T}(\Omega)$.*

To prove Theorem 7.2.3, we shall use the next critical point result.

Theorem 7.2.4. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*
– *Φ is sequentially weakly lower semicontinuous and coercive in X;*
– *Ψ is sequentially weakly continuous in X.*

Assume further that for each $\eta > 0$ *the functional* $J_\eta = \eta\Phi - \Psi$ *satisfies the* $(PS)_c$ *condition for all* $c \in \mathbb{R}$*. Then, for each* $\varrho > \inf_X \Phi$ *and each*

$$\eta > \inf_{u \in \Phi^{-1}(\Sigma_\varrho)} \frac{\sup_{v \in \Phi^{-1}(\Sigma_\varrho)} \Psi(v) - \Psi(u)}{\varrho - \Phi(u)},$$

where $\Sigma_\varrho = (-\infty, \varrho)$*, the following alternative holds: either the functional* J_η *has a strict global minimum which lies in* $\Phi^{-1}(\Sigma_\varrho)$*, or* J_η *has at least two critical points one of which lies in* $\Phi^{-1}(\Sigma_\varrho)$*.*

The above critical point result comes from a joint application of the classical theorem due to P. Pucci and J. Serrin in [212] and of a local minimum result obtained in [224]; see [222, Theorem 6] for a detailed proof.

Proof of Theorem 7.2.3. Fix $\lambda > 0$ and consider the energy functional $\mathcal{J}_\eta : HW^{1,2}_{0,T}(\Omega) \to \mathbb{R}$ defined for every $u \in HW^{1,2}_{0,T}(\Omega)$ by

$$\mathcal{J}_\eta(u) = \frac{1}{\eta}\|u\|^2 - \int_\Omega w(\sigma)F(u)d\mu,$$

where $\eta = 1/2\lambda$. We shall prove that $\mathcal{J}_\eta$ satisfies the assumptions of Theorem 7.2.4, with $X = HW^{1,2}_{0,T}(\Omega)$,

$$\Phi(u) = \|u\|^2, \quad \Psi(u) = \int_\Omega w(\sigma)F(u)d\mu, \quad u \in X.$$

Then, Φ is sequentially weakly lower semicontinuous and coercive, and Ψ is sequentially weakly continuous in $HW^{1,2}_{0,T}(\Omega)$, thanks to the compact embedding in Lemma 7.1.1. Arguing as in the proof of Step 2 of Theorem 7.2.2, it easily seen that there exists $u \in HW^{1,2}_{0,T}(\Omega)$ such that $\mathcal{J}_\eta(su) \to -\infty$ as $s \to \infty$. Moreover, since (f_1') and (f_3) hold, standard arguments ensure that the functional $\mathcal{J}_\eta$ satisfies the compactness $(PS)_c$ condition for all $c \in \mathbb{R}$. In order to conclude the proof, fix $\varrho > 0$, let

$$\frac{1}{\lambda_\star} = 2C_f\left(\frac{c_2}{\sqrt{\varrho}}\|w\|_2 + \frac{c_q^q}{q}\|w\|_\infty \varrho^{q/2-1}\right),$$

and take $\lambda \in (0, \lambda_\star)$, where $c_\wp$, with $\wp \in \{2, q\}$, is the Sobolev constant of the continuous embedding $HW^{1,2}_{0,T}(\Omega) \hookrightarrow L^\wp(\Omega)$. We claim that $\varphi(\varrho) < \eta = 1/2\lambda$, where

$$\varphi(\varrho) = \inf_{u \in \Phi^{-1}(\Sigma_\varrho)} \frac{\sup_{v \in \Phi^{-1}(\Sigma_\varrho)} \Psi(v) - \Psi(u)}{\varrho - \Phi(u)}$$

and $\Sigma_\varrho = (-\infty, \varrho)$. Clearly, the identically zero function 0 is in $\Phi^{-1}(\Sigma_\varrho)$, and $\Phi(0) = \Psi(0) = 0$. Consequently,

$$\varphi(\varrho) = \inf_{u \in \Phi^{-1}(\Sigma_\varrho)} \frac{(\sup_{v \in \Phi^{-1}(\Sigma_\varrho)} \Psi(v)) - \Psi(u)}{\varrho - \Phi(u)} \le \frac{\sup_{v \in \Phi^{-1}(\Sigma_\varrho)} \Psi(v)}{\varrho} = \chi(\varrho).$$

By using (f_1'), it follows that

$$\int_\Omega w(\sigma)F(v)d\mu \le C_f\left(\|w\|_2\|v\|_2 + \frac{\|w\|_\infty}{q}\|v\|_q^q\right)$$
$$\le C_f\left(c_2\|w\|_2\|v\| + \frac{c_q^q}{q}\|w\|_\infty\|v\|^q\right)$$

for every $v \in X$. We deduce that

$$\sup_{v\in\Phi^{-1}(\Sigma_\varrho)} \int_\Omega h(\sigma)F(v)d\mu \le C_f\left(c_2\|w\|_2\sqrt{\varrho} + \frac{c_q^q}{q}\|w\|_\infty \varrho^{q/2}\right).$$

This implies that

$$\chi(\varrho) \le C_f\left(\frac{c_2}{\sqrt{\varrho}}\|w\|_2 + \frac{c_q^q}{q}\|w\|_\infty \varrho^{q/2-1}\right) < \frac{1}{2\lambda} = \eta,$$

as claimed. Hence, Theorem 7.2.4 ensures the existence of two solutions in $HW^{1,2}_{0,T}(\Omega)$ which are critical points for the energy functional $\mathcal{J}_\eta$. Similar arguments as in the proof of Theorem 7.2.2 give that for $\lambda \in (0, \lambda_\star)$, problem (7.6) has at least two solutions in $HW^{1,2}_0(\Omega)$. □

A careful analysis of the proof of Theorem 7.2.3 shows that the main conclusion holds true, provided that

$$\lambda \in \left(0, \frac{q}{2C_f}\sup_{\varrho>0}\frac{\sqrt{\varrho}}{qc_2\|w\|_2 + c_q^q\|w\|_q \varrho^{\frac{q-1}{2}}}\right),$$

see Figure 7.4.

The Rubik-cube technique, developed in [27, Theorem 1.2] and applied to the subgroups $\widehat{T}^{\omega_j}_{N,j} \subset \mathbb{U}(N)$, defined in Section 7.1, allows us to obtain a more precise version of Theorem 7.2.3 in the Heisenberg setting, thanks to Lemma 7.2.1 and Proposition 7.1.4.

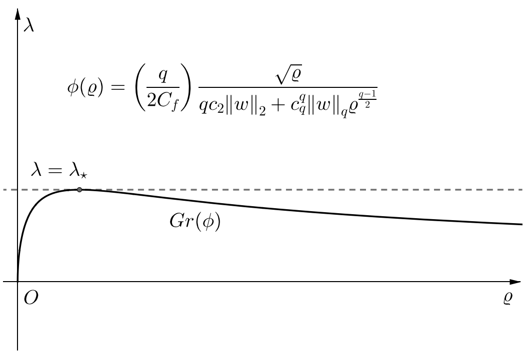

Figure 7.4: The maximal interval $(0, \lambda_\star)$.

Corollary 7.2.5. *Let $\Omega_\psi \subset \mathbb{H}^N$ be as in* (7.3) *and let $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ be a cylindrically symmetric function with* $\operatorname{ess\,inf}_\Omega w > 0$. *Furthermore, let $f \in C(\mathbb{R})$ verify (f_1') and (f_3). Let $T = U(N_1) \times \cdots \times U(N_\ell) \times \{1\}$, where $N = \sum_{k=1}^{\ell} N_k$, with $N_k \geq 1$ and $\ell \geq 1$. Then, the following properties hold*:

(i) *There exists $\lambda_\star > 0$ such that for every $\lambda \in (0, \lambda_\star)$,*

$$\begin{cases} -\Delta_{\mathbb{H}^N} u + u = \lambda w(\sigma) f(u) \text{ in } \Omega_\psi, \\ u = 0 \text{ on } \partial\Omega_\psi \end{cases} \tag{7.9}$$

has at least two solutions localized in

$$HW^{1,2}_{0,T}(\Omega_\psi) = \{u \in HW^{1,2}_0(\Omega_\psi) \,:\, u(z,t) = u(|z_1|, |z_2|, \ldots, |z_\ell|, t)\},$$

where $z_k \in \mathbb{C}^{N_k}$ for all $k = 1, \ldots, \ell$.

(ii) *In addition, if the function f is odd, there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ problem* (7.6) *admits at least* $2([N/2] + 1)$ *solutions in $HW^{1,2}_0(\Omega_\psi)$ with mutually symmetric different structures.*

Proof. (i) This is a direct consequence of Theorem 7.2.3. Indeed, Ω_ψ is $\mathbb{U}(N)$-invariant and an $\mathcal{H}$ domain, as shown in the proof of Theorem 1.1 of [27]. Hence Ω_ψ is a nonempty T-invariant open subset of $\mathbb{H}^N$, satisfying condition ($\mathscr{H}$); see Section 7.1 and Lemmas 3.1 and 3.2 in [27] for additional comments and remarks. Moreover, since T acts isometrically on $HW^{1,2}_0(\Omega_\psi)$ by (7.4), condition (C_Ψ) is easily verified, since $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ is cylindrically symmetric. Then, Theorem 7.2.3 gives the existence of $\lambda_\star > 0$ such that for every $\lambda \in (0, \lambda_\star)$ problem (7.9) admits at least two solutions in $HW^{1,2}_0(\Omega_\psi)$, localized in the symmetric space $HW^{1,2}_{0,T}(\Omega_\psi)$.

(ii) Part (i), with $T = \mathbb{U}(N) = U(N) \times \{1\}$, ensures that there exists $\lambda_\star > 0$ such that, for every $\lambda \in (0, \lambda_\star)$, problem (7.9) admits at least two solutions in $HW^{1,2}_0(\Omega_\psi)$, localized in the symmetric space $HW^{1,2}_{0,\mathrm{cyl}}(\Omega_\psi)$, and the proof is complete when $N = 1$.

Now, let $N \geq 2$. Since f is odd, the energy functional $\mathcal{I}_\eta : HW^{1,2}_0(\Omega_\psi) \to \mathbb{R}$ given by

$$\mathcal{I}_\eta(u) = \frac{1}{\eta}\|u\|^2 - \int_{\Omega_\psi} w(\sigma) F(u)\, d\mu, \quad u \in HW^{1,2}_0(\Omega_\psi),$$

is even. Thus, $\mathcal{I}_\eta$ is $\widehat{T}^{\omega_j}_{N,j}$-invariant for every $j \in J_N = \{1, \ldots, [N/2]\}$, since $\widehat{T}^{\omega_i}_{N,j}$ acts isometrically on $HW^{1,2}_0(\Omega_\psi)$ by virtue of (7.5), and $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ is cylindrically symmetric. Hence, the symmetric criticality Theorem A.1.5 yields that the critical points of the restrictions of $\mathcal{I}_\eta$ to $HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi)$ are also critical points for the functional $\mathcal{I}_\eta$, and so solutions of (7.9).

Then, arguing as in the proof of Theorem 7.2.3, Theorem 7.2.4 ensures that for every $j \in J_N$ there exists $\lambda^{(j)}_\star$ such that the restriction of the functional $\mathcal{I}_\eta$ to the symmetric space E_j admits at least two distinct solutions $(u_k^{(j)})_{k=1}^2$ in $E_j = HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi)$, provided that $\lambda \in (0, \lambda_\star^{(j)})$.

It remains to count the number of distinct solutions of the above type. More precisely, on the basis of Proposition 7.1.4, there are at least $[N/2]$ subspaces $E_j = HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi) \subset HW^{1,2}_0(\Omega_\psi)$ whose mutual intersections contain only the zero function. Put

$$\lambda_0 = \min\{\lambda_\star, \lambda_\star^{(j)} : j \in J_N\}.$$

For every $\lambda \in (0, \lambda_0)$, problem (7.9) admits at least $2([N/2]+1)$ solutions in $HW^{1,2}_0(\Omega_\psi)$ with mutually symmetric different structure. This concludes the proof. □

If the functions ψ_1 and ψ_2 are bounded, the domain Ω_ψ is strongly asymptotically contractive, and the whole space $HW^{1,2}_0(\Omega_\psi)$ is compactly embedded in $L^\wp(\Omega_\psi)$ for every $\wp \in (2, 2_Q^*)$, where $2_Q^* = 2(N+2)/N$. We refer to [27, 169] for further details. In such a case, Theorem 7.2.5 follows by using the embedding result proved by N. Garofalo and E. Lanconelli in [113]. See also I. Schindler and K. Tintarev [229].

We end the section by analyzing problem (7.6) when the nonlinear term f has a sublinear growth at infinity, thus condition (f_3) fails. In this case we assume that the nonlinearity $f \in C(\mathbb{R})$ *verifies the following assumptions*:

(h_1) $f(t) = o(|t|)$ *as* $|t| \to 0$;

(h_2) $f(t) = o(|t|)$ *as* $|t| \to \infty$.

The next multiplicity result holds.

Theorem 7.2.6. *Let* Ω *be a nonempty* T*-invariant open subset of* $\mathbb{G}$*, satisfying condition* $(\mathscr{H})$*, and let* λ *be a positive parameter. Assume that* $w \in L^1(\Omega) \cap L^\infty(\Omega)$ *with* $\operatorname{ess\,inf}_\Omega w > 0$ *and let* $f \in C(\mathbb{R})$ *verify assumptions* (h_1) *and* (h_2)*. Finally, suppose that* (C_Ψ) *holds and* $\sup_{u \in HW^{1,2}_{0,T}(\Omega)} \Psi(u) > 0$*. Then, there exists* $\lambda^\star > 0$ *such that for every* $\lambda > \lambda^\star$ *problem* (7.6) *admits two nontrivial* T*-invariant solutions in* $HW^{1,2}_0(\Omega)$*.*

Proof. Since $\lim_{\|u\| \to 0} \Psi(u)/\|u\|^2 = 0$, the energy functional $\mathcal{J}_\lambda$ given in (7.7) has a local minimum at zero. Taking into account that $\lim_{\|u\| \to \infty} \Psi(u)/\|u\|^2 = 0$, the functional $\mathcal{J}_\lambda$ is also coercive, bounded from below in $HW^{1,2}_{0,T}(\Omega)$, and satisfies the Palais–Smale condition. Thus $\mathcal{J}_\lambda$ has a global minimum in $HW^{1,2}_{0,T}(\Omega)$ with negative energy level for λ sufficiently large. Consequently, the Pucci and Serrin Theorem 1 in [212] gives a third critical point of $\mathcal{J}_\lambda$ in $HW^{1,2}_{0,T}(\Omega)$ for $\mathcal{J}_\lambda$. Finally, similar arguments as in the proof of Theorem 7.2.2 ensure that for $\lambda > \lambda^\star$ problem (7.6) has at least two nontrivial solutions in $HW^{1,2}_0(\Omega)$. □

A more general version of Theorem 7.2.6 can be obtained by arguing as in the proof of Theorem 9.3.2 in Chapter 9, employing the critical point result given in Theorem 9.3.1. However, Theorem 7.2.6 continues to hold even in presence of a sufficiently small nonlinear subcritical perturbation term. Theorem 1.2 of [27], besides other properties, has already shown this extension, but in the Heisenberg setting.

7.3 Dirichlet problems on strips of the Heisenberg group

In this section we turn back to (7.9) when $\lambda = 1$. More precisely, we treat

$$\begin{cases} -\Delta_{\mathbb{H}^N} u + u = w(\sigma) f(u) \text{ in } \Omega_\psi, \\ u = 0 \text{ on } \partial\Omega_\psi, \end{cases} \tag{7.10}$$

when *the nonlinear term* $f \in C(\mathbb{R})$ *verifies*:

(k_1) *For some* $q \in (2, 2_Q^*)$,

$$C_f = \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1 + |t|^{q-1}} < \infty;$$

(k_2) $F_+(t) = o(|t|^2)$ *as* $|t| \to 0$, *where* $F_+ = \max\{F, 0\}$;

(k_3) F *is bounded from below on* $\mathbb{R}_0^+$ *and there exists a sequence* $(t_k)_k \subset \mathbb{R}^+$ *such that*

$$\lim_{k\to\infty} \frac{F(t_k)}{t_k^2} = \infty;$$

(k_4) $f(t)t \geq 2F(t)$ *for every* $t \in \mathbb{R}$;

(k_5) *There exist a constant* $\nu > 2$ *and* $\kappa \in \mathbb{R}$ *such that*

$$f(t)t \geq \nu F(t) \quad \text{for every } t \in \mathbb{R} \text{ such that } F(t) \geq \kappa.$$

Before stating the main result, let us introduce some preliminaries. Following [27], we construct a special test function belonging to $HW_{0,T}^{1,2}(\Omega_\psi)$ which will be useful for the proof of Theorem 7.3.1. Let $N \geq 2$ and let $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ be a cylindrically symmetric nonnegative function as in (7.10). Assume that there exists an open set $\Omega \subset \Omega_\psi$ such that

$$\operatorname*{ess\,inf}_{\Omega} w > 0, \tag{7.11}$$

and set

$$\widehat{\Omega} = \bigcup_{\hat{\tau}\in\mathbb{U}(N)} \{\hat{\tau} * \Omega\}.$$

Since w is cylindrically symmetric,

$$\operatorname*{ess\,inf}_{\Omega} w = \operatorname*{ess\,inf}_{\widehat{\Omega}} w > 0. \tag{7.12}$$

Furthermore, we can find $(z_0, t_0) \in \Omega_\psi$ and R such that

$$0 < R < 2|z_0|(\sqrt{2} - 1), \tag{7.13}$$

and

$$A_R = \{\sigma \in \mathbb{H}^N : \sigma = (z,t), \text{ with } \big||z| - |z_0|\big| \le R, \ |t - t_0| \le R\} \subset \widehat{\Omega}. \tag{7.14}$$

Of course, $A_{\varrho R} \subseteq A_R \subset \widehat{\Omega}$ and $\mu(A_{\varrho R}) > 0$ for every $\varrho \in (0,1]$. Introduce $e_j : \mathbb{C}^j \times \mathbb{C}^j \times \mathbb{R} \to \mathbb{R}$, defined by

$$e_j(z, \tilde{z}, t) = \frac{2}{R}\sqrt{\left(|z|^2 - |z_0| + \frac{R}{2}\right)^2 + |\tilde{z}|^2 + (t - t_0)^2},$$

where z_0, t_0 and $R > 0$ are from (7.13) and (7.14). Set $\varrho \in (0,1)$, and consider $v_{\varrho,j} \in E_j = HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi)$ defined by

$$v_{\varrho,j}(z,t) = \frac{((1 - \max\{e_j(z_1, z_2, t), \varrho\})_+ - (1 - \max\{e_j(z_2, z_1, t), \varrho\})_+)}{1 - \varrho} \tag{7.15}$$

if $j = 2N$ and $z = (z_1, z_2) \in \mathbb{C}^j \times \mathbb{C}^j$, as well as

$$v_{\varrho,j}(z,t) = \frac{((1 - \max\{e_j(z_1, z_3, t), \varrho\})_+ - (1 - \max\{e_j(z_3, z_1, t), \varrho\})_+)}{(1 - \varrho)^2} \times \left(1 - \max\left\{\frac{2}{R}|z_2|, \varrho\right\}\right)_+ \tag{7.16}$$

if $j = 2N$ and $z = (z_1, z_2, z_3) \in \mathbb{C}^j \times \mathbb{C}^{2N} \times \mathbb{C}^j$. For every $\varrho \in (0,1]$, we set:

$$S^j_\varrho = \{(z_1, z_2, t) \in \mathbb{C}^j \times \mathbb{C}^j \times \mathbb{R} : e_j(z_1, z_2, t) \le \varrho \vee e_j(z_2, z_1, t) \le \varrho\} \ \text{ if } N = 2j,$$

$$S^j_\varrho = \Big\{(z_1, z_2, t) \in \mathbb{C}^j \times \mathbb{C}^{N-2j} \times \mathbb{C}^j \times \mathbb{R} : e_j(z_1, z_3, t) \le \varrho \vee e_j(z_3, z_1, t) \le \varrho, \text{ and } |z_2| \le \varrho\frac{R}{2}\Big\} \text{ if } N \ne 2j.$$

Moreover, the following relations hold:

(j_1) $\operatorname{supp}(v^{c_0}_{\varrho,j}) = S^j_1$;

(j_2) $\|v_{\varrho,j}\|_\infty \le 1$;

(j_3) $v_{\varrho,j} = 1$ in S^j_ϱ.

Theorem 7.3.1. *Let $\Omega_\psi \subset \mathbb{H}^N$ be as in* (7.3) *and let $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ be a radially cylindrically nonnegative function such that* (7.11) *holds on some nonempty open set $\Omega \subset \Omega_\psi$. Finally, let $f \in C(\mathbb{R})$ verify (k_1)–(k_5). Then,* (*i*) *Problem* (7.10) *has at least one solution in* $HW_0^{1,2}(\Omega_\psi)$ *localized in*

$$HW_{0,T}^{1,2}(\Omega_\psi) = \{u \in HW_0^{1,2}(\Omega_\psi) \,:\, u(z,t) = u(|z_1|, |z_2|, \dots, |z_\ell|, t)\},$$

where $z_k \in \mathbb{C}^{N_k}$ *for all* $k = 1, \dots, \ell$.

(*ii*) *If* f *is also odd, problem* (7.10) *admits at least* $[N/2] + 1$ *solutions in* $HW_0^{1,2}(\Omega_\psi)$ *with mutually symmetric different structures.*

A key point in the proof of Theorem 7.3.1 is to show the existence of a bounded Palais–Smale sequence at some mountain pass level. More precisely, we shall use the following abstract result proved in [219].

Theorem 7.3.2. *Let* X *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two* C^1 *functionals. Set*

$$J(u) = \Phi(u) - \Psi(u) \quad \textit{for all } u \in X.$$

Assume that

(i_1) Φ *is homogeneous of degree* $p > 1$ *and coercive*;

(i_2) *There exist* $u_1, u_2 \in X$ *such that*

$$\max\{J(u_1), J(u_2)\} < c,$$

where $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ *and*

$$\Gamma = \{\gamma \in C([0,1], X) \,:\, \gamma(0) = u_1 \textit{ and } \gamma(1) = u_2\};$$

(i_3) *The functional* $L(u) = \Psi'(u)u - p\Psi(u) \geq 0$ *for every* $u \in X$;

(i_4) $L(u) \to \infty$ *as* $\Psi(u) \to \infty$.

Then the functional J *admits a bounded* (PS) *sequence at level* c.

Proof of Theorem 7.3.1. Let us consider the energy functional $\mathcal{I} : HW_0^{1,2}(\Omega_\psi) \to \mathbb{R}$ defined for every $u \in HW_0^{1,2}(\Omega_\psi)$ by

$$\mathcal{I}(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega_\psi} w(\sigma)F(u)d\mu. \tag{7.17}$$

(*i*) We shall prove that the restriction of $\mathcal{I}$ to $HW_{0,T}^{1,2}(\Omega_\psi)$, namely $\mathcal{J}$, satisfies the assumptions of Theorem 7.3.2, with $X = HW_{0,T}^{1,2}(\Omega_\psi)$,

$$\Phi(u) = \frac{1}{2}\|u\|^2, \quad \Psi(u) = \int_{\Omega_\psi} w(\sigma)F(u)d\mu, \quad u \in X.$$

Clearly, Φ and Ψ are smooth, with Φ homogeneous of degree 2 and coercive on X. Let us claim that (i_2) holds true. By (k_1), we know that $|F(t)| \leq C_f(|t| + |t|^q)$, so that

$$0 \leq F_+(t) \leq C_f(|t| + |t|^q). \tag{7.18}$$

Now, we shall prove that

$$\mathcal{J}(u) \geq \frac{1}{4}\|u\|^2 \tag{7.19}$$

in some neighborhood of zero in X. Since $\mathcal{J}(0) = 0$ and

$$\mathcal{J}(u) \geq \mathcal{J}_+(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega_\psi} w(\sigma)F_+(u)d\mu,$$

it is enough to show (7.19) for $\mathcal{J}_+$ instead of $\mathcal{J}$. By (k_2), for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$0 \leq F_+(t) \leq \varepsilon|t|^2 \quad \text{for every } t, \text{ with } |t| < \delta_\varepsilon.$$

Thus, (7.18) yields the existence of a positive constant M_ε such that

$$0 \leq F_+(t) \leq \varepsilon|t|^2 + M_\varepsilon|t|^q \quad \text{for every } t \in \mathbb{R}.$$

Hence

$$\begin{aligned} 0 \leq \int_{\Omega_\psi} w(\sigma)F_+(u)d\mu &\leq \|w\|_\infty(\varepsilon\|u\|_2^2 + M_\varepsilon\|u\|_q^q) \\ &\leq \|w\|_\infty(c_2\varepsilon\|u\|^2 + c_q M_\varepsilon\|u\|^q). \end{aligned}$$

Let us fix $\zeta \in (4, \infty)$ and let $\varepsilon = \frac{1}{\zeta c_2\|w\|_\infty}$. The above inequality ensures that

$$\begin{aligned} \mathcal{J}_+(u) &= \frac{1}{2}\|u\|^2 - \int_{\Omega_\psi} w(\sigma)F_+(u)d\mu \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon c_2\|w\|_\infty\|u\|^2 - c_q M_\varepsilon\|w\|_\infty\|u\|^q \\ &= \|u\|^2\left[\frac{1}{2} - \frac{1}{\zeta} - c_q M_\varepsilon\|w\|_\infty\|u\|^{q-2}\right] \\ &\geq \frac{1}{4}\|u\|^2, \end{aligned}$$

provided that $\|u\| < (\frac{\zeta-4}{4c_2\zeta\|w\|_\infty})^{\frac{1}{q-2}}$. This proves the validity of (7.19).

Set $\varrho \in (0, 1)$ and consider the truncation function $u_\varrho \in HW^{1,2}_{0,\mathrm{cyl}}(\Omega_\psi) \subseteq HW^{1,2}_{0,T}(\Omega_\psi)$ given by

$$u_\varrho(\sigma) = \frac{1}{1-\varrho}\left(1 - \max\left\{\frac{||z| - |z_0||}{R}, \frac{||t| - |t_0||}{R}, \varrho\right\}\right)_+, \quad \sigma = (z, t) \in \Omega_\psi.$$

With the above notation, we have:
(j_1') $\operatorname{supp}(u_\varrho) = A_R$;
(j_2') $\|u_\varrho\|_\infty \le 1$;
(j_3') $u_\varrho = 1$ in $A_{\varrho R}$.

Now, the function F is bounded from below on $\mathbb{R}_0^+$ by (k_3), and there exists a sequence $(t_k)_k \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} \frac{F(t_k)}{t_k^2} = \infty. \tag{7.20}$$

Then, (j_1')–(j_3') imply that

$$\begin{aligned}\Psi(t_k u_\varrho) &= \int\limits_{\Omega_\psi} w(\sigma)F(t_k u_\varrho)d\mu = \int\limits_{A_{\varrho R}} w(\sigma)F(t_k u_\varrho)d\mu \\ &= F(t_k)\int\limits_{A_{\varrho R}} w(\sigma)d\mu + \int\limits_{A_R\setminus A_{\varrho R}} w(\sigma)F(t_k u_\varrho)d\mu. \\ &\ge F(t_k)\int\limits_{A_{\varrho R}} w(\sigma)d\mu + \inf_{\mathbb{R}_0^+} F(t)\int\limits_{A_R\setminus A_{\varrho R}} w(\sigma)d\mu.\end{aligned}$$

Since $\mu(A_{\varrho R}) > 0$, formulas (7.12) and (7.20) give

$$\lim_{k\to\infty}\frac{\Psi(t_k u_\varrho)}{t_k^2} = \infty.$$

Consequently,

$$\mathcal{J}(t_k u_\varrho) = \Phi(t_k u_\varrho) - \Psi(t_k \varrho) = t_k^2\left[\frac{\|u_\varrho\|^2}{2} - t_k^{-2}\Psi(t_k u_\varrho)\right] < 0 \tag{7.21}$$

for every $k \in \mathbb{N}$ sufficiently large. Therefore, condition (i_2) is verified taking $u_1 = 0$, $u_2 = t_{\bar{k}} u_\varrho$ and $\bar{k}$ large enough. Indeed, (7.19) and (7.21) give, for some $\bar{k}$ sufficiently large,

$$\max\{\mathcal{J}(u_1), \mathcal{J}(t_{\bar{k}} u_\varrho)\} = 0 < c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

with $\Gamma = \{\gamma \in C([0,1], HW_{0,T}^{1,2}(\Omega_\psi)) : \gamma(0) = u_1 \text{ and } \gamma(1) = t_{\bar{k}} u_\varrho\}$. Now, ($k_4$) yields that

$$\Psi'(u)u - 2\Psi(u) = \int\limits_{\Omega_\psi} w(\sigma)[f(u)u - 2F(u)]d\mu \ge 0,$$

i. e., (i_3) holds true.

Finally, we prove (i_4). Let $(u_k)_k \subset HW^{1,2}_{0,T}(\Omega_\psi)$ be a sequence such that $\Psi(u_k) \to \infty$ as $k \to \infty$. Fix $k \in \mathbb{N}$ and set

$$\Omega^-_\psi = \{\sigma \in \Omega_\psi \,:\, F(u_k(\sigma)) < \kappa\} \quad \text{and} \quad \Omega^+_\psi = \{\sigma \in \Omega_\psi : F(u_k(\sigma)) \geq \kappa\},$$

where κ is given in (k_5). Then, since w is nonnegative in Ω_ψ, it follows that

$$\begin{aligned}\Psi(u_k) &= \int_{\Omega^-_\psi} w(\sigma)F(u_k)d\mu + \int_{\Omega^+_\psi} w(\sigma)F(u_k)d\mu \\ &\leq \kappa\|w\|_1 + \int_{\Omega^+_\psi} w(\sigma)F(u_k)d\mu.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \int_{\Omega^+_\psi} w(\sigma)F(u_k)d\mu = \infty. \tag{7.22}$$

Now, (k_4) and (k_5) yield

$$\begin{aligned}\Psi'(u_k)u_k - 2\Psi(u_k) &= \int_{\Omega^-_\psi} w(\sigma)[f(u_k)u_k - 2F(u_k)]d\mu \\ &\quad + \int_{\Omega^+_\psi} w(\sigma)[f(u_k)u_k - 2F(u_k)]d\mu \\ &\geq (\nu - 2)\int_{\Omega^+_\psi} w(\sigma)F(u_k)d\mu,\end{aligned}$$

which, together with (7.22), gives

$$\lim_{k\to\infty}\big(\Psi'(u_k)u_k - 2\Psi(u_k)\big) = \infty,$$

since $\nu > 2$.

Therefore, an application of Theorem 7.3.2 yields the existence of a bounded sequence $(u_k)_k$ in $HW^{1,2}_{0,T}(\Omega_\psi)$ such that

$$\mathcal{J}(u_k) \to c > 0 \quad \text{and} \quad \|\mathcal{J}'(u_k)\|_{(HW^{1,2}_{0,T})'} \to 0, \tag{7.23}$$

as $k \to \infty$. We prove now that there exists $u_\infty \in HW^{1,2}_{0,T}(\Omega_\psi)$ such that, up to a subsequence,

$$\|u_k - u_\infty\| \to 0 \tag{7.24}$$

as $k \to \infty$. Since $HW^{1,2}_{0,T}(\Omega_\psi)$ is reflexive, up to a subsequence, still denoted by $(u_k)_k$, there exists $u_\infty \in HW^{1,2}_{0,T}(\Omega_\psi)$ such that, thanks also to Lemma 7.1.2,

$$\begin{aligned} u_k \to u_\infty \quad \text{weakly in } HW^{1,2}_{0,T}(\Omega_\psi), \quad u_k \to u_\infty \quad \text{a. e. in } \Omega_\psi, \\ u_k \to u_\infty \quad \text{in } L^{\wp}(\Omega_\psi),\ \wp \in (2, 2^*_Q), \end{aligned} \tag{7.25}$$

as $k \to \infty$. Now

$$\langle \Phi'(u_k), u_k - u_\infty \rangle = \langle \mathcal{J}'(u_k), u_k - u_\infty \rangle + \int_{\Omega_\psi} w(\sigma) f(u_k)(u_k - u_\infty)\, d\mu \tag{7.26}$$

for every $k \in \mathbb{N}$.

Of course, taking into account that $(u_k)_k$ is bounded in $HW^{1,2}_{0,T}(\Omega_\psi)$, by (7.23), it follows that

$$\langle \mathcal{J}'(u_k), u_k - u_\infty \rangle \to 0 \tag{7.27}$$

as $k \to \infty$. On the other hand, since $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$, the Hölder inequality and (k_1) imply that

$$\begin{aligned} \left| \int_{\Omega_\psi} w(\sigma) f(u_k)(u_k - u_\infty)\, d\mu \right| &\le C_f \int_{\Omega_\psi} w(\sigma)(1 + |u_k|^{q-1})|u_k - u_\infty|\, d\mu \\ &\le C_f (\|w\|_{q'} + \|w\|_\infty \|u_k\|_q^{q-1}) \|u_k - u_\infty\|_q, \end{aligned}$$

that is, by (7.25),

$$\int_{\Omega_\psi} w(\sigma) f(u_k)(u_k - u_\infty)\, d\mu \to 0 \tag{7.28}$$

as $k \to \infty$. In conclusion, (7.26), (7.27), and (7.28) yield, as $k \to \infty$,

$$\|u_k - u_\infty\|^2 = \langle u_k, u_k - u_\infty \rangle + o(1) = \langle \Phi'(u_k), u_k - u_\infty \rangle \to 0,$$

which is (7.24).

As a consequence, $\mathcal{J}(u_\infty) = c$ and $\mathcal{J}'(u_\infty) = 0$, that is, u_∞ in $HW^{1,2}_{0,T}(\Omega_\psi)$ is a nontrivial critical point of $\mathcal{J}$. Now, exactly as in the proof of Corollary 7.2.5, the functional $\mathcal{I}$ given in (7.17) is T-invariant. The principle of symmetric criticality, Theorem A.1.5, ensures that $u_\infty \in HW^{1,2}_{0,T}(\Omega_\psi)$ is a critical point of $\mathcal{I}$ in $HW^{1,2}_0(\Omega_\psi)$, i. e., u_∞ is a solution localized in $HW^{1,2}_{0,T}(\Omega_\psi)$ of problem (7.10) set in $HW^{1,2}_0(\Omega_\psi)$.

(*ii*) Part (*i*) ensures that the problem (7.10) admits at least one nontrivial solution in $HW^{1,2}_0(\Omega_\psi)$, localized in the symmetric space $HW^{1,2}_{0,\mathrm{cyl}}(\Omega_\psi)$, and so for $N = 1$ we are done.

Let now $N \geq 2$ and fix $j \in J_N$. Arguing as in part (i), it is possible to prove that the restriction $\mathcal{J}^{(j)}$ of the functional $\mathcal{I}$ to the symmetric space E_j verifies assumptions (i_1)–(i_3) of Theorem 7.3.2. In particular,

$$\mathcal{J}^{(j)}(u) \geq \frac{1}{4}\|u\|^2, \tag{7.29}$$

in some neighborhood of zero in E_j. Furthermore, $\mathcal{J}^{(j)}$ verifies also (i_4). Indeed, by (k_3), the function F is bounded from below on $\mathbb{R}_0^+$ and there exists a sequence $(t_k)_k \subset \mathbb{R}^+$ such that (7.20) holds. Let us consider the symmetric function $v_{\varrho,j} \in E_j$ given in (7.15) and (7.16). Then, since F is even, (j_1)–(j_3) give

$$\begin{aligned}\Psi(t_k v_{\varrho,j}) &= \int_{\Omega_\psi} w(\sigma)F(t_k v_{\varrho,j})d\mu = \int_{S_1^j} w(\sigma)F(t_k v_{\varrho,j})d\mu \\ &= F(t_k)\int_{S_\varrho^j} w(\sigma)d\mu + \int_{S_1^j\setminus S_\varrho^j} w(\sigma)F\big(t_k|v_{\varrho,j}|\big)d\mu. \\ &\geq F(t_k)\int_{S_\varrho^j} w(\sigma)d\mu + \inf_{\mathbb{R}_0^+}F(t)\int_{S_1^j\setminus S_\varrho^j} w(\sigma)d\mu.\end{aligned}$$

Since $\mu(S_\varrho^j) > 0$, then (7.12) and (7.20) imply that

$$\lim_{k\to\infty}\frac{\Psi(t_k v_{\varrho,j})}{t_k^2} = \infty.$$

Consequently,

$$\mathcal{J}^{(j)}(t_k v_{\varrho,j}) = t_k^2\left[\frac{\|v_{\varrho,j}\|^2}{2} - t_k^{-2}\Psi(t_k v_{\varrho,j})\right] < 0 \tag{7.30}$$

for every $k \in \mathbb{N}$ sufficiently large. Therefore, (i_2) is verified taking $u_1 = 0$, $u_2 = t_{\bar{k}}v_{\varrho,j}$, and $\bar{k}$ large enough. Indeed, (7.29) and (7.30) yield for some $\bar{k}$ sufficiently large

$$\max\{\mathcal{J}^{(j)}(u_1), \mathcal{J}^{(j)}(t_{\bar{k}}v_{\varrho,j})\} = 0 < c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}^{(j)}(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C\big([0,1],E_j\big) : \gamma(0) = 0 \text{ and } \gamma(1) = t_{\bar{k}}v_{\varrho,j}\}.$$

Therefore, Theorem 7.3.2 ensures the existence of a bounded sequence $(u_k^{(j)})_k$ in E_j such that $\mathcal{J}^{(j)}(u_k^{(j)}) \to c > 0$ and $\|(\mathcal{J}^{(j)})'(u_k)\|_{E_j'} \to 0$ as $k \to \infty$. Thus, arguing as in part (i), there exists $u_\infty^{(j)} \in E_j$ which is a nontrivial critical point of $\mathcal{J}^{(j)}$. Now, since f is an odd function, the energy functional $\mathcal{I}$ given in (7.17) is even and $\widehat{T}_{N,j}^{\omega_j}$-invariant. In conclusion, Proposition 7.1.4 and Theorem A.1.5 imply that problem (7.10) admits at least $[N/2] + 1$ solutions in $HW_0^{1,2}(\Omega_\psi)$ with mutually symmetric different structures. The proof is now complete. □

Theorem 7.3.1 can be applied to the following model problem in $\Omega_\psi \subset \mathbb{H}^N$, $N \geq 1$:

$$\begin{cases} -\Delta_{\mathbb{H}^N} u + u = 2w(\sigma)u^3\left(2 + 3u^2\left(\sin\frac{1}{u^2} + 1\right) - \cos\frac{1}{u^2}\right) \text{ in } \Omega_\psi, \\ u = 0 \text{ on } \partial\Omega_\psi, \end{cases} \tag{7.31}$$

where $w \in L^1(\Omega_\psi) \cap L^\infty(\Omega_\psi)$ is a cylindrically symmetric nonnegative function such that (7.11) holds on some nonempty open set $\Omega \subset \Omega_\psi$. Indeed, since

$$F(t) = t^6\left(\sin\frac{1}{t^2} + 1\right) + t^4 \quad \text{for every } t \in \mathbb{R},$$

direct computations ensure that all the assumptions of Theorem 7.3.1 are verified; see Figure 7.5. Consequently, part (*ii*) of Theorem 7.3.1 implies that problem (7.31) admits at least $[N/2] + 1$ solutions in $HW_0^{1,2}(\Omega_\psi)$ with mutually symmetric different structures.

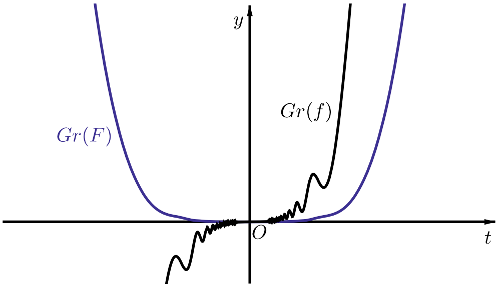

Figure 7.5: The graphs of the functions F and $F' = f$.

The above result can be viewed as an extension of the classical Theorem 4.1 proved by P. Rabier in [219] to problems settled on strip-like domains $\Omega_\psi \subset \mathbb{H}^N$.

Comments on Chapter 7

In [62, Theorems 1.1 and 1.2], G. Citti studies the critical semilinear problem

$$\begin{cases} -\Delta_{\mathbb{H}^N} u + a(\sigma)u = f(\sigma, u) + u^{\frac{N+2}{N}} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.32}$$

where Ω is a smooth bounded domain of the Heisenberg group $\mathbb{H}^N$, $a \in L^\infty(\Omega)$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a suitable subcritical continuous function. The main ingredient in the proof of Theorems 1.1 and 1.2 in [62] is the explicit profile decomposition of the Palais–Smale sequences *à la Lions*. The extension to the Heisenberg context is due to

D. Jerison and J. M. Lee [133]. An interesting open problem is to obtain the results of [62] for critical problems in strip-like domains Ω_ψ of $\mathbb{H}^N$. Certainly, a key point in the new approach will be played by some local weakly lower semicontinuity properties and direct minimization arguments; see [189, 190] for related topics.

8 Arbitrarily many solutions on homogeneous Hadamard manifolds

Taci. Su le soglie
del bosco non odo
parole che dici
umane; ma odo
parole più nuove
che parlano gocciole e foglie
lontane.

Gabriele D'Annunzio
from *La pioggia nel pineto*

In this chapter, using variational methods, we study the following elliptic problem:

$$\begin{cases} -\Delta_g u + u = w(\sigma)[f(u) + \lambda \mathrm{f}(u)] \text{ in } \mathcal{M}, \\ u \geq 0 \text{ in } \mathcal{M}, \quad u \in H_g^1(\mathcal{M}), \end{cases} \tag{P_λ}$$

where Δ_g is the classical Laplace–Beltrami operator on an N-dimensional homogeneous Hadamard manifold $\mathcal{M}$, with $N \geq 3$. In this context, λ is a real parameter, $w : \mathcal{M} \to \mathbb{R}$ is a suitable symmetric positive potential, $f : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function oscillating near the origin or at infinity, and $\mathrm{f} : \mathbb{R}_0^+ \to \mathbb{R}$ is any continuous function, with $\mathrm{f}(0) = 0$.

Problem (P_λ) may be also viewed as a prototype of pattern formation in biology and is in relationship with the steady state problem for a chemotactic aggregation model introduced by E. F. Keller and L. A. Segel [137]. Moreover, (P_λ) also plays an important role in the study of activator inhibitor systems describing biological pattern formations, as proposed by A. Gierer and H. Meinhardt in [118]. Problems of this type, as well as the associated evolution equations, describe super diffusion phenomena. Such models have been studied by P. G. de Gennes [76] to describe long range van der Waals interactions in thin films spread on solid surfaces.

Through variational and topological methods, we show that the number of solutions of (P_λ) is influenced by the value of the real parameter λ. More precisely, a variational construction enforces the use of the principle of symmetric criticality for nonsmooth Szulkin-type functionals defined on certain symmetric subspaces of the Sobolev space $H_g^1(\mathcal{M})$.

The results presented here extend some recent contributions, obtained for equations driven by the Laplace operator on the Euclidean space [143], and Schrödinger–Maxwell systems on Hadamard manifolds [91]. See also the results proved in [87, 147, 149], where competition phenomena are investigated for different classes of elliptic problems.

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8.1 Elements of analysis on Riemannian manifolds

In this section we briefly recall some notions from Riemannian geometry needed in the sequel and then illustrate the functional framework we shall move into. We refer the reader to the classical sources [18–20, 122, 123] for detailed derivations of the geometric quantities, their motivations, and further applications.

Let $\mathcal{M} = (\mathcal{M}, g)$ be an N-dimensional Riemannian manifold, with $N \geq 3$; see Figure 8.1 below. Let g_{ij} be the components of the metric g. Denote by $T_\sigma\mathcal{M}$ the tangent space at $\sigma \in \mathcal{M}$ and by $T\mathcal{M} = \bigcup_{\sigma\in\mathcal{M}} T_\sigma\mathcal{M}$ the tangent bundle. Let $d_g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ be the usual distance function associated with g. Denote by

$$B_g(\sigma_0, r) = \{\sigma \in \mathcal{M} : d_g(\sigma_0, \sigma) < r\}$$

and $\overline{B_g}(\sigma_0, r) = \{\sigma \in \mathcal{M} : d_g(\sigma_0, \sigma) \leq r\}$ the open and closed geodesic balls centered at $\sigma_0 \in \mathcal{M}$ and of radius $r > 0$, respectively.

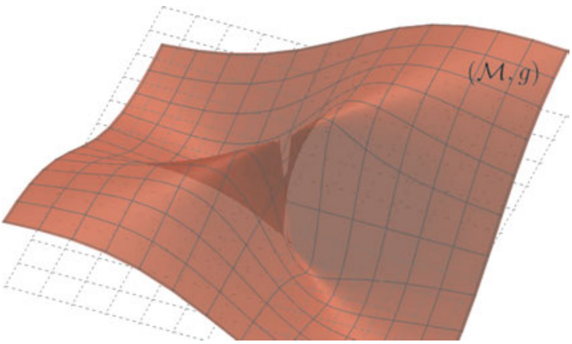

Figure 8.1: An abstract Riemannian manifold $\mathcal{M} = (\mathcal{M}, g)$.

If $C_c^\infty(\mathcal{M})$ denotes, as is customary, the space of real-valued compactly supported smooth functions on $\mathcal{M}$, set

$$\|\varphi\| = \left(\int_{\mathcal{M}} |\nabla_g \varphi|^2 d\sigma_g + \int_{\mathcal{M}} |\varphi|^2 d\sigma_g \right)^{1/2} \tag{8.1}$$

for every $\varphi \in C_c^\infty(\mathcal{M})$, where $\nabla_g\varphi$ is the covariant derivative of φ and $d\sigma_g$ is the Riemannian measure on $\mathcal{M}$, related to the Lebesgue measure dx in $\mathbb{R}^N$ by the formula $d\sigma_g = \sqrt{\mathrm{g}}\, dx$, $\mathrm{g} = \det(g_{ij})$. Put

$$\mathrm{Vol}_g(\Omega) = \int_{\Omega} d\sigma_g$$

for every bounded measurable set $\Omega \subset \mathcal{M}$. For any fixed system of local coordinates $(x_1, \ldots, x_N)$, the gradient $\nabla_g \varphi$ can be represented by

$$(\nabla_g^2 \varphi)_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k},$$

where

$$\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right) g^{lk}$$

are the usual Christoffel symbols and g^{lk} are the elements of the inverse matrix of g. In the last two chapters of the book, Einstein's summation convention is tacitly adopted. The Laplace–Beltrami operator Δ_g is the differential operator $\Delta_g \varphi = \operatorname{div}(\nabla_g \varphi)$ whose local expression is

$$\Delta_g \varphi = g^{ij}\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k} \right) = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_m}\left(\sqrt{g} g^{km} \frac{\partial \varphi}{\partial x_k} \right).$$

The space $H_g^1(\mathcal{M})$ is defined to be the completion of $C_c^\infty(\mathcal{M})$, with respect to the norm (8.1), and it turns out to be a Hilbert space equipped with the inner product

$$\langle u, v \rangle = \int_{\mathcal{M}} \langle \nabla_g u, \nabla_g v \rangle_g d\sigma_g + \int_{\mathcal{M}} uv d\sigma_g \tag{8.2}$$

for every u, $v \in H_g^1(\mathcal{M})$.

A Riemannian manifold $\mathcal{M} = (\mathcal{M}, g)$ with a transitive group of isometries is said to be a *Riemannian homogeneous space*. An *Hadamard manifold* is a Riemannian manifold which is complete, simply connected, and with everywhere nonpositive sectional curvature. The Cartan–Hadamard theorem guarantees that every Hadamard manifold $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^N$, $N = \dim \mathcal{M}$, in striking contrast to the Meyer theorem, which states that any complete Riemannian manifold $\mathcal{M}$ of strictly positive Ricci curvature is compact. Besides the Euclidean space, there exist other interesting geometric objects having the structure of an Hadamard manifold. An Hadamard manifold that is also a homogeneous space is said to be a *homogeneous Hadamard manifold*.

From now on, we always assume that $\mathcal{M} = (\mathcal{M}, g)$ is an N-dimensional homogeneous Hadamard manifold, with $N \geq 3$.

Referring to D. Hoffman and J. Spruck [130], the Sobolev embedding $H_g^1(\mathcal{M}) \hookrightarrow L^{\wp}(\mathcal{M})$ is continuous but not compact for every $\wp \in [2, 2^*]$, where, as usual, $2^* = 2N/(N-2)$ denotes the critical Sobolev exponent. In the light of this result, we indicate by $c_\wp$ the positive constant

$$c_\wp = \sup_{u \in H_g^1(\mathcal{M}) \setminus \{0\}} \frac{\|u\|_\wp}{\|u\|} < \infty,$$

$\| \cdot \|_\wp$ denoting as usual the $L^\wp$-norm on $\mathcal{M}$.

Since the main problem is settled in a noncompact framework, we shall adopt a group-theoretical approach to identify suitable symmetric subspaces of $H^1_g(\mathcal{M})$ for which the compactness of the embedding in $L^{\wp}(\mathcal{M})$ can be regained, when $\wp \in (2, 2^*)$. Denote by $\mathrm{Isom}_g(\mathcal{M})$ the group of isometries of $(\mathcal{M}, g)$ with the natural composition law and let $\mathscr{G}$ be a subgroup of $\mathrm{Isom}_g(\mathcal{M})$.

We say that $u : \mathcal{M} \to \mathbb{R}$ is $\mathscr{G}$-invariant if $u(\tau(\sigma)) = u(\sigma)$ for every $\sigma \in \mathcal{M}$ and $\tau \in \mathscr{G}$, and set

$$\mathrm{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma \in \mathcal{M} : \tau(\sigma) = \sigma \text{ for all } \tau \in \mathscr{G}\}.$$

The natural action $\circledast_{\mathscr{G}} : \mathscr{G} \times H^1_g(\mathcal{M}) \to H^1_g(\mathcal{M})$ of the group $\mathscr{G}$ on the Sobolev space $H^1_g(\mathcal{M})$ is defined, as usual, by

$$\tau \circledast_{\mathscr{G}} u(\sigma) = u(\tau^{-1}(\sigma)) \quad \text{for all } \tau \in \mathscr{G},\ u \in H^1_g(\mathcal{M}),\ \sigma \in \mathcal{M}. \tag{8.3}$$

As a next step, denote by

$$H^1_{\mathscr{G},g}(\mathcal{M}) = \{u \in H^1_g(\mathcal{M}) : \tau \circledast_{\mathscr{G}} u = u \text{ for all } \tau \in \mathscr{G}\}$$

the subspace of the $\mathscr{G}$-invariant functions of $H^1_g(\mathcal{M})$. A recent embedding result *à la Lions* due to L. Skrzypczak and C. Tintarev is decisive in our next arguments. We state it below in a convenient form.

Theorem 8.1.1. *Let $\mathcal{M} = (\mathcal{M}, g)$ be an N-dimensional homogeneous Hadamard manifold, with $N \geq 3$. If $\mathscr{G}$ is a compact connected subgroup of* $\mathrm{Isom}_g(\mathcal{M})$ *and* $\mathrm{Fix}_{\mathscr{G}}(\mathcal{M})$ *is a singleton, then the embedding*

$$H^1_{\mathscr{G},g}(\mathcal{M}) \hookrightarrow\hookrightarrow L^{\wp}(\mathcal{M})$$

is compact for any $\wp \in (2, 2^*)$.

See [232, Theorem 1.3 and Proposition 3.1], as well as [91] for related results.

We conclude this section by constructing a special function which will be useful in the proof of our main theorems. Let a, b be two positive numbers, with $a < b$. Define the annulus domain $A^b_a(\sigma_0)$ centered at $\sigma_0 \in \mathcal{M}$ as

$$A^b_a(\sigma_0) = \{\sigma \in \mathcal{M} \,:\, b - a < d_g(\sigma_0, \sigma) < a + b\}.$$

Moreover, it is useful to recall here that for every fixed $\sigma_0 \in \mathcal{M}$, the eikonal equation

$$|\nabla_g d_g(\sigma_0, \cdot)| = 1 \tag{8.4}$$

is satisfied a. e. in $\mathcal{M} \setminus \{\sigma_0\}$. Now, take r, ρ, with $0 < r < \rho$, and put

$$v_{\rho,r}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \mathcal{M} \setminus A^\rho_r(\sigma_0), \\ 1 & \text{if } \sigma \in A^\rho_{r/2}(\sigma_0), \\ \frac{2(r - |d_g(\sigma_0,\sigma) - \rho|)}{r} & \text{if } \sigma \in A^\rho_r(\sigma_0) \setminus A^\rho_{r/2}(\sigma_0) \end{cases} \tag{8.5}$$

for every $\sigma \in \mathcal{M}$; see Figure 8.2 for details.

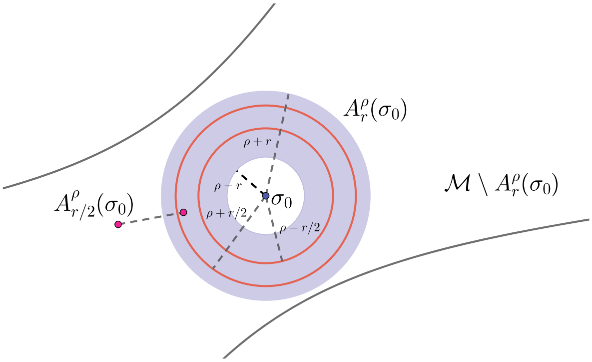

Figure 8.2: The annulus-type domain of the test function $v_{\rho,r}$.

It is clear that $\operatorname{supp} v_{\rho,r} \subset A_r^\rho(\sigma_0)$ and $\|v_{\rho,r}\|_\infty = 1$. By the definition of $v_{\rho,r}$ and exploiting also (8.4), we have

$$\begin{aligned}
\|v_{\rho,r}\|^2 &= \int_{A_r^\rho(\sigma_0)} |\nabla_g v_{\rho,r}|^2 d\sigma_g + \int_{A_r^\rho(\sigma_0)} |v_{\rho,r}|^2 d\sigma_g \\
&= \int_{A_{r/2}^\rho(\sigma_0)} |\nabla_g v_{\rho,r}|^2 d\sigma_g + \int_{A_{r/2}^\rho(\sigma_0)} |v_{\rho,r}|^2 d\sigma_g \\
&\quad + \int_{A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0)} |\nabla_g v_{\rho,r}|^2 d\sigma_g + \int_{A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0)} |v_{\rho,r}|^2 d\sigma_g \\
&\le \operatorname{Vol}_g(A_r^\rho(\sigma_0)) + \frac{4}{r^2} \int_{A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0)} \left|\nabla_g(r - |d_g(\sigma_0,\sigma) - \rho|)\right|^2 d\sigma_g \\
&= \operatorname{Vol}_g(A_r^\rho(\sigma_0)) + \frac{4}{r^2} \int_{A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0)} \left|\nabla_g |d_g(\sigma_0,\sigma) - \rho|\right|^2 d\sigma_g \\
&= \operatorname{Vol}_g(A_r^\rho(\sigma_0)) + \frac{4}{r^2} \operatorname{Vol}_g(A_r^\rho(\sigma_0) \setminus A_{r/2}^\rho(\sigma_0)) \\
&\le \left(1 + \frac{4}{r^2}\right) \operatorname{Vol}_g(A_r^\rho(\sigma_0)).
\end{aligned}$$

For any $t > 0$, we define the function

$$\mathrm{v}_t^{\rho,r} = t v_{\rho,r}, \tag{8.6}$$

where $v_{\rho,r}$ is given in (8.5). It is clear that $\mathrm{v}_t^{\rho,r} \ge 0$ in $\mathcal{M}$. Moreover, if $\mathcal{G}$ is a compact, connected subgroup of $\operatorname{Isom}_g(\mathcal{M})$, with $\operatorname{Fix}_{\mathcal{G}}(\mathcal{M}) = \{\sigma_0\}$, then $\mathrm{v}_t^{\rho,r} \in H^1_{\mathcal{G},g}(\mathcal{M})$. Finally, $\|\mathrm{v}_t^{\rho,r}\|_\infty = t$ and

$$\|\mathrm{v}_t^{\rho,r}\|^2 \le \left(1 + \frac{4}{r^2}\right) \operatorname{Vol}_g(A_r^\rho(\sigma_0)) t^2. \tag{8.7}$$

We also introduce the truncation function $\phi_\eta : \mathbb{R}_0^+ \to \mathbb{R}$ defined as

$$\phi_\eta(t) = \min\{\eta, t\} \tag{8.8}$$

for any $t \geq 0$, where η is the positive constant given in assumption (8.12). Note that ϕ_η is a continuous function in $\mathbb{R}_0^+$.

8.2 An auxiliary elliptic problem on manifolds

In order to solve (P_λ), in this section we introduce the auxiliary equation

$$\begin{cases} -\Delta_g u + u = w(\sigma)\boldsymbol{f}(u) & \text{in } \mathcal{M}, \\ u \geq 0 \quad \text{in } \mathcal{M}, \quad u \in H_g^1(\mathcal{M}). \end{cases} \tag{8.9}$$

Here, we assume that $\boldsymbol{f} : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function satisfying the following conditions:

$$\boldsymbol{f}(0) = 0; \tag{8.10}$$

$$\textit{There exists } M > 0 \textit{ such that } |\boldsymbol{f}(t)| \leq M \textit{ for every } t \in \mathbb{R}_0^+; \tag{8.11}$$

$$\textit{There are } \delta,\ \eta,\ \textit{with } 0 < \delta < \eta,\ \textit{such that } \boldsymbol{f}(t) \leq 0 \textit{ for any } t \in [\delta, \eta]. \tag{8.12}$$

In the sequel, taking into account that (8.10) holds, we extend the function $\boldsymbol{f}$ on the whole real line by taking $\boldsymbol{f}(t) = 0$ for every $t < 0$. For the potential w, we assume

(w) *$w : \mathcal{M} \to \mathbb{R}$ is in $L^1(\mathcal{M}) \cap L^2(\mathcal{M})$; it is positive, continuous, and radially symmetric with respect to $\sigma_0 \in \mathcal{M}$*, i. e., there exists $\psi : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$w(\sigma) = \psi(d_g(\sigma_0, \sigma)) \quad \text{for every } \sigma \in \mathcal{M}. \tag{8.13}$$

In this section we prove the existence of a nonnegative solution for (8.9). Since equation (8.9) is variational, let us introduce the associated energy functional $\mathcal{E}_{\text{aux}} : H_g^1(\mathcal{M}) \to \mathbb{R}$ defined by

$$\mathcal{E}_{\text{aux}}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}(u)\, d\sigma_g\,, \tag{8.14}$$

where $\boldsymbol{F}(t) = \int_0^t \boldsymbol{f}(s) ds$ for any $t \in \mathbb{R}$.

Due to the embedding properties of the space $H_g^1(\mathcal{M})$ into the Lebesgue spaces, it is easy to see that $\mathcal{E}_{\text{aux}}$ is well defined. Indeed, the mean value theorem, (8.11), the Hölder inequality, and (w) yield

$$\int_{\mathcal{M}} w(\sigma)|\boldsymbol{F}(u)| d\sigma_g \leq M\|w\|_2\|u\|_2 < \infty$$

for every $u \in H^1_g(\mathcal{M})$. Moreover, standard arguments show that $\mathcal{E}_{\text{aux}}$ is of class $C^1(H^1_g(\mathcal{M}))$.

Now, according to the notations of Section 8.1, if $\mathscr{G}$ is a compact connected subgroup of $\text{Isom}_g(\mathcal{M})$ such that $\text{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma_0\}$, we denote by

$$H^1_{\mathscr{G},g}(\mathcal{M}) = \{u \in H^1_g(\mathcal{M}) \;:\; \tau(u) = u \text{ for all } \tau \in \mathscr{G}\}$$

the subspace of $\mathscr{G}$-invariant functions of $H^1_g(\mathcal{M})$ and by $\mathcal{E}^{\mathscr{G}}_{\text{aux}}$ the restriction of $\mathcal{E}_{\text{aux}}$ to $H^1_{\mathscr{G},g}(\mathcal{M})$.

In order to find nonnegative solutions of (8.9), we look for nonnegative critical points of the functional $\mathcal{E}^{\mathscr{G}}_{\text{aux}}$. At this purpose, we introduce the set $W_\eta(\mathcal{M})$ defined as follows:

$$W_\eta(\mathcal{M}) = \{u \in H^1_g(\mathcal{M}) \;:\; \|u\|_\infty \le \eta\},$$

and also set

$$W^{\mathscr{G}}_\eta(\mathcal{M}) = W_\eta(\mathcal{M}) \cap H^1_{\mathscr{G},g}(\mathcal{M}),$$

where η is the positive parameter given in (8.12); see Figure 8.3 below.

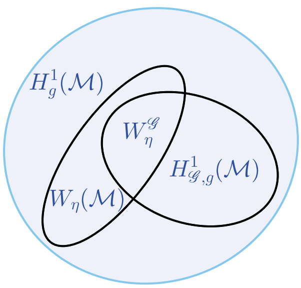

Figure 8.3: The subset $W^{\mathscr{G}}_\eta(\mathcal{M})$.

The main result of the section is given in the following theorem.

Theorem 8.2.1. *Let $\mathcal{M} = (\mathcal{M}, g)$ be a homogeneous Hadamard manifold of dimension $N \ge 3$ and let $\mathscr{G}$ be a compact connected subgroup of* $\text{Isom}_g(\mathcal{M})$ *such that* $\text{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma_0\}$. *Furthermore, let $\boldsymbol{f} : \mathbb{R}^+_0 \to \mathbb{R}$ be a continuous function satisfying conditions* (8.10)–(8.12) *and let $w : \mathcal{M} \to \mathbb{R}$ verify condition* (w). *Then,*

(i) $\mathcal{E}^{\mathscr{G}}_{\text{aux}}$ *is bounded from below on $W^{\mathscr{G}}_\eta(\mathcal{M})$, with infimum attained at some $u^{\mathscr{G}}_\eta \in W^{\mathscr{G}}_\eta(\mathcal{M})$;*

(ii) $u^{\mathscr{G}}_\eta \in [0, \delta]$ *a. e. $\mathcal{M}$, where $\delta > 0$ is given in* (8.12);

(iii) $u^{\mathscr{G}}_\eta$ *is a nonnegative solution of* (8.9).

Proof. The proof is similar to that of [91]; for completeness, we provide its main steps.

(i) Clearly, the set $W_\eta^{\mathcal{G}}(\mathcal{M})$ is convex. Moreover, $W_\eta^{\mathcal{G}}(\mathcal{M})$ is closed in $H^1_{\mathcal{G},g}(\mathcal{M})$. Indeed, if $(u^{\mathcal{G}}_{\eta,k})_k$ in $W_\eta^{\mathcal{G}}(\mathcal{M})$ is such that $u^{\mathcal{G}}_{\eta,k} \to u^{\mathcal{G}}_{\eta,\infty}$ in $H^1_{\mathcal{G},g}(\mathcal{M})$ as $k \to \infty$, then we claim that $u^{\mathcal{G}}_{\eta,\infty} \in W_\eta^{\mathcal{G}}(\mathcal{M})$. Of course, $u^{\mathcal{G}}_{\eta,\infty} \in H^1_{\mathcal{G},g}(\mathcal{M})$. Furthermore, by assumption $(u^{\mathcal{G}}_{\eta,k})_k$ is bounded in $L^\infty(\mathcal{M})$. Since $L^\infty(\mathcal{M})$ is the dual space of $L^1(\mathcal{M})$, which is a separable Banach space, one has $u^{\mathcal{G}}_{\eta,k} \overset{*}{\rightharpoonup} u^{\mathcal{G}}_{\eta,\infty}$ in $L^\infty(\mathcal{M})$ as $k \to \infty$. Hence,

$$\eta \geq \liminf_{k\to\infty} \|u^{\mathcal{G}}_{\eta,k}\|_\infty \geq \|u^{\mathcal{G}}_{\eta,\infty}\|_\infty,$$

that is, $u^{\mathcal{G}}_{\eta,\infty} \in W_\eta^{\mathcal{G}}(\mathcal{M})$, which proves the claim.

Consequently, since $W_\eta^{\mathcal{G}}(\mathcal{M})$ is convex and closed in $H^1_{\mathcal{G},g}(\mathcal{M})$, we get that $W_\eta^{\mathcal{G}}(\mathcal{M})$ is weakly closed in $H^1_{\mathcal{G},g}(\mathcal{M})$.

Now, let us prove that $\mathcal{E}^{\mathcal{G}}_{\text{aux}}$ is sequentially weakly lower semicontinuous in $H^1_{\mathcal{G},g}(\mathcal{M})$. This follows at once if we prove that the functional

$$\Psi(u) = \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}(u)d\sigma_g$$

is sequentially weakly continuous in $H^1_{\mathcal{G},g}(\mathcal{M})$. Otherwise, there exists $(u_k)_k$ in $H^1_{\mathcal{G},g}(\mathcal{M})$ which converges weakly to some u_∞ and such that, for every $k \in \mathbb{N}$,

$$\left|\Psi(u_k) - \Psi(u_\infty)\right| \geq \varepsilon_0,$$

for some appropriate $\varepsilon_0 > 0$. Clearly, $(u_k)_k$ converges strongly to u_∞ in $L^\wp(\mathcal{M})$ for all $\wp \in (2, 2^*)$ by Theorem 8.1.1. Now, fix $\wp \in (2, 2^*)$. Then the mean value theorem, the Hölder inequality, (8.11), and (w), together with the above inequality, give that

$$0 < \varepsilon_0 \leq M \int_{\mathcal{M}} w(\sigma)|u_k - u_\infty| d\sigma_g \leq M\|w\|_{\wp'}\|u_k - u_\infty\|_\wp,$$

thanks to (w), since $1 < \wp' < 2$. But the right-hand side tends to zero as $k \to \infty$, which is the desired contradiction.

Moreover, (8.11) and the definition of $\boldsymbol{F}$ yield that the functional $\mathcal{E}^{\mathcal{G}}_{\text{aux}}$ is bounded from below on $W_\eta^{\mathcal{G}}(\mathcal{M})$. Indeed,

$$\begin{aligned}\mathcal{E}^{\mathcal{G}}_{\text{aux}}(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}(u)\, d\sigma_g \\ &\geq -\int_{\mathcal{M}} w(\sigma)\boldsymbol{F}(u)\, d\sigma_g \geq -M\int_{\mathcal{M}} w(\sigma)|u|\, d\sigma_g \\ &\geq -\eta M\|w\|_1\end{aligned}$$

for any $u \in W_\eta^{\mathcal{G}}(\mathcal{M})$.

Let us denote by $m_\eta^{\mathscr{G}}$ the infimum of $\mathcal{E}_{\text{aux}}^{\mathscr{G}}$ on $W_\eta^{\mathscr{G}}(\mathcal{M})$, that is,

$$m_\eta^{\mathscr{G}} = \inf_{u \in W_\eta^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u) > -\infty . \tag{8.15}$$

It is easily seen that for every $k \in \mathbb{N}$ there exists $u_{\eta,k}^{\mathscr{G}} \in W_\eta^{\mathscr{G}}(\mathcal{M})$ such that

$$m_\eta^{\mathscr{G}} \leq \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_{\eta,k}^{\mathscr{G}}) \leq m_\eta^{\mathscr{G}} + \frac{1}{k}. \tag{8.16}$$

Also, since $u_{\eta,k}^{\mathscr{G}} \in W_\eta^{\mathscr{G}}(\mathcal{M})$ and thanks to (8.11), we get

$$\begin{aligned}
\frac{1}{2}\|u_{\eta,k}^{\mathscr{G}}\|^2 &= \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}(u_{\eta,k}^{\mathscr{G}})d\sigma_g + \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_{\eta,k}^{\mathscr{G}}) \\
&\leq \eta M\|w\|_1 + \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_{\eta,k}^{\mathscr{G}}) \leq \eta M\|w\|_1 + m_\eta^{\mathscr{G}} + \frac{1}{k} \\
&\leq \eta M\|w\|_1 + m_\eta^{\mathscr{G}} + 1
\end{aligned}$$

for every $k \in \mathbb{N}$. Thus,

$$\sup_k \|u_{\eta,k}^{\mathscr{G}}\| \leq \kappa, \quad \text{where } \kappa = \sqrt{2(\eta M\|w\|_1 + m_\eta^{\mathscr{G}} + 1)}.$$

Then, the sequence $(u_{\eta,k}^{\mathscr{G}})_k$ is bounded in $H_{\mathscr{G},g}^1(\mathcal{M})$ and so, up to a subsequence, still denoted by $(u_{\eta,k}^{\mathscr{G}})_k$,

$$u_{\eta,k}^{\mathscr{G}} \rightharpoonup u_\eta^{\mathscr{G}} \quad \text{in } W_\eta^{\mathscr{G}}(\mathcal{M}) \tag{8.17}$$

as $k \to \infty$ for some $u_\eta^{\mathscr{G}} \in W_\eta^{\mathscr{G}}(\mathcal{M})$.

Now, let us show that $u_\eta^{\mathscr{G}}$ is the minimum of $\mathcal{E}_{\text{aux}}^{\mathscr{G}}$ in $W_\eta^{\mathscr{G}}(\mathcal{M})$ we are looking for. Of course, $u_\eta^{\mathscr{G}} \in W_\eta(\mathcal{M})$, since $W_\eta^{\mathscr{G}}(\mathcal{M})$ is weakly closed in $H_{\mathscr{G},g}^1(\mathcal{M})$. Thus, by (8.15),

$$\mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_\eta^{\mathscr{G}}) \geq m_\eta^{\mathscr{G}} . \tag{8.18}$$

On the other hand, thanks to (8.16), (8.17), and the sequential weak lower semicontinuity of $\mathcal{E}_{\text{aux}}^{\mathscr{G}}$, we obtain that

$$m_\eta^{\mathscr{G}} \geq \liminf_{k\to\infty} \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_k^{\mathscr{G}}) \geq \mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_\eta^{\mathscr{G}}).$$

Therefore, (8.18) yields that $\mathcal{E}_{\text{aux}}^{\mathscr{G}}(u_\eta^{\mathscr{G}}) = m_\eta^{\mathscr{G}}$, which, together with (8.15), concludes the proof of statement (*i*); see Figure 8.4.

(*ii*) Let δ be as in (8.12) and define

$$U(u_\eta^{\mathscr{G}}) = \{\sigma \in \mathcal{M} \,:\, u_\eta^{\mathscr{G}} \notin [0,\delta]\}.$$

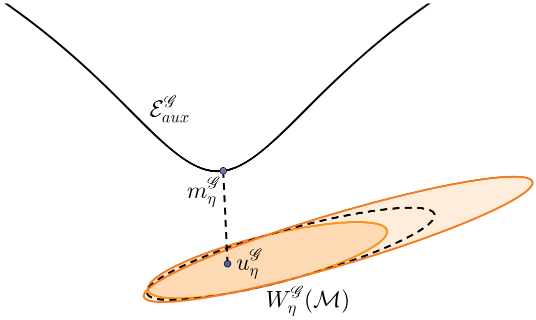

Figure 8.4: The minimization of $\mathcal{E}^{\mathcal{G}}_{\text{aux}}$ on $W^{\mathcal{G}}_{\eta}(\mathcal{M})$.

Assume by contradiction that $\text{Vol}_g(U(w^{\mathcal{G}}_{\eta})) > 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\phi(t) = \min\{t_+, \delta\},$$

where $t_+ = \max\{t, 0\}$. Also, set $\mathrm{u}^{\mathcal{G}}_{\eta} = \phi \circ u^{\mathcal{G}}_{\eta}$, that is,

$$\mathrm{u}^{\mathcal{G}}_{\eta}(\sigma) = \begin{cases} \delta & \text{if } u^{\mathcal{G}}_{\eta}(\sigma) > \delta, \\ u^{\mathcal{G}}_{\eta}(\sigma) & \text{if } 0 \leq u^{\mathcal{G}}_{\eta}(\sigma) \leq \delta, \\ 0 & \text{if } u^{\mathcal{G}}_{\eta}(\sigma) < 0 \end{cases}$$

for a. e. $\sigma \in \mathcal{M}$.

Since ϕ is a Lipschitz function and $\phi(0) = 0$, by [122, Proposition 2.5, p. 24], it follows that $\mathrm{u}^{\mathcal{G}}_{\eta} \in H^1_g(\mathcal{M})$. We claim that $\mathrm{u}^{\mathcal{G}}_{\eta} \in H^1_{\mathcal{G},g}(\mathcal{M})$. Indeed,

$$\begin{aligned} \tau \circledast_{\mathcal{G}} \mathrm{u}^{\mathcal{G}}_{\eta}(\sigma) &= \mathrm{u}^{\mathcal{G}}_{\eta}(\tau^{-1}(\sigma)) = (\phi \circ u^{\mathcal{G}}_{\eta})(\tau^{-1}(\sigma)) \\ &= \phi(u^{\mathcal{G}}_{\eta}(\tau^{-1}(\sigma))) = \phi(u^{\mathcal{G}}_{\eta}(\sigma)) \\ &= \mathrm{u}^{\mathcal{G}}_{\eta}(\sigma) \end{aligned}$$

for all $\tau \in \mathcal{G}$. Moreover, $0 \leq \mathrm{u}^{\mathcal{G}}_{\eta} \leq \delta$ a. e. $\mathcal{M}$. Consequently, $\mathrm{u}^{\mathcal{G}}_{\eta} \in W_{\eta}$, since $\delta < \eta$ by assumption (8.12).

We introduce the sets

$$U_1(u^{\mathcal{G}}_{\eta}) = \{\sigma \in \mathcal{M} : u^{\mathcal{G}}_{\eta}(\sigma) < 0\} \quad \text{and} \quad U_2(u^{\mathcal{G}}_{\eta}) = \{\sigma \in \mathcal{M} : u^{\mathcal{G}}_{\eta}(\sigma) > \delta\}.$$

Thus, $U(u_\eta^{\mathscr{G}}) = U_1(u_\eta^{\mathscr{G}}) \cup U_2(u_\eta^{\mathscr{G}})$ and $\mathrm{u}_\eta^{\mathscr{G}} = u_\eta^{\mathscr{G}}$ a. e. in $\mathcal{M} \setminus U(u_\eta^{\mathscr{G}})$, while $\mathrm{u}_\eta^{\mathscr{G}} = 0$ a. e. in $U_1(u_\eta^{\mathscr{G}})$ and $\mathrm{u}_\eta^{\mathscr{G}} = \delta$ a. e. in $U_2(u_\eta^{\mathscr{G}})$. Thus,

$$\begin{aligned}\mathcal{E}_{\mathrm{aux}}^{\mathscr{G}}(\mathrm{u}_\eta^{\mathscr{G}}) - \mathcal{E}_{\mathrm{aux}}^{\mathscr{G}}(u_\eta^{\mathscr{G}}) &= \frac{1}{2}(\|\mathrm{u}_\eta^{\mathscr{G}}\|^2 - \|u_\eta^{\mathscr{G}}\|^2) \\ &\quad - \int_{\mathcal{M}} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}) - \boldsymbol{F}(u_\eta^{\mathscr{G}}))d\sigma_g \\ &= -\frac{1}{2}\int_{U(u_\eta^{\mathscr{G}})} |\nabla u_\eta^{\mathscr{G}}| d\sigma_g \\ &\quad + \frac{1}{2}\int_{U(u_\eta^{\mathscr{G}})} (|\mathrm{u}_\eta^{\mathscr{G}}|^2 - |u_\eta^{\mathscr{G}}|^2)d\sigma_g \\ &\quad - \int_{U(u_\eta^{\mathscr{G}})} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}) - \boldsymbol{F}(u_\eta^{\mathscr{G}}))d\sigma_g.\end{aligned} \tag{8.19}$$

Moreover,

$$\begin{aligned}\int_{U(u_\eta^{\mathscr{G}})} (|\mathrm{u}_\eta^{\mathscr{G}}|^2 - |u_\eta^{\mathscr{G}}|^2)d\sigma_g &= -\int_{U_1(u_\eta^{\mathscr{G}})} |u_\eta^{\mathscr{G}}|^2 d\sigma_g \\ &\quad + \int_{U_2(u_\eta^{\mathscr{G}})} (\delta^2 - |u_\eta^{\mathscr{G}}|^2)d\sigma_g \leqslant 0.\end{aligned} \tag{8.20}$$

Since $\boldsymbol{f}(t) = 0$ for every $t \le 0$ by definition, we get

$$\int_{U_1(u_\eta^{\mathscr{G}})} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}) - \boldsymbol{F}(u_\eta^{\mathscr{G}}))d\sigma_g = 0. \tag{8.21}$$

Thanks to the mean value theorem, for a. e. $\sigma \in U_2(u_\eta^{\mathscr{G}})$ there exists a number $\theta(\sigma) \in [\delta, u_\eta^{\mathscr{G}}(\sigma)] \subseteq [\delta, \eta]$ such that

$$\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}(\sigma)) - \boldsymbol{F}(u_\eta^{\mathscr{G}}(\sigma)) = \boldsymbol{F}(\delta) - \boldsymbol{F}(u_\eta^{\mathscr{G}}(\sigma)) = \boldsymbol{f}(\theta(\sigma))(\delta - u_\eta^{\mathscr{G}}(\sigma)).$$

Thus, taking into account (8.12) and the definition of $U_2(u_\eta^{\mathscr{G}})$, we have

$$\int_{U_2(u_\eta^{\mathscr{G}})} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}) - \boldsymbol{F}(u_\eta^{\mathscr{G}}))d\sigma_g = \int_{U_2(u_\eta^{\mathscr{G}})} w(\sigma)\boldsymbol{f}(\theta)(\delta - u_\eta^{\mathscr{G}})d\sigma_g \ge 0\,. \tag{8.22}$$

Hence, by (8.21) and (8.22), we get that

$$\int_{U(u_\eta^{\mathscr{G}})} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathscr{G}}) - \boldsymbol{F}(u_\eta^{\mathscr{G}}))d\sigma_g \ge 0.$$

As a consequence, (8.19) and (8.20) yield

$$\mathcal{E}_{\text{aux}}^{\mathcal{G}}(\,\mathrm{u}_\eta^{\mathcal{G}}\,) - \mathcal{E}_{\text{aux}}^{\mathcal{G}}(u_\eta^{\mathcal{G}}\,) \le 0\,. \tag{8.23}$$

On the other hand, it is apparent that $\mathcal{E}_{\text{aux}}^{\mathcal{G}}(\mathrm{u}_\eta^{\mathcal{G}}) \ge \mathcal{E}_{\text{aux}}^{\mathcal{G}}(u_\eta^{\mathcal{G}})$, since $\mathrm{u}_\eta^{\mathcal{G}} \in W_\eta^{\mathcal{G}}(\mathcal{M})$. Therefore, (8.23) gives

$$\mathcal{E}_{\text{aux}}^{\mathcal{G}}(\,\mathrm{u}_\eta^{\mathcal{G}}\,) = \mathcal{E}_{\text{aux}}^{\mathcal{G}}(u_\eta^{\mathcal{G}}).$$

Since all the integrals on the right-hand side of (8.19) are nonpositive, it is easy to see that every integral in (8.19) should be zero. In particular,

$$\int_{\mathcal{M}} w(\sigma)(\boldsymbol{F}(\mathrm{u}_\eta^{\mathcal{G}}) - \boldsymbol{F}(u_\eta^{\mathcal{G}}))d\sigma_g = 0. \tag{8.24}$$

Also, note that the integrals on the right-hand side of (8.20) are both nonpositive. Thus, (8.24) gives

$$\int_{U_1(u_\eta^{\mathcal{G}})} |u_\eta^{\mathcal{G}}|^2 d\sigma_g = \int_{U_2(u_\eta^{\mathcal{G}})} (|u_\eta^{\mathcal{G}}|^2 - \delta^2)d\sigma_g = 0.$$

The definitions of $U_1(u_\eta^{\mathcal{G}})$ and $U_2(u_\eta^{\mathcal{G}})$ provide $\mathrm{Vol}_g(U_1(u_\eta^{\mathcal{G}})) = \mathrm{Vol}_g(U_2(u_\eta^{\mathcal{G}})) = 0$, that is, $\mathrm{Vol}_g(U(u_\eta^{\mathcal{G}})) = 0$. This is impossible and proves (*ii*).

(*iii*) We divide the proof in two steps.

Step 1. $\langle \mathcal{E}_{\text{aux}}'(u_\eta^{\mathcal{G}}), u - u_\eta^{\mathcal{G}} \rangle \ge 0$ *for every* $u \in W_\eta(\mathcal{M})$.

Let ψ_{W_η} be the indicator function of the set $W_\eta(\mathcal{M})$, i. e.,

$$\psi_{W_\eta}(u) = \begin{cases} 0 & \text{if } u \in W_\eta(\mathcal{M}), \\ \infty & \text{if } u \notin W_\eta(\mathcal{M}). \end{cases}$$

The functional $J_\eta : H_g^1(\mathcal{M}) \to \mathbb{R} \cup \{\infty\}$, given by

$$J_\eta = \mathcal{E}_{\text{aux}} + \psi_{W_\eta},$$

is of Szulkin's type, since $\mathcal{E}_{\text{aux}}$ is of class $C^1(H_g^1(\mathcal{M}))$ and ψ_{W_η} is convex, lower semicontinuous, and proper, as $W_\eta(\mathcal{M})$ is closed and convex in $H_g^1(\mathcal{M})$. Now, $W_\eta^{\mathcal{G}}(\mathcal{M}) = W_\eta(\mathcal{M}) \cap H_{\mathcal{G},g}^1(\mathcal{M})$, so that the restriction of ψ_{W_η} to $H_{\mathcal{G},g}^1(\mathcal{M})$ is precisely the indicator function $\psi_{W_\eta^{\mathcal{G}}}$ of the set $W_\eta^{\mathcal{G}}(\mathcal{M})$, i. e.,

$$\psi_{W_\eta^{\mathcal{G}}}(u) = \begin{cases} 0 & \text{if } u \in W_\eta^{\mathcal{G}}(\mathcal{M}), \\ \infty & \text{if } u \notin W_\eta^{\mathcal{G}}(\mathcal{M}). \end{cases}$$

By (i), the function $u_\eta^{\mathscr{G}}$ is a minimum of $\mathcal{E}_{aux}^{\mathscr{G}}$ in $W_\eta^{\mathscr{G}}(\mathcal{M})$. Hence, $u_\eta^{\mathscr{G}}$ a local minimum of the Szulkin functional $\mathcal{J}_\eta^{\mathscr{G}} : H^1_{\mathscr{G},g}(\mathcal{M}) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathcal{J}_\eta^{\mathscr{G}} = \mathcal{E}_{\text{aux}}^{\mathscr{G}} + \psi_{W_\eta^{\mathscr{G}}}.$$

Proposition A.2.1 yields that $u_\eta^{\mathscr{G}}$ is a critical point of $\mathcal{J}_\eta^{\mathscr{G}}$ in $H^1_{\mathscr{G},g}(\mathcal{M})$, that is,

$$0 \in (\mathcal{E}_{\text{aux}}^{\mathscr{G}})'(u_\eta^{\mathscr{G}}) + \partial\psi_{W_\eta^{\mathscr{G}}}(u_\eta^{\mathscr{G}}) \quad \text{in} \quad (H^1_{\mathscr{G},g}(\mathcal{M}))',$$

where $\partial\psi_{W_\eta^{\mathscr{G}}}$ stands for the subdifferential of the convex function $\psi_{W_\eta^{\mathscr{G}}}$.

In order to apply the Palais principle recalled in Theorem A.2.2, the functionals $\mathcal{E}_{\text{aux}}$ and ψ_{W_η} need to be $\mathscr{G}$-invariant. Let us prove that $\mathcal{E}_{\text{aux}}$ is $\mathscr{G}$-invariant. To show this, let $u \in H^1_g(\mathcal{M})$ and $\tau \in \mathscr{G}$ be fixed. Since $\tau \in \mathscr{G}$ is an isometry, on account of (8.3), we get the chain rule as follows:

$$\nabla_g(\tau \circledast_{\mathscr{G}} u)(\sigma) = D\tau_{\tau^{-1}(\sigma)}\nabla_g u(\tau^{-1}(\sigma)), \tag{8.25}$$

for a. e. $\sigma \in \mathcal{M}$, where $D\tau_{\tau^{-1}(\sigma)} : T_{\tau^{-1}(\sigma)}(\mathcal{M}) \to T_\sigma(\mathcal{M})$ denotes the differential of $\tau \in \mathscr{G}$ at the point $\tau^{-1}(\sigma)$. Setting $y = \tau^{-1}(\sigma)$, it follows that

$$\begin{aligned}
\|\tau \circledast_{\mathscr{G}} u\|^2 &= \int_{\mathcal{M}} |\nabla_g(\tau \circledast_{\mathscr{G}} u)(\sigma)|_\sigma^2 d\sigma_g + \int_{\mathcal{M}} |(\tau \circledast_{\mathscr{G}} u)(\sigma)|^2 d\sigma_g \\
&= \int_{\mathcal{M}} |\nabla_g u(\tau^{-1}(\sigma))|^2_{\tau^{-1}(\sigma)} d\sigma_g + \int_{\mathcal{M}} |u(\tau^{-1}(\sigma))|^2 d\sigma_g \\
&= \int_{\mathcal{M}} |\nabla_g u(y)|_y^2 d\sigma_g(y) + \int_{\mathcal{M}} |u(y)|^2 d\sigma_g(y) \\
&= \|u\|^2,
\end{aligned} \tag{8.26}$$

where we have made use of (8.25) and of the fact that the map $D\tau_{\tau^{-1}(\sigma)}$ is inner product preserving.

Moreover, since the weight w is radially symmetric with respect to the point $\sigma_0 \in \mathcal{M}$, thanks to (8.13), for a. e. $\sigma \in \mathcal{M}$ and $\tau \in \mathscr{G}$, we have

$$w(\tau(\sigma)) = \psi(d_g(\sigma_0, \tau(\sigma))) = \psi(d_g(\tau(\sigma_0), \tau(\sigma))) = \psi(d_g(\sigma_0, \sigma)) = w(\sigma),$$

and consequently,

$$\begin{aligned}
\int_{\mathcal{M}} w(\sigma)\left(\int_0^{(\tau \circledast_{\mathscr{G}} u)(\sigma)} \boldsymbol{f}(t)dt\right)d\sigma_g &= \int_{\mathcal{M}} w(\sigma)\left(\int_0^{u(\tau^{-1}(\sigma))} \boldsymbol{f}(t)dt\right)d\sigma_g \\
&= \int_{\mathcal{M}} w(y)\left(\int_0^{u(y)} \boldsymbol{f}(t)dt\right)d\sigma_g(y).
\end{aligned}$$

Therefore, recalling (8.26), we obtain that for every $\tau \in \mathscr{G}$ and $u \in H^1_g(\mathcal{M})$,

$$\mathcal{E}_{\text{aux}}(\tau \circledast_{\mathscr{G}} u) = \frac{1}{2}\|\tau \circledast_{\mathscr{G}} u\|^2 - \int_{\mathcal{M}} w(\sigma)\left(\int_0^{(\tau\circledast_{\mathscr{G}} u)(\sigma)} \boldsymbol{f}(t)dt\right)d\sigma_g = \mathcal{E}_{\text{aux}}(u),$$

which proves the claim.

Moreover, due to the fact that the set $W_\eta(\mathcal{M})$ is $\mathscr{G}$-invariant, the functional ψ_{W_η} is $\mathscr{G}$-invariant as well. Since $H^1_{\mathscr{G},g}(\mathcal{M})$ is exactly the closed subspace of the $\mathscr{G}$-symmetric points of $H^1_g(\mathcal{M})$, from Theorem A.2.2 we obtain

$$0 \in \mathcal{E}'_{\text{aux}}(u^{\mathscr{G}}_\eta) + \partial\psi_{W_\eta}(u^{\mathscr{G}}_\eta) \quad \text{in } (H^1_g(\mathcal{M}))',$$

Consequently, for every $u \in W_\eta(\mathcal{M})$, we have

$$\langle \mathcal{E}'_{\text{aux}}(u^{\mathscr{G}}_\eta), u - u^{\mathscr{G}}_\eta \rangle = \langle \mathcal{E}'_{\text{aux}}(u^{\mathscr{G}}_\eta), u - u^{\mathscr{G}}_\eta \rangle + \psi_{W_\eta}(u) - \psi_{W_\eta}(u^{\mathscr{G}}_\eta) \geq 0,$$

which is exactly what is claimed in Step 1.

Step 2. $u^{\mathscr{G}}_\eta$ *is a solution of* (8.9), *that is,*

$$\begin{cases} \langle u^{\mathscr{G}}_\eta, \varphi\rangle = \int_{\mathcal{M}} w(\sigma)\boldsymbol{f}(u^{\mathscr{G}}_\eta)\varphi\, d\sigma_g & \text{for any } \varphi \in H^1_g(\mathcal{M}), \\ u^{\mathscr{G}}_\eta \in H^1_g(\mathcal{M}). \end{cases}$$

Step 1 ensures that

$$\langle u^{\mathscr{G}}_\eta, u - u^{\mathscr{G}}_\eta \rangle - \int_{\mathcal{M}} w(\sigma)\boldsymbol{f}(u^{\mathscr{G}}_\eta)(u - u^{\mathscr{G}}_\eta)d\sigma_g \geq 0 \tag{8.27}$$

for every $u \in W_\eta(\mathcal{M})$. Let us define the truncation function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \text{sgn}(t)\min(|t|, \eta) \text{ for every } t \in \mathbb{R}.$$

Fix $\varepsilon > 0$ and $\varphi \in H^1_g(\mathcal{M})$ arbitrarily; see Figure 8.5.

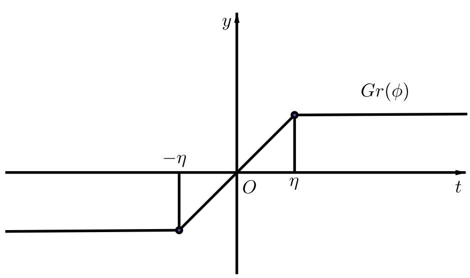

Figure 8.5: The truncation function ϕ.

Since ϕ is Lipschitz continuous, $\mathsf{u}_\phi = \phi \circ (u_\eta^{\mathscr{G}} + \varepsilon\varphi)$ belongs to $H_g^1(\mathcal{M})$, see E. Hebey [122, Proposition 2.5, p. 24]. Set

$$\begin{aligned}
\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\} &= \{\sigma \in \mathcal{M} : u_\eta^{\mathscr{G}}(\sigma) + \varepsilon\varphi(\sigma) < -\eta\},\\
\{|u_\eta^{\mathscr{G}} + \varepsilon\varphi| < \eta\} &= \{\sigma \in \mathcal{M} : |u_\eta^{\mathscr{G}}(\sigma) + \varepsilon\varphi(\sigma)| < \eta\},\\
\{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\} &= \{\sigma \in \mathcal{M} : \eta \le u_\eta^{\mathscr{G}}(\sigma) + \varepsilon\varphi(\sigma)\}.
\end{aligned}$$

The explicit expression of u_ϕ is

$$\mathsf{u}_\phi(\sigma) = \begin{cases} -\eta, & \text{if } \sigma \in \{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\},\\ u_\eta^{\mathscr{G}}(\sigma) + \varepsilon\varphi(\sigma), & \text{if } \sigma \in \{-\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi < \eta\},\\ \eta, & \text{if } \sigma \in \{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}. \end{cases}$$

Therefore, $\mathsf{u}_\phi \in W_\eta(\mathcal{M})$. Taking $u = \mathsf{u}_\phi$ as a test function in (8.27), we easily have

$$\begin{aligned}
0 \le &- \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\}} \{|\nabla u_\eta^{\mathscr{G}}|^2 + u_\eta^{\mathscr{G}}(\eta + u_\eta^{\mathscr{G}}) - w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})(\eta + u_\eta^{\mathscr{G}})\}d\sigma_g\\
&+ \varepsilon \int\limits_{\{|u_\eta^{\mathscr{G}} + \varepsilon\varphi| < \eta\}} \{\langle \nabla u_\eta^{\mathscr{G}}, \nabla\varphi\rangle_g + u_\eta^{\mathscr{G}}\varphi - w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})\varphi\}d\sigma_g\\
&- \int\limits_{\{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}} \{|\nabla u_\eta^{\mathscr{G}}|^2 - u_\eta^{\mathscr{G}}(\eta - u_\eta^{\mathscr{G}}) + w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})(\eta - u_\eta^{\mathscr{G}})\}d\sigma_g.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\varepsilon\Big\{\langle u_\eta^{\mathscr{G}}, \varphi\rangle - \int\limits_{\mathcal{M}} w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})\varphi d\sigma_g\Big\}\\
&\quad\ge \varepsilon \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\} \cup \{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}} \langle \nabla u_\eta^{\mathscr{G}}, \nabla\varphi\rangle_g d\sigma_g\\
&\qquad+ \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\} \cup \{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}} |\nabla u_\eta^{\mathscr{G}}|^2 d\sigma_g\\
&\qquad- \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\}} \{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}}) - u_\eta^{\mathscr{G}}\}(\eta + u_\eta^{\mathscr{G}} + \varepsilon\varphi)d\sigma_g\\
&\qquad- \int\limits_{\{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}} \{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}}) - u_\eta^{\mathscr{G}}\}(-\eta + u_\eta^{\mathscr{G}} + \varepsilon\varphi)d\sigma_g\\
&\quad\ge \varepsilon \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\} \cup \{\eta \le u_\eta^{\mathscr{G}} + \varepsilon\varphi\}} \langle \nabla u_\eta^{\mathscr{G}}, \nabla\varphi\rangle_g d\sigma_g\\
&\qquad- \int\limits_{\{u_\eta^{\mathscr{G}} + \varepsilon\varphi < -\eta\}} \{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}}) - u_\eta^{\mathscr{G}}\}(\eta + u_\eta^{\mathscr{G}} + \varepsilon\varphi)d\sigma_g
\end{aligned}$$

$$-\int_{\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\}}\{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})-u_\eta^{\mathscr{G}}\}(-\eta+u_\eta^{\mathscr{G}}+\varepsilon\varphi)d\sigma_g.$$

Since $u_\eta^{\mathscr{G}}\in[0,\delta]\subset[-\eta,\eta]$ a. e. in $\mathcal{M}$, we have

$$\begin{aligned}\int_{\{u_\eta^{\mathscr{G}}+\varepsilon\varphi<-\eta\}}&\{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})-u_\eta^{\mathscr{G}}\}(\eta+u_\eta^{\mathscr{G}}+\varepsilon\varphi)d\sigma_g\\&\le-\varepsilon\int_{\{u_\eta^{\mathscr{G}}+\varepsilon\varphi<-\eta\}}[M\,w(\sigma)+u_\eta^{\mathscr{G}}]\varphi(\sigma)d\sigma_g\end{aligned}$$

and

$$\int_{\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\}}\{w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})-u_\eta^{\mathscr{G}}\}(-\eta+u_\eta^{\mathscr{G}}+\varepsilon\varphi)d\sigma_g\le\varepsilon M\int_{\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\}}w(\sigma)\varphi d\sigma_g.$$

Using the above estimates and dividing by $\varepsilon>0$, we obtain

$$\begin{aligned}\langle u_\eta^{\mathscr{G}},\varphi\rangle-\int_{\mathcal{M}}w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})\varphi d\sigma_g\ge&\int_{\{u_\eta^{\mathscr{G}}+\varepsilon\varphi<-\eta\}\cup\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\}}\langle\nabla u_\eta^{\mathscr{G}},\nabla\varphi\rangle_g d\sigma_g\\&+\int_{\{u_\eta^{\mathscr{G}}+\varepsilon\varphi<-\eta\}}[M\,w(\sigma)+u_\eta^{\mathscr{G}}]\varphi(\sigma)d\sigma_g\\&-M\int_{\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\}}w(\sigma)\varphi d\sigma_g.\end{aligned}$$

Now, taking into account that $u_\eta^{\mathscr{G}}\in[0,\delta]$ a. e. in $\mathcal{M}$ and $\delta<\eta$ by (8.12), we get

$$\mathrm{Vol}_g(\{u_\eta^{\mathscr{G}}+\varepsilon\varphi<-\eta\})\to0\quad\text{and}\quad\mathrm{Vol}_g(\{\eta\le u_\eta^{\mathscr{G}}+\varepsilon\varphi\})\to0$$

as $\varepsilon\to0^+$. Consequently, the above inequality reduces to

$$\langle\nabla u_\eta^{\mathscr{G}},\nabla\varphi\rangle-\int_{\mathcal{M}}w(\sigma)\boldsymbol{f}(u_\eta^{\mathscr{G}})\varphi d\sigma_g\ge0.$$

Replacing φ by $-\varphi$, we obtain the reverse inequality. Therefore, $u_\eta^{\mathscr{G}}$ is a solution of (8.9). This completes the proof of Step 2 and of the theorem. □

8.3 Competition phenomena for elliptic equations

Let us turn back to (P_λ) and study the number and the behavior of its solutions, when λ is a real parameter, $f:\mathbb{R}_0^+\to\mathbb{R}$ is a continuous function oscillating near the origin or at infinity, and $\mathrm{f} : \mathbb{R}_0^+ \to \mathbb{R}$ is any continuous function, with $\mathrm{f}(0) = 0$. We again assume that the potential $w : \mathcal{M} \to \mathbb{R}$ satisfies condition (w) as before. The analysis is based on variational and topological techniques in addition to the preliminary results presented in Sections 8.1 and 8.2. The main theorems of the section cover two distinct cases, that is, when the nonlinearity f oscillates near the origin or at infinity, respectively.

Oscillation near the origin

In this case we assume the following conditions on f:

$$f \in C(\mathbb{R}_0^+); \tag{8.28}$$

$$\text{There is } (s_k)_k \text{ in } \mathbb{R}^+, \text{with } \lim_{k\to\infty} s_k = 0 \text{ and } f(s_k) < 0 \text{ for any } k \in \mathbb{N}; \tag{8.29}$$

$$-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^2} \le \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty. \tag{8.30}$$

Theorem 8.3.1. *Let $\mathcal{M} = (\mathcal{M}, g)$ be a homogeneous Hadamard manifold of dimension $N \ge 3$ and let $\mathscr{G}$ be a compact connected subgroup of $\mathrm{Isom}_g(\mathcal{M})$ such that $\mathrm{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma_0\}$. Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a function verifying (8.28)–(8.29) and $w : \mathcal{M} \to \mathbb{R}$ a potential satisfying (w). The following facts hold*:

(i_1) *If $\lambda = 0$ there exists a sequence $(u_{0,k}^{\mathscr{G},0})_k \subset H_g^1(\mathcal{M})$ of distinct $\mathscr{G}$-symmetric solutions of (P_0) such that*

$$\lim_{k\to\infty} \|u_{0,k}^{\mathscr{G},0}\| = \lim_{k\to\infty} \|u_{0,k}^{\mathscr{G},0}\|_\infty = 0. \tag{8.31}$$

(i_2) *If $\mathrm{f} \in C(\mathbb{R}_0^+)$, with $\mathrm{f}(0) = 0$, then for every $j \in \mathbb{N}$ there exists $\lambda_j^0 > 0$ such that (P_λ) has at least j distinct $\mathscr{G}$-symmetric solutions in $H_g^1(\mathcal{M})$ whenever $\lambda \in [-\lambda_j^0, \lambda_j^0]$. Moreover, denoted by $(u_{0,i}^{\mathscr{G},\lambda})_{i=1}^j \subset H_g^1(\mathcal{M})$ the j distinct $\mathscr{G}$-symmetric solutions of (P_λ), then*

$$\|u_{0,i}^{\mathscr{G},\lambda}\| \le 1/i \quad \text{and} \quad \|u_{0,i}^{\mathscr{G},\lambda}\|_\infty \le 1/i \quad \text{for every } i = 1, \dots, j, \tag{8.32}$$

provided that $\lambda \in [-\lambda_j^0, \lambda_j^0]$.

Proof. Since the nonlinear terms f and f are continuous, on account of (8.29), there are positive real sequences, $(\delta_k)_k$, $(\eta_k)_k$, and $(\lambda_k)_k$ such that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \eta_k = 0; \tag{8.33}$$

$$\text{for every } k \in \mathbb{N}, \text{ one has } \eta_{k+1} < \delta_k < s_k < \eta_k < 1 \text{ and} \tag{8.34}$$

$$f(t) + \lambda \mathrm{f}(t) \le 0 \text{ for every } t \in [\delta_k, \eta_k] \text{ and } \lambda \in [-\lambda_k, \lambda_k]. \tag{8.35}$$

Bearing in mind the notation given in (8.8), let us consider the real functions f_k, $\mathrm{f}_k : \mathbb{R}_0^+ \to \mathbb{R}$ given by

$$f_k(t) = f(\phi_{\eta_k}(t)) \quad \text{and} \quad \mathrm{f}_k(t) = \mathrm{f}(\phi_{\eta_k}(t)) \tag{8.36}$$

for every $t \in \mathbb{R}_0^+$ and $k \in \mathbb{N}$, where ϕ_{η_k} is defined in (8.8). Now, $f_k(0) = \mathrm{f}_k(0) = 0$ for every $k \in \mathbb{N}$. Indeed, $\mathrm{f}(0) = 0$ by assumption, and (8.29) implies that also $f(0) = 0$. Thus, for every $k \in \mathbb{N}$, we extend continuously the functions f_k and f_k to the whole real line, taking $f_k(t) = \mathrm{f}_k(t) = 0$ for every $t < 0$. Hence, for every $k \in \mathbb{N}$, the explicit expressions of f_k and f_k are

$$f_k(t) = \begin{cases} f(\eta_k) & \text{if } t > \eta_k, \\ f(t) & \text{if } 0 \le t \le \eta_k, \\ 0 & \text{if } t < 0, \end{cases} \quad \text{and} \quad \mathrm{f}_k(t) = \begin{cases} \mathrm{f}(\eta_k) & \text{if } t > \eta_k, \\ \mathrm{f}(t) & \text{if } 0 \le t \le \eta_k, \\ 0 & \text{if } t < 0, \end{cases}$$

see Figure 8.6 below.

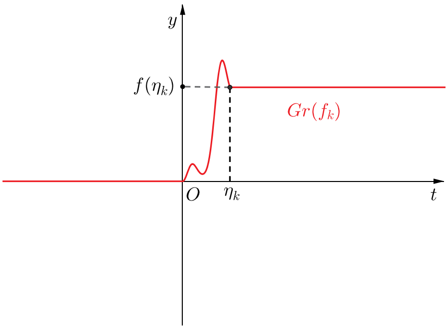

Figure 8.6: The truncation function f_k.

Fix $k \in \mathbb{N}$ and $\lambda \in [-\lambda_k, \lambda_k]$. Let $\boldsymbol{f}_{k,\lambda} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\boldsymbol{f}_{k,\lambda}(t) = f_k(t) + \lambda \mathrm{f}_k(t) \quad \text{and} \quad \boldsymbol{F}_{k,\lambda}(t) = \int_0^t \boldsymbol{f}_{k,\lambda}(s)ds \tag{8.37}$$

for all $t \in \mathbb{R}$. Let us put for simplicity

$$\mathcal{E}_{k,\lambda}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{k,\lambda}(u)\, d\sigma_g \tag{8.38}$$

for all $u \in H_g^1(\mathcal{M})$, cf. (8.14). Clearly, $\mathcal{E}_{k,\lambda}$ is the energy functional associated to (8.9), when $\boldsymbol{f} = \boldsymbol{f}_{k,\lambda}$. The function $\boldsymbol{f}_{k,\lambda}$ verifies all the assumptions of Theorem 8.2.1.

Hence, as a consequence of Theorem 8.2.1, there exists a $\mathscr{G}$-symmetric function $u_{0,k}^{\mathscr{G},\lambda} \in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})$ such that

$$\min_{u \in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k,\lambda}^{\mathscr{G}}(u) = \mathcal{E}_{k,\lambda}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) \quad \text{and} \quad u_{0,k}^{\mathscr{G},\lambda} \in [0, \delta_k] \quad \text{a. e. in } \mathcal{M}, \tag{8.39}$$

where the functional $\mathcal{E}^{\mathscr{G}}_{k,\lambda} : H^1_{\mathscr{G},g}(\mathcal{M}) \to \mathbb{R}$ is given by

$$\mathcal{E}^{\mathscr{G}}_{k,\lambda}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{k,\lambda}(u)\, d\sigma_g, \tag{8.40}$$

and $u^{\mathscr{G},\lambda}_{0,k}$ is a nonnegative solution of

$$\begin{cases} -\Delta_g u + u = w(\sigma)\boldsymbol{f}_{k,\lambda}(u) & \text{in } \mathcal{M}, \\ u \geq 0 \text{ in } \mathcal{M}, \quad u \in H^1_g(\mathcal{M}). \end{cases} \tag{8.41}$$

The definition of ϕ_{η_k}, (8.37), and the fact that $u^{\mathscr{G},\lambda}_{0,k} \leq \delta_k < \eta_k$ a. e. in $\mathcal{M}$ by (8.34) and (8.39) yield

$$\begin{aligned} \boldsymbol{f}_{k,\lambda}(u^{\mathscr{G},\lambda}_{0,k}) &= f(\phi_{\eta_k}(u^{\mathscr{G},\lambda}_{0,k})) + \lambda \mathrm{f}(\phi_{\eta_k}(u^{\mathscr{G},\lambda}_{0,k})) \\ &= f(u^{\mathscr{G},\lambda}_{0,k}) + \lambda \mathrm{f}(u^{\mathscr{G},\lambda}_{0,k}) \quad \text{a. e. in } \mathcal{M}. \end{aligned}$$

Thus, the above relation ensures that $u^{\mathscr{G},\lambda}_{0,k}$ is a nonnegative solution not only of (8.41) but also of (P_λ).

(i_1) Assume $\lambda = 0$. We have to prove that there are infinitely many distinct elements in the sequence $(u^{\mathscr{G},0}_{0,k})_k$ such that (8.31) holds. In order to see this, we first claim that

$$\mathcal{E}^{\mathscr{G}}_{k,0}(u^{\mathscr{G},0}_{0,k}) < 0 \quad \text{for every } k \in \mathbb{N}. \tag{8.42}$$

The right-hand side of (8.30) implies the existence of some $\ell_0 > 0$ and $\varsigma_0 \in (0, \eta_1)$ such that

$$F(t) \geq -\ell_0 t^2 \quad \text{for every } t \in (0, \varsigma_0). \tag{8.43}$$

Set $0 < r < \rho$ and choose $L_0 > 0$ so large that

$$L_0 > \frac{1}{\inf_{\sigma \in A^\rho_r(\sigma_0)} w(\sigma)} \left[\left(\frac{1}{2} + \frac{2}{r^2} \right) + \ell_0 \frac{\|w\|_1}{\mathrm{Vol}_g(A^\rho_r(\sigma_0))} \right], \tag{8.44}$$

where $A^\rho_r(\sigma_0)$ is the annulus type domain given by

$$A^\rho_r(\sigma_0) = \{\sigma \in \mathcal{M} : \rho - r < d_g(\sigma_0, \sigma) < \rho + r\};$$

see Figure 8.7.

Moreover, taking into account the right-hand side of (8.30), there is a sequence $(t_k)_k$ in $(0, \varsigma_0)$ such that $\lim_{k\to\infty} t_k = 0$ and

$$F(t_k) > L_0 t_k^2 \tag{8.45}$$

for $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \delta_k = 0$, we choose a subsequence of $(\delta_k)_k$, still denoted by $(\delta_k)_k$, such that

$$t_k \leq \delta_k \tag{8.46}$$

for every $k \in \mathbb{N}$.

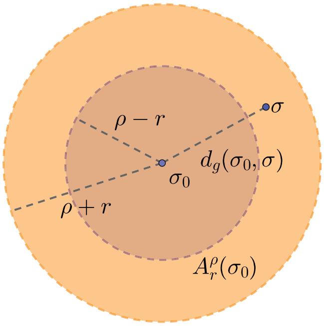

Figure 8.7: The annulus type domain $A_r^\rho(\sigma_0)$.

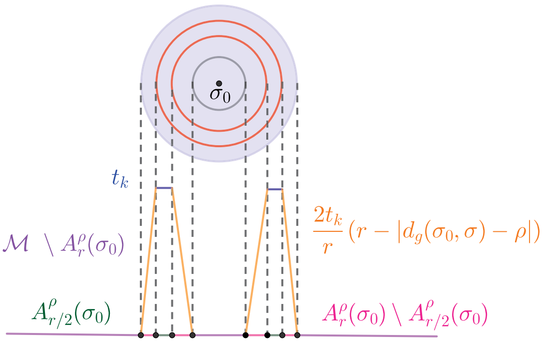

Figure 8.8: The truncation function $\mathrm{u}_{t_k}^{\rho,r}$.

Now, consider the $\mathscr{G}$-invariant function defined in (8.6) with $s = t_k$, that is,

$$\mathrm{u}_{t_k}^{\rho,r}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \mathcal{M} \setminus A_r^\rho(\sigma_0), \\ t_k & \text{if } \sigma \in A_{r/2}^\rho(\sigma_0), \\ \frac{2t_k}{r}(r - |d_g(\sigma_0, \sigma) - \rho|) & \text{if } \sigma \in A_r^\rho(\sigma_0) \setminus A_{r/2}^\rho(\sigma_0), \end{cases} \tag{8.47}$$

for every $\sigma \in \mathcal{M}$; see Figure 8.8. Then, $\mathrm{u}_{t_k}^{\rho,r} \in H^1_{\mathscr{G},g}(\mathcal{M})$ and $\|\mathrm{u}_{t_k}^{\rho,r}\|_\infty = t_k \le \delta_k < \eta_k$ by (8.30) and (8.46). Hence, $\mathrm{u}_{t_k}^{\rho,r} \in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})$ and $0 \le \mathrm{u}_{t_k}^{\rho,r} \le t_k \le \delta_k < \eta_k$ a. e. in $\mathcal{M}$. Consequently,

$$\int_0^{\mathrm{u}_{t_k}^{\rho,r}(\sigma)} \boldsymbol{f}_{k,0}(t)dt = \int_0^{\mathrm{u}_{t_k}^{\rho,r}(\sigma)} f(\phi_{\eta_k}(t))dt = \int_0^{\mathrm{u}_{t_k}^{\rho,r}(\sigma)} f(t)dt$$

for a. e. $\sigma \in \mathcal{M}$. Thus, by (8.7), for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{E}_{k,0}^{\mathscr{G}}(\mathrm{u}_{t_k}^{\rho,r}) &= \frac{1}{2}\|\mathrm{u}_{t_k}^{\rho,r}\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{k,0}(\mathrm{u}_{t_k}^{\rho,r})\, d\sigma_g \\ &= \frac{1}{2}\|\mathrm{u}_{t_k}^{\rho,r}\|^2 - F(t_k) \int_{A_{r/2}^\rho(\sigma_0)} w(\sigma) d\sigma_g \end{aligned}$$

$$- \int_{A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0)} w(\sigma)F(\mathrm{u}_{t_k}^{\rho,r})d\sigma_g$$

$$\leq \left\{\left(\frac{1}{2} + \frac{2}{r^2}\right)\mathrm{Vol}_g(A_r^\rho(\sigma_0)) - L_0\mathrm{Vol}_g(A_r^\rho(\sigma_0)) \inf_{\sigma\in A_r^\rho(\sigma_0)} w(\sigma) + \ell_0\|w\|_1\right\} t_k^2,$$

thanks to (8.43), (8.45), and using the fact that $\mathrm{u}_{t_k}^{\rho,r} < \eta_k < \eta_1$, due to $(\eta_k)_k$ being decreasing by (8.29). Then by (8.44), we get that

$$\mathcal{E}_{k,0}^{\mathcal{G}}(\mathrm{u}_{t_k}^{\rho,r}) < 0 \quad \text{for every } k \in \mathbb{N}.$$

Then, using also (8.39), we obtain for all $k \in \mathbb{N}$,

$$\mathcal{E}_{k,0}^{\mathcal{G}}(u_{0,k}^{\mathcal{G},0}) = \min_{u\in W_{\eta_k}^{\mathcal{G}}(\mathcal{M})} \mathcal{E}_{k,0}^{\mathcal{G}}(u) \leq \mathcal{E}_{k,0}^{\mathcal{G}}(\mathrm{u}_{t_k}^{\rho,r}) < 0, \tag{8.48}$$

which proves (8.42). Inequality (8.48) also guarantees that $u_{0,k}^{\mathcal{G},0} \not\equiv 0$ in $\mathcal{M}$, as $\mathcal{E}_{k,0}^{\mathcal{G}}(0) = 0$.

Now, we claim that

$$\lim_{k\to\infty} \mathcal{E}_{k,0}^{\mathcal{G}}(u_{0,k}^{\mathcal{G},0}) = 0. \tag{8.49}$$

Indeed, for $k \in \mathbb{N}$, the definition of $\boldsymbol{F}_{k,0}$, the fact that $F_k(0) = 0$, (8.39), (8.34), and the mean value theorem give

$$\begin{aligned}\mathcal{E}_{k,0}^{\mathcal{G}}(u_{0,k}^{\mathcal{G},0}) &> -\int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{k,0}(u_{0,k}^{\mathcal{G},0})d\sigma_g \\ &= -\int_{\mathcal{M}} w(\sigma)F_k(u_{0,k}^{\mathcal{G},0})\, d\sigma_g \\ &\geq -\max_{t\in[0,1]}|f(t)| \int_{\mathcal{M}} w(\sigma)|u_{0,k}^{\mathcal{G},0}|d\sigma_g \\ &\geq -\max_{t\in[0,1]}|f(t)|\, \|w\|_1\delta_k.\end{aligned} \tag{8.50}$$

Since $\lim_{k\to\infty} \delta_k = 0$ by (8.33), the above inequality and (8.48) lead to (8.49), and so the claim is proved. Due to (8.36) and (8.39), we notice that

$$\mathcal{E}_{k,0}^{\mathcal{G}}(u_{0,k}^{\mathcal{G},0}) = \mathcal{E}_{1,0}^{\mathcal{G}}(u_{0,k}^{\mathcal{G},0}) \quad \text{for every } k \in \mathbb{N}.$$

Combining the above relation with (8.42) and (8.49), we deduce that the sequence $(u_{0,k}^{\mathcal{G},0})_k$ contains infinitely many distinct elements, that is, (P_λ) has infinitely many distinct $\mathcal{G}$-symmetric solutions.

Finally, it remains to prove relation (8.31). Since $\|u_{0,k}^{\mathcal{G},0}\|_\infty \leq \delta_k$ for $k \in \mathbb{N}$ sufficiently large by (8.39), and $\lim_{k\to\infty} \delta_k = 0$, we easily get that $\lim_{k\to\infty} \|u_{0,k}^{\mathcal{G},0}\|_\infty = 0$.

For the latter limit in (8.31), note that

$$\begin{aligned}\frac{1}{2}\|u_{0,k}^{\mathscr{G},0}\|^2 &< \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{k,0}(u_{0,k}^{\mathscr{G},0})d\sigma_g \\ &= \int_{\mathcal{M}} w(\sigma)F_k(u_{0,k}^{\mathscr{G},0})d\sigma_g \\ &\leq \max_{t\in[0,1]}|f(t)|\,\|w\|_1\delta_k,\end{aligned} \tag{8.51}$$

thanks to (8.28), (8.39), and (8.42).

Thus, by (8.33), inequality (8.51) yields

$$\lim_{k\to\infty}\|u_{0,k}^{\mathscr{G},0}\| = 0,$$

which concludes the proof of (i_1).

(i_2) It remains to prove that for any $j \in \mathbb{N}$ equation (P_λ) admits at least j distinct solutions, namely $(u_{0,i}^{\mathscr{G},\lambda})_{i=1}^{j} \subset H_g^1(\mathcal{M})$, which verify (8.32), provided that λ is suitably small.

Fix $j \in \mathbb{N}$. Let $(\vartheta_k)_k$ be a real sequence such that $\vartheta_k < 0$ and $\lim_{k\to\infty}\vartheta_k = 0$. Up to a subsequence, still denoted by $(\vartheta_k)_k$, we may assume that for all $k \in \mathbb{N}$,

$$\vartheta_k < \mathcal{E}_{k,0}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},0}) \leq \mathcal{E}_{k,0}^{\mathscr{G}}(\mathsf{u}_{t_k}^{\rho,r}) < \vartheta_{k+1}, \tag{8.52}$$

where the function $\mathsf{u}_{t_k}^{\rho,r}$ is given in (8.47) and

$$\delta_k < \frac{1}{k}\min\left\{1, \frac{1}{2k\|w\|_1(\max_{t\in[0,1]}|f(t)| + \max_{t\in[0,1]}|\mathsf{f}(t)|)}\right\}. \tag{8.53}$$

For every $k \in \mathbb{N}$, set

$$\lambda_k' = \frac{\vartheta_{k+1} - \mathcal{E}_{k,0}^{\mathscr{G}}(\mathsf{u}_{t_k}^{\rho,r})}{\|w\|_1(\max_{t\in[0,1]}|f(t)| + 1)} \quad \text{and} \quad \lambda_k'' = \frac{\mathcal{E}_{k,0}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},0}) - \vartheta_k}{\|w\|_1(\max_{t\in[0,1]}|\mathsf{f}(t)| + 1)}. \tag{8.54}$$

Define

$$\lambda_j^0 = \min_{i\in\{1,\ldots,j\}}\{1, \lambda_i, \lambda_i', \lambda_i''\}.$$

Clearly, λ_j^0 is positive on account of (8.52) and (8.54).

From now on we fix $i \in \{1,\ldots,j\}$ and $\lambda \in [-\lambda_j^0, \lambda_j^0]$. We claim that

$$\vartheta_k < \mathcal{E}_{k,\lambda}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) < \vartheta_{k+1}. \tag{8.55}$$

Indeed, by (8.39), the definition of λ_k', and (8.34), it follows that

$$\mathcal{E}_{k,\lambda}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) \leq \mathcal{E}_{k,\lambda}^{\mathscr{G}}(\mathsf{u}_{t_k}^{\rho,r}) = \mathcal{E}_{k,0}^{\mathscr{G}}(\mathsf{u}_{t_k}^{\rho,r}) - \lambda\int_{\mathcal{M}} w(\sigma)\,\mathsf{F}_k(\mathsf{u}_{t_k}^{\rho,r})d\sigma_g < \vartheta_{k+1}. \tag{8.56}$$

On the other hand, since $u_{0,k}^{\mathscr{G},\lambda} \in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})$, bearing in mind the definition of λ_k'' and recalling that $\mathcal{E}_{k,0}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},0}) = \min_{u\in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k,0}^{\mathscr{G}}(u)$ by (8.48), we obtain

$$\begin{aligned}\mathcal{E}_{k,\lambda}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) &= \mathcal{E}_{k,\lambda}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) - \lambda \int_{\mathcal{M}} w(\sigma)\mathsf{F}_k(u_{0,k}^{\mathscr{G},\lambda})\, d\sigma_g \\ &\geq \mathcal{E}_{k,0}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},0}) - \lambda \int_{\mathcal{M}} w(\sigma)\, \mathsf{F}_k(u_{0,k}^{\mathscr{G},\lambda})\, d\sigma_g \\ &> \vartheta_k,\end{aligned} \tag{8.57}$$

on account of (8.34).

Thus, by (8.56) and (8.57), the claim (8.55) is verified. Hence in particular,

$$\mathcal{E}_{1,\lambda}^{\mathscr{G}}(u_{0,1}^{\mathscr{G},\lambda}) < \mathcal{E}_{2,\lambda}^{\mathscr{G}}(u_{0,2}^{\mathscr{G},\lambda}) < \cdots < \mathcal{E}_{j-1,\lambda}^{\mathscr{G}}(u_{0,j-1}^{\mathscr{G},\lambda}) < \mathcal{E}_{j,\lambda}^{\mathscr{G}}(u_{0,j}^{\mathscr{G},\lambda}) \tag{8.58}$$

for all $\lambda \in [-\lambda_j^0, \lambda_j^0]$. Clearly, $u_{0,k}^{\mathscr{G},\lambda} \in W_{\mathscr{G}}^{\eta_1}(\mathcal{M})$, so that

$$\mathcal{E}_{k,0}^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) = \mathcal{E}_1^{\mathscr{G}}(u_{0,k}^{\mathscr{G},\lambda}) \quad \text{for every } i \in \{1, \ldots, j\}. \tag{8.59}$$

Therefore, (8.58) and (8.59) yield

$$\mathcal{E}_{1,\lambda}^{\mathscr{G}}(u_{0,1}^{\mathscr{G},\lambda}) < \mathcal{E}_{1,\lambda}^{\mathscr{G}}(u_{0,2}^{\mathscr{G},\lambda}) < \cdots < \mathcal{E}_{1,\lambda}^{\mathscr{G}}(u_{0,j-1}^{\mathscr{G},\lambda}) < \mathcal{E}_{1,\lambda}^{\mathscr{G}}(u_{0,j}^{\mathscr{G},\lambda}) \tag{8.60}$$

for every $\lambda \in [-\lambda_j^0, \lambda_j^0]$.

Inequalities given in (8.60) ensure that $(u_{0,i}^{\mathscr{G},\lambda})_{i=1}^j \subset H_g^1(\mathcal{M})$ are j distinct $\mathscr{G}$-symmetric solutions of (P_λ).

It remains to show that (8.32) holds. It is clear that (8.39) and (8.53) give

$$\|u_{0,i}^{\mathscr{G},\lambda}\|_\infty < \delta_i < \frac{1}{i}$$

for every $i \in \{1, \ldots, j\}$ and $\lambda \in [-\lambda_j^0, \lambda_j^0]$. In order to prove the second relation in (8.32), let us start by observing that

$$\mathcal{E}_{i,0}^{\mathscr{G}}(u_{0,i}^{\mathscr{G},\lambda}) = \mathcal{E}_1^{\mathscr{G}}(u_{0,i}^{\mathscr{G},\lambda}) < \vartheta_{i+1} < 0 \tag{8.61}$$

for every $i \in \{1, \ldots, j\}$ and $\lambda \in [-\lambda_j^0, \lambda_j^0]$. Due to (8.61), thanks to the mean value theorem, it follows that

$$\begin{aligned}\frac{1}{2}\|u_{0,i}^{\mathscr{G},\lambda}\|^2 &< \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_{i,\lambda}(u_{0,i}^{\mathscr{G},\lambda})d\sigma_g \\ &= \int_{\mathcal{M}} w(\sigma)F_i(u_{0,i}^{\mathscr{G},\lambda})d\sigma_g + \lambda \int_{\mathcal{M}} w(\sigma)\, \mathsf{F}_i(u_{0,i}^{\mathscr{G},\lambda})d\sigma_g \\ &< \frac{1}{2i^2},\end{aligned} \tag{8.62}$$

since $\lambda < \lambda_j^0 \leq \min\{1, \lambda_i\}$ and thanks to (8.34), (8.39), and (8.53). The proof is now complete. □

A model for f in Theorem 8.3.1 is given by

$$f(t) = \begin{cases} \sqrt{t}(\frac{1}{2} + \sin\sqrt{t}) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{8.63}$$

and whose graph is given in Figure 8.9. Note that f is continuous in $\mathbb{R}_0^+$ and

$$F(t) = \begin{cases} \frac{1}{3}t\sqrt{t} + 2(2\sqrt{t}\sin\sqrt{t} + (2-t)\cos\sqrt{t}) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

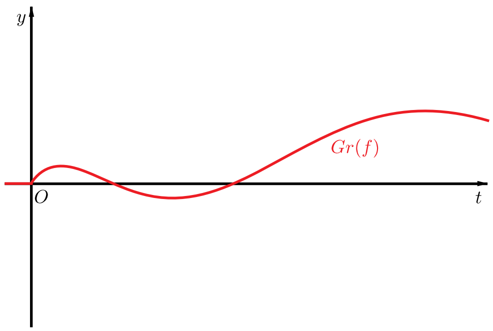

Figure 8.9: The local behavior of f in (8.63) at zero.

Oscillation at infinity

This subsection is devoted to the study of (P_λ) when f oscillates at infinity. In order to prove Theorem 8.3.2 below, we follow more or less the techniques of the previous oscillatory case at 0. However, for completeness, we give all the details.

Precisely, we prove the main result under the following assumptions on the function f:

$$f \in C(\mathbb{R}_0^+) \quad \textit{and} \quad f(0) = 0; \tag{8.64}$$

$$\textit{There is } (s_k)_k \subset \mathbb{R}^+, \textit{with } \lim_{k\to\infty} s_k = \infty \textit{ and } f(s_k) < 0 \textit{ for every } k \in \mathbb{N}; \tag{8.65}$$

$$-\infty < \liminf_{t\to\infty} \frac{F(t)}{t^2} \leq \limsup_{t\to\infty} \frac{F(t)}{t^2} = \infty. \tag{8.66}$$

In this setting, the multiplicity result for (P_λ) is:

Theorem 8.3.2. *Let $\mathcal{M} = (\mathcal{M}, g)$ be a homogeneous Hadamard manifold of dimension $N \geq 3$ and take $\mathcal{G}$ to be a compact connected subgroup of* $\text{Isom}_g(\mathcal{M})$ *such that* $\text{Fix}_{\mathcal{G}}(\mathcal{M}) = \{\sigma_0\}$. *Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a function verifying* (8.64)–(8.66) *and $w : \mathcal{M} \to \mathbb{R}$ is a potential satisfying* (w). *Then, the next facts hold*:

(j_1) *If $\lambda = 0$, there exists a sequence $(u^{\mathscr{G},0}_{\infty,k})_k \subset H^1_g(\mathcal{M})$ of distinct $\mathscr{G}$-symmetric solutions of (P_0) such that*

$$\lim_{k\to\infty} \|u^{\mathscr{G},0}_{\infty,k}\|_\infty = 0. \tag{8.67}$$

(j_2) *If $\mathtt{f} \in C(\mathbb{R}^+_0)$, with $\mathtt{f}(0) = 0$, for every $j \in \mathbb{N}$ there exists $\lambda^\infty_j > 0$ such that for all $\lambda \in [-\lambda^\infty_j, \lambda^\infty_j]$ equation (P_λ) has at least j distinct $\mathscr{G}$-symmetric solutions $(u^{\mathscr{G},\lambda}_{\infty,i})^j_{i=1}$ in $H^1_g(\mathcal{M})$ such that*

$$\|u^{\mathscr{G},\lambda}_{\infty,i}\| > i - 1 \quad \textit{for every } i = 1, \dots, j. \tag{8.68}$$

Proof. The left-hand side of (8.66) ensures that there exist $\ell_\infty > 0$ and $\varsigma_\infty > 0$ such that

$$F(t) \geq -\ell_\infty t^2 \quad \text{for every } \ t \in (\varsigma_\infty, \infty). \tag{8.69}$$

Set $0 < \varrho < \rho$ and choose $L_\infty > 0$ so large that

$$L_\infty > \frac{1}{\inf_{\sigma\in A^\rho_\varrho(\sigma_0)} w(\sigma)} \left[\left(\frac{1}{2} + \frac{2}{\varrho^2} \right) + \ell_\infty \frac{\|w\|_1}{\mathrm{Vol}_g(A^\rho_\varrho(\sigma_0))} \right], \tag{8.70}$$

where $A^\rho_\varrho(\sigma_0)$ is the usual annulus type domain. Taking into account the right-hand side of (8.66), there is a sequence $(t_k)_k \subset \mathbb{R}^+$ such that $\lim_{k\to\infty} t_k = \infty$ and

$$F(t_k) > L_\infty t_k^2 \quad \text{for } k \in \mathbb{N}. \tag{8.71}$$

Since $\lim_{k\to\infty} s_k = \infty$, by (8.65), there is a subsequence $(s_{m_k})_k$ of $(s_k)_k$ such that $t_k \leq s_{m_k}$ for every $k \in \mathbb{N}$. By (8.65) and the continuity of f and $\mathtt{f}$, there are positive real sequences $(\delta_k)_k$, $(\eta_k)_k$, and $(\lambda_k)_k$ such that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \eta_k = \infty; \tag{8.72}$$

$$\text{for every } k \in \mathbb{N} \text{ it results } \delta_k < s_{m_k} < \eta_k < \delta_{k+1} \text{ and} \tag{8.73}$$

$$f(t) + \lambda \mathtt{f}(t) \leq 0 \text{ for every } t \in [\delta_k, \eta_k] \text{ and } \lambda \in [-\lambda_k, \lambda_k]. \tag{8.74}$$

Arguing as in the proof of Theorem 8.3.1, we define f_k, $\mathtt{f}_k : \mathbb{R}^+_0 \to \mathbb{R}$ given for every $k \in \mathbb{N}$ by

$$f_k(t) = f(\phi_{\eta_k}(t)) \quad \text{and} \quad \mathtt{f}_k(t) = \mathtt{f}(\phi_{\eta_k}(t)). \tag{8.75}$$

Clearly, $f_k(0) = \mathtt{f}_k(0) = 0$ for every $k \in \mathbb{N}$, since $f(0) = \mathtt{f}(0) = 0$ by assumptions (8.28) and (j_2). Thus, for every $k \in \mathbb{N}$ we extend continuously the functions f_k and $\mathtt{f}_k$ to the whole real line, putting $f_k(t) = \mathtt{f}_k(t) = 0$ for every $t < 0$.

Fix $k \in \mathbb{N}$ and $\lambda \in [-\lambda_k, \lambda_k]$. Define $\boldsymbol{f}_{k,\lambda} : \mathbb{R}_0^+ \to \mathbb{R}$ by

$$\boldsymbol{f}_{k,\lambda}(t) = f_k(t) + \lambda \mathrm{f}_k(t) \quad \text{and} \quad \boldsymbol{F}_{k,\lambda}(t) = \int_0^t \boldsymbol{f}_{k,\lambda}(s)ds. \tag{8.76}$$

Also, in what follows we denote by $\mathcal{E}_{k,\lambda}^{\mathscr{G}}$ the functional defined in (8.40), with now $\boldsymbol{F}_{k,\lambda}$ given in (8.76). Clearly, $\mathcal{E}_{k,\lambda}$ is the energy functional associated with (8.9), where $\boldsymbol{f} = \boldsymbol{f}_{k,\lambda}$. The function $\boldsymbol{f}_{k,\lambda}$ verifies all the assumptions of Theorem 8.2.1. Hence, Theorem 8.2.1 yields the existence of a $\mathscr{G}$-symmetric function $u_{\infty,k}^{\mathscr{G},\lambda}$ in $W_{\eta_k}^{\mathscr{G}}(\mathcal{M})$ such that

$$\min_{u \in W_{\eta_k}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k,\lambda}^{\mathscr{G}}(u) = \mathcal{E}_k^{\mathscr{G}}(u_{\infty,k}^{\mathscr{G},\lambda}), \quad u_{\infty,k}^{\mathscr{G},\lambda} \in [0, \delta_k] \quad \text{a. e. in } \mathcal{M}, \tag{8.77}$$

and so $u_{\infty,k}^{\mathscr{G},\lambda}$ is a nonnegative solution of

$$\begin{cases} -\Delta_g u + u = w(\sigma)\boldsymbol{f}_{k,\lambda}(u) & \text{in } \mathcal{M}, \\ u \geq 0 \quad \text{in } \mathcal{M}, & u \in H_g^1(\mathcal{M}). \end{cases} \tag{8.78}$$

The definition of ϕ_{η_k}, (8.76), and the fact that $u_{\infty,k}^{\mathscr{G},\lambda} \leq \delta_k < \eta_k$ a. e. in $\mathcal{M}$ by (8.73) and (8.77) yield

$$\boldsymbol{f}_{k,\lambda}(u_{\infty,k}^{\mathscr{G},\lambda}) = f(\phi_{\eta_k}(u_{\infty,k}^{\mathscr{G},\lambda})) + \lambda \mathrm{f}(\phi_{\eta_k}(u_{\infty,k}^{\mathscr{G},\lambda})) = f(u_{\infty,k}^{\mathscr{G},\lambda}) + \lambda \mathrm{f}(u_{\infty,k}^{\mathscr{G},\lambda})$$

a. e. in $\mathcal{M}$. Thus, the above relation and (8.77) ensure that $u_{\infty,k}^{\mathscr{G},\lambda}$ is a nonnegative solution not only of (8.78), but also of (P_λ). We are now in a position to prove the theorem.

(j_1) Assume $\lambda = 0$. We claim that

$$\lim_{k \to \infty} \mathcal{E}_{k,0}^{\mathscr{G}}(u_{\infty,k}^{\mathscr{G},0}) = -\infty. \tag{8.79}$$

Fix $k \in \mathbb{N}$. With the usual notation, take

$$\mathrm{u}_{t_k}^{\rho,r}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \mathcal{M} \setminus A_r^\rho(\sigma_0), \\ t_k & \text{if } \sigma \in A_{r/2}^\rho(\sigma_0), \\ \frac{2t_k}{r}(r - |d_g(\sigma_0, \sigma) - \rho|) & \text{if } \sigma \in A_r^\rho(\sigma_0) \setminus A_{r/2}^\rho(\sigma_0), \end{cases}$$

for a. e. $\sigma \in \mathcal{M}$. Then, $\mathrm{u}_{t_k}^{\rho,r} \in H_{\mathscr{G},g}^1(\mathcal{M})$ and

$$\begin{aligned} \mathcal{E}_{k,0}^{\mathscr{G}}(\mathrm{u}_{t_k}^{\rho,r}) &= \frac{1}{2}\|\mathrm{u}_{t_k}^{\rho,r}\|^2 - \int_{\mathcal{M}} w(\sigma)\boldsymbol{F}_k(\mathrm{u}_{t_k}^{\rho,r})\, d\sigma_g \\ &\leq \left(\frac{1}{2} + \frac{2}{r^2}\right)\mathrm{Vol}_g(A_r^\rho(\sigma_0))t_k^2 - F(t_k) \int_{A_{r/2}^\rho(\sigma_0)} w(\sigma)d\sigma_g \end{aligned}$$

$$- \int_{(A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0))\cap\{\mathrm{u}_{t_k}^{\rho,r}>\varsigma_\infty\}} w(\sigma)F_k(\mathrm{u}_{t_k}^{\rho,r})d\sigma_g$$
$$- \int_{(A_r^\rho(\sigma_0)\setminus A_{r/2}^\rho(\sigma_0))\cap\{\mathrm{u}_{t_k}^{\rho,r}\leq\varsigma_\infty\}} w(\sigma)F_k(\mathrm{u}_{t_k}^{\rho,r})d\sigma_g$$
$$\leq \left\{\left(\frac{1}{2}+\frac{2}{r^2}\right)\mathrm{Vol}_g(A_r^\rho(\sigma_0)) - L_\infty \mathrm{Vol}_g(A_r^\rho(\sigma_0)) \inf_{\sigma\in A_r^\rho(\sigma_0)} w(\sigma) + \ell_\infty\|w\|_1\right\} t_k^2$$
$$+ \|w\|_1 \max_{t\in[0,\varsigma_\infty]}|F(t)|t_k,$$

on account of (8.7), (8.69), and (8.71).

Therefore, since $\lim_{k\to\infty} t_k = \infty$ and (8.70) holds, we have

$$\lim_{k\to\infty} \mathcal{E}_{k,0}^{\mathscr{G}}(\mathrm{u}_{t_k}^{\rho,r}) = -\infty.$$

Recalling that $\mathcal{E}_{k,0}^{\mathscr{G}}(u_{\infty,k}^{\mathscr{G},0}) \leq \mathcal{E}_{k,0}^{\mathscr{G}}(\mathrm{u}_{t_k}^{\rho,r})$ for all k, the claim (8.79) is immediately verified.

Now, (8.79) yields that the sequence $(u_{\infty,k}^{\mathscr{G},0})_k \subset H_g^1(\mathcal{M})$ contains infinitely many distinct solutions of (P_0). Otherwise, the sequence $(u_{\infty,k}^{\mathscr{G},0})_k \subset H_g^1(\mathcal{M})$ contains only a finite number, say k_0, of distinct solutions $\{u_{\infty,k}^{\mathscr{G},0}\}_{k=1}^{k_0}$ of (P_0). This is impossible by (8.79).

Finally, it remains to prove relation (8.67). Arguing again by contradiction, assume that there exists a subsequence $(u_{\infty,k_n}^{\mathscr{G},0})_n$ of $(u_{\infty,k}^{\mathscr{G},0})_k$ such that for some $M > 0$,

$$\|u_{\infty,k_n}^{\mathscr{G},0}\|_\infty \leq M \quad \text{for every } n \quad \text{and} \quad (u_{\infty,k_n}^{\mathscr{G},0})_n \subset W_{\eta_K}^{\mathscr{G}}(\mathcal{M}) \text{ for some } K \in \mathbb{N}.$$

Thus, since $\mathcal{E}_{k_n,0}^{\mathscr{G}} = \mathcal{E}_{K,0}^{\mathscr{G}}$ on $W_{\eta_K}^{\mathscr{G}}(\mathcal{M})$ for every $k_n \geq K$, by (8.73) and (8.75), and for $\eta_{k_n} > \eta_K$, we get

$$\begin{aligned}\mathcal{E}_{K,0}^{\mathscr{G}}(u_{\infty,K}^{\mathscr{G},0}) &= \min_{u\in W_{\eta_K}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{K,0}^{\mathscr{G}}(u) = \min_{u\in W_{\eta_K}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k_n,0}^{\mathscr{G}}(u)\\ &\geq \min_{u\in W_{\eta_{k_n}}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k_n,0}^{\mathscr{G}}(u) = \mathcal{E}_{k_n,0}^{\mathscr{G}}(u_{\infty,k_n}^{\mathscr{G},0})\\ &\geq \min_{u\in W_{\eta_K}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{K,0}^{\mathscr{G}}(u) = \mathcal{E}_{K,0}^{\mathscr{G}}(u_{\infty,K}^{\mathscr{G},0}).\end{aligned}$$

Consequently,

$$\mathcal{E}_{k_n,0}^{\mathscr{G}}(u_{\infty,k_n}^{\mathscr{G},0}) = \mathcal{E}_{K,0}^{\mathscr{G}}(u_{\infty,K}^{\mathscr{G},0}) \quad \text{for every } n \in \mathbb{N}. \tag{8.80}$$

On the other hand, the sequence $(\mathcal{E}_{k,0}^{\mathscr{G}}(u_{\infty,k}^{\mathscr{G},0}))_k$ is nonincreasing. Indeed, by (8.75) and (8.78), it follows that

$$\mathcal{E}_{k+1,0}^{\mathscr{G}}(u_{\infty,k+1}^{\mathscr{G},0}) = \min_{u\in W_{\eta_{k+1}}^{\mathscr{G}}(\mathcal{M})} \mathcal{E}_{k+1,0}^{\mathscr{G}}(u)$$

$$\leq \min_{u \in W^{\mathcal{G}}_{\eta_k}(\mathcal{M})} \mathcal{E}^{\mathcal{G}}_{k+1,0}(u)$$
$$= \min_{u \in W^{\mathcal{G}}_{\eta_k}(\mathcal{M})} \mathcal{E}^{\mathcal{G}}_{k,0}(u)$$
$$= \mathcal{E}^{\mathcal{G}}_{k,0}(w^{\mathcal{G},0}_{\infty,k}).$$

Thus, by (8.80) there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{E}^{\mathcal{G}}_{k,0}(u^{\mathcal{G},0}_{\infty,k}) = \mathcal{E}^{\mathcal{G}}_{K,0}(u^{\mathcal{G},0}_{\infty,K}) \quad \text{for every } k \geq k_0,$$

which clearly contradicts (8.79). This fact concludes the proof of (j_1).

(j_2) Fix $j \in \mathbb{N}$ and λ. Let $(u^{\mathcal{G},\lambda}_{\infty,i})_{i=1}^{j} \subset H^1_g(\mathcal{M})$ be the critical points of $\mathcal{E}^{\mathcal{G}}_{i,\lambda}$, $i = 1, \ldots, j$, constructed in (8.77).

Let $(\vartheta_k)_k$ be a real sequence such that $\vartheta_k < 0$, $\lim_{k\to\infty} \vartheta_k = -\infty$, and

$$\vartheta_{k+1} < \mathcal{E}^{\mathcal{G}}_{k,0}(u^{\mathcal{G},\lambda}_{\infty,k}) \leq \mathcal{E}^{\mathcal{G}}_{k,0}(\mathsf{u}^{\rho,r}_{t_k}) < \vartheta_k \quad \text{and} \quad \delta_k \geq k \tag{8.81}$$

for all k, where the function $\mathsf{u}^{\rho,r}_{t_k}$ is given in (8.47).

Fix $k \in \mathbb{N}$ and set

$$\lambda'_k = \frac{\vartheta_{k+1} - \mathcal{E}^{\mathcal{G}}_{k,0}(\mathsf{u}^{\rho,r}_{t_k})}{\|w\|_1(\max_{t\in[0,\eta_k]} |f(t)| + 1)\eta_k}, \quad \lambda''_k = \frac{\mathcal{E}^{\mathcal{G}}_{k,0}(u^{\mathcal{G},0}_{\infty,k}) - \vartheta_k}{\|w\|_1(\max_{t\in[0,\eta_k]} |\mathsf{f}(t)| + 1)\eta_k}. \tag{8.82}$$

Define

$$\lambda^{\infty}_j = \min_{i\in\{1,\ldots,j\}} \{1, \lambda_i, \lambda'_i, \lambda''_i\}.$$

Clearly λ^{∞}_j is positive by (8.81) and (8.82). Now, for every $i \in \{1, \ldots, j\}$ and $\lambda \in [-\lambda^{\infty}_j, \lambda^{\infty}_j]$, we claim that

$$\vartheta_{i+1} < \mathcal{E}^{\mathcal{G}}_{i,\lambda}(u^{\mathcal{G},\lambda}_{\infty,i}) < \vartheta_i. \tag{8.83}$$

Indeed, by (8.77), the definition of λ'_i, and (8.73), it follows that

$$\mathcal{E}^{\mathcal{G}}_{i,\lambda}(u^{\mathcal{G},\lambda}_{\infty,k}) \leq \mathcal{E}^{\mathcal{G}}_{i,\lambda}(\mathsf{u}^{\rho,r}_{t_i}) = \mathcal{E}^{\mathcal{G}}_{i,0}(\mathsf{u}^{\rho,r}_{t_i}) - \lambda \int_{\mathcal{M}} w(\sigma)\, \mathsf{F}_i(\mathsf{u}^{\rho,r}_{t_i}) d\sigma_g < \vartheta_i, \tag{8.84}$$

since $t_i \leq s_{m_i}$ by construction.

On the other hand, since $u^{\mathcal{G},\lambda}_{0,i} \in W^{\mathcal{G}}_{\eta_i}(\mathcal{M})$, by the definition of λ''_i and by (8.77) when $\lambda = 0$, we have

$$\mathcal{E}^{\mathcal{G}}_{i,\lambda}(u^{\mathcal{G},\lambda}_{\infty,i}) = \mathcal{E}^{\mathcal{G}}_{i,\lambda}(u^{\mathcal{G},\lambda}_{\infty,i}) - \lambda \int_{\mathcal{M}} w(\sigma) \mathsf{F}_i(u^{\mathcal{G},\lambda}_{\infty,i}) d\sigma_g$$

$$\geq \mathcal{E}_{i,0}^{\mathcal{G}}(u_{\infty,i}^{\mathcal{G},0}) - \lambda \int_{\mathcal{M}} w(\sigma)\, \mathsf{F}_i(u_{\infty,i}^{\mathcal{G},\lambda}) d\sigma_g \tag{8.85}$$
$$> \vartheta_{i+1},$$

since again $t_i \leq s_{m_i}$.

Thus, by (8.84) and (8.85), inequality (8.83) is verified for every $i \in \{1, \dots, j\}$ and $\lambda \in [-\lambda_j^\infty, \lambda_j^\infty]$. Hence

$$\mathcal{E}_{k,\lambda}^{\mathcal{G}}(u_{\infty,k}^{\mathcal{G},\lambda}) < \mathcal{E}_{k-1,\lambda}^{\mathcal{G}}(u_{\infty,k-1}^{\mathcal{G},\lambda}) < \cdots < \mathcal{E}_{2,\lambda}^{\mathcal{G}}(u_{\infty,2}^{\mathcal{G},\lambda}) < \mathcal{E}_{1,\lambda}^{\mathcal{G}}(u_{\infty,1}^{\mathcal{G},\lambda}) < 0. \tag{8.86}$$

In particular, $u_{\infty,i}^{\mathcal{G},\lambda} \in W_{\eta_k}^{\mathcal{G}}(\mathcal{M})$ implies that

$$\mathcal{E}_{i,\lambda}^{\mathcal{G}}(u_{\infty,i}^{\mathcal{G},\lambda}) = \mathcal{E}_{1,\lambda}^{\mathcal{G}}(u_{\infty,k}^{\mathcal{G},\lambda}) \ \text{ for every } i \in \{1, \dots, j\}.$$

Therefore, (8.86) gives

$$\mathcal{E}_{j,\lambda}^{\mathcal{G}}(u_{\infty,j}^{\mathcal{G},\lambda}) < \mathcal{E}_{j,\lambda}^{\mathcal{G}}(u_{\infty,j-1}^{\mathcal{G},\lambda}) < \cdots < \mathcal{E}_{j,\lambda}^{\mathcal{G}}(u_{\infty,2}^{\mathcal{G},\lambda}) < \mathcal{E}_{j,\lambda}^{\mathcal{G}}(u_{\infty,1}^{\mathcal{G},\lambda}) < 0 \tag{8.87}$$

for every $\lambda \in [-\lambda_j^\infty, \lambda_j^\infty]$.

Consequently, (8.87) implies that $(u_{0,i}^{\mathcal{G},\lambda})_{i=1}^j \subset H_g^1(\mathcal{M})$ are j distinct $\mathcal{G}$-symmetric solutions of (P_λ).

It remains to show that (8.68) holds. Fix $\lambda \in [-\lambda_j^\infty, \lambda_j^\infty]$. The conclusion is valid for $i = 1$. Indeed, $\mathcal{E}_{1,\lambda}^{\mathcal{G}}(u_{\infty,1}^{\mathcal{G},\lambda}) < \mathcal{E}_{1,\lambda}^{\mathcal{G}}(0) = 0$ gives that $\|u_{\infty,1}^{\mathcal{G},\lambda}\|_\infty > 0$. Let us claim that

$$\|u_{\infty,i}^{\mathcal{G},\lambda}\| > \delta_{i-1} \quad \text{for every } i = 2, \dots, j. \tag{8.88}$$

Otherwise, $\|u_{\infty,i}^{\mathcal{G},\lambda}\| \leq \delta_{i-1}$ for some $i \in \{2, \dots, j\}$. Hence, $u_{\infty,i}^{\mathcal{G},\lambda} \in W_{\eta_{i-1}}^{\mathcal{G}}(\mathcal{M})$, since $\delta_{i-1} < \eta_{i-1}$. Then, (8.75) and (8.77) yield that

$$\mathcal{E}_{i-1,\lambda}^{\mathcal{G}}(u_{\infty,i-1}^{\mathcal{G},\lambda}) = \min_{u \in W_{\mathcal{G}}^{\eta_{i-1}}(\mathcal{M})} \mathcal{E}_{i-1,\lambda}^{\mathcal{G}}(u) \leq \mathcal{E}_{i-1,\lambda}^{\mathcal{G}}(u_{\infty,i}^{\mathcal{G},\lambda}) = \mathcal{E}_{i,\lambda}^{\mathcal{G}}(u_{\infty,i}^{\mathcal{G},\lambda}).$$

This contradicts (8.86), and so the claim (8.88) is proved.

Furthermore, (8.68) follows from (8.88) by (8.81), and the proof is now complete. □

A continuous prototype of f, with oscillations at ∞, is given by

$$f(t) = t^\alpha(\gamma + \sin t^\beta), \quad t \in \mathbb{R}_0^+,$$

where $\alpha > 1$, $\beta > 0$, $\gamma \in (0, 1)$, and $|\alpha - \beta| < 1$. Direct calculations show that f satisfies assumptions (8.64), (8.65), and (8.66) of Theorem 8.3.2.

Comments on Chapter 8

Theorems 8.3.1 and 8.3.2 apply to the N-dimensional Euclidean space endowed by the standard metric under the natural multiplicative action of the orthogonal group $O(N)$. In such a case, Theorems 8.3.1 and 8.3.2 reduce to the results proved in [143]. Moreover, thanks to the famous compact embedding theorems due to P.-L. Lions [160] and W. A. Strauss [236], the main results remain valid in the presence of block-radial symmetries under the natural action of the direct product group $T = \prod_{k=1}^{j} O(N_k)$ on $\mathbb{R}^N$, whenever $\sum_{k=1}^{j} N_k = N$ and $N_k \geq 2$ for every $k = 1, \ldots, j$, with $j \geq 1$. In this framework, Theorems 8.3.1 and 8.3.2 ensure the existence of T-symmetric solutions for problem (P_λ) settled in the Euclidean case, provided that the real parameter λ is sufficiently small. Furthermore, an interesting open problem is to consider equations on unbounded domains of a complete noncompact Hadamard manifold $\mathcal{M}$ involving singular and critical Sobolev nolinearities; see [191, 193] for related topics. After settling the compactness issue by means of a group-theoretical argument as in Theorems 8.3.1 and 8.3.2, the main perspective is to apply minimization arguments on a Nehari manifold decomposition to establish the existence and multiplicity of solutions. In order to handle this kind of problems, the fibering method introduced by S. I. Pohozaev in the seminal papers [205, 206] seems to be essential.

9 Kirchhoff problems on the Poincaré ball model

Lo duca e io per quel cammino ascoso
intrammo a ritornar nel chiaro mondo;
e sanza cura aver d'alcun riposo,
salimmo sù, el primo e io secondo,
tanto ch'i' vidi de le cose belle
che porta 'l ciel, per un pertugio tondo.
E quindi uscimmo a riveder le stelle

Dante
Inferno XXXIV, 133–139

Hyperbolic geometry was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry. In this theory there are different models for the hyperbolic space H^N. Each representation has its own metric, geodesics, isometries, and related properties. In order to understand the relationships among these models, it is helpful to know the geometric properties of the connecting maps. Two of them are the central projection and the stereographic projection from a sphere to a plane; see, e. g., the monograph [37] and references therein.

Recently, some eigenvalue problems in the hyperbolic space framework have been studied; see, for instance, the papers [111, 167, 182], as well as [218, 233, 234]. Motivated by this wide interest in the current literature, in this chapter we deal with the following Kirchhoff problem:

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)\Delta_H u = \lambda w(\sigma) f(u) \quad \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N), \end{cases} \tag{K_λ}$$

settled on the Poincaré ball model $\mathbb{B}^N$, with dimension $N \geq 3$, which is a noncompact manifold of infinite Riemannian measure. The real parameters λ, a, and b are positive, Δ_H denotes the Laplace–Beltrami operator on $\mathbb{B}^N$. The potential w is nontrivial nonnegative of class $L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ and radially symmetric, f is a continuous function. The main results of the chapter may be seen as an extension of multiplicity theorems for different nonlinear elliptic problems obtained in [182, 187].

9.1 The Poincaré ball model $\mathbb{B}^N$

In this section we briefly recall some notions of the hyperbolic geometry needed in the sequel and then illustrate the functional framework we shall use. We refer the reader to the book [37] for detailed derivations of the geometric quantities, their motivations, and further applications. As it is well known, there are several models for the hyperbolic space H^N, for instance, the Poincaré ball $\mathbb{B}^N$. In particular, the Poincaré disk,

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also called the conformal ball, is a model of two-dimensional hyperbolic geometry in which the points of the geometry are inside the unit disk, and the straight lines consist of all segments of circles contained within the disk that are orthogonal to the boundary of the disk, plus all diameters of the disk; see Figure 9.1.

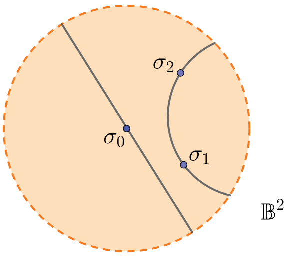

Figure 9.1: The Poincaré disk model.

To be specific, let us set

$$\mathbb{B}^N = \{\sigma = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \;:\; |\sigma| < 1\},$$

endowed with the Riemannian metric given by

$$g_{ij} = \frac{4}{(1 - |\sigma|^2)^2}\delta_{ij}, \quad \sigma \in \mathbb{B}^N,\ i, j = 1, \dots, N,$$

where $|\cdot|$ and δ_{ij} denote the Euclidean distance and the usual Kronecker delta function, respectively. For every $i, j = 1, \dots, N$, set

$$g^{ij} = (g_{ij})^{-1} \quad \text{and} \quad \mathrm{g} = \det(g_{ij}).$$

In this setting, the Laplace–Beltrami operator Δ_H is locally defined as follows:

$$\Delta_H = \frac{1}{\sqrt{\mathrm{g}}} \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sqrt{\mathrm{g}} \sum_{j=1}^N g^{ij} \frac{\partial}{\partial x_j}\right).$$

Now, as usual, let

$$d\mu = \sqrt{\mathrm{g}}dx = \frac{2^N}{(1 - |\sigma|^2)^N} dx$$

be the Riemannian volume element in $\mathbb{B}^N$, where dx denotes the standard Lebesgue measure in the Euclidean space $\mathbb{R}^N$. Hence, if

$$d_H(\sigma) = 2\int_0^{|\sigma|} \frac{dt}{1 - t^2} = \log \frac{1 + |\sigma|}{1 - |\sigma|} \tag{9.1}$$

denotes the geodesic distance of $\sigma \in \mathbb{B}^N$ from the origin $\sigma_0 \in \mathbb{B}^N$, a direct computation ensures that the operator Δ_H has the more convenient form

$$\Delta_H = \frac{1}{4}(1-|\sigma|^2)^2 \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{N-2}{2}(1-|\sigma|^2) \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}.$$

Finally, if (ϱ, θ) are the polar geodesic coordinates in $\mathbb{B}^N \setminus \{0\}$, then in $\mathbb{B}^N \setminus \{0\}$,

$$ds^2 = d\varrho^2 + (\sinh \varrho)^2 d\theta$$

and

$$\Delta_H = \frac{\partial^2}{\partial \varrho^2} + (N-1)\coth \varrho \frac{\partial}{\partial \varrho} + \frac{1}{(\sinh \varrho)^2}\Delta_\theta,$$

where Δ_θ is the Laplace–Beltrami operator on the sphere $\mathbb{S}^{N-1} \hookrightarrow \mathbb{R}^N$.

The hyperbolic distance in the Poincaré ball is given by the formula

$$d_H(\sigma_1, \sigma_2) = \operatorname{arccosh}\left(1 + \frac{2|\sigma_2 - \sigma_1|^2}{(1-|\sigma_1|^2)(1-|\sigma_2|^2)}\right),$$

for every $\sigma_1, \sigma_2 \in \mathbb{B}^N$.

For any $r \in (0,1)$, let us denote by

$$B_r = \{x \in \mathbb{R}^N \,:\, |x| < r\}$$

the open Euclidean ball of radius r centered at $0 \in \mathbb{R}^N$, while

$$\mathbb{B}_\rho = \{\sigma \in \mathbb{B}^N \,:\, d_H(\sigma) < \rho\}$$

means the geodesic ball of radius $\rho > 0$ centered at $\sigma_0 \in \mathbb{B}^N$. Hence,

$$B_r = \mathbb{B}_{\rho(r)}, \quad \rho(r) = \log \frac{1+r}{1-r}.$$

See [218] for additional comments and related facts.

Let $T_\sigma(\mathbb{B}^N)$ be the tangent space at $\sigma \in \mathbb{B}^N$ endowed by the inner product $\langle \cdot, \cdot \rangle_\sigma$ and by $T(\mathbb{B}^N) = \bigcup_{\sigma \in \mathbb{B}^N} T_\sigma(\mathbb{B}^N)$ the tangent bundle on $\mathbb{B}^N$. When no confusion arises, if $X, Y \in T_\sigma(\mathbb{B}^N)$, we simply write $|X|$ and $\langle X, Y\rangle$ instead of the norm $|X|_\sigma$ and inner product $\langle X, Y\rangle_\sigma$, respectively. Let $C_c^\infty(\mathbb{B}^N)$ be the space of real-valued compactly supported smooth functions on $\mathbb{B}^N$. The space $H^1(\mathbb{B}^N)$ is defined to be the completion of $C_c^\infty(\mathbb{B}^N)$ with respect to the Hilbertian norm

$$\begin{aligned} \langle \varphi, \phi \rangle_{H^1} &= \int_{\mathbb{B}^N} (\langle \nabla_H \varphi, \nabla_H \phi \rangle + \varphi\phi)\, d\mu, \\ \|\varphi\|_{H^1} &= \Big(\int_{\mathbb{B}^N} (|\nabla_H \varphi|^2 + |\varphi|^2)\, d\mu \Big)^{1/2}, \end{aligned} \tag{9.2}$$

for every φ, $\phi \in C_c^\infty(\mathbb{B}^N)$, where ∇_H is the covariant derivative and $d\mu$ is the Riemannian measure on $\mathbb{B}^N$. In a direct form,

$$\nabla_H = \left(\frac{1-|\sigma|^2}{2}\right)^2 \nabla \quad \text{and} \quad |\nabla_H u| = \left(\frac{1-|\sigma|^2}{2}\right)^2 |\nabla u|,$$

where ∇ denotes the Euclidean gradient.

As in Chapter 8, referring again to D. Hoffman and J. Spruck [130], the Sobolev embedding $H^1(\mathbb{B}^N) \hookrightarrow L^\wp(\mathbb{B}^N)$ is continuous for every $\wp \in [2, 2^*]$, but not compact. As a biproduct of the results contained in [69], the bottom of the spectrum of $-\Delta_H$ in $\mathbb{B}^N$ is given by

$$\lambda_1 = \lambda_1(-\Delta_H) = \inf_{u \in H^1(\mathbb{B}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} = \frac{(N-1)^2}{4}, \tag{9.3}$$

see also [167] for related topics and direct applications. Consequently, from now on we endow $H^1(\mathbb{B}^N)$ with the equivalent Hilbertian norm

$$\|u\| = \left(\int_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)^{1/2}.$$

The following result will be crucial in the sequel.

Proposition 9.1.1. *Let $\varrho : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and $u \in H^1(\mathbb{B}^N)$. If $\varrho \circ u \in L^2(\mathbb{B}^N)$, then $\varrho \circ u \in H^1(\mathbb{B}^N)$ and*

$$\left|\nabla_H(\varrho \circ u)\right| = \left|\varrho'(u)\right| \cdot |\nabla_H u| \quad a.e. \text{ in } \mathbb{B}^N.$$

In particular, $|u| \in H^1(\mathbb{B}^N)$ and $|\nabla_H |u||$ a.e. in $\mathbb{B}^N$ for all $u \in H^1(\mathbb{B}^N)$.

Proposition 9.1.1 is a corollary of [123, Proposition 2.5, p. 24], and we refer again to [123] for its proof. The result is valid even when $\mathbb{B}^N$ is replaced by a smooth complete Riemannian manifold.

Since (K_λ) is settled in the entire noncompact space $\mathbb{B}^N$, in the next section we shall adopt a group-theoretical approach to identify suitable symmetric subspaces of $H^1(\mathbb{B}^N)$ for which the compactness of the embedding into $L^\wp(\mathbb{B}^N)$, $\wp \in (2, 2^*)$, can be regained.

Let $N \geq 3$ and define the family of subgroups of the special orthogonal group $SO(N)$ given by

$$\begin{aligned} \mathscr{F} = \Big\{ \mathscr{G} \subseteq SO(N) \,:\, \mathscr{G} = \prod_{j=1}^{\ell} SO(N_j),\ \ell \in \mathbb{N}, \\ \text{with } N_j \geq 2,\ j = 1, \dots, \ell, \text{ and } \sum_{j=1}^{\ell} N_j = N \Big\}, \end{aligned} \tag{9.4}$$

where $SO(N_j)$ denotes here the special orthogonal group in dimension N_j for every $j = 1, \dots, \ell$. Now, take $\mathscr{G} \in \mathscr{F}$ and let $\cdot : \mathscr{G} \times \mathbb{B}^N \to \mathbb{B}^N$ be the natural multiplicative action of the group $\mathscr{G}$ on $\mathbb{B}^N$. Fix $\mathscr{G} \in \mathscr{F}$, where $\mathscr{F}$ is the family defined in (9.4). The action $\circledast_{\mathscr{G}} : \mathscr{G} \times H^1(\mathbb{B}^N) \to H^1(\mathbb{B}^N)$ of the subgroup $\mathscr{G}$ on $H^1(\mathbb{B}^N)$ is given, as usual, by

$$g \circledast_{\mathscr{G}} u(\sigma) = u(g^{-1}\sigma) \quad \text{for a. e. } \sigma \in \mathbb{B}^N, \tag{9.5}$$

for every $g \in \mathscr{G}$ and $u \in H^1(\mathbb{B}^N)$. Denote by

$$H^1_{\mathscr{G}}(\mathbb{B}^N) = \{u \in H^1(\mathbb{B}^N) \,:\, g \circledast_{\mathscr{G}} u = u \text{ for every } g \in \mathscr{G}\}$$

the subspace of $\mathscr{G}$-invariant functions of $H^1(\mathbb{B}^N)$.

The next compact embedding result is a particular case of Theorem 8.1.1 given in Chapter 8.

Theorem 9.1.2. *Let $\mathbb{B}^N$ be the N-dimensional homogeneous Poincaré ball. Then the embedding $H^1(\mathbb{B}^N) \hookrightarrow L^{\wp}(\mathbb{B}^N)$ is continuous for every $\wp \in [2, 2^*]$. If $\mathscr{G} \in \mathscr{F}$, then the embedding $H^1_{\mathscr{G}}(\mathbb{B}^N) \hookrightarrow\hookrightarrow L^{\wp}(\mathbb{B}^N)$ is compact for any $\wp \in (2, 2^*)$.*

Now, let us introduce the notations in the hyperbolic variational setting in order to apply the symmetric criticality principle, Theorem A.1.2. For details and comments, we refer to the Appendix, as well as [78] and [63, Section 5]. See also [183, 208] for related topics and results.

A group $\mathscr{H} = (\mathscr{H}, *)$ acts continuously on the real Hilbert space $H^1(\mathbb{B}^N)$ by an application $(\tau, u) \mapsto \tau \circledast_{\mathscr{H}} u$ from $\mathscr{H} \times H^1(\mathbb{B}^N)$ to $H^1(\mathbb{B}^N)$ if $\circledast_{\mathscr{H}}$ *is continuous on* $\mathscr{H} \times H^1(\mathbb{B}^N)$ *and satisfies for every* $u \in H^1(\mathbb{B}^N)$,

(i_1) $\mathrm{id}_{\mathscr{H}} \circledast_{\mathscr{H}} u = u$;

(i_2) $(\tau_1 * \tau_2) \circledast_{\mathscr{H}} u = \tau_1 \circledast_{\mathscr{H}} (\tau_2 \circledast_{\mathscr{H}} u)$ *for every* $\tau_1, \tau_2 \in \mathscr{H}$;

(i_3) $u \mapsto \tau \circledast_{\mathscr{H}} u$ *is linear for every* $\tau \in \mathscr{H}$.

According to the above definition, a group $\mathscr{G} \in \mathscr{F}$ acts continuously on the Hilbert Sobolev space $H^1(\mathbb{B}^N)$ via the map $\circledast_{\mathscr{G}}$ given in (9.5). Finally, as is customary, set

$$\mathrm{Fix}_{\mathscr{G}}(H^1(\mathbb{B}^N)) = \{u \in H^1(\mathbb{B}^N) : \tau \circledast_{\mathscr{G}} u = u \text{ for every } \tau \in \mathscr{G}\}.$$

A functional $I : H^1(\mathbb{B}^N) \to \mathbb{R}$ is said to be $\mathscr{G}$-invariant if

$$I(\tau \circledast_{\mathscr{G}} u) = I(u)$$

for every $u \in H^1(\mathbb{B}^N)$ and $g \in \mathscr{G}$.

9.2 Existence results

Lately, E. Hebey in [126] investigated the existence and compactness properties of the problem

$$\begin{cases} -\left(a + b \int\limits_{\mathcal{M}} |\nabla_g u|^2 d\sigma_g\right)^{\theta_0} \Delta_g u + h(\sigma)u = u^{q-1} \text{ in } \mathcal{M}, \\ u \geq 0 \text{ in } \mathcal{M}, \end{cases} \tag{9.6}$$

where $\mathcal{M} = (\mathcal{M}, g)$ is an N-dimensional, $N \geq 3$, compact manifold, a, b, and θ_0 are positive real parameters, $h \in C^1(\mathcal{M})$, $q \in (2, 2^*)$, and 2^* is the critical Sobolev exponent.

In particular, to prove existence of at least one C^2 solution of (9.6), in [126] E. Hebey required the coercivity of the operator

$$\mathcal{L} = -\Delta_g + \frac{h}{a^{\theta_0}},$$

in addition to the technical assumption $q \neq 2(1 + \theta_0)$. Since the manifold $\mathcal{M}$ is compact and without boundary, the constant functions can be used in order to show the mountain pass geometry of the associated energy functional

$$I_q(u) = \frac{1}{2(1+\theta_0)b}\left(a + b \int\limits_{\mathcal{M}} |\nabla_g u|^2 d\sigma_g\right)^{1+\theta_0} + \frac{1}{2}\int\limits_{\mathcal{M}} h(\sigma)u^2 d\sigma_g - \frac{1}{q}\int\limits_{\mathcal{M}} u_+^q d\sigma_g,$$

where $u_+ = \max\{0, u\}$. Moreover, due to the subcritical assumption on the forcing nonlinear term, standard arguments ensure the validity of the Palais–Smale compactness condition. The classical Ambrosetti–Rabinowitz theorem yields the existence of at least one nontrivial solution; see also [124, 125, 128] for Kirchhoff problems on Riemannian manifolds involving a critical term.

Equation (9.6) is a reasonable generalization of the most studied subcritical elliptic problems, which naturally arise in different branches of mathematics.

Motivated by this wide interest in the current literature, the purpose of the present chapter is to study the existence of solutions for a stationary Kirchhoff equation set on the Poincaré ball model $\mathbb{B}^N$.

A solution of (K_λ) is any function $u \in H^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ such that

$$\left(a + b \int\limits_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right) \int\limits_{\mathbb{B}^N} \langle \nabla_H u, \nabla_H \varphi\rangle_\sigma d\mu - \lambda \int\limits_{\mathbb{B}^N} w(\sigma) f(u)\varphi d\mu = 0$$

for every $\varphi \in H^1(\mathbb{B}^N)$.

Throughout the chapter we assume the following condition on the weight w:

(w) $w \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ *is a nontrivial nonnegative function, which is radially symmetric with respect to the origin* $\sigma_0 \in \mathbb{B}^N$.

In this section we present the main result for the introductory equation (K_λ) and some further generalizations.

Theorem 9.2.1. *Let w satisfy* (w) *and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function verifying*

(f_1) *There are two real sequences* $(\xi_k)_k$ *and* $(\zeta_k)_k$ *such that*

$$\lim_{k\to\infty} \zeta_k = 0, \quad 0 \leq \xi_k < \zeta_k, \quad F(\xi_k) = \sup_{t\in[\xi_k,\zeta_k]} F(t)$$

for every $k \in \mathbb{N}$;

(f_2) *There exist a constant* $M > 0$ *and a sequence* $(\eta_k)_k \subset \mathbb{R}^+$, *with*

$$\lim_{k\to\infty} \eta_k = 0, \quad \lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty \quad \text{and} \quad \inf_{t\in[0,\eta_k]} F(t) \geq -MF(\eta_k)$$

for every $k \in \mathbb{N}$.

Then, for every $\mathscr{G} \in \mathscr{F}$, *where* $\mathscr{F}$ *is defined in* (9.4), *and for each* $\lambda > 0$, *there exists a sequence* $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ *of nontrivial nonnegative* $\mathscr{G}$*-invariant solutions of* (K_λ) *such that*

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0.$$

The approach of the proof of Theorem 9.2.1 is based on variational techniques. In the sequel, we shall describe it briefly. More precisely, it is well known that $H^1(\mathbb{B}^N)$ is not compactly embedded into $L^{\wp}(\mathbb{B}^N)$, $\wp \in (2, 2^*)$, due to the unboundedness of the hyperbolic space. However, by a Lions-type result, the fixed point space of $H^1(\mathbb{B}^N)$ under the action of $\mathscr{G} \in \mathscr{F}$, denoted by $H^1_{\mathscr{G}}(\mathbb{B}^N)$, is compactly embedded into $L^{\wp}(\mathbb{B}^N)$ when $\wp \in (2, 2^*)$; see L. Skrzypczak and C. Tintarev [232].

Instead of (K_λ), we study the auxiliary variational equation (9.11) whose solutions also solve the original equation (K_λ) in the weak sense. If I_λ is the C^1 energy functional associated to (9.11), thanks to a compactness result in [232], the restriction of I_λ to $H^1_{\mathscr{G}}(\mathbb{B}^N)$, denoted by $I_{\mathscr{G},\lambda}$, is weakly sequentially lower semicontinuous in $H^1_{\mathscr{G}}(\mathbb{B}^N)$ and the critical points of $I_{\mathscr{G},\lambda}$ are critical points of I_λ in $H^1(\mathbb{B}^N)$ as well, due to the principle of symmetric criticality, Theorem A.1.2.

The crucial step in the proof argument is the construction of an appropriate sequence of weakly closed subsets $(C_k^{\mathscr{G}})_k$ of $H^1_{\mathscr{G}}(\mathbb{B}^N)$, so that the constrained local minima of $I_{\mathscr{G},\lambda}$ on each $C_k^{\mathscr{G}}$ are actually local minima of $I_{\mathscr{G},\lambda}$ on $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Hence, the constrained critical points are $\mathscr{G}$-invariant solutions of (K_λ). Subsequently, a suitable subsequence of critical points of $I_{\mathscr{G},\lambda}$ can be extracted from the aforementioned constrained local minima and satisfy (9.27). We emphasize that the crucial step described above can be achieved thanks to the continuity of the superposition operator due to M. Marcus and V. Mizel [171, Theorem 1, p. 219] settled in the hyperbolic context instead of the classical Euclidean framework; see also [123, Proposition 2.5, p. 24] for additional comments and remarks in the Riemannian framework.

The main Theorem 9.2.1 complements some results obtained on bounded Euclidean domains, where the elliptic problems with oscillatory nonlinearities have been considered. For instance, among others, Dirichlet problems were studied by G. Anello and G. Cordaro in [16] and G. Molica Bisci and P. Pizzimenti in [184], while Neumann problems have been considered again by G. Anello and G. Cordaro in [17]. We point out that some almost straightforward computations in [16, 17] are adapted here to the hyperbolic setting. Anyway, due to the noncompact framework, the abstract procedure, as well as the setting of Theorem 9.2.1, is different from the results contained in [16, 17], where elliptic problems on bounded smooth domains have been studied.

Furthermore, in [200] P. Omari and F. Zanolin prove the existence of infinitely many solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{9.7}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, with a smooth boundary $\partial\Omega$, under the main assumption

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = 0 \quad \text{and} \quad \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty; \tag{9.8}$$

see also [171, 198, 199, 201] for related topics. Most papers treat odd nonlinearities f in order to apply different variants of the classical Lusternik–Schnirelmann theory. Only a few papers deal with not necessarily odd nonlinearities. Among them, let us cite [16, 200, 226, 228], which are more related to the treatment of the current chapter. We refer the interested reader to the results due to G. Molica Bisci in [181], where a similar multiplicity property has been established for equations settled on the Euclidean sphere $\mathbb{S}^N \hookrightarrow \mathbb{R}^{N+1}$, endowed with the Euclidean induced metric; see Chapter 6.

The Laplacian case was also studied using different methods, and the existence of infinitely many solutions, with the property that the L^2-norm of their gradient go to infinity, was proved by O. Kavian in [135] and M. Struwe in [237, 238]; see also the classical book of P. Rabinowitz [220].

The noncompact hyperbolic setting presents additional difficulties with respect to the aforementioned cases, and suitable geometrical and algebraic tools need to be exploited in order to get the main results. For instance, a crucial ingredient used along the proof of Theorem 9.2.1 is based on a careful analysis of the energy level on $(C_k^{\mathscr{G}})_k$ of some $\mathscr{G}$-invariant functions $v_{\rho,r}^{\varepsilon} \in H^1(\mathbb{B}^N)$ given in (9.22).

Proof of Theorem 9.2.1. Fix $\lambda > 0$ and $t_0 > 0$. Since f is continuous, there exists $\kappa > 0$ such that $|f| \leq \kappa$ in $[0, t_0]$. Moreover, (f_1) and (f_2) yield that $f(0) = 0$. Indeed, by (f_1), the function F attains its maximum in $[\xi_k, \zeta_k]$ at the point ξ_k. Then

$$\lim_{t\to 0^+} \frac{1}{t} \int_{\xi_k}^{\xi_k+t} f(s)ds = f(\xi_k) \leq 0.$$

Hence $\lim_{k\to\infty} f(\xi_k) = f(0) \le 0$, since f is continuous. On the other hand,

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty, \tag{9.9}$$

where $(\eta_k)_k$ is the sequence given in (f_2). We claim that $f(0) \ge 0$. Otherwise, $f(0) < 0$ and by continuity $f < 0$ on some interval $(0, \delta)$, $\delta > 0$. Consequently, $F < 0$ in $(0, \delta)$. Now $\eta_k \to 0^+$ as $k \to \infty$ by (f_2), so that

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} \le 0.$$

Clearly, this contradicts (9.9). In conclusion, the claim is proved, and so $f(0) = 0$.

Without loss of generality, suppose that $\max\{\eta_k, \zeta_k\} \le t_0$ for every $k \in \mathbb{N}$, and define the truncated (continuous) function $\mathtt{f} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\mathtt{f}(t) = \begin{cases} f(t_0) & \text{if } t > t_0, \\ f(t) & \text{if } 0 \le t \le t_0, \\ 0 & \text{if } t < 0. \end{cases}$$

Thanks to (f_1)–(f_2) and (w), the energy functional $J_\lambda : H^1(\mathbb{B}^N) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{\lambda}\left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4\right) - \int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{u(\sigma)} \mathtt{f}(t)dt\right)d\mu \tag{9.10}$$

is well defined and of class $C^1(H^1(\mathbb{B}^N))$ by the continuous embedding result, Theorem 9.1.2. Let us consider the auxiliary equation

$$\begin{cases} -\left(a + b\int_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)\Delta_H u = \lambda w(\sigma)\mathtt{f}(u) & \text{in } \mathbb{B}^N, \\ u \in H^1_{\mathscr{G}}(\mathbb{B}^N). \end{cases} \tag{9.11}$$

Set for all $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$,

$$J_{\mathscr{G},\lambda}(u) = \frac{1}{\lambda}\Phi(u) - \Psi(u), \tag{9.12}$$

where

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 \quad \text{and} \quad \Psi(u) = \int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{u(\sigma)} \mathtt{f}(t)dt\right)d\mu.$$

Let us fix $q \in (2, 2^*)$. Since $\mathtt{f}$ is bounded, $w \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$, and the embedding of $H^1_{\mathscr{G}}(\mathbb{B}^N)$ into $L^q(\mathbb{B}^N)$ is compact, then $J_{\mathscr{G},\lambda}$ is well defined, sequentially weakly lower semicontinuous, and continuously Gâteaux differentiable in the Sobolev space $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Hence, the solutions of (9.11) are exactly the critical points of the C^1 functional $J_{\mathscr{G},\lambda}$. Indeed, a solution of (9.11) is any function $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$ such that

$$\left(a + b\int_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)\int_{\mathbb{B}^N} \langle \nabla_H u, \nabla_H \varphi\rangle d\mu - \lambda \int_{\mathbb{B}^N} w(\sigma)\mathsf{f}(u)\varphi d\mu = 0$$

for every $\varphi \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. Fix $k \in \mathbb{N}$ and define

$$C_k^{\mathscr{G}} = \{u \in H^1_{\mathscr{G}}(\mathbb{B}^N) : 0 \le u \le \zeta_k \text{ a.e. in } \mathbb{B}^N\}.$$

Step 1. The functional $J_{\mathscr{G},\lambda}$ is bounded from below on $C_k^{\mathscr{G}}$ and its infimum on $C_k^{\mathscr{G}}$ is attained at some $u_k^{\mathscr{G}} \in C_k^{\mathscr{G}}$.

Since $C_k^{\mathscr{G}}$ is closed and convex in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, we have that $C_k^{\mathscr{G}}$ is weakly closed in $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Moreover, for every $u \in C_k^{\mathscr{G}}$,

$$J_{\mathscr{G},\lambda}(u) \ge -\Psi(u) \ge -\kappa\|w\|_1\zeta_k, \tag{9.13}$$

since in $C_k^{\mathscr{G}}$ one has

$$\Psi(u) \le \int_{\mathbb{B}^N} w(\sigma)\left|\int_0^{u(\sigma)} \mathsf{f}(t)dt\right| d\mu \le \kappa \int_{\mathbb{B}^N} w(\sigma)u d\mu \le \kappa\|w\|_1\zeta_k.$$

Let $m_k^{\mathscr{G}} = \inf_{u\in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u)$. For every $j \in \mathbb{N}$, there exists $v_j \in C_k^{\mathscr{G}}$ such that

$$m_k^{\mathscr{G}} \le J_{\mathscr{G},\lambda}(v_j) < m_k^{\mathscr{G}} + \frac{1}{j}.$$

Hence, it follows that

$$\begin{aligned}\Phi(v_j) &= \lambda[\Psi(v_j) + J_\lambda(v_j)]\\ &\le \lambda\left[\int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{u(\sigma)} \mathsf{f}(t)dt\right)d\mu + m_k^{\mathscr{G}} + \frac{1}{j}\right]\\ &\le \lambda(\kappa\|w\|_1\zeta_k + m_k^{\mathscr{G}} + 1).\end{aligned}$$

Then $(v_j)_j$ is bounded in $H^1_{\mathscr{G}}(\mathbb{B}^N)$. This implies that there exists a subsequence $(v_{j_n})_n$ weakly convergent to some $u_k^{\mathscr{G}} \in C_k^{\mathscr{G}}$, as $C_k^{\mathscr{G}}$ is weakly closed. Now, the weak sequentially lower semicontinuity of $J_{\mathscr{G},\lambda}$ yields that

$$m_k^{\mathscr{G}} = \inf_{u\in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u) \le J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) \le \liminf_{n\to\infty} J_{\mathscr{G},\lambda}(v_{j_n}) = m_k^{\mathscr{G}}.$$

Hence $J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) = m_k^{\mathscr{G}}$ as claimed.

Step 2. Fix $k \in \mathbb{N}$. We claim that $u_k^{\mathscr{G}} \in [0, \xi_k]$ in a. e. $\mathbb{B}^N$.

Indeed, define $\varrho_k : \mathbb{R} \to \mathbb{R}$ as follows:

$$\varrho_k(t) = \begin{cases} \xi_k & \text{if } t > \xi_k, \\ t & \text{if } 0 \le t \le \xi_k, \\ 0 & \text{if } t < 0, \end{cases}$$

and consider the superposition operator $T_k : H^1_{\mathscr{G}}(\mathbb{B}^N) \to C_k^{\mathscr{G}}$ such that $u \mapsto T_k u$, where for every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$,

$$T_k u = \varrho_k \circ u \quad \text{a. e. in } \mathbb{B}^N. \tag{9.14}$$

We assert that T_k is well posed. Fix $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. Since ϱ_k is Lipschitz continuous, with $\varrho_k(0) = 0$, we have $T_k u \in H^1(\mathbb{B}^N)$ by Proposition 9.1.1. Furthermore, for all $g \in \mathscr{G}$,

$$\begin{aligned} g \circledast_{\mathscr{G}} T_k u(\sigma) &= T_k u(g^{-1}\sigma) = (\varrho_k \circ u)(g^{-1}\sigma) \\ &= \varrho_k(u(g^{-1}\sigma)) = \varrho_k(u(\sigma)) \\ &= T_k u(\sigma) \end{aligned}$$

for a. e. $\sigma \in \mathbb{B}^N$ by (9.5). In other words, $T_k u \in C_k^{\mathscr{G}} \subset H^1_{\mathscr{G}}(\mathbb{B}^N)$, as asserted.

Now, set $v_k^{\mathscr{G}} = T_k u_k^{\mathscr{G}}$ and let

$$X_k^{\mathscr{G}} = \{\sigma \in \mathbb{B}^N : u_k^{\mathscr{G}}(\sigma) \notin [0, \xi_k]\}.$$

If the Riemann measure $\mathrm{Vol}_H(X_k^{\mathscr{G}}) = 0$, the claim of Step 2 is proved. Otherwise, suppose that $\mathrm{Vol}_H(X_k^{\mathscr{G}}) > 0$. Then,

$$\xi_k < u_k^{\mathscr{G}} \le \zeta_k, \quad v_k^{\mathscr{G}} = T_k u_k^{\mathscr{G}} = \xi_k \quad \text{a. e. in } X_k^{\mathscr{G}}. \tag{9.15}$$

On the other hand, assumption (f_1) gives

$$\int_0^{u_k^{\mathscr{G}}(\sigma)} \mathsf{f}(t)dt \le \sup_{s \in [\xi_k, \zeta_k]} \int_0^s \mathsf{f}(t)dt = \int_0^{\xi_k} \mathsf{f}(t)dt = \int_0^{v_k^{\mathscr{G}}(\sigma)} \mathsf{f}(t)dt$$

for a. e. $\sigma \in X_k^{\mathscr{G}}$. Therefore,

$$\begin{aligned} J_{\mathscr{G},\lambda}(v_k^{\mathscr{G}}) - J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) = \frac{1}{\lambda}\left(\frac{a}{2}\|v_k^{\mathscr{G}}\|^2 + \frac{b}{4}\|v_k^{\mathscr{G}}\|^4\right) \\ - \int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{v_k^{\mathscr{G}}} \mathsf{f}(t)dt\right)d\mu \\ - \frac{1}{\lambda}\left(\frac{a}{2}\|u_k^{\mathscr{G}}\|^2 + \frac{b}{4}\|u_k^{\mathscr{G}}\|^4\right) \end{aligned}$$

$$+\int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{u_k^{\mathscr{G}}} \mathsf{f}(t)dt\right)d\mu$$

$$\le -\frac{1}{\lambda}\left(\frac{a}{2}\int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}|^2 d\mu + \frac{b}{4}\left(\int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}|^2 d\mu\right)^2\right)$$

$$-\int_{X_k^{\mathscr{G}}} w(\sigma)\left(\int_{u_k^{\mathscr{G}}}^{v_k^{\mathscr{G}}} \mathsf{f}(t)dt\right)d\mu$$

$$\le -\frac{1}{\lambda}\left(\frac{a}{2}\int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}|^2 d\mu + \frac{b}{4}\left(\int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}|^2 d\mu\right)^2\right),$$

since $|\nabla_H v_k^{\mathscr{G}}| = 0$ a. e. in $X_k^{\mathscr{G}}$. Moreover, $J_{\mathscr{G},\lambda}(v_k^{\mathscr{G}}) \ge J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}})$, thanks to the fact that $v_k^{\mathscr{G}} \in C_k^{\mathscr{G}}$. Hence,

$$\|v_k^{\mathscr{G}} - u_k^{\mathscr{G}}\|^2 = 0,$$

that is,

$$\|v_k^{\mathscr{G}} - u_k^{\mathscr{G}}\|^2 = \int_{\mathbb{B}^N} |\nabla_H(v_k^{\mathscr{G}} - u_k^{\mathscr{G}})|^2 d\mu = \int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}|^2 d\mu = 0.$$

Moreover, $u_k^{\mathscr{G}} = v_k^{\mathscr{G}} \in [0,\xi_k]$ a. e. in $\mathbb{B}^N$, since $\mathrm{Vol}_H(X_k^{\mathscr{G}}) > 0$. Thus, the claim is proved.

Step 3. The function $u_k^{\mathscr{G}}$ is a local minimum point of $J_{\mathscr{G},\lambda}$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$ for every $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. Let

$$Z_{\mathscr{G},k} = \{\sigma \in \mathbb{B}^N \;:\; u(\sigma) \notin [0,\xi_k]\}$$

and let T_k be the operator defined in (9.14). Set

$$v_k(\sigma) = T_k u(\sigma) = \begin{cases} \xi_k & \text{if } u(\sigma) > \xi_k, \\ u(\sigma) & \text{if } 0 \le u(\sigma) \le \xi_k, \\ 0 & \text{if } u(\sigma) < 0, \end{cases}$$

for a. e. $\sigma \in \mathbb{B}^N$. The definition of T_k yields

$$\int_{v_k^\star(\sigma)}^{u(\sigma)} \mathsf{f}(t)dt = 0,$$

if $\sigma \in \mathbb{B}^N \setminus Z_{\mathscr{G},k}$. Furthermore, if $\sigma \in Z_{\mathscr{G},k}$, then the following alternatives hold:

(a) If $u(\sigma) \le 0$, then

$$\int_{v_k(\sigma)}^{u(\sigma)} \mathtt{f}(t)dt = \int_0^{u(\sigma)} \mathtt{f}(t)dt = 0.$$

(b) If $\xi_k < u(\sigma) \le \zeta_k$, then

$$\begin{aligned}\int_{v_k(\sigma)}^{u(\sigma)} \mathtt{f}(t)dt &= \int_0^{u(\sigma)} \mathtt{f}(t)dt - \int_0^{v_k(\sigma)} \mathtt{f}(t)dt \\ &= \int_0^{u(\sigma)} \mathtt{f}(t)dt - \int_0^{\xi_k} \mathtt{f}(t)dt \\ &= \int_0^{u(\sigma)} \mathtt{f}(t)dt - \sup_{s\in[\xi_k,\zeta_k]} \int_0^{s} \mathtt{f}(t)dt \\ &\le 0.\end{aligned}$$

(c) If $u(\sigma) > \zeta_k$, it follows that

$$\int_{v_k(\sigma)}^{u(\sigma)} \mathtt{f}(t)dt = \int_{\xi_k}^{u(\sigma)} \mathtt{f}(t)dt \le \left|\int_{\xi_k}^{u(\sigma)} \mathtt{f}(t)dt\right| \le \kappa\big(u(\sigma) - \xi_k\big).$$

Hence, the constant

$$C = \frac{\kappa}{\|w\|_\infty} \sup_{\xi\ge\zeta_k} \frac{\xi - \xi_k}{(\xi - \xi_k)^q}$$

is finite and

$$\int_{v_k(\sigma)}^{u(\sigma)} \mathtt{f}(t)dt \le C\|w\|_\infty \big|u(\sigma) - v_k(\sigma)\big|^q$$

a. e. in $\mathbb{B}^N$. Then,

$$\int_{\mathbb{B}^N} w(\sigma)\left(\int_{v_k(\sigma)}^{u(\sigma)} \mathtt{f}(t)dt\right)d\mu \le Cc_q^q\|u - v_k\|^q, \tag{9.16}$$

where

$$c_q = \sup_{v\in H^1_{\mathscr{G}}(\mathbb{B}^N)\setminus\{0\}} \frac{\|v\|_q}{\|v\|} < \infty.$$

Therefore, for all $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$,

$$\begin{aligned}
J_{\mathscr{G},\lambda}(u) - J_{\mathscr{G},\lambda}(v_k) &= \frac{1}{\lambda}\left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4\right) - \frac{1}{\lambda}\left(\frac{a}{2}\|v_k\|^2 + \frac{b}{4}\|v_k\|^4\right) \\
&\quad - \int_{\mathbb{B}^N} w(\sigma)\left(\int_{v_k(\sigma)}^{u(\sigma)} \mathsf{f}(t)dt\right)d\mu \\
&= \frac{a}{2\lambda}\int_{Z_k^{\mathscr{G}}} |\nabla_H u|^2 d\mu + \frac{b}{4\lambda}\left(\int_{Z_k^{\mathscr{G}}} |\nabla_H u|^2 d\mu\right)^2 \\
&\quad - \int_{\mathbb{B}^N} w(\sigma)\left(\int_{v_k(\sigma)}^{u(\sigma)} \mathsf{f}(t)dt\right)d\mu \\
&= \frac{a}{2\lambda}\int_{\mathbb{B}^N} \left|\nabla_H(u - v_k)\right|^2 d\mu + \frac{b}{4\lambda}\left(\int_{\mathbb{B}^N} \left|\nabla_H(u - v_k)\right|^2 d\mu\right)^2 \\
&\quad - \int_{\mathbb{B}^N} w(\sigma)\left(\int_{v_k(\sigma)}^{u(\sigma)} \mathsf{f}(t)dt\right)d\mu \\
&\geq \frac{a}{2\lambda}\|u - v_k\|^2 + \frac{b}{4\lambda}\|u - v_k\|^4 - Cc_q^q\|u - v_k\|^q.
\end{aligned}$$

Since $v_k \in C_{k,\mathscr{G}}$, it follows that $J_{\mathscr{G},\lambda}(v_k) \geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}})$. Therefore,

$$\begin{aligned}
J_{\mathscr{G},\lambda}(u) &\geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) + \|u - v_k\|^2\left(\frac{a}{2\lambda} + \frac{b}{4\lambda}\|u - v_k\|^2 - Cc_q^q\|u - v_k\|^{q-2}\right) \\
&\geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) + \|u - v_k\|^2\left(\frac{a}{2\lambda} - Cc_q^q\|u - v_k\|^{q-2}\right)
\end{aligned}$$

for all $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. The operator $T_k : H^1_{\mathscr{G}}(\mathbb{B}^N) \to C_k^{\mathscr{G}}$ is continuous on account of Proposition 9.1.1 and [171, Theorem 1, p. 219]. Hence, fixed $\beta \in (0, \alpha)$, where

$$\alpha = \left(\frac{a}{4\lambda Cc_q^q}\right)^{1/(q-2)},$$

there exists $\delta \in (0, \beta]$ so small that $\|T_k u_k^{\mathscr{G}} - T_k u\| \leq \alpha - \beta \leq \alpha - \delta$ for every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$, with $\|u - u_k^{\mathscr{G}}\| < \delta$. Therefore,

$$\begin{aligned}
\|u - v_k\| &\leq \left\|u - u_k^{\mathscr{G}}\right\| + \left\|u_k^{\mathscr{G}} - v_k\right\| \\
&= \left\|u - u_k^{\mathscr{G}}\right\| + \left\|T_k u_k^{\mathscr{G}} - T_k u\right\| \leq \alpha
\end{aligned}$$

for every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$, with $\|u - u_k^{\mathscr{G}}\| < \delta$. Consequently, if $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$ and $\|u - u_k^{\mathscr{G}}\| < \delta$, then

$$\|u - v_k\|^{q-2} \leq \frac{a}{4\lambda Cc_q^q},$$

since $q > 2$, and so

$$J_{\mathscr{G},\lambda}(u) \geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) + \frac{a}{4\lambda}\|u - v_k\|^2 \geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}),$$

that is, $u_k^{\mathscr{G}}$ is a local minimum of $J_{\mathscr{G},\lambda}$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, as desired.

Step 4. If

$$m_k^{\mathscr{G}} = \inf_{u \in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u), \tag{9.17}$$

then $\lim_{k\to\infty} m_k^{\mathscr{G}} = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = 0$.

Since $u_k^{\mathscr{G}} \in C_k^{\mathscr{G}}$ and $m_k^{\mathscr{G}} = J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}})$, then

$$\begin{aligned} \Phi(u_k^{\mathscr{G}}) &= 2\lambda(\Psi(u_k^{\mathscr{G}}) + J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}})) \\ &= 2\lambda\left(\int_{\mathbb{B}^N} w(\sigma)\left(\int_0^{u(\sigma)} \mathtt{f}(t)dt\right)d\mu + m_k^{\mathscr{G}}\right) \\ &\leq 2\lambda(\kappa\|w\|_1\zeta_k + m_k^{\mathscr{G}}). \end{aligned} \tag{9.18}$$

Now (9.13) holds and

$$-\kappa\|w\|_1\zeta_k \leq m_k^{\mathscr{G}} = \inf_{u \in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u) \leq 0, \tag{9.19}$$

taking into account that the identically zero function $u_0 \equiv 0$ belongs to $C_k^{\mathscr{G}}$ and that $J_{\mathscr{G},\lambda}(0) = 0$. By (9.19), since $\zeta_k \to 0$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} m_k^{\mathscr{G}} = 0. \tag{9.20}$$

Hence, inequality (9.18) yields

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = 0.$$

Step 5. Let $m_k^{\mathscr{G}}$ be given as in (9.17). Then $m_k^{\mathscr{G}} < 0$ for every $k \in \mathbb{N}$.

To prove this, let us fix $k \in \mathbb{N}$. We introduce a class of functions belonging to $H^1_{\mathscr{G}}(\mathbb{B}^N)$ that will be crucial along the proof of the main step.

Since $w \in L^\infty(\mathbb{B}^N) \setminus \{0\}$ is nonnegative in $\mathbb{B}^N$, there are positive real numbers ρ, r, m_0, with $\rho > r$, such that

$$\operatorname*{ess\,inf}_{A_r^\rho} w \geq m_0 > 0. \tag{9.21}$$

Furthermore, for every a, b, with $0 < a < b$, define the following annular domain:

$$A_a^b = \{\sigma \in \mathbb{B}^N : b - a < d_H(\sigma) < a + b\},$$

where d_H is the geodesic distance of the point $\sigma \in \mathbb{B}^N$ from the origin σ_0 of $\mathbb{B}^N$, introduced in (9.1).

With the above notations, fix $\varepsilon \in (0,1)$ and set $v^\varepsilon_{\rho,r} \in H^1(\mathbb{B}^N)$ given by

$$v^\varepsilon_{\rho,r}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \mathbb{B}^N \setminus A^\rho_r, \\ 1 & \text{if } \sigma \in A^\rho_{\varepsilon r}, \\ \frac{1}{(1-\varepsilon)r}(r - |\log(\frac{1+|\sigma|}{1-|\sigma|}) - \rho|) & \text{if } \sigma \in A^\rho_r \setminus A^\rho_{\varepsilon r}, \end{cases} \tag{9.22}$$

for every $\sigma \in \mathbb{B}^N$. Since the group $\mathscr{G}$ is a compact connected subgroup of the isometry group $\mathrm{Isom}_H(\mathbb{B}^N)$ such that $\mathrm{Fix}_{\mathscr{G}}(\mathbb{B}^N) = \{\sigma_0\}$, the function $v^\varepsilon_{\rho,r} \in H^1(\mathbb{B}^N)$, given in (9.22), belongs to $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Direct computations yield

(j_1) $\mathrm{supp}(v^\varepsilon_{\rho,r}) \subset A^\rho_r$;
(j_2) $\|v^\varepsilon_{\rho,r}\|_\infty \leq 1$;
(j_3) $v^\varepsilon_{\rho,r}(\sigma) = 1$ for every $\sigma \in A^\rho_{\varepsilon r}$.

Moreover,

$$\begin{aligned} \|v^\varepsilon_{\rho,r}\|^2 &< \int_{A^\rho_r} |\nabla_H v^\varepsilon_{\rho,r}|^2 d\mu + \int_{A^\rho_r} |v^\varepsilon_{\rho,r}|^2 d\mu = \int_{A^\rho_{\varepsilon r}} |\nabla_H v^\varepsilon_{\rho,r}|^2 d\mu \\ &\quad + \int_{A^\rho_{\varepsilon r}} |v^\varepsilon_{\rho,r}|^2 d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} |\nabla_H v^\varepsilon_{\rho,r}|^2 d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} |v^\varepsilon_{\rho,r}|^2 d\mu \\ &\leq \mathrm{Vol}_H(A^\rho_r) + \frac{1}{(1-\varepsilon)^2 r^2} \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} \left|\nabla_H (r - |d_H(\sigma) - \rho|)\right|^2 d\mu \\ &= \mathrm{Vol}_H(A^\rho_r) + \frac{1}{(1-\varepsilon)^2 r^2} \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} \left|\nabla_H |d_H(\sigma) - \rho|\right|^2 d\mu \\ &= \mathrm{Vol}_H(A^\rho_r) + \frac{1}{(1-\varepsilon)^2 r^2} \mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon r}) \\ &\leq \left(1 + \frac{1}{(1-\varepsilon)^2 r^2}\right) \mathrm{Vol}_H(A^\rho_r), \end{aligned} \tag{9.23}$$

where Vol_H denotes the Lebesgue volume on $\mathbb{B}^N$.

Set $g_\mu : (0,1) \to \mathbb{R}^+$ to be the real function defined by

$$g_\mu(\varepsilon) = \frac{\mathrm{Vol}_H(A^\rho_{\varepsilon r})}{\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon r})}, \quad \varepsilon \in (0,1).$$

Clearly, if $\varepsilon \to 0^+$, then $g_\mu(\varepsilon) \to 0$, as well as $g_\mu(\varepsilon) \to \infty$ if $\varepsilon \to 1^-$. Thus, there exists $\varepsilon_0 \in (0,1)$ such that

$$\frac{\mathrm{Vol}_H(A^\rho_{\varepsilon_0 r})}{\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon_0 r})} = M + 1,$$

where $M > 0$ is given in condition (f_2).

By the former condition of (f_2), there exists $j_0 \in \mathbb{N}$ such that $\eta_j \leq \zeta_k$ and

$$\int_0^{\eta_j} f(t)dt > \frac{M+1}{\lambda\, m_0 \mathrm{Vol}_H(A^\rho_{\varepsilon_0 r})}\Phi(\eta_j v^{\varepsilon_0}_{\rho,r}) \quad \text{for every } j \geq j_0. \tag{9.24}$$

On account of (j_1)–(j_3), the latter condition of (f_2) and (9.21) yield

$$\begin{aligned}
\Psi(\eta_j v^{\varepsilon_0}_{\rho,r}) &= \int_{A^\rho_{\varepsilon_0 r}} w(\sigma)\left(\int_0^{\eta_j} f(t)dt\right)d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon_0 r}} w(\sigma)\left(\int_0^{\eta_j v^{\varepsilon_0}_{\rho,r}(\sigma)} f(t)dt\right)d\mu \\
&\geq m_0 \int_{A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} f(t)dt\right)d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon_0 r}} \inf_{t\in[0,\eta_j]}\left(\int_0^{t} f(s)ds\right)d\mu \\
&\geq m_0\left(\int_{A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} f(t)dt\right)d\mu - M \int_{A^\rho_r \setminus A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} f(t)dt\right)d\mu\right) \\
&= m_0 \frac{\mathrm{Vol}_H(A^\rho_{\varepsilon_0 r})}{M+1} \int_0^{\eta_j} f(t)dt.
\end{aligned}$$

Consequently, recalling (9.24) and that $\Phi(\eta_j v^{\varepsilon_0}_{\rho,r}) > 0$, we have

$$\frac{1}{\lambda} < \frac{\mathrm{Vol}_H(A^\rho_{\varepsilon_0 r})}{M+1}\frac{m_0}{\Phi(\eta_j v^{\varepsilon_0}_{\rho,r})}\int_0^{\eta_j} f(t)dt \leq \frac{\Psi(\eta_j v^{\varepsilon_0}_{\rho,r})}{\Phi(\eta_j v^{\varepsilon_0}_{\rho,r})}$$

for every $j \geq j_0$. Whence $\eta_j v^{\varepsilon_0}_{\rho,r} \in C^{\mathscr{G}}_k$ and $J_{\mathscr{G},\lambda}(\eta_j v^{\varepsilon_0}_{\rho,r}) < 0$ for $j \geq j_0$. Thus, $m_k = \inf_{u\in C^{\mathscr{G}}_k} J_{\mathscr{G},\lambda}(u) < 0$ as claimed.

Now, taking into account that $\|u_k\|_\infty \to 0$ as $k \to \infty$, there exists a subsequence of $(u^{\mathscr{G}}_k)_k \subset H^1_{\mathscr{G}}(\mathbb{B}^N)$, still denoted by $(u_k)_k$, of pairwise distinct elements, with

$$0 \leq \|u^{\mathscr{G}}_k\|_\infty \leq t_0 \text{ for every } k \in \mathbb{N}, \tag{9.25}$$

that weakly solve problem (9.11). Since the fixed point set of $H^1(\mathbb{B}^N)$ under the action of the group $\mathscr{G}$ is exactly $H^1_{\mathscr{G}}(\mathbb{B}^N)$, the symmetric criticality principle recalled in Theorem A.1.2 ensures that $(u^{\mathscr{G}}_k)_k \subset H^1(\mathbb{B}^N)$ is a sequence of critical points for the C^1-functional J_λ for which (9.25) holds, i. e., $(u^{\mathscr{G}}_k)_k \subset H^1(\mathbb{B}^N)$ is a sequence of solutions for problem (K_λ). □

We claim that the functional J_λ defined in (9.10) is $\mathscr{G}$-invariant, so that the key Theorem A.1.2 can be applied. To prove the claim, fix $u \in H^1(\mathbb{B}^N)$ and $g \in \mathscr{G}$. Since

$g \in \mathscr{G} \subseteq SO(N)$ is an isometry, on account of (9.5), we have the following the chain rule:

$$\nabla_H(g \circledast_{\mathscr{G}} u)(\sigma) = Dg_{g^{-1}\sigma} \nabla_H u(g^{-1}\sigma) \tag{9.26}$$

for a. e. $\sigma \in \mathbb{B}^N$, where $Dg_{g^{-1}\sigma} : T_{g^{-1}\sigma}(\mathbb{B}^N) \to T_\sigma(\mathbb{B}^N)$ denotes the differential of $g \in \mathscr{G}$ at the point $g^{-1}\sigma$. Setting $y = g^{-1}\sigma$, we get

$$\begin{aligned} \|g \circledast_{\mathscr{G}} u\|^2 &= \int_{\mathbb{B}^N} |\nabla_H(g \circledast_{\mathscr{G}} u)(\sigma)|_\sigma^2 d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} |\nabla_H u(g^{-1}\sigma)|_{g^{-1}\sigma}^2 d\mu(\sigma) = \int_{\mathbb{B}^N} |\nabla_H u(y)|_y^2 d\mu(y) \\ &= \|u\|^2 \end{aligned}$$

and

$$\begin{aligned} \|g \circledast_{\mathscr{G}} u\|^4 &= \Big(\int_{\mathbb{B}^N} |\nabla_H(g \circledast_{\mathscr{G}} u)(\sigma)|_\sigma^2 d\mu(\sigma) \Big)^2 \\ &= \Big(\int_{\mathbb{B}^N} |\nabla_H u(g^{-1}\sigma)|_{g^{-1}\sigma}^2 d\mu(\sigma) \Big)^2 \\ &= \Big(\int_{\mathbb{B}^N} |\nabla_H u(y)|_y^2 d\mu(y) \Big)^2 \\ &= \|u\|^4, \end{aligned}$$

thanks to (9.26) and to the fact that the map $Dg_{g^{-1}\sigma}$ is inner product preserving. Moreover, since $w \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is radially symmetric with respect to the origin $\sigma_0 \in \mathbb{B}^N$, we obtain

$$\begin{aligned} \int_{\mathbb{B}^N} w(\sigma)\Big(\int_0^{(g\circledast_{\mathscr{G}} u)(\sigma)} \mathtt{f}(t)dt \Big) d\mu(\sigma) &= \int_{\mathbb{B}^N} w(\sigma)\Big(\int_0^{u(g^{-1}\sigma)} \mathtt{f}(t)dt \Big) d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} w(y)\Big(\int_0^{u(y)} \mathtt{f}(t)dt \Big) d\mu(y). \end{aligned}$$

Thus, we conclude that

$$J_\lambda(g \circledast_{\mathscr{G}} u) = \frac{1}{\lambda}\Phi(g \circledast_{\mathscr{G}} u) - \int_{\mathbb{B}^N} w(\sigma)\Big(\int_0^{(g\circledast_{\mathscr{G}} u)(\sigma)} \mathtt{f}(t)dt \Big) d\mu = J_\lambda(u),$$

which proves the claim.

The next is a direct consequence of Theorem 9.2.1.

Corollary 9.2.2. *Let w satisfy (w) and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(i) *There are two real sequences $(\xi_k)_k$ and $(\zeta_k)_k$ such that*

$$\lim_{k\to\infty} \zeta_k = 0, \quad 0 \le \xi_k < \zeta_k, \quad F(\xi_k) = \sup_{t\in[0,\zeta_k]} F(t),$$

for every $k \in \mathbb{N}$;

(ii) $-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^2} \le \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty$,

hold. Then, for every $\mathscr{G} \in \mathscr{F}$ and $\lambda > 0$ there exists a sequence $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ of nontrivial nonnegative $\mathscr{G}$-invariant solutions of (K_λ) such that

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0. \tag{9.27}$$

Proof. If (i) holds, condition (f_1) is automatically verified. On the other hand, (ii) implies (f_2). To prove this, assume that condition (ii) holds. Since $\limsup_{t\to 0^+} F(t)/t^2 = \infty$, there exists a sequence $(\eta_k)_k \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} \eta_k = 0 \quad \text{and} \quad \limsup_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty.$$

Moreover, $\liminf_{t\to 0^+} F(t)/t^2 > -\infty$, so that there exist positive real numbers M and δ such that $F(t) \ge -Mt^2$ for every $t \in (0, \delta)$. Since $\lim_{k\to\infty} \eta_k = 0$, there is $\wp \in \mathbb{N}$ such that $\eta_k \in (0, \delta)$ and $F(\eta_k) \ge -MF(\eta_k)$ for every $k \ge \wp$. Thus, condition (f_2) is verified as claimed. □

The following model equation in $\mathbb{B}^N$, $N \ge 3$, illustrates how Theorem 9.2.1 can be applied:

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{B}^N} |\nabla_H u|^2 \, d\mu\right)\Delta_H u = \lambda\left(\frac{1-|\sigma|^2}{2}\right)^N f(u) & \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N), \end{cases} \tag{9.28}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t) = \begin{cases} 15\sqrt[3]{t^2} \sin \frac{1}{\sqrt[3]{t}} - 3\sqrt[3]{t} \cos \frac{1}{\sqrt[3]{t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

Owing to Theorem 9.2.1, for every $\mathscr{G} \in \mathscr{F}$ and $\lambda > 0$, there exists a sequence $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ of nontrivial nonnegative $\mathscr{G}$-invariant solutions of (9.28) such that

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0.$$

Now, a direct computation ensures that the potential F of f is given by

$$F(t) = \begin{cases} 9\sqrt[3]{t^5}\sin\frac{1}{\sqrt[3]{t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{9.29}$$

so that

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = -\infty;$$

see Figure 9.2 below.

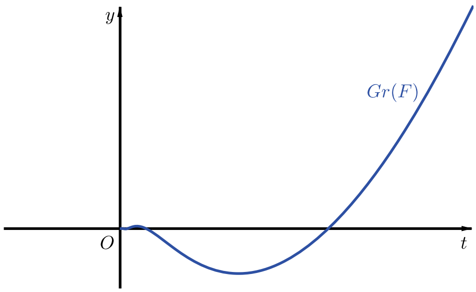

Figure 9.2: The graph of the potential F.

Thus, Corollary 9.2.2 cannot be applied in this case; see also [16, Example 3.1] for additional comments and remarks. However, in the next result we show how assumption (*ii*) of Corollary 9.2.2 implies the existence of a nontrivial solution of (K_λ) on the Poincaré ball model $\mathbb{B}^N$. The main tool used along the proof of this existence theorem is given by a variational principle obtained by B. Ricceri in [226] and recalled in the convenient form below; see [41, Theorem 2.1].

Theorem 9.2.3. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that*

– *Φ is continuous, sequentially weakly lower semicontinuous and coercive in X;*
– *Ψ is sequentially weakly upper semicontinuous in X.*

For every $r > \inf_X \Phi$, put

$$\varphi(r) = \inf_{u\in\Phi^{-1}(\Sigma_r)} \frac{(\sup_{v\in\Phi^{-1}(\Sigma_r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

where $\Sigma_r = (-\infty, r)$.

Then, for each $r > \inf_X \Phi$ and each $\lambda \in (0, 1/\varphi(r))$, the functional $J_\lambda = \Phi - \lambda\Psi$ admits a global minimum in $\Phi^{-1}(\Sigma_r)$, which is a critical point (local minimum) of J_λ in X.

The main result reads as follows.

Theorem 9.2.4. *Let w satisfy (w) and let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function verifying (ii) and the growth condition*

$$C_f=\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{q-1}}<\infty, \tag{9.30}$$

for some $q\in(2,2^*)$. Then, for every $\mathscr{G}\in\mathscr{F}$ there exists $\lambda^\star>0$ such that, for every $\lambda\in(0,\lambda^\star)$, problem (K_λ) admits a $\mathscr{G}$-invariant solution $u_\lambda^{\mathscr{G}}\in H^1(\mathbb{B}^N)$ and

$$\lim_{\lambda\to 0^+}\int_{B_1}\left|\nabla u_\lambda^{\mathscr{G}}(x)\right|^2\frac{2^{N-2}}{(1-|x|^2)^{N-2}}dx=0.$$

Proof. Following [192], the main idea of the proof consists in applying Theorem 9.2.3 to the energy functional

$$\mathscr{J}_\lambda(u)=\Phi(u)-\lambda\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u),\quad u\in H^1_{\mathscr{G}}(\mathbb{B}^N),$$

where

$$\Phi(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4\quad\text{and}\quad\Psi(u)=\int_{\mathbb{B}^N}w(\sigma)F(u)d\mu.$$

Then, the existence of one nontrivial $\mathscr{G}$-symmetric solution of (K_λ) follows from Theorem A.1.2.

The functionals Φ and $\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}$ have the regularity required in Theorem 9.2.3. Moreover, it is also clear that Φ is strongly continuous, coercive in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, and

$$\inf_{u\in H^1_{\mathscr{G}}(\mathbb{B}^N)}\Phi(u)=0.$$

Now, set

$$\lambda^\star=\frac{q}{C_fc_q}\max_{t>0}\left(\frac{t}{q\sqrt{\frac{2}{a}}\|w\|_{q'}+\frac{2^{q/2}c_q^{q-1}\|w\|_\infty}{a^{q/2}}t^{q-1}}\right), \tag{9.31}$$

where $q'=q/(q-1)$, and

$$c_q=\sup_{u\in H^1_{\mathscr{G}}(\mathbb{B}^N)\setminus\{0\}}\frac{\|u\|_q}{\|u\|}.$$

Fix $\lambda\in(0,\lambda^\star)$. By (9.31), there exists $\bar{t}>0$ such that

$$\lambda<\lambda^\star(\bar{t})=\frac{q}{C_fc_q}\cdot\frac{\bar{t}}{q\sqrt{\frac{2}{a}}\|w\|_{q'}+\frac{2^{q/2}c_q^{q-1}\|w\|_\infty}{a^{q/2}}\bar{t}^{q-1}}<\lambda^\star. \tag{9.32}$$

Set $r \in \mathbb{R}^+$ and $\Sigma_r = (-\infty, r)$. Consider the function $\chi : \mathbb{R}^+ \to \mathbb{R}_0^+$ given by

$$\chi(r) = \frac{\sup_{u \in \Phi^{-1}(\Sigma_r)} \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u)}{r}.$$

The growth condition (9.30) yields

$$\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u) \le C_f \int_{\mathbb{B}^N} w(\sigma)|u| d\mu + \frac{C_f}{q} \int_{\mathbb{B}^N} w(\sigma)|u|^q d\mu.$$

Moreover,

$$\|u\|_{H^1_{\mathscr{G}}(\mathbb{B}^N)} < \sqrt{\frac{2r}{a}} \quad \text{for every } u \in H^1_{\mathscr{G}}(\mathbb{B}^N), \text{ with } \Phi(u) < r,$$

which yields, by the Sobolev embedding Theorem 9.1.2,

$$\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u) < C_f c_q \left(\|w\|_{q'} \sqrt{\frac{2r}{a}} + \frac{c_q^{q-1}}{q} \|w\|_\infty \left(\frac{2r}{a} \right)^{q/2} \right)$$

for all $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$, with $\Phi(u) < r$. Hence

$$\sup_{u \in \Phi^{-1}(\Sigma_r)} \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u) \le C_f c_q \left(\|w\|_{q'} \sqrt{\frac{2r}{a}} + \frac{c_q^{q-1}}{q} \|w\|_\infty \left(\frac{2r}{a} \right)^{q/2} \right).$$

Therefore, the above inequality immediately gives

$$\chi(r) \le C_f c_q \left(\|w\|_{q'} \sqrt{\frac{2}{ar}} + \frac{(2/a)^{q/2} c_q^{q-1}}{q} \|w\|_\infty r^{q/2-1} \right) \tag{9.33}$$

for every $r > 0$. Evaluating inequality (9.33) at $r = \bar{t}^2$ and recalling (9.32), we have

$$\chi(\bar{t}^2) \le C_f c_q \left(\sqrt{\frac{2}{a}} \frac{\|w\|_{q'}}{\bar{t}} + \frac{(2/a)^{q/2} c_q^{q-1}}{q} \|w\|_\infty \bar{t}^{q-2} \right) = \frac{1}{\lambda^\star(\bar{t})}. \tag{9.34}$$

Now, put $\Sigma_{\bar{t}^2} = (-\infty, \bar{t}^2)$ and note that the identically zero function 0 is in $\Phi^{-1}(\Sigma_{\bar{t}^2})$, and $\Phi(0) = \Psi(0) = 0$, since clearly $0 \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. Consequently,

$$\begin{aligned}\varphi(\bar{t}^2) &= \inf_{u \in \Phi^{-1}(\Sigma_{\bar{t}^2})} \frac{(\sup_{v \in \Phi^{-1}(\Sigma_{\bar{t}^2})} \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(v)) - \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(u)}{\bar{t}^2 - \Phi(u)} \\ &\le \frac{\sup_{v \in \Phi^{-1}(\Sigma_{\bar{t}^2})} \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(v)}{\bar{t}^2} = \chi(\bar{t}^2).\end{aligned}$$

Thus, (9.34) gives

$$\varphi(\bar{t}^2) \le \chi(\bar{t}^2) \le \frac{1}{\lambda^\star(\bar{t})} < \frac{1}{\lambda}. \tag{9.35}$$

Then

$$\lambda \in (0, \lambda^\star(\bar{t})) \subseteq (0, 1/\varphi(\bar{t}^2)).$$

The main critical point Theorem 9.2.3 ensures that there exists a function $u_\lambda^{\mathscr{G}} \in \Phi^{-1}(\Sigma_{\bar{t}^2})$ such that

$$\Phi'(u_\lambda^{\mathscr{G}}) - \lambda(\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)})'(u_\lambda^{\mathscr{G}}) = 0,$$

and, in particular, $u_\lambda^{\mathscr{G}}$ is a global minimum of the restriction of the functional $\mathcal{J}_\lambda$ to the sublevel $\Phi^{-1}(\Sigma_{\bar{t}^2})$.

Now, we have to show that the solution $u_\lambda^{\mathscr{G}}$ found above is not the trivial (identically zero) function. If $f(0) \neq 0$, then it easily follows that $u_\lambda^{\mathscr{G}} \not\equiv 0$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, since the trivial function is not a solution of (K_λ).

Let us consider the case when $f(0) = 0$. Clearly, $u_\lambda^{\mathscr{G}}$ is a critical point of $\mathcal{J}_\lambda$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$ and has the property that

$$\Phi(u_\lambda^{\mathscr{G}}) < \bar{t}^2, \quad \mathcal{J}_\lambda(u_\lambda^{\mathscr{G}}) \leq \mathcal{J}_\lambda(u) \quad \text{for any } u \in H^1_{\mathscr{G}}(\mathbb{B}^N), \text{with } \Phi(u) < \bar{t}^2. \tag{9.36}$$

Moreover, $u_\lambda^{\mathscr{G}}$ is a solution of (K_λ), due to the $\mathscr{G}$-invariance of the functional J_λ and Theorem A.1.2. In this setting, in order to prove that $u_\lambda^{\mathscr{G}} \not\equiv 0$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, we first claim that there exists a sequence of functions $(v_k)_k$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$ such that

$$\limsup_{k\to\infty} \frac{\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(v_k)}{\Phi(v_k)} = \infty. \tag{9.37}$$

By the assumption on the limsup in (*ii*), there exists a sequence $(t_k)_k \subset \mathbb{R}^+$ such that $t_k \to 0^+$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{F(t_k)}{t_k^2} = \infty. \tag{9.38}$$

Thus for any $M > 0$ and k sufficiently large,

$$F(t_k) > Mt_k^2. \tag{9.39}$$

Define $v_k = t_k v_{\rho,r}^{\varepsilon}$ for any $k \in \mathbb{N}$. Then, $v_k \in H^1_{\mathscr{G}}(\mathbb{B}^N)$ for any $k \in \mathbb{N}$, since $v_{\rho,r}^{\varepsilon} \in H^1_{\mathscr{G}}(\mathbb{B}^N)$. Furthermore, taking into account the algebraic properties of the functions $v_{\rho,r}^{\varepsilon}$ stated in (j_1)–(j_3), since $F(0) = 0$, and using (9.39), we can write

$$\begin{aligned}\frac{\Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(v_k)}{\Phi(v_k)} &= \frac{\int_{A^\rho_{\varepsilon r}} w(\sigma)F(v_k)\,d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} w(\sigma)F(v_k)\,d\mu}{\Phi(v_k)}\\ &= \frac{\int_{A^\rho_{\varepsilon r}} w(\sigma)F(t_k)\,d\mu + \int_{A^\rho_r \setminus A^\rho_{\varepsilon r}} w(\sigma)F(t_k v^\varepsilon_{\rho,r})\,d\mu}{\Phi(v_k)}\end{aligned} \tag{9.40}$$

$$\geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r})t_k^2 + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} F(t_k v^\varepsilon_{\rho,r})\,d\mu}{\frac{at_k^2}{2}\|v^\varepsilon_{\rho,r}\|^2 + \frac{bt_k^4}{4}\|v^\varepsilon_{\rho,r}\|^4}$$

for k sufficiently large. Now we have to distinguish two different cases.

Case 1. Suppose that $\lim_{t\to 0^+} \frac{F(t)}{t^2} = \infty$.

Then, there exists $\rho_M > 0$ such that for any t, with $0 < t < \rho_M$,

$$F(t) \geq Mt^2. \tag{9.41}$$

Since $t_k \to 0^+$ and $0 \leq v^\varepsilon_{\rho,r} \leq 1$ in $\mathbb{B}^N$, one gets $v_k = t_k v^\varepsilon_{\rho,r} \to 0^+$ as $k \to \infty$ uniformly in $\mathbb{B}^N$. Hence, $0 \leq v_k < \rho_M$ in $\mathbb{B}^N$ for k sufficiently large. As a consequence of (9.40) and (9.41), we have

$$\begin{aligned}\frac{\Psi(v_k)}{\Phi(v_k)} &\geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r})t_k^2 + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} F(t_k v^\varepsilon_{\rho,r})\,d\mu}{\frac{at_k^2}{2}\|v^\varepsilon_{\rho,r}\|^2 + \frac{bt_k^4}{4}\|v^\varepsilon_{\rho,r}\|^4}\\ &\geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r}) + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} |v^\varepsilon_{\rho,r}|^2\,d\mu}{\frac{a}{2}\|v^\varepsilon_{\rho,r}\|^2 + \frac{bt_k^4}{4}\|v^\varepsilon_{\rho,r}\|^4}\end{aligned}$$

for k sufficiently large. Thus,

$$\limsup_{k\to\infty} \frac{\Psi(v_k)}{\Phi(v_k)} \geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r}) + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} |v^\varepsilon_{\rho,r}|^2\,d\mu}{\frac{a}{2}\|v^\varepsilon_{\rho,r}\|^2}.$$

This gives (9.37), since $M > 0$ is arbitrary. The claim is so proved.

Case 2. Suppose that $\liminf_{t\to 0^+} \frac{F(t)}{t^2} = \ell \in \mathbb{R}$.

Then, for any $\epsilon > 0$ there exists $\rho_\epsilon > 0$ such that for any t, with $0 < t < \rho_\epsilon$,

$$F(t) \geq (\ell - \epsilon)t^2. \tag{9.42}$$

Arguing as above, we can suppose that $0 \leq v_k = t_k v^\varepsilon_{\rho,r} < \rho_\epsilon$ in $\mathbb{B}^N$ for k large enough. Thus, by (9.40) and (9.42), we get

$$\begin{aligned}\frac{\Psi(v_k)}{\Phi(v_k)} &\geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r})t_k^2 + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} F(t_k v^\varepsilon_{\rho,r})d\mu}{\frac{at_k^2}{2}\|v^\varepsilon_{\rho,r}\|^2 + \frac{bt_k^4}{4}\|v^\varepsilon_{\rho,r}\|^4}\\ &\geq w_0 \frac{M\,\mathrm{Vol}_H(A^\rho_{\varepsilon r}) + (\ell-\epsilon)\int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} |v^\varepsilon_{\rho,r}|^2 d\mu}{\frac{a}{2}\|v^\varepsilon_{\rho,r}\|^2 + \frac{bt_k^4}{4}\|v^\varepsilon_{\rho,r}\|^4},\end{aligned}$$

provided that k is sufficiently large. Choosing $M > 0$ large enough, say

$$M > \max\left\{0, -\frac{2\ell}{\mathrm{Vol}_H(A^\rho_{\varepsilon r})} \int\limits_{A^\rho_r\setminus A^\rho_{\varepsilon r}} \left|v^\varepsilon_{\rho,r}\right|^2 d\mu\right\},$$

and $\epsilon > 0$ so small that

$$\epsilon \int_{A_r^\rho \setminus A_{\varepsilon r}^\rho} |w_{\rho,r}^\varepsilon|^2 \, d\mu < M \frac{\mathrm{Vol}_H(A_{\varepsilon r}^\rho)}{2} + \ell \int_{A_r^\rho \setminus A_{\varepsilon r}^\rho} |v_{\rho,r}^\varepsilon|^2 \, d\mu,$$

we get at once

$$\begin{aligned} \frac{\Psi(v_k)}{\Phi(v_k)} &\geq \frac{w_0}{\Phi(v_k)} \left(M \,\mathrm{Vol}_H(A_{\varepsilon r}^\rho) + \ell \int_{A_r^\rho \setminus A_{\varepsilon r}^\rho} |v_{\rho,r}^\varepsilon|^2 \, d\mu - \epsilon \int_{A_r^\rho \setminus A_{\varepsilon r}^\rho} |v_{\rho,r}^\varepsilon|^2 \, d\mu \right) \\ &> \frac{w_0 M}{a \|v_{\rho,r}^\varepsilon\|^2 + \frac{b t_k^4}{2} \|v_{\rho,r}^\varepsilon\|^4} \cdot \frac{\mathrm{Vol}_H(A_{\varepsilon r}^\rho)}{2}, \end{aligned}$$

for $k \in \mathbb{N}$ large enough. Hence

$$\limsup_{k \to \infty} \frac{\Psi(v_k)}{\Phi(v_k)} \geq \frac{w_0 \,\mathrm{Vol}_H(A_{\varepsilon r}^\rho)}{2a \|v_{\rho,r}^\varepsilon\|^2} M.$$

This gives again (9.37), since M is arbitrary.

Now, $\|v_k\| = t_k \|v_{\rho,r}^\varepsilon\| \to 0$ as $k \to \infty$, so that for large enough k,

$$\frac{a}{2} \|v_k\|^2 + \frac{b}{4} \|v_k\|^4 < \bar{t}^2.$$

Thus $v_k \in \Phi^{-1}(\Sigma_{\bar{t}^2})$ and $\mathcal{J}_\lambda(v_k) = \Phi(v_k) - \lambda \Psi|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}(v_k) < 0$ provided that k is large enough, thanks to (9.37) and the fact that $\lambda > 0$. Consequently,

$$\mathcal{J}_\lambda(u_\lambda^{\mathscr{G}}) \leq \mathcal{J}_\lambda(v_k) < 0 = \mathcal{J}_\lambda(0),$$

since $u_\lambda^{\mathscr{G}}$ is a global minimum of the restriction of $\mathcal{J}_\lambda$ to $\Phi^{-1}(\Sigma_{\bar{t}^2})$ by (9.36). Therefore, $u_\lambda^{\mathscr{G}} \not\equiv 0$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Thus, $u_\lambda^{\mathscr{G}}$ is a nontrivial solution of (K_λ) for any $\lambda \in (0, \lambda^\star)$.

Finally, we prove that $\lim_{\lambda \to 0^+} \|u_\lambda^{\mathscr{G}}\| = 0$. To this end, let us fix $\lambda \in (0, \lambda^\star(\bar{t}))$. By construction,

$$\Phi(u_\lambda^{\mathscr{G}}) = \frac{a}{2} \|u_\lambda^{\mathscr{G}}\|^2 + \frac{b}{4} \|u_\lambda^{\mathscr{G}}\|^4 < \bar{t}^2,$$

that is,

$$\|u_\lambda^{\mathscr{G}}\| < \sqrt{\frac{2}{a}} \bar{t}.$$

Set

$$M_{\bar{t}} = c_q C_f \left(\sqrt{\frac{2}{a}} \|w\|_{q'} \bar{t} + \left(\frac{2}{a} \right)^{q/2} c_q^{q-1} \|w\|_\infty \bar{t}^q \right).$$

The growth condition (9.30) yields

$$\begin{aligned}\left|\int_{\mathbb{B}^N} w(\sigma)f(u_\lambda^{\mathscr{G}})u_\lambda^{\mathscr{G}}\,d\mu\right| &\leq C_f\Big(\int_{\mathbb{B}^N} w(\sigma)|u_\lambda^{\mathscr{G}}|\,d\mu + \int_{\mathbb{B}^N} w(\sigma)|u_\lambda^{\mathscr{G}}|^q\,d\mu\Big)\\ &\leq C_f(\|w\|_{q'}\|u_\lambda^{\mathscr{G}}\|_q + \|w\|_\infty\|u_\lambda^{\mathscr{G}}\|_q^q) \\ &< M_{\bar{t}},\end{aligned} \tag{9.43}$$

since $q \in (2, 2^*)$. Now $u_\lambda^{\mathscr{G}}$ is a critical point of $\mathcal{J}_\lambda$, so that $\langle \mathcal{J}_\lambda'(u_\lambda^{\mathscr{G}}), \varphi\rangle = 0$ for any $\varphi \in H^1_{\mathscr{G}}(\mathbb{B}^N)$ and every $\lambda \in (0, \lambda^\star(\bar{t}))$. In particular, $\langle \mathcal{J}_\lambda'(u_\lambda^{\mathscr{G}}), u_\lambda^{\mathscr{G}}\rangle = 0$, that is,

$$\langle \Phi'(u_\lambda^{\mathscr{G}}), u_\lambda^{\mathscr{G}}\rangle = \lambda \int_{\mathbb{B}^N} w(\sigma)f(u_\lambda^{\mathscr{G}})u_\lambda^{\mathscr{G}}\,d\mu \tag{9.44}$$

for every $\lambda \in (0, \lambda^\star(\bar{t}))$, where

$$\langle \Phi'(u_\lambda^{\mathscr{G}}), u_\lambda^{\mathscr{G}}\rangle = (a + b\|u_\lambda^{\mathscr{G}}\|^2)\|u_\lambda^{\mathscr{G}}\|^2.$$

Then it follows, by (9.43) and (9.44), that

$$0 \leq \|u_\lambda^{\mathscr{G}}\|^2 = \langle \Phi'(u_\lambda^{\mathscr{G}}), u_\lambda^{\mathscr{G}}\rangle = \lambda \int_{\mathbb{B}^N} w(\sigma)f(u_\lambda^{\mathscr{G}})u_\lambda^{\mathscr{G}}\,d\mu < \lambda M_{\bar{t}}$$

for any $\lambda \in (0, \lambda^\star(\bar{t}))$. We get $\lim_{\lambda\to 0^+}\|u_\lambda^{\mathscr{G}}\| = 0$, as claimed. This completes the proof of the main result. □

Theorem 9.2.4 can be applied to the prototype problem

$$\begin{cases} -\Big(a + b\displaystyle\int_{\mathbb{B}^4} |\nabla_H u|^2\,d\mu\Big)\Delta_H u = \lambda\Big(\dfrac{1-|\sigma|^2}{2}\Big)^4(|u|^{r-2}u + |u|^{s-2}u) & \text{in } \mathbb{B}^4, \\ u \in H^1(\mathbb{B}^4), \end{cases}$$

where $1 < r < 2$ and $2 < s < 4$, and ensures that for every $\mathscr{G} \in \mathscr{F}$ there exists $\lambda^\star > 0$ such that for every $\lambda \in (0, \lambda^\star)$ the above equation admits at least one nontrivial $\mathscr{G}$-symmetric solution $u_\lambda^{\mathscr{G}} \in H^1(\mathbb{B}^4)$. Moreover,

$$\lim_{\lambda\to 0^+}\|u_\lambda^{\mathscr{G}}\| = 0.$$

A simple case is displayed in Figure 9.3.

The next result below can be viewed as a natural counterpart of Theorem 9.2.1.

Theorem 9.2.5. *Let w satisfy* (w) *and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that* (f_1) *holds in addition to*

(a) $\liminf_{t\to 0^+} \frac{F(t)}{t^2} = 0$;

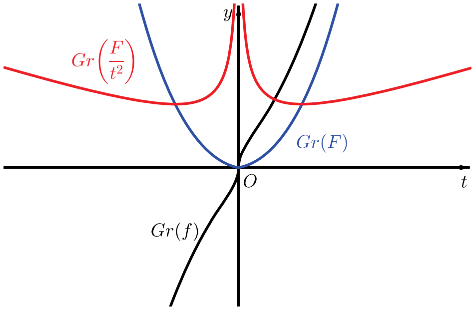

Figure 9.3: The model case $f(t) = |t|^{-1/2}t + |t|t$.

(*b*) $F_0 = \limsup_{t\to 0^+} \frac{F(t)}{t^2} \in \mathbb{R}^+ \cup \{\infty\}$.

Then, for every $\mathscr{G} \in \mathscr{F}$ *and* $\lambda > 0$ *sufficiently large, there exists a sequence* $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ *of nontrivial nonnegative* $\mathscr{G}$*-invariant solutions of* (K_λ) *such that*

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0.$$

Proof. The first steps of the proof are similar to the arguments of the proof of Theorem 9.2.1. The essential difficulty to apply the variational method to study (K_λ) is due to the fact that the Sobolev compact embedding fails on the whole space.

To be precise, the embedding from $H^1(\mathbb{B}^N)$ into $L^\wp(\mathbb{B}^N)$ in not compact when $\wp \in (2, 2^*)$, since $\mathbb{B}^N$ here is equipped with the Poincaré metric, and so it is a complete noncompact Riemannian manifold.

On the other hand, from Theorem 9.1.2 of L. Skrzypczak and C. Tintarev [232], the $\mathscr{G}$-invariant subspace $H^1_{\mathscr{G}}(\mathbb{B}^N)$ of $H^1(\mathbb{B}^N)$ is compactly embedded into $L^\wp(\mathbb{B}^N)$, whenever $\wp \in (2, 2^*)$. Therefore the proof can be sketched as follows. At any subset $C_k^{\mathscr{G}}$, the functional $J_{\mathscr{G},\lambda}$, given in (9.12), attains its infimum at some point $u_k^{\mathscr{G}}$. Then, it is possible to prove that $u_k^{\mathscr{G}}$ is a local minimum point of $J_{\mathscr{G},\lambda}$ on the symmetric subspace $H^1_{\mathscr{G}}(\mathbb{B}^N)$. This partial result is intuitively obvious, since its analogy in basic calculus depends heavily on (f_1). From the Palais Theorem A.1.2, every $u_k^{\mathscr{G}}$ is actually a critical point of the smooth functional J_λ on the entire $H^1(\mathbb{B}^N)$.

A careful analysis of the energy levels

$$m_k^{\mathscr{G}} = \inf_{u\in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u)$$

gives that $\lim_{k\to\infty} m_k^{\mathscr{G}} = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = 0$.

To obtain the conclusion, it is enough to prove that such local minima $(u_k^{\mathscr{G}})_k$ are pairwise distinct. From now on, technical details and methods are different compared to the proof of Theorem 9.2.1. We first claim that

$$m_k^{\mathscr{G}} = \inf_{u\in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u) = J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) < 0$$

for every $k \in \mathbb{N}$. Indeed, fix ρ, r, with $\rho > r > 0$, and $\varepsilon \in (0,1)$. Let $v^{\varepsilon}_{\rho,r}$ be the function given in (9.22) and take κ_0 so that

$$0 < \kappa_0 < \frac{F_0 \int_{A^{\rho}_{\varepsilon r}} w(\sigma) d\mu}{\int_{A^{\rho}_{r} \setminus A^{\rho}_{\varepsilon r}} w(\sigma) |v^{\varepsilon}_{\rho,r}|^2 d\mu}, \tag{9.45}$$

which is possible by (b). Clearly, the right-hand side of (9.45) is trivially satisfied when $F_0 = \infty$. By (a), there exists $\delta > 0$ such that

$$F(t) > -\kappa_0 t^2 \tag{9.46}$$

for every $t \in (0,\delta)$. Thanks to (b), there is a sequence $(s_j)_j \subset \mathbb{R}^+$ such that $\lim_{j\to\infty} s_j = 0$ and

$$\lim_{j\to\infty} \frac{F(s_j)}{s_j^2} = \limsup_{t\to 0^+} \frac{F(t)}{t^2} = F_0 > 0. \tag{9.47}$$

Now, for any $j \in \mathbb{N}$, set

$$\mathrm{v}^{\rho,r}_{s_j} = s_j v^{\varepsilon}_{\rho,r}. \tag{9.48}$$

Observe that

$$\Phi(\mathrm{v}^{\rho,r}_{s_j}) \le \frac{s_j^2}{2}\left(a + \frac{b}{2}\right)\|v^{\varepsilon}_{\rho,r}\|^2 \tag{9.49}$$

for every $j \in \mathbb{N}$ sufficiently large.

Fix $k \in \mathbb{N}$ and $j \in \mathbb{N}$ sufficiently large so that $s_j \le \zeta_k < t_0$ by (9.49). Hence, thanks to (9.46),

$$\begin{aligned}
J_{\mathscr{G},\lambda}(\mathrm{v}^{\rho,r}_{s_j}) &\le \frac{s_j^2}{2\lambda}\left(a + \frac{b}{2}\right)\|v^{\varepsilon}_{\rho,r}\|^2 \\
&\quad - \left(F(s_j) \int_{A^{\rho}_{\varepsilon r}} w(\sigma) d\mu + \int_{A^{\rho}_{r} \setminus A^{\rho}_{\varepsilon r}} w(\sigma) F(\mathrm{v}^{\rho,r}_{s_j}) d\mu\right) \\
&\le \frac{s_j^2}{2\lambda}\left(a + \frac{b}{2}\right)\|v^{\varepsilon}_{\rho,r}\|^2 \\
&\quad - \left(F(s_j) \int_{A^{\rho}_{\varepsilon r}} w(\sigma) d\mu - \kappa_0 s_j^2 \int_{A^{\rho}_{r} \setminus A^{\rho}_{\varepsilon r}} w(\sigma) \left|v^{\varepsilon}_{\rho,r}\right|^2 d\mu\right) \\
&\le \frac{s_j^2}{\lambda}\left[\left(\frac{a}{2} + \frac{b}{4}\right)\|v^{\varepsilon}_{\rho,r}\|^2\right. \\
&\quad \left. - \lambda\left(\frac{F(s_j)}{s_j^2} \int_{A^{\rho}_{\varepsilon r}} w(\sigma) d\mu - \kappa_0 \int_{A^{\rho}_{r} \setminus A^{\rho}_{\varepsilon r}} w(\sigma) \left|v^{\varepsilon}_{\rho,r}\right|^2 d\mu\right)\right].
\end{aligned}$$

Now, (9.46) allows us to take

$$\lambda > \left(\frac{a}{2} + \frac{b}{4}\right) \frac{\|v_{\rho,r}^{\varepsilon}\|^2}{F_0 \int_{A_{\varepsilon r}^{\rho}} w(\sigma) d\mu - \kappa_0 \int_{A_r^{\rho} \setminus A_{\varepsilon r}^{\rho}} w(\sigma) |v_{\rho,r}^{\varepsilon}|^2 d\mu},$$

so that there exists $j_0 \in \mathbb{N}$ such that $s_{j_0} \leq \zeta_k$ and

$$\begin{aligned} J_{\mathscr{G},\lambda}(\mathrm{v}_{s_{j_0}}^{\rho,r}) \leq \frac{s_{j_0}^2}{\lambda} \Bigg[\left(\frac{a}{2} + \frac{b}{4}\right) \|v_{\rho,r}^{\varepsilon}\|^2 \\ - \lambda \left(\frac{F(s_{j_0})}{s_{j_0}^2} \int_{A_{\varepsilon r}^{\rho}} w(\sigma) d\mu - \kappa_0 \int_{A_r^{\rho} \setminus A_{\varepsilon r}^{\rho}} w(\sigma) \left|v_{\rho,r}^{\varepsilon}\right|^2 d\mu \right) \Bigg] < 0. \end{aligned}$$

Thus, the test function $\mathrm{v}_{s_{j_0}}^{\rho,r}$ belongs to $C_k^{\mathscr{G}}$ and $J_{\mathscr{G},\lambda}(\mathrm{v}_{s_{j_0}}^{\rho,r}) < 0$. Hence,

$$m_k^{\mathscr{G}} = \inf_{u \in C_k^{\mathscr{G}}} J_{\mathscr{G},\lambda}(u) = J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) \leq J_{\mathscr{G},\lambda}(\mathrm{v}_{s_{j_0}}^{\rho,r}) < 0,$$

as claimed.

Thanks to (9.20), there exists a subsequence of $(u_k^{\mathscr{G}})_k$, still denoted by $(u_k^{\mathscr{G}})_k$, of pairwise distinct elements that weakly solve the truncated equation (9.11) and such that

$$0 \leq \|u_k^{\mathscr{G}}\|_{\infty} \leq t_0 \tag{9.50}$$

for every $k \in \mathbb{N}$.

Since the fixed point set of $H^1(\mathbb{B}^N)$ under the action of the group $\mathscr{G}$ is exactly $H_{\mathscr{G}}^1(\mathbb{B}^N)$, the symmetric criticality principle, recalled in Theorem A.1.2, ensures that $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ is a sequence of critical points for the C^1-functional J_λ for which (9.50) holds, i. e., $(u_k^{\mathscr{G}})_k \subset H^1(\mathbb{B}^N)$ is a sequence of solutions of (K_λ). □

A concrete application of Theorem 9.2.5 is given by the following example. Let $(a_k)_k$, $(b_k)_k$ be two positive real sequences such that $b_{k+1} < a_k < b_k$ for every $k \geq k_0$ and some $k_0 \in \mathbb{N}$. Assume that

$$\lim_{k\to\infty} b_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{b_k}{a_k} = \infty.$$

Moreover, let $\varphi \in C^1([0,1])$ be a nontrivial nonnegative function such that $\varphi(0) = \varphi(1) = 0$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(t) = \begin{cases} \varphi(\frac{t-b_{k+1}}{a_k-b_{k+1}}) & \text{if } t \in \bigcup_{k\geq k_0}[b_{k+1}, a_k], \\ 0 & \text{otherwise.} \end{cases} \tag{9.51}$$

Furthermore, let

$$0 < \kappa_0 < \frac{\|\varphi\|_{\infty} \int_{A_{\varepsilon r}^{\rho}} w(\sigma) d\mu}{\int_{A_r^{\rho} \setminus A_{\varepsilon r}^{\rho}} w(\sigma) |v_{\rho,r}^{\varepsilon}|^2 d\mu}, \tag{9.52}$$

where w is a nontrivial nonnegative potential, verifying (w). The function $v^{\varepsilon}_{\rho,r}$ is given in (9.22), and the annulus-type domains A^{ρ}_{r} and $A^{\rho}_{\varepsilon r}$ are defined as in the proof of Theorem 9.2.1. By Theorem 9.2.5, for every $\mathscr{G} \in \mathscr{F}$ and for all $\lambda > 0$ sufficiently large, there exists a sequence $(u^{\mathscr{G}}_k)_k \subset H^1(\mathbb{B}^N)$ of nontrivial nonnegative $\mathscr{G}$-symmetric solutions of

$$\begin{cases} -\left(a + b \displaystyle\int_{\mathbb{B}^N} |\nabla_H u|^2 d\mu\right)\Delta_H u = \lambda w(\sigma) u[2\phi(u) + u\phi'(u)] & \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N) \end{cases} \tag{9.53}$$

such that

$$\lim_{k\to\infty} \|u^{\mathscr{G}}_k\| = \lim_{k\to\infty} \|u^{\mathscr{G}}_k\|_\infty = 0.$$

In (9.53), the function ϕ is defined in (9.51) and $f(t) = t[2\phi(t) + t\phi'(t)]$. More precisely, a careful analysis of the proof of Theorem 9.2.5 and inequality (9.23) ensure that the main conclusion holds provided that

$$\lambda > \left(\frac{a}{2} + \frac{b}{4}\right) \frac{\mathrm{Vol}_H(A^{\rho}_r)}{\|\varphi\|_\infty \int_{A^{\rho}_{\varepsilon r}} w(\sigma) d\mu - \kappa_0 \int_{A^{\rho}_r \setminus A^{\rho}_{\varepsilon r}} w(\sigma)|v^{\varepsilon}_{\rho,r}|^2 d\mu} \left(1 + \frac{1}{(1-\varepsilon)^2 r^2}\right),$$

where κ_0 is the constant given in (9.52). We emphasize that Theorem 9.2.1 cannot be applied to (9.53), since in this case $F(t) = t^2\phi(t)$ and so

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} = \limsup_{t\to 0^+} \phi(t) = \|\varphi\|_\infty > 0, \quad \text{while } \liminf_{t\to 0^+} \frac{F(t)}{t^2} = \liminf_{t\to 0^+} \phi(t) = 0.$$

In particular, we can take for all $k \geq 2$,

$$a_k = \frac{1}{k!k}, \quad b_k = \frac{1}{k!}, \quad \text{and} \quad \varphi(t) = Me^4 e^{\frac{1}{t(t-1)}}$$

for $t \in (0,1)$, with $\varphi(0^+) = \varphi(1^-) = 0$, and where M is a positive constant sufficiently large; see Figure 9.4.

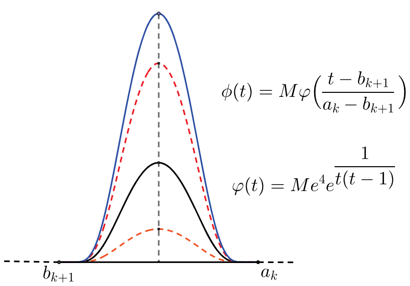

Figure 9.4: The function ϕ in $[b_{k+1}, a_k]$ for different values of M.

9.3 Perturbed Kirchhoff-type problems

The last part of the chapter is devoted to the sublinear case, in which we assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ *is continuous and verifies the assumptions*:

(h_1) $f(t) = o(|t|)$, *as* $|t| \to 0$;
(h_2) $f(t) = o(|t|)$, *as* $|t| \to \infty$;
(h_3) *There exists* $t_0 \in \mathbb{R}$ *such that* $F(t_0) > 0$.

Clearly, (h_2) means exactly that f is sublinear at infinity. Moreover, due to (h_1) and (h_2), the number

$$c_f = \max_{t \neq 0} \frac{|f(t)|}{|t|} \tag{9.54}$$

is well defined and positive. We consider the perturbed form of (K_λ), namely,

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{B}^N} |\nabla_H u|^2 \, d\mu\right)\Delta_H u = \lambda w(\sigma) f(u) + \vartheta \mathtt{w}(\sigma)\mathtt{f}(u) \quad \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N), \end{cases} \tag{$K_{\lambda,\vartheta}$}$$

where $\mathtt{w}$ and $\mathtt{f}$ are of special type. To prove the existence result for $(K_{\lambda,\vartheta})$, we shall use the next general abstract theorem due to B. Ricceri in [226], where, according to the definition given in Section 6.3, the set $\mathcal{W}_X$ is the class of all functionals $I : X \to \mathbb{R}$, with the property that $u_k \rightharpoonup u$ in X and $\liminf_{j\to\infty} I(u_k) \le I(u)$ implies that $u_k \to u$ up to a subsequence.

Theorem 9.3.1. *Let X be a separable reflexive real Banach space. Assume that*

– $\Phi : X \to \mathbb{R}$ *is a coercive, sequentially weakly lower semicontinuous C^1-functional on X, belonging to $\mathcal{W}_X$*;
– Φ *is bounded on each bounded subset of X and its derivative admits a continuous inverse on the dual space X' of X*;
– $\Psi : X \to \mathbb{R}$ *a C^1-functional, with compact derivative*;
– Φ *has a strict local minimum in u_0, with* $\Phi(u_0) = \Psi(u_0) = 0$;
– $\varrho < \chi$, *where*

$$\varrho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{\Psi(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{\Psi(u)}{\Phi(u)}\right\},$$

$$\chi = \sup\left\{\frac{\Psi(u)}{\Phi(u)} \; : \; u \in X \text{ and } \Phi(u) > 0\right\}.$$

Then, for each compact interval $[\lambda_1, \lambda_2] \subset (1/\chi, 1/\varrho)$, *there exists a number* $\eta > 0$, *with the property that for every* $\lambda \in [\lambda_1, \lambda_2]$ *and every C^1-functional* $\Upsilon : X \to \mathbb{R}$, *with compact derivative, there exists* $\delta > 0$ *such that for each* $\vartheta \in [-\delta, \delta]$, *the equation*

$$\Phi' - \lambda\Psi' - \vartheta\Upsilon' = 0$$

admits at least three solutions in X whose norms are strictly less than η.

In order to apply Theorem 9.3.1 to $(K_{\lambda,\vartheta})$, let us introduce

$$\mathscr{W}_q = \left\{ \mathsf{f} \in C(\mathbb{R}) : \sup_{t\in\mathbb{R}\setminus\{0\}} \frac{|\mathsf{f}(t)|}{|t| + |t|^{q-1}} < \infty \right\},$$

where $q \in (2, 2^*)$.

Theorem 9.3.2. *Let w and w satisfy (w) and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying (h_1)–(h_3). Let $\mathsf{f} \in \mathscr{W}_q$ for some $q \in (2, 2^*)$. Then, the following assertions hold:*

(k_1) $(K_{\lambda,0})$ *has only the identically zero solution whenever*

$$0 \le \lambda < a \frac{(N-1)^2}{4c_f \|w\|_\infty},$$

where c_f is given in (9.54);

(k_2) *For every $\mathscr{G} \in \mathscr{F}$, there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$ there is $\delta_\lambda > 0$ with the property that for every $\vartheta \in [-\delta_\lambda, \delta_\lambda]$ equation $(K_{\lambda,\vartheta})$ has at least two distinct nontrivial $\mathscr{G}$-symmetric solutions in $H^1(\mathbb{B}^N)$.*

Proof. The underlying energy functional $\mathcal{J}_\lambda : H^1(\mathbb{B}^N) \to \mathbb{R}$ is defined by

$$\mathcal{J}_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\mathbb{B}^N} w(\sigma)F(u)d\mu - \vartheta \int_{\mathbb{B}^N} \mathsf{w}(\sigma)\,\mathsf{F}(u)d\mu$$

for every $u \in H^1(\mathbb{B}^N)$, where for all $t \in \mathbb{R}$,

$$F(t) = \int_0^t f(s)ds \quad \text{and} \quad \mathsf{F}(t) = \int_0^t \mathsf{f}(s)ds.$$

By Theorem 9.1.2, the functional $\mathcal{J}_\lambda$ is well defined and of class $C^1(H^1(\mathbb{B}^N))$; moreover, its critical points are exactly the solutions of $(K_{\lambda,\vartheta})$.

(k_1) Suppose by contradiction that there exists a solution $u_0 \in H^1(\mathbb{B}^N) \setminus \{0\}$ of $(K_{\lambda,0})$. Thus, taking as test function $\varphi = u_0$, we have

$$\begin{aligned} a\|u_0\|^2 \le (a + b\|u_0\|^2)\|u_0\|^2 &= \lambda \int_{\mathbb{B}^N} w(\sigma)f(u_0)u_0 d\mu \\ &\le \lambda \frac{4}{(N-1)^2} c_f \|w\|_\infty \|u_0\|^2. \end{aligned}$$

Therefore, bearing in mind the assumption on λ, since $a > 0$, we get

$$\|u_0\|^2 \le \lambda \|w\|_\infty \frac{4c_f}{(N-1)^2 a} \|u_0\|^2 < \|u_0\|^2,$$

which is an obvious contradiction. Hence $u_0 = 0$.

(k_2) Let $\mathscr{G} \in \mathscr{F}$. Let us apply Theorem 9.3.1 with the choices $X = H^1_{\mathscr{G}}(\mathbb{B}^N)$ and

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4,$$
$$\Psi(u) = \int_{\mathbb{B}^N} w(\sigma)F(u)d\mu, \quad \Upsilon(u) = \int_{\mathbb{B}^N} \mathsf{w}(\sigma)\,\mathsf{F}(u)d\mu$$

for all $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$.

As a norm-type functional, Φ is coercive and sequentially weakly lower semicontinuous in $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Moreover, Φ belongs to $\mathcal{W}_{H^1_{\mathscr{G}}(\mathbb{B}^N)}$ and is bounded on each bounded subset of $H^1_{\mathscr{G}}(\mathbb{B}^N)$. We claim that its derivative Φ' admits a continuous inverse on the dual space $(H^1_{\mathscr{G}}(\mathbb{B}^N))'$.

To this aim, we identify $(H^1_{\mathscr{G}}(\mathbb{B}^N))'$ with $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Since the Kirchhoff function $K(t) = a + bt$ is nondecreasing in $\mathbb{R}^+_0$, with $K(0) = a > 0$, then the real function $t \mapsto t(a + bt^2)$, with $t \in \mathbb{R}^+_0$, is increasing and onto in its domain, and so there exists a continuous function $\mathsf{g} : \mathbb{R}^+_0 \to \mathbb{R}$ such that

$$\mathsf{g}\big(t(a + bt^2)\big) = t \quad \text{for every } t \in \mathbb{R}^+_0. \tag{9.55}$$

Let $T : H^1_{\mathscr{G}}(\mathbb{B}^N) \to H^1_{\mathscr{G}}(\mathbb{B}^N)$ be the operator defined by

$$T(v) = \begin{cases} \frac{\mathsf{g}(\|v\|)v}{\|v\|} & \text{if } v \neq 0, \\ 0 & \text{if } v = 0 \end{cases}$$

for every $v \in H^1_{\mathscr{G}}(\mathbb{B}^N)$.

Since g is continuous and $\mathsf{g}(0) = 0$, the operator T is continuous in $H^1_{\mathscr{G}}(\mathbb{B}^N)$. Moreover, $K(\|u\|^2) > 0$, and (9.55) yields for each $u \in H^1_{\mathscr{G}}(\mathbb{B}^N) \setminus \{0\}$,

$$\begin{aligned} T\big(\Phi'(u)\big) = T\big(K(\|u\|^2)u\big) &= \frac{\mathsf{g}(K(\|u\|^2)\|u\|)}{K(\|u\|^2)\|u\|}K(\|u\|^2)u \\ &= \frac{\|u\|}{K(\|u\|^2)\|u\|}K(\|u\|^2)u = u. \end{aligned}$$

Thus the derivative Φ' has a continuous inverse on $H^1_{\mathscr{G}}(\mathbb{B}^N)$, as claimed. Furthermore, since $H^1_{\mathscr{G}}(\mathbb{B}^N)$ is a closed subspace of $H^1(\mathbb{B}^N)$ and the embedding $H^1_{\mathscr{G}}(\mathbb{B}^N) \hookrightarrow L^{\wp}(\mathbb{B}^N)$ is compact for every $\wp \in (2, 2^*)$ in force of Theorem 9.1.2, the functionals Ψ and Υ have compact derivatives.

Now, we claim that

$$\lim_{\|u\|\to 0} \frac{\Psi(u)}{\Phi(u)} = \lim_{\|u\|\to\infty} \frac{\Psi(u)}{\Phi(u)} = 0. \tag{9.56}$$

Due to (h_1) and (h_2), for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, 1)$ such that

$$0 \le |f(t)| \le \frac{\varepsilon}{\|w\|_\infty}|t| \quad \text{for every } t, \text{ with either } |t| \le \delta_\varepsilon \text{ or } |t| \ge \delta_\varepsilon^{-1}. \tag{9.57}$$

Let $q \in (2, 2^*)$. It is clear that the real function $t \mapsto t^{1-q} f(t)$ is bounded on $[\delta_\varepsilon, \delta_\varepsilon^{-1}]$. Therefore, for some $q_\varepsilon > 0$, we have

$$0 \leq |f(t)| \leq \frac{\varepsilon}{\|w\|_\infty} |t| + m_\varepsilon |t|^{q-1} \text{ for every } t \in \mathbb{R}. \tag{9.58}$$

Thus, for every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N)$,

$$\begin{aligned} 0 \leq |\Psi(u)| &\leq \int_{\mathbb{B}^N} w(\sigma) |F(u)| d\mu \\ &\leq \int_{\mathbb{B}^N} w(\sigma) \left[\frac{\varepsilon}{2\|w\|_\infty} u^2 + \frac{m_\varepsilon}{q} |u|^q \right] d\mu \\ &\leq \int_{\mathbb{B}^N} \left[\frac{\varepsilon}{2} u^2 + \frac{m_\varepsilon}{q} w(\sigma) |u|^q \right] d\mu \\ &\leq \frac{\varepsilon}{2} \|u\|^2 + \frac{m_\varepsilon}{q} c_q^q \|w\|_\infty \|u\|^q, \end{aligned}$$

where $c_q > 0$ is the best embedding constant in $H^1_{\mathscr{G}}(\mathbb{B}^N) \hookrightarrow L^q(\mathbb{B}^N)$. Thus, for every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N) \setminus \{0\}$,

$$0 \leq \frac{|\Psi(u)|}{|\Phi(u)|} \leq \frac{2|\Psi(u)|}{a\|u\|^2} \leq \frac{\varepsilon}{a} + 2\frac{m_\varepsilon}{aq} c_q^q \|w\|_\infty \|u\|^{q-2}.$$

Since $q > 2$ and $\varepsilon > 0$ is arbitrarily small, the first limit in (9.56) follows at once.

Let $\ell \in (1, 2)$. Since $f \in C(\mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$0 \leq \frac{|f(t)|}{t^{\ell-1}} \leq M_\varepsilon \text{ for every } t \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $\delta_\varepsilon \in (0, 1)$ is from (9.57). The latter relation, together with (9.57), gives that

$$0 \leq |f(t)| \leq \frac{\varepsilon}{\|w\|_\infty} |t| + M_\varepsilon |t|^{\ell-1} \text{ for every } t \in \mathbb{R}.$$

Similarly as above, we get

$$0 \leq |\Psi(u)| \leq \frac{\varepsilon}{2} \|u\|^2 + \frac{M_\varepsilon}{\ell} \|w\|_{\frac{2}{2-\ell}} \|u\|^\ell. \tag{9.59}$$

For every $u \in H^1_{\mathscr{G}}(\mathbb{B}^N) \setminus \{0\}$, we have

$$0 \leq \frac{|\Psi(u)|}{|\Phi(u)|} \leq \frac{2|\Psi(u)|}{a\|u\|^2} \leq \frac{\varepsilon}{a} + 2\frac{M_\varepsilon}{a\ell} \|w\|_{\frac{2}{2-\ell}} \|u\|^{\ell-2}.$$

Since $\varepsilon > 0$ is arbitrary and $\ell \in (1, 2)$, taking the limit as $\|u\| \to \infty$ in $H^1_{\mathscr{G}}(\mathbb{B}^N)$, we obtain the second relation in (9.56).

Moreover, $u_0 = 0$ is a strict global minimum point of the functional Φ, $\Phi(u_0) = \Psi(u_0) = 0$, and (9.56) obviously yields

$$\varrho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{\Psi(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{\Psi(u)}{\Phi(u)}\right\} = 0.$$

Hence it is enough to show that

$$\chi = \sup\left\{\frac{\Psi(u)}{\Phi(u)} : u \in H^1_{\mathscr{G}}(\mathbb{B}^N) \setminus \{0\}\right\} \in \mathbb{R}^+. \tag{9.60}$$

Let $t_0 \in \mathbb{R}$ be the number given in (h_3). Since $w \in L^\infty(\mathbb{B}^N) \setminus \{0\}$ is nonnegative in $\mathbb{B}^N$, there are positive real numbers ρ, r, and w_0, with $\rho > r$, such that condition (9.21) holds.

Fix $\varepsilon \in (0,1)$ and consider the function $\mathrm{v}^\varepsilon_{\rho,r} \in H^1_{\mathscr{G}}(\mathbb{B}^N)$ given by

$$\mathrm{v}^\varepsilon_{\rho,r} = t_0 v^\varepsilon_{\rho,r} \quad \text{in } \mathbb{B}^N, \tag{9.61}$$

where $v^\varepsilon_{\rho,r}$ is defined in (9.22). Direct computations yield

(j_1) $\operatorname{supp}(\mathrm{v}^\varepsilon_{\rho,r}) \subset A^\rho_r$;

(j_2) $\|\mathrm{v}^\varepsilon_{\rho,r}\|_\infty \leq |t_0|$;

(j_3) $\mathrm{v}^\varepsilon_{\rho,r}(\sigma) = 1$ for every $\sigma \in A^\rho_{\varepsilon r}$.

The above properties and the assumptions on the weight w imply that

$$\begin{aligned}\Psi(\mathrm{v}^\varepsilon_{\rho,r}) &= \int_{\mathbb{B}^N} w(\sigma)F(\mathrm{v}^\varepsilon_{\rho,r})d\mu = \int_{A^\rho_r} w(\sigma)F(\mathrm{v}^\varepsilon_{\rho,r})d\mu \\ &= \int_{A^\rho_{\varepsilon r}} w(\sigma)F(\mathrm{v}^\varepsilon_{\rho,r})d\mu + \int_{A^\rho_r\setminus A^\rho_{\varepsilon r}} w(\sigma)F(\mathrm{v}^\varepsilon_{\rho,r})d\mu \\ &\geq w_0F(t_0)\mathrm{Vol}_H(A^\rho_{\varepsilon r}) - \|w\|_\infty \max_{|t|\leq|t_0|}|F(t)|\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon r}).\end{aligned}$$

We claim that there is $\varepsilon_0 > 0$ such that $\Psi(\mathrm{v}^{\varepsilon_0}_{\rho,r}) > 0$. Indeed, set $g_\mu : (0,1) \to \mathbb{R}^+$ be the real continuous function defined by

$$g_\mu(\varepsilon) = \frac{\mathrm{Vol}_H(A^\rho_{\varepsilon r})}{\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon r})}, \quad \varepsilon \in (0,1).$$

Clearly, $g_\mu(\varepsilon) \to \infty$ if $\varepsilon \to 1^-$. Hence $F(t_0) > 0$ by assumption (h_3), and there is $\varepsilon_0 > 0$ such that

$$g_\mu(\varepsilon_0) = \frac{\mathrm{Vol}_H(A^\rho_{\varepsilon_0 r})}{\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon_0 r})} > \|w\|_\infty \frac{\max_{|t|\leq|t_0|}|F(t)|}{w_0F(t_0)}.$$

Thus $\Psi(\mathrm{v}^{\varepsilon_0}_{\rho,r}) > 0$ as affirmed. Consequently, (9.60) immediately holds by (9.56), and the number

$$\lambda^* = \inf\left\{\frac{\Phi(u)}{\Psi(u)} : u \in H^1_{\mathscr{G}}(\mathbb{B}^N) \text{ and } \Psi(u) > 0\right\} < \infty \tag{9.62}$$

is well defined. Moreover, $\lambda^* = \chi^{-1}$.

Applying Theorem 9.3.1, for every $\lambda > \lambda^*$ there exists $\delta_\lambda > 0$ such that for each $\vartheta \in [-\delta_\lambda, \delta_\lambda]$ the functional $\mathcal{J}_\lambda$, restricted to $H^1_{\mathscr{G}}(\mathbb{B}^N)$, has at least three critical points, as desired.

Since $\mathcal{J}_\lambda$ is $\mathscr{G}$-invariant with respect to $\circledast_{\mathscr{G}} : \mathscr{G} \times H^1(\mathbb{B}^N) \to H^1(\mathbb{B}^N)$, defined in (9.5), the symmetric criticality Theorem A.1.2 implies that the critical points of $\mathcal{J}_\lambda|_{H^1_{\mathscr{G}}(\mathbb{B}^N)}$ are also critical points of $\mathcal{J}_\lambda$ in $H^1(\mathbb{B}^N)$, and so $\mathscr{G}$-symmetric solutions of $(K_{\lambda,\vartheta})$. □

In any case we can produce a concrete bound for λ^*. Indeed,

$$\Phi(\mathrm{v}^{\varepsilon_0}_{\rho,r}) = \frac{a}{2}\|\mathrm{v}^{\varepsilon_0}_{\rho,r}\|^2 + \frac{b}{4}\|\mathrm{v}^{\varepsilon_0}_{\rho,r}\|^4,$$

and so (9.23) gives

$$\Phi(\mathrm{v}^{\varepsilon_0}_{\rho,r}) \le \left(1 + \frac{1}{(1-\varepsilon_0)^2 r^2}\right)^2 \frac{\mathrm{Vol}^2_H(A^\rho_r)}{2} t_0^2 \times \left(a + \frac{b}{2}\left(1 + \frac{1}{(1-\varepsilon_0)^2 r^2}\right)^2 \mathrm{Vol}^2_H(A^\rho_r) t_0^2\right).$$

Thanks to the above inequality, a careful analysis of the proof of Theorem 9.3.2 ensures that the main conclusion in (k_2) holds true, provided that

$$\lambda > \frac{t_0^2 \mathrm{Vol}^2_H(A^\rho_r)(1 + \frac{1}{(1-\varepsilon_0)^2 r^2})^2 (a + \frac{b}{2}(1 + \frac{1}{(1-\varepsilon_0)^2 r^2})^2 \mathrm{Vol}^2_H(A^\rho_r) t_0^2)}{2(w_0 F(t_0)\mathrm{Vol}_H(A^\rho_{\varepsilon_0 r}) - \|w\|_\infty \max_{|t|\le|t_0|}|F(t)|\mathrm{Vol}_H(A^\rho_r \setminus A^\rho_{\varepsilon_0 r}))}.$$

Theorem 9.3.2 can be applied in particular to the equation

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{B}^N} |\nabla_H u|^2\, d\mu\right)\Delta_H u = \lambda w(\sigma)\log(1+u^2) \quad \text{in } \mathbb{B}^N, \\ u \in H^1(\mathbb{B}^N), \end{cases} \tag{9.63}$$

where w satisfies (w); see Figure 9.5. Hence, for every $\mathscr{G} \in \mathscr{F}$ there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$ equation (9.63) has at least two nontrivial distinct $\mathscr{G}$-symmetric solutions in $H^1(\mathbb{B}^N)$. Theorem 9.3.2 gives furthermore more precise information, that is, it asserts that the number of solutions is stable with respect to small subcritical perturbations.

In the last results of the section, we turn back to equations settled on a homogeneous Hadamard manifold $\mathcal{M}$ of the type treated in Chapter 8. More precisely, we present an analogue of Theorems 9.2.4 and 9.3.2 in this general framework.

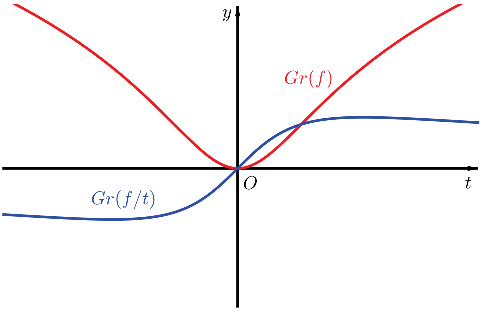

Figure 9.5: The model case $f(t) = \log(1 + t^2)$.

Theorem 9.3.3. *Let $\mathcal{M} = (\mathcal{M}, g)$ be a homogeneous Hadamard manifold of dimension $N \geq 3$ and let $\mathscr{G}$ be a compact connected subgroup of $\mathrm{Isom}_g(\mathcal{M})$ such that $\mathrm{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma_0\}$. Let w satisfy condition (w) of Chapter 8 and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the growth condition*

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1 + |t|^{q-1}} < \infty,$$

for some $q \in (2, 2^*)$ and such that

$$-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^2} \leq \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty.$$

Then, there exists $\lambda^\star > 0$ such that for every $\lambda \in (0, \lambda^\star)$ the equation

$$\begin{cases} (a + b\|u\|^2)(-\Delta_g u + u) = \lambda w(\sigma) f(u) & \text{in } \mathcal{M}, \\ u \in H_g^1(\mathcal{M}), \end{cases} \tag{P_λ}$$

admits a $\mathscr{G}$-invariant solution $u_\lambda^{\mathscr{G}} \in H^1(\mathcal{M})$ and

$$\lim_{\lambda\to 0^+} \|u_\lambda^{\mathscr{G}}\| = 0.$$

In the sublinear case for the perturbed form of (P_λ), namely

$$\begin{cases} (a + b\|u\|^2)(-\Delta_g u + u) = \lambda w(\sigma) f(u) + \vartheta \mathtt{w}(\sigma) \mathtt{f}(u) & \text{in } \mathcal{M}, \\ u \in H_g^1(\mathcal{M}), \end{cases} \tag{$P_{\lambda,\vartheta}$}$$

we get the next result.

Theorem 9.3.4. *Let $\mathcal{M} = (\mathcal{M}, g)$ be a homogeneous Hadamard manifold of dimension $N \geq 3$ and let $\mathscr{G}$ be a compact connected subgroup of $\mathrm{Isom}_g(\mathcal{M})$ such that $\mathrm{Fix}_{\mathscr{G}}(\mathcal{M}) = \{\sigma_0\}$. Let w and $\mathtt{w}$ satisfy condition (w) of Chapter 8, let $f \in C(\mathbb{R})$ be a function verifying (h_1)–(h_3), and let $\mathtt{f} \in \mathscr{W}_q$ for some $q \in (2, 2^*)$. Then, the following assertions hold:*

(k_1) *Equation* $(P_{\lambda,0})$ *has only the trivial solution, whenever*

$$0 \leq \lambda < \frac{a}{c_2^2 C_f \|w\|_\infty}, \text{ where } C_f = \max_{t \neq 0} \frac{|f(t)|}{|t|} \text{ and } c_2 = \sup_{u \in H_g^1(\mathcal{M}) \setminus \{0\}} \frac{\|u\|_2}{\|u\|};$$

(k_2) *There exists* $\lambda^* > 0$ *such that for every* $\lambda > \lambda^*$ *there is* $\delta_\lambda > 0$ *with the property that for every* $\vartheta \in [-\delta_\lambda, \delta_\lambda]$ *equation* $(P_{\lambda,\vartheta})$ *has at least two nontrivial distinct* $\mathcal{G}$*-symmetric solutions in* $H_g^1(\mathcal{M})$*.*

Comments on Chapter 9

An interesting open problem is to extend the methods used to prove the main results of Chapter 9 to the case of degenerate Kirchhoff equations in order to obtain a hyperbolic version of some classical results due to O. Kavian [135], M. Struwe [238, 237], and P. Rabinowitz [220]. More generally, inspired by the heuristic ideas contained in the entire book, in our opinion, a careful analysis of the symmetries on Sobolev spaces associated either to manifolds or to sub-Riemannian structures seems to be a very rich and fruitful theoretical argument in studying existence results and multiplicity phenomena arising from different branches of pure and applied mathematics.

A Appendix – the symmetric criticality principle

An important approach for studying properties of the solution space of nonlinear equations is to restrict attention to manifolds that admit a specified group of symmetries. Viewing these issues strictly from the point of view of action of smooth functionals, R. Palais proved in his celebrated work [202] the so-called *principle of symmetric criticality*, briefly (SC) principle.

The validity of Palais' principle has a powerful impact on applications to nonlinear problems of mathematical physics which are set on noncompact manifolds, and whose associated energy functionals are invariant under the action of suitable groups of isometries.

For instance, this approach, used previously in the context of transverse symmetry group actions, provides a generalization of the well known unimodularity condition that quite naturally arises in spatially homogeneous cosmological physical models.

In this appendix we shortly recall the (SC) principle and some of its extensions for general energy functionals, possibly nonsmooth, associated to variational problems. We refer to the brilliant and detailed discussion on the subject given by J. Kobayashi and M. Ôtani in [138].

A.1 The principle for C^1 functionals

Most of the results in this section are essentially contained in [138, 202] and in [151, Part I, Chapter I]. However, we recall them for the sake of completeness.

Let X be a Banach space and let X' be its dual. As usual, we denote by $\|\cdot\|$ and $\|\cdot\|_{X'}$ the norms on X and X', respectively. Moreover, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between the spaces X and X'.

Let $(\mathscr{G},\cdot)$ be a group, e its identity element, and let π be a representation of $\mathscr{G}$ over X, that is, $\pi(g) : X \to X$ is a linear bounded operator from X into X for each $g \in \mathscr{G}$, such that the following properties hold:

(i) $\pi(e) = \mathrm{id}_X$;

(ii) $\pi(g_1g_2)u = \pi(g_1)(\pi(g_2)u)$ *for every* $g_1, g_2 \in \mathscr{G}$ *and* $u \in X$,

where, as is customary, $\pi(g)u$ denotes the image of $u \in X$ trough the linear bounded operator $\pi(g)$.

This representation π of $\mathscr{G}$ over X induces a canonical representation $\pi_{X'}$ of $\mathscr{G}$ over the dual space X' of X, that is, $\pi_{X'}(g) : X' \to X'$ is a linear bounded operator from X' into X' for each $g \in \mathscr{G}$ such that

(i^*) $\pi_{X'}(e) = \mathrm{id}_{X'}$;

(ii^*) $\pi_{X'}(g_1g_2)v^* = \pi(g_1)(\pi(g_2)v^*)$ *for every* $g_1, g_2 \in \mathscr{G}$ *and* $v^* \in X'$,

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where the functional $\pi_{X'}(g)v^* \in X'$ is defined by duality

$$\langle \pi_{X'}(g)v^*, u\rangle = \langle v^*, \pi(g^{-1})u\rangle \tag{A.1}$$

for every $u \in X$.

In order to simplify the notations, for $g \in \mathscr{G}$, $u \in X$, and $v^* \in X'$, we put

$$gu = \pi(g)u \quad \text{and} \quad gv^* = \pi_{X'}(g)v^*.$$

A functional $I : X \to \mathbb{R}$ is said to be $\mathscr{G}$-invariant if $I(gu) = I(u)$ for every $u \in X$ and $g \in \mathscr{G}$. A subset $S \subseteq X$ is called $\mathscr{G}$-invariant if

$$gS = \{gu : u \in S\} \subseteq S.$$

Let us define the subspace of $\mathscr{G}$-symmetric points of X given by

$$\mathrm{Fix}_{\mathscr{G}}(X) = \{u \in X : gu = u \text{ for every } g \in \mathscr{G}\}.$$

Analogously, let

$$\mathrm{Fix}_{\mathscr{G}}(X') = \{v^* \in X' : gv^* = v^* \text{ for every } g \in \mathscr{G}\},$$

be the subspace of $\mathscr{G}$-symmetric points of X'.

By (A.1), it follows that

$$v^* \in \mathrm{Fix}_{\mathscr{G}}(X') \Longleftrightarrow v^* : X \to \mathbb{R} \text{ is } \mathscr{G}\text{-invariant}.$$

Moreover, $\mathrm{Fix}_{\mathscr{G}}(X)$ and $\mathrm{Fix}_{\mathscr{G}}(X')$ are linear closed subspaces of X and X', respectively, since for every $g \in \mathscr{G}$ the operators $\pi(g)$ and $\pi_{X'}(g)$ are linear and bounded. Consequently, $\mathrm{Fix}_{\mathscr{G}}(X)$ and $\mathrm{Fix}_{\mathscr{G}}(X')$ are Banach spaces with their induced norms. Set

$$C^1_{\mathscr{G}}(X) = \{I : X \to \mathbb{R} : I \in C^1(X) \text{ and } \mathscr{G}\text{-invariant}\},$$
$$(\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} = \{v^* \in X' : \langle v^*, u\rangle = 0 \text{ for every } u \in \mathrm{Fix}_{\mathscr{G}}(X)\}.$$

We have the following result that represents the (SC) principle for smooth functionals in its abstract and general form.

Theorem A.1.1. *The following facts are equivalent*:
(i_1) *If* $I \in C^1_{\mathscr{G}}(X)$ *and* $(I|_{\mathrm{Fix}_{\mathscr{G}}(X)})'(u) = 0$, *then* $I'(u) = 0$;
(i_2) $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} = \{0\}$.

Proof. $(i_1) \Rightarrow (i_2)$ Suppose by contradiction that $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} \neq \{0\}$ and let v^* be a nontrivial element of $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp}$. Define $I_{X'}$ by $I_{X'}(u) = \langle v^*, u\rangle$ for all $u \in X$. It is clear that $I_{X'} \in C^1_{\mathscr{G}}(X)$ and $(I_{X'})' = v^* \neq 0$, so $I_{X'}$ has no critical points in X.

On the other hand, $v^* \in (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp}$ implies that $v^*|_{\mathrm{Fix}_{\mathscr{G}}(X)} = 0$. Thus $(I_{X'}|_{\mathrm{Fix}_{\mathscr{G}}(X)})'(u) = 0$ for every $u \in \mathrm{Fix}_{\mathscr{G}}(X)$, and this contradicts (i_1), as required.

$(i_2) \Rightarrow (i_1)$ Let $u_0 \in \mathrm{Fix}_{\mathscr{G}}(X)$ be a critical point of restriction $I|_{\mathrm{Fix}_{\mathscr{G}}(X)}$ and let us prove that $I'(u_0) = 0$. Since $I(u_0) = I|_{\mathrm{Fix}_{\mathscr{G}}(X)}(u_0)$ and $I(u_0 + u) = I|_{\mathrm{Fix}_{\mathscr{G}}(X)}(u_0 + u)$ for every $u \in \mathrm{Fix}_{\mathscr{G}}(X)$, we obtain

$$\langle I'(u_0), u\rangle = \langle (I|_{\mathrm{Fix}_{\mathscr{G}}(X)})'(u_0), u\rangle_{\mathrm{Fix}_{\mathscr{G}}(X)}$$

for every $u \in \mathrm{Fix}_{\mathscr{G}}(X)$, where $\langle \cdot, \cdot\rangle_{\mathrm{Fix}_{\mathscr{G}}(X)}$ denotes the duality pairing between $\mathrm{Fix}_{\mathscr{G}}(X)$ and its dual $(\mathrm{Fix}_{\mathscr{G}}(X))'$. This implies that $I'(u_0) \in (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp}$. On the other hand, the $\mathscr{G}$-invariance of functional I yields

$$\begin{aligned}\langle I'(gu), v\rangle &= \lim_{t\to 0} \frac{I(gu + tv) - I(gu)}{t} \\ &= \lim_{t\to 0} \frac{I(u + tg^{-1}v) - I(u)}{t} = \langle I'(u), g^{-1}v\rangle \\ &= \langle gI'(u), v\rangle\end{aligned}$$

for every $g \in \mathscr{G}$ and $u, v \in X$. This means that I' is $\mathscr{G}$-equivariant, that is, $I'(gu) = gI'(u)$ for every $g \in \mathscr{G}$ and $u \in X$. Since $u_0 \in \mathrm{Fix}_{\mathscr{G}}(X)$, we obtain $gI'(u_0) = I'(u_0)$ for every $g \in \mathscr{G}$, that is, $I'(u_0) \in \mathrm{Fix}_{\mathscr{G}}(X')$. Hence $I'(u_0) \in \mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} = \{0\}$ by assumption. Therefore $I'(u_0) = 0$, as required. □

In the sequel, we are interested in finding conditions in order to recover the requirement $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} = \{0\}$, so that the (SC) principle is available by Theorem A.1.1. To this aim, we consider two different settings: the so-called *compact* case and the *isometric* case.

The compact case

We recall that for each $u \in X$ there exists a unique element $v \in X$ such that

$$\langle v^*, v\rangle = \int_{\mathscr{G}} \langle v^*, gu\rangle d\mu \quad \text{for every } v^* \in X', \tag{A.2}$$

where $d\mu$ is the normalized Haar measure on $\mathscr{G}$; see [227, Theorem 3.27]. Hence $u \mapsto v = Au$ defines an operator, and Au is actually in $\mathrm{Fix}_{\mathscr{G}}(X)$ by formula (20) on page 68 of Theorem 2.5.13 of [242]. In conclusion, $A : X \to \mathrm{Fix}_{\mathscr{G}}(X)$, and A is called the *averaging operator* on the group $\mathscr{G}$.

We are now in position to deal with the (SC) principle in the compact setting whose original form has been given by R. Palais in [202, Proposition 4.2].

Theorem A.1.2. *Let $\mathscr{G}$ be a compact topological group. Assume that the representation π of $\mathscr{G}$ over X is continuous. Then* $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} = \{0\}$.

Proof. Suppose by contradiction that $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp} \neq \{0\}$ and take $v^* \neq 0$ in $\mathrm{Fix}_{\mathscr{G}}(X') \cap (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp}$. Define the hyperplane

$$H_{v^*} = \{u \in X : \langle v^*, u\rangle = 1\}.$$

Clearly, H_{v^*} is a nonempty, closed, and convex subset of X. Since $v^* \in \mathrm{Fix}_{\mathscr{G}}(X')$, on account of (A.2), for any $u \in H_{v^*}$ we have

$$\begin{aligned}\langle v^*, Au\rangle &= \int_{\mathscr{G}} \langle v^*, gu\rangle d\mu = \int_{\mathscr{G}} \langle g^{-1}v^*, u\rangle d\mu \\ &= \int_{\mathscr{G}} \langle v^*, u\rangle d\mu = \langle v^*, u\rangle \int_{\mathscr{G}} d\mu = \langle v^*, u\rangle = 1.\end{aligned} \tag{A.3}$$

On the other hand, $v^* \in (\mathrm{Fix}_{\mathscr{G}}(X))^{\perp}$. Consequently, $\langle v^*, Au\rangle = 0$ for any $u \in H_{v^*}$. This contradicts (A.3) and completes the proof. □

The isometric case

Let X be a reflexive real Banach space. We say that the group $\mathscr{G}$ *acts isometrically* on X if $\|gu\| = \|u\|$ for every $u \in X$ and $g \in \mathscr{G}$. The following preparatory results hold.

Lemma A.1.3. *If $\mathscr{G}$ acts isometrically on X, then $\mathscr{G}$ acts isometrically on the dual space X', that is, $\|gv^*\|_{X'} = \|v^*\|_{X'}$ for every $v^* \in X'$ and $g \in \mathscr{G}$.*

Proof. Let $v^* \in X'$ and $g \in \mathscr{G}$. We first prove that $\|gv^*\|_{X'} \leq \|v^*\|_{X'}$. Indeed, since $\mathscr{G}$ acts isometrically on X, it follows that

$$\begin{aligned}\|gv^*\|_{X'} &= \sup_{\|u\|=1} |\langle gv^*, u\rangle| = \sup_{\|u\|=1} |\langle v^*, g^{-1}u\rangle| \\ &= \sup_{\|u\|=1} \left|\left\langle v^*, \frac{g^{-1}u}{\|g^{-1}u\|}\right\rangle\right| \|u\|, \qquad \|g^{-1}u\| = \|u\| \\ &\leq \sup_{\|u\|=1} \|v^*\|_{X'} \|u\| = \|v^*\|_{X'}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\|v^*\|_{X'} &= \|g^{-1}(gv^*)\|_{X'} = \sup_{\|u\|=1} |\langle g^{-1}(gv^*), u\rangle| = \sup_{\|u\|=1} |\langle gv^*, gu\rangle| \\ &= \sup_{\|u\|=1} \left|\left\langle gv^*, \frac{gu}{\|gu\|}\right\rangle\right| \|u\| \leq \sup_{\|u\|=1} \|gv^*\|_{X'} \|u\| = \|gv^*\|_{X'}.\end{aligned}$$

Thus, $\|gv^*\|_{X'} = \|v^*\|_{X'}$ for every $v^* \in X'$ and $g \in \mathscr{G}$, as desired. □

Let $F : X \to 2^{X'}$ be the duality map defined for all $u \in X$ by

$$F(u) = \{v^* \in X' : \|v^*\|_{X'} = \|u\| \text{ and } \|v^*\|_{X'}^2 = \langle v^*, u\rangle\},$$

and let $R(F) \subseteq X'$ be the range of the multifunction F.

Lemma A.1.4. *Assume that the group $\mathcal{G}$ acts isometrically on X. Then, the fiber $F^{-1}(v^*)$ is a $\mathcal{G}$-invariant set for every $v^* \in \mathrm{Fix}_{\mathcal{G}}(X') \cap R(F)$, i. e., $gF^{-1}(v^*) \subseteq F^{-1}(v^*)$ for every $g \in \mathcal{G}$.*

If X is also strictly convex, then $F^{-1}(\mathrm{Fix}_{\mathcal{G}}(X'))$ is a subset of $\mathrm{Fix}_{\mathcal{G}}(X)$.

Proof. Let $v^* \in \mathrm{Fix}_{\mathcal{G}}(X') \cap R(F)$ and take $u \in F^{-1}(v^*)$, where

$$F^{-1}(v^*) = \{u \in X : \|u\| = \|v^*\|_{X'} \text{ and } \langle v^*, u\rangle = \|v^*\|^2_{X'}\}.$$

We claim that $gu \in F^{-1}(v^*)$. Indeed, since $u \in F^{-1}(v^*)$ Lemma A.1.3 yields that $\|gu\| = \|u\| = \|v^*\|_{X'}$. Moreover, since $v^* \in \mathrm{Fix}_{\mathcal{G}}(X') \cap R(F)$ and $u \in F^{-1}(v^*)$, we also have

$$\langle v^*, gu\rangle = \langle g^{-1}v^*, u\rangle = \langle v^*, u\rangle = \|v^*\|^2_{X'}.$$

Thus $gF^{-1}(v^*) \subseteq F^{-1}(v^*)$ for every $g \in \mathcal{G}$.

By Theorem 3.4 on page 62 of [61] the reflexivity of X ensures that F is surjective, that is $\bigcup_{u\in X} F(u) = X'$. Assume now that X is also strictly convex. Then by Corollary 1.9 on page 46 of [61], the duality map F is strictly monotone, and so injective. Hence F^{-1} is a single-valued function from X' onto X. Therefore the first part of the lemma shows at once that $gF^{-1}(v^*) = F^{-1}(v^*)$ for every $v^* \in X'$ and $g \in \mathcal{G}$, i. e., $F^{-1}(\mathrm{Fix}_{\mathcal{G}}(X')) \subseteq \mathrm{Fix}_{\mathcal{G}}(X)$, as stated. □

Theorem A.1.5. *Let X be a reflexive and strictly convex real Banach space and let $\mathcal{G}$ be a topological group that acts isometrically on X. Then*

$$\mathrm{Fix}_{\mathcal{G}}(X') \cap \left(\mathrm{Fix}_{\mathcal{G}}(X)\right)^{\perp} = \{0\}.$$

Proof. Assume by contradiction that $\mathrm{Fix}_{\mathcal{G}}(X') \cap (\mathrm{Fix}_{\mathcal{G}}(X))^{\perp} \neq \{0\}$ and let $v^* \in \mathrm{Fix}_{\mathcal{G}}(X') \cap (\mathrm{Fix}_{\mathcal{G}}(X))^{\perp} \setminus \{0\}$. In particular, $v^* \in \mathrm{Fix}_{\mathcal{G}}(X')$ implies that $F^{-1}(v^*) \in \mathrm{Fix}_{\mathcal{G}}(X)$ by Lemma A.1.4. But $v^* \in (\mathrm{Fix}_{\mathcal{G}}(X))^{\perp}$ gives $\|v^*\|_{X'} = \langle v^*, F^{-1}(v^*)\rangle = 0$, which is the desired contradiction. □

A.2 Extensions to nonsmooth functionals of special forms

Let X be a reflexive real Banach space and let $\mathcal{E} : X \to \mathbb{R}$ be a functional of class C^1. Assume that $\psi : X \to \mathbb{R} \cup \{\infty\}$ is a proper, convex, and lower semicontinuous functional. Then, according to [239], we say that $J = \mathcal{E} + \psi$ is a *Szulkin-type functional*.

An element $u \in X$ is named a *critical point of* $J = \mathcal{E} + \psi$ if

$$\langle \mathcal{E}'(u), v - u\rangle + \psi(v) - \psi(u) \geq 0 \quad \text{for every } v \in X. \tag{A.4}$$

The number $J(u)$ is a *critical value* of J.

For every $u \in D(\psi) = \{v \in X : \psi(v) < \infty\}$, the set

$$\partial\psi(u) = \{x^* \in X' : \psi(v) - \psi(u) \geq \langle x^*, v - u\rangle \text{ for every } v \in X\}$$

is called the *subdifferential* of ψ at u. An equivalent formulation for (A.4) is

$$0 \in \mathcal{E}'(u) + \partial\psi(u) \quad \text{in } X'.$$

Proposition A.2.1. *Every local minimum point of $J = \mathcal{E} + \psi$ is a critical point of J in the sense given in* (A.4).

Proof. Let $u \in X$ be a local minimum point of the functional J, so in turn $u \in D(\psi)$. Due to the convexity of ψ, for every $t > 0$ small enough,

$$0 \le \frac{J((1-t)u + tv) - J(u)}{t} \le \frac{\mathcal{E}(u + t(v-u)) - \mathcal{E}(u)}{t} + \big(\psi(v) - \psi(u)\big)$$

for every $v \in X$. Letting $t \to 0^+$, we get (A.4), as claimed. □

Although a slightly more general version is proved by J. Kobayashi and M. Otani in [138], we recall the following form of the *principle of symmetric criticality for Szulkin functionals* used in Chapter 8.

Theorem A.2.2. *Let $J = \mathcal{E} + \psi : X \to \mathbb{R} \cup \{\infty\}$ be a Szulkin-type functional, where X is a reflexive real Banach space. Assume that a compact group $\mathscr{G}$ acts linearly and continuously on X, and that the functionals $\mathcal{E}$ and ψ are $\mathscr{G}$-invariant. If there exists $u \in \Sigma$, where Σ denotes the subspace of $\mathscr{G}$-invariant functions of X, such that*

$$0 \in (\mathcal{E}|_\Sigma)'(u) + \partial(\psi|_\Sigma)(u) \text{ in } \Sigma',$$

then

$$0 \in \mathcal{E}'(u) + \partial\psi(u) \text{ in } X',$$

i. e., u is a critical point of J in the sense given in (A.4).

The proof of Theorem A.2.2 is quite involved, it combines in an ingenious way various methods and notions from convex and functional analysis. The proof is, however, given in details in [138, Theorem 3.16].

List of symbols

The following symbols are frequently used throughout the manuscript:

$\mathbb{R}_0^+$	$\mathbb{R}_0^+ = [0, \infty)$;
$\mathbb{R}^+$	$\mathbb{R}^+ = (0, \infty)$;
$\mathbb{R}^N$	standard N-dimensional Euclidean space;
$\mathbb{S}^N$	unit sphere in $\mathbb{R}^{N+1}$ endowed by the induced Euclidean metric;
$\mathcal{O} \times \mathbb{R}^{N-m}$	strip-like domain in $\mathbb{R}^N$;
$\mathbb{G}$	Carnot group $\mathbb{G} = (\mathbb{G}, \circ)$ endowed by the operation $\circ$;
$\mathfrak{G}$	Lie algebra associated to the Carnot group $\mathbb{G}$;
$\mathbb{H}^N$	Heisenberg group $\mathbb{H}^N = \mathbb{C}^N \times \mathbb{R}$;
Ω_ψ	strip-like domain in Heisenberg group $\mathbb{H}^N$;
H^N	N-dimensional hyperbolic space;
$\mathbb{B}^N$	Poincaré ball model of the hyperbolic space H^N;
$\mathcal{M}$	Riemannian manifold $\mathcal{M} = (\mathcal{M}, g)$ endowed of the metric g;
$O(N)$	orthogonal group in dimension N;
$SO(N)$	special orthogonal group in dimension N;
$U(N)$	unitary group in dimension N;
$\mathcal{M}(\mathbb{R}^N)$	space of $\mathbb{R}$-valued Lebesgue measurable functions in $\mathbb{R}^N$;
$L^\wp(\mathbb{R}^N)$	Lebesgue $L^\wp$ space of $\mathbb{R}$-valued functions in $\mathbb{R}^N$, with $\wp \in [1, \infty]$;
$L^\wp_{\text{loc}}(\mathbb{R}^N)$	space of the locally $L^\wp$ summable of $\mathbb{R}$-valued functions in $\mathbb{R}^N$, with $\wp \in [1, \infty)$;
$L^\wp(\mathbb{R}^N, \lvert x\rvert^{-\wp} dx)$	space of the $L^\wp$ summable weighted functions in $\mathbb{R}^N$, with respect to the weight $\lvert x\rvert^{-\wp}$ and with $\wp \in (1, N)$;
$C^1(\mathbb{R}_0^+)$	space of $\mathbb{R}$-valued functions continuously differentiable in $\mathbb{R}_0^+$;
$W^{1,\wp}(\mathbb{R}^N)$	Sobolev space of weakly differentiable $\mathbb{R}$-valued functions in $\mathbb{R}^N$ with $L^\wp$ weak derivatives;
$H^1(\mathbb{R}^N)$	Hilbertian Sobolev $W^{1,2}$ space of $\mathbb{R}$-valued functions defined over $\mathbb{R}^N$;
$H^1_{\text{rad}}(\mathbb{R}^N)$	subspace of $H^1(\mathbb{R}^N)$ of $O(N)$-invariant functions;
$H^{-1}(\mathbb{R}^N)$	dual space of $H^1(\mathbb{R}^N)$;
$H^1(\mathcal{O} \times \mathbb{R}^{N-m})$	Hilbertian Sobolev $W^{1,2}$ space of $\mathbb{R}$ valued functions defined over $\mathcal{O} \times \mathbb{R}^{N-m}$;
$H^1_0(\mathcal{O} \times \mathbb{R}^{N-m})$	subspace of $H^1(\mathcal{O} \times \mathbb{R}^{N-m})$ of functions with null trace at the boundary of $\mathcal{O} \times \mathbb{R}^{N-m}$;
$W^{k,\wp}(\mathbb{S}^N)$	Sobolev space of the k times weakly differentiable functions in $\mathbb{S}^N$ whose derivatives up to order k are $L^\wp$ summable over $\mathbb{S}^N$;
$H^m(\mathbb{S}^N)$	Hilbertian Sobolev $W^{m,2}$ space of $\mathbb{R}$ valued functions defined over $\mathbb{S}^N$;
$H^m_G(\mathbb{S}^N)$	subspace of $H^m(\mathbb{S}^N)$ of G-invariant functions;
$HW^{1,2}(\Omega_\psi)$	Hilbertian Sobolev $HW^{1,2}$ Folland–Stein space of $\mathbb{R}$-valued functions defined over Ω_ψ;

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$HW_0^{1,2}(\Omega_\psi)$	subspace of $HW^{1,2}(\Omega_\psi)$ of functions with null trace at the boundary of Ω_ψ;
$HW_{0,T}^{1,2}(\Omega_\psi)$	subspace of $HW_0^{1,2}(\Omega_\psi)$ of T-invariant functions;
$HW^{1,2}_{0,\widehat{T}^{\omega_j}_{N,j}}(\Omega_\psi)$	subspace of $HW_0^{1,2}(\Omega_\psi)$ of $\widehat{T}^{\omega_j}_{N,j}$-invariant functions;
$W_g^{k,\wp}(\mathcal{M})$	Sobolev space of the k times weakly differentiable $\mathbb{R}$-valued functions on the manifold $\mathcal{M} = (\mathcal{M}, g)$ whose derivatives up to order k are $L^\wp$ summable over $\mathcal{M}$;
$H_g^1(\mathcal{M})$	Hilbertian Sobolev $W_g^{1,2}$ space of $\mathbb{R}$ valued functions defined over the manifold $\mathcal{M}$;
$H^1_{\mathscr{G},g}(\mathcal{M})$	subspace of $H_g^1(\mathcal{M})$ of $\mathscr{G}$-invariant functions.

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