A Statistical Theory of Gravitating Body Formation in Extrasolar Systems

Alexander Krot

A Statistical Theory of Gravitating Body Formation in Extrasolar Systems

A Statistical Theory of Gravitating Body Formation in Extrasolar Systems

By

Alexander Krot

Cambridge
Scholars
Publishing

A Statistical Theory of Gravitating Body Formation
in Extrasolar Systems

By Alexander Krot

This book first published 2021

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-6222-0
ISBN (13): 978-1-5275-6222-6

Dedicated to my dear parents

CONTENTS

Part II: The Statistical Theory of Gravity in Solar and Extrasolar System Applications

Introduction

Despite the significant successes and achievements in astrophysics and geophysics over recent decades, the problems of the origin of the Solar system and the formation of planets remain important and relevant, if only because there is no general or consistent scenario for the formation of the proto-Sun and the protoplanetary system from a protosolar nebula (molecular cloud).

In particular, in astrophysics, there is the problem of the gravitational condensation of an *infinitely spread* gas-dust cosmic matter which is closely related to the problem of gravitational instability and the well-known Jeans criterion [1]. The main difficulty of the theory of Jeans is associated with the *gravitational paradox*, that is, for an infinite homogeneous substance, there exists no potential of the force of gravity [2]. In other words, due to the absence of the gravitational field inside a spread molecular cloud, gravitational tightening could not arise.

Recently, the general problems of the formation of protoplanetary systems, the study of their dynamical behavior, and the formation and evolution of the planets have begun attracting additional attention among the scientific community in connection with the discovery of *extrasolar planets*, considered one of the greatest achievements of modern astronomy.

Our understanding of our place in the Universe changed measurably in 1995 when Michel Mayor and Didier Queloz of Geneva Observatory in Switzerland announced the discovery of an extrasolar planet around a star, 51 Pegasi, similar to our Sun [3]. Geoff Marcy and Paul Butler in the United States soon confirmed their discovery, and the science of observational extrasolar planetology was born. The field has exploded in recent years, resulting in publications showing numerous planetary systems in 2019 (see http://exoplanet.eu/ and http://exoplanets.org/). Most of these systems contain one or more gas-giant planet close, or very close, to their parent star. In that, they do not resemble our Solar system. In this connection, the recent paper [4] tallies:

> The discovery of the gas-giant planet – named 51 Pegasi b after its parent star, 51 Pegasi – came as a surprise. Gas-giant planets, such as Jupiter, are located in the outer parts of the Solar System. The prevailing theory was, and still is, that the formation of these planets requires icy building blocks that are available only in cold regions far away from stars. Yet Mayor and Queloz found 51 Pegasi b to be orbiting about ten times closer to its host star than Mercury is to the Sun... One possible explanation is that the planet formed farther out and then migrated to its current location.

Nevertheless, earlier detection of planets with masses approximately equal to the mass of our Earth M_{Earth} is evidence that there exist extrasolar planets with low masses. In addition to obtaining important knowledge about the formation and structure of new planetary systems, that is, *exoplanetary systems*, these discoveries provoke genuine interest among the scientific community regarding the prospects for finding life in the Universe.

However, the questions considered in this monograph deal mainly with the problems of cosmogony and only partially touch upon cosmology. *Cosmological* bodies include large-scale space objects (for example, galaxies and their clusters) based on the fact that cosmology is a science that studies the properties and evolution of the Universe as a whole. In this context, *cosmogonical* bodies unite stars, protostars, interstellar molecular clouds, planetary systems, protoplanetary gas-dust disks, planets, protoplanets, and natural satellites of planets.

Generally speaking, the cosmogony, according to O. Yu. Schmidt [5], includes both *planetary cosmogony* and *stellar cosmogony*, that is, such directions have been developing within the framework of various cosmogonical theories.

Several cosmogonical theories are known to explain the formation of the Solar system, the formation of planets and the estimation of planetary orbits [1–16]:

- *electromagnetic theories* based on the works of O. K. Birkeland [17], H. P. Berlage [18], H. Alfvén [9, 19, 20], and others;
- *gravitational theories* based on the works of O. Yu. Schmidt [5, 6, 21], L. E. Gurevich and A. I. Lebedinsky [22, 23], M. M. Woolfson [14, 24], V. S. Safronov [2], S. H. Dole [25], A. V. Vityazev [12], and others;
- *nebular theories* based on the works of C. F. von Weizsäcker [26, 27], G. P. Kuiper [28, 29], F. Hoyle [30, 31], D. Ter Haar [7, 32], T. Nakano [33], A. G. W. Cameron [7, 10] and others;
- *quantum mechanical theories* based on the works of E. Nelson [34, 35], L. Nottale [36, 37], G. Ord [37, 38], M. De Oliveira Neto [15, 39], A. G. Agnese and R. Festa [40], M. S. El Naschie [41, 42], E. G. Sidharth [43] and others.

The state and achievements of cosmogonical theories are described by Stephen G. Brush in his review "Theories of the origin of the Solar system 1956–1985" [11]:

> Attempts to find a plausible naturalistic explanation of the origin of the Solar system began about 350 years ago but have not yet been *quantitatively*[1] successful. The period 1956–1985 includes the first phase of intensive space research; new results from lunar and planetary exploration might be expected to have played a major

[1] Emphasis added.

> role in the development of ideas about lunar and planetary formation. While this is indeed the case for theories of the origin of the moon (selenogony), it was not true for the Solar system in general, where ground-based observations (including meteorite studies) were frequently more decisive. During this period most theorists accepted a monistic scenario: the collapse of a gas-dust cloud to form the sun with a surrounding disk, and condensation of that disk to form planets, were seen as part of a single process. Theorists differed on how to explain the distribution of angular momentum between sun and planets, on whether planets formed directly by condensation of gaseous protoplanets or by accretion of solid planetesimals, on whether the Solar nebula "was ever hot and turbulent enough to vaporize and completely mix its components, and on whether an external cause such as a supernova explosion triggered" the initial collapse of the cloud. Only in selenogony was a tentative consensus reached on a single working hypothesis with quantitative results.

Despite a large amount of research and a huge number of works aimed at studying the formation of the Solar system, because of a lack of *quantitative results* these theories cannot fully explain all the phenomena observed – in particular, the four groups of facts following Ter Haar [6, 44 p. 277]. (The last concerns the distribution of angular momentum: although the Sun has more than 99% of total mass of the Solar system, only 2% of the total angular momentum belongs to Sun, while the remaining 98% belongs to the planets (see also Introduction in Chapter 6).)

In this context, beginning in 1996 the *statistical theory* for a cosmogonical body formation has been developed [45–70] by the author of this monograph, based on the so-called model of a spheroidal body forming through numerous gravitational interactions of its parts and particles (see also recent articles, book, and chapters [16, 71–79]).

The notion of a *spheroidal body* means a sphere-like body whose iso-surfaces (the surfaces of equal mass density) are spheroids (in the case of rotation of this body) or spheres (in the absence of visible motion). The area of investigations within the framework of the statistical theory mainly relates to Newtonian gravity and partly affects Newtonian quantum gravity (this area is highlighted by the arc in Figure I.1 which was proposed on the website of the Bremen University in 2003 (http://www.zarm.uni-bremen.de/2forschung/gravi/gravity_main.htm). In this sense, the present book continues the analytical tradition of Cambridge Scholars, starting with the works of I. Newton, G. Stokes, J. Maxwell, Lord Rayleigh (J. Strutt), J. Jeans, A. Eddington, R. Lyttleton, R. Fowler, and other scientists.

This book has two parts. Part I (Chapters 1–5) seeks to acquaint the readers with the developing statistical theory of gravitating cosmogonical body formation. Within the framework of this theory, the models, as well as the evolutionary equations of the statistical mechanics, are proposed. The well-known problem of gravitational condensation of infinite distributed cosmic substance (in particular, the Jeans gravitational instability) is solved by the proposed statistical model of spheroidal bodies. For the first time, the statistical model of slow-flowing gravitational condensation based on the anti-diffusion process allows a solution to the gravitational paradox for an infinite homogeneous spatial spread substance. With the use of this statistical model, a new nonlinear time-dependent Schrödinger-like undulatory equation describing the processes of cosmogonical body formation is derived.

In detail, in Chapter 1, the problem of the gravitational condensation of the spread cosmic matter is considered for the formation of protoplanets in the gravitational field. In particular, Sections 1.1–1.6 describe the main problems of the theory of gravitational condensation and the theory of gravitational instability applied to the molecular (gas-dust) cloud. In Section 1.7, the general evolutionary equation for the distribution function is obtained [16, 65, 73] which generalizes the well-known Jeans equation characterizing the dynamical behavior of the gas-dust cloud.

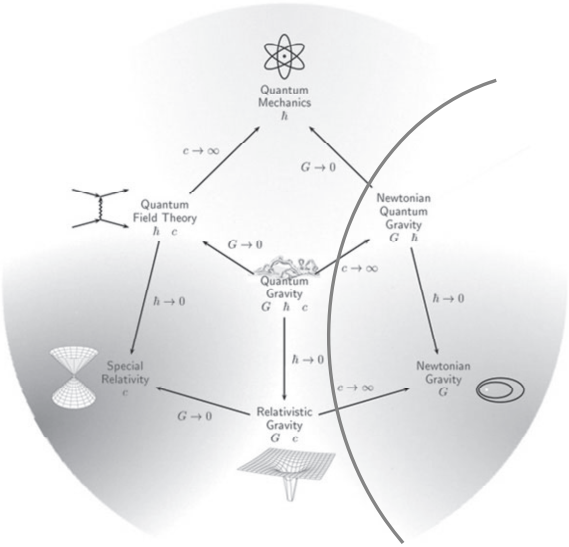


Figure I.1.

As shown in Chapter 2, the particle distribution functions obtained in Section 2.1, as well as the mass density of a *sphere-like gaseous body* (an immovable spheroidal body) [45–61], characterize *the first stage of evolution*: from a molecular cloud (nebula) to a forming core (proto-Sun) together with the outer shell (protosolar nebula). In Sections 2.4 and 2.5 the gravitational potential and the potential energy of a gravitating sphere-like gaseous body, are calculated. The probabilistic interpretation of physical values describing the gravitational interaction of particles in a sphere-like gaseous body is considered in Section 2.6. Under the condition of mechanical equilibrium, the pressure inside a sphere-like gaseous body is calculated in Section 2.8, as well as its internal energy in Section 2.9.

Chapter 3 is devoted to the study of statistical models of a *rotating* and gravitating *spheroidal body* to describe the evolution of a protoplanetary gaseous (gas-dust) cloud around a forming star. In particular, Section 3.1 considers the statistical interpretation of Poincaré's well-known general theorem and the Roche model for a *slowly* rotating and gravitating spheroidal body, that is, for a sphere-like gaseous body. In Section 3.3, the *equilibrium distribution function* of liquid particles relative to the spatial coordinates is derived, and the mass density function is also obtained for a uniformly rotating and gravitating spheroidal body with a small angular velocity. In Section 3.4 the distribution function of the specific angular momentum and the angular momentum density for a uniformly rotating spheroidal body are derived. The average value of specific angular momentum and the total angular momentum of a rotating spheroidal body being in relative mechanical equilibrium are calculated [16, 73]. The determination of the gravitational potential in the case of a uniformly rotating spheroidal body is discussed in Section 3.6. Section 3.7 estimates the potential energy of a uniformly rotating and gravitating spheroidal body. Moreover, as shown here, the disk-shaped spheroidal body does not possess its own gravitational energy. As noted in Section 3.8, the derived mass density function characterizes the flattening process: from initial spherical shapes (for an immovable spheroidal body) through flattened ellipsoidal shapes (for a rotating spheroidal body) to spheroidal disks. So, in Chapter 3 *the second stage of evolution* is described: from the protosolar nebula to the forming protoplanetary gas-dust disk.

Chapter 4 considers equations of a forming spheroidal body (both the centrally symmetric and the axially symmetric spheroidal body) in the process of its initial gravitational condensation. In Section 4.1, the basic *anti-diffusion equation* of the initial gravitational condensation of a non-rotating (or slowly rotating) spheroidal body from an infinitely spread matter is derived, and in Section 4.2, the general differential equations for physical values describing the anti-diffusion process of initial gravitational condensation of a spheroidal body in the vicinity of mechanical equilibrium are obtained. In other words, these equations show that the gravitational field tightening of the molecular cloud (nebula) is preceded by its initial anti-diffusion condensation [16, 65, 68]. Namely, two particular cases of the basic equation of slow-flowing initial gravitational condensation are considered in Sections 4.3 and 4.5. In Section 4.6, the possible dynamical states of the forming of a centrally symmetric spheroidal body from an infinitely spread gas-dust matter are systematized. The *general anti-diffusion equation* for a slowly evolving process of initial gravitational condensation of an axially symmetric spheroidal body (which is formed as a result of its rotation) is derived in Section 4.7.

Chapter 5 is devoted to the derivation of a new time-dependent nonlinear Schrödinger-like equation of a cosmogonical body formation [68, 71, 73, 77, 78] and the development of a scenario for the gravitational field origin based on an avalanche anti-diffusion mass transfer in a forming spheroidal body (see Sections 5.1–5.4), when in addition to the anti-diffusion velocity of particles, the usual (hydrodynamic) velocity arises. Nevertheless, the main result of the research in this chapter is the derivation of the generalized nonlinear Schrödinger-like equation in Section 5.6, describing not only the state of virial mechanical equilibrium and quasi-equilibrium gravitational compression state close to the mechanical equilibrium (with a slowly varying anti-diffusion coefficient) but also gravitational instability states leading to the formation of a cosmogonical body (see Section 5.7). In particular cases, the nonlinear time-dependent generalized Schrödinger-like equation becomes the well-known time-dependent Schrödinger equation or the generalized Schrödinger equation in Nottale's form (see Section 5.5). The *cubic* time-dependent Schrödinger-like equation describing cosmogonical body formation in the state of soliton disturbances is derived in Section 5.7. Since this equation has a soliton solution, the cubic Schrödinger-like equation can describe an evolution of the envelope of a wave packet of Jeans' substantial waves that propagate in a nonlinear and dispersive medium of a forming cosmogonical body (following the gravitational instability theory of Jeans [1]).

Part II of this monograph (Chapters 6–9) explores theoretical and practical approaches to investigating our Solar system and other exoplanetary systems. In particular, a new universal stellar law (USL) for extrasolar planetary systems connecting the temperature, size, and mass of each star is justified. Within the framework of the developed statistical theory, a new law (generalizing the famous law of O. Yu. Schmidt, for example) for the distribution of the planetary distances in our Solar system is proposed.

In detail, *the third stage of evolution* is considered in Chapter 6: from a protoplanetary flattened gas-dust disk to originating protoplanets [16, 65, 73]. The proposed statistical theory is applied primarily to develop *two models* of protoplanet formation (see Sections 6.1 and 6.2) and explaining the distribution law of planetary distances in the Solar system (see subsection 6.1.3), although the results presented in subsection 6.2.2 are also suitable for the construction of models of formation of exoplanetary systems. In more detail, the obtained distribution function of a specific angular momentum for a rotating uniformly spheroidal body (as a gas-dust flattened protoplanetary cloud) is used in

Section 6.1. Since the specific angular momentums (for particles or planetesimals) are averaged during a conglomeration process (under a planetary embryo formation) the specific angular momentum for a protoplanet of the Solar system is found in subsection 6.1.1. As a result, a new law for planetary distances (which generalizes Schmidt's law) is derived theoretically in subsection 6.1.2. Moreover, unlike the well-known planetary distance laws, the proposed law is established by a physical dependence of planetary distances from the value of the specific angular momentum. Within the framework of the second model, Section 6.2 develops an alternative heat emission model of protoplanet formation. As shown in subsection 6.2.1, in the state of relative mechanical equilibrium of particles moving in elliptical orbits in the gravitational field, an equation for the heat distribution function of the specific angular momentum is derived. Within the framework of this model, only 0.8% of the total number of particles of the Solar system composing the protoplanetary cloud has the angular momentum 15.6 times higher than the angular momentum of the remaining 99% of particles in the Solar system. This conclusion is in full agreement with Ter Haar's above-mentioned four facts of a nonuniform distribution of the angular momentum in the Solar system [7, 32].

Chapter 7 investigates the orbits of moving planets and bodies in the centrally symmetric gravitational field of a gravitating and rotating spheroidal body simulating the protostar with the flattened gas-dust disk during the *planetary stage* of its evolution. Though orbits of moving bodies and particles into a flattened rotating spheroidal body are circular initially, they could however be distorted by collisions with planetesimals and gravitational interactions with neighboring originating protoplanets during the evolutionary process of protoplanetary formation. This chapter shows that the orbits of moving particles are formed by the action of the centrally symmetric gravitational field mainly in the final stage of decay of a gravitating and rotating spheroidal body when the particle orbits become Keplerian. In Section 7.1, the estimation of the gravitational potential in the remote zone based on the general solution of the Poisson equation and the general expression for the gravitational potential of an axially symmetric spheroidal body is obtained. Section 7.2 investigates the orbits of moving planets and bodies in the centrally symmetric gravitational field of a gravitating and rotating spheroidal body during the planetary stage of its evolution. In Section 7.3, calculation of the orbit of the planet Mercury as well as estimation of angular displacement of Mercury's perihelion based on the statistical theory of gravitating spheroidal bodies is carried out. As a result, this section shows that according to the proposed statistical theory of gravitating spheroidal bodies the turn of perihelion of Mercury's orbit is equal to 43.93" per century, which is consistent with the conclusions of Einstein's general theory of relativity (his analogous estimation is equal to 43.03") and astronomical observation data ($43.11'' \pm 0.45''$).

In Chapter 8, the statistical theory of gravitating spheroidal bodies is applied to derive and develop a USL for the investigation of extrasolar systems [75, 76]. A preliminary estimate of an average gravitational potential energy of interaction of a particle with the gravitational field of a spheroidal body is given in Section 8.1. Section 8.2 then considers the derivation of the equation of the state of an ideal stellar substance taking into account an extended substance called the *stellar corona*. In other words, the stellar corona together with the star is described through the model of a rotating and gravitating spheroidal body in Section 8.2. Using the virial theorem as well as the theorem of uniform distribution of energy on freedom degrees for each particle inside a rotating and gravitating spheroidal body, the USL for a star including its stellar corona is justified. In the case of the Sun, the verification of USL shows its validity with the relative error equal to 3.37% (see Section 8.3). Section 8.3 then considers the modification of the USL, taking into account the ratio of the temperature of the Solar corona to an effective temperature of the Sun's surface. The verification of the modified USL for other stars belonging to the different spectral classes and types is carried out in Section 8.3. Theoretical estimations of temperatures of stellar coronas for stars belonging to the different spectral classes as well as orbital and thermodynamical characteristics of multi-planet extrasolar systems are investigated in Sections 8.4 and 8.5. So, the knowledge of some characteristics for multi-planet extrasolar systems permits us to refine a star's own parameters. In this context, comparison with estimations of temperatures using the regression dependences for multi-planet extrasolar systems attests the results obtained. In Section 8.6, the known Hertzsprung–Russell dependence is derived from USL directly.

In the final chapter (9) of this monograph, the stability of planetary orbits based on the statistical theory of gravitating spheroidal bodies is investigated [16, 45–79]. Using the obtained USL and its modification connecting temperature, size, and mass of a star the combination of Kepler's 3rd law with the universal stellar law (3KL-USL) is derived in Section 9.1 [79]. As shown in Section 9.1, the combined 3KL-USL law connects among themselves both the mechanical values (the Keplerian angular velocity Ω_K and the major semi-axis a of a planetary orbit) and the statistical (thermodynamic) values (the parameter of gravitational condensation α and the temperature T). The proposed 3KL–USL thus explains the stability of planetary orbits in extrasolar systems. In this context, Section 9.2 investigates the additional periodic force causing the radial and axial *orbital oscillations* (which modify initial circular orbits of bodies) based on the approach of Alfvén and Arrhenius [9, 19, 20]. A prediction of the Alfvén–Arrhenius specific additional periodic force within the framework of the Newtonian theory of gravity is considered in Section 9.3. As shown in Section 9.4, from the point of view of the theory of retarded potentials the *wave gravitational potential* and the Alfvén–Arrhenius specific additional periodic force arise in the remote zone II of the gravitational field under the orbital motion of a body around the central gravitating body. The obtained spectral representations correspond entirely to the analogous spectral expansion derived in the statistical theory of gravitating spheroidal bodies (see Chapter 5). Thus, the proposed statistical *theory of the formation* of planetary systems pointing to the regular and wave gravitational potentials origin is confirmed by the *theories of existence* (Newtonian and retarded potentials). Section 9.5 finds that additional periodic force is similar to Hooke's force which affects free oscillations of a body in orbit. Due to dissipation, these oscillations are damped gradually, so that they need support through the periodic impact of the additional periodic force by analogy with the principle of an anchoring mechanism in a clock. Section 9.6 justifies that the spatial deviation of the gravitational potential of an ellipsoid-like rotating cosmogonical body from the centrally symmetric field $1/r$ – gravitational potential (of a sphere-like body) implies different values of the radial and the axial orbital oscillations.

The investigations presented in this monograph in the field of theoretical statements on the processes of self-organization in a spread gas-dust cosmic media and the development of statistical models for the formation of planetary systems and the origin of planets (including planets in our Solar system and other extrasolar systems) have been widely discussed and reported at several international conferences, in particular, under the aegis of the General Assembly of the European Geosciences Union (EGU) and the European Planetary Science Congress (EPSC). Indeed, the author of this monograph was the organizer (as Convener, co-Convener, Chairman) of sessions: PS15 “Models of Solar system forming” (April 2–7, 2006, Vienna, Austria), PS7.1 “Extrasolar planets and planet formation” (April 16–20, 2007, Vienna, Austria), PS9 “Extrasolar planets and planet formation” (April 13–18, 2008, Vienna, Austria) and PS8 “Extrasolar planets and planet formation, exoplanetary magnetospheres and radio emissions” (April 19–24 2009, Vienna, Austria) of the EGU General Assembly; ON1 “Planetary formation and the origin of the Solar system” (September 18–22, 2006, Berlin, Germany) and OG1 “Origin and evolution” (September 14–18, 2009, Potsdam, Germany) of EPSC; a member of the organizing committee and chairman of international scientific conferences: 2nd International Conference and Exhibition “Satellite & Space Missions” (July 21–23, 2016, Berlin, Germany), the 3rd International Conference and Exhibition “Satellite & Space Missions” (May 11–13, 2017, Barcelona, Spain), the 4th International Conference and Exhibition “Satellite & Space Missions” (June 18–20, 2018, Rome, Italy) and the 12th International Conference “Chaotic Modeling and Simulation (CHAOS-2019)” (June 18–22, 2019, Chania, Crete, Greece).

The topic of this research corresponds to the priority direction of fundamental and applied scientific research of the Republic of Belarus on the mathematical and physical modeling of systems, structures, and processes in nature. This work has been carried out by the author in the Laboratory of Self-Organization System Modeling at the United Institute of Informatics Problems of the National Academy of Sciences of Belarus within the framework of the projects: “Methods of

Mathematical Modeling of Self-organization Processes in Active Media" (2006–2010), "Scientific Foundations and Tools of Space Technologies" (2011–2013), "Investigation of Gas-Dust and Plasma Media and their Impact on the Motion and State of Spacecraft" (2013–2015). This work has also been supported partially by the grant of the President of the Republic of Belarus in science (2019).

The author would like to thank reviewers of this monograph for many stimulating recommendations and comments. The author is grateful to the official reviewers from the National Academy of Sciences of Belarus: Academician, DSc, Professor Victor I. Korzyuk and Head of the Laboratory of theoretical physics at the Institute of Physics, DSc Yury A. Kurochkin, as well as PhD Alexander S. Garkun for their useful comments and recommendations concerning the previous monograph [73]. The author would also like to express his thanks to Dr. Pavel Pintr (Institute of Plasma Physics, Turnov and Palacky University, Olomouc, Czech Republic) and Dr. Vlasta Peřinová (Joint Laboratory of Optics, Palacky University, Olomouc, Czech Republic) for their fruitful comments and discussions concerning the statistical theory of formation of gravitating cosmogonical bodies, in particular, the paper [76].

With gratitude, the author recalls one of the world's outstanding scientists, the Nobel Prize laureate Ilya Prigogine who was a Principal Collaborator of our joint international scientific project B-95 ISTC (1997–2000) and gave his support to developing the author's research. The author is grateful to Professor M. El Naschie (who co-operated closely with Professor I. Prigogine at the International Solvay Institutes for Physics and Chemistry) for his attention and support [16].

The author would like to express special thanks to Mr. Abinash C. Dubey and Dr. Ivan Kadar who were the

Chairs/Editors of the SPIE's Annual International Symposiums "AeroSense" (Orlando, Florida, USA, April 1999–2001) for the financial support to enable the author's participation in the symposium at which he was able to present and develop the first results of this investigation [46–48].

The author expresses thanks to Professor Manuel Grande (Solar System Physics, Institute of Mathematical and Physical Sciences, University of Wales, UK) and Dr. Helmut Lammer (Space Research Institute, Austrian Academy of Sciences, Austria) for their help in organizing the PS15 session "Models of Solar system forming" of the General Assembly of the European Geosciences Union (Vienna, Austria, April 2–7, 2006) where some results of this monograph were presented and discussed [56, 57]. The author would also like to thank Professor Daniel Winterhalter (Jet Propulsion Lab, Caltech, Astrophysics and Space Science, NASA, USA) and Dr. James Cho (Astronomy Unit, Queen Mary University of London, UK) for their assistance in chairing the PS8 session "Extrasolar planets and planet formation, exoplanetary magnetospheres and radio emissions" of General Assembly of the European Geosciences Union, held April 19–24, 2009 in Vienna (Austria).

The author especially thanks Dr. Martin Vermeer (TKK, Finland) and Dr. Christopher Jekeli (School of Earth Sciences, Ohio State University, USA) who were organizers of the IAG Symposium GS002 "Gravity Field" of the XXIV IUGG General Assembly, held in Perugia, Italy, July 2–13, 2007, for their fruitful comments and suggestions on the work [59, 65]. The author would also like to thank Professor Ji-Huan He (Donghua University, Shanghai, China) for his careful review of the author's article [68] and useful recommendations.

The author expresses thanks to Professor Christos H. Skiadas, Conference Chair of "Chaotic Modeling and Simulation International Conference (CHAOS)" on the study of chaotic phenomena and Chief-Editor of the international journal "Chaotic Modeling and Simulation," for his constant attention to these studies.

The author is grateful to the team at the Laboratory of Self-Organization System Modeling at the United Institute of Informatics Problems of the National Academy of Sciences of Belarus, especially to leading research co-worker, PhD Helena B. Minervina who was the inspiration for the writing of this monograph.

I also thank Ms. Helen Edwards (Cambridge Scholars Publishing, UK) for her initiative in writing this book. The author would like to express his thanks to John Ingamells (JMI Editorial, UK) for his fruitful suggestions, editing and proofreading of the text of this monograph. The author also gratefully recalls Liya N. Borisova for her help in editing first works on the statistical theory of gravitation.

Finally, I express my gratitude to my daughter Alexandra (Alice) and my brother Anatoly, and I would like to dedicate this monograph to the memory of my dear parents – my father Mikhail Stepanovich Krot and my mother Polina Adamovna Kulagina for their powerful vital support, attention, and help.

PART I

A STATISTICAL MECHANICS OF THE FORMATION OF GRAVITATING COSMOGONICAL BODIES

Looking at the sky today, we see structures of every scale, from stars and planets to galaxies and galaxy clusters. The investigations of structure formation attempt to model how these structures formed by *gravitational instability* from small early density fluctuations. As we know, the problem of gravitational instability, along with the gravitational condensation problem, was first investigated by Sir James Jeans [1]. Indeed, the linearized theory of gravitational instability leads to the well-known Jeans criterion:

$$\lambda > \lambda_c \tag{I.1}$$

where λ is a wavelength of oscillating disturbances and λ_c is a critical wavelength of the disturbance. Nevertheless, it is well known that an infinite homogeneous non-rotating substance can not be in an equilibrium state. Small disturbances do not, therefore, manage to form any dense bunches. However, the process of planet formation takes a very long time and, in this context, Newtonian consideration of local equilibrium systems becomes preferable, as pointed by Viktor S. Safronov [2].

The main difficulty with Jeans' theory is connected to a *gravitational paradox* [2]: for an infinite homogeneous substance there exists no potential for a gravitational field φ_g in accord with Poisson equation:

$$\nabla^2\varphi_g = \frac{\partial^2\varphi_g}{\partial x^2} + \frac{\partial^2\varphi_g}{\partial y^2} + \frac{\partial^2\varphi_g}{\partial z^2} = 4\pi\gamma\rho\,. \tag{I.2}$$

If a mass density value $\rho \neq 0$ then, according to Eq. (I.2), both gravitational potential φ_g and, therefore, specific gravitational force f_g display *unlimited* growth depending on the distance [2]. Indeed, a constant limit existing for φ_g in spatial infinity leads to the representation that a mass density of a substance tends to zero in infinity. This difficulty is reduced within the framework of Jeans' theory and its next modifications using a supposition that the Poisson equation (I.2) cannot be applied to an infinite substance in whole but to disturbances $\delta\rho$ from a mean value ρ only. It is also supposed that there is no gravitational force in an infinite homogeneous immovable substance because a gradient of acceleration and pressure is absent. Otherwise, it could not be at rest [2].

An important law of statistical mechanics can be obtained by an equation for the evolution of a distribution function Φ of a gas-dust substance known as the Jeans equation [1]:

$$u\frac{\partial\Phi}{\partial x} + \upsilon\frac{\partial\Phi}{\partial y} + w\frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial u}\cdot\frac{\partial\varphi_g}{\partial x} + \frac{\partial\Phi}{\partial \upsilon}\cdot\frac{\partial\varphi_g}{\partial y} + \frac{\partial\Phi}{\partial w}\cdot\frac{\partial\varphi_g}{\partial z} + \frac{\partial\Phi}{\partial t} = 0 \tag{I.3}$$

where $u = \dot{x}, \upsilon = \dot{y}, w = \dot{z}$. However, because of the abovementioned gravitational paradox, the main problem of self-condensation of an infinitely distributed substance was not solved by Jeans' theory.

Ultimately, some problems of gravitational condensation of an infinitely distributed substance, as a molecular cloud, within the framework of Jeans' theory are the following [1-8]:

- a noncontradictory model of gravitational condensation of a molecular cloud is absent;
- a gravitational potential for an immovable molecular cloud is not defined analytically;

- difficulties occur in finding a general (not partial) solution for Jeans' equation (I.3) because of the impossibility of determining an analytical expression for the gravitational potential of a molecular cloud.

Part I of this monograph develops an initial model of a slowly evolving process of the gravitational condensation of a molecular cloud, thus solving the gravitational paradox relative to an infinite homogeneous substance.

CHAPTER ONE

ON THE PROBLEMS OF THE ORIGIN OF THE INITIAL GRAVITATIONAL CONDENSATION OF SPREAD COSMIC MATTER

The Universal Gravitation Law, discovered by Sir Isaac Newton in 1687 [80], has played crucial role in the formation of our knowledge about the Universe and the cosmic space surrounding us. Further studies related to the theory of General Relativity by Albert Einstein [81] advanced our understanding of the origin of the Universe, although many unresolved problems (that are discussed in this chapter) remain. In attempting to understand the formation of galaxies and their clusters, cosmologists are looking for evidence of the accumulation of cosmic matter, that is, cosmic irregularities or textures in the Universe [82–84]. However, the standard cosmological theory of the expansion of space in the early universe (in particular, the "hot universe" model of George Gamov and the "cosmic inflation" model of Alan Guth [85, 86]) is contradicted by the observations of the large-scale structure of the Universe [83].

Developing the hydrodynamic approach to cosmology [87], Neil Turok and Dan N. Spergel proposed the texture theory [83]. At present, modern observational data on the anisotropy of microwave radiation have also been obtained, reducing the popularity of texture theory [88]. Inflation theory predicts the spectrum of the microwave background which is in good agreement with the observational data. Moreover, by invoking an additional hypothesis about the existence of dark matter, it makes it possible even to explain the formation of the large-scale structure of the Universe [89–92].

Although this monograph is primarily devoted to problems of cosmogony, nevertheless, the general approach developed in it consists of the existence of similar stages of matter self-organization in scenarios of the formation of both large-scale space objects (galaxies and their clusters) and smaller-scale objects (stars and planetary systems) from spread cosmic matter [16, 68, 73]. It is well known (see, for example, [12]) that the interstellar cosmic medium contains very little dust and behaves practically as a single-component gaseous medium. Therefore, the main results in the theory of gravitational condensation and gravitational instability were first obtained for such a single-component medium by the famous astrophysicist Sir James Jeans in 1902 [1 pp. 346–348, 95].

Nevertheless, the main difficulty of the Jeans theory is related to the gravitational paradox: for an infinite homogeneous medium, there is no potential for the force of gravity [2]. In addition, there are other fundamental difficulties in the theory of gravitational condensation and the theory of gravitational instability in *infinitely spread media*. For example: the problem of forming a center of spread cosmic matter under its initial gravitational tightening or the other known problems of statistical mechanics of a gas-dust (molecular) cloud; and the impossibility of finding a general but non-partial solution to the Jeans equation due to the difficulty in determining the analytical expression for gravitational potential of a molecular cloud, an infinite mass density on the periphery of a rotating molecular cloud according to the theory of Jeans.

This chapter is devoted to a detailed description of the problems of the initial gravitational tightening origin of spread cosmic matter and to finding a possible way to solve them based on the new evolutionary equation of the statistical mechanics of the molecular (gas-dust) cloud [16, 65, 73] obtained in Section 1.7.

1.1. On Newton's Universal Gravitation Law and the problem of finding the mass center of a spread cosmic matter under its initial gravitational condensation

The Universe, as is now known from observations of the cosmic microwave background radiation, began in a hot, dense, nearly uniform state approximately 13.8 billion years ago [83, 84]. However, looking at the sky today, we see structures of all scales, from stars and planets to galaxies and, on still larger scales, galaxy clusters and sheet-like structures of galaxies separated by enormous voids containing few galaxies. Thus, reviews of the large-scale distribution of galaxies show how our modern Universe is *inhomogeneous* because galaxies tend to congestion (see Figure 1.1) forming layers, clusters, and accumulations surrounding the more sparse domains, that is, voids [83].

Structure formation attempts to model how these structures formed by gravitational instability from small early density fluctuations. In this regard, the investigation of the origin and evolution of the structure of the Universe is one of the most ambitious and urgent problems of modern cosmology. In the new reviews on the distribution of galaxies, huge bubbles and layers extending hundreds of millions of light-years have been found. The most popular models – G. Gamow's "hot" Universe and A. Guth's "cosmic inflation" Universe [85, 86] – successfully describe many aspects of the structure of the Universe but they do not explain the large-scale clustering of the cosmos, that is, all matter in the Universe entirely [83, 84].

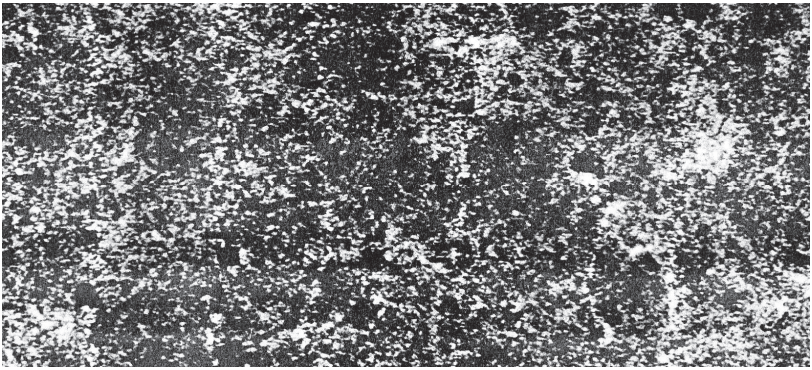


Figure 1.1. Fragment of the sky map with plotted galaxies (indicated by white dots) located at distances of up to 2 billion light-years [83]

In their attempts to understand the formation of galaxies and their clusters, cosmologists are looking for evidence of congestion in the form of cosmic irregularities, that is, textures in the early Universe [83, 84]. However, the standard cosmological theory of the expanding Universe (even taking into account Guth's inflation phase, i.e., a short-term rapid expansion [85, 86]) is contradicted by the observations of the large-scale structure [83]. Nevertheless, the observational data on the anisotropy of microwave radiation obtained at present do not testify in favor of the texture theory of Spergel–Turok, since, according to the latter, the Doppler peak in microwave radiation is suppressed [88]. On the contrary, the inflationary theory predicts the spectrum of the microwave background which is in good agreement with the observational data, and with the usage of an additional hypothesis about the existence of *dark matter*, it even makes it possible to explain the formation of the large-scale structure of the Universe [89–92].

In this regard, a possible answer lies in the existence of similar stages of self-organization of matter in the scenarios of formation of both large-scale space objects (such as galaxies and their clusters) and less large-scale ones (in particular, stars and planetary systems) from spread cosmic matter [16, 73]. Modern concepts of the cosmic dispersed matter based on astrophysical data obtained using radio telescopes say that a cold porous gas-dust medium forms molecular clouds which spread in interstellar space "with some tendency for local clustering are a large number of 'cores'..." [10].

If we conditionally divide this medium into elementary domains, then, using the terminology of hydrodynamics [94], they can be considered to be "liquid particles" by mass m_i (see Fig. 1.2); in this connection, we note that these liquid particles are not elementary but themselves consist of a multitude of elementary particles of mass m_0 [94]. According to the Newton's Universal Gravitation Law, established by the brilliant physicist Sir Isaac Newton [80, 95], these particles must interact with each other through local gravitational forces, the magnitude of which is determined by Newtonian law:

$$F_{ij} = \gamma \frac{m_i m_j}{r_{ij}^2} \tag{1.1.1}$$

where

$\gamma = 6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is the Newtonian gravitational constant,

m_i and m_j are masses of the interacting i-th and j-th liquid particles, and

r_{ij} is a distance between the i-th and j-th particles.

In other words, following the Universal Gravitation Law, a particle m_i attracts another particle m_j with the force F_{ij} defined by the formula (1.1.1). Obviously, with the same magnitude of the force, another particle m_j attracts the particle m_i. The vector, representing the force F_{ij} in its action on the particle m_i, is located on the positive real axis, having an origin in the center of mass m_i of the first particle, and is directed toward the second particle m_j; the vector, which represents the force of attraction of a particle m_j by a particle m_i, has an origin at the center of mass m_j and is located on an oppositely positive real axis directed toward the particle m_i (see Fig. 1.2).

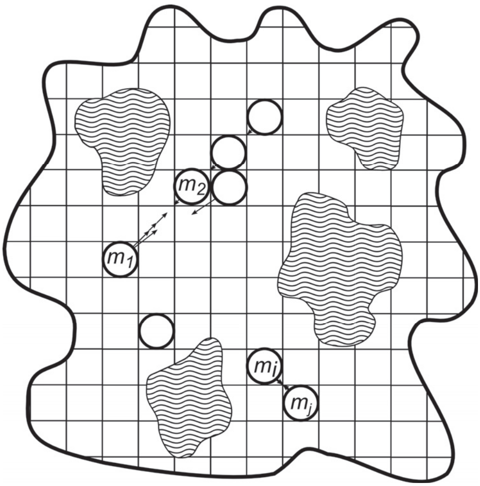


Figure 1.2. The interaction of particles in a porous gas-dust medium

If we consider all the pairwise interactions of liquid particles among themselves then a resulting force of all the gravitational interactions of these particles of cosmic matter among themselves can be calculated as follows:

$$\vec{F}_{\Sigma} = \sum_{ij} \vec{F}_{ij} = -\gamma \sum_{ij} \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} \, . \qquad (1.1.2)$$

The direction of the resulting force $\vec{F}_{\Sigma}$ will be determined by the location of a *larger number* of liquid particles constiting the gas-dust matter, that is, by the location of a densely filled subspace with liquid particles in which there are almost no cavities (voids).

In particular, if we consider an ideal case of the distribution of cosmic matter in the form of a line segment of particles of the same type with diameter $D_i = 1$, that is, as a stretched needle (Fig. 1.3), it is easy to see that the resulting gravitational force will be directed to the geometric center of this needle:

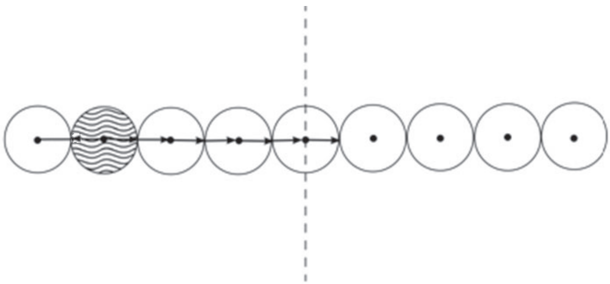


Figure 1.3. The scheme of attraction of particles placed in a line

This is not surprising because of the total number N of particles. The number of particles attracting a selected particle (marked by hatching in Figure 1.3) to the periphery of the needle is 1 while the number of particles attracting this particle to the center is $N-1$. In the case of a large ensemble of particles $(N >> 1)$, the resulting central force $F_c \sim N$, and the resulting peripheral force $F_p \sim 1$, that is, $F_c >> F_p$ occurs with an increase in the total number N of particles.

In the second ideal case, the distribution of cosmic matter in the form of a flat figure, that is, of a circle or disk (see Fig. 1.4), uniformly filled with radially located particles of the same type with $D_i = 1$ (the number of which along the radial direction N approximately determines a diameter of this disk) the number of peripheral particles is approximately equal to the length of a circumference $\text{ent}[\pi N]$ while the number of interior particles is proportional to the area of the inner circle $\text{ent}[\pi((N-2)/2)^2]$. In other words, at very large N a resulting central force $F_c \sim N^2$ and a resulting peripheral force $F_p \sim N$, that is, $F_c >> F_p$ with an unlimited increase in the ensemble of particles. This means that the *peripheral particles* making up the shell *practically do not attract* the interior particles of the disk-shaped body (see later the well-known theorem of Newton [80, 95, 96]).

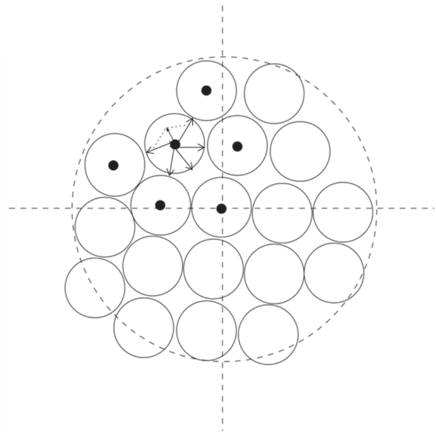


Figure 1.4. The scheme of the interaction of particles inside a disk-shaped figure

Finally, the third case of the distribution of cosmic gas-dust matter refers to the case of a three-dimensional figure which is the sphere uniformly filled with particles of the same type having a diameter equal to the length N of a sequence of radially located particles with $D_i = 1$ (Fig. 1.5).

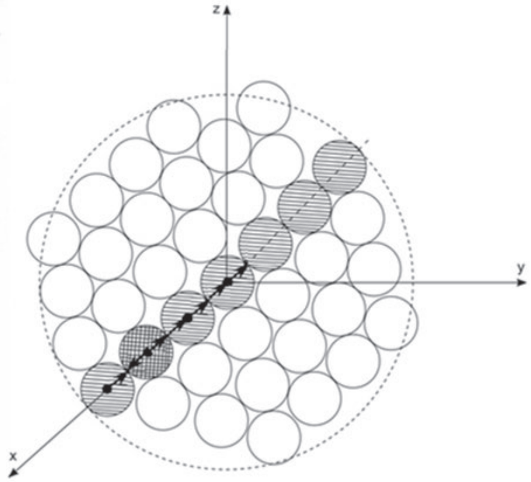


Figure 1.5. The scheme of attraction of particles inside the sphere

If we consider any cross-section of the above-mentioned sphere (passing through its geometric center) and then arbitrarily select the diameter (composed by radially located liquid particles) in it, then we come to the consideration of the first model (Fig. 1.3). Indeed, it can be seen in this figure that lateral resulting forces cannot act on a test particle located on the diameter due to the symmetry of the circular cross-section relative to the diameter of the sphere (since there are approximately equal numbers of particles on either side of the diameter the resulting peripheral forces compensate each other). As a result, only an uncompensated central resulting force acts on the test particle directed to the center along the diameter. When placing a test particle directly in this center of the sphere the resulting central gravitational force is equal to zero due to the attraction of an equal number of particles located in all directions from the center. If we relocate a test particle from the center to the periphery, the resulting central gravitational force becomes greater being proportional to the number of particles contained in the volume of an inner concentric sphere (on the surface of which this test particle is now located). As in the previous case, this means that the peripheral particles (constituting the shell in the form of a spherical layer) do not attract the interior particles of the inner sphere. Indeed, when N is large enough, then by analogy with the second case, the resulting central force $F_c \sim N^3$, and the resulting peripheral force $F_p \sim N^2$, that is, $F_c >> F_p$ at large N; as a result, the *center of gravity* (center of mass) of the cloud of gas-dust matter *coincides with its geometric center*.

Since the problem of determining the center of spread matter as a system of particles (in particular, of a molecular cloud) is important under its initial gravitational condensation, let us consider some mathematical foundations for finding the center of mass of a particle system. As F. Moulton pointed out in [96], the *center of mass* of a system of particles having equal masses, that is, of equal single mass points, is defined as a point, the distance to which from any plane is equal to the average distance of all mass points from this plane. This should occur for three coordinate planes. Indeed, let (x_1, y_1, z_1), (x_2, y_2, z_2), etc. represent the rectangular coordinates of various single mass points while $\bar{x}, \bar{y}, \bar{z}$ are the rectangular coordinates of their center of mass. Then, according to the definition we have:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \quad \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i, \quad \bar{z} = \frac{1}{N}\sum_{i=1}^{N} z_i \qquad (1.1.3)$$

where N is a number of particles in a molecular cloud. If m_0 is a mass of each particle, that is, the mass of the whole system of particles is equal $M = m_0 N$, then Eqs (1.1.3) become:

$$\bar{x} = \frac{m_0 \sum_{i=1}^{N} x_i}{m_0 N} = \frac{1}{M} \sum_{i=1}^{N} m_0 x_i , \quad \bar{y} = \frac{m_0 \sum_{i=1}^{N} y_i}{m_0 N} = \frac{1}{M} \sum_{i=1}^{N} m_0 y_i ,$$

$$\bar{z} = \frac{m_0 \sum_{i=1}^{N} z_i}{m_0 N} = \frac{1}{M} \sum_{i=1}^{N} m_0 z_i . \qquad (1.1.4)$$

It remains to prove that the distance to a point $(\bar{x}, \bar{y}, \bar{z})$ from any other plane is also the *average distance* of mass points from the plane [96]. To this end, we write the equation of an arbitrary plane:

$$ax + by + cz + d = 0 . \qquad (1.1.5)$$

The distance to the point $(\bar{x}, \bar{y}, \bar{z})$ from this plane is determined by the formula:

$$\bar{d} = \frac{a\bar{x} + b\bar{y} + c\bar{z} + d}{\sqrt{a^2 + b^2 + c^2}} , \qquad (1.1.6)$$

and the distance to a certain point (x_i, y_i, z_i) from the same plane is respectively:

$$d_i = \frac{ax_i + by_i + cz_i + d}{\sqrt{a^2 + b^2 + c^2}} . \qquad (1.1.7)$$

Then from Eqs (1.1.3), (1.1.6), and (1.1.7) it follows that:

$$\bar{d} = \frac{a\sum_{i=1}^{N} x_i + b\sum_{i=1}^{N} y_i + c\sum_{i=1}^{N} z_i + Nd}{N\sqrt{a^2 + b^2 + c^2}} = \frac{\sum_{i=1}^{N} (ax_i + by_i + cz_i + d)}{N\sqrt{a^2 + b^2 + c^2}} = \frac{\sum_{i=1}^{N} d_i}{N} , \qquad (1.1.8)$$

Q.E.D. Therefore, the point $(\bar{x}, \bar{y}, \bar{z})$ represented by Eq. (1.1.3) satisfies the definition of the center of mass relative to all planes.

When the particle system contains particles of unequal mass, it is possible to consider two cases in which the masses of these particles are commensurable and incommensurable [96]. In the case in which the masses of the particles are commensurable, a certain mass unit m_0 is selected, for which all N the masses of the particles are divided without remainder. Let us suppose that the first particle has a mass $p_1 m_0$, the second has $p_2 m_0$, and so on, and let $p_1 m_0 = m_1$, $p_2 m_0 = m_2$, etc. Then we can assume that the system of particles consists of $p_1 + p_2 + \ldots$ mass points, each having mass m_0. Then, according to the above regarding Eqs (1.1.4), we directly obtain that:

$$\bar{x} = \frac{\sum_{i=1}^{N} m_0 p_i x_i}{\sum_{i=1}^{N} m_0 p_i} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i, \quad \bar{y} = \frac{\sum_{i=1}^{N} m_0 p_i y_i}{\sum_{i=1}^{N} m_0 p_i} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i,$$

$$\bar{z} = \frac{\sum_{i=1}^{N} m_0 p_i z_i}{\sum_{i=1}^{N} m_0 p_i} = \frac{1}{M}\sum_{i=1}^{N} m_i z_i \qquad (1.1.9)$$

which proves the requirement.

In the case in which the masses of the particles are incommensurable, we can choose an arbitrary mass unit m_0 smaller than each of the N masses of the particles. Then the masses of these particles will be expressed by a product m_0 of an integer plus some residues. If we neglect the residues Eqs. (1.1.9) then give the center of mass. Now let us take as a new unit of mass any share of m_0. As a result, the residues remain the same or decrease (depending on their value) [96]. This share m_0 may be so small that each residue will be less than any given value. It is obvious that these Eqs (1.1.9) are also applicable if m_i are masses of particles minus residues. But since the shares of m_0 tend to zero the sum of the residues also tends to zero, that is, the expressions (1.1.9) tend to the limits in which m_i are masses of single mass points [96]. Therefore, in all cases (commensurable or incommensurable masses of particles) a point is determined by Eqs (1.1.9) that satisfies the definition of the center of mass.

So, to prove the validity of the formulas for the definition of the center of mass of a system of particles, it is sufficient to show that Eqs (1.1.9) do not change, firstly, when the origin of coordinates changes or, secondly, when the rotation occurs around one of these axes. To change the origin of coordinates let us move it along the axis x at a distance a:

$$x = x' + a .$$

Then, taking into account this substitution the first equation in (1.1.9) becomes:

$$\bar{x}' + a = \frac{1}{M}\sum_{i=1}^{N} m_i (x_i' + a) = \frac{1}{M}\sum_{i=1}^{N} m_i x_i' + a\frac{\sum_{i=1}^{N} m_i}{M},$$

whence:

$$\bar{x}' = \frac{1}{M}\sum_{i=1}^{N} m_i x_i',$$

so that this formula is the same as before.

Now let us turn the axes x and y around the axis z at an angle θ. The substitution performing the rotation is a known rotation transformation:

$$x = x'\cos\theta - y'\sin\theta;$$

$$y = x'\sin\theta + y'\cos\theta.$$

After this substitution the first two equations in (1.1.9) take the following form:

$$\bar{x}'\cos\theta - \bar{y}'\sin\theta = \cos\theta\cdot\frac{1}{M}\sum_{i=1}^{N} m_i x_i' - \sin\theta\cdot\frac{1}{M}\sum_{i=1}^{N} m_i y_i';$$

$$\bar{x}'\sin\theta + \bar{y}'\cos\theta = \sin\theta\cdot\frac{1}{M}\sum_{i=1}^{N} m_i x_i' + \cos\theta\cdot\frac{1}{M}\sum_{i=1}^{N} m_i y_i'.$$

Multiplying the first equation by $\cos\theta$ and the second by $\sin\theta$ and then adding and subtracting the resulting equations we find:

$$\bar{x}' = \frac{1}{M}\sum_{i=1}^{N} m_i x_i'; \quad \bar{y}' = \frac{1}{M}\sum_{i=1}^{N} m_i y_i' \ .$$

Thus, the point $(\bar{x}, \bar{y}, \bar{z})$ meets the definition of the center of mass for any plane [96]. Leonard Euler proposed for the center of mass the name of *the center of inertia*.

If the mass points describing a system of particles become more and more numerous and more closely located to each other, then at the limit the system of particles approaches a solid body. Such bodies are characterized by continuous mass distribution of medium. To write the formulas for coordinates of the center of mass of a solid body one must take the limits of the expressions (1.1.9) for which $m_i \ (i = 1,..., N)$ tend to zero. At the limit, the sums go into definite integrals, therefore coordinates of the center of continuous masses are the following:

$$\bar{x} = \frac{\int\limits_M x dm}{\int\limits_M dm}, \quad \bar{y} = \frac{\int\limits_M y dm}{\int\limits_M dm}, \quad \bar{z} = \frac{\int\limits_M z dm}{\int\limits_M dm} \qquad (1.1.10)$$

and the integrals are taken over the whole continuous medium of a solid body.

If the solid body under consideration is nonuniform in its mass distribution, then a *mass density function* $\rho = \rho(x, y, z)$ is introduced, so that a mass element dm in rectangular coordinates is written as follows:

$$dm = \rho dx dy dz \,. \qquad (1.1.11)$$

Given the formula (1.1.11), Eqs (1.1.10) take the form:

$$\bar{x} = \frac{\int_V x\rho dxdydz}{\int_V \rho dxdydz}, \quad \bar{y} = \frac{\int_V y\rho dxdydz}{\int_V \rho dxdydz}, \quad \bar{z} = \frac{\int_V z\rho dxdydz}{\int_V \rho dxdydz} \qquad (1.1.12)$$

where integrals are taken over the whole volume V of a solid body. Similarly, Eqs (1.1.12) written in the Cartesian coordinate system can be represented in curvilinear coordinate systems (mainly in cylindrical and spherical ones).

All the above considerations are valid for the case when a cosmic molecular cloud has *finite dimensions*, so that the question of determining its center, surface, and volume looks quite correct. However, the problem becomes much more complicated when it concerns huge cosmic formations (nebulae) in which a fairly cold porous gas-dust medium spreads in infinite space. The following question is then relevant: *what is the mechanism for the origin of the initial gravitational condensation of infinitely spread cosmic matter?*

First of all, let us consider another conceptual model. We suppose that the texture of a spread porous gas-dust medium can be modeled by a foamed liquid consisting of a collection of bubbles of various sizes (small, medium, and large) tightly adjacent to each other (see Fig. 1.2). Bubbles make up multiple voids where cosmic matter is absent, moreover, the matter itself forms the walls of bubbles and is also located in the inter-bubble space. It is known from the theory of the Newtonian potential [95, 96, 97] that the magnitude of the gravitational force both in a spherical layer of finite thickness and in an infinitely thin spherical layer as well as in cavities of an ellipsoidal shape (infinitely thin and finite thickness) is zero according to the theorem of Newton mentioned above.

Indeed, the attraction of a thin *homogeneous spherical layer* at an interior point, like other simple bodies such as spheres, was first considered by Newton in his “Principia,” Book I, Section XII, Proposition LXX [80]. The following proof essentially coincides with the proof given by him [96].

Let us consider a spherical layer formed by two infinitely close spherical surfaces S and S', and let P be a point of unit mass located inside it (Fig. 1.6).

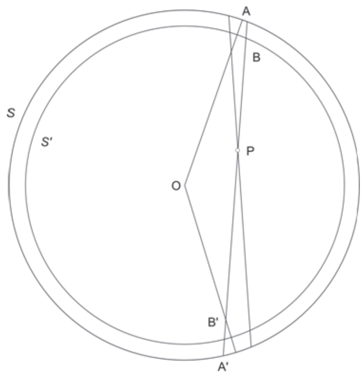


Figure 1.6. The scheme of attraction of a thin homogeneous spherical layer at a point inside it

Let us construct an infinitely small cone with a solid angle o and a vertex at a point P. Let σ be a mass density of the layer. Then the mass of an element of the layer at A is equal to $m = \sigma \cdot \overline{AB} \cdot o \cdot \overline{AP}^2$; similarly, the mass of a layer element in A' is: $m' = \sigma \cdot \overline{A'B'} \cdot o \cdot \overline{A'P}^2$. According to (1.1.1) the forces of attraction of the mass point P by the elements of layer m and m' are respectively equal to:

$$F = \gamma \frac{m \cdot 1}{\overline{AP}^2}; \quad F' = \gamma \frac{m' \cdot 1}{\overline{A'P}^2}.$$

Since $\overline{A'B'} = \overline{AB}$ then $F = \gamma\sigma \cdot \overline{AB} \cdot o = F'$, and this is also true for each infinitely small solid angle with a vertex at a point P. Thus, *a thin homogeneous spherical layer attracts a point inside it equally in opposite directions*, that is, according to (1.1.2) the resultant gravitational force $\vec{F}_{\Sigma} = 0$. This is applicable for any number of thin spherical layers and, therefore, for a spherical layer of a *finite thickness* which proves Newton's theorem.

Similarly, the attraction of a thin *homogeneous ellipsoidal layer* at an interior point can be studied (the corresponding theorem was given in "Principia" [80], Book I, Section XIII, Proposition XCI, Corollary 3). A thin layer enclosed between two similar and similarly placed surfaces of homogeneous ellipsoids is called an *elliptic homeoid* [96]. Let us consider the attraction of an elliptic homeoid bounded by two similar ellipsoids E and E' at an interior point P of a unit mass (Fig. 1.7).

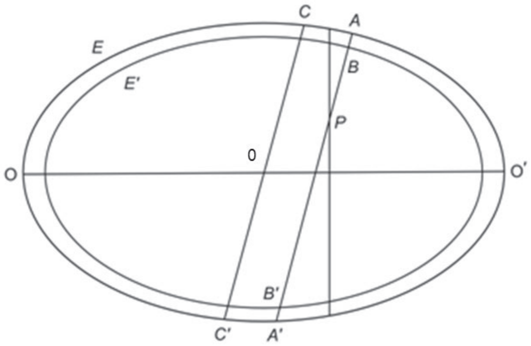


Figure 1.7. The scheme of attraction of a thin homogeneous ellipsoidal layer at a point inside it

Let us construct an infinitely small cone with a solid angle o and with a vertex P. The masses of two infinitely small elements near points A and A' are defined as $m = \sigma \cdot \overline{AB} \cdot o \cdot \overline{AP}^2$ and $m' = \sigma \cdot \overline{A'B'} \cdot o \cdot \overline{A'P}^2$. Accordng to (1.1.1) the attraction forces of the mass point P of a unit mass by the layer m and m' are equal to $F = \gamma m / \overline{AP}^2$ and $F' = \gamma m' / \overline{A'P}^2$ respectively.

Let us construct a diameter $\cdot \overline{CC'}$ parallel to $\overline{AA'}$ at an elliptical cross-section by a plane, passing through the axis of the cone and the center of the ellipse, and then draw its conjugate diameter $\overline{OO'}$. It is the conjugate diameter for both elliptic cross-sections E and E'. Therefore, $\overline{OO'}$ divides in two every chord parallel to $\overline{CC'}$ whence $\overline{A'B'} = \overline{AB}$, therefore, the attractions of the point P by the opposite elements A and A' are equal to each other. Since this is applicable for each infinitely small solid angle, whose vertex is at P, then the resulting gravitational force (1.1.2) is equal to zero: $\vec{F}_{\Sigma} = 0$.

So, the forces of attraction of a thin elliptic homeoid at an interior point are equal in opposite directions. This statement is applicable for any number of thin layers and, as a result, for layers of finite thickness which proves, in general, the Newton theorem [80, 95, 96]:

Theorem 1.1 (the Newton theorem). A homogeneous layer, bounded by two similar and similarly placed concentric ellipsoids, does not exert attraction at a point into the internal cavity of the layer.

P. Dive [98] proved that the inverse conclusion of Newton's theorem is also true.

Corollary 1.1. A potential function (gravitational potential) of a homogeneous layer at an interior point P has a *constant value* into all internal cavity of the layer.

Really, according to Theorem 1.1 since an interior point P does not undergo any attraction from a homogeneous layer, then all components of the resulting gravitational force along coordinate axes are equal to zero The gravitational potential function should, therefore, be constant.

Let us consider a very useful application of Newton's theorem in the case of an *inhomogeneous* ellipsoid [99]. To this end, if we take the main axes of an inhomogeneous ellipsoid E as the coordinate axes we can then write its equation in the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1.1.13}$$

where a,b,c are major semi-axes of the ellipsoid under the assumption $c \le b \le a$. Let us introduce a family $\{E_\kappa\}$ of concentric, similar, and similarly located ellipsoids with a parameter k defined by the equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k^2. \tag{1.1.14}$$

Let be the current coordinates $x^{'},y^{'},z^{'}$ of an interior point M and $\rho(x^{'},y^{'},z^{'})$ be a mass density of an inhomogeneous ellipsoidal body bounded by a surface E at the point M. If for a fixed value k the density ρ has a constant value at all points on the surface E_k, but it changes upon transition from one surface of the family to another, that is, under changing a parameter k, then we can say that the inhomogeneous body T bounded by the surface of an ellipsoid E possesses an *ellipsoidal structure* or has an *ellipsoidal mass density distribution* [99]. In this case, the mass density $\rho(M)$ will be a function of only one k $(0 \le k \le 1)$ and $k=0$ corresponds to the center of the ellipsoidal body whereas the value $k=1$ conforms to its surface [99]. Therefore, we can write that:

$$\rho(M)=\rho(k)=\rho(\frac{x^{'2}}{a^2}+\frac{y^{'2}}{b^2}+\frac{z^{'2}}{c^2}) \tag{1.1.15}$$

where $x^{'}=kx, y^{'}=ky, z^{'}=kz$. To find the gravitational potential function of such an *ellipsoidal body* we proceed as follows: we select from this body T an infinitely thin ellipsoidal layer bounded by two infinitely close surfaces belonging to the family E_k, and we first find an expression for the gravitational potential of this layer (supposing it can be considered as *homogeneous*) at both *exterior* and *interior* point P.

As a consequence of this supposition as well as Theorem 1.1 and Corollary 1.1, the following theorem holds [99]:

Theorem 1.2. A gravitational potential function of a layer bounded by two similar, infinitely close ellipsoids has a constant value at all points of an ellipsoid which is confocal to this layer.

In particular, if an interior point does not undergo any attraction from an *infinitely thin* ellipsoidal layer then the gravitational potential function of this layer has a constant value at an internal confocal ellipsoid in accordance with Corollary 1.1.

So, by splitting the whole ellipsoidal body T into an infinite number of such infinitely thin, homogeneous ellipsoidal layers, we then obtain the gravitational potential of the whole ellipsoidal body as a sum of potentials of all infinitely thin layers [99], that is, as an integral of the gravitational potential of k-layer taken to the parameter k from zero up to one. Thus, first of all, we need to find an expression for the gravitational potential of an infinitely thin layer $T^{'}$ with a constant mass density ρ bounded by two infinitely close, concentric, similar and similarly spaced ellipsoids. To do this, we use the abovementioned Theorems 1.1 and 1.2.

Let $P(x, y, z)$ be a point lying outside a layer $T^{'}$ bounded by a surface $E_1^{'}$ with semi-axes $a^{'}, b^{'}, c^{'}$ and by an infinitely close surface $E_2^{'}$. Then the equation of an ellipsoid passing through the point P and being confocal to an ellipsoid $E_1^{'}$ has the form:

$$\frac{x^2}{a'^2+\lambda'}+\frac{y^2}{b'^2+\lambda'}+\frac{z^2}{c'^2+\lambda'}=1. \qquad (1.1.16)$$

According to Theorem 1.2, the gravitational potential φ' of the layer T' has a constant value at all points of this ellipsoid but upon transition from one confocal ellipsoid to another, that is, under a changing parameter λ', the gravitational potential is also changing. Consequently, the potential φ' of an infinitely thin layer at an *exterior* point of a unit mass is a function only of λ' which in turn is a function of the coordinates of the point P [99]:

$$\varphi' = -\frac{\gamma m'}{2}\int_{\lambda'}^{\infty}\frac{ds'}{R'(s')} \qquad (1.1.17)$$

where λ' is a positive root of Eq. (1.1.15):

$$R'(s') = \sqrt{(a'^2+s')(b'^2+s')(c'^2+s')}, \qquad (1.1.18)$$

and m' is a mass of the infinitely thin layer T', γ is the Newtonian gravitational constant.

As the point P is approaching the surface E_1' of the ellipsoid, the parameter $\lambda' \to 0$. Therefore, taking into account that the gravitational potential is a continuous function in all space, and inside the layer T', it is a constant value following Newton's theorem (see Theorems 1.1, 1.2 and Corollary 1.1) we can obtain the following potential expression for an *interior* point [99]:

$$\varphi' = -\frac{\gamma m'}{2}\int_{0}^{\infty}\frac{ds'}{R'(s')}. \qquad (1.1.19)$$

Now we turn to finding the gravitational potential of a continuous ellipsoidal body T, bounded by a surface E, the mass density of which is determined by the law (1.1.15). Let us consider a family of similar concentric ellipsoids E_k and select a layer $T^{'}$ bounded by an ellipsoid $E_1^{'}$ with semi-axes $a^{'} = ka, \quad b^{'} = kb, \quad c^{'} = kc$ and by an ellipsoid $E_2^{'}$ with semi-axes $a^{''} = (k + dk)a, \quad b^{''} = (k + dk)b, \quad c^{''} = (k + dk)c$. The gravitational potential of the layer $T^{'}$ at an outer point $P(x, y, z)$ is then determined by the formula (1.1.17), and its mass is equal:

$$m^{'} = \frac{4}{3}\pi abc\rho(k^2)[(k + dk)^3 - k^3], \qquad (1.1.20)$$

whence, neglecting the terms higher than the first order of value dk, we obtain:

$$m^{'} \approx dm = 4\pi abc\rho(k^2) \cdot k^2 dk . \qquad (1.1.21)$$

Making the substitution $s^{'} = k^2\sigma$ and denoting the gravitational potential of the layer $T^{'}$ (as an infinitely small part of the gravitational potential of the whole body T) through $d\varphi_g$ instead $\varphi^{'}$, we can write:

$$d\varphi_g = -2\gamma\pi abc\rho(k^2)k^2 dk \int_{\lambda^{'}/k^2}^{\infty} \frac{k^2 d\sigma}{\sqrt{(k^2a^2 + k^2\sigma)(k^2b^2 + k^2\sigma)(k^2c^2 + k^2\sigma)}} =$$

$$= -\gamma\pi abc\rho(k^2)2k^4 dk \int_s^{\infty} \frac{d\sigma}{k^3\sqrt{(a^2 + \sigma)(b^2 + \sigma)(c^2 + \sigma)}} =$$

$$= -\gamma\pi abc\rho(k^2)dk^2 \int_s^{\infty} \frac{d\sigma}{\sqrt{(a^2 + \sigma)(b^2 + \sigma)(c^2 + \sigma)}}, \qquad (1.1.22)$$

where $k^2 s = \lambda^{'}$. Given the designation:

$$R(\sigma) = \sqrt{(a^2 + \sigma)(b^2 + \sigma)(c^2 + \sigma)} \qquad (1.1.23)$$

the expression (1.1.22), therefore, takes the form:

$$d\varphi_g = -\gamma\pi abc\rho(k^2)dk^2 \int_s^{\infty} \frac{d\sigma}{R(\sigma)}, \qquad (1.1.24)$$

and s is determined by the equation derived from Eq. (1.1.16):

$$\frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{c^2 + s} = k^2 . \qquad (1.1.25)$$

Integrating now the equality (1.1.24) with respect to k or k^2, which is the same, from zero to one, we obtain the gravitational potential of the whole ellipsoidal body T:

$$\varphi_g(P) = -\gamma\pi abc \int_0^1 \rho(k^2)dk^2 \int_s^\infty \frac{d\sigma}{R(\sigma)}, \qquad (1.1.26)$$

which can be written shorter, supposing:

$$\varphi_0(s) = \pi abc \int_s^\infty \frac{d\sigma}{R(\sigma)}, \qquad (1.1.27)$$

in the following form [99]:

$$\varphi_g(P) = -\gamma \int_0^1 \varphi_0(s)\rho(k^2)dk^2 . \qquad (1.1.28)$$

To lead the formulas (1.1.26), (1.1.28) to the classical Dirichlet form [99], we consider the function $\rho(k^2)$ as integrable in the interval $0 \le k^2 \le 1$ and assume:

$$\kappa(k^2) = \int_{k^2}^1 \rho(k^2)dk^2 . \qquad (1.1.29)$$

Applying to (1.1.28) the method for integrating by parts we obtain:

$$\varphi_g(P) = -\gamma\left\{ -\varphi_0(s)\kappa(k^2)\Big|_0^1 + \int_0^1 \kappa(k^2)d\varphi_0(s) \right\}. \qquad (1.1.30)$$

However, when $k^2 = 1$ we have $\kappa(1) = 0$, and if $k^2 = 0$, as can be seen from Eq. (1.1.25), we have $s = \infty$, and therefore $\varphi_0(\infty) = 0$, that is, the integrated part disappears:

$$\varphi_0(s)\kappa(k^2)\Big|_0^1 = 0 .$$

In the remaining integral we take the integration variable s instead of k^2. The new boundary will then be ∞ and λ where λ is determined by an equation resulting from Eq. (1.1.25) at $k^2 = 1$, that is, by the following equation:

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \qquad (1.1.31)$$

and the expression for the gravitational potential function takes the final form:

$$\varphi_g(P)=-\gamma\pi abc\int_\lambda^\infty \frac{\kappa(k^2)ds}{R(s)}. \qquad (1.1.32)$$

If $\rho=\text{const}$ then $\kappa=(1-k^2)\rho$, where k^2 is determined by Eq. (1.1.25), so that we obtain the *Dirichlet formula* for the potential of a *homogeneous* ellipsoid at an exterior point [99, 100]:

$$\varphi_g(P)=-\gamma\rho\pi abc\int_\lambda^\infty\left[1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{c^2+s}\right]\frac{ds}{R(s)}. \qquad (1.1.33)$$

Let us find the components of the *specific* gravitational force, that is, when the ellipsoidal body T gravitationally acts on an exterior mass point of a *unit* mass. Differentiating Eq. (1.1.32) with respect to x, for example, we have [99]:

$$\frac{\partial\varphi_g}{\partial x}=-\left(-\frac{\gamma\pi abc}{R(\lambda)}\cdot[\kappa(k^2)]_{s=\lambda}\frac{\partial\lambda}{\partial x}-2\gamma\pi abcx\int_\lambda^\infty\frac{\rho(k^2)ds}{(a^2+s)R(s)}\right)=$$

$$=2\gamma\pi abcx\int_\lambda^\infty\frac{\rho(k^2)ds}{(a^2+s)R(s)}, \qquad (1.1.34)$$

because when $s=\lambda$ we have $k^2=1$ and $\kappa(1)=0$ respectively. Then we can find the component f_x and, similarly, the other two f_y, f_z as the following [99]:

$$f_x=-\frac{\partial\varphi_g}{\partial x}=-2\gamma\pi abcx\int_\lambda^\infty\frac{\rho(k^2)ds}{(a^2+s)R(s)}; \qquad (1.1.35a)$$

$$f_y=-\frac{\partial\varphi_g}{\partial y}=-2\gamma\pi abcy\int_\lambda^\infty\frac{\rho(k^2)ds}{(b^2+s)R(s)}; \qquad (1.1.35b)$$

$$f_z=-\frac{\partial\varphi_g}{\partial z}=-2\gamma\pi abcz\int_\lambda^\infty\frac{\rho(k^2)ds}{(c^2+s)R(s)}. \qquad (1.1.35c)$$

When $\rho = \text{const}$ these formulas become the formulas for the components of the specific gravitational force of a homogeneous ellipsoid.

It is clear that the components of the resulting gravitational force (1.1.2) of the ellipsoidal body T acting on an exterior point having mass m are equal:

$$F_x = mf_x = -2\gamma m\pi abcx\int_\lambda^\infty \frac{\rho(k^2)ds}{(a^2+s)R(s)}; \qquad (1.1.36a)$$

$$F_y = mf_y = -2\gamma m\pi abcy\int_\lambda^\infty \frac{\rho(k^2)ds}{(b^2+s)R(s)}; \qquad (1.1.36b)$$

$$F_z = mf_z = -2\gamma m\pi abcz\int_\lambda^\infty \frac{\rho(k^2)ds}{(c^2+s)R(s)}. \qquad (1.1.36c)$$

We now consider the gravitational potential of a body T with an ellipsoidal density distribution for the case when the attracted point P of unit mass lies *inside* the ellipsoid E [99], that is, when:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1.$$

Let us select from the family of similar ellipsoids E_k the one that passes through a point P, and denote by k_0 $(0<k_0<1)$ the parameter value k corresponding to this ellipsoid (Fig. 1.8).

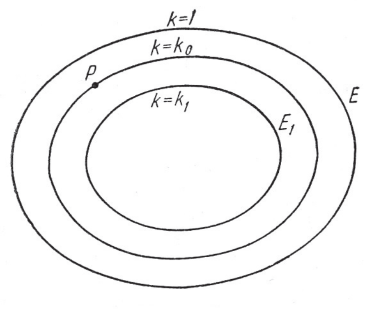


Figure 1.8. The layered representation of a body with an ellipsoidal mass density distribution

Obviously, for all infinitely thin ellipsoidal layers $T^{'}$ lying inside the ellipsoid $E_{k_0}^{'}(0 \le k \le k_0)$ the point P is exterior, and for layers lying outside $E_{k_0}^{'}(k_0 \le k \le 1)$ the point P is interior. Therefore, to get the expression for the full gravitational potential of the body T at the interior point P, it is necessary to integrate the expression (1.1.24) with respect to k^2, from zero to k_0^2, and add to the obtained expression the result of integrating an expression like (1.1.24) with respect to k^2 for the gravitational potential of the layer $T^{'}$ at the interior point, from k_0^2 to one [99].

But the expression of the gravitational potential of the infinitely thin layer at the interior point is obtained from (1.1.24) by replacing the lower integral limit s through zero, and therefore we have [99]:

$$\varphi_g(P) = -\gamma\pi abc \left\{ \int_0^{k_0^2} \rho(k^2)dk^2 \int_s^{\infty} \frac{d\sigma}{R(\sigma)} + \int_{k_0^2}^{1} \rho(k^2)dk^2 \int_0^{\infty} \frac{d\sigma}{R(\sigma)} \right\}. \quad (1.1.37)$$

Applying to the first of these integrals the integration formula by parts and taking into account that if $k^2 = 0$ then $s = \infty$ and if $k^2 = k_0^2$ then $s = 0$, we obtain:

$$\varphi_g(P) = -\gamma \int_0^{k_0^2} \kappa(k^2) d\varphi_0(s) \quad (1.1.38)$$

and passing here to the integration variable s, we finally find:

$$\varphi_g(P) = -\gamma\pi abc \int_0^{\infty} \frac{\kappa(k^2)ds}{R(s)}. \quad (1.1.39)$$

When $\rho = \text{const}$ we obtain the expression of the gravitational potential of a homogeneous ellipsoid at the interior point.

Differentiating (1.1.39) relative to x, y, z we find the components of the specific gravitational force (acting on a particle of a unit mass) inside the whole ellipsoidal body T [99]:

$$f_x = -\frac{\partial \varphi_g}{\partial x} = -2\gamma\pi abcx \int_0^\infty \frac{\rho(k^2)ds}{(a^2+s)R(s)}, \qquad (1.1.40a)$$

$$f_y = -\frac{\partial \varphi_g}{\partial y} = -2\gamma\pi abcy \int_0^\infty \frac{\rho(k^2)ds}{(a^2+s)R(s)}, \qquad (1.1.40b)$$

$$f_z = -\frac{\partial \varphi_g}{\partial z} = -2\gamma\pi abcz \int_0^\infty \frac{\rho(k^2)ds}{(c^2+s)R(s)}. \qquad (1.1.40c)$$

As shown in [99], the gravitational potential of the ellipsoidal body (defined by the formula (1.1.39)) satisfies the Poisson equation [100]:

$$\nabla^2 \varphi_g = 4\pi\gamma\rho. \qquad (1.1.41)$$

Now let us return to the consideration of the abovementioned conceptual model. According to Newton's theorem (see Theorems 1.1, 1.2 and Corollary 1.1) a porous gas-dust medium spread in outer space in the form of a collection of bubbles of various sizes and shapes (spherical and ellipsoidal, see Fig. 1.6 and 1.7) *does not have a gravitational field* inside bubble structures. Consequently, insignificant bursts of gravitational forces can only be associated with cosmic matter contained in inter-bubble gaps. Thus, the resulting gravitational force of the finite segment of such bubble matter will be negligible. When averaging it over an infinitely extended space occupied by bubble matter we obtain a zero value.

So, since the resulting gravitational force, averaged over space, is equal to zero then such *spread matter cannot condense into a denser formation* under the action of just its gravitational forces. The situation does not change significantly if all bubbles are replaced by liquid particles of spread cosmic matter, as in the case of the ideal models considered above (see Figures 1.3–1.5). Thus, within the framework of the traditional field theory [100], it is impossible to find an explanation for the phenomenon of initial gravitational contraction of infinitely distributed cosmic matter in space.

In this regard, it is appropriate to recall the dictum of the great mathematician and physicist Blaise Pascal who stated in his book "Pensées": "Space is an infinite sphere: the center is everywhere but there is no circle anywhere." Similar views were held by Sir Isaac Newton. In his first letter to Dr. Bentley (Dec. 10, 1692) he wrote the following [1 p.352, 101]:

> It seems to me that if the matter of our sun and planets, and all the matter of the universe, were evenly scattered throughout all the heavens, and every particle had an innate gravity toward all the rest, and the whole space throughout which this matter was scattered, was finite, the matter on the outside of this space would by its gravity tend toward all the matter on the inside, and by consequence fall down into the middle of the whole space, and there compose one great spherical mass. But if the matter were evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered great distances from one to another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matters were of a lucid nature.

In this context, as S. Weinberg [101] pointed out, the difficulty of understanding the problems of the dynamics of an *infinite medium* greatly paralyzed further progress until the appearance of Einstein's general theory of relativity (GR). In the framework of GR, Albert Einstein used the existing mathematical theory of non-Euclidean geometry to explain gravity as an effect of the curvature of space and time. Einstein tried to find a solution to his equations that would describe the space-time geometry of the Universe as a whole. Following the cosmological ideas existing at that time, Einstein in particular sought a solution that would be homogeneous, isotropic, and *static* [101]. However, such a solution was not found. To build a model that satisfied these cosmological requirements, Einstein was forced, as Weinberg put it, to "disfigure" his equations by introducing a term, the so-called *cosmological constant*, which "spoiled the elegance of the original theory but could serve to balance the force of gravitational attraction at large distances" [101 p.37]. Moreover, L.D. Landau and E.M. Lifschitz [100], as well as several other scientists, also held a similar negative opinion regarding the introduction of Einstein's cosmological constant. In particular, in [100 p. 544] it is indicated that "the introduction of a constant term into the density of the Lagrangian function, which is completely independent of the field state, would mean ascribing to space-time a fundamentally unremovable curvature that is connected neither matter nor gravitational waves."

We note, however, that Einstein himself, in his work "The problems of cosmology and the general theory of relativity" [102], pointed to the same difficulties inherent in both Newton's theory and Einstein's GR in connection with the *infinitely* spread cosmic matter. Here is what, in particular, he wrote about this [102]:

> It is known that the Poisson differential equation[1]
>
> $$\nabla^2 \varphi_g = 4\pi\gamma\rho$$
>
> in conjunction with the equation of motion of a mass point cannot completely replace the theory of long-range action of Newton. It is necessary to add the condition that the potential φ_g in spatial infinity tends to a certain limit. The situation is similar in the theory of gravity resulting from the general principle of relativity; here also the boundary conditions for spatial infinity should be added to the differential equations if we consider the world as infinitely extended in space.

[1] With author's designations of values.

Fortunately, progress has recently been made in understanding the role and significance of the cosmological constant for modern astrophysics. The physical interpretation of the cosmological constant introduced by Einstein into GR in a somewhat formal way has been formed gradually, decade after decade, beginning with the works of W. de Sitter, G. Lemaître, R.C. Tolman, H. Bondi et al. [103]. As A. Chernin notes in his article [103 p.1154], "it is now considered generally accepted that the cosmological constant describes a cosmic vacuum, i.e. such a state of cosmic energy which has a constant density in time and everywhere in space, and any reference system. According to these properties, the vacuum fundamentally differs from all other forms of cosmic energy, the density of which is heterogeneous in space, decreases with time during cosmological expansion and may be different in other reference systems."

Although the vacuum is called cosmic, it is present everywhere and appears in atomic physics and microphysics where it represents the lowest energy state of quantum fields. This is the same vacuum in which interactions of elementary particles are observed and which directly manifests itself experimentally, for example, in the Lamb shift of the spectral lines of atoms and the Casimir effect [103]. In such experiments, the presence of the vacuum occurs but at the same time, the value of its density is not yielded by measurement. The problem of the vacuum energy density is considered to be the most complex problem of modern fundamental physics [104].

This monograph develops another interpretation of both the Einstein equation for a gravitational potential with a cosmological constant [73] and the phenomenon of the initial gravitational condensation of infinitely spread cosmic matter within the framework of the proposed statistical theory of cosmogonical bodies forming (see

Chapters 2–5 in the monograph as well as Refs.[16,65,73]).

1.2. The virial theorem

The foregoing results, as well as others of a more general kind, may also be obtained from a theorem proved mathematically by Henri Poincaré (Poincaré's Theorem) [1, 105] though this result as a virial theorem was first formulated by Rudolf Clausius in his lectures on thermodynamics [106, 107].

We consider a collection of detached masses (clouds) moving under no force except their mutual gravitational attraction. The masses may be molecules, dust particles, atoms or electrons [1]; for convenience, we shall speak of them as molecules (as well as *molecular clouds* respectively).

Let m_0 be a mass of the particle (molecule) and x, y, z be its coordinates. Let us denote by F_x, F_y, F_z the components of force acting on the particle. According to Newton second law, the equations of motion of a single particle of mass m_0 are:

$$m_0 \frac{d^2 x}{dt^2} = F_x; \quad m_0 \frac{d^2 y}{dt^2} = F_y; \quad m_0 \frac{d^2 z}{dt^2} = F_z. \tag{1.2.1}$$

On the other hand, it is not difficult to see that:

$$\frac{1}{2} \cdot \frac{d^2}{dt^2}\left(m_0 x^2\right) = m_0 \frac{d}{dt}\left(x \frac{dx}{dt}\right) = m_0 x \frac{d^2 x}{dt^2} + m_0 \left(\frac{dx}{dt}\right)^2, \tag{1.2.2}$$

whence:

$$m_0 \frac{d^2 x}{dt^2} = \frac{1}{2x} \cdot \frac{d^2}{dt^2}\left(m_0 x^2\right) - \frac{m_0}{x} \cdot \left(\frac{dx}{dt}\right)^2. \tag{1.2.3}$$

Using the first equality in (1.2.1) as well as the equation (1.2.3) we find that:

$$\frac{1}{2} \cdot \frac{d^2}{dt^2}\left(m_0 x^2\right) = m_0 \left(\frac{dx}{dt}\right)^2 + x F_x \tag{1.2.4a}$$

and by analogy:

$$\frac{1}{2}\cdot\frac{d^2}{dt^2}\left(m_0 y^2\right)=m_0\left(\frac{dy}{dt}\right)^2+yF_y\,, \qquad (1.2.4b)$$

$$\frac{1}{2}\cdot\frac{d^2}{dt^2}\left(m_0 z^2\right)=m_0\left(\frac{dz}{dt}\right)^2+zF_z\,. \qquad (1.2.4c)$$

Summing all the preceding equations (1.2.4a), (1.2.4b) and (1.2.4c), we obtain [106 p.93]:

$$\frac{1}{2}\cdot\frac{d^2}{dt^2}\left(m_0 r^2\right)=m_0\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]+\left(xF_x+yF_y+zF_z\right). \qquad (1.2.5)$$

Let us note that the first addend on the right-hand side of Eq. (1.2.5) is the double kinetic energy of a single particle of mass m_0. If the summation extends over all the molecules of the cloud, so that E_k is the total kinetic energy of translation of the cloud, then Eq. (1.2.5) becomes:

$$\frac{1}{2}\cdot\frac{d^2 I}{dt^2}=2E_k+\sum_i\left(x_i F_{x_i}+y_i F_{y_i}+z_i F_{z_i}\right), \qquad (1.2.6)$$

where I is a moment of inertia of the cloud relative to an origin of coordinates, i.e. to the center of mass of the cloud (see Section 1.1), defined by the formula:

$$I=\sum_i\left(m_{0i} r_i^2\right), \qquad (1.2.7a)$$

and E_k is the total kinetic energy of translation of the cloud (relative to the origin of coordinates):

$$E_k=\frac{1}{2}\sum_i m_{0i}\left[\left(\frac{dx_i}{dt}\right)^2+\left(\frac{dy_i}{dt}\right)^2+\left(\frac{dz_i}{dt}\right)^2\right]. \qquad (1.2.7b)$$

The second addend on the right-hand side of Eq. (1.2.6) is called the *Clausius' virial* [106 p.94]. To estimate the value of virial we call our attention to two selected particles with masses m_{01} and m_{02} located at points (x_1, y_1, z_1) and

(x_2, y_2, z_2). Let a force acting on the first particle relative to the second has components $F_{x_{12}}, F_{y_{12}}$ and $F_{z_{12}}$. Then the force influencing the second particle through the first has the components $-F_{x_{12}}, -F_{y_{12}}$ and $-F_{z_{12}}$ respectively. The contribution in the virial from these two forces is equal to:

$$F_{x_{12}} \cdot (x_1 - x_2) + F_{y_{12}} \cdot (y_1 - y_2) + F_{z_{12}} \cdot (z_1 - z_2)$$

and, therefore, the virial of a particle system (cloud) is defined as follows [106 p.94]:

$$W = \sum_i \sum_j [F_{x_{ij}}(x_i - x_j) + F_{y_{ij}}(y_i - y_j) + F_{z_{ij}}(z_i - z_j)], \qquad (1.2.8)$$

where the summation extends over all pairs of particles (molecules). For a cloud of particles with such a low mass density that it is possible to assume the validity of the laws of ideal gases, all forces (except gravitational) can be ignored [106]. So, we can suppose that $F_{x_{ij}}$, $F_{y_{ij}}$ and $F_{z_{ij}}$ are components of the force of gravity:

$$F_{x_{ij}} = -\gamma \frac{m_i m_j}{r_{ij}^2} \cdot \frac{x_i - x_j}{r_{ij}}; \qquad (1.2.9a)$$

$$F_{y_{ij}} = -\gamma \frac{m_i m_j}{r_{ij}^2} \cdot \frac{y_i - y_j}{r_{ij}}; \qquad (1.2.9b)$$

$$F_{z_{ij}} = -\gamma \frac{m_i m_j}{r_{ij}^2} \cdot \frac{z_i - z_j}{r_{ij}}, \qquad (1.2.9c)$$

where γ is the Newtonian gravitational constant, m_i, m_j are any pair of particles, r_{ij} is their distance apart. Then, according to (1.2.8), the virial is equal to [106 p.94]:

$$W = \sum_i \sum_j -\gamma \frac{m_i m_j}{r_{ij}^2} \cdot \left\{ \frac{(x_i - x_j)^2}{r_{ij}} + \frac{(y_i - y_j)^2}{r_{ij}} + \frac{(z_i - z_j)^2}{r_{ij}} \right\} =$$

$$= \sum_i \sum_j -\gamma \frac{m_i m_j}{r_{ij}^3} r_{ij}^2 = -\sum_i \sum_j \gamma \frac{m_i m_j}{r_{ij}}, \qquad (1.2.10)$$

where the summation extends over all pairs of particles. Thus, the virial W is the total gravitational potential energy E_p of the particle system (molecular cloud) under study [106 p.94]. Therefore, if the only forces which act on the molecules are those arising from their mutual gravitation [1 p.67], we have:

$$F_{x_i} = -\frac{\partial W}{\partial x_i}; \quad F_{y_i} = -\frac{\partial W}{\partial y_i}; \quad F_{z_i} = -\frac{\partial W}{\partial z_i}, \qquad (1.2.11)$$

where W is the total gravitational potential energy of the molecular cloud following the formula (1.2.10). As known (see, for example [108]), for a m-th order homogeneous function f, the Euler's theorem is true, i.e. if $f(kx, ky, ...) = k^m f(x, y, ...)$ then $\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + ... = mf(x, y, ...)$.

Since W is homogeneous in x_i, y_i, z_i and of dimensions $m = -1$, it follows from this theorem that:

$$\sum_i \left(x_i \frac{\partial W}{\partial x_i} + y_i \frac{\partial W}{\partial y_i} + z_i \frac{\partial W}{\partial z_i} \right) = -W. \qquad (1.2.12)$$

Taking into account (1.2.11) and (1.2.12), the equation (1.2.6) now assumes the form:

$$\frac{1}{2} \cdot \frac{d^2 I}{dt^2} = 2E_k + W. \qquad (1.2.13)$$

If the particle system forming a molecular cloud has attained a *steady state*, i.e. $I = \text{const}$, the left-hand member vanishes, and Eq. (1.2.13) becomes:

$$2E_k + W = 0. \qquad (1.2.14)$$

This is known as the virial theorem [1, 105-107]:

Theorem 1.3 (the virial theorem). For self-gravitating masses in a steady state, the kinetic energy of a system of particles is equal to minus ½ of the total gravitational potential energy.

Later A.S. Eddington has remarked in his work [109 p.525] that the virial theorem is not restricted to only states of steady motion of particle systems, i.e. it can be extended in the form (1.2.13). Therefore, for the *unstable motion* of a cloud of particles, the following result should be used [1, 109]:

Theorem 1.4 (the Poincaré–Eddington's theorem). For self-gravitating masses, under the condition of their unstable motion, the sum of the double kinetic energy and the total gravitational potential energy of a system of particles is equal to:

$$2E_k + W = \frac{1}{2} \cdot \frac{d^2 I}{dt^2} \,,$$

where I stands for expression (1.2.7a) of the inertia moment of the particle system relative to its center of mass.

In conclusion of this section, let us apply the virial theorem to a cloud-like configuration of an *ideal* gas being in both mechanical and thermodynamic equilibrium. First of all, as J. Jeans pointed out[2] [1]:

> We can write the kinetic energy (1.2.7b) in the form:
>
> $$E_k = \frac{1}{2} \sum_i m_{0i} v_i^2 ,$$
>
> where $v_i = \sqrt{(dx_i / dt)^2 + (dy_i / dt)^2 + (dz_i / dt)^2}$ is a velocity of translation of a i-th molecule of mass m_{0i}. The potential energy W may similarly be written in the form:
>
> $$W = -\frac{1}{2} \sum_i m_{0i} \varphi_{gi} \,, \qquad (1.2.16)$$

[2] With the author's numeration of formulas and designations

where φ_{gi} is the gravitational potential at the i-th point occupied by the mass m_{0i}. Thus, according to (1.2.15), (1.2.16) the Poincaré's theorem (1.2.14) takes the form that

$$\sum_i m_{0i}\left(v_i^2 - \frac{1}{2}\varphi_{gi}\right) = 0 \qquad (1.2.17)$$

so that, in the steady state, the average value of v_i^2, averaged over all the separate masses, is equal to the average value of $\frac{1}{2}\varphi_g$.

If the system of particles is of total mass M and has a mean radius $\bar{r}$, the average value $\frac{1}{2}\overline{\varphi}_g$ is of the order of magnitude of $\gamma M/\bar{r}$,

so that the average value $\overline{v^2}$ is of this order of magnitude (in accord with (1.2.17)). This provides a convenient rough measure of the average velocity of agitation of a system of gravitating masses in a steady state: it is equally applicable to systems of stars, star clusters, nebulae, and masses of gravitating gas.

If the particles which constitute the system are taken to be the molecules of a gas (or other independently moving units such as atoms, free electrons, etc.) then, as known from the molecular kinetic theory [1, 110], the mean square value of their velocity is equal to:

$$\overline{v^2} = \frac{3k_B T}{m_0} = \frac{3\Re T}{\mu}, \qquad (1.2.18)$$

where T is a temperature of the ideal gas, $k_B = 1.38049 \cdot 10^{-23}$ J/K is the Boltzmann constant, $\Re = k_B \cdot N_A = 8.3169$ J/(mole·K) is the universal gaseous constant ($N_A = 6.023 \cdot 10^{23}$ mole^{-1} is the Avogadro number), $\mu = m_0 \cdot N_A$ is a molar mass of the gaseous substance, m_0 is a mass of the molecule.

Then the mean temperature of the gas is of the order of magnitude:

$$T = \frac{\gamma M}{\bar{r}} \cdot \frac{\mu}{3\Re}, \qquad (1.2.19)$$

where $\gamma = 6.673 \cdot 10^{-11}\ \mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2$ is the Newtonian gravitational constant, so that the mean internal temperatures of different stars are approximately proportional to the values of $\mu M/\bar{r}$ for these stars [1].

In particular, J. Jeans found from this formula that if the Sun is supposed to be formed of hydrogen molecules (for which the relative molecular weight $\mu_r = 2.016$), its mean temperature must be of the order of 15,000,000 degrees [1]. Indeed, it directly follows from the formula (1.2.19) that if the Sun (for which $M_{\mathrm{Sun}} = 1.98 \cdot 10^{30}$ kg and $\bar{r}_{\mathrm{Sun}} = 6.95 \cdot 10^{8}$ m) is predominantly formed from hydrogen ions, deuterium, and helium with a mean molar mass $\mu = \mu_r \cdot 10^{-3} = 2 \cdot 10^{-3} (\mathrm{kg/mole})$, then its mean temperature should be equal to:

$$T = \frac{6.67 \cdot 10^{-11}\ \mathrm{J} \cdot \mathrm{m}/\mathrm{kg}^2 \cdot 1.98 \cdot 10^{30}\ \mathrm{kg}}{6.95 \cdot 10^{8}\,\mathrm{m}} \times$$

$$\times \frac{2 \cdot 10^{-3}\ \mathrm{kg/mole}}{3 \cdot 8.3169\ \mathrm{J}/(\mathrm{mole} \cdot \mathrm{K})} \approx 1{,}47 \cdot 10^{7}\ \mathrm{K}. \qquad (1.2.20)$$

Let us note that modern data on the physical characteristics of the Sun indicate that the temperature of its core is commensurable with the Jeans estimation and is approximately equal 13,500,000 K while the temperature of its corona is ~1,500,000 K only.

1.3. On the gravitational instability of Jeans and the rotational instability of Rayleigh in a gravitating molecular cloud

It is known (see, for example, [12]) that the interstellar medium contains very little dust and practically behaves as a single-component gaseous substance, i.e. the *molecular cloud*. Therefore, the main result in the theory of gravitational instability for such a single-component medium was first obtained by famous astrophysicist Sir James H. Jeans in 1902 whose theory is described below [1 p.346-348]. First of all, J. Jeans noted [1 p.351-352] that:

> "...the general conception of the stars having been formed out of a homogeneous medium by a process of condensation under gravity is of course very old, being indeed almost as old as the law of gravitation itself."

Now we are going to study a system of particles as a single-component molecular cloud. Let us consider any motion of a continuous mass of gas or other compressible matter, this being determined by the Euler hydrodynamic equation in Cartesian coordinates [94, 111]:

$$a_x = f_x - \frac{1}{\rho} \cdot \frac{\partial p}{\partial x}; \quad a_y = f_y - \frac{1}{\rho} \cdot \frac{\partial p}{\partial y}; \quad a_z = f_z - \frac{1}{\rho} \cdot \frac{\partial p}{\partial z}, \qquad (1.3.1)$$

where a_x, a_y, a_z are the components of acceleration of an infinitely small volume of a continuous medium, i.e. the so-called "liquid" particle [94] which is momentarily at x, y, z, f_x, f_y, f_z are the components of specific gravitational force, p is a pressure.

In the case of the *barotropic* motion of a continuous gas medium, the pressure is a function of the density only. This means that a function (potential) of the pressure to be introduced [111]:

$$\wp(\rho) = \int_{p_0}^{p} \frac{dp}{\rho}, \qquad (1.3.2)$$

so that the equations of motion (1.3.1) become:

$$a_x = f_x - \frac{\partial \wp}{\partial x}; \qquad (1.3.3a)$$

$$a_y = f_y - \frac{\partial \wp}{\partial y}; \qquad (1.3.3b)$$

$$a_z = f_z - \frac{\partial \wp}{\partial z}. \quad (1.3.3c)$$

Along with the usual motion of a "liquid" particle, described by Eqs (1.3.3a-c), we consider the modified (variational) motion of this particle under the action of a certain perturbation [1]. Let us compare the motion with a slightly varied motion such that the particle which is at x, y, z at the time t in the original motion is at $x+\xi,\ y+\eta,\ z+\zeta$ at the time t in the varied motion. The acceleration of this particle in the varied motion has components:

$$a_x + d^2\xi / dt^2;\ a_y + d^2\eta / dt^2;\ a_z + d^2\zeta / dt^2,$$

so that the particle which is at x, y, z at time t in the varied motion has components of acceleration [1]:

$$a_x + d^2\xi / dt^2 - \xi\partial a_x / \partial x - \eta\partial a_x / \partial y - \zeta\partial a_x / \partial z;$$

$$a_y + d^2\eta / dt^2 - \xi\,\partial a_y / \partial x - \eta\partial a_y / \partial y - \zeta\partial a_y / \partial z;$$

$$a_z + d^2\zeta / dt^2 - \xi\partial a_z / \partial x - \eta\partial a_z / \partial y - \zeta\partial a_z / \partial z.$$

As a result of varying the motion let the density and the components of specific gravitational force at x, y, z be changed to

$$\rho + \delta\rho,\ f_x + \delta f_x,\ f_y + \delta f_y,\ f_z + \delta f_z.$$

Then, according to (1.3.1) the equations which govern the varied motion are:

$$a_x + \frac{d^2\xi}{dt^2} - \xi\frac{\partial a_x}{\partial x} - \eta\frac{\partial a_x}{\partial y} - \zeta\frac{\partial a_x}{\partial z} = f_x + \delta f_x - \frac{\partial \wp(\rho+\delta\rho)}{\partial x}; \quad (1.3.4a)$$

$$a_y + \frac{d^2\eta}{dt^2} - \xi\frac{\partial a_y}{\partial x} - \eta\frac{\partial a_y}{\partial y} - \zeta\frac{\partial a_y}{\partial z} = f_y + \delta f_y - \frac{\partial \wp(\rho+\delta\rho)}{\partial y}; \quad (1.3.4b)$$

$$a_z + \frac{d^2\zeta}{dt^2} - \xi\frac{\partial a_z}{\partial x} - \eta\frac{\partial a_z}{\partial y} - \zeta\frac{\partial a_z}{\partial z} = f_z + \delta f_z - \frac{\partial \wp(\rho+\delta\rho)}{\partial z} . \quad (1.3.4c)$$

Subtracting corresponding sides of these equations (1.3.4a-c) and Eqs(1.3.3a-c), we obtain [1]:

$$\frac{d^2\xi}{dt^2} - \xi\frac{\partial a_x}{\partial x} - \eta\frac{\partial a_x}{\partial y} - \zeta\frac{\partial a_x}{\partial z} = \delta f_x - \frac{\partial}{\partial x}\left(\frac{\partial \wp}{\partial \rho}\delta\rho\right); \quad (1.3.5a)$$

$$\frac{d^2\eta}{dt^2} - \xi\frac{\partial a_y}{\partial x} - \eta\frac{\partial a_y}{\partial y} - \zeta\frac{\partial a_y}{\partial z} = \delta f_y - \frac{\partial}{\partial y}\left(\frac{\partial \wp}{\partial \rho}\delta\rho\right); \quad (1.3.5b)$$

$$\frac{d^2\zeta}{dt^2} - \xi\frac{\partial a_z}{\partial x} - \eta\frac{\partial a_z}{\partial y} - \zeta\frac{\partial a_z}{\partial z} = \delta f_z - \frac{\partial}{\partial z}\left(\frac{\partial \wp}{\partial \rho}\delta\rho\right), \quad (1.3.5c)$$

where:

$$\frac{\partial \wp}{\partial \rho}\delta\rho = \frac{1}{\rho}\cdot\frac{\partial p}{\partial \rho}\delta\rho . \quad (1.3.6)$$

The obtained three equations (1.3.5a)-(1.3.5c) are *linear* ones in ξ, η, ζ .

To perform some simplifications, we use the continuity equation [94, 111]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial \rho}{\partial t} + v_x\cdot\frac{\partial \rho}{\partial x} + v_y\cdot\frac{\partial \rho}{\partial y} + v_z\cdot\frac{\partial \rho}{\partial z} +$$

$$+\rho\cdot\frac{\partial v_x}{\partial x} + \rho\cdot\frac{\partial v_y}{\partial y} + \rho\cdot\frac{\partial v_z}{\partial z} \equiv 0,$$

which, after multiplying by δt and introducing the Euler's notation $u = v_x, \upsilon = v_y, w = v_z$, $\xi = u\delta t,\ \eta = \upsilon\delta t,\ \zeta = w\delta t$, can be written as follows [73]:

$$\delta t\cdot\frac{\partial \rho}{\partial t} + \xi\cdot\frac{\partial \rho}{\partial x} + \eta\cdot\frac{\partial \rho}{\partial y} + \zeta\cdot\frac{\partial \rho}{\partial z} = -\rho\cdot\frac{\partial \xi}{\partial x} - \rho\cdot\frac{\partial \eta}{\partial y} - \rho\cdot\frac{\partial \zeta}{\partial z} . \quad (1.3.7)$$

Taking into account the fact that $\delta\rho = \delta t \cdot \partial\rho / \partial t + \xi \cdot \partial\rho / \partial x + \eta \cdot \partial\rho / \partial y + \zeta \cdot \partial\rho / \partial z$ from the relation (1.3.7) we finally get that:

$$\delta\rho = -\rho\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right). \tag{1.3.8}$$

Substitution (1.3.8) into Eq. (1.3.6) gives the following formula [1]:

$$\frac{\partial\wp}{\partial\rho}\cdot\delta\rho = -\frac{\partial p}{\partial\rho}\cdot\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right). \tag{1.3.9}$$

The obtained three equations (1.3.5a)-(1.3.5c) can be conveniently written in the form of a single vector equation [73]:

$$\frac{d^2\vec{\chi}}{dt^2} - (\vec{\chi}\cdot\nabla)\vec{a} = \delta\vec{f} - \nabla\left(\delta\rho\cdot\frac{\partial\wp}{\partial\rho}\right), \tag{1.3.10}$$

where $\vec{\chi} = (\xi, \eta, \zeta)$, $\vec{a} = (a_x, a_y, a_z)$, $\delta\vec{f} = (\delta f_x, \delta f_y, \delta f_z)$, moreover, according to (1.3.9) we have [73]:

$$\delta\rho\cdot\frac{\partial\wp}{\partial\rho} = -\nabla\vec{\chi}\cdot\frac{\partial p}{\partial\rho}.$$

The condition that Eqs (1.3.5a)-(1.3.5c) (or the vector equation (1.3.10)) shall have a solution, other than the trivial one $\xi = \eta = \zeta = 0$, is expressed by the vanishing of a quantity which involves only the coefficients of ξ, η, ζ and their differentials in equations Eqs (1.3.5a)-(1.3.5c), etc. In this connection J. Jeans wrote [1 p.346-347]:

> When conditions are such that this quantity vanishes, we are at what may be described as a dynamical point of bifurcation. Indeed, two alternative motions are open, both of which satisfy the dynamical equations of motion. In one the liquid particle which was at $x - dx$, $y - dy$, $z - dz$ at time $t - dt$ moves to x, y, z at time t; in the other, it moves to $x + \xi$, $y + \eta$, $z + \zeta$.

> As in the corresponding statical problem, there may or may not be a transfer of stabilities at a point of bifurcation. To discuss the question of stability we merely have to examine whether small displacements ξ, η, ζ determined by equations (1.3.5a-c) will increase beyond limit or not. If they are found to increase beyond limit, the original unvaried motion was unstable, and the system changes over to the varied motion at the point of bifurcation; in the reverse case the original motion was stable and the displacement ξ, η, ζ, if set up, merely behaves as a small oscillation about a stable state of motion.

Let us note that equations (1.3.5a)-(1.3.5c) are too complex to be solved in the most general case, but we can obtain a knowledge of the general nature of the solution by considering the simple case in which a_x, a_y, a_z are approximately constant throughout a large extent of the medium, this including the case of a medium at rest. In this case, the equations (1.3.5a)-(1.3.5c) assume the form [1]:

$$\frac{d^2\xi}{dt^2} = \delta f_x - \frac{\partial}{\partial x}\left(s\frac{dp}{d\rho}\right); \qquad (1.3.11a)$$

$$\frac{d^2\eta}{dt^2} = \delta f_y - \frac{\partial}{\partial y}\left(s\frac{dp}{d\rho}\right); \qquad (1.3.11b)$$

$$\frac{d^2\zeta}{dt^2} = \delta f_z - \frac{\partial}{\partial z}\left(s\frac{dp}{d\rho}\right), \qquad (1.3.11c)$$

where $s = \delta\rho/\rho$ is a value of condensation of a continuous medium which in accordance with (1.3.8) is defined by the following formula:

$$s = \frac{\delta\rho}{\rho} = -\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right). \qquad (1.3.12)$$

Differentiating each of the equations (1.3.11a)-(1.3.11c) with respect to x, y, z and adding them, we obtain:

$$\frac{d^2(\partial\xi/\partial x+\partial\eta/\partial y+\partial\zeta/\partial z)}{dt^2}=$$

$$=\frac{\partial(\delta f_x)}{\partial x}+\frac{\partial(\delta f_y)}{\partial y}+\frac{\partial(\delta f_z)}{\partial z}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)s\frac{dp}{d\rho}. \qquad (1.3.13a)$$

Taking into account relation (1.3.12) and using the Poisson equation (1.1.41):

$$\frac{\partial(\delta f_x)}{\partial x}+\frac{\partial(\delta f_y)}{\partial y}+\frac{\partial(\delta f_z)}{\partial z}=\nabla(\delta\vec{f})=-\nabla^2\delta\varphi_g=-4\pi\gamma\delta\rho, \qquad (1.3.13b)$$

the equation (1.3.13a) takes the final form [1]:

$$\frac{d^2 s}{dt^2}=4\pi\gamma\rho s+\nabla^2\left(s\frac{dp}{d\rho}\right). \qquad (1.3.14)$$

Unlike Eqs (1.3.11a)-(1.3.11c), this equation involves only the one variable s and "so determines the way in which s changes throughout the motion" [1 p. 347].

On omitting the first term on the right-hand side of Eq. (1.3.14), we obtain the equation of propagation of rarefactions and condensations of the medium, when the gravitational attraction of the medium on itself is neglected [1]. In this case, the equation (1.3.14) reduces to Laplace's equation [94, 111]:

$$\frac{d^2 s}{dt^2}=(dp/d\rho)\nabla^2 s\,, \qquad (1.3.15a)$$

indicating propagation in the form of waves of sound, with the usual speed

$$c=\sqrt{\frac{dp}{d\rho}}\,. \qquad (1.3.15b)$$

To discuss the more general problem in its simplest form, let us confine our attention to a region of space within which $dp/d\rho$ may be treated as *constant* [1], and consider pure wave-motion along the axis of Ox, the value of s being supposed equal to:

$$s(t,x) = S(t)\cos(2\pi x/\lambda), \qquad (1.3.16)$$

so that λ is a wavelength. Taking into account this substitution, Eq. (1.3.14) becomes:

$$\frac{d^2S}{dt^2} = \left[4\pi\gamma\rho - \left(\frac{2\pi}{\lambda}\right)^2 \cdot \frac{dp}{d\rho}\right]S. \qquad (1.3.17)$$

As known, the solution of a harmonic equation of the kind (1.3.17) is a complex-valued exponential function:

$$S(t) = S(t_0)e^{\pm \mathrm{i}\tilde{\omega}t}, \mathrm{i} = \sqrt{-1}, \qquad (1.3.18)$$

where the square of a *new generalized frequency* $\tilde{\omega}$ [73] is:

$$\tilde{\omega}^2 = (2\pi/\lambda)^2 dp/d\rho - 4\pi\gamma\rho. \qquad (1.3.19)$$

It is easily seen that this represents wave-motion along the axis of Ox, with a speed of propagation [1]:

$$\tilde{c} = \tilde{\omega}\cdot\lambda/2\pi = \sqrt{dp/d\rho - 4\pi\gamma\rho\cdot(\lambda/2\pi)^2}. \qquad (1.3.20)$$

Here J. Jeans noted[3] [1 p.348]:

> If we again omit the gravitational term $-4\pi\gamma\rho$, we have a wave-motion which travels with a uniform velocity $(dp/d\rho)^{\frac{1}{2}}$ independently of the wavelength. The restoration of the gravitational term invariably lessens the velocity of propagation, but since the term in question is multiplied by λ^2, we see that the effect of gravitation is inappreciable for waves of short wavelength. For waves of longer wavelength, the gravitational term becomes more important. Finally, a value of λ is reached at which the velocity of propagation, as given by formula (1.3.20), disappears altogether and subsequently becomes imaginary. For such values of λ there can be no proper propagation of waves; the value of $\tilde{\omega}^2$, as given by equation (1.3.19), becomes negative, so that the time factors $e^{\pm \mathrm{i}\tilde{\omega}t}$ assume the form $e^{\pm\gamma t}$, where γ is real. This represents unstable motion, the initial condensations and rarefactions increasing exponentially with the time.

---

[3] With some author's designations and formula numeration

We can see that the equation (1.3.14) determines possible distributions of condensation (s) and rarefaction $(-s)$ which may be superposed on to the original motion. It now appears that all distributions which vary harmonically are unstable if their wavelength λ is greater than a *critical wavelength* λ_c [1] defined by:

$$\lambda_c = \sqrt{\frac{\pi}{\gamma\rho} \cdot \frac{dp}{d\rho}}, \qquad (1.3.21)$$

at which $\tilde{c} = 0$, as it was already noted by Jeans in his analysis of formula (1.3.20). In passing, we note that *in a gravitating medium the speed of a "heavy" sound is less than usual one* [12, 73], according to the same formula (1.3.20). Since the usual speed of sound following (1.3.15b) is equal $c = \sqrt{dp / d\rho}$, then the substitution of this value in (1.3.21) directly leads to the formula that determines the critical wavelength λ_c of the perturbation:

$$\lambda_c = c\sqrt{\frac{\pi}{\gamma\rho}}. \qquad (1.3.22)$$

Having determined the critical wavelength (1.3.22), it is easy to formulate *the Jeans criterion of gravitational instability* [2]:

$$\lambda > \lambda_c. \qquad (1.3.23)$$

Since the speed of wave disturbances in a *non-gravitating* medium is $c = \lambda \cdot \omega / 2\pi$, where ω is a circular frequency of wave disturbances in a non-gravitating medium, we can rewrite the inequality (1.3.23) using (1.3.22) as follows:

$$\lambda > \lambda \cdot (\omega / 2\pi)\sqrt{\pi / \gamma\rho}, \qquad (1.3.24)$$

whence it immediately follows that:

$$\omega^2 < 4\pi\gamma\rho. \qquad (1.3.25)$$

Inequality (1.3.25) is known as Jeans' condition [16, 65, 73] which gives the critical frequency $\omega_c = 2\sqrt{\pi\gamma\rho}$ of the wave perturbation leading to gravitational instability in a gas-dust medium.

The further development of the linearized theory of gravitational instability of Jeans was mainly associated with accounting the role of rotation as well as a magnetic field [2, 12]. Indeed, in 1955 S. Chandrasekhar generalized the Jeans problem in the case of an infinite homogeneous medium with uniform rotation [112]. Then in 1958 N. Bel and E. Schatzman [113] obtained a similar result for a homogeneous in mass density ($\rho = \text{const}$) but non-uniformly rotating medium. They considered disturbances propagating in a plane perpendicular to the axis of rotation z, symmetrical about this axis and independent of z. The condition of gravitational instability in a cylindrical coordinate system (h, ε, z), these authors obtained in the form [2, 113]:

$$\frac{2\Omega}{h}\cdot\frac{d(\Omega h^2)}{dh} + \frac{4\pi^2 c^2}{\lambda^2} + \frac{c^2}{4h^2} < 4\pi\gamma\rho\,, \qquad (1.3.26)$$

where $\Omega = \Omega(h)$ is an angular velocity. As noted in [73], both the Jeans condition (1.3.25) and the condition (1.3.26) can be obtained from the condition of the negative square of generalized frequency:

$$\widetilde{\omega}^2 < 0\,. \qquad (1.3.27)$$

Indeed, substituting into the inequality (1.3.27) the square of the generalized frequency (according to formula (1.3.19)) and taking into account (1.3.15b), we obtain that:

$$(2\pi/\lambda)^2 c^2 < 4\pi\gamma\rho\,, \qquad (1.3.28)$$

whence the Jeans criterion (1.3.25) for a non-rotating one-component medium follows immediately. If we now consider the square of the generalized frequency $\widetilde{\omega}^2$ within the framework of the Bel–Schatzman's model:

$$\tilde{\omega}^2 = \frac{2\Omega}{h} \cdot \frac{d(\Omega h^2)}{dh} + \frac{1}{(2h)^2} \cdot \frac{dp}{d\rho} + \left(\frac{2\pi}{\lambda}\right)^2 \cdot \frac{dp}{d\rho} - 4\pi\gamma\rho, \qquad (1.3.29)$$

then the substitution (1.3.29) into inequality (1.3.27) leads to the condition of gravitational instability (1.3.26) for a medium with non-uniform rotation. As seen, the square of the generalized frequency in the Bel–Schatzman's model (1.3.29) includes the square of the generalized frequency for the Jeans model (1.3.19) together with two additional terms depending on the angular velocity Ω and coordinate h.

In the local approximation ($\lambda << h$), as follows from (1.3.29) and (1.3.15b), the Bel–Schatzman's linearized *dispersion* equation (for axially symmetric radial perturbations) has the form [12]:

$$\tilde{\omega}^2 = \kappa^2 + k^2 c^2 - 4\pi\gamma\rho, \qquad (1.3.30a)$$

where $k = 2\pi / \lambda$ is a wave number, so $\omega = k \cdot c$ is a circular frequency, and κ is an *epicyclic* frequency:

$$\kappa^2 = \frac{2\Omega}{h} \cdot \frac{d(\Omega h^2)}{dh} = \frac{1}{h^3} \cdot \frac{d(\Omega h^2)^2}{dh}. \qquad (1.3.30b)$$

When $\Omega = \text{const}$ the equation (1.3.30a) is reduced to the Chandrasekhar's dispersion equation [12, 112]:

$$\tilde{\omega}^2 = (2\Omega)^2 + (kc)^2 - 4\pi\gamma\rho, \qquad (1.3.31a)$$

and at $\Omega = 0$ it becomes the Jeans equation (1.3.19) for a non-rotating medium:

$$\tilde{\omega}^2 = (kc)^2 - 4\pi\gamma\rho. \qquad (1.3.31b)$$

For all the above cases (1.3.30a)-(1.3.31b), the dependence of the perturbations on the time is taken in the form $e^{i\tilde{\omega}t}$ and, as indicated in (1.3.27), the instability occurs when $\tilde{\omega}^2 < 0$. The obtained expressions refer to media that are infinite both in the radial direction and along the axis of rotation [12].

Therefore, they are not applicable to thin disks with thickness $b << h$.

The dispersion equation for rotating flat systems of finite thickness was obtained by Safronov [114]:

$$\tilde{\omega}^2 = \kappa^2 + (kc)^2 - 4\pi\gamma\rho f(k \cdot b) . \qquad (1.3.32a)$$

As can be seen from equation (1.3.32a), it differs from equation (1.3.30a) by the multiplier $f(kb) < 1$ in the last term on the right-hand side of (1.3.32a), since the gravitational attraction of ring is much less than the attraction of infinite (along z) cylinder. At first, the factor $f(kb)$ was determined numerically [2], but then a simple and at the same time quite satisfactory analytical approximation was found for it [115]. As a result, the Safronov's dispersion equation (1.3.32a) acquired the form:

$$\tilde{\omega}^2 = \kappa^2 + (kc)^2 - 4\pi\gamma\rho(1 + 2/kb)^{-1} . \qquad (1.3.32b)$$

In studies of the stability of the spiral structure of galaxies, a dispersion equation for axially symmetric perturbations in an infinitely thin disk was obtained and developed by A. Toomre [116], P. Goldreich and D. Lynden-Bell [117], P. Goldreich and W.R. Ward [118]:

$$\tilde{\omega}^2 = \kappa^2 + (kc)^2 - 2\pi\gamma k\sigma , \qquad (1.3.32c)$$

where σ is a surface mass density of disk. Let us note that this relation is immediately obtained from (1.3.32b) if we suppose $kb \to 0$ in it [12]. Therefore, it is a satisfactory approximation when disturbance wavelengths exceed the thickness of the disk about the order (or more). For short-wavelength disturbances, it gives a big mistake overstating the gravitational attraction of perturbing ring $(1 + kb/2)$ times and, then so that, underestimating the critical value of the surface mass density at which gravitational instability begins [12]. In contrast, the Bel–Schatzman's equation (1.3.30a) can be obtained from (1.3.32b) when $kb \to \infty$, i.e. it is unfit under the previous condition $kb \leq 1$.

In addition to gravitational instability, the study of *rotational instability* under the formation of a molecular cloud (and, generally speaking, disordered macroscopic motions (turbulence) in it) is a significant problem for clarifying the evolution of a gas-dust medium [2]. The nature of further processes in the cloud should have depended on whether the damping of these initial motions occurred or some kind of stationary turbulent motion of the medium arose.

The idea of turbulence was first introduced by C.F. von Weizsäcker into the cosmogony [26] being a sort of return to the classical vortices of Descartes. Weizsäcker drew attention to the fact that the Reynolds number Re for the cosmic diffuse medium is enormous, i.e. it is much higher than the critical value Re_c [2, 111]. Since then, turbulence has been considered as one of the most widespread states of cosmic matter. Weizsäcker also suggested that turbulence played an important role in the formation of celestial bodies and their systems [26]. In his opinion, planets, stars, galaxies and other structures arose from the turbulent vortices of appropriate scales. In particular, for the protoplanetary cloud its Reynolds number:

$$\mathrm{Re} = \frac{\rho l \upsilon}{\mu}, \qquad (1.3.33)$$

where ρ is a mass density, l is a size, υ is a velocity, μ is a dynamic coefficient of viscosity of flowing gas-dust medium, was found more than 10^{10}, i.e. $\mathrm{Re} \sim 10^{10} - 10^{14} >> \mathrm{Re}_c$ [2, 12]. In order to explain the law of planetary distances, Weizsäcker supposed that turbulent motions in the protoplanetary cloud were a regular system of vortices whose sizes were proportional to the distances from the Sun [26].

However, it should be noted that in rotating hydrodynamic systems the Reynolds number is not the main criterion for the stability of motion [2]. To clarify the basic properties of medium motion in a protoplanetary gas-dust cloud, which is enough flat system, one can use hydrodynamic studies of the motion of liquid between rotating cylinders (the so-called Couette–Taylor flow [94, 119]).

According to the well-known *Rayleigh criterion* [120] proved for an incompressible non-viscous fluid, the necessary and sufficient condition of *stability* of purely rotational motion of ideal incompressible fluid with an angular velocity $\Omega = \Omega(h)$ depending on the radial component h in the cylindrical coordinate system (h, ε, z) is:

$$\frac{d(\Omega h^2)^2}{dh} > 0 \qquad (1.3.34a)$$

throughout the fluid under consideration. The Rayleigh criterion (1.3.34a) was confirmed by theoretical and experimental investigations of Taylor [121]. In particular, it was found that the viscosity of the fluid reinforces its stability. Thus, rotational instability arises if the condition (1.3.34a) is violated somewhere. Since the squared epicyclic frequency (1.3.30b) is a part of the inequality (1.3.34a) then *the Rayleigh condition of rotational instability* looks as follows [73]:

$$\kappa^2 h^3 < 0 . \qquad (1.3.34b)$$

However, Chandrasekhar emphasized that if the Rayleigh criterion (1.3.34a) is fulfilled, then the moving fluid is necessarily stable; if not, then it is not necessarily unstable [122]. He carried out a theoretical consideration of the problem of stability in the more general case when the distance between the cylinders is not small relative to the radius. Calculations performed for two coaxial cylinders (when the radius of the outer one is two times greater than the radius of the inner cylinder) were confirmed by the experiments of R.J. Donnelly and D. Fultz [123, 124].

Thus, according to (1.3.27), (1.3.30a) the Bel–Shatzman's condition (1.3.26) of gravitational instability for a rotating medium includes both the Jeans condition (1.3.25) of gravitational instability for a non-rotating medium and the Rayleigh's condition (1.3.34b) of rotational instability [73]:

$$\kappa^2 + \omega^2 - 4\pi\gamma\rho < 0\,, \quad \kappa^2 < 0\,. \qquad (1.3.35)$$

Entirely, the rotational instability can promote gravitational instability in a rotating molecular cloud following (1.3.35).

Let us note the Rayleigh criterion (1.3.34a) essentially depends on the change of the square of the specific angular momentum λ^2 with respect to the radial coordinate h because:

$$\lambda = \Omega h^2\,. \qquad (1.3.36)$$

Similarly, the square of the epicyclic frequency (1.3.30b) depends on the change λ^2 that is a part of the generalized frequency in accord with the definition (1.3.30a) as well as the Bel–Shatzman's condition (1.3.26) of gravitational instability for a non-uniformly rotating medium. Thus, both the rotational instability of motion (1.3.34b) and the gravitational instability (1.3.26) depend on the law of distribution of the specific angular momentum in a gas-dust medium along the radial coordinate h. In other words, the specific angular momentum is an important quantity characterizing the evolution of gas-dust medium (in particular, a protoplanetary gas-dust cloud, see Chapter 6).

According to the Rayleigh criterion (1.3.34a), the protoplanetary cloud must be stable in rotational motion. Indeed, the specific angular momentum of particle possessing Keplerian circular motion is proportional $\sqrt{h}$, namely, $\lambda = \sqrt{\gamma M h}$ [2, 12], i.e. it grows with increasing h, so that the stability condition (1.3.34a) is fulfilled. The gaseous pressure in the protoplanetary cloud is small, i.e. the motions of its particles should be practically the Keplerian ones [2]. If we neglect the gas pressure gradient dp/dh then the Rayleigh criterion (1.3.34a) as applied to a planar protoplanetary cloud reduces to the stability condition of circular orbits well known in stellar dynamics [2, 125]:

$$\frac{\partial}{\partial h}\left(h^3 \frac{\partial \varphi_g}{\partial h}\right) > 0\,, \qquad (1.3.37)$$

where φ_g is a gravitational potential at a distance h from the axis of rotation (and the axis of symmetry) of the system (protoplanetary gas-dust cloud).

The mass of a protoplanetary cloud is small compared to the mass of a star (the Sun), so that the gravitation is predominantly determined by this central body, i.e. $\partial \varphi_g / \partial h = \gamma M / h^2$. Thus, condition (1.3.37) is fulfilled, and, consequently, circular orbits in the protoplanetary cloud are stable [2].

It is clear that under the conditions (1.3.34a) and (1.3.37) the possibility of convection origin in the cloud is not taken into account, although Weizsäcker [126] tried to justify the existence of turbulence in rotating cosmic gaseous masses (including the protoplanetary cloud) through the condition for the convection origin. However, Weizsäcker neglected the rotation and did not take into account the Rayleigh stability criterion (1.3.34a).

But as known from the Rayleigh criterion (1.3.34a), the fluid motion with axial symmetric rotation is stable relative to small radial perturbations if its specific angular momentum increases with distance h from the rotation axis like the specific angular momentum of a disk with Keplerian rotation for which $\Omega h^2 = \sqrt{\gamma M h}$ [12].

As shown by V.S. Safronov and E.L. Ruskol [127], due to this reason the convection in the radial direction could not occur in the protoplanetary cloud and, therefore, could not be a source of the turbulence itself, as Weizsäcker suggested [26, 126]. Thus, the rotating protoplanetary cloud must be stable relative to small perturbations, therefore convection could not arise in it, and consequently, Weizsäcker's idea relative to cloud turbulence associated with convection is not confirmed.

1.4. Poincaré's general theorem and Roche's model apropos the equilibrium figure for rotating and gravitating continuous medium

Here we consider a general theorem for a rotating continuous medium originally given by Henri Poincaré [1 p.264, 105 p.22].

Let the motion of a continuous mass of gas or liquid (for example, molecular cloud) which is rotating approximately as a rigid (solid) body with angular velocity Ω be referred to axes rotating with angular velocity Ω. The equations of motion of a *rotating* continuous mass are similar to those discussed above (1.3.1) with the addition of components of *specific centrifugal force*. This means that instead of the gravitational potential φ_g we should consider the *general* potential ψ_g of gravitational and inertial (centrifugal) fields (or the potential of weight force [97]):

$$\psi_g = \varphi_g + V_c, \tag{1.4.1}$$

where $V_c = -(1/2)\Omega^2(x^2 + y^2)$ is the potential of centrifugal force. According to Eqs (1.1.40a)- (1.1.40c) we see that the components of the specific gravitational force are equal to:

$$f_{g_x} = -\frac{\partial \varphi_g}{\partial x}, \quad f_{g_y} = -\frac{\partial \varphi_g}{\partial y}, \quad f_{g_z} = -\frac{\partial \varphi_g}{\partial z}. \tag{1.4.2}$$

Similarly, we can also find the components of specific centrifugal force in the case of $\Omega = \text{const}$:

$$f_{cx} = -\frac{\partial V_c}{\partial x} = \Omega^2 x, \; f_{cy} = -\frac{\partial V_c}{\partial y} = \Omega^2 y,$$
$$f_{cz} = -\frac{\partial V_c}{\partial z} = 0. \tag{1.4.3}$$

Let a_x, a_y, a_z be the components of acceleration of an infinitely small volume of a continuous medium ("liquid" particle) and let u, υ, w be the components of velocity $\vec{v} = (u, \upsilon, w)$, which we assume to be small, of any element of the mass relative to these rotating axes [1]. Obviously:

$$a_x = \frac{du}{dt}, \quad a_y = \frac{d\upsilon}{dt}, \quad a_z = \frac{dw}{dt}. \tag{1.4.4}$$

Taking into account three equations in (1.3.1) as well as Eqs (1.4.1)-(1.4.4) we obtain the equations of motion of any small element of the continuous mass which are three Euler hydrodynamic equations in the case of rotation:

$$\frac{du}{dt} = -\frac{\partial \varphi_g}{\partial x} + \Omega^2 x - \frac{1}{\rho} \cdot \frac{\partial p}{\partial x};$$
$$\frac{d\upsilon}{dt} = -\frac{\partial \varphi_g}{\partial y} + \Omega^2 y - \frac{1}{\rho} \cdot \frac{\partial p}{\partial y}; \tag{1.4.5}$$
$$\frac{dw}{dt} = -\frac{\partial \varphi_g}{\partial z} - \frac{1}{\rho} \cdot \frac{\partial p}{\partial z},$$

where p is a pressure, ρ is a mass density. Differentiating these three equations (1.4.5) with respect to x, y, z and adding corresponding sides we get:

$$\frac{d}{dt}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial w}{\partial z}\right)=-\nabla^2\varphi_g+2\Omega^2-$$

$$-\left[\frac{\partial}{\partial x}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}\right)\right]. \qquad (1.4.6)$$

Taking into account the Poisson equation (1.1.41) we see that Eq. (1.4.6) becomes [1]:

$$\frac{d}{dt}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial w}{\partial z}\right)=-4\pi\gamma\rho+2\Omega^2-$$

$$-\left[\frac{\partial}{\partial x}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}\right)\right]. \qquad (1.4.7)$$

Let us multiply both sides of Eq. (1.4.7) by the element of volume $dV = dxdydz$, and integrate throughout the whole of the rotating mass. On transforming the first and last integrals by Gauss–Ostrogradsky' and Green's theorems [1, 128], we obtain:

$$\frac{d}{dt}\iint(\vec{v}\cdot\vec{n})dS=\iiint\left[2\Omega^2-4\pi\gamma\rho\right]dV-\iint\frac{1}{\rho}\cdot\frac{\partial p}{\partial\vec{n}}dS, \qquad (1.4.8)$$

where the surface integrals extend over the whole surface of the mass, $(\vec{n}\cdot\vec{v})=ku+l\upsilon+mw$ is the scalar product, and k,l,m are the direction-cosines of the outward normal $\vec{n}=(k,l,m)$ to this surface at any point, and $\partial/\partial\vec{n}$ denotes differentiation along this normal.

If V is the whole volume of the rotating mass, the surface integral on the left-hand side of Eq. (1.4.8) measures the rate of increase of V, and this integral may be estimated as follows [1]:

$$\iint_S (\vec{v}\cdot\vec{n})dS = \iint_S v_n dS = \iint_S (dr_n / dt)dS =$$

$$= \frac{d}{dt}\left(\iint_S dr_n \cdot dS\right) = \frac{dV}{dt}. \tag{1.4.9}$$

The volume integral on the right-hand side of Eq. (1.4.8) maybe calculated using the theorem on the average [128]:

$$\iiint_V [2\Omega^2 - 4\pi\gamma\rho]dV = (2\Omega^2 - 4\pi\gamma\bar{\rho})V\ , \tag{1.4.10}$$

where $\bar{\rho}$ is the mean mass density of the whole mass. Taking into account Eqs (1.4.9), (1.4.10) we write the equation (1.4.8) in the form:

$$\frac{d^2V}{dt^2} = (2\Omega^2 - 4\pi\gamma\bar{\rho})V - \iint \frac{1}{\rho}\cdot\frac{\partial p}{\partial \vec{n}} dS\ . \tag{1.4.11}$$

Since the pressure p vanishes at the boundary of the mass and must be positive at all interior points, then it is a decreasing function inside the volume V. Consequently, $\partial p / \partial \vec{n} < 0$ is necessarily negative, so that the last term on the right-hand side of Eq. (1.4.11) is necessarily positive:

$$-\iint \frac{1}{\rho}\cdot\frac{\partial p}{\partial \vec{n}} dS > 0\,. \tag{1.4.12}$$

For the mass to be in a state of *steady rotation*, the left-hand member of the equation (1.4.11) must vanish ($d^2V / dt^2 = 0$), so that taking into account the condition (1.4.12) we must have [1, 105 p.22]:

$$\Omega^2 < 2\pi\gamma\bar{\rho}\ . \tag{1.4.13}$$

Thus, the obtained inequality (1.4.13) shows that a general theorem on rotating masses has been proved [1 p.264]:

Theorem 1.5 (the Poincaré's general theorem on rotating masses). Whatever the arrangement of the mass, a rotation of speed greater than that given by the equation $\Omega^2 = 2\pi\gamma\bar{\rho}$ is inconsistent with a steady rotation.

The inequality (1.4.13) is the original Poincaré theorem which determines the upper bound of the angular velocity for a steady rotation of masses. If the inequality (1.4.13) is not satisfied, d^2V/dt^2 must be positive, so that the mass must continually increase its rate of expansion, or, if it is contracting, the contraction will be checked and ultimately replaced by an expansion [1].

The Poincaré theorem is closely related to the *condition for the existence of an equilibrium figure* for a rotating and gravitating mass of a liquid with a convex surface (for example, a rotating gas mass of a molecular cloud). As is known [44, 97, 111], the equilibrium figure of the liquid mass is found from the condition that the external (free) surface is *equipotential*, therefore the equation $\psi_g = C$ defines a family of level surfaces of the general potential of gravity (1.4.1). By generalizing the Poisson equation (1.1.41) we can see that the general potential ψ_g satisfies the following differential equation [44]:

$$\begin{cases} \nabla^2\psi_g = 4\pi\gamma\rho - 2\Omega^2 & \text{inside a volume bounded by an equipotential surface;} \\ \psi_g = C & \text{at the equipotential surface.} \end{cases} \tag{1.4.14}$$

Indeed, let us suppose that the inverse Poincaré's condition [44] holds:

$$\Omega^2 > 2\pi\gamma\bar{\rho}. \tag{1.4.15}$$

Then, according to the first equation from (1.4.14) it should be $\nabla^2\psi_g < 0$. This means that ψ_g is a convex (concave down) function, and the inequality $\psi_g > C$ must be satisfied.

Otherwise, at some point M it would be $\psi_g\big|_M \le C$, i.e. the function ψ_g would reach a minimum inside the region bounded by an equipotential surface [44]. But then at this point, it would be $\nabla^2\psi_g\big|_M > 0$ that is impossible under the condition of the form (1.4.15).

Further, according to the condition of *hydrostatic equilibrium* [94] which is the particular case of Euler hydrodynamic equations (1.4.5) when the acceleration is absent: $\vec{a} = (a_x, a_y, a_z) = 0$, the following relationship holds:

$$\frac{1}{\rho}\nabla p = -\nabla \psi_g \,, \tag{1.4.16}$$

which, taking into account (1.4.14), can be integrated:

$$\int_0^{\vec{r}_{eq}} \frac{1}{\rho}\nabla p d\vec{r} = -\int_0^{\vec{r}_{eq}} \nabla \psi_g d\vec{r} = -C + \psi_g(0) \,, \tag{1.4.17}$$

where $\vec{r}_{eq}$ is a vector-hodograph of the equipotential surface. Applying to the first of these integrals on the left-hand side of Eq. (1.4.17) the integration formula by parts and taking into account that if $\vec{r} = \vec{r}_{eq}$ then $p(\vec{r}_{eq}) = 0$, we obtain:

$$\int_0^{\vec{r}_{eq}} \frac{1}{\rho}\nabla p d\vec{r} = \frac{p}{\rho}\bigg|_0^{\vec{r}_{eq}} - \int_0^{\vec{r}_{eq}} p\nabla\left(\frac{1}{\rho}\right) d\vec{r} = -\frac{p(0)}{\rho(0)} - \int_0^{\vec{r}_{eq}} p\nabla\left(\frac{1}{\rho}\right) d\vec{r} \,. \tag{1.4.18}$$

In both cases of $\rho = \text{const}$ and $\rho(\vec{r})$ is a diminishing function with a maximum in the point $r = 0$ the right-hand side of Eq. (1.4.18) should be negative because $p > 0$ and $\rho > 0$. However, since $\psi_g > C$ in accord with (1.4.15), then $\psi_g(0) - C > 0$ on the right-hand side of Eq. (1.4.17) whereas $\int_0^{\vec{r}_{eq}} \frac{1}{\rho}\nabla p d\vec{r} < 0$ on the left-hand side of Eq. (1.4.17) in accordance with Eq. (1.4.18). Consequently, in this case, the pressure inside the volume must be negative $p < 0$. So, under the inverse condition $\Omega^2 > 2\pi\gamma\overline{\rho}$, the area inside the volume bounded by the equipotential surface will be in a state of tension, and equilibrium cannot be reached. Thus, the condition for the existence of an equilibrium figure of any rotating and gravitating mass of a continuous medium is based on the Poincaré theorem.

As it follows from this proof of the condition for the existence of an equilibrium figure, there are the cases of an *incompressible* continuous medium ($\rho = \text{const}$) and *inhomogeneous* continuous medium (ρ is a diminishing function with a maximum in the center). Following U. Crudeli and W. Nicliborc (for example, see [44], [129], [130]), the condition for the existence of an equilibrium figure with a convex surface (with a positive pressure inside it) for an inhomogeneous continuous medium (fluid or gas-dust continuous medium) can be written as the following inequality:

$$\Omega^2 < \pi\gamma\rho_{\max}, \tag{1.4.19}$$

where $\rho_{\max}$ is the maximum value of mass density inside the inhomogeneous continuous medium. It is well known [1 p.250-251] the various models of an inhomogeneous continuous medium, in particular, the Darwin mass density law, the Shuster mass density law, and the Roche model (see also Section 2.2). In Roche's model, the whole of the mass is supposed to be concentrated at the center, so that Roche's model and the incompressible model form the two limiting cases of the general compressible mass.

In studying the configurations and motion of an incompressible mass, one of the main difficulties was found to lie in the determination of the gravitational potential (for example, see the Dirichlet formula (1.1.33) in Section 1.1). In Roche's model, no such difficulty occurs: the whole mass is collected at a point and the gravitational potential is simply $-\gamma M/r$. To discuss the Roche's model [1 p.252] we consider the problem of single mass M rotating freely in space with an angular velocity Ω, so that its general potential of gravity (following formula (1.4.1)) is equal:

$$\psi_g = \varphi_g + V_c = -\frac{\gamma M}{\sqrt{x^2+y^2+z^2}} - \frac{1}{2}\cdot\Omega^2(x^2+y^2), \qquad (1.4.20)$$

where $\varphi_g = -\gamma M/r$ is the gravitational potential according to the Roche model, $V_c = -(1/2)\Omega^2 h^2$ is the potential of centrifugal force [95, 97], $r = \sqrt{x^2+y^2+z^2}$, $h = \sqrt{x^2+y^2}$.

We have just seen in Eq.(1.4.14) that the boundary of the continuous mass must be one of the equipotential surfaces $\psi_g = \text{const}$, therefore bearing in mind (1.4.20), equation of equipotential surfaces takes the form:

$$\frac{\gamma M}{r} + \frac{1}{2}\cdot\Omega^2(x^2+y^2) = \text{const}. \qquad (1.4.21)$$

On sketching out the forms of the equipotential surfaces $\psi_g = \text{const}$ (see Fig. 1.9), Jeans noted [1 p.252] that:

> ...all possible configurations for a Roche's model must lie on one linear series, and this may in every case be supposed to start from the spherical configuration for which Ω... vanishes. As we proceed along this series, the different boundaries are equipotentials which differ more and more from spheres, until finally it may happen that the equipotential which forms the boundary coincides with one which marks a transition from closed to open equipotentials.

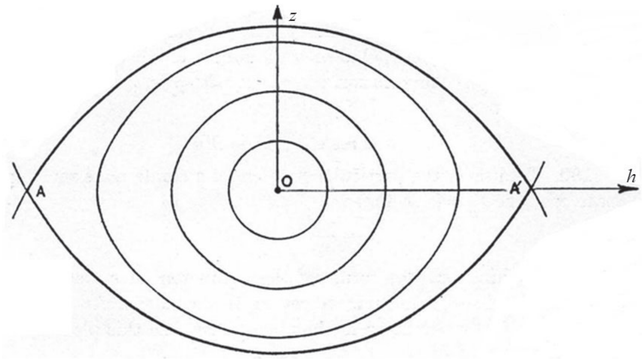


Figure 1.9. Graphic representation of equipotential surfaces in the Roche model

Thus, the condition for a point of bifurcation or a turning point is that there shall be two adjacent configurations of equilibrium, and hence two different boundaries possible for the same value Ω. Then Jeans continued:

> Such a transition must necessarily be through an equipotential which intersects itself, and therefore through an equipotential on which a point of equilibrium occurs. Such a point is determined by the equations:

$$\frac{\partial \psi_g}{\partial x} = \frac{\partial \psi_g}{\partial y} = \frac{\partial \psi_g}{\partial z} = 0 . \qquad (1.4.22)$$

As ψ_g decreases following (1.4.20), the condition for a point of equilibrium (1.4.22) will first be satisfied in the plane Oxy. In this plane the condition becomes:

$$\frac{\partial \psi_g}{\partial x} = \frac{\partial \psi_g}{\partial y} = 0 . \qquad (1.4.23)$$

Substituting (1.4.20) into (1.4.23) we obtain that:

$$\frac{\partial \psi_g}{\partial x} = \left(-\frac{1}{2}\right)\frac{\gamma M}{r^3} \cdot 2x + \frac{1}{2}\Omega^2 \cdot 2x = -\frac{\gamma M}{r^3} \cdot x + \Omega^2 \cdot x = 0 ;$$

$$\frac{\partial \psi_g}{\partial y} = -\frac{\gamma M}{r^3} \cdot y + \Omega^2 \cdot y = 0 ,$$

whence it follows that the condition (1.4.23) is satisfied if:

$$-\frac{\gamma M}{r^3} + \Omega^2 = 0 . \qquad (1.4.24)$$

In other words, if h_0 is the radius of the cross-section in the plane of Oxy then, according to (1.4.24), the equality holds [1]:

$$\gamma M = \Omega^2 \cdot h_0^3 , \qquad (1.4.25a)$$

i.e. the desired radius is equal to:

$$h_0 = (\gamma M / \Omega^2)^{1/3} . \qquad (1.4.25b)$$

The equation (1.4.25a) admits of a very simple interpretation: at a distance h_0 from the nucleus, the gravitation force is $\gamma M / h_0^2$ while centrifugal force is $\Omega^2 \cdot h_0$. So, equality (1.4.25a) expresses that these forces are equal. Thus, a *point of equilibrium* occurs when centrifugal force just balances the force of gravity. The same result follows from the equation (1.4.16) because the condition (1.4.22) leads to:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 , \qquad (1.4.26)$$

which defines the same point in equilibrium, so that gas pressure exerts no force on the matter at the point of equilibrium [1].

According to Eq. (1.4.21) the particular equipotential on which the point of equilibrium occurs is readily found to be:

$$\frac{\gamma M}{r_c} + \frac{1}{2} \cdot \Omega^2 (r_c^2 - z_c^2) =$$

$$= \frac{\gamma M}{(\gamma M / \Omega^2)^{1/3}} + \frac{1}{2} \cdot \Omega^2 \cdot (\gamma M / \Omega^2)^{2/3} - \frac{1}{2} \cdot \Omega^2 z_c^2 =$$

$$= (\gamma M)^{2/3}\Omega^{2/3} + \frac{1}{2}\cdot\Omega^{2-4/3}\cdot(\gamma M)^{2/3} - \frac{1}{2}\cdot\Omega^2 z_c^2 =$$

$$= \frac{3}{2}\cdot(\gamma M\Omega)^{2/3} - \frac{1}{2}\cdot(\Omega\cdot z_c)^2 \ , \qquad (1.4.27)$$

where z_c is a coordinate (along the z-coordinate axis) of the point located on the critical equipotential surface, r_c is a distance from the center of mass, i.e. $r_c = \sqrt{x_c^2 + y_c^2 + z_c^2}$ [73]. In the plane Oxy $r_c = h_0 = (\gamma M / \Omega^2)^{1/3}$ and $z_c = 0$, so that the expression (1.4.27) takes the form:

$$\left.\frac{3}{2}\cdot(\gamma M\Omega)^{2/3} - \frac{1}{2}\cdot(\Omega\cdot z_c)^2\right|_{z_c=0} = \frac{3}{2}\cdot(\gamma M\Omega)^{2/3}. \qquad (1.4.28)$$

Thus, the constant in Eq. (1.4.21) in the case of a critical equipotential surface is determined using formula (1.4.28), so that the equation itself of a critical equipotential surface containing an equilibrium point has the form [1]:

$$\psi_g^{crit} \equiv \frac{\gamma M}{r} + \frac{1}{2}\cdot\Omega^2(x^2+y^2) = \frac{3}{2}\cdot(\gamma M\Omega)^{2/3}. \qquad (1.4.29)$$

The surfaces $\psi_g = \text{const}$ are found to lie as in Fig. 1.9, the critical equipotential defined by Eq.(1.4.29) being drawn thick.

Putting $h = \sqrt{x^2+y^2}$ and using the notation $h_0 = (\gamma M / \Omega^2)^{1/3}$ in accord with (1.4.25b) we can rewrite (1.4.29) as the following [73]:

$$\frac{1}{r} + \frac{1}{2}\Omega^2\cdot\frac{h^2}{h_0^3\Omega^2} = \frac{3}{2}\cdot\frac{\Omega^{2/3}}{(\gamma M)^{1/3}} = \frac{3}{2}\cdot\left(\frac{\Omega^2}{\gamma M}\right)^{1/3} = \frac{3}{2}\cdot\frac{1}{h_0} \ ,$$

whence it follows the equation of the critical equipotential surface [1]:

$$\frac{1}{r}+\frac{1}{2}\cdot\frac{h^2}{h_0^{\ 3}}=\frac{3}{2}\cdot\frac{1}{h_0}\ . \qquad (1.4.30)$$

Taking into account that $r=\sqrt{h^2+z^2}$ we convert equation (1.4.30) to the form:

$$\frac{1}{\sqrt{h^2+z^2}}=\frac{3}{2}\cdot\frac{1}{h_0}-\frac{1}{2}\cdot\frac{h^2}{h_0^{\ 3}}=\frac{3h_0^{\ 2}-h^2}{2h_0^{\ 3}}\ ,$$

where we can express [73]:

$$z^2=\frac{4h_0^6}{(3h_0^2-h^2)^2}-h^2=\frac{4h_0^6-9h_0^4h^2+6h_0^2\cdot h^4-h^6}{(3h_0^2-h^2)^2}=$$

$$=\frac{(h_0^2-h^2)^2\cdot(4h_0^2-h^2)}{(3h_0^2-h^2)^2}. \qquad (1.4.31)$$

Thus, according to (1.4.31), the function describing the critical equipotential surface (1.4.30) (see also Fig. 1.9) assumes the form [1]:

$$z(h)=\pm\frac{h_0^2-h^2}{3h_0^2-h^2}\cdot\sqrt{4h_0^2-h^2}\ , \qquad (1.4.32)$$

i.e. the critical equipotential is bounded by two functions $z_+(h)$ and $z_-(h)$ [73]. In [1 p. 253], the volume bounded by a critical equipotential surface is estimated. Indeed, in the cylindrical coordinate system this volume can be calculated by the following integral [73]:

$$V_{eq}^{crit}\ =\ 2\int_0^{h_0}\int_0^{2\pi} z_+(h)hdhd\varepsilon=4\pi\int_0^{h_0}\frac{h_0^{\ 2}-h^2}{3h_0^{\ 2}-h^2}\cdot\sqrt{4h_0^{\ 2}-h^2}\cdot hdh\ =$$

$$=2\pi h_0^{\ 3}\cdot\int_0^1\frac{1-(h/h_0)^2}{3-(h/h_0)^2}\cdot\sqrt{4-(h/h_0)^2}\cdot d(h/h_0)^2=$$

$$= 2\pi h_0^{\ 3} \cdot \int_0^1 \left(1 - \frac{2}{3-(h/h_0)^2}\right) \cdot \sqrt{4-(h/h_0)^2} \cdot d(h/h_0)^2 =$$

$$= 2\pi h_0^{\ 3} \cdot \{2 \cdot \left(\frac{8}{3} - \sqrt{3}\right) - 2 \cdot \int_0^1 \frac{\sqrt{4-(h/h_0)^2}}{3-(h/h_0)^2} \cdot d(h/h_0)^2\} =$$

$$= 4\pi h_0^{\ 3} \cdot \{2\frac{2}{3} - \sqrt{3} - 2 \cdot \int_{\sqrt{3}}^{2} \frac{t^2}{t^2-1} \cdot dt\} =$$

$$= 4\pi h_0^{\ 3} \cdot \{2\frac{2}{3} - \sqrt{3} - 4 + 2\sqrt{3} - 2 \cdot \int_{\sqrt{3}}^{2} \frac{dt}{t^2-1}\} =$$

$$= 4\pi h_0^{\ 3} \cdot \{\sqrt{3} - 1\frac{1}{3} - 2 \cdot \frac{1}{2} \cdot \ln\left|\frac{t-1}{t+1}\right| \Bigg|_{\sqrt{3}}^{2}\} =$$

$$= 4\pi h_0^{\ 3} \cdot \{\sqrt{3} - \frac{4}{3} - \ln\frac{2+\sqrt{3}}{3}\} = 4\pi h_0^{\ 3} \cdot 0.180371866 \quad \dots (1.4.33)$$

If $\overline{\rho}$ denotes the mean density of all the matter inside this critical equipotential, this volume V_{eq}^{crit} is equal to the mass M divided by $\overline{\rho}$. Hence, taking into account (1.4.25b) let us calculate the following relation [1, 73]:

$$\frac{\Omega^2}{2\pi\gamma\overline{\rho}} = \frac{\Omega^2 V_{eq}^{crit}}{2\pi\gamma M} = \frac{V_{eq}^{crit}}{2\pi h_0^{\ 3}} \approx \frac{4\pi h_0^{\ 3} \cdot 0.180371866}{2\pi h_0^{\ 3}} =$$

$$= 0.360743732. \qquad (1.4.34)$$

Thus, according to the Roche's model the relationship occurs:

$$\Omega^2 = 0.360743732 \cdot 2\pi\gamma\overline{\rho}\,, \qquad (1.4.35)$$

confirming the validity of inequality (1.4.13) implying from Poincaré's general theorem derived from more general conditions (see the proved above Theorem 1.5) [73]. Further Jeans pointed out the following [1 p.254] [4]:

---

[4] With some author's designations and figure numeration

> Let the value of $\Omega^2/2\pi\gamma\overline{\rho}$ increase continuously in a mass of compressible matter in which the distribution of density is approximately that represented in Roche's model. The spherical configuration corresponds to $\Omega^2/2\pi\gamma\overline{\rho}=0$, and as $\Omega^2/2\pi\gamma\overline{\rho}$ increases, the boundary assumes in turn the shape of the different equipotentials shewn in Fig. 1.9, until it reaches the value $\Omega^2/2\pi\gamma\overline{\rho}=0.360744$ at which the series of configurations comes abruptly to an end, through there being no closed equipotentials corresponding to higher values of $\Omega^2/2\pi\gamma\overline{\rho}$.

In this connection, the question naturally arises: when does the value Ω^2 first exceed the critical value $0.360744\cdot 2\pi\gamma\overline{\rho}$? Indeed, when Ω^2 reaches this value, h_0 is determined by the formula (1.4.25b). If Ω^2 increases further, h_0 decreases, since the mass of the central core M remains the same. Thus, there is a *new critical equipotential* of smaller radius, and so of higher density, for which both Ω^2 and $\overline{\rho}$ are increased, but $\Omega^2/2\pi\gamma\overline{\rho}$ retains its original value of 0.360744. Nevertheless, Jeans noted there:

> An increase in Ω can be met by the mass shrinking to this new configuration. But we have already seen that the particles which formed the sharp edge of the original configuration were in pure orbital motion under the gravitational attraction of the central nucleus alone. When the mass shrinking can exercise no grip on these particles, so that they are left revolving in their original orbits with their original angular velocity.
>
> Thus as Ω^2 steadily increases, a rotating mass, formed after Roche's model, will pass through a series of pseudo-spheroidal configurations, rotating as a rigid body, until Ω^2 reaches the critical value $0.360744\times 2\pi\gamma\overline{\rho}$. At this stage, the shape of the mass is that of a lenticular figure with a sharp edge. Beyond this rotation as a rigid body is impossible. As Ω^2 increases further, the central mass shrinks, $\overline{\rho}$ increasing so that $\Omega^2/2\pi\gamma\overline{\rho}$ remains constantly equal to 0.360744. It retains its original lenticular configuration,

> but as it shrinks it leaves behind it successive rings of particles rotating in its equatorial plane. Thus the complete mass at any instant consists of a central lenticular mass rotating as a rigid body with an angular velocity given always by $\Omega^2 / 2\pi\gamma\overline{\rho} = 0.360744$, together with rings of particles occupying the equatorial plane, and rotating at slower speeds.

The above concept was formulated using the well-known nebular hypothesis of Kant and Laplace. P.-S. Laplace believed that ordinary astronomical mass shrunk continually as a result of the emission of radiation from its surface. As a result, its density should continually increase as well as the value of Ω^2, according to the law of conservation of angular momentum. Thus, Laplace consequently supposed that the normal astronomical mass passed through the sequence of the above-described configurations [131 Note VII, Vol. VI, p. 498]. In particular, he believed that Saturn, surrounded by a series of rings, is an example of the final configuration, and Saturn's satellites had been formed by the condensation of similar rings, i.e. the present rings also would in time condense into satellites. Moreover, Laplace hypothesized that the planets and satellites of the Solar system had been produced by the condensation of rings of this type formed as a result of shrinkage of a central cooling mass of the protosolar nebula [1].

According to the ideas of Laplace and Roche the ring of matter which was thrown off from the Sun, and ultimately formed the planets, was rotating at one time as a closed ring with approximately the same angular velocity as the main mass of the Sun. If $\overline{\rho}_{\text{Sun}}$ was the mean density of the Sun, the squared angular velocity of rotation Ω^2 would be estimated by the Roche model (1.4.35):

$$\Omega^2 = 0.360743732 \cdot 2\pi\gamma\overline{\rho}_{\text{Sun}}, \qquad (1.4.36)$$

whence, by using the Poincaré's general theorem, it follows from inequality (1.4.13) that:

$$\bar{\rho} > 0.360743732\bar{\rho}_{\text{Sun}} . \qquad (1.4.37)$$

This inequality (1.4.37) shows that unless the ring condensed at once to have a mass density of at least a third of the mean density of the main mass of the Sun, it could not rotate steadily but would continually expand under the centrifugal forces arising from its rotation [1].

1.5. On the fundamental difficulties of the theory of gravitational instability and the theory of gravitational condensation of an infinitely spread media

The problem of gravitational condensation of an infinitely spread cosmic medium is closely related to the problem of gravitational instability (see Section 1.3 and the numerous works [1, 2, 6, 93, 118]). The linearized theory of gravitational instability, developed for some specific cases, leads to the well-known *Jeans criterion* (1.3.22), (1.3.23) indicating its universality [2 p.60]. At the same time, some works (see, for example, [132, 133]) point to an incorrectness of its derivation because the infinite homogeneous non-rotating medium could not be in an equilibrium state so that small disturbances did not manage to form any sufficiently dense condensations, for example, galaxies [132, 133], in such non-equilibrium (expanding or contracting) system. However, as Victor S. Safronov [2] objected, when studying the formation of stars and, especially planets, there are no difficulties with long times.

Thus, since the process of planet formation is very long in the time, the Newtonian consideration of bounded equilibrium system (like locally equilibrium systems in thermodynamics [134, 135]) becomes preferable, as it pointed by Safronov [2 p.60]. Moreover, one of the important problems is the study of instability in an *infinite homogeneous medium at rest* [2]. As already noticed, concerning infinity I. Newton thought that "some of it would convene into one mass and some into another,...and thus might the sun and fixed stars be formed" [1 p.353].

The Jeans criterion can be considered as the *first* approximation which in the simplest cases gives the correct order of the critical wavelength of perturbation leading to instability. Since the forces opposing the instability origin, it takes into account only the gas pressure in the disturbance wave, this criterion gives the *lower limit* of the critical wavelength [2].

The main difficulty of the theory of Jeans is associated with the *gravitational paradox*: for an infinite homogeneous medium, there exists no potential for the force of gravity [2]. Indeed, from the Poisson equation of the form (1.1.41) (or (I.2)) it follows that if the mass density $\rho \neq 0$ then the gravitational potential φ_g and, therefore, the gravitational force F_g also grows *unlimitedly* with distance. This difficulty is avoided within the framework of Jeans' theory and in its subsequent generalizations using the supposition that the Poisson equation does not apply to the whole infinite medium but only to disturbances $\delta\rho$ (see Eq. (1.3.13b)), i.e. to mass density deviations from its mean value ρ. Here it has just been assumed that in the "true" infinite homogeneous immovable system the gravitational force should be absent since there is no pressure gradient and acceleration in it. Otherwise, it could not be at rest [2 p.61].

Let us note additionally that an infinite system cannot be obtained by passing to the limit $r \to \infty$ from a finite system (for example, spherical one). Such a statement of the problem does not apply to gravitationally coupled finite systems, for which the Poisson equation should be satisfied in Newtonian approximation or its analog in the relativistic consideration [100].

Although the Jeans theory of gravitational instability has been presented in Section 1.3, nevertheless, now we briefly consider the main steps of the Jeans criterion derivation proposed by V.S. Safronov [2]. First of all, let us investigate the forces acting on an element of a continuous medium. When a wave disturbance is propagating two forces arise: the gravitational force, associated with the density perturbation $\delta\rho$, and the gas pressure force depended on the density gradient $\nabla\rho$. In the case of a plane wave, the wavefront is a sphere at the initial point of excitation, so that the gravitational potential at the interior point of this sphere is equal [95, 97]:

$$\varphi_g = -\frac{2}{3}\pi\gamma\rho_0(3R^2 - r^2) = -2\pi\gamma\rho_0 R^2 + \frac{2}{3}\pi\gamma\rho_0 r^2, \qquad (1.5.1)$$

where ρ_0 is a mass density, R is a radius of the sphere, $r \le R$. Relative to the point of excitation ($r = 0$) of the medium with the radius of the spherical wavefront $R = R(t)$, we can calculate the strength of gravitational field or the specific gravitational force $\vec{f}_g$, i.e. the gravitational force $\vec{F}_g$ per mass unit, based on the well-known relation [100]:

$$\vec{f}_g = -\nabla\varphi_g = 4\pi\gamma\rho_0 R(t)\vec{e}_r, \qquad (1.5.2)$$

whence an increment of the specific gravitational force becomes [2]:

$$\delta\vec{f}_g = 4\pi\gamma\rho_0\xi\vec{e}_r, \qquad (1.5.3)$$

where $\xi = \delta R$ is a displacement of a continuous medium, $\vec{e}_r$ is a basic vector along the radial axis. Using the spherical coordinate system the specific surface force f_p of the gas pressure is determined by the ratio [111]:

$$\vec{f}_p = -\frac{\nabla p}{\rho} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial r}\vec{e}_r = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial \rho}\cdot\frac{\partial \rho}{\partial r}\vec{e}_r\,, \qquad (1.5.4)$$

and following the above formula (1.3.15b) its increment accordingly looks like:

$$\delta\vec{f}_p = -\frac{c^2}{\rho}\cdot\frac{\partial \delta\rho}{\partial r}\vec{e}_r\,, \qquad (1.5.5)$$

where $c = \sqrt{(dp/d\rho)_0}$ is the speed of propagation of small perturbations (waves of sound) in a continuous medium [94, 111], $\delta\rho$ is a small perturbation of mass density of a continuous medium, and $\rho = \rho_0 + \delta\rho$, ρ_0 is a mass density of a rest continuous medium [111].

Bearing in mind that the mass density increment $\delta\rho$ does not depend on the angular coordinates θ and ε, we can easily find the relation for it in the spherical coordinate system (r,θ,ε) in accord with the above-derived formulas (1.3.8), (1.3.12):

$$\frac{\delta\rho}{\rho} = -\frac{\partial \xi}{\partial r}\,. \qquad (1.5.6)$$

Since the displacement $\xi = \delta R(t)$ is assumed to be infinitely small then according to formula (1.5.6) the relation (1.5.4) can be represented in the form:

$$\delta\vec{f}_p = -\frac{c^2}{\rho}\cdot\frac{\partial}{\partial r}(\delta\rho)\vec{e}_r = c^2\frac{\partial^2\xi}{\partial r^2}\vec{e}_r + \frac{c^2}{\rho}\cdot\frac{\partial\rho}{\partial r}\cdot\frac{\partial\xi}{\partial r}\vec{e}_r =$$

$$= c^2\frac{\partial^2\xi}{\partial r^2}\vec{e}_r + \frac{c^2}{\rho}\cdot\frac{\partial(\delta\rho)}{\partial r}\cdot\frac{\partial\xi}{\partial r}\vec{e}_r \approx c^2\frac{\partial^2\xi}{\partial r^2}\vec{e}_r\,, \qquad (1.5.7)$$

where the product of derivatives of small quantities (with respect to coordinates) is omitted in the last relation (as the infinitesimal value of higher-order) [111]. According to the condition of hydrostatic equilibrium (1.4.16) of a continuous medium [94] and Eqs (1.5.2), (1.5.4) the well-known condition of *mechanical equilibrium* takes place:

$$\delta\vec{f}_g + \delta\vec{f}_p = 0\,, \tag{1.5.8}$$

whence, taking into account (1.5.3) and (1.5.7), it follows directly:

$$4\pi\gamma\rho_0\xi = -c^2\frac{\partial^2\xi}{\partial r^2}\,. \tag{1.5.9}$$

A sinusoidal perturbation satisfies the equation (1.5.9):

$$\xi = \xi_0 \sin\left(\omega t + \frac{2\pi r}{\lambda}\right). \tag{1.5.10}$$

Indeed, according to Eq. (1.5.10) if we calculate the derivatives of ξ with respect to r we get:

$$\frac{\partial^2\xi}{\partial r^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \cdot \xi_0 \cdot \sin(\omega t + \frac{2\pi r}{\lambda}) = -\left(\frac{2\pi}{\lambda}\right)^2 \cdot \xi\,. \tag{1.5.11}$$

Substituting (1.5.11) into (1.5.9) we find that:

$$4\pi\gamma\rho_0\xi = -c^2 \cdot \left(-\frac{4\pi^2}{\lambda^2}\right)\xi\,, \tag{1.5.12}$$

whence it immediately follows that:

$$\lambda^2 = \frac{\pi c^2}{\gamma\rho_0}\,. \tag{1.5.13}$$

As it is known from Section 1.3, the equation (1.5.13) determines the *critical wavelength* λ_c (1.3.22) of wave disturbance:

$$\lambda_c = c\sqrt{\frac{\pi}{\gamma\rho_0}}\,,$$

since the *instability condition* [2]:

$$\delta f_g > -\delta f_p \qquad (1.5.14)$$

leads to the well-known criterion of Jeans (1.3.23).

So, the Jeans condition (1.3.23) says that a continuous medium filling a certain space is gravitationally unstable if any small mass density perturbations arising in it grow indefinitely with time due to the gravitation and, as a result, violate its equilibrium. Perturbations that have increased due to gravity lead to the formation of separate condensed bunches from a gas-dust continuous medium spread in space, i.e. they stimulate the process of *gravitational condensation* of a molecular cloud. As J. Jeans [1 p. 348] noted:

> A medium of dimensions much greater than λ_c would tend to form condensations whose mean distance apart would be comparable with λ_c.

In the particular *adiabatic* case, when the pressure p and mass density ρ of a continuous medium are connected by a relation of the adiabatic type $p \sim \rho^{\kappa}$, we can find the square of the speed of propagation of wave perturbations [94, 111]:

$$c^2 = \frac{dp}{d\rho} = \kappa \frac{p}{\rho}. \qquad (1.5.15)$$

Substituting the expression for c from (1.5.15) into (1.3.22) we obtain:

$$\lambda_c = \frac{1}{\rho}\sqrt{\frac{\pi \kappa p}{\gamma}}. \qquad (1.5.16)$$

Other modifications of formula (1.3.22) are also possible if we are going to use the results of the molecular kinetic theory [1, 110, 136], in particular:

$$p = \frac{1}{3}\rho \overline{v^2}, \qquad (1.5.17)$$

where $\overline{v^2}$ is the mean of the squared velocity [136] of the particles of which the medium is formed. Substituting (1.5.17) into (1.5.16) we may express λ_c in the equivalent form [1]:

$$\lambda_c = \sqrt{\pi\kappa\overline{v^2}/3\gamma\rho}\ . \tag{1.5.18}$$

Further, J. Jeans noted that "any mass of sufficiently great extent must break up into condensations, but this is obviated by the circumstance that λ_c tends to increase *pari passu* with the size of the mass" [1 p.348]. Indeed, for the most general case of a cloud of particles (say, gas molecules), which form a single mass in a state of mechanical equilibrium, relation (1.2.17) has been obtained using the virial theorem in Section 1.2. According to it the mean of the squared velocity of the particles $\overline{v^2}$ is equal to the average value $\overline{\varphi}_g/2$, and this is of the order of magnitude of $\gamma M/R$, where M is the total mass and $R=\overline{r}$ is the average radius of the system. Taking into account the fact that $\overline{v^2}=\gamma M/R$ we can obtain the following modification of the formula (1.5.18) (or (1.3.22)):

$$\lambda_c = \sqrt{\frac{\pi\kappa M}{3\rho R}} = \frac{2\pi\sqrt{\kappa}}{3}\cdot R\,, \tag{1.5.19}$$

so that $\lambda_c \approx 2R$, i.e. the critical wavelength is approximately equal to the diameter of the molecular cloud. Thus, a single mass of medium in a state of mechanical equilibrium does not tend to break up into condensations at distances apart less than its average diameter. This does not leave room for any subordinate condensation: the mass itself constitutes the one and only condensation possible [1 p. 349].

In general, according to (1.5.18) the value λ^2_c is proportional to $\overline{v^2}$ which in turn is proportional to the temperature T of a continuous medium so that the next modification of (1.5.18) is valid:

$$\lambda_c = \sqrt{\frac{\pi k_B \kappa T}{\gamma m_0 \rho}}\,. \qquad (1.5.20)$$

As follows from (1.5.20), a sudden cooling of a molecular cloud reduces the value of critical wavelength λ_c [1 p.349]. Here J. Jeans defined more exactly:

> Cooling will ultimately result in contraction and by the time the mass has so far contracted as to be again in equilibrium, the value of $2R$, the diameter, will again be equal to λ_c, and no subsidiary condensations can be formed. But if a mass is cooled so rapidly that its linear dimensions cannot keep pace, or for any other reason do not keep pace, with its fall of temperature, then λ^2_c becomes less than the dimensions of the mass, and subsidiary condensations will form at distance apart of the order of λ_c.

The average mass of gas-dust matter in the vicinity of condensation caused by gravitational instability is estimated by the value:

$$M_c \sim \rho \cdot \lambda^3_c\,, \qquad (1.5.21a)$$

and taking into account the formula (1.5.18), it is equal [1]:

$$M_c \sim \rho^{-1/2} \cdot \left(\pi \kappa \overline{v^2} / 3\gamma \right)^{3/2}. \qquad (1.5.21b)$$

If we consider $\overline{v^2}$ as an expression for the mean of the squared velocity (1.2.18) in Section 1.2, then the formula (1.5.21b) takes the form:

$$M_c \sim n^{-1/2} m_0^{-2} \cdot \left(\frac{\pi \kappa k_B T}{\gamma} \right)^{3/2}, \qquad (1.5.22a)$$

and if $\bar{v}^2$ is the square of the average arithmetic velocity $\bar{v} = \sqrt{8 k_B T / \pi m_0}$ [110, 136], then (1.5.21b) is written as follows:

$$M_c \sim \frac{8\sqrt{3}}{9} n^{-1/2} m_0^{-2} \cdot \left(\frac{2 \kappa k_B T}{\gamma} \right)^{3/2}. \qquad (1.5.22b)$$

Another modification of formula (1.3.22) is possible if the molecular cloud is considered an ideal gas for which the known Clapeyron–Mendeleev' equation (or the usual Boyle–Charles law [1]) is true:

$$p \frac{\mu}{\rho} = \Re T, \qquad (1.5.23)$$

where p is a pressure, μ is a molar mass of continuous medium, ρ is a density, T is a temperature, $\Re = 8.3169\, \text{J}/(\text{mole} \cdot \text{K})$ is the universal gaseous constant. Using the equation of state (1.5.23) it is easy to find:

$$c^2 = \frac{dp}{d\rho} = \frac{\Re T}{\mu},$$

where c is the *isothermal* speed of sound:

$$c = \sqrt{\frac{\Re T}{\mu}}. \qquad (1.5.24)$$

Substituting (1.5.24) into (1.3.22) we obtain:

$$\lambda_c = \sqrt{\frac{\pi \Re T}{\gamma \rho \mu}} = \sqrt{\frac{\pi k_B T}{\gamma \rho m_0}} = \frac{\sqrt{\pi k_B / \gamma}}{m_0} \cdot \sqrt{\frac{T}{n}}, \qquad (1.5.25)$$

where $n = \rho / m_0$ is a particle concentration, m_0 is a mass of the particle (molecule), $\mu = m_0 \cdot \mathrm{N_A}$ and $\Re = k_B \cdot \mathrm{N_A}$, and $N_A = 6.023 \cdot 10^{23}$ mole^{-1} is the Avogadro number, $k_B = 1.38049 \cdot 10^{-23}$ J/K is the Boltzmann constant.

As shown in Section 1.3, when $\lambda > \lambda_c$ a small initial perturbation ξ_0 becomes aperiodic and grows exponentially with time: $\xi = \xi_0 e^{-\mathrm{i}\tilde{\omega}t}$ where $\tilde{\omega} = \mathrm{i}\sqrt{-(2\pi/\lambda)^2 dp/d\rho + 4\pi\gamma\rho} = \mathrm{i}\sqrt{4\pi\gamma\rho \cdot [1 - \lambda_c^{\ 2} / \lambda^2]}$ is the imaginary value, following (1.3.19) and (1.3.21). But one such perturbation does not directly lead to the formation of a three-dimensional bunch [12]. It constantly slows down and ends with the appearance of a flat layer. After a new perturbation along with the layer a cylinder is formed, and finally, another perturbation along the cylinder leads to its decay into separate bunches [2].

Consequently, the critical wavelength λ_c does not yet determine the critical mass of bunches formed as a result of gravitational instability [12], since the propagation velocity $\tilde{c}$ of wave disturbances at $\lambda > \lambda_c$, according to (1.3.20), becomes imaginary, i.e. no perturbations can propagate beyond the instability region. However, following Jeans, we can estimate the order of value of the critical mass of bunches based on the relations (1.5.21a) and (1.5.25):

$$M_c \sim \rho \cdot \lambda^3{}_c \sim \frac{T^{3/2} n^{-1/2}}{\gamma^{3/2}}. \qquad (1.5.26)$$

This ratio almost coincides with the condition that the molecular cloud is compressed under the action of gravitational forces if the time of free fall of particles in it is less than the time of propagation of a sound wave through it [137]. As follows from relation (1.5.26) as well as analysis of formula (1.5.20), the colder gas-dust medium (and higher its density), the smaller mass of bunches able to be formed due to gravitational instability.

Let us note that the instability condition (1.5.14) directly implies the *energetic condition* of gravitational binding:

$$\left|E_g\right| > \left|U\right|, \qquad (1.5.27)$$

meaning that a bunch (clot) of the matter can form a gravitationally coupled system if its gravitational potential energy E_g exceeds the internal energy U [73].

As S. Weinberg noted [101], the condition (1.5.27) leads to the determination of the Jeans mass:

$$M_J \sim \frac{\bar{p}^{3/2}}{\gamma^{3/2}\bar{\rho}^2}, \qquad (1.5.28)$$

where $\bar{p}$ is a mean pressure, $\bar{\rho}$ is a mean density, $\gamma = 6.67 \cdot 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$ is the Newton gravitational constant. So, the Jeans mass (1.5.28) is the minimum mass yielding to gravitational binding at a given density and pressure. It is since the force of attraction inside any arising bunch of a substance increases with an increase in the volume of the bunch while the pressure force is proportional to the area of the bunch.

Indeed, the gravitational potential energy of a molecular cloud is equal $E_g = \frac{1}{2}\int_V \rho\varphi_g dV = \frac{1}{2}\bar{\varphi}_g \cdot M$, the internal energy is $U = -\int_V p dV = -\bar{p} \cdot V = -\bar{p}M / \bar{\rho}$ respectively and the average potential, as noted above, is equal $\bar{\varphi}_g = \gamma M / \bar{r}$, and the average radius $\bar{r} = R$ can be estimated as $R = \sqrt[3]{(3/4\pi)M/\bar{\rho}}$ under the assumption of the spherically symmetric molecular cloud. Substitution of these values into inequality (1.5.27) gives the required result (1.5.28) [73].

It is easy to see that the substitution in (1.5.28) of a quantity $\bar{p}$ expressed from the equation of state (1.5.23) transforms relation (1.5.28) into (1.5.26). It should be noted that relation (1.5.28) is more general than the analogous (1.5.26) since it is not limited to the ideal gas model [73]; moreover, (1.5.28) applies to the study of gravitational condensation of molecular clouds of gas-dust matter having a temperature near absolute zero.

As seen from the above, there are fundamental difficulties in the theory of gravitational condensation for a homogeneous non-rotating infinitely spread medium. As for *rotating* infinitely spread media, the Poisson equation (1.1.41) (or (I.2)) also applies only to mass density perturbations $\delta\rho$ (see Eq. (1.3.13b)). Moreover, it is assumed that the undisturbed medium is in equilibrium. But the question of how this equilibrium is realized is usually not considered [2]. However, in contrast to the infinite homogeneous rest medium in a rotating system of particles (for example, in the molecular cloud) there is a *centrifugal* force. It can be assumed that this force is balanced by the gravitational force of a substance enclosed in a cylinder of radius h and infinite along the axis z. This means that Poisson equation (I.2) is applied to a homogeneous medium in the direction of the axis h, but at the same time it cannot be applied in the direction of the axis z for the same reasons as in the Jeans theory (see Section 1.3), i.e. because there are no forces that could counteract the gravity in this direction [2].

The condition of relative equilibrium in the direction h establishes a relationship between the mass density ρ and the angular velocity Ω. According to Poincaré's general theorem on rotating masses (see Theorem 1.5) we have $\Omega = \sqrt{2\pi\gamma\bar{\rho}}$.

Substituting this value Ω into the condition of the Bel–Schatzman' gravitational instability (1.3.26) we obtain, taking into account the value of the wave number $k = 2\pi / \lambda$, that:

$$\frac{2\sqrt{2\pi\gamma\bar{\rho}}}{h}\frac{d(h^2\sqrt{2\pi\gamma\bar{\rho}})}{dh} + k^2c^2 + \frac{c^2}{(2h)^2} < 4\pi\gamma\rho,$$

whence under the condition of local approximation $\lambda << h$ we obtain the following inequality [73]:

$$8\pi\gamma\bar{\rho} + k^2c^2 < 4\pi\gamma\rho. \qquad (1.5.29)$$

Thus, according to inequality (1.5.29), we find that the *critical density* ρ necessary for the origin and development of instability should be at least twice the mean density $\bar{\rho}$ at which relative equilibrium is observed [2]. However, the Bel–Schatzman' gravitational instability (1.3.26) was derived under the condition of *homogeneous* mass density $\rho = \bar{\rho} = \text{const}$. Consequently, when perturbations propagate in a plane perpendicular to the axis z, gravitational instability does not arise in this case, i.e. no gravitational condensation occurs. So, since the condition of gravitational instability (1.3.26) presupposes $\rho = \text{const}$ and $\Omega \neq \text{const}$, then such a system cannot be in equilibrium. To reach equilibrium, R. Simon proposed to place in it additional non-gaseous masses with a mass density ρ_s depending only on h [138], although the instability condition (1.3.26) has not again been satisfied for such a system [2]. As S.Weindenschilling noted later [139], "the critical density is necessary for gravitational instability to occur, but is not sufficient", since "planetesimals form not by gravitational instability, but by collisional coagulation".

Thus, for the rotating systems under consideration, the equilibrium condition based on the application of the Poisson equation (see Eq.(1.4.7) in the proof of the Poincaré's general theorem) is *inconsistent* with the Jeans theory of gravitational instability and, as a consequence, gravitational condensation. A similar result was obtained by Jeans for a finite spherical mass in equilibrium: it cannot disintegrate into separate parts as a result of gravitational instability. So, the theory of gravitational instability in an infinite (along z) rotating medium has mainly a mathematical interest since there are no real systems to which it could be applied [2]. Following V.S. Safronov, the problem of gravitational instability in real astronomical systems of finite sizes, mainly, spherical and flat ones is more relevant (in particular, in gas-dust spherical formations and disks).

Taking into account these difficulties in the theory of gravitational instability and gravitational condensation, we can use another method based on the virial theorem for estimating the critical mass necessary for beginning the collapse of an already formed molecular cloud (see Theorem 1.4 in Section 1.2). According to Poincaré–Eddington's theorem (1.2.13) to compress a molecular cloud, the right-hand side of Eq. (1.2.13) must be negative, i.e. $(1/2)\cdot d^2I/dt^2<0$. So that the particle system to be in equilibrium, it is necessary that $d^2I/dt^2=0$, and then Eq. (1.2.13) goes into the Poincaré's equation (1.2.14). In the simplest case of a homogeneous spherical cloud, the equilibrium mass value practically coincides M_c in relation (1.5.26).

The advantage of expression (1.2.13) is that it can be easily generalized to more complex cases when: the molecular cloud rotates, it has a magnetic field, it is subjected to external pressure, it is heterogeneous in density, etc. [12]. Let us denote trough $\alpha,\beta,\gamma,\delta$ dimensionless relations to the absolute value of the potential energy $|E_g|$ of the cloud, respectively, the total thermal energy $M\overline{v}^2/2$ of the particles of the cloud, its rotational energy $I\Omega^2/2$, the energy of the magnetic field $Mv_A^2/2$ and the energy of external pressure $p_e V$. Then for a homogeneous spherical molecular cloud with mass M and radius R we have [12]:

$$\alpha = M\overline{v}^2/2\left|E_g\right|; \quad \beta = I\Omega^2/2\left|E_g\right|;$$
$$\gamma = Mv_A^2/2\left|E_g\right|; \quad \delta = p_e V/2\left|E_g\right|, \qquad (1.5.30)$$

where $E_g = -3\gamma M^2/5R, \quad I = 3MR^2/5$, and $v_A = H_m\sqrt{4\pi\gamma\rho}$ is a velocity of Alfvén's waves [9], H_m is a strength of the magnetic field, p_e is an external pressure, $V = 4\pi R^3/3$ is a volume of a spherical molecular cloud. The virial equilibrium condition of the molecular cloud following Eqs (1.2.14) and (1.5.30) can be written as follows [12]:

$$2(\alpha+\beta+\gamma-\delta)-1=0. \qquad (1.5.31)$$

Introducing the function $f(R) = \alpha(R)+\beta(R)+$ $+\gamma(R)-\delta(R)-1/2$, one can study the equation (1.5.31): the equilibrium is stable if the derivative is negative, i.e. $f'(R)<0$ [140]. However, for the collapse to be possible, it is necessary $f(R)<0$ while the derivative must be positive: $f'(R)>0$. In the simplest case of an isolated molecular cloud with $\beta=\gamma=\delta=0$, only the internal gaseous pressure obstructs its compression. According to (1.5.26), (1.5.28) and (1.2.13) under medium conditions in interstellar clouds ($n \sim 10\,\text{cm}^{-3}$, $T \sim 100\,\text{K}$), gravitational forces can prevent this pressure only in the case of a very large cloud mass $M > M_c \sim 10^4 M_{\text{Sun}}$ where M_{Sun} is the mass of the Sun [12, 141]. Therefore, for giant complexes of interstellar clouds, this condition is satisfied.

The question of the formation of the complexes themselves is an independent problem. The cloud complexes are located mainly in the spiral branches of the Galaxy. In this regard, some researchers believe that as though the spiral waves of density promoted to the initial compression of gas [12]. Then it intensified as a result of various types of instabilities (Rayleigh's rotational instability (1.3.34b), thermal instability, and others) leading to the general process of star formation (see (1.3.35)) [73]. In the process of star formation, massive stars quickly "burn out" and then they could explode forming compacted gas shells expanding with high velocity. The shells and the shock wave ionization fronts condense the surrounding colder gas so that the term δ is dominant in the expression (1.5.31). In this case $f(R) < 0$, therefore, cold clouds are compressed to the concentrations $n \sim 10^2 - 10^3$ (cm^{-3}) [12].

As the Galaxy rotates large volumes of gas, taking part in its rotation, have a large angular momentum and cannot be compressed to stellar densities. In the expression (1.5.31), the addendum β begins to dominate, and the compression is terminated. Since the critical mass decreases with increasing mass density in Eq. (1.5.26), the cloud disintegrates into smaller fragments. In this case, part of the rotational angular momentum of the cloud passes into the orbital and rotational angular momentums of these fragments [12]. The latter, losing their rotational angular momentums (for example, due to a magnetic field damping [9]), can be compressed further for some time collapsing up to stars. One of these fragments was our protosolar nebula which gave birth to the Sun and the Solar planetary system [12].

This brief semi-qualitative scenario is based not so much on theoretical estimates as on some observational data of the interstellar clouds of our Galaxy. Cloud complexes represent the concentration of gas-dust clouds with a wide variety of masses, mass densities, and temperatures. The temperatures of dark clouds are very low (about 10 K), and their concentrations range from 10^2–10^3 particles per 1 cm^3 up to 10^5–10^6 per 1 cm^3. These dark cold clouds are called *molecular clouds* because they contain CO, H_2O and many other more complex compounds, moreover, hydrogen in them is also in the molecular state H_2 and constitutes the main fraction. The composition of solid particles (dust particles) constitutes about 1% of the whole medium by weight [12]. Cold clouds are in relative equilibrium. When the balance is violated, the collapse of more dense molecular clouds leads to star formation.

Numerical calculations of the gravitational collapse of protostellar nebulae have been carried out repeatedly, and they are continuing at present (see, for example, review articles by W.M. Tscharnuter [142], B. Larson [143], P. Bodenheimer and D.C. Black [144], A.P. Boss [145] and et al.). The initial parameters of molecular clouds have usually been taken from the condition of just beginning the Jeans instability. At a mass close to the solar one and a temperature of 10K, the relations (1.3.22) and (1.5.26) have given $R_0 \sim 10^{17}$ cm, $n_0 \sim 3 \cdot 10^5$ cm^{-3} and $\rho_0 \sim 10^{-18}$ g/cm^3.

In the simplest case of *one-dimensional calculations* of a spherically symmetric collapse of a non-rotating cloud, a sharply expressed *non-homology of compression* has been revealed. The central region has been compressed much faster, and at density $\rho \sim 10^{-12}$ g/cm^3, it has become opaque [145], so that beginning with this moment it has been compressed not isothermally, but adiabatically. Rapid heating has led to stopping compression and the formation of a quasi-equilibrium core with mass $M_c \sim 10^{-2} M_{\text{Sun}}$.The remaining gas mass (shell), being almost transparent, has continued to contract isothermally at $T \sim 10\,\text{K}$ with a speed close to the free-fall velocity. Having dropped to the core, the gas has formed on its surface a shock wavefront. When the mass density $\rho \sim 10^{-7}$ g/cm^3 and temperature $T \sim 2 \cdot 10^3$ K have been reached in the center of the core, the dissociation of H_2 has begun, and the core has collapsed again [145]. After dissociation and ionization at $\rho \sim 10^{-2}$ g/cm^3, $T = 3 \cdot 10^4$ K and $M_c \sim 0.03 M_{\text{Sun}}$, the core has become hydrostatically equilibrium [12, 145].

Most *two-dimensional calculations* (axially symmetric collapse of a rotating molecular cloud) have been carried out for rapidly rotating clouds with parameter values $\beta_0 \sim 10^{-2} - 10^{-1}$, i.e. when the total angular momentum of a cloud $L_0 \geq 10^{54}$ g$\cdot$cm^2/s. Having confirmed the non-homology of compression, they have revealed a new characteristic feature of the process: in the isothermal region around the core in a plane perpendicular to the axis of rotation, a denser *ring structure* has been formed [12].

Very laborious *three-dimensional calculations* have confirmed the formation of a ring in the isothermal region and have revealed its instability relative to non-axially symmetric perturbations [146]. The extrapolation of the data of Bodenheimer and Black [144] on the size of the core at the instant of the appearance of the ring under the condition of small values β has led to the conclusion that at $L \sim 10^{52}$ g$\cdot$cm^2/s the formation of the Sun with a planetary system is possible in principle [147]. At an angular momentum, an order of magnitude greater, a double or multiple stars could most likely be formed due to the violation of the stability of rotating self-gravitating bodies [1, 148] whereas if a momentum less than an order of magnitude then a single star without a planetary system could be formed.

However, within the framework of such a model of evolution and separation during the collapse of the protosolar nebula into the core and the ring structure, it is impossible to solve the main *problem of the distribution of angular momentum* in the Solar system, which was a stumbling-block for all previous cosmogonical hypotheses and models [2, 6, 12]. At $L \sim 10^{52}\ \mathrm{g \cdot cm^2/s}$ gas falls to the center inside the area with a radius of $\sim 0.1\ \mathrm{AU}$, but for the formation of a pre-planetary disk, it needs an effective *transfer of the angular momentum* from this area to the outside.

Thus, the mechanism of a very fast transfer of the angular momentum at which a high concentration of a substance in the disk near its center would be maintained during the entire collapse is unknown up to now [12]. In passing, we note that the value of $L \sim 10^{52}\ \mathrm{g \cdot cm^2/s}$ is obtained from the consideration that the angular velocity of the protosolar nebula was of the same order as the angular velocity of rotation of the Galaxy itself $\Omega \sim 10^{-15}\ \mathrm{s^{-1}}$, so that at $R_0 \sim 10^{17}\ \mathrm{cm}$ and $L_0 \sim 10^{52}\ \mathrm{g \cdot cm^2/s}$ the angular velocity $\Omega = 1.2 \cdot 10^{-15}\ \mathrm{s^{-1}}$ can just be found.

The export of the angular momentum from the core leads to the formation in its equatorial plane of the embryo of a disk with differential rotation around the core. An instability origin is possible in this zone accompanied by gas turbulence. If some source continues to maintain the turbulent state in the disk then the disk can grow to the size of a planetary system in time. The evolution of similar circum-stellar disks has been considered in some papers [149, 150, 151, 152]. However, the fundamentally important question of maintaining turbulence in the gaseous disk remains unresolved. As noted in Section 1.3, due to the validity of the Rayleigh criterion (1.3.34a) for the protoplanetary cloud, convection could not occur in the radial direction there and, therefore, it could not be a source of turbulence, as Weizsäcker suggested [26, 126]. Later attempts by Lin and Papaloizou [153, 154] to construct a model in which turbulence is generated by convection in a z-direction parallel to the axis of rotation also did not lead to noticeable success. Therefore, models with only convective turbulence are scarcely applicable to the solar nebula [12]. In this regard, the *problem of other mechanisms* for the turbulence origin in rotating gaseous disks is still quite actual.

Summarizing the above, it can be stated that a satisfactory solution to the problem of protosolar nebula origin is hardly possible without building a general theory of the formation of single stars, stars with disk-shaped shells and multiple stars. Such a theory will most likely reflect the *probabilistic nature* of the star formation process [12]. In any case, to get rid of the numerous shortages, contradictions and difficulties that have been listed very briefly in this section, such a theory should be developed as self-sufficient as possible, independent of some kind of "first pushes" from the outside, like "mysterious" spiral mass density waves [12] coming from the depths of spiral arms of our Galaxy.

1.6. Fundamental principles and main problems of the statistical mechanics of a molecular cloud

To solve the posed problem of the initial gravitational condensation of the molecular (gas-dust protoplanetary) cloud, it is appropriate to apply the methods of *statistical* mechanics, especially, because the liquid particles in deterministic hydrodynamics themselves consist of an ensemble of elementary particles with mass m_0). Indeed, in complex systems consisting of a large number of particles, it is practically impossible to observe or theoretically determine exactly the behavior of all particles of the system [110, 155]. If we suppose that the positions and velocities of all particles of a molecular cloud are known at some instance, then it would be almost impossible to use conventional mechanics methods to predict the future states of the complex system (in this case, the molecular cloud) due to a computational complexity and a large number of particles. However, in statistical mechanics, there are methods whose application allows us to investigate not the exact behavior of each particle separately, but the behavior of the whole particle system determining the behavior of most particles [110, 155]. So, statistical mechanics does not consider hopeless attempts to predict in advance the exact behavior of each particle as it is limited only by statistical methods studying the behavior of the greater number of particles with similar properties.

The isolated cloud of gas-dust particles left to itself, like the system of gas molecules, is constantly changing, and also tending to the *most probable state*. The difference between the mean values of the squares of any two velocity components of the gas molecules along some axis is not zero; due to collisions of molecules and close passage of them near each other, this difference tends to zero following the exponential law $e^{-\eta\rho t}$ [155]. The time interval during which this difference decreases by a factor e is called the "relaxation time". As known [110, 136], for gas under normal conditions, this value is very small (in particular, for air under normal conditions the relaxation time is about 10^{-10} seconds). On the contrary, for the ensemble of stars, when stars play the role of particles [155-157], the relaxation times are very long (in particular, Charlier in 1917 determined the relaxation time for the Galaxy to be equal to $4 \cdot 10^{16}$ years). This time interval is much longer than the ages of stars accepted in astrophysics, as a result of which there is no statistical equilibrium state in the Galaxy, so that we can talk only about approximation to this state [110, 155, 156]. As Jeans [1 p.318-319] first showed that if V_0 a is the velocity of a star before the encounter began then:

> ...encounters with small value of V_0 are far more effective than those for which V_0 is large. For this reason, on averaging we must take a rather small value for V_0, and shall select $V_0 = 10$ kms. a second $= 10^6$. With these values, the interval of time ... is found to be 7×10^{22} seconds or about 2×10^{15} years...Thus collisions between stars are so rare that they may be disregarded.

In other words, in the Galaxy one can consider the motions of stars as the motions of single mass points in the general force field of the whole system. Thus, the most important difference between a star system and a system of gas particles is that the influence of *collisions can be neglected in a star system* [155-157].

The system of dust and gas particles under study, forming a gas-dust protoplanetary cloud, has a relaxation time longer than the laboratory system of gas molecules, but much smaller than the system of stars (as intermediate in size of system and mass of particles between the first and the second system). As a result, if we neglect the influence of external forces on a given gas-dust protoplanetary cloud, then this cloud left to itself will not noticeably change in shape and its size with the time. Such a state is the *state of mechanical equilibrium* [2, 155, 157]. As noted in [16], the fluctuation interactions of subsystems of this system in the form of a gas-dust protoplanetary cloud can constantly disturb this equilibrium although very slowly (like the effect of irregular forces in stellar systems, following V.A. Ambartsumian [155]). In other words, a gas-dust cloud can be in a state of mechanical equilibrium at each instance, so that the evolution of a gas-dust protoplanetary cloud consists of a permanent change of one equilibrium state by another [47, 145]. It is appropriate to call such a protoplanetary system of gas and dust particles *stationary* one [65, 73], and the period during which the action of fluctuation interactions of subsystems becomes noticeable is the *relaxation time* of the gas-dust protoplanetary cloud [65, 155]. Let us note that without disturbing fluctuation interactions of the subsystems, the same state of mechanical equilibrium would continue indefinitely, i.e. this would be a state of *statistical equilibrium* of a gas-dust protoplanetary cloud.

To present the fundamentals of the statistical mechanics of a protoplanetary system of gas and dust particles, one should dwell on the distinctive features of the statistical mechanics of protoplanetary gas-dust systems from traditional statistical physics [110]. In many respects, they coincide with the distinctions of stellar systems considered by K.F. Ogorodnikoff [156]. In other words, unlike statistical physics and thermodynamics, the complete statistical equilibrium is unattainable in protoplanetary gas-dust systems and all equilibrium internal parameters of these systems depend *not only* on the integral of energy but on integral of angular momentums (or areas) [158]. In this regard, the *ergodic property* [110] does not apply to protoplanetary gas-dust systems, usually assumed for traditional systems of gas molecules within the framework of statistical physics (since for protoplanetary systems the equilibrium parameters depend not only on the energy integral).

Now we introduce a Cartesian frame of reference inside a molecular (gas-dust protoplanetary cloud) and denote the rectangular coordinates of gas and dust particles through x, y, z so that the location of each particle is determined by the radius vector $\vec{r} = (x, y, z)$. We suppose, bypassing all the problems mentioned in Section 1.5, that the protoplanetary cloud has its gravitational field, therefore we choose the center of mass as the origin of a coordinate system (see

Section 1.1). So, assuming a given centrally symmetric gravitational field in the protoplanetary cloud, we can define the strength of the gravitational field or acceleration [100]:

$$\vec{a} = -\partial\varphi_g / \partial\vec{r}\,, \qquad (1.6.1a)$$

where φ_g is the gravitational potential of the protoplanetary gas-dust system. If u, υ, w are components of the velocity vector $\vec{\mathrm{v}} = (u, \upsilon, w)$ of particles moving in the gravitational field relative to the given coordinate system then:

$$u = dx/dt,\ \ \upsilon = dy/dt,\ \ w = dz/dt\,. \qquad (1.6.1b)$$

Since evolution of a gas-dust cloud consists in a replacement of equilibrium states, we can introduce the particle distribution function $\Phi(x, y, z, u, \upsilon, w, t)$ of coordinates x, y, z and velocities u, υ, w [1, 155], such that its value determines a probability to find particles of a gas-dust cloud, possessing coordinates between x and $x + dx$, y and $y + dy$, z and $z + dz$ as well as components of velocities between u and $u + du$, υ and $\upsilon + d\upsilon$, w and $w + dw$. It is naturally assumed that $\Phi = 0$ when $u, \upsilon, w = \pm\infty$ since otherwise the considered protoplanetary cloud would not be real and would immediately disintegrate. The function $\Phi(x, y, z, u, \upsilon, w, t)$ can describe a certain *velocity body*, in particular, an ellipsoid of velocities [155] if the velocity body does not depend on x, y, z, t. If the distribution function Φ does not depend on time t only then such a distribution is called *stationary*, and the system itself is stationary [155].

Now let us consider an *ensemble of particles* whose number is defined by the distribution function Φ multiplied by the total number N of particles into a protoplanetary cloud:

$$N \cdot \Phi(x, y, z, u, \upsilon, w, t)dxdydzdud\upsilon dw\,. \qquad (1.6.2)$$

According to the above, these particles have coordinates x, y, z and velocity components u, υ, w for a given instance t.

Integrating over all u, υ, w we find the number of particles in a unit of volume, or, in other words, the spatial *concentration* n of particles in a given place x, y, z and at a given instance t:

$$n(x,y,z,t) = N \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(x,y,z,u,\upsilon,w,t)\,du\,d\upsilon\,dw\,. \qquad (1.6.3)$$

Using (1.6.3) the mass density ρ of a protoplanetary substance is calculated as follows:

$$\rho = m_0 n\,, \qquad (1.6.4a)$$

if all particles of the protoplanetary cloud have the same mass m_0, and if the particles have different masses m_{0i} then the mass density is equal to:

$$\rho = \sum_i m_{0i} n_i\,. \qquad (1.6.4b)$$

According to Eqs (1.6.3), (1.6.4a,b) in a *stationary system*, in every elementary volume there is always a constant, steady-state distribution of particles in directions and values of their velocities. Consequently, in the place of particles, leaving this volume due to their movement, the equal number of particles is coming with the same distribution of velocities and their directions [155].

Since $\varphi_g(x,y,z,t)$ is the gravitational potential of the whole system of particles, the motion of each of these particles will be determined by the equations in accord (1.4.4), (1.6.1a):

$$a_x = \frac{du}{dt} = -\frac{\partial \varphi_g}{\partial x}\,;\; a_y = \frac{d\upsilon}{dt} = -\frac{\partial \varphi_g}{\partial y}\,;\; a_z = \frac{dw}{dt} = -\frac{\partial \varphi_g}{\partial z}\,.$$

After a time dt the parallel motion of these particles will take them to a position $x + u\,dt, y + \upsilon\,dt, z + w\,dt$, while their gravitational accelerations will have increased their velocity components $u-\frac{\partial\varphi_g}{\partial x}dt,\quad \upsilon-\frac{\partial\varphi_g}{\partial y}dt,\quad w-\frac{\partial\varphi_g}{\partial z}dt$. By analogy with (1.6.2), confining our attention to a small region $dxdydz$ of space, we find that the number of particles within this region, whose velocity components u,υ,w lie within prescribed limits $dud\upsilon dw$, is:

$$N\cdot\Phi(x+udt,y+\upsilon dt,z+wdt,u-\frac{\partial\varphi_g}{\partial x}dt,\upsilon-\frac{\partial\varphi_g}{\partial y}dt,$$
$$w-\frac{\partial\varphi_g}{\partial z}dt,t+dt)dxdydzdud\upsilon dw. \tag{1.6.5}$$

Since we investigate the same ensemble of particles, i.e. the particles "specified in both groups are identical" [1, 155], expression (1.6.5) must be equal to the analogous (1.6.2) whence it follows that:

$$\Phi(x+udt,y+\upsilon dt,z+wdt,u-\frac{\partial\varphi_g}{\partial x}dt,\upsilon-\frac{\partial\varphi_g}{\partial y}dt,w-\frac{\partial\varphi_g}{\partial z}dt,t+dt)-$$
$$-\Phi(x,y,z,u,\upsilon,w,t)=0.$$

By expanding the left-hand side of this equation in Taylor series and limiting to terms of the first order of smallness, we must accordingly have:

$$\frac{\partial\Phi}{\partial x}udt+\frac{\partial\Phi}{\partial y}\upsilon dt+\frac{\partial\Phi}{\partial z}wdt-\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\varphi_g}{\partial x}dt-\frac{\partial\Phi}{\partial\upsilon}\cdot\frac{\partial\varphi_g}{\partial y}dt-$$
$$-\frac{\partial\Phi}{\partial w}\cdot\frac{\partial\varphi_g}{\partial z}dt+\frac{\partial\Phi}{\partial t}dt=0$$

whence it follows the partial differential equation [1, 93, 155]:

$$\frac{\partial\Phi}{\partial t}+u\frac{\partial\Phi}{\partial x}+\upsilon\frac{\partial\Phi}{\partial y}+w\frac{\partial\Phi}{\partial z}-\frac{\partial\varphi_g}{\partial x}\cdot\frac{\partial\Phi}{\partial u}-\frac{\partial\varphi_g}{\partial y}\cdot\frac{\partial\Phi}{\partial\upsilon}-\frac{\partial\varphi_g}{\partial z}\cdot\frac{\partial\Phi}{\partial w}=0. \tag{1.6.6}$$

This equation, first derived in 1915 by J. Jeans, plays a significant role in the dynamics of stellar systems. As Jeans noted, equation (1.6.6) is identical to the Boltzmann equation in the kinetic theory of gases [1]:

> This is the differential equation which must be satisfied by the distribution-function Φ throughout any motion whatever of a system of stars. It will be seen to be identical with the corresponding equation in the kinetic theory of gases, except that the terms arising from collisions are left out.

Indeed, the collisionless kinetic Boltzmann equation states [159]:

$$\frac{\partial \Phi}{\partial t} + \vec{\mathrm{v}} \nabla \Phi + \vec{F} \frac{\partial \Phi}{\partial \vec{p}} = 0\,, \qquad (1.6.6^*)$$

where $\vec{p}$ is a momentum of a particle of mass m_0, $\vec{F} = -\nabla U$ is a force acting on the particle from the external field $U(\vec{r})$ [159]. Taking into account that $\vec{F} = -m_0 \nabla \varphi_g$, $\vec{p} = m_0 \vec{\mathrm{v}}$, $\vec{\mathrm{v}} = (u, \upsilon, w)$, the Jeans equation (1.6.6) directly follows from the Boltzmann kinetic equation relative to a Cartesian frame of reference (x, y, z) whose origin of coordinates coincides with the center of mass of the molecular (gas-dust protoplanetary) cloud. Nevertheless, J. Jeans [1 p.364] indicated to the above important distinction:

> Just as, in the kinetic theory, the gas may be imagined divided up into a system of showers of parallel-moving molecules, so in stellar dynamics, the stars may be imagined divided up into a system of parallel-moving clusters. But there is the essential difference that in stellar dynamics these clusters retain their identity through long periods of time, whereas in gas-theory they do not.

We also note that the Jeans equation (1.6.6) directly follows from the *continuity equation* for the flow of phase trajectories in the state-space $(x, y, z, u, \upsilon, w)$ of the system of particles [73, 155]. Indeed, let us write the Hamilton function *per unit mass* of the molecular cloud:

$$H = \tfrac{1}{2}(u^2 + \upsilon^2 + w^2) + \varphi_g(x, y, z, t) \qquad (1.6.7)$$

It is known [158] that canonical variables q_i (generalized coordinates) and p_i (generalized impulses) satisfy the equations of mechanics in the Hamilton form:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, 3, \qquad (1.6.8)$$

where $q_1 = x, q_2 = y, q_3 = z, p_1 = u, p_2 = \upsilon, p_3 = w$, moreover, from the fundamentals of statistical physics it immediately follows that the Liouville theorem [110] is valid for the distribution function Φ. According to the Liouville theorem for the distribution function $\Phi(q_1, q_2, q_3, p_1, p_2, p_3, t)$, there is a continuity equation showing the invariance of the total number of system states (or the total number of phase points) in 6-dimensional phase space $(q_1, q_2, q_3, p_1, p_2, p_3)$ with the time:

$$\frac{D\Phi}{Dt} = 0, \qquad (1.6.9a)$$

where D / Dt denotes the *Stokes operator* of the form [110, 155]:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{3}\left(\dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i} \right). \qquad (1.6.9b)$$

From Hamilton equations (1.6.8) and Hamilton's functions (1.6.7) directly follow the relations of the kind:

$$\dot{x} = \frac{\partial H}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u^2 + \upsilon^2 + w^2}{2} + \varphi_g(x, y, z, t) \right) = u; \qquad (1.6.10a)$$

$$\dot{u} = -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{u^2 + \upsilon^2 + w^2}{2} + \varphi_g(x, y, z, t) \right) = -\frac{\partial \varphi_g}{\partial x}. \qquad (1.6.10b)$$

The last relations (1.6.10a) and (1.6.10b) show that after denoting by $x = q_1, y = q_2, z = q_3, u = p_1, \upsilon = p_2, w = p_3$ the continuity equation (1.6.9a), (1.6.9b) goes into the Jeans equation (1.6.6), i.e. the Jeans equation directly follows from the continuity equation for the flow of trajectories in the phase space [73]. Let us note that the Jeans equation (1.6.6) can also be considered as the continuity equation (1.3.7) in the real space (x, y, z) if $\varphi_g \equiv 0$ or as the continuity equation with variable masses if $\varphi_g \neq 0$ [94, 111].

To solve the Jeans equation (1.6.6) for Φ, the Lagrange's rule directs us to compose the characteristic equations [1 p.364]:

$$\frac{dx}{u} = \frac{dy}{\upsilon} = \frac{dz}{w} = \frac{du}{-\frac{\partial \varphi_g}{\partial x}} = \frac{d\upsilon}{-\frac{\partial \varphi_g}{\partial y}} = \frac{dw}{-\frac{\partial \varphi_g}{\partial z}} = dt\,, \qquad (1.6.11)$$

thereby, reducing the equation (1.6.6) to a system of ordinary differential equations which are merely the equations of motion of particles in the general gravitational field of the whole system (molecular cloud).

System (1.6.11) has independent integrals $I_1, I_2, ..., I_6$ depending on the form of the potential function φ_g and being the *first integrals* of particle motion. The solution of the equation (1.6.6) is then simply:

$$\Phi(x, y, z, u, \upsilon, w, t) = f(I_1, I_2, I_3, I_4, I_5, I_6)\,, \qquad (1.6.12)$$

where f is any arbitrary function.

To write the general solution (1.6.12) we need to know all six integrals, however, as noted in [1 p.364, 155], it is impossible to find them as long as the *analytical expression* for the gravitational potential $\varphi_g(x, y, z, t)$ *is unknown*. Finding this expression is the most important and complex task of statistical dynamics (both for the system of particles of a molecular cloud and for the star system) [155]. If the analytical expression φ_g is absent, we can draw some conclusions under the condition of certain restrictions superimposed on φ_g [1, 155].

Let us consider *the first case* when the system of particles of a molecular cloud is in a *stationary state*, i.e. the gravitational potential φ_g and also the distribution function Φ does not depend on time. In this case, one such integral can be written down at once, namely the *integral of energy* [155]. Indeed, from Eq. (1.6.11) we directly find that:

$$udu = -\frac{\partial \varphi_g}{\partial x}dx, \quad \upsilon d\upsilon = -\frac{\partial \varphi_g}{\partial y}dy, \quad wdw = -\frac{\partial \varphi_g}{\partial z}dz. \quad (1.6.13)$$

Adding these three relations in (1.6.13) we obtain:

$$\frac{1}{2}\cdot d(u^2+\upsilon^2+w^2) = -\frac{\partial \varphi_g}{\partial x}dx - \frac{\partial \varphi_g}{\partial y}dy - \frac{\partial \varphi_g}{\partial z}dz = -d\varphi_g. \quad (1.6.14)$$

Since the square of the velocity vector $\vec{\mathrm{v}}^2$ is expressed in terms of the sum of squares of its components: $\vec{\mathrm{v}}^2 = u^2+\upsilon^2+w^2$ in the Cartesian frame of reference, then after the integration of Eq. (1.6.14) we find the integral of energy per unit mass [1 p.365]:

$$I_1 = \frac{1}{2}\vec{\mathrm{v}}^2 + \varphi_g = \mathrm{const}. \quad (1.6.15)$$

Other integrals exist in special cases of the considered system of particles of a molecular cloud. If a stationary system is asymmetric then, except I_1, we cannot find other integrals without knowledge of φ_g [1, 155], therefore the following particular solution for the distribution function occurs:

$$\Phi = \Phi(I_1) = \Phi(\tfrac{1}{2}\vec{\mathrm{v}}^2 + \varphi_g) = \Phi(\tfrac{1}{2}\vec{\mathrm{v}}^2 - |\varphi_g|). \quad (1.6.16)$$

In this case, the velocity distribution is spherical since Φ depends on the square $\vec{\mathrm{v}}^2$. As for the mass density distribution, i.e. the distribution of particles for spatial coordinates x, y, z, it can be arbitrary (depending on the analytical expression of function φ_g) [155]. However, there is no spherical velocity distribution in the Galaxy, therefore this particular case is not observed, generally speaking [155]. Nevertheless, there may be other special cases. For example, if for any particle $\mathrm{v}^2 - 2|\varphi_g| \geq 0$, this means that the total energy of this particle is positive or equal to zero. Such a particle should be removed from the considered molecular (gas-dust protoplanetary) cloud. Moreover, its velocity will be greater than or equal to the velocity of escape $\sqrt{2|\varphi_g|}$ (this velocity is also called critical or parabolic) [155]. "Thus the search for systems in steady motion with no symmetry at all has failed; no such motion is possible" [1 p.367].

Let us consider *the second case* corresponding to the stationary particle system of a molecular cloud with *spherical symmetry* [1 p.365]. In this case, the gravitational field is centrally symmetric, so that φ_g is a function only of r, the distance from the center of the molecular cloud:

$$r = \sqrt{x^2 + y^2 + z^2}\,. \tag{1.6.17}$$

Using (1.6.17) it is not difficult to see that:

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2} \cdot 2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}. \tag{1.6.18}$$

In view of (1.6.18), the characteristic equations (1.6.11) assume the form:

$$\frac{dx}{u} = \frac{dy}{\upsilon} = \frac{dz}{w} = \frac{du}{-\frac{x}{r} \cdot \frac{\partial \varphi_g}{\partial r}} = \frac{d\upsilon}{-\frac{y}{r} \cdot \frac{\partial \varphi_g}{\partial r}} = \frac{dw}{-\frac{z}{r} \cdot \frac{\partial \varphi_g}{\partial r}}. \tag{1.6.19}$$

Taking into account Eqs (1.6.19), for example, it is not difficult to get that:

$$-xd\upsilon\cdot\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}=-ydu\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}, \quad (1.6.20a)$$

$$\upsilon dx=udy, \quad (1.6.20b)$$

and also that:

$$-ydw\cdot\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}=-zd\upsilon\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}, \quad (1.6.21a)$$

$$wdy=\upsilon dz \quad (1.6.21b)$$

and finally:

$$-xdw\cdot\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}=-zdu\frac{1}{r}\cdot\frac{\partial\varphi_g}{\partial r}, \quad (1.6.22a)$$

$$wdx=udz\,. \quad (1.6.22b)$$

Indeed, from Eqs (1.6.20a), (1.6.20b) - (1.6.22a), (1.6.22b) it immediately follows the equations:

$$xd\upsilon = ydu, \upsilon dx = udy, \quad (1.6.23a)$$

$$zd\upsilon = ydw, \upsilon dz = wdy, \quad (1.6.23b)$$

$$xdw = zdu, wdx = udz. \quad (1.6.23c)$$

Adding in pairs the relations in (1.6.23a), (1.6.23b), (1.6.23c) and then integrating them, we obtain three integrals:

$$x\upsilon = yu + \text{const}\,,$$

$$z\upsilon = yw + \text{const}\,,$$

$$xw = zu + \text{const}\,,$$

expressing that the moments of momentum I_2, I_3, I_4 per unit mass about the axes of coordinates x, y, z remain constant [1, 155]:

$$I_2 = x\upsilon - yu = \text{const}\,, \quad (1.6.24a)$$

$$I_3 = yw - z\upsilon = \text{const}\,, \quad (1.6.24b)$$

$$I_4 = zu - xw = \text{const}\,. \quad (1.6.24c)$$

Apart from special artificial cases of φ_g [96, 155], there can be no integrals beyond those already mentioned, so that the distribution function Φ must be of the form

$$\Phi = \Phi(I_1, I_2, I_3, I_4), \tag{1.6.25}$$

where I_1, I_2, I_3, I_4 are given by the relations (1.6.15), (1.6.24a)-(1.6.24c). For instance, if $\varphi_g = kr^2/2$, where k is constant, there are additional integrals of the type: $I_5 = (u^2 + kx^2)/2 = \text{const}$, $I_6 = (\upsilon^2 + ky^2)/2 = \text{const}$, $I_7 = (w^2 + kz^2)/2 = \text{const}$, but only two of them are independent because $I_1 = I_5 + I_6 + I_7$, therefore, the corresponding motion being one in which each particle describes a continually repeated elliptic orbit about the center [96].

Obviously, in systems with spherical symmetry the concentration n (or mass density ρ) and the potential φ_g of the whole system are functions of r only, since the concentration n, like the mass density $\rho = \sum_i m_{0i} \cdot n_i$, has a spherical symmetry by condition, and the potential φ_g is connected with the mass density by Poisson equation (1.1.41):

$$\nabla^2 \varphi_g = 4\pi\gamma \sum_i m_{0i} \cdot n_i \,. \tag{1.6.26}$$

Since Eq. (1.6.26) is true, it immediately follows that if the gravitational field is spherically symmetrical then the concentration function n_i is also spherically symmetric and the mass density is arranged in spherical shells. Let us note that the mass density is also obtained by integrating the distribution function (1.6.25) with respect to all values of u, υ, w, i.e. if we substitute the expression (1.6.3) into Poisson equation (1.6.26) instead of n_i, we obtain a condition that the distribution function Φ_i has to satisfy [155]:

$$\nabla^2 \varphi_g = 4\pi\gamma \sum_i m_{0i} \cdot N_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_i(x, y, z, u, \upsilon, w, t)\, du\, d\upsilon\, dw. \tag{1.6.27}$$

The system of dust and gas particles must satisfy both the *statistical condition* (Jeans equation (1.6.6)) and this *dynamical condition* (Poisson equation (1.6.27)). The latter condition is also valid in the case of an external potential φ_g^1 since $\nabla^2\varphi_g^1$ for particles of a molecular cloud is zero (since the function φ_g^1 does not add mass to the particle system).

For the resulting mass density to depend only on r, the distribution function (1.6.25) should depend only on *r*. This means that I_2, I_3, I_4 must enter in Φ as expressions depending only on $r^2 = x^2 + y^2 + z^2$. Such an expression is the *total angular momentum* per unit mass which in accord with (1.6.24a) - (1.6.24c) is equal to:

$$I_2^2 + I_3^2 + I_4^2 = (x\upsilon - yu)^2 + (yw - z\upsilon)^2 + (zu - xw)^2 =$$
$$= x^2(\upsilon^2 + w^2) + y^2(u^2 + w^2) + z^2(u^2 + \upsilon^2) -$$
$$- 2xy \cdot u\upsilon - 2xz \cdot uw - 2yz \cdot \upsilon w =$$
$$= (x^2 + y^2 + z^2) \cdot (u^2 + \upsilon^2 + w)^2 - (x^2u^2 + y^2\upsilon^2 + z^2w^2 +$$
$$+ 2xu \cdot y\upsilon + 2xu \cdot zw + 2y\upsilon \cdot zw) =$$
$$= (x^2 + y^2 + z^2) \cdot (u^2 + \upsilon^2 + w^2) - (xu + y\upsilon + zw)^2 . \quad (1.6.28)$$

With a view to further transformation of (1.6.28), we can note that the time derivative of (1.6.17) is equal to:

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\sqrt{x^2 + y^2 + z^2}\right) =$$
$$= \frac{1}{2} \cdot \frac{2xu + 2y\upsilon + 2zw}{r} = \frac{xu + y\upsilon + zw}{r}, \quad (1.6.29)$$

where v_r denotes the component of the velocity $\vec{\mathrm{v}}$ along the radius r. Taking into account (1.6.17) and (1.6.29), expression (1.6.28) takes the form:

$$I_2^2 + I_3^2 + I_4^2 = \vec{r}^{\,2} \cdot \vec{\mathrm{v}}^2 - \vec{r}^{\,2} v_r^2 = r^2(\mathrm{v}^2 - v_r^2) . \quad (1.6.30)$$

In the spherical coordinates r, θ, ε, $\vec{\mathrm{v}}^2$ can be decomposed into three components:

$$\vec{\mathrm{v}}^2 = v_r^2 + v_\theta^2 + v_\varepsilon^2, \qquad (1.6.31)$$

so that it immediately follows from (1.6.15), (1.6.30), (1.6.31) that:

$$I_2^2 + I_3^2 + I_4^2 = r^2(v_\theta^2 + v_\varepsilon^2),$$

and the distribution function (1.6.25) should be of the form [1, 155]:

$$\Phi = \Phi(\tfrac{1}{2}\vec{\mathrm{v}}^2 + \varphi_g, \vec{r}^{\,2}(v_\theta^2 + v_\varepsilon^2)) = \Phi(\tfrac{1}{2}\mathrm{v}^2 + \varphi_g, r^2\,\mathrm{v}^2 \sin^2\alpha), \qquad (1.6.32)$$

where α is the angle between the directions of $\vec{r}$ and $\vec{\mathrm{v}}$. At any single point in space, the law of distribution Φ depends on v and on α. With this law of distribution, the velocities of particles are not uniformly distributed over all directions in space, for this would require that Φ should depend on v only. Indeed, both components of the velocity v_θ and v_ε, being perpendicular to the radius vector $\vec{r}$, enter symmetrically in the expression (1.6.32), therefore their distribution is *circular* [155]. The asymmetry of the velocity distribution is caused by the component v_r, so that $\Phi = \mathrm{const}$ are surfaces of revolution near the radius r. Thus, the velocity-diagram for the motions of the particles near a given point will not be spherically symmetrical but will be a figure of revolution, having the radius through the point to the center of the system as an origin [1]. It will be directed either the preferred motion of particles on the radial component or, conversely, fewer particles, i.e. stars in the case of a star system, will move in this direction. Respectively, the velocity body will be stretched or, conversely, compressed in the radial direction [155]. In the case of the star system, for example in the Galaxy, the major axis of the velocity ellipsoid is directed approximately to the center of the system. But since our Galaxy Milky Way does not possess spherical symmetry, this case should be rejected again, however, it can occur in spherical star clusters [1].

Now we consider *the third case* corresponding to the stationary system of particles of a molecular cloud with axial symmetry. Let z be the axis of symmetry. The equation of characteristics (1.6.19) gives only two integrals: I_1 and I_2 the latter being obtained relative to the axis z:

$$I_2 = x\upsilon - yu = \text{const} .$$

Indeed, using the cylindrical coordinate system:

$$x = h\cos\varepsilon; \quad y = h\sin\varepsilon; \quad z = z , \tag{1.6.33}$$

where $h = \sqrt{x^2 + y^2}$, we can define the components of the velocity vector $\vec{\text{v}}$ in the cylindrical coordinates (h,ε,z):

$$\begin{aligned} v_h &= \frac{dh}{dt} = \dot{h}; \\ v_\varepsilon &= \frac{d}{dt}(h\varepsilon) = h\dot{\varepsilon}; \\ v_z &= \frac{dz}{dt} = \dot{z}. \end{aligned} \tag{1.6.34}$$

Taking into account the relations (1.6.33) and (1.6.34), we find the relationship between the components of the velocity vector in the Cartesian system (u,v,w) and the cylindrical system $(v_h, v_\varepsilon, v_z)$:

$$\begin{aligned} u &= \frac{dx}{dt} = \dot{h}\cos\varepsilon - h\dot{\varepsilon}\sin\varepsilon = v_h\cos\varepsilon - v_\varepsilon\sin\varepsilon; \\ \upsilon &= \frac{dy}{dt} = \dot{h}\sin\varepsilon + h\dot{\varepsilon}\cos\varepsilon = v_h\sin\varepsilon + v_\varepsilon\cos\varepsilon; \\ w &= \frac{dz}{dt} = v_z. \end{aligned} \tag{1.6.35}$$

So, we consider the system of generalized coordinates (q_1, q_2, q_3) where $q_1 = h, \quad q_2 = \varepsilon, \quad q_3 = z$ and accordingly a system of generalized velocities of the form:

$$\dot{q}_1 = \dot{h}, \quad \dot{q}_2 = \dot{\varepsilon}, \quad \dot{q}_3 = \dot{z}\,. \tag{1.6.36}$$

Let us compose the Lagrange function of a unit mass particle and then use the Euler–Lagrange variational equations in cylindrical coordinates [158]:

$$\begin{aligned} L &= \frac{1}{2}(v_h^2 + v_\varepsilon^2 + v_z^2) - \varphi_g(h,z) = \\ &= \frac{1}{2}(\dot{h}^2 + h^2\dot{\varepsilon}^2 + \dot{z}^2) - \varphi_g(h,z). \end{aligned} \tag{1.6.37}$$

Using (1.6.37) we write the Euler–Lagrange equations relative to the generalized coordinates and velocities (1.6.36):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) = \frac{\partial L}{\partial h}; \tag{1.6.38a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varepsilon}}\right) = \frac{\partial L}{\partial \varepsilon}; \tag{1.6.38b}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z}. \tag{1.6.38c}$$

As follows from Eqs (1.6.38a)-(1.6.38c):

$$\frac{d\dot{h}}{dt} = \frac{dv_h}{dt} = h\dot{\varepsilon}^2 - \frac{\partial \varphi_g}{\partial h}; \tag{1.6.39a}$$

$$\frac{d(h^2\dot{\varepsilon})}{dt} = \frac{d(hv_\varepsilon)}{dt} = -\frac{\partial \varphi_g}{\partial \varepsilon} = 0; \tag{1.6.39b}$$

$$\frac{d\dot{z}}{dt} = \frac{dv_z}{dt} = -\frac{\partial \varphi_g}{\partial z}. \tag{1.6.39c}$$

The second equation (1.6.39b) vanishes due to the axial symmetry which allows us to find the second integral being the angular momentum per unit mass around the axis z [1, 155]:

$$I_2 = h^2 \dot{\varepsilon} = h v_\varepsilon \,. \tag{1.6.40}$$

So, taking the axis of symmetry to be the axis of z, the only integrals of equations (1.6.19) are seen to be the energy integral (1.6.15), and the integral (1.6.40), which expresses that the moment of momentum of a particle about the z-axis remains constant. Hence the only possible state of steady motion is one in which the distribution function (1.6.25) is of the form [1, 155]:

$$\Phi = \Phi(I_1, I_2) = \Phi(\vec{v}^2 + 2\varphi_g, h v_\varepsilon) = \Phi(\tfrac{1}{2}\mathrm{v}^2 + \varphi_g, h^2 \dot{\varepsilon})\,. \tag{1.6.41}$$

From (1.6.41) it can be seen that the distribution with respect to v_h and v_z is symmetrical, while the values v_ε are met more often or scarcer than the previous components [155]. The velocity body will be elongated or compressed in a direction perpendicular to the direction to the center of the system.

In 1922 and a little later, J. Jeans believed that this particular case took place in the Galaxy since the direction of the vertex of stellar motions is $\alpha = 340^\circ$, and according to Kapteyn's discovery of the phenomenon "star-streaming" [1], the direction to the center was obtained $\alpha = 257^\circ$, i.e. perpendicular to the line of vertices. At present, it is known that the direction to the center is $\alpha = 330^\circ$, i.e. it almost coincides with the direction of the vertices deviating from it to 10° [155]. Since there are serious reasons to consider the Galaxy as a system with axial symmetry (due to the rotation of the Galaxy, study of its structure, comparison with other galaxies), the resulting discrepancy can be explained either by the fact that the Galaxy is not in a completely stationary state or that it is not a particular solution (1.6.41) but some other. Let us note that it is impossible to write a general solution without knowing the analytical expression for the potential φ_g [155].

Until now, the Jeans equation (1.6.6) is considered as a differential equation for the distribution function Φ, considering the analytical expression of the potential φ_g to be unknown and only imposing some restrictions on it. Jeans [1] followed this path bypassing the problem that solutions (1.6.16), (1.6.32), (1.6.41) are particular ones, but not general, even for these restrictions discussed above. Meanwhile, the question of the physical significance of these particular solutions is not clear, and the missing independent integrals must exist in a real physical problem.

It is alleged [155] to be more appropriate another way that J.H. Oort [157] and then S.Chandrasekhar [125] followed. Within the framework of this approach, the Jeans equation (1.6.6) is supposed to be considered as a differential equation for the potential φ_g finding, and the distribution function Φ can be given one or another suitable form [155]. In this case, Φ should depend on both distribution of stellar densities (concentrations) n in the Galaxy and distribution of velocities, moreover, the last is supposed to be taken as the Schwarzschild's "ellipsoidal" distribution [1, 155]:

$$\Phi = \text{const}\cdot e^{-k^2u^2+l^2\upsilon^2+m^2w^2}, \tag{1.6.42}$$

resulting from astronomical observations. However, the idea of attaching the distribution function Φ to the so-called "suitable form" cannot be considered rigorously scientifically, since such a function must be obtained by successive mathematical derivations from a physical model describing a system of particles under given conditions [73].

Such an attempt to derive the distribution function Φ, and hence the analytical form of the functions of mass density ρ and particle concentration n was made by J. Jeans within his theory as applied to star systems [1 p.371]. Now we are going to briefly describe the main results of the theory of Jeans, not limited to the system of stars, but considering, in general, the system of particles in the form of some kind of molecular or gas-dust cloud.

As J. Jeans [1 pp.370-371] claimed, a law of distribution of velocity body of the type (1.6.41):

> …will give steady motion except for the disturbing effects of encounters of near stars. Further, this formula has been found to include all possible cases of stable steady motion.
>
> The effect of near encounters will be slowly to change the character of the motion, and after a sufficiently long time... the system of stars will tend to a steady state in which even close encounters do not disturb the statistical specification of motion.
>
> During this process, the form of the function Φ must change, and when the final steady state is attained, the general principles of statistical mechanics indicate* (*see details in* [160 §107, ch.V])[5] that the form of the function Φ must be:

$$\Phi(I_1, I_2) = A \cdot e^{-2\beta m_0 (I_1 + \Omega I_2)}, \qquad (1.6.43)$$

where A, β and Ω are constants. Inserting their values for I_1 and I_2 from relations (1.6.15) and (1.6.24a), the distribution function (1.6.43) becomes:

$$\Phi(I_1, I_2) = A \cdot e^{-\beta m_0 [\vec{v}^2 + 2\varphi_g + 2\Omega(x\upsilon - yu)]} =$$
$$= A \cdot e^{-\beta m_0 [(u^2 + \upsilon^2 + w^2) + 2\varphi_g + 2\Omega(x\upsilon - yu)]} =$$
$$= A \cdot e^{-\beta m_0 [(u - \Omega y)^2 + (\upsilon + \Omega x)^2 + w^2]} \cdot e^{2\beta m_0 [-\varphi_g + \Omega^2 (x^2 + y^2)/2]}. \qquad (1.6.44)$$

As noted in this section (see formulas (1.6.3), (1.6.4a, b), (1.6.27)), the concentration (or mass density) is obtained by integrating the distribution function Φ with respect to all values of the velocity components u, υ, w from $-\infty$ to $+\infty$. Indeed, $\Phi(I_1, I_2)$ is a joint distribution function at spatial coordinates x, y, z and velocity components u, υ, w, so that we can separately consider the distribution function at velocities:

[5] The author’s remark

$$\Phi_{\vec{\mathrm{v}}}(u,\upsilon,w) = B \cdot e^{-\beta m_0[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}, \tag{1.6.45}$$

where B is a normalizing factor, such that $A = B \cdot C$ (compare with the Schwarzschild's distribution function (1.6.42)). This function $\Phi_{\vec{\mathrm{v}}}(u,\upsilon,w)$ satisfies the normalization condition:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{\vec{\mathrm{v}}}\,du\,d\upsilon\,dw = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B \cdot e^{-\beta m_0[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}du\,d\upsilon\,dw = 1,\end{aligned} \tag{1.6.46}$$

therefore, integrating (1.6.44) over all values of u,υ,w from $-\infty$ to $+\infty$ and taking into account (1.6.46), we find that the distribution function at x,y,z must be of the form [1]:

$$\Phi_{\vec{r}}(x,y,z) = C \cdot e^{2\beta m_0[-\varphi_g+\Omega^2(x^2+y^2)/2]}, \tag{1.6.47}$$

where C is a constant. Using the normalization condition (1.6.46), we can estimate the constant B in (1.6.44), (1.6.45):

$$B \cdot \int_{-\infty}^{\infty} e^{-\beta m_0(u-\Omega y)^2}du \cdot \int_{-\infty}^{\infty} e^{-\beta m_0(\upsilon+\Omega x)^2}d\upsilon \cdot \int_{-\infty}^{\infty} e^{-\beta m_0 w^2}dw = 1,$$

whence:

$$B \cdot \left(\frac{1}{\sqrt{\beta m_0}}\right)^3 \cdot \left(\int_{-\infty}^{\infty} e^{-s^2}ds\right)^3 = 1,$$

and taking into account the fact that $\int_{-\infty}^{\infty} e^{-s^2}ds = \sqrt{\pi}$ [128] we obtain:

$$B = \left(\beta m_0 / \pi\right)^{3/2}. \tag{1.6.48}$$

According to (1.6.48) the distribution function (1.6.47), describing both a system of stars and a molecular (gas-dust) cloud takes the form:

$$\Phi_{\vec{r}}(x,y,z) = A\left(\frac{\pi}{\beta m_0}\right)^{3/2} \cdot e^{2\beta m_0[-\varphi_g+\Omega^2(x^2+y^2)/2]}. \tag{1.6.49}$$

Now we will consider a molecular (gas-dust) cloud exclusively. As mentioned above (see Sections 1.2 and 1.5, in particular, the formula (1.2.18)), the mean squared velocity $\overline{v^2}$ in the state of thermodynamic equilibrium can be expressed through the temperature T of a molecular cloud based on molecular kinetic theory [110, 160]. On the other hand, the mean square value of velocity in accordance with (1.6.45) and (1.6.48) can be calculated as follows:

$$\overline{v^2} = \left(\beta m_0 / \pi\right)^{3/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(u-\Omega y)^2 + (\upsilon + \Omega x)^2 + w^2] \times$$

$$\times e^{-\beta m_0 [(u-\Omega y)^2 + (\upsilon+\Omega x)^2 + w^2]} du\, d\upsilon\, dw =$$

$$= \left(\beta m_0 / \pi\right)^{3/2} \cdot \{ \int_{-\infty}^{\infty} (u-\Omega y)^2 e^{-\beta m_0 (u-\Omega y)^2} du \int_{-\infty}^{\infty} e^{-\beta m_0 (\upsilon+\Omega x)^2} d\upsilon \int_{-\infty}^{\infty} e^{-\beta m_0 w^2} dw +$$

$$+ \int_{-\infty}^{\infty} (\upsilon+\Omega x)^2 e^{-\beta m_0 (\upsilon+\Omega x)^2} d\upsilon \int_{-\infty}^{\infty} e^{-\beta m_0 (u-\Omega y)^2} du \int_{-\infty}^{\infty} e^{-\beta m_0 w^2} dw +$$

$$+ \int_{-\infty}^{\infty} w^2 e^{-\beta m_0 w^2} dw \int_{-\infty}^{\infty} e^{-\beta m_0 (u-\Omega y)^2} du \int_{-\infty}^{\infty} e^{-\beta m_0 (\upsilon+\Omega x)^2} d\upsilon \} =$$

$$= 3\left(\beta m_0 / \pi\right)^{3/2} \cdot \frac{1}{\left(\sqrt{\beta m_0}\right)^3} \int_{-\infty}^{\infty} s^2 e^{-s^2} ds \cdot \left(\frac{\sqrt{\pi}}{\sqrt{\beta m_0}}\right)^2 = \frac{3}{2} \cdot \frac{1}{\beta m_0} . \quad (1.6.50)$$

According to (1.6.50) the mean kinetic energy $\overline{E}_k$ of a particle of mass m_0 is equal:

$$\overline{E}_k = \frac{m_0 \overline{v^2}}{2} = \frac{m_0}{2} \cdot \frac{3}{2} \cdot \frac{1}{\beta m_0} = \frac{3}{4} \cdot \frac{1}{\beta}, \qquad (1.6.51a)$$

that is, $\overline{E}_k$ is expressed by the well-known formula of molecular kinetic theory:

$$\overline{E}_k = \frac{3 k_B T}{2}, \qquad (1.6.51b)$$

where k_B is the Boltzmann constant. The comparison (1.6.51a) with (1.6.51b) gives the sought expression for the constant β:

$$\beta = 1/2k_B T\,. \qquad (1.6.52)$$

In turn, the direct substitutions (1.6.48) and (1.6.52) of constants B and β respectively in the formula (1.6.45) gives a Maxwellian law of the kind:

$$\Phi_{\vec{v}}(u,\upsilon,w) = \left(m_0/2\pi k_B T\right)^{3/2}\cdot e^{-\left(m_0/2k_B T\right)\cdot\left[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2\right]}, \quad (1.6.53)$$

and the substitution (1.6.52) in the formula (1.6.49) gives the distribution function at spatial coordinates:

$$\Phi_{\vec{r}}(x,y,z) = A\left(\frac{2\pi k_B T}{m_0}\right)^{3/2}\cdot e^{\left(m_0/k_B T\right)\left[-\varphi_g+\Omega^2\left(x^2+y^2\right)/2\right]}, \quad (1.6.54)$$

where the constant A should be determined from the normalization condition of the function (1.6.54).

According to (1.6.3) and (1.6.54), the concentration and mass density of a rotating mass of gas can be expressed by the formulas [1]:

$$n(x,y,z) = N\Phi_{\vec{r}}(x,y,z) = n_0\cdot e^{\left(m_0/k_B T\right)\cdot\left[-\varphi_g(x,y,z)+\Omega^2\left(x^2+y^2\right)/2\right]}; \quad (1.6.55)$$

$$\rho(x,y,z) = m_0 N\Phi_{\vec{r}}(x,y,z) = \rho_0\cdot e^{\left(m_0/k_B T\right)\cdot\left[-\varphi_g(x,y,z)+\Omega^2\left(x^2+y^2\right)/2\right]}. \quad (1.6.56)$$

As follows from (1.6.55), (1.6.56), just as in a rotating mass of gas (molecular cloud), the surfaces of equal density have equations of the form:

$$\varphi_g(x,y,z) - \frac{1}{2}\Omega^2\left(x^2+y^2\right) = \text{const}, \qquad (1.6.57)$$

coinciding with Eqs (1.4.21) in Section 1.4, which is the well-known fact in the theory of gravitational potential and hydrodynamics [95, 111].

It should be noted [1 pp.371-372], however, that the Jeans law of mass density (1.6.56) as well as Eq. (1.6.49):

> …which must obtain in the final state gives infinite density at an infinite distance from the center except when $\Omega = 0$. Even when $\Omega = 0$, it gives a finite density at all distances from the center, so that the system of stars is of infinite extent in space; it is in fact

arranged like a mass of gas in isothermal equilibrium without rotation.
When Ω is different from zero, the formula shews that there can be no steady state until all the stars have been scattered to infinity. Actually, as we have seen (*Section 1.4 in this monograph*[6]), the surfaces of equal density *(1.6.57)* consist of some closed surfaces and some open surfaces. If the density at the last of the closed surfaces is quite small, then the stars inside it form an *approximately* permanent system, although there is a continual slow loss of stars across this surface. If however the density at the last closed surface, and so also at the first open surface, is not quite small, there will be a rapid loss across these surfaces, the stars streaming off in all directions in their efforts to establish the law of density *(1.6.57)*, and as the velocity of many of these is, by formula *(1.6.44)*, greater than the velocities of escape $\sqrt{2|\varphi_g|}$, a great part of the loss is permanent.

Thus, the Jeans theory, which leads to the law of mass density of the form (1.6.56), refutes the fact of the *long existence* of a molecular (gas-dust) cloud or star system under study (moreover, according to (1.6.45) "in this motion there is no star-streaming" [1]). The infinite mass density at the periphery of a molecular (gas-dust) rotating cloud is one of those difficulties of the Jeans theory concerning the impossibility of determining the gravitational potential for infinitely spread media (see Section 1.5, [73]). Consequently, within the framework of the Jeans theory, in particular, the density law of the form (1.6.56), it is impossible to construct a consistent model of the mass density distribution of a cosmogonical body, therefore, new statistical models of cosmogonical bodies forming are needed.

Such models have been proposed in several works, for example, see [161-166]. Following I. Prigogine and G. Severne [161], a fundamental quantity in the *mean-field theory*, namely the *number density* or concentration $n(\vec{r},t)$, in

[6] The author's remark and numeration

particular, of the gas of stars, called later on the "gravitational plasma" has been introduced. The foundation of the mean-field theory is the collisionless kinetic equation of Vlasov [159] looking like the Boltzmann collisionless equation (1.6.6*) but in which the *self-consistent force* of gravitational interaction [164] (or the mean-field expression of force [166]) is used:

$$\bar{\vec{F}}(\vec{r},t) = \int\int \Phi(\vec{r}_1,\vec{v}_1,t)\vec{F}(|\vec{r}-\vec{r}_1|)d\vec{v}_1\, d\vec{r}_1 =$$

$$= \gamma m_0^2 \int n(\vec{r}_1,t)\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3} d\vec{r}_1 \,. \qquad (1.6.58)$$

Using (1.6.58) in [164], screening of the Newtonian potential of a moving test body in a homogeneous Maxwellian gas of gravitating bodies has been investigated based on the Vlasov collisionless equation of and Poisson equation. The modified potential has been expressed in terms of the test particle velocity and the gravitational susceptibility of the system. Since all bodies in such a system execute a thermal motion, a renormalized gravitational potential in the system has been determined by way of averaging the test-body potential over the body velocities with the Maxwellian distribution function [164]. It has been found that the resultant renormalized potential not only decays faster than the Newtonian potential but also oscillates with the period on order of the Jeans length. Bashkirov and Vityazev have supposed that a dark matter allowance in the system gives rise to a significant decrease in the oscillation period, nevertheless, they have stated that the observable period of these oscillations *enables us to estimate the Jeans wave number for the dark matter* [164].

Using (1.6.58) as well as the Vlasov collisionless equation in [166], it has been noted that the standard results of equilibrium statistical physics do not apply at all in the statistical mechanics of “gravitational plasma”. Yves Pomeau has also explained that the mean-field theory valid for “short time scales” of the order of the orbital period of a star in the mean-field of the stellar cluster. He has also pointed out this mean-field theory is not the last word, because it leaves unspecified some functions and does not truly include irreversible effects due to the close interactions between stars or to resonant interaction between their orbits. In this connection, the concepts of classical statistical mechanics behind Boltzmann–Gibbs equilibrium theory cannot be used in this field because, fundamentally, the *ergodic theory does not make sense* there so that there is no proper way of defining entropy and temperature. The work [167] confirms that a classical *statistical mechanics of gravitation can not be constructed with the Boltzmann–Gibbs distribution* because the integral needed for building up the partition function includes an exponential and thus diverges. In this regard, Y. Pomeau has stated that “therefore, the dynamical problem has to be solved one way or the other” [166].

Thus, although the problem of gravitational self-condensation of an infinite spread cosmic medium is not solved completely within the framework of these modern theories, nevertheless, the statistical theory of the initial gravitational formation of a cosmogonical body from infinitely distributed matter remains highly relevant.

1.7. On the evolutional equation in the statistical mechanics of molecular clouds

The problems of the theory of gravitational condensation of a molecular (gas-dust) cloud are explained in Sections 1.1–1.6, among them, are the following [73]:

– the absence of a clear mechanism of gravitational condensation of the infinitely spread matter, including the

source of origin and supporting waves of gravitational instability;
- the absence of analytical expressions of gravitational potential and force of gravity for an infinitely distributed homogeneous rest medium;
- the impossibility of applying exclusively the deterministic approach to a correct description of the behavior of a large number of particles of a gravitating molecular cloud;
- the impossibility of finding a general (but not particular) solution of the Jeans equation due to the above difficulty of determining the gravitational potential of a molecular (gas-dust) cloud;
- the infinite value of the mass density at the periphery of a rotating molecular (gas-dust) cloud according to the Jeans theory.

In this regard, it is necessary to study in more detail the evolution of the statistical distribution function describing the state of spread matter (in the form of a molecular cloud) in space and time [16, 65, 73]. The starting point in the study of the evolution of the distribution function is the condition of existence of a *point of mechanical equilibrium* (or relative mechanical equilibrium), as a rule, being unstable in time. The unstable mechanical equilibrium of a molecular cloud is gradually replaced by a non-equilibrium state slowly flowing with the time.

First of all, let us consider an almost *immovable* molecular (gas-dust) cloud in a state of unstable mechanical equilibrium. Let $\Phi(x, y, z, t)$ be a probability density function for the location of particles in this immovable cloud, describing the spatial distribution of particles at some instance $t = t_0$. If we choose in a three-dimensional real space $\Re^3$ some fixed point with coordinates (x, y, z), then near this point we can determine the *ensemble of particles* in a small region of space

$dxdydz$, whose the number is currently determined by the distribution function Φ multiplied by $dxdydz$ and by N (the total number of particles in the cloud). However, due to their interactions, after a time interval Δt, these same particles already occupy a position $x+\Delta x, y+\Delta y, z+\Delta z$. Since the number of observable particles of the ensemble is fixed, because we do not consider birth and death process for particles in the time interval Δt, this proves the correctness of the following equality [73]:

$$N\Phi(x+\Delta x, y+\Delta y, z+\Delta z, t_0+\Delta t)dxdydz =$$
$$= N\Phi(x, y, z, t_0)dxdydz\,. \tag{1.7.1}$$

Indeed, as already noted in Section 1.6, equation (1.7.1) is true because *the same ensemble* of particles is considered all the time (see (1.6.2) and (1.6.5)).

Further, under the assumption that an initial evolution of the molecular (gas-dust) cloud is caused by a *slow process* of gravitational tightening (contraction) of its local domains of spread matter (parts of this molecular cloud) [16, 73], the changed distribution function near the point (x, y, z) in the time interval Δt from t_0 to $t_0+\Delta t$ is described by $\Phi(x+\Delta x, y+\Delta y, z+\Delta z, t_0+\Delta t)$. Since this function describes a physical process of gravitational contraction, then it is differentiable, and therefore it can be expanded into a Taylor series in a vicinity of the point (x, y, z, t_0) [16, 65, 73]:

$$\Phi(x+\Delta x, y+\Delta y, z+\Delta z, t_0+\Delta t) = \Phi(x, y, z, t_0) + \frac{\partial \Phi}{\partial x}\Big|_{(x,y,z,t_0)} \cdot \Delta x +$$
$$+\frac{\partial \Phi}{\partial y}\Big|_{(x,y,z,t_0)} \cdot \Delta y + \frac{\partial \Phi}{\partial z}\Big|_{(x,y,z,t_0)} \cdot \Delta z + \frac{\partial^2 \Phi}{\partial x^2}\Big|_{(x,y,z,t_0)} \cdot (\Delta x)^2 + \frac{\partial^2 \Phi}{\partial y^2}\Big|_{(x,y,z,t_0)} \cdot (\Delta y)^2 +$$
$$+\frac{\partial^2 \Phi}{\partial z^2}\Big|_{(x,y,z,t_0)} \cdot (\Delta z)^2 + \ldots + \frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} \cdot \Delta t + \ldots \tag{1.7.2}$$

For the convenience of further considerations, let us put the point (x, y, z) to the origin of a frame of reference and investigate the space-time behavior of Φ in the case of very small distances $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ from the origin of coordinates. Due to the homogeneity and isotropy of space, we suppose that $(\Delta x)^2 \approx (\Delta y)^2 \approx (\Delta z)^2$, i.e. $(\Delta x)^2 \approx \frac{1}{3}(\Delta r)^2$. Taking into account these assumptions, the relation (1.7.2) becomes [16, 73]:

$$\Phi(x+\Delta r/\sqrt{3}, y+\Delta r/\sqrt{3}, z+\Delta r/\sqrt{3}, t_0+\Delta t) - \Phi(x,y,z,t_0) =$$
$$= \frac{\Delta r}{\sqrt{3}} \cdot \left(\frac{\partial \Phi}{\partial x}\Big|_{(x,y,z,t_0)} + \frac{\partial \Phi}{\partial y}\Big|_{(x,y,z,t_0)} + \frac{\partial \Phi}{\partial z}\Big|_{(x,y,z,t_0} \right) + \frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} \cdot \Delta t + \quad (1.7.3)$$
$$+ \frac{1}{3}(\Delta r)^2 \left(\frac{\partial^2 \Phi}{\partial x^2}\Big|_{(x,y,z,t_0)} + \frac{\partial^2 \Phi}{\partial y^2}\Big|_{(x,y,z,t_0)} + \frac{\partial^2 \Phi}{\partial z^2}\Big|_{(x,y,z,t_0)} \right) + \ldots$$

Due to the condition of existence of a point of equilibrium, the function Φ has an extremum in the origin of coordinates, therefore:

$$\frac{\partial \Phi}{\partial x}\Big|_{(x,y,z,t_0)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y,z,t_0)} = \frac{\partial \Phi}{\partial z}\Big|_{(x,y,z,t_0)} = 0, \quad (1.7.4)$$

and, consequently, the relation (1.7.3) takes the form [16, 65, 73]:

$$\Phi(x+\Delta r/\sqrt{3}, y+\Delta r/\sqrt{3}, z+\Delta r/\sqrt{3}, t_0+\Delta t) - \Phi(x,y,z,t_0) =$$
$$= \Delta t \frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} + \frac{1}{3}(\Delta r)^2 \nabla^2 \Phi\Big|_{(x,y,z,t_0)} + \ldots \quad (1.7.5)$$

As far as $(\Delta r)^2 << 1$ and $\Delta t << 1$, we are restricted only by terms of the first and second order of smallness concerning Δt and Δr on the right-hand side of Eq. (1.7.5). Moreover, since the same ensemble of particles is considered, then according to Eq. (1.7.1) we find that:

$$\Phi(x+\Delta r/\sqrt{3}\,, y+\Delta r/\sqrt{3}\,, z+\Delta r/\sqrt{3}\,, t_0+\Delta t) = \\ = \Phi(x, y, z, t_0)\,. \qquad (1.7.6)$$

Taking into account Eq. (1.7.6) the relation (1.7.5) becomes:

$$\frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} + \frac{1}{3}\cdot\frac{(\Delta r)^2}{\Delta t}\nabla^2\Phi\Big|_{(x,y,z,t_0)} \approx 0\,. \qquad (1.7.7)$$

Strict equality in (1.7.7) is attained when $\Delta r \to 0$ and $\Delta t \to 0$:

$$\frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} + \lim_{\substack{\Delta r \to 0 \\ \Delta t \to 0}} \frac{(\Delta r)^2}{3\Delta t}\nabla^2\Phi\Big|_{(x,y,z,t_0)} = 0\,. \qquad (1.7.8)$$

If there exists the limit in Eq. (1.7.8) then we need to find it [16, 73]:

$$\lim_{\substack{\Delta r \to 0 \\ \Delta t \to 0}} \frac{(\Delta r)^2}{3\Delta t} = \mathrm{G}\,, \qquad (1.7.9)$$

and considering formally $\lim\limits_{\substack{\Delta r \to 0 \\ \Delta t \to 0}} \frac{\Delta r}{\Delta t} = \bar{v}$ as a certain velocity of *microscopic* movement of particles, we can write (1.7.9) as follows [16, 73]:

$$\mathrm{G} = \frac{1}{3}\bar{v}\Delta r\,. \qquad (1.7.10)$$

Let us note that the physical meaning of the *gravitational contraction coefficient* G (1.7.10) will be clarified later (see Chapter 4). Taking into account (1.7.9), (1.7.10), the equation (1.7.8) becomes the following:

$$\frac{\partial \Phi}{\partial t}\Big|_{(x,y,z,t_0)} = -\mathrm{G}\nabla^2\Phi\Big|_{(x,y,z,t_0)}\,. \qquad (1.7.11)$$

Because of the arbitrariness of the choice of the point of origin of coordinates and the reference time, it is obvious that equation (1.7.11) is valid not only for a fixed point (x, y, z, t_0) but also for any point (x, y, z, t) [16, 47, 65, 73]:

$$\frac{\partial\Phi}{\partial t} = -G\nabla^2\Phi\,. \tag{1.7.12}$$

In this regard, without loss of generality, we consider Eq. (1.7.12) simply as an *anti-diffusion equation* or an evolutional equation of a *slow-flowing gravitational condensation* (tightening) [16, 47, 48, 65, 73]. The evolutional equation (1.7.12) describing the Gauss–Markov process results from the Chapman–Kolmogorov equation in its limiting case when, following I.Prigogine [134], "between neighboring points there occur very quick jumps ($\Delta t \to 0$) for very small distances ($\Delta r \to 0$)", and the coefficient G being defined by Eq. (1.7.9).

Now let us consider a moving (for example, rotating) molecular cloud in its gravitational field which is in a state of *relative* mechanical equilibrium. In this case, the probability density Φ of particle detection in this molecular is a function of not only the spatial coordinates x, y, z and time t but also the velocity projections u, υ, w in the Cartesian coordinate system, respectively, i.e. $\Phi = \Phi(x, y, z, u, \upsilon, w, t)$. The origin of a gravitational field with a potential φ_g and strength $\vec{a} = -\nabla\varphi_g$ as well as the velocity $\vec{\mathrm{v}} = (u, \upsilon, w)$ and, respectively, momentum $\vec{p} = m_0\,\vec{\mathrm{v}}$ or angular momentum $\vec{L} = [\vec{r} \times \vec{p}]$) of particles leads to reducing the role of slow-flowing anti-diffusion processes in gravitational condensation entirely (since processes in the gravitational field result quickly enough). In this connection, the time evolution of the distribution function $\Phi(x, y, z, u, \upsilon, w, t)$ from the state of relative mechanical equilibrium should be considered at infinitely small intervals dt. In other words, instead of Eq. (1.7.2) we consider the expansion of a function $\Phi(x, y, z, u, \upsilon, w, t)$ in a Taylor series of the form:

$$\Phi(x+udt, y+\upsilon dt, z+wdt, u-\frac{\partial\varphi_g}{\partial x}dt, \upsilon-\frac{\partial\varphi_g}{\partial y}dt, w-\frac{\partial\varphi_g}{\partial z}dt, t+dt)=$$

$$=\Phi(x,y,z,u,\upsilon,w,t)+\frac{\partial\Phi}{\partial x}udt+\frac{\partial\Phi}{\partial y}\upsilon dt+\frac{\partial\Phi}{\partial z}wdt-$$

$$-\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\varphi_g}{\partial x}dt-\frac{\partial\Phi}{\partial \upsilon}\cdot\frac{\partial\varphi_g}{\partial y}dt-\frac{\partial\Phi}{\partial w}\cdot\frac{\partial\varphi_g}{\partial z}dt+\frac{\partial\Phi}{\partial t}dt+... \qquad (1.7.13)$$

Since the same ensemble of particles is considered all the time, then by analogy with Eq. (1.7.1) the following equality is true:

$$N\Phi(x+udt, y+\upsilon dt, z+wdt, u-\frac{\partial\varphi_g}{\partial x}dt, \upsilon-\frac{\partial\varphi_g}{\partial y}dt, w-\frac{\partial\varphi_g}{\partial z}dt,$$

$$t+dt)\cdot dxdydzdud\upsilon dw = N\Phi(x,y,z,u,\upsilon,w,t)\cdot dxdydzdud\upsilon dw\,. \qquad (1.7.14)$$

whence it immediately follows that:

$$\Phi(x+udt, y+\upsilon dt, z+wdt, u-\frac{\partial\varphi_g}{\partial x}dt, \upsilon-\frac{\partial\varphi_g}{\partial y}dt, w-\frac{\partial\varphi_g}{\partial z}dt, t+dt)=$$

$$=\Phi(x,y,z,u,\upsilon,w,t)\,. \qquad (1.7.15)$$

If we are restricted only by the first-order terms of smallness in the series (1.7.13) and take into account Eq. (1.7.15), we can transform Eq. (1.7.13) to the Jeans equation (1.6.6):

$$u\frac{\partial\Phi}{\partial x}+\upsilon\frac{\partial\Phi}{\partial y}+w\frac{\partial\Phi}{\partial z}-\frac{\partial\varphi_g}{\partial x}\cdot\frac{\partial\Phi}{\partial u}-\frac{\partial\varphi_g}{\partial y}\cdot\frac{\partial\Phi}{\partial \upsilon}-\frac{\partial\varphi_g}{\partial z}\cdot\frac{\partial\Phi}{\partial w}+\frac{\partial\Phi}{\partial t}=0 \qquad (1.7.16)$$

Both equations (1.7.12) and (1.7.16) describe *different stages of the evolution* of a molecular cloud. Consequently, there is some more general evolutionary equation [16, 65, 73] which generalizes the derived evolutional equation (1.7.12) and the Jeans equation (1.7.16):

$$\frac{\partial\Phi}{\partial t}+u\frac{\partial\Phi}{\partial x}+\upsilon\frac{\partial\Phi}{\partial y}+w\frac{\partial\Phi}{\partial z}-\frac{\partial\varphi_g}{\partial x}\cdot\frac{\partial\Phi}{\partial u}-$$

$$-\frac{\partial\varphi_g}{\partial y}\cdot\frac{\partial\Phi}{\partial \upsilon}-\frac{\partial\varphi_g}{\partial z}\cdot\frac{\partial\Phi}{\partial w}+\mathrm{G}\nabla^2\Phi=0\,. \qquad (1.7.17)$$

In vector notation, the generalized evolutionary equation (1.7.17) has the form [16, 65, 73]:

$$\frac{\partial \Phi}{\partial t} + \vec{v} \cdot \nabla \Phi - \frac{\partial \Phi}{\partial \vec{v}} \cdot \nabla \varphi_g + G \nabla^2 \Phi = 0 \,, \qquad (1.7.18)$$

where $\vec{v} = (u, \upsilon, w)$. As follows directly from Eq. (1.7.18), in the absence of the particle velocity $(\vec{v} = 0)$ of the molecular cloud, equation (1.7.18) coincides with the evolutional equation (1.7.12), and in the presence of velocity $(\vec{v} \neq 0)$, the Jeans equation (1.7.17) is a special case of Eq. (1.7.18), since the contribution of the second-order term $G\nabla^2\Phi$ on the left-hand side of this evolutionary equation is negligible compared to first-order terms $\vec{v} \cdot \nabla \Phi$ and $\frac{\partial \Phi}{\partial \vec{v}} \cdot \nabla \varphi_g$, i.e.

$$\left| G \nabla^2 \Phi \right| << \left| \vec{v} \cdot \nabla \Phi - \frac{\partial \Phi}{\partial \vec{v}} \cdot \nabla \varphi_g \right| .$$

Conclusion and comments

This chapter is devoted to a review of the problems of the origin of the initial gravitational condensation (tightening) of spread cosmic matter and finding a way to their possible solution. Sections 1.1–1.6 described the main problems of the theory of gravitational condensation and the theory of gravitational instability applied to the molecular (gas-dust) cloud, namely:

- the problem of the formation of a center of spread cosmic matter under its initial gravitational condensation;
- the absence of a clear mechanism of gravitational condensation of infinitely distributed matter, including the source of origin and supporting waves of gravitational instability;

- the absence of analytical expressions of gravitational potential and force of gravity for an infinitely distributed homogeneous rest medium (including the mechanism for the origin of a gravitational field in such media);
- the impossibility of applying exclusively the deterministic approach to a correct description of the behavior of a large number of particles of a gravitating molecular cloud;
- the impossibility of finding a general, and not a particular solution of the Jeans equation due to the difficulty of determining the analytical expression for the gravitational potential of molecular cloud;
- the infinite value of the mass density at the periphery of a rotating molecular cloud according to the Jeans theory.

In Section 1.7, the *anti-diffusion equation* (1.7.12) was derived, that is, an evolutional equation of a *slow-flowing gravitational condensation* (tightening) of a molecular cloud in a state of unstable mechanical equilibrium [16, 47, 48, 65, 73]. Assuming that the evolution of a molecular (gas-dust) cloud is due to the extremely slow process of gravitational tightening of local domains of spread matter (for example, parts of a molecular cloud) [16], the anti-diffusion equation (1.7.12) describes the *initial (first) stage of its evolution*. As shown in Section 1.7, the *second stage of the evolution* of a molecular (gas-dust) cloud, being in a state of *relative* mechanical equilibrium in its gravitational field, obeys the Jeans equation (1.7.16) [1 p. 366–348, 93]. Thus, both Eq. (1.7.12) and Eq. (1.7.16) are evolutionary equations describing different stages of the evolution of a molecular cloud.

In this regard, many of the above-mentioned problems of the theory of gravitational condensation and the theory of the gravitational instability of the molecular (gas-dust) cloud are removed if we take into account the *initial* evolutional equation (1.7.12) along with the traditional Jeans equation (1.7.16). For this reason, the more general equation (1.7.17) of the evolution of a molecular cloud is proposed in Section 1.7, which generalizes the evolutional equations (1.7.12) and (1.7.16) (in vector notation, the generalized evolutionary equation has the form (1.7.18)) [16, 65, 73]). Obviously, in the absence of velocities $(\vec{v} = 0)$ of movement of particles in a molecular (gas-dust) cloud, Eq. (1.7.18) coincides with the evolutional equation (1.7.12), and in the presence of velocities $(\vec{v} \neq 0)$, the evolutional equation of Jeans (1.7.16) is a special case of Eq. (1.7.18).

Thus, the introduction of the anti-diffusion equation (1.7.12) describing the initial stage of the evolution of a molecular (gas-dust) cloud, gives us a foundation to a systematic and consistent study of statistical models of the gravitational formation of cosmogonical bodies.

CHAPTER TWO

THE STATISTICAL MODEL OF INITIAL GRAVITATIONAL INTERACTIONS OF PARTICLES IN A MOLECULAR CLOUD

This chapter discusses the model of the initial gravitational condensation of an isolated interstellar cloud. As noted in Section 1.5 of Chapter 1, the galactic interstellar cloud complexes represent the concentration of *gas-dust clouds* with a wide variety of masses, densities, and temperatures. Dark clouds have very low temperatures (only $T = 10\,\mathrm{K}$) with a small concentration of particles (about $n = 10^1 \div 10^5$ particles per 1 cm^3) and an insignificant quantity of dust-particles (approximately 1% of the mass of the gaseous substance of a cloud). The masses of such clouds vary from $M_c = M_{\mathrm{Sun}}$ to $M_c = 10^4 M_{\mathrm{Sun}}$ where M_{Sun} is the mass of the Sun. Since molecules of H_2O, CO, and other compounds were found in these clouds they are called *molecular clouds* [10], especially because they are based on molecular hydrogen H_2.

According to astrophysical observations, giant cold molecular clouds are approximately in a state of *virial equilibrium* (see details in Section 1.2), that is, the gravitational binding energy of such a cloud is balanced by the kinetic energy of the molecules forming the cloud. Any perturbation of a molecular cloud can violate this state of equilibrium which ultimately leads to the star formation process. As noted in Section 1.5, examples of such disturbances are spiral density waves inside galaxies, shock waves from supernova explosions, as well as a cloud coming nearer to or colliding with another molecular cloud. However, apart from their dependence on the type of perturbation source in the case of the large intensity of perturbation, the forces of gravitational interactions can be greater than the forces due to thermal kinetic energy inside some part of a molecular cloud (see the definition of *Jeans' critical mass* (1.5.26), (1.5.28), and respective comments). As a result of gas compression of a giant molecular cloud, a *protostar* is formed in the interstellar medium. The protostellar phase is an early stage in the process of star formation beginning with the formation of a condensed core in a molecular cloud.

In Chapter 1, the numerous difficulties for theoretical justification of the initial gravitational contraction of cosmic matter were noted, especially in what concerns the impossibility of the gravitational potential finding for an infinitely distributed homogeneous medium as well as the lack of a clear understanding of the nature of emergence of the gravitational instability waves in a molecular cloud. Nevertheless, Section 1.7 showed a possible way to resolve these difficulties and contradictions using the *slowly evolving (anti-diffusion) gravitational condensation model* (see Eq. (1.7.12) and Ref. [16, 47, 65–68, 73]). In this regard, such a scenario is entirely possible when the anti-diffusion condensation of a part of a molecular cloud predetermines the beginning of its further gravitational compression forming a *new body* from an infinitely distributed medium and, thereby, eliminating contradictions relative to the definition of gravitational potential based on the Poisson equation (I.2) (see also Eq. (1.1.41) in Section 1.1).

Thus, the aim of this chapter is to develop a statistical model of the *initial gravitational interaction* of particles in a molecular cloud, in particular, to find the distribution function of a great number of interacting particles in space [45, 46].

Really, since the discovery by Newton of the Universal Gravitation Law [80], and later the creation by Einstein of general relativity (GR) [81, 168, 169] the interest in this area of research has not lessened, as demonstrated by the great number of works dedicated to it. In spite of the considerable successes of GR, the nature of the gravitational interaction has not been completely revealed, especially in what concerns the quantum theory of gravitation. As Stephen Hawking said in [170 p.199]:

> I think it would be fair to say that we do not yet have a fully satisfactory and consistent quantum theory of gravity.

As a consequence, in the 1960s and 1990s, alternative models of the gravitation theory were proposed (e.g. the Brance–Dicke theory [171], the Logunov–Mestvirishvili relativistic theory of gravity [172–174], the Nicolis–Prigogine cosmological model [135], the Nottale scale relativistic theory [175, 176], and others). The area of research within the framework of the developed statistical theory of the formation of cosmogonical bodies [16, 45–71] includes the Newtonian theory of gravity and particularly the Newtonian quantum theory of gravity to investigate an isolated cold molecular cloud.

2.1. The derivation of a function of particle distribution in space based on the statistical model of a molecular cloud

Let us consider the main statements of the statistical theory of gravity [16, 45–71] beginning from the derivation of the distribution function of particles in a space filled in a homogeneous and isotropic dust-gaseous nebula representing a *molecular cloud*. In other words, the question is about the distribution of molecules in space. The statistical aspect of the problem results from the fact that the body being considered, consisting of the gaseous matter, is a system containing a *large number* of molecules (or atoms) interacting among themselves by oscillation in a cosmic vacuum. In microphysics, the cosmic vacuum represents a ground energetic state of quantum fields, and its experimental manifestation is the Casimir effect [103 p. 1154]. Similar oscillations modifying forms of molecule/atom trajectories have been considered by Nelson [34, 35] and later on by Nottale [175, 176]. We can, therefore, comment on their *local oscillatory interactions*.

So, let us consider an immovable molecular cloud as a system of particles in a state of *unstable mechanical equilibrium* at the initial time moment $t = t_0$. Numerous fluctuations of particle concentration caused by their local oscillations do not allow us to predict with certainty the behavior of the system as a whole. We represent the molecular cloud as a gaseous body satisfying the following assumptions [16, 45–47, 73]:

1. The gaseous body is considered in a homogeneous and isotropic space.
2. The gaseous body under consideration is homogeneous in its chemical structure, that is, it consists of N identical particles with the same mass m_0.
3. The gaseous body is isolated, that is, it is not subject to the influence of external fields or other bodies.
4. The gaseous body is isothermal and has a low temperature (as a rule $T \propto 10\,\mathrm{K}$), besides $T_{\deg} < T$ where $T_{\deg} = (h^2 / m_0 k_B) n^{2/3}$ is a degeneration temperature [110], n is a concentration of particles, h is the Planck constant, k_B is the Boltzmann constant.
5. The initial process of the oscillating interaction of particles is slow flowing with time.

Thus, all directions in space are considered to be equally valid, that is, an isotropic space is dealt with. A gaseous body consisting of N similar particles of mass m_0 is placed within it. Inside it, let us choose some spatial coordinates and a direction in space. If we choose the finite solid angle dO we can state that the number of particles sited at finite distances in the direction within dO is equal to

$$\frac{dN}{N} = \frac{dO}{4\pi}, \tag{2.1.1}$$

hence

$$dN = N\frac{dO}{4\pi}. \tag{2.1.2}$$

Formula (2.1.2) states the particle *distribution in the direction* $\vec{r}$ in space. To calculate the solid angle in (2.1.2) we consider the direction of the angles within $[\theta, \theta + d\theta]$ and $[\varepsilon, \varepsilon + d\varepsilon]$ into a sphere of radius r. Then, according to Fig. 2.1, the solid angle "cuts" on the sphere the area element $dS = r^2 \sin\theta d\theta d\varepsilon$ whence

$$dO = \frac{dS}{r^2} = \sin\theta d\theta d\varepsilon, \tag{2.1.3}$$

where θ is an azimuth angle and ε is a polar angle. Substituting (2.1.3) in (2.1.2) we obtain the number of particles the radius-vectors of which have the directions close to the given direction [136]:

$$dN_{\theta,\varepsilon} = N\frac{\sin\theta d\theta d\varepsilon}{4\pi}. \tag{2.1.4}$$

Now we are interested in the following: how many particles have radius-vectors in a certain interval near the given radius-vector $\vec{r}$? This question is about the distribution of particles in space.

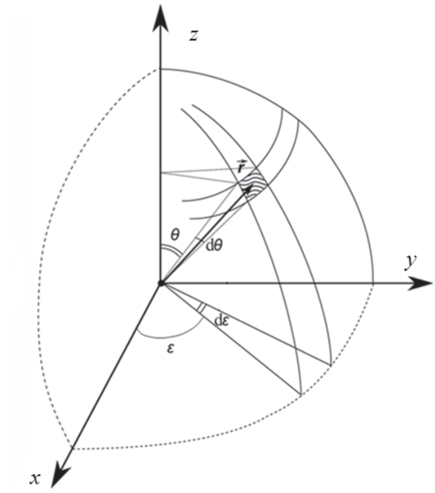


Figure 2.1. The scheme of calculating the solid angle describing direction inside azimuth and polar angular intervals

The statistical aspect of this problem results from the fact that numerous fluctuations of particle concentration caused by their local interactions do not allow us to predict with certainty the behavior of this system. Therefore, we can develop a statistical model similar to the Maxwell velocity distribution of particles [110, 136] for describing the behavior of the system consisting of N particles of mass m_0.

Let a radius-vector $\vec{r}$ with coordinates (x, y, z) be chosen in a three-dimensional space. If in the gaseous body there are N particles, then dN_x of them have coordinates in the interval $[x, x+dx]$, and dN_y and dN_z have coordinates in the intervals $[y, y+dy]$ and $[z, z+dz]$ respectively at a given instant of time. The probabilities of any particle having coordinates in the above-mentioned intervals are equal to:

$$dp_x = \frac{dN_x}{N} = \varphi(x)dx\,;\ dp_y = \frac{dN_y}{N} = \xi(y)dy\,;$$

$$dp_z = \frac{dN_z}{N} = \zeta(z)dz\,, \tag{2.1.5}$$

where $\varphi(x)$, $\xi(y)$ and $\zeta(z)$ are one-dimensional probability densities, that is, shares of particles whose coordinates belong to the elementary intervals close to x, y, and z respectively [16, 45, 46].

Let us introduce a three-dimensional distribution characteristic, that is, a volume density of probability:

$$dp_{x,y,z} = \frac{dN_{x,y,z}}{N} = \Phi(x,y,z)dxdydz\,. \tag{2.1.6}$$

Taking into account the condition of homogeneous and isotropic space with a gaseous body, the volume density of probability has to depend on the value of $r = |\vec{r}|$ only, that is, $\Phi(x,y,z) = \Phi(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$. Hence, Eq. (2.1.6) takes the form [45, 46]:

$$dp_{x,y,z} = \Phi(r)dxdydz\,. \tag{2.1.7}$$

On the other hand, a particle has all the three given coordinates independent of each other. Then, according to the theorem of complex event probability we have:

$$dp_{x,y,z} = dp_x dp_y dp_z = \varphi(x)\xi(y)\zeta(z)dxdydz\,. \tag{2.1.8}$$

Comparing Eq. (2.1.7) with Eq. (2.1.8), we obtain the factorization rule for a *probability volume density function* [45, 46]:

$$\varphi(x)\xi(y)\zeta(z) = \Phi(r)\,. \tag{2.1.9}$$

Proceeding from Eq. (2.1.9) one can define φ, ξ, ζ, and Φ analogously to the scheme formulated in deducing the Maxwell molecule velocity distribution [110, 136]. Indeed, differentiating $\Phi(r)$ as a composite function (with respect to x) we represent the functional equation (2.1.9) as a differential one [16, 45]:

$$\varphi'(x)\xi(y)\zeta(z) = \Phi'(r)\frac{\partial r}{\partial x}. \qquad (2.1.10)$$

It is not difficult to see that

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} 2x = \frac{x}{r}. \qquad (2.1.11)$$

With provision for Eq. (2.1.11) let us divide Eq. (2.1.10) by Eq. (2.1.9):

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{\Phi'(r)}{\Phi(r)} \cdot \frac{x}{r},$$

whence we obtain:

$$\frac{\varphi'(x)}{x\varphi(x)} = \frac{\Phi'(r)}{r\Phi(r)}. \qquad (2.1.12a)$$

Similarly, when differentiating $\Phi(r)$ relative to y and z one can write down that

$$\frac{\xi'(y)}{y\xi(y)} = \frac{\Phi'(r)}{r\Phi(r)}; \qquad (2.1.12b)$$

$$\frac{\zeta'(z)}{z\zeta(z)} = \frac{\Phi'(r)}{r\Phi(r)}. \qquad (2.1.12c)$$

Since in Eqs (2.1.12a)–(2.1.12c) the right-hand parts are the same, the left-hand ones are equal to each other:

$$\frac{\varphi'(x)}{x\varphi(x)} = \frac{\xi'(y)}{y\xi(y)} = \frac{\zeta'(z)}{z\zeta(z)} = -\alpha.$$

These equalities persist when α is some constant (its physical sense is to be defined below). Hence, for example, it follows that

$$\frac{\varphi'(x)}{\varphi(x)} = -\alpha x.$$

Integrating both parts of this equality we obtain the form of a function φ satisfying Eq. (2.1.9):

$$\varphi(x) = Ce^{-\alpha x^2/2},$$

where C is an integration constant. Similar expressions can be written down for $\xi(y)$ and $\zeta(z)$. The normalization condition of the probability density $\int_{-\infty}^{\infty}\varphi(x)dx = 1$ results in the integral convergence which only fulfills with $\alpha > 0$. Moreover, the parameter $\alpha > 0$ because from physical reasoning it is clear that under the increase of x the share of particles is to decrease due to gravitation in accord with the formula obtained. Using the normalization condition one can find the integration constant $C = (\alpha / 2\pi)^{1/2}$ as well as the desired function:

$$\varphi(x) = \sqrt{\frac{\alpha}{2\pi}} \cdot e^{\frac{-\alpha x^2}{2}}. \quad (2.1.13)$$

Since the expressions for $\xi(y)$ and $\zeta(z)$ have a form similar to (2.1.13), it is not difficult, according to (2.1.9), to write down an expression for the probability volume density function [45, 46]:

$$\Phi(r) = \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{\frac{-\alpha}{2}r^2}. \quad (2.1.14)$$

Thus, according to Eqs (2.1.14) and (2.1.6) the probability of a particle having coordinates in the intervals close to x, y, z is equal to:

$$dp_{x,y,z} = \frac{dN_{x,y,z}}{N} = \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} dxdydz. \quad (2.1.15)$$

Now let us find the probability of a value r being confined between r and $r + dr$ for any particle:

$$dp_r = \frac{dN_r}{N} = f(r)dr, \quad (2.1.16)$$

where $f(r)$ is to be a probability density. In spherical coordinates, the volume of the spherical layer at distance r

from the center of coordinates (see Fig. 2.1) is equal to $4\pi r^2 dr$. It is not difficult to see from Eq. (2.1.16) that

$$dp_r = \Phi(r) \cdot 4\pi r^2 dr. \qquad (2.1.17)$$

It follows from Eqs (2.1.14), (2.1.16), and (2.1.17) that the share of particles being at distances close to r is equal to

$$f(r) = \Phi(r) 4\pi r^2 = 4\pi \left(\frac{\alpha}{2\pi} \right)^{3/2} e^{-\frac{\alpha}{2} r^2} r^2. \qquad (2.1.18)$$

According to (2.1.16) and (2.1.18), the share of particles being at distances close to r is equal to

$$\frac{dN_r}{N} = 4\pi \left(\frac{\alpha}{2\pi} \right)^{3/2} e^{-\frac{\alpha}{2} r^2} r^2 dr. \qquad (2.1.19)$$

We note this relationship, in its form, resembles completely the Maxwell molecule velocity distribution law if in Eq. (2.1.19) r is replaced by v.

As follows from Eqs (2.1.4), (2.1.19), and the complex event probability theorem, the share of particles being at distances close to r at a solid angle close to O is equal to [45, 46]:

$$\frac{dN_{r,\theta,\varepsilon}}{N} = \frac{dN_r}{N} \cdot \frac{dN_{\theta,\varepsilon}}{N} =$$

$$= 4\pi \left(\frac{\alpha}{2\pi} \right)^{3/2} e^{-\frac{\alpha}{2} r^2} r^2 dr \cdot \frac{1}{4\pi} \sin\theta \, d\theta \, d\varepsilon =$$

$$= \left(\frac{\alpha}{2\pi} \right)^{3/2} e^{-\frac{\alpha}{2} r^2} r^2 dr \sin\theta d\theta d\varepsilon. \qquad (2.1.20)$$

Relationship (2.1.20) describes the distribution of particles according to the distance from the center and to the direction in space in the spherical coordinates r, θ and ε.

Thus, following Eqs (2.1.18) and (2.1.19), the share of particles at distances $r_1 \le r \le r_2$ from the center is equal to:

$$\frac{N_{r_1 \le r \le r_2}}{N} = \int_{r_1}^{r_2} f(r)dr. \qquad (2.1.21)$$

To reveal the character of distribution we shall investigate the function $f(r)$.

From the normalization condition, it follows that the area under the curve $f(r)$ is finite. According to (2.1.18) at small values of r this function $f(r)$ increases, whereas at large $r \to \infty$ it diminishes abruptly. Consequently, $f(r)$ has a maximum (see Fig. 2.2) when

$$\frac{df(r)}{dr} = 4\pi\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2}(-\alpha r^3 + 2r) = 0,$$

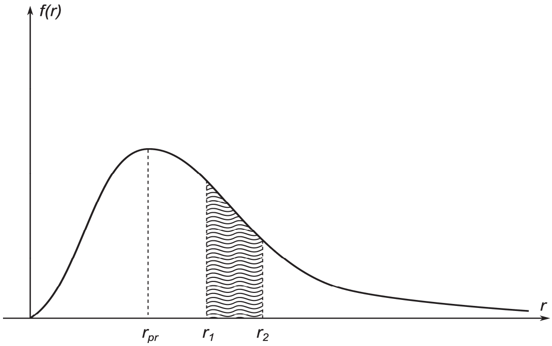


Figure 2.2. The probability density function of finding particles positioned at distances r from the center of masses

whence it follows that the *most probable distance* (where there is the greatest number of particles in its neighborhood) is determined by the formula [45]:

$$r_{pr} = \sqrt{\frac{2}{\alpha}}. \qquad (2.1.22)$$

On the other hand, from (2.1.22) it is not difficult to express the parameter α:

$$\alpha = \frac{2}{r_{pr}^2}. \quad (2.1.23)$$

Taking into account (2.1.23) the probability density can be written as follows:

$$f(r) = \frac{4}{\sqrt{\pi}} \cdot \frac{r^2}{r_{pr}^3} \cdot e^{-r^2/r_{pr}^2}. \quad (2.1.24)$$

Now we can also calculate the *average distance* of particles from the origin of coordinates:

$$\bar{r} = \frac{\int_0^\infty r dN_r}{N} = \int_0^\infty r f(r) dr. \quad (2.1.25)$$

Substituting relationship (2.1.19) into (2.1.25) we obtain:

$$\bar{r} = \frac{4}{\sqrt{\pi}} \cdot \left(\frac{\alpha}{2}\right)^{3/2} \int_0^\infty r^3 e^{\frac{-\alpha}{2} r^2} dr. \quad (2.1.26)$$

Performing the substitution $r = \left(\frac{2}{\alpha}\right)^{1/2} s$ we can transform (2.1.26) to the form [45]:

$$\bar{r} = \frac{4}{\sqrt{\pi}} \cdot \left(\frac{2}{\alpha}\right)^{1/2} \int_0^\infty s^3 e^{-s^2} ds = 2\sqrt{\frac{2}{\pi\alpha}}. \quad (2.1.27)$$

Comparing (2.1.27) with (2.1.22) one can see that

$$\bar{r} = \frac{2}{\sqrt{\pi}} r_{pr}. \quad (2.1.28)$$

It follows from (2.1.28) that the average distance is $\frac{2}{\sqrt{\pi}}$ times larger than the most probable one. Let us calculate the *root-mean-square distance* [45]:

$$r_{sq} = \sqrt{\overline{r^2}} = \left(\int_0^\infty r^2 f(r) dr\right)^{1/2} = \left(\frac{4}{\sqrt{\pi}} \cdot \left(\frac{\alpha}{2}\right)^{3/2} \int_0^\infty r^4 e^{-\frac{\alpha}{2} r^2} dr\right)^{1/2} = \sqrt{\frac{3}{\alpha}}. \quad (2.1.29)$$

From the comparison of (2.1.22), (2.1.27), and (2.1.29) it is not difficult to see that $r_{sq} > \bar{r} > r_{pr}$. However, the main conclusion from the foregoing is the fact that particles, under the influence of their interactions, concentrate at the distance $\frac{\text{const}}{\sqrt{\alpha}}$ from the center of masses, that is, into the volume $V_{pr} = \frac{4}{3}\pi\left(\frac{2}{\alpha}\right)^{3/2}$.

2.2. The distribution of mass density as a result of the initial gravitational interaction of particles in a molecular cloud

Let us consider the relation (2.1.20) describing the distribution density of particles which are at distances close to r with angles close to ε and θ. Taking into account that an elementary volume in spherical coordinates is $dV = r^2 \sin\theta d\theta d\varepsilon dr$ we can transform (2.1.20) into:

$$\frac{dN_{r,\theta,\varepsilon}}{N} = \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} dV. \tag{2.2.1}$$

Now let us rewrite Eq. (2.2.1) as follows:

$$\frac{dN_{r,\theta,\varepsilon}}{dV} = N\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2}. \tag{2.2.2}$$

The value $dN_{r,\theta,\varepsilon} / dV = n_{r,\theta,\varepsilon}$ is a *local concentration* of particles near a point with coordinates (r,θ,ε). Considering this we have:

$$n(r) = N\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2}. \tag{2.2.3}$$

As seen from (2.2.3), the concentration of particles does not depend on directions in space characterized by angles θ and ε but changes depending on distance r only.

If all the particles are alike and have mass m_0 (see the assumption 2) then, by multiplying both sides of relation (2.2.3) by m_0, one obtains [45, 46]:

$$\rho(r) = m_0 N \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} = M\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2}, \qquad (2.2.4)$$

where $\rho(r) = m_0 n(r)$ is a mass density of substance consisting of the particles and $M = m_0 N$ is a mass of the gaseous body composed of these particles. By denoting $\rho_0 = M(\alpha/2\pi)^{3/2}$ the expression (2.2.4) is written down as follows:

$$\rho(r) = \rho_0 e^{-\frac{\alpha}{2}r^2}, \qquad (2.2.5)$$

where α is the abovementioned positive parameter, and the value ρ_0 is written as $\rho_0 = \dfrac{M}{V_0}$ when $V_0 = (2\pi/\alpha)^{3/2} = \left(\sqrt{2\pi/\alpha}\right)^3$ or concerning (2.1.22) ρ_0 can be rewritten down in the form $\rho_0 = \dfrac{4}{3\sqrt{\pi}} \cdot \dfrac{M}{V_{pr}}$ with $V_{pr} = \dfrac{4}{3}\pi r_{pr}{}^3$. It is evident from (2.2.5) that $\rho(r)$ is a diminishing function with a maximum in the point $r = 0$ (Fig. 2.3).

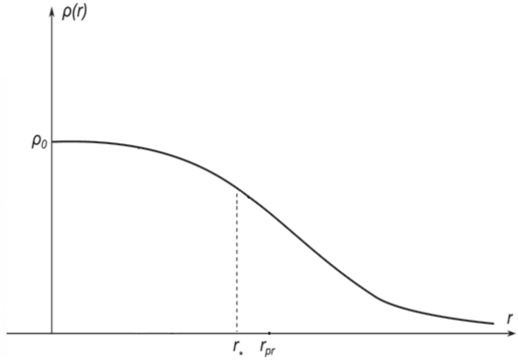


Figure 2.3. The mass density of a gaseous body as a function of distance r

Indeed, from the equation $\frac{d^2\rho(r)}{dr^2} = -\alpha\rho_0 \, e^{-\frac{\alpha}{2}r^2}\left(1-\alpha r^2\right) = 0$ one finds the *point of inflection* of the function $\rho(r)$ [46]:

$$r_* = \frac{1}{\sqrt{\alpha}} = \frac{r_{pr}}{\sqrt{2}}. \tag{2.2.6}$$

As appears from the relation (2.2.5) and Fig. 2.3, under the influence of oscillatory interactions of particles, there arises a substance mass density inhomogeneous along the radial coordinate r. The greatest mass density is concentrated in the interval $[0, r_*]$, where $r_* = 1/\sqrt{\alpha}$ is a point of the mass density bending, outside of which it decreases quickly (Fig. 2.3). For example, at the distance $r = 2r_{pr}$ from the center the mass density decreases in comparison with that in the center $\rho_0 / \rho(2r_{pr}) = e^4 \approx 55$ times. This means that at distances greater than $2r_{pr}$ the density of particles is insignificant.

In such a way, under the action of their own oscillatory interactions as well as originating gravitational forces, a great number of particles are forming a *sphere-like gaseous body* (see Fig. 2.4) whose mass density is uniform in all directions at the same distance from the center of the mass [45, 46]. In this connection, we interpret local oscillatory particle interactions as the *initial gravitational interactions of particles*.

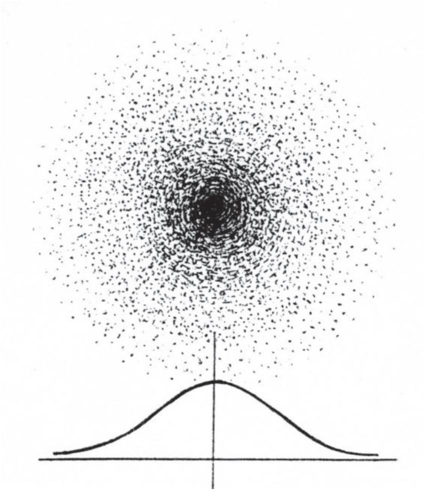


Figure 2.4. The graphic representation of a sphere-like gaseous body formed by a collection of interacting particles

There is a *critical (threshold) value* α_c that if $\alpha \geq \alpha_c$ then a gravitational field arises in the sphere-like gaseous body [16, 59, 65]. Because a mass density value strictly depends on α in the formula (2.2.5) this positive parameter defines a measure of gravitational interactions of particles in a gaseous body. It is, therefore, called *the parameter of gravitational condensation* [16, 47]. As also follows from (2.2.5), the isosurface of mass density (isostere) for such a gaseous cloud is a sphere. Let us note that because of $r_* \sim R$, where R is the mean radius of a gaseous body, then $\alpha \sim 1/R^2$ is a very small positive parameter of gravitational condensation. As a mean radius R of a forming sphere-like gaseous body can also be choosing the most probable distance $r_{pr} = \sqrt{2} \cdot r_*$ close to which there is the greatest number of particles [45, 46, 73].

Thus, we can speak of the process of the gravitational forming of a sphere-like gaseous body whose core is nearly a solid $(r < r_*)$ while the shell is gaseous $(r > 2r_{pr})$. Indeed, the snow mass density (200 kg/m^3) is higher than that of water vapor (0.484 kg/m^3) by about 400 times [177]: a similar decrease of density in the case of the sphere-like gaseous body is observed with the increase of distance from its center about $r = \sqrt{6} r_{pr}$ ($\rho_0 / \rho(\sqrt{6} r_{pr}) = e^6 \approx 404$ times).

Comparing Figs. 2.2 and 2.3, we can see that though particles concentrate near the most probable distance r_{pr}, nevertheless, the maximum mass density is at $r = 0$. In fact, by associating formulas (2.1.14) and (2.2.4) one can easily see that

$$\rho(r) = M \Phi(r), \tag{2.2.7}$$

where $\Phi(r)$ is a volume probability density function having a maximum at $r = 0$ (like the function $\rho(r)$). Moreover, from the normalization condition of the volume probability density:

$$\int_V \Phi(r) dV = 1, \tag{2.2.8}$$

it follows that formula (2.2.7) can be transformed into the well-known relation:

$$\int_V \rho(r) dV = M \int_V \Phi(r) dV = M .$$

Let us note that within the framework of the developed statistical model of initial gravitational particle interaction this work considers both a gaseous *protostellar* (in particular, presolar) *nebula* (a molecular cloud) and a *gas-dust protoplanetary cloud* as gravitating sphere-like gaseous bodies with *different masses and sizes* respectively [16]. If we start from the conception for forming a sphere-like gaseous body as the protostar (in particular, the proto-sun) inside a protostellar (presolar) nebula then the derived distribution function of particles, as well as the mass density of this immovable gaseous body, characterizes the first stage of evolution: from a protostellar molecular cloud (the presolar nebula) to a forming core of protostar (the proto-sun) together with its shell as a gas-dust protoplanetary cloud (the solar nebula). Here we note again that $r_* \sim R$, where R is a radius of a forming protostar (the proto-sun), and $\alpha \sim 1/R^2$ respectively.

As mentioned by Cameron [10], using radio observations, and later the infrared from IRAS data, there has been in recent years a great deal of progress in understanding how stars are generally formed in dense molecular clouds. In particular, he wrote:

> Detailed studies of dense molecular clouds have shown that spread throughout them, with some tendency for local clustering, are a large number of “cores”, in which *the local gas density is typically about 30 times higher than in the average part of the cloud* (where it may typically be about 10^3 molecules cm^{-3}). ... These cores have masses usually within a factor of a few of about 1 M_{Sun}. [10 p. 443]

Using the proposed statistical model it is not difficult to see that the mass density (2.2.5) of a sphere-like body decreases in $\rho_0 / \rho(r) = 30$ times on a distance $r \approx 2.6 r_*$ from its center (here $r_* = 1/\sqrt{\alpha}$ in accordance with (2.2.6)).

It is interesting to note, under this condition ($r \approx 2.6 r_*$), that Kuiper’s famous hypothesis [28] about the mass of the protoplanetary cloud:

$$M_{\textit{protopl. cloud}} \;\geq\; 0.1 M_{\text{Sun}}$$

is also true. In fact, for a sphere-like body with mass density (2.2.5) from the condition [16]:

$$\int_{V_b} \rho dV = 0.9M \,, \tag{2.2.9}$$

where M is a mass of a protostellar (presolar) molecular cloud, it follows directly the relation:

$$\alpha^{3/2}\sqrt{\frac{2}{\pi}}\int_0^{r_b} r^2 e^{-\alpha r^2/2} dr = 0.9 \,,$$

which after the change of variables $s = r\sqrt{\alpha}$ takes the form [16, 73]:

$$\int_0^{s_b} s^2 e^{-s^2/2} ds = 0.9\sqrt{\pi/2} \,, \tag{2.2.10}$$

where $s_b = r_b\sqrt{\alpha}$ respectively. The numerical calculation of the integral (2.2.10) gives us the value $s_b \approx 2.5$ corresponding to the desired distance $r \approx 2.5r_*$ which satisfies Cameron's condition on the mass density decreasing 30 times. Thus, the derived distribution function of particles, as well as the mass density of a sphere-like gaseous body, describe the first (protostellar) stage of evolution: from a presolar molecular cloud to a forming core (the proto-sun) together with its shell (the solar nebula).

Now let us consider briefly the following question: how is the mass density relation (2.2.5) correlated with other known cosmogonical models of mass density? In this connection, we call our attention to the Jeans discussion of special models (see §228, p.250 in [1]):

> Leaving the realm of general principles, we now turn to a discussion of the behaviour of particular models, conforming to special laws of compressibility. There are, of course, an infinite variety of arrangements of compressible matter possible, while the full discussion of even a single case presents a problem of considerable difficulty and complexity...

Compressibility of matter necessarily results in variations of density in the compressible mass, and the greater the compressibility of the matter, the greater these variations of density will be...

In a sense, this problem formed a limiting case of the problem of the motion of a compressible mass. At the other extreme, there will be another limiting case in which the compressibility is so great that infinite variations of density may be expected. Mathematically this limiting case may be specified by the condition that the density is infinite or zero at different places. Physically, this limiting case proves not to be so artificial as its mathematical specification might lead us to suppose.

In particular, the complete law of density obtained by Darwin and others [1, 178] states:

In a mass of gas at rest with the temperature uniform throughout (isothermal equilibrium), the density at great distances from the center falls off as $1/r^2$ ...:

$$\rho(r) = \rho_0 \frac{a^2}{r^2}, \tag{2.2.11}$$

so that, when viewed from a very great distance, the density may be regarded as infinite at the center and zero everywhere else. The total mass is, however, infinite, so that a finite mass of gas in isothermal equilibrium will be of zero density everywhere. [1]

Similarly, for a mass of gas in *adiabatic equilibrium* with the ratio of the specific heat of gas $\kappa = c_p / c_V$ equal to $1\frac{1}{5}$, the law of density first given by Schuster [1, 179] is:

$$\rho(r) = \rho_0 \frac{1}{\left[1 + r^2/a^2\right]^{5/2}}. \tag{2.2.12}$$

Again, when this mass of gas is viewed from a sufficient distance, the value of ρ becomes infinite at the center and zero everywhere else. The same is true for any value of from 1 to $1\frac{1}{5}$. The mass is infinite when $\kappa < 1\frac{1}{5}$ but becomes finite when $\kappa = 1\frac{1}{5}$. [1]

The relation (2.2.12) is also known as the law of mass density decreasing of “degree 5/2” [106].

This same model, in which the density is infinite or very great over a point or small concentrated area but zero everywhere else, has been largely utilized by Roche in his research on cosmogony [180], and was suitably called “Roche’s model” [1 p.251]:

> Roche interpreted it physically as referring to a small and intensely dense solid nucleus surrounded by an atmosphere of negligible density. In Roche’s model, the whole of the mass is supposed concentrated at the center; in this respect, it differs from a mass of gas in isothermal equilibrium, although giving a faithful representation of an adiabatic mass for which $\kappa = 1\frac{1}{5}$.

Let us show that the proposed law of mass density distribution (2.2.5) generalizes the well-known laws of Darwin, Schuster, and Roche (see Section 1.4). For this we present the parameter of gravitational compression α in the form:

$$\alpha = 2k / a^2, \tag{2.2.13}$$

where k and a^2 are some parameters. Immediate substitution of (2.2.13) into (2.2.5) gives the following formula:

$$\rho(r) = \rho_0 e^{-(k/a^2)r^2} = \frac{\rho_0}{e^{(k/a^2)r^2}} = \frac{\rho_0}{\left[e^{r^2/a^2}\right]^k}. \tag{2.2.14}$$

Taking into account that the parameter $\alpha << 1$ for real cosmogonical bodies following (2.2.6) and, on the contrary, the parameter a^2 is very large concerning (2.2.13): $a^2 >> 1$, we can represent the exponent in the denominator of the ratio (2.2.14) by two terms of the Maclaurin series:

$$e^{r^2/a^2} \approx 1 + \frac{r^2}{a^2}. \tag{2.2.15}$$

Substituting (2.2.15) into (2.2.14) we obtain the following expression for the mass density:

$$\rho(r) = \rho_0 \cdot \frac{1}{\left[1 + r^2 / a^2\right]^k} = \rho_0 \cdot \frac{a^{2k}}{\left[a^2 + r^2\right]^k}. \qquad (2.2.16)$$

The relation (2.2.16) generalizes both the Schuster law (2.2.12) when $k = 5/2$ and the Darwin law (2.2.11) under the condition that $k = 1$ and a new value $R = \sqrt{a^2 + r^2}$ defines the above-mentioned "sufficient distance" [1].

So, the derived formula (2.2.5) of mass density in the particular case gives the law (2.2.16), in turn generalizing the Darwin law and the Schuster law of "degree 5/2". Because of the very fast exponential decay of function in (2.2.5), it is clear that the proposed law of mass distribution in a sphere-like gaseous body also generalizes the well-known Roche model.

Thus, the mass distribution formula (2.2.5) obtained within the framework of statistical theory is just a common one allowing it to be used for analyzing the processes of formation and the evolution of both protostellar molecular clouds (protostars) and gas-dust protoplanetary clouds (as well as protoplanets).

2.3. The critical (threshold) value of mass density and gravitational condensation parameter

According to the Jeans theory (see Section 1.3 in Chapter 1), in a gravitating continuous medium, the speed of a "heavy" sound $\widetilde{c}$ is less than the usual speed of sound $c = \sqrt{dp/d\rho}$ and using (1.3.20) it is expressed by the formula:

$$\widetilde{c} = \sqrt{c^2 - \gamma\rho\lambda^2/\pi}\ . \qquad (2.3.1)$$

Accordingly, using the formulas (1.3.19) and (2.3.1), the frequency $\widetilde{\omega}$ of "heavy" sound is equal to

$$\widetilde{\omega} = (2\pi/\lambda)\cdot\widetilde{c} = \sqrt{(2\pi/\lambda)^2 c^2 - 4\pi\gamma\rho}. \qquad (2.3.2)$$

Taking into account formulas (2.3.1) and (2.3.2), Jeans suggested that with increasing wavelength of perturbations to the value of the *critical wavelength* λ_c (1.3.21) the speed of perturbation propagation $\tilde{c}$ tends to zero as well as the frequency of perturbations $\tilde{\omega}$, and they then become imaginary which leads to an increase of gravitational instability and, as a result, to the gravitational tightening of the gaseous substance [1]. According to (1.3.21) the critical value of the wavelength λ_c (when $\tilde{c}=0$ and $\tilde{\omega}=0$) is determined by the formula:

$$\lambda_c = c\sqrt{\pi / \gamma\rho}\ . \qquad (2.3.3)$$

However, in his theory, Jeans did not answer the two essential questions: Why can the perturbation wavelength increase with the time? What is the source (or mechanism) of the amplification of gravitational perturbations?

On the contrary, the statistical theory of initial gravitational interactions shows that (due to the *initial gravitational interactions* of particles) the mass density of a molecular cloud evolves from the almost uniform distribution law (uniform mass density) to the Gaussian mass distribution law (see Sections 2.1, 2.2 and (2.2.5)):

$$\rho(r) = \rho_0 e^{-\frac{\alpha}{2}r^2}\ , \qquad (2.3.4)$$

where $\rho_0 = M \cdot (\alpha / 2\pi)^{3/2}$ and α is a parameter of gravitational condensation of a forming sphere-like gaseous body that can depend on the time, that is, $\alpha = \alpha(t)$.

Indeed, if in the initial state of a molecular (gas-dust) cloud $\alpha \to 0$, then according to (2.3.4) the mass density becomes homogeneous: $\rho = \rho_0$ as it was originally assumed. Although the mechanism and model of the initial gravitational interaction of particles will be discussed in detail in subsequent chapters of this monograph (see Chapters 4 and 5), here we note that the elementary premises to justify the evolutionary process of a slow-flowing gravitational condensation have been given in Section 1.7 under derivation of the *anti-diffusion equation* (1.7.12) and analyzing evolutionary equations (1.7.17) and (1.7.18) of the statistical mechanics of a molecular cloud (see also [16, 73]).

Thus, unlike the Jeans theory in the framework of the developed theory of initial particle interactions, the main reason for the speed and frequency of disturbances first becoming zero ($\widetilde{c}=0$, $\widetilde{\omega}=0$) and then imaginary is the existence of a *critical (threshold) mass density value* ρ_c [73]. So, due to the evolutionary process of initial gravitational particle interactions in a certain part (usually at the geometric center) of a molecular cloud, the critical mass density value ρ_c is reached when $\widetilde{c}=0$ and $\widetilde{\omega}=0$. According to (2.3.1) and (2.3.2) it means that

$$\gamma\rho_c\lambda^2/\pi = c^2$$

whence the *critical mass density value* immediately follows:

$$\rho_c = \pi c^2/\gamma\lambda^2 . \tag{2.3.5a}$$

Moreover, according to the foregoing, the wavelength of the disturbances λ is assumed to be constant λ_c. In turn, this means that $\lambda_c = 2\pi c/\omega_c$, so that the formula (2.3.5a) takes the form:

$$\rho_c = \frac{\omega_c^2}{4\pi\gamma}. \tag{2.3.5b}$$

Let us note that the relation (2.3.5b) defines the *critical frequency* $\omega_c = 2\sqrt{\pi\gamma\rho_c}$ leading to the gravitational instability in accord with the formula (1.3.25) (compare with (3.1.26)).

Taking into account that the mass density (2.3.4) uniquely depends on the gravitational condensation parameter α it is quite possible to determine the critical value of the parameter α_c when the initial particle interactions within the process of anti-diffusion are *sharply amplified* and replaced by gravitational compression due to an originating gravitational field. Initially, when an anti-diffusion process arises, the parameter α is very small, that is, $\alpha \to 0$. It then increases to a certain critical value α_c (see details in Section 5.2 or works [16, 47, 65]), so that a core of a sphere-like gaseous body with a radius R begins to form. Taking into account that $\alpha_c \sim 1/R^2$ we also find that $\alpha_c << 1$. In this regard, near the center of a sphere-like gaseous body being formed $e^{-\alpha_c r^2/2} \to 1$ at $\alpha_c \to 0$, that is, $\rho_c = \rho_0$, and the formula (2.3.5b) takes the form:

$$M\left(\frac{\alpha_c}{2\pi}\right)^{3/2} = \frac{\omega_c^2}{4\pi\gamma}. \tag{2.3.6}$$

Using the relation (2.3.6) it is easy to estimate the desired *critical value* of the gravitational condensation parameter:

$$\alpha_c \sim \frac{\omega_c^{4/3}}{\gamma^{2/3} M^{2/3}}, \tag{2.3.7a}$$

which we can also express more exactly as follows:

$$\alpha_c = \left(\frac{\pi}{2}\right)^{1/3} \cdot \left(\frac{\omega_c^2}{\gamma M}\right)^{2/3}. \tag{2.3.7b}$$

Thus, during the formation of the core of a sphere-like gaseous body (when the gravitational condensation parameter reaches a certain critical value α_c) wave gravitational perturbations that had previously freely propagated in a gaseous substance cease to propagate due to their deceleration in an increased gravitational field and a partial reflection of these waves from a spherical surface of the isostere of mass density bending. According to the formula (2.2.6) an equation of the critical isostere of mass density bending is the following:

$$R_c = r_*(\alpha_c) = 1/\sqrt{\alpha_c} = (2/\pi)^{1/6} \cdot (\gamma M/\omega_c^2)^{1/3} \sim (\gamma M/\omega_c^2)^{1/3}. \quad (2.3.8)$$

Because the frequency of these perturbations (2.3.2) becomes imaginary it makes it possible to change the wave mode of propagation of initial gravitational disturbances to an aperiodic mode of their amplification [1]. In this connection, a sphere with a radius $\sim R_c$ becomes a peculiar *spherical resonator* of gravitational oscillations [73]. Thus, in the framework of the statistical theory of initial gravitational interactions, the Jeans criterion (see Section 1.3) has acquired a new meaning.

Let us note that the stage of the initial anti-diffusion process can last long enough while an intensity of initial gravitational interactions of particles (characterized by the parameter α) gradually increases up to the value α_c defined by the formula (2.3.7a). Obviously, at the stage of the anti-diffusion process, global coherent (consistent) gravitational interactions of particles do not occur (although local ones can take place). The Newtonian gravitational constant, therefore, γ is assumed to be zero and that, in turn, leads to equalities $\tilde{c} = c$ and $\tilde{\omega} = \omega$ in accordance with the formulas (2.3.1) and (2.3.2).

Using (2.3.1) and (2.3.4) we can estimate a dependence of the speed of "heavy" sound $\tilde{c}$ on the gravitational condensation parameter α for which we calculate the following derivatives:

$$\frac{d\tilde{c}}{d\rho} = \frac{1}{2} \cdot \frac{(-\gamma\lambda^2/\pi)}{\sqrt{c^2 - \gamma\rho\lambda^2/\pi}} = -\frac{\gamma\lambda^2}{2\pi} \cdot \frac{1}{\tilde{c}}; \quad (2.3.9a)$$

$$\frac{d\tilde{c}}{d\alpha} = \frac{d\tilde{c}}{d\rho} \cdot \frac{d\rho}{d\alpha} = -\frac{\gamma\lambda^2}{2\pi} \cdot \frac{1}{\tilde{c}} \left\{ \frac{d\rho_0}{d\alpha} e^{-\frac{\alpha}{2}r^2} + \rho_0 \left(-r^2/2\right) e^{-\frac{\alpha}{2}r^2} \right\} =$$

$$= -\frac{\gamma\lambda^2}{2\pi\tilde{c}} \cdot \left\{ \frac{3}{2} M \frac{\alpha^{1/2}}{(2\pi)^{3/2}} e^{-\frac{\alpha}{2}r^2} - \frac{r^2}{2} \rho_0 e^{-\frac{\alpha}{2}r^2} \right\} =$$

$$= -\frac{\gamma\rho\lambda^2}{4\pi\tilde{c}} \cdot \left\{ \frac{3}{\alpha} - r^2 \right\}. \qquad (2.3.9b)$$

As follows from (2.3.9b), the extreme (minimal) value of the speed of “heavy” sound depending on α is reached when

$$\alpha = \frac{3}{r^2}. \qquad (2.3.10)$$

According to (2.1.29) the condition (2.3.10) (when the wave disturbances are damped) is fulfilled only in the case of

$$r = r_{sq}\,. \qquad (2.3.11)$$

Thus, the speed of gravitational perturbations attains its minimum, that is, the zero speed when $r = r_{sq}$. So, the wave disturbances attenuate on isostere of the root-mean-square distance (due to their deceleration in the formed gravitational field of the main mass of sphere-like gaseous body).

2.4. The strength and potential of the gravitational field of a sphere-like gaseous body formed by a collection of interacting particles

As mentioned in Sections 2.2 and 2.3, there are the critical (threshold) values of mass density ρ_c (2.3.5a, b) and the gravitational condensation parameter α_c (2.3.7a, b) when a weak gravitational field arises in a sphere-like gaseous body that is forming.

Supposing $\alpha \geq \alpha_c$ let us calculate the characteristics of the gravitational field produced by a collection of interacting particles in the form of a sphere-like gaseous body. We shall use the gravitational field equation in nonrelativistic mechanics written down in the form of Poisson equation [99, 100] (see also Eq. (1.1.41)):

$$\nabla^2 \varphi_g = 4\pi\gamma\rho\,, \qquad (2.4.1)$$

where ∇^2 is the Laplace operator, $\gamma = 6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is the Newtonian gravitational constant, φ_g is a gravitational field potential, ρ is a body mass density.

Taking into account that following (2.2.5) the mass density $\rho(r) = \rho_0 e^{-\alpha r^2/2}$ is a function of distance r alone in Eq. (2.1.4) the gravitational potential does not depend on a direction in space for a sphere-like gaseous body. In compliance with this fact, we consider only the radial part of the Laplacian in the spherical system of coordinates, so that Eq. (2.1.4) becomes the following [16, 45, 46, 65, 73]:

$$\frac{1}{r^2}\left[\frac{d}{dr}\left(r^2 \frac{d\varphi_g(r)}{dr}\right)\right] = 4\pi\gamma\rho_0 e^{-\frac{\alpha}{2}r^2}, \qquad (2.4.2)$$

whence

$$\frac{d}{dr}\left(r^2 \frac{d\varphi_g}{dr}\right) = 4\pi\gamma\rho_0 r^2 e^{-\frac{\alpha}{2}r^2}. \qquad (2.4.3)$$

Integrating Eq. (2.4.3) over r one obtains:

$$\frac{d\varphi_g(r)}{dr} = 4\pi\gamma\rho_0 \frac{\int\limits_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx}{r^2}. \qquad (2.4.4)$$

On the other hand, the derivative $\dfrac{d\varphi_g(r)}{dr}$ determines the gradient value:

$$\vec{a} = -\operatorname{grad} \varphi_g. \qquad (2.4.5)$$

Moreover, the gradient is also termed the *strength* of the gravitational field [100]. Indeed, the gradient expression in spherical coordinates (under the mentioned condition $\dfrac{\partial \varphi_g}{\partial \theta} = \dfrac{\partial \varphi_g}{\partial \varepsilon} = 0$) transforms into

$$\vec{a}(r) = -\,\text{grad}\ \varphi_g(r) = -\frac{d\varphi_g(r)}{dr}\cdot\vec{e}_r = -\frac{d\varphi_g(r)}{dr}\cdot\frac{\vec{r}}{r}. \qquad (2.4.6)$$

Resulting from (2.4.4)–(2.4.6) the strength of the gravitational field produced by a collection of interacting particles is expressed in the following relation [45, 46]:

$$\vec{a}(r) = -4\pi\gamma\rho_0 \frac{\int_0^r x^2 e^{-\frac{\alpha}{2}x^2}dx}{r^2}\frac{\vec{r}}{r}. \qquad (2.4.7)$$

The strength (2.4.7) determines the *field of accelerations* acquired by bodies under the influence of the gravitational field produced by a collection of particles. As seen from (2.4.7), the vector of the gravitational field strength is in the opposite direction to vector $\vec{r}$, that is, it is directed to the center of masses of a sphere-like gaseous body. Further, we shall be interested in the strength value (acceleration of bodies) [45, 46]:

$$a(r) = 4\pi\gamma\rho_0 \frac{\int_0^r x^2 e^{-\frac{\alpha}{2}x^2}dx}{r^2}, \qquad (2.4.8a)$$

which, taking into account that $\rho_0 = M(\alpha/2\pi)^{3/2}$, is presented as follows [45, 46]:

$$a(r) = \frac{\gamma M}{r^2}\,\text{erf}_2\left(r\sqrt{\alpha/2}\right), \qquad (2.4.8b)$$

where $\text{erf}_2\left(r\sqrt{\alpha/2}\right)$ is a nonelementary function of the type: $\text{erf}_2(x) = \frac{4}{\sqrt{\pi}}\int_0^x s^2 e^{-s^2}ds$. The tables of special distribution probabilities can be used for calculating $\text{erf}_2(x)$, in particular, the table of the error function erf(x) [128, 136]. Thus, we shall use two formulas, (2.4.8a) and (2.4.8b), for calculating the strength value, the former being convenient for analysis, the latter for numerical calculations.

According to Newton's second law, the acceleration (the gravitational field strength) being known, the force acting in this field on a body of mass m is equal to

$$\vec{F}_g = m \cdot \vec{a}(r) = -m \cdot \operatorname{grad} \varphi_g(r). \tag{2.4.9}$$

In accordance with (2.4.8a) and (2.4.8b) the gravitation force value is determined using the following relations [45, 46]:

$$F_g = 4\pi\gamma\rho_0 m \cdot \frac{\int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx}{r^2}, \tag{2.4.10a}$$

$$F_g = \frac{\gamma Mm}{r^2} \operatorname{erf}_2\left(r\sqrt{\alpha/2}\right). \tag{2.4.10b}$$

The relations thus obtained, (2.4.10a) and (2.4.10b), need to be compared with the results of classical physics. According to the well-known universal gravitation law of Newton [80], two-point bodies, material points, or two spherical bodies, attract to each other with the force equal to

$$F_g = \gamma \frac{Mm}{r^2}, \tag{2.4.11}$$

where

γ is the Newtonian gravitational constant,

M and m are masses of interacting particles (spherical bodies), and

r is a distance between their mass centers.

Let, as before, M be the mass of a sphere-like gaseous body formed by a collection of interacting particles, and m be the mass of some particle (or a spherical body). We assume that they are at a distance $r >> 2r_{pr}$ from each other. Let r increase infinitely. Then from the foregoing, it follows that

$$\lim_{r\to\infty} \operatorname{erf}_2\left(r\sqrt{\alpha/2}\right) = \frac{4}{\sqrt{\pi}}\int_0^\infty s^2 e^{-s^2} ds = 1\,. \qquad (2.4.12)$$

On account of (2.4.12) with large r the formula (2.4.10b) coincides with (2.4.11), that is, at infinitely large distances, $r \to \infty$, the gravitational interaction forces tend to zero just as $1/r^2$. Similarly, in the case of *large* r the gravitational field strength value (2.4.8b) will take the form:

$$a(r) = \gamma\frac{M}{r^2}\,. \qquad (2.4.13)$$

Now let us consider another case, a limited one of (2.4.8a,b), with small r. It is known [100] that the strength value inside a homogeneous sphere of constant density ρ_0 is equal to:

$$a(r) = \frac{4\pi}{3}\gamma\rho_0 r\,. \qquad (2.4.14)$$

In the case of small r, the function $e^{-\frac{\alpha}{2}x^2} \approx 1 - \frac{\alpha}{2}x^2$ is in the subintegral expression in (2.4.8a). Because of this formula (2.4.8a) transforms into

$$a(r) = 4\pi\gamma\rho_0\frac{\int_0^r x^2\left(1-\frac{\alpha}{2}x^2\right)dx}{r^2} = 4\pi\gamma\rho_0\frac{\frac{1}{3}r^3 - \frac{\alpha}{10}r^5}{r^2} \approx \frac{4\pi\gamma\rho_0}{3}r. \qquad (2.4.15)$$

In (2.4.15) values greater than the second order of smallness of r have been ignored. Thus, formulas (2.4.15) and (2.4.14) coincide. As follows from them, $a(r) \to 0$ with $r \to 0$, that is, there is no field in the center of the body.

In this way, the relation obtained, (2.4.8a) (or (2.4.8b)), for the strength of gravitational field of a sphere-like gaseous body, the body having been formed from a large number of interacting particles, includes the known results of (2.4.13) and (2.4.14) as particular cases with r values large and small, respectively. It should be noted that classical formulas (2.4.13) and (2.4.14) can not be used if there is no preliminary information about the value r. Indeed, according to (2.4.14) $a(r) \to 0$ when $r \to 0$, whereas according to (2.4.13) $a(r) \to \infty$ when $r \to 0$, which is absurd. On the contrary, according to (2.4.13) $a(r) \to 0$ with $r \to \infty$. At the same time, according to (2.4.14), $a(r) \to \infty$ with $r \to \infty$. It makes no sense. It is obvious that (2.4.14) is valid for a *small* r only, while (2.4.13) is only for a *large* one. But in the case of *medium-size* r-s we should use the formulas obtained, (2.4.8a) or (2.4.8b); these show that the relations (2.4.13) and (2.4.14) conform just as solutions are "sewn together" at domain boundaries in mathematical physics problems.

As an example let us consider the plotted dependence [177] of gravity acceleration g on the distance r to the center of the Earth (Fig. 2.5). As seen from this figure, the function $g(r)$ maximum is reached when $r = R$ is the Earth's radius. As pointed out above, the relation (2.4.8a) (or (2.4.8b)) includes (2.4.13) and (2.4.14) in the cases of large and small r, and even medium r, that is, the relation obtained can be used directly, and it does not require any agreement of results with different r.

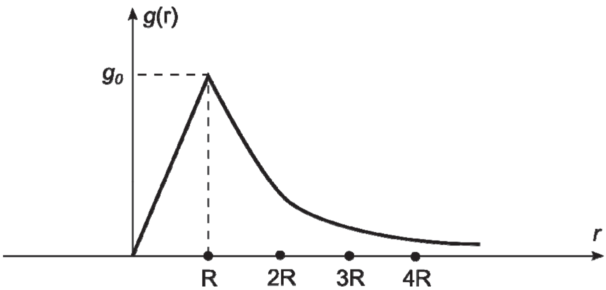


Figure 2.5. The gravity acceleration dependence on the distance r to the Earth's center diagram (R is a mean radius of the Earth)

Let us investigate the type of dependence of field strength $a = a(r)$ according to (2.4.8a, b). As mentioned above, since $a(r) \to 0$ both with $r \to 0$ and with $r \to \infty$, the function $a(r)$ has a maximum which is determined by the following equation:

$$\frac{da(r)}{dr} = 4\pi\gamma\rho_0 \frac{d}{dr}\left(\frac{1}{r^2}\int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx\right) = 0\,. \qquad (2.4.16)$$

Hence it is not difficult to see that

$$\int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx = \frac{r^3}{2} e^{-\frac{\alpha}{2}r^2}\quad . \qquad (2.4.17)$$

To find $r_{\max}$ we have to differentiate the integral equation (2.4.17):

$$r^2 e^{-\frac{\alpha}{2}r^2} = \frac{3}{2} r^2 e^{-\frac{\alpha}{2}r^2} + \frac{r^3}{2} e^{-\frac{\alpha}{2}r^2}\left(-\alpha r\right). \qquad (2.4.18)$$

It follows from (2.4.18) (58) that

$$r_{\max} = \frac{1}{\sqrt{\alpha}} = r_*\,. \qquad (2.4.19)$$

Comparing (2.4.19) with (2.2.6) one can see that the strength maximum of a gravitational field is reached at the point of mass density bending (Fig. 2.6). The course of the plotted $a(r)$ dependence in Fig. 2.6 resembles that of $g(r)$ in Fig. 2.5 obtained in classical theory.

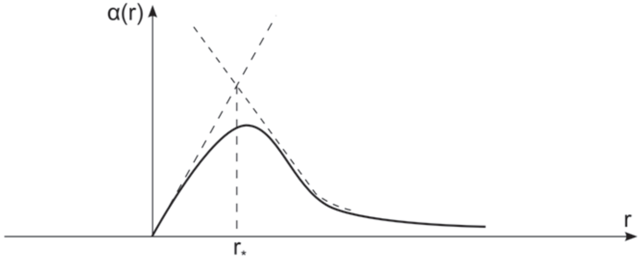


Figure 2.6. The sphere-like gaseous body's gravitational field strength dependence on distance *r* diagram

In such a way, a mass density overfall of a sphere-like gaseous body gives rise to a maximum value of the gravitational field strength produced by this body. It should be noted that the maximum value of the distribution function is reached when $r_{pr} = \sqrt{2} r_*$.

Now let us return to the formula (2.4.4) and calculate the gravitational potential [45, 46]:

$$\varphi_g(r) = 4\pi\gamma\rho_0 \int \frac{1}{r^2} \int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dxdr + C , \qquad (2.4.20)$$

where C is an integration constant defined from the condition that the potential is equal to zero on the infinity: $\varphi_g(\infty) = 0$. To simplify (2.4.20) we can transform the indefinite integral based on the integration formula by parts:

$$\int \frac{1}{r^2} \int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dxdr = -\frac{1}{r} \int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx - \int \left(-\frac{1}{r}\right) r^2 e^{-\frac{\alpha}{2}r^2} dr =$$

$$= -\frac{1}{r} \int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx - \frac{1}{\alpha} e^{-\frac{\alpha}{2}r^2} . \qquad (2.4.21)$$

To simplify the last expression we shall calculate (using the integrating rule by parts) the following integral:

$$\int_0^r e^{-\frac{\alpha}{2}x^2}dx = re^{-\frac{\alpha}{2}r^2} + \alpha\int_0^r x^2 e^{-\frac{\alpha}{2}x^2}dx\,,$$

whence

$$\int_0^r x^2 e^{-\frac{\alpha}{2}x^2}dx = \frac{1}{\alpha}\int_0^r e^{-\frac{\alpha}{2}x^2}dx - \frac{1}{\alpha}re^{-\frac{\alpha}{2}r^2}\,. \qquad (2.4.22)$$

Substituting (2.4.22) into (2.4.21) we obtain

$$\int \frac{1}{r^2}\int_0^r x^2\, e^{-\frac{\alpha}{2}x^2}dx\,dr = -\frac{1}{r}\left[\frac{1}{\alpha}\int_0^r e^{-\frac{\alpha}{2}x^2}dx - \frac{1}{\alpha}r\,e^{-\frac{\alpha}{2}r^2}\right] - \frac{1}{\alpha}e^{-\frac{\alpha}{2}r^2} =$$

$$= -\frac{1}{\alpha r}\int_0^r e^{-\frac{\alpha}{2}x^2}dx\,. \qquad (2.4.23)$$

On account of (2.4.23) the formula (2.4.20) takes the form:

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r e^{-\frac{\alpha}{2}x^2}dx + C. \qquad (2.4.24)$$

From the condition $\varphi_g(\infty) = 0$ and the formula (2.4.24) we obtain:

$$-\frac{4\pi\gamma\rho_0}{\alpha}\lim_{r\to\infty}\frac{1}{r}\int_0^r e^{-\frac{\alpha}{2}x^2}dx + C = 0. \qquad (2.4.25)$$

Since $\int_0^\infty e^{-\frac{\alpha}{2}x^2}dx = \left(\frac{2}{\alpha}\right)^{1/2}\int_0^\infty e^{-s^2}ds = \left(\frac{2}{\alpha}\right)^{1/2}\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2\alpha}}$ then $C = 0$. Thus, the gravitational potential is determined by the following relation [16, 45, 46]:

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r e^{-\frac{\alpha}{2}x^2}dx\,. \qquad (2.4.26)$$

Using the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$ [128] let us transform (2.4.26) into:

$$\varphi_{\mathrm{g}}(r) = -\frac{4\pi\gamma\rho_0}{\alpha r}\sqrt{\frac{2}{\alpha}}\int_0^{r\sqrt{\alpha/2}} e^{-s^2}\,ds = -\left(\frac{2\pi}{\alpha}\right)^{3/2}\frac{\gamma\rho_0}{r}\mathrm{erf}\left(r\sqrt{\alpha/2}\right) =$$

$$= -\frac{\gamma \mathrm{M}}{r}\mathrm{erf}\left(r\sqrt{\alpha/2}\right). \qquad (2.4.27)$$

Since $\lim_{r\to\infty}\mathrm{erf}\left(r\sqrt{\alpha/2}\right) = 1$ then for large r the last expression turns into

$$\varphi_g(r) = -\frac{\gamma M}{r}. \qquad (2.4.28)$$

Relation (2.4.28), as known from [100], describes the gravitational potential of a field produced by one particle (or a spherical body) of mass M. However, as noted correctly in [100], "potential of its field in the point $r = 0$ turns into infinity." Thus, according to the classical field theory, the particle "should have infinite 'own' energy and, consequently, infinite mass. Physical nonsense of this result shows the main principles ... lead to that its applicability should be bounded by specific limits" [100 p.121].

In another way, the situation is as in the case of the developed statistical theory of the initial gravitational interactions of particles. Indeed, in the case of small r, the function $e^{-\frac{\alpha}{2}r^2} \approx 1 - \frac{\alpha}{2}r^2$ leads to the transformation of the formula (2.4.26) using the designation (2.1.22):

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r\left(1 - \frac{\alpha}{2}x^2\right)dx = -\frac{4\pi\gamma\rho_0}{\alpha r}\left(r - \frac{\alpha}{6}r^3\right) \approx$$

$$\approx -\frac{4\pi\gamma\rho_0}{\alpha} + \frac{2\pi\gamma\rho_0}{3}r^2 = -2\pi\gamma\rho_0 r_{pr}^2 + (2\pi\gamma\rho_0/3)r^2. \qquad (2.4.29)$$

Though, in the expression (2.4.29), values higher than the second order of smallness of r have been ignored.

Nevertheless, the gravitational potential in the point $r = 0$ is a finite value: $\varphi_g(0) = -\frac{4\pi\gamma\rho_0}{\alpha}$.

On the other hand, as is known from the classical theory of potentials [95, 97], the potential of a sphere at an interior point is equal to

$$\varphi_g(r) = -\frac{2}{3}\pi\gamma\sigma \cdot (3R^2 - r^2), \qquad (2.4.30)$$

where R is a radius of the sphere and σ is a density of mass. Obviously, formulas (2.4.29) and (2.4.30) coincide if $R = r_{pr}$ and $\sigma = \rho_0$ (for example, for the Earth as a sphere-like body its radius R can be estimated by $R = r_{pr}$). Moreover, as seen from (2.4.29), at small distances from the center ($r \to 0$), the field potential is proportional to the mass density near the center ρ_0 and the sphere area of radius r_*: $S_* = \frac{4\pi}{\alpha} 4\pi r^2{}_*$.

Thus, the expression (2.4.29) describes the potential in the near zone of the gravitational field while (2.4.28) describes that in the remote one.

2.5. The potential energy of a gravitating sphere-like gaseous body

The potential energy of a particle in a gravitational field is equal to its mass multiplied by the potential of the field. The potential energy of any distribution of masses is, therefore, described by the expression [100]:

$$E_g = \frac{1}{2}\int_V \rho\varphi_g dV. \qquad (2.5.1)$$

In the spherical system of coordinates the expression (2.5.1) has the form:

$$E_g = \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \rho\varphi_g r^2 \sin\theta d\theta d\varepsilon dr\,. \tag{2.5.2}$$

Since $\rho = \rho(r)$ and $\varphi_g = \varphi_g(r)$ are functions independent of angle variables θ and ε then, having done the integration over θ and ε in (2.5.2), we obtain:

$$E_g = 2\pi\int_0^{\infty} \rho(r)\varphi_g(r) r^2 dr\,. \tag{2.5.3}$$

Substituting expressions (2.2.5) and (2.4.26) for $\rho(r)$ and $\varphi_g(r)$ in the (2.5.3) we obtain:

$$E_g = -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha}\int_0^{\infty} re^{-\frac{\alpha}{2}r^2}\int_0^{r} e^{-\frac{\alpha}{2}x^2}dxdr\,. \tag{2.5.4}$$

To calculate E_g we can use the formula of integrating by parts in the relation (2.5.4):

$$E_g = -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha}\int_0^{\infty} re^{-\frac{\alpha}{2}r^2}\int_0^{r} e^{-\frac{\alpha}{2}x^2}dxdr = -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha}\left[-\frac{1}{\alpha}e^{-\frac{\alpha}{2}r^2}\int_0^{r} e^{-\frac{\alpha}{2}x^2}dx\Big|_0^{\infty} - \right.$$

$$\left. -\int_0^{\infty}\left(-\frac{1}{\alpha}\right)e^{-\frac{\alpha}{2}r^2}e^{-\frac{\alpha}{2}r^2}dr \right] = -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha^2}\int_0^{\infty} e^{-\alpha r^2}dr = -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha^2\alpha^{1/2}}\int_0^{\infty} e^{-s^2}ds =$$

$$= -\frac{8\gamma\pi^2{\rho_0}^2}{\alpha^{5/2}}\frac{\sqrt{\pi}}{2} = -4\gamma{\rho_0}^2\left(\frac{\pi}{\alpha}\right)^{5/2}. \tag{2.5.5}$$

Taking into account that $\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}$ we transform the formula (2.5.5) into the following [45, 46]:

$$E_g = -4\gamma{\rho_0}^2\left(\frac{\pi}{\alpha}\right)^{5/2} = -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha}{\pi}}. \tag{2.5.6}$$

From (2.5.6) it is not difficult to see that

$$\alpha = \frac{4\pi}{\gamma^2} \cdot \frac{E_g{}^2}{M^4} = 4\pi \left(\frac{E_g}{\gamma M^2} \right)^2 . \qquad (2.5.7)$$

Since α is proportional to $E_g{}^2$, then, although $E_g < 0$, the parameter $\alpha > 0$.

The following distances have been deduced before:

$r_* = \frac{1}{\sqrt{\alpha}}$ is a point of mass density overfall;

$r_{pr} = \sqrt{\frac{2}{\alpha}}$ is a most probable distance;

$\bar{r} = 2\sqrt{\frac{2}{\pi\alpha}}$ is an average distance; and

$r_{sq} = \sqrt{\frac{3}{\alpha}}$ is a root-mean-square distance.

According to (2.5.6) we can deduce one more distance, that is, an *effective radius* of the body:

$$r_+ = \sqrt{\frac{\pi}{\alpha}}, \qquad (2.5.8)$$

close to r_{sq} in value. On account of (2.5.8) one obtains [45, 46]:

$$E_g = -\frac{\gamma M^2}{2r_+}. \qquad (2.5.9)$$

Now let us introduce some physical value:

$$\bar{k} = \frac{\gamma}{r_+}, \qquad (2.5.10)$$

the measuring unit of which is J/kg^2. Substituting (2.5.10) into (2.5.9) we obtain:

$$E_g = -\frac{\bar{k} M^2}{2}. \qquad (2.5.11)$$

According to (2.5.11), value $\bar{k}$ is the proportionality coefficient between the potential energy and the square of body mass. Let us find what $\bar{k}$ depends on. We shall use formula (2.5.6) transformed on account of the fact that $\sqrt{\alpha} = 1/r_*$:

$$E_g = -\frac{\gamma M^2}{2}\frac{1}{\sqrt{\pi}r_*} = -\frac{\gamma M^2}{\sqrt{4\pi r_*^2}} = -\frac{\gamma M^2}{\sqrt{S_*}}, \qquad (2.5.12)$$

where S_* is an area of a sphere of radius r_*. Comparing (2.5.11) with (2.5.12) one can see that

$$\bar{k} = \frac{2\gamma}{\sqrt{S_*}}. \qquad (2.5.13)$$

From (2.5.13) one can see that the smaller the sphere area on which the overfall of body mass density occurs, the greater the proportionality coefficient between mass and energy. In other words, the higher the curvature of the mass density overfall sphere, the greater the potential energy with the same mass.

As an example let us consider a homogeneous spherical body, a sphere of radius a and constant mass density. Using (2.5.1), it is not difficult to calculate the potential energy of a gravitating sphere which is equal to [100]:

$$E_g^s = -\frac{3\gamma m^2}{5a}, \qquad (2.5.14)$$

where m is a mass of the sphere. From (2.5.14) and (2.5.11) it is clear that the proportionality coefficient in the case of the sphere is equal to:

$$\bar{k} = \frac{6\gamma}{5a}. \qquad (2.5.15)$$

We consider now a gravitating sphere-like gaseous body of mass M and a spherical body of mass m and radius a.

From the condition of their energies being equal (see relations (2.5.9) and (2.5.14)) one obtains:

$$\frac{M^2}{2r_+} = \frac{3m^2}{5a}, \qquad (2.5.16)$$

whence

$$\frac{M^2}{m^2} = \frac{6}{5}\cdot\frac{r_+}{a}. \qquad (2.5.17)$$

It follows from (2.5.17) that if, instead of a sphere-like gaseous body, one chooses a homogeneous spherical one, equivalent to the former both in energy and in mass, then such a body must have a radius equal to:

$$a = \frac{6}{5}r_+ = \frac{6}{5}\sqrt{\frac{\pi}{\alpha}}. \qquad (2.5.18)$$

It is quite clear because a sphere-like gaseous body also contains particles of its substance outside distance r_+, that is, a spherical body of a larger radius is required for the masses m and M to be equal.

Let us evaluate the relationship between the potential energy of particles, the own potential energy, and the potential energy of the interaction of these particles, the mutual potential energy. Taking it that particles are spherical bodies of mass m_0 and radius a_0, their total own potential energy, according to (2.5.14), is equal to

$$E_\Sigma = NE_g^s = -\frac{3\gamma m_0 M}{5a_0}. \qquad (2.5.19)$$

On the other hand, the total potential energy of a gravitating body is determined by relation (2.5.9). On account of (2.5.9) and (2.5.19), the potential energy of the interaction of these particles in a sphere-like gaseous body, the mutual potential energy, is equal to

$$E_{\text{int}\Sigma} = E_g - E_\Sigma = -\gamma \frac{M}{2}\left(\frac{M}{r_+} - \frac{6}{5}\frac{m_0}{a_0}\right) = -\gamma \frac{Mm_0}{2}\left(\frac{N}{r_+} - \frac{6}{5a_0}\right). \quad (2.5.20)$$

One can see from (2.5.20) that the energy of interaction is proportional to the mass of the body and to that of the particle.

Let us consider a single particle of mass m_0 in the collective gravitational field of a sphere-like gaseous body of mass M situated at distance r from the body's center. We now evaluate the potential energy of the interaction of the particle with the sphere-like body. It is clear that on part of the field the particle is affected by a gravitational force $\vec{F}_g(\vec{r})$ which tends to move the particles to the center of masses. While at an infinite distance from the center, the potential energy of interaction is equal to zero. Therefore, the potential energy of a particle situated at distance r from the center is equal (with an accuracy to the sign) to the work done by the gravitational force in moving the particle from infinity to the given point:

$$E_{\text{int}}(r) = -A(r) = -\int_{\infty}^{\vec{r}} \vec{F}_g(\vec{r})d\vec{r} \ . \quad (2.5.21)$$

On account of (2.4.6) and (2.4.9) the formula (2.5.21) takes the form:

$$E_{\text{int}}(r) = -\int_{\infty}^{r}\left(-m_0 \frac{d\varphi_g(r)}{dr}\right)dr = m_0 \int_{\varphi(\infty)}^{\varphi(r)} d\varphi_g =$$

$$= m_0(\varphi_g(r) - \varphi_g(\infty)) = m_0\varphi_g(r). \quad (2.5.22)$$

Using relations (2.5.22) and (2.4.26) one can easily calculate the energy of the interaction of a sphere-like gaseous body and a particle placed at different distances from the center of masses. Since energy depends on the distance at which a particle is, and particles themselves are distributed over space, one can determine the *mean potential energy of* *interaction* of a particle with a sphere-like gaseous body formed by a collection of such particles [45, 46]:

$$\overline{E} = \int_0^\infty E_{\text{int}}(r) f(r) dr = m_0 \int_0^\infty \varphi_g(r) f(r) dr . \qquad (2.5.23)$$

Let us use the relation (2.4.26) for calculating $\varphi_g(r)$; on account of it and (2.1.18) the relation (2.5.23) takes the form:

$$\overline{E} = -\frac{4\pi\gamma\rho_0 m_0}{\alpha} \int_0^\infty \frac{1}{r} \int_0^r e^{-\alpha x^2/2} dx \cdot 4\pi \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\alpha r^2/2} r^2 dr =$$

$$= -\frac{4\pi\gamma\rho_0 m_0}{\alpha} \frac{4\pi\alpha^{3/2}}{(2\pi)^{3/2}} \int_0^\infty r e^{-\alpha r^2/2} \int_0^r e^{-\alpha r^2/2} dx dr. \qquad (2.5.24)$$

The integral entering the relation (2.5.24) has been calculated in (2.5.5), it is equal to

$$\int_0^\infty r e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx dr = \frac{1}{\alpha} \int_0^\infty e^{-\alpha r^2} dr = \frac{1}{\alpha^{3/2}} \frac{\sqrt{\pi}}{2} . \qquad (2.5.25)$$

With (2.5.25) the relation (2.5.24) takes the form:

$$\overline{E} = -\frac{4\pi\gamma\rho_0 m_0}{\alpha} \frac{4\pi\pi^{3/2}}{(2\pi)^{3/2}} \frac{1}{\alpha^{3/2}} \frac{\sqrt{\pi}}{2} = -\frac{4\pi\gamma\rho_0 m_0}{\alpha\sqrt{2}} . \qquad (2.5.26)$$

Substituting $\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}$ in (2.5.26) we obtain [45, 46]:

$$\overline{E} = -\frac{4\pi\gamma\rho_0 m_0}{\alpha\sqrt{2}} = -\frac{\gamma M m_0 \sqrt{\alpha}}{\sqrt{\pi}} = -\frac{\gamma m_0 M}{r_+} . \qquad (2.5.27)$$

Let $m_0 = dM$ in (2.5.27). Taking this into account, the formula (2.5.27) is transformed into

$$d\overline{E} = -\frac{\gamma M dM}{r_+} . \qquad (2.5.28)$$

By integrating both parts of Eq.(2.5.28) we obtain again the formula (2.5.9):

$$\int d\overline{E} = -\frac{\gamma M^2}{2r_+} = E_g . \qquad (2.5.29)$$

Thus, according to (2.5.9), (2.5.27), and (2.5.29), the potential energy of a gravitating sphere-like gaseous body is only then equal to the total mean potential energy of the gravitational interaction of particles when these particles have *infinitely small masses* [45, 46, 73]. Indeed, in this case, particles do not possess their own gravitational energy because, according to (2.5.9) and (2.5.14), it is a value of the second order of smallness with respect to dM. In fact, with putting the question in this way, we deal with massless particles whose gravitational energy is the potential energy of the interactions of particles between one another [68]. Further, from (2.5.27) it follows that

$$r_+ = \frac{\gamma m_0 M}{-\overline{E}} , \qquad (2.5.30)$$

$$\alpha = \frac{\pi \overline{E}^2}{\gamma^2 m_0{}^2 M^2} = \pi \left(\frac{\overline{E}}{\gamma m_0 M} \right)^2 . \qquad (2.5.31)$$

Using the sense of parameter α (defined by (2.5.31)) let us investigate the character of distributing particles near distance r depending on the physical values in the formula (2.5.31). According to (2.1.22) the most probable distance r_{pr} near which one can find the greatest share of particles can be calculated using (2.5.31):

$$r_{pr} = \sqrt{\frac{2}{\alpha}} = \sqrt{\frac{2}{\pi}} \cdot \frac{\gamma m_0 M}{\overline{E}} = \sqrt{\frac{2}{\pi}} \cdot \frac{\gamma m_0{}^2 N}{\overline{E}} . \qquad (2.5.32)$$

As r_{pr} is the maximum of the probability density function $f(r)$, then, on account of (2.5.32), the formula (2.1.24) becomes [46]:

$$f(r)=\frac{4}{\sqrt{\pi}}\cdot\frac{r^2}{r_{pr}{}^3}e^{-r^2/r_{pr}{}^2}=\sqrt{2}\cdot\pi\left(\frac{\overline{E}}{\gamma m_0 M}\right)^3 r^2 e^{-\frac{\pi}{2}\left(\frac{\overline{E}}{\gamma m_0 M/r}\right)^2}. \quad (2.5.33)$$

As seen from (2.5.32) and (2.5.33), the type of function $f(r)$ is determined by parameters $\overline{E}$, m_0, and M (see Fig.2.7).

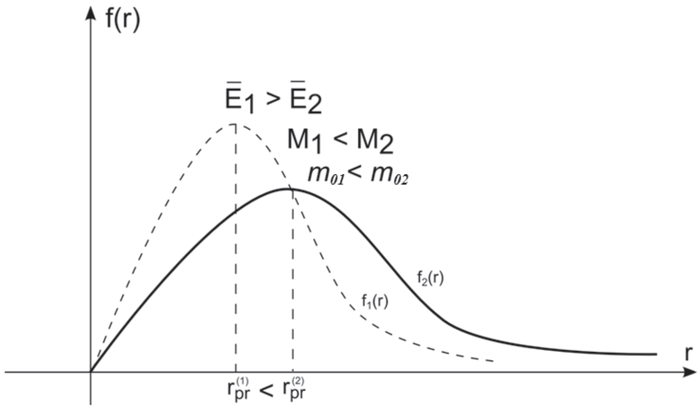


Figure 2.7. The diagrams of probability densities as functions of distance r depending on the mean potential energy $\overline{E}$ of gravitational interaction of particles in a sphere-like gaseous body, M is a mass of the sphere-like gaseous body, m_{0i} is a particle mass

Indeed, according to (2.5.32) the maximum of this function is nearer to the origin of coordinates, the greater the mean energy of the gravitational interaction $\overline{E}$, and the smaller the mass of the body M and mass m_0 of particles the body consists of. So, for the two functions $f_1(r)$ and $f_2(r)$ in Fig. 2.7 the abscissa of the $r_{pr}^{(1)}$ maximum of the former function is less than that of the $r_{pr}^{(2)}$ maximum of the latter, if $\overline{E}_1 > \overline{E}_2$, or $M_1 < M_2$, or $m_{01} < m_{02}$. If the condition $m_{01} < m_{02}$ is valid then the condition $M_1 < M_2$ is fulfilled automatically provided the two sphere-like gaseous bodies under consideration contain the same number of particles N (the more so, if in the first body there are fewer of them: $N_1 < N_2$).

Similar reasoning can also be applied to the function of the density distribution of mass $\rho(r)$. Indeed, substituting (2.5.31) into (2.2.5) one obtains [45, 46]:

$$\rho(r) = \rho_0 e^{-\frac{\pi}{2}\left(\frac{\bar{E}}{\gamma m_0 M / r}\right)^2}, \qquad (2.5.34)$$

where $\rho_0 = \frac{1}{2\sqrt{2}M^2}\left(\frac{\bar{E}}{\gamma m_0}\right)^3$. Since, according to (2.2.6) the bending point for $\rho(r)$ is equal to $r_* = r_{pr}/\sqrt{2}$, then, taking into account Eq. (2.5.32), the mass density of a sphere-like gaseous body is more concentrated near the center of masses, the greater $\bar{E}$ and the smaller M and m_0 [45, 46].

Thus, a sphere-like gaseous body has a “strict” (distinct) outline if the potential energy of gravitational interaction of the body particles is sufficiently great and the body mass itself and its particle masses are relatively small [45, 46, 73]. Ordinary macroscopic bodies possess distinct outlines due to their relatively small masses and to sufficiently great energies of the interaction of particles the bodies consist of. On the contrary, giant cosmic objects (star formations, nebulae, etc.) have fuzzy contours because of their huge masses and the enormous number of particles forming them [45, 46, 73].

2.6. The probability interpretation of physical values describing the gravitational interaction of particles in a sphere-like gaseous body

In dealing with the statistical model of the gravitational interaction of particles it has been supposed that:

- particles are isolated from other bodies and exterior fields;
- particles are at sufficiently low temperatures;
- time is an almost fixed value for slow-flowing processes of initial gravitational condensation, that is, the initial gravitational interactions are considered in quasi-statics.

Since the statistical model has been taken as the basis of describing the gravitational interactions of particles, probability functions should be expected to be involved in physical values as well. Indeed, as follows from (2.2.7), the mass density of a sphere-like gaseous body is proportional to the volume probability density:

$$\rho(r) = M\Phi(r),$$

where $\int_V \Phi(r)dV = 4\pi\int_0^\infty \Phi(r)r^2 dr = 1$. The gravitational field strength value according to (2.4.8a), on account of $f(r) = 4\pi\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} r^2$ and $\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}$, is written down as follows [73]:

$$a(r) = 4\pi\gamma\rho_0 \cdot \frac{1}{r^2}\int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx = \frac{\gamma M}{r^2}\int_0^r f(x)dx = \frac{\gamma M}{r^2}P(x \le r), \quad (2.6.1)$$

where $P(x \le r)$ is a probability of share of particles being at distances $\le r$ from the mass center of a sphere-like gaseous body. As results from (2.6.1), the strength (acceleration) is directly proportional to the probability of finding particles in the interval $0 \le x \le r$, that is, with $r = 0$ the strength $a(r) = 0$ because $P(0) = 0$. At large distances $r \to \infty$ the probability $P(x < \infty) = 1$ and $a(\infty) = \lim_{r\to\infty} \gamma M / r^2 = 0$.

On account of (2.6.1), the gravitational force value is equal to

$$F_g(r) = ma(r) = \frac{\gamma m M}{r^2} P(x \le r). \tag{2.6.2}$$

In other words, the gravitational force is a force due to the probability of the greatest number of particles being in a given point of space [45, 73].

Taking into account that $\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}$ and also the type of function $\Phi(r)$, the volume probability density according to (2.1.14), the formula (2.4.26) for the gravitational potential is transformed into:

$$\varphi_g(r) = -\frac{4\pi\gamma M}{\alpha r}\left(\frac{\alpha}{2\pi}\right)^{3/2} \int_0^r e^{-\frac{\alpha}{2}x^2} dx = -\frac{4\pi\gamma M}{\alpha r}\int_0^r \Phi(x) dx. \tag{2.6.3}$$

On the other hand, the gravitational potential (2.4.20), on account of $C = 0$ and notations for $f(r)$ and ρ_0, takes the form [73]:

$$\varphi_g(r) = \gamma M \int \frac{1}{r^2} P(x \le r) dr. \tag{2.6.4}$$

Using the integration by parts formula the relation (2.6.4) becomes:

$$\varphi_g(r) = \gamma M\left\{\left(-\frac{1}{r}\right)P(x \le r) - \int\left(-\frac{1}{r}\right)\frac{d}{dr}P(x \le r) dr\right\} =$$

$$= -\frac{\gamma M}{r}P(x \le r) + \gamma M \int \frac{1}{r} f(r) dr.$$

If one substitutes the type of function $f(r)$ in the last addend then, on account of the formula (2.1.14), it is written as follows:

$$\gamma M \int \frac{1}{r} f(r) dr = \gamma M \cdot 4\pi\left(\frac{\alpha}{2\pi}\right)^{3/2} \int e^{-\frac{\alpha}{2}r^2} r dr =$$

$$= \frac{4\pi\gamma M}{2\cdot\left(-\frac{\alpha}{2}\right)}\cdot\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} = -\frac{4\pi\gamma M}{\alpha}\Phi(r). \qquad (2.6.5)$$

Using the simple formula (2.2.7), the relation (2.6.5) takes the form:

$$\gamma M\int\frac{1}{r}f(r)dr = -\frac{4\pi\gamma}{\alpha}\rho(r). \qquad (2.6.6)$$

Thus, it follows from (2.6.4)–(2.6.6) that

$$\varphi_g(r) = -\frac{\gamma M}{r}P(x\le r) - \frac{4\pi\gamma}{\alpha}\rho(r) = -\frac{\gamma M}{r}P(x\le r) - \frac{4\pi\gamma M}{\alpha}\Phi(r). \qquad (2.6.7)$$

Comparing (2.6.3) with (2.6.7) one obtains the expressions for the gravitational potential [45, 46, 73]:

$$\varphi_g(r) = -\frac{\gamma M}{r}P(x\le r) - \frac{4\pi\gamma M}{\alpha}\Phi(r) = -\frac{4\pi\gamma M}{\alpha r}\int_0^r\Phi(x)dx. \qquad (2.6.8)$$

Thus, all the values depending on r: mass density $\rho(r)$; gravitational strength $a(r)$; gravitational force $F_g(r)$; and gravitational potential $\varphi_g(r)$ are of *probability nature* due, possibly, to the fact that a sphere-like gaseous body formed under the influence of initial gravitational interactions does not have distinctly outlined borders, but rather fuzzy borders. It is the absence of clearly visible covering of such a body at small distances from its iso-surface of mass density (isostere) $r = r_*$ which can only be seen, with a certain amount of truth, at considerable distances that result in the probability character of gravitational values.

Therefore, it is the application of averaged gravitational values, for example, a mean-field strength value:

$$\overline{a} = \int_0^\infty a(r)f(r)dr, \qquad (2.6.9)$$

that makes it possible to exclude the probability effect. Now let us calculate an average value $\overline{a}$ of gravitational field strength. Substituting the formulas (2.1.18) and (2.4.8a) into (2.6.9) and considering the relation (2.4.22) one can find:

$$\bar{a} = 4\pi\gamma\rho_0 \int_0^\infty \frac{1}{r^2} \int_0^r x^2 e^{-\frac{\alpha}{2}x^2} dx \cdot 4\pi \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2} r^2 dr =$$

$$= (4\pi)^2 \gamma M (\alpha / 2\pi)^3 \int_0^\infty e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr =$$

$$= \frac{2\gamma M}{\pi} \frac{\alpha^3}{\alpha} \cdot \left\{ \int_0^\infty e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx dr + \frac{1}{2\alpha} \int_0^\infty e^{-\alpha r^2} d\left(-\alpha r^2\right) \right\} =$$

$$= \frac{2\gamma M \alpha^2}{\pi} \left\{ \int_0^\infty e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx dr \right\} + \frac{\gamma M \alpha}{\pi} e^{-\alpha r^2} \Big|_0^\infty .$$

The integral in the curly brackets is calculated because

$$\int_0^\infty f'(r) f(r) dr = \int_0^\infty f(r) df(r) = \frac{1}{2} f^2(r) \Big|_0^\infty ,$$

whence

$$\bar{a} = \frac{2\gamma M \alpha^2}{\pi} \frac{1}{2} \left(\int_0^r e^{-\alpha x^2/2} dx \right)^2 \Bigg|_0^\infty + \frac{\gamma M \alpha}{\pi} (0 - 1) =$$

$$= \frac{\gamma M \alpha^2}{\pi} \frac{1}{\alpha/2} \left(\int_0^\infty e^{-u^2} du \right)^2 - \frac{\gamma M \alpha}{\pi} = \gamma M \alpha \left(\frac{1}{2} - \frac{1}{\pi} \right) = \frac{\gamma M}{r_+^{\,2}} \cdot \frac{\pi - 2}{2} .$$

Thus, an average value

$$\bar{a} = \frac{\gamma M}{r_+^{\,2}} \cdot \left(\frac{\pi}{2} - 1 \right) \tag{2.6.10}$$

is equal to the mean strength value of the gravitational field of a sphere-like gaseous body.

As the final result of this section, let us formulate the following theorem [73]:

Theorem 2.1 (an analog of the Newton general theorem in statistical interpretation). Spherical layer, bounded by two similar and similarly placed concentric spherical surfaces, inside a sphere-like gaseous body, does not exert attraction at a point into the internal domain of this layer.

Proof: let us insert inside a sphere-like gaseous body (around its center of mass) a spherical layer, bounded by two similar and similarly placed concentric spherical surfaces at distances R and the sphere of radius $R+\Delta R$ from the mass center of a sphere-like gaseous body. As follows from the derived formula (2.6.1), the strength is directly proportional to the probability of finding particles in the interval $0 \leq x \leq r$:

$$a(r,\alpha) = \frac{\gamma M}{r^2} P_\alpha(x \leq r). \qquad (2.6.11)$$

Choosing $R = r + dr$ we can see that the strength at the point M (placed at a distance r from the mass center) depends on all $x \leq r$, that is, it does not depend on $R > r$. Consequently, the value of gravitational field strength at the point M depends exclusively on attractions of mass points belonging to the sphere of radius r. Let us choose a new sphere of radius $R+\Delta R$. According to (2.6.11) the value of gravitational field strength at the interior point M of this sphere does not depend on attractions of mass points belonging to a spherical layer, bounded by two similar concentric spherical surfaces of radii R and $R+\Delta R$ respectively. The theorem is proved.

2.7. The statistical model of gravitation treated from the point of view of Einstein's general relativity

According to (2.2.5) the distribution law of mass density of the sphere-like body in its own gravitational field is determined by the relation:

$$\rho(r) = \rho_0 e^{-\alpha r^2/2},$$

where $\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2} = \frac{M}{V_0}$ and $V_0 = \left(\sqrt{\frac{2\pi}{\alpha}}\right)^3$. Further, we assume the following principle of equivalence: the distribution of mass in a space of given volume may be changed by changing the mass density (with the volume being fixed) or by changing, or rather curving the volume (with the mass density remaining unchanged) [16, 45, 46]. Following it, a nonuniformity of mass density distribution is due to space curvature as a result of which the volume of a curved space is expressed by the following function [45, 46, 73]:

$$V = \frac{M}{\rho(r)} = V_0 e^{\alpha r^2/2}, \qquad (2.7.1)$$

where $V_0 = M/\rho_0$ is a volume which would be occupied by a body if it possessed constant density and mass equal to that of a sphere-like gaseous body.

As we know [100], the geometric space volume element is given in curvilinear coordinates u_1, u_2, and u_3 not by $dV = du_1 du_2 du_3$ itself but by the product $\sqrt{\hat{g}}dV$ where $\hat{g}$ is space metric tensor determinant.

The sphere-like gaseous body under consideration is characterized by a centrally symmetric distribution of substance, it produces a gravitational field possessing the central symmetry. The centrally symmetric gravitational field is also typical of a spherical body, and of a body whose particles are in centrally symmetric motion, the velocity of every particle being directed along the radius [100]. If one uses spherical space coordinates r, θ and ε then the "centrally symmetric expression" for the interval ds^2 is determined within the framework of the GR as follows [100, 168, 169, 181, 182, 183]:

$$ds^2 = e^{\nu} c^2 dt^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varepsilon^2\right) - e^{\lambda} dr^2, \qquad (2.7.2)$$

where

c is the speed of light,
t is time, and
ν and λ are some functions of r, t.

Here the coordinates of the four-dimensional curvilinear space are $u_0 = ct,\ u_1 = r, u_2 = \theta, u_3 = \varepsilon$. It follows from (2.7.2) that for metric tensor components other than zero we obtain the following expressions:

$$g_{00} = e^{\nu}, g_{11} = -e^{\lambda}, g_{22} = -r^2, g_{33} = -r^2 \sin\theta . \qquad (2.7.3)$$

The space metric, being a particular case of the space-time metric (2.7.2), is determined by the expression for spatial distance element:

$$dl^2 = e^{\lambda}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varepsilon^2 . \qquad (2.7.4)$$

The spatial volume element in the metric (2.7.2) or (2.7.4) is [100]:

$$dV = 4\pi r^2 e^{\lambda/2} dr = e^{\lambda/2} dV_0 , \qquad (2.7.5)$$

where $dV_0 = 4\pi r^2 dr$. Using the metric tensor values (2.7.3), Christoffel symbols $\Gamma^i{}_{kl}$ are calculated, followed by the calculation of Ricci tensor components $R^k{}_i$ [81, 100, 168, 169, 181, 182, 183]; as a result, Einstein's equations for the centrally symmetric gravitational field have the form [100]:

$$\frac{8\pi\gamma}{c^4} T_1^1 = -e^{-\lambda}\left(\frac{1}{r}\cdot\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \qquad (2.7.6)$$

$$\frac{8\pi\gamma}{c^4} T_2^2 = \frac{8\pi\gamma}{c^4} T_3^3 = -\frac{1}{2}\cdot e^{-\lambda}\cdot\left[\frac{\partial^2\nu}{\partial r^2} + \frac{1}{2}\left(\frac{\partial\nu}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\right) - \right.$$

$$\left. -\frac{1}{2}\cdot\frac{\partial\nu}{\partial r}\cdot\frac{\partial\lambda}{\partial r}\right] + \frac{1}{2}\cdot e^{-\nu}\left[\frac{\partial^2\lambda}{\partial (ct)^2} + \frac{1}{2}\left(\frac{\partial\lambda}{\partial (ct)}\right)^2 - \frac{1}{2}\cdot\frac{\partial\lambda}{\partial (ct)}\cdot\frac{\partial\nu}{\partial (ct)}\right], \qquad (2.7.7)$$

$$\frac{8\pi\gamma}{c^4} T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{1}{r}\cdot\frac{\partial\lambda}{\partial r}\right) + \frac{1}{r^2}, \qquad (2.7.8)$$

$$\frac{8\pi\gamma}{c^4}T_0^1 = -e^{-\lambda}\cdot\frac{1}{r}\cdot\frac{\partial\lambda}{\partial(ct)}, \tag{2.7.9}$$

where T_i^k are energy-impulse tensor components. Analyzing the Einstein equations (2.7.6)–(2.7.9) one can put forward some general considerations on the centrally symmetric gravitational field inside gravitating masses [100, 183]. It is seen from (2.7.8) that *with* $r \to 0$ λ *must also turn into zero, at least like* r^2; otherwise, the right-hand side of the equation, with $r \to 0$, would turn into infinity, that is, T_0^0 would have in $r = 0$ a singular point [100].

Because of the positive definiteness of energy $T_0^0 \geq 0$, it follows from (2.7.8) that

$$\frac{8\pi\gamma}{c^4}T_0^0 = \frac{1}{r^2}\left(1 - e^{-\lambda}\right) + \frac{e^{-\lambda}}{r}\cdot\frac{\partial\lambda}{\partial r} \geq 0. \tag{2.7.10a}$$

From the condition of the inequality (2.7.10a) being fulfilled it follows that $e^{\lambda} \geq 1$, that is,

$$\lambda \geq 0. \tag{2.7.10b}$$

Integrating the Einstein equation (2.7.8) formally, with the boundary condition $\lambda\big|_{r=0} = 0$, one obtains [100]:

$$e^{-\lambda} = 1 - \frac{8\pi\gamma}{c^4 r}\int_0^r T_0^0 r^2 dr, \tag{2.7.11a}$$

whence

$$\lambda = -\ln\left(1 - \frac{8\pi\gamma}{c^4 r}\int_0^r T_0^0 r^2 dr\right). \tag{2.7.11b}$$

In the case of *static* distribution of the substance $T_0^0 = \varsigma$ is an energy density, so that the formula (2.7.11a) and (2.7.11b) take the form:

$$e^{-\lambda} = 1 - \frac{8\pi\gamma}{c^4 r}\int_0^r \varsigma r^2 dr\ ; \tag{2.7.12a}$$

$$\lambda = -\ln\left(1 - \frac{8\pi\gamma}{c^4 r}\int_0^r \varsigma r^2 dr\right). \qquad (2.7.12b)$$

At very large distances $r \to \infty$, that is, far from the masses producing the field, the space metric is automatically the Galilean one: $\lambda \to 0, \quad \nu \to 0$ [100]. In fact, the energy density of the gravitational field $T_0^0 = \varsigma \to 0$ with $r \to \infty$, so that Eq. (2.7.8) becomes the following:

$$e^{-\lambda} - re^{-\lambda} \cdot \frac{\partial \lambda}{\partial r} - 1 = 0\,. \qquad (2.7.13a)$$

Integrating this Eq. (2.7.13a) we have:

$$e^{-\lambda} = 1 + \frac{\text{const}}{r}, \qquad (2.7.13b)$$

where the second addend const/r within the framework of GR is chosen to obtain Newton's the universal gravitation law at large distances [100], that is, to be equal to $\text{const}/r = (2/c^2)\cdot\varphi_g(r) = (-2\gamma M/c^2)/r$ in the remote zone of field of gravitating masses M. Taking into account the integration constant $\text{const} = -2\gamma M/c^2$ the formula (2.7.13b) becomes [100]:

$$e^{-\lambda} = 1 - \frac{2\gamma M}{rc^2}\,. \qquad (2.7.13c)$$

Using (2.7.13c) we can also see that for large r the space metric (2.7.4) is transformed into Schwarzschild's space metric [100,183]:

$$dl^2 = \frac{dr^2}{1 - \frac{r_g}{r}} + r^2\left(\sin^2\theta d\varepsilon^2 + d\theta^2\right), \qquad (2.7.13d)$$

where $r_g = \dfrac{2\gamma M}{c^2}$ is the so-called gravitational radius of the body. Comparing (2.7.12a) with (2.7.13c), it is not difficult to find

$$M = \frac{4\pi}{c^2}\int_0^r \varsigma r^2 dr = \int_0^r \frac{\varsigma}{c^2}\cdot 4\pi r^2 dr\,. \qquad (2.7.14)$$

As mentioned above, the spatial volume element in the metric (2.7.2) is defined by the formula (2.7.5) within the framework of GR, that is, it is equal to $dV = 4\pi r^2 e^{\lambda/2} dr$. The formula (2.7.14) can, therefore, be written as follows [73]:

$$M = \int_0^r \frac{\varsigma}{c^2}\cdot e^{-\lambda/2}\cdot e^{\lambda/2} 4\pi r^2 dr = \int_V \frac{\varsigma}{c^2}\cdot e^{-\lambda/2} dV\,. \qquad (2.7.15)$$

In the case of the *initial* process of gravitational condensation of the cosmogonical body, the energy density of gravitating masses ς is small enough, so that the uniform distribution law of energy density $\varsigma = \varsigma_0$ occurs. Since the energy density $\varsigma_0 = \rho_0 c^2$ in GR the formula (2.7.15) takes the simple form [73]:

$$M = \int_V \rho_0 \cdot e^{-\lambda/2} dV. \qquad (2.7.16)$$

For the same reason, we assume that $\varsigma = \varsigma_0 << 1$ and the value r is limited by the radius R of a gravitational instability domain (an area of gravitational field forming), that is, $r \le R$. In this connection the formula (2.7.12b) goes over to the following:

$$\lambda = \frac{8\pi\gamma}{c^4 r}\int_0^r \varsigma r^2 dr = \frac{8\pi\gamma\varsigma_0}{3c^4}\cdot r^2\,, \qquad (2.7.17)$$

whence under the choice of $\alpha = 8\pi\gamma\varsigma_0 / 3c^4$ [73] we obtain

$$\lambda = \alpha r^2\,. \qquad (2.7.18)$$

Substitution of (2.2.18) into (2.2.16) gives the distribution function of mass density $\rho(r)$ in accordance with the following formula obtained within the framework of GR:

$$M = \int_V \rho_0 \cdot e^{-\alpha r^2/2} dV . \qquad (2.7.19)$$

It coincides absolutely with the function of mass density (2.2.5) derived in the statistical theory of the formation of cosmogonical bodies [16, 46, 54].

Further, let us consider the derivation of the function of mass density $\rho(r)$ based on the proposed *principle of equivalence* (2.7.1). Indeed, one can obtain the same formula (2.7.18) from the comparison of (2.7.1) with (2.7.5). Summing up, the formulas (2.7.1), (2.7.5), and (2.7.18) show that the expression for the space volume element obtained by the statistical model of gravitation and by GR coincide [45, 46, 54].

It is easy to see from (2.7.18) that, with $r \to 0$, the value $\lambda = \alpha r^2$ becomes zero according to the r^2 law, as was pointed out in analyzing Einstein's equations (2.7.6)–(2.7.9). It is evident, too, that $\lambda\big|_{r=0} = \left(\alpha r^2\right)\big|_{r=0} = 0$ and $\alpha r^2 \geq 0$ according to the inequality (2.7.10b). Thus, the results obtained based on the statistical model of initial gravitation agree, on the whole, with those of Einstein GR [16, 45, 46, 54].

At last, let us try to obtain the relations (2.1.14) and (2.2.5) from the point of view of GR, jointly with the proposed principle of equivalence [16, 45, 54]. According to the statistical model, there are a fixed number N of identical particles with mass m_0 into the sphere-like gaseous body, that is,

$$M = m_0 N = \text{const} .$$

Because of the mass of a sphere-like gaseous body is $M = \text{const}$ then

$$dM = d(\rho V) = 0\,. \tag{2.7.20}$$

It follows from Eq. (2.7.20) directly that $V d\rho + \rho\, dV = 0$ whence

$$\frac{d\rho}{\rho} = -\frac{dV}{V}. \tag{2.7.21}$$

Integrating Eq. (2.7.21) we obtain [16, 54, 73]:

$$\int_{\rho_0}^{\rho(r)} \frac{d\rho}{\rho} = -\int_{V_0}^{V(r)} \frac{dV}{V} \tag{2.7.22}$$

and then

$$\ln\frac{\rho(r)}{\rho_0} = -\ln\frac{V(r)}{V_0}, \tag{2.7.23}$$

where $\rho_0 = \rho(0)$ and $V_0 = V(0)$. Thus, it follows directly from Eqs (2.7.5) and (2.7.18):

$$V(r) = e^{\lambda/2} V(0) = e^{\alpha r^2/2} V_0\,. \tag{2.7.24}$$

Taking into account Eqs (2.7.23) and (2.7.24) we obtain:

$$\ln\frac{\rho(r)}{\rho_0} = -\alpha r^2/2\,, \tag{2.7.25}$$

whence the relation (2.2.5) follows immediately (and then (2.1.14)) which has been derived already within the framework of the statistical model of initial gravity (see Sections 2.1 and 2.2).

2.8. The pressure in a gravitating sphere-like gaseous body formed by a collection of interacting particles

As far back as the 17th century, Sir Isaac Newton in particular stressed the role of pressure inside the body in the gravitational interaction of particles forming it, since for gravitation to result in gathering a substance into separate clots it must overcome the substance pressure as well as radiation pressure associated with it [101].

Let us consider the sphere-like body formed by the initial collective gravitational field of interacting particles as a continuous medium (liquid or gaseous). Further, we assume that the liquid or gas is in *mechanical equilibrium* in the gravitational field. For a liquid or a gas at rest $(\vec{v}=0)$ the Euler equation [94, 111]:

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\nabla)\vec{v}=-\frac{\nabla p}{\rho}+\vec{a}$$

takes in the form:

$$\text{grad}p=\rho\vec{a}, \tag{2.8.1}$$

where

p is a pressure,

ρ is a density,

$\vec{a}$ is a strength of the gravitational field, and

∇ is the Hamiltonian.

Let us derive an equilibrium equation for a very big mass of liquid forming a sphere-like body part of which is kept together by the forces of gravitation (a star) [94]. Let φ_g be the gravitational potential of a field produced by a sphere-like body which satisfies the differential equation in the Poisson form in accordance with (2.4.1):

$$\nabla^2\varphi_g=4\pi\gamma\rho\ . \tag{2.8.2}$$

According to (2.4.5), the strength of the gravitational field $a(r)=-\text{grad}\varphi_g$, so that the condition of mechanical equilibrium (2.8.1) assumes the form:

$$\text{grad}p=-\rho\cdot\text{grad}\varphi_g\,. \tag{2.8.3}$$

Dividing Eq. (2.8.3) by ρ, applying to both its sides the operation div and using Eq. (2.8.2), we obtain [94]:

$$\mathrm{div}\left(\frac{1}{\rho}\mathrm{grad}\,p\right) = -4\pi\gamma\rho. \tag{2.8.4}$$

As was stressed in [94], only mechanical equilibrium is dealt with here, the existence of complete *thermal equilibrium* in Eq. (2.8.4) *is not presupposed* at all. However, it should be noted that pressure and density definitely determine the temperature at a given point of a liquid. Since density is only a function of the radial ordinate r, and pressure (according to (2.8.4)) as well, the temperature should be a function of r only. Otherwise, with the temperature being different in different places of a liquid at the same altitude, mechanical equilibrium in it is impossible. Thus, given mechanical equilibrium in the gravitational field, density and pressure distribution and temperature depend on r alone.

If the body does not rotate then in equilibrium it will have a spherical form (it also follows from (2.2.5)) and the density and pressure distribution in it will be centrally symmetric [94]. Under this condition Eq. (2.8.4) written down in the spherical coordinates will assume the form:

$$\frac{1}{r^2}\cdot\frac{d}{dr}\left(\frac{r^2}{\rho}\cdot\frac{dp}{dr}\right) = -4\pi\gamma\rho. \tag{2.8.5}$$

Taking into account that $\rho(r) = \rho_0 e^{-\alpha r^2/2}$ based on the relation (2.2.5) and Eq. (2.8.5) we can obtain:

$$\frac{1}{\rho(r)}\cdot\frac{dp(r)}{dr} = -4\pi\gamma\rho_0\frac{\int\limits_0^r x^2 e^{-\alpha x^2/2}dx}{r^2}, \tag{2.8.6}$$

whence

$$\frac{dp(r)}{dr} = -4\pi\gamma\rho_0{}^2\cdot\frac{1}{r^2}\cdot e^{-\alpha r^2/2}\int\limits_0^r x^2 e^{-\alpha x^2/2}dx. \tag{2.8.7}$$

Integrating (2.8.7) we shall have [47, 49]:

$$p(r) = -4\pi\gamma\rho_0{}^2 \int \frac{1}{r^2} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr + C\,, \qquad (2.8.8)$$

where C is an integration constant which can be found from the condition that pressure is equal to zero at an infinite distance from a sphere-like gaseous body's center: $p(\infty) = 0$.

The integral in (2.8.8) will be calculated using the rule of integrating by parts [49]:

$$\int \frac{dr}{r^2} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx = -\frac{1}{r} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx +$$

$$+ \int \frac{1}{r} \cdot (-\alpha r) e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr + \int \frac{1}{r} \cdot e^{-\alpha r^2/2} r^2 e^{-\alpha r^2/2} dr =$$

$$= -\frac{1}{r} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx - \alpha \int e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} + \int e^{-\alpha r^2} r dr =$$

$$= -\frac{1}{r} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx - \alpha \int e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr - \frac{1}{2\alpha} \cdot \int e^{-\alpha r^2} d(-\alpha r^2) =$$

$$= -\frac{1}{r} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx - \alpha \int e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr - \frac{1}{2\alpha} \cdot e^{-\alpha r^2}\,. \qquad (2.8.9)$$

Let us simplify the second integral in the given expression (2.8.9) using relation (2.4.22) [47, 73]:

$$\int e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr = \int e^{-\alpha r^2/2} \frac{1}{\alpha} \int_0^r e^{-\alpha x^2/2} dx dr +$$

$$+ \int e^{-\alpha r^2/2} \left(-\frac{1}{\alpha} r e^{-\alpha r^2/2} \right) dr = \frac{1}{\alpha} \int e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx dr +$$

$$+ \frac{1}{2\alpha^2} \int e^{-\alpha r^2} d(-\alpha r^2) = \frac{1}{2\alpha} \left(\int_0^r e^{-\alpha x^2/2} dx \right)^2 + \frac{1}{2\alpha^2} e^{-\alpha r^2}. \qquad (2.8.10)$$

Substituting (2.8.10) in (2.8.9) we obtain, as a result:

$$\int \frac{1}{r^2} e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx dr =$$

$$= -\frac{1}{r} e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx - \frac{1}{2}\left(\int_0^r e^{-\alpha x^2/2} dx\right)^2 - \frac{1}{2\alpha} e^{-\alpha r^2} - \frac{1}{2\alpha} e^{-\alpha r^2} =$$

$$= -\frac{1}{r} \cdot e^{-\alpha r^2/2} \int_0^r x^2 e^{-\alpha x^2/2} dx - \frac{1}{2} \cdot \left(\int_0^r e^{-\alpha x^2/2} dx\right)^2 - \frac{1}{\alpha} \cdot e^{-\alpha r^2} \qquad (2.8.11)$$

For simplifying (2.8.11) we shall again use the relation (2.4.22) with provision for which we obtain:

$$-\frac{1}{r} e^{-\alpha r^2/2}\left(\frac{1}{\alpha}\int_0^r e^{-\alpha x^2/2} dx - \frac{1}{\alpha} r e^{-\alpha r^2/2}\right) - \frac{1}{2}\left(\int_0^r e^{-\alpha x^2/2} dx\right)^2 - \frac{1}{\alpha} e^{-\alpha r^2} =$$

$$= -\frac{1}{\alpha r} \cdot e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx + \frac{1}{\alpha} e^{-\alpha r^2} - \frac{1}{2} \cdot \left(\int_0^r e^{-\alpha x^2/2} dx\right)^2 - \frac{1}{\alpha} \cdot e^{-\alpha r^2} =$$

$$= -\frac{1}{\alpha r} \cdot e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx - \frac{1}{2} \cdot \left(\int_0^r e^{-\alpha x^2/2} dx\right)^2 . \qquad (2.8.12)$$

At last, substituting (2.8.12) into (2.8.8) we obtain an expression for pressure [47,73]:

$$p(r) = 4\pi\gamma\rho_0{}^2\left[\frac{1}{\alpha r} \cdot e^{-\alpha r^2/2} \int_0^r e^{-\alpha x^2/2} dx + \frac{1}{2} \cdot \left(\int_0^r e^{-\alpha x^2/2} dx\right)^2\right] + C =$$

$$= 4\pi\gamma\rho_0{}^2 \int_0^r e^{-\alpha x^2/2} dx \left[\frac{1}{\alpha r} \cdot e^{-\alpha r^2/2} + \frac{1}{2} \cdot \int_0^r e^{-\alpha x^2/2} dx\right] + C . \qquad (2.8.13)$$

Considering that $p(\infty) = 0$ we can determine C:

$$C = -4\pi\gamma\rho_0{}^2 \int_0^\infty e^{-\alpha x^2/2} dx \cdot \frac{1}{2} \int_0^\infty e^{-\alpha x^2/2} dx . \qquad (2.8.14)$$

Taking into account that $\int_0^\infty e^{-\alpha x^2/2}dx = \sqrt{\frac{2}{\alpha}}\int_0^\infty e^{-s^2}ds = \sqrt{\frac{\pi}{2\alpha}}$ the relation (2.8.14) will be written down in the form:

$$C = -4\pi\gamma\rho_0{}^2\frac{1}{2}\cdot\frac{\pi}{2\alpha} = -\frac{\gamma(\pi\rho_0)^2}{\alpha}. \qquad (2.8.15)$$

Substituting (2.8.15) in (2.8.13) we shall finally obtain the expression for pressure inside a sphere-like gaseous body [47, 49, 73]:

$$p(r) = 4\pi\gamma\rho_0{}^2\int_0^r e^{-\alpha x^2/2}dx\left[\frac{1}{\alpha r}\cdot e^{-\alpha r^2/2} + \frac{1}{2}\int_0^r e^{-\alpha r^2/2}dx\right] - \frac{\gamma(\pi\rho_0)^2}{\alpha} =$$

$$= 4\pi\gamma\rho_0^2\left[\frac{1}{2}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 + \frac{1}{\alpha r}\cdot e^{-\alpha r^2/2}\int_0^r e^{-\alpha x^2/2}dx - \frac{\pi}{4\alpha}\right]. \qquad (2.8.16)$$

Using the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$ [128] we shall transform (2.8.16) into the form:

$$p(r) = 4\pi\gamma\rho_0{}^2\left[\frac{1}{2}\cdot\frac{2}{\alpha}\cdot\left(\int_0^{r\sqrt{\alpha/2}} e^{-s^2}ds\right)^2 + \frac{1}{\alpha r}\cdot e^{-\alpha r^2/2}\sqrt{\frac{2}{\alpha}}\cdot\int_0^{r\sqrt{\alpha/2}} e^{-s^2}ds - \frac{\pi}{4\alpha}\right] =$$

$$= \frac{\gamma(\pi\rho_0)^2}{\alpha}\cdot\left[\mathrm{erf}^2\left(r\sqrt{\alpha/2}\right) + \frac{2}{r}\cdot\sqrt{\frac{2}{\pi\alpha}}\cdot e^{-\alpha r^2/2}\mathrm{erf}\left(r\sqrt{\alpha/2}\right) - 1\right].$$

With provision for the relation

$$\frac{2}{\sqrt{\pi}}\cdot e^{-\alpha r^2/2} = \frac{d}{d\left(r\sqrt{\alpha/2}\right)}\cdot\mathrm{erf}\left(r\sqrt{\alpha/2}\right),$$

we shall finally obtain [47, 73]:

$$p(r) = \frac{\gamma(\pi\rho_0)^2}{\alpha}\cdot[\mathrm{erf}^2\left(r\sqrt{\alpha/2}\right) +$$

$$+ \frac{1}{r\sqrt{\alpha/2}}\cdot\left(\mathrm{erf}\left(r\sqrt{\alpha/2}\right)\right)'\cdot\mathrm{erf}\left(r\sqrt{\alpha/2}\right) - 1]. \qquad (2.8.17)$$

Thus, the formula:

$$p(r)=\frac{\gamma(\pi\rho_0)^2}{\alpha}\cdot Q\left(r\sqrt{\alpha/2}\right) \qquad (2.8.18)$$

is valid where $Q(x)=\text{erf}^2(x)+(1/x)\cdot(d\,\text{erf}(x)/dx)\cdot\text{erf}(x)-1$.

Let us investigate the pressure as a function $p=p(r)$. For this we shall find a derivative of p relative to r, using (2.8.16):

$$\frac{dp(r)}{dr}=4\pi\gamma\rho_0{}^2\cdot\frac{1}{\alpha r^2}\cdot e^{-\alpha r^2/2}\cdot\left[re^{-\alpha r^2/2}-\int_0^r e^{-\alpha x^2/2}dx\right]. \qquad (2.8.19)$$

From the condition of extremum, it is not difficult to see that

$$\frac{1}{\alpha r^2}\cdot e^{-\alpha r^2/2}=0 \text{ with } r\to\infty,$$

and also

$$re^{-\alpha r^2/2}=\int_0^r e^{-\alpha x^2/2}dx,$$

whence

$$\alpha r^2 e^{-\alpha r^2/2}=0 \text{ with } r=0.$$

Let us find pressure values at the extremum points $r=0$ and $r=\infty$:

$$p(0)=4\pi\gamma\rho_0{}^2\cdot\left[\frac{1}{2}\lim_{r\to 0}\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2+\lim_{r\to 0}\frac{\int_0^r e^{-\alpha x^2/2}dx}{\alpha r}-\frac{\pi}{4\alpha}\right]=$$

$$=\frac{\pi\gamma\rho_0{}^2}{\alpha}\cdot(4-\pi); \qquad (2.8.20)$$

$$p(\infty)=4\pi\gamma\rho_0{}^2\left[\frac{1}{2}\left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2+\int_0^\infty e^{-\alpha x^2/2}dx\lim_{r\to\infty}\frac{e^{-\alpha r^2/2}}{\alpha r}-\frac{\pi}{4\alpha}\right]=$$

$$= 4\pi\gamma\rho_0^{\,2}\left[\frac{1}{2}\left(\sqrt{\frac{\pi}{2\alpha}}\right)^2 - \frac{\pi}{4\alpha}\right] = 0. \qquad (2.8.21)$$

As follows directly from (2.8.21), the relation is true:

$$\frac{\pi}{4\alpha} = (1/2)\cdot\left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2. \qquad (2.8.22)$$

Thus, $p = p(r)$ is a nonnegative function: $p(r) \geq 0$, because $p_{\max} = p(0) > 0$ (since $\rho_0 > 0, \alpha > 0$) and $p_{\min} = p(\infty) = 0$.

Let us find the inflection point of the given function, for which we shall seek the condition when $d^2 p(r)/dr^2 = 0$. It is not difficult to see from (2.8.19) that

$$\frac{d^2 p(r)}{dr^2} = 4\pi\gamma\rho_0^{\,2}\left[\frac{2}{\alpha r^3}\cdot e^{-\alpha r^2/2}\int_0^r e^{-\alpha x^2/2}dx + \frac{1}{r}\cdot e^{-\alpha r^2/2}\int_0^r e^{-\alpha x^2/2}dx - \right.$$

$$\left. - \frac{2}{\alpha r^2}\cdot e^{-\alpha r^2} - 2e^{-\alpha r^2}\right],$$

hence the condition of the second derivative being equal to zero means that

$$\frac{1}{r}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)\cdot\left(1 + \frac{2}{\alpha r^2}\right) = 2e^{-\alpha r^2/2}\left(1 + \frac{1}{\alpha r^2}\right) \qquad (2.8.23)$$

or

$$\frac{\int_0^r e^{-\alpha x^2/2}dx}{e^{-\alpha r^2/2}} = r\cdot\frac{\alpha r^2 + 1}{\alpha r^2 + 2}. \qquad (2.8.24)$$

Expressions (2.8.23) and (2.8.24) become identities when $r = \infty$ and $r = 0$ because all the functions are nonnegative ($\alpha r^2 + 1$ and $\alpha r^2 + 2$ vanish only at imaginary values of r which is impossible since r is a distance). Consequently, the inflection points coincide with those of maximum and minimum of the function $p(r)$.

Thus, the pressure is a monotonically diminishing positively defined function of r (see Fig. 2.8) which corresponds to its physical sense completely.

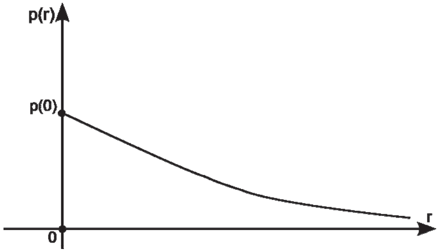


Figure 2.8. The graphic dependence of pressure on distance r into a sphere-like gaseous body

It should be noted that in the framework of GR one cannot always succeed in getting a pressure function agreeing with its physical sense. Thus, Sir A. S. Eddington, considering Schwartzschield's solution of the Einstein equations for a homogeneous liquid sphere of radius a, pointed out [183 p.318–319, 320]:

> Pressure vanishes at $r = a$ and would become negative if we tried to extend the solution to the boundary $r = a$. Therefore, a sphere is a boundary of a liquid. If one finds it necessary to extend the solution to the region outside the sphere, one should proceed from a different ds^2 (*interval*[1]) which corresponds to the vacuum equations… As long as the dimensions of a sphere are small, this difference does not result in great discrepancy; but for big spheres, pressure near the center is very large, and both solutions may differ (from each other) greatly. It is easily proved that for big spheres where $a > (5/9\alpha)^{1/2}$ the Schwartzschield solution gives in the

[1] The author's remark (see Section 2.7)

central point a negative value for T_i^k (*energy-impulse tensor*[2]). Therefore, even without reaching the boundary $a = (8/9\alpha)^{1/2}$, the solution does not seem to have any physical sense…

We note that since Schwartzshield's solution for a sphere of the non-compressible liquid has the form $\lambda = -\ln\left(1-\alpha r^2\right)$ where α is a certain constant, for small r's it turns into $\lambda = \alpha r^2$ which corresponds completely to the provisions of the statistical model proposed [45, 73] (see also Section 2.7).

2.9. The internal energy of a gravitating sphere-like gaseous body

Let us use (2.9.1) and the formula (2.4.26) for the gravitational potential of a sphere-like gaseous body:

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r e^{-\alpha x^2/2}dx, \qquad (2.9.1)$$

in order to transform expression (2.8.16) for $p(r)$ [47, 73]:

$$p(r) = \frac{4\pi\gamma\rho_0^{\,2}}{\alpha r}\cdot e^{-\alpha r^2/2}\int_0^r e^{-\alpha x^2/2}dx + 4\pi\gamma\rho_0^{\,2}\cdot\left[\frac{1}{2}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \frac{\pi}{4\alpha}\right] =$$

$$= -\left(-\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r e^{-\alpha x^2/2}dx\right)\cdot\rho_0 e^{-\alpha r^2/2} +$$

$$+ 4\pi\gamma\rho_0^{\,2}\cdot\left[\frac{1}{2}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \frac{1}{2}\cdot\left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2\right] =$$

$$= -\rho(r)\cdot\varphi_g(r) + 2\pi\gamma\rho_0^{\,2}\cdot\left[\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2\right]. \qquad (2.9.2)$$

---

[2] The author's comment (see Section 2.7)

Considering that a gravitating sphere-like gaseous body is at the temperature close to the absolute zero, that is, at $T \to 0$, the basic thermodynamic relation [94, 110] will assume the following form:

$$dU_g = -pdV\,, \tag{2.9.3}$$

where dU_g is an infinitesimal change of internal gravitational energy of a gravitating sphere-like gaseous body.

Substituting (2.9.2) into (2.9.3) we obtain [47, 73]:

$$dU_g = \rho(r)\cdot\varphi_g(r)dV - 2\pi\gamma\rho_0^{\,2}\left[\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2\right]dV\,,$$

hence the internal energy of a gravitating sphere-like body is equal to

$$U_g = \int_V \rho(r)\varphi_g(r)dV - 2\pi\gamma\rho_0^{\,2}\int_V\left[\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2\right]dV =$$

$$= U_{g1} + U_{g2}\,. \tag{2.9.4}$$

In (2.9.4) the first addend U_{g1} is, as we known [100], a double of the potential energy of any distribution of masses:

$$E_g = \frac{1}{2}\int_V \rho\varphi_g dV\,, \tag{2.9.5}$$

that is, $U_{g1} = 2E_g$. In the case of a gravitating sphere-like gaseous body for calculating E_g and U_{g2} it is relevant to use the spherical system of coordinates in which $dV = r^2 \sin\theta drd\theta d\varepsilon$. According to (2.5.6) the potential energy of a gravitating sphere-like body [45, 46]:

$$E_g = -4\gamma\rho_0^{\,2}\left(\frac{\pi}{\alpha}\right)^{5/2} = -\frac{\gamma M^2}{2}\cdot\sqrt{\frac{\alpha}{\pi}}\,. \tag{2.9.6}$$

Consequently, the addend U_{g1} is equal to:

$$U_{g1} = -8\gamma\rho_0^{\,2}\left(\frac{\pi}{\alpha}\right)^{5/2} = -\gamma M^2\sqrt{\frac{\alpha}{\pi}}. \qquad (2.9.7)$$

Let us calculate the second addend in (2.9.4) [47, 49, 73]:

$$U_{g2} = -2\pi\gamma\rho_0^2\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\left[\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2\right]r^2\sin\theta d\theta d\varepsilon dr =$$

$$= 8\pi^2\gamma\rho_0^{\,2}\int_0^{\infty} r^2\left[\left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^r e^{-\alpha x^2/2}dx\right)^2\right]dr\,. \qquad (2.9.8)$$

We can calculate the integral in (2.9.8) using the method of integrating by parts [47, 49, 73]:

$$U_{g2} = 8\gamma(\pi\rho_0)^2\int_0^{\infty} r^2\left[\left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^r e^{-\alpha x^2/2}dx\right)^2\right]dr =$$

$$= 8\gamma(\pi\rho_0)^2\left\{\frac{r^3}{3}\cdot\left[\left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^r e^{-\alpha x^2/2}dx\right)^2\right]\Bigg|_0^{\infty} -\right.$$

$$\left. -\int_0^{\infty}\frac{r^3}{3}\cdot\left(-2\int_0^r e^{-\alpha x^2/2}dx\right)\cdot e^{-\alpha r^2/2}dr\right\}. \qquad (2.9.9)$$

Let us apply the L'Hospital rule for calculating the first addend in braces in (2.9.9):

$$\frac{r^3}{3}\cdot\left[\left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^r e^{-\alpha x^2/2}dx\right)^2\right]\Bigg|_0^{\infty} =$$

$$= \lim_{r\to\infty}\frac{1}{3}\cdot\frac{\left(\int_0^{\infty} e^{-\alpha x^2/2}dx\right)^2 - \left(\int_0^r e^{-\alpha x^2/2}dx\right)^2}{1/r^3} -$$

$$-\lim_{r\to 0}\frac{1}{3}\cdot r^3\left[\left(\int_0^\infty e^{-\alpha x^2/2}dx\right)^2-\left(\int_0^r e^{-\alpha x^2/2}dx\right)^2\right]=$$

$$=\lim_{r\to\infty}\frac{1}{3}\cdot\frac{-2e^{-\alpha r^2/2}\int_0^r e^{-\alpha x^2/2}dx}{-3\cdot 1/r^4}-0=$$

$$=\lim_{r\to\infty}\frac{r^4}{e^{\alpha r^2/2}}\cdot\frac{2}{9}\int_0^\infty e^{-\alpha x^2/2}dx=0\,. \qquad (2.9.10)$$

With provision for (2.9.10) the relation (2.9.9) will assume the form [47, 49, 73]:

$$U_{g2}=\frac{16\gamma(\pi\rho_0)^2}{3}\int_0^\infty r^3e^{-\alpha r^2/2}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)dr\,. \qquad (2.9.11)$$

For calculating (2.9.11) let us use the rule of integrating by parts:

$$U_{g2}=\frac{16\gamma(\pi\rho_0)^2}{3}\int_0^\infty r^3e^{-\alpha r^2/2}\cdot\left(\int_0^r e^{-\alpha x^2/2}dx\right)dr=$$

$$=\frac{16\gamma(\pi\rho_0)^2}{3}\cdot\left[\int_0^r e^{-\alpha x^2/2}dx\cdot\left(\int r^3e^{-\alpha r^2/2}dr\right)\Bigg|_0^\infty-\right.$$

$$\left.-\int_0^\infty e^{-\alpha r^2/2}\cdot\left(\int r^3e^{-\alpha r^2/2}dr\right)dr\right]. \qquad (2.9.12)$$

To calculate (2.9.12) it is necessary to find the indefinite integral placed in round brackets. With this aim we shall use the rule of integration by parts for finding the following indefinite integral:

$$\int r\cdot e^{-\alpha r^2/2}dr=\frac{1}{2}\cdot r^2\cdot e^{-\alpha r^2/2}+\frac{\alpha}{2}\cdot\int r^3\cdot e^{-\alpha r^2/2}dr\,,$$

hence

$$\int r^3e^{-\alpha r^2/2}dr=\frac{1}{\alpha}\cdot\left[2\int re^{-\alpha r^2/2}dr-r^2e^{-\alpha r^2/2}\right]=$$

$$=\frac{1}{\alpha}\cdot\left[\int e^{-\alpha r^2/2}dr^2-r^2e^{-\alpha r^2/2}\right]=-\frac{1}{\alpha}\cdot e^{-\alpha r^2/2}\left(\frac{2}{\alpha}+r^2\right). \quad (2.9.13)$$

Substituting (2.9.13) into (2.9.12) we obtain [47, 49, 73]:

$$U_{g2}=\frac{16\gamma(\pi\rho_0)^2}{3}\cdot\left[-\frac{1}{\alpha}\cdot e^{-\alpha r^2/2}\left(\frac{2}{\alpha}+r^2\right)\int_0^r e^{-\alpha x^2/2}dx\Big|_0^\infty+\right.$$

$$\left.+\int_0^\infty\frac{1}{\alpha}\cdot e^{-\alpha r^2/2}\left(\frac{2}{\alpha}+r^2\right)\cdot e^{-\alpha r^2/2}dr\right]=\frac{16\gamma(\pi\rho_0)^2}{3\alpha}\cdot\int_0^\infty e^{-\alpha r^2}\left(\frac{2}{\alpha}+r^2\right)dr=$$

$$=\frac{16\gamma(\pi\rho_0)^2}{3\alpha}\cdot\left[\int_0^\infty r^2e^{-\alpha r^2}dr+\frac{2}{\alpha}\cdot\int_0^\infty e^{-\alpha r^2}dr\right]. \quad (2.9.14)$$

Using the rule of integration by parts let us express the integrals in (2.9.14), one through the other:

$$\int_0^\infty e^{-\alpha r^2}dr=re^{-\alpha r^2}\Big|_0^\infty-\int_0^\infty r\cdot(-2\alpha r)\cdot e^{-\alpha r^2}dr=2\alpha\int_0^\infty r^2e^{-\alpha r^2}dr,$$

whence

$$\int_0^\infty r^2e^{-\alpha r^2}dr=\frac{1}{2\alpha}\cdot\int_0^\infty e^{-\alpha r^2}dr. \quad (2.9.15)$$

Substituting (2.9.15) into (2.9.14) we obtain [47, 49, 73]:

$$U_{g2}=\frac{16\gamma(\pi\rho_0)^2}{3\alpha}\cdot\left(\frac{1}{2\alpha}+\frac{2}{\alpha}\right)\int_0^\infty e^{-\alpha r^2}dr=\frac{16\gamma(\pi\rho_0)^2}{3\alpha^2}\cdot\frac{5}{2}\int_0^\infty e^{-\alpha r^2}dr=$$

$$=\frac{16\gamma(\pi\rho_0)^2}{3\alpha^2}\cdot\frac{5}{2}\cdot\frac{1}{\sqrt{\alpha}}\int_0^\infty e^{-s^2}ds=\frac{40\gamma(\pi\rho_0)^2}{3\alpha^2}\cdot\frac{1}{\sqrt{\alpha}}\cdot\frac{\sqrt{\pi}}{2}=$$

$$=\frac{20\gamma\rho_0^{\,2}}{3}\cdot\left(\frac{\pi}{\alpha}\right)^{5/2}=6\frac{2}{3}\gamma\rho_0^{\,2}\left(\frac{\pi}{\alpha}\right)^{5/2} \quad (2.9.16)$$

As a result, according to (2.9.4) and with provision for (2.9.7) and (2.9.16), the internal energy of a gravitating body is equal to [47, 49, 73]:

$$U_g = U_{g1} + U_{g2} = -8\gamma\rho_0{}^2\left(\frac{\pi}{\alpha}\right)^{5/2} + 6\frac{2}{3}\gamma\rho_0{}^2\left(\frac{\pi}{\alpha}\right)^{5/2} =$$

$$= -1\frac{1}{3}\gamma\rho_0{}^2\left(\frac{\pi}{\alpha}\right)^{5/2} = -\frac{4}{3}\gamma\rho_0{}^2\left(\frac{\pi}{\alpha}\right)^{5/2}. \tag{2.9.17}$$

Taking into account that $\rho_0 = M(\alpha / 2\pi)^{3/2}$ we can transform (2.9.17) into the form [47, 49, 73]:

$$U_g = -\frac{4}{3}\gamma M^2\left(\frac{\alpha}{2\pi}\right)^3 \cdot \left(\frac{\pi}{\alpha}\right)^{5/2} = -\frac{1}{6}\gamma M^2 \cdot \sqrt{\frac{\alpha}{\pi}}. \tag{2.9.18}$$

It is easy to see from (2.9.18) that

$$\alpha = \frac{36\pi}{\gamma^2} \cdot \frac{U_g{}^2}{M^4} = \pi \cdot \left(\frac{6U_g}{\gamma M^2}\right)^2. \tag{2.9.19}$$

Formula (2.9.19) expresses the dependence of α on the internal energy of a gravitating sphere-like body. Similarly to (2.5.11), the internal energy can be expressed through the value $\bar{k} = \gamma\sqrt{\alpha / \pi}$ as follows:

$$U_g = -\frac{\bar{k}M^2}{6}. \tag{2.9.20}$$

The internal energy (2.9.18) in magnitude is less than the potential one (2.9.6) of a gravitating sphere-like gaseous body:

$$E_g - U_g = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} - \left(-\frac{\gamma M^2}{6} \cdot \sqrt{\frac{\alpha}{\pi}}\right) = -\frac{\gamma M^2}{3} \cdot \sqrt{\frac{\alpha}{\pi}}. \tag{2.9.21}$$

Namely, according to (2.9.6) and (2.9.18) the internal energy due to the pressure inside a gravitating sphere-like gaseous body is three times less than the potential energy of this body:

$$\frac{|E_g|}{|U_g|} = \frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \Bigg/ \frac{\gamma M^2}{6} \cdot \sqrt{\frac{\alpha}{\pi}} = 3. \tag{2.9.22}$$

Integrating the formula (2.9.3) we obtain:

$$U_g = -\int_V p dV \,. \tag{2.9.23}$$

Analogously, if we represent the potential energy E_g of a gravitating sphere-like gaseous body in the form:

$$E_g = \int_V w dV \,, \tag{2.9.24}$$

where w is an energy density of gravitational potential energy [47], then as follows from (2.9.22), (2.9.23), and (2.9.24), the pressure p itself is three times less than the gravitational potential energy density $|w|$ of a sphere-like gaseous body [73]. This conclusion about the relationship between p and w is in complete agreement with the results of Einstein's GR [81, 100] and with the Nicolis–Prigogine cosmological model [135 p.322] of the irreversible process of particle formation due to gravitational energy, according to which $p = |w|/3$.

2.10. The Jeans mass and the number of particles needed for gravitational binding of a sphere-like gaseous body

In order for a substance bunch to form a gravitationally coupled system, it is necessary that its gravitational energy exceeds the internal one. The gravitational potential energy of a bunch which is a sphere-like gaseous body of mass M is determined by relation (2.9.6), that is,

$$E_g = -\frac{\gamma M^2}{2r_+} \,, \tag{2.10.1}$$

where r_+ is an effective radius of a sphere-like gaseous body. The internal energy of a sphere-like gaseous body (2.9.17) can be expressed through the maximum pressure (2.8.20) and the effective radius by the following relation:

$$U_g = -\frac{4}{3(4-\pi)} \cdot p_0 \left(\frac{\pi}{\alpha}\right)^{3/2} = -\frac{4}{3(4-\pi)} \cdot p_0 r_+{}^3, \qquad (2.10.2)$$

where $p_0 = p(0)$. Consequently, according to the formula (1.5.27) (see Section 1.5) the gravitational binding (compression) will prevail if

$$\left|E_g\right| > \left|U_g\right|, \qquad (2.10.3)$$

that is, with provision for (2.10.1) and (2.10.2) we have:

$$\gamma M^2 / 2r_+ > \frac{4}{3(4-\pi)} p_0 r_+{}^3,$$

whence

$$\gamma M^2 > \frac{8}{3(4-\pi)} p_0 r_+{}^4. \qquad (2.10.4)$$

Taking into account that the maximum density $\rho_0 = M(\alpha / 2\pi)^{3/2} = \left(M / 2^{3/2}\right) \cdot \left(1 / r_+{}^3\right)$, it is not difficult to see that

$$r_+ = \frac{1}{\sqrt{2}} \cdot \left(\frac{M}{\rho_0}\right)^{1/3}. \qquad (2.10.5)$$

Using (2.10.4) and (2.10.5) we rewrite the condition of gravitational tightening in the form:

$$\gamma M^2 > \frac{2}{3(4-\pi)} \cdot p_0 \left(\frac{M}{\rho_0}\right)^{4/3},$$

from where

$$M > \left(\frac{2}{3(4-\pi)}\right)^{3/2} \cdot \frac{p_0{}^{3/2}}{\gamma^{3/2} \rho_0{}^2}. \qquad (2.10.6)$$

Following formula (1.5.28) from Section 1.5, on the right-hand side of (2.10.6), the value known in the literature as the *Jeans mass* [101]:

$$M_J \sim \frac{p^{3/2}}{\gamma^{3/2}\rho^2}\,. \tag{2.10.7}$$

As noted in Section 1.5 (see (1.5.21a,b), (1.5.22a,b), (1.5.26), and (1.5.28)), the Jeans mass is the minimum mass to be gravitationally binding at given density and pressure. It is caused by the fact that the force of gravitation inside any originating substance bunch increases with the increase of the bunch size whereas the pressure does not depend on the size [101]. Let us note that according to (1.5.16) the average mass (1.5.21a) of a gas-dust bunch becomes equal to

$$M_c \sim \rho\,\frac{1}{\rho^3}\cdot\left(\frac{\pi\kappa p}{\gamma}\right)^{3/2} \sim \frac{\kappa^{3/2}p^{3/2}}{\gamma^{3/2}\rho^2}\,, \tag{2.10.8}$$

that is, it practically coincides with the *Jeans mass*: $M_c \sim \kappa^{3/2} M_J$ in accordance with formulas (2.10.7) and (2.10.8).

Now if we take into account that $M = m_0 N$ in (2.10.6) then we get:

$$N > \left(\frac{2}{3(4-\pi)}\right)^{3/2} \cdot \frac{p_0^{\ 3/2}}{\gamma^{3/2}\rho_0^{\ 2} m_0}\,. \tag{2.10.9a}$$

From (2.10.9a), up to an insignificant numerical factor, we determine the number of particles necessary for the gravitational binding estimated by the internal state values ρ_0 and p_0 in the center of a sphere-like gaseous body [73]:

$$N_J^{(0)} \sim \frac{p_0^{\ 3/2}}{\gamma^{3/2}\rho_0^{\ 2} m_0}\,. \tag{2.10.9b}$$

As seen from (2.10.6) and (2.10.9b), for a sphere-like gaseous body, the estimations of the Jeans mass or the Jeans number can be determined by pressure and density *in the center* of this body.

The value $N_J^{(0)}$ will be called the Jeans number depending on ρ_0 and p_0. Thus, the gravitational binding in a substance clot of mass M (having the number of particles N) occurs with the following condition fulfilled:

$$M > M_J^{(0)},$$

where $M_J^{(0)}$ is determined according to (2.10.7) (this condition for the same substance being $N > N_J^{(0)}$).

On the other hand, it is intuitively comprehensible that any sphere-like gaseous body has a mass no less than the Jeans mass. Indeed, substituting into (2.10.7) maximum values of pressure p_0 and density ρ_0 we obtain [73]:

$$M_J^{(0)} \sim \frac{{p_0}^{3/2}}{\gamma^{3/2}{\rho_0}^2} = \frac{(\pi\gamma/\alpha)^{3/2}{\rho_0}^3(4-\pi)^{3/2}}{\gamma^{3/2}{\rho_0}^2} = \left(\frac{\pi}{\alpha}\right)^{3/2}\rho_0(4-\pi)^{3/2} =$$

$$= \left(\frac{\pi}{\alpha}\right)^{3/2} M\left(\frac{\alpha}{\pi}\right)^{3/2}\frac{1}{2^{3/2}}(4-\pi)^{3/2} = M\left(\frac{4-\pi}{2}\right)^{3/2},$$

whence

$$M \sim \left(\frac{2}{4-\pi}\right)^{3/2} M_J^{(0)}. \qquad (2.10.10)$$

According to (3.3.10) the mass M of a sphere-like gaseous body is $\left(2/(4-\pi)\right)^{3/2} \approx 3$ times larger than the Jeans mass $M_J^{(0)}$ (or is comparable to it). In the most general case, a similar conclusion can be obtained [73]. Indeed, to calculate the Jeans mass M_J we use the relations (2.9.6) and (2.9.18) under the following condition:

$$\left|E_g\right| = \left|U_g\right|, \qquad (2.10.11)$$

that is, when the gravitational potential energy is compared with the internal energy. Under this condition, the process of gravitational tightening (binding) can begin, so that the inequality (2.10.3) becomes valid.

So, according to the condition (2.10.11) and the relations (2.10.1) and (2.10.2) we have:

$$\gamma M_J^2 / 2r_+ = [4/3(4-\pi)] \cdot p_0 r_+{}^3,$$

whence, taking into account the value $p_0 = p(0)$ (2.8.20), we obtain:

$$M_J^2 = \frac{8}{3} \cdot \frac{\pi \rho_0^2}{\alpha} \cdot r_+{}^4 . \qquad (2.10.12)$$

Then, substituting in (2.10.12) the value of the effective radius (2.5.8) of the sphere-like gaseous body we obtain the equality:

$$M_J^2 = \frac{8}{3} \cdot \left(\frac{\pi}{\alpha}\right)^3 \cdot \rho_0^2 ,$$

which after substitution of the maximum density value $\rho_0 = M \cdot (\alpha / 2\pi)^{3/2}$ becomes:

$$M_J^2 = \frac{1}{3} M^2 . \qquad (2.10.13)$$

Obviously, the relation (2.10.13) has been obtained yet in accordance with (2.9.22).

Thus, according to the result obtained (2.10.13), a gravitational field is present in a *formed* sphere-like gaseous body, causing a process of gravitational contraction, since the mass M of the formed sphere-like body exceeds $\sqrt{3}$ times the Jeans critical mass M_J [73]:

$$M = \sqrt{3} M_J . \qquad (2.10.14)$$

A comparison of (2.10.10) with (2.10.14) shows that the formation of a gravitating sphere-like gaseous body with parameters near the center gives a Jeans mass $M_J^{(0)}$ $\sqrt{3}$ times smaller than the Jeans mass M_J formed over the sphere-like body as a whole. As mentioned in Section 2.3, there is a critical value α_c of the gravitational condensation parameter defined by the formulas (2.3.7a, b) when the initial gravitational interactions of particles within the anti-diffusion process are amplified coherently and form a gravitational field leading to the gravitational compression of a sphere-like gaseous body. According to formulas (2.10.10) and (2.10.14) the field gravitational interactions of particles cover not all particles, that is, the total mass M of a sphere-like gaseous body, but only its part is determined by the Jeans mass M_J. Moreover, if the particles are chosen near the center of a sphere-like gaseous body where the density and pressure are maximal, that is, particle interactions are more intense, then the Jeans number $N_J^{(0)}$ of particles according to (2.10.9b), and therefore the Jeans mass $M_J^{(0)}$, is about three times less than the total number N of particles and, accordingly, the total mass M of a sphere-like gaseous body. On average, the Jeans mass M_J (and, therefore, the Jeans number N_J) is $\sqrt{3}$ times smaller than the total mass M (total number of particles N) of a sphere-like gaseous body according to formula (2.10.14).

The approximate dimension of a *domain of field gravitational interactions* of particles can be estimated using formula (2.3.8). So, near the center of a sphere-like gaseous body, the domain of gravitational field interactions (whose mass is equal $M_J^{(0)}$) is characterized by a sphere of the following radius [73]:

$$R_J^{(0)} = (2/\pi)^{1/6} \cdot (\gamma M_J^{(0)} / \omega_c^2)^{1/3} \approx (2/\pi)^{1/6} \cdot (\gamma M / 3\omega_c^2)^{1/3} =$$
$$= (2/9\pi)^{1/6} \cdot (\gamma M / \omega_c^2)^{1/3}. \qquad (2.10.15a)$$

Inside a sphere-like gaseous body, the domain of field gravitational interactions with mass M_J is described by a sphere of the respective radius [73]:

$$R_J = (2/\pi)^{1/6} \cdot (\gamma M_J / \omega_c^2)^{1/3} = (2/\pi)^{1/6} \cdot (\gamma M / \sqrt{3} \cdot \omega_c^2)^{1/3} =$$
$$= (2/3\pi)^{1/6} \cdot (\gamma M / \omega_c^2)^{1/3}. \qquad (2.10.15b)$$

According to (2.3.8), (2.10.15a), and (2.10.15b), the ratio of the radii $R_J^{(0)}$, R_J to the radius R_c shows that the linear size of the domain of field gravitational interactions is $\sqrt[3]{3}$ times smaller (near the center) and $\sqrt[6]{3}$ times smaller (on the average) relative to the linear size of the area of the initial gravitational interactions of particles of a sphere-like gaseous body.

Consequently, inside the domain of field interactions of particles the parameter of gravitational condensation $\alpha_J^{(0)}$ (or α_J) is higher than inside a sphere-like gaseous body entirely (being formed as a result of the initial gravitational interactions of particles). In this regard, it is important to investigate the equations of the initial gravitational interaction of particles (see Chapters 4 and 5) to explain the mechanism of the origin of field gravitational interactions inside a sphere-like gaseous body. As the final result, let us consider the following general theorem [73]:

Theorem 2.2 (the necessary and sufficient conditions of mass distribution of an isolated immovable molecular cloud in its gravitational field).

A gravitating isolated immovable molecular cloud is in a state of virial equilibrium if and only if its mass density distribution satisfies the law:

$$\rho(r) = \rho_0 e^{-\frac{\alpha}{2} r^2}, \qquad \alpha = \frac{4\pi\gamma m_0 \sigma}{3k_B T}, \qquad (2.10.16)$$

where

$\rho_0 = \rho(0)$,

α is a parameter of gravitational condensation,

γ and k_{B} are the constants of Newton and Boltzmann respectively,
m_0 is a mass of a molecule,
T is a temperature, and
σ is an initial mass density of an isolated immovable molecular cloud.

Proof: first of all, let us derive the mass density distribution of particles of an isolated immovable molecular cloud in its gravitational field under the condition of virial mechanical equilibrium. Under the condition of mechanical (hydrostatic) equilibrium of a gas (or liquid) in a field of force, the Euler equation (2.8.1) is true. Within the framework of the molecular kinetic theory, according to the known Clapeyron–Mendeleev equation of state of an *ideal* gas [110, 136] (or the Boyle–Charles law [1]), it follows that

$$p = nk_B T\,; \qquad (2.10.17a)$$

$$\rho = m_0 n\,, \qquad (2.10.17b)$$

where $n = n(x, y, z) = n(\vec{r})$ is a concentration of molecules of an ideal gas and m_0 is a mass of a molecule. Substituting the formulas (2.10.17a) and (2.10.17b) into Eq. (2.8.1) we obtain:

$$\mathrm{grad}(nk_B T) = nm_0 \vec{a}\,, \qquad (2.10.18)$$

where $\vec{a}$ is an acceleration, that is, the gravitational field strength $\vec{a} = -\mathrm{grad}\,\varphi_{\mathrm{g}}$. Taking into account that, following assumption 4, the molecular cloud is *isothermal* and has a low ($T \propto 10\,\mathrm{K}$) temperature (being in a state of thermodynamic equilibrium) T can be considered as independent on the spatial coordinates $\vec{r} = (x, y, z)$. In this connection, the differential operator of the gradient is equal to:

$$\mathrm{grad}(nk_B T) = k_B T \frac{\partial n}{\partial \vec{r}}. \quad (2.10.19)$$

Taking into account (2.4.5) and (2.10.19), Eq. (2.10.18) takes the form:

$$k_B T \frac{\partial n}{\partial \vec{r}} = -nm_0 \frac{\partial \varphi_{\mathrm{g}}}{\partial \vec{r}}, \quad (2.10.20a)$$

whence

$$k_B T \frac{dn}{n} = -m_0 d\varphi_{\mathrm{g}}. \quad (2.10.20b)$$

To find the change of the *interior* gravitational potential $d\varphi_{\mathrm{g}}$ of an isolated immovable molecular cloud let us use the formula (2.4.30) neglecting the exact shape of this cloud because the wavefront of arising gravitational perturbations is a sphere [95, 97]:

$$d\varphi_g = \frac{4}{3}\pi\gamma\sigma r dr, \quad (2.10.21)$$

where σ is an initial mass density of an immovable molecular cloud. Substituting the formula (2.10.21) in Eq. (2.10.20b) we obtain:

$$k_B T \frac{dn}{n} = -\frac{4}{3}\pi\gamma m_0 \sigma r dr. \quad (2.10.22)$$

Integrating Eq. (2.10.22) we find:

$$\int_{n_0}^{n} \frac{dn}{n} = -\frac{4\pi\gamma m_0 \sigma}{3k_B T} \int_0^r r dr \quad (2.10.23a)$$

and then

$$\ln \frac{n}{n_0} = -\frac{4\pi\gamma m_0 \sigma}{3k_B T} \frac{r^2}{2}. \quad (2.7.23b)$$

By denoting $\alpha = \frac{4\pi\gamma m_0 \sigma}{3k_B T}$, the concentration of molecules of an isolated immovable molecular cloud can be written in accordance with (2.7.23b):

$$n(r) = n_0 e^{-\frac{\alpha}{2}r^2}, \qquad \alpha = \frac{4\pi\gamma m_0 \sigma}{3k_B T}. \tag{2.10.24}$$

With provision for the relation (2.10.17b), we shall finally obtain the formula (2.10.16) of mass density for an isolated immovable molecular cloud in its gravitational field. The necessary condition is proved.

Now we are going to prove that the formula (2.10.16) corresponds to the mass density of gravitating isolated immovable molecular cloud is in a state of virial equilibrium. As shown in Section 2.1 (see the derivation based on formulas from (2.1.1) to (2.1.15)) and Section 2.2 (see the respective formulas from (2.2.1) to (2.2.5)), due to their own oscillatory interactions as well as originating gravitational forces a great number of particles form an isolated *isothermal* sphere-like gaseous body whose mass density is uniform in all directions at the same distance from the mass center:

$$\rho(r) = \rho_0 e^{-\frac{\alpha}{2}r^2}, \tag{2.10.25a}$$

where

$\rho_0 = \rho(0) = M(\alpha / 2\pi)^{3/2}$,

α is a parameter of gravitational condensation, and

M is a mass of a sphere-like gaseous body.

As noted there, if the parameter of gravitational condensation α exceeds a critical (threshold) value α_c then a gravitational field arises in the sphere-like gaseous body. Indeed, as shown in Section 2.3, initially oscillatory interactions of particles can lead to the gravitational instability of the sphere-like gaseous body with a critical wavelength λ_c. Gravitational instabilities

($\lambda \geq \lambda_c$) in a sphere-like gaseous body then form a gravitationally coupled bunch with the Jeans critical mass $M_c \propto \kappa^{3/2} M_J = \kappa^{3/2} M / \sqrt{3}$, as mentioned above, relative to the relations (2.10.8) in this section. As the final result, a gravitational field is originating in the sphere-like gaseous body under the given low temperature, so that the mass density of a *gravitating* sphere-like gaseous body satisfies the following formula in accordance with (2.3.7b), (2.10.14), (2.10.25a):

$$\rho(r) = \rho_0 e^{-\frac{\alpha}{2} r^2}, \qquad \alpha \geq \alpha_c, \tag{2.10.25b}$$

where $\alpha_c \propto \left(\omega_c^2 / \gamma M\right)^{2/3}$. As follows from the formula (2.10.25b), the parameter of gravitational condensation α can vary with the time. Therefore, under the condition of a steady-state virial equilibrium this parameter α can be expressed by the given temperature T according to formula (1.5.26) of a gravitating sphere-like gaseous body (by analogy, see the so-called USL in Chapter 8). As a result, the formula (2.10.16) follows directly from (2.10.25b). The theorem is proved.

Corollary 2.1. The evolution of the mass density of a gravitating sphere-like gaseous body with the time is described by the formula:

$$\rho(r,t) = \rho_0 e^{-\frac{\alpha(t)}{2} r^2}, \alpha(t) \geq \alpha_c,\ \alpha_c \propto \left(\omega_c^2 / \gamma M\right)^{2/3}, \tag{2.10.26}$$

whose the particular case is the formula (2.10.16) describing a state of virial equilibrium, where $\rho_0 = M(\alpha / 2\pi)^{3/2}$, α is a parameter of gravitational condensation, γ is the Newtonian gravitational constant, M is a mass of a molecular cloud, and ω_c is a critical frequency leading to the gravitational instability.

Corollary 2.2. The dynamics of the evolution process of a gravitating sphere-like gaseous body includes multivariate states of virial equilibrium.

Corollary 2.3. Gravitating sphere-like gaseous bodies can be the model of a gravitationally coupled bunch (planetesimal).

In fact, according to (2.10.8) a gravitationally coupled bunch possesses the Jeans critical mass $M_c \propto \kappa^{3/2} M_J = \kappa^{3/2} M / \sqrt{3}$.

Conclusion and comments

The principal conclusion resulting from the statistical model considered is that gravitating bodies have indistinct contours. It is necessary to note that some arguments in favor of the existence of fuzzy borders were partly expressed in some works. Thus, A. S. Eddington [183 p.320], considering Schwarzschild's solution to Einstein's equations for a homogeneous liquid sphere, pointed out that

> For large spheres … Schwarzschild's solution provides, in the central point, a negative value for T_i^k (*the energy-impulse tensor*[3]). Therefore, even before approaching the border $a = \sqrt{\dfrac{8}{9\alpha}}$, the solution seems to cease possessing any physical sense. It is most regretful that it is for large spheres that the solution stops being real, for *the existence of the upper border* for spheres is one of the most interesting points of the whole problem.

Note that, since Schwarzschild's solution for a sphere of an uncompressed liquid has the form: $\lambda = -\ln\left(1 - \alpha r^2\right)$, where α is a certain constant, for small r it transforms to $\lambda = \alpha r^2$, which corresponds completely to provisions of the proposed

---

[3] The author's remark (see Section 2.7)

statistical model (see, e.g., (2.7.18)). Further, since $a = \sqrt{\frac{8}{9\alpha}} \approx \bar{r}$ is an average distance (2.1.27), then for all $r < \bar{r}$ the statistical model agrees with that of Schwarzschild obtained in the framework of GR (see Section 2.7). It should also be noted that in both models the value α is constant. This is also confirmed by the results obtained by J.R. Oppenheimer and H. Snyder [184].

However, as was mentioned above by Eddington, Schwarzschild's model provides a noncontradictory description for small-size spheres and fits poorly in the case of large-size ones. But it is the latter ones that can be the basis for modeling the structure and evolution of stars. In connection with this, Ya. B. Zeldovich and I. D. Novikov, investigating in [182 p.355] the statistical characteristics of star substance distribution, noted that

> …the definition being unsatisfactory is the fact that a sphere *with a distinct border* is taken. The result is greatly affected by a random position of objects. It is necessary to introduce a weight function equal to 1 inside the sphere and decreasing smoothly to the edges, "blurring" thereby the distinctness of the border.

For investigating the distribution of objects on a given scale of the order R they determined the mass of a body as function $M(R) = 4\pi \int_0^\infty \rho \cdot e^{-r^2/R^2} r^2 dr$. Comparing this formula with (2.2.5), one can see that in the case $\rho = \rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}$, the value $\alpha = 2/R^2$. Though the definition $M(R)$ was introduced artificially, however,

> …it is very important that in the new definition the distinct border is absent. [182]

Finally, A. G. W. Cameron [10 p.443], considering the possible scenarios for star cores forming in dense molecular clouds, also pointed out that

> ...it is difficult to define a radius for these cores since their densities drop off smoothly from the central value toward the background, but most measurements would place the radius at about 0.1 parsec or less.

As shown in Section 2.2, within the framework of the proposed statistical theory the mass density (2.2.5) of a sphere-like body decreases in $\rho_0 / \rho(r) = 30$ times on a distance $r \approx 2.6 r_*$ from its center where $r_* = 1/\sqrt{\alpha}$ (in accord with (2.2.6)). This result satisfies Cameron's condition on the mass density decreasing 30 times away from a forming stellar core [10]. In this connection, the Kuiper hypothesis that the mass of a protoplanetary cloud $M_{protopl.\ cloud} \geq 0.1 M_{\text{Sun}}$ is also true. Thus, the derived functions of particle distribution and the mass density of a sphere-like gaseous body describe the first (protostellar) stage of evolution: from a molecular cloud to a forming core with its shell (protostar).

As noted in Section 2.5, according to the statistical model of an initial gravitational interaction of molecular cloud particles, with a fixed dynamic solution in time, the notion of "distinctness" or "blurring" of borders of gravitating bodies is highly relative. Thus, a gravitating body has a distinctly outlined shape, if the potential energy of the gravitational interaction of its particles is sufficiently great, the body mass itself and the masses of particles forming it being relatively small (in the framework of the given model the existence of massless particles is admitted) [45, 46]. So, ordinary macroscopic bodies have distinct contours due to their relatively small masses, whereas giant cosmic objects have fuzzy ones because of their huge masses and a vast number of particles forming them.

In this connection, in Section 2.6 the probabilistic interpretation of physical values describing the gravitational interaction of particles in a sphere-like gaseous body is considered. As a corollary of this approach, the analog of Newton's general theorem in statistical interpretation (Theorem 2.1) is proved:

> A spherical layer, bounded by two similar and similarly placed concentric spherical surfaces, inside a sphere-like gaseous body, does not exert attraction at a point into the internal domain of this layer.

Under the condition of mechanical equilibrium, the pressure (2.8.16) inside a sphere-like gaseous body [47, 49] is calculated in Section 2.8, while Section 2.9 shows that the internal energy associated with it is three times less than the potential energy of the gravitating sphere-like gaseous body (see the formula (2.9.22)). Section 2.10 is devoted to estimating the Jeans mass as well as the number of particles necessary for the gravitational binding of a sphere-like gaseous body. According to the result obtained (2.10.14), a gravitational field is present in the *formed* sphere-like gaseous body, causing a process of gravitational contraction, and the mass M of the formed sphere-like gaseous body exceeds the Jeans critical mass M_J by a factor of $\sqrt{3}$ times.

Resuming the results of this chapter, in Section 2.10 the general Theorem 2.2 is proved, which says:

> a gravitating isolated immovable molecular cloud is in a state of virial equilibrium if and only if its mass density distribution satisfies the law:
>
> $$\rho(r) = \rho_0 e^{-\frac{\alpha}{2}r^2}, \qquad \alpha = \frac{4\pi\gamma m_0 \sigma}{3k_B T},$$
>
> where

$\rho_0 = \rho(0)$,

α is a parameter of gravitational condensation,

γ and k_B are the constants of Newton and Boltzmann respectively,

m_0 is a mass of a molecule,

T is a temperature, and

σ is an initial mass density of an isolated immovable molecular cloud.

Thus, the statistical model has been developed to study in detail the complex dynamical picture of the gravitational interaction of particles. It, therefore, describes a certain fixed dynamical solution only. So, the principal provisions of the statistical model of particle gravitational interaction can be used as a basis for investigating the structure and evolution of large cosmic objects (stars and planetary systems).

CHAPTER THREE

FORMATION OF COSMOGONICAL BODIES BASED ON A STATISTICAL MODEL OF A ROTATING AND GRAVITATING SPHEROIDAL BODY

The previous chapter clarified the statistical theory of a non-rotating and slowly compressible sphere-like gaseous body formed by a set of interacting particles (isolated from the influence of external fields and bodies) in its gravitational field. According to Corollary 2.2 of Theorem 2.2, the general form of the probability volume density function (2.1.14) of a sphere-like gaseous body:

$$\Phi(r) = \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}r^2},$$

is not determined by the presence of gravitational field exclusively, that is, it is valid before the gravitational field origin when $\alpha < \alpha_c$, $\alpha_c \propto \left(\omega_c^2 / \gamma M\right)^{2/3}$. So, like the well-known Gibbs distribution [110]:

$$p(E_n) = \frac{1}{Z} e^{-\frac{E_n}{k_B T}},$$

where

$$Z = \sum_n e^{-\frac{E_n}{k_B T}},$$

E_n is an energy,

T is a temperature of a gas, and

k_B is the Boltzmann constant,

the proposed distribution (2.1.14) is, generally speaking, true both in macroscopic and microscopic applications. In the case of *microscopic application*, interacting molecules or atoms of gas can form new aggregate nano- and micro-particles called *colloidal* or *liquid* particles (in hydrodynamic meaning [94]). As pointed out in [94], any liquid particle having a small element of a medium volume is still considered to be large enough to contain many molecules. In turn, the liquid particles constitute a molecular cloud at low temperatures.

This chapter explores the formation of a predominantly slowly rotating and gravitating cosmogonical body from the molecular cloud based on the statistical model of the so-called *spheroidal body* [16, 73]. Here, as in the previous chapter, it is initially assumed that a rotating gravitating spheroidal body is homogeneous in chemical composition and isolated from the influence of other fields and bodies, being in a state of *relative mechanical equilibrium*. As a rule, the temperature of a rotating spheroidal body is supposed to be both close to absolute zero (with relatively small masses of spheroidal bodies, in particular, a protoplanetary cloud or disk) and high enough (for huge masses, for example, when describing stellar systems).

In connection with the foregoing, the following assumptions are used in this chapter:

1. The gaseous body (molecular cloud) under consideration is homogeneous in its chemical structure, that is, it consists of N identical liquid particles of the mass m_0;
2. The gaseous cloud is isolated one, that is, as a rule it is not subjected to influence from external fields and bodies, except for the model of the formation of a rotating disk;

3. The initial molecular cloud is isothermal and has a low temperature T (as a rule $T \propto 10\,\mathrm{K}$), or it has a sufficiently high temperature (for huge masses of gravitating gas in the state of virial equilibrium under star formation);
4. The initial process of the oscillating interaction of liquid particles is slow-flowing with the time;
5. Except for the initial rotation case and the model of a disk-shaped body, a rotating gaseous cloud is found in the relative mechanical equilibrium (the case of observed particle velocities $\vec{\mathrm{v}} \neq 0$), that is, the evolutional gaseous body is uniformly rotating as a whole.

Using these assumptions and the results of the previous chapter, this chapter is devoted to the study of statistical models of a rotating and gravitating spheroidal body to describe the evolution of a protoplanetary gas-dust cloud around a forming star (in particular, the proto-sun) based on the flatness process modeling from the initial sphere-like body (in the case of a non-rotating spheroidal body) through flattened ellipsoidal forms (in the case of a rotating and gravitating spheroidal body) to the protoplanetary disk (disk-shaped spheroidal body).

3.1. Poincaré's general theorem and Roche's model in statistical interpretation for a slowly rotating and gravitating cosmogonical body

As shown in the previous section, under the action of its own oscillatory interactions, a great number of particles form a sphere-like gaseous body from the molecular cloud. In this section, we consider a *slowly rotating* and gravitating gaseous body.

As noted in the introduction to this chapter, interacting molecules or atoms of gas form new aggregate nano- and micro-particles called the *liquid* particles [94]. These liquid particles constitute a slowly rotating sphere-like gaseous body that possesses angular velocity.

We assume that at any *initial time instant* of rotation of a sphere-like gaseous body, all liquid particles composing it have a variation in the values of angular velocity, and each of the liquid particles has its angular velocity vector $\vec{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ oriented randomly in space. Then, in the process of rotation of the sphere-like gaseous body, some equalization of the angular velocities of liquid particles takes place that allows us to consider the rotation of this body in the steady state as a whole.

So, according to the theorem on multiplying the probabilities of independent events, the probability that any liquid particle inside a rotating sphere-like gaseous body has coordinates in the vicinity of x, y, z and simultaneously the components of the vector of angular velocity lying in the intervals near $\Omega_x, \Omega_y, \Omega_z$ is equal to:

$$\begin{aligned} dp_{x,y,z,\Omega_x,\Omega_y,\Omega_z} &= dp_{x,y,z} \cdot dp_{\Omega_x,\Omega_y,\Omega_z} = \\ &= \Phi(x,y,z)\Phi(\Omega_x,\Omega_y,\Omega_z)dxdydzd\Omega_x d\Omega_y d\Omega_z . \end{aligned} \tag{3.1.1}$$

According to (3.1.1), we should introduce a joint probability volume density function $\Phi(\vec{r}, \vec{\Omega})$, which is the six-dimensional probability density to locate a liquid particle in a rotating sphere-like gaseous body:

$$\begin{aligned} \Phi(\vec{r}, \vec{\Omega}) &= \Phi(x, y, z, \Omega_x, \Omega_y, \Omega_z) = \\ &= \Phi(x, y, z)\Phi(\Omega_x, \Omega_y, \Omega_z) = \Phi(\vec{r})\Phi(\vec{\Omega}), \end{aligned} \tag{3.1.2}$$

so that

$$dp_{x,y,z,\Omega_x,\Omega_y,\Omega_z} = \Phi\left(x,y,z,\Omega_x,\Omega_y,\Omega_z\right) dxdydzd\Omega_x d\Omega_y d\Omega_z = \\ = \Phi\left(\vec{r},\vec{\Omega}\right) dV_\Omega \, dV = \Phi_\Omega^{eff}\left(\vec{r}\right) dV, \tag{3.1.3}$$

where

$\Phi_\Omega^{eff}(\vec{r})$ is an *effective* probability volume density,

$dV = dxdydz$, and

$dV_\Omega = d\Omega_x d\Omega_y d\Omega_z$ [52, 73].

To find the effective probability volume density $\Phi_\Omega^{eff}(\vec{r})$ in the case of a slowly rotating sphere-like gaseous body, it is necessary to determine $\Phi(\vec{r},\vec{\Omega})$. To this end, we should consider the phase space composed of the components of angular velocity $(\Omega_x,\Omega_y,\Omega_z)$. In connection with the foregoing, we assume that at an initial instant of rotation dN_{Ω_x} liquid particles have angular velocity x-components in the elementary interval $[\Omega_x,\Omega_x + d\Omega_x]$; dN_{Ω_y} liquid particles have angular velocity y-components in the elementary interval $[\Omega_y,\Omega_y + d\Omega_y]$; and dN_{Ω_z} liquid particles have angular velocity z-components in the elementary interval $[\Omega_z,\Omega_z + d\Omega_z]$.Since the sphere-like gaseous body contains N liquid particles, the probabilities that any liquid particle has angular velocity components in the indicated intervals equal, respectively:

$$dp_{\Omega_x} = \frac{dN_{\Omega_x}}{N} = \varphi(\Omega_x)d\Omega_x, \qquad dp_{\Omega_y} = \frac{dN_{\Omega_y}}{N} = \varphi(\Omega_y)d\Omega_y,$$

$$dp_{\Omega_z} = \frac{dN_{\Omega_z}}{N} = \varphi(\Omega_z)d\Omega_z, \tag{3.1.4}$$

where $\varphi(\Omega_x)$, $\varphi(\Omega_y)$ and $\varphi(\Omega_z)$ are the probability densities characterizing the share of liquid particles whose angular velocities belong to the elementary intervals near Ω_x,Ω_y and

Ω_z respectively. Similarly to formula (2.1.6) from Section 2.1, we can introduce the three-dimensional probability density as follows:

$$dp_{\Omega_x,\Omega_y,\Omega_z} = \frac{dN_{\Omega_x,\Omega_y,\Omega_z}}{N} = \Phi(\Omega_x,\Omega_y,\Omega_z)d\Omega_x d\Omega_y d\Omega_z . \quad (3.1.5)$$

Taking into account that $d\Omega_x d\Omega_y d\Omega_z = dV_\Omega$ is an element of volume in the phase space of angular velocities, and $\Omega = \sqrt{\Omega_x{}^2 + \Omega_y{}^2 + \Omega_z{}^2}$ is a value of angular velocity, we rewrite formula (3.1.5) in the form:

$$dp_{\Omega_x,\Omega_y,\Omega_z} = \Phi(\Omega)dV_\Omega . \quad (3.1.6)$$

It is clear that a rotating liquid particle has all three components $\Omega_x, \Omega_y, \Omega_z$ of the angular velocity and independently of each other. According to the theorem on multiplying the probabilities of independent events we obtain:

$$\begin{aligned} dp_{\Omega_x,\Omega_y,\Omega_z} &= dp_{\Omega_x} dp_{\Omega_y} dp_{\Omega_z} = \\ &= \varphi(\Omega_x)\varphi(\Omega_y)\varphi(\Omega_z)dV_\Omega . \end{aligned} \quad (3.1.7)$$

Comparing (3.1.7) with (3.1.6) it is not difficult to see that:

$$\Phi(\Omega) = \varphi(\Omega_x)\varphi(\Omega_y)\varphi(\Omega_z) . \quad (3.1.8)$$

Using (3.1.8) we can define the type of functions φ and Φ by analogy with the scheme described in Section 2.1 under derivation of the distribution function of particles in space (see also [16, 45, 46, 73]). Representing the functional equation (3.1.8) in the form of three partial differential equations with respect to Ω_x, Ω_y and Ω_z we obtain:

$$\frac{\varphi'(\Omega_x)}{\Omega_x \varphi(\Omega_x)} = \frac{\varphi'(\Omega_y)}{\Omega_y \varphi(\Omega_y)} = \frac{\varphi'(\Omega_z)}{\Omega_z \varphi(\Omega_z)} = -\beta . \quad (3.1.9)$$

Taking into account Eqs (3.1.9) let us note that these equalities are valid for independent variables Ω_x, Ω_y and Ω_z.

They must, therefore, be identically equal to some constant β. Whence it immediately follows that:

$$\varphi(\Omega_x) = \sqrt{\frac{\beta}{2\pi}} \cdot e^{-\frac{\beta\Omega_x^2}{2}}, \quad \varphi(\Omega_y) = \sqrt{\frac{\beta}{2\pi}} \cdot e^{-\frac{\beta\Omega_y^2}{2}},$$

$$\varphi(\Omega_z) = \sqrt{\frac{\beta}{2\pi}} \cdot e^{-\frac{\beta\Omega_z^2}{2}}, \tag{3.1.10}$$

where the parameter $\beta > 0$ in accordance with the normalization condition for probability densities $\varphi(\Omega_x)$, $\varphi(\Omega_y)$ and $\varphi(\Omega_z)$. According to (3.1.10) and (3.1.8), the three-dimensional probability density (or volume one in the phase space of angular velocities) of detecting a liquid particle with a given angular velocity Ω is determined by the formula [52]:

$$\Phi(\Omega) = \left(\frac{\beta}{2\pi}\right)^{3/2} \cdot e^{-\frac{\beta\Omega^2}{2}}, \tag{3.1.11}$$

because the normalizing factor is $C = \left(\beta / 2\pi\right)^{3/2}$.

In the case of small values of the angular velocity Ω and finite value of the parameter β, that is, when $\frac{\beta}{2}\Omega^2 < 1$, the three-dimensional probability density (3.1.11) can be represented by the first terms of the Taylor series [52, 73]:

$$\Phi(\Omega) = \frac{1}{V_{\Omega 0}} \cdot \left(1 - \frac{\beta\Omega^2}{2}\right), \tag{3.1.12}$$

where $V_{\Omega 0} = 4\pi\Omega_0{}^3 / 3$ is the volume of a small sphere with a radius $\Omega_0 < 1$ whose center is the origin of 3-dimensional space of angular velocities $(\Omega_x, \Omega_y, \Omega_z)$. The normalizing factor in formula (3.1.12) is $C = 1 / V_{\Omega 0}$. Indeed, according to the normalization condition we have:

$$\int_{V_{\Omega 0}} \Phi(\Omega) dV_\Omega = \frac{1}{V_{\Omega 0}} \cdot \int_0^{\Omega_0} \left(1 - \frac{\beta\Omega^2}{2}\right) 4\pi\Omega^2 d\Omega =$$

$$= \frac{1}{4\pi\Omega_0^{\ 3}/3} \cdot \left(\frac{4}{3}\pi\Omega_0^{\ 3} - \beta\frac{2}{5}\pi\Omega_0^{\ 5}\right) = 1 - \beta\frac{3}{10}\Omega_0^{\ 2} \underset{\Omega_0<1}{\rightarrow} 1.$$

Taking into account that $\Omega = \sqrt{\Omega_x^{\ 2} + \Omega_y^{\ 2} + \Omega_z^{\ 2}}$, the probability that a liquid particle inside a rotating sphere-like gaseous body has projections of the angular velocity vector within the intervals $[\Omega_x, \Omega_x + d\Omega_x]$, $[\Omega_y, \Omega_y + d\Omega_y]$, $[\Omega_z, \Omega_z + d\Omega_z]$ is equal to:

$$dp_{\Omega_x,\Omega_y,\Omega_z} = \Phi(\Omega) dV_\Omega = \left(1 - \frac{\beta\Omega^2}{2}\right) \cdot \frac{dV_\Omega}{V_{\Omega 0}}. \qquad (3.1.13)$$

Obviously, $dV_\Omega \subset V_{\Omega 0}$ but because of $\Omega_0 < 1$ then $V_{\Omega 0} \rightarrow dV_\Omega$, so that the formula (3.1.13) goes into the following:

$$dp_{\Omega_x,\Omega_y,\Omega_z} = 1 - \frac{\beta\Omega^2}{2}, \quad |\Omega| < \Omega_0, \quad \Omega_0 < 1. \qquad (3.1.14)$$

The plot of the function $\Phi(\Omega)$ on the interval $[0, \Omega_0]$ is presented in Figure 3.1.

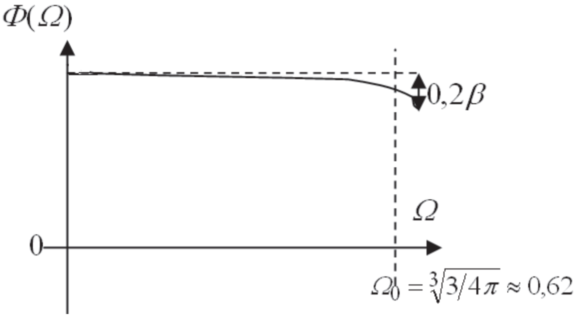


Figure 3.1. Graphic representation of the probability density function $\Phi(\Omega)$ in the case of small angular velocity $|\Omega| < \Omega_0, \quad \Omega_0 < 1$

As seen in Fig. 3.1, at this interval, $\Phi(\Omega)$ is almost constant with a slight decline $\delta = 0.2\beta$ if $\beta < 1$. This means that for small values of angular velocity $|\Omega| < \Omega_0, \quad \Omega_0 < 1$ the function $\Phi(\Omega)$ can be approximated by a *uniform* law of probability density distribution.

So, taking into account the above-mentioned inequality:

$$\frac{\beta}{2}\Omega^2 < 1 \qquad (3.1.15)$$

the three-dimensional probability density $\Phi(\Omega)$ is described by the formula (3.1.12) so that in this case the effective probability volume density $\Phi_\Omega^{eff}(\vec{r})$ following (5.1.1)–(5.1.3) takes the form [52, 73]:

$$\Phi_\Omega^{eff}(\vec{r}) = \Phi(\vec{r}) \cdot \left(1 - \frac{\beta\Omega^2}{2}\right) = \Phi(\vec{r}) - \frac{\beta\Phi(\vec{r})}{2}\Omega^2, \qquad (3.1.16)$$

where β is some unknown positive parameter ($\beta > 0$). To clarify the physical meaning of this parameter in the case of a slowly rotating sphere-like gaseous body let us use the Poisson equation.

As shown in Section 2.2, the mass density ρ of a non-rotating sphere-like gaseous body is expressed through its volume probability density Φ by the very simple relation (2.2.7), that is, $\rho = M\Phi$. So, substituting (2.2.7) into the Poisson equation (2.4.1) we obtain [73]:

$$\nabla^2\varphi_g = 4\pi\gamma M\Phi\,, \tag{3.1.17}$$

where

φ_g is a potential of the gravitational field,

γ is the gravitational constant of Newton, and

M is a mass of the sphere-like gaseous body.

As shown in Section 1.4 (see formula (1.4.1)), if the sphere-like gaseous body is rotating freely in space with an angular velocity $\vec{\Omega} = \vec{\Omega}(t)$, then the general potential of gravity $\psi_g = \psi_g(\vec{r}, \vec{\mathrm{v}}, t)$ should be considered instead of $\varphi_g(\vec{r}, t)$ since $\psi_g(\vec{r}, \vec{v}, t) = \varphi_g(\vec{r}, t) - \frac{1}{2}\vec{\mathrm{v}}^2(\vec{r}, t)$ is a function of not only $\vec{r}$, but also velocity $\vec{\mathrm{v}} = [\vec{\Omega} \times \vec{r}]$ [48, 50]. In other words, in this particular case of rotational motion, the generalized potential of the gravitational field is called the *general potential of gravity* $\psi_g(\vec{r}, \vec{v}, t) = \varphi_g(\vec{r}, t) - \frac{1}{2}[\vec{\Omega}(t) \times \vec{r}]^2$ because its addend $V_c = -\frac{1}{2}[\vec{\Omega}(t) \times \vec{r}]^2$ is known as the *potential of centrifugal force* [95, 97]. So, according to the above, instead of Eq. (3.1.17) we use Poisson equation for the general potential of gravity ψ_g (see Eq. (1.4.14)) which for a rotating sphere-like gaseous body takes the form [50, 53, 73]:

$$\nabla^2\psi_g = 4\pi\gamma M\Phi - 2\Omega^2 = 4\pi\gamma M(\Phi - \Omega^2 / 2\pi\gamma M)\,. \tag{3.1.18}$$

Finally, taking into account the Poisson equations (3.1.17) and (3.1.18) in parallel with the formula (3.1.16), we can estimate the *effective probability volume density* that we are looking for as [52, 73]:

$$\Phi_{\Omega}^{eff}(r) = \Phi(r) - \frac{\Omega^2(t)}{2\pi\gamma \mathrm{M}}, \tag{3.1.19}$$

which is essentially a probability volume density to locate a particle in a rotating sphere-like gaseous body. Moreover, comparing the right-hand sides of Poisson equations (3.1.17) and (3.1.18) we find that the distribution function of particles inside a *slowly* rotating sphere-like gaseous body undergoes a perturbation due to rotation and, as a result, it can be represented as:

$$\Phi_{\Omega}^{eff}(r) = \Phi(r) + \delta\Phi(r,\Omega) = \Phi(r) - \frac{\Omega^2(t)}{2\pi\gamma \mathrm{M}},$$

$$\frac{\Omega^2}{2\pi\gamma \mathrm{M}} << 1\,. \tag{3.1.20}$$

where $\Phi(r)$ is a stationary isotropic spatially homogeneous distribution function, unperturbed by the field of gravity, and $\delta\Phi(r,\Omega)$ is its small deviation under the influence of the field of centrifugal force which, as follows from (3.1.20), is equal to:

$$\delta\Phi(r,\Omega) = -\frac{\Omega^2(t)}{2\pi\gamma \mathrm{M}}\,. \tag{3.1.21}$$

Let us note that although formulas (3.1.16) and (3.1.20) are obtained using completely different theories (the first in the framework of statistical representations whereas the second based on the deterministic theory), nevertheless, they lead to the same result which allows us to estimate a small perturbation of distribution function $\delta\Phi(r,\Omega)$:

$$-\frac{\beta\Phi(r)}{2}\Omega^2 = -\frac{\Omega^2}{2\pi\gamma \mathrm{M}}, \tag{3.1.22}$$

whence the meaning of the desired parameter β directly follows in the case of a slowly rotating sphere-like gaseous body [52, 73]:

$$\beta = \frac{1}{\pi\gamma \mathrm{M}\Phi(r)} = \frac{1}{\pi\gamma\rho(r)} . \qquad (3.1.23)$$

Since the three-dimensional probability density $\Phi(\Omega)$ and the effective probability volume density $\Phi_\Omega^{eff}(\vec{r})$, described by formulas (3.1.12) and (3.1.16) respectively, are obtained under the assumption that the parameter $\eta = \beta\Omega^2/2$ is small following inequality (3.1.15), then substituting the derived formula (3.1.23) into this inequality (3.1.15) we obtain Poincaré's well-known general theorem on rotating masses (Theorem 1.5) [1, 105 p.22] (see also inequality (1.4.13) in Section 1.4):

$$\frac{\Omega^2}{2\pi\gamma\rho(r)} < 1 . \qquad (3.1.23)$$

Thus, the obtained inequality (3.1.23) proves Poincaré's general theorem in statistical interpretation for a slowly rotating and gravitating sphere-like gaseous body [52, 73]:

Theorem 3.1 (an analog of Poincaré's general theorem in statistical interpretation). The steady motion of a rotating and gravitating sphere-like gaseous body is possible only with small values of the argument $\eta = \beta\Omega^2/2$ of the distribution function in space of angular velocities of liquid particles constituent it, that is, at $\eta < 1$.

As shown in Section 1.4, Theorem 1.5 is closely related to the condition (1.4.14) for the existence of an equilibrium figure for rotating and gravitating masses of a liquid. So, the condition for the existence of the equilibrium figure of a slowly rotating and gravitating sphere-like gaseous body (as well as any rotating and gravitating gaseous masses of a molecular cloud) is based on Theorem 3.1.

In Section 2.2, various models of a gas-dust medium in cosmogony problems have been considered. As noted there, for a gaseous bunch at rest with a uniformly distributed temperature throughout its volume (the case of isothermal equilibrium) the Darwin mass density law (2.2.11) is valid, whereas in the case of adiabatic equilibrium with the ratio $\kappa = c_p / c_V = 6/5$ Schuster's mass density law (2.2.12) is true, similar to the well-known Roche model in which the mass density has an infinitely large value in the center of the core and a very small value in the peripheral region [1, 180]. As shown in Section 2.2, all the above models (including the Roche model) are specific cases of the proposed sphere-like gaseous body model with the law of mass density (2.2.5) [16, 73].

As J. Jeans noted [1 p. 255], both Roche's model and the well-known model of incompressible homogeneous fluid [1, 148] form the two limiting cases of the general compressible mass changing the spatial configuration with increasing angular velocity of rotation: Roche's model breaking up through the shedding of successive rings of matter from its equator (see (1.4.37)), the incompressible mass breaking by fission into two parts.

Moreover, in the case of an incompressible continuous medium under increasing angular velocity of its rotation, a *linear series* of continuous configurations of equilibrium (stable or unstable) is observed: spherical, spheroidal, ellipsoidal, and pear-shaped [1, 148]. The violation of stability and the transition to different configurations occur through the bifurcation points. In turn, the condition for a *point of bifurcation* [135] is that there shall be two adjacent configurations of equilibrium, and hence two different boundaries (equipotential surfaces) are possible for the same values of angular velocity Ω. For example, the transition from the Maclaurin spheroid to the Jacobi ellipsoid occurs under the following condition [1]:

$$\Omega^2 / 2\pi\gamma\bar{\rho} = 0.18712 . \qquad (3.1.24)$$

So, in the case being discussed of an incompressible continuous medium, the Maclaurin spheroid becomes unstable when the angular velocity of rotation Ω satisfies this condition (3.1.24).

As for Roche's model, which, on the contrary, represents the extreme limit of compressibility, the angular velocity of rotation Ω at which the mass begins to break-up is given by the above condition (1.4.35):

$$\Omega^2 / 2\pi\gamma\bar{\rho} > 0.360744 . \qquad (3.1.25)$$

In this regard, Jeans proposed a *generalized Roche model* [1], which combines some of the properties of both of the two models so far discussed, namely, the generalized Roche model consisting of a homogeneous incompressible mass of finite size and finite density ρ_0, surrounded by an atmosphere of negligible density. Thus, the generalized Roche model, in general, resembles the model of sphere-like gaseous body proposed in Chapter 2 [16, 45, 46, 73].

Let us note that the results obtained by the Roche model are also fully confirmed within the framework of the developed statistical theory. Indeed, as shown above in Section 2.3, the finding of the critical values of the mass density ρ_c and the parameter of gravitational compression α_c of a sphere-like gaseous body results in the radius of *critical iso-surface* of mass density bending in accordance with the formula (2.3.8):

$$R_c = r_*(\alpha_c) = 1/\sqrt{\alpha_c} = (2/\pi)^{1/6} \cdot (\gamma M / \omega_c^2)^{1/3} ,$$

where $\omega_c = 2\sqrt{\pi\gamma\rho_c}$ is the critical frequency of gravitational perturbations leading to the gravitational instability. On the other hand, as shown when considering Roche's model in

Section 1.4, the radius of *critical equipotential surface* (1.4.29) is defined by formula (1.4.25b):

$$h_0 = (\gamma M / \Omega^2)^{1/3} .$$

So, directly comparing formulas (1.4.25b) and (2.3.8) under the assumption that $h_0 = R_c$ we obtain:

$$\Omega = \sqrt[4]{\frac{\pi}{2}} \cdot \omega_c \approx 1.11952 \cdot \omega_c . \qquad (3.1.26)$$

According to (3.1.26) the break-up angular velocity Ω is 12% higher than the critical frequency ω_c (that is, almost coincides) of gravitational perturbations leading to the gravitational instability in a sphere-like gaseous body. Indeed, when the gravitational condensation parameter reaches the critical value α_c the breaking of plane waves of initial gravitational perturbations takes place. As a result, the propagation wave mode of initial gravitational perturbations is changed by an aperiodic mode of their amplification due to the *formation of a core* with the radius $\sim R_c$ in a sphere-like gaseous body. Moreover, the initial wave disturbances with frequency ω_c as a result of their interference can lead to a rotation of the formed core with the angular velocity $\omega_c < \Omega$.

Thus, within the framework of the theory of rotating and gravitating sphere-like gaseous bodies, both Poincaré's general theorem in the statistical interpretation (3.1.23) is derived and the main results of Roche's model (1.4.25b), (3.1.26) is also obtained.

3.2. The nonequilibrium particle distribution function for spatial coordinates in a sphere-like gaseous body during its initial rotation

To describe the *initial rotation* of a sphere-like gaseous body, we introduce two frames of reference: an immovable (inertial) system of coordinates $x' \, y' \, z'$; and a moving (rotating) coordinate system $x \, y \, z$ which is assumed to be rigidly connected with the sphere-like gaseous body. It is convenient to match the origins of both inertial and rotating frames of reference with the center of inertia of the sphere-like gaseous body (see Fig. 3.2).

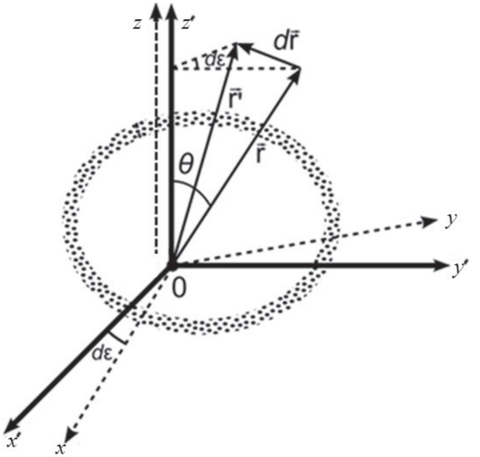


Figure 3.2. Graphic representation of the model of initial rotation of a sphere-like gaseous body

The radius-vectors $\vec{r}$ and $\vec{r}\,'$ of the same point of the sphere-like gaseous body in two different reference systems $x \, y \, z$ and $x' \, y' \, z'$, of which the first rotates relative to the second at an infinitesimal angle $d\varepsilon$, are connected to each other by the Galilean transformation [158]:

$$\vec{r}\,' = \vec{r} + [d\vec{\varepsilon}(t) \times \vec{r}\,] \;, \quad t' = t \,, \qquad (3.2.1)$$

where $d\vec{\varepsilon}$ is a vector of infinitesimal rotation whose absolute value is equal to the angle $d\varepsilon$ of rotation, and its direction coincides with the axis of rotation (and so that the direction of rotation corresponds to the screw rule with respect to the direction $d\vec{\varepsilon}$), and t and t' is the time in the reference systems $x\,y\,z$ and $x'\,y'\,z'$ [158]. Under such a rotation, an increment of the radius vector drawn from the common origin of frames of reference (located on the axis of rotation) to a given point of the sphere-like gaseous body is equal:

$$d\vec{r} = [d\vec{\varepsilon} \times \vec{r}] \ ,$$

since, as can be seen from Fig. 3.2, the movement of the end of the radius vector is related to the angle by the relation [158]:

$$\left| d\vec{r} \right| = r \sin\theta \, d\varepsilon \ .$$

So, according to (3.2.1) the radius vector $\vec{r}'$ of a liquid particle relative to the system $x'\,y'\,z'$ is added by the radius vector $\vec{r}$ relative to the system $x\,y\,z$ and the increment of the radius vector $d\vec{r}$ as a result of the rotation of the liquid particle together with the system $x\,y\,z$.

As known from Section 2.1 [16, 45, 46], according to formula (2.1.14), the probability volume density function of detecting a liquid particle with a given radius vector $\vec{r}'$ inside a gravitating sphere-like gaseous body is equal:

$$\Phi(\vec{r}') = (\alpha / 2\pi)^{3/2} \cdot e^{-\alpha\, \vec{r}'^2/2} \ . \qquad (3.2.2)$$

Substituting (3.2.1) into (3.2.2) and then taking into account that $\vec{r} \cdot [d\vec{\varepsilon} \times \vec{r}] = 0$ as well as $[d\vec{\varepsilon} \times \vec{r}]^2 = d\varepsilon^2\, r^2 \sin^2\theta$, we obtain [73]:

$$\Phi(\vec{r}, d\vec{\varepsilon}) = (\alpha / 2\pi)^{3/2} e^{-\alpha\, (\vec{r} + [d\vec{\varepsilon} \times \vec{r}])^2/2} =$$

$$= (\alpha / 2\pi)^{3/2} e^{-\alpha\, \vec{r}^2/2} \cdot e^{-\alpha\, \vec{r} \cdot [d\vec{\varepsilon} \times \vec{r}]} \cdot e^{-\alpha\, [d\vec{\varepsilon} \times \vec{r}]^2/2} =$$

$$= (\alpha / 2\pi)^{3/2} e^{-\alpha\, \vec{r}^2/2} \cdot e^{-\alpha\, \vec{r}^2 d\vec{\varepsilon}^2 \sin^2\theta/2} \ . \qquad (3.2.3)$$

So, according to (3.2.3) we have [73]:

$$\Phi(r, \theta, d\varepsilon) = (\alpha / 2\pi)^{3/2} \cdot e^{-\alpha r^2 (1 + d\varepsilon^2 \sin^2\theta)/2} \ . \qquad (3.2.4)$$

In the general case, the angular velocity of rotation of a sphere-like gaseous body is equal to

$$\vec{\Omega} = d\vec{\varepsilon} / dt \ , \tag{3.2.5}$$

so that according to (2.1.14), (3.2.4), and (3.2.5) the function $\Phi(r, \theta, d\varepsilon)$ describes the distribution of liquid particles for the spatial coordinates in a sphere-like gaseous body at *the beginning* of its rotation, that is, in the *nonequilibrium* case [73]:

$$\Phi(r, \theta, \Omega \cdot dt) = (\alpha / 2\pi)^{3/2} e^{-\alpha r^2/2} \cdot e^{-\alpha r^2 \Omega^2 \sin^2\theta \, dt^2/2} \approx$$

$$\approx \Phi(r) \cdot \left(1 - \frac{\alpha r^2 \sin^2\theta \, dt^2}{2} \cdot \Omega^2\right). \tag{3.2.6}$$

On the other hand, according to (3.1.16), formula (3.2.6) describes the effective probability volume density $\Phi_\Omega^{eff}(r)$ of detecting a liquid particle at the initial instant of rotation of a sphere-like gaseous body, if we assume the initial value of the parameter β to be equal $\beta_0 = \alpha r^2 \sin^2\theta \, dt^2$ under this nonequilibrium rotation.

It is desirable to bridge over, if possible, the wide gap between these two different approaches in Section 3.1 and Section 3.2. To some extent, a bridge is formed by the consideration of a nonequilibrium volume probability density function for spatial coordinates in a sphere-like gaseous body during its rotation [73]. Omitting any details of derivation (as its analog will be presented in the following section), as shown in [73], the nonequilibrium probability volume density function to locate a liquid particle inside a rotating sphere-like gaseous body is equal [57, 73]:

$$\Phi(r, \theta, \varepsilon(t)) = \frac{\sqrt{2}}{\operatorname{erf}(\varepsilon_0 r \cdot \sin\theta \sqrt{\alpha/2})} \cdot \frac{\alpha^{3/2}}{\pi^{3/2}} \cdot e^{-\alpha r^2 (1+\varepsilon^2 \sin^2\theta)/2} \tag{3.2.7}$$

where ε_0 is an azimuth angular parameter. Comparing (3.2.7) with (3.2.4), it is easy to see that the formula (3.2.7) for a nonequilibrium probability volume density of the liquid particle detection in a rotating sphere-like gaseous body generalizes the formula obtained (3.2.4) for the probability volume density for a sphere-like gaseous body at the very beginning of its rotation.

3.3. Derivation of the equilibrium distribution function of liquid particles in space and mass density functions based on the statistical model of a uniformly rotating and gravitating spheroidal body with a small angular velocity

As shown in the previous chapter, interacting molecules or atoms of gas form new aggregate nano- and micro-particles called *colloidal* or *liquid* particles (in hydrodynamic meaning [94]). As pointed out in [94], any liquid particle having a small element of a medium volume is still considered to be large enough to contain many molecules. The liquid particles (which constitute a molecular cloud at low temperatures) also have oscillatory interactions among themselves. In reality, in *macrophysics* (when liquid particles can be regarded as planetesimals [139]) it is alleged that the cosmological constant [102] describes the cosmic vacuum [103, 104]. Its experimental manifestations on cosmic scales are, therefore, the fluctuations stipulated by Alfvén–Arrhenius oscillations [9, 19]. Moreover, we know [9, 19] that due to the radial and the axial oscillations the moving solid bodies in the gravitational field of a central body have elliptic and inclined orbits.

Let us consider an *initially immovable* molecular cloud in a state of *unstable* mechanical equilibrium at the instant $t = t_0$. The statistical aspect of this problem results from the abovementioned fact that numerous fluctuations of liquid (colloidal) particle concentration caused by their local initial oscillations do not allow us to predict with certainty the behavior of the system as a whole. We are now interested in the following: how many liquid particles have radius-vectors in an interval near the given radius vector $\vec{r}$?

To answer this, we investigate the evolution of a gaseous cloud consisting of N similar liquid particles of mass m_0 in an initially isotropic space. Let a radius vector $\vec{r}$ with coordinates (h, ε, z) be chosen in three-dimensional space, that is, we use a cylindrical frame of reference with its origin in the geometrical center of a gaseous cloud:

$$x = h \cdot \cos\,\varepsilon;\; y = h \cdot \sin\,\varepsilon;\; z = z, \tag{3.3.1}$$

where $h = \sqrt{x^2 + y^2}, 0 \le h < \infty, -\infty \le z \le \infty, 0 \le \varepsilon \le 2\pi$. The choice of the *rotating* cylindrical frame of reference gives us an advantage in the case of rotation because a rotating uniformly gaseous body (with a constant angular velocity $\vec{\Omega}$) remains relatively *immovable* in this system of coordinates. For a plane of rotation, we choose the plane Oxy, that is, we consider Oz to be the axis of rotation of this gaseous cloud. Then, with reasoning similar to the derivation of the particle distribution function relative to coordinates for a non-rotating sphere-like gaseous body in Section 2.1, we can introduce the probabilities that a particle at a given time has coordinates in intervals $[h, h+dh]$, $[\varepsilon, \varepsilon + d\varepsilon]$, $[z, z+dz]$. Concretely, if in the gaseous body there are N particles, then dN_h have coordinates in the interval $[h, h+dh]$, dN_ε and dN_z have coordinates in the intervals $[\varepsilon, \varepsilon + d\varepsilon]$ and $[z, z+dz]$ respectively at the given instant $t = t_0$. The probabilities of any liquid particle having coordinates in these intervals are equal respectively [16, 65, 73] to:

$$dp_h = \varphi(h)dh = \frac{dN_h}{N}\,; \quad dp_\varepsilon = \eta(\varepsilon)hd\varepsilon = \frac{dN_\varepsilon}{N}\,;$$

$$dp_z = \zeta(z)dz = \frac{dN_z}{N}\,, \qquad (3.3.2)$$

where $\varphi(h), \eta(\varepsilon), \zeta(z)$ are one-dimensional probability densities, characterizing shares of liquid particles whose coordinates belong to the elementary intervals close to h, ε, and z respectively.

Let us introduce a three-dimensional probability [16, 55, 73]:

$$dp_{h,\varepsilon,z} = \frac{dN_{h,\varepsilon,z}}{N} = \Phi(h,\varepsilon,z)hdh\,d\varepsilon dz\,, \qquad (3.3.3)$$

where $\Phi(h,\varepsilon,z)$ is a volume density of probability to locate a liquid particle inside a gaseous cloud in the cylindrical system of coordinates.

On the other hand, a liquid particle has all the three given coordinates independent of each other in the initial (non-rotating) instant $t = t_0$ as well as in any instant of *uniform* rotating (since within the uniform rotating gaseous cloud the liquid particles do not move relative to the rotating frame of reference with a constant angular velocity $\Omega = \text{const}$). Then, according to the theorem of complex event probability we have [16, 55, 65, 73]:

$$dp_{h,\varepsilon,z} = \varphi(h)\eta(\varepsilon)\zeta(z)hdhd\varepsilon dz\,. \qquad (3.3.4)$$

Comparing Eq. (3.3.3) with Eq. (3.3.4) we obtain the factorization rule for the probability volume density function [16, 55, 65]:

$$\Phi(h,\varepsilon,z) = \varphi(h)\eta(\varepsilon)\zeta(z)\,. \qquad (3.3.5)$$

Taking into account the condition of *initial isotropy and homogeneity* of space with a gaseous body (in the initial instant $t = t_0$), the volume density of probability has to depend on the value of the distance $r = \left|\vec{r}\right|$ from a given particle to its center [16, 45, 46, 55, 73], that is, $\Phi(h, \varepsilon, z) = \Phi(r)$, where

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{h^2 \cos^2 \varepsilon + h^2 \sin^2 \varepsilon + z^2} = \\ = \sqrt{h^2 + z^2}\,. \quad (3.3.6)$$

Hence, we assume that Eq. (3.3.5) takes the same form in the closest instant $t = t_0 + dt$ at the beginning of rotation [16, 73]:

$$\Phi(r) = \varphi(h)\eta(\varepsilon)\zeta(z)\,. \quad (3.3.7)$$

Starting from the functional equation (3.3.7) we can define the form of function Φ. Differentiating Φ as a composite function with respect to h, let us represent Eq. (3.3.7) in the differential equation form [16, 65, 73]:

$$\Phi'(r)\cdot\frac{\partial r}{\partial h} = \varphi'(h)\eta(\varepsilon)\zeta(z)\,. \quad (3.3.8)$$

Then let us calculate the partial derivative $\partial r / \partial h$:

$$\frac{\partial r}{\partial h} = \frac{\partial}{\partial h}\left(\sqrt{h^2 + z^2}\right) = \frac{h}{r}\,. \quad (3.3.9)$$

With provision for Eq. (3.3.9) the equation (3.3.8) becomes:

$$\Phi'(r)\cdot\frac{h}{r} = \varphi'(h)\eta(\varepsilon)\zeta(z)\,. \quad (3.3.10)$$

Dividing Eq. (3.3.10) by Eq. (3.3.7) we obtain [16, 55, 73]:

$$\frac{\varphi'(h)}{h\varphi(h)} = \frac{\Phi'(r)}{r\Phi(r)}\,. \quad (3.3.11)$$

Analogously we can derive a differential ratio relative to the space coordinate z:

$$\frac{\zeta'(z)}{z\zeta(z)} = \frac{\Phi'(r)}{r\Phi(r)}\,. \quad (3.3.12)$$

Now from Eq. (3.3.7) let us find the differential equation with respect to the *angular* coordinate ε:

$$\varphi(h)\eta'(\varepsilon)\zeta(z) = \Phi'(r)\frac{\partial r}{\partial \varepsilon}. \qquad (3.3.13)$$

It is to be noted that the right-hand part of Eq. (3.3.13) becomes zero at the initial instant $t = t_0$ because $\partial r / \partial \varepsilon = 0$ according to Eq. (3.3.6), i.e. $\eta'(\varepsilon) = 0$ respectively in the left-hand part of Eq. (3.3.13) at the initial instant $t = t_0$. Consequently, the differential equation (3.3.13) with respect to ε has nonzero both right and left-hand parts of Eq. (3.3.13) at the next instant $t = t_0 + dt$ only. In other words, beginning with this instant, the *interference* of the orthogonal radial and axial initial oscillations can lead to the rotational motion of liquid particles [73].

To calculate the partial derivative $\partial r / \partial \varepsilon$ let us consider a motion of a liquid particle (situated distance h from the axis Oz) over the plane of rotation. From the beginning $t = t_0$ of a nonuniform (unsteady-state) rotation of the gaseous body with an angular velocity $\vec{\Omega} = \vec{\Omega}(t)$, a liquid particle begins to move inside it in the opposite direction to this rotation owing to an inertia force action. In this connection, if the frame of reference with a gaseous body turns on a small angle ε then particles rotate on $-\varepsilon$ the next instant $t = t_0 + dt$. If any particle moves along an arc of a circle on a small angle $-\varepsilon$ then the length of its span is equal to $l = \sqrt{h^2 + h^2 - 2h \cdot h \cdot \cos(-\varepsilon)} = h\sqrt{2 \cdot (1 - \cos\varepsilon)} = 2h \cdot \sin(\varepsilon / 2)$. Since the angle ε is small enough ($|\varepsilon| < 1$), the distance r (from the origin of coordinates to a moving liquid particle) can be estimated as:

$$r = \sqrt{h^2 + l^2 + z^2} = \sqrt{h^2 + h^2\varepsilon^2 + z^2}\ . \qquad (3.3.14)$$

According to Eq. (3.3.14) the desired partial derivative is equal [16, 55, 73]:

$$\frac{\partial r}{\partial \varepsilon} = \frac{\frac{1}{2}h^2 2\varepsilon}{\sqrt{h^2 + h^2\varepsilon^2 + z^2}} = \frac{h^2\varepsilon}{r}. \qquad (3.3.15)$$

Let us note though that, according to the new definition (3.3.14) of distance, $\partial r / \partial h = h(1+\varepsilon^2)/r$. But the previous formula for the partial derivative (3.3.9) remains correct because terms $O(\varepsilon^2)$ are neglected. As follows from formulas (3.3.14) and (3.3.15), the direct dependence r on ε means modifying the form of the gaseous body through its rotation, that is, such dependence defines a *non-stationarity form* of the gaseous body in the rotating frame of reference under initial rotation at $t = t_0 + dt$.

Substituting Eq. (3.3.15) into Eq. (3.3.13) we obtain [16, 55, 73]:

$$\frac{\eta'(\varepsilon)}{h^2\varepsilon \cdot \eta(\varepsilon)} = \frac{\Phi'(r)}{r\Phi(r)}. \qquad (3.3.16)$$

Since in Eqs (3.3.11), (3.3.12), and (3.3.16) the right-hand parts are the same, the left-hand parts are then equal among themselves:

$$\frac{\varphi'(h)}{h\varphi(h)} = \frac{\zeta'(z)}{z\zeta(z)} = \frac{\eta'(\varepsilon)}{h^2\varepsilon \cdot \eta(\varepsilon)}. \qquad (3.3.17)$$

The left, central and right-hand parts of Eqs (3.3.17) are functions of either h or z, or h and ε, that is, according to the independence of these coordinates it takes place only in the case of the constancy of these parts individually [16, 55-57, 73]:

$$\varphi'(h)/h\varphi(h) = \zeta'(z)/z\zeta(z) = (1/h^2)\eta'(\varepsilon)/\varepsilon\eta(\varepsilon) = -\alpha, \qquad (3.3.18)$$

where α is a constant called, as before in Section 2.2, the *parameter of gravitational condensation* [16, 73].

Since Ω is an increasing function with a tendency toward stabilization $\Omega = \text{const}$ during an infinitesimal interval dt, we need to take into account some peculiarities of integration of

Eqs (3.3.18) concerning the angular coordinate ε at the *relative mechanical equilibrium* of the rotating gaseous body. So, integrating Eqs (3.3.18) under the condition of stabilization of the variable ε to its constant value ε_0 (which is the upper limit of ε), we obtain:

$$\int \varphi'(h)dh/\varphi(h) = -\alpha \int hdh; \qquad (3.3.19a)$$

$$\int \zeta'(z)dz/\zeta(z) = -\alpha \int zdz; \qquad (3.3.19b)$$

$$\int_{\eta(\varepsilon)}^{\eta_0} \eta'(\varepsilon)d\varepsilon/\eta(\varepsilon) = -\alpha h^2 \int_{\varepsilon}^{\varepsilon_0} \varepsilon d\varepsilon . \qquad (3.3.19c)$$

Taking into account the preliminary condition that terms $O(\varepsilon^2)$ can be neglected in the third Eq. (3.3.19c) because $\varepsilon \to 0$ (excepting the upper limit value $|\varepsilon_0| < 1$), we can rewrite Eqs (3.3.19a–c) in the form:

$$\varphi(h) = c_1 \cdot e^{-\alpha h^2/2}; \qquad (3.3.20a)$$

$$\zeta(z) = c_2 \cdot e^{-\alpha z^2/2}; \qquad (3.3.20b)$$

$$\eta(\varepsilon) = \eta_0 \cdot e^{\alpha h^2 \varepsilon_0^2/2} \qquad (\varepsilon \to 0). \qquad (3.3.20c)$$

According to Eqs (3.3.7) and (3.3.20a–c) it is not difficult to write down a general expression for the *equilibrium* function of probability volume density to locate a liquid particle relative to the rotating frame of reference [16, 73]:

$$\Phi(h, z, \varepsilon_0) = \varphi(h)\zeta(z)\eta(\varepsilon_0) = C \cdot e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}, \qquad (3.3.21)$$

where $C = c_1 c_2 \eta_0$ is a constant of integration. The parameter $\alpha > 0$ because from physical reasoning it is clear that under increasing $r = \sqrt{h^2 + z^2}$ the share of liquid particles, following Eq. (3.3.21), is to decrease due to gravitational interactions, therefore α is the parameter of gravitational condensation. Moreover, the normalization condition of the probability volume density $\int_V \Phi dV = 1$ results in the integral convergence which only fulfills when $\alpha > 0$. Starting from the normalization condition for the probability volume density (3.3.21):

$$\int_V \Phi \, dV = C \cdot 2\pi \int_0^\infty e^{-\alpha h^2 (1-\varepsilon_0^2)/2} h dh \int_{-\infty}^{\infty} e^{-\alpha z^2/2} dz = 1,$$

we can find $C = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2)$ and, hence, derive the probability volume density function describing a liquid particle distribution into a rotating gaseous body (being in a *state of relative mechanical equilibrium*) in the cylindrical coordinates [16, 65, 73]:

$$\Phi(h, z) = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2) e^{-\alpha (h^2 (1-\varepsilon_0^2) + z^2)/2}, \qquad (3.3.22a)$$

as well as in the Cartesian coordinates:

$$\Phi(x, y, z) = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2) e^{-\alpha (x^2 (1-\varepsilon_0^2) + y^2 (1-\varepsilon_0^2) + z^2)/2}, \qquad (3.3.22b)$$

and in the spherical coordinates:

$$\Phi(r, \theta) = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2) e^{-\alpha r^2 (1-\varepsilon_0^2 \sin^2 \theta)/2}, \qquad (3.3.22c)$$

where θ and ε are polar and azimuth angles and ε_0 is a constant of stabilization of the angular variable ε.

When $\varepsilon_0^2 \to 0$ the formula (3.3.22c) goes to the formula (2.1.14), describing the equilibrium function of probability volume density for the *non-rotational* case (or *slowly rotational* one) [16, 45, 46, 73]:

$$\Phi(r) = (\alpha / 2\pi)^{3/2} e^{-\alpha r^2/2}.$$

Now let us note that the relationship (3.3.3) describes the distribution of liquid particles along the distance from the geometrical center and the direction in space in cylindrical coordinates h, ε, and z. Taking into account that an elementary volume in cylindrical coordinates is $dV = hdhd\varepsilon dz$ we can transform Eq. (3.3.3) into

$$\frac{dN_{h,\varepsilon,z}}{dV} = N\Phi(h,\varepsilon,z) = \\ = N(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}. \quad (3.3.23)$$

The value $\frac{dN_{h,\varepsilon,z}}{dV} = n_{h,\varepsilon,z}$ is a *local concentration* of liquid particles near a point with coordinates (h,ε,z). Considering (3.3.23) we have:

$$n(h,z) = N(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}. \quad (3.3.24)$$

If all liquid particles are like and have mass m_0, then, by multiplying both sides of relation (3.3.24) by m_0, we obtain:

$$\rho(h,z) = m_0 N(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2} = \\ = M(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2} = M\Phi(h,z), \quad (3.3.25)$$

where $\rho = m_0 n$ is a mass density of the gaseous substance and $M = m_0 N$ is a mass of the gaseous body composed of the liquid particles.

By denoting $\rho_0 = M(\alpha/2\pi)^{3/2}$ the mass density for a rotating and gravitating gaseous body, being in a state of relative mechanical equilibrium, can be written in cylindrical, Cartesian, and spherical coordinate systems respectively [16, 55, 56, 73]:

$$\rho(h,z) = \rho_0(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}, \quad (3.3.26a)$$

$$\rho(x,y,z) = \rho_0(1-\varepsilon_0^2)e^{-\alpha(x^2(1-\varepsilon_0^2)+y^2(1-\varepsilon_0^2)+z^2)/2}, \quad (3.3.26b)$$

$$\rho(r,\theta) = \rho_0(1-\varepsilon_0^2)e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2}. \quad (3.3.26c)$$

The iso-surfaces (isostere) of the mass density (3.3.26a–c) are flattened ellipsoidal ones and ε_0^2 is a parameter of their flatness (ε_0 is the eccentricity of an ellipse). As a rule $|\varepsilon_0| < 1$, so that these mass density iso-surfaces become *spheroidal surfaces*.

Thus, under the influence of the initial oscillations of liquid particles an isolated gaseous cloud can be transformed into the *spheroid-like gaseous body* or, simply put, the ***spheroidal body*** [16, 56, 65, 73].

Let us note the important particular case of spheroidal bodies which are *sphere-like* gaseous bodies (see Section 2.2). Indeed, we can see if $\varepsilon_0^2 \to 0$ then the equation (3.3.26c) becomes the mass density function (2.2.5) for a slowly rotating or *immovable spheroidal body* [16, 45, 46–48, 73]:

$$\rho(r) = \rho_0 e^{-\alpha r^2/2}. \tag{3.3.27}$$

On the contrary, if the squared eccentricity $\varepsilon_0^2 \to 1$ then the equation (3.3.26a) can describe the mass density of a flattened gaseous disk [16, 73]:

$$\rho(h,z) = \rho_c(h) e^{-\alpha z^2/2}, \tag{3.3.28}$$

where $\rho_c(h) = \lim\limits_{\substack{\varepsilon_0^2 \to 1 \\ M \to \infty}} M(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha h^2(1-\varepsilon_0^2)/2}$ is a value of mass density in a *central* flat of this gaseous disk, M is a mass of star plus mass of gaseous protoplanetary disk, that is, the total mass of an initial prestellar molecular cloud (a more rigorous justification of formula (3.3.28) will be given in below in Section 3.8).

It is interesting to note that this formula (3.3.28) completely coincides with the known *barometric formula* (for a flat rotating disk) obtained with the usage of the hydrostatic mechanical equilibrium condition [2 p.36, 23 p.769] (or with the same formula of mass density distribution in the disk "standard" model derived using the hydrostatic equilibrium condition jointly with the ideal gas state equation [12 p.19]). Moreover, L.E. Gurevich and A.I. Lebedinsky used the designation $\rho_{\max}(h)$ instead of $\rho_c(h)$.

In sum, the obtained function of mass density (3.3.26a–c) characterizes a flatness process: from initial spherical forms (for a non-rotational spheroidal body case (3.3.27)) through flattened ellipsoidal forms (for a rotating and gravitating spheroidal body (3.3.26c)) to fuzzy contour disks (3.3.28) when the squared eccentricity ε_0^2 varies from 0 to 1. This means that the derived function (3.3.26a–c) is appropriate to describe the evolution of a protoplanetary gaseous (gas-dust) disk around a star.

As noted in Section 1.7, one of the main difficulties has been found to lie in the determination of the analytical expression for the gravitational potential of a cosmogonical body including the rotating and gravitating spheroidal body model (3.3.26a–c). Although the derivation of gravitational potential of a rotating and gravitating spheroidal body will be given below in Sections 3.6 and 3.7, let us nevertheless substantiate the fact that the equation of mass density iso-surfaces of a rotating spheroidal body in relative mechanical equilibrium can be directly obtained from both the equilibrium distribution function of its mass density and the equilibrium hydrodynamic equations in the potential field of its internal forces. To do this, we prove an auxiliary statement [73]:

Lemma 3.1. The iso-surface of the mass density of a spheroidal body being in absolute or relative mechanical equilibrium coincides with the equipotential surface of the field of potential forces.

Proof: we first consider the spheroidal body as an absolutely rest continuous medium in full accordance with the property of its light mobility [111]. In this case, the tangential components of the stress tensor are zero, and the total value of normal stresses at a given point of the medium, taken with a minus sign, is equal to the pressure p at this point. Then the equilibrium condition of a continuous medium is described by the Euler equation of statics of medium [111]:

$$\rho \vec{f} = \text{grad}\, p \,, \tag{3.3.29}$$

where $\vec{f}$ is a specific force (field strength), ρ is a mass density (density of the medium).

It is possible to exclude the mass density ρ and pressure p from equation (3.3.29); for this, we take from both its parts the operation of curl, that is, rot . Then p is excluded because $\text{rot grad}\, p \equiv 0$. As a result, equation (3.3.29) takes the form:

$$\text{rot}(\rho \vec{f}) = 0 \,. \tag{3.3.30}$$

Further, opening the brackets in Eq. (3.3.30) in accord with the well-known vector analysis rule [100, 111], we obtain:

$$\rho\, \text{rot}\, \vec{f} + [\text{grad}\, \rho \times \vec{f}] = 0 \,, \tag{3.3.31}$$

and, scalar multiplying both sides of equality (3.3.31) on $\vec{f}$, we have:

$$\vec{f} \cdot \text{rot}\, \vec{f} = 0 \,. \tag{3.3.32}$$

Equation (3.3.32) determines the general restriction imposed on the class of forces, under the action of which the absolute equilibrium of a continuous medium with light mobility is possible. On the other hand, Eq. (3.3.32) defines the condition for the existence of surfaces $\Pi(x, y, z) = \text{const}$ normal to the force lines of the field, that is, *equipotential surfaces*.

Indeed, let us intersect the force lines of the field with a family of surfaces $\Pi(x, y, z) = \text{const}$ and require that these surfaces be orthogonal to the force lines of the field. For this, what is normal at any point on the surface must coincide in direction with the strength $\vec{f}$ at that point, that is, equality is required:

$$\vec{f} = \lambda\, \text{grad}\, \Pi \,. \tag{3.3.33}$$

Taking from both sides of this equality the operation of a vortex rot , we have following the known formulas of vector analysis [100, 111] that:

$$\operatorname{rot}\vec{f} = \operatorname{rot}(\lambda \operatorname{grad}\Pi) = \lambda \operatorname{rot}\operatorname{grad}\Pi + [\operatorname{grad}\lambda \times \operatorname{grad}\Pi] = \\ = [\operatorname{grad}\lambda \times \operatorname{grad}\Pi]. \qquad (3.3.34)$$

Then, using the formula (3.3.33), we transform (3.3.34) to the form:

$$\operatorname{rot}\vec{f} = \frac{1}{\lambda} \cdot [\operatorname{grad}\lambda \times \vec{f}], \qquad (3.3.35)$$

and hence, due to the perpendicularity of the vector product to its multipliers the condition (3.3.32) is carried out immediately. Thus, equation (3.3.32) determines *the condition for the existence of equipotential surfaces* of the field (normal cross-sections of the force lines of the force field).

The forces satisfying the above condition (3.3.32) include forces having a potential, in particular, the potential of a gravitational field φ_g. Since the strength of the gravitational field (specific gravitational force) $\vec{f}_g$ is equal to:

$$\vec{f}_g = -\operatorname{grad}\varphi_g ,$$

then:

$$\operatorname{rot}\vec{f}_g = 0 . \qquad (3.3.36)$$

According to (3.3.31) in this case, the following condition is true:

$$[\operatorname{grad}\rho \times \vec{f}_g] = -[\operatorname{grad}\rho \times \operatorname{grad}\varphi_g] = 0 , \qquad (3.3.37)$$

whence it follows that under condition of absolute mechanical equilibrium of a continuous medium in a spheroidal body, the force lines of the gravitational field are orthogonal to the iso-surfaces (surfaces of equal mass density) and also that the iso-surfaces of a rest spheroidal body coincide with the equipotential surfaces of the gravitational field of a spheroidal body in mechanical equilibrium.

Now let us consider a gravitating spheroidal body as a *relatively resting* continuous medium, that is, such a continuous medium being at relative rest for some coordinate system uniformly rotating with an angular velocity $\vec{\Omega}$ [111]. In this case, equation (3.3.32) determines the condition of the relative mechanical equilibrium of a continuous medium in a rotating spheroidal body if $\vec{f}$ is a total strength of the gravitational field with potential φ_g and the inertial field of centrifugal forces with potential V_c:

$$\vec{f} = \vec{f}_g + \vec{f}_c \,, \tag{3.3.38a}$$

where

$$\vec{f}_g = -\operatorname{grad} \varphi_g \,; \tag{3.3.38b}$$

$$\vec{f}_c = -\operatorname{grad} V_c \,. \tag{3.3.38c}$$

According to Eqs (3.3.38a)–(3.3.38c) the following equality takes place:

$$\operatorname{rot} \vec{f} = \operatorname{rot} \vec{f}_g + \operatorname{rot} \vec{f}_c = 0 \,, \tag{3.3.39}$$

and in accordance with relations (3.3.31) and (3.3.39) the following condition is true:

$$[\operatorname{grad} \rho \times \vec{f}\,] = -[\operatorname{grad} \rho \times \operatorname{grad}(\varphi_g + V_c)] = 0 \,, \tag{3.3.40}$$

which states that under the condition of relative mechanical equilibrium of a continuous medium of a spheroidal body, *the iso-surfaces of the mass density of a rotating spheroidal body coincide with the equipotential surfaces of the total potential field* of gravitational and centrifugal forces of a spheroidal body in relative mechanical equilibrium.

Summarizing the above, it is not difficult to see that the iso-surface equation:

$$\rho(\vec{r}, \vec{\Omega}) = \text{const}$$

coincides with the equipotential surface of the total potential $\phi = \varphi_g + V_c$ of the field of gravitational and centrifugal forces of a rotating spheroidal body:

$$\phi(\vec{r}, \vec{\Omega}) = \text{const},$$

which proves the lemma.

Theorem 3.2. For a gravitating spheroidal body to be in absolute or relative mechanical equilibrium under the action of a potential field of forces, it is necessary and sufficient that the equipotential surfaces of the field coincide with mass density iso-surfaces (isostere) and isobars.

Proof: necessity directly follows from the fact that when the medium is in equilibrium, the force lines of the field are perpendicular to the isobars (surfaces of equal pressure) following Eqs (3.3.29) and (3.3.33). According to Lemma 3.1, at the equilibrium of a continuous medium in a gravitating spheroidal body under the action of a potential field of forces, the equipotential surfaces of the field coincide with mass density iso-surfaces and, therefore, isobars with regard to Eq. (3.3.29).

To prove the sufficiency, it is necessary to prove the inverse statement: if the isobars coincide with the iso-surfaces of mass density, then the equilibrium of the continuous medium of a spheroidal body is possible only in the case of a potential field of forces. Indeed, from the condition:

$$[\text{grad}\, p \times \text{grad}\, \rho] = 0, \tag{3.3.41}$$

because of Eq. (3.3.29), it immediately follows that:

$$[\vec{f} \times \text{grad}\, \rho] = 0, \tag{3.3.42}$$

so that we can conclude based on Eq. (3.3.31) that:

$$\text{rot}\, \vec{f} = 0, \tag{3.3.43}$$

that is, $\vec{f} = -\text{grad}\, \phi$. Thus, the theorem is proved [56, 73].

Remark 3.1. The gravitational field ($\phi = \varphi_g$) of a resting spheroidal body or the total (gravitational and inertial of centrifugal force) field ($\phi = \varphi_g + V_c$) in the case of a rotating spheroidal body has been considered as a potential field of forces acting on a continuous medium of a spheroidal body.

Corollary 3.1. The iso-surface of the mass density of a spheroidal body is a sphere in space if and only if the potential field of forces is its gravitational field of the resting spheroidal body being in absolute mechanical equilibrium.

Proof: the necessity of this statement easily follows from the fact that in accordance with the formula (2.2.5) from Chapter 2 (or (3.3.26c) if $\varepsilon_0 = 0$), the mass density of a resting spheroidal body could be constant over the observation interval $t \in [t_0, t_0 + T_{obs}]$, that is:

$$\rho(\vec{r}) = \rho_0 e^{-\alpha \vec{r}^2/2} = \text{const} \tag{3.3.44}$$

in case:

$$\vec{r}^2 = x^2 + y^2 + z^2 = \text{const} \tag{3.3.45}$$

(of course, we assume that $\alpha(t) = \text{const}$ on the observation interval $[t_0, t_0 + T_{obs}]$). Hence, if the mass density iso-surface (3.3.44) is a sphere (3.3.45) then according to Lemma 3.1 and Theorem 3.2 the equipotential surface of the potential field of forces must be a sphere.

On the other hand, according to (3.3.29) the equilibrium equation for a continuous medium of a spheroidal body in the potential field of gravitational forces can be written in the form:

$$-\rho \cdot \text{grad}\, \varphi_g = \text{grad}\, p\,. \tag{3.3.46}$$

According to Theorem 3.2, $\rho = \text{const}$ on the equipotential surface of the potential field of forces, which is the gravitational field, so that it directly follows from Eq. (3.3.46) that:

$$p + \rho\varphi_g = \text{const}\,. \tag{3.3.47}$$

But according to the same Theorem 3.2 on the equipotential surface (since it coincides with the isobar) $p = \text{const}$ which, in turn, allows writing (3.3.47) in the form of:

$$\varphi_g = \text{const}. \qquad (3.3.48)$$

Substituting in (3.3.48) the formula (2.4.26) for the gravitational potential of a resting ($\varepsilon_0 = 0$) spheroidal body [16, 47, 73], we obtain:

$$\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r e^{-\alpha r^2/2} dr = \text{const} \qquad (3.3.49)$$

which, under the invariability $\alpha(t)$ on the observation interval $t \in [t_0, t_0 + T_{obs}]$, gives the desired equation of the sphere (3.3.45). The corollary is proven.

Corollary 3.2. The iso-surface of the mass density of a spheroidal body is described by a flattened ellipsoid (spheroid) in space if and only if the potential field of forces is the total potential field of gravitational and centrifugal forces of a rotating spheroidal body in relative mechanical equilibrium.

Proof: the necessity of this statement directly follows from the fact that, in accordance with formulas (3.3.26a–c), the mass density of a rotating spheroidal body (being in relative mechanical equilibrium) could be constant:

$$\rho(\vec{r}) = \rho_0(1-\varepsilon_0^2)e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2} = \text{const}, \qquad (3.3.50)$$

in case:

$$\frac{x^2}{2/\alpha(1-\varepsilon_0^2)} + \frac{y^2}{2/\alpha(1-\varepsilon_0^2)} + \frac{z^2}{2/\alpha} = \text{const} \qquad (3.3.51)$$

and if $\alpha(t) = \text{const}$ on the observation interval $t \in [t_0, t_0 + T_{obs}]$. According to (3.3.51) an iso-surface of the mass density of a spheroidal body being in relative mechanical equilibrium is a flattened ellipsoid of rotation, that is, the spheroid since $\alpha << 1$ and $|\varepsilon_0| < 1$.

On the other hand, according to Eq. (3.3.29) and Eqs (3.3.38a–c), the equation of relative equilibrium of a continuous medium of a rotating spheroidal body in the total potential field of gravitational and centrifugal forces is written in the form:

$$-\rho \cdot \mathrm{grad}(\varphi_g + V_c) = \mathrm{grad}\, p \,. \tag{3.3.52}$$

But according to Theorem 3.2, $\rho = \mathrm{const}$ on the equipotential surface of the potential field of forces, which is the total potential field of gravitational and centrifugal forces, so that it directly follows from Eq. (3.3.52) that:

$$p + \rho \cdot (\varphi_g + V_c) = \mathrm{const} \,. \tag{3.3.53}$$

According to the same Theorem 3.2 $p = \mathrm{const}$ on an equipotential surface (because it coincides with the isobar), so that Eq. (3.3.53) becomes:

$$\phi = \varphi_g + V_c = \mathrm{const} \,. \tag{3.3.54}$$

Since the iso-surface of mass density (3.3.50) is a spheroid (3.3.51), then according to Lemma 3.1 and Theorem 3.2, the equipotential surface of the potential field of forces (3.3.54) must also be a spheroid, which proves corollary.

Remark 3.2. As noted in Section 3.1, the total potential of the common field of the gravitational and centrifugal forces is called *the general potential of gravity* $\psi_g = \varphi_g + V_c$ [95, 97]. Taking into account that $V_c = -\frac{1}{2}[\vec{\Omega} \times \vec{r}]^2 = -\frac{1}{2}\Omega^2(x^2 + y^2)$, the equipotential surface equation (3.3.54) of the potential field of forces, which is the total potential field of gravitational and centrifugal forces, takes the form:

$$\psi_g = \varphi_g - \Omega^2(x^2 + y^2)/2 = \mathrm{const} \,. \tag{3.3.55}$$

In the case of a very slow rotation of the spheroidal body, when $\Omega << 1$, equation (3.3.54) degenerates into the equipotential surface equation of the gravitational field of a slowly rotating spheroidal body:

$$\varphi_g = \text{const},$$

which in space, according to Corollary 3.2, must also be described by Eq. (3.3.51). This conclusion will be confirmed by the analytical formula of the gravitational potential of a spheroidal body in Section 3.6.

3.4. Derivation of the distribution function of the specific angular momentum value and angular momentum density for a uniformly rotating spheroidal body in a state of relative mechanical equilibrium

The distribution function of the specific angular momentum value for a uniformly rotating spheroidal body ($\Omega = \text{const}$) is easily derived based on the obtained (in Section 3.3) relation for the probability volume density $\Phi(h, z)$ in a cylindrical coordinate system (we assume that the axis of rotation of a spheroidal body coincides with the axis Oz). Indeed, let us use the probability volume density (3.3.22a) of detecting particles at distances close to h from the axis Oz in a rotating spheroidal body in the *relative mechanical equilibrium state* (Fig. 3.3).

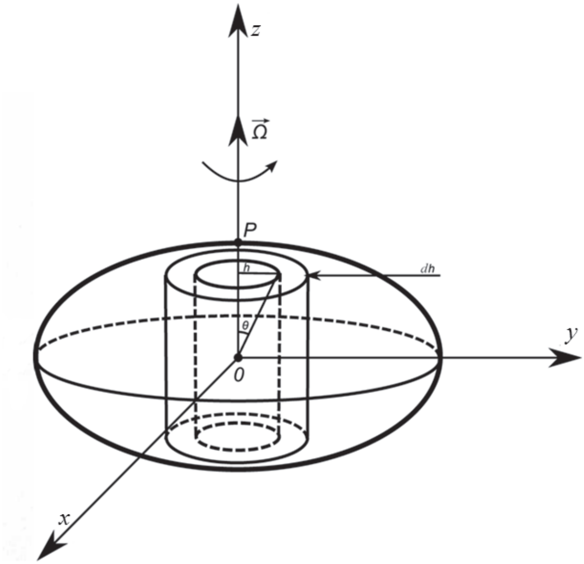


Figure 3.3. The scheme for calculating a share of particles having the same value of specific angular momentum in a uniformly rotating spheroidal body

Let us note that, following (2.1.16), the probability that the coordinate h belongs to the interval $[h, h+dh]$ for a particle is equal [16, 73]:

$$dp_h = \frac{dN_h}{N} = f(h)dh, \quad h \in \left[h, h+dh\right], \qquad (3.4.1)$$

where dN_h / N is a share of particles located at distances (from the axis of rotation Oz) close to h, and $f(h)$ is a one-dimensional probability density to locate a particle at distance h from the axis of rotation.

On the other hand, according to (3.3.22a), the probability dp_h can be calculated by integrating the probability volume density function $\Phi(h, z)$ with respect to the coordinates z and

ε of the cylindrical system (except for the coordinate h) [16, 56, 73]:

$$dp_h = \int_{-\infty}^{\infty}\int_{0}^{2\pi}\Phi(h,z)hdhdzd\varepsilon =$$

$$= (\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)\cdot 2\pi hdh\cdot\int_{-\infty}^{\infty}e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}dz =$$

$$= \frac{\alpha^{3/2}}{\sqrt{2\pi}}(1-\varepsilon_0^2)\cdot\int_{-\infty}^{\infty}e^{-\alpha z^2/2}dz\cdot e^{-\alpha(1-\varepsilon_0^2)h^2/2}hdh =$$

$$= \alpha(1-\varepsilon_0^2)e^{-\alpha(1-\varepsilon_0^2)h^2/2}hdh \qquad (3.4.2)$$

From a comparison of (3.4.1) and (3.4.2), we obtain the share of particles, located at distances close to h from the axis of rotation Oz, equal to [16, 56, 73]:

$$\frac{dN_h}{N} = \alpha(1-\varepsilon_0^2)e^{-\alpha(1-\varepsilon_0^2)h^2/2}hdh. \qquad (3.4.3)$$

Now we can see in Fig. 3.3 that the share of particles located near the distance close to h from the axis Oz of rotation, that is, into a volume of an annular cylindrical layer $[h, h+dh]$, is equal to the share of particles rotating with a constant angular velocity $\vec{\Omega}$ around the axis Oz and having the values of specific angular momentum value in the interval $[\lambda,\ \lambda+d\lambda]$:

$$dN_h/N = dN_\lambda/N,\ h\in[h, h+dh],$$
$$\lambda\in[\lambda,\ \lambda+d\lambda]. \qquad (3.4.4)$$

Evidently, the relation (3.4.4) is valid for a rotating spheroidal body in the state of *relative mechanical equilibrium* [16, 73]. Under this condition, the moving particles being at distance h from the axis of rotation Oz have *circular orbits* into a uniformly rotating spheroidal body and, therefore, the value of the z-projection of angular momentum acting on a particle of mass m_0 is equal [158]:

$$L_{0z} = m_0 h^2 \Omega\,. \tag{3.4.5}$$

As follows from Eq. (3.4.5), the value of the z-projection of specific angular momentum:

$$\lambda = L_{0z} / m_0 = \Omega\, h^2 \tag{3.4.6}$$

is directly proportional to the square of the distance from the axis of rotation Oz into a uniformly rotating spheroidal body.

Consequently, the share of particles having a specific angular momentum value in the interval $[\lambda,\ \lambda + d\lambda]$, following Eqs (3.4.4) and (3.4.6), is equal to:

$$\begin{aligned} &dN_\lambda / N = dN_h / N = \\ &= [\alpha(1-\varepsilon_0^2)/2\Omega]\cdot e^{-\alpha(1-\varepsilon_0^2)\Omega h^2/2\Omega} d(\Omega\, h^2) = \\ &= [\alpha(1-\varepsilon_0^2)/2\Omega]\cdot e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} d\lambda, \end{aligned} \tag{3.4.7}$$

where λ is a value of specific angular momentum [16, 73]. Similar to (3.4.1), the probability that the value of specific angular momentum belongs to the interval $[\lambda,\ \lambda + d\lambda]$ is equal:

$$f(\lambda) d\lambda = dp_\lambda = \frac{dN_\lambda}{N} = \frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\cdot e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} d\lambda\,, \tag{3.4.8}$$

that is, a desired probability density function $f(\lambda)$, expressing a mass distribution by values of specific angular momentum, is described by the formula [16, 73]:

$$f(\lambda) = \frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\cdot e^{-\frac{\alpha(1-\varepsilon_0^2)\lambda}{2\Omega}}\,. \tag{3.4.9}$$

This function (3.4.9) satisfies the normalization condition as a function of the distribution of specific angular momentum:

$$\int_0^\infty f(\lambda) d\lambda = 1, \tag{3.4.10a}$$

because

$$\int f(\lambda)d\lambda = \int e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} \cdot \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} d\lambda = -e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega}. \quad (3.4.10b)$$

Let us calculate an average value of specific angular momentum based on the integration by the parts with the usage of formulas (3.4.10a–b):

$$\overline{\lambda} = \int_0^\infty \lambda\, f(\lambda)d\lambda = \lambda \int f(\lambda)d\lambda \Big|_0^\infty - \int_0^\infty d\lambda \int f(\lambda)d\lambda =$$

$$= -\lambda \cdot e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} \Big|_0^\infty - \int_0^\infty (-e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega})d\lambda =$$

$$= \frac{2\Omega}{\alpha(1-\varepsilon_0^2)} \int_0^\infty e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} \cdot \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} d\lambda =$$

$$= \frac{2\Omega}{\alpha(1-\varepsilon_0^2)} \int_0^\infty f(\lambda)d\lambda = \frac{2\Omega}{\alpha(1-\varepsilon_0^2)}. \quad (3.4.11)$$

According to (3.4.8) and (3.4.9), the quantity of particles having values of specific angular momentum close to λ is equal to:

$$dN_\lambda = N \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} \cdot e^{-\alpha(1-\varepsilon_0^2)\lambda/2\Omega} d\lambda = Nf(\lambda)d\lambda. \quad (3.4.12)$$

Using (3.4.12), it is not difficult to find a total angular momentum of a uniformly rotating spheroidal body being in relative mechanical equilibrium [16, 73]:

$$L = \int_0^N m_0 \lambda\, dN_\lambda = m_0 N \int_0^\infty \lambda\, f(\lambda)d\lambda = M\overline{\lambda}, \quad (3.4.13)$$

where M is a mass of a spheroidal body (for example, the total mass of a star and protoplanetary gas-dust disk [16]). Substituting (3.4.11) into (3.4.13), we obtain that the value of the total angular momentum of a uniformly rotating spheroidal body is expressed by the formula [16, 73]:

$$L = \frac{2\Omega M}{\alpha(1-\varepsilon_0^2)}. \tag{3.4.14}$$

In the particular case of a *slowly* rotating spheroidal body when $\varepsilon_0^2 \to 0$, that is, without a deformation variation of the spherical iso-surface of mass density, the value of its total angular momentum is equal:

$$L = \frac{2\Omega M}{\alpha}, \tag{3.4.15}$$

which follows naturally from the just derived formula (3.4.14) at $\varepsilon_0 = 0$.

As we know [12], generally speaking, the total angular momentum of a rotating axial symmetric body relative to the cylindrical coordinate system is determined by the following formula:

$$L = \int_V \rho(h,z)\Omega(h,z)h^2 \cdot hdhd\varepsilon dz =$$

$$= 2\pi\int_0^\infty \int_{-\infty}^\infty \rho(h,z)\Omega(h,z)h^3 dhdz, \tag{3.4.16}$$

where $\rho = \rho(h,z)$ is a function of mass density and $\Omega = \Omega(h,z)$ is a function of angular velocity. As applied to a uniformly rotating spheroidal body, $\Omega = \Omega(h,z) = \text{const}$ in the state of relative mechanical equilibrium. Expression (3.4.16), therefore, goes into the following:

$$L = 2\pi\Omega\int_0^\infty \int_{-\infty}^\infty \rho(h,z)h^3 dhdz. \tag{3.4.17}$$

Substituting into (3.4.17) formula (3.3.26a) for the mass density of a uniformly rotating spheroidal body and taking into account the result of integration (3.2.13) we obtain:

$$L = 2\pi\Omega M\left(\frac{\alpha}{2\pi}\right)^{3/2}(1-\varepsilon_0^2)\int_0^\infty \int_{-\infty}^\infty e^{-\alpha h^2(1-\varepsilon_0^2)/2} h^3 dh \cdot e^{-\alpha z^2/2} dz =$$

$$=\frac{1-\varepsilon_0^2}{\sqrt{2\pi}}\Omega M\alpha^{3/2}\int_{-\infty}^{\infty}e^{-\alpha z^2/2}dz\int_0^{\infty}h^3e^{-\alpha(1-\varepsilon_0^2)h^2/2}dh=$$

$$=\frac{1-\varepsilon_0^2}{\sqrt{2\pi}}\Omega M\alpha^{3/2}\sqrt{\frac{2\pi}{\alpha}}\cdot\frac{1}{\alpha(1-\varepsilon_0^2)}\left[-\frac{2}{\alpha(1-\varepsilon_0^2)}\cdot e^{-\alpha(1-\varepsilon_0^2)h^2/2}-h^2e^{-\alpha(1-\varepsilon_0^2)h^2/2}\right]_0^{\infty}=$$

$$=\frac{2\Omega M}{\alpha(1-\varepsilon_0^2)} \qquad (3.4.18)$$

So, although formula (3.4.18) for the total angular momentum of a uniformly rotating spheroidal body has been derived in a different way than (3.4.14), nevertheless, both of them lead to the same result. Indeed, the formula (3.4.14) has also been obtained under the assumption of the relative mechanical equilibrium of a spheroidal body when $\Omega=\text{const}$. Thus, in the state of relative mechanical equilibrium, the initial formulas (3.4.13) and (3.4.16) coincide. Indeed, the z-projection of specific angular momentum is expressed by formula (3.4.6), so that with regard for (3.4.6) expression (3.4.16) takes the form:

$$L=2\pi\int_0^{\infty}\int_{-\infty}^{\infty}\rho(h,z)\lambda(h)hdhdz=$$

$$=2\pi M\int_0^{\infty}\int_{-\infty}^{\infty}\Phi(h,z)\lambda(h)hdhdz\,. \qquad (3.4.19)$$

On the other hand, according to (3.4.1)–(3.4.4), (3.4.7) and (3.4.8), the following equalities are true [16, 73]:

$$f(\lambda)d\lambda=f(h)dh=\int_{-\infty}^{\infty}dz\int_0^{2\pi}\Phi(h,z)hdhd\varepsilon=$$

$$=2\pi\int_{-\infty}^{\infty}\Phi(h,z)hdhdz\,, \qquad (3.4.20)$$

whence we can see that formula (3.4.19) goes to (3.4.13). Obviously, formula (3.4.16) is more general than formula

(3.4.17) or (3.4.13), because it takes into account the nonequilibrium case when the angular velocity Ω is a function of spatial coordinates called the *vorticity* $\omega(h,z)$ [111]:

$$\Omega = \tfrac{1}{2}\,\omega(h,z). \tag{3.4.21}$$

Denoting by the function of the specific angular momentum:

$$\lambda(h,z) = 2h^2\omega(h,z)\,, \tag{3.4.22}$$

we can rewrite the formula (3.4.16) in the form :

$$L = 2\pi\int_0^\infty\int_{-\infty}^\infty \rho(h,z)\lambda(h,z)hdhdz\,. \tag{3.4.23}$$

The obtained formula (3.4.23) as well as the formula for the mass distribution:

$$M = 2\pi\int_0^\infty\int_{-\infty}^\infty \rho(h,z)hdhdz \tag{3.4.24}$$

are appropriate to use in the models of forming cosmogonical bodies (planetesimals, planetary embryos, protoplanets, and protostars) [73].

Obviously, in the stationary case of relative mechanical equilibrium, formula (3.4.24) gives a constant value $M = \text{const}$ which directly follows under substitution of formula (3.3.26a) for mass density into (3.4.24):

$$M = 2\pi\int_0^\infty\int_{-\infty}^\infty M(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)\cdot e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2}hdhdz =$$

$$= M\sqrt{\alpha/2\pi}\int_0^\infty e^{-\alpha(1-\varepsilon_0^2)h^2/2}d[\alpha(1-\varepsilon_0^2)h^2/2]\cdot\int_{-\infty}^\infty e^{-\alpha z^2/2}dz =$$

$$= M\sqrt{\alpha/2\pi}\cdot\sqrt{2\pi/\alpha} = M = \text{const}\,.$$

Using (5.4.17), it is easy to find the angular momentum distribution function in space $l(h,z)$, that is, the density of angular momentum in the case of a uniformly rotating spheroidal body [73]:

$$L = \int_V l(h,z)dV = \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} \Omega\rho(h,z)h^2 \cdot hdhd\varepsilon dz =$$

$$= \int_V \Omega h^2 \rho(h,z)dV \ , \qquad (3.4.25)$$

hence the density of angular momentum is equal [73]:

$$l(h,z) = \Omega h^2 \rho(h,z) \, . \qquad (3.4.26)$$

Substituting into (3.4.26) the mass density formula (3.3.26a) we get:

$$l(h,z) = M\Omega(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)h^2 e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2} . \qquad (3.4.27)$$

On the other hand, from (3.4.13) it immediately follows that the average specific angular momentum is equal to:

$$\overline{\lambda} = \frac{L}{M} , \qquad (3.4.28)$$

so that from (3.4.25), (3.4.26), and (3.4.28) we have:

$$\overline{\lambda} = \int_V \frac{l(h,z)}{M} dV = \int_V ƛ(h,z)dV \ , \qquad (3.4.29)$$

where $ƛ(h,z)$ is a density of the average specific angular momentum in space [73]. Taking into account that the specific angular momentum in a uniformly rotating spheroidal body in accordance with (3.4.6) is equal $\lambda = \Omega h^2$, from (3.4.26), (3.4.27), and (3.4.29) we obtain [73]:

$$l(h,z) = \lambda(h)\rho(h,z) \, ; \qquad (3.4.30a)$$

$$\overline{\lambda} = \int_V \lambda \cdot (\alpha/2\pi)^{3/2}(1-\varepsilon_0^2) \cdot e^{-\alpha[(1-\varepsilon_0^2)h^2+z^2]/2} dV =$$

$$= \int_V \lambda(h)\Phi(h,z)dV \ , \qquad (3.4.30b)$$

which is fully confirmed by formula (3.4.19). Bearing in mind that $\lambda = \lambda(h)$ is a function of h only, but not of z, the formula (3.4.30b) can be simplified [73]:

$$\overline{\lambda} = \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} \lambda(h)\cdot\left(\alpha/2\pi\right)^{3/2}(1-\varepsilon_0^2)\cdot e^{-\alpha(1-\varepsilon_0^2)h^2/2} e^{-\alpha z^2/2} h\, dh\, d\varepsilon\, dz =$$

$$= 2\pi\cdot\frac{\alpha^{3/2}(1-\varepsilon^2)}{\left(2\pi\right)^{3/2}}\int_{-\infty}^{\infty} e^{-\alpha z^2/2}dz\int_0^{\infty}\lambda(h)\cdot e^{-\alpha(1-\varepsilon_0^2)h^2/2}h\, dh =$$

$$= \alpha(1-\varepsilon_0^2)\int_0^{\infty}\lambda(h)\, h\, e^{-\alpha(1-\varepsilon_0^2)h^2/2}\, dh = \int_0^{\infty}\lambda(h)\, f(h)\, dh\,, \quad (3.4.31)$$

where

$$f(h) = \alpha(1-\varepsilon_0^2)h\, e^{-\alpha(1-\varepsilon_0^2)h^2/2}. \quad (3.4.32)$$

According to (3.4.2), (3.4.3), as well as (3.4.31) and (3.4.32), $f(h)$ is a distribution function of the specific angular momentum on distance. The graphic dependence of the distribution function (3.4.32) of the specific angular momentum on distance h recalls the gravitational field strength dependence $a = a(r)$ on distance r diagram (see Fig. 2.6) for a sphere-like gaseous body [73]. Indeed, let us find the point of extremum of function $f(h)$, for which we calculate:

$$f'(h) = \alpha(1-\varepsilon_0^2)e^{-\alpha(1-\varepsilon_0^2)h^2/2}[1-\alpha(1-\varepsilon_0^2)h^2].$$

By analogy with Eq. (2.4.16), equality $f'(h) = 0$ means the presence of a maximum point of the distribution function of the specific angular momentum when:

$$h_* = \frac{1}{\sqrt{\alpha(1-\varepsilon_0^2)}}. \quad (3.4.33)$$

According to (3.4.27), the formula for the density of angular momentum at a fixed value $z = z_0$ has the form:

$$l(h, z_0) = M\Omega(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha z_0{}^2/2}\cdot h^2 e^{-\alpha(1-\varepsilon_0^2)h^2/2}. \quad (3.4.34)$$

The similar function $l(h)\Big|_{z_0} = l(h, z_0)$ graph is shown in Fig. 2.2, corresponding to the probability density function $f(r)$ for finding particles located at distances r from the center. Analogously to (2.1.22), we first calculate:

$$l'(h)\Big|_{z_0} = M\Omega(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)\,h\,e^{-\alpha\frac{(1-\varepsilon_0^2)h^2+z_0{}^2}{2}}[2-\alpha(1-\varepsilon_0^2)h^2]$$

and then, using $l'(h)\Big|_{z_0} = 0$, we can find the maximum point of the density of angular momentum:

$$h_{pr} = \sqrt{\frac{2}{\alpha(1-\varepsilon_0^2)}} = \sqrt{2}h_* \ . \qquad (3.4.35)$$

Considering the maximum points (3.4.33) and (3.4.35) we can affirm that an "export" of the specific angular momentum by particles from the axis of rotation to the region $h = h_*$ occurs in a uniformly rotating spheroidal body.

3.5. The distribution function of particles in space for a rotating and gravitating spheroidal body from the point of view of the general relativity theory

Unlike the point discussed in Section 2.7, the gravitational field of a rotating body within the framework of the general relativity (GR) theory is characterized by an axially symmetric stationary Kerr metric [100]:

$$ds^2 = \left(1 - \frac{r_g r}{\delta^2}\right)c^2 dt^2 - \frac{\delta^2}{\Delta}dr^2 - \delta^2 d\theta^2 -$$

$$-\left(r^2 + a^2 + \frac{r_g r a^2}{\delta^2}\cdot\sin^2\theta\right)\sin^2\theta\, d\varepsilon^2 + \frac{2r_g r a}{\delta^2}\sin^2\theta\, d\varepsilon\, dt, \quad (3.5.1)$$

where the following notations are introduced:

$$\Delta = r^2 - r_g r + a^2, \quad \delta^2 = r^2 + a^2\cos^2\theta \ , \quad a = L/cM \ ,$$

and $r_g = \dfrac{2\gamma M}{c^2}$ is a gravitational radius of the body, L is an angular momentum, M is a mass of the body, γ is the gravitational constant of Newton, and c is the speed of light. As follows from (3.5.1), the components of the metric space-time tensor are defined by the expressions:

$$g_{00} = 1 - \frac{r_g r}{\delta^2}, g_{11} = -\frac{\delta^2}{\Delta}, g_{22} = -\delta^2,$$
$$g_{33} = -\left(r^2 + a^2 + \frac{r_g r a^2}{\delta^2}\sin^2\theta\right)\sin^2\theta \ , \qquad (3.5.2)$$
$$g_{03} = g_{30} = \frac{r_g r a}{\delta^2}\sin^2\theta .$$

The pure spatial Kerr metric is determined by the expression for the spatial distance element [100]:

$$dl^2 = \frac{\delta^2}{\Delta}dr^2 + \delta^2 d\theta^2 + \frac{\Delta\sin^2\theta}{1 - r_g r/\delta^2}d\varepsilon^2 , \qquad (3.5.3)$$

that is, $\hat{g}_{11} = \dfrac{\delta^2}{\Delta}$, $\hat{g}_{22} = \delta^2$, $\hat{g}_{33} = \dfrac{\Delta\sin^2\theta}{1 - r_g r/\delta^2}$. Taking into account that:

$$\delta^2 = r^2 + a^2\cos^2\theta = (r^2 + a^2) - a^2\sin^2\theta \ , \qquad (3.5.4)$$

we can represent the value Δ as follows:

$$\Delta = (r^2 + a^2) - r_g r = \delta^2 + a^2\sin^2\theta - r_g r . \qquad (3.5.5)$$

As noted in [100] (and it will be shown below), the determinant of the metric space-time tensor for the Kerr metric (3.5.1) is equal:

$$-g = \delta^4\sin^2\theta \ , \qquad (3.5.6)$$

while the determinant of the metric spatial three-dimensional tensor can be found by (3.5.3) in the form [73]:

$$\hat{g} = \frac{\delta^4 \sin^2\theta}{1 - r_g r/\delta^2}. \quad (3.5.7)$$

Taking into account the modified relations for δ^2 and Δ in (3.5.4) and (3.5.5), we transform the components $g_{11}, \hat{g}_{11}$ from (3.5.2) and (3.5.3) as follows:

$$g_{11} = -\frac{\delta^2}{\Delta} = -\frac{\delta^2}{\delta^2 + a^2\sin^2\theta - r_g r} =$$

$$= -\frac{1}{1 - \frac{r_g}{\delta^2/r} + \frac{a^2}{\delta^2}\sin^2\theta}, \quad (3.5.8a)$$

$$\hat{g}_{11} = \frac{1}{1 - \frac{r_g}{\delta^2/r} + \frac{a^2}{\delta^2}\sin^2\theta}. \quad (3.5.8b)$$

By analogy with the derivation of the Schwarzschild metric [100] for a centrally symmetric gravitational field (see formulas (2.7.2)–(2.7.5) in Section 2.7), we assume that at *large* distances:

$$e^{\lambda} \approx \frac{1}{1 - \frac{r_g}{\delta^2/r}} = 1 + \frac{r_g}{\delta^2/r}. \quad (3.5.9)$$

This means that according to (3.5.8a) and (3.5.9) we have [73]:

$$e^{\lambda-\mu} \approx \frac{1}{1 - \left(\frac{r_g}{\delta^2/r} - \frac{a^2}{\delta^2}\sin^2\theta\right)} = -g_{11} = \hat{g}_{11} = \frac{\delta^2}{\Delta}. \quad (3.5.10)$$

According to (3.5.9) and (3.5.10) we can generalize the spatial Kerr metric under consideration of the following components of the three-dimensional metric tensor [54, 73]:

$$g_{11} = -e^{\lambda-\mu}, g_{22} = -\delta^2,$$

$$g_{33} = -e^{\lambda}\Delta\sin^2\theta = -e^{\lambda}\frac{\delta^2}{e^{\lambda-\mu}}\sin^2\theta = -e^{\mu}\delta^2\sin^2\theta \ , \quad (3.5.11)$$

$$\hat{g}_{11} = e^{\lambda-\mu}, \hat{g}_{22} = \delta^2, \hat{g}_{33} = e^{\mu}\delta^2\sin^2\theta \ .$$

From (3.5.11) it immediately follows that the determinant of this type of three-dimensional (spatial) metric tensor is equal [73]:

$$\hat{g} = e^{\lambda-\mu}\delta^2 e^{\mu}\delta^2\sin^2\theta = \delta^4 e^{\lambda}\sin^2\theta \ . \quad (3.5.12)$$

It is obvious that the formula (3.5.12) generalizes (3.5.7). Further, the spatial volume element in the generalized Kerr metric is determined in accordance with (3.5.12) by the relation [54, 73]:

$$dV = \sqrt{\hat{g}}dV_0' = e^{\lambda/2}\cdot\delta^2\sin\theta\, drd\theta d\varepsilon = e^{\lambda/2}dV_0, \quad (3.5.13)$$

where $dV_0 = \delta^2\sin\theta drd\theta d\varepsilon$ is an element of spatial volume in the flattened spherical coordinate system and $dV_0' = drd\theta d\varepsilon$ is a product of the differentials of three spatial coordinates r, θ, ε. The expression for dV_0 in the flattened spherical coordinate system will be obtained below.

Let us note that relation (3.5.13) can be directly derived from the expression for the spatial distance element (3.5.3) in the Kerr metric:

$$dV = \sqrt{\frac{\delta^2}{\Delta}\cdot dr^2\delta^2 d\theta^2\cdot\frac{\Delta\sin^2\theta}{1-r_g r/\delta^2}d\varepsilon^2} =$$

$$= \frac{\delta^2\sin\theta}{\sqrt{1-r_g r/\delta^2}}drd\theta d\varepsilon \quad (3.5.14)$$

with the subsequent generalization of relation (3.5.14) based on (3.5.9):

$$dV = e^{\lambda/2}\cdot\delta^2\sin\theta\, drd\theta\, d\varepsilon = e^{\lambda/2}dV_0, \quad (3.5.15)$$

where, as will be shown below, $dV_0 = \delta^2 \sin\theta dr d\theta d\varepsilon$ is an element of spatial volume in the flattened spherical coordinate system.

Similar to the case of the Schwarzschild metric [100], the element of spatial volume according to (2.7.5) is equal to:

$$dV = e^{\lambda/2} \cdot r^2 \sin\theta \, dr d\theta d\varepsilon = e^{\lambda/2} dV_0 , \qquad (3.5.16)$$

where $dV_0 = r^2 \sin\theta \, dr d\theta d\varepsilon$ is a volume element in the traditional spherical coordinate system.

According to the formula (2.7.18) from Section 2.7 (as well as from the comparison of (3.5.15) with (3.5.16)), it follows that for small distances the function λ takes the form [16, 73]:

1) for the Schwarzschild metric

$$\lambda = \alpha r^2 , \qquad (3.5.17a)$$

2) for the Kerr metric

$$\lambda = \alpha\delta^2 = \alpha[(r^2 + a^2) - a^2 \sin^2\theta] . \qquad (3.5.17b)$$

Using (3.5.17b) we can introduce a new distance:

$$\mathrm{r}^2 = r^2 + a^2 . \qquad (3.5.18)$$

By substituting (3.5.17b) and (3.5.18) into (3.5.15) we obtain the following relation [73]:

$$dV = e^{\alpha \mathrm{r}^2 \cdot \frac{1 - \frac{a^2}{\mathrm{r}^2} \sin^2\theta}{2}} dV_0 . \qquad (3.5.19)$$

Starting from the relation (3.5.19) between the elements of spatial volume in the generalized Kerr metric, we can obtain the form of equilibrium function of the probability volume density for a uniformly rotating spheroidal body following the scheme proposed in Section 2.7 (see also [16, 73]). Indeed, since a uniformly rotating spheroidal body contains a fixed number N of particles identical in mass m_0, that is, $M = m_0 N = \text{const}$, then

$$dM = d(\rho V) = 0 . \qquad (3.5.20)$$

From (3.5.20) it directly follows that:

$$\rho dV = -Vd\rho\,,$$

from where:

$$\frac{d\rho}{\rho} = -\frac{dV}{V}\,. \qquad (3.5.21)$$

Integrating (3.5.21) we obtain [73]:

$$\int_{\rho_0}^{\rho(\vec{r})} \frac{d\rho}{\rho} = \int_{V_0}^{V(\vec{r})} \frac{dV}{V}, \qquad (3.5.22)$$

whence:

$$\ln\frac{\rho(\vec{r})}{\rho_0} = -\ln\frac{V(\vec{r})}{V_0}\,, \qquad (3.5.23)$$

where $\vec{r} = \vec{r}(\mathsf{r},\theta)$ and $\rho_0 = \rho(0)$, $V_0 = V(0)$. According to (3.5.19) there is a relationship between V_0 and $V(\vec{r})$ in the curvilinear Kerr space. Let us note, however, following quantum mechanics and statistical physics, that the volume dV_0 cannot be infinitely small and must be bounded below by the elementary volume of the quantum mechanical cell [110]:

$$V_0 \geq V_{quant} = \frac{h^3}{(2\pi m_0 k_B T)^{3/2}}\,. \qquad (3.5.24)$$

Then, as follows from (3.5.19) and (3.5.24), the next relationship holds:

$$V(\mathsf{r},\theta) = e^{\alpha\,\mathsf{r}^2 \cdot \frac{1-\frac{a^2}{\mathsf{r}^2}\sin^2\theta}{2}} V_0\,, \qquad (3.5.25)$$

where $\mathsf{r} = \sqrt{r^2 + a^2}$, and $a = L/cM$ is a constant. Taking into account the relation (3.5.25), equality (3.5.23) takes the form:

$$\ln\frac{\rho(\mathsf{r},\theta)}{\rho_0} = -\alpha\,\mathsf{r}^2 \cdot \frac{1-\frac{a^2}{\mathsf{r}^2}\sin^2\theta}{2}\,, \qquad (3.5.26)$$

whence:

$$\rho(\mathsf{r},\theta)=\rho_0\cdot e^{-\alpha\,\mathsf{r}^2[1-(a/\mathsf{r})^2\sin^2\theta]/2}. \tag{3.5.27}$$

According to (2.2.7), the mass density (3.5.27) of a rotating spheroidal body is directly related to the probability volume density $\Phi(\mathrm{r},\theta)$ by the simple relation:

$$\rho(\mathsf{r},\theta)=M\cdot\Phi(\mathsf{r},\theta), \tag{3.5.28}$$

where M is a mass of a spheroidal body. From (3.5.27) and (3.5.28) it immediately follows that from the point of view of GR, the volume density of the probability of detecting a particle in a uniformly rotating spheroidal body is equal [54, 73]:

$$\Phi(\mathsf{r},\theta)=(\rho_0/M)\cdot e^{-\alpha\,\mathsf{r}^2[1-(a/\mathsf{r})^2\sin^2\theta]/2}. \tag{3.5.29}$$

Comparing (3.5.29) with a similar relation (3.3.22c) obtained in the framework of the statistical theory (see Section 3.3), we can see that:

$$\frac{\rho_0}{M}=(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2); \tag{3.5.30a}$$

$$\frac{a}{\mathsf{r}}\to\varepsilon_0 \text{ at } \mathsf{r}\to r,\ \varepsilon_0\ll 1, \tag{3.5.30b}$$

where

$a=\mathrm{I}\Omega/Mc$,

I is a moment of inertia, and

ε_0 is a stabilization constant of the angular azimuth variable.

In conclusion, as already mentioned above, we derive formulas (3.5.6) and (3.5.15), that is, initially we find the value of the determinant of the metric space-time tensor for the Kerr metric (3.5.1):

$$g=\det\begin{bmatrix} 1-r_g r/\delta^2 & 0 & 0 & (r_g ra/\delta^2)\cdot\sin^2\theta \\ 0 & -\delta^2/\Delta & 0 & 0 \\ 0 & 0 & -\delta^2 & 0 \\ (r_g ra/\delta^2)\cdot\sin^2\theta & 0 & 0 & -\left(r^2+a^2+(r_g ra/\delta^2)\cdot\sin^2\theta\right)\cdot\sin^2\theta \end{bmatrix}=$$

$$=-\left(1-\frac{r_g r}{\delta^2}\right)\cdot\frac{\delta^4}{\Delta}\cdot\left(r^2+a^2+\frac{r_g ra^2}{\delta^2}\cdot\sin^2\theta\right)\cdot\sin^2\theta-$$

$$-\frac{r_g ra}{\delta^2}\cdot\sin^2\theta\cdot\det\begin{bmatrix} 0 & -\delta^2/\Delta & 0 \\ 0 & 0 & -\delta^2 \\ (r_g ra/\delta^2)\cdot\sin^2\theta & 0 & 0 \end{bmatrix}=$$

$$=-\left(1-\frac{r_g r}{\delta^2}\right)\cdot\frac{\delta^4}{\Delta}\cdot\left(r^2+a^2+\frac{r_g ra^2}{\delta^2}\cdot\sin^2\theta\right)\cdot\sin^2\theta-\frac{r_g ra}{\delta^2}\cdot\sin^2\theta\cdot\frac{r_g ra}{\delta^2}\cdot\sin^2\theta\cdot\frac{\delta^4}{\Delta}=$$

$$=-\frac{\delta^4}{\Delta}\cdot\sin^2\theta\cdot\left[r^2+a^2+\frac{r_g ra^2}{\delta^2}\cdot\sin^2\theta-\frac{r_g r}{\delta^2}\cdot\left(r^2+a^2\right)\right]=$$

$$=-\frac{\delta^4}{\Delta}\cdot\sin^2\theta\cdot\left[r^2+a^2-\frac{r_g ra^2}{\delta^2}\cdot\cos^2\theta-\frac{r_g r}{\delta^2}\cdot r^2\right]=$$

$$=-\frac{\delta^4}{\Delta}\cdot\sin^2\theta\cdot\left[r^2+a^2-\frac{r_g r}{\delta^2}\cdot\left(a^2\cos^2\theta+r^2\right)\right]=$$

$$=-\frac{\delta^4}{\Delta}\cdot\sin^2\theta\cdot\left[r^2+a^2-r_g r\right]=-\delta^4\sin^2\theta,$$

which completely coincides with the formula (3.5.6).

We now make some important clarifications regarding the above. Let us note that in accordance with (3.5.4), (3.5.15), (3.5.18), and (3.5.19) we have:

$$dV=e^{a\mathrm{r}^2\frac{1-\frac{a^2}{\mathrm{r}^2}\sin^2\theta}{2}}\left(\mathrm{r}^2-a^2\sin^2\theta\right)\cdot\sin\theta\,drd\theta\,d\varepsilon=$$

$$=e^{a\mathrm{r}^2\frac{1-\frac{a^2}{\mathrm{r}^2}\sin^2\theta}{2}}\left(1-\frac{a^2}{\mathrm{r}^2}\sin^2\theta\right)\cdot\mathrm{r}^2\sin\theta\,drd\theta\,d\varepsilon. \qquad (3.5.31)$$

Obviously, under the condition $\mathrm{r} \to \infty$ (or $r \to \infty$), the Kerr spatial volume element (3.5.31) tends to the Schwarzschild spatial volume element (3.5.16), since:

$$\frac{a^2}{\mathrm{r}^2}\sin^2\theta \to 0 \text{ and } \mathrm{r} \to r \text{ at } r \to \infty . \qquad (3.5.32)$$

In the absence of mass, the Kerr metric (3.5.1) should be reduced to Galilean one [100]. Indeed, under condition $M = 0$ provided that $r_g = 0$, the Kerr metric (3.5.1) goes into the following:

$$ds^2 = c^2dt^2 - \frac{\delta^2}{\mathrm{r}^2}dr^2 - \delta^2 d\theta^2 - \mathrm{r}^2\sin^2\theta\, d\varepsilon^2 \qquad (3.5.33)$$

in accordance with (3.5.18), which is a Galilean metric written in flattened spherical spatial coordinates:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 . \qquad (3.5.34)$$

The transformation of these coordinates into the Cartesian is carried out by the formulas [100]:

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\varepsilon = \mathrm{r}\sin\theta\cos\varepsilon \ ; \qquad (3.5.35a)$$

$$y = \sqrt{r^2 + a^2}\sin\theta\sin\varepsilon = \mathrm{r}\sin\theta\sin\varepsilon \ ; \qquad (3.5.35b)$$

$$z = r\cos\theta , \qquad (3.5.35c)$$

and the surfaces $r = \text{const}$ are flattened ellipsoids of rotation:

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 . \qquad (3.5.36)$$

Finally, let us estimate the volume element in flattened spherical coordinates, for which we calculate the Lamé coefficients in this coordinate system (3.5.35a–c):

$$H_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} =$$

$$= \sqrt{\frac{r^2}{r^2 + a^2}\sin^2\theta\cdot\left(\cos^2\varepsilon + \sin^2\varepsilon\right) + \cos^2\theta} =$$

$$=\sqrt{\frac{r^2+a^2\cos^2\theta}{r^2+a^2}}\,; \tag{3.5.37a}$$

$$H_\theta=\sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2+\left(\frac{\partial y}{\partial\theta}\right)^2+\left(\frac{\partial z}{\partial\theta}\right)^2}=$$

$$=\sqrt{\left(r^2+a^2\right)\cos^2\theta\cdot\left(\cos^2\varepsilon+\sin^2\varepsilon\right)+r^2\sin^2\theta}=$$

$$=\sqrt{r^2+a^2\cos^2\theta}\,; \tag{3.5.37b}$$

$$H_\varepsilon=\sqrt{\left(\frac{\partial x}{\partial\varepsilon}\right)^2+\left(\frac{\partial y}{\partial\varepsilon}\right)^2+\left(\frac{\partial z}{\partial\varepsilon}\right)^2}=$$

$$=\sqrt{\left(r^2+a^2\right)\sin^2\theta\cdot\left(\sin^2\varepsilon+\cos^2\varepsilon\right)}=$$

$$=\sqrt{r^2+a^2}\cdot\sin\theta\,. \tag{3.5.37c}$$

Using (3.5.37a–c), it is easy to see that the volume element in the flattened spherical coordinate system is equal to:

$$dV=H_rH_\theta H_\varepsilon drd\theta d\varepsilon=\left(r^2+a^2\cos^2\theta\right)\cdot\sin\theta\,drd\theta d\varepsilon\,, \tag{3.5.38}$$

so that with regard to the notation (3.5.4) it is described by the formula [73]:

$$dV=\delta^2\sin\theta\,drd\theta d\varepsilon\,. \tag{3.5.39}$$

As mentioned above, this formula has been used in determining relations (3.5.13) and (3.5.15).

3.6. The strength and potential of the gravitational field of a uniformly rotating spheroidal body

As mentioned in Sections 2.2 and 2.3 in Chapter 2, there is the critical value α_c of the gravitational condensation parameter (2.3.7a, b) when a weak gravitational field arises in a forming sphere-like gaseous body. Supposing the condition of the gravitational field arising is valid: $\alpha\geq\alpha_c$, we intend to find the gravitational potential and strength of a uniformly rotating spheroidal body. To this end, let us use Eqs (1.1.40a)–

(1.1.40c) from Chapter 1 (see also [99]) for finding the gravitational strength $\vec{a}$ (or specific gravitational force $\vec{f}_g$) of a uniformly rotating spheroidal body.

So, substituting into Eqs (1.1.40a–c) the analytical form of the mass density (3.3.26b) with regard to Eq. (3.3.51), that is, assuming:

$$a = b = \sqrt{2} / \sqrt{\alpha(1-\varepsilon_0^2)}\ ; \qquad c = \sqrt{2} / \sqrt{\alpha}$$

in Eqs (1.1.40a–c), we finally obtain [70, 74]:

$$f_{g\,x} = -\frac{2\gamma Mx}{\sqrt{\pi}} \int_0^\infty e^{-\frac{x^2+y^2}{2/\alpha(1-\varepsilon_0^2)+s}} e^{-\frac{z^2}{2/\alpha+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\ ; \qquad (3.6.1a)$$

$$f_{g\,y} = -\frac{2\gamma My}{\sqrt{\pi}} \int_0^\infty e^{-\frac{x^2+y^2}{2/\alpha(1-\varepsilon_0^2)+s}} e^{-\frac{z^2}{2/\alpha+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\ ; \qquad (3.6.1b)$$

$$f_{g\,z} = -\frac{2\gamma Mz}{\sqrt{\pi}} \int_0^\infty e^{-\frac{x^2+y^2}{2/\alpha(1-\varepsilon_0^2)+s}} e^{-\frac{z^2}{2/\alpha+s}} \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}\ . \qquad (3.6.1c)$$

Since the x - projection of the gravitational field strength in accordance with Eq. (2.4.5) is

$$a_x = f_{g\,x} = -\frac{\partial \varphi_g}{\partial x},$$

of course, the gravitational potential φ_g can be calculated by integrating relation (3.6.1a) with respect to x:

$$\varphi_g = \int \frac{\partial \varphi_g}{\partial x} dx + C = -\int f_{g\,x} dx + C = \frac{2}{\sqrt{\pi}} \gamma M \int x dx \int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} \times$$

$$\times e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+C=$$

$$=-\frac{2\gamma M}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \int e^{-\frac{x^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} d\left(\frac{-x^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}\right)\times$$

$$\times\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}+C=$$

$$=-\frac{\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}+C \quad .(3.6.2a)$$

Similarly, the gravitational potential is found through the y -projection (3.6.1b) of specific gravitational force:

$$\varphi_g=\int\frac{\partial\varphi_g}{\partial y}dy+C=-\int f_{g\,y}dy+C=$$

$$=-\frac{\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}+C. \quad (3.6.2b)$$

As for the z -projection (3.6.1c), with some difference in the integration process over z, we then obtain the same result:

$$\varphi_g=\int\frac{\partial\varphi_g}{\partial z}dz+C=-\int f_{g\,z}dz+C=\frac{2\gamma M}{\sqrt{\pi}}\int zdz\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}\times$$

$$\times e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} + C =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} d\left(-\frac{z^2}{\frac{2}{\alpha}+s}\right) \times$$

$$\times \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} + C =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} + C. \quad (3.6.2c)$$

Thus, according to Eqs (3.6.2a–c) the gravitational potential of a rotating spheroidal body is described by the expression [74, 78]:

$$\varphi_g = -\frac{\gamma M}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} + C =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \alpha^{3/2}(1-\varepsilon_0^2) \int_0^{\infty} e^{-\frac{\alpha(1-\varepsilon_0^2)(x^2+y^2)}{2+s\alpha(1-\varepsilon_0^2)}} e^{-\frac{\alpha z^2}{2+s\alpha}} \frac{ds}{\sqrt{2+s\alpha}\left(2+s\alpha(1-\varepsilon_0^2)\right)} + C \quad (3.6.3)$$

So, formula (3.6.3) is true only if the components f_{gx}, f_{gy}, f_{gz} of the specific gravitational force have been calculated correctly. To verify this, we need to find the divergence of the specific gravitational force:

$$\operatorname{div}\vec{f}_g = \frac{\partial f_{g\,x}}{\partial x} + \frac{\partial f_{g\,y}}{\partial y} + \frac{\partial f_{g\,z}}{\partial z} =$$

$$= -\frac{2\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2} -$$

$$-\frac{2\gamma Mx}{\sqrt{\pi}}\int_0^\infty \frac{(-2x)}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2} -$$

$$-\frac{2\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2} +$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^\infty \frac{y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2} -$$

$$-\frac{2\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} +$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^\infty \frac{z^2}{\frac{2}{\alpha}+s} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} =$$

$$=-\frac{4\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{\frac{x^2+y^2}{2}}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}e^{-\frac{\frac{z^2}{2}}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{4\gamma M}{\sqrt{\pi}}\int_0^\infty\frac{(x^2+y^2)}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}e^{-\frac{\frac{x^2+y^2}{2}}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}\times$$

$$\times e^{-\frac{\frac{z^2}{2}}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}-\frac{2\gamma M}{\sqrt{\pi}}\int_0^\infty e^{-\frac{\frac{x^2+y^2}{2}}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}e^{-\frac{\frac{z^2}{2}}{\frac{2}{\alpha}+s}}\frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^\infty\frac{z^2}{\frac{2}{\alpha}+s}e^{-\frac{\frac{x^2+y^2}{2}}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}e^{-\frac{\frac{z^2}{2}}{\frac{2}{\alpha}+s}}\frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}. \qquad (3.6.4)$$

Let us calculate the auxiliary integral using integration by parts:

$$\int_0^\infty\frac{ds}{\left[2+s\alpha(1-\varepsilon_0^2)\right]\cdot\left[2+s\alpha\right]^{3/2}}=\frac{1}{2+s\alpha(1-\varepsilon_0^2)}\frac{(2+s\alpha)^{-1/2}}{-1/2}\cdot\frac{1}{\alpha}\Bigg|_0^\infty-$$

$$-\int_0^\infty\left(-\frac{2}{\alpha}\right)\cdot\left(2+s\alpha\right)^{-1/2}\cdot\left(-\frac{\alpha(1-\varepsilon_0^2)}{(2+s\alpha(1-\varepsilon_0^2))^2}\right)ds=$$

$$=\frac{1}{\alpha\sqrt{2}}-2(1-\varepsilon_0^2)\int_0^\infty\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}. \qquad (3.6.5)$$

According to (3.6.5) the following identity holds [74, 185]:

$$2\cdot(1-\varepsilon_0^2)\int_0^\infty\frac{ds}{\sqrt{2+s\alpha}\left(2+s\alpha(1-\varepsilon_0^2)\right)^2}+$$

$$+\int_0^\infty\frac{ds}{(2+s\alpha)^{3/2}\left(2+s\alpha(1-\varepsilon_0^2)\right)}=\frac{1}{\alpha\sqrt{2}}. \qquad (3.6.6)$$

Taking into account properties (3.6.5), (3.6.6) let us transform equation (3.6.4). For this purpose, we calculate separately the first integral from the right-hand side of Eq. (3.6.4) using the integration method by parts:

$$-\frac{4\gamma M}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}=$$

$$=-e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} \frac{4\gamma M}{\sqrt{\pi}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \int \frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\Bigg|_0^{\infty}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^{\infty}\left\{\left(-x^2-y^2\right)\frac{-1}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} - z^2\cdot\frac{-1}{\left(\frac{2}{\alpha}+s\right)^2}\cdot e^{-\frac{z^2}{\frac{2}{\alpha}+s}} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}\right\}\times$$

$$\times\int\frac{ds'}{\sqrt{\frac{2}{\alpha}+s'}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s'\right)^2}ds=-\frac{4\gamma M}{\sqrt{\pi}}\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\right]_0^{\infty}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds'ds}{\sqrt{\frac{2}{\alpha}+s'}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s'\right)^2}=$$

$$=-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+S}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}\right]_0^{\infty}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2\int_0^\infty\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}\times$$

$$\times e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}e^{-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot(2+s'\alpha(1-\varepsilon_0^2))^2}ds. \qquad (3.6.7)$$

As we did in (3.6.5), we need to note the same property for indefinite integrals:

$$\int\frac{ds}{(2+s\alpha(1-\varepsilon_0^2))\cdot(2+s\alpha)^{3/2}}=$$

$$=-\frac{2}{\alpha}\cdot\frac{1}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}-2(1-\varepsilon_0^2)\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2} \qquad (3.6.8)$$

from which the analog of identity (3.6.6) directly follows [74, 185]:

$$\int\frac{ds}{(2+s\alpha(1-\varepsilon_0^2))\cdot(2+s\alpha)^{3/2}}+2(1-\varepsilon_0^2)\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}=$$

$$=-\frac{2/\alpha}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}. \qquad (3.6.9)$$

By analogy with (3.6.7), we can also compute by integration by parts the third integral in the right-hand side of equation (3.6.4):

$$-\frac{2\gamma M}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)} =$$

$$= -\frac{2\gamma M}{\sqrt{\pi}}\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \cdot \int \frac{ds}{\left(\frac{2}{\alpha}+s\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}\right]\Bigg|_0^{\infty} +$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\int_0^{\infty}\left\{\left(-x^2-y^2\right)\cdot\frac{-1}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)+s}\right)^2} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}} - z^2\cdot\frac{-1}{\left(\frac{2}{\alpha}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{2}{\alpha}+s}}\right\}\times$$

$$\times\int \frac{ds'}{\left(\frac{2}{\alpha}+s'\right)^{3/2}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s'\right)^{3/2}} ds = \frac{-2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\cdot\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \int \frac{ds}{(2+s\alpha)^{3/2}(2+s\alpha(1-\varepsilon_0^2))}\right]\Bigg|_0^{\infty} +$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}\left[\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right]\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \times$$

$$\times\int \frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))} ds\,. \qquad (3.6.10)$$

According to (3.6.4), let us calculate separately the sum of the first and third integrals, for which we need to add (3.6.7) and (3.6.10), taking into account the identity (3.6.9):

$$\frac{-2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\left[2(1-\varepsilon_0^2)\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}+\int\frac{ds}{(2+s\alpha)^{3/2}(2+s\alpha(1-\varepsilon_0^2))}\right]_0^{\infty}+$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\left[\frac{x^2+y^2}{\left(2/\alpha(1-\varepsilon_0^2)+s\right)^2}+\frac{z^2}{\left(2/\alpha+s\right)^2}\right]\times$$

$$\times\left\{2\cdot(1-\varepsilon_0^2)\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot\left(2+s'\alpha(1-\varepsilon_0^2)\right)^2}+\int\frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))}\right\}ds=$$

$$=-\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\left.\frac{-2/\alpha}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\right|_0^{\infty}+$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\left[\frac{x^2+y^2}{\left(2/\alpha(1-\varepsilon_0^2)+s\right)^2}+\frac{z^2}{\left(2/\alpha+s\right)^2}\right]\times$$

$$\times\frac{(-2/\alpha)ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=\frac{-2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\cdot\frac{-2/\alpha}{\infty}+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}}-\frac{z^2}{\frac{2}{\alpha}}}\frac{-2/\alpha}{\sqrt{2}\cdot 2}-$$

$$-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_0^{\infty}\left[\frac{x^2+y^2}{\left(2/\alpha(1-\varepsilon_0^2)+s\right)^2}+\frac{z^2}{\left(2/\alpha+s\right)^2}\right]\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+S}-\frac{z^2}{\frac{2}{\alpha}+S}}\cdot\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}.\qquad(3.6.11)$$

Finally, substituting (3.6.11) in the right-hand side of equation (3.6.4), we get:

$$\mathrm{div}\vec{f}_g=-\sqrt{\frac{2}{\pi}}\gamma M\alpha^{3/2}(1-\varepsilon_0^2)e^{-\frac{\alpha(1-\varepsilon_0^2)(x^2+y^2)}{2}-\frac{\alpha z^2}{2}}-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$

$$\times\int_0^{\infty}\left[\frac{x^2+y^2}{\left(2/\alpha(1-\varepsilon_0^2)+s\right)^2}+\frac{z^2}{\left(2/\alpha+s\right)^2}\right]e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^{\infty}\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}+$$

$$+\frac{4\gamma M}{\sqrt{\pi}}\int_0^{\infty}\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{ds}{\sqrt{\frac{2}{\alpha}+s}\cdot\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)}=$$

$$=-4\pi\left(\frac{\alpha}{2\pi}\right)^{3/2}\gamma M(1-\varepsilon_0^2)\cdot e^{-\alpha[(1-\varepsilon_0^2)(x^2+y^2)+z^2]/2}. \qquad (3.6.12)$$

So, according to (3.6.12) the following identity is true:

$$\operatorname{div}\vec{f}_g=-4\pi\gamma\rho(x,y,z), \qquad (3.6.13)$$

where $\rho(x,y,z)$ is the mass density (3.3.26b) of a rotating spheroidal body: $\rho(x,y,z)=\rho_0(1-\varepsilon_0^2)e^{-\alpha(x^2(1-\varepsilon_0^2)+y^2(1-\varepsilon_0^2)+z^2)/2}$, $\rho_0=M\left(\alpha/2\pi\right)^{3/2}$. Taking into account Eq. (2.4.5), we can transform the obtained equation (3.6.13) to the well-known Poisson equation (2.4.1):

$$\operatorname{div}(-\operatorname{grad}\varphi_g)=-\nabla^2\varphi_g=-4\pi\gamma\rho(x,y,z) \qquad (3.6.14)$$

so that correctness of Poisson equation (3.6.14) means that the specific gravitational force $\vec{f}_g$ (or the gravitational field strength) of a uniformly rotating spheroidal body satisfies Eq. (3.6.13).

On the other hand, identity (3.6.13) and Poisson equation (3.6.14) imply that formula (3.6.3) looks highly like a function of the gravitational potential of a uniformly rotating spheroidal body with the mass density (3.3.26b) under the condition $C\to 0$, because $\varphi_g\to 0$ when $x\to\infty, y\to\infty, z\to\infty$:

$$\varphi_g\left(x,y,z\right)=-\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$

$$\times\int_0^\infty e^{-\frac{\alpha(1-\varepsilon_0^2)(x^2+y^2)}{2+s\alpha(1-\varepsilon_0^2)}-\frac{\alpha z^2}{2+s\alpha}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}. \quad (3.6.15a)$$

The formulas for the gravitational potential of a uniformly rotating spheroidal body have the form in the cylindrical and spherical coordinate systems respectively [78, 185]:

$$\varphi_g\left(h,z\right)=-\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$

$$\times\int_0^\infty e^{-\frac{\alpha(1-\varepsilon_0^2)h^2}{2+s\alpha(1-\varepsilon_0^2)}-\frac{\alpha z^2}{2+s\alpha}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}; \quad (3.6.15b)$$

$$\varphi_g\left(r,\theta\right)=-\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$

$$\times\int_0^\infty e^{-\alpha r^2\left[\frac{(1-\varepsilon_0^2)\sin^2\theta}{2+s\alpha(1-\varepsilon_0^2)}-\frac{\cos^2\theta}{2+s\alpha}\right]}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}. \quad (3.6.15c)$$

Let us make sure directly that the gravitational potential (3.6.15a) of a uniformly rotating spheroidal body satisfies the Poisson equation of the kind (3.6.14) or (2.4.1). Taking into account the type of Laplace operator [95, 96, 99], we first calculate the second derivatives of the function of gravitational potential φ_g for the Cartesian coordinates $x_i,\quad i=1,2,3$, that is, $x_1=x, x_2=y, x_3=z$:

$$\frac{\partial^2}{\partial x^2}\left(e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\right)=\frac{\partial}{\partial x}\left(-\frac{2x}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\right)=$$

$$=\left[-\frac{2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}+\frac{4x^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\right]e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}. \quad (3.6.16)$$

Because of (3.6.15a) and (3.6.16), the left-hand side of the Poisson equation (2.4.1) takes the form:

$$\nabla^2\varphi_g=-\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_0^{\infty}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\left[-\frac{4}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{2}{\frac{2}{\alpha}+s}+\right.$$

$$\left.+\frac{4(x^2+y^2)}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{4z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right]\cdot\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=$$

$$=\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2\int_0^{\infty}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}+$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{ds}{(2+s\alpha)^{3/2}(2+s\alpha(1-\varepsilon_0^2))}-$$

$$-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_0^{\infty}\left[\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right]\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\times$$

$$\times\frac{ds}{\sqrt{2+s\alpha}\,(2+s\alpha(1-\varepsilon_0^2)}. \quad (3.6.17)$$

Let us calculate the first integral in the right-hand side of equation (3.6.17) using the method of integration by parts:

$$\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2\int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}=$$

$$=\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2 e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}\Bigg|_0^{\infty}-$$

$$-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)^2\int_0^{\infty}\left\{(-x^2-y^2)\frac{-1}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}\cdot e^{-\frac{z^2}{\frac{2}{\alpha}+s}}-\right.$$

$$\left.-z^2\cdot\frac{-1}{\left(\frac{2}{\alpha}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}}\cdot e^{-\frac{z^2}{\frac{2}{\alpha}+s}}\right\}\cdot\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot(2+s'\alpha(1-\varepsilon_0^2))^2}ds=$$

$$=\frac{4\gamma M\alpha^{5/2}(1-\varepsilon_0^2)^2}{\sqrt{\pi}}\cdot\left\{e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}\right\}\Bigg|_0^{\infty}-$$

$$-\frac{4\gamma M\alpha^{5/2}(1-\varepsilon_0^2)^2}{\sqrt{\pi}}\cdot\int_0^{\infty}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\times$$

$$\times\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot(2+s'\alpha(1-\varepsilon_0^2))^2}ds\,. \qquad (3.6.18)$$

Similarly, we can calculate the second integral in the right-hand side of equation (3.6.17) using the integration method by parts:

$$\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty} e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \cdot \frac{ds}{(2+s\alpha)^{3/2}\left(2+s\alpha(1-\varepsilon_0^2)\right)} =$$

$$= \frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \cdot \int \frac{ds}{(2+s\alpha)^{3/2}\left(2+s\alpha(1-\varepsilon_0^2)\right)} \right]\Bigg|_0^{\infty} -$$

$$-\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}\left\{(-x^2-y^2)\cdot\frac{-1}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} \cdot e^{-\frac{z^2}{\frac{2}{\alpha}+s}} -\right.$$

$$\left. -z^2\cdot\frac{-1}{\left(\frac{2}{\alpha}+s\right)^2}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} \cdot e^{-\frac{z^2}{\frac{2}{\alpha}+s}}\right\}\cdot\int_0^{\infty}\frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))}ds =$$

$$= \frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\left[e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}} \int \frac{ds}{(2+s\alpha)^{3/2}\left(2+s\alpha(1-\varepsilon_0^2)\right)} \right]\Bigg|_0^{\infty} -$$

$$-\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\int_0^{\infty}\left\{-\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\times$$

$$\times\int\frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))}ds\,, \tag{3.6.19}$$

Finally, substituting (3.6.18) and (3.6.19) into equation (3.6.17) and taking into account the identity (3.6.9), we obtain:

$$\nabla^2\varphi_g=\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\cdot\left\{e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot 2(1-\varepsilon_0^2)\int\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))^2}\right\}\Bigg|_0^{\infty}-$$

$$-\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}2(1-\varepsilon_0^2)\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot(2+s'\alpha(1-\varepsilon_0^2))^2}ds+$$

$$+\frac{2\gamma M}{\sqrt{\pi}}\alpha^{5/2}(1-\varepsilon_0^2)\cdot\left\{e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\int\frac{ds}{(2+s\alpha)^{3/2}\cdot(2+s\alpha(1-\varepsilon_0^2))}\right\}\Bigg|_0^{\infty}-$$

$$-\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\int\frac{ds'}{(2+s'\alpha)^{3/2}\cdot(2+s'\alpha(1-\varepsilon_0^2))}ds-$$

$$-\frac{4\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_0^{\infty}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=$$

$$=\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\cdot\left\{e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\left[\int\frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))}+2(1-\varepsilon_0^2)\int\frac{ds'}{\sqrt{2+s'\alpha}(2+s'\alpha(1-\varepsilon_0^2))^2}\right]\right\}\Bigg|_0^{\propto}-$$

$$-\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\propto}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\times$$

$$\times\left[\int\frac{ds'}{(2+s'\alpha)^{3/2}(2+s'\alpha(1-\varepsilon_0^2))}+2(1-\varepsilon_0^2)\int\frac{ds'}{\sqrt{2+s'\alpha}\cdot(2+s'\alpha(1-\varepsilon_0^2))^2}\right]ds-$$

$$-\frac{4\gamma M\alpha^{3/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\propto}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=$$

$$=\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\left\{e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\frac{(-2/\alpha)}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\right\}\Bigg|_0^{\propto}-$$

$$-\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\propto}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\cdot\frac{(-2/\alpha)ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}-$$

$$-\frac{4\gamma M\alpha^{3/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\int_0^{\propto}\left\{\frac{x^2+y^2}{\left(\frac{2}{\alpha(1-\varepsilon_0^2)}+s\right)^2}+\frac{z^2}{\left(\frac{2}{\alpha}+s\right)^2}\right\}\cdot e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}-\frac{z^2}{\frac{2}{\alpha}+s}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=$$

$$=\frac{2\gamma M\alpha^{5/2}(1-\varepsilon_0^2)}{\sqrt{\pi}}\cdot\left\{0-e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}}-\frac{z^2}{\frac{2}{\alpha}}}\cdot\left(-\frac{2/\alpha}{\sqrt{2}\cdot 2}\right)\right\}=$$

$$=\sqrt{\frac{2}{\pi}}\gamma M\alpha^{3/2}(1-\varepsilon_0^2)e^{-\alpha[(1-\varepsilon_0^2)(x^2+y^2)+z^2]/2}. \qquad (3.6.20)$$

The result obtained (3.6.20) proves the validity of the Poisson equation (2.4.1) in the case of mass density described by the function (3.3.26b). The solvability of the Poisson equation for the function (3.6.15a) of the gravitational potential of a rotating spheroidal body is thus justified.

3.7. The potential energy of a uniformly rotating and gravitating spheroidal body

Since the potential energy of a particle in a gravitational field is equal to its mass multiplied by the potential of the field [94], then the potential energy of any distribution of gravitating masses is described by expression (2.5.1):

$$E_g=\frac{1}{2}\int_V \rho\varphi_g dV, \qquad (3.7.1)$$

where ρ and φ_g are supposed to be the mass density (3.3.26b) and the gravitational potential (3.6.15a) of a uniformly rotating spheroidal body respectively, $dV=dxdydz$. So, gravitational energy (3.7.1) is calculated by integrating the expression (3.6.15a) for φ_g multiplied by $\frac{1}{2}\cdot\rho$ over the total volume of a uniformly rotating spheroidal body:

$$E_g=\int_V \rho\varphi_g dV=-\frac{1}{2}\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_V \rho_0(1-\varepsilon_0^2)e^{-\frac{\alpha(1-\varepsilon_0^2)x^2}{2}}\times$$

$$\times e^{-\frac{\alpha(1-\varepsilon_0^2)y^2}{2}}e^{-\frac{\alpha z^2}{2}}\int_0^{\infty} e^{-\frac{\alpha(1-\varepsilon_0^2)x^2}{2+\alpha(1-\varepsilon_0^2)s}}e^{-\frac{\alpha(1-\varepsilon_0^2)y^2}{2+\alpha(1-\varepsilon_0^2)s}}e^{-\frac{\alpha z^2}{2+\alpha s}}\frac{dsdV}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}=$$

$$=-\frac{\gamma M}{2\sqrt{\pi}}\alpha^{3/2}\rho_0(1-\varepsilon_0^2)^2\int_0^{\infty}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\int_{-\infty}^{\infty} e^{-z^2\cdot\frac{\alpha}{2}\left(1+\frac{1}{1+s\alpha/2}\right)}dz\times$$

$$\times\int_{-\infty}^{\infty} e^{-x^2\cdot\frac{\alpha(1-\varepsilon_0^2)}{2}\cdot\left(1+\frac{1}{1+s\alpha(1-\varepsilon_0^2)/2}\right)}dx\int_{-\infty}^{\infty} e^{-y^2\frac{\alpha(1-\varepsilon_0^2)}{2}\left(1+\frac{1}{1+s\alpha(1-\varepsilon_0^2)/2}\right)}dy=$$

$$=-\frac{\gamma M}{2\sqrt{\pi}}\alpha^{3/2}\rho_0(1-\varepsilon_0^2)^2\int_0^{\infty}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\int_{-\infty}^{\infty} e^{-x^2\cdot\frac{\alpha\left(1-\varepsilon_0^2\right)\left(2+s\alpha(1-\varepsilon_0^2)/2\right)}{2+s\alpha(1-\varepsilon_0^2)}}dx\times$$

$$\times\int_{-\infty}^{\infty} e^{-y^2\cdot\frac{\alpha\left(1-\varepsilon_0^2\right)\left(2+s\alpha(1-\varepsilon_0^2)/2\right)}{2+s\alpha(1-\varepsilon_0^2)}}dy\int_{-\infty}^{\infty} e^{-z^2\cdot\frac{\alpha(2+s\alpha/2)}{2+s\alpha}}dz \qquad (3.7.2)$$

Let us carry out a change of variables:

$$x^{'}=ax;\;\; y^{'}=by;\;\; z^{'}=cz, \qquad (3.7.3)$$

where $a=b=\sqrt{\frac{\alpha\left(1-\varepsilon_0^2\right)\left(2+s\alpha(1-\varepsilon_0^2)/2\right)}{2+s\alpha(1-\varepsilon_0^2)}};\; c=\sqrt{\frac{\alpha\left(2+s\alpha/2\right)}{2+s\alpha}}$.

According to the change (3.7.3), the right-hand side of equation (3.7.2) becomes:

$$E_g=-\gamma\pi\sqrt{2}\rho_0^2(1-\varepsilon_0^2)^2\int_0^{\infty}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\int_{-\infty}^{\infty} e^{-x'^2}\frac{dx^{'}\sqrt{2+s\alpha(1-\varepsilon_0^2)}}{\sqrt{\alpha\left(1-\varepsilon_0^2\right)\left(2+s\alpha(1-\varepsilon_0^2)/2\right)}}\times$$

$$\times\int_{-\infty}^{\infty} e^{-y'^2}\frac{dy^{'}\sqrt{2+s\alpha(1-\varepsilon_0^2)}}{\sqrt{\alpha\left(1-\varepsilon_0^2\right)\left(2+s\alpha(1-\varepsilon_0^2)/2\right)}}\int_{-\infty}^{\infty} e^{-z'^2}\cdot\frac{dz^{'}\sqrt{2+s\alpha}}{\sqrt{\alpha(2+s\alpha/2)}}=$$

$$=-\sqrt{2}\pi\gamma\rho_0^2(1-\varepsilon_0^2)^2\int_0^{\infty}\frac{ds}{\sqrt{\alpha(2+s\alpha/2)}\cdot\alpha(1-\varepsilon_0^2)(2+s\alpha(1-\varepsilon_0^2)/2)}\cdot\sqrt{\pi}\cdot\sqrt{\pi}\cdot\sqrt{\pi}=$$

$$=-\gamma\pi\sqrt{2}\rho_0^2\cdot(1-\varepsilon_0^2)\cdot 2^{3/2}\alpha^{-3/2}\pi^{3/2}\int_0^{\infty}\frac{ds}{\sqrt{4+s\alpha}\cdot(4+s\alpha(1-\varepsilon_0^2))}=$$

$$=-\frac{4\gamma\rho_0^2(1-\varepsilon_0^2)\pi^{5/2}}{\alpha^{3/2}}\int_0^{\infty}\frac{ds}{\sqrt{4+s\alpha}\cdot(4+s\alpha(1-\varepsilon_0^2))}=$$

$$=-4\gamma\rho_0^2(1-\varepsilon_0^2)\cdot\frac{\pi^{5/2}}{\alpha^{3/2}}\int_2^{\infty}\frac{d\left(\sqrt{4+s\alpha}\right)}{4+s\alpha-\varepsilon_0^2\alpha s}\cdot\frac{2}{\alpha}. \qquad (3.7.4)$$

Introducing the following designations:

$$\tau = \sqrt{4 + s\alpha}\ ;$$
$$s\alpha = \tau^2 - 4\ ;$$
$$q = \tau\sqrt{1-\varepsilon_0^2}\ ,$$

it is not difficult to see that Eq. (3.7.4) takes the form:

$$E_g = -8\gamma\rho_0^2(1-\varepsilon_0^2)\cdot\left(\frac{\pi}{\alpha}\right)^{5/2}\int_2^{\infty}\frac{d\tau}{\tau^2 - \varepsilon_0^2(\tau^2-4)} =$$

$$= -8\gamma\rho_0^2(1-\varepsilon_0^2)\left(\frac{\pi}{\alpha}\right)^{5/2}\int_2^{\infty}\frac{d\tau}{(1-\varepsilon_0^2)\tau^2 + (2\varepsilon_0)^2} =$$

$$= -8\gamma\rho_0^2\left(\frac{\pi}{\alpha}\right)^{5/2}\cdot\sqrt{1-\varepsilon_0^2}\int_{2\sqrt{1-\varepsilon_0^2}}^{\infty}\frac{dq}{q^2+(2\varepsilon_0)^2}\,. \qquad (3.7.5)$$

Bearing in mind the table integral:

$$\int\frac{dq}{q^2+a^2} = \frac{1}{a}\arctan\frac{q}{a} + C \qquad (3.7.6a)$$

and using the trigonometric property:

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

or

$$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x \qquad (3.7.6b)$$

we calculate the definite integral (3.7.5) as

$$E_g = -8\gamma\rho_0^2\left(\frac{\pi}{\alpha}\right)^{5/2}\cdot\sqrt{1-\varepsilon_0^2}\cdot\frac{1}{2\varepsilon_0}\arctan\frac{q}{2\varepsilon_0}\Bigg|_{2\sqrt{1-\varepsilon_0^2}}^{\infty} =$$

$$= -4\gamma\rho_0^2\left(\frac{\pi}{\alpha}\right)^{5/2}\cdot\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\left(\frac{\pi}{2} - \arctan\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\right) =$$

$$= -4\gamma\rho_0^2\left(\frac{\pi}{\alpha}\right)^{5/2}\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\operatorname{arccot}\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\,. \qquad (3.7.7)$$

Finally, taking into account that $\rho_0 = M(\alpha / 2\pi)^{3/2}$, the value (3.7.7) of gravitational energy of a uniformly rotating spheroidal body is equal [79]:

$$E_g = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \cdot \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} \operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}. \qquad (3.7.8)$$

In the particular case $\varepsilon_0 \to 0$ formula (3.7.8) gives an estimation of the gravitational energy of a non-rotating (weakly rotating) spheroidal body:

$$\lim_{\varepsilon_0 \to 0} E_g = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \lim_{\varepsilon_0 \to 0} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} \operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} =$$

$$= -\frac{\gamma M^2}{2} \sqrt{\frac{\alpha}{\pi}} \lim_{\varepsilon_0 \to 0} \sqrt{1-\varepsilon_0^2} \lim_{\varepsilon_0 \to 0} \frac{\operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}}{\varepsilon_0} =$$

$$= -\frac{\gamma M^2}{2} \sqrt{\frac{\alpha}{\pi}} \lim_{\varepsilon_0 \to 0} \frac{\operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}}{\varepsilon_0}. \qquad (3.7.9)$$

To reveal the uncertainty $\frac{0}{0}$ in (3.7.9) following the rule of L'Hospital, let us calculate the derivative:

$$\left(\operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} \right)'_{\varepsilon_0} = -\frac{1}{1+\frac{1-\varepsilon_0^2}{\varepsilon_0^2}} \cdot \left(\frac{1}{2\varepsilon_0} \cdot \frac{-2\varepsilon_0}{\sqrt{1-\varepsilon_0^2}} - \frac{1}{\varepsilon_0^2} \sqrt{1-\varepsilon_0^2} \right) =$$

$$= -\varepsilon_0^2 \cdot \left(-\frac{1}{\sqrt{1-\varepsilon_0^2}} - \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0^2} \right) = \frac{\varepsilon_0^2}{\sqrt{1-\varepsilon_0^2}} + \sqrt{1-\varepsilon_0^2}. \qquad (3.7.10)$$

Substituting (3.7.10) into (3.7.9) we finally obtain:

$$\lim_{\varepsilon_0 \to 0} E_g = -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha}{\pi}} \lim_{\varepsilon_0 \to 0} \frac{\dfrac{\varepsilon_0^2}{\sqrt{1-\varepsilon_0^2}} + \sqrt{1-\varepsilon_0^2}}{1} =$$

$$= -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha}{\pi}} \cdot \frac{0+1}{1} = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} . \qquad (3.7.11)$$

The obtained relation (3.7.11) coincides completely with the formula (2.5.6) of gravitational energy in the case of a *spherically symmetric* spheroidal body or sphere-like gaseous body (see Section 2.5 of Chapter 2 and [46, 73, 76]).

We now investigate another extreme case of a uniformly rotating spheroidal body when $\varepsilon_0 \to 1$, that is, let us estimate the gravitational energy of a *flattened* spheroidal body:

$$\lim_{\varepsilon_0 \to 1} E_g = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \lim_{\varepsilon_0 \to 1} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} \operatorname{arccot} \frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} =$$

$$= -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \cdot \frac{0}{1} \cdot \operatorname{arccot} 0 = -\frac{\gamma M^2}{2} \cdot \sqrt{\frac{\alpha}{\pi}} \cdot 0 \cdot \frac{\pi}{2} = 0. \qquad (3.7.12)$$

As follows from (3.7.12), the *disk-shaped* (flattened) spheroidal body does not possess its gravitational energy, that is, in fact, there is no gravitational interaction of particles inside a disk-shaped body; therefore, particles inside it move along independent Keplerian orbits.

Indeed, according to the condition of the *statistical* equilibrium of a gravitating body [110]:

$$\varphi_g + \frac{\mu}{m_0} = \text{const}, \qquad (3.7.13)$$

where μ is a chemical potential. We know [110] that $\upsilon = \left(\dfrac{\partial \mu}{\partial p}\right)_T$ is the volume assigned to a single particle under the condition of constant temperature $T = \text{const}$. Taking into account that $\upsilon = \frac{m_0}{\rho}$, where m_0 is a mass of a particle, ρ is a mass density, we obtain:

$$\mu = \int_{p_0}^{p} \upsilon dp = \int_{p_0}^{p} \frac{m_0}{\rho} dp = m_0 \int_{p_0}^{p} \frac{dp}{\rho} = m_0 \wp , \qquad (3.7.14)$$

where $\wp = \int_{p_0}^{p} \frac{dp}{\rho}$ is a pressure function [111]. Taking into account (3.7.14), the statistical equilibrium condition (3.7.13) in the layers of a gravitating body becomes:

$$\varphi_g + \wp = \text{const} . \qquad (3.7.15)$$

In the case of a disk-shaped spheroidal body, the result (3.7.12) means that it is actually $\wp \equiv 0$ inside a disk-shaped body, which confirms the above idea that the particles do not interact there. In other words, when $\varepsilon_0 \to 1$, the decay of the spheroidal body takes place as such. Indeed, under the condition of the relative mechanical equilibrium of a uniformly rotating and gravitating body, equation (3.7.15) is replaced by a more general one [111]:

$$\varphi_g + \wp + V_c = \text{const} , \qquad (3.7.16)$$

where $V_c = -\frac{1}{2}\Omega^2 r^2 = -\frac{1}{2}\left[\vec{\Omega} \times \vec{r}\right]^2$ is a potential of centrifugal force. Therefore, in the case of a flattened ($\varepsilon_0 \to 1$) rotating spheroidal body, equation (3.7.16) degenerates into the following:

$$\varphi_g + V_c = \text{const} , \qquad (3.7.17)$$

which means that inside a rotating disk-shaped spheroidal body there is no pressure gradient, and the particles rotate independently along their orbits in the plane of the disk under the condition of equality of the gravitational and centrifugal forces.

The fact that the rotating spheroidal body's own gravitational energy decreases as it flattens can be well illustrated by the estimation (3.7.8) in the case of $0 < \varepsilon_0 < 1$. For example, let $\varepsilon_0 = 1/\sqrt{2}$, then:

$$\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\operatorname{arccot}\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0} = \frac{\sqrt{1-1/2}}{1/\sqrt{2}}\operatorname{arccot}\frac{\sqrt{1-1/2}}{1/\sqrt{2}} =$$

$$= \operatorname{arccot} 1 = \frac{\pi}{4} < 1 . \qquad (3.7.18)$$

Taking into account (3.7.18) the body's own gravitational energy (3.7.8) is equal to:

$$E_g\Big|_{\varepsilon_0 = 1/\sqrt{2}} = -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha}{\pi}}.\frac{\pi}{4} = -\frac{\gamma M^2}{8}\sqrt{\pi\alpha} , \qquad (3.7.19)$$

which is approximately 21% less than the gravitational energy (3.7.11) of a spherically symmetric spheroidal body. Thus, as a result of the rotation of a gravitating spheroidal body, the internal processes of flattening its shape lead to a decrease in its gravitational energy, contributing to the transition of its constituent parts to Keplerian orbital motion.

3.8. The mass density of a rotating flattened (disk-shaped) spheroidal body and the model of formation of a rotating disk

According to formula (2.2.4) from Section 2.2, the mass density of a non-rotating gravitating spheroidal body (or sphere-like gaseous body) is described by the expression [45, 46]:

$$\rho^{(0)}(r) = M(\alpha / 2\pi)^{3/2} \cdot e^{-\alpha r^2/2} , \qquad (3.8.1)$$

and in accordance with the formula (3.3.26c) from Section 3.3, the mass density of a uniformly rotating spheroidal body is characterized by the relation [16, 55]:

$$\rho^{(1)}(r,\theta) = M(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)\cdot e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2} . \quad (3.8.2)$$

If we assume, as in Sections 3.2 and 3.3, that as a result of the action of inertial forces during the initial nonuniform rotation of a spheroidal body, the initial formation of the *core* and *shell* regions takes place, then from relation (3.8.2) it follows that in the case of steady uniform rotation of a spheroidal body, M is a total mass of its core (having mass M_0) and shell (having mass M_1), that is:

$$M = M_0 + M_1 . \quad (3.8.3)$$

Obviously, in formula (3.8.1) $M = M_0$ due to the fact that, in a non-rotating spheroidal body, there is no explicit separation into the core and the shell (in the form of a gas-dust cloud).

Comparing (3.8.2) with (3.8.1) we obtain:

$$\rho^{(1)}(r,\theta) = \rho^{(0)}(r)\cdot(1-\varepsilon_0^2)e^{\varepsilon_0^2\alpha r^2\sin^2\theta/2} , \quad (3.8.4a)$$

where:

$$\rho^{(0)}(r) = (M_0+M_1)(\alpha/2\pi)^{3/2}e^{-\alpha r^2/2} . \quad (3.8.4b)$$

If a rotating spheroidal body of mass $M = M_0 + M_1$ interacts with a moving (or immovable) gas-dust cloud of mass M_2 in its motion, then, by analogy with (3.8.4a) and (3.8.4b), it should be expected that the mass density of the combined body (rotating spheroidal body together with captured gas-dust bunch) is equal to:

$$\rho^{(2)}(r,\theta) = \rho^{(1)}(r,\theta)\cdot(1-\varepsilon_1^2)e^{\varepsilon_1^2\alpha r^2\sin^2\theta/2} , \quad (3.8.5a)$$

where

$$\rho^{(1)}(r,\theta) = (M_0+M_1+M_2)(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2} . \quad (3.8.5b)$$

Substituting (3.8.5b) into (3.8.5a), we obtain:

$$\rho^{(2)}(r,\theta) = (\alpha/2\pi)^{3/2}\sum_{i=0}^{2}M_i\prod_{l=0}^{1}(1-\varepsilon_l^2)\,e^{-\alpha r^2(1-\sum_{l=0}^{1}\varepsilon_l^2\sin^2\theta)/2} . \quad (3.8.6)$$

Summarizing (3.8.6), it is easy to see that if a rotating spheroidal body repeatedly captures gas-dust bunches, then its mass density can be expressed by the following formula [73]:

$$\rho^{(n)}(r,\theta) = (\alpha/2\pi)^{3/2}\sum_{i=0}^{n} M_i \cdot e^{-\alpha r^2/2}\prod_{l=0}^{n-1}\left\{(1-\varepsilon_l^2)\cdot e^{\varepsilon_l^2 \alpha r^2 \sin^2\theta/2}\right\} =$$

$$= (\alpha/2\pi)^{3/2}\sum_{i=0}^{n} M_i \prod_{l=0}^{n-1}(1-\varepsilon_l^2)\cdot e^{-\alpha r^2 (1-\sum_{l=0}^{n-1}\varepsilon_l^2 \sin^2\theta)/2}. \qquad (3.8.7)$$

According to (3.8.7), as a result of a series of captures by a rotating spheroidal body of gas-dust bunches (with different angular momentum), an increase in the eccentricity of the combined body occurs:

$$\varepsilon_\Sigma^2 = \sum_{l=0}^{n-1}\varepsilon_l^2 . \qquad (3.8.8)$$

Indeed, in a spherical coordinate system, surfaces of equal mass density (3.8.2) of a uniformly rotating spheroidal body are described by the equation:

$$\alpha r^2 - \varepsilon_0^2 \alpha r^2 \sin^2\theta = 1 ,$$

which, in the Cartesian coordinate system, appears as:

$$\alpha(1-\varepsilon_0^2)x^2 + \alpha(1-\varepsilon_0^2)y^2 + \alpha z^2 = 1 . \qquad (3.8.9)$$

Taking into account the change $r_* = 1/\sqrt{\alpha}$, equation (3.8.9) is reduced to the usual equation of an ellipsoid flattened along the axis of rotation Oz, that is, a spheroid:

$$\frac{x^2}{r_*^2/(1-\varepsilon_0^2)} + \frac{y^2}{r_*^2/(1-\varepsilon_0^2)} + \frac{z^2}{r_*^2} = 1 . \qquad (3.8.10)$$

As we know, the *geometrical eccentricity* of a flattened ellipsoid with a major semi-axis a and a minor semi-axis b is defined as follows:

$$e = \sqrt{a^2 - b^2}/a , \qquad (3.8.11)$$

and a value $c = \sqrt{a^2 - b^2}$ is called the focal length. It is easy to see that in the case of a spheroid (3.8.10) the geometric eccentricity is equal to:

$$e = \sqrt{r_*^2/(1-\varepsilon_0^2) - r_*^2} \Big/ \left(r_* / \sqrt{1-\varepsilon_0^2}\right) =$$
$$= \sqrt{1-\varepsilon_0^2} \cdot \sqrt{1/(1-\varepsilon_0^2) - 1} = \varepsilon_0. \qquad (3.8.12)$$

In addition to the geometric eccentricity, as we know from the theory of equilibrium figure of rotating planets, a parameter of *relative flattening* is determined [44]:

$$e_c = (a-b)/a \ , \qquad (3.8.13)$$

where a is an equatorial radius and b is a polar radius. In the case of a slightly flattened ellipsoid or spheroid (3.8.10), the relative flattening e_c is equal to:

$$e_c = \left(r_* / \sqrt{1-\varepsilon_0^2} - r_*\right) \Big/ \left(r_* / \sqrt{1-\varepsilon_0^2}\right) = 1 - \sqrt{1-\varepsilon_0^2} \ . \qquad (3.8.14)$$

If we take into account the condition of weak flattening $\varepsilon_0^2 << 1$, then expression (3.8.14) becomes:

$$e_c = 1 - \sqrt{1-\varepsilon_0^2} \approx 1 - \left(1 - \varepsilon_0^2/2\right) = \varepsilon_0^2/2 = e^2/2 \, . \qquad (3.8.15)$$

If $\varepsilon_0^2 = 0$ (the case of a sphere), then also $e_c = e = 0$, and if $\varepsilon_0^2 = 1$ (the case of a disk), then $e_c = e = 1$ as well.

So, taking into account the condition of the derivation of the model of a uniformly rotating spheroidal body in Section 3.3, the quantity $\varepsilon_0^2 << 1$, that is, $0 \le \varepsilon_0 < 1$, and in its sense ε_0^2 is the *square of the geometric eccentricity* in accordance with (3.8.12). Thus, as noted in [16, 55], the function of mass density (3.8.4a) characterizes the flattening process: from the initial spherical forms (in the case of a non-rotating spheroidal body with mass density $\rho^{(0)}(r)$) through flattened ellipsoidal forms (in the case of a rotating and gravitating spheroidal body with mass density $\rho^{(1)}(r,\theta)$) to the disk (a disk-shaped spheroidal body with mass density $\rho^{(n)}(r,\theta)$) when the squared eccentricity ε_0^2 varies from 0 to 1. This means that the mass density function studied $\rho^{(1)}(r,\theta)$ is suitable to the description of the evolution of a protoplanetary gaseous (gas-dust) disk around a star (in particular, the Sun) [16, 55, 73].

Let us consider the above in more detail in the case of a *flattened* rotating spheroidal body. According to (3.8.7) and (3.8.8), as a result of the capture of gas-dust clots by a rotating spheroidal body, there is an increase in the geometric eccentricity as well as in the mass of the whole body. A diagram of the flattening of a rotating spheroidal body is shown in Fig. 3.4.

$$\begin{array}{ccccccc} \rho^{(0)}(r) & \dashrightarrow & \rho^{(1)}(r,\theta) & \dashrightarrow & \rho^{(2)}(r,\theta) & \dashrightarrow \ldots \dashrightarrow & \rho^{(n)}(r,\theta) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M=M_0 & \dashrightarrow & M=M_0+M_1 & \dashrightarrow & M=M_0+M_1+M_2 & \dashrightarrow \ldots \dashrightarrow & M=\sum_{i=0}^{n} M_i \end{array}$$

Figure 3.4. Diagram of the formation of a flattened (disk-shaped) rotating spheroidal body

We suppose that at each stage of the flattening of a rotating spheroidal body the square of the geometric eccentricity ε_l^2 satisfies the following inequalities:

$$\varepsilon_l^2 < 1, \quad \varepsilon_{l+m}^2 << 1, \quad m > 1. \tag{3.8.16}$$

According to (3.8.16) we take into account the smallness value $O(\varepsilon_l^2)$ which allows us to make some simplifications, for example:

$$\lim_{\varepsilon_l^2 \to 0} \prod_{l=0}^{n-1} (1-\varepsilon_l^2) = 1-\varepsilon_0^2-\varepsilon_1^2-\ldots-\varepsilon_{n-1}^2 = 1-\sum_{l=0}^{n-1}\varepsilon_l^2 . \tag{3.8.17}$$

If we assume that during the subsequent stages of capture of gas-dust clots by a forming flattened spheroidal body, its eccentricity is changed as follows:

$$\varepsilon_{l+1}^2 / \varepsilon_l^2 = \varepsilon_0^2 \ , \qquad (3.8.18a)$$

then the eccentricity of the combined body (3.8.8) is equal to:

$$\varepsilon_\Sigma^2 = \sum_{l=0}^{n-1} \varepsilon_l^2 = \varepsilon_0^2 + \varepsilon_0^4 + \varepsilon_0^6 + ... + \varepsilon_0^{2+2(n-1)} =$$

$$= \varepsilon_0^2 \sum_{i=0}^{n-1} \varepsilon_0^{2i} = \varepsilon_0^2 (\varepsilon_0^{2n} - 1)/(\varepsilon_0^2 - 1) \ . \qquad (3.8.18b)$$

If n is large enough ($n \to \infty$) and the ratio ε_0^2 of geometric eccentricities is small $(\varepsilon_0^2 < 1)$ in accordance with (3.8.16), then the formula (3.8.18b) takes the form:

$$\varepsilon_\Sigma^2 = \lim_{\substack{n \to \infty \\ \varepsilon_0^2 < 1}} (\varepsilon_0^2 / (\varepsilon_0^2 - 1)) \cdot (\varepsilon_0^{2n} - 1) = \varepsilon_0^2 / (1 - \varepsilon_0^2) \ . \qquad (3.8.19)$$

Let us note that condition (5.7.18a) is consistent with inequalities (5.7.16), because if, for example, $\varepsilon_0^2 = 1/2 < 1$, $\varepsilon_1^2 < 1$, then $\varepsilon_5^2 = 1/2^6 << 1$. Consequently,

$$\lim_{n \to \infty} \varepsilon_n^2 = \lim_{n \to \infty} \varepsilon_0^{2(n+1)} = 0 \ , \quad \varepsilon_0^2 < 1 \ . \qquad (3.8.20)$$

Taking into account Eq. (3.8.20) the relation (3.8.17) can be rewritten as follows:

$$\lim_{n \to \infty} \prod_{l=0}^{n-1} (1 - \varepsilon_l^2) = 1 - \lim_{n \to \infty} \sum_{l=0}^{n-1} \varepsilon_l^2 = 1 - \lim_{n \to \infty} \varepsilon_\Sigma^2 \ , \qquad (3.8.21)$$

where the designation ε_Σ^2 is introduced following (3.8.8) and (3.8.18b).

If the number of the captures n of gas-dust bunches is sufficiently large ($n \to \infty$), then taking into account (3.8.21), the mass density of a rotating spheroidal body (together with gas-dust bunches) takes the form:

$$\rho(r, \theta) = \lim_{n \to \infty} \rho^{(n)}(r, \theta) =$$

$$= \lim_{n\to\infty}\left\{\sum_{i=0}^{n} M_i \lim_{n\to\infty}\prod_{l=0}^{n-1}(1-\varepsilon_l^2)\cdot(\alpha/2\pi)^{3/2} e^{-\alpha r^2 (1-\sum_{l=0}^{n-1}\varepsilon_l^2 \sin^2\theta)/2}\right\} =$$

$$= \lim_{n\to\infty}\left\{\sum_{i=0}^{n} M_i\right\}\cdot(1-\varepsilon_\Sigma^2)\cdot(\alpha/2\pi)^{3/2} e^{-\alpha r^2 (1-\varepsilon_\Sigma^2 \sin^2\theta)/2} \quad . \qquad (3.8.22)$$

However, given the reasoning about the condition (3.8.20), taking $\varepsilon_0^2 \to 1/2$ from (3.8.19) we obtain:

$$\lim_{\substack{n\to\infty \\ \varepsilon_0^2\to 1/2}} \varepsilon_\Sigma^2 = \lim_{\varepsilon_0^2\to 1/2} \varepsilon_0^2/(1-\varepsilon_0^2) = 1 . \qquad (3.8.23)$$

Further, denoting the total mass M of a flattened spheroidal body together with captured gas-dust bunches as $M = \sum_{i=1}^{n} M_i$ we find that:

$$\lim_{n\to\infty}\sum_{i=1}^{n} M_i = \lim_{M\to\infty} M \quad . \qquad (3.8.24)$$

As a result, using the conditions (3.8.23) and (3.8.24), we can obtain from (3.8.22) the mass density of a rotating *disk-shaped* spheroidal body [73]:

$$\rho(r,\theta) = \lim_{\substack{M\to\infty \\ \varepsilon_\Sigma^2\to 1}} \left\{M(\alpha/2\pi)^{3/2}(1-\varepsilon_\Sigma^2)\cdot e^{-\alpha r^2 (1-\varepsilon_\Sigma^2 \sin^2\theta)/2}\right\} =$$

$$= \lim_{\substack{M\to\infty \\ \varepsilon_\Sigma^2\to 1}} M(1-\varepsilon_\Sigma^2)\cdot(\alpha/2\pi)^{3/2} \lim_{\varepsilon_\Sigma^2\to 1}\left\{e^{-\alpha r^2 (1-\varepsilon_\Sigma^2 \sin^2\theta)/2}\right\} =$$

$$= \lim_{\substack{M\to\infty \\ \varepsilon_\Sigma^2\to 1}} \left\{M(1-\varepsilon_\Sigma^2)\right\}\cdot(\alpha/2\pi)^{3/2} e^{-\alpha r^2 \cos^2\theta/2} \quad . \qquad (3.8.25)$$

Introducing the limit value of mass in the disk-shaped spheroidal body:

$$m = \lim_{\substack{M\to\infty \\ \varepsilon_\Sigma^2\to 1}} M(1-\varepsilon_\Sigma^2) = \lim_{\substack{N\to\infty \\ \varepsilon_\Sigma^2\to 1}} m_0 N(1-\varepsilon_\Sigma^2) \qquad (3.8.26)$$

and taking into account $z = r\cos\theta$ in relation (3.8.25), we shall again write formula (3.3.28) of the mass density of the disk-shaped spheroidal body [16, 73]:

$$\rho(z) = m(\alpha / 2\pi)^{3/2} \cdot e^{-\alpha z^2 / 2}, \tag{3.8.27}$$

which is a function of only a single coordinate z in the case of a spheroidal disk.

As noted, in the case of a spheroidal disk $\varepsilon_\Sigma^2 \to 1$, the geometric eccentricity and compression are equal to the unity, that is, $e_c = e = 1$. In other words, the disk can be considered as a strongly flattened spheroidal body with an eccentricity equal to 1. Indeed, from (3.8.7) and (3.8.22) it follows that in the cylindrical coordinate system the mass density of a flattened spheroidal body is:

$$\rho(h, z) = \sum_{i=0}^{n} M_i (\alpha / 2\pi)^{3/2} \cdot (1 - \varepsilon_\Sigma^2)\, e^{-\alpha\left[(1-\varepsilon_\Sigma^2)h^2 + z^2\right]/2} . \tag{3.8.28}$$

It is clear when $n \to \infty$ and $\varepsilon_\Sigma^2 \to 1$ the relation (3.8.28) naturally goes into (3.8.27). Under these limiting conditions, the diagram in Fig. 3.4 may describe the process of the formation of thin rings of the type around Saturn in a protoplanetary cloud around a protoplanetary embryo.

As already mentioned in Section 3.3, the obtained formula (3.8.27) for the mass density of a disk-shaped spheroidal body completely coincides with the so-called *barometric formula* for plane rotating systems [2]. As known [2], the plane systems in astrophysics are called systems whose thickness B is much less than the distance h from the center of a system rotating with angular velocity Ω, for example, these are preplanetary clouds. The thickness of a preplanetary cloud is determined by the thermal velocities of particles and can be found from an expression similar to the barometric formula for the Earth's atmosphere. In this connection, let us consider the following general theorem [73]:

Theorem 3.3 (the necessary and sufficient conditions of mass distribution of an isolated rotating disk in the gravitational field).

A rotating and gravitating gaseous disk is in a state of relative mechanical equilibrium if and only if its mass density distribution satisfies the barometric formula:

$$\rho(z) = \rho_0 e^{-\frac{\alpha}{2}z^2}, \qquad \alpha = 3\Omega^2 / \overline{v^2}, \tag{3.8.29}$$

where

$\rho_0 = \rho(0)$,

α is a parameter of gravitational condensation,

Ω is an angular velocity, and

$\overline{v^2}$ is a mean of the squared heat velocities of the particles of which the disk is formed.

Proof: first of all, we intend to prove this theorem by analogy with Theorem 2.2 from Chapter 2. We start, therefore, to derive the mass density function of an isolated rotating disk being in its gravitational field under the condition of relative equilibrium. Let us assume that the preplanetary cloud consists of a single-component rotating laminar gas. Its *relative equilibrium* in a *radial* direction h (perpendicular to the axis of rotation) is supported mainly by rotation. The gas pressure gradient along $|z| \leq B/2$ is very small [26], so that the rotation is almost Keplerian, that is, according to Eq. (1.4.5) gravitational acceleration is balanced by specific centrifugal force:

$$a_h + \Omega^2 h = 0, \tag{3.8.30}$$

where a_h is a gravitational acceleration in the direction of the axis Oh and $h = \sqrt{x^2 + y^2}$. It is created by the gravitational attraction of a central body (a star, for example, the Sun) and the gravitational attraction of a cloud. The latter does not play a significant role so that it can be neglected if the cloud density is several times lower than the critical value when a gravitational instability arises in the cloud (see Sections 1.3 and 1.5). Consequently, determining $a_h = f_{g\,h} = -\gamma M_c / h^2$ and substituting it into Eq. (3.8.30), we obtain:

$$\Omega^2 = \gamma M_c / h^3 , \qquad (3.8.31)$$

where M_c is a mass of a central body (in particular, the Sun). In contrast, the equilibrium in the direction z (perpendicular to the central plane) is maintained by a pressure gradient [2], so that the Euler equation [94, 111]:

$$\partial \vec{\mathrm{v}} / \partial t + (\vec{\mathrm{v}}\,\nabla)\,\vec{\mathrm{v}} = -\nabla p / \rho + \vec{a}$$

under the condition of hydrostatic equilibrium $(\vec{\mathrm{v}} = 0)$ takes the form:

$$dp / dz = \rho \cdot a_z , \qquad (3.8.32)$$

where a_z is a gravitational acceleration in the direction of the axis Oz. Bearing in mind that the gravitational attraction of the cloud can be neglected, we take into account only the gravitational attraction of the central body whose gravitational potential is expressed by the formula:

$$\varphi_g(h, z) = -\gamma M_c / \sqrt{h^2 + z^2} , \qquad (3.8.33a)$$

whence gravitational acceleration $\vec{a} = -\nabla\varphi_g$ in the direction Oz should be equal to:

$$a_z = -\partial\varphi_g / \partial z = -\gamma M_c z / (h^2 + z^2)^{3/2} \approx -\gamma M_c z / h^3 , \qquad (3.8.33b)$$

where h is a distance from the axis of rotation. Taking into account Eq. (3.8.31) the formula (3.8.33b) becomes:

$$a_z = -\Omega^2 z . \qquad (3.8.34)$$

Substituting (3.8.34) into (3.8.32) we obtain the following equation:

$$(1/\rho) \cdot dp / dz = -\Omega^2 z . \qquad (3.8.35)$$

Since the preplanetary cloud has a constant temperature and contains the same particles, it follows from the molecular kinetic theory [110, 136, 160] that:

$$p = (1/3)\rho \overline{v^2}\,, \tag{3.8.36}$$

where $\overline{v^2}$ is the mean of the squared velocity of a particle. From the condition of constant temperature and identity of particles, we evaluate that the means of the squared velocities of particles do not depend on z. Taking into account (3.8.36) this allows us to write equation (3.8.35) as follows:

$$(1/3)\overline{v^2}\cdot(1/\rho)\cdot d\rho/dz = -\Omega^2 z\,,$$

whence after integration of this equation, it is easy to find the mass density of the gaseous disk [2]:

$$\rho(z) = \rho_0 \cdot e^{-3\Omega^2 z^2/2\overline{v^2}}\,, \tag{3.8.37}$$

where ρ_0 is a mass density in the central part of the gaseous disk when $z = 0$. Denoting through:

$$\alpha = 3\Omega^2/\overline{v^2} \tag{3.8.38}$$

and substituting (3.8.38) in relation (3.8.37) we finally obtain formula (3.8.29) of mass density for an isolated rotating disk in the gravitational field. So, the necessary condition is proved.

Now we need to prove that formula (3.8.29) corresponds to the mass density of an isolated gravitating and rotating disk in the gravitational field being in a state of both relative mechanical and hydrostatic equilibrium. As shown in Section 3.3 (see the derivation based on formulas from (3.3.2) to (3.3.25)), due to their own oscillatory interactions as well as originating gravitational forces a great number of liquid particles form an isolated uniformly rotating spheroidal body whose mass density is described by formulas (3.3.26a–c). Following Corollary 3.2 since the iso-surfaces of mass density

(3.3.26a–c) of a spheroidal body are described by spheroids (3.3.51) then the total potential field of gravitational and centrifugal forces of a rotating spheroidal body is being in *relative equilibrium* in accordance with (3.3.54). In reality, according to formula (3.6.15a) from Section 3.6, the equation of the equipotential surface of the gravitational field of a uniformly rotating spheroidal body is also described by spheroid (3.3.51). According to Theorem 3.2, the state of relative mechanical equilibrium of a spheroidal body presupposes the state of *hydrostatic equilibrium* when the equipotential surfaces of its gravitational field coincide with isobars. Using the diagram in Fig. 3.4 we can again describe the process of formation of an isolated rotating gaseous disk based on interactions of the rotating spheroidal body with captured gas-dust bunches. Formally reasoning, if we assume $\varepsilon_0^2 \to 1$ in Eq. (3.3.26a) from Section 3.3 (or $\varepsilon_\Sigma^2 \to 1$ in Eq. (3.8.22)) then we obtain equation (3.3.28) (or (3.8.27)) describing the mass density of an isolated gravitating and rotating gaseous disk [16, 73]:

$$\rho(z) = \rho_0 e^{-\alpha z^2/2}, \tag{3.8.39}$$

where, according to formulas (3.8.26) and (3.8.27), the value of mass density in a central plane of this gaseous disk is equal to:

$$\rho_0 = (\alpha / 2\pi)^{3/2} \lim_{\substack{\varepsilon_0^2 \to 1 \\ M \to \infty}} M(1-\varepsilon_0^2) = m(\alpha / 2\pi)^{3/2}. \tag{3.8.40}$$

Comparing formula (3.8.37) with a similar relation (3.8.39), obtained exclusively within the framework of the statistical theory of gravity, we can see their complete identity suggesting that

$$\rho_0 = \rho(0) = m(\alpha / 2\pi)^{3/2} \ . \tag{3.8.41}$$

Thus, this theorem is proved.

Corollary 3.3. The dynamics of the evolution process of formation of an isolated gravitating and rotating gaseous disk include multivariate states of relative mechanical equilibrium.

Proof: as noted by Jeans [1 p.338], "for any rotating mass whatever, the shape of figure is determined approximately by the value $\Omega^2 / 2\pi\gamma\overline{\rho}$, and depends on nothing else," because the geometrical eccentricity of a rotating ellipsoid can be approximately expressed by the angular velocity Ω as:

$$e = \frac{3}{4}\frac{\Omega^2}{2\pi\gamma\overline{\rho}}. \qquad (3.8.42)$$

where $\overline{\rho}$ is a mean density of the mass. In conformity with the process of formation of an isolated rotating gaseous disk in Fig. 3.4, formula (3.8.42) means that eccentricity ε_l at l-stage of formation of a flattened spheroidal body can be found as follows:

$$\varepsilon_l^2 = \frac{3}{4}\frac{\Omega_l^2}{2\pi\gamma\overline{\rho}_l}. \qquad (3.8.43)$$

According to Theorem 3.3 (in particular, see formula (3.8.37)) the angular velocity Ω_l determines the mass density ρ_l at l-stage of formation of an isolated rotating spheroidal body being in the l-state of relative mechanical equilibrium where $l = 0,1,...,n-1$. The corollary is proven.

Since according to formula (1.2.18) from Section 1.2 of Chapter 1, the mean of the squared heat velocities of particles $\overline{v^2}$ is proportional to the mean internal temperature T of a gaseous disk (in accordance with assumption 3 of this Chapter the rotating spheroidal disk is an isothermal one being at a temperature $T \to 0$), then the derived formulas (3.8.38) and (3.8.41) clearly explain the cause for the increase in both the gravitational compression parameter α (with decreasing temperature T) and the mass density ρ_0 in the central part of the gaseous disk. Such a conclusion is in complete agreement with the theory of the gravitational instability of Jeans [1], described in Sections 1.3 and 1.5 of Chapter 1. Namely, according to Jeans' theory, the cooling of a gas-dust cloud reduces the value of critical wave-length λ_c which, in turn, leads to further contraction of the cloud owing to a gravitational condensation process (see formulas (1.5.20) and (1.5.26)).

Let us note that according to (3.8.37), the *surface* mass density of a flat disk can be found by the formula:

$$\sigma = \int_{-\infty}^{\infty} \rho(z) dz \ ,$$

whereas the *thickness* B of a homogeneous disk-shaped rotating system is equal [2]:

$$B \equiv \sigma / \rho_0 = (1/\rho_0) \cdot \int_{-\infty}^{\infty} \rho(z) dz \ . \tag{3.8.44}$$

Substituting (3.8.37) into (3.8.44), it is not difficult to find B [73]:

$$B = (1/\rho_0) \int_{-\infty}^{\infty} \rho_0 \cdot e^{-3\Omega^2 z^2 / 2\overline{v^2}} dz = \int_{-\infty}^{\infty} e^{-(3\Omega^2/\overline{v^2}) \cdot z^2/2} dz =$$

$$= \sqrt{\frac{2\pi}{3\Omega^2 / \overline{v^2}}} = \sqrt{\frac{2\pi}{3}} \cdot \frac{\sqrt{\overline{v^2}}}{\Omega} . \tag{3.8.45}$$

To simplify formula (3.8.45), we use the connection between the mean square of the velocity $\overline{v^2}$ and the square of the arithmetic mean velocity $\bar{v}^2$ of the heat motion of particles taking place in the Maxwellian distribution of velocities. This connection can be easily revealed using the comparison of formulas (2.1.27) and (2.1.29) from Section 2.1, meaning $\bar{v}$ instead of $\bar{r}$ and $\overline{v^2}$ instead of $\overline{r^2}$ in them:

$$\overline{v^2} = \frac{3\pi}{8}\bar{v}^2. \tag{3.8.46}$$

Taking into account the relation (3.8.46), the formula (3.8.45) becomes:

$$B = \frac{1}{\Omega}\cdot\sqrt{\frac{2\pi}{3}}\cdot\sqrt{\frac{3\pi}{8}\bar{v}^2} = \frac{\pi}{2}\cdot\frac{\bar{v}}{\Omega}, \tag{3.8.47}$$

where $\bar{v}$ is an arithmetic mean velocity of particles. So, formulas (3.8.45) and (3.8.47) show that the thickness of the preplanetary cloud (disk) is determined by the heat velocities of the particles [2]. The accounting of relation (3.8.46) allows us to transform the formula (3.8.38) for estimating the parameter of gravitational condensation α in a rotating spheroidal disk [73]:

$$\alpha = \frac{8}{\pi}\cdot\left(\frac{\Omega}{\bar{v}}\right)^2. \tag{3.8.48}$$

Finally, let us try to estimate the square of the total geometric eccentricity ε_Σ^2 of a rotating disk-shaped spheroidal body. For this purpose, we use the model of formation of a flattened (disk-shaped) rotating spheroidal body (see Fig. 3.4) and note that the above arguments have some common features with the Roche model discussed in Sections 1.4 and 3.1. It especially concerns the changing of the shape of equipotential surfaces (see Fig. 1.9) as the value $\Omega^2 / 2\pi\gamma\bar{\rho}$ increases until it reaches its limit value of 0.360744 in accordance with (1.4.34), so that when Ω^2 exceeds the critical value $0.360744\cdot 2\pi\gamma\bar{\rho}$ for the first time, then its further increase can lead to a loss of mass in the volume of this new equipotential configuration [1].

Indeed, the change in the shape of a rotating spheroidal body (see Fig. 3.4) within the framework of the developed model for the formation of a spheroidal disk is associated with an increase in the square of the geometric eccentricity following (3.8.8) and (3.8.23), while according to the Roche model as well as Corollary 3.3 the change in the shape of a rotating body is stipulated by the increase in angular velocity Ω. In this regard, we shall find the dependence ε_{Σ}^2 on the magnitude Ω.

In the framework of the Roche model, equation (1.4.21) is considered for the equipotential surface of a rotating solitary gas-dust cloud which at the pole $r = r_*$ (in the direction z) of the chosen equipotential surface takes the form:

$$\frac{\gamma M}{r_*} = \text{const} \tag{3.8.49}$$

which, in turn, allows us to clarify the constant in the equation under consideration (1.4.21) and rewrite it as follows:

$$\frac{\gamma M}{r} + \frac{1}{2} \cdot \Omega^2 (x^2 + y^2) = \frac{\gamma M}{r_*} . \tag{3.8.50}$$

Dividing the left-hand side of Eq. (3.8.50) by its right-hand side, we obtain:

$$\frac{r_*}{r} = 1 - \frac{1}{2} \cdot \frac{r_*}{\gamma M} \Omega^2 r^2 \sin^2 \theta , \tag{3.8.51}$$

where the next equation immediately follows [73]:

$$r = r_* \left[1 - \frac{1}{2} \cdot \frac{\Omega^2 r_*}{\gamma M} \cdot r^2 \sin^2 \theta \right]^{-1} . \tag{3.8.52}$$

In the theory of a rotating planet figure (in particular, Earth) a small dimensionless quantity $\frac{\Omega^2 r_*}{a_*} << 1$ is considered in the hydrostatic equilibrium approximation (for

example, for the Earth rotating with an angular velocity of $\Omega \approx 1/13700\,\mathrm{s}^{-1}$, it is equal to the value $\approx 1/300$ [44, 186]).
Taking into account that $a_* = a(r_*) = \gamma M / r_*^{\,2}$ is a value of gravitational acceleration at the pole, this small dimensionless quantity is equal:

$$\frac{\Omega^2 r_*}{a_*} = \frac{\Omega^2 r_*^{\,3}}{\gamma M}\,. \tag{3.8.53}$$

According to the Roche model, an equilibrium figure for the solitary non-rotating gas-dust cloud should be the sphere [44, 111]:

$$r = r_*\,. \tag{3.8.54}$$

For a gas-dust cloud freely rotating with an infinitely small angular velocity Ω, equation (3.8.52) also transforms into the sphere equation (3.8.54), that is:

$$r \to r_* \qquad \text{at} \qquad \Omega^2 r_*^{\,3} / \gamma M \to 0\,. \tag{3.8.55}$$

So, taking into account (3.8.53)–(3.8.55) due to the smallness of the parameter $\Omega^2 r_*^{\,3} / \gamma M << 1$ (due to the very small rotation of the solitary gas-dust cloud), the equation of the equipotential surface (3.8.52) is approximately described by an equation of the spheroid:

$$r \approx r_* \left[1 + \frac{1}{2} \cdot \frac{\Omega^2 r_*^{\,3}}{\gamma M} \cdot \sin^2\theta \right]. \tag{3.8.56}$$

On the other hand, according to formula (3.8.10) and (3.3.51) from Section 3.3, the iso-surface of the mass density of a spheroidal body in relative mechanical equilibrium is a flattened ellipsoid of rotation whose equation in the spherical coordinate system has the form:

$$\alpha\, r^2 (1 - \varepsilon_0^2 \sin^2\theta) / 2 = \text{const}\,. \tag{3.8.57}$$

Putting the constant equal $\text{const} = 1/2$ in equation (3.8.57) and introducing the traditional designation $r_* = 1/\sqrt{\alpha}$ in accordance with the formula (2.2.6), we can express the coordinate r from this equation:

$$r = r_* \left[1 - \varepsilon_0^2 \sin^2 \theta\right]^{-1/2},$$

where, because of the smallness of the parameter ε_0^2, that is, $\varepsilon_0^2 << 1$, we obtain an approximate equation of the spheroid:

$$r \approx r_* \left[1 + \frac{1}{2}\varepsilon_0^2 \sin^2 \theta\right]. \qquad (3.8.58)$$

Following Corollary 3.2 of Theorem 3.2 and Remark 3.2 from Section 3.3, we conclude (because of the coincidence of equations (3.8.56) and (3.8.58)) that

$$\varepsilon_0^2 = \frac{\Omega^2 r_*^{\,3}}{\gamma M} = \frac{\Omega^2 r_*}{a_*}. \qquad (3.8.59)$$

Generalizing (3.8.59) we find:

$$\varepsilon_l^2 = \frac{\Omega_l^2 r_*^{(l)\,3}}{\gamma M} = \frac{\Omega_l^2 r_*^{(l)}}{a_*^{(l)}}, l = 0,1,...,n-1. \qquad (3.8.60)$$

Let us note that choosing the mean volume $M / \overline{\rho}_l = \overline{V}_l = 4\pi r_*^{(l)^3} / 3$ we can transform formula (3.8.60) to the kind (3.8.43). Finally, using (3.8.60) as well as (3.8.8) we obtain:

$$\varepsilon_\Sigma^2 = \sum_{l=0}^{n-1} \frac{\Omega_l^2 r_*^{(l)}}{a_*^{(l)}}, \qquad (3.8.61)$$

where $\Omega_0 = \Omega$, $r_*^{(0)} = r_*$, $a_*^{(0)} = a_*$. Formulas (3.8.59)–(3.8.61) give an explicit form of the dependence of the square of geometric eccentricity ε_l^2 and the total geometric eccentricity ε_Σ^2 on the parameters of a rotating disk-shaped spheroidal body.

Conclusion and comments

This chapter has been occupied with an investigation into the configurations assumed by masses rotating freely in space under their gravitational forces [1]. Before leaving the theoretical discussion, and turning our attention to the actual problems of astrophysics, it may be profitable to summarize the main theoretical results which have been obtained.

First of all, since this chapter is devoted to the study of statistical models of a rotating and gravitating *spheroidal body* to describe the evolution of a protoplanetary gaseous (gas-dust) cloud around a forming star, Section 3.1 considers the statistical interpretation of Poincaré's well-known general theorem and the Roche model for a *slowly* rotating and gravitating spheroidal body, that is, for a sphere-like gaseous body. In particular, the distribution function (3.1.11) is introduced in the space of angular velocities. Moreover, the *effective* probability volume density (3.1.16) is essentially a probability volume density $\Phi_\Omega^{eff}(\vec{r})$ of detecting a particle in a rotating spheroidal body. Since the effective probability volume density is obtained under the assumption of the smallness of the parameter $\eta = \beta\Omega^2/2$, this condition automatically leads to Poincaré's well-known general theorem (3.1.23) in statistical interpretation for a slowly rotating and gravitating spheroidal body, that is, Theorem 3.1 says [52, 73]:

> Stable movement of a rotating and gravitating spheroidal body is possible only for small values of the argument $\eta = \beta\Omega^2/2$ of the distribution function in space of angular velocities of its constituent liquid particles, i.e. when $\eta < 1$.

The Poincaré theorem is closely related to the *condition for the existence of an equilibrium figure* for a rotating and gravitating mass of a liquid with a convex surface (for example, a rotating gaseous mass of a molecular cloud). Since the liquid mass equilibrium figure is found from the condition that the external (free) surface should be equipotential, the equation of the critical equipotential surface (1.4.29) is considered within the framework of the Roche model (see also Section 1.4). As we know, the Roche model leads to the *limit relation* (3.1.25) confirming the general conclusion that follows from the Poincaré theorem. Section 3.1 also notes that the results obtained using the Roche model are fully confirmed within the framework of the developed theory of forming spheroidal bodies. In particular, according to formula (3.1.26), when the wave propagation of initial gravitational perturbations changes to an aperiodic mode of their amplification (due to the formation of the core of a spheroidal body), the wave perturbations lead to the self-rotation of the forming core as a result of their interference. Let us note that the break-up angular velocity Ω of rotation is 12% higher than the critical frequency ω_c of the wave perturbations. Thus, within the framework of the theory of rotating and gravitating spheroidal bodies, both the Poincaré general theorem is derived in the statistical interpretation (3.1.23) (see Theorem 3.1) and the main result (1.4.25b) of the Roche model is also obtained by formula (2.3.8).

In Section 3.2, using the proposed model of initial rotation of a spheroidal body, the evolution of a *nonequilibrium* function of distribution of liquid particles for the spatial coordinates in a gaseous body at the beginning of its rotation is considered. Namely, a nonequilibrium function of the form (3.2.4) (or (3.2.6)) is obtained. As noted, formula (3.2.6) describes the effective probability volume density $\Phi_{\Omega}^{eff}(r)$ to locate a liquid particle at the initial instant of rotation of a sphere-like gaseous body.

In Section 3.3, the *equilibrium distribution function* of liquid particles with respect to the spatial coordinates is derived, and the mass density function is also obtained for a uniformly rotating and gravitating spheroidal body with a small angular velocity $\Omega = \text{const}$. Although in a uniformly rotating spheroidal body the particles do not move relative to the rotating cylindrical frame of reference itself, however, from the beginning of a nonuniform (unsteady-state) rotation of gaseous body with an angular velocity $\vec{\Omega} = \vec{\Omega}(t)$ a liquid particle begins to move inside it in the opposite direction to this rotation owing to an inertia force action (in fact, the uniform rotation of the body always precedes unsteady nonequilibrium initial rotation, as noted in Section 3.2). That is why in the subsequent reasoning there are distinctive features of the derivation related to the integration process of the third differential equation (3.3.18) along the angular coordinate ε. Namely, to derive the distribution function of particles inside a spheroidal body being in *relative mechanical equilibrium* (i.e., relative to a coordinate system rotating with a constant angular velocity Ω), we take into account that the variable ε in equation (3.3.19c) stabilizes after some time to its constant value ε_0 which is the upper limit value of ε. As a result, in Section 3.3, expressions for the probability volume density function (3.3.22a–c) and mass density (3.3.26a–c) of a rotating spheroidal body in the state of relative mechanical equilibrium are obtained [16, 73]. In Section 3.3, Lemma 3.1 is proved, according to which the iso-surface of the mass density of a spheroidal body being in absolute or relative mechanical equilibrium coincides with the equipotential surface of the field of potential forces, and also Theorem 3.2, which says [73]:

> In order for a gravitating spheroidal body to be in absolute or relative mechanical equilibrium under the action of a potential field of forces, it is necessary and sufficient that the equipotential surfaces of the field coincide with mass density iso-surfaces (isostere) and isobars.

Corollary 3.2 is also considered, according to which the iso-surface of the mass density of a spheroidal body is described by a *flattened ellipsoid* (*spheroid*) in space if and only if the potential field of forces is the total potential field of gravitational and centrifugal forces of a rotating spheroidal body being in relative mechanical equilibrium. So, according to Lemma 3.1 and Theorem 3.2, in the case of a very slow rotation of a spheroidal body ($\Omega^2 << 1$), the equation of the equipotential surface of the gravitational field of a slowly rotating spheroidal body must also be described in space by a *flattened* ellipsoid of the form (3.3.51).

Using the relation (3.3.22a) for the probability volume density function $\Phi(h, z)$ of detecting particles in a rotating spheroidal body in the state of relative mechanical equilibrium (Fig. 3.3), in Section 3.4 the distribution function (3.4.9) of the *specific angular momentum* $f(\lambda)$ and the angular momentum distribution function in space $l(h, z)$, that is, *angular momentum density* (3.4.27), for a uniformly rotating spheroidal body are derived. The average value of specific angular momentum (3.4.11) and the total angular momentum (3.4.13) of a rotating spheroidal body being in relative mechanical equilibrium are calculated [16, 73]. The distribution function (3.4.32) of the specific angular momentum $f(h)$ on distance h from the axis of rotation Oz is found. It is noted that the presence of maximum points of the distribution functions of the specific angular momentum and the density of angular momentum means that an "export" of the specific angular momentum by particles from the axis of rotation to the region $h = h_*$ occurs in a uniformly rotating spheroidal body.

Section 3.5 provides the derivation of a function of the spatial distribution of particles for a rotating and gravitating spheroidal body within the framework of the GR theory when the gravitational field of a rotating body is characterized by an axially symmetric stationary Kerr' metric [54, 73]. It is shown that, from the point of view of GR, the probability volume density function of detecting a particle in a uniformly rotating spheroidal body is described by the relation (3.5.29). Comparing (3.5.29) with a similar relation (3.3.22a) obtained within the framework of the statistical theory of gravity (see Section 3.3), we can see a certain similarity when the conditions (3.5.30a) and (3.5.30b) are fulfilled.

The determination of the gravitational potential based on general Eqs (1.1.40a)–(1.1.40c) from Chapter 1 for finding the specific gravitational force $\vec{f}_g$ (or gravitational strength $\vec{a}$) in the case of a *uniformly rotating* spheroidal body is discussed in Section 3.6. It is shown that the gravitational potential of a uniformly rotating spheroidal body is described by the expressions (3.6.15a–c) in the different coordinates, and it satisfies the Poisson equation (3.6.14). The form of the obtained formulas (3.6.15a–c) fully confirms the main conclusion of Corollary 3.2 of Theorem 3.2 and Remark 3.2 from Section 3.3 that the equipotential surface equation of the gravitational field of a slowly rotating (i.e. $\varepsilon_0^2 << 1$) spheroidal body should also be described in space by a flattened ellipsoid of the form (3.3.51).

Section 3.7 shows that the potential energy of a uniformly rotating and gravitating spheroidal body is estimated by formula (3.7.8). In the particular case $\varepsilon_0 \to 0$, formula (3.7.8) gives an estimation of the gravitational energy (3.7.11) of a spherically symmetric spheroidal body, that is, sphere-like gaseous body (see formula (2.5.6) in Sections 2.5 of Chapter 2). In the other particular case $\varepsilon_0 \to 1$, formula (3.7.8) demonstrates that the gravitational energy of a *disk-shaped* (flattened) spheroidal body is equal to zero in accordance with (3.7.12), that is, the disk-shaped spheroidal body does not possess its gravitational energy. This means that inside a rotating disk-shaped spheroidal body there is no pressure gradient, and the particles move independently along their orbits in the plane of the disk under condition (3.7.17) of equality of the gravitational and centrifugal forces.

In Section 3.8, the mass density of a flattened rotating spheroidal body (see formula (3.8.7)) is investigated, and a model for the formation of a rotating spheroidal disk is proposed. Within the framework of the model being developed, as a result of a series of captures by a rotating spheroidal body of gas-dust clots, both the mass and the eccentricity of the combined body increase following the formula (3.8.8). A diagram of the flattening of a rotating spheroidal body and the formation of a disk-shaped body is shown in Fig. 3.4. Under conditions (3.8.23), (3.8.24), and (3.8.26), the mass density of a rotating *spheroidal disk* of the form (3.8.27) is derived. This result is a part of the proved Theorem 3.3, which says:

> A rotating and gravitating gaseous disk is in a state of relative mechanical equilibrium if and only if its mass density distribution satisfies the barometric formula:
>
> $$\rho(z) = \rho_0 e^{-\frac{\alpha}{2}z^2}, \qquad \alpha = 3\Omega^2 / \overline{v^2},$$
>
> where $\rho_0 = \rho(0)$, α is a parameter of gravitational condensation, Ω is an angular velocity, and $\overline{v^2}$ is a mean of the squared heat velocities of the particles of which the disk is formed.

According to Theorem 3.3, formula (3.8.27) completely coincides with the barometric formula (3.8.37) for plane rotating systems being under the condition of hydrostatic mechanical equilibrium. A comparison of these formulas gives us a possibility to find an analytical dependence (3.8.48) of the parameter of gravitational condensation α of a rotating spheroidal disk on the angular velocity Ω and arithmetic mean heat velocity $\bar{v}$ of a particle. According to the proved Corollary 3.3, the dynamics of the evolution process of formation of an isolated gravitating and rotating gaseous disk includes *multivariate states* of relative mechanical equilibrium.

Let us summarize some results. In Sections 2.1–2.10 of Chapter 2, the proposed statistical theory proceeds from the concept of forming a sphere-like gaseous body (a spheroidal body at $\varepsilon_0 \to 0$) as an initial protoplanetary system (including a proto-sun within a protosolar nebula) from a nebula. As shown there, the obtained functions of particle distribution and mass density of an immovable (or slowly rotating when $\varepsilon_0 \to 0$) spheroidal body characterize the *first stage of evolution*: from a molecular cloud (nebula) to a forming core (proto-Sun) together with the outer shell (protosolar nebula).

In Sections 3.1–3.8 of this chapter, the *second stage* of evolution is described: from the protosolar nebula to the forming protoplanetary gas-dust disk based on the obtained distribution functions (nonequilibrium and equilibrium) and the mass density of rotating spheroidal body. As noted in Section 3.8, the resulting mass density function characterizes the flattening process: from initial spherical shapes (for an immovable spheroidal body) through flattened ellipsoidal shapes (for a rotating spheroidal body) to spheroidal disks when the square of the geometric eccentricity ε_0^2 varies from 0 to 1. The resulting formulas (3.3.26a–c) and (3.8.27) may describe possible scenarios of the formation of both a star and a protoplanetary gas-dust disk around it (in particular, the Sun together with its protoplanetary gas-dust disk).

CHAPTER FOUR

EQUATIONS AND STATE PARAMETERS OF A FORMING SPHEROIDAL BODY IN THE PROCESS OF INITIAL GRAVITATIONAL CONDENSATION

The phenomenon of gravitation concerns all the branches of physics, both traditional and newly emerging. In this connection, G. Nicolis and I. Prigogine in the book [135] pointed out, in particular, that “the self-organization theory cannot be completed without involving the force of gravitation, the most universal of all the known forces.”

In Chapters 2 and 3, statistical models of the gravitational interaction of particles were proposed (the immovable and rotating cases respectively). In the framework of these models, cosmogonical bodies have fuzzy contours and are represented by *spheroidal* forms [16, 45, 46, 73]. In particular, in Chapter 2 a centrally symmetric distribution of particles in space was used whose form is analogous to the one describing the Maxwell velocity distribution law.

The present chapter, being the continuation of Chapters 2 and 3, is aimed at the investigation of a slow-flowing-in time process of gravitational condensation experienced by a chemically homogeneous *spheroidal body* [47, 73], unaffected by other fields and bodies, and having a temperature close to absolute zero. In other words, in Chapter 3, both intensive parameters of the state of a spheroidal body (for example, pressure, temperature or chemical potential) were investigated, and differential equations describing the dynamics of a forming spheroidal body in the process of slow-flowing initial gravitational condensation were derived.

As follows from the results of statistical mechanics of irreversible processes [134], the condition of a not too large concentration gradient (the substance mass density ρ does not change noticeably through a distance of the order of the particle-free mean path, that is, the interphase boundaries within the body volume are absent) "implies the proximity of locally treated distribution functions of impulses and particle mutual positions to the Maxwell–Boltzmann equilibrium distribution." The foregoing means that the approach being described here, as well as that in Chapters 2 and 3, is in complete agreement with the statistical mechanics of irreversible processes. In addition, the *anti-diffusion equation* (1.7.12) predicted in Section 1.7 of Chapter 1 is also derived in this chapter within the framework of the more general model of a slowly compressible spheroidal body [16, 47, 65, 73]. As mentioned above, due to initial oscillatory interactions of colloidal particles, a spheroidal (cosmogonical) body can be formed in an immovable isolated molecular cloud.

In connection with the above remarks, in this chapter, as a rule, the following assumptions are used [16, 47, 49, 73]:

1. The spheroidal body considered is homogeneous, that is, it consists of like particles of mass m_0.
2. The spheroidal body is not affected by external fields and bodies.
3. The initial spheroidal body is isothermal, being at a temperature T close to absolute zero. Moreover $T > T_{\text{deg}}$, where T_{deg} is the temperature of degeneration [110] (this does not apply to the subsequent *virial equilibrium states* considered in Section 4.1 when $T \neq 0$ following the reasoning from Section 1.2).

4. The concentration gradient is not too great in the sense that the interphase boundaries inside the spheroidal body are absent.
5. The spheroidal body is in a state of mechanical equilibrium (there is no flow of mass), or in a state close to equilibrium (a weak mass flow), that is, *anti-diffusion process of the initial gravitational condensation* is a slow-flowing-in time one, as a rule [16, 47, 49, 73] (although, as shown in Section 4.6, the differential equations obtained in Section 4.2 for physical values are quite general, allowing us to describe, among other things, the avalanche gravitational compression due to the gravitational field that arises in the spheroidal body).

4.1. The main anti-diffusion equation of initial gravitational condensation of a spheroidal body with a centrally symmetric distribution of masses from an infinitely spread matter

Let us find the differential equation describing the process of gravitational condensation of a non-rotating or slowly rotating spheroidal body at $\varepsilon_0 \to 0$ having a centrally symmetric distribution of mass density (sphere-like gaseous body) near an equilibrium state (in the vicinity of mechanical equilibrium) [16, 47, 48, 73]. Henceforth, we will consider that parameter $\alpha > 0$ is a slowly changing function of time starting from a certain instant t_0, that is, $\alpha = \alpha(t)$ at $t > t_0$. Moreover, as follows from the derivation of formula (2.1.13), the function $\alpha = \alpha(t)$ is a positively defined monotonically increasing in time function (Fig. 4.1).

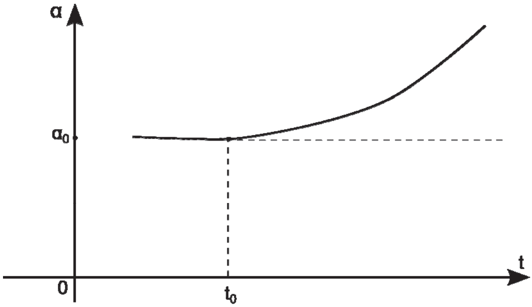


Figure 4.1. The parameter α dependence on time t

Indeed, with the increase of function $\alpha = \alpha(t)$ the maximum of the probability density function of revealing particles:

$$f(r,\alpha) = \sqrt{2/\pi}\,\alpha^{3/2} e^{-\alpha r^2/2} r^2 \qquad (4.1.1)$$

shifts to the left and increases in amplitude, as is shown in Fig.4.2.

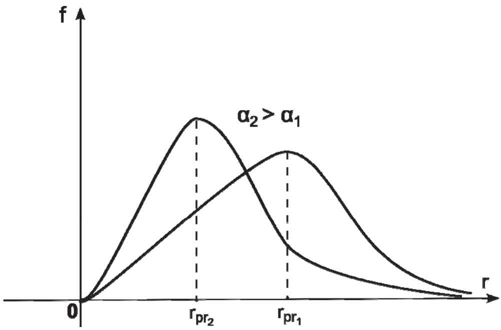


Figure 4.2. The probability densities as a function of distance r depending on the parameter α

This means that the diagram of the spheroidal body mass density function:

$$\rho(r,\alpha) = M\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\alpha r^2/2} \qquad (4.1.2)$$

has a steeper slope to the axis of abscissa, with α increasing (see Fig. 4.3), which, in turn, results in concentrating a mass of a non-rotating (or slowly rotating) spheroidal body near its center, that is, in gravitational condensation (compression).

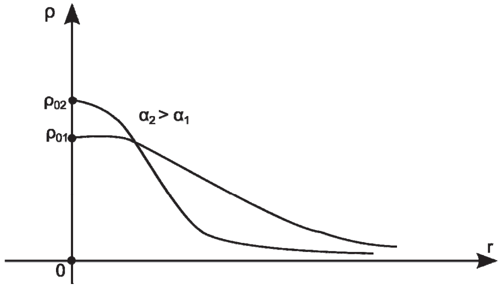


Figure 4.3. The mass density of a spheroidal body as a function of distance r depending on the parameter α

It is important to note here that, despite the seeming decrease of spheroidal body size (Fig. 4.4), the number of particles comprising it, N, does not decrease, that is, the total spheroidal body mass remains constant: $M = \text{const}$.

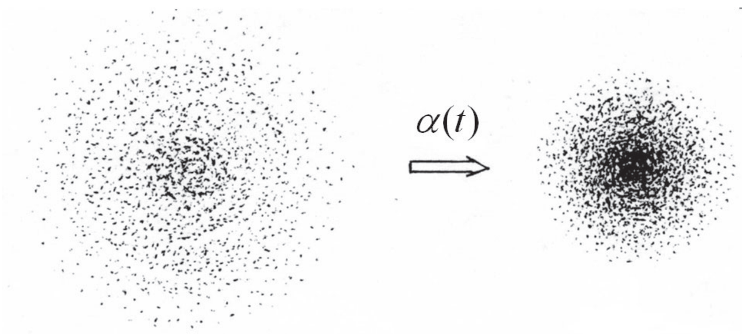


Figure 4.4. The graphic representation of gravitational condensation of a spheroidal body with the time

Let us consider the mass density ρ as a function of two variables r and t. Since the dependence on t is expressed by a composite function, we shall calculate partial derivative ρ with respect to t:

$$\frac{\partial\rho}{\partial t}=\frac{\partial\rho}{\partial\alpha}\cdot\frac{d\alpha}{dt}. \qquad (4.1.3)$$

As follows from (4.1.3), it is necessary first to calculate the derivative of ρ with respect to α, applying relation (4.1.2) for that purpose [16, 47, 48, 73]:

$$\frac{\partial\rho}{\partial\alpha}=M\frac{3}{2}\cdot\alpha^{1/2}\cdot\frac{1}{(2\pi)^{3/2}}e^{-\alpha r^2/2}+M\left(\frac{\alpha}{2\pi}\right)^{3/2}\left(-\frac{r^2}{2}\right)\cdot e^{-\alpha r^2/2}=$$

$$=\frac{Me^{-\alpha r^2/2}}{(2\pi)^{3/2}}\cdot\frac{\alpha^{1/2}}{2}\cdot\left(3-\alpha r^2\right). \qquad (4.1.4)$$

Let us now calculate the Laplacian operator of a scalar function ρ, taking into account that the spheroidal body has a centrally symmetric distribution of mass density. With provision for this, the Laplacian in the spherical system of coordinates has the form:

$$\nabla^2\rho=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho(r,\alpha)}{\partial r}\right)\right]=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(-\alpha M\left(\frac{\alpha}{2\pi}\right)^{3/2}r^3e^{-\alpha r^2/2}\right)\right]=$$

$$=-\alpha M\left(\frac{\alpha}{2\pi}\right)^{3/2}e^{-\alpha r^2/2}\cdot\left(3-\alpha r^2\right). \qquad (4.1.5)$$

Comparing (4.1.4) and (4.1.5) one can see that these relations coincide to the accuracy of a multiplier $-1/2\alpha^2$, that is, the following relation is valid:

$$\frac{\partial\rho}{\partial\alpha}=-\frac{1}{2\alpha^2}\nabla^2\rho. \qquad (4.1.6)$$

Substituting (4.1.6) into (4.1.5), we obtain [16, 47, 48, 73]:

$$\frac{\partial\rho}{\partial t}=-\left(\frac{1}{2\alpha^2}\frac{d\alpha}{dt}\right)\nabla^2\rho. \qquad (4.1.7)$$

The relation obtained (4.1.7) serves as a reminder, in its form, of the *anti-diffusion equation* (the sign '$-$' in the right-hand part testifies that the initial density disturbance will not decrease but rather increase $\left(\frac{1}{2\alpha^2}\frac{d\alpha}{dt} > 0\right)$, while, at the same time, it principally differs from the diffusion equation. As follows from (4.1.7), introducing the *gravitational compression function* (GCF) as a function of time t through the formula [16, 47, 48, 73]:

$$\mathrm{G}(t) = \frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt}, \qquad (4.1.8)$$

we can write the basic anti-diffusion equation of the *initial gravitational condensation* of a non-rotating (or slowly rotating at $\varepsilon_0 \to 0$) spheroidal body in the form [16, 47, 48, 73]:

$$\frac{\partial \rho}{\partial t} = -\mathrm{G}(t)\nabla^2 \rho\,. \qquad (4.1.9a)$$

Since $\alpha(t)$ is a monotonically increasing function of time, then $\mathrm{G}(t) > 0$. This means that, according to (4.1.9a), the initial perturbation of the mass density of a spheroidal body will not damp (as in the diffusion equation [94]), but increase. By (2.2.7) $\rho(r) = M\Phi(r)$, so that a similar equation also holds for the volume density of probability for detecting particles in a gaseous matter of a spheroidal body [16, 47, 48, 73]:

$$\frac{\partial \Phi}{\partial t} = -\mathrm{G}(t)\nabla^2 \Phi\,. \qquad (4.1.9b)$$

Rewriting (4.1.8) in the form:

$$\frac{d\alpha}{\alpha^2} = 2\mathrm{G}(t)dt, \qquad (4.1.10)$$

and then integrating (4.1.10), we obtain:

$$\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha^2} = 2\int_{t_0}^{t} \mathrm{G}(t)dt,$$

where $\alpha_0 = \alpha(t_0)$. Hence it is not difficult to see that:

$$\alpha = \frac{1}{-2\int_{t_0}^{t} \mathrm{G}(t)dt + \frac{1}{\alpha_0}} = \frac{\alpha_0}{1 - 2\alpha_0 \int_{t_0}^{t} \mathrm{G}(t)dt}. \tag{4.1.11}$$

Substituting (4.1.11) into (2.2.4), we then obtain:

$$\rho(r,t) = M(2\pi/\alpha_0 - 4\pi\int_{t_0}^{t} \mathrm{G}(t)dt)^{-3/2} \cdot e^{\frac{r^2}{4\int_{t_0}^{t} \mathrm{G}(t)dt - 2/\alpha_0}}. \tag{4.1.12}$$

Relation (4.1.12) is a solution of the derived differential equation (4.1.9a). If a function $\mathrm{G}(t)$ of the form (4.1.8) takes a constant value $\mathrm{G} = \mathrm{const}$, then the basic equation of gravitational compression (4.1.9b) of a spheroidal body generalizes the anti-diffusion equation (1.7.12), derived in Section 1.7. Thus, the model of a slowly flowing gravitational contraction of a spheroidal body near a state of unstable mechanical equilibrium adequately describes the evolutionary processes of the gravitational contraction of a molecular (gas-dust) cloud having a rather low temperature and being solitary in space.

As we know from Section 1.2 (in particular, Theorem 1.4), for a solitary molecular (gas-dust) cloud being in *unstable motion*, the Poincaré–Eddington virial theorem is valid (see formula (1.2.13)). This virial theorem in the Poincaré–Eddington interpretation applied to a radially moving spheroidal body with $\varepsilon_0 \to 0$ reads as follows [73]:

Theorem 4.1 (the Poincaré–Eddington virial theorem applied to a spheroidal body with a centrally symmetric distribution of masses being in unstable radial motion). For a self-gravitating spheroidal body with a centrally symmetric distribution of masses under the condition of their unstable radial motion the sum of the double kinetic energy and the total gravitational potential energy of this sphere-like gaseous system of particles is equal:

$$2E_k + E_g = -2M\dot{\mathrm{G}}(t)\,, \tag{4.1.13}$$

where

E_k and E_g are respectively the kinetic and gravitational potential energy of the particles forming a spheroidal body at $\varepsilon_0 \to 0$,

$\dot{\mathrm{G}}(t)$ is a derivative of GCF $\mathrm{G}(t)$ of the spheroidal body with a centrally symmetric distribution of masses, and

M is its total mass.

Proof: We can now apply Theorem 1.4 (the virial theorem in the Poincaré–Eddington interpretation) to the sphere-like gaseous system of particles being in unstable motion:

$$2E_k + E_g = \frac{1}{2}\frac{d^2 I(t)}{dt^2}, \tag{4.1.14}$$

where $I(t)$ is a moment of inertia of a spheroidal body at $\varepsilon_0 \to 0$ which can be easily calculated using the spherically symmetric distribution law of its mass density (2.2.5):

$$I(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x^2 + y^2)dxdydz =$$

$$= \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \rho \cdot r^2 \sin^2\theta \cdot r^2 \sin\theta dr d\theta d\varepsilon =$$

$$= 2\pi\int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta\int_0^{\infty} r^4 \rho dr =$$

$$= 2\pi\int_{-1}^{1}(1-\cos^2\theta)d\cos\theta \cdot \int_0^{\infty} r^4\rho dr = \frac{8\pi}{3}\int_0^{\infty} r^4 \rho dr. \tag{4.1.15}$$

As we know [16, 47, 65, 73], in the process of gravitational condensation of a spheroidal body, the parameter α increases monotonically with the time, that is, $\alpha = \alpha(t)$ is a positive definite monotonically increasing function of time. This means that the function of the mass density:

$$\rho(r,\alpha(t)) = M\left(\frac{\alpha(t)}{2\pi}\right)^{3/2} \cdot e^{-\frac{\alpha(t)r^2}{2}} \qquad (4.1.16)$$

has a steeper descent to the radial axis r under increasing $\alpha(t)$ which ultimately leads to compaction of the mass of spheroidal body near its center. The change in mass density along with r leads to a change in the value of the moment of inertia of a spheroidal body with the time:

$$I(t) = \frac{8\pi}{3}\int_0^\infty r^4 \rho(r,\alpha(t))dr = \frac{2}{3}\cdot\sqrt{\frac{2}{\pi}}M\cdot(\alpha(t))^{3/2}\int_0^\infty r^4 e^{-\frac{\alpha(t)r^2}{2}}dr. \qquad (4.1.17)$$

Using the integration rule by parts (see (2.9.15)), it is easy to show that the following relations hold:

$$\int_0^\infty r^4 e^{-\frac{\alpha(t)r^2}{2}}dr = \frac{3}{\alpha}\int_0^\infty r^2 e^{-\frac{\alpha r^2}{2}}dr\,;$$

$$\int_0^\infty r^2 e^{-\frac{\alpha(t)r^2}{2}}dr = \frac{1}{\alpha}\int_0^\infty e^{-\frac{\alpha r^2}{2}}dr\,.$$

Then taking into account these relation expressions, (4.1.17) takes the form:

$$I(t) = \frac{2}{3}\cdot\sqrt{\frac{2}{\pi}}M(\alpha(t))^{\frac{3}{2}}\cdot\frac{3}{\alpha^2}\int_0^\infty e^{-\frac{\alpha(t)r^2}{2}}dr =$$

$$= \frac{2}{3}\cdot\sqrt{\frac{2}{\pi}}M(\alpha(t))^{3/2}\frac{3}{\alpha^{5/2}}\cdot\sqrt{\frac{\pi}{2}} = \frac{2M}{\alpha(t)}. \qquad (4.1.18)$$

On the other hand, the relation (4.1.15) can be directly calculated in the cylindrical coordinate system (h,ε,z):

$$I(t)=\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi} h^2\rho(h,z,\alpha(t))hdhdzd\varepsilon = M\left(\frac{\alpha(t)}{2\pi}\right)^{3/2}\cdot 2\pi\int_0^{\infty} h^3 e^{-\frac{\alpha r^2}{2}}dh\cdot 2\int_0^{\infty} e^{-\frac{\alpha z^2}{2}}dz=$$

$$=M\frac{(\alpha(t))^{3/2}}{\sqrt{2\pi}}\cdot\frac{2}{\alpha^2}\cdot 2\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\alpha^{1/2}}=\frac{2M}{\alpha(t)}$$

which completely coincides with (4.1.18). Substituting (4.1.18) into Eq. (4.1.14), we obtain:

$$2E_k + E_g = M\frac{d^2(1/\alpha(t))}{dt^2}. \tag{4.1.19}$$

Taking into account (4.1.8) we transform the right-hand side of equation (4.1.19) to the form:

$$M\frac{d^2(1/\alpha(t))}{dt^2} = M\frac{d}{dt}\left(-\frac{1}{\alpha^2}\cdot\frac{d\alpha}{dt}\right)=$$

$$=-2M\frac{d}{dt}\left(\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}\right)=-2M\frac{d\mathrm{G}(t)}{dt}=-2M\dot{\mathrm{G}}(t), \tag{4.1.20}$$

where $\dot{\mathrm{G}}(t)$ is a derivative of the function $\mathrm{G}(t)$ with respect to the time. Finally, substituting expression (4.1.20) into the right-hand side of Eq. (4.1.19) we obtain, as a result, Eq. (4.1.13). The theorem is proved.

Remark 4.1. According to (4.1.18), in the case of a slowly and uniformly rotating spheroidal body when $\varepsilon_0 \to 0$ and $\Omega = \text{const}$, the value of its total angular momentum $L = I\Omega$ becomes equal to:

$$L = I\Omega = \frac{2\Omega M}{\alpha(t)},$$

which fully coincides with the earlier derived formula (3.4.15) in Section 3.4.

Corollary 4.1. In a state of mechanical equilibrium of a spheroidal body with a centrally symmetric distribution of masses, GCF $\mathrm{G}(t)$ should be a constant gravitational contraction coefficient:

$$\mathrm{G}(t) = \mathrm{G}_{\mathrm{s}} = \mathrm{const}\,. \tag{4.1.21}$$

Proof: In reality, as we know from Section 1.2, in the state of *mechanical equilibrium*, the right-hand side of Eq. (4.1.13) following the Theorem 1.3 (the virial theorem) and formula (1.2.14), must be equal to zero, that is, $\dot{\mathrm{G}}(t) = 0$. This means that equality (4.1.21) occurs, and the basic anti-diffusion equation (4.1.9b) of the initial gravitational condensation of a spheroidal body with a centrally symmetric distribution of masses becomes Eq. (1.7.12) of slow-flowing gravitational condensation. This corollary is proved [73].

Summarizing the above, we conclude that an approximate graph of the GCF change from the initial instant t_0 of the anti-diffusion process of the beginning of gravitational condensation to the instant t_s of the stabilization of the slow-flowing gravitational condensation process is shown in Fig. 4.5,a. Subject to the constancy of the gravitational contraction coefficient (4.1.21), beginning from the instant t_s, the formula (4.1.11) takes the form [73]:

$$\alpha(t) = \frac{\alpha_s}{1 - 2\alpha_s \mathrm{G}_{\mathrm{s}} \cdot (t - t_s)} = \alpha_s \left[1 - 2\alpha_s \mathrm{G}_s (t - t_s)\right]^{-1}, \tag{4.1.22}$$

where $\alpha_s = \alpha(t_s)$ is a value of the parameter of gravitational condensation corresponding to the time instant t_s of stabilization of GCF.

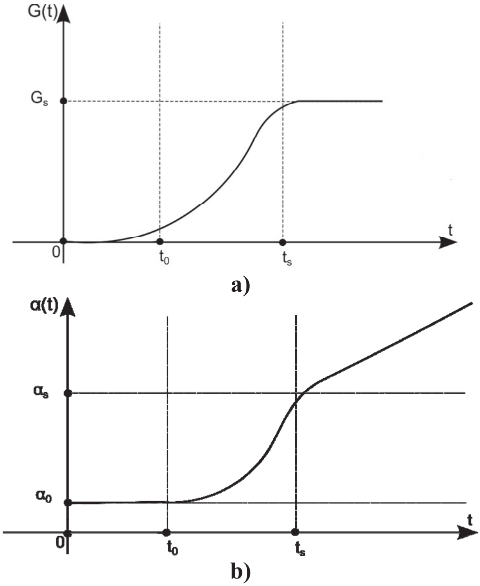


Figure 4.5. The diagrams of dependence on the time for GCF $\mathrm{G}(t)$ (a) and the parameter $\alpha(t)$ of gravitational condensation (b) from the beginning of the anti-diffusion process of the gravitational compression of a spheroidal body

Corollary 4.2 (Analog of Einstein's formula in the Brownian motion theory [136]). The mean-square distance of displacement of a colloidal particle in an immovable or slowly rotating spheroidal body during the time interval $\tau = t - t_s$ is equal:

$$\overline{\delta r^2} = 6\mathrm{G_s}\tau \,. \tag{4.1.23}$$

Proof: taking into account Corollary 4.1, let us rewrite formula (4.1.22) as follows:

$$\frac{1}{\alpha(t)} = \frac{1}{\alpha_s} - 2\mathrm{G_s}(t - t_s) \,. \tag{4.1.24}$$

According to formula (2.1.29) from Section 2.1, in the state of relative mechanical equilibrium, the mean-square distance of displacement of a colloidal particle in an immovable (or slowly rotating) spheroidal body is equal to:

$$\overline{r^2} = \frac{3}{\alpha}. \qquad (4.1.25)$$

Using the new notation (4.1.25) formula (4.1.24) becomes:

$$\overline{r_s^2} - \overline{r^2} = 6\mathrm{G}_s\tau,$$

where $\overline{r_s^2} = 1/\alpha_s$ and $\tau = t - t_s$, that is, the mean-square distance of displacement of the colloidal particle $\overline{\delta r^2} = \overline{r_s^2} - \overline{r^2}$ can be found by formula (4.1.23). The corollary is proved.

Remark 4.2. The formula (4.1.23) of Corollary 4.2 points to the existence of *dark matter* owing to which molecules (atoms) and, generally speaking, the colloidal (liquid) particles interact and form spheroidal bodies, that is, cosmogonical objects.

Taking into account $|\alpha_s| < 1$ the formula (4.1.22) goes to the following:

$$\alpha(t) \approx \alpha_s[1 + 2\alpha_s \mathrm{G}_s(t - t_s)], \qquad (4.1.26)$$

which approximately describes the linear law of increasing the parameter of gravitational condensation with the time. Consequently, starting from the instant t_s of the stabilization of GCF, formula (4.1.22), due to the smallness of α_s, describes the *almost linear law* of increasing $\alpha = \alpha(t)$ with the time t, which is illustrated in Fig. 4.5,b. Generally speaking, the initial instant t_0 of the beginning of the anti-diffusion process of the initial gravitational interactions of particles also corresponds to a certain state of the *initial virial* (unstable mechanical) equilibrium, so that in this case formula (4.1.26) takes the form:

$$\alpha(t) \approx \alpha_0[1 + 2\alpha_0 G_0(t - t_0)], \tag{4.1.27}$$

and this directly follows from (4.1.11) when $t = t_0 + dt$. In other words, there are *many states* of virial (unstable relative mechanical) equilibrium in the process of gravitational condensation of a spheroidal body with a centrally symmetric distribution of masses. Thus, we can formulate the following:

Corollary 4.3. The dynamics of the condensation process of a spheroidal body with a centrally symmetric mass distribution include multivariate states of unstable mechanical equilibrium.

As follows from Corollaries 2.2, 3.3, and 4.1, Corollary 4.3 is also true. This means that instants t_0 and t_S are repeated, that is, $t_0^{(1)} = t_0 + T_1$ and $t_s^{(1)} = t_s + T_1$, then $t_0^{(2)} = t_0^{(1)} + T_2$ and $t_s^{(2)} = t_s^{(1)} + T_2$,…. We observe, as noted in Section 1.2, that the virial theorem applied to large cloud-like configurations of ideal gas being in mechanical equilibriums (stellar systems, nebulae, and interstellar gas masses) can lead to a significant increase in the average temperature of the gravitating gas [1].

Under the condition of infinite smallness of the initial parameter of gravitational condensation ($|\alpha_0| << 1$) as well as under the assumption $t_0 \to 0$, formula (4.1.12) with regard for (4.1.22) takes the form:

$$\rho(r,t) \approx M\left(\frac{\alpha_0}{2\pi}\right)^{3/2} \{1 + 3\alpha_0 G_0 \cdot t\} \cdot e^{-\alpha_0 r^2 [1+2\alpha_0 G_0 t]/2}. \tag{4.1.28}$$

So, according to the above graphs in Fig. 4.5a and b, at the beginning of the anti-diffusion process the parameter α is constant ($\alpha \to \alpha_0$) and, respectively, $G(t) \to 0$ in accordance with (4.1.8). Then $G(t)$ and $\alpha(t)$ increase until the instant t_S of stabilization $G(t) = G_s$, which corresponds to almost linear growth of $\alpha(t)$ with $t > t_s$, that is, in the state of virial mechanical equilibrium the parameter α in the first approximation linearly depends on t.

Thus, in a state close to mechanical equilibrium, the parameter of gravitational condensation α of a spheroidal body *almost linearly* increases with the time t (see formulas (4.1.22), (4.1.26), and (4.1.27)). Two special cases of the anti-diffusion equation (4.1.9a) of slow-flowing initial gravitational condensation will be considered in Section 4.3: quasi-equilibrium gravitational condensation ($\alpha = t/2G_1$) and initial gravitational condensation ($\alpha = t^2/G_2$) of a weakly gravitating spheroidal body formed from an infinitely spread matter [47].

4.2. General differential equations for physical values describing the anti-diffusion process of an initial gravitational condensation of a centrally symmetric spheroidal body near mechanical equilibrium

Let us find the form of differential equations that are satisfied by other physical values that describe an *anti-diffusion process of the initial gravitational condensation* of a spheroidal body with centrally symmetric mass density (sphere-like gaseous body). As noted in Section 2.6, the physical values describing the gravitational interaction of particles have a probabilistic interpretation. Indeed, the analysis of relations (2.6.1)–(2.6.3) demonstrates that physical values considered within the framework of the statistical model of gravity [16, 45, 46, 73] contain either an exponential function $e^{-\alpha x^2/2}$-like as a multiplier, or its integral (the integral of probability theory).

Using the reasoning given in Section 4.1, we similarly assume that $\alpha = \alpha(t)$ is a positively defined monotonically increasing function of time (Fig. 4.1). Indeed, as the function

$\alpha = \alpha(t)$ increases, the maximum of probability density function (4.1.1) for detecting particles shifts to the left and increases in amplitude (see Fig. 4.2), and the mass density function (4.1.2) also increases in amplitude and has a steeper descent to the abscissa axis (see Fig. 4.3) which leads to the concentration of mass of a spheroidal body (sphere-like gaseous body) near its center, that is, to the anti-diffusion gravitational condensation of the spheroidal body based on the initial gravitational interactions of particles.

Another important remark concerns the virial equilibrium state considered in Section 4.1, or rather *one of the states* of unstable mechanical equilibrium. Although the parameter of gravitational condensation α increases linearly with time t, the corresponding *anti-diffusion process does not disturb the virial equilibrium* of a spheroidal body, since in this state GCF is constant: $\mathrm{G}(t) = \mathrm{G}_s = \text{const}$ in accordance with (4.1.21).

Now we are going to obtain the form of a differential equation which satisfies function (4.1.1) of the probability density of particles being there. For that purpose $f(r,\alpha)$ will be expressed in terms of mass density $\rho(r,\alpha)$ using (4.1.1) and (4.1.2):

$$f(r,\alpha) = \frac{4\pi r^2}{M}\rho(r,\alpha)\,, \tag{4.2.1}$$

whence:

$$\rho(r,\alpha) = \frac{M}{4\pi r^2} f(r,\alpha)\,. \tag{4.2.2}$$

Substituting (4.2.2) in (4.1.6), we obtain:

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{r^2} f(r,\alpha)\right) = -\frac{1}{2\alpha^2}\nabla^2\left(\frac{1}{r^2} f(r,\alpha)\right). \tag{4.2.3}$$

Developing the form of the Laplacian operator in the case of the centrally symmetric solution in (4.2.3), we will have:

$$\frac{1}{r^2}\cdot\frac{\partial f(r,\alpha)}{\partial\alpha}=\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial}{\partial r}\left(\frac{1}{r^2}\left[-\frac{f(r,\alpha)}{2\alpha^2}\right]\right)\right)\right),$$

whence:

$$\frac{\partial f(r,\alpha)}{\partial\alpha}=-\frac{1}{2\alpha^2}\left\{\frac{\partial}{\partial r}\left(r^2\left[-\frac{2}{r^3}f(r,\alpha)+\frac{1}{r^2}\cdot\frac{\partial f(r,\alpha)}{\partial r}\right]\right)\right\}=$$

$$=-\frac{1}{2\alpha^2}\left\{\frac{\partial}{\partial r}\left(-\frac{2}{r}f(r,\alpha)+\frac{\partial f(r,\alpha)}{\partial r}\right)\right\}=$$

$$=-\frac{1}{2\alpha^2}\left\{\frac{2}{r^2}f(r,\alpha)-\frac{2}{r}\cdot\frac{\partial f(r,\alpha)}{\partial r}+\frac{\partial^2 f(r,\alpha)}{\partial r^2}\right\}.$$

Thus, finally, we have [48, 73]:

$$\frac{\partial f(r,\alpha)}{\partial\alpha}=-\frac{1}{2\alpha^2}\left(\frac{\partial^2 f(r,\alpha)}{\partial r^2}-\frac{2}{r}\cdot\frac{\partial f(r,\alpha)}{\partial r}+\frac{2}{r^2}\cdot f(r,\alpha)\right). \quad (4.2.4)$$

Expressing in the right-hand part of (4.2.4) the differential operators through the Laplacian of the scalar function written down in the spherical coordinates:

$$\nabla^2 f(r,\alpha)=\frac{1}{r^2}\cdot\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r,\alpha)}{\partial r}\right)\right]=\frac{2}{r}\cdot\frac{\partial f(r,\alpha)}{\partial r}+\frac{\partial^2 f(r,\alpha)}{\partial r^2},$$

we obtain [48, 73]:

$$\frac{\partial f(r,\alpha)}{\partial\alpha}=-\frac{1}{2\alpha^2}\left(\nabla^2 f(r,\alpha)-\frac{4}{r}\cdot\frac{\partial f(r,\alpha)}{\partial r}+\frac{2}{r^2}\cdot f(r,\alpha)\right). \quad (4.2.5)$$

Using the formula (4.1.8) and Eq. (4.2.5) the differential equation for probability density can be written down in the general case [48, 73]:

$$\frac{\partial f(r,t)}{\partial t}=-\mathrm{G}(t)\cdot\left(\nabla^2 f(r,t)-\frac{4}{r}\cdot\frac{\partial f(r,t)}{\partial r}+\frac{2}{r^2}\cdot f(r,t)\right). \quad (4.2.6)$$

Let us find the differential equation form satisfied by the value of gravitational field strength $a(r,\alpha(t))$ in the anti-diffusion process of gravitational condensation. We shall begin with expressing $a(r,\alpha(t))$ through $\rho(r,\alpha)$, applying (2.4.8a) and (4.1.2):

$$a(r,\alpha)=\frac{4\pi\gamma}{r^2}\int_0^r x^2\rho(x,\alpha)dx\,. \qquad (4.2.7)$$

It is not difficult to obtain from (4.2.7) that:

$$\rho(r,\alpha)=\frac{1}{4\pi\gamma}\cdot\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 a(r,\alpha))\,. \qquad (4.2.8)$$

Substituting (4.2.8) in equation (4.1.6) we obtain:

$$\frac{\partial}{\partial\alpha}\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 a(r,\alpha))\right]=-\frac{1}{2\alpha^2}\nabla^2\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 a(r,\alpha))\right]. \qquad (4.2.9)$$

Applying the expression for the Laplacian operator of a scalar function in spherical coordinates: $\nabla^2[\bullet]=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[\bullet]\right)\right]$, expression (4.2.9) will be written down in the form:

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial a(r,\alpha)}{\partial\alpha}\right)=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\left\{-\frac{a(r,\alpha)}{2\alpha^2}\right\}\right)\right]\right)$$

whence:

$$\frac{\partial a(r,\alpha)}{\partial\alpha}=\frac{\partial}{\partial r}\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\left\{-\frac{a(r,\alpha)}{2\alpha^2}\right\}\right)\right]. \qquad (4.2.10)$$

Simplifying (4.2.5), the strength value differential equation is presented in the form:

$$\frac{\partial a(r,\alpha)}{\partial\alpha}=-\frac{1}{2\alpha^2}\cdot\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a(r,\alpha))\right]. \qquad (4.2.11)$$

Removing the parentheses from the right-hand part of (4.2.11) we obtain:

$$\frac{\partial a(r,\alpha)}{\partial\alpha}=-\frac{1}{2\alpha^2}\cdot\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(2ra(r,\alpha)+r^2\frac{\partial a(r,\alpha)}{\partial r}\right)\right]=$$

$$= -\frac{1}{2\alpha^2} \cdot \frac{\partial}{\partial r}\left[\frac{2}{r}a(r,\alpha) + \frac{\partial a(r,\alpha)}{\partial r}\right] =$$

$$= -\frac{1}{2\alpha^2} \cdot \left[-\frac{2}{r^2}a(r,\alpha) + \frac{2}{r}\frac{\partial a(r,\alpha)}{\partial r} + \frac{\partial^2 a(r,\alpha)}{\partial r^2}\right]. \qquad (4.2.12)$$

Thus, according to (4.2.12) the differential equation for the value $a(r,\alpha)$ is finally produced [48, 73]:

$$\frac{\partial a(r,\alpha)}{\partial \alpha} = -\frac{1}{2\alpha^2} \cdot \left[\frac{\partial^2 a(r,\alpha)}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial a(r,\alpha)}{\partial r} - \frac{2}{r^2} \cdot a(r,\alpha)\right]. \qquad (4.2.13)$$

On the other hand, taking into consideration that the *irrotational* (potential) gravitational field strength $\vec{a}(r,\alpha)$ is a vector function, its Laplacian operator has the form [128]:

$$\nabla^2 \vec{a}(r,\alpha) = \text{grad}\left(\text{div}\, \vec{a}(r,\alpha)\right). \qquad (4.2.14)$$

In spherical coordinates, relation (4.2.14) for $\vec{a}(r,\alpha) = -a(r,\alpha)\vec{e}_r$ is written down as follows:

$$\nabla^2 \vec{a}(r,\alpha) = -\frac{\partial}{\partial r}\left(\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 a(r,\alpha)\vec{e}_r\right)\right]\right), \qquad (4.2.15)$$

where $\vec{e}_r$ is a unit basis vector along coordinate r. Comparing Eq. (4.2.15) with Eq. (4.2.11), the gravitational field strength equation is written down in the vector form:

$$-\frac{\partial \vec{a}(r,\alpha)}{\partial \alpha} = -\frac{1}{2\alpha^2} \cdot \nabla^2(-\vec{a}(r,\alpha)). \qquad (4.2.16)$$

Taking into account that $\alpha = \alpha(t)$ is a monotonically increasing time function, $\partial\vec{a}/\partial t$ is calculated using (4.1.8) and (4.2.16):

$$\frac{\partial \vec{a}}{\partial t} = \frac{\partial \vec{a}}{\partial \alpha} \cdot \frac{d\alpha}{dt} = -\left(\frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt}\right) \cdot \nabla^2 \vec{a} = -\text{G}(t) \cdot \nabla^2 \vec{a},$$

whence we obtain finally [48, 73]:

$$\frac{\partial \vec{a}}{\partial t} = -\mathrm{G}(t)\nabla^2 \vec{a} \,. \tag{4.2.17}$$

As seen from (4.2.17), the anti-diffusion equation for gravitational field strength describes the dynamics of the strength vector of the gravitational field of a centrally symmetric spheroidal body in space and in time. Following [73] if we determine:

$$\mathrm{G}(t) = \mathrm{i}\,\mathrm{G}_s = \mathrm{i}\hbar / 2m_0 \,, \tag{4.2.17*}$$

where the constant $\hbar$ is identified with Planck constant divided by 2π, i is the imaginary unit, equation (4.2.17) describes the Schrödinger linear equation [187]. As we know [188], the solution of equation (4.2.17) can determine the initial conditions for the solution of a nonlinear equation such as the Schrödinger cubic equation:

$$\mathrm{i}\frac{\partial \vec{a}}{\partial t} = \mathrm{G}\nabla^2 \vec{a} + \mathrm{D}\vec{a}\left|\vec{a}\right|^2 ,$$

where $\vec{a}$ is, generally speaking, a complex-valued strength vector, G and D are constants. The analog of Schrödinger's nonlinear (cubic) equation in nonlinear optics has a soliton solution describing the self-focusing of an electromagnetic flux in a nonlinear media whose dielectric permeability depends on the electric field strength [188], that is, the envelopes of solutions of this equation in the form of a traveling nonlinear waves are *solitons* (or solitary waves) [188, 189]. In principle, a model of nonequilibrium gravitational compression can also lead to a sharp increase in the intensity of the anti-diffusion flow (see Eqs (4.6.22) and (5.7.18) in Chapter 5), so that the solution of equation (4.2.17) determines the *initial conditions* for the soliton solution of the nonlinear anti-diffusion equation.

For the case of quasi-equilibrium compression ($\alpha = t/2G$) [47, 65] considered in the next section, the spatial form of the *initial* gravitational strength soliton $a(r,t)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\gamma M}{r^2}\cdot\left(\frac{t}{G}\right)^{3/2}\int_0^r r^2 e^{-\frac{r^2 t}{4G}}dr$ is shown in Fig. 4.6.

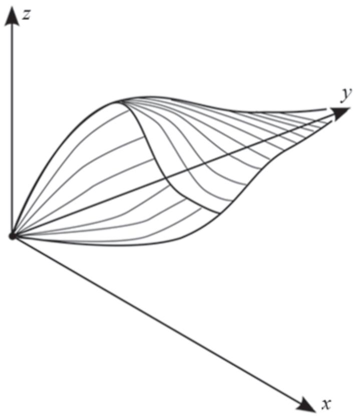


Figure 4.6. The spatial representation of the initial strength soliton of the gravitational field of a spheroidal body

Outwardly, the initial gravitational strength soliton $a(r,t)$ looks like an initial probabilistic one $f(r,t)$, although it has "sharper" vertices. In the process of propagation, the initial soliton $a(r,t)$ is deformed to become "steeper," so that its geometrical center is displaced towards the center of coordinates (see Fig. 4.6).

Applying formula (2.6.1) from Section 2.6, according to which $a(r,\alpha)=\frac{\gamma M}{r^2}P_\alpha(x\le r)$, a differential equation for probability $P_\alpha(x\le r)$ is obtained by (4.2.13):

$$\frac{\gamma M}{r^2}\cdot\frac{\partial P_\alpha}{\partial\alpha}(x\le r)=-\frac{\gamma M}{2\alpha^2}\left[\frac{\partial^2}{\partial r^2}\left(\frac{1}{r^2}P_\alpha(x\le r)\right)+\right.$$

$$\left.+\frac{2}{r}\cdot\frac{\partial}{\partial r}\left(\frac{1}{r^2}P_\alpha(x\le r)\right)-\frac{2}{r^4}P_\alpha(x\le r)\right]=$$

$$=-\frac{\gamma M}{2\alpha^2}\cdot\left[\frac{\partial}{\partial r}\left(-\frac{2}{r^3}P_\alpha(x\le r)+\frac{1}{r^2}\cdot\frac{\partial P_\alpha(x\le r)}{\partial r}\right)+\right.$$

$$\left.+\frac{2}{r}\cdot\left(-\frac{2}{r^3}P_\alpha(x\le r)+\frac{1}{r^2}\cdot\frac{\partial P_\alpha(x\le r)}{\partial r}\right)-\frac{2}{r^4}P_\alpha(x\le r)\right]=$$

$$=-\frac{\gamma M}{2\alpha^2}\cdot\frac{1}{r^2}\left[\frac{\partial^2 P_\alpha(x\le r)}{\partial r^2}-\frac{2}{r}\cdot\frac{\partial P_\alpha(x\le r)}{\partial r}\right],$$

whence it finally results [48, 73] in:

$$\frac{\partial P_\alpha(x\le r)}{\partial\alpha}=-\frac{1}{2\alpha^2}\left[\frac{\partial^2 P_\alpha(x\le r)}{\partial r^2}-\frac{2}{r}\cdot\frac{\partial P_\alpha(x\le r)}{\partial r}\right]. \qquad (4.2.18)$$

Checking the validity of equation (4.2.18) we calculate the first derivative of $P_\alpha(x\le r)$ with respect to α following (2.6.1):

$$\frac{\partial P_\alpha(x\le r)}{\partial\alpha}=\frac{\partial}{\partial\alpha}\left[\sqrt{\frac{2}{\pi}}\cdot\alpha^{3/2}\int_0^r x^2e^{-\alpha x^2/2}dx\right]=\frac{\partial}{\partial\alpha}\left[\frac{4}{\sqrt{\pi}}\int_0^{r\sqrt{\alpha/2}}s^2e^{-s^2}ds\right]=$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{\sqrt{\alpha}}{2}r^3\cdot e^{-\alpha r^2/2}$$

and then we find the second derivative with respect to r:

$$\frac{\partial^2 P_\alpha(x\le r)}{\partial r^2}-\frac{2}{r}\cdot\frac{\partial P_\alpha(x\le r)}{\partial r}=$$

$$=\frac{\partial}{\partial r}\left[\sqrt{\frac{2}{\pi}}\alpha^{3/2}r^2e^{-\alpha r^2/2}\right]-\frac{2}{r}\sqrt{\frac{2}{\pi}}\cdot\alpha^{3/2}r^2e^{-\alpha r^2/2}=$$

$$= \sqrt{\frac{2}{\pi}} \cdot \alpha^{3/2} 2re^{-\alpha r^2/2} + \sqrt{\frac{2}{\pi}} \cdot \alpha^{3/2} r^2 (-\alpha r) e^{-\alpha r^2/2} -$$

$$-\frac{2}{r}\sqrt{\frac{2}{\pi}} \cdot \alpha^{3/2} r^2 e^{-\alpha r^2/2} = -\sqrt{\frac{2}{\pi}} \cdot \alpha^{5/2} r^3 \cdot e^{-\alpha r^2/2}$$

Comparing these relations, one can see that they coincide up to the multiplier $-1/2\alpha^2$, that is, equation (4.2.18) is valid.

Let us write down a differential equation for probability $P_\alpha(x \le r)$ characterizing the gravitational condensation dynamics in time [48, 73]. To this end, the partial derivative of probability with the time will be calculated, with provision for relations (4.1.8) and (4.2.18):

$$\frac{\partial P_t(x \le r)}{\partial t} = -\frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt}\left[\frac{\partial^2 P_t(x \le r)}{\partial r^2} - \frac{2}{r} \cdot \frac{\partial P_t(x \le r)}{\partial r}\right] =$$

$$= -\mathrm{G}(t)\left[\frac{\partial^2 P_t(x \le r)}{\partial r^2} - \frac{2}{r} \cdot \frac{\partial P_t(x \le r)}{\partial r}\right],$$

whence we obtain the desired differential equation [48, 73]:

$$\frac{\partial P_t(x \le r)}{\partial t} = -\mathrm{G}(t)\left[\frac{\partial^2 P_t(x \le r)}{\partial r^2} - \frac{2}{r} \cdot \frac{\partial P_t(x \le r)}{\partial r}\right]. \quad (4.2.19)$$

Thus, the probability $P_t(x \le r)$ of a certain share of particles in a spheroidal body being at distances $\le r$ from the center of masses at a certain instant t satisfies equation (4.2.19). Let us note that the equation (4.2.19), in the general form, looks like the Fokker–Planck differential equation [190, 191], while its partial form resembles the differential *equation for a random walk* of a particle describing the Markov process [134]:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial r^2} - 2\zeta \frac{\partial P}{\partial r}, \quad (4.2.20)$$

where

P is a probability of a particle being in the interval $[r, r+dr]$ at the instant t,

D is a diffusion coefficient, and

ζ is a "drift" coefficient.

The equation (4.2.20) describing the Gauss–Markov process results from the Chapman–Kolmogorov equation in its limiting case when "between neighbouring points there occur very quick jumps ($\Delta t \to 0$) for very small distances ($\Delta r \to 0$)" [134], the diffusion coefficients being equal to:

$$D = \lim_{\substack{\Delta r \to 0 \\ \Delta t \to 0}} \frac{\Delta r^2}{2\Delta t}, \tag{4.2.21}$$

and the "drift" coefficient is accordingly:

$$\zeta = -\lim_{\substack{\Delta r \to 0 \\ \Delta t \to 0}} \frac{(q-p)\Delta r}{2\Delta t} = -\lim_{\Delta r \to 0} \frac{(q-p)D}{\Delta r}, \tag{4.2.22}$$

where q is a probability of a particle moving in the positive direction and p is a probability of a particle moving in the negative direction [134]. Let us note that for the Brownian motion of a particle $q = p = 1/2$, that is, the "drift" coefficient $\zeta = 0$. In other words, the "drift" coefficient is due to the difference between probabilities of transition of p and q.

A similar gravitational "drift" coefficient ζ_g is available in equation (4.2.19), describing the slow-flowing gravitational condensation of a centrally symmetric spheroidal body [48, 73]. Like the "drift coefficient" (4.2.22) in the problem of *random walk* (4.2.20) is related to the diffusion coefficient (4.2.21), ζ_g is similarly expressed through GCF (4.1.8) or the gravitational contraction coefficient $\mathrm{G_s}$ in the particular case of unstable mechanical equilibrium (see formula (1.7.9) analogous to (4.2.21)), as follows directly from equation (4.2.19):

$$\zeta_g(t,r) = \frac{G(t)}{r}. \qquad (4.2.23)$$

As follows from the comparison of (4.1.8) with (4.2.21) and (4.2.23) with (4.2.22), coefficients D and ζ are constants, while $G(t)$ and $\zeta_g(t,r)$ are variables. Besides, equations (4.2.19) and (4.2.20) differ in sign in their right-hand parts. This means that despite the outward similarity of differential equations (4.2.19) and (4.2.20), the anti-diffusion process of the initial gravitational condensation is radically different from those of diffusion (though certain conclusions obtained for the diffusion processes might also be applied in the statistical model of gravity). In particular, in [134 p. 248] it is shown that in an equilibrium thermodynamic state "the distribution of fluctuations is a Gauss one." Indeed, according to the statistical theory [190], the Fokker–Planck equation, a special case of which is (4.2.20), under the condition of *infinitely distant boundaries* has a fundamental solution, which is described by normal law. Strictly speaking, it should be noted that a similar situation also occurs in the slow gravitational condensation in the vicinity of unstable mechanical equilibrium (see Sections 2.1 and 2.6, and also [16, 45, 47, 49, 73]).

With provision for the notation of (4.2.23) and of the form of Laplacian for a scalar function written down in spherical coordinates, equation (4.2.19) is presented as follows [48, 73]:

$$\frac{\partial P_t(x \le r)}{\partial t} = -G(t)\nabla^2 P_t(x \le r) + 4\zeta_g(t,r)\cdot\frac{\partial P_t(x \le r)}{\partial r}. \qquad (4.2.24)$$

The equation obtained (4.2.24) describes the dynamics of the probability of finding particles during t at distances $\le r$ to the center of mass.

Let us find now the form of a differential equation satisfied by a gravitational field potential $\varphi_g(r,\alpha)$. To do this we are going to use formula (2.4.26) from which we shall express $\varphi_g(r,\alpha)$ through $\rho_g(r,\alpha)$ and vice versa:

$$\varphi_g(r,\alpha) = -\frac{4\pi\gamma}{\alpha r} \cdot \int_0^r \rho(x,\alpha)dx, \tag{4.2.25}$$

$$\rho(r,\alpha) = -\frac{\alpha}{4\pi\gamma} \cdot \frac{\partial}{\partial r}(r\varphi_g(r,a)). \tag{4.2.26}$$

Substituting (4.2.26) into equation (4.1.6) we obtain:

$$\frac{\partial}{\partial \alpha}\left[-\frac{\alpha}{4\pi\gamma} \cdot \frac{\partial}{\partial r}(r\varphi_g(r,\alpha))\right] =$$

$$= -\frac{1}{2\alpha^2}\nabla^2\left[-\frac{\alpha}{4\pi\gamma} \cdot \frac{\partial}{\partial r}(r\varphi_g(r,\alpha))\right]. \tag{4.2.27}$$

To further simplify this relation, we note that according to the Poisson equation [100]:

$$\nabla^2\varphi_g = 4\pi\gamma\rho(r,\alpha), \tag{4.2.28}$$

so that taking into account (4.2.26) we obtain [73]:

$$\nabla^2\varphi_g = -\alpha\frac{\partial}{\partial r}\left(r\varphi_g(r,\alpha)\right). \tag{4.2.29}$$

Using (4.2.29), expression (4.2.27) can be represented as:

$$\frac{\partial}{\partial \alpha}\left[\nabla^2\varphi_g(r,\alpha)\right] = -\frac{1}{2\alpha^2}\nabla^2\left[\nabla^2\varphi_g(r,\alpha)\right],$$

whence it follows that

$$\frac{\partial\varphi_g(r,\alpha)}{\partial\alpha} = -\frac{1}{2\alpha^2}\nabla^2\varphi_g(r,\alpha). \tag{4.2.30}$$

It is possible to verify the validity (4.2.30) by directly calculating the Laplacian and the derivative of the gravitational potential (4.2.25) with respect to α:

$$\nabla^2\varphi_g(r,\alpha)=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\varphi_g(r,\alpha)}{\partial r}\right)\right]=$$

$$=\frac{\partial^2\varphi_g(r,\alpha)}{\partial r^2}+\frac{2}{r}\cdot\frac{\partial\varphi_g(r,\alpha)}{\partial r}=\sqrt{\frac{2}{\pi}}\cdot\gamma M\alpha^{3/2}e^{-\alpha r^2/2};$$

$$\frac{\partial\varphi_g(r,\alpha)}{\partial\alpha}=-\sqrt{\frac{2}{\pi}}\cdot\frac{\gamma M}{r}\cdot\frac{1}{2\cdot\alpha^{1/2}}\int_0^r e^{-\alpha x^2/2}dx-$$

$$-\sqrt{\frac{2}{\pi}}\cdot\frac{\gamma M\alpha^{1/2}}{r}\int_0^r\left(-x^2/2\right)\cdot e^{-\alpha x^2/2}dx=-\sqrt{\frac{2}{\pi}}\gamma M\cdot\frac{1}{2\alpha^{1/2}}e^{-\alpha r^2/2}.$$

With provision for formula (4.1.8) and equation (4.2.30), let us write the *anti-diffusion equation* characterizing the dynamics of change in the gravitational potential with time:

$$\frac{\partial\varphi_g}{\partial t}=\frac{\partial\varphi_g}{\partial\alpha}\cdot\frac{d\alpha}{dt}=-\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}\cdot\nabla^2\varphi_g=-\mathrm{G}(t)\cdot\nabla^2\varphi_g,$$

whence this relation takes on the form [73]:

$$\nabla^2\varphi_g+\frac{1}{\mathrm{G}(t)}\cdot\frac{\partial\varphi_g}{\partial t}=0. \qquad (4.2.31)$$

The resulting equation determines the potential of the gravitational field outside the center of mass of a centrally symmetric spheroidal body since it does not take into account the mass density function $\rho(r,t)$. On the contrary, the Poisson equation of the type (4.2.28) characterizes the potential of the gravitational field depending on the mass distribution function $\rho(r,t)$ in space. Considering that both the homogeneous equation (4.2.31) and the Poisson equation (4.2.28) are linear, their joint solution will satisfy a non-homogeneous equation of the form [73]:

$$\nabla^2\varphi_g+\frac{1}{\mathrm{G}(t)}\cdot\frac{\partial\varphi_g}{\partial t}=4\pi\gamma\rho. \qquad (4.2.32)$$

Indeed, as we know [192], a solution of a non-homogeneous linear equation of the type (4.2.32) can be represented as the sum of a solution of the homogeneous equation (4.2.31) and a particular integral of the same equation with the right-hand side of the form (4.2.28). The resulting equation is important in problems of initial gravidynamics, as is the well-known D'Alembert equation used in electrodynamics [100], for determining the potential of the electromagnetic field created by moving charges:

$$\nabla^2 \varphi_e - \frac{1}{c^2} \cdot \frac{\partial^2 \varphi_e}{\partial t^2} = -4\pi\rho_e , \tag{4.2.33}$$

where

c is the speed of light,

ρ_e is a charge density, and

φ_e is an electromagnetic potential.

However, the obtained Eq. (4.2.32), in contrast to Eq. (4.2.33), is a diffusion (or rather parabolic) type equation, while Eq. (4.2.33) is a wave equation (of the hyperbolic type).

In deriving formula (4.2.16), relation (4.2.14) has been used, which is a consequence of the condition of the *potentiality* of the strength (acceleration) field $\vec{a}(\vec{r},t)$ of a weakly gravitating spheroidal body. The potential (irrotational) character of the acceleration field results directly from relation (2.4.8a) (or (4.2.7)) describing the gravitational field strength of a centrally symmetric spheroidal body. Indeed, according to (2.4.8a) and the results obtained in Section 2.4, the strength vector of the *quasistatic* (in the case $\alpha \approx \mathrm{const}$) gravitational field of a centrally symmetric spheroidal body (sphere-like gaseous body) is:

$$\vec{a}(\vec{r}) = -\sqrt{\frac{2}{\pi}} \cdot \gamma M \alpha^{3/2} \cdot \frac{\int_0^r x^2 e^{-\frac{\alpha x^2}{2}} dx}{r^2} \cdot \frac{\vec{r}}{r} =$$

$$= -4\pi\gamma\rho_0 \frac{\int_0^r x^2 e^{-\frac{\alpha x^2}{2}} dx}{r^2} \cdot \vec{e}_r \,, \tag{4.2.34}$$

where $\rho_0 = M(\alpha/2\pi)^{3/2}$. Taking into account the condition of spherical symmetry for this spheroidal body and the relation (2.2.4), we can calculate the divergence of the vector function (4.2.34):

$$\operatorname{div}\vec{a}(\vec{r}) = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 a_r\right)\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(-4\pi\gamma\rho_0\int_0^r x^2 e^{-\frac{\alpha x^2}{2}} dx\right)\right] =$$

$$= -\frac{4\pi\gamma\rho_0}{r^2} r^2 e^{-\frac{\alpha r^2}{2}} = -4\pi\gamma\rho_0 e^{-\frac{\alpha r^2}{2}} = -4\pi\gamma\rho(\vec{r}) \,.$$

On the other hand, for a constant in time gravitational field, Poisson equation (4.2.28) is valid which, taking into account that $\vec{a} = -\operatorname{grad}\varphi_g$ [100] as well as $\nabla^2\varphi_g = \operatorname{div}(\operatorname{grad}\varphi_g) = -\operatorname{div}\vec{a}$, we can represent in the form:

$$-\operatorname{div}\vec{a} = 4\pi\gamma\rho \,, \tag{4.2.35}$$

coinciding with the previous relation (see also Eq. (3.6.13) from Chapter 3).

Let us note that in the general case $\alpha = \alpha(t)$, that is, in expressions (4.2.34) and (4.2.35) $\vec{a} = \vec{a}(\vec{r},t)$ and $\rho = \rho(\vec{r},t)$ are functions of time. To take into account the dynamics of changes in the strength of the potential (irrotational) gravitational field of a centrally symmetric spheroidal body, equation (4.2.32) can be represented as:

$$-\operatorname{div}\vec{a} + \frac{1}{\mathrm{G}(t)} \cdot \frac{\partial\varphi_g}{\partial t} = 4\pi\gamma\rho \,. \tag{4.2.36}$$

It is clear that Eq. (4.2.36) generalizes Eq. (4.2.35) in the case of a variable (in time) *irrotational* gravitational field. Taking from both sides of Eq. (4.2.36) the operation of a gradient, we get:

$$\text{grad div}\,\vec{a} - \frac{1}{\text{G}(t)} \cdot \frac{\partial}{\partial t}\text{grad}\ \varphi_g = -4\pi\gamma\,\nabla\rho\,. \tag{4.2.37}$$

Taking into account that the relation (4.2.14) is valid for an irrotational gravitational field and, as consequence, $\vec{a}(\vec{r},t) = -\text{grad}\ \varphi_g(\vec{r},t)$, expression (4.2.37) can be represented as follows:

$$\nabla^2\vec{a} + \frac{1}{\text{G}(t)} \cdot \frac{\partial\vec{a}}{\partial t} = -4\pi\gamma\,\nabla\rho\,. \tag{4.2.38}$$

The non-homogeneous equation (4.2.38) obtained naturally generalizes the homogeneous equation (4.2.17) which determines the strength of an irrotational (vortex-free) gravitational field far from the iso-surface of the mass density bending when $\nabla\rho \to 0$ (see Section 2.2). It also generalizes equation (4.2.35) which characterizes the irrotational static gravitational field created by immovable masses. As well as equation (4.2.32) for gravitational potential, equation (4.2.38) for an irrotational gravitational field strength belongs to the class of second-order differential equations of parabolic type.

4.3. Special cases of the basic equation of slow-flowing initial gravitational condensation and its solution near the state of mechanical equilibrium of a centrally symmetric spheroidal body

To write down a more explicit form of Eq. (4.1.7) it is necessary to determine function $\alpha = \alpha(t)$ such that $\alpha > 0$ at $t > t_0$. With this aim one should recall the sense of parameter α as defined by formula (2.5.7) in Sections 2.5:

$$\alpha = \frac{4\pi}{\gamma^2} \cdot \frac{E_g{}^2}{M^4} = \pi\left(\frac{2E_g}{\gamma M^2}\right)^2, \tag{4.3.1}$$

where

E_g is a potential energy of a centrally symmetric gravitating spheroidal body,

M is a mass of a spheroidal body, and

γ is the Newton's gravitational constant.

According to (2.5.31) there is another formula for determining α:

$$\alpha = \frac{\pi}{\gamma^2} \cdot \frac{\overline{E}^2}{m_0{}^2 M^2} = \pi \left(\frac{\overline{E}}{\gamma m_0 M} \right)^2, \qquad (4.3.2)$$

where $\overline{E}$ is a mean potential energy of interaction of a particle with the centrally symmetric gravitating spheroidal body and m_0 is a mass of a particle.

Relations (4.3.1) and (4.3.2) are obtained in Chapter 2, with α being supposed to be constant. But now that $\alpha = \alpha(t)$ is a positively defined monotonically increasing function of time t, it is necessary to find out which of the physical values involved in (4.3.1) and (4.3.2) depend on t. As already mentioned, the mass $M = \mathrm{const}$, therefore it cannot depend on t. Consequently, in the process of gravitational compression, it is the potential gravitational energy that changes: $E_g = E_g(t)$ in formula (4.3.1) and $\overline{E} = \overline{E}(t)$ in (4.3.2). To further the calculations, we use, for example, formula (4.3.2).

Considering that the gravitational condensation process occurs in an unstable vicinity of mechanical equilibrium at $t = t_0$, let us expand function $\alpha(t)$ into a Taylor series:

$$\alpha(t) = \alpha(t_0) + \frac{d\alpha(t_0)}{dt} \cdot (t - t_0) + \frac{1}{2} \cdot \frac{d^2\alpha(t_0)}{dt^2} \cdot (t - t_0)^2 + \ldots + \qquad (4.3.3)$$

Using (4.3.2) one can easily see, for example, that:

$$\frac{d\alpha(t_0)}{dt} = \frac{\partial\alpha}{\partial\overline{E}} \cdot \frac{d\overline{E}(t_0)}{dt} = \frac{2\pi}{(\gamma m_0 M)^2} \cdot \overline{E}(t_0)\overline{E}'(t_0),$$

whence it follows that relation (4.3.3) can be written down in the form [47, 73]:

$$\alpha(t) = \frac{\pi}{(\gamma m_0 M)^2}\Big\{\overline{E}^2(t_0) + 2\overline{E}(t_0)\overline{E}'(t_0)(t - t_0) + \\ + \Big(\big(\overline{E}'(t_0)\big)^2 + \overline{E}(t_0)\overline{E}''(t_0)\Big)(t - t_0)^2 + \ldots\Big\}. \quad (4.3.4)$$

Expression (4.3.4) allows several important particular cases; let us consider some of them [47]:

a) The case of infinitely small removal (in time) from an unstable mechanical equilibrium of a centrally symmetric gravitating spheroidal body (quasi-equilibrium condensation).

Limiting ourselves in (4.3.4) to values not higher than those of the first order of smallness with respect to $t - t_0$ we obtain:

$$\alpha(t) = \frac{\pi}{(\gamma m_0 M)^2} \cdot \left\{\overline{E}^2(t_0) + 2\overline{E}(t_0) \cdot \overline{E}'(t_0) \cdot (t - t_0)\right\} = \\ = \frac{\pi}{(\gamma m_0 M)^2}\left\{\overline{E}^2(t_0) - 2\overline{E}(t_0) \cdot \overline{E}'(t_0)t_0 + 2\overline{E}(t_0) \cdot \overline{E}'(t_0)t\right\} \approx \\ \approx \frac{2\pi\overline{E}(t_0)\overline{E}'(t_0)}{(\gamma m_0 M)^2}t. \quad (4.3.5a)$$

In (4.3.5a) it has been taken into account that for a small-in-value gravitational potential energy $\overline{E}^2(t_0) - 2\overline{E}(t_0)\overline{E}'(t_0)t_0 \approx 0$, that is, the value $\alpha_0 = \frac{\pi}{(\gamma m_0 M)^2}\left\{\overline{E}^2(t_0) - 2\overline{E}(t_0)\overline{E}'(t_0)t_0\right\}$ can be neglected. With provision for (4.3.5a) the expression for $\mathrm{G}(t)$ (in round brackets in Eq. (4.1.7)) can be written as follows:

$$\mathrm{G}(t) = \frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt} = \frac{(\gamma m_0 M)^4}{2\left(2\pi \overline{E}(t_0)\overline{E}'(t_0)\right)^2 t^2} \cdot \frac{2\pi \overline{E}(t_0)\overline{E}'(t_0)}{(\gamma m_0 M)^2} =$$

$$= \frac{(\gamma m_0 M)^2}{4\pi \overline{E}(t_0)\overline{E}'(t_0)} \cdot \frac{1}{t^2}. \tag{4.3.6a}$$

Substituting (4.3.6a) in Eq. (4.1.7) and introducing a coefficient of gravitational compression [47, 73]:

$$G = \frac{(\gamma m_0 M)^2}{4\pi \overline{E}(t_0)\overline{E}'(t_0)}, \tag{4.3.7a}$$

we obtain the following equation [47, 73]:

$$t^2 \frac{\partial \rho}{\partial t} = -G\nabla^2 \rho. \tag{4.3.8a}$$

The given differential equation describes the process of quasi-equilibrium slow-flowing gravitational condensation in time so that, from now on, the anti-diffusion coefficient G is named *gravitational compression factor* [47, 73]. Gravitational condensation along with diffusion and thermoconductivity are the examples of evolutionary processes which "cannot be described nontrivially without introducing directly to the direction of time" [134, 135]. Indeed, in reversing time in the equations describing diffusion and heat conduction, or in the gravitational condensation obtained Eq. (4.3.8a), one comes to quite different laws. Thus, because of the unidirectional nature in time of the diffusion and heat conduction processes, and of the slow-flowing gravitation one as well, time is also unidirectional (in contrast with Newton's classical mechanics, Maxwell's electrodynamics, and Einstein's relativity theory [100, 158]).

Thus, the centrally symmetric mass density of a spheroidal body in the quasi-equilibrium state satisfies the differential *equation of a slow-flowing gravitational condensation* (4.3.8a). In connection with some mathematical analogy in the processes of gravitational condensation and diffusion (heat conduction), one can assume that gravitational interaction among bodies is due to the necessity of equalizing the distribution of mass densities over space according to the law (4.1.2) which, with provision for the gravitational compression factor G introduced (4.3.7a) and the relation obtained from (4.3.5a) and (4.3.7a):

$$\alpha = \frac{t}{2G}, \tag{4.3.9a}$$

takes on the form [47, 73]:

$$\rho(r,t) = \frac{M}{8(\pi G)^{3/2}} t^{3/2} e^{-r^2 t/4G}. \tag{4.3.10a}$$

The obtained formula (4.3.9a) (as well as (4.3.10a)) confirms *linear dependence* (4.1.26) of the gravitational condensation parameter α on the time t, derived in Section 4.1 under the condition of the state of unstable mechanical equilibrium (see also (4.1.28)).

The form of relations (4.3.8a)–(4.3.10a) remains complete if one applies (4.3.1) expressing the dependence of α on the potential energy E_g of a centrally symmetric gravitating spheroidal body. In this case (as with (4.3.4)) we consider that:

$$\alpha(t) = \frac{4\pi}{\gamma^2 M^4}\left\{E_g{}^2(t_0) + 2E_g(t_0)\cdot E_g'(t_0)\cdot(t-t_0) + \right.$$
$$\left. + \left(\left(E_g'(t_0)\right)^2 + E_g(t_0)\cdot E_g''(t_0)\right)(t-t_0)^2 + \ldots\right\},$$

whence, taking into account the terms of not higher than the *first order of smallness* with respect to $t - t_0$, we obtain the linear law of increasing the parameter of gravitational condensation α with time:

$$\alpha = \frac{8\pi}{\gamma^2} \cdot \frac{E_g(t_0) E_g'(t_0)}{M^4} \cdot t. \tag{4.3.11a}$$

After which, using (4.3.11a), we calculate:

$$\mathrm{G}(t)=\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}=\frac{\gamma^2 M^4}{16\pi E_g(t_0)E_g'(t_0)t^2}=\frac{G}{t^2},$$

where:

$$G=\frac{(\gamma M^2/2)^2}{4\pi E_g(t_0)E_g'(t_0)}. \qquad (4.3.12a)$$

b) The case of initial gravitational condensation (the formation of a centrally symmetric spheroidal body [47, 73]*).*

Let us assume that the process of forming a centrally symmetric spheroidal body starts, arbitrarily, at the instance $t_0=0$, and that, at $t_0=0$, the gravitational interaction of particles is absent, that is, $\overline{E}(0)=0$ is an unstable zero equilibrium state. Then, as follows from (4.3.4), we obtain:

$$\alpha(t)=\frac{\pi\left(\overline{E}'(0)\right)^2}{(\gamma m_0 M)^2}\cdot t^2. \qquad (4.3.5b)$$

With provision for (4.3.5b), the expression for $\mathrm{G}(t)$ in round brackets in Eq. (4.1.7) takes the form:

$$\mathrm{G}(t)=\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}=\frac{(\gamma m_0 M)^4}{2\pi^2\left(\overline{E}'(0)\right)^4 t^4}\cdot\frac{2\pi\left(\overline{E}'(0)\right)^2 t}{(\gamma m_0 M)^2}=$$

$$=\frac{1}{\pi}\cdot\left(\frac{\gamma m_0 M}{\overline{E}'(0)}\right)^2\cdot\frac{1}{t^3}. \qquad (4.3.6b)$$

Substituting (4.3.6b) (67b) into Eq. (4.1.7) and introducing the following *gravitational compression factor* [47, 73]:

$$G=\frac{1}{\pi}\cdot\left(\frac{\gamma m_0 M}{\overline{E}'(0)}\right)^2, \qquad (4.3.7b)$$

we obtain the differential equation for the *initial gravitational condensation* [47, 73]:

$$t^3 \frac{\partial \rho}{\partial t} = -G\nabla^2 \rho \,. \tag{4.3.8b}$$

In contrast with Eq. (4.3.8a), the given Eq. (4.3.8b) is reversible in time, that is, under reflection $t \to -t$, it retains the form. In this sense, a similar situation occurs in describing gravitation in Einstein's theory of GR [81, 100]. It should also be noted that a situation in which one equation (4.3.8a) describes only the direct gravitational condensation process, and the other one (4.3.8b) describes both the direct and the reverse gravitational condensation processes does occur in the formalism of other physical phenomena. Indeed, in [134] it is pointed out that "in inhomogeneous, or anisotropic media, in membranes or liquid phases, for instance, in the presence of actually nonlinear chemical reactions, diffusion flows may have the reverse direction. Then the substance is transferred against the concentration gradient." This means that the corresponding differential equation describing the nonlinear diffusion process allows as a solution both the direct and the reverse diffusion flow. A similar situation in considering the initial gravitational condensation Eq. (4.3.8b) is likely to be accounted for by an equiprobable possibility of both the gravitational condensation process (compression) and the gravitational rarefaction process (expansion) arising in a particular point of space at the initial stage of gravitational interaction of particles [47, 73]. Most probably, the predominance of one process over the other is determined by the presence of the substance density fluctuations in a particular point of space at a particular instant.

From the comparison of (4.3.5b) with (4.3.7b), it is not difficult to see that:

$$\alpha = \frac{t^2}{G} \,. \tag{4.3.9b}$$

The substitution of (4.3.9b) into relation (4.1.2) results in the following expression for mass density [47, 73]:

$$\rho(r,t) = \frac{M}{(2\pi G)^{3/2}} t^3 \cdot e^{-r^2 t^2/2G} . \qquad (4.3.10b)$$

Likewise, if one uses (4.3.1) expressing the dependence of α on the potential energy E_g of the centrally symmetric gravitating spheroidal body one again obtains the relations (4.3.8b)–(4.3.10b). But parameters α and G involved in them will be somewhat different. As with (4.3.5b), substituting (4.3.1) into (4.3.3), one can easily see that:

$$\alpha = \frac{4\pi}{\gamma^2} \cdot \frac{\left(E_g'(0)\right)^2 t^2}{M^4} = \pi \cdot \left(\frac{2E_g'(0)}{\gamma M^2}\right)^2 \cdot t^2 . \qquad (4.3.11b)$$

whence:

$$\mathrm{G}(t) = \frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt} = \frac{\gamma^2 M^4}{4\pi\left(E_g'(0)\right)^2 \cdot t^3} = \frac{G}{t^3} ,$$

also:

$$G = \frac{1}{\pi} \cdot \left(\frac{\gamma M^2}{2E_g'(0)}\right)^2 . \qquad (4.3.12b)$$

As is evident from (4.3.12b), the gravitational compression factor is directly proportional to the fourth power of the spheroidal body mass and inversely proportional to the square of the gravitational potential energy rate-of-change of this body. Thus, in deriving equations (4.3.8a, b) of a slow-flowing quasi-equilibrium and initial gravitational condensation, the expressions for the variable α described by relations (4.3.9a, b) have been obtained.

Let us note that if α is determined by the expression:

$$\alpha = \frac{1}{2Dt} ,$$

where D is a certain positive constant whose sense will be cleared up below and t is time, then the differential equation (4.1.6) with respect to α is transformed into the linear one with respect to t. Indeed, according to (4.1.7) it is not difficult to see that:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2\left(1/4D^2t^2\right)} \cdot \left(-\frac{1}{2Dt^2}\right)\nabla^2 \rho \,,$$

whence:

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \,. \tag{4.3.13}$$

The equation obtained coincides completely with that of *diffusion*, that is, the constant D has the sense of the diffusion factor. Mass density ρ satisfying (4.3.13) has the form:

$$\rho\left(r,t\right) = \frac{M}{8\left(\pi D t\right)^{3/2}} \cdot e^{-r^2/4Dt} \,,$$

which is in complete agreement with what is stated in [94]. The present relation for the "diffusion" mass density, to the accuracy of the change $t \to 1/t$, is inverse to the relation (4.3.10a) describing particle gravitational interaction in the proximity of equilibrium state. Consequently, *in the linear approximation* of the gravitational condensation function $\alpha(t)$ with respect to t, the case *a)* of slow-flowing quasi-equilibrium gravitational condensation is qualitatively like an anti-diffusion process (see Section 1.7 as well as [16, 65]).

The behavior of the initial gravitational condensation (case *b)*) is different. Here the function $\alpha(t)$ is, already in the initial approximation, a square function of time in accordance with (4.3.5b), and the mass density under such gravitational condensation is now satisfying the relation (4.3.10b). It is clear that the diffusion flow cannot now compensate for the initial gravitational one. Let us note from (4.3.10b) that, as a result of evolution at separate fixed points of a centrally symmetric spheroidal body, condensation may be followed by rarefaction $\rho(r,t) \to 0$ at $t \to \infty$, that is, there may occur cavities (or voids, as indicated in Section 1.1) inside it. This fact is completely confirmed by up-to-date astrophysical observations of cosmic space and modern theories according to which forming cosmogonical bodies (in particular, molecular clouds) are porous (see Section 1.1), that is, there are numerous voids in them [10, 83].

The obtained differential equations of quasi-equilibrium slow-flowing and initial gravitational condensation can be combined in one equation of the following form:

$$t^n \frac{\partial \rho}{\partial t} = -G\nabla^2 \rho \,, \tag{4.3.14}$$

Where $n = 2, 3$. But in Section 4.5, we will consider predominantly Eq. (4.3.14) with $n = 3$, that is, the process of the initial gravitational condensation of a centrally symmetric spheroidal body.

4.4. The gravity–thermodynamic relationship for a centrally symmetric gravitating spheroidal body

To derive some thermodynamic expressions applicable to a centrally symmetric spheroidal body, we shall use the Gibbs relation in the case of thermodynamic (heat) equilibrium [110]:

$$TdS = dU + pdV - \mu dm \,, \tag{4.4.1}$$

where

T is a temperature,
S is an entropy,
U is an internal energy,
p is a pressure,
V is a volume,
μ is a chemical potential, and

m is a mass.

Following equilibrium thermodynamics, we have in a heat equilibrium state [134]:

$$\mu = \mu^*(T, p) + k_B T \cdot \ln\rho, \tag{4.4.2}$$

where

k_B is the Boltzmann constant,

ρ is a mass density, and

μ^* is a mean value of μ.

Formula (4.4.2) was derived in thermodynamics for the case of a highly rarefied system, that is, for a mixture under conditions of small concentration gradients [134].

Let us consider an entropy change dS over a time interval dt. As shown by I. Prigogine [108, 134], it can be divided into the sum of two contributions:

$$dS = d_e S + d_i S, \tag{4.4.3}$$

where $d_e S$ is an external entropy flow due to the energy and substance exchange with the environment and $d_i S$ is an entropy production inside the system due to *irreversible processes* such as gravitational condensation, diffusion, thermal conductivity, or chemical reactions.

According to the second law of thermodynamics:

$$d_i S \geq 0. \tag{4.4.4}$$

Moreover, the equality sign in (4.4.4) corresponds to *thermodynamic equilibrium* in the system. It is clear that for an isolated system $d_e S = 0$, that is, $dS = d_i S \geq 0$. On the contrary, for an open system expression (4.4.3) for entropy change contains the term $d_e S$ corresponding to exchange, with $d_e S > 0$ or $d_e S < 0$ [134].

As before, we assume that forming cosmogonical bodies (in the absence of rotation*) are described by the model of a sphere-like gaseous body (or centrally symmetric spheroidal body) in a state of mechanical equilibrium (or relative mechanical equilibrium in the case of rotation). As indicated in [94, 106], the mechanical equilibrium of the system does not at all imply relaxation toward thermodynamic equilibrium.

To derive a gravithermodynamic relation for a centrally symmetric spheroidal body, let us outline *irreversible processes* that may, in principle, take place in it. As already mentioned, these processes are initial gravitational condensation, diffusion, heat conduction, also those based on reactions (chemical, nuclear, etc.). We assume that concentration gradients in a centrally symmetric spheroidal body are not too large, that is, the process of gravitational condensation goes on slowly in time, which, in turn, means a state close to equilibrium (mechanical and thermal).

We note that the process of slow-flowing gravitational condensation is inverse concerning the diffusion process (see Section 4.3, (4.3.10a) and (4.3.13)), characterized in (4.4.1) by chemical potential $\mu > 0$. Consequently, the slow-flowing gravitational condensation can be similarly described through *gravitational thermodynamic potential* $g < 0$ [47, 49]. In connection with this we add to formula (4.4.1) a term containing the gravitational thermodynamic potential [47, 49, 73]:

$$TdS = dU + pdV - \mu dm - gdm . \quad (4.4.5)$$

Relation (4.4.5) describes the locally equilibrium thermodynamic state of a gravitating spheroidal body being

* In the presence of rotation, the sphere-like gaseous body (centrally symmetric spheroidal body) is transformed into an axially symmetric spheroidal one (or simply, a spheroidal body).

simultaneously in a state of mechanical (or relative mechanical) equilibrium. We can rewrite relation (4.4.5) relative to the internal energy:

$$dU = TdS - pdV + \mu dm + gdm\,. \tag{4.4.6}$$

Let us take advantage of the term gdm having the dimension of energy (due to gravitational condensation), so as to find the relation describing g. As before, just as in Chapter 2 and [16, 45, 65], supposing that the spheroidal body has a temperature close to absolute zero $\left(T \to 0\,\mathrm{K}\right)$, we can simplify expression (4.4.6) by canceling the terms associated with thermal conduction and diffusion (the diffusion factor in the linear approximation is, as we know [136], proportional to T). With provision for these simplifications, relation (4.4.6) is written down in the form:

$$dU_g = -pdV + gdm\,. \tag{4.4.7}$$

As we know from (4.4.7), the infinitesimal change of the internal gravitational energy of a centrally symmetric spheroidal body consists of two terms, the first of which $dU_g^p = -pdV$ is associated with the *substance pressure* (see relation (2.9.3)), and the second $dU_g^m = gdm$ is related to the *mass transfer* due to gravitational condensation. Assuming, as above, that gravitational condensation goes on slowly ($\mathrm{G}(t) = \mathrm{const}$ according to (4.1.21)), we consider the gravitating spheroidal body in *mechanical equilibrium*. In this case, integrating (4.4.7) with respect to volume, we obtain:

$$U_g = U_g{}^p + U_g{}^m = -\int_V pdV + \int_V g\rho dV\,. \tag{4.4.8}$$

The first term (4.4.8) has been calculated in Section 2.9: according to relation (2.9.18) it is determined by:

$$U_g{}^p = -\frac{1}{6}\gamma M^2 \sqrt{\frac{\alpha}{\pi}}\,. \tag{4.4.9}$$

The second term (4.4.8) should be calculated using the spherical system of coordinates:

$$U_g{}^m = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} g(r)\,\rho_0 e^{-\alpha r^2/2} r^2 \sin\theta d\theta d\varepsilon dr =$$

$$= 4\pi\rho_0 \int_0^{\infty} r^2 \cdot g(r) \cdot e^{-\alpha r^2/2} dr =$$

$$= 4\pi M \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \int_0^{\infty} g(r) \cdot r^2 \cdot e^{-\alpha r^2/2} dr\,, \qquad (4.4.10)$$

where $g(r)$ is an unknown function formalizing the gravitational thermodynamic potential. It is clear that under the above-mentioned conditions of the absolutely low temperature and mechanical equilibrium, the potential energy of the centrally symmetric gravitating spheroidal body is consumed in overcoming the substantial pressure and in the mass transfer due to gravitational condensation. Equilibrium in a continuous-in-time process of gravitational condensation sets in when the gravitating spheroidal body's potential energy becomes equal to its internal one:

$$E_g = U_g\,. \qquad (4.4.11)$$

On the one hand, the centrally symmetric spheroidal body's internal energy, in absolute value, is the greater the more substance gets through the fixed spherical surface surrounding its center. On the other hand, it also increases with the increase of pressure inside the volume confined by this surface.

Thus, under the equilibrium condition (4.4.11) with regard to (4.4.8) one can see that:

$$U_g{}^m = E_g - U_g{}^p\,, \qquad (4.4.12)$$

and with provision for relations (2.9.21), (4.4.10) and (4.4.12), we obtain that:

$$-\frac{\gamma M^2}{3}\sqrt{\frac{\alpha}{\pi}}=\sqrt{\frac{2}{\pi}}\cdot M\alpha^{3/2}\int_0^\infty g(r)\cdot r^2 e^{-\alpha r^2/2}dr\ ,$$

whence:

$$\int_0^\infty g(r)\cdot r^2\cdot e^{-\alpha r^2/2}dr=-\frac{\gamma M}{3\sqrt{2}\alpha}\,. \tag{4.4.13}$$

Relation (4.4.13) is an integral (with respect to $g(r)$) equation having an infinite number of solutions. We shall seek the solution of Eq. (4.4.13) in the form of the power function:

$$g(r)=C\cdot r^n\ , \tag{4.4.14}$$

where n is an integer and C is a constant. With provision for (4.4.14), equation (4.4.13) takes the form:

$$\int_0^\infty r^{n+2}\cdot e^{-\alpha r^2/2}dr=-\frac{\gamma M}{3\sqrt{2}C}\cdot\frac{1}{\alpha}\ , \tag{4.4.15}$$

with $n=...,-2,-1,\ 0,\ 1,\ 2,...$. Taking advantage of relations (2.4.22), (2.9.13), (2.9.15) and (4.1.18) and using the integration rule by parts, one can calculate the integral in the left-hand part of Eq. (4.4.15) for some odd and even n:

$$\int_0^\infty e^{-\alpha r^2/2}dr=\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\alpha^{1/2}},\qquad \int_0^\infty re^{-\alpha r^2/2}dr=\frac{1}{\alpha};$$

$$\int_0^\infty r^2e^{-\alpha r^2/2}dr=\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\alpha^{3/2}},\quad \int_0^\infty r^3e^{-\alpha r^2/2}dr=\frac{2}{\alpha^2};$$

$$\int_0^\infty r^4e^{-\alpha r^2/2}dr=\sqrt{\frac{\pi}{2}}\cdot\frac{3}{\alpha^{5/2}},\int_0^\infty r^5e^{-\alpha r^2/2}dr=\frac{8}{\alpha^3}; \tag{4.4.16}$$

$$\int_0^\infty r^6e^{-\alpha r^2/2}dr=\sqrt{\frac{\pi}{2}}\cdot\frac{15}{\alpha^{7/2}},\quad \int_0^\infty r^7e^{-\alpha r^2/2}dr=\frac{48}{\alpha^4};$$

Substituting the integral values (4.4.16) into (4.4.15) we obtain the form of function (4.4.14) for different values of n:

$$g(r) = -\frac{\gamma M}{3\sqrt{\pi\alpha}} \cdot \frac{1}{r^2}; \qquad g(r) = -\frac{\gamma M}{3\sqrt{2}} \cdot \frac{1}{r};$$

$$g(r) = -\frac{\gamma M}{3} \cdot \sqrt{\frac{\alpha}{\pi}}; \qquad g(r) = -\frac{\gamma M}{6\sqrt{2}} \cdot \alpha r;$$

$$g(r) = -\frac{\gamma M}{9\sqrt{\pi}} \cdot \alpha^{3/2} r^2; \; g(r) = -\frac{\gamma M}{24\sqrt{2}} \cdot \alpha^2 r^3; \qquad (4.4.17)$$

$$g(r) = -\frac{\gamma M}{45\sqrt{\pi}} \cdot \alpha^{5/2} r^4; \; g(r) = -\frac{\gamma M}{144\sqrt{2}} \cdot \alpha^3 r^5;$$

Analyzing relations (4.4.17) one sees that the solutions $g(r)$ obtained for positive $n = 1, 2, 3, 4, 5, \ldots$, do not fit since the gravitational thermodynamic potential cannot increase as a power function of r which is the distance to the spheroidal body center of masses. It is also evident that the thermodynamic gravitational potential cannot be constant (the case $n = 0$) because at infinity it must be equal to zero. To clear up which solutions with negative values of n fit as expressions for the gravitational thermodynamic potential, we shall turn to the relation (2.4.27) formalizing the gravitational field potential of a centrally symmetric spheroidal body [16, 45, 46]:

$$\varphi_g(r) = -\frac{\gamma M}{r} \cdot \operatorname{erf}\left(r\sqrt{\alpha/2}\right), \qquad (4.4.18)$$

where $\operatorname{erf}\left(r\sqrt{\alpha/2}\right) = \frac{2}{\sqrt{\pi}} \cdot \int_0^{r\sqrt{\alpha/2}} e^{-s^2} ds$ is the error function. It follows from the comparison of (4.4.18) with (4.4.17) that the solution in the case of $n = -2$ is canceled, as well as all others obtained for $n < -2$. Thus, the only expression possible for the gravitational thermodynamic potential of a centrally symmetric spheroidal body is the following [47, 49, 73]:

$$g(r) = -\frac{1}{3\sqrt{2}} \cdot \frac{\gamma M}{r}. \qquad (4.4.19)$$

As follows from (4.4.19), the gravitational thermodynamic potential is a decreasing, with the increase of r, function, a hyperbola, with $g(\infty) = 0$. Near the center of masses (with $r \to 0$) the value of the gravitational thermodynamic potential becomes infinitely large (see Fig. 4.7).

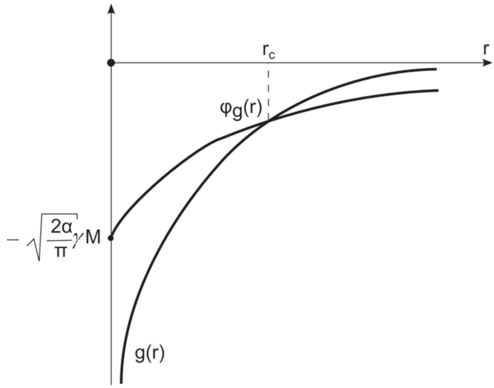


Figure 4.7. The graphic dependences of the gravitational potential φ_g and the gravitational thermodynamic potential g on the distance r

According to (2.4.28)–(2.4.29) for large values r expression (4.4.18) turns into the formula of Newtonian gravitational potential produced by one particle (or a spherical body) of mass M:

$$\varphi_g(r) = -\frac{\gamma M}{r}, \qquad (4.4.20)$$

and for small values r (4.4.18) becomes the formula of the interior gravitational potential in the center of a sphere of mass M:

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha} = -\sqrt{\frac{2}{\pi}}\gamma M\alpha^{1/2} \qquad (4.4.21)$$

(see Section 2.4). The diagram for $\varphi_g(r)$ is presented in Fig. 4.7 which shows that the gravitational thermodynamic potential becomes equal to the gravitational field potential of a centrally symmetric spheroidal body under the condition:

$$\mathrm{erf}\left(r\sqrt{\frac{\alpha}{2}}\right) = \frac{1}{3\sqrt{2}} = 0.2357\ldots.$$

Using the error function table [128], one can easily see that $\mathrm{erf}(0.212) \approx 0.2357$, whence:

$$r_c \approx 0.212 \cdot \sqrt{\frac{2}{\alpha}} = 0.212 \cdot r_{pr}, \qquad (4.4.22)$$

where r_{pr} is the most probable distance [45, 46].

With the decrease of r (the distance to the center of masses, beginning with r_c) the absolute value $g(r)$ becomes much larger than $\varphi_g(r)$. This means that a more intensive anti-diffusion mass transfer of substance begins (when $r < r_c$) leading to a process of condensation of a centrally symmetric spheroidal body (if $r > r_c$ then the process of condensation of the spheroidal body is less significant because a gravity-field process prevails).

Comparing (4.4.19) with (4.4.18), it is not difficult to express the gravitational field potential in terms of the gravitational thermodynamic one [47, 73]:

$$\varphi_g(r) = 3\sqrt{2}g(r)\,\mathrm{erf}\left(r\sqrt{\alpha/2}\right). \qquad (4.4.23)$$

Revealing the sense of parameter α through, say, formula (4.3.9b), one can write down relation (4.4.23) as follows:

$$\varphi_g(r) = 3\sqrt{2}g(r) \cdot \mathrm{erf}\left(\frac{rt}{\sqrt{2G}}\right), \qquad (4.4.24)$$

where G is a gravitational compression factor (4.3.7b).

4.5. The mass density and internal energy flows for slow-flowing and initial gravitational condensation of a centrally symmetric spheroidal body

We are going to use the general equation (4.3.14) of slow-flowing and initial gravitational condensation, for this we shall rewrite it as follows:

$$\frac{\partial \rho}{\partial t} = -\frac{G}{t^n} \cdot \nabla^2 \rho , \tag{4.5.1}$$

where ∇ is the Hamilton differential operator and $n = 2, 3$. Since mass density ρ is a scalar value, then $\nabla \rho = \mathrm{grad}\rho$. According to (4.3.7a) and (4.3.7b), the gravitational compression factor G does not depend on the spatial variable r. Eq. (4.5.1) can, therefore, be rewritten as follows:

$$\frac{\partial \rho}{\partial t} = -\nabla\left(\frac{G}{t^n} \cdot \nabla \rho\right) = -\mathrm{div}\left(\frac{G}{t^n} \cdot \mathrm{grad}\rho\right), \tag{4.5.2}$$

whence:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\frac{G}{t^n} \cdot \mathrm{grad}\rho\right) = 0 . \tag{4.5.3}$$

Relation (4.5.3) reminds us completely that in the continuity equation [94] expressing the law of conservation of mass in a nonrelativistic system:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{j} = 0 ,$$

where $\vec{j}$ is a mass flow density of the continuous medium. In this connection, the value in round brackets in Eq. (4.5.3) has the sense of *anti-diffusion flow density* $\vec{j}$ arising at the gravitational condensation of a centrally symmetric spheroidal body [47, 73]:

$$\vec{j} = \frac{G}{t^n} \cdot \mathrm{grad}\rho . \qquad (4.5.4)$$

Moreover, relation (4.5.4) with $n = 2$ describes the flow density for the quasi-equilibrium slow-flowing gravitational condensation and that, with $n = 3$, it corresponds to the flow density for the initial gravitational condensation.

Since ρ is a function of the spatial variable r, then in the spherical system of coordinates $\mathrm{grad}\rho = \frac{d\rho}{dr} \cdot \vec{e}_r = \frac{d\rho}{dr} \cdot \frac{\vec{r}}{r}$. Taking into account the fact that according to (4.1.2) the mass density ρ is an exponentially decreasing function, its derivative $\frac{d\rho}{dr} < 0$. Consequently, the direction of the anti-diffusion flow density vector $\vec{j}$ is directly opposite to the basis vector $\vec{e}_r$, that is, the vector $\vec{j}$ is directed to a centrally symmetric spheroidal body center.

Let us note that relations (4.3.14) and (4.5.2)–(4.5.4) have been obtained under the conditions of *small concentration gradients* and a sufficiently *slow gravitational condensation*, which allowed us to consider the coefficients α and G to be independent of r. It is clear that, at large concentration gradients, GCF G is a function of both t and r, that is, equation (4.5.3) is transformed into the following:

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}\left(\mathrm{G}(t, r) \cdot \mathrm{grad}\rho\right). \qquad (4.5.5)$$

In this connection, similarly to (4.5.4), a *diffusion flow* of a substance A is introduced in the same way as in the book by G. Nicolis and I. Prigogine [134 p. 230]:

$$\vec{j}_{\mathrm{A}} = -D_{\mathrm{A}} \frac{\partial \rho_{\mathrm{A}}}{\partial \vec{r}}, \qquad (4.5.6)$$

where ρ_{A} is a mass density of the substance A which, in the framework of the trimolecular model, changes in space and in time (inside a system with semipermeable walls) according to the equation [134 p.141]:

$$\frac{\partial \rho_{\mathrm{A}}}{\partial t} = -\rho_{\mathrm{A}} + D_{\mathrm{A}} \frac{\partial^2 \rho_{\mathrm{A}}}{\partial r^2} \quad (0 \le r \le l), \qquad (4.5.7)$$

where D_{A} is a diffusion factor of the substance A and the boundary conditions are: $\rho_{\mathrm{A}}(0) = \rho_{\mathrm{A}}(l) = \overline{\mathrm{A}}$. As noted in [108, 134], at large density gradients, the conductive or diffusion flow (4.5.6) becomes nonlinear, as a result of which the diffusion exchange processes proceed fairly quickly; in other words, D_{A} becomes a function of both r and t, that is, $D_{\mathrm{A}} = D_{\mathrm{A}}(r,t)$. A similar situation is observed in the case of anti-diffusion gravitational condensation in accordance with equation (4.5.5) (details are presented in Chapter 5).

Thus, according to the definition of the flow density [94], the quantity of mass transported by it is equal to:

$$dm = \vec{j}\, d\vec{S}\, dt, \qquad (4.5.8)$$

where $d\vec{S} = \vec{n} dS$ is an oriented elementary site ($\vec{n}$ is an external normal to the elementary site dS) and dt is an infinitesimal time increment. Revealing the sense of the scalar product of vectors in (4.5.8) we obtain:

$$dm = j dS \cos\theta' dt, \qquad (4.5.9)$$

where θ' is an angle between the external normal to the site and the flow density vector. Since the flow enters inside the body, the angle $\pi/2 \le \theta' \le \pi$. We are going to calculate dm in accordance with (4.5.9). Let us outline, arbitrarily, two concentric spheres of radii r and $r + dr$ inside the spheroidal body confining the volumes V and $V + dV$, respectively (Fig. 4.8a).

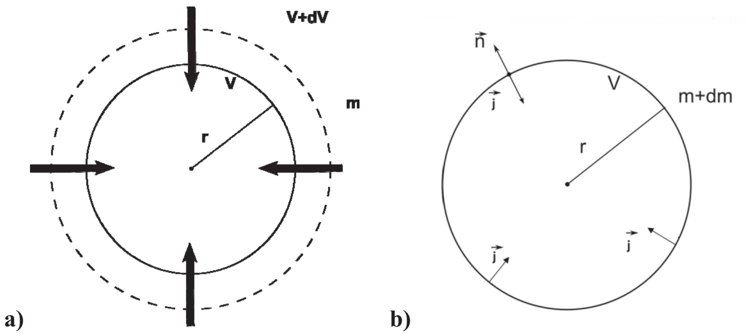


Figure 4.8. The scheme of gravitational compaction of a centrally symmetric spheroidal body:
a) by reducing the allocated volume $V + dV \to V$; b) due to the influx of mass into a fixed volume region V

The process of the slow-flowing initial gravitational condensation of a centrally symmetric spheroidal body involves decreasing the concentric sphere volumes $V + dV \to V$, with the number of particles in them remaining the same (see Fig.4.8a). It is equivalent to the process of the mass dm "inflow" into the sphere of volume V at the expense of the anti-diffusion flow $\vec{j}$ in accordance with (4.5.4) (see Fig. 4.8b).

Taking into account that the radius r of a sphere of the volume V is fixed and that a sphere of the volume $V + dV$ is situated at an infinitely small distance from the first sphere, one can see that the angle between the external normal $\vec{n}$ to the spheres mentioned and the flow direction $\vec{j}$ persists and is equal to $\theta' = \pi$. The quantity of mass dm transported by the flow (4.5.9) through the surface of the sphere with radius r and area $S = 4\pi r^2$ is then equal to:

$$dm = -j 4\pi r^2 dt \, . \qquad (4.5.10)$$

With provision for (4.5.4) relation (4.5.10) is written down in the form [47, 49, 73]:

$$dm = -\frac{4\pi G}{t^n} \cdot \frac{\partial \rho}{\partial r} r^2 dt , \tag{4.5.11}$$

where $\rho = \rho(r,t)$. Taking into consideration that $dV = r^2 \sin\theta d\theta d\varepsilon dr$ is the elementary volume in the spherical system of coordinates, one can calculate the mass of the sphere of radius r before the beginning of gravitational condensation, that is, before the flow $\vec{j}$ in-coming:

$$m = \int_V \rho(r,t) dV = \int_0^r \int_0^\pi \int_0^{2\pi} \rho(r,t) r^2 \sin\theta d\theta d\varepsilon dr =$$

$$= 4\pi \int_0^r \rho(r,t) r^2 dr . \tag{4.5.12}$$

After the flow $\vec{j}$ in-coming, in time dt, to the sphere of radius r (see Fig. 4.8b), the mass of the latter is changed by the value dm and is equal to:

$$m + dm = \int_V \rho(r, t + dt) dV = 4\pi \int_0^r \rho(r, t + dt) r^2 dr . \tag{4.5.13}$$

Comparing (4.5.11)–(4.5.13) it is not difficult to see that:

$$4\pi \int_0^r \rho(r, t + dt) r^2 dr =$$

$$= 4\pi \int_0^r \rho(r,t) r^2 dr - \frac{4\pi G}{t^n} \cdot \frac{\partial \rho(r,t)}{\partial r} r^2 dt , \tag{4.5.14}$$

whence, after differentiating with respect to r, both parts of Eq. (4.5.14) it is easy to see that:

$$r^2 \left(\rho(r, t + dt) - \rho(r,t) \right) =$$

$$= -\frac{G}{t^n} \cdot \left(r^2 \cdot \frac{\partial^2 \rho(r,t)}{\partial r^2} + 2r \cdot \frac{\partial \rho(r,t)}{\partial r} \right) dt . \tag{4.5.15}$$

Taking into consideration that the Laplacian of ρ, according to (4.1.5), is equal to $\nabla^2\rho = \frac{\partial^2\rho}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial\rho}{\partial r}$, and $\frac{\partial\rho}{\partial t} = \frac{\rho(r,t+dt)-\rho(r,t)}{dt}$, we obtain from (4.5.15) equation (4.3.14) of slow-flowing initial gravitational condensation [47, 73]:

$$\frac{\partial\rho(r,t)}{\partial t} = -\frac{G}{t^n}\cdot\nabla^2\rho(r,t).$$

The present equation is obtained proceeding not from the function $\rho = \rho(r,t)$, but from the model of slow-flowing initial gravitational condensation. Applying this model for the case $n = 3$, that is, for initial gravitational condensation, we shall now calculate the mass transfer, and the internal energy change in the whole centrally symmetric spheroidal body, caused by it in some finite time Δt. With that aim, on the sphere of radius r, we isolate an elementary parallelepiped of volume $dV = r^2 \sin\theta d\theta d\varepsilon dr$, as shown in Fig. 4.9. The upper part of the elementary parallelepiped is situated on the sphere of radius $r + dr$, the lower one is located on the radius r sphere.

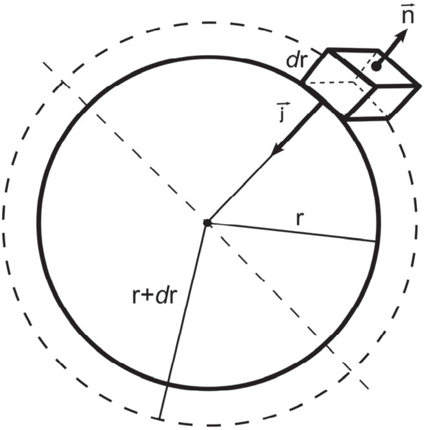


Figure 4.9. Diagram of the passage of the anti-diffusion mass flow $\vec{j}$ through two concentric spheres located at an infinitely small distance from each other

The respective areas of these facets, elementary sites on the spheres with radii $r + dr$ and r, are equal to:

$$dS_1 = (r + dr)^2 \cdot \sin\theta d\theta d\varepsilon \approx \left(r^2 + 2rdr\right) \cdot \sin\theta d\theta d\varepsilon ,$$

$$dS_2 = r^2 \cdot \sin\theta d\theta d\varepsilon . \qquad (4.5.16)$$

According to (4.5.9), in time dt, through the site dS_1, an elementary mass arrives inside the elementary parallelepiped:

$$dm_1(r, \theta, \varepsilon, t) = j\cos\theta' dS_1 dt =$$

$$= j\cos\theta' \left(r^2 + 2rdr\right) \sin\theta d\theta d\varepsilon dt , \qquad (4.5.17)$$

and through dS_2 an elementary mass flows out:

$$dm_2(r, \theta, \varepsilon, t) = j\cos\theta' r^2 \sin\theta d\theta d\varepsilon dt . \qquad (4.5.18)$$

The elementary mass that entered into the elementary parallelepiped during dt is equal [47, 49] to:

$$dm(r, \theta, \varepsilon, t) = dm_1 - dm_2 = 2 j\cos\theta' \sin\theta \cdot rdrd\theta d\varepsilon dt . \qquad (4.5.19)$$

Let us consider the two upper hemispheres of radii $r + dr$ and r, which confine the elementary volume dV being examined. This means that the interval of angle θ change is equal to $\pi/2$, that of angle ε is 2π. Inside the radius $r + dr$ sphere, let us consider several concentric hemispheres separated from one another by infinitely small distances dr. The mass exchange inside the elementary parallelepipeds confined between the concentric hemispheres is determined by the formula (4.5.19). Because the direction of normal $\vec{n}$ to the site coincides with that of the radius-vector $\vec{r}$ in the spherical system of coordinates, as is seen from Fig. 4.10, the angles θ' and θ are bound by the following relation: $\theta' = \pi - \theta$.

As already mentioned in examining the model in Fig. 4.8, the directions of the external normals to dS_1 and dS_2 practically coincide because of the infinitely small distance between them. However, with moving nearer and nearer to the center of masses of the centrally symmetric spheroidal body, the curvature of the concentric hemispheres increases, which causes incoincidence of the directions of external normals to them, and finally results in the change of θ' from π to $\pi/2$ ($0 \le \theta \le \pi/2$ respectively).

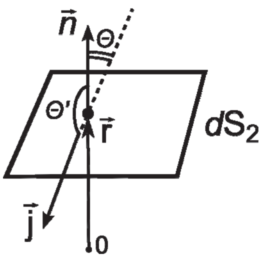


Figure 4.10. The scheme of motion of anti-diffusion mass flow $\vec{j}$ through an elementary site located at a distance r from the center of a centrally symmetric spheroidal body

According to the foregoing, from (4.5.19) we obtain:

$$dm(r,t) = 2jrdrdt\int_0^{\pi/2}(-\cos\theta)\cdot\sin\theta d\theta\int_0^{2\pi}d\varepsilon =$$

$$= 2\,jrdrdt \cdot \left(-\frac{1}{2}\right) \cdot 2\pi = -2\pi jrdrdt\ . \qquad (4.5.20)$$

With provision for (4.5.4), for the case $n = 3$, formula (4.5.20) takes on the form:

$$dm(r,t) = -2\pi \cdot \frac{G}{t^3} \cdot \frac{\partial \rho}{\partial r} rdrdt\ . \qquad (4.5.21)$$

Taking into account form (4.3.10b) of the mass density function $\rho(r,t)$, let us calculate the partial derivative $\partial\rho/\partial r$:

$$\frac{\partial \rho}{\partial r} = \frac{Mt^3}{(2\pi G)^{3/2}} \cdot \left(-\frac{rt^2}{G}\right) \cdot e^{-r^2t^2/2G} = -\frac{Mt^5 r}{(2\pi G)^{3/2} G} \cdot e^{-r^2t^2/2G}. \qquad (4.5.22)$$

Substituting (4.5.22) into (4.5.21), we obtain [73]:

$$dm(r,t) = \frac{Mt^2 r^2}{(2\pi)^{1/2} G^{3/2}} \cdot e^{-r^2t^2/2G} drdt\ . \qquad (4.5.23)$$

Formula (4.5.23) describes the mass “inflow” in time dt into an infinitely thin hemisphere layer situated near the radius r hemisphere. The same formula is also valid for the hemisphere layer situated diametrically opposite the one examined in Fig. 4.9. Consequently, “the total mass inflow”, caused by gravitational compression, into the dr-thick sphere layer, situated at distance r from the center of masses of a centrally symmetric spheroidal body, in infinitely small time dt, is equal to [47, 49, 73]:

$$dm(r,t) = \frac{2Mt^2 r^2}{(2\pi)^{1/2} G^{3/2}} \cdot e^{-r^2t^2/2G} drdt =$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{Mt^2 r^2}{G^{3/2}} \cdot e^{-r^2t^2/2G} drdt\ . \qquad (4.5.24)$$

According to (4.4.7) and (4.4.19) the sphere layer internal energy change, associated with the inflow of mass (4.5.24) in the process of the slow-flowing gravitational condensation of a centrally symmetric spheroidal body, is formalized by the relation [47, 49]:

$$dU_g(r,t) = g(r)dm(r,t) = -\frac{1}{3\sqrt{2}} \cdot \frac{\gamma M}{r} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{Mr^2t^2}{G^{3/2}} e^{-r^2t^2/2G} drdt =$$

$$= -\frac{\gamma M^2}{3\pi^{1/2}G^{3/2}} \cdot rt^2 e^{-r^2t^2/2G} drdt . \qquad (4.5.25)$$

Having summed up (4.5.24) over all the concentrically situated sphere layers, one can find the quantity of mass transported inside the centrally symmetric spheroidal body in the process of gravitational condensation in infinitely small time dt:

$$dm(t) = \sqrt{\frac{2}{\pi}} \cdot \frac{Mt^2dt}{G^{3/2}} \int_0^\infty r^2 e^{-r^2t^2/2G} dr . \qquad (4.5.26)$$

To calculate the definite integral in the right-hand part of (4.5.26) we shall use (4.4.16) considering $\alpha = t^2 / G$:

$$dm(t) = \sqrt{\frac{2}{\pi}} \cdot \frac{Mt^2dt}{G^{3/2}} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{1}{(t^2/G)^{3/2}} = M\frac{dt}{t} . \qquad (4.5.27)$$

Similarly to (4.5.27), we shall calculate the centrally symmetric spheroidal body internal energy change in an infinitely small time dt in the process of gravitational condensation, having summed up (4.5.25) over all the concentrically situated sphere layers, and then having used (4.4.16):

$$dU_g(t) = -\frac{\gamma M^2 t^2 dt}{3\pi^{1/2}G^{3/2}} \int_0^\infty r e^{-r^2t^2/2G} dr =$$

$$= -\frac{\gamma M^2 t^2 dt}{3\pi^{1/2}G^{3/2}} \cdot \frac{1}{t^2/G} = -\frac{\gamma M^2 dt}{3\sqrt{\pi G}} . \qquad (4.5.28)$$

Expressing dt from (4.5.27), we can find the connection between $dU_g(t)$ and $dm(t)$:

$$dU_g(t) = \frac{-\gamma M^2}{3\sqrt{\pi G}} \cdot \frac{tdm(t)}{M} = -\frac{\gamma Mt}{3\sqrt{\pi G}} \cdot dm(t). \qquad (4.5.29)$$

Using the sense of α from (4.3.9b) we rewrite (4.5.29) as follows:

$$dU_g(t) = -\frac{\gamma M dm(t)}{3}\sqrt{\frac{\alpha}{\pi}} . \qquad (4.5.30)$$

Formula (4.5.30) establishes the connection between the internal energy change and the change of mass due to its displacement in the process of slow-flowing condensation of a centrally symmetric spheroidal body. As follows from (4.5.28), the internal energy change, associated with the mass displacement caused by the slow-flowing gravitational condensation of a centrally symmetric spheroidal body in some finite time Δt, is expressed by relation [47, 49, 73]:

$$\Delta U_g = U_g(\Delta t) = \frac{-\gamma M^2 \Delta t}{3\sqrt{\pi G}} . \qquad (4.5.31)$$

On the other hand, as shown in Section 4.4, under the conditions of mechanical equilibrium, the internal gravitational energy associated with the substance mass transfer is expressed by the formula:

$$U_g{}^m = -\frac{\gamma M^2}{3} \cdot \sqrt{\frac{\alpha}{\pi}} , \qquad (4.5.32)$$

and with provision for the sense of parameter α according to (4.3.9b), relation (4.5.32) is written in the form:

$$U_g{}^m = -\frac{\gamma M^2 t}{3\sqrt{\pi G}} . \qquad (4.5.33)$$

As follows from (4.5.33), the change in the time interval $\Delta t = t_2 - t_1$ of the internal gravitational energy owing to the substance mass-transfer is expressed by the formula:

$$\Delta U_g{}^m = U_g{}^m(t_2) - U_g{}^m(t_1) = -\frac{\gamma M^2 \Delta t}{3\sqrt{\pi G}} . \qquad (4.5.34)$$

Comparing (4.5.34) with relation (4.5.31) obtained in a different way, one is convinced of their complete identity [47, 49, 73].

Let us calculate the internal energy flow density due to the slow gravitational mass-transfer. According to (4.4.7), the infinitely small change of centrally symmetric spheroidal body internal gravitational energy is expressed by the relation:

$$dU_g = -pdV + gdm,$$

whence, introducing the *internal gravitational energy density* ε_g [47, 73], we obtain:

$$\varepsilon_g dV = -pdV + g\rho dV$$

or finally:

$$\varepsilon_g = -p + g\rho. \tag{4.5.35}$$

Differentiating (4.5.35) with respect to time t, and taking into consideration that, according to (4.4.19), g does not depend on t, we have [47, 49]:

$$\frac{\partial \varepsilon_g}{\partial t} = -\frac{\partial p}{\partial t} + g\cdot\frac{\partial \rho}{\partial t}. \tag{4.5.36}$$

Making use of the continuity equation (4.5.3) and the mass flow density expression (4.5.4), from (4.5.36) we obtain:

$$\frac{\partial \varepsilon_g}{\partial t} = -\frac{\partial p}{\partial t} - g\cdot\mathrm{div}\vec{j}. \tag{4.5.37}$$

Transforming (4.5.37) with the aid of the familiar vector analysis formula [128]: $\nabla\left(g\vec{j}\right) = g\nabla\vec{j} + \left(\nabla g\right)\cdot\vec{j}$, it is not difficult to see that:

$$\frac{\partial \varepsilon_g}{\partial t} = -\frac{\partial p}{\partial t} - \mathrm{div}\left(g\vec{j}\right) + \left(\mathrm{grad}\, g\right)\vec{j}. \tag{4.5.38}$$

To reveal the sense of the value $g\vec{j}$ we shall make use of relations (4.4.7) and (4.5.8). Combining them, it is not difficult to ascertain that the infinitely small internal energy change associated with anti-diffusion mass-transfer is expressed by the formula:

$$dU_g^{\ m} = g\vec{j}d\vec{S}dt\,. \tag{4.5.39}$$

Similarly to relation (4.5.8), determining the mass flow density, relation (4.5.39) establishes the sense of the *internal energy flow density* due to anti-diffusion mass-transfer:

$$u_g^{\ m} = g\vec{j}\,. \tag{4.5.40}$$

Further, it is also evident that the internal gravitational energy density change becomes both from pressure and from anti-diffusion mass-transfer, that is,

$$\frac{\partial \varepsilon_g}{\partial t} = \frac{\partial \varepsilon_g^{\ p}}{\partial t} + \frac{\partial \varepsilon_g^{\ m}}{\partial t} = -\frac{\partial p}{\partial t} + \frac{\partial \varepsilon_g^{\ m}}{\partial t}\,. \tag{4.5.41}$$

With provision for (4.5.40) and (4.5.41), relation (4.5.38) takes on the form:

$$\frac{\partial \varepsilon_g^{\ m}}{\partial t} = -\mathrm{div} u_g^{\ m} + \vec{j}\,\mathrm{grad}\, g\,. \tag{4.5.42}$$

Now let us rewrite (4.5.42) as follows [47, 73]:

$$\frac{\partial \varepsilon_g^{\ m}}{\partial t} + \mathrm{div} u_g^{\ m} - \vec{j}\,\mathrm{grad}\, g = 0\,. \tag{4.5.43}$$

Formula (4.5.43) establishing the relationship between the internal gravitational energy density caused by anti-diffusion mass-transfer, $\varepsilon_g^{\ m}$, and the internal energy flow density associated with anti-diffusion mass-transfer, $u_g^{\ m}$, expresses the law of conservation of internal energy for the slow-flowing gravitational condensation of a centrally symmetric spheroidal body.

4.6. Dynamical states of a forming centrally symmetric spheroidal body near the points of mechanical equilibrium

We consider a centrally symmetric spheroidal body being formed, the dynamical states of which depend exclusively on the parameter of gravitational condensation $\alpha = \alpha(t)$ [16, 47, 73], as a dynamical system with *one degree of freedom* in the space of states. In the general case, the behavior (motion) of the *one-dimensional* dynamical system is described by a single variable $\mathrm{A}(t)$ that satisfies a differential equation of the form:

$$\frac{d\mathrm{A}(t)}{dt} = f(\mathrm{A}) . \tag{4.6.1}$$

We suppose that the function $f(\mathrm{A})$ is analytic on the axis of abscissa, with the exception, perhaps, of a finite set of points. The *state-space* of the considered dynamical system (4.6.1) is a *straight line* (Fig. 4.11), and the motion of this system corresponds to the translation of the imaging point $\mathrm{A}(t)$ along this straight line.

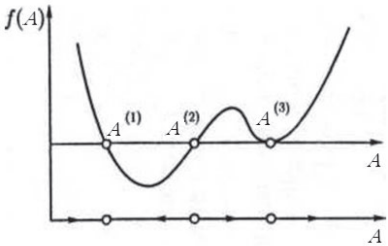


Figure 4.11. Model of a dynamical system with one degree of freedom

According to the Cauchy theorem [193], the differential equation (4.6.1) under given initial conditions $\mathrm{A}(t_0)$ has a unique solution. The equilibrium state (position) of the system, that is, singular point of the equation (4.6.1), is obtained from the solution of the equation:

$$f(\mathrm{A}_0) = 0\,. \tag{4.6.2}$$

Let the solutions of this equation be points [193]:

$$\mathrm{A}_0 = \mathrm{A}^{(k)}, \quad k = 0,1,...,n\,. \tag{4.6.3}$$

By equation (4.6.2) and the condition $(d/dt)\mathrm{A}^{(k)} = 0$, the equilibrium state:

$$\mathrm{A}(t) = \mathrm{A}^{(k)} \tag{4.6.4}$$

is also a solution of the equation of motion (4.6.1) for initial conditions $\mathrm{A}(0) = \mathrm{A}^{(k)}$.

It follows from the Cauchy uniqueness theorem that if the motion of the system did not begin from an equilibrium state, then the system cannot reach this state in a finite time. Otherwise, despite the Cauchy theorem, it is for the equilibrium state that, along with the solution $\mathrm{A}(t) = \mathrm{A}^{(k)}$, there will be a second solution [193].

Thus, the imaging point can only asymptotically tend to the equilibrium state when $t \to \infty$. In Fig. 4.11 shows an example of the course of a function $f(\mathrm{A})$ and the corresponding straight line of states with the indicated direction of the trajectories [193]. It can be expected that the singular point $\mathrm{A}^{(1)}$ is stable, and the singular points $\mathrm{A}^{(2)}$ and $\mathrm{A}^{(3)}$ are unstable. To study stability using the Lyapunov criterion, we restrict ourselves to small deviations from equilibrium:

$$\alpha(t) = \mathrm{A}(t) - \mathrm{A}_0\,. \tag{4.6.5}$$

By decomposing the function $f(\mathrm{A})$ in a Taylor series in a neighborhood A_0, from (4.6.1), (4.6.5) we obtain:

$$\frac{d\alpha(t)}{dt} = f(\mathrm{A}_0) + f'(\mathrm{A}_0)\alpha + \frac{1}{2}\cdot f''(\mathrm{A}_0)\alpha^2 + \dots. \tag{4.6.6}$$

Taking into account the condition (4.6.2) and equation (4.6.6), we find in the linear approximation that:

$$\frac{d\alpha(t)}{dt} = f^{'}(\mathrm{A}_0)\alpha \,. \qquad (4.6.7)$$

As we know, the solution to such a linear equation (4.6.7) is:

$$\alpha(t) = \alpha(0) \cdot e^{f^{'}(\mathrm{A}_0)\cdot t} \,. \qquad (4.6.8)$$

In particular, for $f^{'}(\mathrm{A}_0) < 0$ we have:

$$\left|\alpha(t)\right| = \left|\mathrm{A}(t) - \mathrm{A}_0\right| \to 0 \quad \text{at } t \to \infty \,.$$

In other words, the solution A_0 is *asymptotically stable* in the sense of Lyapunov, while when $f^{'}(\mathrm{A}_0) > 0$, there is instability.

Consideration of an example in Fig. 4.11 shows that for the left equilibrium state $f^{'}(\mathrm{A}^{(1)}) < 0$, that is, this state is stable, while the state $\mathrm{A}^{(2)}$ is unstable because $f^{'}(\mathrm{A}^{(2)}) > 0$ [193]. For a stationary state $\mathrm{A}^{(3)}$, taking into account that $f^{'}(\mathrm{A}^{(3)}) = 0$, for small deviations, we can write the equation:

$$\frac{d\alpha(t)}{dt} = \frac{1}{2} \cdot f^{''}(\mathrm{A}_0)\alpha^2 \,, \qquad (4.6.9)$$

whose solution is the function [193]:

$$\alpha(t) = \frac{1}{(1/\alpha_0) - [f^{''}(\mathrm{A}^{(3)})/2] \cdot t} \,. \qquad (4.6.10)$$

If the initial perturbation is such that $\alpha_0 < 0$, then the system returns to a stationary state $\alpha = 0$, if so $\alpha_0 > 0$, then the system moves away from the stationary state. Therefore, the point $\mathrm{A}^{(3)}$ is *unstable* in the sense of Lyapunov [193].

Summing up the brief analysis, it can be argued that in a system with *one degree of freedom* the following typical situations are possible [73]:

a) *asymptotic stability* when $f^{'}(\mathrm{A}_0) < 0$ (exponential decay of deviations from the equilibrium state);
b) *instability* when $f^{'}(\mathrm{A}_0) > 0$ (exponential increase in deviations from the stationary state); and
c) *unstable equilibrium* requiring additional analysis in the case of $f^{'}(\mathrm{A}_0) = 0$ [193].

It should be noted in particular that the above typical situations have already been partially considered in the present and previous sections of this monograph, as well as in the works [16, 47, 48, 65, 73]. Indeed, in Section 4.3, the case of an infinitely small removal (in time) from an *unstable* mechanical equilibrium state of a centrally symmetric gravitating spheroidal body (a quasi-equilibrium condensation, in particular, the special case of initial instability of type *b*) was investigated in detail; in Section 2.3, the critical (threshold) values of mass density ρ_c and gravitational condensation parameter α_c were found, at which the gravitational interactions of particles due to the anti-diffusion process become sharply amplified and replaced by *fast-flowing gravitational compression* (contraction) resulting in *gravitational field origin* (the above case of an exponentially increasing instability of type b); in Section 4.1, the case of *unstable virial equilibrium* of the anti-diffusion process of gravitational condensation of a centrally symmetric spheroidal body (the above case of unstable mechanical equilibrium of type c) was also considered. Indeed, a comparison of formulas (4.1.22) and (4.6.10) shows their identity.

In this regard, it is advisable to systematize the possible dynamical states of a forming centrally symmetric spheroidal body from infinitely spread gas-dust matter [73]:

1) The *initial equilibrium state* of a gas-dust nebula (molecular cloud), when the parameter of gravitational

condensation $\alpha(t) = \text{const}$, GCF becomes the gravitational compression coefficient $\mathrm{G}(t) = \frac{1}{2\alpha^2} \cdot \frac{d\alpha}{dt} = 0$, and the basic anti-diffusion equation of initial gravitational condensation (4.1.9a) is degenerate, that is:

$$\frac{\partial \rho}{\partial t} = 0. \tag{4.6.11}$$

2) A particular case of *initial instability* at infinitesimal removal (in time) from an unstable equilibrium state of a gas-dust nebula is a quasi-equilibrium gravitational condensation [47, 48], when, according to (4.3.9a), the parameter of gravitational condensation $\alpha = t/2G$, the gravitational compression factor $G = \text{const}$, GCF $\mathrm{G}(t) = G/t^2$ in accordance with (4.3.6a), (4.3.7a), so that the basic anti-diffusion equation of gravitational condensation (4.1.9a) goes into equation (4.3.8a) of the quasi-equilibrium gravitational condensation, that is:

$$\frac{\partial \rho}{\partial t} = -(G/t^2) \cdot \nabla^2 \rho. \tag{4.6.12}$$

3) *Gravitational instability*, that is, an avalanche-like gravitational compression due to the arising gravitational field of a centrally symmetric spheroidal body, when according to (2.3.7a, b), the parameter of gravitational condensation reaches its critical value α_c as a result of an exponential increase $\alpha(t) = \alpha(0) \cdot e^{f'(\mathrm{A}_0) \cdot t}$ following (4.6.8), whereas GCF can be found as:

$$\mathrm{G}(t) = \frac{1}{2\left(\alpha(0) \cdot e^{f'(\mathrm{A}_0) \cdot t}\right)^2} \cdot \frac{d}{dt}\left(\alpha(0) \cdot e^{f'(\mathrm{A}_0) \cdot t}\right) =$$

$$= \frac{f'(\mathrm{A}_0)}{2\alpha(0)} \cdot e^{-f'(\mathrm{A}_0) \cdot t}, \tag{4.6.13a}$$

and the basic anti-diffusion equation of gravitational condensation (4.1.9a) takes the form:

$$\frac{\partial \rho}{\partial t} = -(f'(\mathrm{A}_0)/2\alpha(t))\cdot \nabla^2 \rho \,. \tag{4.6.13b}$$

4) *Unstable virial mechanical equilibrium* of the anti-diffusion process of gravitational condensation of a centrally symmetric spheroidal body [16, 63, 64, 65] when, according to (4.1.21), GCF is constant, that is, $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$, the parameter of gravitational condensation $\alpha(t) = \alpha_s \left[1 - 2\alpha_s \mathrm{G}_s (t - t_s)\right]^{-1}$ increases almost linearly with the time in accord with (4.1.22) and (4.1.23) where $\alpha_s = \alpha(t_s)$, and the basic anti-diffusion equation of gravitational condensation (4.1.9a) becomes the following virial equilibrium equation:

$$\frac{\partial \rho}{\partial t} = -\mathrm{G}_\mathrm{s} \cdot \nabla^2 \rho \,. \tag{4.6.14}$$

Let us note that from a direct comparison of formula (4.1.22) with (4.6.10) it follows that:

$$\mathrm{G}_\mathrm{s} = f''(\mathrm{A}^{(3)})/4 \,, \tag{4.6.15}$$

if $t_s = 0$, that is, $\alpha_s = \alpha(t_s) = \alpha_0$. In other words, knowing GCF G_s, from (4.6.15) it is easy to find the value of the second derivative of a function $f(\mathrm{A})$ at the point of virial equilibrium:

$$f''(\mathrm{A}^{(3)}) = 4\mathrm{G}_\mathrm{s} \,. \tag{4.6.16}$$

Similarly, we can find the value of the first derivative of a function $f(\mathrm{A})$ at the point of the initial equilibrium state. To do this, we first consider the particular case of initial instability at an infinitesimal removal in time from an unstable equilibrium state (a quasi-equilibrium gravitational condensation), for which the parameter of gravitational condensation is $\alpha = t/2G$ following (4.3.9a). Then, according to (4.1.8), GCF is easily found:

$$\mathrm{G}(t) = \frac{1}{2(t/2G)^2} \cdot \frac{d}{dt}(t/2G) = \frac{G}{t^2} \,, \tag{4.6.17}$$

which completely coincides with the previously derived formula (4.3.6a) or (4.3.12a), according to which:

$$G = \frac{(\gamma m_0 M)^2}{4\pi \overline{E}(t_0)\overline{E}'(t_0)} = \frac{(\gamma M^2/2)^2}{4\pi E_g(t_0)E_g'(t_0)}. \tag{4.6.18}$$

Assuming an infinitesimal removal in time ($t \to 0$) from the equilibrium state for the case of initial gravitational instability, we can approximate the function (4.6.8) by the Maclaurin series in the linear approximation:

$$\alpha(t) = \alpha_0 \cdot e^{f'(\mathrm{A}_0)\cdot t} \approx \alpha_0 \cdot [1 + f'(\mathrm{A}_0)\cdot t]. \tag{4.6.19}$$

Since $|\alpha_0| << 1$ and $|f'(\mathrm{A}_0)| >> 1$, in (4.6.19) we restrict ourselves to the consideration of only the second term:

$$\alpha(t) \approx \alpha_0 \cdot f'(\mathrm{A}_0)\cdot t. \tag{4.6.20}$$

Comparing formula (4.6.20) with the one mentioned above (4.3.9a), we obtain:

$$G = 1/(2\alpha_0 \cdot f'(\mathrm{A}_0)),$$

where the desired value of the first derivative calculated at a point A_0 immediately follows:

$$f'(\mathrm{A}_0) = 1/(2\alpha_0 \cdot G). \tag{4.6.21}$$

Taking into account (4.6.21), the approximate equation ($t \to 0$) of an avalanche-type gravitational compression (4.6.13b) takes the form:

$$\frac{\partial \rho}{\partial t} = -[4G \cdot \alpha_0 \alpha(t)]^{-1} \cdot \nabla^2 \rho, \tag{4.6.22}$$

where $\alpha(t) = \alpha_0 \cdot e^{t/2\alpha_0 \cdot G}$. Knowing the values of the derivatives of a function $f(\mathrm{A})$ at the equilibrium points according to (4.6.16) and (4.6.21), we can try to reconstruct this function as well as the differential equation (4.6.1). This means a more accurate description of the dynamical states of a forming centrally symmetric spheroidal body depending on general function $\mathrm{A} = \mathrm{A}(t)$ (instead of the "small deviations" function $\alpha = \alpha(t)$ being the parameter of gravitational condensation) permitting us to find a general form of the mass density function $\rho = \rho(\vec{r}, \mathrm{A})$.

4.7. The derivation of the general anti-diffusion equation for a slowly evolving process of gravitational condensation of a rotating axially symmetric spheroidal body

Now let us derive the differential equation for a formation process of a spheroidal body having an *axially symmetric* distribution of mass density (or, simply speaking, a spheroidal body) when its mass density iso-surface is evolving from the sphere to a spheroid. In this connection, using the cylindrical frame of reference (h, ε, z) we consider the mass density function (3.3.26a) of a *rotating* spheroidal body in a vicinity of *relative* mechanical equilibrium (see Section 3.3 and [16, 73]).

Let us calculate derivative of ρ with respect to the spatial coordinates h and z as well as the parameters α and ε_0 supposing these are to be slowly changing functions with the time, that is, $\alpha = \alpha(t)$ and $\varepsilon_0 = \varepsilon_0(t)$ [79]:

$$\frac{\partial \rho}{\partial \alpha} = \frac{3}{2} M \left(\frac{\alpha}{2\pi} \right)^{1/2} \cdot \frac{1-\varepsilon_0^2}{2\pi} e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2} +$$

$$+ M \left(\frac{\alpha}{2\pi} \right)^{3/2} (1-\varepsilon_0^2) \cdot [-\frac{(1-\varepsilon_0^2)h^2 + z^2}{2}] e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2} =$$

$$= \frac{\rho}{2\alpha}[3 - \alpha(h^2(1-\varepsilon_0^2) + z^2)]; \qquad (4.7.1)$$

$$\frac{\partial \rho}{\partial \varepsilon_0} = \rho_0(-2\varepsilon_0) e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2} +$$

$$+ \rho_0(1-\varepsilon_0^2)\{-\frac{\alpha h^2}{2}(-2\varepsilon_0)\} \cdot e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2} =$$

$$= -\frac{2\varepsilon_0}{1-\varepsilon_0^2}\cdot\rho[1-\frac{\alpha}{2}h^2(1-\varepsilon_0^2)]; \qquad (4.7.2)$$

$$\nabla_h^2\rho = \frac{1}{h}\cdot\frac{\partial}{\partial h}\left(h\frac{\partial\rho}{\partial h}\right) = -\alpha(1-\varepsilon_0^2)^2\rho_0\cdot\{\frac{1}{h}\cdot\frac{\partial}{\partial h}(h^2 e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2})\} =$$

$$= -\alpha(1-\varepsilon_0^2)\cdot\rho[2-\alpha(1-\varepsilon_0^2)h^2]; \qquad (4.7.3)$$

$$\nabla_z^2\rho = \frac{\partial^2\rho}{\partial z^2} = \rho_0(1-\varepsilon_0^2)\frac{\partial}{\partial z}\{-\alpha z\cdot e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2}\} =$$

$$= -\alpha\rho_0(1-\varepsilon_0^2)e^{-\alpha[h^2(1-\varepsilon_0^2)+z^2]/2}(1-\alpha z^2) = -\alpha\rho\cdot[1-\alpha z^2]; \qquad (4.7.4)$$

$$\nabla^2\rho = \nabla_h^2\rho + \nabla_z^2\rho =$$

$$= -\alpha\rho[3-\alpha(h^2(1-\varepsilon_0^2)+z^2)] + 2\varepsilon_0^2\alpha\rho[1-\frac{\alpha}{2}(1-\varepsilon_0^2)h^2] \qquad (4.7.5)$$

So, taking into account (4.7.1) we can see that:

$$\rho[3-\alpha(h^2(1-\varepsilon_0^2)+z^2)] = 2\alpha\frac{\partial\rho}{\partial\alpha} \qquad (4.7.6)$$

and according to (4.7.2) we find:

$$\rho[1-\frac{\alpha}{2}(1-\varepsilon_0^2)h^2] = -\frac{1-\varepsilon_0^2}{2\varepsilon_0}\cdot\frac{\partial\rho}{\partial\varepsilon_0}. \qquad (4.7.7)$$

With regard for (4.7.6), (4.7.7) equation (4.7.5) becomes [79]:

$$\nabla^2\rho = -2\alpha^2\frac{\partial\rho}{\partial\alpha} - \alpha\varepsilon_0(1-\varepsilon_0^2)\frac{\partial\rho}{\partial\varepsilon_0}. \qquad (4.7.8)$$

Now we suppose that an evolution of the mass density of a rotating spheroidal body with the time can be expressed by a composite function of $\alpha = \alpha(t)$ and $\varepsilon_0 = \varepsilon_0(t)$. In the first place, we obtain the basic anti-diffusion equation (4.1.9a) in the special case of the fixed parameter ε_0 [16, 73]:

$$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial\alpha}\cdot\frac{d\alpha}{dt} = \left(-\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}\right)\cdot\nabla^2\rho = -\mathrm{G}(t)\cdot\nabla^2\rho. \qquad (4.7.9a)$$

On the other hand, the following equation is true under consideration of the fixed parameter α [77, 79]:

$$\frac{\partial\rho}{\partial t}=\frac{\partial\rho}{\partial\varepsilon_0}\cdot\frac{d\varepsilon_0}{dt}=\left(-\frac{1}{\alpha\varepsilon_0(1-\varepsilon_0^2)}\cdot\frac{d\varepsilon_0}{dt}\right)\cdot\nabla^2\rho\,. \qquad (4.7.9b)$$

Intending to study a temporal evolution of a solution to Eq. (4.7.8) we need to investigate the functional dependence $\varepsilon_0=\varepsilon_0(\alpha)$ occurring when $\alpha\geq\alpha_c$, where $\alpha_c=\alpha(t_c)$ and t_c is a moment of rotation origin. However, it is impossible under the initial statement (3.3.18) about the independence α of the coordinates h,ε,z. Consequently, the total derivative of the mass density function ρ (with respect to the time) can be represented by the following relation:

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial\alpha}\cdot\frac{d\alpha}{dt}+\frac{\partial\rho}{\partial\varepsilon_0}\cdot\frac{d\varepsilon_0}{dt}\,. \qquad (4.7.10)$$

To find $\partial\rho/\partial\varepsilon_0$ let us use Eq. (4.7.8) at the fixed parameter α whence the desired partial derivative is equal to:

$$\frac{\partial\rho}{\partial\varepsilon_0}=-\frac{1}{\alpha\varepsilon_0(1-\varepsilon_0^2)}\nabla^2\rho\,. \qquad (4.7.11a)$$

Analogously, if the parameter ε_0 is fixed then the above-mentioned basic equation relatively α (similar to (4.1.6)) follows directly from Eq. (4.7.8):

$$\frac{\partial\rho}{\partial\alpha}=-\frac{1}{2\alpha^2}\cdot\nabla^2\rho\,. \qquad (4.7.11b)$$

Substituting (4.7.11a) and (4.7.11b) into (4.7.10) leads to the following *general equation of anti-diffusion* concerning a deformation of a spheroidal body as a result of its rotation [77, 79]:

$$\frac{d\rho}{dt}=-\tilde{\mathrm{G}}(t)\nabla^2\rho\,, \qquad (4.7.12)$$

where $\tilde{\mathrm{G}}(t)$ is an anti-diffusion function, that is, the *generalized GCF*, taking account of flattening process into the rotating axially symmetric spheroidal body [77, 79]:

$$\widetilde{G}(t) = \frac{1}{2\alpha^2(t)} \cdot \frac{d\alpha}{dt} + \frac{1}{\alpha\varepsilon_0(1-\varepsilon_0^2)} \cdot \frac{d\varepsilon_0}{dt} . \qquad (4.7.13)$$

In the case of the finite value of $\dot{\alpha}$ and $\dot{\varepsilon}_0$ the anti-diffusion function $\widetilde{G}(t)$ can increase unlimitedly when $\alpha \to 0$ (at the so-called initial anti-diffusion condensation) and when $\varepsilon_0 \to 0$ (at the initial flattening). Therefore, the anti-diffusion condensation instant and the flattening instant can be the same, but it is possible that they can be inconsistent in general. Probably, the flattening occurs when the gravitational field arises in a spheroidal body, that is, in the case if $\alpha(t)$ exceeds its threshold value α_c [16, 65, 73].

According to (3.3.25) $\rho = M\Phi$, so that the analogous equation (4.7.12) of general anti-diffusion condensation of gaseous substance is true for the function of volume probability density Φ to locate liquid particles in an axially symmetric spheroidal body [77, 79]:

$$\frac{d\Phi}{dt} = -\widetilde{G}(t)\nabla^2\Phi . \qquad (4.7.14)$$

When the parameter ε_0 becomes finite, $\varepsilon_0 \neq 0$ then a centrally symmetric spheroidal body (sphere-like gaseous body) begins to deform (to be flattened) implying a *bifurcation* on the diagram of dynamical states of this body. As noticed by Jeans [1 p.188, p.190, p.191],

> On continually varying some parameter (*say* ε_0 , *is allowed slowly to vary*[1]) we pass through a whole series of continuous configurations of equilibrium, which form what Poincaré has called a “linear series”. …Every point on a linear series is a configuration of equilibrium; a question which is of the utmost importance in cosmogonical problem is whether this equilibrium is stable or

[1] Author’s remark

> unstable. … Thus we see that there is an *exchange of stabilities* at the point of bifurcation.

In this connection, if we suppose that $\alpha = \alpha(t)$ is a variable of one-dimensional state-space of a spheroidal body then ε_0 can be considered as a *control* parameter [119, 135]. In reality, "the conditions of secular stability assume a somewhat different form for a mass rotating freely in space. In this case, the rate of rotation is not constant, but changes as the moment of inertia of the mass changes … Secular stability is lost at a 'turning point' or 'point of bifurcation'" [1 p.199-201]. Jeans then clarified [1 p.207, p.209] that:

> If $\Omega = 0$,…so that the configuration must be spherical. If Ω is small, although not actually zero, a spherical surface does not satisfy the condition, the term $\frac{1}{2}\Omega^2(x^2 + y^2)$ destroying the spherical symmetry. In this case, as we shall see almost immediately, the configuration is that of an oblate spheroid of small ellipticity…Two linear series of equilibrium configuration, which are spheroidal and ellipsoidal respectively. The configurations which form the first series are commonly known as Maclaurin's spheroids; those which form the second as Jacobi's ellipsoids… .

So, according to Eq. (4.7.12), a variation of the form of an axially symmetric spheroidal body is caused by dissipation, that is, by the gravitational energy changing due to the internal energy of the particles of a gaseous cloud. According to (3.3.26a) the flattening process cannot decrease the anti-diffusion condensation in the axial direction, but it can reduce the anti-diffusion condensation in the plane of rotation of an axially symmetric spheroidal body.

Conclusion and comments

According to the statistical models of a slowly flowing gravitational interaction of gas-dust body particles considered in Chapters 2 and 3, under the influence of initial oscillations of the particles an isolated gaseous cloud can be condensed to a centrally symmetric spheroidal body (see Chapter 2) or axially symmetric spheroidal body (see Chapter 3). Each of these models developed for a detailed study of a complex dynamical picture of the gravitational interaction of particles describes only a certain dynamical solution near an equilibrium state.

In Section 4.1 of this chapter, the basic anti-diffusion equation (4.1.9a, b) of the initial gravitational condensation of a non-rotating (or slowly rotating) spheroidal body from infinitely spread matter was derived, and in Section 4.2, the general differential equations for physical values describing the anti-diffusion process of the initial gravitational condensation of a spheroidal body near the state of mechanical equilibrium were obtained. An important result obtained here says that the *anti-diffusion process does not violate the virial equilibrium of a centrally symmetric spheroidal body* since according to Corollary 4.1 in the unstable mechanical equilibrium state GCF $\mathrm{G}(t)$ remains constant: $\mathrm{G}(t) = \mathrm{G}_s = \text{const}$, although the parameter of gravitational condensation α increases almost linearly with the time t following formulas (4.1.22), (4.1.26), and (4.1.27).

In Section 4.1, the Poincaré–Eddington virial theorem was applied to a spheroidal body with a centrally symmetric distribution of masses (being in unstable radial motion), and as a result Theorem 4.1 was proved:

> For a self-gravitating spheroidal body with a centrally symmetric distribution of masses under the condition of their unstable radial motion the sum of the double kinetic energy and the total

gravitational potential energy of this sphere-like gaseous system of particles is equal:

$$2E_k + E_g = -2M\dot{\mathrm{G}}(t),$$

where E_k and E_g are respectively the kinetic and potential energy of the particles forming a spheroidal body at $\varepsilon_0 \to 0$, $\dot{\mathrm{G}}(t)$ is a derivative of GCF $\mathrm{G}(t)$ of the spheroidal body with a centrally symmetric distribution of masses, M is its total mass.

By analogy with Einstein's formula in the Brownian motion theory, Corollary 4.2 is also derived which states:

the mean-square distance of displacement of a colloidal particle in an immovable or slowly rotating spheroidal body during the time interval $\tau = t - t_s$ is equal:

$$\overline{\delta r^2} = 6\mathrm{G}_s \tau .$$

According to Corollary 4.2 displacement and then interaction of colloidal particles (as well as molecules and atoms) can be caused by the existence of a *dark matter*. As a result, centrally symmetric or axially symmetric spheroidal bodies are formed.

Section 4.2 also shows that both the strength and potential of a weak gravitational field of a slowly contracting centrally symmetric spheroidal body satisfy second-order differential equations of parabolic type, that is, anti-diffusion equations of the form (4.2.17) or (4.2.31) respectively.

Two particular cases of the basic equation of slow-flowing initial gravitational condensation were considered in Section 4.3, namely, equations (4.3.8a) and (4.3.8b) of a slow-flowing gravitational condensation of a centrally symmetric spheroidal body near an unstable mechanical equilibrium state: *quasi-equilibrium* gravitational condensation ($\alpha = t/2G_1$) and *initial* gravitational condensation ($\alpha = t^2/G_2$) [47, 73].

In Section 4.4, the gravity–thermodynamic relation (4.4.6) was introduced (for a centrally symmetric gravitating spheroidal body). Using this the gravitational thermodynamic potential (4.4.19) was determined. As shown in Sections 4.4 and 4.5, about 2/3 of the gravitational potential energy is spent on anti-diffusion mass transfer of matter inside a centrally symmetric gravitating spheroidal body, determined by the gravitational thermodynamic potential $g(r)$ [47, 49, 73]. In Section 4.5, the densities of mass flow (4.5.4) and internal energy flow (4.5.40) are introduced for the process of slow-flowing initial gravitational condensation of a centrally symmetric spheroidal body.

In Section 4.6, the possible dynamical states of forming a centrally symmetric spheroidal body from an infinitely spread gas-dust matter are systematized:

- the *initial equilibrium state* of a gas-dust nebula (molecular cloud), when the parameter of gravitational condensation $\alpha(t) = \text{const}$, and GCF $\mathrm{G}(t) = 0$;
- the case of *initial instability* at infinitesimal removal (in time) from an unstable equilibrium state of a gas-dust nebula being a *quasi-equilibrium gravitational condensation* [47, 48], when, according to (4.3.9a), the parameter of gravitational condensation $\alpha = t/2G$, the gravitational compression factor $G = \text{const}$, and GCF $\mathrm{G}(t) = G/t^2$ following (4.3.6a), (4.3.7a);
- the case of an *avalanche-like gravitational compression* due to the arising gravitational field of a centrally symmetric spheroidal body, when according to (4.6.8) the parameter of gravitational condensation is exponentially increasing $\alpha(t) = \alpha(0) \cdot e^{f'(A_0)\cdot t}$, and GCF is $\mathrm{G}(t) = \dfrac{f'(A_0)}{2\alpha(0)} \cdot e^{-f'(A_0)\cdot t}$ in accordance with (4.6.13a); and

- the case of *unstable mechanical equilibrium* of the anti-diffusion process of gravitational condensation of a centrally symmetric spheroidal body [16, 63–65] when the parameter of gravitational condensation $\alpha(t) = \alpha_s [1 - 2\alpha_s \mathrm{G}_s (t - t_s)]^{-1}$ is almost linearly increasing with the time following (4.1.22), (4.1.23), where $\alpha_s = \alpha(t_s)$ and GCF is constant $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$ in accordance with (4.1.21).

In Section 4.7, the *general anti-diffusion equation* (4.7.12) for a slowly evolving process of initial gravitational condensation of an *axially symmetric* spheroidal body (which is formed in result of its rotation) was derived. The basic anti-diffusion equation (4.1.9a) of the initial gravitational condensation of an immovable (or slowly rotating at $\varepsilon_0 \to 0$) centrally symmetric spheroidal body was obtained as a special result (see Eq. (4.7.11b)). Since the evolution of the mass density of a rotating spheroidal body includes both anti-diffusion and flattening processes, a *generalized* GCF $\widetilde{\mathrm{G}}(t)$ is introduced in accordance with (4.7.13). Obviously, GCF (4.1.8) is the particular case of generalized GCF (4.7.13) when the flattening process is absent ($\dot{\varepsilon}_0 \to 0, \varepsilon_0 \to 0$). As indicated, since $\alpha = \alpha(t)$ is a variable of one-dimensional state-space of an axially symmetric spheroidal body then ε_0 can be considered as a control parameter within the framework of the self-organization theory [135].

So, summing up, we note that in [135] gravitation was treated from the point of view of the self-organization theory (synergetics), first of all, as "a primary irreversible process." Such an interpretation was mainly because evolution processes "cannot be described by some nontrivial means, without involving time direction" [134]. Indeed, heat conduction, diffusion, chemical reactions, biological and physiological processes, and so forth, are examples of *dissipation processes*, that is, the ones connected with consumption (dissipation) of energy (e.g., heat). As a result of this, in the reversing of time, say, in diffusion equations describing these processes (see e.g., (4.3.13)), "we obtain quite different laws" [135]. Thus, one of the principal results of the development of synergetics is a "rediscovery" of time in physics [194, 195], so that synergetics marked a passage from *physics of being to physics of becoming* with its "time arrow" (a pattern suggested by A. Eddington) [196, 197].

In connection with this, there have recently appeared many papers of philosophical character devoted to the attempts to gain new insight into "the views of the nature, properties, and structure of time" [194–198]. The unsatisfactory description by physics of natural phenomena was formulated by I. Prigogine thus: "Reversibility of laws of dynamics, as well as of laws of quantum mechanics and relativity theory, expresses … a radical negation of time" [194 p.7]. "The disparity between time being unidirectional, which results from the second law of thermodynamics, and the reversibility of the rest of fundamental physical laws worried A. Einstein very much" [195]. He considered "the time arrow to be always connected with thermodynamic conditions," but "while we possess, mainly, direct knowledge of elementary processes, there is a corresponding reverse process for each of them" [199 p.58].

However, it should be noted, that gravitation, despite being an *evolutional process*, cannot be classified completely as a dissipative process because it is connected with accumulation rather than with the dissipation of energy. As a result of Section 4.3, the phenomenon of the slow-flowing gravitational condensation of a centrally symmetric spheroidal body is described by, at least, *two anti-diffusion equations*. One of them, characterizing the process of the initial slow-flowing gravitational condensation (see (4.3.8b)), is reversible in time, while the other, describing the quasi-equilibrium slow-flowing gravitational condensation (see (4.3.8a)), is irreversible in time [16, 47, 73]. This means that the evolution in time of the complex physical phenomenon under study is presented through several differential equations, some of which are reversible in time, others are irreversible. Consequently, the same complex phenomenon cannot be characterized based on reversibility in time alone [73].

In connection with the above-noted disparity of time being unidirectional, resulting from the second law of thermodynamics, and the reversibility of the principal fundamental laws of physics, it is reasonable to accept the point of view of H. Bergson, V. Vernadsky, and I. Prigogine on the necessity of distinguishing (alongside physical time) "the so-called second time characterizing the process of the system change in the course of its irreversible development, or its age" [195, 197]. Such a point of view is, at present, the acceptable one, since there is a hypothesis of S. Weinberg, according to which, energy dissipation, on the scale of the expanding Universe, may someday become reversible [101]. This means that the kinetic energy and radiation of galaxies that are moving away from one another are not devalued in the thermodynamic sense but are accumulated in the fields of gravitation, which, in turn, will inevitably result in the compression of the Universe.

In conclusion, we will point out that for all the varied points of view and mathematical models that have been presented to explain a physical phenomenon, fortunately, they frequently agree as far as the main deductions and evaluations are concerned. Thus, according to the proposed statistical model of gravity, pressure inside a spheroidal body under mechanical equilibrium is three times less than the potential energy density of gravitating the spheroidal body (formula

(2.9.22) as well as (2.5.6) and (4.4.9)), which agrees with the deductions of Einstein's general relativity theory [100], and also with the Nicolis–Prigogine cosmological model [135] of the irreversible process of particles origin at the expense of gravitational energy. It means that in the course of development of our knowledge, the most essential provisions and deductions resulting from various theoretical constructions will, undoubtedly, remain, and the seeming contradictions will complement each other in the unique comprehension of the essence of physical phenomena.

Chapter Five

On the Generalized Nonlinear Schrödinger-like Equation Describing the Evolution of a Forming Cosmogonical Body

This chapter considers statistical theory in order to derive and develop a new generalized nonlinear Schrödinger-like equation of a cosmogonical body formation [68, 71, 73, 77, 78]. In the previous chapters, the statistical theory for a cosmogonical body formation (the so-called spheroidal body with fuzzy boundaries) was proposed (see also [16, 45–67, 73]). Within the framework of this theory, interactions of oscillating particles inside a spheroidal body lead to a gravitational condensation increasing with time. Moreover, under the condition of critical values of mass density (or the parameter of gravitational condensation α) the centrally symmetric gravitational field arises in such a spheroidal body [16, 65, 73].

In this connection, the main problem in understanding the statistical model is what mechanism of particle interactions leads to the slow-flowing process of an initial *gravitational condensation* of a spheroidal body. As shown in Chapter 2, interacting molecules or atoms of gas form new aggregate nano- and micro-particles called *colloidal* or *liquid* particles (in hydrodynamic meaning [94]). As was pointed out in [94], any liquid particle having a small element of a medium volume is still considered to be large enough to contain many molecules. The liquid particles (constituting a molecular cloud at low temperatures) also have oscillatory interactions among themselves. In reality, in *macrophysics* it is alleged that the cosmological constant [102] describes the cosmic vacuum [103, 104] and therefore, its experimental manifestation on cosmic scales is the fluctuations stipulated by Alfvén–Arrhenius oscillations [9, 19, 20]. Moreover, as we know [9, 19, 20] that due to the radial and the axial oscillations the moving solid bodies in the gravitational field of a central body have elliptic and inclined orbits.

Recently, L. Nottale [36, 37, 175, 176, 200, 201] has developed his theory of the scale relativity to describe both deterministic and stochastic behavior of a particle in the gravitational field of a cosmogonical body. In Nottale's model, both direct and reverse Wiener stochastic processes are considered in parallel, which leads to the introduction of a twin Wiener (backward and forward) process as a single complex process [175, 176]. For the first time, the backward and the forward derivatives for the Wiener process were introduced within the framework of the statistical mechanics of Nelson [34, 35]. Both Nelson's statistical mechanics and Nottale's scale relativistic theory investigate families of virtual particle trajectories that are continuous but nondifferentiable. The important point in Nelson's works [34, 35] is that a diffusion process can be described in terms of a *Schrödinger-type equation* with help of the hypothesis that any particle in the empty space, under the influence of any interaction field, is also subject to a universal Brownian motion (i.e., from the mathematical viewpoint, a Markov–Wiener process) based on the hypothesis on the quantum nature of space-time or quantum fluctuations on cosmic scale [35–43]. However, despite the important role of Nelson's and Nottale's theories the general equation of gravitational condensation has not been obtained.

In this connection, we derive the generalized nonlinear Schrödinger-like equation describing the evolution of dynamical states of a forming cosmogonical body within the framework of the proposed statistical theory of gravitating spheroidal bodies [68, 71, 73]. This equation proves to be more general than analogous equations obtained in Nelson's stochastic mechanics [34] and Nottale's scale relativistic theory [175, 176]. This chapter investigates different dynamical states of a gravitating spheroidal body and respective forms of the generalized nonlinear time-dependent Schrödinger-like equation [77, 78]. In particular, this equation also explains an initial slowly evolving process of gravitational condensation of a cosmogonical body from an infinitely distributed gaseous substance and hence solves the gravitational paradox problem.

5.1. The density of anti-diffusion mass flow and the anti-diffusion velocity into a gravitational compressible spheroidal body

At first, we are going to use equation (4.1.9a) (or (4.7.9a)) of the initial gravitational condensation of a *centrally symmetric* spheroidal body (sphere-like cosmogonical body) under the assumption $\varepsilon_0 \to 0$. By analogy with Section 4.5, we shall also rewrite it taking into account that GCF $\mathrm{G}(t)$ does not depend on the spatial variable *r*, therefore:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\big(\mathrm{G}(t)\,\mathrm{grad}\,\rho\big) = 0\,. \tag{5.1.1}$$

The relation (5.1.1) fully reminds us of the continuity equation expressing the law of conservation of mass in a nonrelativistic system [94]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0\,, \tag{5.1.2}$$

where $\vec{j}$ is a mass flow density of a continuous medium. In this connection, the value in round brackets of Eq. (5.1.1) has the sense of a *mass flow density* (like a conductive flow [108, 134]) $\vec{j}$ arising at the gravitational compression of a spheroidal body [47–49]:

$$\vec{j} = \mathrm{G}(t)\,\mathrm{grad}\rho\,. \qquad (5.1.3)$$

For the first time, the conductive (owing to diffusion or thermal conductivity) flows in dissipative systems were investigated by I. Prigogine (see, for example, [108, 134]). As it follows from Eq. (5.1.3) directly, there exists an *anti-diffusion mass flow density* in a slowly compressible gravitating spheroidal body [16, 47, 73]. Applying the equation of continuity (5.1.2) to this anti-diffusion flow density (5.1.3), we obtain again the mentioned linear anti-diffusion equation (4.1.9a). Since ρ is a function of the spatial variable r, then in the spherical system of coordinates $\mathrm{grad}\;\rho = \frac{\partial\rho}{\partial r}\vec{e}_r = \frac{\partial\rho}{\partial r}\cdot\frac{\vec{r}}{r}$. Taking into account the fact that according to (4.1.2) the mass density ρ is an exponentially decreasing function, then its derivative $\frac{\partial\rho}{\partial r} < 0$. Consequently, the direction of the anti-diffusion flow density vector $\vec{j}$ is directly opposite to the basis vector $\vec{e}_r$, that is, the vector $\vec{j}$ is directed to the spheroidal body center. Equations obtained (5.1.1) and (5.1.3) generalize analogous Eqs (4.5.3) and (4.5.4) in the case of the slow-flowing initial gravitational condensation of a *centrally symmetric* spheroidal body.

Like the elementary particle momentum operator $\hat{p} = i\hbar\nabla$ in quantum mechanics [202–204] we can introduce from Eq. (5.1.3) a *velocity operator* in the case of *unobservable* velocities of particles [48, 53, 68]:

$$\hat{v} = \mathrm{G}(t)\nabla\,, \tag{5.1.4}$$

that is, $\hat{v}$ is an operator of unobservable anti-diffusion velocity. Taking into account this Eq. (5.1.4) the anti-diffusion mass flow density (5.1.3) of slow-flowing gravitational contraction of a spheroidal body (with unobservable velocities of particles) can be written as follows [48, 53, 68]:

$$\vec{j} = \hat{v}\rho\,. \tag{5.1.5}$$

According to Eq. (5.1.5) the continuity equation (5.1.2) takes the form [68, 73]:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\hat{v}\rho) = 0\,. \tag{5.1.6}$$

As has been mentioned above, I. Prigogine, G. Nicolis, and P. Glansdorff studied the so-called *conductive* (diffusive and thermal conductive) flows [108, 134] satisfying equations analogous to Eqs (5.1.2) and (5.1.6). In this connection, along with the velocity operator $\hat{v}$ let us introduce a conductive velocity for the anti-diffusion mass flow density or, simply put, the *anti-diffusion velocity* (unlike that of the ordinary hydrodynamic velocity $\vec{\mathrm{v}}$) for an immovable or a slowly rotating spheroidal body [64, 67, 68, 73]:

$$\vec{\mathrm{u}} = \mathrm{G}(t)\frac{\nabla\rho}{\rho} = \mathrm{G}(t)\frac{\nabla(\rho/\rho_0)}{\rho/\rho_0} = \mathrm{G}(t)\,\mathrm{grad}\,\ln(\rho/\rho_0)\,. \tag{5.1.7}$$

Obviously, the anti-diffusion velocity $\vec{\mathrm{u}}$ of anti-diffusion mass flow density satisfies the well-known continuity equation of the kind:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\,\vec{\mathrm{u}}) = 0\,. \tag{5.1.8}$$

Using this continuity equation (5.1.8) we can calculate the partial derivative of the anti-diffusion velocity (5.1.7) with respect to the time [64, 67, 68, 73]:

$$\frac{\partial \vec{u}}{\partial t} = \frac{dG(t)}{dt}\,\mathrm{grad}\ln\rho + G(t)\,\mathrm{grad}\left[\frac{1}{\rho}\cdot\frac{\partial\rho}{\partial t}\right] =$$

$$= \frac{dG(t)}{dt}\left[\frac{1}{G(t)}\vec{u}\right] + G(t)\nabla\left[\frac{1}{\rho}(-\mathrm{div}(\rho\vec{u}))\right] =$$

$$= \frac{d\ln G(t)}{dt}\vec{u} - G(t)\nabla\left[\nabla\vec{u} + \vec{u}\frac{\nabla\rho}{\rho}\right] =$$

$$= -G(t)\,\mathrm{grad}(\mathrm{div}\,\vec{u}) - \mathrm{grad}(\vec{u}^2) + \frac{d\ln G(t)}{dt}\vec{u}. \qquad (5.1.9)$$

An advantage of the anti-diffusion velocity notion (5.1.7) versus the velocity operator notion (5.1.4) to be introduced is contained in the fact that the anti-diffusion velocity of liquid particles inside a slow-flowing gravitational compressible spheroidal body can become *observable* if the mass density of a centrally symmetric spheroidal body is very small. Indeed, according to Eq. (5.1.7) if the mass density $\rho \to 0$ then the anti-diffusion velocity $\vec{u} \to \infty$ (under the condition that $\mathrm{grad}\rho$ is to be finitary). The condition of smallness for the mass density ρ takes place in the molecular clouds of spread gas-dust matter in space [10]. Thus, as a result of spheroidal body formation from an initial weakly condensed gas-dust cloud, it might be a *sharp increase* of the anti-diffusion velocity of particles into the forming spheroidal body under the condition of the finiteness of the mass density gradient. Indeed, as shown in Section 4.6, if $f'(A_0) > 0$ in formula (4.6.8) then gravitational instability occurs, which is accompanied by an *avalanche-like gravitational compression* due to the gravitational field arising in a centrally symmetric spheroidal body when the parameter of gravitational condensation reaches its critical value α_c as a result of an exponential increase $\alpha(t) = \alpha(0)\cdot e^{f'(A_0)\cdot t}$ in accordance with (4.6.8).

In this case it is reasonable to rewrite Eq. (5.1.9) based on the familiar formulas of vector analysis [94, 111]:

$$\frac{1}{2}\,\mathrm{grad}\,\vec{u}^2 = \left(\vec{u}\cdot\nabla\right)\vec{u} + \left[\vec{u}\times \mathrm{rot}\,\vec{u}\right]; \tag{5.1.10a}$$

$$\nabla^2\,\vec{u} = \mathrm{grad}(\mathrm{div}\,\vec{u}) - \mathrm{rot}(\mathrm{rot}\,\vec{u})\,. \tag{5.1.10b}$$

Taking into account Eq. (5.1.7) one can see that $\mathrm{rot}\,\vec{u} = 0$, so that Eqs (5.1.10a, b) become respectively:

$$\mathrm{grad}\,\vec{u}^2 = 2\left(\vec{u}\cdot\nabla\right)\vec{u}\,; \tag{5.1.11a}$$

$$\nabla^2\,\vec{u} = \mathrm{grad}(\mathrm{div}\,\vec{u})\,. \tag{5.1.11b}$$

Substituting Eqs (5.1.11a, b) into Eq. (5.1.9) we obtain [68, 73]:

$$\frac{\partial\vec{u}}{\partial t} = -\mathrm{G}(t)\nabla^2\,\vec{u} - 2\left(\vec{u}\cdot\nabla\right)\vec{u} + \frac{d\ln \mathrm{G}(t)}{dt}\vec{u}\,. \tag{5.1.12}$$

Taking into account Eq. (5.1.11a) again, the equation (5.1.12) can be written as follows:

$$\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\nabla\right)\vec{u} = -\,\mathrm{grad}(\vec{u}^2/2) - \mathrm{G}(t)\nabla^2\,\vec{u} + \frac{d\ln \mathrm{G}(t)}{dt}\vec{u}\,. \tag{5.1.13}$$

The equation obtained, (5.1.13), is similar to the Navier–Stokes equation of motion of a viscous liquid [94, 111] under the condition that a gas-dust substance of a spheroidal body is isolated from the influence of external fields and $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$.

Now let us estimate the anti-diffusion velocity (5.1.7) of particles into a spherically symmetric gravitational compressible spheroidal body taking account of its mass density function (2.2.5):

$$\vec{u}(\vec{r},t) = \mathrm{G}(t)\nabla\ln\left(\frac{\rho}{\rho_0}\right) = \mathrm{G}(t)\nabla\left(-\alpha(t)\vec{r}^{\,2}/2\right) = -\mathrm{G}(t)\alpha(t)\vec{r}\,. \tag{5.1.14}$$

We can see that the anti-diffusion velocity $\vec{u}$ is expressed by the very simple relation (5.1.14) in the case of a sphere-like cosmogonical body. The obtained Eq. (5.1.14) recalls the formula of the *velocity of autowave front propagation* [66, 73] for gravitational strength magnitude in the remote zone of a slowly compressible gravitating spheroidal body, according to which: $\dot{r}_* = \frac{1}{2} \cdot \frac{\dot{G}(t)}{G(t)} \cdot r_* = -\frac{1}{2} \cdot G(t)\alpha(t) \cdot r_*$ if $t_0 = 0$ and $\ddot{\alpha}(t) = 0$ [66, 73].

Along with the anti-diffusion velocity, $\vec{u}$ there exists an ordinary *hydrodynamic velocity* $\vec{v}$ (or a convective velocity [108]). In principle, the hydrodynamic velocity $\vec{v}$ of mass flow arises as a result of powerful gravitational contraction of a centrally symmetric spheroidal body on the next (field) stages of its evolution. When $\alpha > \alpha_c$ the growing magnitude of gravitational field strength $\vec{a}$ induces the significant (i.e., *observable*) value of the hydrodynamic velocity $\vec{v}$ of mass flows moving into a spheroidal body. This means that the value of anti-diffusion velocity (5.1.7) becomes much less than the value of hydrodynamic velocity, that is:

$$|\vec{u}| << |\vec{v}| \ . \tag{5.1.15}$$

Under this condition (5.1.15), a common (hydrodynamic and anti-diffusion) mass flow density inside a centrally symmetric spheroidal body satisfies the hydrodynamic equation of continuity [94, 111]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \ . \tag{5.1.16}$$

Taking into account Eq. (5.1.16) we can also calculate the partial derivative of the anti-diffusion velocity (5.1.7) with respect to time [68, 73] in accordance with the condition (5.1.15):

$$\frac{\partial \vec{u}}{\partial t} = \frac{dG(t)}{dt} \text{grad} \ln \rho + G(t) \, \text{grad} \left\{ \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right\} =$$

$$= \frac{dG(t)}{dt} \left\{ \frac{1}{G(t)} \vec{u} \right\} + G(t) \nabla \left\{ \frac{1}{\rho} (-\text{div}(\rho \vec{v})) \right\} =$$

$$= \frac{d \ln G(t)}{dt} \vec{u} - G(t) \nabla \left\{ \nabla \vec{v} + \vec{v} \cdot \frac{\nabla \rho}{\rho} \right\} =$$

$$= -G(t) \, \text{grad}(\text{div} \vec{v}) - \text{grad}(\vec{v} \cdot \vec{u}) + \frac{d \ln G(t)}{dt} \vec{u}. \qquad (5.1.17)$$

As we know from a fluid-like description [94, 111], the complete time-derivative of the common (hydrodynamic plus anti-diffusion) velocity $\vec{v} + \vec{u}$ inside a centrally symmetric spheroidal body defines the common acceleration (or gravitational field strength) including the partial time-derivatives and convective derivatives [68, 73]:

$$\vec{a} = \frac{d(\vec{v} + \vec{u})}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \,. \qquad (5.1.18)$$

Taking into account Eq. (5.1.13) as well as Eq. (5.1.11a), the complete acceleration (5.1.18) can be represented in the form [68, 71, 73]:

$$\vec{a} = \frac{d(\vec{v} + \vec{u})}{dt} =$$

$$= \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} - (\vec{u} \cdot \nabla) \vec{u} - G(t) \nabla^2 \vec{u} + \frac{d \ln G(t)}{dt} \vec{u} \,. \qquad (5.1.19)$$

Let us note that since the mass density of a spheroidal body is directly proportional to the probability volume density function Φ according to Eq. (2.2.7), the anti-diffusion velocity (5.1.7) (or (5.1.14)) can be defined through the probability volume density function:

$$\vec{u} = G(t) \frac{\nabla \Phi}{\Phi} = G(t) \, \text{grad} \ln \Phi \,. \qquad (5.1.20)$$

Obviously, the anti-diffusion velocity (5.1.20) of probability volume flow density also satisfies Eqs (5.1.9), (5.1.13), (5.1.14), and (5.1.17)–(5.1.19).

Secondly, in the case of an *axially symmetric* spheroidal body $\varepsilon_0 \neq 0$, therefore using the general equation of anti-diffusion mass transfer (4.7.12) we can estimate an anti-diffusion velocity of liquid particles inside a rotating ellipsoid-like cosmogonical body [73, 77] taking into account its mass density formula (3.3.26a):

$$\vec{\mathrm{u}} = \widetilde{\mathrm{G}}(t)\frac{\nabla\rho}{\rho} = \widetilde{\mathrm{G}}(t)\,\mathrm{grad}\ln\left(\rho(h,z)/\rho_0\right) =$$
$$= \widetilde{\mathrm{G}}(t)\,\mathrm{grad}\left\{\ln\left(1-\varepsilon_0^2\right) - \alpha\left[h^2\left(1-\varepsilon_0^2\right)+z^2\right]/2\right\} =$$
$$= -\widetilde{\mathrm{G}}(t)\alpha\left\{\left(1-\varepsilon_0^2\right)h\cdot\vec{e}_h + z\cdot\vec{e}_z\right\} = \mathrm{u}_h\cdot\vec{e}_h + \mathrm{u}_z\cdot\vec{e}_z\,, \qquad (5.1.21)$$

where $\vec{e}_h$ and $\vec{e}_z$ are the basis vectors of cylindrical frame of reference and u_h and u_z are the radial h-projection and the axial z-projection of the anti-diffusion velocity respectively [73, 77]:

$$\mathrm{u}_h = -\widetilde{\mathrm{G}}(t)\alpha\left(1-\varepsilon_0^2\right)h\,; \qquad (5.1.22a)$$

$$\mathrm{u}_z = -\widetilde{\mathrm{G}}(t)\alpha z\,. \qquad (5.1.22b)$$

Taking into account that $\mathrm{rot}\,\vec{\mathrm{u}} = 0$ for the anti-diffusion velocity defined by Eq. (5.1.21) we can see that Eqs (5.1.11a, b) are also true. Thus, if we replace GCF $\mathrm{G}(t)$ on the generalized GCF $\widetilde{\mathrm{G}}(t)$ in the Eqs (5.1.9), (5.1.13), and (5.1.17)–(5.1.20) they remain valid in the general case of a rotating and gravitating ellipsoid-like cosmogonical body. For example, analogs of Eqs (5.1.17) and (5.1.19) in the case of an axially symmetric spheroidal body have the form:

$$\frac{\partial\vec{\mathrm{u}}}{\partial t} = -\widetilde{\mathrm{G}}(t)\,\mathrm{grad}(\mathrm{div}\,\vec{\mathrm{v}}) - \mathrm{grad}(\vec{\mathrm{v}}\cdot\vec{\mathrm{u}}) + \frac{d\ln\widetilde{\mathrm{G}}(t)}{dt}\vec{\mathrm{u}}\,; \qquad (5.1.23)$$

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} - (\vec{u}\cdot\nabla)\vec{u} - \widetilde{G}(t)\nabla^2\vec{u} + \frac{d\ln\widetilde{G}(t)}{dt}\vec{u}\,. \qquad (5.1.24)$$

5.2. The initial potential of an arising gravitational field, the initial gravitational strength induced by the anti-diffusion velocity, and the characterizing number K as a control parameter of dynamical states of a forming spheroidal body

Using the definitions (5.1.3) and (5.1.7) of the mass flow density $\vec{j}$ and the anti-diffusion velocity $\vec{u}$ let us investigate explicitly the case of an avalanche gravitational compression of a *centrally symmetric* spheroidal body as a consequence of an arising gravitational field in it (see Section 4.6, formula (4.6.8)). Firstly, we suppose that in the case of an *initial* gravitational instability (the quasi-equilibrium gravitational condensation under the condition of *unobservable* velocities of particles [16, 53, 68, 73]) the hydrodynamic velocity $\vec{v}$ of moving particles is absent practically, that is, $\vec{v}=0$. In addition, its partial and convective derivatives are also equal to zero:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = 0\,. \qquad (5.2.1)$$

Taking into account Eqs (5.1.7) and (5.2.1) as well as the simplified formulas of vector analysis (5.1.11a), (5.1.11b) and relation (5.1.19) takes the form [73, 77]:

$$\vec{a} = -\left[(\vec{u}\cdot\nabla)\vec{u} + G(t)\nabla^2\vec{u} - \frac{d\ln G(t)}{dt}\vec{u}\right] =$$

$$= -\nabla\left[\frac{\vec{u}^2}{2} + G(t)\,\mathrm{div}\vec{u} - \frac{d\ln G(t)}{dt}\cdot G(t)\ln\left(\frac{\rho}{\rho_0}\right)\right] =$$

$$= -\nabla\left[\frac{\vec{u}^2}{2} + G(t)\,\mathrm{div}\vec{u} - \dot{G}(t)\ln\left(\frac{\rho}{\rho_0}\right)\right], \qquad (5.2.2)$$

whence using Eqs (2.2.5) and (5.1.14) we can establish that an induced acceleration is calculated by the formula:

$$\vec{a} = -\nabla\left[\frac{1}{2}\mathrm{G}^2(t)\alpha^2(t)\vec{r}^{\,2} - 3\,\mathrm{G}^2(t)\alpha(t) + \dot{\mathrm{G}}(t)\left(\frac{1}{2}\alpha(t)r^2\right)\right] =$$

$$= -\alpha(t)\cdot\left[\mathrm{G}^2(t)\alpha(t) + \dot{\mathrm{G}}(t)\right]\vec{r}\ . \qquad (5.2.3)$$

According to the field theory [100], after a gravitational field becoming its strength $\vec{a}$ can be calculated directly by the gravitational potential φ_g: $\vec{a} = -\mathrm{grad}\varphi_g$, therefore, as follows from Eq. (5.2.3), the *arising* gravitational potential has to be equal to:

$$\varphi_g = \frac{1}{2}\mathrm{G}^2(t)\alpha^2(t)\left[r^2 - \frac{6}{\alpha(t)}\right] + \frac{1}{2}\dot{\mathrm{G}}(t)\alpha(t)r^2 \qquad (5.2.4a)$$

or:

$$\varphi_g = \frac{1}{2}\left[\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)\right]r^2 - 3\mathrm{G}^2(t)\alpha(t)\,. \qquad (5.2.4b)$$

As can be seen from Eq. (5.2.4b), the gravitational potential φ_g of a forming centrally symmetric spheroidal body is the sum of a *regular part* $\varphi_g^* = \frac{1}{2}\mathrm{G}^2(t)\alpha^2(t)r^2 - 3\mathrm{G}^2(t)\alpha(t)$ and a *fluctuation part* $\delta\varphi_g = \frac{1}{2}\dot{\mathrm{G}}(t)\alpha(t)r^2$. Moreover, if $\left|\dot{\mathrm{G}}(t)\right| << \left|\mathrm{G}(t)\right|$ then Eq. (5.2.4a) becomes the following:

$$\varphi_g = \varphi_g^* = \frac{1}{2}\mathrm{G}^2(t)\alpha^2(t)\left[r^2 - \frac{6}{\alpha(t)}\right]. \qquad (5.2.5)$$

Taking into account the definition of GCF (4.1.8), we can transform Eq. (5.2.5) for the arising gravitational potential of an initial gravitational field of a forming centrally symmetric spheroidal body [73, 77]:

$$\varphi_g^*(r,t) = \frac{1}{2}\mathrm{G}^2(t)\alpha^2(t)\left[r^2 - \frac{6}{\alpha(t)}\right] =$$

$$= \frac{1}{8}\cdot\frac{1}{\alpha^2(t)}\cdot\left(\frac{d\alpha}{dt}\right)^2\cdot\left[r^2 - \frac{6}{\alpha(t)}\right] = \frac{1}{4}\cdot\mathrm{G}(t)\dot{\alpha}(t)\cdot\left[r^2 - \frac{6}{\alpha(t)}\right]. \quad (5.2.6)$$

On the other hand, after obtaining a *critical* (threshold) value $\alpha = \alpha_c$, as mentioned in Sections 2.3 and 2.4 of Chapter 2, the gravitational field arises inside a sphere-like gaseous body (or a centrally symmetric spheroidal body), so that we can estimate the *inner* gravitational potential of the centrally symmetric spheroidal body (in the *near* zone I of the gravitational field) based on Eq. (2.4.29):

$$\varphi_g^{(\mathrm{I})} = -\frac{4\pi\gamma\rho_0}{\alpha r}\int_0^r\left(1 - \frac{\alpha}{2}x^2\right)dx = -\frac{4\pi\gamma\rho_0}{\alpha r}\left(r - \frac{\alpha}{6}r^3\right) =$$

$$= \frac{2\pi\gamma\rho_0}{3}\left(r^2 - \frac{6}{\alpha}\right), \quad (5.2.7)$$

where $\rho_0 = M(\alpha/2\pi)^{3/2}$ in accordance with formula (2.2.5). If we consider in Eq. (5.2.6) that

$$\alpha(t) \to \alpha_c \qquad \text{at} \qquad t \to t_c, \quad (5.2.8)$$

where $\alpha_c = \alpha(t_c)$ is a *critical value* of the parameter of gravitational condensation at the beginning of *gravitational field origin* $t = t_c$, then formulas (5.2.6) and (5.2.7) become practically identical to each other. This means that the origin of the gravitational field inside a forming centrally symmetric spheroidal body occurs at the time $t = t_c$ when the monotonically increasing function $\alpha(t)$ reaches its critical value α_c following the condition (5.2.8). By comparing Eq. (5.2.6) with Eq. (5.2.7) we obtain the equation [73]:

$$\frac{1}{8}\cdot\frac{1}{\alpha^2(t)}\cdot\left(\frac{d\alpha}{dt}\right)^2=\frac{2\pi\gamma\rho_0(\alpha_c)}{3}\,,\quad t\geq t_c\,, \tag{5.2.9}$$

which takes place if the following differential equation is true:

$$\left(\frac{\dot{\alpha}(t)}{\alpha(t)}\right)^2=C\,, \tag{5.2.10}$$

where $C=\dfrac{16\pi\gamma\rho_0}{3}$. As follows directly from Eq. (5.2.10):

$$\frac{\dot{\alpha}(t)}{\alpha(t)}=\pm\sqrt{C}=\pm4\sqrt{\frac{\pi\gamma\rho_0}{3}}\,. \tag{5.2.11}$$

Separating variables and integrating this differential equation (5.2.11) we obtain:

$$\int_{\alpha_c}^{\alpha}\frac{d\alpha}{\alpha}=\pm4\sqrt{\frac{\pi\gamma\rho_0}{3}}\cdot\int_{t_c}^{t}dt\,,$$

whence:

$$\ln\frac{\alpha(t)}{\alpha_c}=\pm4\sqrt{\frac{\pi\gamma\rho_0}{3}}\cdot(t-t_c)\,, \tag{5.2.12}$$

where $\alpha_c=\alpha(t_c)$. Ultimately, as is obvious from (5.2.12) the solutions of Eq. (5.2.10) are the following [73]:

$$\alpha(t)=\alpha_c\cdot e^{4\sqrt{\pi\gamma\rho_0/3}\cdot(t-t_c)}\,; \tag{5.2.13a}$$

$$\alpha(t)=\alpha_c\cdot e^{-4\sqrt{\pi\gamma\rho_0/3}\cdot(t-t_c)}\,. \tag{5.2.13b}$$

So, the temporal evolution of the parameter of gravitational condensation at the instant of gravitational field origin inside a forming centrally symmetric spheroidal body is described by dependencies (5.2.13a, b). If we consider a forming centrally symmetric spheroidal body like the 1D dynamical system (see Section 4.6) then both obtained solutions (5.2.13a) and (5.2.13b) are solutions of the Cauchy linearized problem of the kind (4.6.7):

$$\alpha(t) = \alpha(0) \cdot e^{f'(\mathrm{A}_0) \cdot t}, \qquad (5.2.14)$$

where $f'(\mathrm{A}_0) = 4\sqrt{\pi\gamma\rho_0 / 3}$ or $f'(\mathrm{A}_0) = -4\sqrt{\pi\gamma\rho_0 / 3}$ if $t_c = 0$. In particular, when $f'(\mathrm{A}_0) < 0$ then the solution is asymptotically stable in Lyapunov's meaning. Otherwise, when $f'(\mathrm{A}_0) > 0$ the *gravitational instability* occurs if $t \geq t_c$, that is the *avalanche gravitational compression* as a consequence of an arising gravitational field into a centrally symmetric spheroidal body [73]. In this case, according to Eq. (5.2.13a), the parameter of gravitational condensation $\alpha(t)$ increases exponentially up to its *stabilization value*:

$$\alpha_s = \alpha_c \cdot e^{4\sqrt{\pi\gamma\rho_0/3}\cdot(t_s - t_c)}, \qquad (5.2.15)$$

where:

$$\rho_0(\alpha_c) = M(\alpha_c / 2\pi)^{3/2} \qquad (5.2.16)$$

and $\alpha_s = \alpha(t_s)$, t_s is a stabilization instant. Thus, the process of avalanche gravitational compression forms a dense bunch (core) with the central mass density:

$$\rho_s = \rho_0(\alpha_s) = M(\alpha_s / 2\pi)^{3/2}, \quad t \geq t_s \qquad (5.2.17)$$

as well as the *inner* gravitational potential $\varphi_g^{(\mathrm{I})}$ following the formula (5.2.7):

$$\varphi_g^{(\mathrm{I})} = \frac{2\pi\gamma\rho_0(\alpha_s)}{3} \cdot \left(r^2 - \frac{6}{\alpha_s} \right), \quad t \geq t_s. \qquad (5.2.18a)$$

Generalizing (5.2.18a) we can also estimate the gravitational potential φ_g in accordance with the formula (2.4.27) [16, 73]:

$$\varphi_g(r, \alpha_s) = -\frac{4\pi\gamma\rho_0(\alpha_s)}{\alpha_s r} \sqrt{\frac{2}{\alpha_s}} \int_0^{r\sqrt{\alpha_s/2}} e^{-s^2} ds, \quad t \geq t_s. \qquad (5.2.18b)$$

Similar reasoning takes place in the case of a rotating *axially symmetric* spheroidal body, that is, when $\varepsilon_0 \neq 0$ we also suppose that in the case of an *initial* gravitational instability (under condition of *unobservable* velocities of particles [68, 73, 77]) the hydrodynamic velocity $\vec{\mathrm{v}}$ of moving particles and its partial and convective derivatives are equal to zero in accordance with (5.2.1). Taking into account Eqs (5.1.21) and (5.2.1) as well as the simplified formulas of vector analysis (5.1.11a,b), the relation (5.1.24) takes the form [73, 77]:

$$\vec{a} = -\left[\left(\vec{\mathrm{u}}\cdot\nabla\right)\vec{\mathrm{u}} + \widetilde{\mathrm{G}}(t)\nabla^2\,\vec{\mathrm{u}} - \frac{d\ln\widetilde{\mathrm{G}}(t)}{dt}\vec{\mathrm{u}} \right] =$$

$$= -\nabla\left[\frac{\vec{\mathrm{u}}^2}{2} + \widetilde{\mathrm{G}}(t)\,\mathrm{div}\vec{\mathrm{u}} - \frac{d\ln\widetilde{\mathrm{G}}(t)}{dt}\cdot\widetilde{\mathrm{G}}(t)\ln\left(\frac{\rho}{\rho_0}\right) \right] =$$

$$= -\nabla\left[\frac{\vec{\mathrm{u}}^2}{2} + \widetilde{\mathrm{G}}(t)\,\mathrm{div}\vec{\mathrm{u}} - \dot{\widetilde{\mathrm{G}}}(t)\ln\left(\frac{\rho}{\rho_0}\right) \right]. \qquad (5.2.19)$$

Using estimations of the radial h-projection (5.1.22a) and the axial z-projection (5.1.22b) of anti-diffusion velocity (5.1.21) of particles as well as Eqs (3.3.26a), an acceleration (or initial gravitational field strength) induced by the anti-diffusion process inside an axially symmetric spheroidal body (ellipsoid-like cloud) is calculated by the formula [73, 77]:

$$\vec{a} = -\nabla[\mathrm{u}_h^2/2 + \mathrm{u}_z^2/2 + \widetilde{\mathrm{G}}(t)\cdot\mathrm{div}\left(\mathrm{u}_h\cdot\vec{e}_h + \mathrm{u}_z\cdot\vec{e}_z\right) - \dot{\widetilde{\mathrm{G}}}(t)\ln\left(\rho/\rho_0\right)] =$$

$$= -\,\mathrm{grad}[\,\frac{1}{2}\widetilde{\mathrm{G}}^2(t)\alpha^2\left\{h^2\left(1-\varepsilon_0^2\right)^2 + z^2\right\} - \widetilde{\mathrm{G}}^2(t)\alpha\left\{2\left(1-\varepsilon_0^2\right)+1\right\} +$$

$$+\frac{1}{2}\dot{\widetilde{\mathrm{G}}}(t)\left\{\alpha(h^2(1-\varepsilon_0^2)+z^2) - 2\ln\left(1-\varepsilon_0^2\right)\right\}]. \qquad (5.2.20)$$

According to Eq. (5.2.20), when a gravitational field is arising, its strength $\vec{a}$ can be calculated directly through the gravitational potential φ_g: $\vec{a} = -\mathrm{grad}\varphi_g$, therefore:

$$\varphi_g = \frac{1}{2}\widetilde{\mathrm{G}}^2(t)\alpha^2(t)\left[\left(1-\varepsilon_0^2\right)^2 h^2 + z^2 - \frac{4(1-\varepsilon_0^2)+2}{\alpha(t)}\right] +$$

$$+\frac{1}{2}\dot{\tilde{G}}(t)\alpha(t)\left\{(1-\varepsilon_0^2)h^2+z^2-\frac{2\ln\left(1-\varepsilon_0^2\right)}{\alpha(t)}\right\}. \qquad (5.2.21)$$

Analogously to Eq. (5.2.4b), the *arising* gravitational potential φ_g of a forming axially symmetric spheroidal body (5.2.21) is the sum of a *regular* part

$$\varphi_g^* = \frac{1}{2}\tilde{G}^2(t)\alpha^2(t)\left[\left(1-\varepsilon_0^2\right)^2 h^2+z^2-\frac{4(1-\varepsilon_0^2)+2}{\alpha(t)}\right]$$

and a fluctuation part

$$\delta\varphi_g = \frac{1}{2}\dot{\tilde{G}}(t)\alpha(t)\left\{(1-\varepsilon_0^2)h^2+z^2-\frac{2\ln\left(1-\varepsilon_0^2\right)}{\alpha(t)}\right\}$$

if $\left|\dot{\tilde{G}}(t)\right| << \left|\tilde{G}(t)\right|$, so that in this case Eq. (5.2.21) takes the form [77]:

$$\varphi_g = \varphi_g^* = \frac{1}{2}\tilde{G}^2(t)\alpha^2(t)\left[\left(1-\varepsilon_0^2\right)^2 h^2+z^2-\frac{4(1-\varepsilon_0^2)+2}{\alpha(t)}\right]. \qquad (5.2.22)$$

Obviously, if $\varepsilon_0 \to 0$ then formula (5.2.22) becomes the formula of the arising gravitational potential (5.2.5) (or (5.2.18a)) of a *forming centrally symmetric* spheroidal body.

Let us note that according to the Jeans criterion (1.3.25) developed within the framework of his linearized theory of gravitational instability [1 p. 345–350] we can estimate the critical value of mass density ρ_c through the critical wavelength λ_c or the critical circular frequency ω_c of propagation of condensations and rarefactions in a medium:

$$\rho_c = \frac{\omega_c^2}{4\pi\gamma}. \qquad (5.2.23)$$

By comparing Eq. (5.2.23) with Eq. (5.2.16) we can express the critical parameter of gravitational condensation α_c

through the critical circular frequency ω_c of *propagation of disturbances* ("the initial condensations and rarefactions increasing exponentially with the time" [1 p. 348]) in a forming centrally symmetric spheroidal body (see the same formula (2.3.7b) in Section 2.3 of Chapter 2):

$$\alpha_c = \left(\frac{\pi}{2}\right)^{1/3} \cdot \left(\frac{\omega_c{}^2}{\gamma M}\right)^{2/3} . \qquad (5.2.24)$$

As the parameter of gravitational condensation $\alpha(t)$ reaches its critical value α_c, the propagation of gravitational disturbances with the circular frequency $\omega < \omega_c$ (or the wavelength $\lambda > \lambda_c$) ceases to be wave-motion and leads to an unstable motion according to the Jeans criterion (1.3.25). In this case, the parameter of gravitational condensation increases exponentially with the time in accordance with Eq. (5.2.13a) whereas GCF decreases exponentially with the time [73]:

$$\mathrm{G}(t) = \frac{1}{2\left(\alpha_c \cdot e^{4\sqrt{\pi\gamma\rho_0/3}\cdot t}\right)^2} \cdot \frac{d}{dt}\left(\alpha_c \cdot e^{4\sqrt{\pi\gamma\rho_0/3}\cdot t}\right) =$$

$$= \frac{2\sqrt{\pi\gamma\rho_0/3}}{\alpha_c} \cdot e^{-4\sqrt{\pi\gamma\rho_0/3}\cdot t} , \qquad (5.2.25a)$$

so that the anti-diffusion equation (4.1.9a) of gravitational compression of an immovable (of slowly rotating) centrally symmetric spheroidal body takes the form:

$$\frac{\partial \rho}{\partial t} = -\frac{2\sqrt{\pi\gamma\rho_0/3}}{\alpha(t)} \nabla^2 \rho . \qquad (5.2.25b)$$

Thus, the above-mentioned arguments under the derivation of Eqs (5.1.1)–(5.1.20) in Section 5.1 confirm entirely the scenario of a gravitational field arising (see Eqs (5.2.2)–(5.2.6)) based on a transfer of the anti-diffusion velocity of particles motion into a forming centrally symmetric spheroidal body. The same concerns the scenario of the gravitational field arising into a forming axially symmetric spheroidal body (see Eqs (5.2.19)–(5.2.22)).

Let us consider a general approach to investigate different dynamical states of a forming spheroidal body. Using Eqs (5.1.17), (5.1.19) and the simplified formulas of vector analysis (5.1.11a,b), we can carry out an *analysis* of dynamical states of a forming centrally symmetric spheroidal body by introducing the scales of physical values $T, L, V, U, F, \mathrm{G}_s$, and the respective dimensionless variables $\tau, \vec{\xi}, \vec{v}, \vec{u}, \vec{f}, g$ as follows:

$$t = T\tau; \quad \vec{r} = L\vec{\xi}; \quad \vec{\mathrm{v}} = V\vec{v}; \quad \vec{\mathrm{u}} = U\vec{u};$$
$$\vec{a} = F\vec{f}; \quad \mathrm{G}(t) = \mathrm{G}_s g(t). \tag{5.2.26}$$

By substituting Eqs (5.2.26) into Eqs (5.1.17) and (5.1.19) and taking into account simplified formulas (5.1.11a, b) we obtain:

$$\frac{U}{T}\frac{\partial \vec{u}}{\partial \tau} = -\mathrm{G}(t)\frac{V}{L^2}\mathrm{grad}(\mathrm{div}\vec{v}) - \frac{VU}{L}\mathrm{grad}(\vec{v}\cdot\vec{u}) +$$
$$+\frac{U}{T}\frac{d\ln \mathrm{G}(\tau)}{d\tau}\vec{u}; \tag{5.2.27a}$$

$$\frac{V}{T}\frac{\partial \vec{v}}{\partial \tau} = F\vec{f} - \frac{V^2}{L}(\vec{v}\cdot\nabla)\vec{v} + \frac{U^2}{L}\mathrm{grad}(\vec{u}^2/2) +$$

$$+\mathrm{G}(t)\frac{U}{L^2}\mathrm{grad}(\mathrm{div}\vec{u}) - \frac{U}{T}\frac{d\ln \mathrm{G}(t)}{dt}\vec{u}\cdot \tag{5.2.27b}$$

Similarly to [111], dividing Eq. (5.2.27b) by V^2/L and Eq. (5.2.27a) by VU/L we derive the following dimensionless equations:

$$\mathrm{Sh}\frac{\partial \vec{u}}{\partial \tau} = -\frac{\mathrm{G}_s}{\nu}\cdot\frac{1}{\mathrm{K}\cdot\mathrm{Re}}\mathrm{grad}(\mathrm{div}\vec{v}) - \mathrm{grad}(\vec{v}\cdot\vec{u}) +$$

$$+\,\mathrm{Sh}\,\frac{d\ln g(t)}{dt}\vec{u}\,; \qquad (5.2.28a)$$

$$\mathrm{Sh}\,\frac{\partial\vec{v}}{\partial\tau}=\frac{1}{\mathrm{Fr}}\vec{f}-(\vec{v}\cdot\nabla)\vec{v}+\mathrm{K}^2\mathrm{grad}(\vec{u}^2/2)+$$

$$+\frac{\mathrm{G}_s}{\nu}\cdot\frac{\mathrm{K}}{\mathrm{Re}}g(t)\,\mathrm{grad}(\mathrm{div}\vec{u})-\mathrm{Sh}\cdot\mathrm{K}\,\frac{d\ln g(t)}{dt}\vec{u}\,, \qquad (5.2.28b)$$

where

$\mathrm{Sh}=L/VT$ is the Strouhal number,
$\mathrm{Fr}=V^2/FL$ is the Froude number,
$\mathrm{Re}=VL/\nu$ is the Reynolds number (ν is a kinematic coefficient of viscosity of flow of liquid particles [111]), and
$\mathrm{K}=U/V$ is a new number of similarity.

The new number of similarity is a measure of the values $|\vec{\mathrm{u}}|$ versus $|\vec{\mathrm{v}}|$ prevailing:

$$\mathrm{K}=\frac{|\vec{\mathrm{u}}|}{|\vec{\mathrm{v}}|}. \qquad (5.2.29)$$

When this similarity number exceeds unity ($\mathrm{K}>>1$) then the anti-diffusion condensation of a centrally symmetric spheroidal body occurs exclusively, so that the value of hydrodynamic velocity is negligible ($|\vec{\mathrm{v}}|\approx 0$) because a gravitational field is absent in practice. If the similarity number becomes close to unity ($\mathrm{K}\approx 1$) then the hydrodynamic velocity $\vec{\mathrm{v}}$ of mass flow arises as a result of a gravitational contraction of a centrally symmetric spheroidal body on the field stage of its evolution. As mentioned with regard to Eq. (5.1.15) in Section 5.1, the value of anti-diffusion velocity (5.1.7) becomes much less than the value of hydrodynamic velocity $|\vec{\mathrm{u}}|<<|\vec{\mathrm{v}}|$ when $\mathrm{K}<<1$. This means that the growing magnitude of powerful gravitational field strength $\vec{a}$ induces the significant value of the hydrodynamic velocity $\vec{\mathrm{v}}$ of mass flows moving into a centrally symmetric spheroidal body. Thus, like the Mach number M [111] the new number of similarity K is a control parameter of dynamical states of a forming centrally symmetric spheroidal body.

As shown in [77], in the case of a rotating axially symmetric spheroidal body ($\varepsilon_0 \neq 0$) we can also obtain equations analogous to Eqs (5.2.28a) and (5.2.28b):

$$\mathrm{Sh}\frac{\partial \vec{u}}{\partial \tau} = -\frac{\widetilde{\mathrm{G}}_s}{\nu}\cdot\frac{1}{\mathrm{K}\cdot\mathrm{Re}}\mathrm{grad}(\mathrm{div}\vec{v}) - \mathrm{grad}(\vec{v}\cdot\vec{u}) + \\ + \mathrm{Sh}\frac{d\ln g(t)}{dt}\vec{u}\,; \qquad (5.2.30a)$$

$$\mathrm{Sh}\frac{\partial \vec{v}}{\partial \tau} = \frac{1}{\mathrm{Fr}}\vec{f} - (\vec{v}\cdot\nabla)\vec{v} + \mathrm{K}^2\mathrm{grad}(\vec{u}^2/2) + \\ + \frac{\widetilde{\mathrm{G}}_s}{\nu}\cdot\frac{\mathrm{K}}{\mathrm{Re}}g(t)\,\mathrm{grad}(\mathrm{div}\vec{u}) - \mathrm{Sh}\cdot\mathrm{K}\frac{d\ln g(t)}{dt}\vec{u}\,. \qquad (5.2.30b)$$

In particular, in the special case $\mathrm{K} >> 1$ corresponding to the *initial quasi-equilibrium* gravitational condensation state the dimensionless Eqs (5.2.28a, b) are reduced to one dimensionless equation of the kind:

$$\mathrm{K}\mathrm{grad}(\vec{u}^2/2) + \frac{\mathrm{G}_s}{\nu}\cdot\frac{1}{\mathrm{Re}}g(t)\,\mathrm{grad}(\mathrm{div}\vec{u}) = \mathrm{Sh}\cdot\frac{\partial\vec{u}}{\partial\tau}, \qquad (5.2.31)$$

which corresponds to the following equation:

$$\mathrm{grad}(\vec{\mathrm{u}}^2/2) + \mathrm{G}(t)\,\mathrm{grad}(\mathrm{div}\vec{\mathrm{u}}) = \frac{\partial\vec{\mathrm{u}}}{\partial t}. \qquad (5.2.32)$$

Except for the anti-diffusion solution, the equation (5.2.32) has a *wave* solution in the vicinity of equilibrium state when $\mathrm{G}_s = \mathrm{const}$ and $|\vec{\mathrm{u}}| < 1$:

$$\vec{\mathrm{u}} = \vec{\mathrm{u}}_0\, e^{\pm \mathrm{i}\vec{k}\vec{r} - \vec{k}^2\mathrm{G}_s t} = -\mathrm{G}_s\alpha_s\vec{r}_0 e^{\pm \mathrm{i}\vec{k}\vec{r} - \vec{k}^2\mathrm{G}_s t},\ \mathrm{i} = \sqrt{-1}. \qquad (5.2.33)$$

In the initial quasi-equilibrium gravitational condensation state, the wave solution (5.2.33) is generated. Moreover, it induces specific wave force:

$$\vec{f} = -\mathrm{G}(t)\,\mathrm{grad}(\mathrm{div}\,\vec{\mathrm{u}}) = -\mathrm{G}(t)\mathrm{G}_s\alpha_s\vec{k}^2 e^{\pm \mathrm{i}\vec{k}\vec{r} - \vec{k}^2\mathrm{G}_s t}\vec{r}_0 . \quad (5.2.34)$$

Like Eq. (4.2.17*), if we determine G_s as iG_s, the wave solution (5.2.33) determines an additional periodic force of the Alfvén–Arrhenius kind [9, 19]:

$$\vec{f}_a = \mp\,\mathrm{G}(t)\,\mathrm{G}_s\,\alpha_s\vec{k}^2 e^{\pm \mathrm{i}(\vec{k}\vec{r} - \vec{k}^2\,\mathrm{G}_s\,t)}\vec{r}_0 \quad (5.2.35)$$

and respective spatial oscillations in the different domains of a forming centrally symmetric spheroidal body (see the next section).

5.3. The equilibrium dynamical states after the origin of a gravitational field inside a forming spheroidal body

Let us consider a frequency interpretation of the gravitational potential and the gravitational strength of a forming centrally symmetric spheroidal body (*sphere-like* cosmogonical body). As shown in Section 5.2, the acceleration (5.2.3) induced by anti-diffusion velocity in a forming centrally symmetric spheroidal body is calculated by the formula:

$$\vec{a}(\vec{r}) = -\alpha(t)\cdot[\mathrm{G}^2(t)\alpha(t) + \dot{\mathrm{G}}(t)]\,\vec{r} . \quad (5.3.1)$$

According to Newton's second law, the equation of motion of a particle under action of specific force $\vec{a}$ into a forming centrally symmetric spheroidal body is the following:

$$\frac{d^2\vec{r}}{dt^2} + \alpha(t)\cdot[\mathrm{G}^2(t)\alpha(t) + \dot{\mathrm{G}}(t)]\,\vec{r} = 0 . \quad (5.3.2)$$

Since Eq. (5.3.2) is a harmonic oscillator equation, then the inducible acceleration $\vec{a}$ based on the anti-diffusion process has to be *the oscillating specific force* (because $\vec{a}(0) = 0$) with the circular frequency of the *radial* oscillations:

$$\omega(t) = \sqrt{\alpha(t)\cdot[\mathrm{G}^2(t)\alpha(t) + \dot{\mathrm{G}}(t)]}\,, \quad (5.3.3)$$

so that, at the stage of a gravitational field formation, the following representation for $\omega(t)$ in accordance with the formula (5.3.3) is true [77]:

$$\omega^2(t) = \omega^{*2}(t) + (\delta\varpi)^2(t), \qquad (5.3.4)$$

where:

$$\omega^{*2}(t) = \mathrm{G}^2(t)\alpha^2(t); \qquad (5.3.5a)$$

$$(\delta\varpi)^2(t) = \dot{\mathrm{G}}(t)\alpha(t) \qquad (5.3.5b)$$

and $\delta\varpi$, generally speaking, is a generalized circular frequency since the value $\dot{\mathrm{G}}(t)$ can be negative ($\dot{\mathrm{G}}(t) < 0$), that is, $\delta\varpi(t) = \delta\omega(t) + \mathrm{i}\,\delta\gamma(t)$, $\mathrm{i} = \sqrt{-1}$. This representation (5.3.4) is equivalent to the mentioned expansion of gravitational potential (5.2.4a, b) of a forming centrally symmetric spheroidal body as the sum $\varphi_g = \varphi_g^* + \delta\varphi_g$ of the regular part $\varphi_g^* = \mathrm{G}^2(t)\alpha^2(t)r^2/2 - 3\mathrm{G}^2(t)\alpha(t)$ and the fluctuation part $\delta\varphi_g = \dot{\mathrm{G}}(t)\alpha(t)r^2/2$. Consequently, according to Eq. (5.3.4), the regular part for $\omega^2(t)$ can be a squared main circular frequency of radial oscillations (or angular velocity of rotation) $\omega^{*2}(t)$ whereas its fluctuation part is a squared generalized circular frequency $(\delta\varpi)^2(t)$ of fluctuations. Indeed, by substituting Eqs (5.3.5a, b) into Eq. (5.3.1) we obtain:

$$\vec{a} = [\omega^{*2}(t) + (\delta\varpi)^2(t)]\vec{r} = \vec{\omega}^{*2}(t)\vec{r} + (\delta\varpi)^2(t)\vec{r} =$$
$$= [\vec{\omega}^*(t)\times[\vec{r}\times\vec{\omega}^*(t)]] + (\vec{\omega}^*(t)\cdot\vec{r})\vec{\omega}^*(t) + (\delta\varpi)^2(t)\vec{r}, \qquad (5.3.6)$$

because:

$$\vec{\omega}^{*2}\,\vec{r} = [\vec{\omega}^*\times[\vec{r}\times\vec{\omega}^*]] + (\vec{\omega}^*\cdot\vec{r})\,\vec{\omega}^* = \vec{f}_c + (\vec{\omega}^*\cdot\vec{r})\,\vec{\omega}^*, \qquad (5.3.7)$$

where $\vec{f}_c$ is a specific (per mass unit) centrifugal force.

As follows directly from Eqs (5.3.6) and (5.3.7), there exists no absolutely immovable spheroidal body *after the emergence of a gravitational field* but there are the very slow or slow rotating spheroidal bodies. Here we intend to consider the cases of mechanical equilibrium and quasi-equilibrium for a *slowly rotating spheroidal body*, that is, for a centrally symmetric spheroidal body.

Let us consider two important cases of equilibrium dynamical states for a centrally symmetric spheroidal body [205].

1) An induced acceleration $\vec{a}$ in Eq. (5.3.1) becomes the *regular gravitational field strength* $\vec{a}^* = \vec{f}_{\mathrm{g}}$ under the condition of *mechanical equilibrium* when $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$:

$$\vec{a}^* = -\operatorname{grad}\varphi_{\mathrm{g}}\Big|_{\mathrm{equil}} = -\operatorname{grad}\varphi_{\mathrm{g}}^*\Big|_{\mathrm{equil}} = -\mathrm{G}_s^2\,\alpha^2\vec{r}\,, \qquad (5.3.8)$$

because $\dot{\mathrm{G}}(t) = 0$ and $(\delta\varpi)^2(t) = 0$ based on Eq. (5.3.5b). According to (5.3.3), (5.3.4), and (5.3.5a, b) the main circular frequency of radial oscillations being an angular velocity of rotation $\omega^* = \Omega$ in the case of full mechanical equilibrium is equal to the following:

$$\Omega = \left|\vec{\Omega}\right| = \mathrm{G}_s\,\alpha\,. \qquad (5.3.9)$$

In the mechanical quasi-equilibrium state $\vec{\Omega} \perp \vec{r}$ under *stable* rotation, that is, $(\vec{\Omega}\cdot\vec{r})\vec{\Omega} = 0$ in the expression: $\vec{\Omega}^2\,\vec{r} = [\vec{\Omega}\times[\vec{r}\times\vec{\Omega}]] + (\vec{\Omega}\cdot\vec{r})\vec{\Omega}$, so that Eq. (5.3.8) becomes:

$$\vec{a}^* = -\Omega^2\,\vec{r} = -[\vec{\Omega}\times[\vec{r}\times\vec{\Omega}]] = -\vec{f}_{\mathrm{c}}\,. \qquad (5.3.10)$$

Therefore the regular gravitational acceleration $\vec{a}^*$ (or the specific force of gravity $\vec{f}_{\mathrm{g}}$) is compensated by the specific centrifugal force $\vec{f}_{\mathrm{c}}$ completely: $\vec{f}_{\mathrm{g}} + \vec{f}_{\mathrm{c}} = 0$.

As shown in Section 4.1 (see (4.1.22) and (4.1.27)), while GCF remains a fixed value under condition of mechanical equilibrium: $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$, the parameter of gravitational condensation $\alpha(t)$ monotonically increases with time as a linear function approximately:

$$\alpha(t) = \alpha_s \left[1 - 2\alpha_s \mathrm{G}_s (t - t_s)\right]^{-1} \approx \alpha_s [1 + 2\alpha_s \mathrm{G}_s (t - t_s)], \qquad (5.3.11)$$

where $\alpha_s = \alpha(t_s)$ is a *stabilization value* of the parameter of gravitational condensation at the instant of GCF stabilization $t = t_s$. Then the basic anti-diffusion Eq. (4.1.9a) of gravitational condensation of a slowly rotating spheroidal body takes the form in the case of virial equilibrium:

$$\frac{\partial \rho}{\partial t} = -\mathrm{G}_s \nabla^2 \rho . \qquad (5.3.12)$$

2) Under the condition of *mechanical quasi-equilibrium*, the small fluctuations of gravitational field strength $\vec{a}$ in a centrally symmetric spheroidal body are possible, so that, generally speaking, although $\dot{\mathrm{G}}(t) \neq 0$, $\mathrm{G}(t) \approx \mathrm{G}_s$. Therefore, taking into account Eq. (5.3.1) as well as Eq. (5.3.8), we obtain:

$$\vec{a} = -\,\mathrm{grad}\,\varphi_{\mathrm{g}}\Big|_{\mathrm{quasiequil}} = -\,\mathrm{grad}(\varphi^*_{\mathrm{g}}\Big|_{\mathrm{equil}} + \delta\varphi_{\mathrm{g}}) =$$
$$= \vec{a}^* + \delta\vec{a} = -\mathrm{G}_{\mathrm{s}}^2 \alpha^2 \vec{r} - \dot{\mathrm{G}}(t)\alpha\vec{r} \, . \qquad (5.3.13)$$

According to Eqs (5.3.1), (5.3.9), and (5.3.10) the relation (5.3.13) becomes [77]:

$$\vec{a} = \vec{a}^* + \delta\vec{a} = -\Omega^2 \vec{r} - \dot{\mathrm{G}}(t)\alpha\vec{r} =$$
$$= -[\vec{\Omega} \times [\vec{r} \times \vec{\Omega}]] - (\delta\varpi)^2 (t)\vec{r} = -\vec{f}_c + \vec{f}_a , \qquad (5.3.14)$$

where $\vec{f}_{\mathrm{c}}$ is the mentioned (in the previous case) specific centrifugal force and $\vec{f}_a$ is a specific additional periodic force of Alfvén–Arrhenius type (5.2.35) respectively:

$$\vec{f}_c = \Omega^2 \vec{r} = [\vec{\Omega} \times [\vec{r} \times \vec{\Omega}]] ; \qquad (5.3.15a)$$

$$\vec{f}_a = -\dot{\mathrm{G}}(t)\alpha\vec{r} = -(\delta\varpi(t))^2 \vec{r} , \qquad (5.3.15b)$$

whence we conclude that the gravitational acceleration $\vec{a}$, more exactly, the specific force of gravity $\vec{f}_g$ is balanced by the vector sum of $\vec{f}_c$ and $\vec{f}_a$, that is, $\vec{f}_g + \vec{f}_c = \vec{f}_a$.

The obtained Eq. (5.3.15b) shows that a small temporal deviation (from $\mathrm{G_s}$) of GCF $\mathrm{G}(t)$ for a spherically symmetric spheroidal body under the condition of mechanical quasi-equilibrium is determined by an *oscillation* behavior of its derivative $\dot{\mathrm{G}}(t)$. This implies the special cases both $\dot{\mathrm{G}}(t) < 0$ and $\dot{\mathrm{G}}(t) > 0$, that is, $(\delta\varpi)^2 < 0$ and $(\delta\varpi)^2 > 0$. Therefore, if the squared *generalized circular frequency* $(\delta\varpi)^2 < 0$ then, according to Eq. (5.3.15b), the additional periodic force $\vec{f}_a$ becomes oriented *opposite* the gravitational force $\vec{f}_{\mathrm{g}}$ (see also Sections 9.2, 9.5 in the next Chapter 9).

According to Eq. (5.2.35) and (5.3.9) the specific additional periodic force of Alfvén–Arrhenius can be written in the form:

$$\vec{f}_a = -(\delta\varpi(t))^2 \vec{r} = -\Omega\,\omega(t) e^{\mathrm{i}(\vec{k}\vec{r} - \omega t)} \vec{r}\,, \qquad (5.3.16)$$

where $\omega(t) = \mathrm{G}(t)\vec{k}^2$ is a main circular frequency of radial oscillations. Since $\omega = A\Omega$, $A = \text{const}$, moreover, $\omega \approx \Omega$ in Eq. (5.3.16) under the condition of equilibrium, then the squared generalized circular frequency $(\delta\varpi)^2(t)$ of fluctuations is equal to:

$$(\delta\varpi(t))^2 \approx A\Omega^2 e^{\mathrm{i}(\vec{k}\vec{r} - \Omega t)}\,. \qquad (5.3.17)$$

We should note that the mechanical quasi-equilibrium case corresponds to the virial theorem in Poincaré–Eddington's form applied to a centrally symmetric spheroidal body being in unstable radial motion (see Theorem 4.1) [73]:

$$2E_{\mathrm{k}} + E_{\mathrm{g}} = -2M\dot{\mathrm{G}}(t)\,, \qquad (5.3.18)$$

where

E_{k} is the total kinetic energy,

E_{g} is the total gravitational potential energy of a centrally symmetric spheroidal body in the form of a collection (cloud) of particles moving under no forces except their mutual gravitational attraction,

$\dot{\mathrm{G}}(t)$ is a derivative of GCF $\mathrm{G}(t)$ of a centrally symmetric spheroidal body, and

M is its total mass.

According to the reasoning relative to Eqs (5.3.13)–(5.3.15a, b), the GCF derivative changing $\dot{\mathrm{G}}(t)$ has an *oscillating character* in Eq. (5.3.18), namely, in the case of a *quasi-equilibrium* state. The more simple mechanical equilibrium occurs if $\dot{\mathrm{G}}(t) = 0$, that is, when the Poincaré–Eddington virial theorem becomes the Poincaré virial theorem (see Theorem 1.3 in Section 1.2 and Section 4.1).

In spite of the almost linear increasing parameter of gravitational condensation (5.3.11) we can note that its average integral value remains stable during the main period T of oscillations of a gravitational field:

$$\alpha_s = \overline{\alpha(t)} = \frac{1}{T}\int_{t_\mathrm{s}}^{t_\mathrm{s}+T} \alpha(t)dt \ , \tag{5.3.19a}$$

where t_s is a stabilization instant of time, as well as in quasi-equilibrium state the following equation is true:

$$\overline{\mathrm{G}(t)} = \mathrm{G}_s = \mathrm{const}\,. \tag{5.3.19b}$$

Indeed, the following theorem results [205]:

Theorem 5.1. The average integral value of the parameter of gravitational condensation does not depend on the duration of the period averaging both in equilibrium and in quasi-equilibrium states of a centrally symmetric (or slowly rotating) spheroidal body:

$$\overline{\alpha} = \frac{1}{T}\int_{t_\mathrm{s}}^{t_\mathrm{s}+T} \alpha(t)dt = \mathrm{const}\ ,$$

where T is a period averaging and t_s is a stabilization instant of the equilibrium state of a centrally symmetric spheroidal body.

Proof: *a*) as we know from from the Poincaré virial theorem, in the mechanical *equilibrium state* of a centrally symmetric spheroidal body the right-hand side of Eq. (5.3.18) must be equal to zero, that is, $\dot{\mathrm{G}}(t)=0$. This means that GCF becomes a constant coefficient: $\mathrm{G}(t)=\mathrm{G}_s=\mathrm{const}$.

Let us calculate an indefinite integral of the function (5.3.11):

$$\int \alpha(t)dt=\int \frac{\alpha_s}{1-2\alpha_s\,\mathrm{G}_s(t-t_s)}dt=-\frac{1}{2\mathrm{G}_s}\ln\left|1-2\alpha_s\,\mathrm{G}_s(t-t_s)\right|\approx$$

$$\approx-\frac{1}{2\mathrm{G}_s}(-2\alpha_s\,\mathrm{G}_s(t-t_s))=\alpha_s(t-t_s)\ , \qquad (5.3.20)$$

because $\alpha_s<<1$. It follows directly from Eq. (5.3.20) that $\int_{t_s}^{t_s+T}\alpha(t)dt=\alpha_s(t-t_s)\Big|_{t_s}^{t_s+T}=\alpha_s T$, that is, Eq. (5.3.19a) is true for $\forall T<\infty$ in the case of a centrally symmetric (or slowly rotating) spheroidal body being in the equilibrium state.

b) Generally speaking, the Poincaré–Eddington virial theorem is described by Eq. (5.3.18) in conformity to a centrally symmetric (or slowly rotating) compressible spheroidal body in *quasi-equilibrium state*. In this case of a quasi-equilibrium state, the GCF derivative changing $\dot{\mathrm{G}}(t)$ has an oscillating character, therefore $\overline{\dot{\mathrm{G}}(t)}=0$, that is, Eq. (5.3.19b) occurs. Taking into account the definition of GCF (4.1.8), equation (5.3.19b) becomes:

$$\overline{\mathrm{G}(t)}=\frac{1}{T}\int_{t_s}^{t_s+T}\frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}dt=\mathrm{G}_s\,. \qquad (5.3.21)$$

By rewriting Eq. (5.3.21) and then integrating it we find that:

$$\frac{1}{T}\int_{\alpha_s}^{\alpha(t_s+T)}\frac{d\alpha}{\alpha^2}=2\frac{1}{T}\int_{t_s}^{t_s+T}\mathrm{G}(t)dt=2\overline{\mathrm{G}(t)}=2\mathrm{G}_s\,, \tag{5.3.22}$$

whence:

$$\frac{\alpha^{-1}(t_s)-\alpha^{-1}(t_s+T)}{T}=2\mathrm{G}_s=\text{const}\,. \tag{5.3.23}$$

As follows from Eq. (5.3.23), the next equation is true:

$$\alpha(t_s+T)=\frac{\alpha_s}{1-2\alpha_s\mathrm{G}_s T}=\frac{\alpha_s}{1-2\alpha_s\overline{\mathrm{G}(t)}T}\,, \tag{5.3.24}$$

where $\alpha_s=\alpha(t_s)$ is a stabilization value of the parameter of gravitational condensation when GCF stabilization first occurs. Since T is an arbitrary temporal period then Eq. (5.3.24) becomes Eq. (5.3.11) if $t=t_s+T$. Consequently, we consider again the equilibrium case (*a*). So, repeating analogous arguments relating to Eq. (5.3.20), we prove this theorem for the quasi-equilibrium case when $\alpha(t)=\alpha_s\left[1-2\alpha_s\overline{\mathrm{G}(t)}(t-t_s)\right]^{-1}$.

Thus, GCF $\mathrm{G}(t)$ can be a *periodic* function (in general, a *quasiperiodic* function [119]) in the case of quasi-equilibrium state of a compressible slowly rotating spheroidal body, therefore it can be expanded by mean of Fourier series:

$$\mathrm{G}(t)=\mathrm{G}_s+\sum_{n=1}^{\infty}\sqrt{A_n^2+B_n^2}\,\sin(n\omega t+\varphi_n)\,, \tag{5.3.25}$$

where

$$A_n=\frac{2}{T}\int_{-T/2}^{T/2}\mathrm{G}(t)\cos(n\omega t)dt;\; B_n=\frac{2}{T}\int_{-T/2}^{T/2}\mathrm{G}(t)\sin(n\omega t)dt;$$

$$\varphi_n=\arctan\frac{A_n}{B_n};\;\omega=\frac{2\pi}{T};\; t\ge t_s+T/2$$

[206].Using Eqs (5.3.9), (5.3.19a), and Theorem 5.1 we can also note that the average integral value of the angular velocity of rotation is equal [205]:

$$\Omega_s = \overline{\Omega(t)} = \frac{1}{T}\int_{t_s}^{t_s+T}\Omega(t)dt = \mathrm{G}_s\,\overline{\alpha(t)} = \mathrm{G}_s\,\alpha_s \quad . \qquad (5.3.26a)$$

Taking into account Eq. (5.3.5a) as well as Eqs (5.3.19b) and (5.3.26a), we find that the average integral value of circular frequency of the radial oscillations coincides with the value of the angular velocity of rotation in the case of quasi-equilibrium state:

$$\omega_s = \overline{\omega(t)} = \overline{\Omega(t)} = \Omega_s = \mathrm{G}_s\,\alpha_s \ . \qquad (5.3.26b)$$

In this connection, supposing that ω in Eq. (5.3.25) can be the same average integral value of circular frequency in Eq. (5.3.26b), that is, $\omega = \omega_{\mathrm{s}}$, we can differentiate Eq. (5.3.25) to find the GCF derivative:

$$\dot{\mathrm{G}}(t) = \sum_{n=1}^{\infty} n\omega_{\mathrm{s}}\sqrt{A_n^2 + B_n^2}\cos(n\omega_{\mathrm{s}}t + \varphi_n) . \qquad (5.3.27a)$$

The right-hand part of Eq. (5.3.27a) describes the derivative of quasi-equilibrium GCF since

$$\overline{\dot{\mathrm{G}}(t)} = \omega_{\mathrm{s}}\sqrt{A_1^2 + B_1^2}\cdot\overline{\cos(\omega_{\mathrm{s}}t + \varphi_1)} + 2\omega_{\mathrm{s}}\sqrt{A_2^2 + B_2^2}\cdot\overline{\cos(2\omega_{\mathrm{s}}t + \varphi_2)} +$$
$$+3\omega_{\mathrm{s}}\sqrt{A_3^2 + B_3^2}\cdot\overline{\sin(3\omega_{\mathrm{s}}t + \varphi_3)} + \ldots = 0 . \qquad (5.3.27b)$$

Let us note after finishing the process of avalanche gravitational compression and forming a dense core with the central mass density $\rho_0(\alpha_{\mathrm{s}})$ in accordance with Eq. (5.2.17) *the average integral value of the gravitational potential* (5.2.4b) of a slowly rotating spheroidal body ($\varepsilon_0^2 \to 0$) in *quasi-equilibrium state* is equal to:

$$\overline{\varphi}_{\mathrm{g}} = \frac{1}{2}\left(\mathrm{G}_s^2\,\overline{\alpha^2(t)} + \overline{\dot{\mathrm{G}}(t)\alpha(t)}\right)r^2 - 3\mathrm{G}_{\mathrm{s}}^2\,\overline{\alpha(t)} = \frac{1}{2}\mathrm{G}_s^2\,\alpha_s^2\cdot\left(r^2 - \frac{6}{\alpha_s}\right), (5.3.28)$$

$t \geq t_s$. That completely *coincides with the inner gravitational potential* (5.2.18a). Comparing Eq. (5.3.28) with Eq. (5.2.5) under the condition $\varepsilon_0^2 \to 0$, we can also see that the average integral value of the gravitational potential $\overline{\varphi}_{\mathrm{g}}$ is the regular part of the gravitational potential φ_{g}^* of a slowly rotating spheroidal body in the *equilibrium* state.

Now let us estimate the frequencies of the radial and axial oscillations of a rotating *axially symmetric* spheroidal body (or a rotating ellipsoid-like cosmogonical body) when $\varepsilon_0 \neq 0$. Using Eq. (5.2.20) we can obtain the induced acceleration of an initial gravitational field:

$$\vec{a} = -\widetilde{\mathrm{G}}^2(t)\alpha^2(t)\left\{\left(1-\varepsilon_0^2\right)^2 h \cdot \vec{e}_h + z \cdot \vec{e}_z\right\} - \\ - \dot{\widetilde{\mathrm{G}}}(t)\alpha(t)\left\{\left(1-\varepsilon_0^2\right)h \cdot \vec{e}_h + z \cdot \vec{e}_z\right\}. \qquad (5.3.29)$$

According to Newton's second law [158], the equation of motion of a particle under action of specific force $\vec{a}$ into a forming axially symmetric spheroidal body is the following [77]:

$$\frac{d^2(h \cdot \vec{e}_h + z \cdot \vec{e}_z)}{dt^2} + \alpha(t)\left(1-\varepsilon_0^2\right)[\widetilde{\mathrm{G}}^2(t)\alpha(t)\left(1-\varepsilon_0^2\right) + \\ + \dot{\widetilde{\mathrm{G}}}(t)]h \cdot \vec{e}_h + \alpha(t)[\widetilde{\mathrm{G}}^2(t)\alpha(t) + \dot{\widetilde{\mathrm{G}}}(t)]z \cdot \vec{e}_z = 0. \qquad (5.3.30)$$

Unlike Eq. (5.3.2), equation (5.3.30) is a sum of *two* harmonic oscillatory equations, so that the inducible acceleration $\vec{a}$ leads to the *oscillating motion* of particles. According to Eq. (5.3.30) we can obtain the radial h-projection and the axial z-projection of this vector equation of motion of a particle under action of a specific force (5.3.29) into an axially symmetric spheroidal body [77]:

$$\frac{d^2 h}{dt^2} + [\widetilde{\mathrm{G}}^2(t)\alpha^2(t)\left(1-\varepsilon_0^2\right)^2 + \dot{\widetilde{\mathrm{G}}}(t)\alpha(t)\left(1-\varepsilon_0^2\right)]h = 0; \qquad (5.3.31a)$$

$$\frac{d^2 z}{dt^2} + [\widetilde{\mathrm{G}}^2(t)\alpha^2(t) + \dot{\widetilde{\mathrm{G}}}(t)\alpha(t)]z = 0\,. \qquad (5.3.31b)$$

It follows from Eq. (5.3.31a) that the circular frequency of the *radial oscillations* is expressed by the formula [77]:

$$\omega_h(t) = \sqrt{\widetilde{\mathrm{G}}^2(t)\alpha^2(t)\left(1-\varepsilon_0^2\right)^2 + \dot{\widetilde{\mathrm{G}}}(t)\alpha(t)\left(1-\varepsilon_0^2\right)}\,, \qquad (5.3.32)$$

so that, at the stage of formation of an ellipsoid-like cosmogonical body, the following representation for $\omega_h(t)$ in accordance with the formula (5.3.32) is true:

$$\omega_h^2(t) = \omega_h^{*2}(t) + (\delta\varpi_h)^2(t)\,, \qquad (5.3.33)$$

where:

$$\omega_h^{*2}(t) = \widetilde{\mathrm{G}}^2(t)\alpha^2(t)\left(1-\varepsilon_0^2\right)^2; \qquad (5.3.34a)$$

$$(\delta\varpi_h)^2(t) = \dot{\widetilde{\mathrm{G}}}(t)\alpha(t)\left(1-\varepsilon_0^2\right) \qquad (5.3.34b)$$

and $\delta\varpi_h$, generally speaking, is a generalized circular frequency of the radial oscillations since $\dot{\widetilde{\mathrm{G}}}(t)$ can be a negative value ($\dot{\widetilde{\mathrm{G}}}(t) < 0$). This representation (5.3.33) is equivalent to the mentioned expansion of gravitational potential (5.2.21) of a forming axially symmetric spheroidal body as a sum of the regular part φ_g^* and the fluctuation part $\delta\varphi_g$. Consequently, according to Eq. (5.3.33) the regular part of ω_h^2 can be a squared angular velocity of rotation $\Omega^2(t) = \vec{\omega}_h^{*2}(t)$ in the equatorial $(x, y)-$plane whereas its fluctuation part is a squared generalized circular frequency $(\delta\varpi_h)^2$ of perturbations. Indeed, by substituting Eqs (5.3.34a, b) into Eq. (5.3.29) we obtain [77]:

$$a_h = -[\omega_h^{*2}(t) + (\delta\varpi_h)^2(t)]\, h =$$
$$= -\Omega^2(t)\, h - (\delta\varpi_h)^2(t)\, h = -f_c + f_{ah}\,, \qquad (5.3.35)$$

where f_c is a specific (per mass unit) centrifugal force and f_{ah} is a h-projection of specific additional periodic force of Alfvén–Arrhenius (see analogous formulas (5.3.15b) and (5.3.16)).

It follows from Eq. (5.3.31b) the circular frequency of the *axial oscillations* is expressed by the formula [77]:

$$\omega_z(t) = \sqrt{\tilde{G}^2(t)\alpha^2(t) + \dot{\tilde{G}}(t)\alpha(t)}\,, \tag{5.3.36}$$

so that according to Eq. (5.3.36):

$$\omega_z^2(t) = \omega_z^{*2}(t) + (\delta\varpi_z)^2(t)\,, \tag{5.3.37}$$

where:

$$\omega_z^{*2}(t) = \tilde{G}^2(t)\alpha^2(t)\,; \tag{5.3.38a}$$

$$(\delta\varpi_z)^2(t) = \dot{\tilde{G}}(t)\alpha(t) \tag{5.3.38b}$$

and $\delta\varpi_z$ is a generalized circular frequency of the axial oscillations.

So, the gravitational acceleration $\vec{a}$ (or the specific force of gravity $\vec{f}_g$) is balanced by the vector sum of specific centrifugal force $\vec{f}_c$ and specific additional periodic force $\vec{f}_a$ of Alfvén–Arrhenius (see analogous Eq. (5.3.14)):

$$\vec{a} = -\vec{f}_c + \vec{f}_a\,, \tag{5.3.39}$$

where:

$$\vec{f}_c = \Omega^2(t)\vec{r} = [\vec{\Omega}(t) \times [\vec{r} \times \vec{\Omega}(t)]]\,; \tag{5.3.40a}$$

$$\vec{f}_a = -(\delta\varpi(t))^2\,\vec{r} = -\Omega(t)\,\omega(t)e^{i(\vec{k}\vec{r} - \omega t)}\vec{r}\,. \tag{5.3.40b}$$

According to Eqs (5.3.29), (5.3.32), (5.3.34a), (5.3.36), and (5.3.38a), in the particular case of *relative mechanical equilibrium*, that is, under the condition of stabilization of the generalized GCF $\tilde{G}(t) = \tilde{G}_s = \text{const}$, the induced acceleration of a stationary gravitational field is equal:

$$\vec{a} = -\widetilde{G}_s^2 \alpha^2 \left\{ \left(1 - \varepsilon_0^2\right)^2 h \cdot \vec{e}_h + z \cdot \vec{e}_z \right\}, \qquad (5.3.41)$$

so that the circular frequencies of the radial ω_h and the axial oscillations ω_z inside an ellipsoid-like cosmogonical body are described respectively by the formulas [73, 77]:

$$\omega_h = \omega_h^* = \widetilde{G}_s \, \alpha \left(1 - \varepsilon_0^2\right); \qquad (5.3.42a)$$

$$\omega_z = \omega_z^* = \widetilde{G}_s \, \alpha. \qquad (5.3.42b)$$

According to (5.3.42a) and (5.3.42b), we can see that in the case of relative mechanical equilibrium of a rotating *axially symmetric* spheroidal body the following inequality is true:

$$\omega_z > \omega_h , \qquad (5.3.43)$$

and that this fully confirms the analogous conclusion of Alfvén and Arrhenius [9, 19] (see also Section 9.6 of Chapter 9).

5.4. The dynamical states after the decay of a rotating spheroidal body and the formation of protoplanetary shells

Taking into account the definition of the average integral value of angular velocity (the main circular frequency of the oscillations) (5.3.26a) under the condition of *mechanical quasi-equilibrium* $\overline{G(t)} = G_s = \text{const}$ of a *centrally symmetric* (*slowly* rotating) spheroidal body we can rewrite Eq. (5.3.28) as follows:

$$\overline{\varphi_g(r)} = \frac{1}{2} \Omega_s^{\ 2} \left[r^2 - \frac{6}{\alpha_s} \right] = \frac{1}{2} \Omega_s^{\ 2} \left[r^2 - 3 r_{pr}^2 \right], \qquad (5.4.1)$$

where $r_{pr} = \sqrt{2/\alpha_s}$ is the most probable distance (2.1.22) for particle distribution in space [46]. Let us note if $\overline{\varphi}_g$ in the left-hand part of Eq. (5.4.1) is the stabilization value of inner gravitational potential $\varphi_g^{(\mathrm{I})}$ following the formula (5.2.18a) then by choosing $r = 2r_{pr}$ the right-hand part of Eq. (5.4.1) determines the absolute value of a centrifugal potential of the kind:

$$\left|V_c(2r_{pr})\right| = \frac{1}{2}\Omega_s^2 r_{pr}^2, \tag{5.4.2}$$

where $V_c(r) = -\frac{1}{2}\Omega_s^2 r^2 = -\frac{1}{2}[\vec{\Omega}_s \times \vec{r}]^2$ is the centrifugal potential. This equality of the absolute values of averaged gravitational and centrifugal potentials: $V_c + \overline{\varphi}_g = 0$ when $r = 2r_{pr}$ means that *decay* of a centrally symmetric spheroidal body (or collapse of a gas-dust cloud [11]) occurs at distances $r > r_{pr}$. Let us investigate in detail an initial separation of a slowly rotating spheroidal body on a core and exterior shells.

First of all, we note that the stability of rotating configurations has long been studied (see review in [1, 207], for example with Maclaurin spheroids, where the density ρ is supposed constant or with the Roche model, which assumes an infinite central condensation (see Section 1.4)). As A. Maeder noted [207 p. 19–20],

> In the case of the Maclaurin spheroids, the equilibrium configurations flatten for high rotation. For extremely high angular momentum, it tends toward an infinitely thin circular disk. The maximum value of the angular velocity Ω (supposed to be *constant* in the body) is $\Omega_{\max}^2 = 0.4494\pi\gamma\rho$. In reality, some instabilities would occur before this limit is reached. In the case of the Roche model with constant Ω (this is not a necessary assumption), the equilibrium figure also flattens to reach a ratio of 2/3 between the polar and the equatorial radii, with a maximum angular velocity $\Omega_{\max}^2 = 0.7215\pi\gamma\overline{\rho}$, where $\overline{\rho}$ is the mean density... Except for the academic case of stars with constant density or nearly constant density, the Roche approximation better corresponds to the stellar reality. Recent results from long-baseline interferometry ...

support the application of the Roche model in the cases of Altair and Achernar, which both rotate very fast close to their break-up velocities....

In the considered case of a slowly rotating spheroidal body model with the *constant* value of angular velocity Ω_s throughout the nebula interior, we can see (comparing Eq. (5.2.7) with Eq. (5.3.28) and taking into account (5.3.26a, b) as well as the virial theorem: $2V_c + \overline{\varphi}_g = 0$) that a maximum (break-up) angular velocity is:

$$\Omega^2_{\max} = \frac{2\pi\gamma\rho_0}{3} = 0.6667\pi\gamma\rho_0 , \qquad (5.4.3)$$

where $\rho_0 = M(\alpha/2\pi)^{3/2}$ is a density in the center of a slowly rotating spheroidal body. In other words, the maximum values of the angular velocity in the Roche model and the spheroidal body model almost coincide, that is, following A. Maeder both model approximations "better correspond to the stellar reality." We also state that "more elaborate models consider differential rotation, in particular the case of the so-called shellular rotation…" [207 p. 20, 29–34]. Unlike the *mechanical equilibrium* case when the mentioned models were applied to a solid-body rotation with Ω = const throughout the stellar interior, in the case of *differential rotation* of a centrally symmetric spheroidal body we start from the equality of averaged gravitational and centrifugal potentials ($V_c + \overline{\varphi}_g = 0$) and the equivalence of Eq. (5.2.7) and Eq. (5.3.28) respectively:

$$\Omega^2_s = \mathrm{G}^2_s\,\alpha^2_s = \frac{4\pi\gamma\rho_0}{3} . \qquad (5.4.4)$$

By comparing Eq. (5.2.18) with Eq. (5.3.28) and taking into account the notation (5.3.26a, b) we can see that:

$$\Omega_s^2 = \mathrm{G}_s^2\,\alpha_s^2 = \frac{4\pi\gamma\rho_0(\alpha_s)}{3} = \frac{\gamma M}{3}\sqrt{\frac{2\alpha_s^3}{\pi}}\,, \qquad (5.4.5)$$

whence we can obtain the following *analog* of Kepler 3[rd] law [77]:

$$\Omega_s^2\,R_s^3 = \gamma M \qquad (5.4.6)$$

under the supposition that a *characteristic orbital radius* is equal:

$$R_s = \frac{\sqrt[3]{3\sqrt{\pi/2}}}{\sqrt{\alpha_s}}\,. \qquad (5.4.7)$$

In reality, as mentioned relative to Eq. (5.4.2), equality of averaged gravitational and centrifugal potentials ($V_c + \overline{\varphi}_g = 0$) means a *decay*, that is, the separation of a centrally symmetric spheroidal body, at distances $r > r_{pr}$ because of $R_s > r_{pr}$ where $R_s \approx 1.554988/\sqrt{\alpha_s}$ and $r_{pr} = \sqrt{2/\alpha_s}$ [73, 77].

Taking into account that the Keplerian angular velocity of the orbital motion is:

$$\Omega_{\mathrm{K}} = \sqrt{\frac{\gamma M}{a^3}} \qquad (5.4.8)$$

with the period $T_K = 2\pi/\Omega_K$ and the orbital semi-major axis a [158], the expression (5.4.8) after substitution (5.4.6) can be represented as follows:

$$\Omega_{\mathrm{K}} = \Omega_s(R_s/a)^{3/2}\,, \qquad (5.4.9)$$

where Ω_s is a main circular frequency of oscillations.

The separation process of a centrally symmetric spheroidal body leads to the formation of its inner zone I (a stellar core) and remote zone II (an exterior shell). Respectively, we consider the *inner* gravitational potential $\varphi_g^{(\mathrm{I})}$ (see Eq. (5.2.18a)):

$$\varphi_g^{(\mathrm{I})}(r,\alpha_s)=\frac{2\pi\gamma\rho_0(\alpha_s)}{3}\cdot\left(r^2-\frac{6}{\alpha_s}\right),\ t\ge t_s \qquad (5.4.10a)$$

as well as the *exterior* gravitational potential $\varphi_g^{(\mathrm{II})}$ for large r (see Eq. (5.2.18b)):

$$\varphi_g^{(\mathrm{II})}(r,\alpha_s)=-\frac{\gamma M}{r}\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right)\approx-\frac{\gamma M}{r}\ ,\ t\ge t_s\,, \qquad (5.4.10b)$$

where $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$ is the error function [128]. The evolution of the core and exterior hull describes the processes of the formation of a central cosmogonical body (a protostar) and numerous shells (a disk with embryos of forming protoplanets).

Analogously, the decay of an *axially symmetric* spheroidal body on its core and its exterior shell leads to its *fragmentation stage* describing the process of formation of a *fast* rotating cosmogonical body. In the case of a rapidly rotating spheroidal body, the axial rotation of the spheroidal body creates a flattening of its core. As shown in Section 3.6 of Chapter 3, the gravitational potential (3.6.15b) for a rotating and gravitating axially symmetric spheroidal body is determined in cylindrical coordinates (h,ε,z) as follows:

$$\varphi_g(h,z)=-\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$

$$\times\int_0^\infty e^{-\frac{\alpha(1-\varepsilon_0^2)h^2}{2+s\alpha(1-\varepsilon_0^2)}-\frac{\alpha z^2}{2+s\alpha}}\frac{ds}{\sqrt{2+s\alpha}\cdot(2+s\alpha(1-\varepsilon_0^2))}\,.$$

As follows from this equation, the gravitational potential in a *near zone* of a uniformly rotating axially symmetric spheroidal body can be described by the following expression [77, 79]:

$$\varphi_g^{(\mathrm{I})}(h,z)=2\pi\gamma\rho_0\frac{1-\varepsilon_0^2}{2(1-\varepsilon_0^2)^2+1}\left[(1-\varepsilon_0^2)^2h^2+z^2-\frac{4(1-\varepsilon_0^2)+2}{\alpha}\right]. \quad (5.4.11)$$

The process of avalanche gravitational compression of an axially symmetric spheroidal body forms a dense bunch (core) with the central mass density (5.2.17):

$$\rho_0(\alpha_s)=M(\alpha_s/2\pi)^{3/2},\ \ t\geq t_s,$$

so that formula (5.4.11) for the *inner* gravitational potential $\varphi_g^{(\mathrm{I})}$ takes the form:

$$\varphi_g^{(\mathrm{I})}(h,z,\alpha_s)=2\pi\gamma\rho_0(\alpha_s)\frac{1-\varepsilon_0^2}{2(1-\varepsilon_0^2)^2+1}\times$$

$$\times\left[(1-\varepsilon_0^2)^2h^2+z^2-\frac{4(1-\varepsilon_0^2)+2}{\alpha_s}\right],\quad t\geq t_s. \quad (5.4.12)$$

Obviously, if $\varepsilon_0=0$ then Eq. (5.4.12) becomes Eq. (5.4.10a). In other words, when a forming ellipsoid-like cosmogonical body reaches the relative mechanical equilibrium state ($\widetilde{\mathrm{G}}(t)\to\widetilde{\mathrm{G}}_s=\mathrm{const}$) at $t\to t_s$ then the equivalence of Eq. (5.4.12) and Eq. (5.2.22) takes place [77], whence:

$$\frac{1}{2}\widetilde{\mathrm{G}}_s^2\alpha_s^2=2\pi\gamma\rho_0(\alpha_s)\frac{1-\varepsilon_0^2}{2(1-\varepsilon_0^2)^2+1}, \quad (5.4.13)$$

where $\rho_0(\alpha_s)$ is a central mass density at the stabilization instant t_s. As follows from Eq. (5.4.13),

$$\widetilde{\mathrm{G}}_s^2\alpha_s^2[2(1-\varepsilon_0^2)^2+1]=4\pi\gamma M(\alpha_s/2\pi)^{3/2}(1-\varepsilon_0^2). \quad (5.4.14)$$

Taking into account the definitions (5.3.42a) and (5.3.42b) we can find that:

$$2\,\omega_h^{*2}+\omega_z^{*2}=\gamma M(1-\varepsilon_0^2)\sqrt{\frac{2\alpha_s^3}{\pi}}. \quad (5.4.15)$$

Introducing a *total circular frequency of oscillations* for a rotating and gravitating axially symmetric spheroidal body:

$$\widetilde{\Omega}_s = \sqrt{(2\,\omega_h^{*\,2} + \omega_z^{*\,2})/3}\,, \tag{5.4.16}$$

we can rewrite Eq. (5.4.15) as follows:

$$\widetilde{\Omega}_s^2 = \frac{\gamma M\left(1-\varepsilon_0^2\right)}{3}\sqrt{\frac{2\alpha_s^3}{\pi}}. \tag{5.4.17}$$

In the case of $\varepsilon_0 = 0$ Eq. (5.4.17) becomes Eq. (5.4.5), while the total circular frequency of oscillations $\widetilde{\Omega}_s$ coincides with the main circular frequency of oscillations Ω_s. By analogy with (5.4.6) we obtain a version of Kepler's third law for the case of an axially symmetric spheroidal body [77]:

$$\widetilde{\Omega}_s^2\, R_s^3 = \gamma M\,, \tag{5.4.18}$$

where $\widetilde{R}_s$ is a respective characteristic orbital radius:

$$\widetilde{R}_s = \frac{\sqrt[3]{3/(1-\varepsilon_0^2)\cdot\sqrt{\pi/2}}}{\sqrt{\alpha_s}}\,. \tag{5.4.19}$$

In reality, according to Eq. (5.4.13), the condition of equality of gravitational and centrifugal potentials ($V_c + \overline{\varphi}_g = 0$) means the collapse of an axially symmetric spheroidal body at distances $r \geq \widetilde{R}_s$ where $\widetilde{R}_s \approx 1.554988/\sqrt[3]{(1-\varepsilon_0^2)}\sqrt{\alpha_s}$. The separation process of an axially symmetric spheroidal body leads to the formation of its inner zone I (a stellar core) and its remote zone II (an exterior shell). The evolution of core and exterior shell describes the processes of formation of a central ellipsoid-like cosmogonical body (a protostar) and numerous shells (a disk with embryos of forming protoplanets) [208].

As a rule $\varepsilon_0 << 1$, therefore $\omega_z^* \approx \omega_h^* = \Omega_s \approx \widetilde{\Omega}_s$ in Eqs (5.4.15)–(5.4.18), so that we can investigate without any simplifications the case of a *centrally symmetric* (slowly rotating) spheroidal body. We consider later on the inner gravitational potential $\varphi_g^{(\mathrm{I})}$ and the exterior gravitational potential $\varphi_g^{(\mathrm{II})}$ in accordance with formulas (5.4.10a) and (5.4.10b) respectively. Due to the decay of a gravitating spheroidal body and formation of its remote zone II (where the equality of gravitational and centrifugal potentials occurs) the equality of respective strengths takes place at least when $r = a$. Indeed, it directly follows from (5.4.8):

$$f_c(a) = \Omega_K^2\, a = \frac{\gamma M}{a^2} = \left.\frac{\gamma M}{r^2}\right|_{r=a} = f_g(a)\,, \qquad (5.4.20)$$

where $\vec{f}_g = -\operatorname{grad}\varphi_g^{(\mathrm{II})}$ is a value of gravity strength in the remote zone II following Newton's third law.

According to Eqs (5.3.5a, b), (5.3.6), and (5.3.8), in the first case of the *mechanical equilibrium* ($\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$) the specific force of gravity $\vec{f}_g$ in the remote zone II ($V_c + \varphi_g^{(\mathrm{II})} = 0$) is exactly compensated by the centrifugal specific force $\vec{f}_c$ exclusively because $\vec{f}_a = 0$, that is, equality of gravitational and centrifugal strengths takes place for the all equidistant $\vec{r}$. The simplest type of motion of particle with a constant velocity v_0 around the center of a spheroidal body is, therefore, a *circular orbit* with radius R_0:

$$r = \left|\vec{r}\right| = R_0\,. \qquad (5.4.21)$$

Taking into account Eq. (5.4.5) we obtain:

$$3\sqrt{\frac{\pi}{2}}\,\mathrm{G}_s^2\sqrt{\alpha_s} = \gamma M\,. \qquad (5.4.22)$$

As seen from Eq. (5.4.22), in this case, the specific force of gravity $\vec{f}_g$ is equal to:

$$\vec{f}_g = -\frac{\gamma M}{r^2}\cdot\frac{\vec{r}}{r}\bigg|_{\text{equil}} = -\frac{\gamma M}{r^2}\cdot\vec{e}_r\bigg|_{R_0} = -3\sqrt{\frac{\pi\alpha_s}{2}}\cdot\frac{\mathrm{G}_\mathrm{s}^2}{R_0^{\ 2}}\cdot\vec{e}_r\,. \quad (5.4.23)$$

So, when a body moves exclusively along an undisturbed circular orbit $|\vec{r}| = R_0$ around the core of a slowly rotating spheroidal body the specific force of gravity f_g is exactly compensated by the specific centrifugal force f_c so that the equation is true:

$$3\sqrt{\frac{\pi\alpha_s}{2}}\cdot\frac{\mathrm{G}_\mathrm{s}^2}{R_0^{\ 2}} = \Omega_\mathrm{K}^2\cdot R_0\,,$$

from which expressions (5.4.8) and (5.4.9) are immediately obtained if $a = R_0$.

As seen from Eqs (5.3.13)–(5.3.15a, b), the specific force of gravity $\vec{f}_g$ in the second case of the *mechanical quasi-equilibrium* ($\overline{\mathrm{G}(t)} = \mathrm{G}_s\,, \dot{\overline{\mathrm{G}}}(t) = 0$) for the remote zone II ($V_c + \overline{\varphi_g^{(\mathrm{II})}} = 0$) is equal to:

$$\vec{f}_g = -\operatorname{grad}\varphi_g^{(\mathrm{II})}\Big|_{\text{quasiequil}} = -\operatorname{grad}(\overline{\varphi_g^{(\mathrm{II})}} + \delta\varphi_g^{(\mathrm{II})}) =$$
$$= -\operatorname{grad}(-V_c + \delta\varphi_g^{(\mathrm{II})}) = -\vec{f}_c + \vec{f}_a\,. \qquad (5.4.24)$$

So, the decay of a rotating and gravitating spheroidal body and the formation of its remote zone II leads to the equality $\vec{f}_c + \vec{f}_g = \vec{f}_a$ of respective field strengths (see also Eq. (9.2.2) in Section 9.2 of Chapter 9).

Now let us separate the average and fluctuation parts in the quasi-equilibrium case being considered. According to Eqs (5.3.14) and (5.3.15a), the specific force of gravity $\vec{f}_g$ inside a slowly rotating spheroidal body, that is, into the near zone I of the gravitational field $\vec{f}_g = -\operatorname{grad}\varphi_g^{(\mathrm{I})}$, in the case mentioned of mechanical quasi-equilibrium is determined by the analogous formula:

$$\vec{f}_g = -\operatorname{grad} \varphi_g^{(\mathrm{I})}\Big|_{\text{quasiequil}} = -\left[\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)\right]\vec{r}\,. \quad (5.4.25)$$

On the other hand, using formula (5.2.7), we can calculate the gravitational field strength of a centrally symmetric spheroidal body in the zone I of the gravitational field in general [46, 77]:

$$\vec{f}_g = -\operatorname{grad}\varphi_g^{(\mathrm{I})} = -\frac{4\pi\gamma\rho_0}{3}\vec{r}\,. \quad (5.4.26)$$

According to Eqs (5.3.28) and (5.2.17), the comparison of Eqs (5.4.25) and (5.4.26) leads to the following averaged relation [205]:

$$\overline{\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)} = \frac{4\pi\gamma}{3}\overline{\rho_0(t)} = \frac{4\pi\gamma}{3}\rho_0(\alpha_s) =$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\gamma M}{3}\alpha_s^{3/2} = \sqrt{\frac{2}{\pi}}\cdot\frac{\gamma M}{3}\overline{\alpha^{3/2}(t)}\,. \quad (5.4.27)$$

Taking into account Eq. (5.3.27b), we note that Eq. (5.4.27) generalizes Eq. (5.4.22) because the case of mechanical quasi-equilibrium is more general than the case of mechanical equilibrium, that is, *the averaged quasi-equilibrium is equilibrium*. Hence, we obtain the quasi-equilibrium analog (5.4.27) of formula (5.4.22). Using Eq. (5.4.27) we can modify formula (5.2.18b) for the averaged gravitational potential $\varphi_g(r,\alpha_s) = \overline{\varphi_g(r,\alpha(t))}$ when $t \geq t_s$ [205]:

$$\overline{\varphi_g} = -\frac{4\pi\gamma\rho_0(\alpha_s)}{\alpha_s r}\sqrt{\frac{2}{\alpha_s}}\int_0^{r\sqrt{\alpha_s/2}} e^{-s^2}ds = -\frac{(2\pi)^{3/2}\gamma\overline{\rho_0(\alpha(t))}}{\alpha_s^{3/2} r}\sqrt{\frac{2}{\pi}}\int_0^{r\sqrt{\alpha_s/2}} e^{-s^2}ds =$$

$$= -3\sqrt{\frac{\pi}{2}}\cdot\frac{\overline{\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)}}{\alpha_s^{3/2} r}\operatorname{erf}\left(r\sqrt{\alpha_s/2}\right). \quad (5.4.28)$$

Consequently, the gravitational potential, in the case of mechanical *quasi-equilibrium*, is equal to

$$\varphi_g\Big|_{\text{quasiequil}} = -3\sqrt{\frac{\pi}{2}}\cdot\frac{\mathrm{G}^2(t)\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t)}{\alpha_s^{3/2}r}\,\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right). \quad (5.4.29)$$

Now let us transform expression in the numerator of (5.4.29):

$$\mathrm{G}^2(t)\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t) = \left[\frac{1}{2\alpha^2(t)}\frac{d\alpha}{dt}\right]^2\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t) =$$

$$= \frac{1}{2}\cdot\mathrm{G}(t)\frac{d\alpha}{dt}+\dot{\mathrm{G}}(t)\alpha(t). \quad (5.4.30)$$

We are then going to use the Fourier series representation of GCF (5.3.25) and its derivative (5.3.27a) as periodic functions (in the case of a quasi-equilibrium state of a slowly rotating spheroidal body [205, 209]) in Eq. (5.4.30):

$$\mathrm{G}^2(t)\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t) = \frac{1}{2}\cdot\mathrm{G}(t)\dot{\alpha}(t)+\dot{\mathrm{G}}(t)\alpha(t) =$$

$$= \frac{1}{2}\cdot\dot{\alpha}(t)\left[\mathrm{G}_s+\sum_{n=1}^{\infty}\sqrt{A_n^2+B_n^2}\sin(n\omega_s t+\varphi_n)\right]+\alpha(t)\sum_{n=1}^{\infty}n\omega_s\sqrt{A_n^2+B_n^2}\cos(n\omega_s t+\varphi_n) =$$

$$= \frac{\mathrm{G}_s}{2}\cdot\dot{\alpha}+\mathrm{G}_s\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\times$$

$$\times\left\{\frac{1}{2}\cdot\dot{\alpha}\cos(n\omega_s t+\varphi_n+\frac{3\pi}{2})+n\omega_s\alpha\cdot\cos(n\omega_s t+\varphi_n)\right\}, \quad (5.4.31)$$

where:

$$a_n = A_n/\mathrm{G}_s = \frac{2}{T}\int_{-T/2}^{T/2}(\mathrm{G}(t)/\mathrm{G}_s)\cos(n\omega_s t)dt;\; b_n = B_n/\mathrm{G}_s =$$

$$= \frac{2}{T}\int_{-T/2}^{T/2}(\mathrm{G}(t)/\mathrm{G}_s)\sin(n\omega_s t)dt;\; \varphi_n = \arctan\frac{a_n}{b_n}.$$

In the parenthesis of the right-hand side of Eq. (5.4.31), the sum of two harmonic oscillations of the same frequency is written, therefore according to the formula of the addition of two harmonic oscillations the resulting oscillation can be found by the parallelogram rule [206], so that we have:

$$\mathrm{G}^2(t)\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t)=$$

$$=\frac{1}{2}\cdot\mathrm{G}_s\dot{\alpha}\left[1+\sum_{n=1}^{\infty}c_n\sqrt{1+4n^2\omega_s^2[\alpha/\dot{\alpha}]^2}\cos(n\omega_s t+\psi_n)\right], \quad (5.4.32)$$

where $c_n=\sqrt{a_n^2+b_n^2}$ and

$$\psi_n=\arctan\frac{-\cos\varphi_n+(2n\omega_s\alpha/\dot{\alpha})\sin\varphi_n}{\sin\varphi_n+(2n\omega_s\alpha/\dot{\alpha})\cos\varphi_n}=\arctan\frac{-1+(2n\omega_s\alpha/\dot{\alpha})\tan\varphi_n}{2n\omega_s\alpha/\dot{\alpha}+\tan\varphi_n}.$$

According to Eq. (5.3.24), from the beginning instant of time t_s of stabilization of GCF, the parameter of gravitational condensation, in the case of a quasi-equilibrium state, is $\alpha(t)\approx\alpha_s[1+2\alpha_s\overline{\mathrm{G}(t)}(t-t_s)]$ for the finite time intervals $t-t_s$, so that the derivative:

$$\dot{\alpha}(t)=2\alpha_s^2\mathrm{G}_s\,,\ \mathrm{G}_s=\overline{\mathrm{G}(t)}\,. \quad (5.4.33)$$

Then, substituting Eq. (5.4.33) into Eq. (5.4.32) and taking into account formula (5.3.26b) we find [205, 209]:

$$\mathrm{G}^2(t)\alpha^2(t)+\dot{\mathrm{G}}(t)\alpha(t)=$$

$$=\mathrm{G}_s^2\alpha_s^2\left[1+\sum_{n=1}^{\infty}\sqrt{(a_n^2+b_n^2)(1+[n\alpha(t)/\alpha_s]^2)}\cos(n\omega_s t+\psi_n)\right]=$$

$$=\mathrm{G}_s^2\alpha_s^2\left[1+\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\right], \quad (5.4.34a)$$

where:

$$C_n(t)=c_n\sqrt{1+n^2[\alpha(t)/\alpha_s]^2}\,;\ c_n=\sqrt{a_n^2+b_n^2}\ \text{ and}$$

$$\psi_n=\arctan\frac{1+n(\alpha(t)/\alpha_s)\tan\varphi_n}{n\alpha(t)/\alpha_s+\tan\varphi_n}\,. \quad (5.4.34b)$$

Let us note that the substitution of Eq. (5.4.34a) into Eq. (5.4.28) gives the identity because $\overline{\cos(n\omega_s t+\psi_n)}=0$. Moreover, this substitution in Eq. (5.4.29) shows (with regard for formula (5.4.22)) that:

$$\varphi_g\Big|_{\text{quasiequil}} = -3\sqrt{\frac{\pi}{2}}\cdot\frac{\mathrm{G}_s^2\sqrt{\alpha_s}\left[1+\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\right]}{r}\,\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right) =$$

$$= -\frac{\gamma M}{r}\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right)\left[1+\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\right]. \qquad (5.4.35)$$

For large r ($r \to \infty$), the error function $\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right) \to 1$, therefore, in the remote zone II Eq. (5.4.35) becomes [205, 209]:

$$\varphi_g^{(\mathrm{II})}\Big|_{\text{quasiequil}} = \varphi_g^{(\mathrm{II})}\Big|_{\text{equil}} + \delta\varphi_g^{(\mathrm{II})} =$$

$$= -\frac{\gamma M}{r} - \frac{\gamma M}{r}\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n). \qquad (5.4.36)$$

Comparing Eq. (5.4.36) with Eq. (5.4.24) we can see that:

$$\varphi_g^{(\mathrm{II})}\Big|_{\text{equil}} = \overline{\varphi_g^{(\mathrm{II})}} = -\frac{\gamma M}{r},$$

$$\delta\varphi_g^{(\mathrm{II})} = -\frac{\gamma M}{r}\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n); \qquad (5.4.37a)$$

as well as:

$$\vec{f}_g\Big|_{\text{equil}} = -\mathrm{grad}\,\overline{\varphi_g^{(\mathrm{II})}} = -\frac{\gamma M}{r^2}\cdot\frac{\vec{r}}{r},$$

$$\vec{f}_a = -\mathrm{grad}\,\delta\varphi_g^{(\mathrm{II})} = -\frac{\gamma M}{r^2}\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\cdot\frac{\vec{r}}{r}. \qquad (5.4.37b)$$

According to Eq. (5.4.24) and Eq. (5.4.37b), we can estimate the specific force of gravity in the remote zone II of the quasi-equilibrium gravitational field as follows [205, 209]:

$$\vec{f}_g = -\frac{\gamma M}{r^2}\cdot\frac{\vec{r}}{r}\Big|_{\text{quasiequil}} = -\frac{\gamma M\left[1+\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\right]}{r^2}\cdot\frac{\vec{r}}{r} =$$

$$= \vec{f}_g\Big|_{\text{equil}} + \vec{f}_a = -\vec{f}_c + \vec{f}_a. \qquad (5.4.38)$$

Evidently, formula (5.4.38) generalizes (5.4.23) in that case if $\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t+\psi_n)\neq 0$, so that the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ causing the radial and axial oscillations (which modify an initially circular orbit [9, 19]) can be calculated by the following relation [205, 209]:

$$\vec{f}_a = -\frac{\gamma M}{r^2}\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t+\psi_n)\cdot\frac{\vec{r}}{r} =$$

$$= -3\sqrt{\frac{\pi\alpha_s}{2}}\cdot\frac{\mathrm{G}_s^2}{r^2}\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t+\psi_n)\cdot\frac{\vec{r}}{r}, \qquad (5.4.39)$$

where ω_s is the average integral value of the main circular frequency of radial oscillations inside a forming core of a centrally symmetric (slowly rotating) spheroidal body (a central cosmogonical body).

It follows directly from Eq. (5.4.39) that the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ is the sum of spectral components with multiple ordered frequencies to the average main circular frequency [205, 209]:

$$\vec{f}_a = \vec{f}_a^{(1)}(\omega_s t,\vec{r}) + \vec{f}_a^{(2)}(2\omega_s t,\vec{r}) +$$

$$+ \vec{f}_a^{(3)}(3\omega_s t,\vec{r}) + \ldots + \vec{f}_a^{(l)}(l\omega_s t,\vec{r}) + \ldots, \qquad (5.4.40a)$$

where:

$$\vec{f}_a^{(l)}(l\omega_s t,\vec{r}) = -\gamma M\cdot\frac{C_l(t)\cos(l\omega_s t+\psi_l)}{r^2}\cdot\frac{\vec{r}}{r}, \quad l=1,2,3,\ldots \qquad (5.4.40b)$$

Let us note that similar results occur in the case of a rotating *axially symmetric* spheroidal body. In particular, the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ is the sum of spectral components (5.4.40b) with multiple ordered frequencies to the circular frequencies of both radial ω_h and axial ω_z oscillations. Indeed, owing to the fluctuation part

$\delta\varphi_g$ of gravitational potential (5.2.21) of a forming axially symmetric spheroidal body in its quasi-equilibrium state, the radial and the axial oscillations of the orbital motion of particles also exist under the action of a specific additional periodic force in the remote zone. So, following Alfvén and Arrhenius [9, 19], the circular orbit of moving particles in the gravitational field can be modified by both the radial and the axial oscillations.

5.5. Interconnections of the proposed statistical theory of gravitating spheroidal bodies with Nelson's statistical mechanics and Nottale's scale relativistic theory

Let us consider some results obtained in Sections 5.1–5.4 from the point of view of Nelson's statistical mechanics and Nottale's scale relativistic theory. Within the framework of Nelson's statistical mechanics [34, 35], a particle (for example, an atom or electron) in an external field is considered. The atom is regarded as a point particle of mass m_0 in the sense of Newtonian mechanics. Nelson's basic assumption was that any particle of mass m_0 constantly undergoes a universal Brownian motion with diffusion coefficient D inversely proportional to m_0 [34]:

$$D = \hbar / 2m_0 , \qquad (5.5.1)$$

where the constant $\hbar$ (having the dimensions of action) is identified with the Planck constant divided by 2π, that is, as usual $\hbar = h/2\pi$. As in the theory of macroscopic Brownian motion, the influence of the external force is expressed by Newton's second law: $\vec{F} = m_0\vec{a}$, where $\vec{a}$ is a mean acceleration of the colloidal particle. On the other hand, the particle (in particular, electron) is in dynamical equilibrium between the random force causing the Brownian motion and the external force (attractive Coulomb force of nucleus), so *its trajectory is very irregular* (analogous behaviour of a particle into a colloidal suspension also occurs in dynamical equilibrium between osmotic forces and gravity) [34]. This means that the dependence of a coordinate x of the moving particle on time t can be described by a *stochastic process* $x(t)$. However, it is well known [210–212] that for many stochastic processes $x(t)$ is not differentiable. Following E. Nelson [34], the both *mean forward* and *mean backward derivatives* of the stochastic process are to be introduced to this end:

$$\frac{d_+x(t)}{dt} = \lim_{\Delta t \to 0+} \langle \frac{x(t+\Delta t)-x(t)}{\Delta t} \rangle_t ; \qquad (5.5.2a)$$

$$\frac{d_-x(t)}{dt} = \lim_{\Delta t \to 0-} \langle \frac{x(t)-x(t-\Delta t)}{\Delta t} \rangle_t , \qquad (5.5.2b)$$

where $\langle \ \rangle_t$ denotes the conditional expectation (average) given the state of the system with the time t. It should be from comparing Eq. (5.5.2a) with Eq. (5.5.2b) that the mean backward derivative can be obtained from the mean forward derivative by the reflection [200]: $\Delta t \to -\Delta t$.

Let $w(\mathbf{x},t)$ be the probability density of a *vector-valued* stochastic process $\mathbf{x}(t)$ where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ denotes the position of a Brownian particle with the time t. Then the probability density satisfies both the *forward* and *backward* Fokker–Planck equations [34, 190, 191]:

$$\frac{\partial w}{\partial t} = -\mathrm{div}(w\mathbf{v}_+) + D\nabla^2 w ; \qquad (5.5.3a)$$

$$\frac{\partial w}{\partial t} = -\mathrm{div}(w\mathbf{v}_-) - D\nabla^2 w , \qquad (5.5.3b)$$

where $\mathbf{v}_+$ and $\mathbf{v}_-$ are vector-valued functions on space-time called *the mean forward* and *backward velocities* [34] in the sense of the definitions (5.5.2a,b):

$$\mathbf{v}_+ = \frac{d_+\mathbf{x}(t)}{dt}; \qquad (5.5.4a)$$

$$\mathbf{v}_- = \frac{d_-\mathbf{x}(t)}{dt}. \qquad (5.5.4b)$$

Using the definitions (5.5.4a, b), E. Nelson introduced two averaged velocities, the so-called the *current* (or ordinary regular) *velocity* $\vec{\mathrm{v}}$ as well as the *osmotic velocity* $\vec{\mathrm{u}}$ [34]:

$$\vec{\mathrm{v}} = \frac{1}{2}(\mathbf{v}_+ + \mathbf{v}_-); \qquad (5.5.5a)$$

$$\vec{\mathrm{u}} = \frac{1}{2}(\mathbf{v}_+ - \mathbf{v}_-). \qquad (5.5.5b)$$

Let us note that by comparing Eq. (5.5.3a) with Eq. (5.5.3b) we find that:

$$\frac{\partial w}{\partial t} = \nabla\{-w\mathbf{v}_+ + D\nabla w\} = \nabla\{-w\mathbf{v}_- - D\nabla w\},$$

whence Eq. (5.5.5b) becomes:

$$\vec{\mathrm{u}} = \frac{1}{2}(\mathbf{v}_+ - \mathbf{v}_-) = D\frac{\nabla w}{w} = D\,\mathrm{grad}\ln w, \qquad (5.5.6)$$

that is, in other words, by subtracting Eq. (5.5.3a) from Eq. (5.5.3b) and taking into account Eq. (5.5.5b) we obtain Eq. (5.5.6). Nelson noted in [34] that:

> According to Einstein's theory [46] of Brownian motion, (26) [1], *i.e. our (5.5.6)*, is the velocity acquired by a Brownian particle, in equilibrium with respect to an external force, to balance the osmotic force.

Moreover, according to Eq. (5.5.5a), the average of Eq. (5.5.3a) and Eq. (5.5.3b) yields the ordinary equation of continuity (see Eq. (5.1.16)):

[1] The formula numeration corresponds to Nelson's paper [34].

$$\frac{\partial w}{\partial t} = -\mathrm{div}(w\vec{\mathrm{v}}) \,. \qquad (5.5.7)$$

Using the equation of continuity (5.5.7), we can compute the partial derivative of the osmotic velocity (5.5.6) with respect to the time:

$$\frac{\partial \vec{\mathrm{u}}}{\partial t} = D\,\mathrm{grad}\left\{\frac{1}{w}\frac{\partial w}{\partial t}\right\} = D\nabla\left\{\frac{1}{w}(-\mathrm{div}(w\vec{\mathrm{v}}))\right\} =$$

$$= D\nabla\left\{-\nabla\vec{\mathrm{v}} - \vec{\mathrm{v}}\frac{\nabla w}{w}\right\} = -D\,\mathrm{grad}(\mathrm{div}\vec{\mathrm{v}}) - \mathrm{grad}(\vec{\mathrm{v}}\cdot\vec{\mathrm{u}}) \,. \qquad (5.5.8)$$

Let us note that Nelson's equation (5.5.8) is the *particular case* of equation (5.1.17) obtained within the framework of the statistical theory of spheroidal bodies (see Section 5.1 of this chapter). Indeed, Eq. (5.1.17) goes over to Eq. (5.5.8) and the anti-diffusion velocity (5.1.20) becomes Nelson's osmotic velocity (5.5.6) when $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$ and, moreover, the value G_s should be equal to $D = \hbar / 2m_0$ in accordance with Eq. (5.5.1), that is, the proposed statistical theory of spheroidal bodies generalizes some statements of Nelson's statistical mechanics.

Following Nelson, the mean acceleration is defined by the mean second derivative of a stochastic process [34]:

$$\vec{a} = \frac{1}{2}\left(\frac{d_+\mathbf{v}_-}{dt} + \frac{d_-\mathbf{v}_+}{dt}\right) =$$

$$= \frac{\partial\vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}}\cdot\nabla)\,\vec{\mathrm{v}} - (\vec{\mathrm{u}}\cdot\nabla)\,\vec{\mathrm{u}} - D\nabla^2\,\vec{\mathrm{u}} \,. \qquad (5.5.9)$$

This Nelson equation (5.5.9) is a particular case of the equation (5.1.19) already obtained in Section 5.1 based on the statistical theory of spheroidal bodies. Indeed, if $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$. Moreover, the value G_s is equal to $D = \hbar / 2m_0$ due to Eq. (5.5.1), then Eq. (5.1.19) becomes Eq. (5.5.9).

E. Nelson also showed in his paper [34] that both Eqs (5.5.8) and (5.5.9) are equivalent to the time-dependent Schrödinger equation [187]. Taking into account the above-mentioned remarks relative to the considered Eqs (5.1.17) and (5.5.8), Eqs (5.1.20) and (5.5.6), and Eqs (5.1.19) and (5.5.9), we conclude that the main results of Nelson's statistical mechanics can be obtained within the framework of the proposed statistical theory of spheroidal bodies. Moreover, Nelson's statistical mechanics is based on the *hypothesis* that particles in empty space (or the ether) are subject to a universal Brownian motion [34, 35]. However, the nature of such motion was not explained. On the contrary, the developing statistical theory of spheroidal bodies does not use any additional hypothesis because the behavior of particles in a spheroidal body is explored through the above proposed statistical model (see Section 3.3 in Chapter 3).

The scale of the relativistic theory of L. Nottale [36, 37, 175, 176, 200, 201] is the furthest development of Nelson's statistical mechanics. This theory extends Einstein's principle of relativity to scale laws because relativity had only been applied to motion laws up to then [200]. Nottale's theory of scale-relativity is based on three hypotheses [201].

Indeed, within the framework of Nottale's approach, both direct and reverse processes in parallel are considered. That leads to the introduction of a *twin* Wiener (backward and forward) process described in terms of a *single complex process* [37, 175, 176, 200]. Then, in terms of this global tool, reversibility is recovered. Following Nelson [34], the mean forward and backward derivatives (5.5.2a, b) are also considered in Nottale's theory. From these quantities and the properties of the "double-Wiener" process, L. Nottale introduced a *complex derivative operator* [200]:

$$\frac{\dot{\mathrm{d}}}{dt} = \frac{d_+ + d_-}{2dt} - \mathrm{i}\frac{d_+ - d_-}{2dt}\,. \qquad (5.5.10)$$

Applying Eq. (5.5.10) to the position vector $\mathbf{x}$ and bearing in mind Eqs (5.5.4a, b) as well as Nelson's definitions (5.5.5a, b), it yields the above-mentioned complex velocity:

$$\dot{\vec{\mathrm{V}}} = \frac{\dot{\mathrm{d}}\mathbf{x}}{dt} = \frac{d_+\mathbf{x} + d_-\mathbf{x}}{2dt} - \mathrm{i}\frac{d_+\mathbf{x} - d_-\mathbf{x}}{2dt} =$$

$$= \frac{\mathbf{v}_+ + \mathbf{v}_-}{2} - \mathrm{i}\frac{\mathbf{v}_+ - \mathbf{v}_-}{2} = \vec{\mathrm{v}} - \mathrm{i}\vec{\mathrm{u}}\,. \qquad (5.5.11)$$

Taking into account Nelson's Eqs (5.5.5a), (5.5.6), and (5.5.9), Nottale's complex derivative operator (5.5.10) with the usage of Eq. (5.5.11) can be represented in the form [37, 175, 176, 200, 201]:

$$\frac{\dot{\mathrm{d}}}{dt} = \frac{\partial}{\partial t} + \vec{\mathrm{v}}\cdot\nabla - \mathrm{i}\vec{\mathrm{u}}\cdot\nabla - \mathrm{i}\,D\nabla^2 = \frac{\partial}{\partial t} + \dot{\vec{\mathrm{V}}}\cdot\nabla - \mathrm{i}\,D\nabla^2, \qquad (5.5.12)$$

where D is a parameter characterizing the *fractal behavior* of trajectories [201]. Since the mean velocity $\dot{\vec{\mathrm{V}}}$ is complex-valued, the same is true of a Lagrange function, then of the generalized action $\dot{S}$ [37, 175, 176, 200, 201]. The complex velocity $\dot{\vec{\mathrm{V}}}$ is a gradient of the complex action $\dot{S}$:

$$\dot{\vec{\mathrm{V}}} = \nabla\dot{S}/m_0\,. \qquad (5.5.13)$$

Nottale also introduced a complex wave function Ψ:

$$\Psi = \mathrm{e}^{\mathrm{i}\dot{S}/2m_0 D}\,. \qquad (5.5.14)$$

According to Eqs (5.5.13) and (5.5.14), this is related to the complex velocity $\dot{\vec{\mathrm{V}}}$ as follows [176, 200]:

$$\dot{\vec{\mathrm{V}}} = -2\mathrm{i}\,D\nabla(\ln\Psi)\,. \qquad (5.5.15)$$

Let us consider now a particle with mass m_0 moving in a gravitational field and subjected to strong chaos [200]. Using the Nottale's complex derivative operator (5.5.12) as well as the complex velocity $\dot{\vec{\mathrm{V}}}$ notion (5.5.11), Newton's fundamental equation of dynamics becomes [200]:

$$m_0 \frac{\dot{\mathrm{d}}}{dt}\dot{\vec{\mathrm{V}}} = -m_0 \nabla \varphi_g \,, \qquad (5.5.16)$$

where φ_g is Newtonian potential. Taking into account Eqs (5.5.12) and (5.5.15) one can write the fundamental equation of dynamics (5.5.16) in terms of the new quantity Ψ [176]:

$$-2\mathrm{i}\, D m_0 \frac{\dot{\mathrm{d}}}{dt}\big(\nabla(\ln\Psi)\big) = -2\mathrm{i}\, D m_0 \nabla\{\frac{\partial}{\partial t}\ln\Psi - \mathrm{i}\, D\frac{\nabla^2\Psi}{\Psi}\} = -m_0 \nabla \varphi_g \,.$$

Integrating this equation finally yields the generalized time-dependent Schrödinger equation [175, 176, 200, 201]:

$$\mathrm{i}\, 2 m_0 D \frac{\partial \Psi}{\partial t} = (-2 m_0 D^2 \nabla^2 + m_0 \varphi_g)\Psi \,, \qquad (5.5.17)$$

whence taking into account the representation $\Psi(\vec{r},t) = \psi(\vec{r}) e^{-\mathrm{i}\, Et/2m_0 D}$ this follows directly the generalized stationary Schrödinger equation in the form of Nottale:

$$(-2 m_0 D^2 \nabla^2 + m_0 \varphi_g)\psi = E\psi \,. \qquad (5.5.18)$$

The meaning of Ψ can be understood by setting $w = \Psi\Psi^*$ where w can be interpreted as giving the probability density of the particle positions. Thus, within the framework of Nottale's approach, the Newtonian equation of dynamics can be transformed and integrated in terms of the generalized Schrödinger equations (5.5.17) and (5.5.18). Thus, Nottale's theory of scale-relativity was initially developed to refound quantum mechanics on first principles, although the scale-relativistic approach can be applied not only at small scales but also at *very large space-time scales* [176]. In this case, Eqs (5.5.17) and (5.5.18) must be independent of the test-particle mass m_0 and, as a consequence, the parameter D takes the form [201]:

$$D = \gamma M / 2\upsilon \,, \qquad (5.5.19)$$

where γ is the Newtonian gravitational constant and υ is a fundamental constant that has the dimension of velocity. In addition, the ratio υ / c plays the role of a gravitational coupling constant [201, 213].

Unlike Nottale's scale relativistic theory (when $D = \gamma M / 2\upsilon$) or Nelson's statistical mechanics (when $D = \hbar / 2m_0$), within the framework of the proposed statistical theory of spheroidal bodies, the gravitational compression function $\mathrm{G}(t)$ can be a constant $\mathrm{G}(t) = \mathrm{G}_s = \text{const}$ only in the very particular cases if a spheroidal body is being in states of mechanical or relative mechanical equilibrium [16, 47, 68, 71, 73] (it follows from a comparison of the obtained Eqs (5.1.17) and (5.1.19) with the relevant Eqs (5.5.8) and (5.5.9)). Moreover, within the framework of Nottale's scale relativistic theory an irregular behavior of trajectories of particles is explained by the fractal properties of the space-time continuum exclusively [201] whereas, as noted in Section 3.3 of Chapter 3, such irregular trajectories are caused by the initial oscillations of interacting particles in accordance with the anti-diffusion process into a gravitational compressible spheroidal body.

5.6. The derivation of the generalized nonlinear Schrödinger-like equation in the statistical theory of gravitating spheroidal bodies

As shown in Sections 4.1 and 4.7, initially the probability density for observing an oscillating colloidal (or liquid) particle satisfies the *anti-diffusion equation* (4.1.9a) or (4.7.12) when $\varepsilon_0 \neq 0$, that is, consideration in Sections 3.3, 4.1, and 4.7 points to an *initial quasi-equilibrium gravitational* condensation occurring in a forming spheroidal body. On the other hand, a sharp *increase* of the anti-diffusion velocity

(5.1.7) when $\rho \to 0$, as noted in Section 5.1, can lead to the coherent displacement of particles inside a spheroidal body and, as a consequence, to a resonance increase of the parameter of gravitational condensation $\alpha(t)$ (see respective formula (5.2.13a) in Section 5.2). This means that *nonlinear phenomena* arise as a result of self-organization processes [135] into a spheroidal body under its formation. These nonlinear phenomena induce nonlinear autowaves satisfying a *nonlinear* undulatory Schrödinger-like equation [68, 71, 73].

Now let us note that the well-known linear Schrödinger equation [187], as well as its generalization (5.5.17), were mentioned in Section 5.5 in connection with Nelson's statistical mechanics and Nottale's scale relativity. Moreover, both these equations of Schrödinger are derived in the special case of a constant $\mathrm{G}(t)$ in the anti-diffusion equation (4.1.9b). Nevertheless, this chapter studies the general case of $\mathrm{G}(t)$ which can be different from Nelson's and Nottale's considerations. This is the reason why we must return to the derived (in Section 5.1) equations for calculating the partial derivatives (relatively to t) of anti-diffusion velocity and ordinary hydrodynamic velocity to obtain a *nonlinear* generalized Schrödinger-like equation by analogy with Nelson's and Nottale's theories.

So, now let us consider again Eqs (5.1.17) and (5.1.19) derived within the framework of the statistical theory of a gravitating *centrally symmetric* spheroidal body. Taking into account the simple formulas (5.1.11a, b) and (5.1.20), then Eqs (5.1.17) and (5.1.19) can be rewritten in the form:

$$\frac{\partial \vec{\mathrm{u}}}{\partial t} = -\mathrm{G}(t)\,\mathrm{grad}(\mathrm{div}\,\vec{\mathrm{v}}) - \mathrm{grad}(\vec{\mathrm{v}}\cdot\vec{\mathrm{u}}) + \frac{d\ln \mathrm{G}(t)}{dt}\vec{\mathrm{u}}\,; \quad (5.6.1\mathrm{a})$$

$$\frac{\partial \vec{\mathrm{v}}}{\partial t} = \vec{a} - (\vec{\mathrm{v}}\cdot\nabla)\,\vec{\mathrm{v}} + \mathrm{grad}(\vec{\mathrm{u}}^2/2) +$$

$$+\,G(t)\,\mathrm{grad}(\mathrm{div}\vec{u}) - \frac{d\ln G(t)}{dt}\vec{u}\,. \qquad (5.6.1b)$$

Let us investigate some special solutions of Eqs (5.6.1a, b) in the case that the acceleration (or gravitational field strength) coming from a gravitational field potential of a spheroidal body, in other words,

$$\vec{a} = -\,\mathrm{grad}\varphi_g\,, \qquad (5.6.2)$$

under the assumption that the hydrodynamic velocity $\vec{v}$ is a gradient of a statistical action $\Im$ which is a potential of velocity [68, 71, 73]:

$$\vec{v} = 2G(t)\,\mathrm{grad}\,\Im\,. \qquad (5.6.3)$$

In the special case of a constant $G(t)$ as $\hbar/2m_0$ Eq. (5.6.3) becomes the Nelson formula [34]: $\vec{v} = \frac{\hbar}{m_0}\mathrm{grad}\,\Im$. In this connection, $\mathrm{rot}\vec{v} = 0$, thus, $(\vec{v}\cdot\nabla)\,\vec{v} = \mathrm{grad}(\vec{v}^2/2)$. Since $\vec{u}$ is also a gradient due to Eq. (5.1.20), as well as $\vec{a}$ and $\vec{v}$ according to Eqs (5.6.2) and (5.6.3), then Eqs (5.6.1a, b) become the following:

$$\mathrm{grad}\frac{\partial(G(t)\ln\Phi)}{\partial t} = -G(t)\,\mathrm{grad}(\mathrm{div}\vec{v}) - \mathrm{grad}(\vec{v}\cdot\vec{u}) +$$

$$+\,\{d\ln G(t)/dt\}G(t)\,\mathrm{grad}\ln\Phi\;; \qquad (5.6.4a)$$

$$\mathrm{grad}\frac{\partial(2G(t)\Im)}{\partial t} = -\,\mathrm{grad}\,\varphi_g - \mathrm{grad}(\vec{v}^2/2) + \mathrm{grad}(\vec{u}^2/2) +$$

$$+\,G(t)\,\mathrm{grad}(\mathrm{div}\vec{u}) - \{d\ln G(t)/dt\}G(t)\,\mathrm{grad}\ln\Phi\,. \qquad (5.6.4b)$$

Integrating these Eqs (5.6.4a, b) and considering a simplification $\{d\ln G(t)/dt\}\cdot G(t) = d\,G(t)/dt$, we can find that:

$$\frac{\partial(G(t)\ln\Phi)}{\partial t} = -G(t)\mathrm{div}\,\vec{v} - \vec{v}\cdot\vec{u} + \frac{d\,G(t)}{dt}\ln\Phi\;; \qquad (5.6.5a)$$

$$\frac{\partial(2\mathrm{G}(t)\Im)}{\partial t}=-\varphi_g-\frac{\vec{\mathrm{v}}^2}{2}+\frac{\vec{\mathrm{u}}^2}{2}+\mathrm{G}(t)\mathrm{div}\vec{\mathrm{u}}-\frac{d\,\mathrm{G}(t)}{dt}\ln\Phi\,. \quad (5.6.5b)$$

Let us carry out a change of dependent variable:

$$\Re=\frac{1}{2}\ln\Phi\,; \quad (5.6.6a)$$

$$\Psi=\mathrm{e}^{\Re+\mathrm{i}\,\Im}\,, \quad (5.6.6b)$$

where $\Im$ is defined by Eq. (5.6.3), $\mathrm{i}=\sqrt{-1}$. It follows directly from Eqs (5.6.6a, b) that:

$$\Psi=\sqrt{\Phi}\cdot\mathrm{e}^{\mathrm{i}\,\Im}\,, \quad (5.6.7)$$

so that $\Phi=\Psi\Psi^*=\left|\Psi\right|^2$ as usual. According to the first change (5.6.6a) it is not difficult to see that:

$$\frac{\partial(2\mathrm{G}(t)\,\Re)}{\partial t}=-2\mathrm{G}^2(t)\nabla^2\Im-$$

$$-4\mathrm{G}^2(t)\nabla\Re\cdot\nabla\Im+2\frac{d\,\mathrm{G}(t)}{dt}\Re\,; \quad (5.6.8a)$$

$$\frac{\partial(2\mathrm{G}(t)\Im)}{\partial t}=-\varphi_g+2\mathrm{G}^2(t)(\nabla\Re)^2-2\mathrm{G}^2(t)(\nabla\Im)^2+$$

$$+2\mathrm{G}^2(t)\nabla^2\Re-2\frac{d\,\mathrm{G}(t)}{dt}\Re\,. \quad (5.6.8b)$$

Let us rewrite these two Eqs (5.6.8a, b) as one. To this end, after the multiplication of the second Eq. (5.6.8b) on an imaginary unit and with the addition of Eqs (5.6.8a, b), we find [68, 71, 73]:

$$\frac{\partial}{\partial t}\left[2\mathrm{G}(t)(\Re+\mathrm{i}\,\Im)\right]=-\mathrm{i}\,\varphi_g+\mathrm{i}\,2\mathrm{G}^2(t)(\nabla^2\Re+\mathrm{i}\,\nabla^2\Im)+$$

$$+\mathrm{i}\left[\sqrt{2}\mathrm{G}(t)\nabla(\Re+\mathrm{i}\,\Im)\right]^2+2(1-\mathrm{i})\frac{d\,\mathrm{G}(t)}{dt}\Re\,. \quad (5.6.9)$$

Taking into account the second change (5.6.6b) we can see that

$$\Re+\mathrm{i}\,\Im=\ln\Psi;$$

$$2\Re = \ln\Psi + \ln\Psi^* = \ln|\Psi|^2;$$
$$\nabla(\Re + \mathrm{i}\,\Im) = \nabla\ln\Psi = \nabla\Psi/\Psi;$$
$$\nabla^2(\Re + \mathrm{i}\,\Im) = \nabla^2\Psi/\Psi - (\nabla\Psi)^2/\Psi^2,$$

so that Eq. (5.6.9) takes the form:

$$\frac{\partial}{\partial t}\left[2\mathrm{G}(t)\ln\Psi\right] = -\mathrm{i}\varphi_g + \mathrm{i}\,2\mathrm{G}^2(t)\frac{\nabla^2\Psi}{\Psi} + (1-\mathrm{i})\frac{d\,\mathrm{G}(t)}{dt}\ln|\Psi|^2 \cdot \quad (5.6.10)$$

After some transformations and simplifications Eq. (5.6.10) can be represented as follows:

$$\mathrm{i}\,2\mathrm{G}(t)\frac{\partial\Psi}{\partial t} = \varphi_g\Psi - 2\mathrm{G}^2(t)\nabla^2\Psi + 2\,\mathrm{i}(1-\mathrm{i})\frac{d\,\mathrm{G}(t)}{dt}\Psi\ln|\Psi| -$$
$$-\,2\,\mathrm{i}\frac{d\,\mathrm{G}(t)}{dt}\Psi\ln\Psi\,, \qquad (5.6.11)$$

whence we can obtain a nonlinear time-dependent generalized Schrödinger-like equation of the kind [68, 71, 73]:

$$\mathrm{i}\,2\mathrm{G}(t)\frac{\partial\Psi}{\partial t} = \left[-2\mathrm{G}^2(t)\nabla^2 + \varphi_g\right]\Psi + 2\frac{d\,\mathrm{G}(t)}{dt}\left[\ln|\Psi| - \mathrm{i}\ln\frac{\Psi}{|\Psi|}\right]\Psi \cdot \quad (5.6.12)$$

In the case of a rotating *axially symmetric* spheroidal body when $\varepsilon_0 \neq 0$, as noted in Section 5.1, we can replace GCF $\mathrm{G}(t)$ on the generalized GCF $\widetilde{\mathrm{G}}(t)$ in the Eqs (5.1.9), (5.1.13), and (5.1.17)–(5.1.20), so that Eqs (5.6.1a, b) as well as (5.6.12) remain valid in the general case of a rotating axially symmetric spheroidal body. So, we can obtain the nonlinear time-dependent generalized Schrödinger-like equation describing the formation of a rotating and gravitating ellipsoid-like cosmogonical body [77]:

$$\mathrm{i}\,2\widetilde{\mathrm{G}}(t)\frac{\partial\Psi}{\partial t} = \left[-2\widetilde{\mathrm{G}}^2(t)\nabla^2 + \psi_g\right]\Psi + 2\frac{d\,\widetilde{\mathrm{G}}(t)}{dt}\left[\ln|\Psi| + \arg\Psi\right]\Psi, \quad (5.6.13)$$

where ψ_g is a general potential of gravitational and inertial (centrifugal) fields (see Section 1.4).

Let us note that $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$ or $\widetilde{\mathrm{G}}(t) = \widetilde{\mathrm{G}}_s = \mathrm{const}$ in the *relative mechanical equilibrium* states of a spheroidal body [16, 68, 71, 73], so the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or (5.6.13)) becomes linear in these special cases: for example, the time-dependent Schrödinger equation [187] is a particular case of Eq. (5.6.12) if $\mathrm{G}(t)$ satisfies the Nelson basic assumption (5.5.1), as well as the time-dependent Schrödinger equation in the form of Nottale (5.5.17), and is a special case of Eq. (5.6.12). So, Nelson's and Nottale's considerations are appropriate, mainly in the case of gravitational interaction of particles in a spheroidal body being in a virial equilibrium state. We can note that the derived generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or (5.6.13)) also describes the *gravitational instability state* with an increase of gravitational compression leading to the formation of a core of a cosmogonical body. Since it is difficult to find a general solution to the generalized nonlinear Schrödinger-like equation (5.6.12) (or (5.6.13)), we intend to consider its important particular cases below.

5.7. Some particular cases of the generalized nonlinear Schrödinger-like equation describing different dynamical states of a gravitating spheroidal body

Let us consider different dynamical states of a gravitating centrally symmetric spheroidal body as well as the respective forms of the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or (5.6.13)) in the case of the axially symmetric spheroidal body). Indeed, the derived equation (5.6.12) describes not only the mentioned state of *mechanical equilibrium* [68, 71, 73] when GCF $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const} \in \mathbf{R}$ and $\Psi \in \mathbf{R}$ or $\Psi \in \mathbf{C}$:

$$\mathrm{i}\,\mathrm{G}_s \frac{\partial \Psi}{\partial t} = \left[-\mathrm{G}_s{}^2 \nabla^2 + \frac{1}{2} \varphi_g \right] \Psi \tag{5.7.1}$$

and the *quasi-equilibrium* gravitational condensation state [68, 71, 73] with a slowly (periodically) varying GCF increment when $\mathrm{G}(t) = \mathrm{G}_s [1 + \delta \cos \omega_s t] \in \mathbf{R}$ and $\Psi \in \mathbf{R}$ or $\Psi \in \mathbf{C}$:

$$\mathrm{i}\,\mathrm{G}(t) \frac{\partial \Psi}{\partial t} = \left[-\mathrm{G}^2(t) \nabla^2 + \frac{1}{2} \varphi_g \right] \Psi + \frac{d\,\mathrm{G}(t)}{dt} \ln|\Psi| \cdot \Psi , \tag{5.7.2}$$

but also the *initial* equilibrium gravitational condensation state [16, 73] occurring in a forming gas-dust protoplanetary cloud:

$$\mathrm{i} \frac{\partial \Psi}{\partial t} = -\mathrm{G}_s \nabla^2 \Psi \tag{5.7.3}$$

as well as the *soliton disturbances state* [77] arising in a spheroidal body under formation:

$$\mathrm{i} \frac{\partial \Psi}{\partial t} = \left[-\mathrm{G}(t) \nabla^2 + \frac{1}{2} \frac{d \ln \mathrm{G}(t)}{dt} |\Psi|^2 \right] \Psi \tag{5.7.4}$$

and the *gravitational instability states* [71, 73] when GCF $\mathrm{G}(t) \in \mathbf{C}$ and $\Psi = |\Psi| e^{\mathrm{i} \arg \Psi} \in \mathbf{C}$:

$$\mathrm{i}\,\mathrm{G}(t) \frac{\partial \Psi}{\partial t} = \left[-\mathrm{G}^2(t) \nabla^2 + \frac{1}{2} \varphi_g \right] \Psi + \frac{d\,\mathrm{G}(t)}{dt} \left[\ln|\Psi| + \arg \Psi \right] \Psi , \tag{5.7.5}$$

including the increase of gravitational compression of a spheroidal body providing the formation of a core of cosmogonical body if $0 \le \arg \Psi < 2\pi$ (the case of unlimited gravitational compression leading to a collapse occurs when $\arg \Psi \to \arg \Psi \pm 2\pi n,\, n \in \mathbf{Z}$).

Let us note that according to the relation (5.6.7) the probability density function $\Phi = \Psi^* \Psi$ satisfies the anti-diffusion equation of the type (4.1.9b) while this wave function Ψ satisfies the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12). However, in the case of a constant G(t) in Eq. (4.1.9b) the equivalence between Eq. (4.1.9b) and Eq. (5.6.12) becomes possible because the derived equation (5.6.12) (or (5.7.5)) goes over the well-known linear time-dependent Schrödinger equation (5.7.1). Thus, the generalized nonlinear Schrödinger-like equation describes different dynamical states of a gravitating spheroidal body.

Now let us consider some wave solutions of the generalized nonlinear Schrödinger-like equation taking into account its important particular cases (5.7.3) and (5.7.4).

The *initial equilibrium* gravitational condensation state is realized in a forming gas-dust protoplanetary cloud when the initial gravitational field φ_g is absent ($\varphi_g = 0$) and $\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$ so that the generalized nonlinear Schrödinger-like equation (5.6.12) becomes the *linearized* Schrödinger equation (5.7.3). This equation has a *wave* solution in the vicinity of the equilibrium state when $\mathrm{G}_s = \mathrm{const}$:

$$\Psi(\vec{r},t) = \Psi_0 e^{-\mathrm{i}(\omega_s t \pm \vec{k}\vec{r} + \varepsilon^0)},\ \mathrm{i} = \sqrt{-1}. \tag{5.7.6}$$

Indeed, let us calculate the derivatives of Ψ with respect to the spatial vector $\vec{r}$ and the time t:

$$\frac{\partial \Psi}{\partial \vec{r}} = -\mathrm{i}(\pm k)\Psi_0 e^{-\mathrm{i}(\omega_s t \pm \vec{k}\vec{r} + \varepsilon^0)} = \mp \mathrm{i}\, k\Psi; \tag{5.7.7a}$$

$$\frac{\partial^2 \Psi}{\partial \vec{r}^2} = -(\mp k)^2 \Psi_0 e^{-\mathrm{i}(\omega_s t \pm \vec{k}\vec{r} + \varepsilon^0)} = -k^2\Psi; \tag{5.7.7b}$$

$$\frac{\partial \Psi}{\partial t} = -\mathrm{i}\,\omega_s \Psi_0 e^{-\mathrm{i}(\omega_s t \pm \vec{k}\vec{r} + \varepsilon^0)} = -\mathrm{i}\,\omega_s \Psi. \tag{5.7.7c}$$

So, comparing (5.7.7b) and (5.7.7c) we can see that:

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = -\frac{\omega_s}{k^2}\frac{\partial^2 \Psi}{\partial \vec{r}^2}. \tag{5.7.8}$$

Since Eq. (5.7.8) coincides with Eq. (5.7.3), the following formula is true:

$$\omega_s = \mathrm{G}_s k^2 . \qquad (5.7.9)$$

Taking into account formula (5.7.9) the wave solution (5.7.6) of Eq. (5.7.3) is rewritten in the form:

$$\Psi(\vec{r},t) = \Psi_0 e^{-\mathrm{i}(\mathrm{G}_s k^2 t \pm \vec{k}\vec{r} + \varepsilon^0)} . \qquad (5.7.10)$$

According to Eqs (5.2.33), (5.2.35) the analogous wave solutions occur, for example, for the anti-diffusion velocity $\vec{\tilde{\mathrm{u}}} = \vec{\mathrm{u}}_0\, e^{-\mathrm{i}(\pm\vec{k}\vec{r} + \vec{k}^2 \mathrm{G}_s t + \varepsilon^0_{\mathrm{\tilde{u}}})}$. In other words, in the quasi-equilibrium gravitational condensation state with a periodically varying GCF $\mathrm{G}(t) = \mathrm{G}_s[1+\delta\cos\omega_s t]$, the wave solutions $\vec{\tilde{\mathrm{u}}}$ are generated, moreover, they induce specific periodic forces (5.2.35), (5.3.16), or (5.4.39) and respective spatial oscillations (like the radial and the axial oscillations of Alfvén–Arrhenius [9, 19]) in the different domains of a forming spheroidal body (see Sections 5.2-5.4 and 9.2).

We will now investigate *nonlinear wave solutions* of the generalized nonlinear Schrödinger-like equation. As shown in Sections 5.1 and 5.2, as a result of the formation of a core of a cosmogonical body (based on a model of a spheroidal body) from an initial weakly condensed molecular cloud, a *sharp increase* in the anti-diffusion velocity of particles inside the cloud is highly likely leading to the gravitational field origin, subsequently. In this connection, we consider a possible scenario of transition from solutions of Eq. (5.7.3) in the form of plane waves of the type (5.7.10) (corresponding to the state of initial gravitational condensation of the molecular cloud) to *nonlinear wave solutions* of the generalized nonlinear Schrödinger-like equation (5.7.5) for the case $\mathrm{K} >> 1$ (see Eqs (5.2.31)–(5.2.34) in Section 5.2).

In other words, let us pass from equation (5.7.3) to a more general equation with a time-varying GCF $\mathrm{G}(t)$ under the condition that the absolute value of the wave function is still small, that is, $\left|\Psi(t)\right| < 1$. To this end, we use the generalized nonlinear Schrödinger-like equation (5.7.5) of a spheroidal body formation from a molecular cloud in a state of gravitational instability but under the condition of the smallness of the initial gravitational field φ_g and the absolute value $\left|\Psi(t)\right|$ respectively:

$$\mathrm{i}\,2\mathrm{G}(t)\frac{\partial\Psi}{\partial t} = \left[-2\mathrm{G}^2(t)\nabla^2 + \varphi_g\right]\Psi + \frac{d\,\mathrm{G}(t)}{dt}\left[\ln\left|\Psi\right|^2 + 2\arg\Psi\right]\Psi. \quad (5.7.11)$$

In this case, the logarithmic function in the right-hand side of Eq. (5.7.11) can be decomposed into a Taylor series and restricted to the first term of smallness [214]:

$$\ln\left|\Psi\right|^2 = \ln(1+[\left|\Psi\right|^2-1]) = [\left|\Psi\right|^2-1] - [\left|\Psi\right|^2-1]^2/2 + \ldots \approx \left|\Psi\right|^2 - 1. \quad (5.7.12)$$

Using (5.7.12) equation, (5.7.11) takes the form:

$$\mathrm{i}\,2\mathrm{G}(t)\frac{\partial\Psi}{\partial t} = -2\mathrm{G}^2(t)\nabla^2\Psi + \frac{d\,\mathrm{G}(t)}{dt}\left|\Psi\right|^2\Psi + \left[\varphi_g + \frac{d\,\mathrm{G}(t)}{dt}(2\arg\Psi - 1)\right]\Psi. \quad (5.7.13)$$

Dividing both sides of Eq. (5.7.13) by $2\mathrm{G}(t)$ we obtain:

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = -\mathrm{G}(t)\nabla^2\Psi + \frac{\dot{\mathrm{G}}(t)}{2\mathrm{G}(t)}\left|\Psi\right|^2\Psi + \frac{1}{2\mathrm{G}(t)}\left[\varphi_g - \dot{\mathrm{G}}(t) + 2\dot{\mathrm{G}}(t)\arg\Psi\right]\Psi. \quad (5.7.14)$$

According to Eq. (5.6.3) and (5.6.7) the argument of the wave function is the statistical action $\Im$ so that the hydrodynamic velocity is to be its gradient:

$$\vec{\mathrm{v}} = 2\mathrm{G}(t)\,\mathrm{grad}\,\Im. \quad (5.7.15)$$

Since there is no practically hydrodynamic velocity for a motionless molecular cloud $(|\vec{v}| \to 0)$, the value of statistical action is also negligible:

$$\Im = \arg\Psi \to 0 . \tag{5.7.16}$$

Taking into account the condition (5.7.16), Eq. (5.7.14) goes to the following [214]:

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = -\mathrm{G}(t)\nabla^2 \Psi + \frac{1}{2}\frac{d \ln \mathrm{G}(t)}{dt}|\Psi|^2 \Psi + + \frac{1}{2\mathrm{G}(t)}\left[\varphi_g - \dot{\mathrm{G}}(t)\right]\Psi . \tag{5.7.17}$$

When a spheroidal body is forming from an initial weakly condensed molecular cloud, its initial gravitational potential φ_g is proportional to $\dot{\mathrm{G}}(t)$, as noted in Sections 5.2 and Ref.[77]. So, taking this circumstance into account, Eq. (5.7.17) is noticeably simplified [214]:

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = -\mathrm{G}(t)\nabla^2 \Psi + \frac{1}{2}\frac{d \ln \mathrm{G}(t)}{dt}|\Psi|^2 \Psi . \tag{5.7.18}$$

We can note that the obtained equation (5.7.18) fully corresponds to the announced nonlinear Schrödinger-like equation (5.7.4) of a spheroidal body forming in the state of soliton disturbances. Indeed, denoting in Eq. (5.7.18) by $\beta = \mathrm{G}(t)$, $\gamma = \frac{1}{2}\frac{d \ln \mathrm{G}(t)}{dt}$, $A = \Psi$ we obtain the well-known nonlinear (cubic) Schrödinger equation (NSE) [188]:

$$\mathrm{i}\frac{\partial A}{\partial t} = -\beta\nabla^2 A + \gamma|A|^2 A , \tag{5.7.19}$$

where $A = A(\vec{r}, t)$ is an amplitude of the envelope of the wave packet and β, γ are some values.

NSE, a nonlinear second-order partial differential equation describing the wave packet envelope in a medium with dispersion and cubic nonlinearity, is one of the key equations playing an important role in the theory of nonlinear waves, in particular, in nonlinear optics and plasma physics [188], [215–217]. Using Maxwell equations, as well as equations of a medium, in the case of a slowly varying amplitude $A = |\vec{\mathrm{E}}|$ of a linearly polarized wave:

$$\vec{\mathrm{E}}(x, y, z, t) = \frac{1}{2} A(x, y, z, t) \exp[\mathrm{i}(kx - \omega_0 t)] \vec{\mathrm{e}}_x \quad (5.7.20)$$

in the reference system of a moving electromagnetic pulse (z, $t = t_{lab} - z / \upsilon_g(\omega_0)$ where $\upsilon_g(\omega_0) \equiv \partial\omega / \partial k|_{\omega_0}$ is the group velocity, a scalar equation of the NSE-type (5.7.19) can be obtained within the framework of the paraxial approximation [217]. In this case, the cubic term in the right-hand side of Eq. (5.7.19) describes the optical effect of Kerr, that is, a change of the refractive index of optical material is proportional to the second power of strength of the acting electric field.

Since NSE (5.7.19) completely corresponds to the cubic generalized Schrödinger-like equation (5.7.18) for the state of soliton perturbations, this means that just as NSE (5.7.19) describes an evolution of the envelope of a wave packet of electromagnetic waves propagating in *nonlinear dispersible media*, the *cubic* generalized Schrödinger-like equation (5.7.18) describes an evolution of the envelope of a wave packet of *Jeans' substantial waves* that propagate in a nonlinear and dispersive medium of a forming cosmogonical body (following the theory of gravitational instability of Jeans [1] (see also Section 1.3)).

Using a suitable choice of parameters, Eq. (5.7.18) (or NSE (5.7.19)) can be reduced to a *standard dimensionless* form of the *one-dimensional* cubic generalized Schrödinger-like equation [215]:

$$\mathrm{i}\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + \kappa |\Psi|^2 \Psi = 0, \qquad (5.7.21)$$

where, in the general case, $\Psi(x,t)$ is a complex-valued function. The particular solution of Eq. (5.7.21) in the form of a traveling nonlinear wave satisfying the condition $\Psi \to 0$ at $|x| \to \infty$ is the following [215, 216]:

$$\Psi(x,t) = a\sqrt{2\kappa}\,\frac{\exp\{-\mathrm{i}[2\upsilon x + 4(\upsilon^2 - a^2)\,t - \phi_0]\}}{\mathrm{ch}\,2a(x + 4\upsilon t - x_0)}, \qquad (5.7.22)$$

where a, υ and ϕ_0, x_0 are arbitrary constants. As we know [188, 215, 216], the envelopes of the NSE solution in the form of a traveling nonlinear wave (5.7.22) are also called *solitons* (see Fig. 5.1).

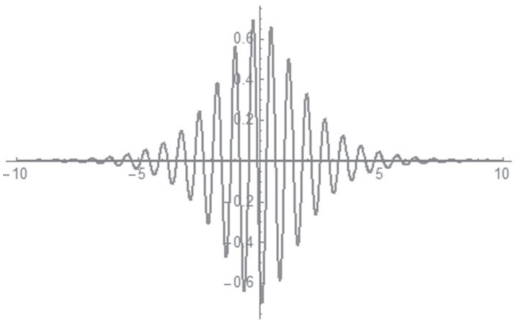


Figure 5.1. Soliton solution of a one-dimensional cubic generalized Schrödinger-like equation of a forming spheroidal body

Thus, this feature of the solution behavior in Fig. 5.1 has predetermined the title of equation (5.7.4).

5.8. Derivation of the reduced model in the state-space of a nonlinear dynamical system describing the behavior of the cubic generalized Schrödinger-like equation

Considering the one-dimensional partial differential equation (5.7.21) we will obtain a system of ordinary differential equations (ODEs) in state-space like the well-known Lorenz system [218]. To this end, we intend to rewrite solution (5.7.22) in the form:

$$\Psi(x,t)=\Psi_0(x,t)\cdot \mathrm{e}^{\mathrm{i}\Im(x,t)}\ , \qquad (5.8.1)$$

where, according to formula (5.6.7), $\Psi_0(x,t)=\sqrt{\Phi(x,t)}$ and $\Phi(x,t)$ is a one-dimensional probability density function [71, 214]. In this case, for the derivatives in Eq. (5.7.21) the following expressions are valid:

$$\frac{\partial\Psi}{\partial t}=\dot{\Psi}_0\mathrm{e}^{\mathrm{i}\Im}+\mathrm{i}\,\Psi_0\dot{\Im}\mathrm{e}^{\mathrm{i}\Im}\,,\quad \frac{\partial\Psi}{\partial x}=\Psi_0'\mathrm{e}^{\mathrm{i}\Im}+\mathrm{i}\,\Psi_0\Im'\mathrm{e}^{\mathrm{i}\Im}, \qquad (5.8.2a)$$

$$\frac{\partial^2\Psi}{\partial x^2}=\Psi_0''\mathrm{e}^{\mathrm{i}\Im}+2\mathrm{i}\,\Psi_0'\Im'\mathrm{e}^{\mathrm{i}\Im}+\mathrm{i}\,\Psi_0\Im''\mathrm{e}^{\mathrm{i}\Im}-\Psi_0\Im'^2\mathrm{e}^{\mathrm{i}\Im}\,, \qquad (5.8.2b)$$

where the dot means differentiation with respect to the time while the dash is differentiation relative to the coordinate in (5.8.2a,b) (the arguments of functions are omitted for brevity).

Substitution of (5.8.2a) and (5.8.2b) into Eq. (5.7.21) leads to the following equation:

$$-\Psi_0\dot{\Im}+\mathrm{i}\,\dot{\Psi}_0=-\Psi_0''-2\mathrm{i}\,\Psi_0'\Im'-\mathrm{i}\,\Psi_0\Im''+\Psi_0\Im'^2-\kappa\Psi_0^3\,, \qquad (5.8.3)$$

so that after separation of the real and imaginary parts we obtain:

$$\begin{cases}-\Psi_0\dot{\Im}=-\Psi_0''+\Psi_0\Im'^2-\kappa\Psi_0^3;\\ \quad\dot{\Psi}_0=-2\Psi_0'\Im'-\Psi_0\Im''.\end{cases} \qquad (5.8.4)$$

Let us represent the system of two equations (5.8.4) in the form [214]:

$$\begin{cases} \dot{\Im} = \dfrac{\Psi_0''}{\Psi_0} - \Im'^2 + \kappa\Psi_0^2; \\ \dot{\Psi}_0 = -2\Psi_0'\Im' - \Psi_0\Im''. \end{cases} \tag{5.8.5}$$

Relations (5.8.5) are a system of nonlinear equations leading to the *reduced model* like the Lorenz model allowing chaotic dynamics in state-space [218]. With a view to further transformation of the system (5.8.5), we assume that the amplitude $\Psi_0(x,t)$ depends on the coordinate rather weakly that initially takes place in the molecular cloud (when $\alpha \to 0$). This assumption permits us to neglect the term Ψ_0'' / Ψ_0 in the first equation of the system (5.8.5). As a result, we obtain the following system of equations [214]:

$$\begin{cases} \dot{\Im} = -\Im'^2 + \kappa\Psi_0^2; \\ \dot{\Psi}_0 = -2\Psi_0'\Im' - \Psi_0\Im''. \end{cases} \tag{5.8.6}$$

In (5.8.6), two variables x and t still appear in explicit form. To use only one variable (temporal) t we can apply the Galerkin's method known in hydrodynamics for flow stability problem solving [119]. According to this method, we are going to look for the functions Ψ_0 and $\Im$ in the form of expansions in a set of orthogonal basic functions:

$$\Psi_0(x,t) = \sum_n (A_n(t)\sin nkx + B_n(t)\cos nkx);$$

$$\Im(x,t) = \sum_n (G_n(t)\sin nkx + H_n(t)\cos nkx). \tag{5.8.7}$$

Choosing the concrete expansions (5.8.7), then substituting them into (5.8.6) and grouping the terms associated with the different components of these expansions we obtain various ODE systems of the kind:

$$\dot{q}_i = f(q_1, q_2, ... q_n), \tag{5.8.8}$$

where q_i are amplitudes in expansions (5.8.7), that is, A_n, B_n, and so forth, and the function $f(q_1, q_2, ...q_n)$ is a polynomial one in the case under consideration.

So, the nonlinearity in this reduced mathematical model is associated respectively with the nonlinear terms in equations of the system (5.8.6), and it is manifested when the multiplication of two trigonometric functions of the series (5.8.7) gives the third one, also presenting in the given decomposition. Later on, we consider expansions involving *second-order harmonics* only [214].

With a view to simplification, we can additionally assume a weak dependence of phase $\Im$ on the time leading to the condition $\dot{\Im} \approx 0$, that is consonant with the above-mentioned condition (5.7.16). This means that we can pass from system (5.8.6) to a single nonlinear equation relative to the function $\Psi_0(x,t)$. In this case, we obtain an expression for the coordinate derivative of Ψ_0 from the first equation of system (5.8.6):

$$\Im' = \sqrt{\kappa}\Psi_0, \tag{5.8.9}$$

which after substitution into the second equation of this system leads to a simple nonlinear differential equation for Ψ_0 [214]:

$$\dot{\Psi}_0 = -2\Psi_0'\sqrt{\kappa}\Psi_0 - \Psi_0\sqrt{\kappa}\Psi_0' = -3\sqrt{\kappa}\Psi_0\Psi_0'. \tag{5.8.10}$$

To eliminate the coordinate derivative from Eq. (5.8.10) and obtain the model in a reduced form (just as it has been done in the works [218, 219, 220] concerning problems of Rayleigh–Benard (convection in the heated layer [119]), Couette–Taylor (flows between coaxial rotating cylinders [121], [221]), Görtler (flows past a concave wall [222])) we suppose that the function $\Psi_0(x,t)$ is periodic with respect to x

so that we can represent it in the form of a decomposition in a trigonometric series leaving the first and second harmonics:

$$\Psi_0(x,t) = A(t)\sin kx + B(t)\cos kx + C(t)\sin 2kx + D(t)\cos 2kx. \quad (5.8.11)$$

Then for the derivative with respect to the coordinate, we obtain the expression:

$$\Psi_0'(x,t) = A(t)k\cos kx - B(t)k\sin kx + C(t)2k\cos 2kx - D(t)2k\sin 2kx. \quad (5.8.12)$$

After substituting (5.8.11) and (5.8.12) into Eq. (5.8.10) we have:

$$\dot{A}(t)\sin kx + \dot{B}(t)\cos kx + \dot{C}(t)\sin 2kx + \dot{D}(t)\cos 2kx =$$
$$= -3\sqrt{\kappa}(A^2k\sin kx\cos kx - ABk\sin kx\sin kx + A2kC\sin kx\cos 2kx -$$
$$- A2kD\sin kx\sin 2kx + BAk\cos kx\cos kx - B^2k\cos kx\sin kx +$$
$$+ B2kC\cos kx\cos 2kx - B2kD\cos kx\sin 2kx + CAk\sin 2kx\cos kx -$$
$$- CBk\sin 2kx\sin kx + C^2 2k\sin 2kx\cos 2kx - C2kD\sin 2kx\sin 2kx +$$
$$+ DAk\cos 2kx\cos kx - DBk\cos 2kx\sin kx + D2kC\cos 2kx\cos 2kx -$$
$$- D^2 2k\cos 2kx\sin 2kx),$$

whence, after separation of the terms associated with various components of the decomposition (5.8.11), we obtain the following system of ODEs:

$$\begin{aligned}
\dot{A} &= \frac{3k\sqrt{\kappa}}{2}(AC + BD);\\
\dot{B} &= \frac{3k\sqrt{\kappa}}{2}(AD + BC);\\
\dot{C} &= \frac{3k\sqrt{\kappa}}{2}(B^2 - A^2);\\
\dot{D} &= -3k\sqrt{\kappa}AB.
\end{aligned} \quad (5.8.13)$$

Renaming coefficients A, B, C, D with the preceding notation q_1, q_2, q_3, q_4 in Eq. (5.8.8) and introducing the control parameter $a = 3k\sqrt{\kappa}/2$ we obtain the following reduced model [214]:

$$\begin{aligned}\dot{q}_1 &= a(q_1 q_3 + q_2 q_4);\\ \dot{q}_2 &= a(q_1 q_4 + q_2 q_3);\\ \dot{q}_3 &= a(q_2{}^2 - q_1{}^2);\\ \dot{q}_4 &= -2a q_1 q_2.\end{aligned} \tag{5.8.14}$$

The obtained system (5.8.14) is an ODE system with quadratic nonlinearity, so in this sense, it is similar to the logistic parabola model [119] as well as the Lorenz model [218] and the model describing dynamical behavior of flow with a curvature of streamlines [219]:

$$\begin{aligned}\dot{q}_1 &= a q_2 - q_1 - q_2 q_3;\\ \dot{q}_2 &= q_1 q_3 + b q_1 + c q_2;\\ \dot{q}_3 &= q_1 q_2 + d q_3,\end{aligned} \tag{5.8.15}$$

however, unlike the final case, it contains four instead of three equations.

As seen from a comparison of Eq. (5.7.18) with Eq. (5.7.21), the value κ in Eq. (5.7.21) is proportional to $\dot{\mathrm{G}}(t)$ under consideration of a one-dimensional version of the cubic generalized Schrödinger-like equation (5.7.18) of a forming spheroidal body in the state of soliton perturbations [214]. This means that the control parameter a of the reduced model (5.8.14) in the state-space of the nonlinear dynamical system (describing the behavior of the cubic generalized Schrödinger-like equation (5.7.18)) is determined by the values of $\sqrt{\dot{\mathrm{G}}(t)}$.

Conclusion and comments

As already pointed out, inside a forming spheroidal body a colloidal particle undergoes attractions from originating numerous cores causing an *oscillatory character* of particle dynamics [68, 73]. The main contribution of this chapter has been to show that interactions of oscillating particles lead to increasing gravitational condensation (as a consequence, to the gravitational formation of a cosmogonical body) as well as the *evolution of dynamical states* of a forming cosmogonical body described by the generalized nonlinear Schrödinger-like equation.

Section 5.1 considered the density of anti-diffusion mass flows into a slow-flowing gravitational compressible spheroidal body. Here the notions of *anti-diffusion velocity* (5.1.7) and (5.1.21) both a *centrally symmetric* spheroidal body and an *axially symmetric* spheroidal body were introduced respectively. Equations (5.1.9), (5.1.17) for calculating the partial derivative of the anti-diffusion velocity with respect to time (in the cases of absence or presence of the ordinary hydrodynamic velocity inside a centrally symmetric spheroidal body) were obtained. Equation (5.1.19) relative to the complete time-derivative of the common (hydrodynamic plus anti-diffusion) velocity was derived. In the case of an axially symmetric spheroidal body, the analogous Eqs (5.1.23), (5.1.24) were also considered.

Section 5.2 investigated the initial gravitational field potential and the strength induced by anti-diffusion velocity into forming centrally and axially symmetric spheroidal bodies. This section derived equations (5.2.4a,b) and (5.2.21) of *arising gravitational potential* of an initial gravitational field for both forming spheroidal bodies. Within the framework of the statistical theory of gravitating spheroidal bodies, it showed that the gravitational potential φ_g of a forming spheroidal body is the sum of an average (statistical regular) part $\overline{\varphi}_g$ and a fluctuation part $\delta\varphi_g$. If the parameter of gravitational condensation $\alpha(t)$ reaches its critical value α_c then the propagation of gravitational disturbances with the circular frequency $\omega < \omega_c$ (or the wavelength $\lambda > \lambda_c$) ceases to be wave-motion and leads to an unstable motion according to the Jeans criterion (1.3.23). In this case, the parameter of gravitational condensation increases exponentially with the time in accordance with Eq. (5.2.13a), that is, an *avalanche gravitational compression* occurs as a consequence of the arising gravitational field of a spheroidal body. The complete analysis of dynamical states of a centrally symmetric spheroidal body can be carried out by respective dimensionless Eqs (5.2.28a, b) using the similarity numbers (for example, the introduced characterizing K number (5.2.29)). Since the new number K is a measure of the prevailing value of anti-diffusion velocity $|\vec{\mathrm{u}}|$ relative to the value of hydrodynamic velocity $|\vec{\mathrm{v}}|$ the number K can be considered as a *control parameter* of dynamical states of a centrally symmetric spheroidal body and an axially symmetric spheroidal body (see respective Eqs (5.2.30a, b)).

In Section 5.3, a frequency interpretation of the gravitational potential (as well as the gravitational strength) of forming centrally and axially symmetric spheroidal body was considered. By analogy with the mentioned expansion of gravitational potential (5.2.4b), we found in Eq. (5.3.4) that statistical regular part of the circular frequency of *radial* oscillations $\omega(t)$ is the angular velocity $\Omega(t)$ of the rotational motion of a particle whereas its fluctuation part is a generalized circular frequency $\delta\varpi(t)$ of fluctuations of the particle inside a centrally symmetric spheroidal body. As consequence, the inducible gravitational acceleration (5.3.14) is the sum of a specific centrifugal force $\vec{f}_c$ and a specific additional periodic force of Alfvén–Arrhenius $\vec{f}_a$ [9, 19] in the case of *quasi-equilibrium state* of a centrally symmetric (or slowly rotating) spheroidal body. This result points to a possibility of the presence of *statistical oscillations* of motion in planetary orbits, that is, oscillations of the major semi-axis a and the orbital angular velocity Ω of rotation of bodies and planets around stars (see Sections 9.1 and 9.5).

The spatial deviation of the gravitational potential function (5.2.21) of an axially symmetric spheroidal body implies a difference in the values of the *radial* ω_h and the *axial* ω_z oscillations in accordance with Eqs (5.3.42a, b) and (5.3.43) even in the case of its relative mechanical equilibrium $(\widetilde{\mathrm{G}}(t) = \widetilde{\mathrm{G}}_s = \mathrm{const})$. Indeed, such a conclusion is confirmed by existing Alfvén–Arrhenius radial and axial orbital oscillations of moving bodies [9, 19].

Generally, two important cases of mechanical *equilibrium* and *quasi-equilibrium* states for centrally and axially symmetric spheroidal bodies are considered, but the GCF derivatives $\dot{\mathrm{G}}(t)$ and $\dot{\widetilde{\mathrm{G}}}(t)$ have an *oscillating* character only in the case of quasi-equilibrium state, therefore average integral values were introduced in Section 5.3. Here, Theorem 5.1 was proved, which says:

> The average integral value of the parameter of gravitational condensation does not depend on the duration of the period averaging both in equilibrium and in quasi-equilibrium states of a centrally symmetric (or slowly rotating) spheroidal body:
>
> $$\overline{\alpha} = \frac{1}{T}\int_{t_s}^{t_s+T} \alpha(t)\,dt = \mathrm{const}\,,$$
>
> where T is a period averaging, t_s is a stabilization instant of the equilibrium state of a centrally symmetric spheroidal body.

Using Theorem 5.1 we found that in the *quasi-equilibrium state* the average integral value of circular frequency of the radial oscillations coincides with the value of angular velocity of rotation. At the same time, the average integral value $\overline{\varphi}_g$ of the gravitational potential (5.2.4b) coincides with the inner gravitational potential (5.2.18a) of a slowly rotating spheroidal body ($\varepsilon_0^2 \to 0$), that is, $\overline{\varphi}_g$ is the regular part φ_{g}^* of gravitational potential in the *equilibrium* state.

Section 5.4 describes the dynamical states after the decay of a rotating spheroidal body and the formation of a protoplanetary system. Starting from the equality of averaged gravitational and centrifugal potentials ($V_c + \overline{\varphi}_g = 0$) we obtain the analog of Kepler's third law (5.4.6) in the case of decay of a centrally symmetric spheroidal body or the analog of Kepler's third law (5.4.18) in the case of decay of an axially symmetric spheroidal body. The separation process of centrally and axially symmetric spheroidal bodies leads to the formation of its inner zone I (a stellar core) and remote zone II (an exterior shell). In this connection, we consider the *inner* gravitational potential $\varphi_g^{(\mathrm{I})}$ (see Eq. (5.4.10a) or Eq. (5.4.14)) as well as the *exterior* gravitational potential $\varphi_g^{(\mathrm{II})}$ for large r (see Eq. (5.4.10b)). Using the spectral representation of GCF (5.3.25) and its derivative (5.3.27a) as periodic functions (in the case of quasi-equilibrium state of a slowly rotating spheroidal body) we present the gravitational potential $\varphi_g\big|_{\mathrm{quasiequil}}$ of the kind (5.4.29) through Fourier series (5.4.35). For large r we also have the spectral representation (5.4.36) of the exterior gravitational potential $\varphi_g^{(\mathrm{II})}\big|_{\mathrm{quasiequil}}$ as well as the spectral representation (5.4.38) of the specific force of gravity $\vec{f}_g\big|_{\mathrm{quasiequil}} = \vec{f}_g\big|_{\mathrm{equil}} + \vec{f}_a$ in the remote zone II of quasi-equilibrium gravitational field. This means that the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ is also the sum (5.4.40a) of spectral components $\vec{f}_a^{(l)}(l\omega_s t, \vec{r})$ with multiple ordered frequencies to the average main circular frequency ω_s. Similar results occur in the case of a rotating axially symmetric spheroidal body. Namely, the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ is the sum of analogous spectral components (5.4.40b) with multiple ordered frequencies to the circular frequencies of both radial ω_h and axial ω_z oscillations [77]. That is why the circular orbit of moving particles in the remote zone II of the gravitational field can be modified by both the radial and the axial oscillations (following Alfvén and Arrhenius [9, 19]).

Thus, Sections 5.3 and 5.4 showed that the *temporal deviation* of GCF $\mathrm{G}(t)$ of a spheroidal body under the condition of its *mechanical quasi-equilibrium* leads to it becoming an Alfvén–Arrhenius additional periodic force modifying forms of circular orbits to slightly elliptical orbits of moving bodies. The temporal deviation of GCF $\mathrm{G}(t)$ is determined by an oscillation behavior of its derivative $\dot{\mathrm{G}}(t)$ that implies the special case when $\dot{\mathrm{G}}(t) < 0$ and, therefore, $(\delta\varpi)^2 < 0$ as well as $\vec{f}_a^{(l)}(l\omega_s t, \vec{r}) < 0$, that is, according to Eq. (5.3.15b) or Eq. (5.4.39) the additional periodic force $\vec{f}_a$ becomes oriented opposite the gravitational force $\vec{f}_g$ (hence, realizing the so-called principle of an anchoring mechanism in planetary systems [77]).

In Section 5.5, interconnections of the proposed statistical theory of spheroidal bodies with Nelson's statistical mechanics and Nottale's scale relativistic theory were investigated. As pointed out in this section, both Nelson's statistical mechanics and Nottale's scale relativistic theory introduce so-called mean forward and mean backward derivatives. It is remarkable that, in the proposed statistical theory of spheroidal bodies, the main equations (relative to the *anti-diffusion velocity*) have been obtained *without introducing* any *mean forward* nor *mean backward derivatives* of stochastic processes. In this regard, the proposed statistical theory differs profoundly from Nelson's stochastic mechanics [34, 35] as well as from Nottale's scale relativistic theory [36, 37, 175, 176, 200, 201]. Moreover, the obtained main Eqs (5.1.17) and (5.1.19) are *more general* than analogous Eqs (5.5.8) and (5.5.9) in Nelson's stochastic mechanics. Indeed, within the framework of the proposed statistical theory of spheroidal bodies the generalized Schrödinger equations (5.5.17) and (5.5.18) can also be derived as in Nottale's scale relativistic theory. So, Nelson's and Nottale's considerations are appropriate mainly in the case of gravitational interaction of particles in a spheroidal body being in a mechanical *equilibrium* state.

In Section 5.6, the generalized nonlinear time-dependent Schrödinger-like equation describing a gravitational formation of a cosmogonical body was derived. The derived nonlinear time-dependent generalized Schrödinger-like equation (5.6.12) for the case of a centrally symmetric spheroidal body (or Eq. (5.6.13) in the case of axially symmetric spheroidal body) describes not only the mentioned states of mechanical equilibrium ($\mathrm{G}(t) = \mathrm{G}_s = \mathrm{const}$ or $\widetilde{\mathrm{G}}(t) = \widetilde{\mathrm{G}}_s = \mathrm{const}$) and quasi-equilibrium gravitational compression state close to the mechanical equilibrium with a slowly varying anti-diffusion coefficient [68, 71, 73] but gravitational instability states leading to formation of a cosmogonical body. So, this equation predicts gravitational instability states in a forming spheroidal body. When GCF $\mathrm{G}(t)$ is a constant, the nonlinear time-dependent generalized Schrödinger-like equation (5.6.12) becomes Eq. (5.7.1) corresponding to the well-known time-dependent Schrödinger equation [187] or the generalized Schrödinger equation in Nottale's form (5.5.17).

Indeed, the next Section 5.7 considered different dynamical states of a gravitating spheroidal body and respective forms of the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or Eq. (5.6.13)): the *initial equilibrium* gravitational condensation state (5.7.3); the mechanical *equilibrium* case (5.7.1); the *quasi-equilibrium* case (5.7.2); the *soliton disturbances state* (5.7.4); and the *gravitational instability* case (5.7.5). As mentioned here, the last case includes the *avalanche gravitational compression* increasing among them the case of unlimited gravitational compression leading to a collapse of a spheroidal body. Thus, an evolution of dynamical states of a spheroidal body can be investigated using the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or Eq. (5.6.13)). In particular, when a cosmogonical body is formed in the state of soliton perturbations nonlinear waves of various types arise there including soliton-like waves. The cubic time-dependent Schrödinger-like equation describing cosmogonical body formation in the state of soliton disturbances was derived in Section 5.7. The soliton solution of the cubic generalized Schrödinger-like equation of a forming spheroidal body was considered, hence the propagation of soliton waves of Schrödinger-type during the formation of the core of a cosmogonical body was justified.

In Section 5.8, the reduced model representing the dynamical system (5.8.14) of four ordinary nonlinear differential equations (with quadratic nonlinearity) into the state-space of the cubic generalized Schrödinger-like equation was derived.

The obtained result relative to the generalized nonlinear time-dependent Schrödinger-like equation (5.6.12) (or (5.6.13)) has been suggested in accordance with similar conclusions by I. Prigogine, E. Rössler, M. El Naschie [41], and Ord [223, 224] that the Schrödinger equation could be universal. In other words, it may have a large domain of applications, but with interpretations different from that of standard quantum mechanics.

On the whole, it should be mentioned that the derived nonlinear generalized Schrödinger-like equation is a *macroscopic* equation (5.6.12) (or (5.6.13)) unlike the quantum mechanical Schrödinger equation [187]. In this connection, we emphasize that the quantum mechanical Schrödinger equation describes the *microscopic* behavior of a particle. Due to the Casimir effect, the oscillation behavior of particles (before the origin of the gravitational field in a molecular cloud) is described by the time-dependent Schrödinger equation for the quantum-mechanical harmonic oscillator [187, 225]:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t)=\left(-\frac{\hbar^2}{2m_0}\nabla^2+\frac{k}{2}\mathbf{x}^2\right)\Psi(\mathbf{x},t),\quad k=m_0\omega^2, \quad \text{(A5.1)}$$

where

its solution is a wave function $\Psi(\mathbf{x},t)$,

m_0 is a mass of a particle,

ω is its angular frequency of oscillations,

$\mathbf{x}^2=\Delta q_1^2+\Delta q_2^2+\Delta q_3^2$, and

Δq_i is a displacement of a particle from an equilibrium position.

To construct a realistic physical picture when a wave is localized in a finite region of the space, the concept of a *wave packet* was introduced (in which amplitudes are localized in some spatial domain). Thus, by the wave packet one means a *superposition* of a sufficient number of wave functions of different frequencies and amplitudes:

$$\Psi(\mathbf{x},t)=\sum_{n=0}^{\infty}a_n\Psi_n(\mathbf{x},t). \quad \text{(A5.2)}$$

As a result, we can describe a particle (moving under the influence of spontaneous harmonic forces from local cores inside a molecular cloud) not as a "point mass" but as a "wave packet" (A5.2). It has been reported in [225] that a particle described by the three-dimensional harmonic oscillator is characterized also as an *oscillating expectation value* of a three-dimensional Gaussian wave packet and also an *oscillating width* of this packet. In the *one-dimensional* case ($x = \Delta q_1$), an evolution of the probability density to observe a particle described by a quantum mechanical oscillator with the initial expectation values of position x_0 and momentum $p_0 = 0$, the width of the initial wave packet σ_{x_0} with time t is also characterized by a Gaussian wave packet [225]:

$$w(x,t) = |\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\sigma_x(t)} e^{-\frac{(x - x_0 \cos \omega t)^2}{2\sigma_x^2(t)}} =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2\sigma_{x_0}}{\sqrt{4\sigma_{x_0}^4 \cos^2 \omega t + \sigma_0^4 \sin^2 \omega t}} \times$$

$$\times \exp\left[- \frac{2\sigma_{x_0}^2 (x - x_0 \cos \omega t)^2}{4\sigma_{x_0}^4 \cos^2 \omega t + \sigma_0^4 \sin^2 \omega t} \right] \qquad \text{(A5.3)}$$

with the oscillating expectation value $x_0(t) = x_0 \cos \omega t$ and the oscillating width $\sigma_x(t) = \sqrt{4\sigma_{x_0}^4 \cos^2 \omega t + \sigma_0^4 \sin^2 \omega t} \Big/ (2\sigma_{x_0})$ and $\sigma_0 = \sqrt{\hbar / m_0 \omega}$.

Taking into account that oscillations of three-dimensional Gaussian wave packet are independent (they are also orthogonal to one another), their resulting oscillation has an *elliptical trajectory of motion*. This means that the shape of an oscillating particle is described by an ellipsoid changing with the time, just as the trajectory of motion of this particle in the space is elliptical [225].

During a slowly evolving process of initial gravitational condensation of a forming spheroidal body from an infinitely distributed substance (a molecular cloud) the parameter of gravitational condensation $\alpha = \alpha(t)$ increases with the time t that leads to a growth of the potential gravitational energy (see formula (2.5.6) in Section 2.5):

$$E_{\mathrm{g}}(t) = \frac{1}{2}\int_V \rho(\vec{r},t)\varphi_{\mathrm{g}}(\vec{r},t)dV = -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha(t)}{\pi}}, \qquad \text{(A5.4)}$$

where M is a mass of a forming spheroidal body, and ρ and φ_{g} are mass density (2.2.5) and gravitational potential (2.4.26) respectively (in the case of a centrally symmetric spheroidal body). When an essential growth of the potential gravitational energy (A5.4) occurs then *nonlinear phenomena* arise owing to self-organization processes [134, 193] of interactions of oscillating particles into a spheroidal body under its formation. These nonlinear phenomena induce nonlinear waves satisfying a *nonlinear* undulatory Schrödinger-like equation (5.6.12) (or (5.6.13)), in particular Eq. (5.7.4) (in which except a wave function $\Psi(\vec{r},t)$ there is a temporal function called GCF $\mathrm{G}(t)$). So, the GCF (4.1.8) or (4.7.13) is a *measure of interactions* of oscillating particles into a forming spheroidal body. Really, in the case of a *constant* $\mathrm{G}(t) = \mathrm{G}_s$ the derived nonlinear Schrödinger-like equation (5.6.12) becomes similar to the time-dependent Schrödinger equation (A5.1) if we might assume formally that $\vec{r} = \mathbf{x}, \mathrm{G}(t) = \mathrm{G}_s = \hbar / 2m_0$ and $\varphi_g = m_0\omega^2\mathbf{x}^2/2$.

Despite the derived nonlinear Schrödinger-like equation (5.6.12) (or (5.6.13)) being a macroscopic equation while the quantum mechanical Schrödinger equation (A5.1) is a microscopic equation, they are associated among themselves like the macroscopic hydrodynamic equation of Euler and the microscopic kinetic equation of Boltzmann because the Euler equation can be derived through the Boltzmann microscopic equation [193].

PART II:

THE STATISTICAL THEORY OF GRAVITY IN SOLAR AND EXTRASOLAR SYSTEM APPLICATIONS

According to considered statistical theory, the formation of cosmogonical bodies (see Chapters 1–3) as part of the process of planetary system formation—from a *protoplanetary cloud* to the star with planets—can be described by a multi-scale control parameter of dynamical states of a spheroidal body, called *the parameter of gravitational condensation* α . For example, the nebular origin of our Solar system permits us to describe the proto-Sun together with a flattened gas-dust protoplanetary cloud as the model of a rotating and gravitating spheroidal body. According to Hoyle's theory, an evolutionary process for the formation of the Solar system based on the nebular origin leads to the compression of the proto-Sun and, as a consequence, rotational instabilities when the radius of the proto-Sun became equal to $R_{\text{proto-Sun}} = 3 \cdot 10^{10}$ m. If we investigate the proto-Sun inside the protoplanetary cloud separately as a spheroidal body then the parameter of gravitational condensation of the proto-Sun is equal to:

$$\alpha_{\text{proto-Sun}} \approx \frac{1}{R_{\text{proto-Sun}}^{\;2}} \approx 1.11 \cdot 10^{-21} (\text{m}^{-2}) . \qquad \text{(II.1)}$$

Then, bearing in mind that a primary presolar cloud has a mass density of 1 atom per sm^3 and an angular velocity of $\Omega \sim 10^{-15}$ s^{-1} (as for our Galaxy on the whole), Hoyle found that the angular momentum value of the presolar nebula (cloud) has to be equal $L = 4 \cdot 10^{44}$ kg m^2 /s, that is, the mean value of specific angular momentum for the forming Solar system is $\overline{\lambda} = L / M = 2.012 \cdot 10^{14}$ m^2/s if the mass of the presolar cloud is approximately $M = 1.988 \cdot 10^{30}$ kg. Using our formula, $L = 2\Omega\, M / \alpha(1-\varepsilon_0^2)$, it is not difficult to see that the parameter of gravitational condensation of the flattened cloud can be estimated by the same value, that is, $\alpha_{\text{proto-cloud}} = \alpha_{\text{proto-Sun}} \approx 1.11 \cdot 10^{-21} (\text{m}^{-2})$.

On the other hand, the usage of this statistical theory, as well as laws of celestial mechanics in conformity to the stars, requires us to take into account an extended substance called a *stellar corona*. Therefore, we have to consider the stellar corona (in particular, the solar one) because the Sun is assumed to be present as the solar disk (for which the equatorial radius is $R_{\text{Sun}} = 6.955 \cdot 10^8$ m) embedded into the solar corona following this statistical theory. In this connection, the stellar corona together with its star might be described using the model of a gravitating spheroidal body, that is, the parameter of gravitational condensation α of such a spheroidal body can be estimated by the linear size of its core and the thickness of a visible part of the stellar corona. In particular, the parameter of gravitational condensation α of a spheroidal body in the case of the Sun with its corona has been estimated by the following value [72, 73]:

$$\alpha_{\text{Sun\&corona}} \approx \frac{1}{(3R_{\text{Sun}})^2} \approx 2.29701177718 \cdot 10^{-19} (\text{m}^{-2}). \qquad \text{(II.2)}$$

From the point of view of the general statements of the statistical theory of gravitating cosmogonical bodies, its application to the concrete cosmogonical body (protoplanetary cloud, proto-star, star, planet, or satellite) requires the use of the respective estimations (similar to (II.1), (II.2), etc.) of the parameter of gravitational condensation α .

CHAPTER SIX

ON THE MODELS OF PROTOPLANETARY FORMATION AND THE LAWS OF PLANETARY DISTANCES IN THE SOLAR SYSTEM AND OTHER EXOPLANETARY SYSTEMS

Understanding conformities in the Solar system is connected with solving the problem of the origin and evolution of our Solar system or other extrasolar planetary (exoplanetary) systems in the Universe. This problem, starting with Descartes, has attracted the attention of natural scientists. As we know, many models of the Solar system formation are based on nebular theories, theories of the capture of celestial bodies and interstellar gas-dust matter, theories of plasma-dust interactions, and so forth (see, for example [6–14]).

In 1766, Johann Titius von Wittenberg outlined his famous commentary on planetary distances for the Solar system in the form of a geometric progression with the factor of 2 (the so-called Titius–Bode law) on the pages of the German translation of Charles Bonnet's book "Contemplation de la Nature." Then, in 1772, Johann Bode revealed this commentary and included it in the text of his book [8].

The idea underlying the Titius–Bode law is related with Johann Kepler, who tried to explain the observable relative sizes of planetary orbits using celestial spheres and regular polygons (he also drew attention to a gap existing between the orbits of Mars and Jupiter, where there should be one small "invisible" planet) [226]. With his discovery of the Universal gravitation law [80], Isaac Newton gave us an explanation not only of the dynamics of the orbital motion of the planets based on gravitational forces, but also proposed a theoretical basis for the derivation of Kepler's laws. Christian Wolf, who believed that the planetary orbits obeyed some regularity [8], also conveyed cosmogonical ideas to his brilliant pupil Immanuel Kant, who published his theory of the Universe in 1755 [227].

Thus, scientific cosmogony originates from the famous works of Kant (1755) and Laplace (1796) [227, 228]. Both of them came up with the unique idea of the formation of planets from matter scattered in outer space, that is, from a protoplanetary cloud, which in its general form occurs in modern cosmogonical theories. Kant and Laplace built their hypotheses for the formation of the Solar system based on the general idea of the simultaneous origin of both the Sun and planets from matter scattered in outer space [227, 228]. Nevertheless, it is now well known that the main difficulty of the Kant–Laplace hypothesis was the problem of the *appearance of angular momentum* in the Solar system.

The Titius–Bode law was very successful in discovering Uranus, asteroids and their orbital locations. In addition, the mathematician Karl Gauss developed the now famous least squares method for solving problems in celestial mechanics. Various modifications and generalizations to the Titius–Bode law have been proposed by Wurm, Gilbert, Challis, Kirkwood, Chambers, Charlier, Blagg [229], Richardson [230], and others. These modifications were purely empirical expressions for the observable distribution of planetary distances in the Solar system or distance distributions for the moons of Jupiter and Saturn. As a rule, all these generalized laws of planetary distances were described by a geometric progression with a factor of 1.73 or 1.89 (and not 2), whose terms were multiplied by a periodic function expressing a deviation from the geometric progression itself.

The next significant achievement in solving the problems of cosmogony and its further development was Jean's theory [1, 93] (see also Sections 1.3, 1.5 of this monograph). Beginning in the 1940s, a great number of works were carried out in the field of cosmogony: in particular, in 1943, von Weizsäcker proposed a nebular theory to explain the evolutionary processes of the formation of our Solar system [26, 27]. Later other evolutionary theories were developed by, for example, O. Yu. Schmidt [6, 21], Kuiper [28, 29], Hoyle [30, 31], Ter Haar [7, 32], and Cameron [7, 10]. According to Ter Haar (1948), a viable cosmogonical theory should explain the following four groups of facts [6, 44 p.277]:

- Group "A" is the law of the orbits: the planetary orbits are almost circular; they lie in the same plane; the rotation occurs in one direction; and the Sun rotates in the same direction. Moreover, the equatorial plane of the Sun close to the same orbital plane;
- Group "B" is the law of planetary distances: the planets are not randomly distributed; there is a conformity in their distances empirically formulated by Bode (in the form of the Titius–Bode law);
- Group "C" is the separation of the planets into two different groups: the inner planets (Mercury, Venus, Earth, and Mars) are relatively small, but with high density, a rather slow rotation around the axis, and small numbers of satellites; the outer planets (Jupiter, Saturn, Uranus, Neptune) are large, with lower density, high velocity of rotation, and numerous satellites (Pluto is not included because it is located on the edge of the Solar system and may not fit into the law);
- Group "D" is the distribution of angular momentum: though the Sun has more than 99% of the total mass of the Solar system, it accounts for less than 2% of the angular

momentum, while the remaining 98% belong to the planets.

Using the Titius–Bode law, Otto Yu. Schmidt proposed his model of the origin of planets and satellites suggesting they were dependent on the distribution of specific angular momentum in the Solar system [6]. However, Schmidt's model was not able to explain the origin of the angular momentum itself in a gas-dust cloud orbiting the Sun, and his hypothesis that the moving gas-dust cloud is captured by the Sun does not look convincing enough since the process of permanent gas-dust cloud capture by other stars during the formation of exoplanetary systems seems unlikely. Up to now, the essential question remains unanswered: *Does the Titius–Bode law have a physical meaning?*

Our understanding of our place in the Universe changed radically in 1995 when Michel Mayor and Didier Queloz of Geneva Observatory in Switzerland announced the discovery of an extrasolar planet around a star similar to our Sun [3]. Geoff Marcy and Paul Butler in the United States soon confirmed their discovery, and the science of observational extrasolar planetology was born. The field has expanded significantly in recent years, resulting in numerous publications on planetary systems in 2019 (see http://exoplanet.eu/ and http://exoplanets.org/ for an up-to-date list). Most of these systems contain one or more gas giant planet close, or very close, to their parent star, and thus, do not resemble our Solar system. Nevertheless, in 2010, an international group of astronomers, using the HARPS spectrograph of the European Southern Observatory (ESO) in La Silla (Chile), reported the discovery of new planets by the variations of the solar-type HD 10180 star. This exoplanetary system comprises at least five Neptune-like planets with minimum masses ranging from 12 to 25 M_{Earth}, orbiting the solar-type star HD 10180 at separations between 0.06 and 1.4

AU [231]. Moreover, their orbits are *almost circular*. Using data about this system together with information about other planetary systems, the researchers found a certain analog of the Titius–Bode rule for exoplanetary systems, which may be a reflection of the general regularity of the process of planetary formation.

Various theories were proposed to explain the Titius–Bode law [8], although, generally speaking, they were intended directly for estimating planetary distances and studying planetary orbits. These theories, overlapping each other by research methods, are conventionally divided into five categories [16]:

- electromagnetic;
- gravitational;
- nebular;
- quantum mechanical; and
- statistical theories.

We note once again that the theories under consideration are fairly general since their tasks are broader than exploring only the Titius–Bode law. In particular, Alfvén's *electromagnetic* theory is found to be very important because it reveals the mechanism of the damping action of the magnetic field in transferring angular momentum from the Sun to an ionizing gas around the Sun [9]. Based on such a process of momentum transmission to massive ions, Alfvén derived a relation linking masses and distances corresponding to the Titius–Bode law.

Schmidt's *gravitational* theory [6, 21] is based on the idea of the capture of a dust-gas cloud by the Sun and showed the importance of the process of a conglomeration of dust particles in the formation of planets. Indeed, this process leads to the Titius–Bode law due to the phenomenon of the "scooping out" of the gas-dust matter of the cloud by the neighboring protoplanets owing to the difference between specific angular momentums for these protoplanets. Schmidt's theory was actively continued by his colleagues and pupils: L.E. Gurevich and A.I. Lebedinsky [22, 23], V.S. Safronov [2], A.V. Vityazev, and others [12]. In particular, Viktor S. Safronov was the first to suggest the so-called planetesimal hypothesis stating that planets were formed from dust grains that collided and stuck to form larger and larger bodies, that is, planetesimals which then began to interact gravitationally between themselves. Dole [25] also developed and modeled the dust accumulation (around a planetary core or planetary embryo) process describing the absorption of gas by planets with "critical sizes" and, hence, leading to the formation of giant planets.

Von Weizsäcker's *nebular* theory [26, 27] provoked the greatest interest among scientists. It pointed out the importance of turbulent processes in the formation of a protosolar cloud. Von Weizsäcker showed that turbulent motions lead to the formation of vortex cells located in ring regions, and then to condensation of a substance between the rings. As a result, the Titius–Bode law can be predicted. The initial hypothesis of von Weizsäcker has been modified by many scientists [8]. The development and continuation of modern nebular theory is traced to Kuiper [28, 29], who not only suggested that the protoplanetary nebula was significantly more massive than the present-day sum of planetary masses, perhaps exceeding 0.1 of Sun's mass M_S but also proposed that the gas giant planets are the result of gaseous accretion of solid protoplanetary cores (see [232 p. 495]).

Some recent works have been devoted to investigating the possibility of describing planetary orbits based on *quantum mechanical approaches* [15, 36–40, 233–235]. A brief review of the set of these theoretical studies has been made by De Oliveira Neto and co-workers [15]. One of the interesting results found refers to the prediction of a fundamental radius given by r = 0.05AU, also predicted by Nottale [36, 37], and Agnese and Festa [40, 233] in their studies. In particular, Agnese and Festa [40] described the Solar system as a gravitational atom (these authors also proposed a formula describing the distances to the bodies of the Solar system). Following the Bohr–Sommerfeld atomic theory foundations, quantum mechanics emerged with its Schrödinger equation. A derivation of the Schrödinger equation from Newtonian mechanics was given in the works of E. Nelson [34, 35] (see details in Section 5.5). The important point in Nelson's works is that a diffusion process can be described in terms of a Schrödinger-type equation, with the help of the hypothesis that any particle in the empty space, under the influence of any interaction field, is also subject to a universal Brownian motion [41] based on the quantum nature of space-time in quantum gravity theories or on quantum fluctuations on a cosmic scale [37, 38, 43]. In this context, the possibility of describing a classical process like the formation of the Solar system in terms of quantum mechanics can be considered seriously [15]. As for macroscopic bodies, the chaotic behavior of the Solar system during its formation and evolution [236, 237] indicates a diffusion process to be described in terms of a Schrödinger-type equation. The description of the planetary system using a Schrödinger-type diffusion equation has been presented in [15]. As shown in [235], an analysis of the relationship to the electrostatic and gravitational forces has revealed the utility of the Newton gravity constant for the estimation of planetary orbits. A review of simple quantization procedures, a utility of a Schrödinger-type equation and Newton's dimensionless constant of gravity has been provided [238]. Recently, a membrane model has been proposed to explain regularities in the distribution of distances of bodies in the Solar system [239]. As shown in Chapter 5, the most common nonlinear time-dependent Schrödinger-like equation of the form (5.6.13) has been derived [68, 71, 77], describing the formation of a cosmogonical body (see details in Section 5.6).

In spite of a great number of works aimed at exploring the formation of the Solar system and significant efforts by many brilliant scientists, the these theories have not been able to explain completely all phenomena occurring in the Solar system. In this connection, the *statistical* theory (presented in this monograph as well as in [16, 45–71]) for the formation of a cosmogonical body (the so-called spheroidal body) using numerous gravitational interactions of its parts (particles) has been proposed. The domain of investigations within the framework of the proposed statistical theory of gravity includes Newtonian gravity and partly Newtonian quantum gravity. As we know from Chapters 2–5 of this monograph, the proposed theory starts from the conception for forming a spheroidal body inside a gas-dust protoplanetary nebula enabling it to derive the form of distribution functions, mass density, gravitational potentials, and strengths for an immovable and rotating spheroidal body [45–56] and also to find the distribution function of specific angular momentum [16, 56–60, 65] for a uniformly rotating spheroidal body.

Chapter 6 discusses the statistical theory to develop models of the formation of the solar and exoplanetary systems. It investigates a gas-dust protoplanetary cloud as a rotating and gravitating spheroidal body having the specific angular momentum distribution function. Since the specific angular momentums of moving particles of a gas-dust cloud are averaged during the conglomeration process, the mean specific angular momentum for a planet of the Solar system (as well as a planetary distance) can be found through such procedure. This chapter presents a new law on planetary distances in the Solar system [16, 65, 73] which generalizes the well-known Schmidt law.

6.1. Evolution equations of the distribution of the specific angular momentum in the protoplanetary cloud and the laws on planetary distances

The results obtained in the previous chapters now allow us to proceed to the consideration of the next stage of evolution: from a flattened protoplanetary gas-dust disk to the originating protoplanets. Let us note that the formulas (3.3.26a–c), and (3.8.27) point to a *common scenario* of the formation of both a star and a protoplanetary gas-dust disk around it (in particular, the Sun and the solar protoplanetary gas-dust disk) because M is considered here as the total mass of the star (Sun) and the protoplanetary (solar) gas-dust disk. Indeed, as noted in the work [240]:

> Measurements of the composition of the Earth, Moon, and meteorites support a common origin for the Sun and planets (e.g., Harris 1976; Anders & Grevesse 1989).

So, in this section, let us consider a statistical model of the origin of protoplanets embedded in a flattened gas-dust protoplanetary disk (see Section 3.4) based on the distribution function (3.4.9) of the s*pecific angular momentum* for a uniformly rotating spheroidal body (as a flattened gas-dust protoplanetary clouds) [16, 65, 73].

Schmidt's cosmogonical hypothesis on the origin of the Solar system as a result of the evolution of gas-dust swarm was used by L.E. Gurevich and A.I. Lebedinsky in their studies [22, 23] to show that the swarm condensation process takes place necessarily (even if there were no initial bunches or protoplanetary embryos and the cloud itself consisted only of dust and gas), thanks to the following scenario [6 p.27, 21]:

a) as a result of collisions, the relative velocities of particles decrease, so that the system flattens and thereby becomes denser, hence, increasing the frequency of collisions;

b) after reaching a certain critical value of the density, the system cannot remain in its previous state: under the influence of the forces of gravity, the intensive formation of condensations (the so-called planetesimals) begins [2, 12];
c) these condensations (planetesimals) have a flattened shape and masses of the order of the masses of asteroids;
d) in turn, the condensations are forced to collide (due to the small mean free path) and merge into a small number of large bodies called protoplanets.

In its entirety, the scenario described is confirmed by numerous results of computational modeling. Indeed, the last four decades of research in this area have shown that numerical modeling has become an important part of understanding the evolution of the Solar system. A contemporary understanding holds that the evolution from dust to planet can be divided into *three successive phases* (see for example [208, 241 p. 267]). The *first main stage* is the coagulation of micrometer-sized grains into kilometer-sized planetesimals (see [2, 118, 139, 242, 243, 244]). The first is that planetesimals form as the result of gravitational instability in the solar nebula, in which solids are sufficiently concentrated to enable planetesimals to form purely by self-gravity (e.g., [118, 245, 246]). The second is that planetesimals form by the direct collisional accretion between colliding particles (e.g., [242, 247, 248]), that is, grains are assumed to stick together if an impact occurs at a critical, threshold velocity [249]. Due to the stochastic nature of growth, not all planetesimals grow at the same rate and some will become more massive than others. More massive bodies are more effectively able to accrete the surrounding planetesimals. This quickly leads to a *runaway* accretion process (e.g., [250, 251, 252, 253]) beginning the *second main phase*. In a swarm of planetesimals, the relative velocity v_{rel}

is governed by their frequent encounters with one another, and given their small gravity, is kept low [208]. Runaway growth is stalled somewhat when the planetary embryos grow large enough that their gravitational perturbations on the planetesimals become the dominant influence on v_{rel}. As this occurs, planetesimals decouple from gaseous disk and start to interact gravitationally with each other. This catalyzes the second phase completely: a runaway growth leading to the formation of a *planetary embryo* with masses 1%–10% of that of Earth M_{Earth} [254, 255]. These embryos accrete material locally and form a dense population distributed throughout the Solar system. In the *third* (and final) *stage* of terrestrial planet accretion, the gravitational effect of the planetesimals begins to fade as their numbers decrease, and the planetary embryos begin to perturb one another onto crossing orbits. Planets then begin to grow from collisions between embryos and the accretion of remaining planetesimals. This stage is characterized by relatively violent, stochastic large collisions as compared to the previous stages, where the continual accretion of small bodies dominates [208, 256].

In contrast to the numerical models for planet formation with enormous numerical efforts, the present chapter relies on analytical principles only. Although the numerical models seem to be quite capable already, it is always useful to check at least with one from the mentioned theoretical approaches, whether the results are still reasonable.

Chapter 6 shows that the evolution of a flattened rotating and gravitating spheroidal body, in particular, and, generally speaking, the proposed statistical theory proves to be useful for explaining the origin of the Solar system [16, 65, 73]. Indeed, let us briefly consider the evolution of two neighboring bunches (protoplanetary embryos) being in the growth stage. As Schmidt noted [6 p. 32–33]:

> If their orbits are very close, they will quickly exhaust the supply of bodies and particles moving in the region between their orbits. If two planetary embryos do not unite into one, then in the future they will acquire mass and momentum already predominantly from bodies turning from the outer sides of the exhausted zone. In this case, the momentum per unit mass of one planet will decrease, the other will increase, and the radii of the orbits of two planets will begin to diverge. Thus, in the process of growth of the planets at the expense of bodies and particles, it lies the principle of adjusting the distances between them.

Following the logic of Schmidt's reasoning, one can find the law of planetary distances between bunches (protoplanetary formations) in a rotating spheroidal body, bypassing the detailed kinetics of the process. Namely, his model (explaining the law of planetary distances) is based on the hypothesis that *each law of distribution of a specific angular momentum λ of particles with a distribution function $f(\lambda)$ corresponds to its law of planetary distances* [6]. It follows from the fact that when planets are formed (for example, in the Solar system) each particle (or a conglomerate of particles, generally speaking, a planetesimal [2, 12, 242, 245, 246]) in a gas-dust protoplanetary cloud (or a swarm of planetesimals) has the greatest chance of hitting in that protoplanet, the specific angular momentum which differs least of all from the specific angular momentum of the particle (or the planetesimal). Although individual particles may not fall into their bunch, however, following O. Yu. Schmidt [6 p. 33],

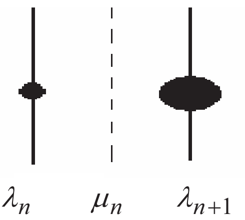
> These deviations are mutually compensated so that for the calculation it can be assumed that the particles are precisely distributed over the "areas" outlined on the axis of specific angular momentum for each planet. The boundary of the area we will consider the value of specific momentum, equidistant from the specific momentum of two neighboring planets.

However, O. Yu. Schmidt could not analytically derive the form of the distribution function $f(\lambda)$ of the specific angular momentum within the framework of his model, noting only that $f(\lambda)$ changes with the time in the process of cloud evolution, but this is still an unsolved problem [6 p. 35–36].

6.1.1. Review of particular cases of the distribution function of the specific angular momentum of forming protoplanets and the law of planetary distances from the point of view of the statistical theory of spheroidal bodies

Using Schmidt's cosmogonical hypothesis of evolution from the flattened protoplanetary gas-dust disk to the emerging protoplanets [6, 21], let us consider a flattened protoplanetary gas-dust cloud based on the model of a rotating and gravitating spheroidal body with the distribution function of the specific angular momentum $f(\lambda)$ described by formula (3.4.9) from Section 3.4.

Let μ_n be a value of specific angular momentum corresponding to the boundary between the domains of n-th and $(n+1)$-th protoplanets (bunches or protoplanetary embryos in the flattened gas-dust protoplanetary cloud), whose specific angular momentums are equal to λ_n and λ_{n+1} respectively (Fig. 6.1).



λ_n μ_n λ_{n+1}

Figure 6.1. Graphic representation of the conditional border between the regions of n-th and $(n+1)$-th forming protoplanets

Then, following the hypothesis of Schmidt, we can see that:

$$\mu_n = \frac{\lambda_n + \lambda_{n+1}}{2}. \quad (6.1.1)$$

During the process of a conglomeration of particles of gas-dust media in a bunch, their specific angular momentums are averaged and the specific angular momentum of the bunch (planetesimal) as a forming protoplanet is, therefore, the ratio [6]:

$$\lambda_n = \frac{\int_{\mu_{n-1}}^{\mu_n} \lambda f(\lambda) d\lambda}{\int_{\mu_{n-1}}^{\mu_n} f(\lambda) d\lambda}. \quad (6.1.2)$$

According to Eq. (3.4.9) obtained in the statistical theory [16, 73], the probability density distribution function of the specific angular momentum is:

$$f(\lambda) = \frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega} \cdot e^{\frac{-\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda}. \quad (6.1.3)$$

In the assumption of the smallness of an inverse average value of the specific angular momentum $\overline{\lambda}^{-1}$ since $\alpha << 1$ in accordance with (3.4.11):

$$\overline{\lambda}^{-1} = \frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega} << 1, \quad (6.1.4)$$

we obtain the function $f(\lambda)$ to be approximated by a *uniform distribution law* of the kind:

$$f(\lambda) \approx \frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega} = \text{const}. \quad (6.1.5)$$

Taking into account that $\lambda = \Omega h^2$ as well as Eq. (3.4.4) we obtain:

$$f(\lambda)d\lambda = \frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\cdot e^{\frac{-\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda}d\lambda =$$

$$= \frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\cdot e^{\frac{-\alpha(1-\varepsilon_0{}^2)}{2\Omega}\Omega h^2}d(\Omega h^2) =$$

$$= \alpha(1-\varepsilon_0{}^2)\cdot e^{\frac{-\alpha(1-\varepsilon_0{}^2)}{2}h^2}hdh = f(h)dh, \qquad (6.1.6)$$

where $f(h) = \alpha(1-\varepsilon_0{}^2)\cdot he^{\frac{-\alpha(1-\varepsilon_0{}^2)}{2}h^2}$ is the particle distribution function (3.4.3) with respect to the radial coordinate h in a rotating (with uniform angular velocity $\Omega = \text{const}$) spheroidal body, that is, one-dimensional probability density of finding a particle in a uniformly rotating spheroidal body at the distance h from the axis of rotation.

On the other hand, a one-dimensional probability density function $f(h)$ of particle detection along the radial coordinate h can be found through the volume probability density function $\Phi(h,z)$ based on the following relation:

$$f(h) = \int_{-\infty}^{\infty}\int_{0}^{2\pi}\Phi(h,z)hdzd\varepsilon\,. \qquad (6.1.7)$$

Substituting (6.1.7) into Eq. (6.1.6) we obtain:

$$f(\lambda)d\lambda = hdh\int_{-\infty}^{\infty}\int_{0}^{2\pi}\Phi(h,z)dzd\varepsilon\,. \qquad (6.1.8)$$

Taking into account that $\lambda = \Omega h^2$ and that, moreover, $\Omega = \text{const}$, equation (6.1.8) can be rewritten as follows:

$$2\Omega f(\lambda) = \int_{-\infty}^{\infty}\int_{0}^{2\pi}\Phi(h,z)dzd\varepsilon\,. \qquad (6.1.9)$$

When the *condition of smallness* (6.1.4) is carried out, the function $f(\lambda)$ is approximated by the uniform law (6.1.5), so that Eq. (6.1.9) takes the form:

$$\int_{-\infty}^{\infty}\int_{0}^{2\pi}\Phi(h,z)dzd\varepsilon = 2\Omega\cdot\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega} = \alpha(1-\varepsilon_0{}^2)\,, \qquad (6.1.10a)$$

that is, the relation is true:

$$\int_{-\infty}^{\infty}\Phi(h,z)dz = \frac{\alpha\cdot(1-\varepsilon_0{}^2)}{2\pi}\,. \qquad (6.1.10b)$$

Taking into account that $\rho(h,z)dz = M\Phi(h,z)$ is the mass density of a spheroidal body we obtain the requirement:

$$\int_{-\infty}^{\infty}\rho(h,z)dz = M\frac{\alpha\cdot(1-\varepsilon_0{}^2)}{2\pi}\,. \qquad (6.1.11)$$

Moreover, the function of mass density ρ must satisfy this requirement under the condition of the *uniform law* (6.1.5) of angular momentum distribution $f(\lambda)$. As is well known [6, 12, 16], in the case of the uniform distribution law (6.1.5), Schmidt's law of planetary distances occurs:

$$\sqrt{a_n} = c_1 + c_2\cdot n\,, \qquad (6.1.12)$$

where c_1, c_2 are some constants. Thus, *if the mass density of a spheroidal body satisfies the condition* (6.1.11) *then Schmidt's law* (6.1.12*) takes place*. This result generalizes and refines the conclusion of J. Laskar [257]: for a constant mass density (constant distribution) $\rho(a)$ the orbital major semi-axes a_n of formed extrasolar planets satisfy the relation: $\sqrt{a_n} = c_1 + c_2\cdot n$, that is, Schmidt's law of planetary distances (6.1.12).

Indeed, taking into account:

$$\rho(h,z) = M\Phi(h,z) = M\left(\frac{\alpha}{2\pi}\right)^{3/2}\cdot(1-\varepsilon_0{}^2)\cdot e^{-\alpha[h^2(1-\varepsilon_0{}^2)+z^2]/2} =$$

$$= \sigma(h)\cdot\sqrt{\frac{\alpha}{2\pi}}\cdot e^{-\alpha z^2/2}\,, \qquad (6.1.13)$$

where $\sigma(h)$ is a *surface mass density*, we establish from Eq. (6.1.10a) that:

$$\int_{-\infty}^{\infty}\int_{0}^{2\pi}\sigma(h)\cdot\sqrt{\alpha}\cdot e^{\frac{-\alpha z^2}{2}}dz d\varepsilon = M\alpha(1-\varepsilon_0{}^2),$$

whence, bearing in mind that $\int_{-\infty}^{\infty} e^{\frac{-\alpha z^2}{2}}dz=\sqrt{\frac{2\pi}{\alpha}}$, we derive *the condition of Laskar* (constancy of the surface mass density):

$$\sigma(h) = M\cdot\alpha(1-\varepsilon_0{}^2) = \text{const}, \qquad (6.1.14)$$

because h is a radial coordinate axis along which the major semi-axes a of planet orbits are measured.

Now we consider another case investigated by J. Laskar [257], namely when:

$$\sigma(a) = c\cdot a^{-3/2}. \qquad (6.1.15)$$

Indeed, formula (6.1.15) can be obtained as a special case of (6.1.13) representing $\sigma(h)$ in the form:

$$\sigma(h) = M\frac{\alpha}{2\pi}\cdot(1-\varepsilon_0{}^2)\cdot[e^{\frac{\alpha h^2(1-\varepsilon_0{}^2)}{3}}]^{-\frac{3}{2}} =$$

$$= M\cdot\frac{\alpha(1-\varepsilon_0{}^2)}{2\pi}\cdot a^{-3/2}(h), \qquad (6.1.16)$$

that is, by introducing a variable:

$$a = e^{\alpha h^2(1-\varepsilon_0{}^2)/3}, \qquad (6.1.17a)$$

we can write Eq. (6.1.16) in the form:

$$\sigma(a) = M\cdot\frac{\alpha(1-\varepsilon_0{}^2)}{2\pi}\cdot a^{-3/2}. \qquad (6.1.17b)$$

Moreover,

$$h^2 = \frac{3}{\alpha(1-\varepsilon_0{}^2)}\ln a. \qquad (6.1.17c)$$

Taking into account that $\lambda = \Omega h^2$ let us rewrite (6.1.17a) in relation to the variable λ, choosing the value of the specific angular momentum equal to $\lambda = \lambda_n$:

$$a_n = e^{\frac{\alpha(1-\varepsilon_0{}^2)\lambda_n}{3\Omega}}, \tag{6.1.18a}$$

whence:

$$\ln a_n = \frac{\alpha(1-\varepsilon_0{}^2)}{3\Omega}\lambda_n\,. \tag{6.1.18b}$$

So, we can determine the value of specific angular momentum along the border between n-th and $(n+1)$-th protoplanets:

$$\begin{cases} \lambda_n = \dfrac{3\Omega}{\alpha(1-\varepsilon_0{}^2)}\ln a_n; \\ \mu_n = \dfrac{3\Omega}{\alpha(1-\varepsilon_0{}^2)}\ln a'_n. \end{cases} \tag{6.1.18c}$$

Substituting (6.1.17a)–(6.1.17c) into Eq. (6.1.6) we have:

$$f(\lambda)d\lambda = \frac{\alpha(1-\varepsilon_0{}^2)}{2}\cdot a^{-3/2}\cdot d\left(\frac{3}{\alpha(1-\varepsilon_0{}^2)}\cdot\ln a\right) =$$

$$= \frac{3}{2}a^{-3/2}\cdot\frac{1}{a}da = \frac{3}{2}\cdot a^{-5/2}da\,; \tag{6.1.19a}$$

$$\lambda f(\lambda)d\lambda = \Omega\frac{3}{\alpha(1-\varepsilon_0{}^2)}\ln a\cdot\frac{3}{2}a^{-5/2}da =$$

$$= \frac{9\Omega}{2\alpha(1-\varepsilon_0{}^2)}\cdot a^{-5/2}\ln a da\,. \tag{6.1.19b}$$

Given (6.1.19a, b), let us calculate the integrals in Eq. (6.1.2):

$$\int_{\mu_{n-1}}^{\mu_n} f(\lambda)d\lambda = \int_{a'_{n-1}}^{a'_n}\frac{3}{2}\cdot a^{-5/2}da = \frac{3}{2}\cdot\frac{a^{-5/2+1}}{-3/2}\Bigg|_{a'_{n-1}}^{a'_n} =$$

$$= a'^{\,-3/2}_{n-1} - a'^{\,-3/2}_{n}\,; \tag{6.1.20a}$$

$$\int_{\mu_{n-1}}^{\mu_n} \lambda f(\lambda) d\lambda = \frac{9\Omega}{2\alpha(1-\varepsilon_0{}^2)} \int_{a'_{n-1}}^{a'_n} a^{-5/2} \ln a da =$$

$$= \frac{9\Omega}{2\alpha(1-\varepsilon_0{}^2)} \cdot \left\{ -\frac{2}{3} \ln a \cdot a^{-3/2} \Bigg|_{a'_{n-1}}^{a'_n} + \frac{2}{3} \int_{a'_{n-1}}^{a'_n} a^{-5/2} da \right\} =$$

$$= \frac{3\Omega}{\alpha(1-\varepsilon_0{}^2)} \cdot (a'^{-3/2}_{n-1} \ln a'_{n-1} - a'^{-3/2}_n \ln a'_n) +$$

$$+ \frac{2\Omega}{\alpha(1-\varepsilon_0{}^2)} \cdot (a'^{-3/2}_{n-1} - a'^{-3/2}_n). \tag{6.1.20b}$$

Substituting (6.1.20a and b) into (6.1.2) we yield:

$$\lambda_n = \frac{\frac{3\Omega}{\alpha(1-\varepsilon_0{}^2)} \cdot \left\{ a'^{-3/2}_{n-1} \ln a'_{n-1} - a'^{-3/2}_n \ln a'_n \right\}}{a'^{-3/2}_{n-1} - a'^{-3/2}_n} +$$

$$+ \frac{2\Omega}{\alpha(1-\varepsilon_0{}^2)} \cdot \frac{a'^{-3/2}_{n-1} - a'^{-3/2}_n}{a'^{-3/2}_{n-1} - a'^{-3/2}_n} =$$

$$= \frac{3\Omega}{\alpha(1-\varepsilon_0{}^2)} \cdot \frac{a'^{-3/2}_{n-1} \ln a'_{n-1} - a'^{-3/2}_n \ln a'_n}{a'^{-3/2}_{n-1} - a'^{-3/2}_n} + \frac{2\Omega}{\alpha(1-\varepsilon_0{}^2)}. \tag{6.1.21}$$

According to Eq. (3.4.11) (or condition (6.1.4)):

$$\overline{\lambda} = \frac{2\Omega}{\alpha(1-\varepsilon_0{}^2)} \tag{6.1.22}$$

and also taking into account that:

$$\frac{a'^{-3/2}_{n-1} \ln a'_{n-1} - a'^{-3/2}_n \ln a'_n - a'^{-3/2}_{n-1} \ln a'_{n-1} - a'^{-3/2}_n \ln a'_{n-1}}{a'^{-3/2}_n (\ln a'_{n-1} - \ln a'_n)} \Bigg| \frac{a'^{-3/2}_{n-1} - a'^{-3/2}_n}{\ln a'_{n-1}},$$

that is,

$$\frac{a_{n-1}'^{-3/2}\ln a_{n-1}' - a_n'^{-3/2}\ln a_n'}{a_{n-1}'^{-3/2} - a_n'^{-3/2}} = \ln a_{n-1}' + \frac{a_n'^{-3/2}\ln\dfrac{a_{n-1}'}{a_n'}}{a_{n-1}'^{-3/2} - a_n'^{-3/2}} =$$

$$= \ln a_{n-1}' + \frac{\ln\dfrac{a_n'}{a_{n-1}'}}{1-\left(\dfrac{a_n'}{a_{n-1}'}\right)^{3/2}},$$

we obtain:

$$\lambda_n \cdot \overline{\lambda}^{-1} = \frac{3}{2}\ln a_{n-1}' + \frac{\dfrac{3}{2}\ln\dfrac{a_n'}{a_{n-1}'}}{1-\left(\dfrac{a_n'}{a_{n-1}'}\right)^{3/2}} + 1. \qquad (6.1.23)$$

Carrying out the long division on the left-hand side of Eq. (6.1.23) we find:

$$\begin{array}{c|c} \dfrac{3}{2}\ln(a_n'/a_{n-1}') & \dfrac{1-(a_n'/a_{n-1}')^{3/2}}{\dfrac{3}{2}\ln(a_n'/a_{n-1}')} \\ -\dfrac{\dfrac{3}{2}\ln(a_n'/a_{n-1}') - \dfrac{3}{2}(a_n'/a_{n-1}')^{3/2}\ln(a_n'/a_{n-1}')}{(a_n'/a_{n-1}')^{3/2}\ln(a_n'/a_{n-1}')^{3/2}} & \end{array},$$

that allows us to write Eq. (6.1.23) in the form:

$$\lambda_n \cdot \overline{\lambda}^{-1} = \frac{3}{2}\ln a_{n-1}' + \frac{3}{2}\ln\frac{a_n'}{a_{n-1}'} + \frac{\left(\dfrac{a_n'}{a_{n-1}'}\right)^{3/2}\ln\left(\dfrac{a_n'}{a_{n-1}'}\right)^{3/2}}{1-\left(\dfrac{a_n'}{a_{n-1}'}\right)^{3/2}} + 1. \qquad (6.1.24)$$

According to (6.1.18c), (6.1.22), and (6.1.24), it follows directly that:

$$\frac{3\Omega}{\alpha(1-\varepsilon_0{}^2)}\cdot\ln a_n\cdot\overline{\lambda}^{-1}=\frac{3}{2}\ln a'_{n-1}+\frac{\frac{3}{2}\ln a'_n-\frac{3}{2}\ln a'_{n-1}}{1-\left(\frac{a'_n}{a'_{n-1}}\right)^{3/2}}+1, \qquad (6.1.25a)$$

whence:

$$\ln a_n=\ln a'_{n-1}+\frac{\ln a'_{n-1}-\ln a'_n}{\left(\frac{a'_n}{a'_{n-1}}\right)^{3/2}-1}+\frac{2}{3}. \qquad (6.1.25b)$$

Using (6.1.1), (6.1.18c), and (6.1.22) and taking into account that $\lambda_n=\frac{3}{2}\overline{\lambda}\cdot\ln a_n$ and $\mu_n=\frac{3}{2}\overline{\lambda}\cdot\ln a'_n$ we obtain:

$$\ln a'_{n-1}=\frac{\ln a_{n-1}+\ln a_n}{2}. \qquad (6.1.26)$$

Substituting (6.1.26) into Eq. (6.1.25b) this equation becomes:

$$2\ln a_n=\ln a_{n-1}+\ln a_n+\frac{4}{3}+\frac{\ln a_{n-1}+\ln a_n-\ln a_n-\ln a_{n+1}}{\left(\frac{\sqrt{a_n\cdot a_{n+1}}}{\sqrt{a_{n-1}\cdot a_n}}\right)^{3/2}-1}, \qquad (6.1.27a)$$

whence:

$$\ln a_n=\ln a_{n-1}+\frac{4}{3}+\frac{\ln a_{n-1}-\ln a_{n+1}}{\left(\frac{a_{n+1}}{a_{n-1}}\right)^{3/4}-1}. \qquad (6.1.27b)$$

Taking into account that:

$$\left|\left(\frac{a_{n+1}}{a_{n-1}}\right)^{3/4}-1\right|>>\left|\ln\frac{a_{n+1}}{a_{n-1}}\right|, \qquad (6.1.28)$$

equation (6.1.27b) with sufficient accuracy is approximated as follows:

$$\ln a_n=\ln a_{n-1}+\frac{4}{3} \qquad (6.1.29)$$

or:

$$\ln\frac{a_n}{a_{n-1}} = \frac{4}{3}, \quad n = 1,2,3,\dots\ . \tag{6.1.30}$$

Exponentiating equation (6.1.30) we find that:

$$a_n = a_{n-1}\cdot e^{4/3}, \quad n = 1,2,3\dots, \tag{6.1.31a}$$

that is:

$$\begin{aligned} a_1 &= a_0\cdot e^{4/3}, \\ a_2 &= a_1\cdot e^{4/3}, \\ a_3 &= a_2\cdot e^{4/3},\dots, \end{aligned} \tag{6.1.31b}$$

whence:

$$\begin{aligned} a_1 &= a_0\cdot e^{1\cdot\frac{4}{3}}, \\ a_2 &= a_0\cdot e^{2\cdot\frac{4}{3}}, \\ a_3 &= a_0\cdot e^{3\cdot\frac{4}{3}},\dots, \end{aligned} \tag{6.1.31c}$$

Summarizing Eqs (6.1.31c) we find:

$$a_n = a_0\cdot e^{\frac{4n}{3}}, \quad n = 1,2,3,\dots \tag{6.1.32}$$

Taking the logarithm of (6.1.32) we obtain the relation:

$$\log_b a_n = \log_b a_0 + \frac{4}{3}\log_b e\cdot n, \tag{6.1.33a}$$

that in the case of constants selecting $c_1 = \log_b a_0$ and $c_2 = \frac{4}{3}\log_b e$ passes into the law of planetary distances of Gurevich and Lebedinsky [23] and other authors:

$$\log_b a_n = c_1 + c_2\cdot n, \tag{6.1.33b}$$

which proves Laskar's proposition [257]. It is clear that Eq. (6.1.33b) corresponds to the exponential type of law:

$$a_n = C_1\cdot b^{c_2 n}, \tag{6.1.33c}$$

where $C_1 = b^{c_1}$. Obviously, Eq. (6.1.33c) generalizes several empirical laws proposed by Murray and Dermott (1999) [258, 259], Poveda and Lara (2008) [260, 261], Flores-Gutierrez and Garcia-Guerra (2011) [262], and others.

6.1.2. The general equation of distribution of the specific angular momentum of forming protoplanets

Now let us consider the general derivation of the law of planetary distances within the framework of the statistical theory of spheroidal bodies (without any restriction like (6.1.4)).

Using Eqs (6.1.2), (6.1.3) we can calculate:

$$\lambda_n = \frac{\int_{\mu_{n-1}}^{\mu_n} \lambda f(\lambda) d\lambda}{\int_{\mu_{n-1}}^{\mu_n} f(\lambda) d\lambda} = \frac{\lambda \int f(\lambda) d\lambda \Big|_{\mu_{n-1}}^{\mu_n} - \int_{\mu_{n-1}}^{\mu_n} d\lambda \int f(\lambda) d\lambda}{\int_{\mu_{n-1}}^{\mu_n} f(\lambda) d\lambda} =$$

$$= \frac{-\lambda e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} \Big|_{\mu_{n-1}}^{\mu_n} - \int_{\mu_{n-1}}^{\mu_n} \left(-e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} \right) d\lambda}{-e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} \Big|_{\mu_{n-1}}^{\mu_n}} =$$

$$= \frac{\lambda \cdot e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} \Big|_{\mu_{n-1}}^{\mu_n} - \int_{\mu_{n-1}}^{\mu_n} e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} d\lambda}{e^{-\frac{\alpha(1-\varepsilon_0{}^2)}{2\Omega}\lambda} \Big|_{\mu_{n-1}}^{\mu_n}}. \tag{6.1.34a}$$

Taking into account that Eq. (6.1.22) (see also (6.1.4)) we obtain:

$$\lambda_n = \frac{\lambda \cdot e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n} - \int\limits_{\mu_{n-1}}^{\mu_n} e^{-\frac{\lambda}{\overline{\lambda}}} d\lambda}{e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n}} = \frac{\lambda \cdot e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n} + \overline{\lambda} \cdot e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n}}{e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n}} =$$

$$= \frac{\lambda \cdot e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n}}{e^{-\frac{\lambda}{\overline{\lambda}}}\Big|_{\mu_{n-1}}^{\mu_n}} + \overline{\lambda} = \overline{\lambda} + \frac{\mu_n \cdot e^{-\frac{\mu_n}{\overline{\lambda}}} - \mu_{n-1} \cdot e^{-\frac{\mu_{n-1}}{\overline{\lambda}}}}{e^{-\frac{\mu_n}{\overline{\lambda}}} - e^{-\frac{\mu_{n-1}}{\overline{\lambda}}}}. \qquad (6.1.34b)$$

Performing the long division in the right-hand side of Eq. (6.1.34b) we obtain:

$$\begin{array}{r|l} \mu_n \cdot e^{-\frac{\mu_n}{\overline{\lambda}}} - \mu_{n-1} \cdot e^{-\frac{\mu_{n-1}}{\overline{\lambda}}} & e^{-\frac{\mu_n}{\overline{\lambda}}} - e^{-\frac{\mu_{n-1}}{\overline{\lambda}}} \\ - \quad \mu_n \cdot e^{-\frac{\mu_n}{\overline{\lambda}}} - \mu_n \cdot e^{-\frac{\mu_{n-1}}{\overline{\lambda}}} & \hline \\ \hline (\mu_n - \mu_{n-1}) \cdot e^{-\frac{\mu_{n-1}}{\overline{\lambda}}} & \mu_n \end{array},$$

that allows us to write Eq. (6.1.34b) in the form:

$$\lambda_n = \overline{\lambda} + \mu_n + \frac{(\mu_n - \mu_{n-1}) \cdot e^{-\frac{\mu_{n-1}}{\overline{\lambda}}}}{e^{-\frac{\mu_n}{\overline{\lambda}}} - e^{-\frac{\mu_{n-1}}{\overline{\lambda}}}} =$$

$$= \overline{\lambda} + \mu_n + \frac{\mu_n - \mu_{n-1}}{e^{-\frac{(\mu_n - \mu_{n-1})}{\overline{\lambda}}} - 1}. \qquad (6.1.35)$$

Multiplying both sides of Eq. (6.1.35) by $\overline{\lambda}^{-1}$ we obtain:

$$\lambda_n \cdot \overline{\lambda}^{-1} = 1 + \mu_n \cdot \overline{\lambda}^{-1} + \frac{(\mu_n - \mu_{n-1}) \cdot \overline{\lambda}^{-1}}{e^{-(\mu_n - \mu_{n-1})\overline{\lambda}^{-1}} - 1}. \qquad (6.1.36)$$

To limit $\overline{\lambda}^{-1} \to 0$ in equation (6.1.36) let us use L'Hospital's rule:

$$0 = 1 + \lim_{\overline{\lambda}^{-1} \to 0} \frac{(\mu_n - \mu_{n-1}) \cdot \overline{\lambda}^{-1}}{e^{-(\mu_n - \mu_{n-1})\overline{\lambda}^{-1}} - 1} = 1 + \lim_{\overline{\lambda}^{-1} \to 0} \frac{\mu_n - \mu_{n-1}}{-(\mu_n - \mu_{n-1}) \cdot e^{-(\mu_n - \mu_{n-1})\overline{\lambda}^{-1}}} =$$

$$= 1 - \frac{\mu_n - \mu_{n-1}}{\mu_n - \mu_{n-1}} = 1 - 1 = 0,$$

that is, we obtain the identity. Using (6.1.1) for Eq. (6.1.35) we obtain:

$$\lambda_n = \overline{\lambda} + \frac{\lambda_n + \lambda_{n+1}}{2} + \frac{1}{2} \cdot \frac{\lambda_n + \lambda_{n+1} - \lambda_{n-1} - \lambda_n}{e^{-(\lambda_{n+1} - \lambda_{n-1}) \cdot \overline{\lambda}^{-1}/2} - 1}, \qquad (6.1.37)$$

from which we obtain the *general difference equation*:

$$\lambda_n = 2\overline{\lambda} + \lambda_{n+1} + \frac{\lambda_{n+1} - \lambda_{n-1}}{e^{-(\lambda_{n+1} - \lambda_{n-1})/2\overline{\lambda}} - 1}. \qquad (6.1.38)$$

Let us consider some particular cases of the equation (6.1.38):

a) if the condition (6.1.4) is true then we can represent the exponential function in Eq. (6.1.38) by Maclaurin's series in the *linear* approximation:

$$e^{-(\lambda_{n+1} - \lambda_{n-1})/2\overline{\lambda}} = 1 - (\lambda_{n+1} - \lambda_{n-1}) \cdot \frac{1}{2\overline{\lambda}}. \qquad (6.1.39a)$$

Substituting (6.1.39a) into the equation (6.1.38) we obtain:

$$\lambda_n = 2\overline{\lambda} + \lambda_{n+1} + \frac{\lambda_{n+1} - \lambda_{n-1}}{-(\lambda_{n+1} - \lambda_{n-1})/2\overline{\lambda}},$$

whence follows a *uniform law* of distribution of the specific angular momentum:

$$\lambda_n - \lambda_{n+1} = 0 \Rightarrow \lambda_{n+1} = \lambda_n = \text{const}; \qquad (6.1.39b)$$

b) taking into account the condition (6.1.4) let us represent the exponential function in Eq. (6.1.38) by Maclaurin's series in the *quadratic* approximation:

$$e^{-(\lambda_{n+1} - \lambda_{n-1})/2\overline{\lambda}} = 1 - \frac{\lambda_{n+1} - \lambda_{n-1}}{2\overline{\lambda}} + \frac{(\lambda_{n+1} - \lambda_{n-1})^2}{2! \cdot 4\overline{\lambda}^2}. \qquad (6.1.40a)$$

Substituting (6.1.40a) into the Eq. (6.1.38) we obtain:

$$\lambda_n = 2\overline{\lambda} + \lambda_{n+1} + \frac{\lambda_{n+1} - \lambda_{n-1}}{-(\lambda_{n+1} - \lambda_{n-1})/2\overline{\lambda} + (\overline{\lambda}_{n+1} - \lambda_{n-1})^2/8\overline{\lambda}^2} =$$

$$= 2\overline{\lambda} + \lambda_{n+1} - \frac{2\overline{\lambda}}{1 - \overline{\lambda}^{-1}(\lambda_{n+1} - \lambda_{n-1})/4} \approx$$

$$\approx 2\overline{\lambda} + \lambda_{n+1} - 2\overline{\lambda}\left[1 + \overline{\lambda}^{-1} \cdot \frac{\lambda_{n+1} - \lambda_{n-1}}{4}\right] = \lambda_{n+1} - \frac{\lambda_{n+1} - \lambda_{n-1}}{2} = \frac{\lambda_{n+1} + \lambda_{n-1}}{2},$$

that is:

$$\lambda_{n+1} - 2\lambda_n + \lambda_{n-1} = 0 . \qquad (6.1.40b)$$

The characteristic equation for the second order difference equation (6.1.40b) has the form ($Z^n = \lambda_n$):

$$Z^2 - 2Z + 1 = (Z - 1)^2 = 0 , \qquad (6.1.41)$$

whose solutions are two identical roots $Z_1 = Z_2 = 1$, that is, the one with a multiplicity of $d = 2$. This means that the linear difference equation of the form (6.1.40b) has a solution in the form of an arithmetic progression:

$$\lambda_n = (A + B \cdot n) \cdot [Z_1]^n = A + B \cdot n . \qquad (6.1.42)$$

Based on the logical deduction of (6.1.40b) and (6.1.42) we conclude that the following constants A and B should be chosen:

$$A = \lambda_0 ; \qquad (6.1.43a)$$

$$B = \lambda_{n+1} - \lambda_n = \lambda_n - \lambda_{n-1} = d , \qquad (6.1.43b)$$

where d is a difference.

Using the condition of decay ($V_c + \overline{\varphi}_g = 0$) of spheroidal body (see Section 5.4) and Eq. (7.3.43) for n-th protoplanet (see Section 7.3 in the next Chapter 7) we find that:

$$\lambda_n = \sqrt{\gamma M a_n \cdot (1 - e_n^2)} , \qquad (6.1.44)$$

where a_n are the major semi-axes of protoplanets, and e_n are the geometric eccentricities of orbits. Substitution of (6.1.43a, b), and (6.1.44) into Eq. (6.1.42) gives *Schmidt's law* of planetary distances [6]:

$$\sqrt{R_n} = \sqrt{a_n(1-e_n^2)} = \frac{A}{\sqrt{\gamma M}} + \frac{Bn}{\sqrt{\gamma M}} = \frac{\lambda_0}{\sqrt{\gamma M}} + \frac{d}{\sqrt{\gamma M}} n . \quad (6.1.45)$$

If $n = 0$ then it follows directly from Eq. (6.1.45) that:

$$\lambda_0 = \sqrt{\gamma M a_0 \cdot (1-e_0^2)}\,; \quad (6.1.46)$$

c) concerning the condition (6.1.4) let us represent the exponential function in Eq. (6.1.38) by Maclaurin's series in the *cubic* approximation:

$$e^{-(\lambda_{n+1}-\lambda_{n-1})/2\overline{\lambda}} =$$

$$= 1 - \frac{\lambda_{n+1}-\lambda_{n-1}}{1\cdot 2\overline{\lambda}} + \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{2!\cdot 4\overline{\lambda}^2} - \frac{(\lambda_{n+1}-\lambda_{n-1})^3}{3!\cdot 8\overline{\lambda}^3} . \quad (6.1.47)$$

Substituting (6.1.47) into equation (6.1.38) we obtain:

$$\lambda_n = 2\overline{\lambda} + \lambda_{n+1} +$$

$$+ \frac{\lambda_{n+1}-\lambda_{n-1}}{-(\lambda_{n+1}-\lambda_{n-1})/2\overline{\lambda} + (\lambda_{n+1}-\lambda_{n-1})^2/8\overline{\lambda}^2 - (\lambda_{n+1}-\lambda_{n-1})^3/48\overline{\lambda}^3} =$$

$$= 2\overline{\lambda} + \lambda_{n+1} - \frac{2\overline{\lambda}}{1 - \frac{\lambda_{n+1}-\lambda_{n-1}}{4\overline{\lambda}} + \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{24\overline{\lambda}^2}}, \quad (6.1.48a)$$

and taking into account the approximation:

$$\frac{1}{1 - \frac{\lambda_{n+1}-\lambda_{n-1}}{4}\overline{\lambda}^{-1} + \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{24}\overline{\lambda}^{-2}} \approx$$

$$\approx 1 + \frac{\lambda_{n+1}-\lambda_{n-1}}{4}\overline{\lambda}^{-1} + \left(\left[\frac{\lambda_{n+1}-\lambda_{n-1}}{4}\right]^2 - \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{24}\right)\overline{\lambda}^{-2} =$$

$$= 1 + \frac{\lambda_{n+1}-\lambda_{n-1}}{4}\overline{\lambda}^{-1} + \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{48}\overline{\lambda}^{-2} \quad (6.1.48b)$$

we have:

$$\lambda_n \approx 2\overline{\lambda} + \lambda_{n+1} - 2\overline{\lambda}\left[1 + \frac{\lambda_{n+1}-\lambda_{n-1}}{4}\overline{\lambda}^{-1} + \frac{(\lambda_{n+1}-\lambda_{n-1})^2}{48}\overline{\lambda}^{-2}\right] =$$

$$= 2\overline{\lambda} + \lambda_{n+1} - 2\overline{\lambda} - \frac{\lambda_{n+1} - \lambda_{n-1}}{2} - \frac{(\lambda_{n+1} - \lambda_{n-1})^2}{24\overline{\lambda}} =$$

$$= \frac{\lambda_{n+1} + \lambda_{n-1}}{2} - \frac{(\lambda_{n+1} - \lambda_{n-1})^2}{24\overline{\lambda}},$$

whence:

$$\lambda_{n+1} - 2\lambda_n + \lambda_{n-1} - \frac{\overline{\lambda}^{-1}}{12}(\lambda_{n+1} - \lambda_{n-1})^2 = 0. \qquad (6.1.49)$$

The characteristic equation for the difference equation (6.1.49) has the form:

$$Z^{n+1} - 2Z^n + Z^{n-1} - \frac{\overline{\lambda}^{-1}}{12} \cdot (Z^{n+1} - Z^{n-1})^2 = 0$$

or:

$$Z^{n-1}(Z-1)^2 - \frac{\overline{\lambda}^{-1}}{12} Z^{2(n-1)}(Z^2 - 1)^2 = 0,$$

whence we find that:

$$(Z-1)^2 \cdot \left[1 - \frac{\overline{\lambda}^{-1}}{12} Z^{n-1}(Z+1)^2\right] = 0. \qquad (6.1.50)$$

The characteristic equation (6.1.50) is reduced to two equations:

$$(Z-1)^2 = 0; \qquad (6.1.51a)$$

$$(Z+1)^2 Z^{n-1} = 12\overline{\lambda}. \qquad (6.1.51b)$$

The solution of Eq. (6.1.51a) is the root $Z_1^{(1)} = 1$ of a multiplicity of $d = 2$, and the solutions of Eq. (6.1.51b) are $(n+1)$-th power roots $Z_k^{(2)} = \sqrt[n+1]{12\overline{\lambda}} \cdot e^{-\frac{i2\pi}{n+1}k}$, $k = 0,1,...,n$ with sufficient accuracy (under condition $\overline{\lambda}^{-1} \to 0$, that is, $\overline{\lambda} \to \infty$). As a result, the nonlinear difference equation (6.1.49) at the *linear approximation* ($\overline{\lambda}^{-1} \to 0$) has a general solution [16, 73]:

$$\lambda_n = (A + B \cdot n) \cdot [Z_1^{(1)}]^n + \sum_{k=0}^{n} C_k [Z_k^{(2)}]^n =$$

$$= A + B \cdot n + \sum_{k=0}^{n} C_k \left(\sqrt[n+1]{12\bar{\lambda}} \right)^n \cdot e^{-\frac{i\,2\pi n}{n+1}k} . \qquad (6.1.52)$$

Taking into account (6.1.44), from Eq. (6.1.52) the law of planetary distances of the kind [73] follows directly:

$$\sqrt{R_n} = \sqrt{a_n \cdot (1 - e_n^2)} = \frac{A}{\sqrt{\gamma M}} + \frac{B}{\sqrt{\gamma M}} n + \left(12\bar{\lambda}\right)^{\frac{n}{n+1}} \sum_{k=0}^{n} \frac{C_k}{\sqrt{\gamma M}} e^{-\frac{i2\pi n}{n+1}k} =$$

$$= a + bn + 2\left(12\bar{\lambda}\right)^{\frac{n}{n+1}} \sum_{k=0}^{ent[n/2]} c_k \cos\left(\frac{2\pi n}{n+1}k\right), \qquad (6.1.53)$$

if $c_{n+1-k} = c_k^*,\ k = 0,1,...,\mathrm{ent}[n/2]$, $c_k = C_k / \sqrt{\gamma M}$, $a = A / \sqrt{\gamma M}$, $b = B / \sqrt{\gamma M}$.

6.1.3. A statistical model of evolution for rotating and gravitating spheroidal bodies and its application to the problem of distribution of planetary distances in the Solar system

Let us now use the cosmological hypothesis of Schmidt [6, 21] at develop a statistical model of a rotating and gravitating spheroidal body in the stage of evolution from a protoplanetary flattened gas-dust disk to the formation of protoplanets. As noted in Sections 6.1.1 and 6.1.2, this theory starts from that fact that during the process of the origin of a protoplanet, each particle (or generally speaking, planetesimal) in a gas-dust protoplanetary cloud (in a swarm of planetesimals) has a chance to land on the protoplanet whose specific angular momentum value is the same as one for the particle/planetesimal (or it differs less than all values for other protoplanets).

We consider a flattened gas-dust protoplanetary cloud as a uniformly rotating and gravitating spheroidal body with the specific angular momentum distribution function (6.1.3). Analogously to Sections 6.1.1 and 6.1.2, taking into account that the inverse parameter $\bar{\lambda}^{-1}$ for the specific angular momentum of a spheroidal body is very small in accordance with (6.1.4), we can also represent the function of specific angular momentum $f(\lambda)$ by Maclaurin's series [16, 73]:

$$f(\lambda)=\frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\left\{1-\frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\lambda+\frac{1}{2}\left[\frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\lambda\right]^2-\ldots\right\}. \qquad (6.1.54)$$

Let us limit the series (6.1.54) by the zeroth term, that is, we suppose that

$$f(\lambda)\approx\alpha(1-\varepsilon_0^2)/2\Omega=\text{const}. \qquad (6.1.55)$$

Taking into account Eq. (6.1.55), the formula (6.1.2) becomes:

$$\lambda_n=\Big(\int_{\mu_{n-1}}^{\mu_n}\lambda\,(\alpha(1-\varepsilon_0^2)/2\Omega)d\lambda\Big)/\Big(\int_{\mu_{n-1}}^{\mu_n}(\alpha(1-\varepsilon_0^2)/2\Omega)d\lambda\Big)=$$

$$=(\mu_n+\mu_{n-1})/2\,. \qquad (6.1.56)$$

Substituting Eq. (6.1.1) into Eq. (6.1.56) we obtain the difference equation (see the case of *quadratic* approximation b) in Sections 6.1.2):

$$\lambda_n=(\lambda_{n+1}+\lambda_{n-1})/2\quad . \qquad (6.1.57)$$

It is clear that Eq. (6.1.57) describes the well-known property of an arithmetic progression whose n-th term is calculated by the formula:

$$\lambda_n=\lambda_0+d\cdot n, \qquad (6.1.58)$$

where d is the difference and λ_0 is the first (the zeroth) term of an arithmetic progression.

Taking into account Eq. (6.1.58) and formula (6.1.44) for the relation of the specific angular momentum λ_n with the square root of radius R_n for orbit for the n-th protoplanet (under the condition of the *circular* character of planetary orbit when $e_n \to 0$):

$$\lambda_n = \sqrt{\gamma M}\sqrt{R_n} \quad , \tag{6.1.59}$$

we conclude that zeroth approximation of function $f(\lambda)$ leads to Schmidt's well-known law:

$$\sqrt{R_n} = a + b \cdot n \, , \tag{6.1.60}$$

where a, b are some constants. In 1944, O. Schmidt derived his law for planetary distances. However, for the defined constants a and b this law (6.1.60) does not permit us to estimate correctly planetary distances for all planets of the Solar system. In connection with this Schmidt [6, 21] proposed using his law (6.1.60) in a combination, that is, separately for the planets of the Earth group on the one hand and for the planets of the Jupiter group on the other. Moreover, the distribution function of a specific angular momentum of a gas-dust cloud was not derived analytically within the framework of Schmidt's model [16, 73].

In our view, the cause of the above-mentioned problem with Schmidt's lies in the distribution function being too simplified as a *uniform* kind (6.1.55) for a specific angular momentum. As shown in Sections 6.1.2, to modify the evolution model for planet formation let us use the linear approximation of the function $f(\lambda)$, taking into account both zeroth and first terms [16, 73]:

$$f(\lambda) = \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} \cdot \left[1 - \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} \cdot \lambda \right]. \tag{6.1.61}$$

Bearing in mind Eq. (6.1.61), we can obtain from formula (6.1.2) the following:

$$\lambda_n = \frac{\dfrac{(\mu_n+\mu_{n-1})}{2} - \dfrac{\alpha(1-\varepsilon_0^2)}{6\Omega}(\mu_n^2+\mu_n\mu_{n-1}+\mu_{n-1}^2)}{1-\dfrac{\alpha(1-\varepsilon_0^2)}{4\Omega})(\mu_n+\mu_{n-1})}. \quad (6.1.62)$$

and then with the use of Eq. (6.1.22) and Eq. (6.1.1) we finally have the difference equation (see the analogous case *c)* of *cubic* approximation in Sections 6.1.2):

$$\lambda_n = \frac{\lambda_{n+1}+2\lambda_n+\lambda_{n-1} - \dfrac{\bar{\lambda}^{-1}}{3}[(\lambda_{n+1}+2\lambda_n+\lambda_{n-1})^2-(\lambda_{n+1}+\lambda_n)(\lambda_n+\lambda_{n-1})]}{4-\bar{\lambda}^{-1}(\lambda_{n+1}+2\lambda_n+\lambda_{n-1})}. \quad (6.1.63)$$

To find its solution let us carry out the following substitution:

$$\lambda_n = Z^n, \quad n=1,\ 2, 3,\ \ldots . \quad (6.1.64)$$

As a result of this, we obtain from Eqs (6.1.63) and (6.1.64) the following characteristic equation [16, 73]:

$$(Z-1)^2 \cdot \left[(\bar{\lambda}^{-1}/3)Z^{n-1}(Z+1)^2-1\right] = 0 \quad (6.1.65)$$

which is reduced to two equations:

$$(Z-1)^2 = 0 \quad (6.1.66a)$$

and:

$$Z^{n-1}(Z+1)^2 = 3\bar{\lambda}\,. \quad (6.1.66b)$$

Taking into account that $\bar{\lambda} \to \infty$ when $\alpha \to 0$ in the right-hand part of Eq. (6.1.66b), that is, in the left-hand part $Z^{n-1}(Z+1)^2 \to Z^{n+1}$ at $\alpha \to 0$ (see (II.1)), the characteristic equation (6.1.66b) can be simplified to the form [16, 73]:

$$Z^{n+1} = 3\bar{\lambda} \quad (6.1.67)$$

Equation (6.1.66a) has one root with double multiplicity:

$$Z_1^{(1)} = 1, \quad d_1 = 2 \quad (6.1.68a)$$

while equation (6.1.67) has $n+1$ roots:

$$Z_k^{(2)} = \sqrt[n+1]{3\bar{\lambda}}\ e^{-\mathrm{i}2\pi k/(n+1)}, \quad k=0,\ldots,n \quad (\bar{\lambda} >> 1)\,. \quad (6.1.68b)$$

According to the roots (6.1.68a) and (6.1.68b), the general solution of the difference equation (6.1.63) has the form [16, 73]:

$$\lambda_n = (A + Bn)[Z_1^{(1)}]^n + \sum_{k=0}^{n} A_k [Z_k^{(2)}]^n =$$

$$= A + Bn + (3\bar{\lambda})^{n/(n+1)} \left[A_0 + 2 \sum_{k=1}^{ent[n/2]} A_k \cos(2\pi nk/(n+1)) \right], \quad (6.1.69)$$

where it has been taken into consideration that $A_k = A_{n+1-k} = A^*_k$, under derivation of Eq. (6.1.69). The obtained formula (6.1.69) points to a quantization of the specific angular momentum λ_n as well as a possibility to represent the specific angular momentum as a wave packet: $\lambda_n = \sum_{k=0}^{n} C_k \exp\{i\, 2\pi nk/(n+1)\}$. A similar conclusion has been proposed within the framework of the above-mentioned quantum mechanical approach [15, 34–42, 233–235, 238].

Taking into account Eq. (6.1.59), the solution (6.1.69) permits us to obtain a new law for the square root of planetary distances (see also Eq. (6.1.53)) in the form [16, 65, 73]:

$$\sqrt{R_n} = a + bn + (3\bar{\lambda})^{\frac{n}{n+1}} \left[a_0 + 2 \sum_{k=1}^{ent[n/2]} a_k \cos\left(\frac{2\pi nk}{n+1} \right) \right], \quad (6.1.70)$$

where $a, b, a_0, ..., a_{[n/2]}$ are coefficients to be sought for a planetary system (the Solar system) in the form of dependence on $\bar{\lambda}$. The proposed law for planetary distances (6.1.70) generalizes Schmidt's well-known law (6.1.60). Let us note that the quantization of specific angular momentum λ_n leads to the quantization of planetary orbits following the quantum mechanical approach [15, 34–42, 200, 201, 233–235, 238].

Now let us consider the evolution of our Solar system based on a nebular origin following the theory of Hoyle [30, 31]. Undoubtedly, the nebular origin of the Solar system permits us to describe the proto-Sun together with a flattened gas-dust protoplanetary cloud as a model of a rotating and gravitating spheroidal body. According to Hoyle's theory [8, 30], we consider an angular momentum value of the presolar nebula (cloud) being equal to $L = 4 \cdot 10^{51} (\text{g} \cdot \text{cm}^2/\text{s}) = 4 \cdot 10^{44} (\text{kg} \cdot \text{m}^2/\text{s})$. This value was obtained by Hoyle bearing in mind that a primary presolar cloud has a mass density 1 atom per 1 sm^3 and angular velocity $\Omega \sim 10^{-15}\ \text{s}^{-1}$ as for our Galaxy as a whole [8, 10]. It is well known that the mass of the presolar cloud is approximately $M = 1.988 \cdot 10^{30} (\text{kg})$. Then it is not difficult to see that the mean value of specific angular momentum for the forming Solar system is $\bar{\lambda} = L / M = 2.012 \cdot 10^{14} (\text{m}^2 / \text{s})$.

Let us apply the proposed law (6.1.70) to the estimation of the planetary distance in the forming Solar system [16, 73]. Supposing the case $n = 0$ corresponds to the planet of Mercury, $n = 1$ conforms to Venus, $n = 2$ corresponds to the Earth, ..., $n =$ 8 corresponds to Pluto we can calculate the value $(3\bar{\lambda})^{n/(n+1)}$ in (6.1.70) for different values of n (see Table 6.1).

The proposed law for planetary distances (6.1.70) gives us the calculated formulas to estimate the square root of distances for all planets of the Solar system [16, 73]:

$$\sqrt{R_0} = a + a_0;$$

$$\sqrt{R_1} = a + b + (3\bar{\lambda})^{1/2} (a_0 - 2a_1);$$

$$\sqrt{R_2} = a + 2b + (3\bar{\lambda})^{2/3} (a_0 - a_1);$$

$$\sqrt{R_3} = a + 3b + (3\bar{\lambda})^{3/4} a_0;$$

$$\sqrt{R_4} = a + 4b + (3\bar{\lambda})^{4/5} (a_0 - 2a_1 \cos\frac{3\pi}{5}); \qquad (6.1.71)$$

$$\sqrt{R_5} = a + 5b + (3\bar{\lambda})^{5/6} (a_0 + a_1 - a_2);$$

$$\sqrt{R_6} = a + 6b + (3\bar{\lambda})^{6/7} [a_0 - 2(a_1 \cos\frac{5\pi}{7} + a_2 \cos\frac{3\pi}{7} + a_3 \cos\frac{\pi}{7})];$$

$$\sqrt{R_7} = a + 7b + (3\bar{\lambda})^{7/8} [a_0 + \sqrt{2}(a_1 - a_3) - 2a_4];$$

$$\sqrt{R_8} = a + 8b + (3\bar{\lambda})^{8/9}[a_0 - 2(a_1\cos\frac{7\pi}{9} + a_2\cos\frac{5\pi}{9} + \frac{1}{2}a_3 + a_4\cos\frac{\pi}{9})],$$

where the coefficients are to be equal to a = 0.622173; b = 0.228316; $a_0 = -0.597348\cdot10^{-12}$; $a_1 = 2.773092\cdot10^{-12}$; $a_2 = 1.534506\cdot10^{-12}$; $a_3 = 0.925297\cdot10^{-12}$; $a_4 = 0.816924\cdot10^{-12}$ (see Table 6.1).

Taking advantage of coefficients and substituting theirs in Eqs (6.1.71), we can find distance square root estimations for the planets of the Solar system. The preliminary calculations of $\sqrt{R_n}, n = 0,1,...,8$ in accordance with Eqs (6.1.71) and Table 6.1 give us a good agreement with the observable data (see Table 6.2), excepting the case of $n = 8$ for Pluto. To decrease the error of estimation $\sqrt{R_8}$ we can introduce an *additional* coefficient a_5 in the formulas (6.1.71).

In other words, instead of the previous formula $\sqrt{R_8}$ for Pluto in (6.1.71), we have to use the following [16, 73]:

$$\sqrt{R_8} = a + 8b + (3\bar{\lambda})^{8/9}\times$$
$$\times[a_0 - 2(a_1\cos\frac{7\pi}{9} + a_2\cos\frac{5\pi}{9} + \frac{1}{2}a_3 + (a_4 + a_5)\cos\frac{\pi}{9})], \qquad (6.1.72)$$

where $a_5 = 0.7679326\cdot10^{-12}$. However, such a decision violates the proposed law (6.1.70) because the necessary number of coefficients is equal to $ent[n/2] = 4$ only. In this connection, there is no reason to include the data (6.1.72) for Pluto in Table 6.2.

Table 6.2 presents the values of the square root of planetary distances calculated following the proposed law. In Table 6.3, a comparative analysis of different laws is presented [6, 8, 12]. As follows from Table 6.3 and Fig. 6.2, the proposed law gives the best results in the prediction of planetary distances for the Solar system.

In reality, for all planets of the Solar system, excepting the Earth and Pluto, the absolute estimation error (and, naturally, the relative one) is equal to 0% (for the Earth, the absolute estimation error is 11% and the relative one is 10%). Thus, the mean error of estimation of planetary distances in the Solar system is equal to 1.4% in accordance with the proposed law. For comparison, Table 6.3 also presents both the well-known Titius–Bode law and its modification of Wurm [8]:

$$R_n = a + b \cdot 2^n, n = -\infty, 0, 1, 2, ..., 7, \qquad (6.1.73)$$

where $a = 0.387$ and $b = 0.293$. However, formula (6.1.73) indicates additional planets between Mercury and Venus. On this question of "the lost planet," the answer can be found in the form of the proposition of J. Bailey [263]: "The Moon may be a former planet." According to this proposition, the Moon had an unstable orbit from which it was captured by the Earth [8]. However, the modern point of view considers the Moon's origin to be result of a collision of a former planet with the proto-Earth [264–269]. Moreover, the Titius–Bode law does predict the existence of a "trans-martian" planet between Mars and Jupiter [8] (see Table 6.3). In fact, the area is occupied by the asteroid belt, with the dwarf planet, Ceres, the largest body in the belt.

The empirical formula of Blagg [229]:

$$R_n = A \cdot (1.7275)^n [B + f(\alpha + \beta \cdot n)], \quad n = -2, -1, 0, 1, 2, ..., 7, \qquad (6.1.74)$$

where

A and B are constants,

f is a periodic function, and

α and β are constant angles);

as well as the empirical formula of Richardson [230]:

$$R_n = A \cdot (1.728)^n F_n(\theta_n), n = 1, 2, ..., 10, \qquad (6.1.75)$$

where $\theta_n = (4\pi/13)n$ and F_n is a periodic function, also predict the existence of the so-called "trans-Martian" planet. This produces a "gap" between Mars and Jupiter because Ceres is not considered a planet. Although the formulas of Blagg and Richardson are pure heuristic and having no theoretical base, nevertheless, they revealed *the presence of a periodic function* in the planetary distance laws (6.1.74) and (6.1.75). This fact is fully confirmed by the proposed theoretical law (6.1.70).

Unlike Blagg's and Richardson's empirical formulas, Schmidt's law (6.1.60) was founded using scientific hypothesis on a correspondence between the specific angular momentum distribution law of gas-dust protoplanetary cloud and the planetary distances law. Schmidt's law, (Table 6.3) (6.1.60) estimates the distances for outer planets (beginning from Jupiter) at $a = 2.28$, $b = 1.0$ and separately for inner planets (beginning from Mercury) at $a = 0.62$ and $b = 0.20$. In fact, Schmidt proposed two laws of the Solar system's origin [6, 8]. Nevertheless, because Schmidt's law (6.1.60) is a particular case of the law (6.1.70) then the proposed law connects these two laws as a whole. Indeed, this law surely predicts the planetary distances for the Solar system, except for the Earth and Pluto.

Let us note that this exception is not a deficiency in the proposed theory since the theories of von Weizsäcker [26, 27] and Ter Haar and Cameron [7] also revealed an exceptional peculiarity for the Earth and Pluto. In particular, Ter Haar and Cameron supposed that the best factor in the geometrical progression for planetary distances is equal to 1.89 because this number was obtained within the framework of von Weizsäcker's theory [26, 27] (see also the like estimation of Murray and Dermott [258, 259]). Namely, this geometrical progression gives a divergence with observable planetary data for the Earth and the Pluto only. Thus, the proposed law and von Weizsäcker's and Ter Haar–Cameron's theories lead to a similar result.

Table 6.1. The values of coefficients and parameters estimated in accordance with the proposed law

Solar system planets	n	$(3\overline{\lambda})^{n/(n+1)}$	$ent[n/2]$	$a, b, a_0, \ldots, a_{[n/2]}$
Mercury	0	1.0	0	a=0.622173; a_0=–0.597348·10^{-12}
Venus	1	2.456827·10^{7}	1	a=0.622173; b=0.228316; a_0=– 0.597348·10^{-12}; a_1=2.773092·10^{-12}
Earth	2	7.142213·10^{9}	1	a=0.622173; b=0.228316; a_0=–0.597348·10^{-12}; a_1=2.773092·10^{-12}
Mars	3	1.21776·10^{11}	2	a=0.622173; b=0.228316; a_0= – 0.597348·10^{-12}
Jupiter	4	6.677277·10^{11}	2	a=0.622173; b=0.228316; a_0=–0.597348·10^{-12}; a_1=2.773092·10^{-12}
Saturn	5	2.076304·10^{12}	3	a=0.622173; b=0.228316; a_0= –0.597348·10^{12}; a_1=2.773092·10^{-12}; a_2=1.534506·10^{-12}
Uranus	6	4.668889·10^{12}	3	a=0.622173; b=0.228316; a_0=–0.597348·10^{12}; a_1=2.773092·10^{-12}; a_2=1.534506·10^{-12}; a_3=0.925297·10^{-12}
Neptune	7	8.573449·10^{12}	4	a=0.622173; b=0.228316; a_0=–0.597348·10^{12}; a_1=2.773092·10^{-12}; a_2=1.534506·10^{-12}; a_3=0.925297·10^{-12}; a_4=0.816924·10^{-12}
Pluto	8	1.375447·10^{13}	4	a=0.622173; b=0.228316; a_0=–0.597348·10^{12}; a_1=2.773092·10^{-12}; a_2=1.534506·10^{-12}; a_3=0.925297·10^{-12} a_4=0.816924·10^{-12}

Table 6.2. The values of the square root of the Solar system planetary distances calculated following the proposed law

Solar system planets	n	$ent[n/2]$	$\sqrt{R_n}$ (AU)	Error, δ (%)
Mercury	0	0	0.622173	0%
Venus	1	1	0.850338	0%
Earth	2	1	1.054732	5%
Mars	3	2	1.234377	0%
Jupiter	4	2	2.280965	0%
Saturn	5	3	3.095158	0%
Uranus	6	3	4.375043	0%
Neptune	7	4	5.495270	0%

Table 6.3. The comparative analysis for different laws of prediction of planetary distances (AU) for the Solar system

Solar system planets	Observable distances	Titius–Bode law	Wurm's modification of Titius – Bode law	Empirical formulas		Schmidt law	Proposed law
				Blagg	Richardson		
Mercury	0.3871	0.4	0.387	0.387	0.3869	0.3844	0.3871
Venus	0.7233	0.7	0.68	0.723	0.7240	0.6724	0.7231
Earth	1.0000	1.0	0.973	1.000	0.9994	1.0404	1.1124
Mars	1.5237	1.6	1.559	1.524	1.5252	1.4884	1.5237
Ceres	_	2.8	2.731	2.67	2.8695	_	_
Jupiter	5.2028	5.2	5.075	5.200	5.1935	5.1984	5.2028
Saturn	9.580	10.0	9.763	9.550	9.5053	10.7584	9.580
Uranus	19.141	19.6	19.139	19.23	19.2104	18.3184	19.1410
Neptune	30.198	38.8	37.891	30.13	30.3005	27.8784	30.1980

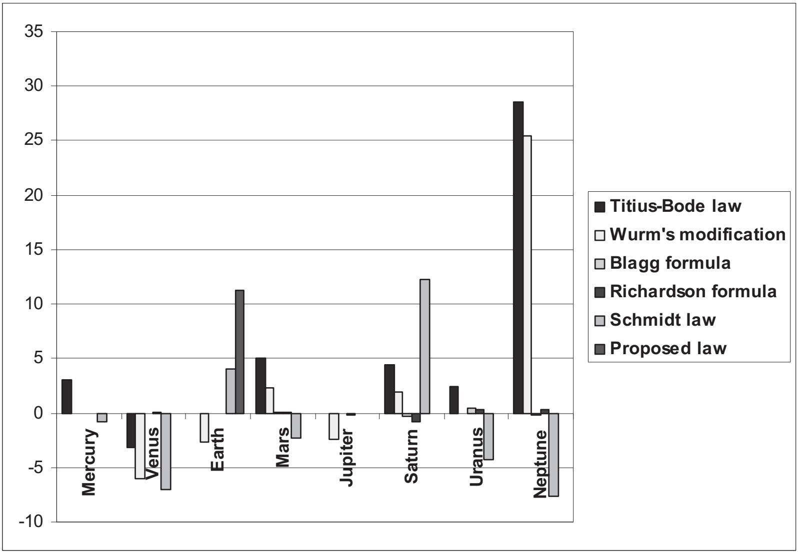


Figure 6.2. The diagram of measure of deviation $\delta = (R_n^{th} - R_n^{obs}) / R_n^{obs}$,% for planets of the Solar system where R_n^{obs} is a value of real (observable) planetary distances in the Solar system, R_n^{th} is an estimation of planetary distances following theoretical laws, n is a planet number

6.2. The thermal emission model of protoplanetary cloud formation

Schmidt's cosmogonical hypothesis on the origin of the Solar system as a result of the evolution of gas-dust meteorite swarm can be put forward on a basis of the *emission model* of evolutionary development from the flattened protoplanetary gas-dust disk to the emerging protoplanets [6, 21]. As Schmidt noted, a swarm of relatively large bodies and small particles exists for quite a long time before being merged into large bodies, that is, into protoplanets [6 p.31]:

> During this time, these bodies and particles mixed and interacted. The magnitude of the momentum was different in different parts of the swarm, but the direction of the cumulative momentum of one or another fairly large part of the swarm could not differ cardinally for the parts that went to the formation of individual planets. Therefore, the momentums of the planets should be approximately parallel. This is the explanation for the coplanarity of the planetary orbits: all the planets move very close to the constant (Laplace) plane and in the same direction.
>
> Thus, all the laws of planetary orbits – motion almost in one plane, in one direction and almost in circles – are explained simply and naturally, based on the idea of the formation of planets by combining a very large number of bodies.

As mentioned in some recent works (see, for example, [232, 241, 270, 271]) within the last decade, a coherent scenario has arisen for the formation of the terrestrial planets from an initial dusty protoplanetary disk. The modern insight is the result of numerous *N*-body computational simulations of the Solar system formation (see [242, 247, 251, 272-277) and numerical high-resolution hydrodynamic as well as magneto-hydrodynamic calculations ([145, 256, 278–282]).

6.2.1. The distribution functions of moving particles in the gravitational field of a spheroidal body due to heat emission of particles in the outer protoplanetary shell under formation

As already noted, the main challenge of modern cosmogonical theories is the problem of the distribution of angular momentum in the Solar system: while the Sun constitutes more than 99% of the total mass of the Solar system (the total mass of all planets is equal to only 1/745 or 0.13% of the Sun's mass), it fits less than 2% of the total angular momentum only, that is, the remaining 98% belong to the planet exclusively [44 p. 277]. In our opinion [73], a possible explanation for the angular momentum removal to the periphery of the Solar system is caused by the thermal and gravitational instabilities of a forming spheroidal body. Indeed, the rapid increase of the gravitational field in the instable gravitating spheroidal body leads to disturbances of the particle distribution function of the rotating spheroidal body in passing from one virial equilibrium state to another (at the given temperature).

As shown in Sections 3.3 and 3.4, the *equilibrium* distribution functions (volume density of probability function $\Phi(h, z)$ and the distribution function of specific angular momentum on the distance $f(h)$) are described by formulas (3.3.22a) and (3.4.32) of the kind:

$$\Phi(h, z) = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2) \cdot e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}; \qquad (6.2.1a)$$

$$f(h) = \alpha(1 - \varepsilon_0^2) \cdot h e^{-\alpha(1-\varepsilon_0^2)h^2/2}. \qquad (6.2.1b)$$

Jeans found that under the transition to a new state of *virial equilibrium* it is possible that the temperature of gravitating gas masses (nebulae) increases due to the energy of the gravitational field [1 p.68]. This means that the unstable state of virial equilibrium can be violated *with the increasing temperature* of a spheroidal body. Conversely, the temperature increase leads to an increase in the kinetic energy of the thermal motion of particles, so that many of them acquire the mean square velocity of thermal movement becoming greater than the escape velocity from the spheroidal body, that is, $\sqrt{\bar{v^2}} \geq \sqrt{2|\varphi_g|}$. Leaving the spheroidal body these particles begin to move on Keplerian elliptical orbits (see [158] or (7.2.29) in the next Chapter 7):

$$r = \frac{a \cdot \sqrt{1 - e^2}}{1 + e \cdot \cos(\varepsilon - \varepsilon_*)}, \qquad (6.2.2)$$

where a is a major semi-axes of protoplanets, and e is an eccentricity of the orbit of a particle.

Using the formula of the gravitational potential of a rotating spheroidal body in a remote zone $r >> r_*$ (see formula (7.2.20) in Chapter 7 and [72, 73]):

$$\varphi_g(r,\theta_0)|_{r>>r_*} \approx -\frac{\gamma M}{r\sqrt{1-\varepsilon_0^2 \sin^2\theta_0}}, \qquad (6.2.3)$$

we can find the *critical velocity* of the escape of particles from this spheroidal body (excluding the potential of centrifugal force):

$$\mathrm{v}_c = \sqrt{2|\varphi_g|} = \frac{\sqrt{2\gamma M}}{\sqrt{r}\cdot\sqrt[4]{1-\varepsilon_0{}^2 \sin^2\theta_0}}. \qquad (6.2.4a)$$

If we take into account the potential of the centrifugal force $V_c = -(1/2)[\vec{\Omega}\times\vec{r}]^2$ [53, 95, 97], the escape velocity of particles from a rotating spheroidal body can be calculated through the potential of gravity $\psi_g = \varphi_g + V_c$ as follows:

$$\mathrm{v}_c = \sqrt{2\left(|\varphi_g| + V_c\right)} = \sqrt{2|\varphi_g(r,\theta)| - \Omega^2 r^2\cdot\sin^2\theta} =$$
$$= \sqrt{2|\varphi_g| - \Omega^2 h^2}\ . \qquad (6.2.4b)$$

In connection with this, we need to estimate the number of particles leaving a rotating spheroidal body (due to their thermal chaotic motions) and moving along Keplerian elliptical trajectories in its gravitational field. According to [1 p. 364] the particle distribution function of gas-dust protoplanetary cloud (system of stars) *in its gravitational field* obeys the Jeans equation. As shown by Jeans [1 p. 371] (see also Section 1.6), the joint distribution function of the spatial coordinates x, y, z and the velocity components u, υ, w for such particles is described by the expression (1.6.44):

$$\Phi(\vec{r},\vec{\mathrm{v}}) = \Phi_{\vec{r}}(x,y,z)\cdot\Phi_{\vec{\mathrm{v}}}(u,\upsilon,w) =$$
$$= a\cdot e^{2\beta m_0[-\varphi_g+\Omega^2(x^2+y^2)/2]}\cdot e^{-\beta m_0[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]} =$$
$$= b\cdot e^{-\beta m_0[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}\cdot c\cdot e^{-2\beta m_0[\varphi_g-\Omega^2(x^2+y^2)/2]}. \qquad (6.2.5)$$

Substituting the integration constants $b = (\beta m_0 / \pi)^{3/2}$ and $\beta = 1/2k_{\rm B}T$ into Eq. (6.2.5) in accordance with (1.6.48) and (1.6.52) gives us the form of the Jeans joint distribution function of spatial coordinates as well as velocity components for particles in a gravitational field:

$$\Phi(\vec{r},\vec{\rm v}) = \Phi_{\vec{r}}(x,y,z)\cdot\Phi_{\vec{\rm v}}(u,\upsilon,w) =$$

$$= c\cdot\left(\frac{m_0}{2\pi k_B T}\right)^{3/2}\cdot e^{-(m_0/2k_BT)\cdot[2\varphi_g - \Omega^2(x^2+y^2)]}\times$$

$$\times e^{-(m_0/2k_BT)\cdot[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}, \qquad (6.2.6)$$

where $k_{\rm B} = 1.38\cdot 10^{-23}\,{\rm J\cdot K^{-1}}$ is the Boltzmann constant. In particular, as follows from Eq. (6.2.6), the Jeans distribution function of the spatial coordinates of particles in the gravitational field of a uniformly rotating spheroidal body with the gravitational potential (5.4.11) at an interior point:

$$\varphi_g(h,z) = 2\pi\gamma\rho_0\frac{1-\varepsilon_0^2}{2(1-\varepsilon_0^2)^2+1}\left[(1-\varepsilon_0^2)^2h^2 + z^2 - \frac{4(1-\varepsilon_0^2)+2}{\alpha}\right]\approx$$

$$\approx -\frac{4\pi\gamma\rho_0}{\alpha} + \frac{2\pi\gamma\rho_0 r^2}{3}$$

becomes the following:

$$\Phi_{\vec{r}}(x,y,z) = c\cdot e^{-\left(\frac{m_0}{2k_BT}\right)\cdot[-\frac{8\pi\gamma\rho_0}{\alpha}+(\frac{4\pi\gamma\rho_0}{3})r^2-\Omega^2(x^2+y^2)]} =$$

$$= c\cdot e^{\left(\frac{4\pi\gamma m_0\rho_0}{\alpha k_BT}\right)}\cdot e^{-\left(\frac{2\pi\gamma m_0\rho_0}{3k_BT}\right)\cdot[r^2-(\frac{3\Omega^2}{4\pi\gamma\rho_0})\cdot h^2]}. \qquad (6.2.7)$$

Comparing Eq. (6.2.7) with the formula (6.2.1a) for the equilibrium probability volume density function to locate a particle in a uniformly rotating gravitating spheroidal body we can see their full identity allowing us to determine the basic parameters in the state of *virial equilibrium* of a rotating spheroidal body (see also Theorem 1.5, 2.2, 3.1 and 3.3):

$$\alpha = 2\pi\gamma m_0\rho_0/3k_BT; \qquad (6.2.8a)$$

$$\varepsilon_0^2 = 3\Omega^2 / 4\pi\gamma\rho_0 ; \tag{6.2.8b}$$

$$c = (\alpha / 2\pi)^{3/2} (1 - \varepsilon_0^2) \cdot e^{-6} . \tag{6.2.8c}$$

Assuming $\rho_0 = M / (4\pi r_*^3 / 3)$, $r_* = 1/\sqrt{\alpha}$ the formulas (6.2.8a) and (6.2.8b) obtain the following:

$$\alpha = \gamma m_0 M / 2k_B T r_*^3 ; \tag{6.2.9a}$$

$$\varepsilon_0^2 = \Omega^2 r_*^3 / \gamma M . \tag{6.2.9b}$$

As follows from (6.2.9a), the parameter of gravitational condensation α is directly proportional to the potential energy $E_{g_{\text{int}}}(r_*) = \gamma m_0 M / r_*$ of a particle in the gravitational field at distance r_* from the center and inversely proportional to the temperature T of a rotating spheroidal body in the virial equilibrium state. Let us note that if $\varepsilon_0^2 = 1$ in (6.2.9b) then $r_* = (\gamma M / \Omega^2)^{1/3} = h_0$ is the radius of the critical (*cross-section*) equipotential surface in Roche's model [1 p. 252] (see also Section 1.4, (1.4.25b)), therefore $\varepsilon_0^2 = r_*^3 / h_0^3 < 1$ as a rule, that is, Eq. (6.2.9b) is valid.

We will now estimate the number $\aleph_1$ of particles leaving a rotating spheroidal body being in a state of virial equilibrium with temperature T. According to the Poincaré virial theorem [1 p. 68, 105] (see also Section 1.2) applying to the gravitating spheroidal body as a cloud-like configuration of ideal gas, the mean value of square velocity of thermal particle motion $\overline{v^2}$, averaged over all the separate masses, is equal to half the average value of the gravitational potential $-\frac{1}{2}\overline{\varphi}_{\text{g}}$ of gaseous cloud in the steady state (see Theorem 1.3), or the absolute value of average potential energy of interaction $\overline{E}_{\text{g}}$ of a particle is equal to the double average kinetic energy $\overline{E}_{\text{k}}$ of a moving particle. Using the Maxwell velocity distribution law [110] we can find the share of particles whose velocities are close to the given speed v of the heat motion:

$$\frac{dN_{\mathrm{v}}}{N}=\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\cdot e^{-\frac{m_0\,\mathrm{v}^2}{2k_{\mathrm{B}}T}}. \qquad (6.2.10a)$$

However, the magnitude of the resulting velocity of the moving particle into a uniformly rotating spheroidal body consists of orbital velocity $\left|[\vec{\Omega}\times\vec{r}]\right|=\Omega\cdot h$ and heat speed v, so that when we estimate the number $\aleph_1$ of particles leaving a rotating spheroidal body at temperature T, we should use the Jeans distribution function of velocity components $\Phi_{\vec{\mathrm{v}}}(u,\upsilon,w)$ incoming in Eq. (6.2.6) instead of Eq. (6.2.10a):

$$\frac{dN_{\mathrm{v}}}{N}=\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\cdot e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}=$$

$$=\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\cdot e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[(u^2+\upsilon^2+w^2)+2\Omega(\upsilon x-uy)+\Omega^2(x^2+y^2)]}=$$

$$=\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\cdot e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[\mathrm{v}^2+2\Omega\dot{\varepsilon}h^2+\Omega^2h^2]}=$$

$$=\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\cdot e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[\mathrm{v}^2+(2\Omega\dot{\varepsilon}+\Omega^2)h^2]}. \qquad (6.2.10b)$$

When the critical escape velocity $\mathrm{v}\geq\sqrt{2|\varphi_g|}$, according to (6.2.4a), the number $\aleph_1$ of particles leaving a spheroidal body due to thermal chaotic motion can be expressed by the formula [73]:

$$\aleph_1=\int_0^{\aleph_1}dN_{\mathrm{v}}=N\left(\frac{m_0}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\iiint\limits_{\sqrt{(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2}\geq\sqrt{2|\varphi_{\mathrm{g}}|}} e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[(u-\Omega y)^2+(\upsilon+\Omega x)^2+w^2]}du\,d\upsilon\,dw=$$

$$= N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} \iiint\limits_{\sqrt{u'^2+\upsilon'^2+w^2}\geq\sqrt{2|\varphi_g|}} e^{-\frac{m_0}{2k_B T}\cdot[u'^2+\upsilon'^2+w^2]} du'd\upsilon'dw =$$

$$= 4\pi\cdot N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2} \int\limits_{v'\geq\sqrt{2|\varphi_g|}} e^{-\frac{m_0}{2k_B T}\cdot v'^2}\, v'^2\, d\,v' =$$

$$= N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2}\cdot 4\pi \int\limits_{\sqrt{2|\varphi_g|}}^{\infty} v'^2\, e^{-\frac{m_0}{2k_B T}\cdot v'^2}\, d\,v' =$$

$$= N\left(\frac{m_0}{2\pi k_B T}\right)^{3/2}\cdot 4\pi\cdot\frac{k_B T}{m_0}\left[\int\limits_{\sqrt{2|\varphi_g|}}^{\infty} e^{-\frac{m_0}{2k_B T}\cdot v'^2}\, d\,v' - v'\cdot e^{-\frac{m_0}{2k_B T}\cdot v'^2}\Bigg|_{\sqrt{2|\varphi_g|}}^{\infty}\right] =$$

$$= 2N\cdot\sqrt{\frac{m_0}{2\pi k_B T}}\cdot\left[\int\limits_{\sqrt{2|\varphi_g|}}^{\infty} e^{-\frac{m_0}{2k_B T}\cdot v'^2}\, d\,v' + \sqrt{2|\varphi_g|}\cdot e^{-\frac{m_0|\varphi_g|}{k_B T}}\right]. \qquad (6.2.11)$$

Let us estimate the number of particles having the critical thermal velocity $v_c = \sqrt{2|\varphi_g|}$ in the case of the Solar system. Taking into account that the Sun and the solar corona were formed mainly of hydrogen atoms with the mass $m_{0\,H} = 1.6734\cdot 10^{-27}$ (kg) [206], the temperature of the solar corona is approximately $T \approx 1.5\cdot 10^6$(K) [283], the radius of the solar disk is equal to $R = 6.955\cdot 10^8$ (m), the thickness of the visible part of the solar corona is estimated by the value $\Delta = 2R$ (see Section 7.3 in the next Chapter 7) and, therefore, $r_* = R + \Delta = 3R = 2.0865\cdot 10^9$ (m) [76], the squared geometric eccentricity of the solar disk with a visible part of the solar corona is $\varepsilon_0^2 \approx 1.799992\cdot 10^{-5}$ in accordance with formula (7.3.42) from Section 7.3 (see also [72, 73]), we can estimate the values in formula (6.2.11):

$$\mathrm{v_c} = \sqrt{2\left|\varphi_g\right|}\Bigg|_{\theta_0 \to \frac{\pi}{2}} = \frac{\sqrt{2\gamma M}}{\sqrt{r_*}\cdot\sqrt[4]{1-\varepsilon_0{}^2}} \approx 3.5651\cdot 10^5\ \text{(m/s)}; \qquad (6.2.12a)$$

$$\sqrt{\frac{m_0}{2\pi k_\mathrm{B} T}} \approx 3.5869\cdot 10^{-6}\ \text{(s/m)}; \qquad (6.2.12b)$$

$$\frac{m_0\left|\varphi_g\right|}{k_\mathrm{B}T} = \frac{m_0\,\mathrm{v}_c{}^2}{2k_\mathrm{B}T} \approx 5.1375\,. \qquad (6.2.12c)$$

To estimate the value of the integral in Eq. (6.2.11) in the case of the Solar system we transform it into the form [73]:

$$\aleph_1 = 2N\cdot\sqrt{\frac{m_0}{2\pi k_\mathrm{B}T}}\cdot\left[\int\limits_{\sqrt{2|\varphi_g|}}^{\infty} e^{-\left(\mathrm{v}'\sqrt{\frac{m_0}{k_\mathrm{B}T}}\right)^2\Big/2} d\,\mathrm{v}' + \sqrt{2\left|\varphi_g\right|}\cdot e^{-\frac{m_0|\varphi_g|}{k_\mathrm{B}T}}\right] =$$

$$= 2N\cdot\left[\frac{1}{\sqrt{2\pi}}\int\limits_{\sqrt{2|\varphi_g|m_0/k_\mathrm{B}T}}^{\infty} e^{-s^2/2}ds + \sqrt{\frac{\left|\varphi_g\right|m_0}{\pi k_\mathrm{B}T}}\cdot e^{-\frac{m_0|\varphi_g|}{k_\mathrm{B}T}}\right] =$$

$$= 2N\cdot\left[\frac{1}{\sqrt{2\pi}}\int\limits_{\mathrm{v_c}\sqrt{m_0/k_\mathrm{B}T}}^{\infty} e^{-s^2/2}ds + \mathrm{v_c}\sqrt{\frac{m_0}{2\pi k_\mathrm{B}T}}\cdot e^{-\frac{m_0\,\mathrm{v_c}^2}{2k_\mathrm{B}T}}\right] \qquad (6.2.13a)$$

and then use the values of (6.2.12a, b) and the table to calculate the probability $\Pr[s > s_*] = \frac{1}{\sqrt{2\pi}}\int\limits_{s_*}^{\infty} e^{-s^2/2}ds$ that the observed value of a standard normal random variable exceeds the preassigned value [284 p.500] when $s^* = \mathrm{v}_c\sqrt{m_0/k_\mathrm{B}T} = 3.2054693$:

$$\aleph_1 \approx 2N\cdot\left[\frac{1}{\sqrt{2\pi}}\int\limits_{3.2054693}^{\infty} e^{-s^2/2}ds + 1.278781\cdot e^{-5.14}\right] \approx$$

$$\approx 2N\cdot\left[0.000674212 + 1.278781\cdot 0.00585769\right] =$$

$$= 2N \cdot [0.0006742 + 0.0074907] \approx 0.0082 \cdot N . \qquad (6.2.13b)$$

Thus, according to (6.2.13b) in the virial equilibrium state at the temperature $T \approx 1.5 \cdot 10^6\,(\mathrm{K})$, the number $\aleph_1$ of particles leaving the solar corona by heat emission can be up to 0.8% of the total number N of particles of the Solar system. This qualitative estimation is entirely consistent with the above-mentioned fact that more than 99% of the total mass of the Solar system is concentrated in the Sun. Let us note that the part $\aleph_1$ of all particles leaving a spheroidal body to collide with the $\aleph_2$ particles of the *peripheral region* of the rotating spheroidal body. As a result, some particles leaving the spheroidal body come back. So, only a small fraction of the particles that have left can be moving on elliptical orbits in the gravitational field of a spheroidal body.

By analogy with Eq. (6.2.13b) the number $\aleph_2$ of particles from the peripheral region is also easily estimated through the distribution function of specific angular momentum (6.2.1b) and the maximum point value $h = h_* = 1/\sqrt{\alpha(1-\varepsilon_0^2)}$ of this distribution function [73]:

$$\aleph_2 = \int_0^{\aleph_2} dN_h = N\int_{h_*}^{\infty} f(h)dh = N\int_{h_*}^{\infty} \alpha(1-\varepsilon_0^2)\cdot he^{-\alpha(1-\varepsilon_0^2)h^2/2}\,dh =$$

$$= N \int_{\alpha(1-\varepsilon_0^2)h_*^2/2}^{\infty} e^{-\alpha(1-\varepsilon_0^2)h^2/2}\, d[\alpha(1-\varepsilon_0^2)h^2/2] = Ne^{-\alpha(1-\varepsilon_0^2)h_*^2/2} =$$

$$= Ne^{-1/2} \approx 0.60653N . \qquad (6.2.14)$$

Because $\aleph_2 >> \aleph_1$ according to (6.2.13b) and (6.2.14), the scattering of $\aleph_1$ particles leaving the spheroidal body due to the thermal emission on $\aleph_2$ particles from the periphery of rotating spheroidal body takes place, leading to a reduction in the number of particles really leaving the spheroidal body (in the case of the Solar system from 0.8% to 0.13% of the total number of particles).

The Jeans distribution function (6.2.10b) of particles leaving the spheroidal body can be regarded as a *joint distribution function* of both components of the thermal velocity $\vec{\mathrm{v}} = (u, \upsilon, w)$ and the coordinates h, ε of the plane motion (or components of the orbital velocity $[\vec{\Omega}_1 \times \vec{r}]$, where $\vec{\Omega}_1{}^2 = 2\Omega\dot{\varepsilon} + \Omega^2$) for particles *in the gravitational field* of a spheroidal body [73]:

$$\Phi(\vec{\mathrm{v}}, [\vec{\Omega}_1 \times \vec{r}]) = \left(\frac{m_0}{2\pi k_{\mathrm{B}} T}\right)^{3/2} e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot\vec{\mathrm{v}}^2} \cdot A e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[2\Omega\dot{\varepsilon}+\Omega^2]h^2} =$$

$$= \left(\frac{m_0}{2\pi k_{\mathrm{B}} T}\right)^{3/2} e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot\vec{\mathrm{v}}^2} \cdot A e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[\vec{\Omega}_1\times\vec{r}]^2}, \qquad (6.2.15)$$

after carrying out the respective normalization of this function to find A. Thus, the joint distribution function of particles in the gravitational field of a spheroidal body can be written as

$$\Phi(\vec{\mathrm{v}}, [\vec{\Omega}_1 \times \vec{r}]) = \left(\frac{m_0}{2\pi k_{\mathrm{B}} T}\right)^{3/2} e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot\mathrm{v}^2} \cdot A e^{-\frac{m_0\Omega_1{}^2}{2k_{\mathrm{B}}T}\cdot h^2} =$$

$$= \Phi(\vec{\mathrm{v}}) \cdot \Phi([\vec{\Omega}_1 \times \vec{r}]) . \qquad (6.2.16a)$$

Moreover, the normalization condition of the function $\Phi([\vec{\Omega}_1 \times \vec{r}])$ looks like a two-dimensional analog of the normalization condition of the function $\Phi(\vec{\mathrm{v}})$:

$$\int_0^\infty\int_0^\infty \Phi([\vec{\Omega}_1 \times \vec{r}]) d[\vec{\Omega}_1 \times \vec{r}]_x d[\vec{\Omega}_1 \times \vec{r}]_y =$$

$$= A \cdot \int_0^\infty\int_0^{2\pi} e^{-\frac{m_0\Omega_1{}^2}{2k_{\mathrm{B}}T}\cdot h^2} \Omega_1 h d(\Omega_1 h) d\varepsilon = 1, \qquad (6.2.16b)$$

whence it follows that:

$$A = \frac{m_0}{2\pi k_{\mathrm{B}} T}. \qquad (6.2.16c)$$

Taking into account (6.5.16a)–(6.5.16c) the distribution function of the particle components of the orbital velocity $[\vec{\Omega}_1 \times \vec{r}]$, where $\left|\vec{\Omega}_1\right| = \sqrt{\Omega^2 + 2\Omega\dot{\varepsilon}}$, can be written as [73]:

$$\Phi([\vec{\Omega}_1 \times \vec{r}]) = \frac{m_0}{2\pi k_{\mathrm{B}} T} \cdot e^{-\frac{m_0}{2k_{\mathrm{B}}T}\cdot[\vec{\Omega}_1 \times \vec{r}]^2} = \frac{m_0}{2\pi k_{\mathrm{B}} T} \cdot e^{-\frac{m_0 \Omega_1^{\,2}}{2k_{\mathrm{B}}T}\cdot h^2}. \qquad (6.2.17a)$$

In a state of the relative mechanical equilibrium of orbiting particles in the gravitational field of a spheroidal body, we assume that $\dot{\varepsilon} = \Omega \cdot (a/r)^2, \dot{\varepsilon} \to \Omega$, that is, $\vec{\Omega}_1^{\,2} \to 3\vec{\Omega}^2$, so that the formula (6.2.17a) becomes the following [73]:

$$\Phi([\vec{\Omega}_1 \times \vec{r}]) = \frac{m_0}{2\pi k_{\mathrm{B}} T} \cdot e^{-\frac{3m_0 \Omega^2}{2k_{\mathrm{B}}T}\cdot h^2} = \frac{m_0}{2\pi k_{\mathrm{B}} T} \cdot e^{-\frac{3m_0}{2k_{\mathrm{B}}T}\cdot[\vec{\Omega} \times \vec{r}]^2}. \qquad (6.2.17b)$$

As we know [158] that the projection of angular momentum L_0 on the axis $\vec{n}$ (passing through the center) is preserved under the motion of a particle with a mass m_0 in the field with central symmetry:

$$L_{0\vec{n}} = m_0 h^2 \dot{\varepsilon} = \mathrm{const}.$$

The value of the normal projection of the specific angular momentum is then equal to

$$\lambda = L_{0\vec{n}} / m_0 = \dot{\varepsilon}\, h^2.$$

In the state of *relative mechanical equilibrium* of individual particles moving in elliptical orbits (in the gravitational field of a spheroidal body with a constant angular velocity $\Omega_1 = \dot{\varepsilon}_1 = \mathrm{const}$ around the axis Oz) and having a specific angular momentum value $\lambda = \Omega_1 h^2$, the equation (6.2.17a) can be transformed to find the distribution function of the specific angular momentum as a result of heat emission, that is, briefly speaking, the *heat* (*thermal*) *distribution function* of the specific angular momentum $f_T(\lambda)$. For this we note that the share $dN_\lambda / \aleph_1$ of particles rotating in elliptical orbits with a constant angular velocity $\Omega_1 = \dot{\varepsilon}_1 = \text{const}$ around the axis Oz and having a specific angular momentum in the interval $[\lambda,\ \lambda + d\lambda]$ is equal to the share $dN_{\left|[\vec{\Omega}_1 \times \vec{r}]\right|} / \aleph_1$ of particles moving due to thermal emission with the orbital velocities close to $\left|[\vec{\Omega}_1 \times \vec{r}]\right| = \Omega_1 \cdot h$ which can be estimated using (6.2.17a) as follows [73]:

$$f_T(\lambda)d\lambda = \frac{dN_\lambda}{\aleph_1} = \frac{dN_{\Omega_1 \cdot h}}{\aleph_1} = \frac{m_0}{2\pi k_B T} \cdot \int_0^{2\pi} e^{-\frac{m_0 \Omega_1{}^2}{2k_B T} \cdot h^2} \Omega_1 h d(\Omega_1 h) d\varepsilon =$$

$$= \frac{m_0 \Omega_1}{2k_B T} \cdot e^{-\frac{m_0 \Omega_1{}^2}{2k_B T} \cdot h^2} d(\Omega_1 h^2) = \frac{m_0 \Omega_1}{2k_B T} \cdot e^{-\frac{m_0 \Omega_1}{2k_B T} \cdot \lambda} d\lambda, \qquad (6.2.18a)$$

whence we can see that:

$$f_T(\lambda) = \frac{m_0 \Omega_1}{2k_B T} \cdot e^{-\frac{m_0 \Omega_1}{2k_B T} \cdot \lambda}. \qquad (6.2.18b)$$

Furthermore, we can calculate the average specific angular momentum as a result of heat emission through integration by parts [73]:

$$\overline{\lambda_T} = \int_0^\infty \lambda\ f_T(\lambda) d\lambda = \lambda \int f_T(\lambda) d\lambda \Big|_0^\infty - \int_0^\infty d\lambda \int f_T(\lambda) d\lambda =$$

$$= -\lambda \cdot e^{-m_0 \Omega_1 \lambda / 2k_B T} \Big|_0^\infty - \int_0^\infty (-e^{-m_0 \Omega_1 \lambda / 2k_B T}) d\lambda =$$

$$= \frac{2k_B T}{m_0 \Omega_1} \int_0^\infty f_T(\lambda) d\lambda = \frac{2k_B T}{m_0 \Omega_1}. \qquad (6.2.19)$$

According to Eq. (6.2.18a), the number of particles having values of the specific angular momentum close to λ resulting from thermal emission is equal to:

$$dN_\lambda = \aleph_1 \cdot \frac{m_0\Omega_1}{2k_B T} \cdot e^{-m_0\Omega_1\lambda/2k_B T} d\lambda = \aleph_1 f_T(\lambda) d\lambda \,. \qquad (6.2.20)$$

Starting from (6.2.20) it is easy to calculate the total angular momentum of the rotating *outer shell* formed by particles leaving the spheroidal body due to thermal emission in relative mechanical equilibrium [73]:

$$L_T = \int_0^{\aleph_1} m_0 \lambda \, dN_\lambda = m_0 \aleph_1 \int_0^\infty \lambda \, f_T(\lambda) d\lambda = m_0 \aleph_1 \overline{\lambda}_T \,, \qquad (6.2.21)$$

where $\aleph_1$ is a number of particles leaving the spheroidal body due to thermal chaotic motion. Substituting (6.2.19) into (6.2.21) we find that the value of total angular momentum of the uniformly rotating outer shell formed by the particles that have left as a result of heat emission is expressed by the formula [73]:

$$L_T = \frac{2k_B T \aleph_1}{\Omega_1} = \frac{2k_B T \iota N}{\Omega_1} = \frac{2\iota k_B T M}{m_0 \Omega_1} \,, \qquad (6.2.22)$$

where $\iota = \aleph_1 / N$ is a share of the total number of particles leaving the spheroidal body due to thermal chaotic motion (according to Eq. (6.2.13b) $\iota \approx 0.0082$ for the Solar system).

For comparison, we note that the value of the total angular momentum of a uniformly rotating spheroidal body is given by formula (3.4.14) from Section 3.4 [16 p.1493]:

$$L = \frac{2\Omega M}{\alpha(1-\varepsilon_0^2)} \,. \qquad (6.2.23)$$

Let us compare the value of the total angular momentum (6.2.23) of a uniformly rotating spheroidal body with the total angular momentum value (6.2.22) of a uniformly rotating outer shell (protoplanetary cloud) in the case of the Solar system. As in formula (6.2.17b), we suppose that $\dot{\varepsilon} \to \Omega$, $\dot{\varepsilon} = \Omega \cdot (a/r)^{3/2}$ in a state of relative mechanical equilibrium of a rotating protoplanetary cloud in the gravitational field of the Sun, so that $\Omega_1 = \sqrt{\Omega^2 + 2\Omega\dot{\varepsilon}} \to \Omega\sqrt{3}$.

Taking into account that the Sun was formed mainly from hydrogen atoms with mass $m_{0\mathrm{H}} = 1.6734 \cdot 10^{-27}$ (kg) [206], the average temperature T of Sun is approximately $1.47 \cdot 10^7\,(\mathrm{K})$ [1], the angular speed of rotation of the externally visible layers of the Sun (at the equator) is equal to $\Omega \approx 2.91 \cdot 10^{-6}$ (s^{-1}) [283], the parameter of gravitational condensation of spheroidal body in the case of the Sun is estimated by the value $\alpha \approx 2.297 \cdot 10^{-19}\,(\mathrm{m}^{-2})$ (see formula (7.3.41) from Section 7.3 of Chapter 7 as well as Chapter 8 and [76 p. 13]), the square geometric eccentricity of the solar disk is $\varepsilon_0^2 \approx 1.799992 \cdot 10^{-5}$ (see formula (7.3.42) from Section 7.3 and [72, 73]), we can estimate the values of the total angular momentums of the protoplanetary cloud and the Sun based on Eqs (6.2.22) and (6.2.23):

$$L = \frac{2\Omega M_{\mathrm{Sun}}}{\alpha(1-\varepsilon_0^2)} = \frac{2 \cdot 2.91 \cdot 10^{-6}}{2.297 \cdot 10^{-19} \cdot (1 - 1.799992 \cdot 10^{-5})} \cdot M_{\mathrm{Sun}} \approx$$

$$\approx 2.534 \cdot 10^{13} \cdot M_{\mathrm{Sun}}\,(\mathrm{kg \cdot m^2/s}); \tag{6.2.24a}$$

$$L_T = \frac{2\iota k_B T M_{\mathrm{Sun}}}{m_0 \Omega_1} \propto \frac{2 \cdot 1.38 \cdot 10^{-23} \cdot 1.47 \cdot 10^7 \cdot 0.0082}{\sqrt{3} \cdot 1.6734 \cdot 10^{-27} \cdot 2.91 \cdot 10^{-6}} \cdot M_{\mathrm{Sun}} \approx$$

$$\approx 3.945 \cdot 10^{14} \cdot M_{\mathrm{Sun}}\,(\mathrm{kg \cdot m^2/s}), \tag{6.2.24b}$$

where M_{Sun} is the mass of the Sun. These quality evaluations (6.2.24a) and (6.2.24b) provide useful although rough estimations of the total angular momentums of the Sun and the protoplanetary cloud showing that

$$L_T / L \propto 15.57\,. \tag{6.2.25}$$

What is surprising is the fact that only 0.8% of the total number of particles of the Solar system composing the protoplanetary cloud have the angular momentum that is 15.6

times higher than the angular momentum of the remaining 99% [73].

Thus, according to the estimations (6.2.24a, b), only 6% of the total angular momentum belongs to the Sun, and the remaining 94% fits on the protoplanetary cloud. These qualitative estimations in the case of the Solar system, in general, confirm the known fact of nonuniform distribution of the angular momentum noted by Ter Haar [7, 32]: only 2% of the total angular momentum belongs to the Sun, while the remaining 98% belongs to the planets. They also point to the possibility of usage of the model considered above of "removal" of the maximal specific angular momentum from spheroidal body by particles due to their thermal emissions.

A more appropriate model of removal of angular momentum by particles as a result of the thermal emission from the spheroidal body and, therefore, more accurate estimations of the total angular momentum of Sun and the protoplanetary cloud can be obtained by taking into account *the multiple of states of virial equilibrium* of a spheroidal body at different temperatures T_l which is also confirmed by numerical modeling of the protostellar hydrodynamic collapse of stars under their formation [144, 145].

Let us note that, according to (3.8.7) from Section 3.8, there is a representation of the mass density of an oblate spheroidal body:

$$\rho^{(n)}(r,\theta) = \sum_{i=0}^{n} M_i \cdot (\alpha/2\pi)^{\frac{3}{2}} \prod_{l=0}^{n-1} (1-\varepsilon_l^2) \cdot e^{\frac{-\alpha r^2 (1-\varepsilon_l^2 \sin^2\theta)}{2}}, \quad (6.2.26)$$

which then leads to a factorization of the volume probability density $\Phi^{(n)}(r,\theta)$ since the latter is related to the mass density $\rho^{(n)}(r,\theta) = \sum_{i=0}^{n} M_i \cdot \Phi^{(n)}(r,\theta)$:

$$\Phi^{(n)}(r,\theta) = (\alpha/2\pi)^{3/2} e^{-\alpha r^2/2} \prod_{l=0}^{n-1} (1-\varepsilon_l^2) \cdot e^{\alpha r^2 \varepsilon_l^2 \sin^2\theta/2}. \quad (6.2.27)$$

According to (6.2.27), the volume probability density $\Phi^{(n)}(r,\theta)$ is represented as a product of the following factors:

$$\begin{aligned}
&\Phi^{(0)}(r)=(\alpha/2\pi)^{3/2}e^{-\alpha r^2/2}\ ;\\
&\varphi_0(r,\theta)=(1-\varepsilon_0^2)\cdot e^{\varepsilon_0^2\alpha r^2\sin^2\theta/2}\ ;\\
&\varphi_1(r,\theta)=(1-\varepsilon_1^2)\cdot e^{\varepsilon_1^2\alpha r^2\sin^2\theta/2}\ ;\\
&\ldots\ldots\ldots\ldots\ldots\\
&\varphi_{n-1}(r,\theta)=(1-\varepsilon_{n-1}^2)\cdot e^{\varepsilon_{n-1}^2\alpha r^2\sin^2\theta/2}\ .
\end{aligned}\tag{6.2.28}$$

We can note that the obtained distribution function (6.2.17a) of the components of orbital velocity for the particles from the *peripheral region* remains entirely a flattening cofactor $\phi_l=\varphi_l^{-1}$ of the probability volume density (6.2.28) of a spheroidal body under its rotation:

$$\phi_l(h)=\varphi_l^{-1}(h)=(1-\varepsilon_l{}^2)^{-1}\cdot e^{-\varepsilon_l{}^2\alpha h^2/2},\tag{6.2.29a}$$

whence:

$$\Phi^{(0)}(h,z)=\phi_0(h)\Phi^{(1)}(h,z)\,.\tag{6.2.29b}$$

This is the first stage of factorization of the probability volume density of an *immovable* spheroidal body being in the state virial equilibrium at the temperature T_1, that is, the first step of the flattening of a spheroidal body due to its rotation. In fact, the l-th factorization step of the probability volume density of a rotating spheroidal body :

$$\Phi^{(0)}(h,z)=\phi_0(h)\ldots\phi_{l-1}(h)\Phi^{(l)}(h,z)\ ,\tag{6.2.30a}$$

Means the l-th step of the flattening of rotating spheroidal body under transition to the state of virial equilibrium at the temperature T_l when:

$$\lim_{\substack{\varepsilon_l^2\to 1\\ l\to\infty}}\Phi^{(l)}(h,z)=\Phi_0(h)e^{-\alpha z^2/2}\,,\tag{6.2.30b}$$

where $\Phi_0(h)$ is a value of probability volume density in a *central* flat of the oblate spheroidal body representing a gaseous disk. Moreover, according to the known *barometric formula* (3.8.37): $\rho(z) = \rho_0 \cdot e^{-3\Omega^2 z^2 / 2\overline{v^2}}$ [2 p.36, 12], the mass density $\rho(z)$ of a protoplanetary flattened gaseous (gas-dust) disk has the same cofactor $3\Omega^2 / \overline{v^2} = m_0\Omega^2 / k_B T$ in the argument of the exponential function as in formula (6.2.17a). This means a new sense of the parameter of gravitational condensation $\alpha = 3\Omega^2 / \overline{v^2}$ applied to *the external gas-dust bunches* originating as a result of the thermal emission of particles from a rotating spheroidal body at its transition from one virial equilibrium state to another. We also note that the definition of the meaning of the parameter of gravitational condensation of a spheroidal body in a state of virial equilibrium, according to (6.2.8a), as well as its square geometric eccentricity (6.2.8b) allows us to describe the cofactor (6.2.29a) of l-th step factorization of the probability volume density $\Phi^{(l)}(h,z)$ of an oblate spheroidal body as the following:

$$\phi_l\ (h) = (1 - 3\Omega^2 / 4\pi\gamma\rho_0)^{-1} \cdot e^{-(m_0\Omega^2 / 2k_B T) h^2 / 2} \quad . \qquad (6.2.30c)$$

Thus, under the separation of a spheroidal body to the core and the outer shells due to the thermal emission of particles and gravitational (rotational) instabilities, the physical meaning of the values describing a spheroidal body as a whole can vary significantly concerning the outer gas-dust shells of a forming protoplanetary cloud.

6.2.2. An application of a statistical model of particles moving in the gravitational field of a spheroidal body due to heat emission to the problems of the formation of exoplanetary systems

The discovery of a planet orbiting the star 51 Pegasi (Mayor & Queloz, 1995) marked the birth of a new field of astronomy, the study of extrasolar planetary systems around Main Sequence stars. Since then, many planets outside our own Solar system have been discovered [285]. These planets most closely resemble the gas giant planets, with masses in the range 20–3000 M_{Earth}, where M_{Earth} is the mass of Earth, but many of them are either in highly eccentric or very small (0.1–50.02 AU) orbits. The latter have surface temperatures up to 2,000 K, and are hence known as "Hot Jupiters." The existence of Hot Jupiters can be explained by the inward migration of planets formed at larger distances from their star, most likely due to tidal interactions with the circumstellar disk.

Studies have revealed that Hot Jupiters preferentially form around higher metallicity stars; almost 15% of solar-type stars with metallicity greater than 1/3 that of the Sun possess at least one planet of Saturn mass or larger, and the lowest mass exoplanets range from 5 to 7 M_{Earth}. Moreover, many of the moving extrasolar planets have large eccentricities and angles of inclination in their orbits, which together comprise the main distinguishing features of exoplanetary systems in comparison with our Solar system. However, at present, the methods of observational extrasolar planetology continue to develop and improve, so that soon more and more extrasolar planets with dimensions and masses similar to our Earth will be observable. Recently, two extrasolar planets with masses and radii close to those of the Earth have been discovered: CoRoT-7b [286, 287] and GJ 1214 b [288].

High-precision observations of the radial velocity of motion now allow us to find extrasolar planets with minimal masses of the order 1.9 M_{Earth} [289]. Preliminary results of surveys based on the HARPS spectrograph revealed a large population of planets like Neptune and super-Earths at distances of approximately 0.5 AU from stars of the solar type. Moreover, hundreds of "potential planets" of small radii have been also announced by a group of researchers known as "Kepler" [290]. So, the study of populations of extrasolar planets of small masses has become the main direction of research for the coming years.

A high-precision radial-velocity survey of approximately 400 of the brightest stars adjacent to the Sun is currently being carried out using the HARPS spectrograph [291]. Observation results have already shown the presence of small orbital objects, that is, extrasolar planets of small mass around a series of stars: HD 160691 [292, 293], HD 69830 [294], HD 4308 [295], HD 40307 [296], HD 47186, HD 181433 [297] and HD 90156.

In 2010, an international team of astronomers revealed traces of the presence of extrasolar planets through oscillations of the Sun-like star HD 10180 using the HARPS spectrograph at the ESO in La Silla [231]. At least five extrasolar planets have been found around the star HD 10180, resembling Neptune and moving in almost circular orbits: their masses range from 13 to 25 times the Earth's mass (M_{Earth}), and the period of their orbital motion varies from 6 to 600 days. The distance from them to the star HD 10180 is respectively from 0.06 to 1.4 AU.

From an observational point of view, it should be stated that exoplanetary systems exhibit a huge variety of their properties indicating complex processes of their formation. Figure 6.3 shows the values of the major semi-axes of ellipsoidal orbits of extrasolar planets, that is, distances (in AU) from them to the star, for 15 exoplanetary systems (including our Solar system) having more than three extrasolar planets [231]. Using data on these exoplanetary systems, this section investigates the Titius–Bode law (or its analog) for exoplanetary systems reflecting the general laws of the formation of extrasolar planets.

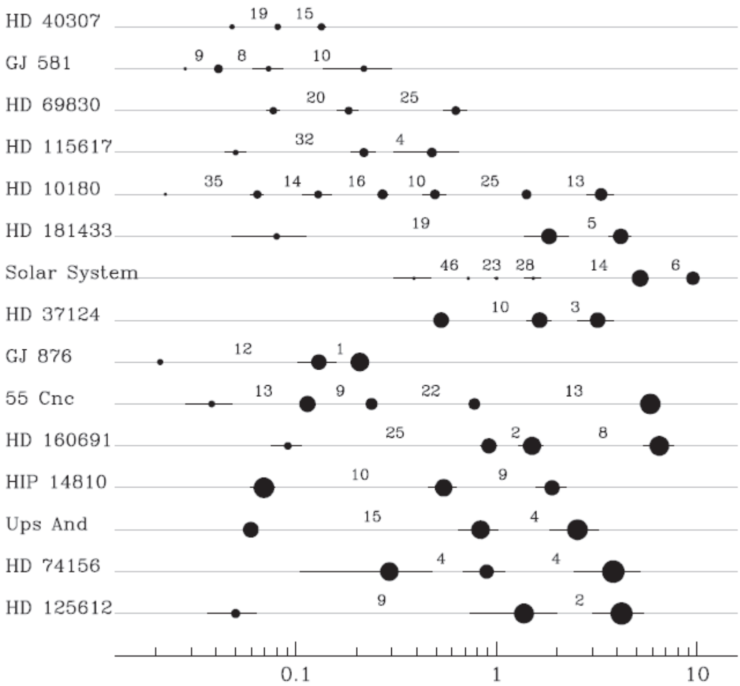


Figure 6.3. The values of major semi-axes a_n in logarithmic scale (AU) of the ellipsoidal orbits of extrasolar planets for the 15 planetary systems with at least three known planets as of May 2010 (the numbers give the minimal distance between adjacent planets expressed in mutual Hill radii) [231]

As already noted here, in the exoplanetary systems that have been studied, the moving extrasolar planets often have more significant eccentricities e and angles of inclination i of their orbits in comparison with the planets of the Solar system. This means that the statistical model of moving particles proposed in the previous subsection 6.2.1 due to *thermal emission* in the gravitational field of a spheroidal body can be applied to solving the problems of the formation of extrasolar planets.

According to (6.2.13b), in the state of virial equilibrium of a certain Sun-like star (at its corona temperature $T \approx 1.5 \cdot 10^6 (\mathrm{K})$), the number of particles $\aleph_1$ leaving this star as a result of thermal emission and moving along inclined elliptical orbits in its gravitational field can be up to 0.8% of the total number N of particles of the exoplanetary system. Moreover, as noted in Section 5.4 and subsection 6.2.1 when a spheroidal body is divided as a result of the thermal emission of particles or gravitational (rotational) instabilities, the physical meaning of values describing the spheroidal body as a whole can change *after its decay* and the formation of the external gas-dust protoplanetary shells. Indeed, from the reasoning in Section 7.2 (see formula (7.2.43)), it immediately follows that the double areal velocity C of the orbital motion of a particle in the gravitational field of a rotating spheroidal body completely coincides with the value of the specific angular momentum of the particle λ:

$$C = \lambda = \frac{\sqrt{\gamma M a (1 - e^2)}}{\sqrt[4]{1 - {\varepsilon_0}^2 \sin^2 \theta_0}}. \tag{6.2.31}$$

Let us note that in Sections 3.4, 6.1.1, and 6.1.2, the variable λ was used to denote the value of specific angular momentum *inside a uniformly rotating spheroidal body* in the state of relative mechanical equilibrium, in particular, the average specific angular momentum $\overline{\lambda}$ has been calculated by the formula (3.4.28). In this regard, to denote the value of the specific angular momentum of a particle *in the gravitational field of a rotating spheroidal body*, it is advisable to use the variable C instead of λ. Such a difference in notation is related to the fundamental differences in the nature of their origin, as well as in the quantities and directions characterizing the *orbital* angular momentum of a particle and the *inner* angular momentum of a particle in a uniformly rotating spheroidal body (for example, the average specific orbital angular momentum $\overline{C}$ of a *stream of particles* cannot be found by formula (3.4.28) since, instead of L and M, L_T and $m_0\aleph_1$ should be used following (6.2.21)).

As already noted in Section 6.1, the formation of planets is possible not only based on a gas-dust protoplanetary substance (for example, a protostellar nebula simulated by a uniformly rotating spheroidal body), but also by capturing and merging particles and bodies moving in close orbits (meteorites, asteroids, planetesimals, etc.) in the star's gravitational field in accordance with Schmidt's model [6, 21]. Then, for a body (or a particle) of the *exoplanetary* system (including the solar one), moving in an orbit with a large semi-axis a and eccentricity e, the value of the orbital specific angular momentum can be determined using the formula (6.2.31) taking into account that $\varepsilon_0^2 << 1$ and $\left|\sin\theta_0\right| \leq 1$:

$$C = \sqrt{\gamma Ma(1-e^2)}\,. \qquad (6.2.32)$$

where C is a double areal velocity of the orbital motion of a particle.

According to the Schmidt approach [6, 8, 21], the distributions of orbits and masses of moving bodies (meteorites, planetesimals, planetary embryos) with orbital specific angular momentum belonging to some interval of values, is described by the formulas:

$$a = \frac{l}{2}\cdot\frac{1+e}{1-e}, \qquad (6.2.33a)$$

$$dm = \frac{m}{2}\cdot de, \qquad (6.2.33b)$$

where l is a limiting distance at which the capture is carried out and which is constant for all systems [21] (in fact, l is a parameter of the parabolic orbit which becomes an elliptic one when capture occurs, which is then $l/2 = R_p$ and $a = R_a$ in accordance with (7.2.38)). According to formulas (6.2.32) and (6.2.33a), we establish that the orbital specific angular momentum of a moving body is equal:

$$C = \sqrt{\gamma Ml/2} \cdot (1+e) . \qquad (6.2.34)$$

According to (6.1.1), let μ_n be a value of the orbital specific angular momentum corresponding to the *border* between the regions of n-th and $(n+1)$-th extrasolar protoplanets (or planetary embryos), whose orbital specific angular momentums are equal respectively to C_n and C_{n+1}. Then for a moving body at the boundary distance, for which the value e'_n is known, the orbital specific angular momentum can be found by the relation:

$$\mu_n = \sqrt{\gamma Ml/2} \cdot (1+e'_n) . \qquad (6.2.35)$$

Equating the orbital angular momentum of the n-extrasolar planet to the total orbital angular momentum of bodies moving in close orbits (particles due to thermal emission or meteoritic matter [8, 21]), we have:

$$m_n C_n = \int_{e'_{n-1}}^{e'_n} \sqrt{\gamma Ml/2} \cdot (1+e) \frac{m}{2} \cdot de =$$

$$= \sqrt{\gamma Ml/2} \cdot \frac{m}{2} [(e'_n - e'_{n-1}) + \frac{1}{2}({e'}^2_n - {e'}^2_{n-1})] =$$

$$= \sqrt{\gamma Ml/2} \cdot \frac{m}{2} (e'_n - e'_{n-1}) \cdot [1 + \frac{1}{2}(e'_n + e'_{n-1})] . \qquad (6.2.36)$$

Using the designation:

$$m_n = \frac{m}{2} \cdot (e'_n - e'_{n-1}) , \qquad (6.2.37)$$

the previous equality (6.2.36) (in view of (6.2.37)) takes the form:

$$m_n C_n = m_n \sqrt{\frac{\gamma Ml}{2}} \cdot \frac{(1+e'_n) + (1+e'_{n-1})}{2} = \frac{m_n}{2} (\mu_{n-1} + \mu_n) . \qquad (6.2.38)$$

As shown in Section 6.1 (see analogous formula (6.1.40b)), taking into account (6.1.1) we can establish from the derived formula (6.2.38) that:

$$C_n = \frac{C_{n+1} + C_{n-1}}{2} \tag{6.2.39a}$$

and then:

$$\sqrt{a_n} = \frac{\sqrt{a_{n+1}} + \sqrt{a_{n-1}}}{2}\,. \tag{6.2.39b}$$

The solution of the difference equation (6.2.39b) is the well-known Schmidt law of planetary distances (6.1.12):

$$\sqrt{a_n} = \mathrm{c}_1 + \mathrm{c}_2 n\,, \tag{6.2.40}$$

where c_1 and c_2 are some constants. As a result, according to the considered model of Schmidt, orbits of the formed extrasolar planets should be close to circular and a_n should be a radius of the circular orbit R_n. Then, according to formula (6.1.44), the value of orbital specific angular momentum of the formed protoplanet is equal to $C_n^* = \sqrt{\gamma M a_n}$ which is observed in some exoplanetary systems (for example, in HD 10180) and in our Solar system.

So, according to (6.2.40), the major semi-axes a_n of orbits of the extrasolar planets being formed can satisfy Schmidt's law of planetary distances. Indeed, a closer examination of Fig. 6.4 shows that the exponential laws of planetary distances of Titius–Bode may indeed occur in some exoplanetary systems. As shown in subsection 6.1.1 (see formula (6.1.15)), following J. Laskar [257], the exponential law of the form $\log a_n = \mathrm{c}_1 + \mathrm{c}_2 n$ is obtained when the initial distribution density $\rho(a)$ of planetesimals is approximated by a function $a^{-3/2}$, while at a constant density $\rho(a)$, the major semi-axes a_n of the orbits of extrasolar planets satisfy the relation of the form $\sqrt{a_n} = c_1 + c_2 n$, that is, Schmidt's law of planetary distances (6.2.40).

However, this conclusion can be drawn with confidence when all extrasolar planets are discovered in the exoplanetary systems under study, especially low-mass planets. At the present level of development of the observational technique based on the HARPS spectrograph, the discovery of new extrasolar planets of small mass within 1 AU is no simple task. Therefore, in the conditions of limited data, it is not worth thinking about some "missing" planets, possibly included in the interplanetary distances, so that these data fully comply with the Titius–Bode law of planetary distances.

However, even the quickest glance at Fig. 6.3 shows that a fairly regular interval between adjacent extrasolar planets is observed mainly for low-mass exoplanetary systems HD 40307, HD 69830, and HD 10180 and to a lesser extent for GJ 581. Among massive exoplanetary systems (for example, 55 Cnc) there is also an almost regular interval of planetary distances [259], but the presence of gas giant planets in this system makes the interactions of extrasolar planets among themselves much stronger and, perhaps, indicates a different scenario of formation of massive exoplanetary systems in comparison with low-mass exoplanetary systems. In this regard, concentrating on considering only low-mass exoplanetary systems, Fig. 6.4 shows the graphs of correspondence of the observed distributions of the major semi-axes a_n of the orbits of extrasolar planets to the exponential law of planetary distances $a_n = c_1 \cdot c_2^{\,n}$ (see also [258, 259]) as a function of the number n of extrasolar planets, starting with $n = 1$ [231].

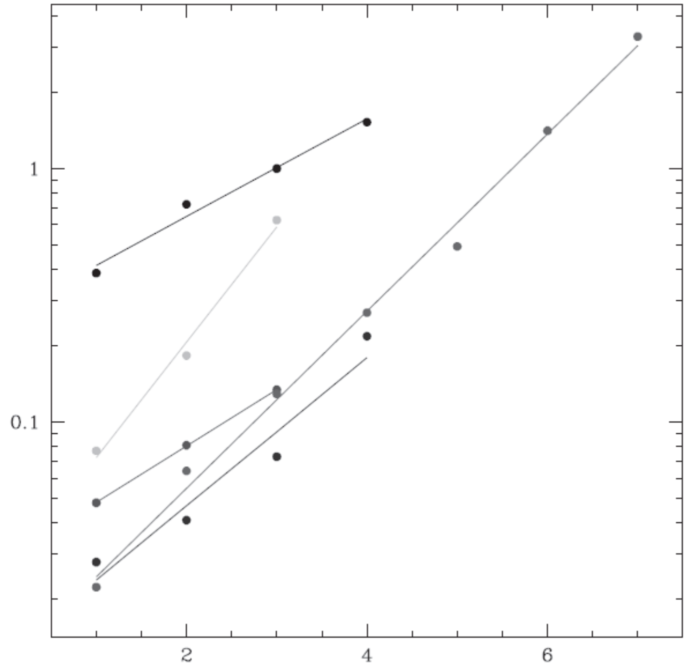


Figure 6.4. The fit of exponential laws to semi-major axes a_n as a function of planet number n for the inner Solar system (black), HD 40307 (red), GJ 581 (blue), HD 69830 (green) and HD 10180 (magenta) [211]

Satisfactory compliance has been obtained for low-mass exoplanetary systems HD 40307, HD 69830, and HD 10180 with a relative standard deviation error of 0.57%, 10.2%, and 12.0%, respectively. Compliance with the exponential law of Titius–Bode (or Murray–Dermott [258, 259]) of the observed distributions of planetary distances for the inner planets in our Solar system, as we know, gives a relative error of deviation of 8.0%. For the exoplanetary system GJ 581, this correspondence is less convincing, since the variance of the relative deviation error is 21.0% (perhaps, due to the existence of an additional body there between the third and fourth planets in this system).

We emphasize that one should not consider the Titius–Bode laws of planetary distances in any other meaning, except in the sense of a possible consequence of the features (a kind of "signature") of the processes of planetary formation, since these laws are characteristic only of certain types of planetary systems, for example, low mass configurations of many bodies, that is, low-mass exoplanetary systems. Massive exoplanetary systems, on the other hand, apparently experienced a more *chaotic history in their formation* [231, 236, 237], which indicates the possible applicability of the statistical model of randomly moving particles due to thermal emissions in the gravitational field of the spheroidal body (see subsection 6.2.1) to explain the scenario of the formation of protoplanets in massive exoplanetary systems. Moreover, even for not all low-mass systems, the exponential laws of planetary distances of the Titius–Bode type are valid (for example, as in the case of GJ 581), since the physics of planetary formation itself is so complex and diverse that it is difficult to expect the existence of any universal ordering rule for planets (see for comparison the model of formation of protoplanets based on the evolution of a flattened rotating spheroidal body and by capturing and merging particles and bodies moving in close orbits, described in Section 3.8, subsection 6.1.1, and this subsection).

So, in the general case, when considering of a model of formation of protoplanets based on the capture and combination of bodies and particles moving in close orbits due to thermal emission, which mainly takes place in *massive exoplanetary systems*, the value of the orbital angular momentum of n-th protoplanet with mass m_n according to (6.2.32), that is, by formula (6.2.31) under the assumption $\varepsilon_0^2 << 1$ and $|\sin\theta_0| \le 1$, is equal to:

$$C_n = m_n \sqrt{\gamma M a_n} \cdot \sqrt{1 - e_n^2} \, . \tag{6.2.41}$$

Since the orbit of each particle leaving a spheroidal body due to chaotic thermal motion has a *chaotic inclination* of its elliptical orbit, the averaged orbit of the n-th planet of the exoplanetary system is also characterized by an average angle of inclination i_n, so it is advisable to consider the z-projection of the angular momentum of the n-th planet:

$$C_{nz} = m_n \sqrt{\gamma M a_n} \cdot \sqrt{1 - e_n^2} \cdot \cos i_n = m_n \sqrt{\gamma M} \cdot \sqrt{a_n (1 - e_n^2)} \cdot \cos i_n \, . \tag{6.2.42}$$

Following P. Laplace [131 vol. XI p.49], let us consider a value called *the deficit of the angular momentum*:

$$\Delta C_n = C_n^* - C_{nz} = m_n \sqrt{\gamma M a_n} \cdot \left(1 - \sqrt{1 - e_n^2} \cdot \cos i_n\right) =$$
$$= \Lambda_n \cdot \left(1 - \sqrt{1 - e_n^2} \cdot \cos i_n\right), \tag{6.2.43}$$

where $\Lambda_n = m_n \sqrt{\gamma M a_n}$ are the quantities that are essentially constants in averaged equations relative to the average longitude, as Laskar [257] noted, that is, the magnitudes of the specific angular momentum C_n^* in the case of circular orbits. Using the value of the angular momentum deficit (6.2.43), Laplace proved that variations of eccentricities and inclinations of the orbits of forming planets are limited to the first order. In particular, if the total deficit of angular momentum $\Delta C = \sum_{n=0}^{n_p - 1} \Delta C_n$, where n_p is the number of all planets, is equal to zero for the planetary system:

$$\Delta C = \sum_{n=0}^{n_p - 1} \Lambda_n \left(1 - \sqrt{1 - e_n^2} \cdot \cos i_n\right) = 0 \, , \tag{6.2.44}$$

then the averaged motion of the planets is flat and circular, and also stable in time [257]. As Laskar pointed out, large values of the angular momentum deficit always lead to *chaotic system behavior* [236, 237], which is observed in some exoplanetary systems [257, 285]. Thus, the deficit of angular momentum is a measure of the amplitude of nonlinearity present in the averaged planetary system. Indeed, the planetary system is stable if the total angular momentum deficit $\Delta C = \sum_{n=0}^{n_p-1} \Delta C_n$ is not so significant in accordance with (6.2.43) that collisions of the planets would be possible [257]. In the process of collisions of protoplanets (planetary embryos, planetesimals, and particles), the local deficit of angular momentum decreases, leading to a reduction in the total deficit of angular momentum $\Delta C = \sum_{n=0}^{n_p-1} \Delta C_n$, so that the trajectories of motion of protoplanets are averaged. As a result, collisions cease immediately, as soon as the total deficit of the angular momentum becomes so small that planetary collisions cannot be possible.

Thus, when the temperature in a rotating spheroidal body (in the state of virial equilibrium) becomes so significant that some of the particles leave the body as a result of *thermal emission*, the total deficit of the angular momentum of the particles leaving the spheroidal body will be sufficiently large due to their motion along the Keplerian elliptical trajectories with large inclinations. In this situation, collisions of particles become inevitable, accompanied by a decrease in the total deficit of angular momentum. As a result, their trajectories converge and are averaged, leading to the formation of protoplanetary embryos, having a maximum specific angular momentum in comparison with the remaining particles of a spheroidal body (see formulas (3.4.33)–(3.4.35) in Section 3.4 as well as formulas (6.2.24a) and (6.2.24b) in subsection 6.2.1)) due to the "thermal removal" of the maximal specific angular momentum from the boundary region $h = h_*$ of the spheroidal body (for example, from the stellar corona).

Unfortunately, at the present level of development of the tools of observational extrasolar planetology, a detailed study of stellar coronas and their influence on the formation of exoplanetary systems is difficult. In this regard, we confine ourselves to the study of some Sun-like stars assuming them to have similar solar coronas (see Section 7.3 and the next Chapter 8).

Conclusion and comments

There exist a number of theories for exploring the formation of the Solar system and estimating planetary orbits [1–44, 127, 237, 298–300]:

- electromagnetic theories by, for example, Birkeland (1912) and Alfvén (1942);
- gravitational theories by, for example, (Schmidt (1944), Gurevich and Lebedinsky (1950), Woolfson (1964, 2000), and Safronov (1969);
- nebular theories by, for example, von Weizsäcker (1943, 1947), Berlage (1948), Kuiper (1949, 1951), Hoyle (1960, 1963), Ter Haar (1963, 1972), and Cameron (1963, 1988); and
- quantum mechanical theories by, for example, Nelson (1966, 1985), Nottale (1993, 1996), De Oliveira Neto (1996, 2004), and Agnese and Festa (1997).

In spite of a great amount of work aimed at exploring the formation of the Solar system, the theories mentioned are not able to explain all phenomena. In this connection, in 1996, *the statistical theory* for the formation of a cosmological body (the so-called spheroidal body model) was proposed [45-79]. The present monograph develops this statistical theory relative to our Solar system and other exoplanetary systems formation.

As shown in Chapters 2 and 3 of this monograph, the proposed theory starts from the conception for the formation of a spheroidal body as a protoplanetary system from a protoplanetary nebula (or proto-Sun inside a presolar nebula). In particular, in Chapter 2, the derived distribution functions of particles as well as the mass density of an immovable spheroidal body were used to characterize the first stage of evolution: from a presolar molecular nebula to a forming core (proto-Sun) together with its shell (the solar nebula).

Chapter 3 described the second stage of evolution: from the solar nebula to a forming protoplanetary gas-dust disk based on the derived distribution function and the density mass function for a rotating spheroidal body. As shown in Chapter 3, the derived function of mass density (3.3.26a–c) characterizes a flatness process: from initial spherical forms (for a non-rotational spheroidal body case (3.3.27)) through flattened ellipsoidal forms (for a rotating spheroidal body (3.3.26c)) to fuzzy contour disks (3.3.28) when the squared eccentricity ε_0^2 varies from 0 to 1. The obtained formulas, (3.3.26a–c) and (3.3.28), can describe a possible scenario of the formation both of a star and of a protoplanetary gas-dust disk around it (in particular, the Sun and the solar protoplanetary gas-dust disk).

The differential equations describing the process of gravitational condensation of a spheroidal body (from an infinitely distributed substance) in the vicinity of mechanical equilibrium were derived in Chapters 4 and 5. In particular, Sections 1.7, 4.1, and 4.7 also considered a problem of gravitational condensation of a gas-dust protoplanetary cloud with a view to protoplanetary formation in its gravitational field. Section 4.1 (as well as 4.7) derived a more general evolutionary equation (relative to a distribution function) which generalizes the Jeans equation characterizing protoplanetary system behavior. In Chapter 5, the generalized nonlinear time-dependent Schrödinger-like equation describing a common scenario of gravitational formation of a cosmogonical body was derived.

In the present Chapter 6, the next stage of evolution (from a protoplanetary flattened gas-dust disk to originating protoplanets) has been considered. To this end, the distribution function (3.4.9) of a specific angular momentum for a rotating uniformly spheroidal body (as a gas-dust flattened protoplanetary cloud) was used. As the specific angular momentums (for particles or planetesimals) are averaged during a conglomeration process (under a planetary embryo formation) the specific angular momentum for a protoplanet of the Solar system was found in subsection 6.1.1. As a result, a new law (6.1.53) for planetary distances (which generalizes Schmidt's law) was derived theoretically in subsection 6.1.2. Moreover, unlike the well-known planetary distance laws, the proposed law was established by a physical dependence of planetary distances from the value of the specific angular momentum.

Subsection 6.1.3 considered an application of the proposed law for planetary distances to our Solar system. As shown in this subsection (see Tables 6.2 and 6.3), this new law has given a very good estimation of real planetary distances in the Solar system (0% for the relative error of estimation and 1.4% for the absolute error). In addition, its maximal value is equal to 11% for the Earth, but for Pluto, the proposed law gives too high an error according to the derived rule $ent[n/2]$ determining the *maximal* number of necessary coefficients a_k in the law (6.1.70).

Thus, this chapter has shown:

1. The proposed law of the planetary distances based on a model of a spheroidal body is found in agreement with the Solar system's observable planetary distances.
2. The analysis of Tables 6.2 and 6.3 points to two possible scenarios: a capture of the Moon by the Earth (known as Bailey's proposition [8, 263]) or the Moon forming from rocky debris after the collision of a former

planet with the proto-Earth (the leading modern Hartmann–Davis hypothesis and its development within the framework of the *giant impact theory* [264–269]).

3. There is no gap between the orbits of Mars and Jupiter and we cannot, therefore, say that there is a planet missing in that region (see Table 6.3).
4. The ninth planet Pluto did not form as part of our Solar system forming. It is more probable that it was attracted by the Solar system (this proposition was stated by von Weizsäcker, Schmidt, Ter Haar, and Cameron).

According to the *first* conclusion, this chapter has shown that the proposed law for predicting the distance between the Sun and a planet of the Solar system is relatively accurate for most planets (see Figure 6.2) but it is not very good for Earth and particularly poor for Pluto. The *fourth* conclusion presented also at sessions ST0/PS0 “Plasma processes at Earth and other Solar system bodies” and PS15 “Models of Solar system forming” of the General Assembly of the European Geosciences Union in Vienna, Austria, 2–7 April 2006 [56, 57] was confirmed by the decision of the 26th General Assembly of the International Astronomical Union in Prague, Czech Republic, 14–17 August 2006 (see http://www.astronomy2006.com/).

Section 6.2 developped an alternative *heat emission model* of the formation of protoplanets. As shown in subsection 6.2.1, in the state of relative mechanical equilibrium of particles moving in elliptical orbits in the gravitational field of a spheroidal body and having a specific angular momentum value $\lambda = \Omega_1 h^2$, equation (6.2.18b) for the *heat distribution function* $f_T(\lambda)$ of the specific angular momentum was derived. Within the framework of this model, only 0.8% of the total number of particles in the Solar system composing the protoplanetary cloud have angular momentum 15.6 times higher than the angular momentum of the remaining 99% of particles of the Solar system. This conclusion agrees entirely with the known fact of a nonuniform distribution of the angular momentum noted by Ter Haar [7, 32]: only 2% of the total angular momentum belongs to the Sun, while the remaining 98% part belongs to the planets.

The discovery of extrasolar planets is one of the greatest achievements of modern astronomy, and the recent discovery of planets with masses comparable to the mass of the Earth indicates that extrasolar planets of low mass also exist. In subsection 6.2.2, an application of a statistical model of particles moving in the gravitational field of a spheroidal body due to heat emission to the problems of the formation of exoplanetary systems is considered. As pointed out, the exponential laws of planetary distances of Titius–Bode (or Murray–Dermott) may occur in some exoplanetary systems. Indeed, satisfactory compliance with the exponential law of Titius–Bode has been obtained for low-mass exoplanetary systems HD 40307, HD 69830, and HD 10180 with a relative standard deviation error of 0.57%, 10.2%, and 12.0%, respectively.

Let us note that the proposed simple statistical approach to the investigation of our Solar system as well as the formation of exoplanetary systems describes only a natural self-evolution or an inner process of development of protoplanets from a dust-gas cloud. However, this approach naturally does not include any dynamics like collisions and giant impacts of protoplanets with large cosmic bodies. Henceforth, the presented statistical theory will only be able to predict with certainty the protoplanets' positions according to the proposed $ent[n/2]$ rule (see Eq. (6.1.70)), that is, the findings in this chapter are useful for predicting if the position of a planet *today* coincides with its protoplanet's location or not.

CHAPTER SEVEN

ON THE CALCULATION OF PLANETARY ORBIT AND THE INVESTIGATION OF ITS FORMS IN PLANETARY SYSTEMS

As pointed in Chapter 6, despite great successes in recent decades in the fields of both astrophysics and geophysics, many problems in relation to the formation of the Solar system (as well as other exoplanetary systems) remain unresolved, in part because there is now no single general and uncontested scenario for the formation of a proto-sun and protoplanetary system from a protosolar nebula (a molecular cloud). At present, in cosmogony, there are electromagnetic, gravitational, nebular, quantum mechanical, and statistical theories [16, 73]. In spite of a great many works aimed at solving the problems of planetary formation, the theories mentioned above are not able to fully explain all phenomena occurring in our Solar system and other exoplanetary systems. Within the framework of the proposed *statistical* theory for a cosmogonical body forming (see Chapters 2–5), the conception of the evolution of the so-called *spheroidal body* inside a gas-dust protoplanetary nebula has been developed. This permits us to derive the form of distribution functions as particles in a space, the mass density, the gravitational potentials and the strengths for both immovable and rotating spheroidal bodies (Chapter 2). In addition we can find the distribution function of specific angular momentum (Chapter 3) and the general differential equations for the physical values describing the anti-diffusion process of the initial gravitational condensation of immovable and rotating spheroidal bodies (Chapter 4). As the specific angular momentums are averaged during the conglomeration process, the specific angular momentum for a protoplanet in a forming planetary system (as well as a planetary distance) can be found using such procedures (Chapters 5–6).

As pointed in Section 6.1, the proposed statistical approach to the formation of the Solar system describes only the internal process of self-organization of protoplanets from a gas-dust cloud and therefore does not include such additional dynamic processes as collisions and the huge influences of protoplanets or other big cosmic bodies (asteroids and a swarm of planetesimals or meteorites). Hence, the proposed statistical theory of a protoplanet's origin based on the evolution of a flattened rotating and gravitating spheroidal body cannot precisely predict the position of planets in the case of giant impacts and, as a consequence, it is also unable to authentically estimate their orbits. Indeed, the orbits of moving particles inside a flattened rotating and gravitating spheroidal body are initially *circular*. However, during the evolution of a spheroidal body at the formation of protoplanets these orbits can be deformed a little due to collisions with other particles or gravitational influences of forming adjacent planetesimals. As the famous astrophysicist V.S. Safronov remarked [2 p.145]:

> The assumption of initial movement of particles on circular orbits looks natural. At small masses of bodies, their gravitational variations were weak, and particles moved on the orbits close to circular. In the process of growth of a planet, deviations of orbits from circular increased, and all bodies of a zone had an opportunity to be joined in one planet.

So, the process of evolution of a rotating and gravitating spheroidal body at first leads to its flattening, and the process then results in its disintegration (decay) on forming protoplanets (see Sections 5.4 and 3.8, Eq. (3.8.8)). Hence, the orbits of moving particles at later stages of evolution of a rotating and gravitating spheroidal body are formed mainly under the influence of its central gravitational field, that is, in essence, they are *Keplerian* [158]. In this context, we shall consider in more detail the calculation of orbits of moving bodies and planets in a centrally symmetric gravitational field of a rotating and gravitating spheroidal body on *the planetary stage of its evolution*.

On the other hand, this chapter shows that the orbits of moving particles are formed by the action of the centrally symmetric gravitational field mainly in the later stages of evolution of a gravitating and rotating spheroidal body, that is, when the particle orbits become Keplerian. In this context, this chapter also investigates the orbits of moving planets and bodies in the centrally symmetric gravitational field of a gravitating and rotating spheroidal body during the planetary stage of its evolution.

7.1. Calculation of the gravitational potential in a remote zone of a uniformly rotating spheroidal body

Let us consider some statements of the statistical theory of the formation of cosmological bodies (see Chapters 2–3), beginning with the distribution function of particles in a space in a homogeneous gaseous (dust-like) nebula. As shown in Section 3.3, the mass density for a rotating and gravitating spheroidal body being in a state of relative mechanical equilibrium can be written in the cylindrical, Cartesian, and spherical coordinate system by the formulas (3.3.26a)–(3.3.26c) respectively:

$$\rho(h,z) = M(\alpha/2\pi)^{3/2}(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2} =$$

$$= \rho_0(1-\varepsilon_0^2)e^{-\alpha(h^2(1-\varepsilon_0^2)+z^2)/2}, \qquad (7.1.1a)$$

$$\rho(x,y,z)=\rho_0(1-\varepsilon_0^2)e^{-\alpha(x^2(1-\varepsilon_0^2)+y^2(1-\varepsilon_0^2)+z^2)/2}, \quad (7.1.1b)$$

$$\rho(r,\theta)=\rho_0(1-\varepsilon_0^2)e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2}, \quad (7.1.1c)$$

where $M = m_0 N$ is the mass of a rotating and gravitating spheroidal body. Obviously, the iso-surfaces of the mass density (7.1.1a–c) are flattened ellipsoidal ones and ε_0^2 is a parameter of their flatness (a squared eccentricity of an ellipse). As pointed out in Section 3.3, the function of mass density (7.1.1a–c) characterizes a flatness process: from initial spherical forms (in the case of a non-rotational spheroidal body) through flattened ellipsoidal forms (for a rotating and gravitating spheroidal body) to fuzzy contour disks when the squared eccentricity ε_0^2 varies from 0 to 1. This means that the derived function (7.1.1a–c) is appropriate to describe the evolution of a protoplanetary gaseous (gas-dust) disk around a star (in particular, the Sun).

As shown in Sections 2.2 and 2.3, in the non-rotational case there is a *threshold value* of the parameter of gravitational condensation α_c that, if $\alpha \geq \alpha_c$, then a weak gravitational field with gravitational potential φ_g arises in a spheroidal body. According to (7.1.1a–c) we can try to seek a solution φ_g *in the remote zone* bearing in mind the Poisson equation [72, 73].

As we know from the theory of potential [95, 97], the general solution of the Poisson equation:

$$\nabla^2\varphi_g = 4\pi\gamma\rho \quad (7.1.2)$$

has the form:

$$\varphi_g(\vec{r}) = -\gamma\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV', \quad (7.1.3)$$

where

$\rho(\vec{r}')$ is a mass density of a gravitating body,

$\vec{r}'$ is a radius vector of a volume element of the body (distance from the center of mass of the body to a given volume element), and

$\vec{r}$ is a radius vector of observations of the gravitational field.

Initially, we apply the formula (7.1.3) to the calculation of the potential of a *non-rotating* spheroidal body with mass density (2.2.5):

$$\rho(\vec{r}) = \rho_0 \cdot e^{-\alpha \vec{r}^2/2}, \tag{7.1.4}$$

where $\rho_0 = M(\alpha/2\pi)^{3/2}$. To do this, first note (see [44, 95]) that:

$$\left|\vec{r} - \vec{r}'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\psi}\ , \tag{7.1.5a}$$

where:

$$\cos\psi = \cos\theta \cdot \cos\theta' + \sin\theta \cdot \sin\theta' \cdot \cos(\varepsilon - \varepsilon')\ . \tag{7.1.5b}$$

We note that if $\varepsilon' \to \varepsilon$ then we obtain $\cos\psi = \cos\theta \cdot \cos\theta' + \sin\theta \cdot \sin\theta' = \cos(\theta - \theta')$, that is, $\psi = \theta - \theta'$. Taking into account this notion let us find the gravitational potential in the *remote* zone II:

$$\varphi_g(\vec{r}) = -\gamma \int_0^r \int_0^\pi \int_0^{2\pi} \frac{\rho_0 e^{-\alpha r'^2/2} r'^2 \sin\theta'\, dr' d\theta' d\varepsilon'}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}} =$$

$$= -\gamma\rho_0 \int_0^r r'^2 e^{-\alpha r'^2/2} \int_0^\pi \int_0^{2\pi} \frac{\sin\theta'\, d\theta' d\varepsilon'}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}} dr'\ . \tag{7.1.6}$$

We will now consider the case $r' << r$ by selecting *a spherical volume of radius* r' around the origin of coordinates. To estimate the gravitational potential of the spherical volume:

$$\varphi_g(r)\Big|_{r>>r'} = \lim_{r'/r \to 0} -\gamma\rho_0 \int_0^{r'} r'^2 e^{\frac{-\alpha r'^2}{2}} \int_0^\pi \int_0^{2\pi} \frac{\sin\theta'\, d\theta' d\varepsilon'}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}} dr', \tag{7.1.7a}$$

let us calculate separately the inner integral in this expression:

$$\lim_{r'/r\to 0}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta'\,d\theta' d\varepsilon'}{\sqrt{r^2+r'^2-2rr'\cos\psi}}=\lim_{r'/r\to 0}\frac{1}{r}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta'\,d\theta' d\varepsilon'}{\sqrt{1+\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\cos\psi}}=$$

$$=\lim_{r'/r\to 0}\frac{1}{r}\int_0^{\pi}\int_0^{2\pi}\sin\theta'\,d\theta' d\varepsilon'=\frac{4\pi}{r}. \qquad (7.1.7b)$$

Substituting (7.1.7b) into (7.1.7a) we obtain [72, 73]:

$$\varphi_g(\vec{r})\Big|_{r>>r'}=\lim_{r'/r\to 0}\left\{-\gamma\rho_0\cdot\frac{4\pi}{r}\int_0^{r'} r'^2 e^{-\alpha\frac{r'^2}{2}}dr'\right\}=$$

$$=-4\pi\gamma\rho_0\lim_{r'/r\to 0}\frac{1}{r}\left\{\frac{1}{\alpha}\left[\int_0^{r'} e^{-\alpha\frac{r'^2}{2}}dr'-r'\cdot e^{-\alpha\frac{r'^2}{2}}\Bigg|_0^{r'}\right]\right\}=$$

$$=-\frac{4\pi\gamma\rho_0}{\alpha}\lim_{r'/r\to 0}\left\{\frac{1}{r}\int_0^{r'} e^{-\alpha\frac{r'^2}{2}}dr'-\frac{r'}{r}\cdot e^{-\alpha\frac{r'^2}{2}}\Bigg|_0^{r'}\right\}=$$

$$=-\frac{4\pi\gamma\rho_0}{\alpha}\cdot\frac{1}{r}\int_0^{r'} e^{-\alpha\frac{r'^2}{2}}dr'. \qquad (7.1.8)$$

So, if $r>>r'$ then (7.1.8) becomes formula (2.4.26) for the gravitational potential of a non-rotating spheroidal body [16, 45, 46]. The second case $r'\propto r>>r_*$ leads to the same result if instead of the limit $r'/r\to 0$ in (7.1.8) we consider the limit $r'\to r$ under the condition $r>>r_*$.

Thus, the formula (7.1.6) coincides with the expression for the gravitational potential of a non-rotating spheroidal body (2.4.26). Moreover, the magnitude of the potential $\varphi_g(r)$ of the spheroidal body is determined by the mass of an inner ball with the radius r (see Eq. (2.6.11) and Theorem 2.1). Similarly, we attempt to derive the potential of the gravitational field in a *remote zone* for the case of a uniformly rotating spheroidal body based on the general solution (7.1.3) of the Poisson equation [72, 73].

More exactly, using the general solution (7.1.3) of the Poisson equation (7.1.2), let us calculate the estimation of the gravitational potential of a *uniformly rotating* spheroidal body under the following conditions [62, 72, 73]:

1) a distance r' from the center of mass of a spheroidal body to a volume element dV in the process of integration in this area V is not greater than the distance r from the center to the point of observation of the gravitational field (the test body in the point M):
$$r' \le r\,; \tag{7.1.9a}$$
2) a distance r from the center of mass O to the observation point M is much larger than the distance $r_* = 1/\sqrt{\alpha}$ from the center to the point of density inflection (a conditional shell) of a spheroidal body or the extremum point of gravitational field strength:
$r_* << r$ (the condition of the remote zone); (7.1.9b)
3) an evaluation of the gravitational potential φ_g is carried out by accounting for the terms of first-order relative to small quantity ε_0^2, that is,
$$\varphi_g = O\left(\varepsilon_0^2\right). \tag{7.1.9c}$$

We now find the gravitational potential of a uniformly rotating spheroidal body in a remote zone of the gravitational field based on the conditions (7.1.9a–c); to this end let us extract initially some ellipsoidal volume around the origin of coordinates (Fig. 7.1).

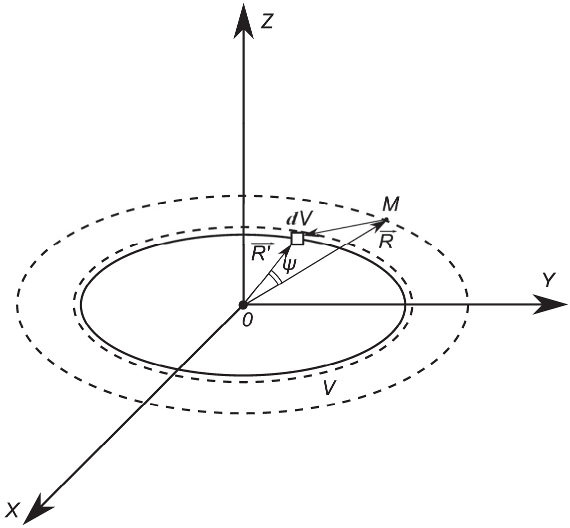


Figure 7.1. Scheme for calculating the gravitational potential of a uniformly rotating spheroidal body in a remote zone of the gravitational field

The integration over the volume V should be then carried out on the elements dV that are disposed concentrically on oblate ellipsoidal surfaces. These confocal oblate ellipsoidal surfaces are *isosteres* of a uniformly rotating spheroidal body with a mass density of the kind (7.1.1b).

As a result of such constructions, the radius vector $\overrightarrow{R'}$ of a volume element dV of the body is a hodograph whose endpoint moves on a given ellipsoidal surface (see Fig. 7.1). Similarly, we assume that the endpoint of the radius vector $\vec{R}$ of the observation point of the field belongs to an ellipsoidal surface limiting the volume V of a rotating spheroidal body. So, $\overrightarrow{R'}$ are the radius-vectors of points of the ellipsoidal surfaces in a Cartesian coordinate system:

$$\frac{x^2}{2/\left(1-\varepsilon_0{}^2\right)\cdot\alpha}+\frac{y^2}{2/\left(1-\varepsilon_0{}^2\right)\cdot\alpha}+\frac{z^2}{2/\alpha}=1. \qquad (7.1.10)$$

Moreover, $\overrightarrow{R'}$ is a radius vector of a point on the surface of the ellipsoid bounding the volume V. Let us note that according to the Newton theorem (see Theorem 1.2 in Section 1.1 and [95]) a homogeneous body, bounded by two similar and similarly placed concentric ellipsoids, has no attraction at points inside the inner cavity V. Since the mass density $\rho(x,y,z)$ of a spheroidal body (under the condition (7.1.9b)) is insignificant and can be considered homogeneous, then following Newton's theorem an area external to the volume V of spheroidal body has practically no influence on the magnitude of potential φ_g that also follows from the derivation of formula (7.1.8) in the case of a non-rotating spheroidal body. In a spherical coordinate system:

$$\begin{cases} x = r\sin\theta\cos\varepsilon; \\ y = r\sin\theta\sin\varepsilon; \\ z = r\cos\theta \end{cases} \qquad (7.1.11a)$$

the argument of the function of mass density (7.1.1b) takes the form:

$$\left(1-\varepsilon_0{}^2\right)\cdot x^2 + \left(1-\varepsilon_0{}^2\right)\cdot y^2 + z^2 = \left(1-\varepsilon_0{}^2\right)\cdot r^2\sin^2\theta + r^2\cdot\cos^2\theta =$$
$$= r^2\cdot(1-\varepsilon_0{}^2\sin^2\theta)\,. \qquad (7.1.11b)$$

If, however, we use the elliptical, or more exactly, the *spheroidal* coordinate system (in the flattened spherical coordinates [72, 73]):

$$\begin{cases} x = \dfrac{1}{\sqrt{1-\varepsilon_0{}^2}} R\sin\Theta\cos\mathrm{E}; \\ y = \dfrac{1}{\sqrt{1-\varepsilon_0{}^2}} R\sin\Theta\sin\mathrm{E}; \\ z = R\cos\Theta, \end{cases} \qquad (7.1.12a)$$

then the above-mentioned argument of the function of the mass density (10.1.32a) can be written more simply [62, 72, 73]:

$$\left(1-\varepsilon_0{}^2\right)\cdot x^2+\left(1-\varepsilon_0{}^2\right)\cdot y^2+z^2=R^2\sin^2\Theta+R^2\cos^2\Theta=R^2. \quad (7.1.12b)$$

Comparing (7.1.11b) and (7.1.12b), it is not difficult to see that:

$$R^2=r^2\cdot\left(1-\varepsilon_0{}^2\sin^2\theta\right). \quad (7.1.13)$$

In the flattened spherical coordinates, the length of a vector $\vec{R}$ according to (7.1.12b) is defined by the relation:

$$\left|\vec{R}\right|=\sqrt{\left(1-\varepsilon_0{}^2\right)\cdot x^2+\left(1-\varepsilon_0{}^2\right)\cdot y^2+z^2}\,, \quad (7.1.14a)$$

and the distance between two vectors $\vec{R}$ and $\vec{R'}$ is found by the cosine theorem:

$$\left|\vec{R}-\vec{R'}\right|=\sqrt{R^2+R'^2-2RR'\cos\psi}=$$

$$=R\sqrt{1+\left(\frac{R'}{R}\right)^2-2\left(\frac{R'}{R}\right)\cos\psi}\;. \quad (7.1.14b)$$

Moreover, because of (7.1.12b) the function of mass density (7.1.1b) in the flattened spherical coordinates has a very simple form [62, 72, 73]:

$$\rho\left(\vec{R}\right)=M\left(\frac{\alpha}{2\pi}\right)^{3/2}\left(1-\varepsilon_0{}^2\right)\cdot e^{-\alpha\vec{R}^2/2} \quad (7.1.15)$$

reminding us of the mass density function (7.1.4) for a non-rotating spheroidal body (up to a factor $1-\varepsilon_0{}^2$). In this regard, the gravitational potential of a uniformly rotating spheroidal body in the remote zone can be calculated similarly to the above-mentioned approach (7.1.6)–(7.1.8) for a non-rotating spheroidal body.

Now let us calculate the Lamé coefficients for the flattened spherical coordinates in the general form (7.1.12a):

$$H_R = \sqrt{\left(\frac{\partial x}{\partial R}\right)^2 + \left(\frac{\partial y}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2} = \sqrt{\frac{1-\varepsilon_0{}^2\cos^2\Theta}{1-\varepsilon_0{}^2}}\,; \qquad (7.1.16a)$$

$$H_\Theta = \sqrt{\left(\frac{\partial x}{\partial \Theta}\right)^2 + \left(\frac{\partial y}{\partial \Theta}\right)^2 + \left(\frac{\partial z}{\partial \Theta}\right)^2} = R\sqrt{\frac{1-\varepsilon_0{}^2\sin^2\Theta}{1-\varepsilon_0{}^2}}\,; \qquad (7.1.16b)$$

$$H_\mathrm{E} = \sqrt{\left(\frac{\partial x}{\partial \mathrm{E}}\right)^2 + \left(\frac{\partial y}{\partial \mathrm{E}}\right)^2 + \left(\frac{\partial z}{\partial \mathrm{E}}\right)^2} = \frac{R\sin\Theta}{\sqrt{1-\varepsilon_0{}^2}}\,. \qquad (7.1.16c)$$

Using (7.1.16a–c) let us find the volume element dV in the flattened spherical coordinates [62, 72, 73]:

$$dV = H_R H_\Theta H_\mathrm{E}\, dR\, d\Theta\, d\mathrm{E} = \frac{R^2\sin\Theta}{(1-\varepsilon_0{}^2)^{3/2}} \times$$

$$\times\sqrt{(1-\varepsilon_0{}^2\sin^2\Theta)\cdot(1-\varepsilon_0{}^2\cos^2\Theta)}\, dR\, d\Theta\, d\mathrm{E}\,. \qquad (7.1.17)$$

Finally, taking into account formulas (7.1.14b), (7.1.15), (7.1.17) and conditions (7.1.9a–c) let us calculate an estimation of the gravitational potential (7.1.3) in a remote zone of a uniformly rotating spheroidal body in the flattened spherical coordinates [62, 72, 73]:

$$\varphi_g(R)\Big|_{R>>R^*} = -\gamma\int_V \frac{\rho\left(\overrightarrow{R'}\right)dV'}{R\sqrt{1+\left(R'/R\right)^2 - 2\left(R'/R\right)\cos\psi}} =$$

$$= -\frac{\gamma M(\alpha/2\pi)^{3/2}\left(1-\varepsilon_0{}^2\right)}{\left(1-\varepsilon_0{}^2\right)^{3/2}}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{R'^2 e^{-\alpha R'^2/2}}{R\sqrt{1+\left(R'/R\right)^2-2\left(R'/R\right)\cos\psi}}\cdot\sin\Theta'\times$$

$$\times\sqrt{\left(1-\varepsilon_0{}^2\sin^2\Theta'\right)\left(1-\varepsilon_0{}^2\cos^2\Theta'\right)}\,dR'\,d\Theta'\,d\mathrm{E}' =$$

$$= -\frac{\gamma M(\alpha/2\pi)^{3/2}}{\sqrt{1-\varepsilon_0{}^2}}\int_0^R\frac{R'^2 e^{-\alpha R'^2/2}dR'}{R\sqrt{1+\left(R'/R\right)^2-2\left(R'/R\right)\cos\psi}}\times$$

$$\times 2\pi\int_0^\pi\sqrt{\left(1-\varepsilon_0{}^2\sin^2\Theta'\right)\left(1-\varepsilon_0{}^2\cos^2\Theta'\right)}\sin\Theta'\,d\Theta'. \qquad (7.1.18)$$

Let us calculate separately the integrals in (7.1.18). As in the case of a non-rotating spheroidal body, that is, according to derivations (7.1.7a, b) and (7.1.8), to calculate the integral by R' first we *select an ellipsoidal volume* with the radius vector $\overrightarrow{R'}$ around the origin of coordinates (see Fig.7.1) and then apply the limiting condition for the denominator:

$$\lim_{R'/R\to 0}\int_0^{R'}\frac{R'^2 e^{-\alpha R'^2/2}dR'}{R\sqrt{1+(R'/R)^2-2(R'/R)\cos\psi}}=$$

$$=\lim_{R'/R\to 0}\frac{1}{R}\int_0^{R'}R'^2 e^{-\alpha R'^2/2}dR'=$$

$$=\lim_{R'/R\to 0}\frac{1}{\alpha}\left\{\frac{1}{R}\int_0^{R'}e^{-\alpha R'^2/2}dR'-\left(\frac{R'}{R}\right)\cdot e^{-\alpha R'^2/2}\Bigg|_0^{R'}\right\}=$$

$$=\frac{1}{\alpha R}\int_0^{R'}e^{-\alpha R'^2/2}dR'. \qquad (7.1.19)$$

Since the relation (7.1.19) holds for all concentric ellipsoidal volumes with the length of the radius vector $\left|\vec{R}'\right| << \left|\vec{R}\right|$, then according to (7.1.9a) it is also suitable for the considered ellipsoidal area of volume V when $\left|\vec{R}'\right| \le \left|\vec{R}\right|$ (in other words, for $\left|\vec{R}'\right| < \left|\vec{R}\right|$ and even $\left|\vec{R}'\right| = \left|\vec{R}\right|$). Indeed, at $R'=R$ the function $e^{-\alpha R^2/2}\to 0$ because $R >> R_*$ according to the condition of a remote zone (7.1.9b). Thus, starting from (7.1.9a) and (7.1.9b), the required integral in (7.1.18) is equal [62, 72, 73]:

$$\lim_{\substack{R_*/R\to 0\\ R'\le R}}\int_0^{R}\frac{R'^2 e^{-\alpha R'^2/2}dR'}{R\sqrt{1+(R'/R)^2-2(R'/R)\cos\psi}}=$$

$$= \frac{1}{\alpha R} \int_0^R e^{-\alpha R'^2/2} dR' . \tag{7.1.20}$$

To calculate the second integral with respect to Θ', belonging to Eq. (7.1.18), we use the substitution $s = \varepsilon_0 \cos \Theta'$:

$$\int_0^\pi \sqrt{(1-\varepsilon_0{}^2 \sin^2 \Theta') \cdot (1-\varepsilon_0{}^2 \cos^2 \Theta')} \sin\Theta' \, d\Theta' =$$

$$= \frac{1}{\varepsilon_0} \int_{-\varepsilon_0}^{\varepsilon_0} \sqrt{[(1-\varepsilon_0{}^2)+s^2]\cdot[1-s^2]} ds = \frac{2}{\varepsilon_0} \int_0^{\varepsilon_0} \sqrt{[(1-\varepsilon_0{}^2)+s^2]\cdot[1-s^2]} ds \tag{7.1.21}$$

The integral (7.1.21) can be expressed in terms of the elliptic integrals of the first and second kind [301 p.262]. However, taking into account the third condition (7.1.9c) the desired integral (7.1.21) can be calculated much more easily [72, 73]:

$$\int_0^\pi \sqrt{\left(1-\varepsilon_0{}^2 \sin^2 \Theta'\right)\cdot\left(1-\varepsilon_0{}^2 \cos^2 \Theta'\right)} \sin \Theta' \, d\Theta' \approx$$

$$\approx \int_0^\pi \sqrt{1-\varepsilon_0{}^2 \cdot \left(\sin^2 \Theta' + \cos^2 \Theta'\right)} \sin\Theta' \, d\Theta' = 2\sqrt{1-\varepsilon_0{}^2} . \tag{7.1.22}$$

Here the terms of the fourth order of smallness $O(\varepsilon_0{}^4)$ are neglected under the derivation of (7.1.22).

Thus, bearing in mind the integrals (7.1.20) and (7.1.22) calculated under the conditions (7.1.9a–c), the estimation of the gravitational potential in a remote zone (7.1.18) for a uniformly rotating spheroidal body is equal [72, 73]:

$$\varphi_g(R)\Big|_{R>>R_*} = -\frac{\gamma M \alpha^{3/2}}{\sqrt{2\pi(1-\varepsilon_0^2)}} \cdot 2\sqrt{1-\varepsilon_0{}^2} \cdot \frac{1}{\alpha R} \int_0^R e^{-\alpha R'^2/2} dR' =$$

$$= -\sqrt{\frac{2\alpha}{\pi}} \cdot \gamma M \cdot \frac{1}{R} \int_0^R e^{-\alpha R'^2/2} dR' . \tag{7.1.23}$$

Under derivations (7.1.8) and (7.1.23), an idea about the maximal exclusion of dependencies on spatial coordinates on equipotential surfaces has been exploited (in particular, on the spheres in the first case, and on the flattened ellipsoids in the second case). If we use an *eigen coordinate system* (for example, spherical or ellipsoidal) then a canonical coordinate is to be r for a sphere or R in the case of an oblate ellipsoid (instead of r and θ in accordance with (7.1.13)). In this connection, using (7.1.13), we can estimate the gravitational potential in a remote zone of uniformly rotating spheroidal body in the *spherical coordinates* [72, 73]:

$$\varphi_g(r,\theta)\Big|_{r>>r_*} = -\sqrt{\frac{2\alpha}{\pi}}\cdot\gamma M\cdot\frac{1}{r\sqrt{1-\varepsilon_0{}^2\sin^2\theta}}\times$$
$$\times\int_0^{r\sqrt{1-\varepsilon_0{}^2\sin^2\theta}} e^{-\alpha r'^2/2}dr'. \qquad (7.1.24)$$

Now we can verify the accuracy of the derived estimation (7.1.24) by the usage of the general expression (3.6.15a) for the gravitational potential of an axially symmetric spheroidal body (see Section 3.6). Using (3.6.15a) let us estimate the gravitational potential in a remote zone of the gravitational field of a rotating spheroidal body when:

$$r >> r_{pr}, \qquad (7.1.25a)$$

where $r_{pr} = \sqrt{\dfrac{2}{\alpha(1-\varepsilon_0^2)}}$. Since the points at r_{pr} belong to some equipotential surface of ξ-spheroid, then the above-mentioned condition (7.1.25a) means that:

$$(1-\varepsilon_0^2)x^2 + (1-\varepsilon_0^2)y^2 + z^2 \to [2+\xi\alpha(1-\varepsilon_0^2)]/\alpha. \qquad (7.1.25b)$$

Taking into account the conditions (7.1.25a) and (7.1.25b), the formula for the gravitational potential in the remote zone becomes:

$$\varphi_g\Big|_{r>>r_{pr}} = -\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\times$$
$$\times\int_\xi^\infty e^{-\frac{x^2+y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+s}} e^{-\frac{z^2}{\frac{z}{\alpha}+s}} \frac{ds}{\sqrt{2+\alpha s}(2+\alpha s(1-\varepsilon_0^2))}, \quad (7.1.26a)$$

where value ξ is a positive root of equation [99]:

$$\frac{x^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+\xi}+\frac{y^2}{\frac{2}{\alpha(1-\varepsilon_0^2)}+\xi}+\frac{z^2}{\frac{2}{\alpha}+\xi}=1. \quad (7.1.26b)$$

Taking into account (7.1.25a) and (7.1.25b), let us transform (7.1.26a):

$$\varphi_g\Big|_{r>>r_{pr}} = -\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_\xi^\infty e^{-\alpha\frac{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2}{2+s\alpha(1-\varepsilon_0^2)}} e^{-\frac{\alpha z^2}{2+s\alpha}} \frac{\sqrt{2+s\alpha(1-\varepsilon_0^2)}}{\sqrt{2+s\alpha}}\times$$

$$\times\frac{ds}{[2+s\alpha(1-\varepsilon_0^2)]^{3/2}} = -\frac{\gamma M}{\sqrt{\pi}}\alpha^{3/2}(1-\varepsilon_0^2)\int_{1/\sqrt{2+\xi\alpha(1-\varepsilon_0^2)}}^0 e^{-\alpha\frac{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2}{2+s\alpha(1-\varepsilon_0^2)}} e^{-\alpha\frac{z^2}{2+s\alpha}}\times$$

$$\times\sqrt{1-\varepsilon_0^2\frac{s\alpha}{s\alpha+2}}\left(-\frac{2}{\alpha(1-\varepsilon_0^2)}\right)d\left(\frac{1}{\sqrt{2+s\alpha(1-\varepsilon_0^2)}}\right). \quad (7.1.27)$$

Using the representation:

$$\frac{1}{2+s\alpha}=\frac{1}{2+s\alpha(1-\varepsilon_0^2)}-\frac{\varepsilon_0^2 s\alpha}{(2+s\alpha)(2+s\alpha(1-\varepsilon_0^2))},$$

we can rewrite Eq. (7.1.27) as follows:

$$\varphi_g\Big|_{r>>r_{pr}} = \frac{2\gamma M}{\sqrt{\pi}}\sqrt{\alpha}\int_{1/\sqrt{2+\xi\alpha(1-\varepsilon_0^2)}}^0 e^{-\alpha\frac{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2}{2+s\alpha(1-\varepsilon_0^2)}}\times$$

$$\times\sqrt{1-\varepsilon_0^2\frac{s\alpha}{s\alpha+2}}\cdot e^{\alpha\frac{z^2}{2+s\alpha(1-\varepsilon_0^2)}\cdot\frac{\varepsilon_0^2 s\alpha}{2+s\alpha}} d\left(\frac{1}{\sqrt{2+s\alpha(1-\varepsilon_0^2)}}\right). \quad (7.1.28)$$

Applying the condition (7.1.25a) we have:

$$\varepsilon_0{}^2 \frac{s\alpha}{s\alpha+2} << 1,$$

so that relation (7.1.28) can be approximated by the relation:

$$\varphi_g\Big|_{r>>r_{pr}} \approx -\frac{2\gamma M}{\sqrt{\pi}}\sqrt{\alpha}\int_0^{1/\sqrt{2+\xi\alpha(1-\varepsilon_0^2)}} e^{-\alpha\frac{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2}{1/\sigma^2}} d\sigma, \quad (7.1.29)$$

where the following designation is used in (7.1.29):

$$\sigma = \frac{1}{\sqrt{2+s\alpha(1-\varepsilon_0^2)}}\cdot$$

According to (7.1.14a), $(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2=R^2$, so that relation (7.1.29) becomes:

$$\varphi_g\Big|_{r>>r_{pr}} = -2\sqrt{\frac{\alpha}{\pi}}\gamma M\frac{1}{R}\int_0^{R/\sqrt{2+\xi\alpha(1-\varepsilon_0^2)}} e^{-\alpha\sigma^2R^2} d(\sigma R)\,. \quad (7.1.30)$$

Using a change of dependent variable $r'/\sqrt{2}=\sigma R$ in (7.1.30) we obtain:

$$\varphi_g\Big|_{r>>r_{pr}} = -2\sqrt{\frac{\alpha}{\pi}}\gamma M\frac{1}{\sqrt{2}R}\int_0^{R\sqrt{2}/\sqrt{2+\xi\alpha(1-\varepsilon_0^2)}} e^{-\alpha r'^2/2}dr' =$$

$$= -\sqrt{\frac{2\alpha}{\pi}}\frac{\gamma M}{R}\int_0^{R/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2}dr'. \quad (7.1.31)$$

So, according to (7.1.31) the estimation of the gravitational potential in the *remote zone* of a rotating spheroidal body takes the form:

$$\varphi_g\Big|_{r>>r_{pr}} = -\sqrt{\frac{2\alpha}{\pi}}\frac{\gamma M}{\sqrt{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2}}\times$$

$$\times\int_0^{R/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2}dr' =$$

$$= -\frac{2^{3/2}}{2^{3/2}}\frac{\pi\alpha}{\pi\alpha}\sqrt{\frac{2\alpha}{\pi}}\frac{\gamma M}{\sqrt{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2}}\times$$

$$\times\int_0^{R/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2} dr' = -4\pi\gamma\rho_0 \frac{1}{\alpha\sqrt{(1-\varepsilon_0^2)x^2+(1-\varepsilon_0^2)y^2+z^2}} \times$$

$$\times\int_0^{R/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2} dr'. \qquad (7.1.32a)$$

In cylindrical and spherical coordinate systems, the gravitational potential of a rotating spheroidal body in a remote zone is determined by the following expressions respectively:

$$\varphi_g\Big|_{r>>r_{pr}} = -4\pi\gamma\rho_0 \frac{1}{\alpha\sqrt{(1-\varepsilon_0^2)h^2+z^2}} \times$$

$$\times\int_0^{\sqrt{(1-\varepsilon_0^2)h^2+z^2}/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2} dr'; \qquad (7.1.32b)$$

$$\varphi_g\Big|_{r>>r_{pr}} = -4\pi\gamma\rho_0 \frac{1}{\alpha r\sqrt{1-\varepsilon_0^2\sin^2\theta}} \times$$

$$\times\int_0^{r\sqrt{1-\varepsilon_0^2\sin^2\theta}/\sqrt{1+\xi\alpha(1-\varepsilon_0^2)/2}} e^{-\alpha r'^2/2} dr'. \qquad (7.1.32c)$$

When choosing $\xi = 0$, formula (7.1.32c) gives the derived estimation (7.1.24) of the gravitational potential of a rotating spheroidal body in a remote zone [72]. In the case of $\varepsilon_0 = 0$ and $\xi = 0$, we obtain the above estimation (7.1.8) in a remote zone of the gravitational potential for an immovable spheroidal body.

Substituting (7.1.24) into the left-hand side of Poisson equation (7.1.2), as well as (7.1.1c) into the right-hand side, we can see that the absolute error of estimation of the Laplacian of the gravitational potential (7.1.24) of a uniformly rotating spheroidal body is expressed by the following relation [72, 73]:

$$\Delta^{abs}(\theta,\varepsilon_0,\rho) = \frac{\varepsilon_0^2\cos^2\theta}{1-\varepsilon_0^2\sin^2\theta}\cdot\frac{4\pi\gamma\rho(r,\theta)}{1-\varepsilon_0^2}. \qquad (7.1.33a)$$

The relative error in estimating the Laplacian of the gravitational potential (7.1.24) is given by relation [72, 73]:

$$\Delta^{rel}(\theta, \varepsilon_0) = \frac{\varepsilon_0^2}{1-\varepsilon_0^2} \cdot \frac{\cos^2\theta}{1-\varepsilon_0^2 \sin^2\theta}. \qquad (7.1.33b)$$

So, absolute and relative errors depend on the angular coordinate θ and the value of oblateness ε_0^2. In particular, if $\varepsilon_0^2 \to 0$ then there are no errors: $\Delta^{abs}(\theta, \varepsilon_0, \rho) \to 0$, $\Delta^{rel}(\theta, \varepsilon_0) \to 0$. Thus, for the case of a *weakly flattened* spheroidal body ($\varepsilon_0^2 << 1$), the obtained formula (7.1.24) of estimation of the gravitational potential in a remote zone is exact enough since the maximum relative error in calculating the Laplacian of the gravitational potential tends to zero: $\Delta^{rel}_{\max}(0, \varepsilon_0) = \varepsilon_0^2 / (1-\varepsilon_0^2) << 1$.

7.2. The calculation of the orbits of planets and bodies of the Solar system in a centrally symmetric gravitational field of a rotating spheroidal body based on Binet's differential equation

As is well known [96, 158], the general method for finding an orbit consists of the integration of the differential equations of motion with the subsequent exception of time. Often it is a very complex process and, therefore, a natural question arises as to whether it is possible to exclude time before integration so that integration has given the required orbit directly. In particular, as shown in [96], it is possible in that case, when force does not depend on time. So, we consider the motion of the material point subject to the action of the *central force* of a gravitational attraction[1].

[1] This material was previously published in Solar System: Structure, Formation and Exploration, edited by Matteo de Rossi in 2012 [72] and is being reproduced with permission from Nova Science Publishers, Inc.

Let f_g be a *specific value* of gravitational force, that is, an acceleration to which the point is subjected. By definition of the central force, directions of this force always pass through a fixed point (or center) which we shall accept for the origin of coordinates. If O is the center of force, then P is some position of a moving point in a plane XY whose rectangular coordinates are x and y, and polar coordinates are r and ε accordingly (Fig. 7.2). The projections of accelerations on axes x and y are also then accordingly equal to $-f_g \cos\varepsilon$ and $-f_g \sin\varepsilon$, and the differential equations of motion become:

$$\frac{d^2x}{dt^2} = -f_g \cos\varepsilon = -f_g \frac{x}{r}; \frac{d^2y}{dt^2} = -f_g \sin\varepsilon = -f_g \frac{y}{r}. \quad (7.2.1)$$

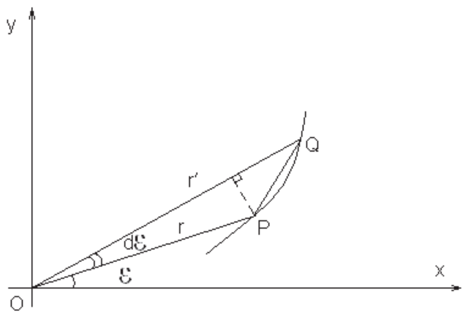


Figure 7.2. Graphic representation of the motion of a material point in a field of the central force

Multiplying the first equation in (7.2.1) by $-y$ and the second equation on x, then adding them we obtain:

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0. \quad (7.2.2)$$

Adding and subtracting the value $\frac{dx}{dt}\cdot\frac{dy}{dt}$ in the left-hand part of the equation (7.2.2) we transform it to the kind:

$$x\frac{d^2y}{dt^2}+\frac{dx}{dt}\cdot\frac{dy}{dt}-\frac{dy}{dt}\cdot\frac{dx}{dt}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=0. \quad (7.2.3)$$

Integrating the equation (7.2.3) we find that

$$x\frac{dy}{dt}-y\frac{dx}{dt}=C, \quad (7.2.4)$$

where C is a constant of integration. To find out the sense of a constant of integration in (7.2.4) we shall consider the motion of a material point in a plane XY for a time interval Δt (Fig. 7.2). Let ΔS designate the area of triangle OPQ limited by radius vector in Fig. 7.2 for a time interval Δt:

$$\Delta S=\frac{1}{2}r'r\sin(\Delta\varepsilon)=\frac{r'r}{2}\cdot\frac{\sin(\Delta\varepsilon)}{\Delta\varepsilon}\cdot\Delta\varepsilon. \quad (7.2.5)$$

If an angle $\Delta\varepsilon$ decreases unlimitedly then the area of triangle OPQ tends to the area of the sector. Moreover, the limit r' is r. Passing to the limit at $\Delta t\to 0$ in (7.2.5) we obtain:

$$dS=\lim_{\Delta t\to 0}\left(\frac{r'r}{2}\cdot\frac{\sin(\Delta\varepsilon)}{\Delta\varepsilon}\cdot\Delta\varepsilon\right)=\frac{1}{2}r^2d\varepsilon,$$

whence it directly follows that:

$$\frac{dS}{dt}=\frac{1}{2}r^2\cdot\frac{d\varepsilon}{dt}. \quad (7.2.6)$$

The value (7.2.6) is called an *areal velocity* of a moving point [96]. By substitution:

$$r=\sqrt{x^2+y^2}, \qquad \varepsilon=\arctan\left(\frac{y}{x}\right),$$

let us write the following expression for the areal velocity in rectangular coordinates:

$$\frac{dS}{dt}=\frac{1}{2}(x^2+y^2)\cdot\frac{d}{dt}[\arctan\left(\frac{y}{x}\right)]=$$

$$= \frac{1}{2}(x^2 + y^2) \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{d}{dt}\left(\frac{y}{x}\right) =$$

$$= \frac{1}{2} x^2 \cdot \frac{\dot{y}x - y\dot{x}}{x^2} = \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right). \qquad (7.2.7)$$

So, comparing now equations (7.2.4) and (7.2.7), we can see that the required constant C is expressed by the areal velocity [96]:

$$C = 2\dot{S} = r^2 \dot{\varepsilon}\,, \qquad (7.2.8)$$

where S is an area limited by radius vector with the time t. According to (7.2.8) if the origin of coordinates is chosen by a suitable way then motion obeys the *law of the areas*: $r^2 \dot{\varepsilon} = \text{const}$. Integrating equation (7.2.8) we obtain:

$$S = \frac{1}{2} \cdot Ct + c\,,$$

that is, the area S changes directly proportional to time (*Kepler's second law* [96, 158]).

To derive the equation of an orbit of a material point let us return to equations of movement (7.2.1). According to a primary assumption, the function f_g does not depend on time then t is considered relative to derivatives only. However, for the exception of t it is necessary first to reduce the order of derivatives. For convenience, we transform equations (7.2.1) in the polar coordinates [96].

As in polar coordinates $x = r\cos\varepsilon$ and $y = r\sin\varepsilon$, the velocity components in rectangular coordinates are then expressed through components in polar coordinates:

$$v_x = \frac{dx}{dt} = \frac{dr}{dt}\cos\varepsilon - r\sin\varepsilon\frac{d\varepsilon}{dt} = v_r\cos\varepsilon - v_\varepsilon\sin\varepsilon\,; \qquad (7.2.9a)$$

$$v_y = \frac{dy}{dt} = \frac{dr}{dt}\sin\varepsilon + r\cos\varepsilon\frac{d\varepsilon}{dt} = v_r\sin\varepsilon + v_\varepsilon\cos\varepsilon\,, \qquad (7.2.9b)$$

where $v_r = dr/dt$ and $v_\varepsilon = r \cdot d\varepsilon/dt$ are the polar components of velocity on radius vector and along a perpendicular to it. To find acceleration components let us differentiate the formulas (7.2.9a) and (7.2.9b):

$$a_x = \frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\varepsilon}{dt}\right)^2\right] \cdot \cos\varepsilon -$$

$$- \left[r\frac{d^2\varepsilon}{dt^2} + 2\frac{dr}{dt} \cdot \frac{d\varepsilon}{dt}\right] \cdot \sin\varepsilon\,; \qquad (7.2.10a)$$

$$a_y = \frac{d^2y}{dt^2} = \left[r\frac{d^2\varepsilon}{dt^2} + 2\frac{dr}{dt} \cdot \frac{d\varepsilon}{dt}\right] \cdot \cos\varepsilon +$$

$$+ \left[\frac{d^2r}{dt^2} - r\left(\frac{d\varepsilon}{dt}\right)^2\right] \cdot \sin\varepsilon\,, \qquad (7.2.10b)$$

whence by analogy with (7.2.9a) and (7.2.9b), the polar components of acceleration a_r and a_ε (on radius vector and along a perpendicular to it) can be calculated through formulas [96]:

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\varepsilon}{dt}\right)^2; \qquad (7.2.11a)$$

$$a_\varepsilon = r\frac{d^2\varepsilon}{dt^2} + 2\frac{dr}{dt} \cdot \frac{d\varepsilon}{dt} = \frac{1}{r} \cdot \frac{d}{dt}\left(r^2\frac{d\varepsilon}{dt}\right). \qquad (7.2.11b)$$

According to Fig. 7.2 the polar components of acceleration on radius vector a_r and along a perpendicular a_ε are equal to $-f_g$ and 0. Then following formulas (7.2.11a) and (7.2.11b) the differential equations of movement become:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varepsilon}{dt}\right)^2 = -f_g\,; \qquad (7.2.12a)$$

$$\frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\varepsilon}{dt}\right)=0. \qquad (7.2.12b)$$

But according to the formula (7.2.8), the integral for the second of these equations is equal:

$$r^2\frac{d\varepsilon}{dt}=C,$$

so, excepting $d\varepsilon/dt$ from equation (7.2.12a) and using this integral we obtain:

$$\frac{d^2r}{dt^2}=\frac{C^2}{r^3}-f_g\,. \qquad (7.2.13)$$

Supposing $r=1/q$ we calculate, in view of (7.2.8), the following derivatives:

$$\frac{dr}{dt}=-\frac{1}{q^2}\cdot\frac{dq}{dt}=-\frac{1}{q^2}\cdot\frac{dq}{d\varepsilon}\cdot\frac{d\varepsilon}{dt}=-C\cdot\frac{dq}{d\varepsilon}; \qquad (7.2.14a)$$

$$\frac{d^2r}{dt^2}=-C\cdot\frac{d}{dt}\left(\frac{dq}{d\varepsilon}\right)=-C\cdot\frac{d^2q}{d\varepsilon^2}\cdot\frac{d\varepsilon}{dt}=-C^2q^2\cdot\frac{d^2q}{d\varepsilon^2}\,. \qquad (7.2.14b)$$

Substituting (7.2.14b) into equation (7.2.13) we obtain the differential equation of the second-order relatively q [96]:

$$f_g=C^2q^2\cdot\left(q+\frac{d^2q}{d\varepsilon^2}\right). \qquad (7.2.15)$$

As the integral of Eq. (7.2.15) expresses q and consequently r as a function of ε, this equation, after integration, gives a relation between ε and r. On the other hand, equation (7.2.15) can be used for a finding of the law of the central force that undergoes a material point to move along the given curve. For this purpose, it is necessary to write only the equation of a curve in polar coordinates and then to calculate the right-hand part of Eq. (7.2.15). This problem is much easier than a direct problem of a finding of an orbit when the law of force is given [96].

Taking into account that $\vec{f}_g = -\text{grad}\,\varphi_g$, that is, for the considered flat motion $f_g = \left|\vec{f}_g\right| = d\varphi_g / dr$ and also that $r = 1/q$, we present equation (7.2.15) in the form of Binet's formula [96, 200] allowing us to define an orbit of a moving material point (planet) in the central gravitational field:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{r^2}{C^2}\cdot\varphi_g^{'}(r), \qquad (7.2.16)$$

where $\varphi_g(r)$ is a function of gravitational potential and $C = r^2\dot{\varepsilon}$ is an areal constant.

As shown in Section 7.1, the gravitational potential of a rotating spheroidal body in a remote zone is described by the formula (7.1.24), that is,

$$\varphi_g(r,\theta)\Big|_{r>>r_*} = -\sqrt{\frac{2\alpha}{\pi}}\cdot\gamma M\cdot\frac{1}{r\sqrt{1-\varepsilon_0^{\,2}\sin^2\theta}}\cdot\int\limits_0^{r\sqrt{1-\varepsilon_0^{\,2}\sin^2\theta}} e^{-\alpha r'^2/2}dr'. \quad (7.2.17)$$

To take advantage of the Binet equation (7.2.16) for the definition of an orbit of a planet moving in the central gravitational field of a rotating spheroidal body, we shall carry out an approximation of potential (7.2.17) under the condition of $r/r_* \to \infty$ where, as designated above, $r_* = 1/\sqrt{\alpha}$.

As a first approximation, the gravitational potential (7.2.17) of a rotating spheroidal body in the *remote zone* can be estimated by the formula [72]:

$$\varphi_g(r,\theta)\Big|_{r>>r_*} = -\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\gamma M}{r\sqrt{1-\varepsilon_0^{\,2}\sin^2\theta}}\times$$

$$\times\int\limits_0^{r\sqrt{\alpha\left(1-\varepsilon_0^{\,2}\sin^2\theta\right)/2}} e^{-\left(r\sqrt{\alpha/2}\right)^2}\, d(r\sqrt{\alpha/2})\Big|_{r/r_*\to\infty} =$$

$$= -\frac{2}{\sqrt{\pi}} \cdot \frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta}} \int_0^{r\sqrt{\alpha\left(1-\varepsilon_0{}^2\sin^2\theta\right)/2}} e^{-s^2}\, ds\Big|_{r\sqrt{\alpha}\to\infty} =$$

$$= -\frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta}} \cdot \frac{2}{\sqrt{\pi}} \lim_{r\sqrt{\alpha}\to\infty} \int_0^{r\sqrt{\alpha}\cdot\sqrt{\left(1-\varepsilon_0{}^2\sin^2\theta\right)/2}} e^{-s^2}\, ds =$$

$$= -\frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta}}. \tag{7.2.18}$$

Supposing the condition of smallness $\varepsilon_0{}^2 << 1$ formula (7.2.18) can be approximated also as follows:

$$\varphi_g(r,\theta)\Big|_{r>>r_*} \approx -\frac{\gamma M}{r} \cdot \left(1 + \frac{\varepsilon_0{}^2}{2}\sin^2\theta\right). \tag{7.2.19}$$

Taking into account that the orbit of each forming planet moving in the *central* gravitational field of a rotating spheroidal body lies entirely in the same plane, $\theta = \theta_0 = \text{const}$, near to an equatorial plane of the protoplanetary gas-dust cloud ($\theta = \pi/2$), that is, $\theta_0 \propto \pi/2$, formula (7.2.18) becomes:

$$\varphi_g(r,\theta_0)\Big|_{r>>r_*} = -\frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}. \tag{7.2.20}$$

Substituting (7.2.20) into Binet's formula (7.2.16), we derive the equation of an orbit of a planet in a remote zone of a rotating spheroidal body [70, 72]:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}, \tag{7.2.21}$$

where $C = r^2\dot{\varepsilon}$ is an areal constant and $\sin^2\theta_0$ is a parameter. Carrying out the usual substitution $r = 1/q$ in equation (7.2.21) we obtain:

$$\frac{d^2q}{d\varepsilon^2} = -q + \frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}. \qquad (7.2.22)$$

Multiplying both parts (7.2.22) by $2\frac{dq}{d\varepsilon}$ following the solution offered in [96]:

$$2\frac{dq}{d\varepsilon}\cdot\frac{d^2q}{d\varepsilon^2} = -2q\frac{dq}{d\varepsilon} + 2\frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\cdot\frac{dq}{d\varepsilon},$$

we transform this equation into the kind:

$$\frac{d}{d\varepsilon}\left[\left(\frac{dq}{d\varepsilon}\right)^2\right] = -\frac{d}{d\varepsilon}\left[q^2\right] + 2\frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\cdot\frac{dq}{d\varepsilon}. \qquad (7.2.23)$$

Integrating (7.2.23) we find the first integral of the given equation:

$$\left(\frac{dq}{d\varepsilon}\right)^2 = -q^2 + \frac{2\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\cdot q + c_1$$

(c_1 is a constant of integration) whence we obtain:

$$\frac{dq}{d\varepsilon} = \pm\sqrt{-q^2 + \frac{2\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\cdot q + c_1} =$$

$$= \pm\sqrt{-\left(q - \frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\right)^2 + \frac{(\gamma M/C^2)^2}{1-\varepsilon_0{}^2 \sin^2\theta_0} + c_1}$$

and at last,

$$d\varepsilon = \pm\frac{dq}{\sqrt{\frac{(\gamma M/C^2)^2}{1-\varepsilon_0{}^2 \sin^2\theta_0} + c_1 - \left(q - \frac{\gamma M}{C^2\sqrt{1-\varepsilon_0{}^2 \sin^2\theta_0}}\right)^2}}. \qquad (7.2.24)$$

Let $(\gamma M / C^2)^2 / (1 - \varepsilon_0{}^2 \sin^2 \theta_0) + c_1 = k^2$ and $q - (\gamma M / C^2 \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0}) = s$. Choosing the bottom sign in the equation (7.2.24) we write the given equation in the form [96]:

$$d\varepsilon = -\frac{ds}{\sqrt{k^2 - s^2}} = -\frac{1}{k} \cdot \frac{ds}{\sqrt{1 - (s/k)^2}} = -\frac{d(s/k)}{\sqrt{1 - (s/k)^2}}. \quad (7.2.25)$$

Then, integrating (7.2.25), we obtain:

$$\varepsilon = \arccos\left(\frac{s}{k}\right) + c_2,$$

whence:

$$s = k \cos(\varepsilon - c_2), \quad (7.2.26)$$

where c_2 is a constant of integration. Returning to a variable q and then to r, we can write (7.2.26) in the form:

$$\frac{1}{r} - \frac{\gamma M}{C^2 \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0}} = k \cdot \cos(\varepsilon - c_2),$$

whence:

$$r = \frac{(C^2 / \gamma M) \cdot \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0}}{1 + (C^2 / \gamma M) \cdot \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0} \cdot k \cos(\varepsilon - c_2)}. \quad (7.2.27)$$

Substituting the parameter $k = (\gamma M / C^2 \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0}) \cdot \sqrt{1 + c_1 \cdot (C^2 / \gamma M)^2 \cdot (1 - \varepsilon_0{}^2 \sin^2 \theta_0)}$ in equation (7.2.27) we obtain:

$$r = \frac{(C^2 / \gamma M) \cdot \sqrt{1 - \varepsilon_0{}^2 \sin^2 \theta_0}}{1 + \sqrt{1 + c_1 \cdot (C^2 / \gamma M)^2 \cdot (1 - \varepsilon_0{}^2 \sin^2 \theta_0)} \cdot \cos(\varepsilon - c_2)}. \quad (7.2.28)$$

At the choice of a constant as $c_1 = -(\gamma M / C^2)^2$, that directly follows from the designation k^2, *equation* (7.2.28) *of the orbit of a planet in a remote zone* of the gravitational field of a rotating spheroidal body goes over to the following [70, 72]:

$$r = \frac{(C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}{1+\left|\varepsilon_0\sin\theta_0\right|\cdot\cos\left(\varepsilon - c_2\right)}\,. \qquad (7.2.29)$$

If we assume that $\theta_0 = \pi/2 - i$ where i is an angle of an inclination of an orbital plane then the equation (7.2.29) becomes:

$$r = \frac{(C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\cos^2 i}}{1+\left|\varepsilon_0\cos i\right|\cdot\cos\left(\varepsilon - c_2\right)}\,. \qquad (7.2.30)$$

So, comparing (7.2.29) with the polar equation of conic section with a focus in the origin of coordinates $r = \dfrac{p}{1+e\cos\varphi}$ [96, 158] we find that:

$$p = (C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}\,,\; e = \varepsilon_0\left|\sin\theta_0\right|, \qquad (7.2.31)$$
$$\varphi = \varepsilon - c_2$$

where p is a parameter of an orbit, e is an eccentricity of orbits and $c_2 = \varepsilon_*$ is a constant. Moreover, ε_* is an angle between the polar axis and the endpoint of the big axis directed to vertex [96]. Constants C^2, ε_0 and θ_0 also are defined by initial conditions, and in turn, they define p and e using (7.2.31). If $e<1$ then the conic section is an ellipse; if $e=1$ then the conic section is a parabola; if $e>1$ then the conic section is a hyperbole; if $e=0$ then the conic section is a circle.

As $\varepsilon_0^2 << 1$ and $\left|\sin\theta_0\right| \le 1$ then according to formula (7.2.31) $e << 1$, that is, the conic section is an ellipse with small eccentricity. In other words, formula (7.2.30) expresses the equation of an ellipse in polar coordinates with the origin in focus, that is, planets of the Solar system move in elliptic orbits, in one of whose focus there is the Sun (*the Kepler first law*) [96, 158].

Let us note that elliptic orbits of planets of the Solar system are almost circular, namely $e=0$ at $\theta_0=0$ and $e=\varepsilon_0$ at $\theta_0=\pi/2$, that is, (7.2.30) is the equation of a circle at $\theta_0=0$ or $i=\pi/2$, and, accordingly, at $\theta_0=\pi/2$ or $i=0$, relation (7.2.30) is the equation of an ellipse with small eccentricity.

According to the derived formula (7.2.31) and known formulas of analytical geometry the major and minor semi-axes of an ellipse are accordingly equal [72]:

$$a=\frac{p}{1-e^2}=\frac{(C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}{1-\varepsilon_0{}^2\sin^2\theta_0}=\frac{C^2}{\gamma M\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}, \quad (7.2.32a)$$

$$b=\frac{p}{\sqrt{1-e^2}}=\frac{(C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}{\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}=\frac{C^2}{\gamma M}. \quad (7.2.32b)$$

The least distance, called the *perihelion*, of an orbit and the greatest distance, called the *aphelion*, of orbit up to the center of the field (which is the focus) are accordingly defined by expressions [72]:

$$r_{\min}=\frac{p}{1+e}=a(1-e)=\frac{C^2\cdot(1-\varepsilon_0|\sin\theta_0|)}{\gamma M\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}=$$

$$=\frac{C^2}{\gamma M}\cdot\sqrt{\frac{1-\varepsilon_0|\sin\theta_0|}{1+\varepsilon_0|\sin\theta_0|}}, \quad (7.2.33a)$$

$$r_{\max}=\frac{p}{1-e}=a(1+e)=\frac{C^2\cdot(1+\varepsilon_0|\sin\theta_0|)}{\gamma M\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}=$$

$$=\frac{C^2}{\gamma M}\cdot\sqrt{\frac{1+\varepsilon_0|\sin\theta_0|}{1-\varepsilon_0|\sin\theta_0|}}. \quad (7.2.33b)$$

The time it takes a planet with mass m to circle in an elliptic orbit, that is, the period T of its motion, can be conveniently defined using the law of conservation of the angular momentum in the form of «integral of the areas» [158]:

$$L = L_z = mr^2\dot{\varepsilon} = 2m\dot{S} = \text{const}\,, \tag{7.2.34}$$

where $\dot{S}$ is the above-mentioned value of areal velocity (7.2.6) of a moving body. Integrating this equality on time from 0 up to T we obtain [158]:

$$2mS = LT\,, \tag{7.2.35}$$

where S is an area of an orbit. For an ellipse $S = \pi ab$, so using formulas (7.2.8), (7.2.31), and (7.2.32a, b) we can find that:

$$T = \frac{2mS}{L} = \frac{2\pi ab}{L/m} = \frac{2\pi a^2\sqrt{1-e^2}}{2\dot{S}} = \frac{2\pi a^2\sqrt{p/a}}{C} =$$

$$= \frac{2\pi a^{3/2}\sqrt{p}}{C} = \frac{2\pi a^{3/2}\sqrt{(C^2/\gamma M)\cdot\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}}{C} =$$

$$= a^{3/2}\cdot\frac{2\pi\cdot\sqrt[4]{1-\varepsilon_0{}^2\sin^2\theta_0}}{\sqrt{\gamma M}} = a^{3/2}\cdot 2\pi\sqrt{\frac{\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}}{\gamma M}}\,. \tag{7.2.36}$$

The fact that the square of a period must be proportional to the cube of a linear size of orbit expresses *the Kepler third law* [96, 158], namely, the ratio of the cubes of the major semi-axes of orbits to the squares of the orbital times for all planets of the Solar system is the same [72]:

$$\frac{a^3}{T^2} = \frac{\gamma M}{4\pi^2\sqrt{1-\varepsilon_0{}^2\sin^2\theta_0}} = \text{const}\,. \tag{7.2.37}$$

Thus, moving bodies (conglomerates of particles, planetesimals, planetary embryos, and planets) in a remote zone of a rotating spheroidal body have trajectories in the form of ellipses with the origin in focus. In other words, the orbits of the Solar system's planets, distant enough from the Sun ($r/r_* \to \infty$), are described by ellipses with small eccentricities (this fact occurs for all planets beginning with Venus). Indeed, the value of the *geometrical eccentricity* of orbit $e = \sqrt{a^2 - b^2} / a$ can be defined easily by the condition of derivation of the equation (7.2.29) of the planetary orbit in a remote zone of the gravitational field of a uniformly rotating spheroidal body, that is, according to the formula (7.2.31) $0 \le e << 1$. Besides the geometrical ones, in astrophysics [12]:

$$e_o = (R_a - R_p)/(R_a + R_p) \ , \qquad (7.2.38)$$

is considered an *orbital eccentricity*, where $R_a = r_{\max}$ is an aphelion of orbit and $R_p = r_{\min}$ is its perihelion. According to formulas (7.2.33a, b), the orbital eccentricity (7.2.38) is equal to:

$$e_o = \big(a(1+e) - a(1-e)\big) / \big(a(1+e) + a(1-e)\big) = e \,, \qquad (7.2.39)$$

that is, it coincides with geometrical eccentricity for the type of orbits of planets moving in a remote zone of the gravitational field of a uniformly rotating spheroidal body.

Let us note in particular that the derived expression (7.2.37) for Kepler's third law generalizes the known relation obtained in the theory of Newton [80, 96, 158] in the sense that the constant in the right-hand side includes, besides γ and M, additional parameters ${\varepsilon_0}^2$ and θ_0. Thus, the ratio of cubes of the major semi-axes of orbits to squares of the rotation periods for any n-th planet of the Solar system is equal. Moreover, a constant coincides with the constant of Newton $\gamma M / 4\pi^2$ up to a very small value $(1/2) \cdot {\varepsilon_0}^2 \sin^2 \theta_0 << 1$:

$$\frac{{a_n}^3}{{T_n}^2} = \frac{\gamma M}{4\pi^2} \cdot \frac{1}{\sqrt{1 - {\varepsilon_0}^2 \sin^2 \theta_0}} = \text{const} \,. \qquad (7.2.40)$$

The specific angular momentum of the n-th planet following formulas (7.2.32a, b) and (7.2.35), that is, the relation $b_n = a_n\sqrt{1-e_n^{\ 2}}$, is equal:

$$\lambda_n = \frac{L_n}{m_n} = \frac{2S_n}{T_n} = \frac{2\pi a_n b_n}{T_n} = \frac{2\pi a_n^{\ 2}\sqrt{1-e_n^{\ 2}}}{T_n}. \qquad (7.2.41)$$

It follows from Kepler's third law in the formulation (7.2.40) that:

$$T_n = a_n^{\ 3/2}\cdot\frac{2\pi\cdot\sqrt[4]{1-(\varepsilon_0\sin\theta_0)^2}}{\sqrt{\gamma M}}. \qquad (7.2.42)$$

Substituting (7.2.42) into (7.2.41) we find the value of specific angular momentum of the n-th planet:

$$\lambda_n = \frac{2\pi a_n^{\ 2}\sqrt{1-e_n^{\ 2}}}{a_n^{\ 3/2}\cdot 2\pi\cdot\sqrt[4]{1-(\varepsilon_0\sin\theta_0)^2}\,/\sqrt{\gamma M}} = \frac{\sqrt{\gamma M a_n(1-e_n^{\ 2})}}{\sqrt[4]{1-(\varepsilon_0\sin\theta_0)^2}}. \qquad (7.2.43)$$

Let us note that the value of specific angular momentum of the n-th planet (7.2.43) obtained using the statistical theory of spheroidal bodies generalizes (when $\varepsilon_0 \neq 0$) the analogous formula $\lambda_n = \sqrt{\gamma M a_n(1-e_n^{\ 2})}$ [6, 8] derived within the framework of Newton's theory up to very small value $(1/4)\cdot\varepsilon_0^{\ 2}\sin^2\theta_0 << 1$ [72]:

$$\lambda_n = \sqrt{\gamma M a_n(1-e_n^{\ 2})}[1-(\varepsilon_0\sin\theta_0/2)^2]. \qquad (7.2.44)$$

Taking into account that $e_n^{\ 2} = \varepsilon_0^{\ 2}\sin^2\theta_0$, in accordance with (7.2.31), the formula (7.2.43) becomes:

$$\lambda_n = \frac{\sqrt{\gamma M a_n(1-e_n^{\ 2})}}{\sqrt[4]{1-e_n^{\ 2}}} = \sqrt{\gamma M a_n\sqrt{1-e_n^{\ 2}}}. \qquad (7.2.45)$$

According to the formula (7.2.45), the value of specific angular momentum λ_n of the n-th planet depends on the eccentricity value e_n of orbit (as the function $\sqrt[4]{1-e_n^{\,2}}$) more weakly than the analogous value λ_n depends first of all on e_n (as $\sqrt{1-e_n^{\,2}}$ within the framework of the Newtonian theory) due to the account of influence of a flattening parameter ε_0 and the initial value of a polar angle θ_0 (angle of an inclination i of the orbital plane). Thus, within the framework of the statistical theory of spheroidal bodies orbits of forming planets are closest to circular when, according to formula (6.1.59), $\lambda_n = \sqrt{\gamma M R_n}$ (R_n is a radius of a circular orbit) that takes place in our Solar system. As shown by Schmidt [6, 21], the formation of planets is possible not only based on gas-dust protoplanetary substance (protosolar nebula) but also through the capture of moving bodies (meteorites, asteroids, etc.) in close orbit to the gravitational field of a star (see subsection 6.2.2). For a body moving in an orbit (with the major semi-axis a and the eccentricity e) in the Solar system, Shmidt defined the above-mentioned value of specific angular momentum following Newton's theory:

$$\lambda = \sqrt{\gamma M a(1-e^2)}\,. \tag{7.2.46}$$

If, instead of (7.2.46), we take advantage of the result (7.2.45) obtained within the framework of the statistical theory of spheroidal bodies then the value of specific angular momentum is equal:

$$\lambda = \sqrt{\gamma M a\sqrt{1-e^2}}\,, \tag{7.2.47}$$

then the analogous substitution of (6.2.33a) into (7.2.47) gives us:

$$\lambda = \sqrt{\gamma M l/2}\cdot(1+e)^{3/4}/(1-e)^{1/4}\,. \tag{7.2.48}$$

Thus, according to (7.2.48) the law of planetary distances in the case of the formation of planets through the capture of bodies in close orbits will have more complex dependence.

7.3. Calculation of the orbit of the planet Mercury and estimation of the angular displacement of Mercury's perihelion based on the statistical theory of gravitating spheroidal bodies

As shown in Section 7.2, moving solid bodies in a remote zone of a rotating spheroidal body have elliptic trajectories of the kind (7.2.29). This means that orbits for planets sufficiently remote from the Sun in the Solar system ($r / r_* \to \infty$) are described by ellipses with a focus on the origin of coordinates and with small eccentricities. The planet nearest to the Sun, Mercury, has a more complex trajectory. Namely, in the case of Mercury, the angular displacement of a Newtonian ellipse is observed during one circle in the orbit[2] (Fig. 7.3), that is, *a regular* (*secular*) *shift of the perihelion of Mercury's orbit occurs* [100 p.391].

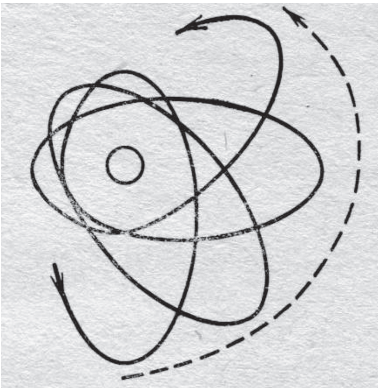


Figure 7.3. The graphic representation of shift of the perihelion of the Mercury' orbit

[2] This material was previously published in Solar System: Structure, Formation and Exploration, edited by Matteo de Rossi in 2012 [72] and is being reproduced with permission from Nova Science Publishers, Inc.

Inspired by his successful explanation of the deviation of the orbit of Uranus, which led to the discovery of the new planet Neptune, the famous astronomer U.J.J. Leverrier engaged in the analysis of the orbit of Mercury. In 1859, regarding the anomalous additional advance of the perihelion of Mercury at 31" in the century, he argued [8]:

> The conclusion is clear enough. In the vicinity of Mercury, more precisely between it and the Sun, undoubtedly, there is still unknown matter. Is it a single planet, or several small planets, or asteroids, or, finally, cosmic dust – the theory cannot give an answer to this..

As we know, by the specific use of Newton's law, a solution to the problem of Mercury was established using the theory of general relativity. On this occasion, in 1917, Einstein in his work "On the special and general theory of relativity" wrote [302 p.561]:

> Indeed, astronomers found that the theory of Newton is not sufficient to calculate the observable motion of Mercury with an accuracy which can be reached under observation at present. After all disturbing influences of other planets on the motion of Mercury were taken into account, it was found (Leverrier, 1859; Newcomb, 1895) what remains unexplained is the motion of the perihelion of the orbit of Mercury, the velocity of which does not differ noticeably from the above-mentioned +43 arc-seconds per century. The error of this empirical result is only a few seconds.

In connection with this statement (concerning the secular displacement of the perihelion of Mercury's orbit), we note that from the general position of the statistical theory of gravitating spheroidal bodies both Leverrier's point of view (on the existence of unknown matter) and Einstein's (on the theory of Newton) are almost the same. Indeed, there exists a plasma and gas-dust substance around the core of a rotating spheroidal body (in this case, the Sun) and, taking into account the fact that forming cosmogonical bodies have fuzzy outlines and are represented by spheroidal forms, it requires some clarification of Newton's law concerning a gravitating spheroidal body [16, 47, 48, 65, 73].

So, to find the trajectory of Mercury within the framework of the statistical theory of gravitating spheroidal bodies, it is necessary to estimate the gravitational potential at a nearby distance from the Sun, that is, *in a remote zone of the gravitational field and the immediate vicinity of the core of a rotating spheroidal body* [48, 70, 72, 73]. As well as in Section 7.2, let us use formula (7.2.18):

$$\varphi_g(r,\theta)\Big|_{r>>r_*} = -\frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta}}, \qquad (7.3.1)$$

which because of the smallness of parameter $\varepsilon_0{}^2 << 1$ and $|\sin\theta| < 1$ becomes the following:

$$\varphi_g(r,\theta)\Big|_{r>>r_*} = -\frac{\gamma M}{r}\cdot\left(1+\frac{\varepsilon_0{}^2}{2}\sin^2\theta\right), \qquad (7.3.2)$$

where $r_* = 1/\sqrt{\alpha}$ and α is a parameter of gravitational condensation of a spheroidal body [47, 65]. Let us note that formula (7.3.2), as well as (7.2.19), can be obtained by a Maclaurin series expansion of function (7.2.17) on degrees of a small parameter $\varepsilon_0{}^2$ in linear approximation and under the condition $r/r_* \to \infty$.

To estimate the gravitational potential (7.3.2) at a closer distance from the core of a rotating spheroidal body, we take into account the fact that Mercury (being the closest planet to the Sun) circles the Sun in a rather strongly elongated and inclined elliptic orbit (its eccentricity is equal to 0.205, and the orbital inclination to the ecliptic plane is 7°, that is, approximately 6.3° to the main plane of the Solar system). Although, as noted in Section 7.2, all the formed planets in the remote zone of a rotating spheroidal body have the elliptic and inclined orbits of the kind (7.2.30), the eccentricities and angles of inclination of orbits for all other planets of the Solar system are very small (their eccentricities are in the interval 0.0017–0.093, and the inclinations of orbits to the main plane of the Solar system belong to the interval 0.3°–2.2°).

The above-noted feature of the highly elongated elliptic orbit of Mercury leads to the fact that in the perihelion of its orbit Mercury is more than one and a half times closer to the Sun than in the aphelion. Taking into account its closest proximity to the Sun and the essential inclination of its orbit, we conclude that *the projection of the perihelion point of Mercury's orbit directly falls into a nearby vicinity of the Sun, namely, in the visible part of the solar corona* [70, 72], interpreted as the core of a rotating and gravitating spheroidal body (see Fig.7.4).

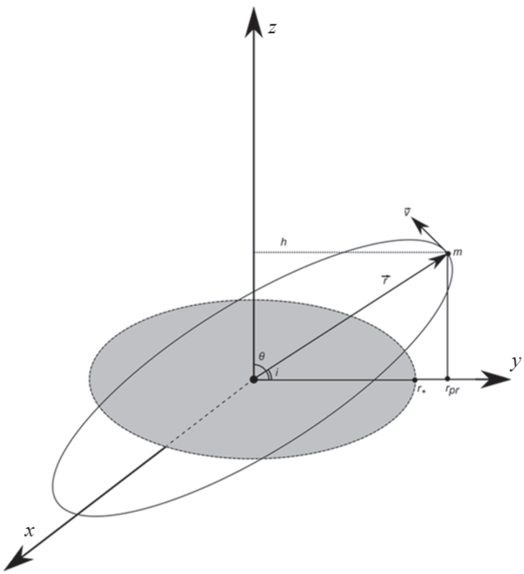


Figure 7.4. Graphic representation of the orbit of a moving planet near to an equatorial plane of the core of a gravitating and rotating spheroidal body

Indeed, if i is an angle of inclination of the orbital plane of a planet then the distance h from the moving planet to an axis of rotation of the core of a spheroidal body (perpendicular to the equatorial plane of a spheroidal body) is expressed through the distance r from the planet up to the center of a rotating and gravitating spheroidal body by the simple relation:

$$h = r\cos i\,. \qquad (7.3.3)$$

As the orbit of each planet (moving in the central gravitational field of a rotating spheroidal body) entirely lays in one plane characterized by a constant polar angle $\theta = \text{const}$ (Fig. 7.4) then given a relation $\theta = \pi/2 - i$ formula (7.3.3) takes the form:

$$h = r\sin\theta\,. \qquad (7.3.4)$$

As already noted, for the planet Mercury which is located closest to a visible part of the Solar corona (or the core of a rotating and gravitating spheroidal body whose equatorial plane is allocated by the gray color in Fig. 7.4), we estimate with sufficient accuracy that $h = r_{pr}$, where $r_{pr} = \sqrt{2}\cdot r_*$ (see (2.2.6) in Section 2.2). Therefore, it follows from (7.3.4) directly that:

$$\sin\theta = \frac{r_{pr}}{r} = \frac{\sqrt{2}\cdot r_*}{r}\,. \qquad (7.3.5)$$

So, supposing $\sin\theta = r_{pr}/r = \sqrt{2}\cdot r_*/r$, then according to (7.3.5), the formula for estimation of gravitational potential (7.3.2) at closer distance from the core of a rotating spheroidal body (in vicinities of Mercury' orbit) can be presented as follows:

$$\varphi_g(r)|_{r>>r_*} = -\frac{\gamma M}{r}\cdot\left(1+\frac{\varepsilon_0^{\;2} r_*^{\;2}}{r^2}\right) = -\frac{\gamma M}{r}\cdot\left(1+\frac{\delta_0^{\;2}}{r^2}\right), \qquad (7.3.6)$$

where $\delta_0 = \varepsilon_0 r_* = \varepsilon_0 / \sqrt{\alpha}$ is a value with the dimension of length which is small in comparison with distance from the moving planet up to the center of a rotating and gravitating spheroidal body (taking into account that $r_* << r$ and $\varepsilon_0{}^2 << 1$). Let us note that an analogous formula (7.3.6) was obtained by H. Alfvén and G. Arrhenius [9 p. 45] bearing in mind the following: "The axial rotations (spins) of the planets change the shape of these bodies from spherical to ellipsoidal. We can consider their gravitation to consist of a Coulomb field from a sphere, on which is superimposed the field from the 'equatorial bulge'".

Substituting (7.3.6) into the Binet formula (7.2.16) we obtain the equation of the *disturbed orbit* of a planet (the Mercury) in the vicinity of the core of a rotating spheroidal body:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{r^2}{C^2}\cdot\frac{\gamma M}{r^2}\left(1+\frac{3\delta_0{}^2}{r^2}\right)=\frac{\gamma M}{C^2}\cdot\left(1+\frac{3\delta_0{}^2}{r^2}\right), \qquad (7.3.7)$$

where $C = r^2\dot{\varepsilon}$. After the traditional substitution $q = 1/r$ equation (7.3.7) becomes the following:

$$\frac{d^2 q}{d\varepsilon^2} = -q + \frac{\gamma M}{C^2} + \frac{3\gamma M \delta_0{}^2}{C^2} q^2 . \qquad (7.3.8)$$

Multiplying both parts (7.3.8) on $2\frac{dq}{d\varepsilon}$ we can transform this equation into the kind:

$$\frac{d}{d\varepsilon}\left[\left(\frac{dq}{d\varepsilon}\right)^2\right]=-\frac{d}{d\varepsilon}\left[q^2\right]+\frac{2\gamma M}{C^2}\cdot\frac{dq}{d\varepsilon}+\frac{2\gamma M\delta_0{}^2}{C^2}\cdot\frac{dq}{d\varepsilon}\left[q^3\right]. \qquad (7.3.9)$$

Integrating (7.3.9), we calculate the first integral of this equation:

$$\left(\frac{dq}{d\varepsilon}\right)^2 = -q^2 + \frac{2\gamma M}{C^2}\cdot q + \frac{2\gamma M\delta_0{}^2}{C^2}\cdot q^3 + c_1 =$$

$$= c_1 + \frac{2\gamma M}{C^2} \cdot q - q^2 + \frac{2\gamma M \delta_0{}^2}{C^2} \cdot q^3, \qquad (7.3.10)$$

whence we find that:

$$\frac{dq}{d\varepsilon} = \pm \sqrt{\frac{2\gamma M}{C^2} q \cdot \left(\delta_0{}^2 q^2 + 1\right) - \left(q^2 - c_1\right)}.$$

Separating variables in the given equation we obtain:

$$d\varepsilon = \pm \frac{dq}{\sqrt{\frac{2\gamma M}{C^2} q \cdot \left([\delta_0 q]^2 + 1\right) - \left(q^2 - c_1\right)}}. \qquad (7.3.11)$$

Supposing $c_1 = -1/\delta_0{}^2$, $\kappa = \dfrac{C^2}{2\gamma M \delta_0}$, let us rewrite (7.3.11) in the form:

$$d\varepsilon = \pm \frac{dq}{\sqrt{\frac{2\gamma M}{C^2} q \cdot \left([\delta_0 q]^2 + 1\right) - \left(q^2 + \frac{1}{\delta_0{}^2}\right)}} =$$

$$= \pm \frac{dq}{\sqrt{\left([\delta_0 q]^2 + 1\right) \cdot \left(\frac{2\gamma M}{C^2} q - \frac{1}{\delta_0{}^2}\right)}} = \pm \frac{\sqrt{\kappa} \cdot dq}{\sqrt{[\delta_0 q]^2 + 1} \cdot \sqrt{\delta_0 q - \kappa}}$$

$$= \pm \frac{(2\sqrt{\kappa} / \delta_0) \cdot d(\sqrt{\delta_0 q - \kappa})}{\sqrt{[\delta_0 q]^2 + 1}}. \qquad (7.3.12)$$

Introducing the designation $s = \sqrt{\delta_0 q - \kappa}$ and then integrating (7.3.12) we obtain:

$$\varepsilon = \pm \frac{2\sqrt{\kappa}}{\delta_0} \int \frac{ds}{\sqrt{[s^2 + \kappa]^2 + 1}} + c_2 =$$

$$= \pm \frac{2\sqrt{\kappa}}{\delta_0} \cdot \int \frac{ds}{\sqrt{s^4 + 2\kappa s^2 + (\kappa^2 + 1)}} + c_2, \qquad (7.3.13)$$

where c_2 is a constant of integration.

The relation (7.3.13) led to the elliptic integral of 1-st kind which expresses ε as a function from s [301 p. 274]. Taking inverse functions and values, we can express r as a Jacobi elliptic function from ε. As a result, the equation of the disturbed orbit of a planet (near to the core of a rotating spheroidal body) describes spirals for which a circle passing through the origin of coordinates and a circle with the origin in the center are limiting cases [96], that is, *the disturbed trajectories of the finite motion of a planet are not closed* [158 p. 46].

To obtain an equation of the disturbed orbit of the planet Mercury in an explicit form we shall attempt to make some simplification of the initial equation (7.3.10). First of all, let us note that a similar equation (7.3.10) was derived by Einstein within the framework of his GR theory. Indeed, if we introduce designations $2\gamma M\delta_0{}^2 / C^2 = A$, $2\gamma M / C^2 = A / B^2$, and $c_1 = 2D / B^2$ then this equation (7.3.10) becomes exactly equation (11)[3] from Einstein's work [303] (a similar equation (18) was also deduced by K. Schwarzschild in his work [302], [304 p. 206]). As Einstein indicated in his work, this equation differs from the corresponding equation of Newton's theory only in the last term, Aq^3, in the right-hand part which allowed him to replace, with sufficient accuracy, the solution of this equation in the form of an elliptic integral by *pseudo-elliptic* integral (which could be calculated using elementary functions) [301 p. 105].

[3] With the original numeration of formula used in the work [303]

In this context, within the framework of the proposed statistical theory of gravitating spheroidal bodies, we can also introduce convenient notations $2\gamma M\delta_0^{\ 2}/C^2 = \sigma$ and $2\gamma M/C^2 = \sigma/\delta_0^{\ 2}$ with the aim of expressing a polynomial $K(q) = \sigma q^3 - q^2 + (\sigma/\delta_0^{\ 2})q + c_1$ in the right-hand part of the equation (7.3.10) through a corresponding polynomial $N(q) = -q^2 + (\sigma/\delta_0^{\ 2})q + c_1$ in the right-hand part of equation of Newton's theory based on an algorithm for division of two polynomials with a residue:

$$K(q) = L(q)N(q) + R(q)\,, \qquad (7.3.14)$$

where $R(q) \equiv K(q) \bmod N(q)$ is a polynomial residue, besides $\deg R(q) < \deg N(q)$ [305]. Direct application of the algorithm of the division of polynomials with the residue (procedure of division by "corner") gives [72, 73, 74]:

$$L(q) = -\sigma q + 1 - \sigma^2/\delta_0^{\ 2}\,, \qquad (7.3.15a)$$

$$R(q) = (\sigma c_1 - \sigma^3/\delta_0^{\ 4})q + c_1 \cdot \sigma^2/\delta_0^{\ 2}\,. \qquad (7.3.15b)$$

So, according to formulas (7.3.14) and (7.3.15a, b) the polynomial in the right-hand part of the equation (7.3.10) can be expressed as follows [72, 73, 74]:

$$K(q) = (1 - \sigma q - \frac{\sigma^2}{\delta_0^{\ 2}}) \cdot N(q) + (\sigma c_1 - \frac{\sigma^3}{\delta_0^{\ 4}})q + c_1 \cdot \frac{\sigma^2}{\delta_0^{\ 2}}\,. \qquad (7.3.16)$$

Using the results in Section 7.2, we suppose that the constant of integration c_1 should be proportional to the value $-(\gamma M/C^2)^2 = -(\sigma/2\delta_0^{\ 2})^2 = -\sigma^2/4\delta_0^{\ 4}$. Because of that $\sigma << 1$, we are limited by terms not higher than the second order of smallness relative to σ in this formula, that is, $O(\sigma^2)$:

$$K(q) \approx (1 - \sigma q - \frac{\sigma^2}{\delta_0^{\ 2}}) \cdot N(q) = (1 - \sigma q - \frac{\sigma^2}{\delta_0^{\ 2}}) \cdot (-q^2 + \left(\frac{\sigma}{\delta_0^{\ 2}}\right)q + c_1)\,. \qquad (7.3.17)$$

At last, substituting the formula (7.3.17) into equation (7.3.10), we obtain [72, 74]:

$$\frac{dq}{d\varepsilon} = \pm\sqrt{K(q)} = \pm\sqrt{\left(1-\sigma q-\frac{\sigma^2}{\delta_0{}^2}\right)\cdot\left(-q^2+\left(\frac{\sigma}{\delta_0{}^2}\right)q+c_1\right)}, \quad (7.3.18)$$

whence, after separation of variables and integration of equation (7.3.18), we find that:

$$d\varepsilon = \pm\frac{dq}{\sqrt{\left(1-\sigma\left[q+\frac{\sigma}{\delta_0{}^2}\right]\right)\cdot\left(-q^2+\left(\frac{\sigma}{\delta_0{}^2}\right)q+c_1\right)}}. \quad (7.3.19)$$

Decomposing function $(1-\sigma[q+\sigma^2/\delta_0{}^2])^{-1/2}$ in the Maclaurin series we can write integral (7.3.19) in the form:

$$\varepsilon = \pm\int\frac{\{1+(\sigma/2)\cdot[q+\sigma/\delta_0{}^2]\}dq}{\sqrt{-q^2+(\sigma/\delta_0{}^2)q+c_1}}+c_2 =$$

$$= \pm\left(1+\frac{\sigma^2}{2\delta_0{}^2}\right)\cdot\int\frac{dq}{\sqrt{-q^2+(\sigma/\delta_0{}^2)q+c_1}} \pm$$

$$\pm\frac{\sigma}{2}\cdot\int\frac{qdq}{\sqrt{-q^2+(\sigma/\delta_0{}^2)q+c_1}}+c_2, \quad (7.3.20)$$

where c_2 is a constant of integration. By analogy with the transformation of the radical in denominator of the right-hand part of equation (7.2.24), we shall write down radicals in denominators of subintegral expressions in the right-hand part of the equation (7.3.20):

$$\varepsilon = \pm\left(1+\frac{\sigma^2}{2\delta_0{}^2}\right)\cdot\int\frac{dq}{\sqrt{\left((\sigma/2\delta_0{}^2)^2+c_1\right)-\left(q-\sigma/2\delta_0{}^2\right)^2}} \pm$$

$$\pm\frac{\sigma}{2}\cdot\int\frac{qdq}{\sqrt{\left((\sigma/2\delta_0^{\ 2})^2+c_1\right)-\left(q-\sigma/2\delta_0^{\ 2}\right)^2}}+c_2. \qquad (7.3.21)$$

Introducing the previous notations (as in Section 7.2) $k^2=(\sigma/2\delta_0^{\ 2})^2+c_1$ and $s=q-\sigma/2\delta_0^{\ 2}$ we can note that integrals in the right-hand part of the equation (7.3.21) have a meaning only if $k^2>s^2$, that is, $\left|\frac{s}{k}\right|<1$. Taking this into account and choosing the bottom sign in equation (7.3.21) we obtain [73]:

$$\varepsilon=-\left(1+\frac{\sigma^2}{2\delta_0^{\ 2}}\right)\cdot\int\frac{ds}{\sqrt{k^2-s^2}}-\frac{\sigma}{2}\cdot\int\frac{(s+\sigma/2\delta_0^{\ 2})ds}{\sqrt{k^2-s^2}}+c_2=$$

$$=\left(1+\frac{\sigma^2}{2\delta_0^{\ 2}}\right)\cdot\arccos\left(\frac{s}{k}\right)+\frac{\sigma^2}{4\delta_0^{\ 2}}\cdot\arccos\left(\frac{s}{k}\right)-\frac{\sigma}{2}\cdot\left(-\sqrt{k^2-s^2}\right)+c_2=$$

$$=\left(1+\frac{3\sigma^2}{4\delta_0^{\ 2}}\right)\cdot\arccos\left(\frac{s}{k}\right)+\frac{\sigma}{2}\cdot\sqrt{k^2-s^2}+c_2. \qquad (7.3.22)$$

Using the above-mentioned condition $|s/k|<1$, let us present the function $\sqrt{k^2-s^2}$ belonging to the right-hand part of the equation (7.3.22) in the form:

$$\sqrt{k^2-s^2}=k\sqrt{1-(s/k)^2}=k\sin\left[\arccos\left(\frac{s}{k}\right)\right],$$

and then expand it in the Maclaurin series:

$$\sqrt{k^2-s^2}=k\sin\left[\arccos\left(\frac{s}{k}\right)\right]=$$

$$=k\cdot\arccos\left(\frac{s}{k}\right)-k\cdot\frac{\left[\arccos(s/k)\right]^3}{3!}+\dots \qquad (7.3.23)$$

Being restricted by the first term in the Maclaurin series (7.3.23) and substituting it in Eq. (7.3.22) we can simplify this equation essentially [72, 74]:

$$\varepsilon = \left(1 + \frac{\sigma \cdot k}{2} + \frac{3\sigma^2}{4{\delta_0}^2}\right) \cdot \arccos\left(\frac{s}{k}\right) + c_2 , \qquad (7.3.24)$$

whence:

$$s = k \cdot \cos\left(\frac{\varepsilon - c_2}{1 + k\sigma/2 + 3\sigma^2/4{\delta_0}^2}\right). \qquad (7.3.25)$$

Returning to the variable q and then r we can write (7.3.25) in the form:

$$\frac{1}{r} - \frac{\sigma}{2{\delta_0}^2} = k \cdot \cos\left(\frac{\varepsilon - c_2}{1 + k\sigma/2 + 3\sigma^2/4{\delta_0}^2}\right),$$

whence:

$$r = \frac{2{\delta_0}^2/\sigma}{1 + (2k{\delta_0}^2/\sigma) \cdot \cos\left(\dfrac{\varepsilon - c_2}{1 + k\sigma/2 + 3\sigma^2/4{\delta_0}^2}\right)}. \qquad (7.3.26)$$

Substituting into Eq. (7.3.26) the value of the parameter $k = \sqrt{(\sigma/2{\delta_0}^2)^2 + c_1}$ we obtain [73]:

$$r = \frac{2{\delta_0}^2/\sigma}{1 + \dfrac{2{\delta_0}^2}{\sigma}\sqrt{\left(\dfrac{\sigma}{2{\delta_0}^2}\right)^2 + c_1} \cdot \cos\left(\dfrac{\varepsilon - c_2}{1 + \dfrac{\sigma}{2}\sqrt{\left(\dfrac{\sigma}{2{\delta_0}^2}\right)^2 + c_1} + \dfrac{3\sigma^2}{4{\delta_0}^2}}\right)} =$$

$$=\frac{2\delta_0^{\ 2}/\sigma}{1+\sqrt{1+c_1\cdot\left(\frac{2\delta_0^{\ 2}}{\sigma}\right)^2}\cdot\cos\left(\frac{\varepsilon-c_2}{1+\left[3+\sqrt{1+c_1\cdot\left(\frac{2\delta_0^{\ 2}}{\sigma}\right)^2}\right]\cdot\frac{\sigma^2}{4\delta_0^{\ 2}}}\right)}. \qquad (7.3.27)$$

Because of the smallness of the parameter σ, that is, $\sigma \ll 1$, equation (7.3.27) becomes:

$$r=\frac{2\delta_0^{\ 2}/\sigma}{1+\sqrt{1+c_1\left(\frac{2\delta_0^{\ 2}}{\sigma}\right)^2}\cos\left(\left[1-[3+\sqrt{1+c_1\left(\frac{2\delta_0^{\ 2}}{\sigma}\right)^2}\,]\frac{\sigma^2}{4\delta_0^{\ 2}}\right](\varepsilon-c_2)\right)}. \qquad (7.3.28)$$

Taking into account the notation $\sigma=2\gamma M\delta_0^{\ 2}/C^2$, accepted above, we can determine the values $2\delta_0^{\ 2}/\sigma=C^2/\gamma M$ and $\sigma^2/4\delta_0^{\ 2}=(\gamma M\delta_0/C^2)^2$, so that equation (7.3.28) for a disturbed orbit of the planet Mercury takes the form [70, 72-74]:

$$r=\frac{C^2/\gamma M}{1+\sqrt{1+c_1\left(\frac{C^2}{\gamma M}\right)^2}\cos\left(\left[1-[3+\sqrt{1+c_1\left(\frac{C^2}{\gamma M}\right)^2}\,]\left(\frac{\gamma M\delta_0}{C^2}\right)^2\right](\varepsilon-c_2)\right)} \qquad (7.3.29)$$

As a result, comparing (7.3.29) with the polar equation of conic section with a focus in the origin of coordinates $r=\frac{p}{1+e\cos\varphi}$ [96, 158] we find that:

$$p=C^2/\gamma M; \quad e=\sqrt{1+c_1\cdot(C^2/\gamma M)^2}\ ;$$

$$\varphi = \left[1 - [3 + \sqrt{1 + c_1 \left(\frac{C^2}{\gamma M} \right)^2} \,] \cdot \left(\frac{\gamma M \delta_0}{C^2} \right)^2 \right] \cdot \left(\varepsilon - c_2 \right), \quad (7.3.30)$$

where

p is a parameter of an orbit,

e is an eccentricity of orbits,

and $c_2 = \varepsilon_*$ is a constant.

In addition, ε_* is an angle between the polar axis and the endpoint of the big axis directed to vertex [96]. As mentioned above in Section 7.2, the constants C^2, c_1, and c_2 are defined by initial conditions, and in turn, they define p and e using (7.3.30).

Taking into account the smallness of the parameter $\sigma = 2\gamma M \delta_0{}^2 / C^2$, equation (7.3.28) of the disturbed orbit of a planet cannot differ so considerably from equation (7.2.28) of orbit for a planet in a remote zone of a gravitational field of a rotating spheroidal body. In connection with that for the undisturbed orbit of a planet in a remote zone of a gravitational field, the constant of integration c_1 (in Section 7.2) is chosen to be equal, $c_1 = -(\gamma M / C^2)^2$. There is a reason to suppose that $c_1 \to -(\gamma M / C^2)^2$, besides $|c_1| < (\gamma M / C^2)^2$ and $c_1 < 0$. This means that, according to (7.3.30), the eccentricity e of the disturbed orbit should be $e = \sqrt{1 + c_1 \cdot (C^2 / \gamma M)^2} < 1$, that is, formula (7.3.29) expresses the equation of the "disturbed" ellipse in polar coordinates with the origin of coordinates in focus [72, 73, 74]:

$$r = \frac{C^2 / \gamma M}{1 + e \cdot \cos\left([1 - (3 + e) \cdot \delta_0{}^2 \cdot \left(\frac{\gamma M}{C^2} \right)^2] \cdot (\varepsilon - \varepsilon_*) \right)}, \quad (7.3.31)$$

that is, the planet Mercury is moving in a *precessing elliptic orbit* since there is a modulating multiplier of a phase (an azimuth angle $\varepsilon - \varepsilon_*$) in equation (7.3.31):

$$\eta = 1 - (3+e)\delta_0{}^2 \cdot \left(\frac{\gamma M}{C^2}\right)^2 . \qquad (7.3.32)$$

Moreover, the multiplier η is close to unity because $(\gamma M \delta_0 / C^2)^2 = \sigma^2 /(2\delta_0)^2 << 1$ [73]. Let us note that according to the derived formula (7.3.31) and the formula of analytical geometry the major semi-axis a of an ellipse is equal to:

$$a = \frac{C^2 / \gamma M}{1 - e^2},$$

so that we find the known formula [6, 8]:

$$C^2 = \gamma M a (1 - e^2) . \qquad (7.3.33)$$

Substituting (7.3.33) into formula (7.3.32) we obtain:

$$\eta = 1 - (3+e) \cdot \left(\frac{\delta_0}{a(1-e^2)}\right)^2 . \qquad (7.3.34)$$

So, according to formulas (7.3.31) and (7.3.34), at the full circle of a planet in the disturbed orbit the increment of a phase is given by:

$$\Delta\varphi = 2\pi\eta = 2\pi - \frac{2\pi(3+e)\cdot\delta_0{}^2}{a^2(1-e^2)^2} . \qquad (7.3.35)$$

The second summand represents the required angular movement of a Newtonian ellipse during one turn of a planet in the disturbed orbit, that is, displacement of the perihelion of orbit for the period is equal to the following angle [72, 73]:

$$\delta\varepsilon = \frac{2\pi(3+e)\cdot\delta_0{}^2}{a^2(1-e^2)^2} . \qquad (7.3.36)$$

Taking into account that the value $\delta_0 = \varepsilon_0 r_*$ according to (7.3.6), formula (7.3.36) becomes [70, 72-74]:

$$\delta\varepsilon = \frac{2\pi(3+e)\cdot\varepsilon_0^{\ 2}}{\alpha\cdot a^2(1-e^2)^2}, \qquad (7.3.37)$$

where through a and e the major semi-axis and the eccentricity of the orbit of Mercury are designated respectively, α is a parameter of gravitational compression and ε_0 is a geometrical eccentricity of the core of a rotating and gravitating spheroidal body (the Sun) [16]. Taking into account Kepler's third law [96, 158] and expressing from it a square of the major semi-axis, formula, (7.3.37) takes the form [70, 72-74]:

$$\delta\varepsilon = \frac{8\pi^3 a(3+e)\cdot\varepsilon_0^{\ 2}}{\alpha\cdot\gamma MT^2(1-e^2)^2}, \qquad (7.3.38)$$

where T is a period of the turn of Mercury around the Sun.

Before we begin to consider the calculation of the displacement of Mercury's perihelion according to the obtained formula (7.3.37), let us note that a similar formula of an estimation of the angular shift of Mercury' perihelion by period has been offered by L. Nottale [200]:

$$\delta\varepsilon = \frac{6\pi \in^2}{a^2(1-e^2)^2}, \qquad (7.3.39)$$

where $\in^2 = \delta A/2M$. Moreover, a value δA is treated as the difference of polar and equatorial moments of inertia for the oblate object, namely, the Solar system.

Moreover, in the above-mentioned work [303 p. 446], Einstein showed that under the condition of the whole turn of Mercury's perihelion moves on an angle:

$$\delta\varepsilon = 3\pi\frac{A}{a(1-e^2)} = \frac{6\pi\gamma M}{c^2 a(1-e^2)}, \qquad (7.3.40)$$

where c is the speed of light; in this connection he noted:

> Calculation gives the planet Mercury a turn of perihelion of 43" per century, while astronomers indicate $45'' \pm 5''$ as an inexplicable difference between observations and Newton's theory. This means

full agreement with observations.

In other his work "On the special and general theory of relativity," on the occasion of displacement of Mercury's perihelion Einstein wrote [302 p. 195–196]:

> If we calculate a gravitational field up to higher-order values and with the corresponding accuracy to calculate movement in an orbit of a mass point with infinitesimal mass then the following deviation from the laws of movement of planets of Kepler–Newton is obtained. The elliptic orbit of a planet undergoes the slow rotation in a direction of motion of the planet which is equal to
>
> $$\delta\varepsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1-e^2)}$$
>
> during one full turn of the planet. In this formula a means the major semi-axis, c is the speed of light in usual units, e is the eccentricity of the orbit, T is the period of a turn of the planet in seconds.
>
> For the planet Mercury, the rotation of an orbit equal to 43" a the century is obtained that precisely corresponds to the value established by astronomers (Leverrier). Astronomers have found that some part of the general movement of the perihelion of this planet is not explained by disturbing action of other planets and is equal to the pointed out value.

So, following formula (7.3.37), let us calculate the displacement of the perihelion of the orbit of Mercury based on the statistical theory of gravitating spheroidal bodies [72, 73]. First of all, according to (7.3.37), it is necessary to estimate α (which is a parameter of gravitational condensation of a spheroidal body, that is, of the Sun) based on an estimation of the linear size of its core, that is, *of the thickness of a visible part of the solar corona* (see Fig. 7.5).

As we know, the solar corona represents the external layers of an atmosphere of the Sun, and it extends, at least, up to the borders of our Solar system in the form of "a solar wind," that is, our Earth, along with the other planets of the

Solar system, could be said to lie within the solar corona. In this connection, *the Sun together with the solar corona can be described by the model of a rotating and gravitating spheroidal body* [72, 73].

The spectrum of the solar corona consists of three components called L-, K-, and F-components (see Fig. 7.6). In particular, the K-component represents a continuous spectrum of the corona, and in its background, one can see the emission L-component within the apparent interval 9'–10' from a visible edge of the Sun. Beginning with the apparent height at around 3' and above we see the Fraunhoffer's spectrum constituting the F-component of the solar corona.

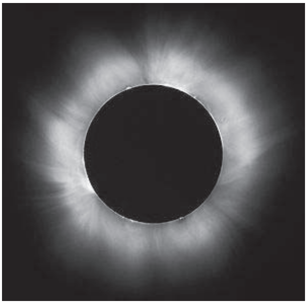


Figure 7.5. The solar corona imprinted during a solar eclipse in 1999

The F-spectrum of the corona is formed as a result of sunlight scattering on *particles of interplanetary dust*. In immediate proximity to the Sun the dust cannot exist, therefore the F-spectrum starts to dominate in the spectrum of the corona at some distance (at the apparent height 20') from the Sun.

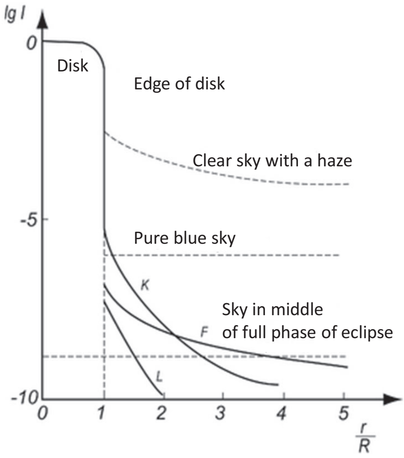


Figure 7.6. Graphic dependence of relative brightness of components of the spectrum of the solar corona on distance up to edge of a disk

As we can see in Fig. 7.6, a recession of plots of dependencies of the relative brightness of components of the spectrum of the Solar corona occurs at the distance of 3–3.5 radii from the center, that is, at 2–2.5 radii from the edge of the solar disk. Indeed, the well-known astronomer at NASA, S. Odenwald in his investigation "How thick is the solar corona?" wrote [306]:

> The corona actually extends throughout the entire solar system as a "wind" of particles, however, the densest parts of the corona are usually seen not more than about 1–2 solar radii from the surface or about 690,000 to 1.5 million kilometers at the equator. Near the poles, it seems to be a bit flatter....

Thus, accepting the thickness of a *visible part of the solar corona* as equal, $\Delta = 2R$ (here R is a radius of the solar disk), we find that $r_* = R + \Delta = 3R$. In other words, the parameter of gravitational condensation $\alpha = 1/r_*^2$ of a spheroidal body in the case of the Sun with its corona (for which the equatorial radius of disk $R = 6.955 \cdot 10^8$ m) can be estimated by the value [72, 73, 76]:

$$\alpha = \frac{1}{(3R)^2} \approx 2.29701177718 \cdot 10^{-19} \, (\mathrm{m}^{-2}). \qquad (7.3.41)$$

According to formula (3.8.14) in Section 3.8, if a and b are equatorial and polar radii of the disk then a relative flattening $e_c = (a-b)/a$ of a spheroid (a flattened ellipsoid) can be expressed through its geometrical eccentricity $\varepsilon_0 = \sqrt{a^2 - b^2}/a$ as $e_c = 1 - \sqrt{1-\varepsilon_0^2}$ whence, in view of the value of flatness for the Sun $e_c = 9 \cdot 10^{-6}$, we find that the square of geometrical eccentricity of the core of a rotating spheroidal body, that is, the solar disk together with a visible part of the solar corona, is equal:

$$\varepsilon_0^2 = 1 - (1 - e_c)^2 \approx 1.7999919 \cdot 10^{-5}. \qquad (7.3.42)$$

At last, taking into account that the major semi-axis of the orbit of the planet Mercury is

$$a = 5.7909068 \cdot 10^{10} \, (\mathrm{m}), \qquad (7.3.43)$$

and its orbital eccentricity is equal to

$$e = 0.20530294, \qquad (7.3.44)$$

let us substitute all the mentioned values (7.3.41)–(7.3.44) into formula (7.3.37) and calculate the angular displacement of the perihelion of Mercury's orbit for one circle [72-74]:

$$\delta\varepsilon = \frac{6.28318530718 \cdot 3.20530294 \cdot 1.7999919 \cdot 10^{-5}}{2.29701177718 \cdot 10^{-19} \cdot 3.35346015663 \cdot 10^{21} \cdot 0.91747796891} \approx$$
$$\approx 0.10580169298''. \qquad (7.3.45)$$

Taking into account that the sidereal period of a turn of Mercury is equal to 87.969 terrestrial days then during one terrestrial year Mercury performs 4.15214450545 orbits around the Sun so that angular displacement of the perihelion of its orbit for one terrestrial year is to be $4.15214450545 \cdot \delta\varepsilon = 0.43930391816''$. Thus, according to the statistical theory of gravitating spheroidal bodies *the turn* *of the perihelion of Mercury's orbit is equal to 43.93 angular seconds per century* which is consistent with the conclusions of Einstein's GR and astronomical observations (see Table 7.1) because an accuracy of modern astronomical measurements of movement of the perihelion of Mercury is 0.4500" per century from Clemence's analysis [168, 307].

Let us note that the calculated values of the parameter of gravitational compression (7.3.41) and the square of geometrical eccentricity (7.4.42) of a rotating spheroidal body allow us to estimate a characteristic length δ_0:

$$\delta_0 = \varepsilon_0 r_* = \sqrt{\varepsilon_0^2 / \alpha} = 8.852249876 \cdot 10^6 \,(\mathrm{m}), \qquad (7.3.46)$$

which appears considerably smaller than the average distance from the Sun as far as Mercury $r = 5.79 \cdot 10^{10}\,(\mathrm{m})$, as it was supposed initially at the derivation of formula (7.3.6). At the same time, the value $r_* = 1/\sqrt{\alpha} = 2.0865 \cdot 10^9$ (m) is almost 28 times smaller than the average distance r from the Sun to Mercury. This made it possible to legitimately use it in the above calculations for the estimation (7.3.1) of the gravitational potential in the remote zone of a rotating spheroidal body. In this connection, the formula obtained, (7.3.37), is suitable for calculating angular displacements of perihelia of the subsequent planets (after Mercury) though it is obvious that because of even greater deviation of the characteristic length δ_0 and the average distance r from the Sun up to any of these planets, angular displacements of the perihelia of their orbits will be negligible. As Einstein remarked [303]:

> For the Earth and Mars astronomers point to the turn 11" and 9" a century accordingly whereas our formula gives only 4" and 1". However, owing to small eccentricity of orbits of these planets observational data are insufficiently exact.

Table 7.1. Data on angular shifts of perihelia of orbits of the planets of the Solar system calculated according to proposed theories and modern astronomical observations

Planet	The angle of displacement of the perihelion of the orbit of a planet (in angular seconds,")		
	General relativity theory of Einstein	Astronomical observations	Statistical theory of gravitating spheroidal bodies
Mercury	43.03	43.11 ± 0.45	43.93
Venus	8.3	3.4 ± 4.8	4.24
Earth	3.8	5.0 ± 1.2	1.36
Mars	1.4	1.1 ± 0.3	0.34

Indeed, modern data [302 p. 447] in Table 7.1 about turns of perihelia in the orbits of the neighboring planets to Mercurfy testify in favor of both Einstein's GR and the proposed statistical theory of gravitating spheroidal bodies [72, 73]. Unlike Einstein's formula (7.3.40), the proposed formula (7.3.37) depends on both the parameter of flatness ε_0 of a star (the Sun) and its *evolutionary* parameter gravitational condensation $\alpha = \alpha(t)$, that is, formula (7.3.37) (or (7.3.38)) takes into account the dynamics of the star.

Conclusion and Comments

In this chapter, we have considered the statistical theory of gravitating spheroidal bodies to develop and apply a model of formation and self-organization in the case of the Solar system. In this context, this chapter also investigated the orbits of moving planets and bodies in the centrally symmetric gravitational field of a gravitating and rotating spheroidal body during the *planetary stage* of its evolution.

Though orbits of moving bodies and particles into a flattened rotating spheroidal body are initially circular, they could be distorted by collisions with planetesimals and gravitational interactions with neighboring originating protoplanets during the evolutionary process of protoplanetary formation. In reality, at first, the process of evolution of gravitating and rotating spheroidal body leads to their flattening. After that, the evolutionary process results in their decay into forming protoplanets. This chapter showed that the orbits of moving particles are formed by the action of a centrally symmetric gravitational field mainly on the later stages of the evolution of a gravitating and rotating spheroidal body, that is, when the particle orbits become Keplerian.

In Section 7.1, the proposed theory started with the conception for forming a rotating spheroidal body as the protoplanetary system from a protoplanetary nebula (or proto-Sun inside a presolar nebula) using the above-derived distribution function and the density mass function (7.1.1a)–(7.1.1c). The estimation of the gravitational potential (7.1.24) in a remote zone of a uniformly rotating spheroidal body based on the general solution (7.1.3) of the Poisson equation was then obtained. Section 7.1 also verified the accuracy of the derived estimation (7.1.24) by the use of the general expression (3.6.15a) for the gravitational potential of an axially symmetric spheroidal body (see Section 3.6). Indeed, using (3.6.15a), as shown in Section 7.1, the gravitational potential in a remote zone of the gravitational field of a rotating spheroidal body (when $r >> r_{pr}$) is estimated by formula (7.1.32c) whose particular case is the estimation explained above (7.1.24) at $\xi = 0$.

In Section 7.2, the planetary stage of evolution of a gravitating and rotating spheroidal body was considered. This section investigated the orbits of moving planets and bodies in the centrally symmetric gravitational field of a gravitating and rotating spheroidal body during the planetary stage of its evolution. To this end, the calculation of the orbit of a planet (for example, one belonging to our Solar system) in the centrally symmetric gravitational field based on Binet's differential equation (7.2.16) was carried out. In particular, using the obtained approximation of the gravitational potential in the remote zone (7.2.18) and solving the respective Binet equation (7.2.21), relation (7.2.29) for the Keplerian orbit of a planet in the gravitational field of a rotating spheroidal body was derived. As noted here, the formation of planets is possible not only based on particles of the protoplanetary gas-dust cloud (modeled by a gravitating and rotating spheroidal body) but also through the capture and joining of bodies moving in close orbits (meteorites, asteroids, etc.) in a gravitational field of a star (the Sun).

In Section 7.3, calculation of the orbit of the planet Mercury, as well as estimation of angular displacement of Mercury's perihelion based on the statistical theory of gravitating spheroidal bodies was carried out. As noted here, because of its close proximity to the Sun and the essential inclination of its orbit, the projection of the perihelion point of Mercury's orbit falls directly into the close vicinity of the Sun, namely, in the visible part of the solar corona, interpreted as the core of a rotating and gravitating spheroidal body. In this connection, as is well known, the angular displacement of a Newtonian ellipse is observed during one rotation of Mercury in orbit, that is, the secular shift of the perihelion of Mercury's orbit occurs. Using estimation (7.3.6) of the gravitational potential in the case of closer distance from the core of a rotating spheroidal body (in the vicinities of Mercury' orbit) and solving Binet's "disturbed" equation (7.3.7), an equation for *precessing elliptic* orbit (7.3.31) of the planet Mercury was derived. Taking into account equation (7.3.31), formula (7.3.37) for calculating the displacement of the perihelion of Mercury'orbit for the period was proposed in Section 7.3. As a result, this section shows that according to the proposed statistical theory of gravitating spheroidal bodies the turn of the perihelion of Mercury's orbit is equal to 43.93" per century. That is consistent with the conclusions of Einstein's GR theory (his analogous estimation was equal to 43.03") and astronomical observation data (43.11" ± 0.45").

CHAPTER EIGHT

THE DERIVATION OF THE UNIVERSAL STELLAR LAW FOR EXTRASOLAR SYSTEMS

Between 1911 and 1924, the astronomers Russell, Hertzsprung, and Eddington established that for stars of the Main Sequence there is a dependence of luminosity of a star on the temperature of its stellar surface (the diagram of Hertzsprung–Russell). There is also a connection between luminosity L and mass M of star (the diagram of mass luminosity) [1, 109]. According to this diagram for stars of the Main Sequence, the mass-luminosity dependence looks like $L \propto M^{s}$, where $s = 2.6$ for stars of small mass ($-1.1 < \lg M / M_{Sun} < -0.2$), $s = 4.2$ for stars of medium mass ($-0.2 < \lg M / M_{Sun} < 0.4$), and $s = 3.3$ for stars of large mass ($0.6 < \lg M / M_{Sun} < 1.7$), M_{Sun} is the mass of the Sun. In Sections 2.10, 6.2, and the monograph [73 p. 416], the different versions of invariant relations between the temperature T, the concentration n, and the parameter of gravitational condensation α of Sun-like stars have been derived using the models of rotating and gravitating spheroidal bodies (see Theorem 2.2, formulas (2.10.16) and (6.2.8a)–(6.2.8c)). Recently, P. Pintr et al. have found heuristic regression dependences, that is, they have studied the regression dependence of the distance of planets a_n from the central stars on the parameter of specific angular momentum $a_n v_n$ (a_n is a planetary distance and v_n is a planetary velocity) and then they have applied the regression analysis to other physical parameters of stars, namely, $a_n T_{\text{eff}}$, $a_n L$, and $a_n J$, where T_{eff} is an effective temperature of a stellar surface, L is a luminosity of a star, and J is a stellar irradiance for the multi-planet extrasolar systems [308]. This raises a question as to *whether there exists, as in the Kepler laws, a universal law for the planetary systems connecting the temperature, size, and mass of each star?*

According to the considered statistical theory of gravitating spheroidal bodies (see Chapters 1–5) under the application of laws of celestial mechanics in conformity with cosmogonical bodies (especially, to the stars), it is necessary to take into account an extended substance called the *stellar corona*. In this connection, the stellar corona together with the star's core can be described by the model of a rotating and gravitating spheroidal body. Moreover, the parameter of gravitational condensation α of a spheroidal body (describing the Sun, in particular) has been estimated through the linear size of its core, that is, by *the thickness of a visible part of the solar corona* (see Section 7.3 of the previous Chapter 7). Indeed, NASA astronomer Dr. S. Odenwald in his notice "How thick is the solar corona?" wrote [306]:

> The corona actually extends throughout the entire solar system as a "wind" of particles, however, the densest parts of the corona is usually seen not more than about 1–2 solar radii from the surface or about 690,000 to 1.5 million kilometers at the equator. Near the poles, it seems to be a bit flatter....

A recession of plots of dependencies of the relative brightness of components of the spectrum of the solar corona occurs at a distance of 3–3.5 radii from the center, that is, at 2–2.5 radii from the edge of the solar disk (see Fig. 7.6). Thus, accepting the thickness of a visible part of the solar corona equal to $\Delta = 2R$ (here R is the radius of the solar disk) we find that $r_* = R + \Delta = 3R$, where

$r_* = 1/\sqrt{\alpha}$. In other words, the parameter of gravitational condensation $\alpha = 1/r_*^2$ of a spheroidal body in the case of the Sun with its corona (for which the equatorial radius of the disk $R = 6.955 \cdot 10^8$ m) can be estimated by the value following formula (7.3.41) from Section 7.3 (see also formula (II.2) and Ref. [72, 73]):

$$\alpha = \frac{1}{(3R)^2} \approx 2.29701177718 \cdot 10^{-19}\,(\text{m}^{-2}).$$

So, the procedure for finding α is based on the known 3σ-rule in the statistical theory, where $\sigma = 1/\sqrt{\alpha}$ is a root-mean-square deviation of a random variable.

Indeed, taking into account the solar corona in calculating the perturbed orbit of Mercury allows us to estimate the displacement of the perihelion of Mercury's orbit per period within the framework of the statistical theory of gravitating spheroidal bodies. As we know, Newton's law, using the theory of general relativity (GR), found a solution to the problem of Mercury [81, 303]. Nevertheless, we note that from the general position of the statistical theory of gravitating spheroidal bodies, the points of view of both Leverrier (on the existence of unknown matter) and Einstein (on the insufficiency of the Newton's theory) are almost the same (see Section 7.3). Indeed, there exists plasma as well as a gas-dust substance around the core of a cosmogonical body (in particular, the solar corona in the case of the Sun), that is, the account of the circumstances that forming cosmogonical bodies have no precise outlines and are represented by means of spheroidal forms demands some specification in Newton's law in connection with a gravitating spheroidal body.

Using the Binet formula the equation of the disturbed orbit of a planet (Mercury) in the vicinity of the core of a rotating and gravitating spheroidal body has been derived (see Section 7.3). The obtained relationship expresses the equation of the so-called "disturbed" ellipse in polar coordinates with the origin of coordinates in focus, that is, the planet Mercury is moving in a *precessing elliptic orbit* since there is a modulating multiplier of the phase (or azimuth angle). So, within the framework of the statistical theory of gravitating spheroidal bodies the required angular motion of a Newtonian ellipse during one turn of Mercury in the disturbed orbit (or the displacement of the perihelion of its orbit in the period) was estimated in Section 7.3 (see formula (7.3.37)) and [72, 73]:

$$\delta\varepsilon = \frac{2\pi(3+e)\cdot{\varepsilon_0}^2}{\alpha \cdot a^2(1-e^2)^2},$$

where through a and e a major semi-axis and an eccentricity of the disturbed orbit are designated respectively, α is a parameter of gravitational condensation, and ε_0 is a geometrical eccentricity of the core of a rotating and gravitating spheroidal body (the Sun). Thus, according to the proposed formula (7.3.45), the turn of the perihelion of Mercury's orbit is equal to 43.93" per century (see Table 7.1 in Chapter 7). This is consistent with conclusions of Einstein's theory of GR theory (his analogous estimation is 43.03") and astronomical observation data (43.11 ± 0.45") [72, 73].

This chapter also considers the solar corona in connection with so-called universal stellar law (USL) [75, 76] introduced in Section 8.2. Then it is taking into account in calculating the ratio of the temperature of the solar corona to an effective temperature of the Sun's surface and the modification of the USL in Section 8.3. To test the accuracy of USL for different types of stars, the temperature of the stellar corona is estimated in Section 8.4. Section 8.5 shows that knowledge of some characteristics for *multi-planet extrasolar systems* permits us to refine own parameters of stars. In reality, numerous papers are devoted to recent investigations of the exoplanetary systems (for example, [15, 76, 200, 201, 231, 233, 235, 238–240, 257, 259–261, 285–299, 308–318]). In this context, comparison with estimations of temperatures using the above-mentioned regression dependences for multi-planet extrasolar systems fully justifies the results obtained.

8.1. On the potential and potential energy of the gravitational field of a spheroidal body

According to the statistical theory of gravitating spheroidal bodies (see Chapters 1–4 and also [16, 73]), a mass density function (3.3.26c) of a uniformly rotating spheroidal body has the form:

$$\rho(r,\theta) = \rho_0 (1-\varepsilon_0^2) e^{-\alpha r^2 (1-\varepsilon_0^2 \sin^2\theta)/2}, \qquad (8.1.1)$$

and in the case of $\varepsilon_0^2 \to 0$ the mass density function of a slowly rotating spheroidal body becomes:

$$\rho(r) = \rho_0 \cdot e^{-\alpha r^2/2}, \qquad (8.1.2)$$

where

$\rho_0 = M(\alpha/2\pi)^{3/2}$ is a density in the center of a spheroidal body,

M is a mass of a spheroidal body, and

α is a parameter of gravitational condensation.

As follows from (8.1.2), the probability density function of a particle having distance r being confined between r and $r+dr$ from the center of a slowly rotating (immovable) spheroidal body can be expressed by the formula (2.1.18):

$$f(r) = 4\pi \left(\frac{\alpha}{2\pi}\right)^{3/2} r^2 e^{-\frac{\alpha}{2}r^2}. \qquad (8.1.3)$$

Let us calculate the characteristics of the gravitational field produced by a collection of isolated particles in the form of a spheroidal body. As shown in Section 3.6, the gravitational potential of a uniformly rotating spheroidal body with the mass density (3.3.26c) is described by formula (3.6.15c). In the particular case of $\varepsilon_0 \to 0$, formula (3.6.15c) gives the following result:

$$\varphi_g(r)\Big|_{\varepsilon_0 \to 0} = -\gamma M \frac{\alpha^{3/2}}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{\alpha}{2+\alpha s} r^2} \frac{ds}{(2+\alpha s)^{3/2}} \qquad (8.1.4)$$

where $\gamma = 6.673 \cdot 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$ is the Newtonian constant of gravitation. Using the sequence of transformations of Eq. (8.1.4) we can obtain expression of the form:

$$\varphi_g(r)\Big|_{\varepsilon_0 \to 0} = -\gamma M \frac{\alpha^{3/2}}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{\alpha}{2+\alpha s} r^2} \frac{ds}{(2+\alpha s)^{3/2}} =$$

$$= -\frac{2\gamma M \alpha^{3/2}}{\alpha\sqrt{\pi}} \frac{1}{r\sqrt{2}} \int_0^{r} e^{-\frac{\alpha r^2}{2+\alpha s}} d\left(\frac{r\sqrt{2}}{\sqrt{2+s\alpha}}\right). \qquad (8.1.5)$$

In other words, by changing variables:

$$r' = r\sqrt{\frac{2}{2+s\alpha}} \qquad (8.1.6)$$

in (8.1.5), we obtain the resulting formula:

$$\varphi_g(r)\Big|_{\varepsilon_0 \to 0} = -\sqrt{\frac{2}{\pi}} \gamma M \alpha^{3/2} \frac{1}{\alpha r} \int_0^{r} e^{-\alpha r'^2/2} dr' =$$

$$= -4\pi\gamma M \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \cdot \frac{1}{\alpha r} \int_0^{r} e^{-\alpha r'^2/2} dr' = -4\pi\gamma\rho_0 \frac{1}{\alpha r} \int_0^{r} e^{-\alpha r'^2/2} dr', \qquad (8.1.7)$$

that is, we derive formula (2.4.26) for the gravitational potential of a slowly rotating (immovable) spheroidal body (see Section 2.4):

$$\varphi_g(r) = -\frac{4\pi\gamma\rho_0}{\alpha r} \int_0^{r} e^{-\frac{\alpha}{2} r^2} dr. \qquad (8.1.8)$$

Unlike the resulting formula (8.1.8), where integration is performed by coordinate r (or distance from the center of a weakly rotating spheroidal body to the observation point), in formula (8.1.4), integration is carried out with respect to the *state parameter* s of the weakly rotating spheroidal body. Indeed, according to (8.1.4), we have:

$$\alpha(s) = \frac{\alpha}{2 + \alpha s} = \frac{\alpha / 2}{1 + (\alpha / 2)s}. \quad (8.1.9)$$

As shown in Section 4.1, in the state of virial equilibrium of an immovable spheroidal body (when $\mathrm{G}(t) = \mathrm{G_s} = \mathrm{const}$) formula (4.1.22) for the gravitational condensation parameter $\alpha(t)$ is valid:

$$\alpha(t) = \frac{\alpha_s}{1 - 2\alpha_s \mathrm{G_s} \cdot (t - t_s)}, \quad (8.1.10)$$

where $\alpha_s = \alpha(t_s)$ is the parameter corresponding to the stabilization. In the general case of an arbitrary gravitational compression function (GCF) $\mathrm{G}(t)$, a formula similar to (4.1.22) takes the form:

$$\alpha(t) = \frac{\alpha_0}{1 - 2\alpha_0 \int_{t_0}^{t} \mathrm{G}(t)dt}, \quad (8.1.11)$$

where $\alpha_0 = \alpha(t_0)$ is an initial parameter. If we rewrite formula (8.1.11) in the following form:

$$\alpha(t) = \frac{\alpha_0}{1 + 2\alpha_0 \int_{t}^{t_0} \mathrm{G}(t)dt}, \quad (8.1.12)$$

and compare it with (8.1.9), then the equivalence condition for these formulas means that:

$$\alpha_0 = \alpha / 2; \quad (8.1.13a)$$

$$s = 2\int_{t}^{t_0} \mathrm{G}(t)dt. \quad (8.1.13b)$$

In other words, the state parameter s of a gravitating spheroidal body is an integral time function of GCF, showing the history of the states of gravitational contraction of a spheroidal body: from the state of an infinitely spread molecular cloud ($s=0$ at $t=t_0$) to the state of unlimitedly compressed spheroidal body to a point ($s=\infty$ at $t=-\infty$). If a gravitationally compressible spheroidal body passes through an infinitely large number of exclusively equilibrium states, then rewriting formula (8.1.10) as follows:

$$\alpha(t)=\frac{\alpha_s}{1+2\alpha_s \mathrm{G}_s(t_s-t)}, \tag{8.1.14}$$

and then comparing it with (8.1.9), we obtain:

$$\alpha_s=\alpha/2; \tag{8.1.15a}$$

$$s=2\mathrm{G}_s(t_s-t). \tag{8.1.15b}$$

From (8.1.15b) it directly follows that the state parameter s is proportional to the inverse time, but according to the initial formula (8.1.10), the time of evolution of a gravitating spheroidal body goes from an unstable equilibrium state of a molecular cloud to a state of unlimited compression to a point. In this connection, it is advisable to rewrite formula (8.1.9) on the contrary, replacing s by $-s$, so that the state parameter would be proportional to the time of evolution of the spheroidal body:

$$\alpha(s)=\frac{\alpha}{2-\alpha s}=\frac{\alpha/2}{1-(\alpha/2)s}, \tag{8.1.16}$$

where $s=-\infty,...,-1,0$. In this form, formula (8.1.16) is completely equivalent to the original formulas (8.1.10) or (8.1.11) with the state $s=-\infty$ corresponding to the state of a spread molecular cloud, and the state $s=0$ respecting to the unlimited compression of the spheroidal body to a point. In this connection, formula (8.1.4) of the gravitational potential of a weakly rotating spheroidal body takes form:

$$\varphi_g\Big|_{\varepsilon_0\to 0}=-\gamma M\frac{\alpha^{3/2}}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-\frac{\alpha}{2-\alpha s}r^2}\frac{ds}{(2-\alpha s)^{3/2}}. \tag{8.1.17}$$

Using (8.1.17) we can estimate the gravitational potential of an immovable spheroidal body in the *near zone* $(r \to 0)$, for which we expand the exponential function in the Maclaurin series limiting it by the first terms:

$$\varphi_g\Big|_{\varepsilon_0 \to 0} = -\gamma M \frac{\alpha^{3/2}}{\sqrt{\pi}} \int_{-\infty}^{0} \left(1 - \frac{\alpha}{2-\alpha s} r^2\right) \frac{ds}{(2-s\alpha)^{3/2}} =$$

$$= -\gamma M \frac{\alpha^{3/2}}{\sqrt{\pi}} \{\int_{-\infty}^{0} \frac{ds}{(2-\alpha s)^{3/2}} - \alpha r^2 \int_{-\infty}^{0} \frac{ds}{(2-\alpha s)^{5/2}}\} =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \alpha^{3/2} \left\{\left(-\frac{1}{\alpha}\right) \int_{\infty}^{0} \frac{d(-\alpha s)}{(2-\alpha s)^{3/2}} + r^2 \int_{\infty}^{0} \frac{d(-\alpha s)}{(2-\alpha s)^{5/2}}\right\} =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \alpha^{3/2} \left\{\frac{1}{\alpha} \int_{2}^{\infty} \frac{d(2-\alpha s)}{(2-\alpha s)^{3/2}} - r^2 \int_{2}^{\infty} \frac{d(2-\alpha s)}{(2-\alpha s)^{5/2}}\right\}$$

Introducing the change of variables $t = 2 - \alpha s$ and then integrating the last expression we obtain:

$$\varphi_g\Big|_{\varepsilon_0 \to 0} = -\frac{\gamma M}{\sqrt{\pi}} \alpha^{3/2} \left(\frac{1}{\alpha} \cdot \frac{t^{-1/2}}{-1/2}\Bigg|_2^{\infty} - r^2 \frac{t^{-3/2}}{-3/2}\Bigg|_2^{\infty} \right) =$$

$$= -\frac{\gamma M}{\sqrt{\pi}} \alpha^{3/2} \left(\frac{2}{\alpha\sqrt{2}} - \frac{2r^2}{3 \cdot 2^{3/2}} \right) = -\frac{\gamma M}{\sqrt{2\pi}} \alpha^{3/2} \left(\frac{2}{\alpha} - \frac{r^2}{3} \right) =$$

$$= -2\pi \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \gamma M \left(\frac{2}{\alpha} - \frac{r^2}{3} \right) = -2\pi\gamma\rho_0 \left(\frac{2}{\alpha} - \frac{r^2}{3} \right). \quad (8.1.18a)$$

The same result is obtained using formula (8.1.8) in the case of small $r \ll 1$ (see formula (2.4.29) in Section 2.4 and also [16, 46, 73]):

$$\varphi_g(r)\Big|_{r \ll 1} = -\frac{4\pi\gamma\rho_0}{\alpha r} \int_0^r \left(1 - \frac{\alpha}{2} r^2\right) dr = -\frac{4\pi\gamma\rho_0}{\alpha r} \left(r - \frac{\alpha}{6} r^3 \right) =$$

$$= -2\pi\gamma\rho_0 \left(\frac{2}{\alpha} - \frac{r^2}{3} \right). \quad (8.1.18b)$$

Using the error function $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$ [128], we can transform (8.1.8) into (2.4.27):

$$\varphi_{\mathrm{g}}(r)=-\frac{4\pi\gamma\rho_0}{\alpha r}\sqrt{\frac{2}{\alpha}}\int_0^{r\sqrt{\frac{\alpha}{2}}}e^{-s^2}ds=-\frac{\gamma M}{r}\mathrm{erf}\left(r\sqrt{\frac{\alpha}{2}}\right). \qquad (8.1.19)$$

Since $\lim\limits_{r\to\infty}\mathrm{erf}\left(r\sqrt{\alpha/2}\right)=1$ then for large r the last expression turns into:

$$\varphi_{\mathrm{g}}(r)=-\frac{\gamma M}{r}. \qquad (8.1.20)$$

Relation (8.1.20), as we know, describes the gravitational potential of a field produced by one particle (or a spherical body) of mass M. Thus, expression (8.1.18b) describes the gravitational potential in the near zone of the field, while Eq. (8.1.20) describes that in the remote one.

The potential energy of a particle in a gravitational field is equal to its mass multiplied by the potential of the field. The potential energy of any distribution of masses is described by the expression [100]:

$$E_{\mathrm{g}}=\frac{1}{2}\int_V \rho\varphi_g dV, \qquad (8.1.21)$$

where ρ and φ_g are supposed to be the mass density (3.3.26b) and the gravitational potential (3.6.15a) of a uniformly rotating spheroidal body respectively. As shown in Section 3.7, the value of gravitational energy (3.7.8) of a uniformly rotating spheroidal body is equal to:

$$E_g=-\frac{\gamma M^2}{2}\cdot\sqrt{\frac{\alpha}{\pi}}\cdot\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}\mathrm{arccot}\frac{\sqrt{1-\varepsilon_0^2}}{\varepsilon_0}. \qquad (8.1.22)$$

In the particular case of $\varepsilon_0 \to 0$, formula (8.1.22) gives an estimation of the gravitational energy of a non-rotating (weakly rotating) spheroidal body:

$$E_{\mathrm{g}} = -\frac{\gamma M^2}{2}\sqrt{\frac{\alpha}{\pi}}. \qquad (8.1.23)$$

From (8.1.23) it is not difficult to see that:

$$\alpha = \frac{4\pi}{\gamma^2} \cdot \frac{E_{\mathrm{g}}^{\ 2}}{M^4}. \qquad (8.1.24)$$

According to (8.1.23) one can deduce a distance called the effective radius of the body (see Section 2.5):

$$r_+ = \sqrt{\frac{\pi}{\alpha}}. \qquad (8.1.25)$$

On account of (8.1.25) we obtain formula (2.5.9) [46, 73]:

$$E_{\mathrm{g}} = -\frac{\gamma M^2}{2r_+}. \qquad (8.1.26)$$

Let us consider a single small body (a test particle) of mass m in the gravitational field of its own collective body of mass M, situated at distance r from the body center. Now evaluate the potential energy of interaction of the particle and the spheroidal body:

$$E_{\mathrm{g_{int}}}(r) = m\varphi_{\mathrm{g}}(r). \qquad (8.1.27)$$

Using relations (8.1.2), (8.1.3), (8.1.21), and (8.1.27) one can easily calculate the energy of interaction of a spheroidal body and a test particle placed at different distances from the center of masses. Since energy depends on distance at which a test particle is, and particles themselves are distributed over space, one can determine the *average gravitational potential energy of interaction* of a test particle with a spheroidal body formed by a collection of such particles:

$$\overline{E}_{\text{g}} = \int_0^{\infty} E_{\text{g}_{\text{int}}}(r) f(r) dr = m \int_0^{\infty} \varphi_{\text{g}}(r) f(r) dr . \qquad (8.1.28)$$

As shown in Section 2.5, relation (8.1.28) takes the form (see formula (2.5.27) and [46, 73]):

$$\overline{E}_{\text{g}} = -\frac{4\pi\gamma\rho_0 m}{\alpha\sqrt{2}} = -\frac{\gamma m M}{r_+} . \qquad (8.1.29)$$

Let be $m = dM$ in (8.1.29). Taking this into account formula (8.1.29) is transformed into:

$$d\overline{E}_{\text{g}} = -\frac{\gamma M dM}{r_+} . \qquad (8.1.30)$$

By integrating both parts of (8.1.30) we obtain the formula (8.1.26), that is:

$$\int d\overline{E}_{\text{g}} = -\frac{\gamma M^2}{2r_+} = E_{\text{g}} . \qquad (8.1.31)$$

Thus, according to (8.1.26), (8.1.29), and (8.1.31), the potential energy of the gravitating spheroidal body is only then equal to the total average potential energy of the gravitational interaction of particles when these particles have infinitely small masses [46, 73]. Indeed, in this case, particles do not possess their own gravitational energy because, according to (8.1.26), it is a value of the second-order of smallness with respect to dM. In fact, putting the question in this way, we deal with massless particles whose gravitational energy is the potential energy of interaction of particles between one another (see Section 2.5). Further, supposing $m = m_0$ for each particle of one-component gas it follows from (8.1.25) and (8.1.29) that:

$$\alpha = \pi \left(\frac{\overline{E}_{\text{g}}}{\gamma m_0 M} \right)^2 , \qquad (8.1.32)$$

where $\overline{E}_g$ is an average gravitational potential energy of interaction of a particle m_0 with the gravitational field of a spheroidal body.

Thus, a spheroidal body has a "strict" (distinct) outline if the potential energy of the gravitational interaction of the body particles is sufficiently great, and the body mass itself and its particle masses are relatively small. Ordinary macroscopic bodies possess distinct outlines due to their relatively small masses and to sufficiently great energies of the interaction of particles the bodies consist of. On the contrary, giant cosmic objects (star formations, nebulae, etc.) have fuzzy contours because of their huge masses and enormous numbers of particles forming them. For instance, the Earth's atmosphere has an indistinct outline, the temperature being different at various altitudes, so does the Sun's photosphere which also lacks temperature balance.

8.2. Derivation of the universal stellar law

As shown in Section 6.2 (as well as the monograph [73 p.398]), under a condition of virial balance of a *rotating* and gravitating spheroidal body its parameter of gravitational condensation α is expressed by the formula (6.2.8a) :

$$\alpha = \frac{2\pi\gamma m_0 \rho_0}{3k_B T}, \qquad (8.2.1)$$

where γ and k_B are the constants of Newton and Boltzmann respectively, and m_0 is a mass of a particle.

Taking into account (8.1.2), that is, that according to the statistical theory of gravitating spheroidal bodies (see Chapter 2 and 3) the density in the center of a spheroidal body is equal to:

$$\rho_0 = M\left(\frac{\alpha}{2\pi}\right)^{3/2}, \tag{8.2.2}$$

let us rewrite formula (8.2.1) for the parameter of gravitational condensation:

$$\alpha = 2\pi \cdot \left(\frac{3k_\mathrm{B}T}{\gamma m_0 M}\right)^2. \tag{8.2.3}$$

It is not difficult to see that invariant relations follow from Eqs (8.2.1) and (8.2.3) directly (see also [73 p.416]):

$$\frac{\alpha T}{m_0\rho_0} = \mathrm{const} = \frac{2\pi\gamma}{3k_\mathrm{B}}; \tag{8.2.4a}$$

$$\sqrt{\alpha}\cdot\frac{m_0 M}{T} = \mathrm{const} = 3\sqrt{2\pi}\cdot\frac{k_\mathrm{B}}{\gamma}. \tag{8.2.4b}$$

Let us try to justify them. In reality, within a framework of the statistical theory of gravitating spheroidal bodies, that is, under supposition that the geometrical eccentricity of the core of a rotating and gravitating spheroidal body $\varepsilon_0 \to 0$, in particular, for the solar disk $\varepsilon_0^2 \approx 1.7999919\cdot 10^{-5}$ (see formula (7.3.42) in Section 7.3 and [72, 73]), the similar formula (8.1.32) has been derived for *one-component* gas in the previous Section 8.1.

The following result, as well as others of a more general kind, may also be obtained from a *virial* theorem of Poincaré [105] (see Theorem 1.3):

$$2E_\mathrm{k} + E_\mathrm{g} = 0, \tag{8.2.5}$$

where E_k is the total kinetic energy of translation and E_g is the total gravitational potential energy of a *steady state* system in the form of a collection of detached masses moving under no force except their own mutual gravitational attraction.

Let us apply Poincaré's virial theorem to a cloud-like configuration of ideal gas as a gravitating spheroidal body in the steady state. As shown in Section 1.2, we can write the kinetic energy E_{k} in the form [1]:

$$E_{\mathrm{k}} = \frac{1}{2}\sum_l m_{0l}\overline{v_l^2} = \sum_l \overline{E}_{\mathrm{k}l}, \tag{8.2.6}$$

where v_l is a velocity of translation of a l-th particle of mass m_{0l}, and $\overline{E}_{\mathrm{k}l}$ is an average kinetic energy of a moving l-th particle. The gravitational potential energy E_{g} may similarly be written in the form [1]:

$$E_{\mathrm{g}} = \frac{1}{2}\sum_l m_{0l}\overline{\varphi}_{\mathrm{g}l} = \sum_l \overline{E}_{\mathrm{g}l}, \tag{8.2.7}$$

where $\varphi_{\mathrm{g}l}$ is a gravitational potential at the l-th point occupied by mass m_0, and $\overline{E}_{\mathrm{g}l}$ is an average potential energy of interaction of a particle with a cloud. Taking into account (8.2.6) and (8.2.7), Poincaré's theorem (8.2.5) becomes:

$$\sum_l m_{0l}\left(\overline{v_l^2} + \frac{1}{2}\overline{\varphi}_{\mathrm{g}l}\right) = \sum_l \left(2\overline{E}_{\mathrm{k}l} + \overline{E}_{\mathrm{g}l}\right) = 0, \tag{8.2.8}$$

so that, in the steady state, the average value of $\overline{v_l^2}$, averaged over all the separate masses, is equal to the average value of $-\frac{1}{2}\overline{\varphi}_{\mathrm{g}l}$ [1], or the absolute value of average potential energy of interaction $\overline{E}_{\mathrm{g}l}$ of a particle is equal to the double average kinetic energy $\overline{E}_{\mathrm{k}l}$ of a moving particle [107] (because of the arbitrariness of l-th particle, we will further omit the index l in (8.2.8)).

As Sir J. Jeans remarked, this virial theorem of the kind (8.2.8) "provides a convenient rough measure of the average velocity of agitation of a system of gravitating masses in a steady state: it is equally applicable to systems of stars, star-clusters, nebulae, and masses of gravitating gas" [1 p.68].

Taking into account the condition of mechanical balance and using the virial theorem (8.2.8) as well as the theorem of uniform distribution of energy on freedom degrees for each particle [110] (under a condition $k_{\mathrm{B}}T >> h\nu$, where k_{B} and h are the constants of Boltzmann and Planck respectively, T is a temperature, ν is a frequency) we obtain:

$$-\overline{E}_{\mathrm{g}} = 2\overline{E}_{\mathrm{k}} = 2 \cdot \frac{i}{2} \cdot k_{\mathrm{B}}T = ik_{\mathrm{B}}T \, , \qquad (8.2.9a)$$

where $\overline{E}_{\mathrm{k}}$ is the kinetic energy of the heat movement of a particle, and i is a number of freedom degree of a moving particle. In particular $i = 3$ for the translational movement of a particle (for example, an atom of hydrogen H with mass $m_0 = m_{\mathrm{H}}$), so:

$$\overline{E}_{\mathrm{g}} = 2 \cdot \frac{3}{2} k_{\mathrm{B}}T = 3k_{\mathrm{B}}T \, . \qquad (8.2.9b)$$

If the particles which constitute the system are taken to be the molecules of a gas, or other independently moving units such as atoms, free electrons, ions, and so forth, then according to (8.2.9b) $\overline{v^2}$ is equal to $3k_{\mathrm{B}} / m_{\mathrm{H}}\overline{\mu}_{\mathrm{r}}$ times the temperature of the gas, where $\overline{\mu}_{\mathrm{r}}$ is its mean relative molecular weight. Thus, according to the virial theorem (8.2.8), the mean temperature of the gas is of the order of magnitude of:

$$-\frac{1}{2}\overline{\varphi}_{\mathrm{g}}\left(\frac{m_{\mathrm{H}}\overline{\mu}_{\mathrm{r}}}{3k_{\mathrm{B}}}\right) \propto \frac{\gamma M}{r}\left(\frac{m_{\mathrm{H}}\overline{\mu}_{\mathrm{r}}}{3k_{\mathrm{B}}}\right), \qquad (8.2.10)$$

so that the mean internal temperatures of different stars are approximately proportional to the values of $\overline{\mu}_{\mathrm{r}} M / r$ for these stars [1 p.68], that is, stellar temperatures are very high as a consequence of their enormous masses M.

From (8.1.32) and (8.2.9a) it follows directly that the parameter of gravitational condensation α of a spheroidal body, being formed by a collection of particles of a one-component ideal gas, is equal [75, 76]:

$$\alpha = \pi \cdot \left(\frac{ik_{\mathrm{B}}T}{\gamma m_0 M} \right)^2 = \pi \left(\frac{k_{\mathrm{B}}}{\gamma} \cdot \frac{i}{m_0} \cdot \frac{T}{M} \right)^2 . \qquad (8.2.11)$$

Nevertheless, even if the spheroidal body was initially formed exclusively based on a cloud of molecular hydrogen (with $\mu_{\mathrm{r}} = 2$), then as a result of gravitational heating (8.2.10), ionization and later nuclear reactions other atoms appear (for example, atoms of helium He, carbon C, oxygen O, etc.). Thus, in this case of the formation of a spherical body based on multi-component ionized gas, formula (8.2.11) becomes:

$$\alpha = \pi \left(\frac{k_{\mathrm{B}}}{\gamma} \cdot \left\langle \frac{i}{m_0} \right\rangle \cdot \frac{T}{M} \right)^2 , \qquad (8.2.12)$$

where < > is an operation of statistical averaging. In reality, in the process of a rise in temperature, the composition of existing particles becomes simpler in the stellar atmosphere. Spectral analysis of stars belonging to the spectral classes O, B, and A (with temperatures from 52,500 K to 7,550 K) shows lines of the ionized hydrogen and helium as well as ions of metals in their atmospheres, whereas in the spectral class K (4,050–5,250 K) radicals are already found out, and there even exist molecule oxides in the spectral class of M (2,500–3,850 K). For stars belonging to the first four classes, hydrogen and helium lines prevail, but in the process of the temperature falling, lines of other elements also appear. Besides, there even appear the lines pointing to the existence of chemical compounds, though these compounds are still very simple (CH, OH, NH, CH_2, C_2, C_3, CaH, etc.). External layers of stars consist mainly of hydrogen so, on average, there are only 1,000 atoms of helium, five atoms of oxygen and less than one atom of other elements per 10,000 atoms of hydrogen.

As follows directly from (8.2.12), the equation [75, 76] is true:

$$\frac{\sqrt{\alpha}\cdot M}{T}=\sqrt{\pi}\cdot\frac{k_{\mathrm{B}}}{\gamma}\cdot\left\langle\frac{i}{m_0}\right\rangle. \qquad (8.2.13)$$

The operation of statistical averaging in (8.2.13) means that:

$$\left\langle\frac{i}{m_0}\right\rangle=\frac{\langle i\rangle}{\langle m_0\rangle}, \qquad (8.2.14a)$$

where $\langle i\rangle$ is an average number of degrees of freedom for a particle with averaged mass $\langle m_0\rangle$. Following S. Chandrasekhar [106], it is convenient to express the averaged mass of a particle $\langle m_0\rangle$ through a mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$ of a highly ionized stellar substance and the mass of proton m_{p}:

$$\langle m_0\rangle=\bar{\mu}_{\mathrm{r}}\cdot m_{\mathrm{p}}, \qquad (8.2.14b)$$

where $m_{\mathrm{p}}=1.67248\cdot10^{-27}$ kg (instead of m_{H} though $m_{\mathrm{H}}\approx m_{\mathrm{p}}$). Similarly to (8.2.14b), the average number of degrees of freedom of a particle with averaged mass $\langle m_0\rangle$ is equal:

$$\langle i\rangle=\bar{i}\cdot i_{\mathrm{p}}, \qquad (8.2.14c)$$

where $i_{\mathrm{p}}=3$ is the number of translation degrees of freedom of a proton with mass m_{p}, and $\bar{i}$ is an average number of all degrees of freedom for a particle with a mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$.

Substituting (8.2.14b) and (8.2.14c) into (8.2.13) allows us to write down the following equation connecting macroscopic and microscopic physical values:

$$\frac{\sqrt{\alpha}\cdot M}{T}=\sqrt{\pi}\cdot\frac{k_{\mathrm{B}}}{\gamma}\cdot\frac{3\bar{i}}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}. \qquad (8.2.15)$$

Now let us introduce a new constant called *the universal stellar constant* [75, 76]:

$$\kappa=3\sqrt{\pi}\cdot\frac{k_{\mathrm{B}}}{\gamma}\approx 3\sqrt{3.14159265}\cdot\frac{1.38049\cdot 10^{-23}\ \mathrm{J/K}}{6.673\cdot 10^{-11}\ \mathrm{m^3/kg\cdot s^2}}=$$

$$=1.10003963\cdot 10^{-12}(\mathrm{kg^2/K\cdot m}). \qquad (8.2.16)$$

Taking (8.2.15) and (8.2.16) into account, we obtain *the equation of state of an ideal stellar substance* [75, 76]:

$$\sqrt{\alpha}\cdot M=\frac{\kappa\bar{i}}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}\cdot T, \qquad (8.2.17)$$

named so by analogy to the known Clapeyron–Mendeleev equation of the state of an ideal gas (or the usual Boyle–Charles law [1]). Stars that obey the equation of the state of an ideal stellar substance (8.2.17) we can call *ideal*.

Having rewritten the equation of the state of an ideal stellar substance (8.2.17) in the form:

$$\sqrt{\alpha}\cdot\frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}}\cdot\frac{m_{\mathrm{p}}M}{T}=\kappa, \qquad (8.2.18)$$

we obtain *the universal stellar law* (*USL*) [75, 76]:

$$\sqrt{\alpha}\cdot\frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}}\cdot\frac{m_{\mathrm{p}}M}{T}=\mathrm{const}. \qquad (8.2.19)$$

Obviously, a verification of USL (8.2.19) for different stars requires estimating their parameters $\sqrt{\alpha}$, M, T and also $\bar{\mu}_{\mathrm{r}}$, that is, chemical composition of stars.

8.3. Estimation of mean relative molecular weight of a highly ionized stellar substance and verification of the universal stellar law

As already noted, the spectral analysis of different stars reveals lines mainly of ionized hydrogen and helium. There are stars having the heightened content of a certain chemical element (carbon stars, silicon stars, iron stars, etc.). Stars with an abnormal compositions of chemical elements are numerous enough. At the same time, the stellar chemical composition depends on where the star is situated the Galaxy. Old stars in the spherical part of the Galaxy contain few atoms of heavy elements; by contrast, there are many heavy elements in stars belonging to the peripheral spiral branches of the Galaxy as well as its flat part where new stars are arising. Therefore, it is possible to connect the presence of heavy elements with features of the chemical evolution characterizing the life of a star.

It is well-known [106] that a mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$ of a highly ionized stellar substance can be found with the formula:

$$\bar{\mu}_{\mathrm{r}} = \frac{1}{\sum_{Z} x_Z \cdot \bar{\nu}_Z}, \tag{8.3.1}$$

where x_Z is the relative content of an element with an atomic serial number Z in a mass unit of a stellar matter, and $\bar{\nu}_Z$ is the number of free particles per unit of atomic weight A_Z produced by each atom of an element as a result of its ionization.

As shown by Chandrasekhar [106], as a *first approximation* under the condition of *full ionization* for an element with atomic serial number Z and relative atomic weight A_Z, the value $\bar{\nu}_Z$ is equal:

$$\bar{\nu}_Z = \frac{Z+1}{A_Z}. \tag{8.3.2}$$

As follows from the periodic system of chemical elements, under the condition of excluding the easiest elements (namely, hydrogen H and helium He) the ratio $\bar{\nu}_Z$ is equal to approximately ½ following (8.3.2) since a serial number Z of elements defines a total number of electrons in an atom.

Hence, if we suppose that there are x_{H} grams of hydrogen, x_{He} grams of helium, and $x_Z = 1 - x_{\mathrm{H}} - x_{\mathrm{He}}$ grams of "heavy elements" in 1 gram of stellar matter then we can find that:

$$\bar{\nu}_{\mathrm{H}} = 2; \quad \bar{\nu}_{\mathrm{He}} = 3/4; \quad \bar{\nu}_Z = 1/2,$$

that allows us to define the mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$ of a highly ionized stellar substance as the first approximation according to formula (8.3.1):

$$\bar{\mu}_{\mathrm{r}} = \frac{1}{2x_{\mathrm{H}} + \frac{3}{4}x_{\mathrm{He}} + \frac{1}{2}\left(1 - x_{\mathrm{H}} - x_{\mathrm{He}}\right)} = \frac{2}{1 + 3x_{\mathrm{H}} + 0.5x_{\mathrm{He}}}. \tag{8.3.3}$$

The *second approximation* takes into account the ionization state of a stellar substance so that the value of $\bar{\nu}_Z$ can be more precisely calculated by the Strömgren's formula [319]:

$$\bar{\nu}_Z = \frac{1}{A_Z} \cdot \left[1 + \sum_n \frac{2n^2}{1 + \frac{N_{\mathrm{e}}}{G(T)} \cdot e^{\frac{\chi_n^{(Z)}}{k_{\mathrm{B}} T}}} \right], \tag{8.3.4}$$

where $\chi_n^{(Z)}$ is an average ionized potential of various electron layers defined according to the Bohr's theory as:

$$\chi_n^{(Z)} = \frac{2\pi^2 \, \mathrm{e}^4 \, m_{\mathrm{e}} Z^2}{n^2 h^2}; \tag{8.3.5a}$$

where

n is the main quantum number ($n=1$ for the K-electron, $n=2$ for the L-electron, $n=3$ for the M-electron, etc.),
N_e is a number of free electrons in a volume unit,
$G(T)$ is calculated by the formula of statistical physics [320]:

$$G(T) = 2 \cdot \frac{(2\pi m_e k_B T)^{3/2}}{h^3}, \tag{8.3.5b}$$

m_e is a mass of an electron,
h is the Planck constant, and
other designations have the usual sense.

Comparing (8.3.4) with (8.3.2) we can see that the value in the square brackets in formula (8.3.4) is the corrected (by Strömgren) number of free particles per one nucleus with charge Ze [319]. Then, according to (8.3.1), a mean relative molecular weight $\bar{\mu}_r$ of a mixture of chemical elements with the values $\bar{\nu}_Z$ (calculated using (8.3.4)) and the determined values of abundance x_Z as previously, it is equal:

$$\bar{\mu}_r = \frac{1}{\sum_Z x_Z \cdot \bar{\nu}_Z} = \frac{1}{\bar{\nu}}, \tag{8.3.6a}$$

where

$$\bar{\nu} = \sum_Z x_Z \cdot \bar{\nu}_Z . \tag{8.3.6b}$$

Having used (8.3.4), (8.3.5a, b), and (8.3.6a, b), Strömgren calculated $\bar{\mu}_r$ for the so-called "Russell's mixture" in which elements O, (Na+Mg), Si, (K+Ca), and Fe meet in the weight proportion 8:4:1:1:2 at the preassigned values T and $G(T)/N_e$. In particular, when $T = 10^7$ K and $\ln[G(T)/N_e] = 5$ he found that $\bar{\nu}_R = 0.52$ and $\bar{\mu}_R = \bar{\nu}_R^{-1} = 1.92$ [319].

Reasoning in accordance with the above-mentioned, that is, that 1 gram of a stellar substance contains x_{H} grams of H, x_{He} grams of He and $(1 - x_{\mathrm{H}} - x_{\mathrm{He}})$ grams of the "Russell's mixture," we establish that the required mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$ is estimated by Strömgren's formula:

$$\bar{\mu}_{\mathrm{r}} = \frac{1}{2x_{\mathrm{H}} + \frac{3}{4}x_{\mathrm{He}} + \bar{\nu}_R(1 - x_{\mathrm{H}} - x_{\mathrm{He}})}. \qquad (8.3.7)$$

which is more exact in comparison with formula (8.3.3).

Using his method [106], Strömgren calculated the values $\bar{\mu}_{\mathrm{r}}$ and deduced the conclusions relatively of the hydrogen content of those stars for which there were reliable data concerning their luminosity L, mass M, and radius R. Thus, estimating the hydrogen content of the Sun he established that:

$$\ln\left[\frac{G(T)}{N_{\mathrm{e}}}\right] = 3; \qquad (8.3.8a)$$

$$x_{\mathrm{H}} = 0.36; \qquad (8.3.8b)$$

$$\bar{\mu}_{\mathrm{r}} = 1.00. \qquad (8.3.8c)$$

Concerning the hydrogen content of the Capella A (or α Aurigae, HD340029=HIP24608) he found that $\ln\left[\frac{G(T)}{N_{\mathrm{e}}}\right] = 7$, $x_{\mathrm{H}} = 0.30$ and accordingly $\bar{\mu}_{\mathrm{r}} = 1.01$.

In reality, according to the modern data on the relative quantity content of atoms in the stars the photospheric composition of the Sun includes hydrogen H (73.46 %), helium He (24.85 %), oxygen O (0.77 %), carbon C (0.29 %), iron Fe (0.16 %), neon Ne (0.12 %), nitrogen N (0.09 %), silicon Si (0.07 %), magnesium Mg (0.05 %), sulfur S (0.04 %) [283]. The relative content of hydrogen x_{H}, helium x_{He} and "heavy elements" $x_Z = 1 - x_{\mathrm{H}} - x_{\mathrm{He}}$ in a mass unit of the stellar substance of the Sun is then estimated by the values [75, 76]:

$$x_{\mathrm{H}} = 0.3527;$$
$$x_{\mathrm{He}} = 0.4772;$$
$$x_Z = 0.17.$$

As noticed by Chandrasekhar [106] in the assumption that the helium content can be neglected, formulas (8.3.3) and (8.3.7) give us an approximately identical estimation of the mean relative molecular weight of a stellar matter:

$$\overline{\mu}_{\mathrm{r}} = \frac{2}{1+3x_{\mathrm{H}}} \approx \frac{1}{2x_{\mathrm{H}} + \overline{\nu}_R \cdot (1 - x_{\mathrm{H}})}. \quad (8.3.9a)$$

Then according to (8.3.9a) in the case of the Sun, we can establish that

$$\overline{\mu}_{\mathrm{r}} = \frac{2}{1+3\cdot 0.3527} = 0.97, \quad (8.3.9b)$$

which practically coincides with the Strömgren's estimation (8.3.8c). According to [321 p.51] the mean relative atomic weight of *full ionized* stellar plasma may be estimated by the smaller value $\overline{\mu}_{\mathrm{r}} = 0.60$. However, there are no data about the average number of degrees of freedom $\bar{i}$ in this case.

So, being guided by the Strömgren's estimation $\overline{\mu}_{\mathrm{r}} = 1.00$ and the proposition $\bar{i} = 1$, let us verify the equation of the state of an ideal stellar substance (8.2.17) as well as the USL (8.2.18) in the case of the Sun [75, 76]:

$$\sqrt{\alpha_{\mathrm{Sun}}} \cdot \frac{m_{\mathrm{p}} \cdot M_{\mathrm{Sun}}}{T_{\mathrm{corSun}}} = \kappa. \quad (8.3.10)$$

Relatively T_{corSun}, the analysis of references [322]–[325] shows "…that the temperature of the corona was mostly recorded in a range of 1,000,000 to 2,000,000 K, whereas the temperatures of the other layers were more exact numbers. This is because the temperature drops slowly as you move from the corona into space. The variance in temperature is also due to the fact that the Sun's corona has no defined boundary…" [326]. This means that the *average* temperature of the Sun's corona can be chosen as 1,500,000 K [283, 326].

According to the left-hand part of (8.3.10) and taking into account that the mass of the Sun $M_{\text{Sun}} = 1.9891 \cdot 10^{30}\,\text{kg}$, the average temperature of the Solar corona $T_{\text{corSun}} = 1.5 \cdot 10^{6}\,\text{K}$ [283, 326], and the parameter of gravitational condensation of the Sun $\alpha_{\text{Sun}} = 2.29701 \cdot 10^{-19}\,\text{m}^{-2}$, we calculate:

$$\sqrt{\alpha_{\text{Sun}}} \cdot \frac{m_{\text{p}} M_{\text{Sun}}}{T_{\text{corSun}}} = \sqrt{2.29701 \cdot 10^{-19}} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 1.9891 \cdot 10^{30}}{1.5 \cdot 10^{6}} =$$

$$= 1.062937513 10^{-12} (\text{kg}^2 / \text{K} \cdot \text{m}). \qquad (8.3.11)$$

Comparing (8.3.11) with (8.2.16) shows a coincidence up to the relative error equal to [75, 76]

$$\delta = \frac{\kappa - \sqrt{\alpha_{\text{Sun}}} \cdot \dfrac{m_{\text{p}} M_{\text{Sun}}}{T_{\text{corSun}}}}{\kappa} \cdot 100\% = 3.37\%,$$

which testifies to the validity of USL for the Sun which is a star of type G2V. Starting from formula (8.3.11) it is possible to verify USL for other Sun-like stars since in our Milky Way alone there are more than 100 billion stars of type G2.

Indeed, the average value of the effective temperature of the surface of the Sun is $T_{\text{eff Sun}} = 5.778 \cdot 10^{3}\,\text{K}$ [283] whereas the temperature of its corona is accordingly equal to $T_{\text{cor Sun}} = 1.5 \cdot 10^{6}\,\text{K}$, that is, after finding the ratio:

$$\frac{T_{\text{cor Sun}}}{T_{\text{eff Sun}}} = \frac{1.5 \cdot 10^{6}}{5.778 \cdot 10^{3}} \approx 2.596 \cdot 10^{2}, \qquad (8.3.12)$$

we can use the effective temperature $T_{\text{eff Sun}}$ instead of the temperature of its corona $T_{\text{cor Sun}}$ in the USL (8.2.18). Then, expressing T_{cor} through T_{eff} for a star belonging to the spectral class G (in particular, the Sun) we obtain the modified USL (relatively T_{eff}):

$$\sqrt{\alpha}\cdot\frac{m_{\text{p}}M}{T_{\text{eff}}}=\Xi\,, \tag{8.3.13}$$

where

$$\Xi=\kappa\cdot 2.596\cdot 10^{2}=2.856296878\cdot 10^{-10}(\text{kg}^{2}/\text{K}\cdot\text{m})\,. \tag{8.3.14}$$

Besides the Solar system, we will now focus on the study of the multi-planet extrasolar systems Kepler-20 [285, 318], HD10180 [231], HIP14810 [285], 61Virginis [285], 55Cnc [259, 285], Alpha Centauri [285], Upsilon Andromedae [285, 309], and Gliese 876 [285, 310] whose stars belong to the different spectral classes G, F, K, and M.

We would like to verify the correctness of the modified USL (8.3.13) for these multi-planet systems [75, 76]. First of all, we note that the satisfiability of equality (8.3.13) for the Sun belonging to the spectral type G2V is confirmed by the calculations in (8.3.11), that is, the modified USL (8.3.13) is carried out for the Sun with the relative accuracy $\delta=3.37\%$ [76].

Example 8.3.1. For the second representative of this spectral class of stars, namely for the star Kepler-20 of type G8, we know [285, 308, 318] that:

$$M_{\text{Kepler-20}}=0.912M_{\text{Sun}}=0.912\cdot 1.9891\cdot 10^{30}\,\text{kg}=1.814059\cdot 2\cdot 10^{30}\,\text{kg};$$

$$R_{\text{Kepler-20}}=0.944R_{\text{Sun}}=0.944\cdot 6.955\cdot 10^{8}\,\text{m}=6.56552\cdot 10^{8}\,\text{m}\,;$$

and

$$T_{\text{eff Kepler-20}}=5466\,\text{K}\,.$$

that is, the square root of the parameter of gravitational condensation for the star Kepler-20 is equal:

$$\sqrt{\alpha_{\text{Kepler-20}}}\approx\frac{1}{3R_{\text{Kepler-20}}}=\frac{1}{3\cdot 6.56552\cdot 10^{8}}=5.07702868\cdot 10^{-10}(\text{m}^{-1})\,.$$

Then, according to (8.3.13) we obtain:

$$\sqrt{\alpha_{\text{Kepler-20}}}\cdot\frac{m_{\text{p}}M_{\text{Kepler-20}}}{T_{\text{eff Kepler-20}}}=5.07702868\cdot10^{-10}\cdot\frac{1.67248\cdot10^{-27}\cdot1.8140592\cdot10^{30}}{5.466\cdot10^{3}}=$$

$$=2.8180739\cdot10^{-10}(\text{kg}^2/\text{K}\cdot\text{m}). \qquad (8.3.15)$$

Comparing (8.3.15) with (8.3.14) shows that the modified USL (8.3.13) is carried out with relative accuracy $\delta=\frac{2.8562969-2.8180739}{2.8562969}\cdot100\%=1.34\%$ for the star Kepler-20 of type G8.

Example 8.3.2. Let us test the realizability of the modified USL (8.3.13) for a representative of spectral class G, namely for the star HD10180 of type G1V [231, 285]:

$M_{\text{HD10180}}=1.06\cdot M_{\text{Sun}}=1.06\cdot1.9891\cdot10^{30}\text{ kg}=2.108446\cdot10^{30}\text{ kg}$;

$R_{\text{HD10180}}=R_{\text{Sun}}=6.955\cdot10^{8}\text{ m}$; and

$T_{\text{eff HD10180}}=5911\text{K}$,

whence

$$\sqrt{\alpha_{\text{HD10180}}}\approx\frac{1}{3R_{\text{HD10180}}}\approx\frac{1}{3\cdot6.955}\cdot10^{-8}\text{ m}^{-1}=4.792715\cdot10^{-10}\text{ m}^{-1}.$$

Substituting the estimated parameter for HD10180 into the left-hand part of Eq. (8.3.13) we have:

$$\sqrt{\alpha_{\text{HD10180}}}\cdot\frac{m_{\text{p}}M_{\text{HD10180}}}{T_{\text{eff HD10180}}}=\frac{4.792715\cdot10^{-10}\cdot1.67248\cdot10^{-27}\cdot2.108446\cdot10^{30}}{5.911\cdot10^{3}}=$$

$$=2.8591969\cdot10^{-10}(\text{kg}^2/\text{K}\cdot\text{m}). \qquad (8.3.16)$$

Thus, according to (8.3.16) the modified USL (8.3.13) is carried out with the relative error $\delta=\frac{2.8562969-2.8591969}{2.8562969}=-0.1\%$ for the star HD10180 of type G1V.

Example 8.3.3. Now let us verify the modified USL (8.3.13) for a representative of spectral class G which is the star HIP14810 of type G5 [285]:

$$M_{\text{HIP14810}} = 0.99 \cdot M_{\text{Sun}} = 0.99 \cdot 1.9891 \cdot 10^{30}\,\text{kg} = 1.969209 \cdot 10^{30}\,\text{kg};$$

$$R_{\text{HIP14810}} = R_{\text{Sun}} = 6.955 \cdot 10^{8}\,\text{m}; \text{ and}$$

$$T_{\text{eff HIP14810}} = 5485\,\text{K},$$

whence

$$\sqrt{\alpha_{\text{HIP14810}}} = \sqrt{\alpha_{\text{Sun}}} \approx \sqrt{2.29701 \cdot 10^{-19}} = 4.7927132 \cdot 10^{-10}\,\text{m}^{-1}.$$

Then, according to (8.3.13) we obtain:

$$\sqrt{\alpha_{\text{HIP14810}}} \cdot \frac{m_{\text{p}} M_{\text{HIP14810}}}{T_{\text{eff HIP14810}}} = 4.7927132 \cdot 10^{-10} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 1.969209 \cdot 10^{30}}{5.485 \cdot 10^{3}} =$$

$$= 2.87778 \cdot 10^{-10} (\text{kg}^2/\text{K} \cdot \text{m}). \qquad (8.3.17)$$

Comparing (8.3.17) with (8.3.14) shows that the modified USL (8.3.13) is carried out with relative accuracy $\delta = \frac{2.8562969 - 2.8777798}{2.8562969} \cdot 100\% = -0.75\%$ for the star HIP14810 of class G5.

Example 8.3.4. For one more representative of spectral class G, namely for the star 61Virginis of type G5V, we know [285] that:

$$M_{\text{61Vir}} = 0.95 M_{\text{Sun}} = 0.95 \cdot 1.9891 \cdot 10^{30}\,\text{kg} = 1.889645 \cdot 10^{30}\,\text{kg};$$

$$R_{\text{61Vir}} = 0.94 R_{\text{Sun}} = 0.94 \cdot 6.955 \cdot 10^{8}\,\text{m} = 6.5377 \cdot 10^{8}\,\text{m}; \text{ and}$$

$$T_{\text{eff 61Vir}} = 5531\,\text{K},$$

that is, the square root of the parameter of gravitational condensation for the star 61Virginis is equal:

$$\sqrt{\alpha_{\text{61Vir}}} \approx \frac{1}{3R_{\text{61Vir}}} = \frac{1}{3 \cdot 6.5377 \cdot 10^{8}} = 5.098633056 \cdot 10^{-10} (\text{m}^{-1}).$$

Then, according to (8.3.13) we obtain:

$$\sqrt{\alpha_{\text{61Vir}}} \cdot \frac{m_{\text{p}} M_{\text{61Vir}}}{T_{\text{eff 61Vir}}} = 5.09863306 \cdot 10^{-10} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 1.889645 \cdot 10^{30}}{5.531 \cdot 10^{3}} =$$

$$= 2.9133406 \cdot 10^{-10} (\text{kg}^2/\text{K} \cdot \text{m}). \qquad (8.3.18)$$

Comparing (8.3.18) with (8.3.14) shows that the modified USL (8.3.13) is carried out with relative accuracy $\delta = \frac{2.8562969 - 2.9133406}{2.8562969} \cdot 100\% = -1.997\%$ for the star 61Virginis of type G5V. The quite high accuracy of this law for the given *stellar class of* G can most likely be explained by the good approximation T_{cor} for these stars based on the formula (8.3.12).

Example 8.3.5. Let us investigate the law (8.3.13) for a representative of spectral class K which is the star 55 Cancri (55Cnc=HD75732=HIP43580) of type KOIV-V. According to the paper [259] as well as the catalog [285] the mass and the effective temperature of the stellar surface of the star 55 Cancri are equal respectively:

$$M_{55\text{Cnc}} = 0.905 M_{\text{Sun}} = 1.800136 \cdot 10^{30}\ \text{kg};$$

$$T_{\text{eff }55\text{Cnc}} = 5196\,\text{K},$$

and the radius is

$$R_{55\text{Cnc}} = 0.943 R_{\text{Sun}} = 0.943 \cdot 6.955 \cdot 10^{8} m = 6.558565 \cdot 10^{8}\ \text{m}.$$

Let us find an estimation of the value:

$$\sqrt{\alpha_{55\text{Cnc}}} \approx \frac{1}{3R_{55\text{Cnc}}} = \frac{1}{3 \cdot 6.558565 \cdot 10^{8}} = 5.082412591 \cdot 10^{-10}\,(\text{m}^{-1})$$

and then calculate the left-hand part of the equation (8.3.13):

$$\sqrt{\alpha_{55\text{Cnc}}} \cdot \frac{m_{\text{p}} M_{55\text{Cnc}}}{T_{\text{eff}\,55\text{Cnc}}} = 5.0824126 \cdot 10^{-10} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 1.8001355 \cdot 10^{30}}{5.196 \cdot 10^{3}} =$$

$$= 2.9448753 10^{-10}\,(\text{kg}^2/\text{K} \cdot \text{m}). \qquad (8.3.19)$$

The modified USL (8.3.13) for the star 55Cancri of type KOIV-V is carried out with a relative error $\delta = \frac{2.8562969 - 2.9448753}{2.8562969} \cdot 100\% = -3.1\%$ that confirms precise enough estimations of the effective temperature, mass, and radius of this star.

Example 8.3.6. For the second representative of this spectral class K, namely for the star α Centauri of type K1V, it we know [285] that:

$$M_{\alpha\mathrm{Cent}} = 0.934 M_{\mathrm{Sun}} = 0.934 \cdot 1.9891 \cdot 10^{30}\ \mathrm{kg} = 1.8578194 \cdot 10^{30}\ \mathrm{kg};$$

$$R_{\alpha\mathrm{Cent}} = 0.863 R_{\mathrm{Sun}} = 0.863 \cdot 6.955 \cdot 10^{8}\ \mathrm{m} = 6.002165 \cdot 10^{8}\ \mathrm{m};$$

and

$$T_{\mathrm{eff}\,\alpha\mathrm{Cent}} = 5214\ \mathrm{K}.$$

that is, the square root of the parameter of gravitational condensation for the star α Centauri is equal:

$$\sqrt{\alpha_{\alpha\mathrm{Cent}}} \approx \frac{1}{3R_{\alpha\mathrm{Cent}}} = \frac{1}{3 \cdot 6.002165 \cdot 10^{8}} = 5.5535516 \cdot 10^{-10}\,(\mathrm{m}^{-1}).$$

Then, according to (8.3.13), we obtain:

$$\sqrt{\alpha_{\alpha\mathrm{Cent}}} \cdot \frac{m_{\mathrm{p}} M_{\alpha\mathrm{Cent}}}{T_{\mathrm{eff}\,\alpha\mathrm{Cent}}} = 5.553552 \cdot 10^{-10} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 1.8578194 \cdot 10^{30}}{5.214 \cdot 10^{3}} =$$

$$= 3.309514 \cdot 10^{-10}\,(\mathrm{kg}^2/\mathrm{K}\cdot\mathrm{m}). \qquad (8.3.20)$$

The modified USL (8.3.13) is carried out with a relative accuracy $\delta = \frac{2.8562969 - 3.3095139}{2.8562969} \cdot 100\% = -15.87\%$ following (8.3.20) for the star α Centauri of type K1V which is caused, probably, by an *inexact estimation of temperature of its corona* $T_{55\mathrm{Cnc}}$ for this class K, that is, through a directly proportional dependence (8.3.14) as in the special case of G2V.

Example 8.3.7. Now we shall consider the modified USL (8.3.13) for a star of new class F using an example of the Ups Andromedae of type F8V [285, 308, 309]:

$$M_{\upsilon\mathrm{And}} = 1.27 M_{\mathrm{Sun}} = 2.526157 \cdot 10^{30}\ \mathrm{kg};$$

$$R_{\upsilon\mathrm{And}} = 1.631 R_{\mathrm{Sun}} = 1.1343605 \cdot 10^{9}\ \mathrm{m};\ \text{and}$$

$$T_{\mathrm{eff}\,\upsilon\mathrm{And}} = 6212\ \mathrm{K}.$$

To this end we shall find the following value:

$$\sqrt{\alpha_{\upsilon \mathrm{And}}} \approx \frac{1}{3R_{\upsilon \mathrm{And}}} = \frac{1}{3.4030815 \cdot 10^{9}} = 2.9385132 \cdot 10^{-10} (\mathrm{m}^{-1}),$$

after which we shall be able to estimate the left-hand part of the equation (8.3.13):

$$\sqrt{\alpha_{\upsilon \mathrm{And}}} \cdot \frac{m_{\mathrm{p}} \cdot M_{\upsilon \mathrm{And}}}{T_{\mathrm{eff}\,\upsilon \mathrm{And}}} = 2.938513 \cdot 10^{-10} \cdot \frac{1.67248 \cdot 10^{-27} \cdot 2.526157 \cdot 10^{30}}{6.212 \cdot 10^{3}} =$$

$$= 1.998561 \cdot 10^{-10} (\mathrm{kg}^2 / \mathrm{K} \cdot \mathrm{m}). \qquad (8.3.21)$$

According to (8.3.21) for the star υ Andromedae of type F8V, the law (8.3.13) is carried out approximately with a relative error $\delta = 30\%$ because of the estimation of the corona temperature of the star υ Andromedae based on (8.3.12) or (8.3.14) is rough for stars of class F.

Example 8.3.8. At last, we shall consider characteristics of the star Gliese 876 of type M4V [261, 285, 310]:

$M_{\mathrm{Gl}876} = 0.334 M_{\mathrm{Sun}} = 0.334 \cdot 1.9891 \cdot 10^{30}\ \mathrm{kg} = 6.643594 \cdot 10^{29}\ \mathrm{kg}$;

$R_{\mathrm{Gl}876} = 0.36 R_{\mathrm{Sun}} = 0.36 \cdot 6.955 \cdot 10^{8}\ \mathrm{m} = 2.5038 \cdot 10^{8}\ \mathrm{m}$; and

$T_{\mathrm{eff\,Gl}876} = 3350\,\mathrm{K}$,

and then also check them for conformity to the law (8.3.13) for which we will first calculate:

$$\sqrt{\alpha_{\mathrm{Gl}876}} \approx \frac{1}{3R_{\mathrm{Gl}876}} = \frac{1}{7.5114} \cdot 10^{-8} = 1.3313097 \cdot 10^{-9} (\mathrm{m}^{-1})$$

and after that we shall find:

$$\sqrt{\alpha_{\mathrm{Gl}876}} \cdot \frac{m_{\mathrm{p}} \cdot M_{\mathrm{Gl}876}}{T_{\mathrm{eff\,Gl}876}} = 1.3313097 \cdot 10^{-9} \cdot \frac{1.67248 \cdot 10^{-27}}{3.35 \cdot 10^{3}} \cdot 6.643594 \cdot 10^{29} =$$

$$= 4.4156873 \cdot 10^{-10} (\mathrm{kg}^2 / \mathrm{K} \cdot \mathrm{m}). \qquad (8.3.22)$$

For the star Gliese 876, the modified USL (8.3.13) is carried out very approximately with the high relative error $\delta = -54.59\%$ as this star belongs to the class M (type M4V) for which the relation between the temperature of its corona

$T_{\text{cor Gl 876}}$ and effective temperature of its surface $T_{\text{eff Gl 876}}$ is not known exactly (because the expression (8.3.12) is true mainly for the class G).

The low accuracy of this law for the stars belonging to the spectral classes F or M can most likely be explained by the approximation T_{cor} which is too rough for the more bright or dim stars based on the formulas (8.3.12) or (8.3.14) [75, 76].

Stellar parameters being estimated in the above-mentioned examples are placed in Table 8.1. This table also contains data relative to other stars belonging to the high order spectral classes O, B, and A (for simplification we suppose that $\bar{\mu}_{\text{r}} = 1$ and $\bar{i} = 1$ for all types of stars). Indeed, it is well-known that $\bar{\mu}_{\text{r}} \approx 1$ for numerous stars (in particular, $\bar{\mu}_{\text{r}} = 1.01$ for Capella A [106]).

The investigation of the character of the function of relative error δ (see Fig. 8.1) depending on different types of stars reveals determinate regularity for very bright and dim stars; the low accuracy of the modified USL (8.3.13) for stars belonging to the high order spectral classes O, B, and A ($\delta \approx +50\%$) as well as for stars belonging to the last spectral class M ($\delta \approx -60\%$). This fact can be explained only by simple linear dependence $T_{\text{cor}}(T_{\text{eff}})$ according to (8.3.12) in the case of the Sun. Indeed, the Sun belongs to the spectral class G. We can, therefore, reach a good approximation T_{cor} for the stars of this spectral class (see Example 8.3.1–8.3.4 and Table 8.1) using the formula (8.3.12) [76]. Thus, finding the dependence $T_{\text{cor}}(T_{\text{eff}})$ is an important task.

As seen in Table 8.1 and Fig. 8.1, there also exists an abnormal group of intermediate-mass red giants (belonging to the spectral class K mainly) as 24 Sextanis, 18 Delphini, Capella A, 14 Andromedae, γ Cephei, β Ceti, ξ Aquilae, and 11 Comae [312, 313, 316] whose chemical composition includes even *radicals* except ions and electrons. In this connection our preliminary assumption (8.3.9b), that is, $\bar{\mu}_{\mathrm{r}} \approx 1$, gives us a very high relative error $\delta \approx 75\%$ here, therefore following A. Eddington [1, 106], it is reasonable to suppose that $\bar{\mu}_{\mathrm{r}} \approx 2$ for this group of red giants (see Table 8.2). In other words, the ratio $\bar{\mu}_{\mathrm{r}} / \bar{i}$ can be an increasing function in the case of stars belonging to the spectral class K following the modified USL (8.3.13), as shown in Table 8.2 [76].

Table 8.3 contains the data from Tables 8.1 and 8.2 together with the $\pm$ errors in measurements of mass, radius, and effective temperature as well as estimations for the modified USL with relative errors. To recalculate the estimations obtained in Tables 8.1 and 8.2 for modified USL relative to different classes of stars, the following simple operations are used:

$$(a \pm x) + (b \pm y) = (a + b) \pm (x + y);$$
$$(a \pm x) \cdot (b \pm y) = ab \pm (ay + bx); \text{ and}$$
$$(a \pm x)^{-1} = 1/a \mp (1/a)^2 x,$$

where x and y are the instrumental errors in measurements of the physical values a and b, so that $|x| << a$ and $|y| << b$. As seen in Table 8.3, the instrumental errors in measurements of mass, radius, and effective temperature define the lower limit of relative error $\delta, \%$ testifying to the validity of the modified USL at the level of 3% for ideal stars, that is, for the stars obeying the equation of state of an ideal stellar substance (8.2.17).

Table 8.1. Verification of the modified USL for stars belonging to the different spectral classes and types

Stars	Spectral class and type	Mass M, kg	Radius R, m	Effective tempera ture T_{eff}, K	Estimation of constant Ξ in the modified USL, $\text{kg}^2/\text{K}\cdot\text{m}$	Relative error δ, %
ξ Persei	O7.5III	$7.16076\cdot10^{31}$	$9.737\cdot10^{9}$	35000	$1.1714009\cdot10^{-10}$	58.9
τ Scorpii	B0.2V	$2.98365\cdot10^{31}$	$4.52075\cdot10^{9}$	29850	$1.2326299\cdot10^{-10}$	56.8
γ Pegasi	B2IV	$1.770299\cdot10^{31}$	$3.3384\cdot10^{9}$	21179	$1.3958619\cdot10^{-10}$	51.1
α Andromedae	A3V	$7.16076\cdot10^{30}$	$1.87785\cdot10^{9}$	13800	$1.5404897\cdot10^{-10}$	46.1
Sirius A	A1V	$4.017982\cdot10^{30}$	$1.190001\cdot10^{9}$	9940	$1.8937129\cdot10^{-10}$	33.7
WASP-12	G0	$2.685285\cdot10^{30}$	$1.091935\cdot10^{9}$	6300	$2.1761695\cdot10^{-10}$	23.8
υ Andromedae	F8V	$2.526157\cdot10^{30}$	$1.134361\cdot10^{9}$	6212	$1.9985613\cdot10^{-10}$	30
KOI-94	None	$2.486375\cdot10^{30}$	$1.151748\cdot10^{9}$	6116	$1.9678019\cdot10^{-10}$	31.1
HD 74156	G0	$2.466484\cdot10^{30}$	$1.09889\cdot10^{9}$	6039	$2.0720429\cdot10^{-10}$	27
Kepler-60	None	$2.18801\cdot10^{30}$	$1.04325\cdot10^{9}$	5915	$1.9767231\cdot10^{-10}$	30.8
HD 10180	G1V	$2.108446\cdot10^{30}$	$6.955\cdot10^{8}$	5911	$2.8591969\cdot10^{-10}$	-0.1
Kepler-33	None	$2.567928\cdot10^{30}$	$1.26581\cdot10^{9}$	5904	$1.9156125\cdot10^{-10}$	32.9
HD 155358	G0	$1.829972\cdot10^{30}$	$6.955\cdot10^{8}$	5900	$2.4861938\cdot10^{-10}$	12.9
47UrsaeMajoris	G0V	$2.048773\cdot10^{30}$	$8.6242\cdot10^{8}$	5892	$2.2477705\cdot10^{-10}$	21
Sun	G2V	$1.9891\cdot10^{30}$	$6.955\cdot10^{8}$	5778	$2.760955\cdot10^{-10}$	3.37
HD 1461	G0V	$2.148228\cdot10^{30}$	$7.615725\cdot10^{8}$	5765	$2.727781\cdot10^{-10}$	4.5
μ Andromedae	G3IV-V	$2.148228\cdot10^{30}$	$8.658975\cdot10^{8}$	5700	$2.4264909\cdot10^{-10}$	15

Kepler-11	G	$1.889645 \cdot 10^{30}$	$7.6505 \cdot 10^{8}$	5680	$2.4242742 \cdot 10^{-10}$	15.1
HAT-P-13	G4	$2.426702 \cdot 10^{30}$	$1.08498 \cdot 10^{9}$	5638	$2.2116139 \cdot 10^{-10}$	22.6
HD 37124	G4V	$1.810081 \cdot 10^{30}$	$5.7031 \cdot 10^{8}$	5610	$3.1540156 \cdot 10^{-10}$	-10.4
KOI-730	None	$2.128337 \cdot 10^{30}$	$7.6505 \cdot 10^{8}$	5590	$2.7744599 \cdot 10^{-10}$	2.87
61 Virginis	G5V	$1.889645 \cdot 10^{30}$	$6.5377 \cdot 10^{8}$	5531	$2.9133406 \cdot 10^{-10}$	-1.99
HIP 14810	G5	$1.969209 \cdot 10^{30}$	$6.955 \cdot 10^{8}$	5485	$2.8777798 \cdot 10^{-10}$	-0.75
Kepler-20	G8	$1.814059 \cdot 10^{30}$	$6.56552 \cdot 10^{8}$	5466	$2.8180739 \cdot 10^{-10}$	1.34
α Centauri	K1V	$1.857819 \cdot 10^{30}$	$6.002165 \cdot 10^{8}$	5214	$3.3095139 \cdot 10^{-10}$	-15.87
55 Cancri	K0IV-V	$1.800136 \cdot 10^{30}$	$6.558565 \cdot 10^{8}$	5196	$2.9448753 \cdot 10^{-10}$	-3.1
24 Sextanis	G5	$3.063214 \cdot 10^{30}$	$3.40795 \cdot 10^{9}$	5098	$9.8293312 \cdot 10^{-11}$	65.6
18 Delphini	G6III	$4.57493 \cdot 10^{30}$	$5.91175 \cdot 10^{9}$	4979	$8.6649477 \cdot 10^{-11}$	69.7
Capella A	K0III	$5.350679 \cdot 10^{30}$	$8.4851 \cdot 10^{9}$	4940	$7.1164706 \cdot 10^{-11}$	75.1
14 Andromedae	K0III	$4.37602 \cdot 10^{30}$	$7.6505 \cdot 10^{9}$	4813	$6.6254182 \cdot 10^{-11}$	76.8
γ Cephei	K1III-V	$2.78474 \cdot 10^{30}$	$3.40795 \cdot 10^{9}$	4800	$9.4905172 \cdot 10^{-11}$	66.8
β Ceti	K0III	$5.56948 \cdot 10^{30}$	$1.167049 \cdot 10^{10}$	4797	$5.5461999 \cdot 10^{-11}$	80.6
ξ Aquilae	G9III	$4.37602 \cdot 10^{30}$	$8.346 \cdot 10^{9}$	4780	$6.1152286 \cdot 10^{-11}$	78.6
11 Comae	G8 III	$5.37057 \cdot 10^{30}$	$1.32145 \cdot 10^{10}$	4742	$4.7780179 \cdot 10^{-11}$	83.3
HIP 57274	K5V	$1.452043 \cdot 10^{30}$	$4.7294 \cdot 10^{8}$	4640	$3.6888851 \cdot 10^{-10}$	-29.15
Groombridge34	M1.5V	$8.035964 \cdot 10^{29}$	$2.635945 \cdot 10^{8}$	3730	$4.5565116 \cdot 10^{-10}$	-59.5
Gliese 581	M2.5V	$6.16621 \cdot 10^{29}$	$2.0865 \cdot 10^{8}$	3498	$4.7099829 \cdot 10^{-10}$	-64.9
Gliese 876	M4V	$6.643594 \cdot 10^{29}$	$2.5038 \cdot 10^{8}$	3350	$4.4156873 \cdot 10^{-10}$	-54.59

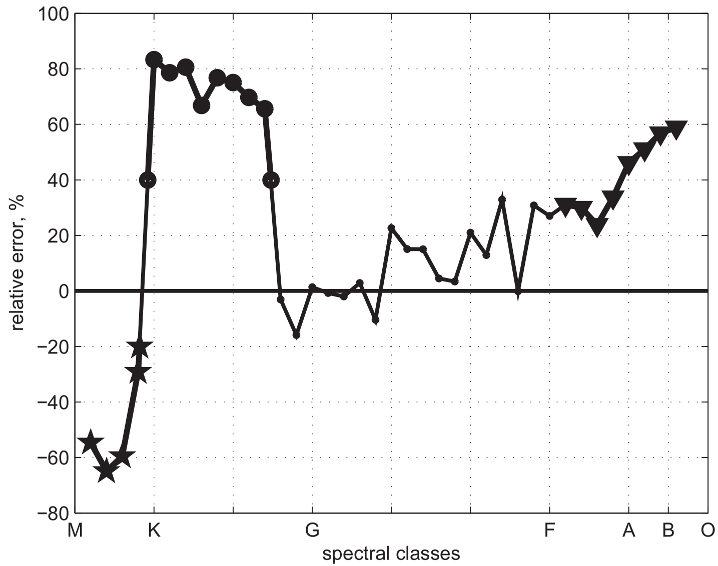


Figure 8.1. The plot of relative error δ, % of constant estimating Ξ in the modified USL for stars belonging to the different spectral classes

Table 8.2. Refinement of the modified USL for the group of intermediate-mass red giants

Red giant stars	**Effective temperature** T_{eff}, K	**Mean relative molecular weight** μ	**Mass** M, kg	**Radius** R, m	**Estimation of constant** Ξ **in the modified USL,** $\mathrm{kg}^2/\mathrm{K}\cdot\mathrm{m}$	**Relative error** δ, %
24 Sextanis	5098	2	$3.063214\cdot10^{30}$	$3.40795\cdot10^{9}$	$1.9658662\cdot10^{-10}$	31.17
18 Delphini	4979	2	$4.57493\cdot10^{30}$	$5.91175\cdot10^{9}$	$1.7329895\cdot10^{-10}$	39.33
Capella A	4940	2	$5.350679\cdot10^{30}$	$8.4851\cdot10^{9}$	$1.4232941\cdot10^{-10}$	50.17
14 Andromedae	4813	2	$2.148228\cdot10^{30}$	$8.658975\cdot10^{8}$	$1.3250836\cdot10^{-10}$	53.61
γ Cephei	4800	2	$2.78474\cdot10^{30}$	$3.40795\cdot10^{9}$	$1.8981034\cdot10^{-10}$	33.55
β Ceti	4797	2	$5.56948\cdot10^{30}$	$1.167049\cdot10^{10}$	$1.1092399\cdot10^{-10}$	61.17
ξ Aquilae	4780	2	$4.37602\cdot10^{30}$	$8.346\cdot10^{9}$	$1.2230457\cdot10^{-10}$	57.18
11 Comae	4742	2	$5.37057\cdot10^{30}$	$1.32145\cdot10^{10}$	$9.5560358\cdot10^{-11}$	66.54
42 Draconis	4200	2	$1.949318\cdot10^{30}$	$1.53219\cdot10^{10}$	$3.3774696\cdot10^{-11}$	88.18

Table 8.3. Estimation of the modified USL for the different classes of stars with regard to the errors in measurements

Stars	Mass M, kg	Radius R, m	Effective temperature T_{eff}, K	Estimation of constant Ξ in the modified USL, $\text{kg}^2/\text{K}\cdot\text{m}$	Relative error δ, %
υ Andromedae	$(2.526157 \pm 0.119346)\cdot 10^{30}$	$(1.134361 \pm 0.009737)\cdot 10^{9}$	$(6.212 \pm 0.08)\cdot 10^{3}$	$(1.9985613 \pm 0.051527)\cdot 10^{-10}$	30.0 ∓ 1.8
HD 10180	$(2.108446 \pm 0.099455)\cdot 10^{30}$	$6.955\cdot 10^{8}$	$(5.911 \pm 0.019)\cdot 10^{3}$	$(2.8591969 \pm 0.125677)\cdot 10^{-10}$	-0.1 ∓ 4.4
61 Virginis	$(1.889645 \pm 0.0597)\cdot 10^{30}$	$(6.5377 \pm 0.202)\cdot 10^{8}$	$5.531\cdot 10^{3}$	$(2.913305 \pm 0.002027)\cdot 10^{-10}$	-1.99 ∓ 0.07
HIP 14810	$(1.969209 \pm 0.079564)\cdot 10^{30}$	$(6.955 \pm 0.4173)\cdot 10^{8}$	$(5.485 \pm 0.044)\cdot 10^{3}$	$(2.87778089 \mp 0.079478)\cdot 10^{-10}$	-0.75 ± 2.78
Kepler-20	$(1.8140592 \pm 0.069619)\cdot 10^{30}$	$(6.56552 \pm 0.660725)\cdot 10^{8}$	$(5.466 \pm 0.093)\cdot 10^{3}$	$(2.81807391 \mp 0.223396)\cdot 10^{-10}$	1.34 ± 7.8
α Centauri	$(1.857819 \pm 0.011935)\cdot 10^{30}$	$6.002165\cdot 10^{8}$	$(5.214 \pm 0.033)\cdot 10^{3}$	$(3.3095139 \pm 0.000314)\cdot 10^{-10}$	-15.9 ∓ 0.01
55 Cancri	$(1.8001355 \pm 0.029836)\cdot 10^{30}$	$(6.558565 \pm 0.06955)\cdot 10^{8}$	$(5.196 \pm 0.024)\ )\cdot 10^{3}$	$(2.9448753 \pm 0.003979)\cdot 10^{-10}$	-3.1 ∓ 0.14

24 Sextanis	$(3.063214 \pm 0.159128) \cdot 10^{30}$	$(3.40795 \pm 0.05564) \cdot 10^{9}$	$(5.098 \pm 0.044) \cdot 10^{3}$	$(1.96586625 \pm 0.05306) \cdot 10^{-10}$	31.17 ∓ 1.86
18 Delphini	$4.57493 \cdot 10^{30}$	$5.91175 \cdot 10^{9}$	$(4.979 \pm 0.018) \cdot 10^{3}$	$(1.7329895387 \mp 0.006265) \cdot 10^{-10}$	39.33 ± 0.22
γ Cephei	$(2.78474 \pm 0.238692) \cdot 10^{30}$	$3.40795 \cdot 10^{9}$	$(4.8 \pm 0.1) \cdot 10^{3}$	$(1.8981034 \pm 0.123151) \cdot 10^{-10}$	33.55 ∓ 4.31
11 Comae	$(5.37057 \pm 0.59673) \cdot 10^{30}$	$(1.32145 \pm 0.1391) \cdot 10^{10}$	$(4.742 \pm 0.1) \cdot 10^{3}$	$(9.5560358 \pm 0.884368) \cdot 10^{-11}$	66.5 ∓ 3.1
Gliese 876	$(6.643594 \pm 0.59673) \cdot 10^{29}$	$2.5038 \cdot 10^{8}$	$(3.35 \pm 0.3) \cdot 10^{3}$	$(4.415687396 \pm 0.001184) \cdot 10^{-10}$	-54.6 ∓ 0.04

8.4. Estimation of the temperature of the stellar corona

To test the accuracy of USL for other stars, it is necessary to estimate the temperature of their stellar coronas T_{cor} using the value of the effective radiative temperature T_{eff} of the stars' surfaces. To measure the real (thermodynamic) temperature T of a body through its effective radiative temperature T_{eff} the following relation can be used:

$$T = \frac{T_{\text{eff}}}{\sqrt[4]{A_T}}, \qquad (8.4.1)$$

where a factor A_T is an integral absorptivity of a body. Moreover, for a real body $A_T < 1$ [206] the radiative temperature is always less than the real temperature. Indeed, A_T is the ratio of the powers of integral radiant emittance of the given body (E_T) and the perfect black body (ε_T) at a temperature T following Kirchhoff's law:

$$A_T = E_T / \varepsilon_T . \qquad (8.4.2)$$

A_T, therefore, has the meaning of the power of the blackness of a body: $0 \le A_T \le 1$, that is, for the black body $A_T = 1$, and for the mirror body $A_T = 0$. According to the Stefan-Boltzmann law, the power of integral radiant emittance of the perfect black body is equal:

$$\varepsilon_T = \sigma \cdot T_{\text{eff}}^4 , \qquad (8.4.3)$$

where $\sigma = 5.6686 \cdot 10^{-8} (\text{W/m}^2 \cdot \text{K}^4)$ is the Stefan–Boltzmann constant.

Taking into account (8.4.1), we write USL (8.2.17) for an arbitrary case of a remote star [76]:

$$\sqrt{\alpha} \cdot \frac{\bar{\mu}_{\text{r}}}{\bar{i}} \cdot \frac{m_{\text{p}} \cdot M}{T_{\text{eff}}} = \frac{\kappa}{\sqrt[4]{A_T}} . \qquad (8.4.4)$$

Let us note that except for the temperature of a body the value

A_T in (8.4.4) depends on its chemical composition as well as the form and the surface condition [206].

As a first approximation, the value $\sqrt[4]{A_T}$ for each considered star depends on its temperature T, that is, on the above-mentioned value $\bar{i}$. Nevertheless, in the general case A_T is also a function of the star's chemical composition defining the mentioned parameter $\overline{\mu}_{\mathrm{r}}$, the star's form determining its radius R and star's surface condition depending on its stellar belonging to the different spectral classes {..., F, G, K,...}, that is, $A_T = A_T(\overline{\mu}_{\mathrm{r}}, R, \bar{i}, \{...,\mathrm{F},\mathrm{G},\mathrm{K},...\})$. In this context, it should be reasonable to consider the dependencies on the spectral classes directly as $A_{T\mathrm{F}} = A_{T\mathrm{F}}(\bar{i}, \overline{\mu}_{\mathrm{r}}, R)$, $A_{T\mathrm{G}} = A_{T\mathrm{G}}(\bar{i}, \overline{\mu}_{\mathrm{r}}, R)$, $A_{T\mathrm{K}} = A_{T\mathrm{K}}(\bar{i}, \overline{\mu}_{\mathrm{r}}, R)$.

In particular (for the Sun), the effective radiative temperature of its solar surface is equal:

$$T_{\mathrm{effSun}} = 5.778 \cdot 10^3\ \mathrm{K},$$

whereas the temperature of its corona is:

$$T_{\mathrm{corSun}} = 1.5 \cdot 10^6\ \mathrm{K}.$$

Using (8.4.1), we can estimate a 4-th degree root of an integral absorptivity of the solar corona:

$$\sqrt[4]{A_{T\,\mathrm{cor\,Sun}}} = \frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{cor\,Sun}}} = \frac{5.778 \cdot 10^3}{1.5 \cdot 10^6} = 3.852 \cdot 10^{-3}. \qquad (8.4.5)$$

Since, for any other star, the temperature of its stellar corona is unknown, it is not possible to estimate $\sqrt[4]{A_T}$ directly as for the Sun. So, let us, therefore, consider relation (8.4.2) for the Sun separately and any other star belonging to the spectral class G:

$$A_{T\,\mathrm{corSun}} = E_{T\,\mathrm{corSun}} / \varepsilon_{T\,\mathrm{corSun}}; \qquad (8.4.6a)$$

$$A_{T\,\mathrm{cor\,G}} = E_{T\,\mathrm{cor\,G}} / \varepsilon_{T\,\mathrm{cor\,G}}. \qquad (8.4.6b)$$

Dividing (8.4.6b) on (8.4.6a) we obtain:

$$\frac{A_{T\,\mathrm{cor\,G}}}{A_{T\,\mathrm{corSun}}} = \frac{E_{T\,\mathrm{cor\,G}}}{E_{T\,\mathrm{corSun}}} \cdot \frac{\varepsilon_{T\,\mathrm{corSun}}}{\varepsilon_{T\,\mathrm{cor\,G}}} . \qquad (8.4.7)$$

Taking into account the Stefan–Boltzmann law (8.4.3), relation (8.4.7) passes into the following:

$$\frac{A_{T\,\mathrm{cor\,G}}}{A_{T\,\mathrm{corSun}}} \cdot \frac{E_{T\,\mathrm{corSun}}}{E_{T\,\mathrm{cor\,G}}} = \left(\frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{eff\,G}}} \right)^4 \qquad (8.4.8)$$

As follows from equation (8.4.8), choosing $E_{T\,\mathrm{corSun}} / E_{T\,\mathrm{cor\,G}} = \left(T_{\mathrm{eff\,Sun}} / T_{\mathrm{eff\,G}} \right)^{4+s}$ the ratio (8.4.7) becomes:

$$\frac{A_{T\,\mathrm{cor\,G}}}{A_{T\,\mathrm{corSun}}} = \left(\frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{eff\,G}}} \right)^{-s} = \left(\frac{T_{\mathrm{eff\,G}}}{T_{\mathrm{eff\,Sun}}} \right)^{s} , \qquad (8.4.9)$$

whence

$$\sqrt[4]{A_{T\,\mathrm{cor\,G}}} = \sqrt[4]{A_{T\,\mathrm{corSun}}} \cdot \left(\frac{T_{\mathrm{eff\,G}}}{T_{\mathrm{eff\,Sun}}} \right)^{s/4} . \qquad (8.4.10)$$

Substituting (8.4.10) into (8.4.4) leads to the equation:

$$\sqrt{\alpha} \cdot \frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}} \cdot \frac{m_{\mathrm{p}} M}{T_{\mathrm{eff\,G}}} = \frac{\kappa}{\sqrt[4]{A_{T\,\mathrm{corSun}}}} \cdot \left(\frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{eff\,G}}} \right)^{s/4} , \qquad (8.4.11)$$

which allows USL to be presented in the form:

$$\sqrt{\alpha} \cdot \frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}} \cdot \frac{m_{\mathrm{p}} M}{T_{\mathrm{eff\,G}}^{1-s/4}} = \kappa \cdot \left(\frac{T_{\mathrm{eff\,Sun}}^{s}}{A_{T\,\mathrm{corSun}}} \right)^{1/4} . \qquad (8.4.12)$$

Choosing some value s and also $\sqrt{\alpha} \approx 1/3R$ (R is a star radius) according to the mentioned 3σ-rule, we can derive the *empirical* variant of USL [76]:

$$\frac{\bar{\mu}_{\mathrm{r}}}{3R\bar{i}} \cdot \frac{m_{\mathrm{p}} M}{T_{\mathrm{eff\,G}}^{1-s/4}} = \kappa \cdot \left(\frac{T_{\mathrm{eff\,Sun}}^{s}}{A_{T\,\mathrm{corSun}}} \right)^{1/4} . \qquad (8.4.13)$$

If the parameters of a star are given in units relative to the

Sun's parameters:

$$R = k_R \cdot R_{\text{Sun}}; \tag{8.4.14a}$$

$$M = k_M \cdot M_{\text{Sun}}, \tag{8.4.14b}$$

then having divided Eq. (8.4.11) by the similar equation concerning the Sun we obtain:

$$\frac{\dfrac{1}{3 \cdot k_R \cdot R_{\text{Sun}}} \cdot \dfrac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \dfrac{m_{\text{p}} \cdot k_M \cdot M_{\text{Sun}}}{T_{\text{eff G}}}}{\dfrac{1}{3 \cdot R_{\text{Sun}}} \cdot \dfrac{1}{1} \cdot \dfrac{m_{\text{p}} \cdot M_{\text{Sun}}}{T_{\text{eff Sun}}}} = \frac{\dfrac{\kappa}{\sqrt[4]{A_{T\,\text{corSun}}}} \cdot \left(\dfrac{T_{\text{eff Sun}}}{T_{\text{eff G}}}\right)^{s/4}}{\dfrac{\kappa}{\sqrt[4]{A_{T\,\text{corSun}}}}},$$

whence

$$\frac{1}{k_R} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot k_M \cdot \frac{T_{\text{eff Sun}}}{T_{\text{eff G}}} = \left(\frac{T_{\text{eff Sun}}}{T_{\text{eff G}}}\right)^{s/4}.$$

In other words, the relation:

$$\frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{k_M}{k_R} = \left(\frac{T_{\text{eff Sun}}}{T_{\text{eff G}}}\right)^{\frac{s}{4}-1} \tag{8.4.15}$$

is valid. Moreover, it allows finding as a first approximation:

$$s \approx 4. \tag{8.4.16}$$

The more exact value s can be found by taking the logarithm of equation (8.4.15):

$$\ln \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{k_M}{k_R} = \left(\frac{s}{4} - 1\right) \ln\left(\frac{T_{\text{eff Sun}}}{T_{\text{eff G}}}\right),$$

whence it directly follows that

$$s = 4 + 4 \cdot \frac{\ln \dfrac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \dfrac{k_M}{k_R}}{\ln\left(\dfrac{T_{\text{eff Sun}}}{T_{\text{eff G}}}\right)} = 4\left[1 + \frac{\ln \overline{\mu}_{\text{r}} - \ln \overline{i} + \ln k_M - \ln k_R}{\ln T_{\text{eff Sun}} - \ln T_{\text{eff G}}}\right]. \tag{8.4.17}$$

Coming from formulas (8.4.1), (8.4.10), and (8.4.17), it is easy to calculate the temperature of a stellar G-corona through the temperature of a solar corona [76]:

$$T_{\mathrm{cor\,G}} = \frac{T_{\mathrm{eff\,G}}}{\sqrt[4]{A_{T\,\mathrm{Sun}}} \cdot \left(\dfrac{T_{\mathrm{eff\,G}}}{T_{\mathrm{eff\,Sun}}} \right)^{1+\frac{\ln\left(\frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}} \cdot \frac{k_M}{k_R}\right)}{\ln\left(\frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{eff\,G}}}\right)}}} =$$

$$= T_{\mathrm{cor\,Sun}} \cdot \left(\frac{T_{\mathrm{eff\,Sun}}}{T_{\mathrm{eff\,G}}} \right)^{\frac{\ln(\bar{\mu}_{\mathrm{r}}/\bar{i}) + \ln(k_M/k_R)}{\ln(T_{\mathrm{eff\,Sun}}/T_{\mathrm{eff\,G}})}} . \qquad (8.4.18a)$$

Using formula (8.4.18a) we can calculate the temperature of stellar corona $T_{\mathrm{cor\,G}}$ accurately enough for stars belonging to the spectral class G. This formula can be applied approximately to the different spectral classes (see Table 8.4, the values $\bar{\mu}_{\mathrm{r}}$ for the Xi Persei and the Tau Scorpio were calculated in accordance with [106]). Table 8.4 shows that functional dependence $T_{\mathrm{cor\,G}}$ on stellar spectral classes M, K, G,.., O is described by monotonically increasing function except for the group of red giants (K0III- K1III-V). This can probably be explained by the origin of *radicals* and other chemical elements in the stellar substance of red giants, that is, by chemical composition changing from $\bar{\mu}_{\mathrm{r}} = 1$ to $\bar{\mu}_{\mathrm{r}} = 2$ approximately. In other words, if $\bar{i} = 1$ as usual then $\bar{\mu}_{\mathrm{r}} / \bar{i} = 2$. This means that a new virial equilibrium takes place with a new temperature state of the stellar corona $T_{\mathrm{cor\,G}}$ in the case of *red giant* stars. There are also radicals in the stellar substance of the spectral class M. However, in this case $\bar{\mu}_{\mathrm{r}} = 2$ and $\bar{i} = 2$ simultaneously due to the temperature of stellar corona of class M becoming higher than for red giants (K0III- K1III-V).

Obviously, if for each spectral class we can find a representative star (such as the Sun for the spectral class G) then the relation (8.4.18a) determines a sufficiently exact dependence between T_{cor} of the given spectral class and T_{eff} of its representative (for example, between $T_{\text{cor F}}$ and $T_{\text{eff}\,\upsilon\text{And}}$ if the last value is supposed to be known). Evidently, the formula (8.4.18a) does not have a high relative accuracy of estimation T_{cor} for all distant spectral classes from G.

Let us note that the formula (8.4.18a) can be easily transformed into the simpler form. In reality, according to the Eqs (8.4.17) and (8.4.18a) and taking into account (8.4.15) we obtain [76]:

$$T_{\text{cor G}} = T_{\text{cor Sun}} \cdot \left(\frac{T_{\text{eff Sun}}}{T_{\text{eff G}}} \right)^{\frac{s}{4}-1} = T_{\text{cor Sun}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{k_M}{k_R},$$

that is, a relation in the form:

$$T_{\text{cor G}} = T_{\text{cor Sun}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{k_M}{k_R} \,. \qquad (8.4.18b)$$

This form does not comprise $T_{\text{eff G}}$ or $T_{\text{eff Sun}}$ at all. Analogously, the dependences, say, $T_{\text{cor K}}\,(T_{\text{eff 55Cnc}})$ and $T_{\text{cor M}}\,(T_{\text{eff Gl 876}})$ can be written. That is why the estimation of $T_{\text{cor F}}, T_{\text{cor K}}$ or $T_{\text{cor M}}$ through $T_{\text{cor Sun}}$ (in accordance with (8.4.18a, b)) leads to more big errors.

8.5. Comparison with estimations of temperatures based on regression dependences for multi-planet extrasolar systems

In this section, following Pintr et al. [308], we shall focus mainly on the spectral classes of stars F, G, K, and M. This is justified since the life spans of spectral classes of stars O, B, and A are so short that the complex life could never form on the planets associated with them [308].

According to Schneider's catalogue [285], the spectral classes of interest here can be characterized as:

– spectral class F with T_{eff} between 6,000 and 7,500 K;

– spectral class G with T_{eff} between 5,200 and 6,000 K;

– spectral class K with T_{eff} between 3,700 and 5,200 K; and

– spectral class M with T_{eff} less than 3,700 K.

Recently Pintr et al. obtained the regression dependences of the effective temperature of a stellar surface T_{eff} from specific angular momentum $a_n v_n$ (a_n is a planetary distance and v_n is a planetary velocity) for the different spectral classes of star [308]:

$$a_n T_{\text{eff F}} \approx 4 \cdot 10^{-17} (a_n v_n)^{1.9963} \qquad (8.5.1a)$$

with the coefficient of determination R^2=0.997 for stars of the spectral class F;

$$a_n T_{\text{eff G}} \approx 1 \cdot 10^{-16} (a_n v_n)^{1.976} \qquad (8.5.1b)$$

with the coefficient of determination of regression R^2=0.995 for stars belonging to the spectral class G;

$$a_n T_{\text{eff K}} \approx 4 \cdot 10^{-14} (a_n v_n)^{1.807} \qquad (8.5.1c)$$

with the coefficient of determination R^2=0.776 for the stellar spectral class K; and

$$a_n T_{\text{eff M}} \approx 5 \cdot 10^{-14} (a_n v_n)^{1.814} \qquad (8.5.1d)$$

with the coefficient of determination R^2=0.974 for stars of the spectral class M.

In reality, according to Kepler's 3rd law:

$$a_n^3 \Omega_n^2 = \gamma M , \qquad (8.5.2)$$

where a_n is a major semi-axis of the planetary orbit, and Ω_n is an angular velocity of motion of a planet in its orbit. Supposing that $v_n = a_n \Omega_n$ is Kepler's velocity of the movement of a planet, let us use Kepler's law in the form of Utting [8]:

$$a_n \cdot v_n^2 = \gamma M \ . \tag{8.5.3}$$

In turn, the right-hand part of Eq. (8.5.3) can be expressed from the equation (8.2.15) of the state of an ideal stellar substance taking into account a designation (8.2.16):

$$\gamma M = \frac{3\sqrt{\pi}k_{\mathrm{B}}}{m_{\mathrm{p}}} \cdot \frac{\bar{i}}{\bar{\mu}_{\mathrm{r}}} \cdot \frac{T}{\sqrt{\alpha}} \approx \frac{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}} \cdot RT \ . \tag{8.5.4}$$

Comparing (8.5.3) with (8.5.4), we obtain:

$$T = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}\sqrt{\alpha}}{3\sqrt{\pi}k_{\mathrm{B}}\bar{i}} \cdot a_n v_n^2 \approx \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R} \cdot a_n v_n^2 \ , \tag{8.5.5}$$

whence

$$a_n T = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}\sqrt{\alpha}}{3\sqrt{\pi}k_{\mathrm{B}}\bar{i}} \cdot (a_n v_n)^2 \ . \tag{8.5.6}$$

Substituting (8.4.1) into the formula (8.5.6) we rewrite it as follows:

$$a_n T_{\mathrm{eff}} = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}\sqrt[4]{A_T}\sqrt{\alpha}}{3\sqrt{\pi}k_{\mathrm{B}}\bar{i}} \cdot (a_n v_n)^2 \ . \tag{8.5.7}$$

Introducing the following notation:

$$\mathrm{K} = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}\sqrt[4]{A_T}\sqrt{\alpha}}{3\sqrt{\pi}k_{\mathrm{B}}\bar{i}} \ , \tag{8.5.8}$$

we obtain a theoretical dependence confirming the mentioned regression equations (8.5.1a)–(8.5.1d) of Pintr et al. (PPLP equations) as a whole:

$$a_n T_{\mathrm{eff}} = \mathrm{K} \cdot (a_n v_n)^2 \ , \tag{8.5.9a}$$

though comparing (8.5.1c) and (8.5.1d) with (8.5.7) reveals an approximation lack in the degree of $a_n v_n$. Taking into account the estimation (8.4.5) let us calculate the value of the theoretical coefficient K for the Sun which is a representative of stars of the spectral class G:

$$\mathrm{K_{Sun}} = \frac{1.67248\cdot 10^{-27}\cdot 1\cdot 3.852\cdot 10^{-3}\sqrt{2.297\cdot 10^{-19}}}{3\sqrt{3.14159265}\cdot 1.38049\cdot 10^{-23}\cdot 1} \approx$$

$$\approx 0.4206\cdot 10^{-16}\left(\mathrm{K\cdot s^2/m^3}\right). \qquad (8.5.9b)$$

The derived theoretical estimation (8.5.9b) corresponds closely to the heuristic PPLP dependence (8.5.1a) for stars of the spectral class F and corresponds satisfactorily (8.5.1b) for stars of the spectral class G.

On the other hand, taking into account (8.5.3) and (8.5.5), it is not difficult to see that for any star:

$$T = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}\cdot\gamma M\ ,$$

so that

$$T_{\mathrm{eff}} = \frac{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}\cdot\sqrt[4]{A_T}\cdot\gamma M\ . \qquad (8.5.10)$$

Using (8.5.3), let us represent the heuristic PPLP dependences (8.5.1a)–(8.5.1d) by analogy with (8.5.10) in the form:

$$T_{\mathrm{eff\ F}} \approx \frac{4\cdot 10^{-17}}{(a_n v_n)^{0.0037}}\cdot\gamma M\ ; \qquad (8.5.11a)$$

$$T_{\mathrm{eff\ G}} \approx \frac{1\cdot 10^{-16}}{(a_n v_n)^{0.024}}\cdot\gamma M\ ; \qquad (8.5.11b)$$

$$T_{\mathrm{eff\ K}} \approx \frac{4\cdot 10^{-14}}{(a_n v_n)^{0.193}}\cdot\gamma M\ ; \qquad (8.5.11c)$$

$$T_{\mathrm{eff\ M}} \approx \frac{5\cdot 10^{-14}}{(a_n v_n)^{0.186}}\cdot\gamma M\ . \qquad (8.5.11d)$$

Comparing (8.5.10) with (8.5.11a)–(8.5.11d) shows that the following heuristic estimations are valid:

$$\sqrt[4]{A_{T\,\mathrm{F}}} \approx \frac{4\cdot 10^{-17}}{(a_n v_n)^{0.0037}} \cdot \frac{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.12a)$$

$$\sqrt[4]{A_{T\,\mathrm{G}}} \approx \frac{1\cdot 10^{-16}}{(a_n v_n)^{0.024}} \cdot \frac{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.12b)$$

$$\sqrt[4]{A_{T\,\mathrm{K}}} \approx \frac{4\cdot 10^{-14}}{(a_n v_n)^{0.193}} \cdot \frac{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.12c)$$

$$\sqrt[4]{A_{T\,\mathrm{M}}} \approx \frac{5\cdot 10^{-14}}{(a_n v_n)^{0.186}} \cdot \frac{9\sqrt{\pi}k_{\mathrm{B}}\bar{i}R}{m_{\mathrm{p}}\bar{\mu}_{\mathrm{r}}}. \qquad (8.5.12d)$$

Calculating the common constant in (8.5.12a)–(8.5.12d) separately:

$$\frac{9\sqrt{\pi}k_{\mathrm{B}}}{m_{\mathrm{p}}} = \frac{9\cdot\sqrt{3.14159265}\cdot 1.38049\cdot 10^{-23}}{1.67248\cdot 10^{-27}} = 1.31670892\cdot 10^{5}\,(\mathrm{J/K\cdot kg}),$$

let us rewrite the formulas (8.5.12a)–(8.5.12d) in the form:

$$\sqrt[4]{A_{T\,\mathrm{F}}} \approx \frac{5.266835678\cdot 10^{-12}}{(a_n v_n)^{0.0037}} \cdot \frac{\bar{i}R}{\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.13a)$$

$$\sqrt[4]{A_{T\,\mathrm{G}}} \approx \frac{1.31670892\cdot 10^{-11}}{(a_n v_n)^{0.024}} \cdot \frac{\bar{i}R}{\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.13b)$$

$$\sqrt[4]{A_{T\,\mathrm{K}}} \approx \frac{5.266835678\cdot 10^{-9}}{(a_n v_n)^{0.193}} \cdot \frac{\bar{i}R}{\bar{\mu}_{\mathrm{r}}}; \qquad (8.5.13c)$$

$$\sqrt[4]{A_{T\,\mathrm{M}}} \approx \frac{6.5835446\cdot 10^{-9}}{(a_n v_n)^{0.186}} \cdot \frac{\bar{i}R}{\bar{\mu}_{\mathrm{r}}}. \qquad (8.5.13d)$$

Using (8.4.1) and (8.5.13a)–(8.5.13d), we can estimate the average temperature of the stellar corona for the spectral classes of stars F, G, K, and M [76]:

$$\bar{T}_{\mathrm{cor\,F}} = T_{\mathrm{eff\,F}} \cdot \frac{\overline{(a_n v_n)}^{0.0037}}{5.266835678\cdot 10^{-12}} \cdot \frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}R}; \qquad (8.5.14a)$$

$$\overline{T}_{\text{cor G}} = T_{\text{eff G}} \cdot \frac{\overline{(a_n v_n)}^{0.024}}{1.31670892 \cdot 10^{-11}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i} R}; \qquad (8.5.14b)$$

$$\overline{T}_{\text{cor K}} = T_{\text{eff K}} \cdot \frac{\overline{(a_n v_n)}^{0.193}}{5.266835678 \cdot 10^{-9}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i} R}; \qquad (8.5.14c)$$

$$\overline{T}_{\text{cor M}} = T_{\text{eff M}} \cdot \frac{\overline{(a_n v_n)}^{0.186}}{6.5835446 \cdot 10^{-9}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i} R}. \qquad (8.5.14d)$$

Using heuristic dependence (8.5.11b) we can suppose that this relation is also valid for the Sun belonging to the spectral class G, that is,

$$T_{\text{eff Sun}} = \frac{1 \cdot 10^{-16}}{(a_n v_n)^{0.024}} \cdot \gamma M_{\text{Sun}}. \qquad (8.5.15)$$

Dividing both parts of Eq. (8.5.11b) on the respective parts of (8.5.15) we obtain:

$$\frac{T_{\text{eff G}}}{T_{\text{eff Sun}}} = \frac{M}{M_{\text{Sun}}} = k_M. \qquad (8.5.16)$$

According to Eq. (8.5.14b) we can estimate the average value of the temperature of the solar corona because the Sun belongs to the spectral class G:

$$\overline{T}_{\text{cor Sun}} = T_{\text{eff Sun}} \cdot \frac{\overline{(a_n v_n)}^{0.024}}{1.31670892 \cdot 10^{-11}} \cdot \frac{1}{1 \cdot R_{\text{Sun}}}. \qquad (8.5.17)$$

Analogously dividing Eq. (8.5.14b) by Eq. (8.5.17) we find [76]:

$$\frac{\overline{T}_{\text{cor G}}}{\overline{T}_{\text{cor Sun}}} = \frac{T_{\text{eff G}}}{T_{\text{eff Sun}}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{R_{\text{Sun}}}{R} = \frac{T_{\text{eff G}}}{T_{\text{eff Sun}}} \cdot \frac{\overline{\mu}_{\text{r}}}{\overline{i}} \cdot \frac{1}{k_R}. \qquad (8.5.18)$$

Taking into account (8.5.16), we again derive from (8.5.18) the above-mentioned formula (8.4.18b). Thus, the regression dependences for multiplanet extrasolar systems confirm the result obtained in Section 8.4 completely [76].

Table 8.4. Theoretical estimations of temperatures of stellar corona T_{cor} for stars belonging to the different spectral classes

Stars	**Spectral class and type**	k_M	k_R	$\overline{\mu}_r / \overline{i}$	T_{eff}, K	T_{cor}, K
Xi Persei	O7.5III	26	14	0.764	35000	$2.1288 \cdot 10^6$
Tau Scorpio	B0.2V	15	6.5	0.7296	31440	$2.5255 \cdot 10^6$
Sirius A	A1V	2.02	1.711	1	9940	$1.7886 \cdot 10^6$
UpsAndromedae	F8V	1.27	1.631	1	6212	$1.1680 \cdot 10^6$
HD10180	G1V	1.06	1	1	5911	$1.5899 \cdot 10^6$
Sun	G2V	1	1	1	5778	$1.5 \cdot 10^6$
Kepler-20	G8	0.912	0.944	1	5466	$1.4492 \cdot 10^6$
55 Cancri	K0IV-V	0.905	0.943	1	5196	$1.4396 \cdot 10^6$
Capella A	K0III	2.69	12.2	2	4940	$6.6148 \cdot 10^5$
14 Andromedae	K0III	2.2	11	2	4813	$5.9999 \cdot 10^5$
Gamma Cephei	K1III-V	1.4	4.9	2	4800	$8.5714 \cdot 10^5$
Groombridge 34	M1.5V	0.404	0.379	1	3730	$1.5989 \cdot 10^6$
Gliese 876	M4V	0.334	0.36	1	3350	$1.3917 \cdot 10^6$

Table 8.5. Orbital and thermodynamic characteristics of multi-planet extrasolar systems

Extrasolar system	Spectral class and type	k_M	k_R	$\bar{\mu}_r / \bar{i}$	T_{eff}, K	Theoretical dependence T_{cor}, K	Number of planets n	Average specific angular momentum $\overline{a_n v_n}$, m^2/s	Regression dependence $\bar{T}_{cor}$, K
Ups And	F8V	1.27	1.63	1	6212	$1.1679 \cdot 10^6$	4	$6.35 \cdot 10^{15}$	$1.1896 \cdot 10^6$
HD10180	G1V	1.06	1	1	5911	$1.5899 \cdot 10^6$	9	$3.723 \cdot 10^{15}$	$1.5261 \cdot 10^6$
Solar	G2V	1	1	1	5778	$1.5 \cdot 10^6$	8	$5.316 \cdot 10^{15}$	$1.5045 \cdot 10^6$
Kepler-20	G8	0.912	0.944	1	5466	$1.4492 \cdot 10^6$	5	$1.415 \cdot 10^{15}$	$1.4606 \cdot 10^6$
55 Cnc	K0IV-V	0.905	0.943	1	5196	$1.4396 \cdot 10^6$	5	$3.599 \cdot 10^{15}$	$1.5124 \cdot 10^6$
Gliese 876	M4V	0.334	0.36	1	3350	$1.3916 \cdot 10^6$	4	$9.92 \cdot 10^{14}$	$1.2512 \cdot 10^6$

Example 8.5.1. Let us estimate the temperature of stellar corona for the star υ Andromedae which is a representative of the spectral type F8. Taking into account that $\overline{a_n v_n} \approx 6.35 \cdot 10^{15} (\mathrm{m}^2/\mathrm{s})$, $n = 4$ [308] we obtain in accordance with (8.5.14a) [76]:

$$\overline{T}_{\mathrm{cor\,F}} = 6212 \cdot \frac{(6.35 \cdot 10^{15})^{0.0037}}{5.266835678 \cdot 10^{-12}} \cdot \frac{1}{1.1343605 \cdot 10^{-9}} =$$
$$= 1.189599699 \cdot 10^{6} (\mathrm{K}) .$$

Since the theoretical estimation of the temperature of stellar corona (8.4.18a) for the star υ Andromedae is equal, $T_{\mathrm{cor}\,\upsilon\mathrm{And}} = 1.167995095 \cdot 10^{6}$ K (see Table 8.4), then the relative error of discrepancy $\delta_{T\mathrm{cor}} = \left[\left(T_{\mathrm{cor}\,\upsilon\mathrm{And}} - \overline{T}_{\mathrm{cor\,F}}\right) / T_{\mathrm{cor}\,\upsilon\mathrm{And}}\right] \cdot 100\% = 1.85\%$. Of course, the heuristic estimation $\overline{T}_{\mathrm{cor\,F}}$ is more exact under comparison with the preliminary estimation in Example 8.3.7 where $\widetilde{T}_{\mathrm{cor}\,\upsilon\mathrm{And}} \approx 6.212 \cdot 10^{3} \cdot 2.596 \cdot 10^{2} = 1.6126352 \cdot 10^{6}$ (K) (in this case $\delta_{T\,\mathrm{cor}} = -38\%$).

Example 8.5.2. Let us calculate the estimation of the temperature of stellar corona for the star Kepler-20 which is a representative of the spectral class G (type G8). According to the formula (8.5.14b), Example 8.3.1 and the average value for the Kepler-20 $\overline{a_n v_n} \approx 1.4148 \cdot 10^{15} (\mathrm{m}^2/\mathrm{s})$, $n = 5$ [308] we obtain [76]:

$$\overline{T}_{\mathrm{cor\,G}} = 5466 \cdot \frac{(1.4148 \cdot 10^{15})^{0.024}}{1.31670892 \cdot 10^{-11}} \cdot \frac{1}{6.56552 \cdot 10^{8}} =$$
$$= 1.460586904 \cdot 10^{6} (\mathrm{K}) .$$

Taking into account that, according to Table 8.4, the theoretic estimation of the temperature of its stellar corona $T_{\mathrm{cor\,Kepler\text{-}20}} = 1.449 \cdot 10^{6}$ K, we find that a relative error of discrepancy is $\delta_{T\text{cor}} = \left[\left(T_{\text{cor Kepler-20}} - \overline{T}_{\text{cor G}}\right)/T_{\text{cor Kepler-20}}\right]\cdot 100\% = -0.8\%$; for comparison, for a preliminary estimation in Example 8.3.1 $\widetilde{T}_{\text{cor Kepler-20}} \approx 5466\cdot 2.596\cdot 10^2 = 1.41897\cdot 10^6\,(\text{K})$, the relative error is equal $\delta_{T\,\text{cor}} = 2.1\%$.

Example 8.5.3. Let us estimate the temperature of stellar corona for the star 55Cnc belonging to the spectral class K (type KOIV-V). Taking into account that $\overline{a_n v_n} \approx 3.5998\cdot 10^{15}\,(\text{m}^2/\text{s}),\ n = 5$ [308], as well as Example 8.3.5, we obtain following (8.5.14c) [76]:

$$\overline{T}_{\text{cor K}} = 5196\cdot\frac{(3.5998\cdot 10^{15})^{0.193}}{5.266835678\cdot 10^{-9}}\cdot\frac{1}{6.558565\cdot 10^{8}} = \\ = 1.512419515\cdot 10^6\,(\text{K}).$$

Taking into account Table 8.4, the theoretic estimation of the temperature of its stellar corona $T_{\text{cor 55Cnc}} = 1.439554612\cdot 10^6$ K we can find a relative error:

$$\delta_{T\text{cor}} = \left[\left(T_{\text{cor 55Cnc}} - \overline{T}_{\text{cor K}}\right)/T_{\text{cor 55Cnc}}\right]\cdot 100\% = -5.1\%.$$

According to Example 8.3.5, a preliminary estimation is: $\widetilde{T}_{\text{cor 55Cnc}} \approx 5196\cdot 2.596\cdot 10^2 = 1.34899\cdot 10^6\,(\text{K})$, so that the respective error $\delta_{T\,\text{cor}} = 6.3\%$.

Example 8.5.4. Let us calculate the estimation of the temperature of stellar corona for the star Gliese 876 belonging to the spectral class M (type M4V). According to the formula (8.5.14d), Example 8.3.8 and the average value for the Gliese 876 $\overline{a_n v_n} \approx 9.92\cdot 10^{14}\,(\text{m}^2/\text{s}),\ n = 4$ [308] we obtain [76]:

$$\overline{T}_{\text{cor M}} = 3350\cdot\frac{(9.92\cdot 10^{14})^{0.186}}{6.5835446\cdot 10^{-9}}\cdot\frac{1}{2.5038\cdot 10^{8}} = \\ = 1.251228307\cdot 10^6\,(\text{K}).$$

Since the theoretic estimation of the temperature of stellar corona (8.4.18a) for the star Gliese 876 is equal $T_{\mathrm{cor\,Gl\,876}} = 1.391666667 \cdot 10^6$ K (see Table 8.4), then the relative error of discrepancy $\delta_{T\mathrm{cor}} = \left[\left(T_{\mathrm{cor\ Gl\,876}} - \overline{T}_{\mathrm{cor\,M}}\right) / T_{\mathrm{cor\ Gl\,876}}\right] \cdot 100\% = 10\%$.
Nevertheless, the heuristic estimation $\overline{T}_{\mathrm{cor\,M}}$ is more exact under comparison with the preliminary estimation in Example 8.3.8 where $\widetilde{T}_{\mathrm{cor\,Gl\,876}} \approx 3350 \cdot 2.596 \cdot 10^2 = 8.6966 \cdot 10^5$ (K) (in this case $\delta_{T\,\mathrm{cor}} = 37.5\%$).

Thus, the heuristic estimations (8.5.14a)–(8.5.14d) confirm satisfactorily enough the derived theoretic formulas (8.4.18a, b) for estimation of the temperature of stellar corona of multi-planet extrasolar systems (see Table 8.5 as well as the derivation (8.4.18b) based on (8.5.15)–(8.5.18)).

8.6. Derivation of Hertzsprung–Russell's dependence based on the USL

Let us note that a variant of the Hertzsprung–Russell dependence can be obtained from (8.3.13) directly [76]. Indeed, if we use the Stefan–Boltzmann law and calculate the luminosity of a star in the form:

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4, \tag{8.6.1}$$

where R is the stellar radius and σ is the Stefan–Boltzmann constant, we can formulate the modified USL (8.3.13) through L:

$$\sqrt{\alpha R} \cdot \frac{m_{\mathrm{p}} M}{\sqrt[4]{L}} = \frac{\Xi / \sqrt{2}}{\sqrt[4]{\pi\sigma}}. \tag{8.6.2}$$

Taking into account that according to (II.2) and (7.3.41) the square root of the parameter of gravitational condensation is approximately equal $\sqrt{\alpha} \approx 1/3R$, we can establish from (8.6.2) that:

$$L \approx 4\pi\sigma\left(\frac{m_{\mathrm{p}}}{3\Xi}\right)^4 \cdot \frac{M^4}{R^2}. \quad (8.6.3)$$

Remarking that $(\sqrt{\alpha})^3 = (2\pi)^{3/2}\rho_0 / M$ according to (8.1.2), where ρ_0 is a density in the center of a spheroidal body, we obtain from (8.6.3) the variant of Hertzsprung–Russell's dependence [76]:

$$L \approx \frac{8\pi^2}{9}\cdot\sigma\rho_0^{2/3}\left(\frac{m_{\mathrm{p}}}{\Xi}\right)^4 \cdot M^{10/3} = \frac{4\pi}{9}\cdot\sigma\alpha\left(\frac{m_{\mathrm{p}}}{\Xi}\right)^4 \cdot M^4. \quad (8.6.4)$$

As follows from (8.6.4), the obtained dependence confirms Hertzsprung–Russell's law completely in the case of s lying between 3.3 and 4 (see the introduction to this chapter), that is, for stars of medium and large masses ($\lg M / M_{\mathrm{Sun}} < 1.7$). As concerns the other types of star (small and medium masses), the initial USL (8.2.19) should be used (not its approximate version in the form of modified USL (8.3.13)) together with the extended variants of formulas (8.4.18a, b) for estimation of temperature of the stellar corona T_{cor} for not only the spectral class G but for F, K, M, and so forth.

Conclusion and comments

In this chapter, we have considered the statistical theory of gravitating spheroidal bodies to derive and develop a universal stellar law for extrasolar systems. In the previous chapters, we proposed the statistical theory for a cosmogonical body forming (so-called spheroidal body). In conformity with stars, the proposed theory takes into account an extended substance called the *stellar corona*. That is why the stellar corona, together with the star's core, can be described by the model of a rotating and gravitating spheroidal body. In this context, Section 8.1 considers the different forms of the potential and potential energy of the gravitational field of spheroidal bodies.

Section 8.2 derived equation of state (8.2.17) of an ideal stellar substance based on the conception of a gravitating spheroidal body in as much as cosmic interstellar space is not empty (in particular, the mean mass density of substance in the neighborhood of the Sun is $6 \cdot 10^{-24}\ \mathrm{g/cm^3}$ whereas in interstellar space it is $3 \cdot 10^{-24}\ \mathrm{g/cm^3}$) [75, 76]. Using this equation, *the universal stellar law* (USL) for planetary systems [76] (connecting the temperature, size, and mass of stars) was obtained in Section 8.2. Analysis of the USL (8.2.19) and its version (8.3.13) has shown that most of them correspond to a category of ideal (or classical) stars (and, respectively, planetary systems) independent of the spectral belonging to O, B, A, F, G, K, and M classes (see Table 8.1, Table 8.3, and Fig. 8.1). The ordinary classical stars satisfy USL relative to their temperatures, sizes, and masses and possess maximal mass densities in the star centers according to (8.1.1) or (8.1.2). Nevertheless, there exists a subclass of stars called the group of red giants (for example, 18 Delphini, ξ Aquilae, HD 81688, 4 Ursae Majoris, Betelgeuse, etc.) for which some characteristics, among them the temperature of the stellar corona (see Table 8.2), differ essentially from analogous characteristics of classical stars. In this context, due to the unstable process of stellar diameters, it is assumed that cavities form inside them (it is well-known that the diameter of Betelgeuse (α Orionis) has decreased systematically by 15% between 1993 and 2009 [327]).Section 8.3 also considered the solar corona in connection with the USL. To modify the USL [76], it takes into account the effective temperature T_{eff} of the Sun's surface in calculating the ratio

(8.3.12) of the temperature of the solar corona T_{cor} to T_{eff}. To test the justice of the modified USL (8.3.13) for different types of star entirely, the temperature of the stellar corona T_{cor} was estimated approximately in Section 8.4 (see the formulas (8.4.18a), (8.4.18b) and Table 8.4).

Section 8.5 showed that knowledge of some characteristics for *multi-planet extrasolar systems* permits us to refine a star's own parameters [76]. In this context, comparison with estimations of temperatures using the above-mentioned regression dependences for multi-planet extrasolar systems testifies to the results obtained. In Section 8.6, the known Hertzsprung–Russell dependence was derived from the USL directly.

Using the modified USL (8.3.13) some predictions of a star's parameters can be made. In particular, for the star HD 181433 (see [285]), the modified USL (8.3.13) gives the following estimation of unknown radius: $R_{\mathrm{HD181433}} \approx 6.102823 \cdot 10^{8}\,\mathrm{m}$ as well as for τ Gem [285] the modified USL (8.3.13) permits us to find the unknown effective temperature $T_{\mathrm{eff}\ \tau\,\mathrm{Gem}} \approx 5582\,\mathrm{K}$. Moreover, using the formula (8.4.18a, b) it becomes possible to estimate the temperature of its stellar corona as $T_{\mathrm{cor}\ \tau\,\mathrm{Gem}} \approx 1.553 \cdot 10^{6}\,\mathrm{K}$.

CHAPTER NINE

THE EXPLANATION OF THE ORIGIN OF THE ALFVÉN–ARRHENIUS OSCILLATING FORCE MODIFYING FORMS OF PLANETARY ORBITS IN THE SOLAR SYSTEM AND OTHER EXOPLANETARY SYSTEMS

We develop the statistical theory of gravitating spheroidal bodies to explain the stability of the orbital motions of planets as well as the forms of planetary orbits with regard to the Alfvén–Arrhenius oscillating force [9, 19, 20] in our Solar system and other exoplanetary systems. The statistical theory of the formation of gravitating spheroidal bodies was proposed in Chapters 1–5 (see also our previous works [16, 45–79]). Starting from the concept of forming a spheroidal body inside a gas-dust protoplanetary nebula, this theory solves the problem of gravitational condensation of a gas-dust protoplanetary cloud with a view to planet formation in its gravitational field and derives the law of planetary distances in the Solar system generalizing the well-known laws [16, 65, 73]. Within the framework of the statistical theory of gravitating spheroidal bodies, the new universal stellar law (USL) [75, 76] connecting the temperature, size, and mass of each star was derived in Chapter 8.

In this chapter, we also consider the USL to explain the stability of planetary orbits in extrasolar systems. Since the USL is based on applying the Poincaré virial theorem to a cloud-like configuration of ideal gas as the stellar corona together with the star in a *mechanical equilibrium state* then the question arises as to how long the gravitational field of the star remains stable. Naturally, the stabilities, as well as the forms of planetary orbits, depend directly on the constancy of a gravitational field level around a star. As Alfvén and Arrhenius noted [9, 19 pp. 343–344]:

> The typical orbits of satellites and planets are circles in certain preferred planes. In satellite systems, the preferred planes tend to coincide with equatorial planes of the central bodies. In the planetary system, the preferred plane is essentially the orbital plane of Jupiter (because this is the biggest planet), which is close to the plane of the ecliptic. The circular motion with period T is usually modified by superimposed oscillations. Radial oscillations (in the preferred plane) with period $\approx T$ change the circle into an ellipse with eccentricity e. Axial oscillations (perpendicular to the preferred plane), also with a period $\approx T$, make the orbit inclined at an angle i to this plane.

Thus, due to the radial (axial) oscillating force, the orbits of moving planets in the Solar system are described by ellipses with focuses on the origin of coordinates and small eccentricities (inclinations). In this connection, the following question arises: *What is the cause of the radial and the axial oscillations as well as the nature of the periodic radial and the periodic axial forces?*

In this chapter, we explain the origin of these oscillating forces [9, 19, 20] which modify forms of planetary orbits within the framework of the statistical theory of gravitating spheroidal bodies. A justification of the stability of orbital motions of planets based on the USL as well as on the forms of planetary orbits with regard to the Alfvén–Arrhenius oscillating forces in our Solar system and other exoplanetary systems is considered. Concretely, as shown in Chapter 5, a temporal deviation of the gravitational compression function (GCF) of a spheroidal body (at first modeling a gas-dust protoplanetary cloud) induces the additional periodic forces.

Then after the decay of the spheroidal body (see Section 5.4), the same additional periodic forces make the planetary orbits elliptic ones [79].

Indeed, as alleged earlier, orbits of moving particles inside a flattened rotating and gravitating gas-dust protoplanetary cloud are initially circular. However, during the evolution of this protoplanetary cloud at the formation of protoplanets these orbits can be deformed a little due to collisions with other particles or gravitational influences of forming adjacent planetesimals. In particular, V.S. Safronov remarked [2 p.145]:

> The assumption of initial motion of particles in circular orbits looks natural. At small masses of bodies, their gravitational variations were weak, and particles moved in orbits close to the circular ones. In the process of a planet growth, deviations of orbits from the circular increased, and all bodies of a zone had an opportunity to be joined in one planet.

However, it should be noted that orbits of the moving bodies at the *later stages of evolution* of a gas-dust protoplanetary cloud are formed mainly under the influence of its centrally symmetric gravitational field. The solution of Binet's equation, therefore, determines the elliptic (or Keplerian) forms of planetary orbits [96]. Moreover, both the Newtonian theory of gravity [80] and the consequent Laplacean celestial mechanics [228] explain the elliptic orbits based on centrally symmetric gravitational forces exclusively and do not consider the processes of formation (including collisions, giant impacts, accretions, or gravitational influences of other bodies). This means that such modification (from the circular orbit to the elliptic one) cannot be explained by a process of formation only since the possible reason consists of a temporal deviation (pulsations) of the gravitational field of a central body (a star) into a protoplanetary cloud. These small pulsations of compression induce the radial and the axial oscillations (with the circular frequencies ω_h and ω_z respectively) of orbital body motion.

As Alfvén and Arrhenius noted, the motion in a centrally symmetric gravitational field "is degenerate in the sense that $\omega_h = \omega_z$... This is due to the fact that there is no preferred direction" [9, 19]. On the contrary, we confirm here that a spatial deviation of the gravitational potential from the centrally symmetric one defines a difference in the values of the radial and the axial orbital oscillations (when $\omega_h \neq \omega_z$) for a rotating ellipsoid-like spheroidal body. That is why an interference of these orbital oscillations can lead to the nonuniform rotation of the stellar layers at different latitudes of a star.

In this chapter, we will also show that the stability of parameters of planetary orbits is determined by a constancy of the specific entropy in conformity with the principles of self-organization in complex systems [79]. We note that a temporal deviation of GCF leads periodically to the special cases when the additional periodic force becomes a counterbalance to the gravitational force, that is, the principle of an anchoring mechanism occurs in planetary systems (Chapter 5). Owing to this principle, the stability of planetary orbits is realized in our Solar system and other exoplanet systems.

Besides the Solar system, here we will focus on the study of the multi-planet extrasolar systems as such Kepler-20 [285, 318], HD10180 [231], HIP14810 [285], 61Virginis [285], 55Cnc [259, 285, 308], Alpha Centauri [285], Upsilon Andromedae [257, 285, 309], Gliese 876 [285, 310] as well as 24 Sextanis, 18 Delphini, Capella A, 14 Andromedae, γ Cephei, β Ceti, ξAquilae, 11 Comae [312, 313, 316, 327] etc. whose stars belong to the different spectral classes F,G, K, and M.

9.1. The derivation of the combination of Kepler's 3rd law with the universal stellar law (3KL-USL) and an explanation of the stability of planetary orbits through 3KL-USL

For a one-component gaseous cloud (for a spheroidal body formed by a collection N of similar particles with the masses m_0 so that $M = m_0 N$ is its mass) let us use the virial theorem of Poincaré (see Theorem 1.3 in Section 1.2) [1, 105, 106]:

$$2E_k + E_g = 0, \tag{9.1.1}$$

where E_k is the total kinetic energy and E_g is the total gravitational potential energy of a *steady state* system in the form of a collection of particles moving under no forces except their mutual gravitational attraction.

According to Boltzmann's molecular kinetic theory, the total kinetic energy of the heat movement of particles E_k connects with the interior energy U of ideal gas. As noticed by S. Chandrasekhar [106], for a cloud-like configuration of an ideal gas the following formula is true:

$$E_k = \frac{3}{2}k_{\text{B}}TN = \frac{3}{2}(\iota - 1)U, \tag{9.1.2}$$

where

U is the interior energy of a cloud-like configuration of an ideal gas,

ι is the polytropic exponent, and

$k_{\text{B}} = 1.38049 \cdot 10^{-23}$ J/K is the Boltzmann constant.

It follows directly from the Poincaré virial theorem (9.1.1) and formula (9.1.2) that:

$$3(\iota - 1)U + E_{\text{g}} = 0. \tag{9.1.3}$$

We note that in the case of Eq. (4.4.11) the polytropic index is equal to $\iota = 2/3$ (see Sections 2.9 and 4.4). Let $E_g + U = E$ be the *total energy* of a cloud-like configuration of the ideal gas. Then, according to Eq. (9.1.3) we can establish that:

$$E = -(3\imath - 4)U = \frac{3\imath - 4}{3\imath - 3} E_g . \tag{9.1.4}$$

As shown by Chandrasekhar [106], a gaseous sphere is *stable* when $\imath > 4/3$. Let ΔE and ΔU be a change of the total energy and a change of the interior energy respectively. Then, according to (9.1.4), the quantity of energy lost by *radiation* $-\Delta E$ during the compression process of a cloud-like configuration is equal to:

$$-\Delta E = -\frac{3\imath - 4}{3\imath - 3} \Delta E_g > 0 , \tag{9.1.5}$$

whereas the *interior energy* increases by the following quantity:

$$\Delta U = -\frac{1}{3\imath - 3} \Delta E_g > 0 . \tag{9.1.6}$$

It follows from Eqs (9.1.4)–(9.1.6) as well as Eq. (8.1.22), a share of the gravitational potential energy in the form of the work $\left|\Delta E_g\right|$ done by the gravitational compression is only partly (in $\frac{3\imath - 4}{3(\imath - 1)}$ times) scattered in space through radiation whereas a remaining part ($1 - \frac{3\imath - 4}{3(\imath - 1)} = \frac{1}{3(\imath - 1)}$) is spent to increase the temperature T of a cloud-like gaseous configuration [106].

On the other hand, if a gravitating spheroidal body is the evolutionary model of a star then a significant part of its gravitational potential energy during the compression process goes over to the particle heat motions into it [79, 328]. In this connection, the question arises*: how long and why is a stable level of the gravitational field in fixed points of space (for example, around stars) supported?*

The preliminary reply follows from Eqs (8.1.22)-(8.1.24), and (8.1.32) (in Section 8.1) under the consideration that the parameter of gravitational condensation is changing with the time, that is, $\alpha = \alpha(t)$. Namely, owing to the slowly increasing parameter of gravitational condensation of a spheroidal body:

$$\alpha(t_2) = \alpha(t_1) + \delta\alpha, \quad t_2 > t_1, \qquad (9.1.7)$$

the absolute value of gravitational potential energy is also growing:

$$E_g(t_2) = E_g(t_1) + \delta E_g, \quad t_2 > t_1. \qquad (9.1.8)$$

In other words, the speed of dissipation E_g is equal to the speed of α changing in the steady state of the virial equilibrium of a spheroidal body. To answer the formulated question quantitatively let us use the universal stellar law (USL) from the previous Section 8.2 [75, 76].

First of all, let us note that the USL (8.2.18) connects the temperature, the size, and the mass of a star [76]:

$$\sqrt{\alpha} \cdot \frac{\bar{\mu}_{\mathrm{r}}}{\bar{i}} \cdot \frac{m_{\mathrm{p}} \cdot M}{T} = \kappa, \qquad (9.1.9)$$

where

$\kappa = 3\sqrt{\pi} \cdot \dfrac{k_{\mathrm{B}}}{\gamma} \approx 1.10003963 \cdot 10^{-12} (\mathrm{kg}^2 / \mathrm{K} \cdot \mathrm{m})$ is the universal stellar constant (8.2.16),

$m_{\mathrm{p}} = 1.67248 \cdot 10^{-27}$ kg is the mass of a proton,

$\bar{i}$ is an average *relative* number of all degrees of freedom for the particle m_0 of a highly ionized stellar substance with a mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$, that is, $m_0 = \bar{\mu}_{\mathrm{r}} \cdot m_{\mathrm{p}}$, and

$i = 3\bar{i}$ is a number of all degrees of freedom.

As shown in Section 8.3, by calculating the left-hand part of Eq. (9.1.9) for the Sun and comparing it with the universal stellar constant κ, we find a coincidence up to the relative error equal to $\delta = 3.37\%$ that testifies to the validity of the USL for the Sun.

To verify the USL for other stars, we need approximation T_{cor} through the effective temperature T_{eff} of the stellar surface of these stars, that is, the modification of USL. This modified USL (8.3.13) can be verified by calculating the left-hand part of Eq. (9.1.9) with the usage of parameters for the different types of stars (see Table 8.1 and Ref. [75, 76]).

First of all, we can note the satisfiability of Eq. (9.1.9) for the stars belonging to the spectral class G, like the Sun: namely, Table 9.1 shows that the modified USL applies with the relative accuracy $\delta = 1.34\%$ for the star Kepler-20 of type G8, it is applied with the small relative error $\delta = -0.1\%$ for HD10180 of type G1V and with high relative accuracy $\delta = -0.75\%$ for the star HIP14810 of class G5 [76]. On the other hand, the modified USL applies with $\delta = -15.87\%$ for the star α Centauri of type K1V which is caused, probably, by an *inexact estimation of temperature of its corona* (see T_{55Cnc} for the respective class K), that is, through a directly proportional dependence T_{cor} on T_{eff} as in the special case of the Sun (for type G2V). Thus, the low accuracy of this law for the stars belonging to the spectral classes F or M, most likely, can be explained by too rough an approximation T_{cor} for the more bright or dim stars [76, 79].

For an abnormal group of intermediate-mass red giants—24 Sextanis, 18 Delphini, Capella A, 14 Andromedae, γ Cephei, β Ceti, ξ Aquilae, 11 Comae [312, 313, 316, 327] (whose chemical composition includes radicals, ions, and electrons)—we suppose $\overline{\mu}_{\text{r}} \approx 2$ in Table 9.1 (unlike Table 8.1).

Table 9.1.Verification of the modified USL for stars belonging to the different spectral classes and types

Stars	**Spectral class and type**	**Mass** M, kg	**Ratio** $\overline{\mu}_r / \overline{i}$	**Radius** R, m	**Effective temperature** T_{eff}, K	**Relative error** δ, %
ξ Persei	O7.5III	$7.16076 \cdot 10^{31}$	1/1	$9.737 \cdot 10^{9}$	35000	58.9
γ Pegasi	B2IV	$1.770299 \cdot 10^{31}$	1/1	$3.3384 \cdot 10^{9}$	21179	51.1
Sirius A	A1V	$4.017982 \cdot 10^{30}$	1/1	$1.190001 \cdot 10^{9}$	9940	33.7
υ Andromedae	F8V	$2.526157 \cdot 10^{30}$	1/1	$1.134361 \cdot 10^{9}$	6212	30
HD 74156	G0	$2.466484 \cdot 10^{30}$	1/1	$1.09889 \cdot 10^{9}$	6039	27
HD 10180	G1V	$2.108446 \cdot 10^{30}$	1/1	$6.955 \cdot 10^{8}$	5911	-0.1
HD 155358	G0	$1.829972 \cdot 10^{30}$	1/1	$6.955 \cdot 10^{8}$	5900	12.9
Sun	G2V	$1.9891 \cdot 10^{30}$	1/1	$6.955 \cdot 10^{8}$	5778	3.37
HD 1461	G0V	$2.148228 \cdot 10^{30}$	1/1	$7.615725 \cdot 10^{8}$	5765	4.5
μ Andromedae	G3IV-V	$2.148228 \cdot 10^{30}$	1/1	$8.658975 \cdot 10^{8}$	5700	15
HD 37124	G4V	$1.810081 \cdot 10^{30}$	1/1	$5.7031 \cdot 10^{8}$	5610	-10.4
61 Virginis	G5V	$1.889645 \cdot 10^{30}$	1/1	$6.5377 \cdot 10^{8}$	5531	-1.99
HIP 14810	G5	$1.969209 \cdot 10^{30}$	1/1	$6.955 \cdot 10^{8}$	5485	-0.75

Kepler-20	G8	$1.814059 \cdot 10^{30}$	1/1	$6.56552 \cdot 10^{8}$	5466	1.34
α Centauri	K1V	$1.857819 \cdot 10^{30}$	1/1	$6.002165 \cdot 10^{8}$	5214	-15.87
55 Cancri	K0IV-V	$1.800136 \cdot 10^{30}$	1/1	$6.558565 \cdot 10^{8}$	5196	-3.1
24 Sextanis	G5	$3.063214 \cdot 10^{30}$	2/1	$3.40795 \cdot 10^{9}$	5098	31.17
18 Delphini	G6III	$4.57493 \cdot 10^{30}$	2/1	$5.91175 \cdot 10^{9}$	4979	39.33
Capella A	K0III	$5.350679 \cdot 10^{30}$	2/1	$8.4851 \cdot 10^{9}$	4940	50.17
14 Andromedae	K0III	$4.37602 \cdot 10^{30}$	2/1	$7.6505 \cdot 10^{9}$	4813	53.61
γ Cephei	K1III-V	$2.78474 \cdot 10^{30}$	2/1	$3.40795 \cdot 10^{9}$	4800	33.55
β Ceti	K0III	$5.56948 \cdot 10^{30}$	2/1	$1.167049 \cdot 10^{10}$	4797	61.17
ξ Aquilae	G9III	$4.37602 \cdot 10^{30}$	2/1	$8.346 \cdot 10^{9}$	4780	57.18
11 Comae	G8 III	$5.37057 \cdot 10^{30}$	2/1	$1.32145 \cdot 10^{10}$	4742	66.54
HIP 57274	K5V	$1.452043 \cdot 10^{30}$	2/2	$4.7294 \cdot 10^{8}$	4640	-29.15
Groombridge 34	M1.5V	$8.035964 \cdot 10^{29}$	2/2	$2.635945 \cdot 10^{8}$	3730	-59.5
Gliese 876	M4V	$6.643594 \cdot 10^{29}$	2/2	$2.5038 \cdot 10^{8}$	3350	-54.59

Supposing the mass of a spheroidal body is to be a constant, that is, $M = \text{const}$, let us apply USL (9.1.9) for two *different states* of a spheroidal body under the assumption that $\alpha = \alpha(t)$ is a variable in a dynamical state-space of this spheroidal body:

$$\sqrt{\alpha_1} \cdot \frac{\bar{\mu}_{\mathrm{r}1}}{\bar{i}_1} \cdot \frac{m_{\mathrm{p}} M}{T_1} = \kappa, \qquad (9.1.10a)$$

$$\sqrt{\alpha_2} \cdot \frac{\bar{\mu}_{\mathrm{r}2}}{\bar{i}_2} \cdot \frac{m_{\mathrm{p}} M}{T_2} = \kappa, \qquad (9.1.10b)$$

where $\alpha_l = \alpha(t_l)$, $\bar{\mu}_{\mathrm{r}l} = \bar{\mu}_{\mathrm{r}}(t_l)$, $\bar{i}_l = \bar{i}(t_l)$ and $T_l = T(t_l)$, $l = 1, 2$.
Dividing Eq. (9.1.10b) into Eq. (9.1.10a) we obtain:

$$\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} = \frac{T_2}{T_1} \cdot \frac{\bar{i}_2}{\bar{i}_1} \cdot \frac{\bar{\mu}_{\mathrm{r}1}}{\bar{\mu}_{\mathrm{r}2}}. \qquad (9.1.11)$$

As noted in [79], as a rule, $\bar{i}_1 = \bar{i}_2 = 1$ for a highly ionized stellar substance, therefore following Eq. (9.1.11) we conclude that:

$$\frac{\alpha_2}{\alpha_1} = \left(\frac{T_2}{T_1} \cdot \frac{\bar{\mu}_{\mathrm{r}1}}{\bar{\mu}_{\mathrm{r}2}} \right)^2. \qquad (9.1.12)$$

The ratio shows that the parameter of gravitational condensation α increases when the temperature T of the shell of a spheroidal body (called the stellar corona) grows whereas the mean relative molecular weight $\bar{\mu}_{\mathrm{r}}$ reduces, that is, α is directly proportional to the squared T and inversely proportional to the squared $\bar{\mu}_{\mathrm{r}}$. The finding $\bar{\mu}_{\mathrm{r}}$ is no simple task in the case of a highly ionized stellar substance [106], so that formula (9.1.12) can be used for calculating $\bar{\mu}_{\mathrm{r}}$ in the process of the evolution of a star:

$$\bar{\mu}_{\mathrm{r}2} = \bar{\mu}_{\mathrm{r}1} \cdot \frac{T_2}{T_1} \cdot \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}. \qquad (9.1.13)$$

Now with the aim of answering the main question on how a stable level of the gravitational potential φ_g is supported around a star in an exoplanet system and how, because of this, the stability of planetary orbit occurs, let us consider jointly USL (9.1.9) and Kepler's 3rd law (3KL) [206]:

$$\frac{a^3}{\mathrm{T}_\mathrm{K}^2} = \frac{\gamma M}{4\pi^2}, \qquad (9.1.14)$$

where a is a major semi-axis of the planetary orbit, and T_K is a Keplerian period of motion of the planet around its star (belonging to our Solar or exoplanet system).

Taking into account the formula of the universal stellar constant (8.2.16) let us rewrite USL (9.1.9) in the following way:

$$\sqrt{\frac{\pi}{\alpha}} \cdot \frac{3\bar{i}}{m_\mathrm{p}\bar{\mu}_\mathrm{r}} \cdot k_\mathrm{B} T = \gamma M \,. \qquad (9.1.15)$$

Then substituting the left-hand part of (9.1.15) into the right-hand part of Eq. (9.1.14) we obtain *the combined Kepler 3rd law with the universal stellar law* (3KL-USL) [79, 328]:

$$\frac{a^3}{\mathrm{T}_\mathrm{K}^2} = \frac{3}{4\pi^{3/2} m_\mathrm{p}} \cdot \frac{\bar{i}}{\bar{\mu}_\mathrm{r}} \cdot \frac{\theta}{\sqrt{\alpha}}\;, \qquad (9.1.16)$$

where $\theta = k_\mathrm{B} T$ is a statistical temperature [110]. The combined law (9.1.16) shows that the stability of parameters of planetary orbits is determined by the constancy of the value:

$$\frac{\theta}{\sqrt{\alpha}} \cdot \frac{\bar{i}}{\bar{\mu}_\mathrm{r}} = \mathrm{const}\,, \qquad (9.1.17)$$

that is fully confirmed by Eq. (9.1.11).

Introducing an angular velocity $\Omega_\mathrm{K} = 2\pi/\mathrm{T}_\mathrm{K}$ of rotational motion of a planet around a star, the combined law (9.1.16) can be written in the following form:

$$a^3\Omega_{\mathrm{K}}^2 = s\cdot\frac{\bar{i}}{\bar{\mu}_{\mathrm{r}}}\cdot\frac{T}{\sqrt{\alpha}}, \qquad (9.1.18)$$

where:

$$s = 3\sqrt{\pi}\cdot\frac{k_{\mathrm{B}}}{m_{\mathrm{p}}} \qquad (9.1.19a)$$

is a constant with the value equal to:

$$s \approx 3\sqrt{3.14159265}\cdot\frac{1.38049\cdot10^{-23}\ \mathrm{J/K}}{1.67248\cdot10^{-27}\ \mathrm{kg}} = 4.3890297\cdot10^4\,(\mathrm{J/kg\cdot K}).$$

Let us note that this constant has a physical measure of the specific heat capacity c_V or the specific entropy s [206], that is, according to Eqs (9.1.18) and (9.1.19a) the stability of the parameters of planetary orbits is determined by a *constancy of the specific entropy* s in conformity with the principles of self-organization in complex systems [134, 135]. Indeed, the constant s can be represented as the following [79]:

$$s = 3\sqrt{\pi}\cdot\frac{k_{\mathrm{B}}}{\mu_{\mathrm{p}}}\cdot N_A = 3\sqrt{\pi}\cdot\frac{\Re}{\mu_{\mathrm{p}}} = \frac{3\Re}{2}\cdot\frac{2\sqrt{\pi}}{\mu_{\mathrm{p}}} = c_{\mu V}^{(1)}\cdot\frac{2\sqrt{\pi}}{\mu_{\mathrm{p}}}, \qquad (9.1.19b)$$

where

$c_{\mu V}^{(1)}$ is a molar heat capacity of one-atomic gas under the condition of its constant volume V,

N_A is the Avogadro constant, and

$\Re$ is the universal gaseous constant [206].

Then, according to Eqs (9.1.14), (9.1.18) and (9.1.19b) we obtain:

$$\gamma M = a^3\Omega_{\mathrm{K}}^2 = s\cdot\frac{\bar{i}}{\bar{\mu}_{\mathrm{r}}}\cdot\frac{T}{\sqrt{\alpha}} = c_{\mu V}^{(1)}\cdot\frac{2\sqrt{\pi}}{\mu_{\mathrm{p}}}\cdot\frac{\bar{i}}{\bar{\mu}_{\mathrm{r}}}\cdot\frac{T}{\sqrt{\alpha}} =$$

$$= 2\sqrt{\pi}\cdot\frac{c_{\mu V}^{(i)}}{\mu}\cdot\frac{T}{\sqrt{\alpha}} = \frac{2\sqrt{\pi}}{\sqrt{\alpha}}\cdot c_V T = \frac{2\sqrt{\pi}}{\sqrt{\alpha}}\cdot u, \qquad (9.1.20a)$$

where

$u = c_V T$ is a specific value of the interior energy of ideal gas,

$c_V = c_{\mu V}^{(i)} / \mu$ is a specific heat capacity, and

$c_{\mu V}^{(i)} = \bar{i} c_{\mu V}^{(1)} = \bar{i} 3\Re / 2 = i\Re / 2$ is a molar heat capacity of an ideal gas.

Moreover, $\mu = \bar{\mu}_{\rm r} \mu_{\rm p}$. Since the specific value of the interior energy is $u = dU / dm$ (where U is interior energy of a spheroidal configuration of an ideal gas, and dm is an elementary mass) we can suppose that $dU / dm \approx U / M$ at the initial stage of formation of a spheroidal body with the total mass M. Then taking into account formula (8.1.23) from Section 8.1 as well as (9.1.20a) we establish that:

$$U \approx \sqrt{\frac{\alpha}{\pi}} \cdot \frac{\gamma M^2}{2} = \left| E_g \right|, \qquad (9.1.20b)$$

whence $\iota \approx 4/3$ following Eq. (9.1.3), that is, the gravitational potential energy is spent by the interior energy under the *condition of virial equilibrium* of a spheroidal body. As A.Ritter showed in 1878 [106], the polytropic process with $\iota \approx 4/3$ has a special "cosmogonical" interest.

The combined law connects the mechanical values a and $\Omega_{\rm K}$ in the left-hand part of Eq. (9.1.18) and the statistical (thermodynamic) values α, T, $\bar{i}$, and $\bar{\mu}_{\rm r}$ in the right-hand part of this equation. It means that the stability of the mechanical values (including the angular velocity $\Omega_{\rm K}$ and the major semi-axis a of a planetary orbit) depends on a statistical regularity of the right-hand part of Eq. (9.1.18). Thus, we conclude there is a possibility of the presence of *statistical oscillations* of motion in planetary orbit [79, 328], that is, the oscillations of the major semi-axis a and the orbital angular velocity $\Omega_{\rm K}$ of the rotational motion of planets and bodies around stars. Indeed, this conclusion is fully confirmed by the existing radial and axial orbital oscillations of bodies described for the first time by Alfvén and Arrhenius [9, 19].

9.2. On the Alfvén–Arrhenius specific additional periodic force modifying circular orbits of bodies

At the simplest type of body motion with a constant velocity v_0 around a central gravitating body in a circular orbit with radius R_0, the specific force (per mass unit) of gravity f_g is exactly compensated for by the centrifugal specific force f_c:

$$f_g = f_c , \tag{9.2.1}$$

whereas an additional specific force acts upon the body at the modification of the circular orbit:

$$f_a = f_c - f_g . \tag{9.2.2}$$

As we know [96], the orbits of bodies moving in a centrally symmetric gravitational field of central body can be calculated by the equation in the form:

$$\frac{d^2r}{dt^2} = \frac{C^2}{r^3} - f_g , \tag{9.2.3}$$

where C is an areal velocity [96, 158]:

$$C = r^2\dot{\varepsilon} . \tag{9.2.4}$$

Comparing Eq. (9.2.2) with Eq. (9.2.3) we establish that:

$$f_c = \frac{C^2}{r^3} ; \tag{9.2.5a}$$

$$f_a = \frac{d^2r}{dt^2} . \tag{9.2.5b}$$

Following Alfvén and Arrhenius [9, 19], a body's circular orbit can be modified by both the radial and the axial oscillations. We can, therefore, consider separately the radial h- and the axial z-projection of the force $\vec{f}_a$. As usual, the body location in orbit can be estimated by the radius-vector $\vec{r}$ with length equal to:

$$r = \sqrt{h^2 + z^2}\ . \tag{9.2.6}$$

Initially, a body is moving along an unperturbed circular orbit whose equation is:

$$r = R_0, \tag{9.2.7}$$

where $R_0 = \sqrt{H_0^2 + Z_0^2} = H_0$ because $Z_0 = 0$, that is, it is assumed that the unperturbed circular orbit is situated in the plane Oxy.

Thus, if the body is displaced radially from $h = R_0$ to $h = R_0 + \Delta h$, it is acted upon by the force (per mass unit):

$$f_{a_h}(h) \approx f_{a_h}(R_0) + \left.\frac{\partial f_{a_h}}{\partial h}\right|_{R_0} \cdot \Delta h, \tag{9.2.8}$$

which can be represented by the Taylor series in linear approximation (9.2.8) as far as $\Delta h << R$. Taking (9.2.1) into account, we note this force has to be equal to zero $f_{a_h}(R_0) = 0$ under the condition (9.2.7) so that according to Eqs (9.2.2), (9.2.5a), and (9.2.8) we obtain:

$$f_{a_h} = \left.\frac{\partial f_{a_h}}{\partial h}\right|_{R_0} \cdot \Delta h = \frac{\partial}{\partial r}\left(\frac{C^2}{r^3} - f_g\right)\left.\frac{\partial r}{\partial h}\right|_{R_0} \cdot \Delta h =$$

$$= \left.\left[-\left(\frac{3C^2}{r^4} + \frac{\partial f_g}{\partial r}\right)\cdot\frac{h}{r}\right]\right|_{R_0} \cdot \Delta h = -\left.\left[\frac{3C^2}{r^4} + \frac{\partial f_g}{\partial r}\right]\right|_{R_0} \cdot \Delta h. \tag{9.2.9}$$

On the other hand, the substitution (9.2.8) into Eq. (9.2.5b) with regard to $f_{a_h}(R_0) = 0$ and $h = R_0 + \Delta h$ yields a harmonic oscillator equation [73]:

$$\frac{d^2(\Delta h)}{dt^2} - \left.\frac{\partial f_{a_h}}{\partial h}\right|_{R_0} \cdot \Delta h = 0, \tag{9.2.10}$$

that is, a circular frequency ω_h of the harmonic oscillator is $\sqrt{-\left.\frac{\partial f_{a_h}}{\partial h}\right|_{R_0}}$. Thus, the body oscillates in the radial h-direction around the circle with the angular frequency [9, 19]:

$$\omega_h = \sqrt{-\left.\frac{\partial f_{a_h}}{\partial h}\right|_{R_0}} = \sqrt{\left[\frac{3C^2}{r^4}+\frac{\partial f_g}{\partial r}\right]_{R_0}} = \sqrt{\frac{3C^2}{R_0^4}+\left.\frac{\partial f_g}{\partial r}\right|_{R_0}} \cdot \quad (9.2.11)$$

Taking into account the condition (9.2.1) for the circular orbit and Eq. (9.2.5a), expression (9.2.11) for the frequency of the radial oscillations takes the form:

$$\omega_h = \sqrt{\frac{3f_g(R_0)}{R_0}+\left.\frac{\partial f_g}{\partial r}\right|_{R_0}}\ . \quad (9.2.12)$$

If the body is displaced in the z-direction (axial direction), it is acted upon by the force f_{a_z} which can be defined like the expansion (9.2.8):

$$f_{a_z}(z) = f_{a_z}(0) + \left.\frac{\partial f_{a_z}}{\partial z}\right|_0 \cdot \Delta z = \left.\frac{\partial f_{a_z}}{\partial z}\right|_0 \Delta z\ . \quad (9.2.13)$$

According to Eq. (9.2.5b) and the formula (9.2.13) the axial oscillations in the z-direction are described by the same form (9.2.10) of harmonic oscillator equation, namely [73]:

$$\frac{d^2(\Delta z)}{dt^2} - \left.\frac{\partial f_{a_z}}{\partial z}\right|_0 \cdot \Delta z = 0\ . \quad (9.2.14)$$

As to the angular ε-component of the additional specific force f_{a_ε}, Alfvén and Arrhenius supposed that $f_{a_\varepsilon} \to 0$. Because $\operatorname{div}\vec{f}_a = 0$ for a periodic force and the same divergence of $\vec{f}_a$ in the cylindrical coordinate system is:

$$\operatorname{div}\ \vec{f}_a = \frac{1}{h}\cdot\frac{\partial(hf_{a_h})}{\partial h} + \frac{1}{h}\cdot\frac{\partial(f_{a_\varepsilon})}{\partial \varepsilon} + \frac{\partial f_{a_z}}{\partial z} = 0\,,$$

then:

$$\frac{\partial f_{a_z}}{\partial z} = -\frac{\partial f_{a_h}}{\partial h} - \frac{f_{a_h}}{h}. \tag{9.2.15}$$

Substituting (9.2.15) into Eq. (9.2.14) we find the circular frequency of the axial oscillations [9, 19]:

$$\omega_z = \sqrt{-\frac{\partial f_{a_z}}{\partial z}\bigg|_{Z_0=0}} = \sqrt{\frac{\partial f_{g_z}}{\partial z}\bigg|_{Z_0=0}} = \sqrt{\left[\frac{\partial f_{a_h}}{\partial h} + \frac{f_{a_h}}{h}\right]\bigg|_{Z_0=0}} =$$

$$= \sqrt{-\frac{\partial f_{g_h}}{\partial h}\bigg|_{Z_0=0} + \frac{f_{c_h} - f_{g_h}}{\sqrt{r^2 - z^2}}\bigg|_{Z_0=0}} = \sqrt{-\frac{\partial f_g}{\partial r}\bigg|_{R_0} - \frac{f_g}{R_0}}, \tag{9.2.16}$$

where it was considered in Eq. (9.2.16) that the specific centrifugal force $\vec{f}_c$ does not contribute to the z-projection of the specific periodic force: $f_{a_z} = -f_{g_z}$ because $f_{c_z} = 0$ as well as $R_0 = H_0$ when $Z_0 = 0$, $f_{c_h}\big|_{Z_0=0} = 0$. As follows from (9.2.12) and (9.2.16), the equation holds [9, 19]:

$$\omega_h^2 + \omega_z^2 = \frac{3f_g(R_0)}{R_0} + \frac{\partial f_g}{\partial r}\bigg|_{R_0} - \frac{\partial f_g}{\partial r}\bigg|_{R_0} - \frac{f_g(R_0)}{R_0} = \frac{2f_g(R_0)}{R_0}. \tag{9.2.17}$$

Let us note that when a body is moving in a circular orbit of the radius R_0 with the constant velocity v_0 the force of gravitational attraction is exactly compensated for by the centrifugal force (9.2.5a), so that:

$$f_g = \frac{v_0^2}{R_0}, \tag{9.2.18}$$

that is, the orbital angular velocity is:

$$\Omega_K = \frac{v_0}{R_0} = \sqrt{\frac{f_g}{R_0}} \tag{9.2.19}$$

and the period of this motion $T_K = 2\pi/\Omega_K$ we know as the *Keplerian period* [96, 158]. Substituting (9.2.19) into Eq. (9.2.17) we obtain [9, 19]:

$$\omega_h^2 + \omega_z^2 = 2\Omega_K^2 . \tag{9.2.20}$$

It is well known [80, 96] that moving bodies draw elliptical orbits (6.2.2) in the gravitational field of a central body, and the central body is in the orbital focus according to Kepler's 1st law:

$$r = \frac{a(1-e^2)}{1+e\cos\varepsilon}, \tag{9.2.21}$$

where e is an eccentricity of the elliptical orbit, and a is an orbital semi-major axis of this ellipse. When $\varepsilon = 0$ we have the perihelion of this orbit $\varepsilon_{\min} = a(1-e)$ and when $\varepsilon = \pi$ we obtain the aphelion $\varepsilon_{\max} = a(1+e)$. It is clear that if $e = 0$ then Eq. (9.2.21) describes a circular orbit (9.2.7):

$$r = a, \tag{9.2.22}$$

where $R_0 = a$ is a radius of the circular orbit. According to Eq. (9.2.21) the location coordinate h in the orbit is given by:

$$h = \frac{a(1-e^2)\sin\theta}{1+e\cos\varepsilon} = \frac{a(1-e^2)\cos i}{1+e\cos\varepsilon} \tag{9.2.23a}$$

and the orbital position coordinate z is found accordingly:

$$z = \frac{a(1-e^2)\cos\theta}{1+e\cos\varepsilon} = \frac{a(1-e^2)\sin i}{1+e\cos\varepsilon}, \tag{9.2.23b}$$

where θ is a polar angle, i is an angle of inclination ($i=\pi/2-\theta$). Since $C^2 = \gamma Ma\cdot(1-e^2)$ then the values $a\cdot(1-e^2)\cos i = (C^2/\gamma M)\cos i$ in Eq. (9.2.23a) and $a\cdot(1-e^2)\sin i = (C^2/\gamma M)\sin i$ in Eq. (9.2.23b) indicate the projections of the specific angular momentum on the axis Oz and the plane xOy, that is, we can talk about the angular momentum deficit $C\cdot\cos i$ and $C\cdot\sin i$ introduced by Laplace firstly [228, 257].

If $e << 1$ then, taking into account only the first order term on e in Eq. (9.2.21), we obtain the equation of an *almost* circular orbit:

$$r = a \cdot [1 - e\cos\varepsilon]. \qquad (9.2.24)$$

As in Eqs (9.2.23a, b) the projections of Eq. (9.2.24) on the axis Oz and the plane xOy with regard for the angle of inclination $i << 1$ respectively take the forms [9, 19]:

$$h = a\cos i\,[1 - e\cos\varepsilon_h] \approx a\,[1 - e\cos\varepsilon_h]; \qquad (9.2.25a)$$

$$z = a\sin i\,\sin\varepsilon_z \approx ai\,\sin\varepsilon_z, \qquad (9.2.25b)$$

where in Eqs (9.2.25a) and (9.2.25b), we have taken into account that $\sin i \to i$, $\cos i \to 1$ at $i \to 0$. In view of the identity of Eqs (9.2.24) and (9.2.25a) the index h can be replaced by r.

According to the obtained Eqs (9.2.25a) and (9.2.25b), the deviation of the elliptical orbit (with a small eccentricity $e << 1$) from the circular one is described by the *radial* harmonic oscillation with the amplitude $ea << a$ (here $a = R_0$) and the circular frequency ω_h following (9.2.11):

$$h - a = -ea \cdot \cos\varepsilon_h = -eR_0\cos(\omega_h t - \varepsilon_h^0), \qquad (9.2.26a)$$

and the deviation of this coplanar orbit from the plane xOy is described by the *axial* harmonic oscillation with the amplitude $ia << a$ (where $a = R_0$) and the circular frequency ω_z in accordance with (9.2.16):

$$z = ia \cdot \sin\varepsilon_z = iR_0\sin(\omega_z t - \varepsilon_z^0), \qquad (9.2.26b)$$

where ε_h^0, ε_z^0 are initial phase angles of the radial and the axial oscillations.

Taking into account the dependence (9.2.20) of the circular frequencies ω_h, ω_z, and Ω_K, let us introduce the following angular velocity [9, 19]:

$$\Omega_P = \Omega_K - \omega_h. \qquad (9.2.27)$$

Using Eq. (9.2.27) we can write the above-mentioned relation (9.2.26a) as follows:

$$\xi = h - R_0 = -eR_0 \cos\left(\Omega_K t - \Omega_P t - \varepsilon_h^0\right). \qquad (9.2.28)$$

In this context, H. Alfvén proposed an approximate description of Kepler's motion using the *guiding-center method* [9, 19, 329] (a similar idea was first formulated within the framework of the Ptolemaic system of the world). According to this method which was developed in plasma physics [19 p. 344, 329], let us place a moving frame of reference with the origin at a point moving along the unperturbed orbit (circle) with the angular velocity Ω_K (see Figure 9.1). The origin of such a system is called "the guiding-center," and the axis $O\xi$ points in the radial direction and the axis $O\eta$ in the forward tangential direction.

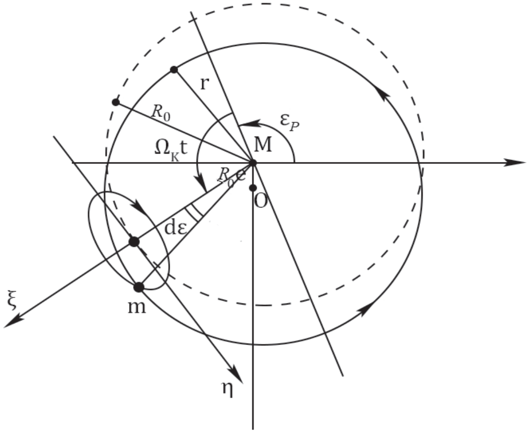


Figure 9.1. An approximate description of the motion using the guiding-center method

According to Fig. 9.1 the guiding-center moves with a constant velocity along the dashed circle with a radius R_0 in the center of which there is an attracting mass M. In turn, the body with a mass m moves in an "epicycle" (around the guiding-center) which is an ellipse with a minor axis eR_0. Thus, the pericenter moves (more precisely, it precesses) with the angular velocity Ω_{P} defined by formula (9.2.27). Indeed, it follows from Eqs (9.2.12) and (9.2.27) directly that rotation with the frequency Ω_{P} is opposite to the Keplerian rotation with Ω_{K}. The linear tangential velocity of motion along the axis $O\eta$ is equal to a difference in the values of the azimuthal velocities along with the slightly elliptical and the circular orbits:

$$v_\eta = \frac{d\eta}{dt} = v_\varepsilon - v_0 = r\dot{\varepsilon} - R_0\Omega_{\text{K}} . \qquad (9.2.29)$$

Taking into account formula (9.2.4) and Eqs (9.2.24), (9.2.25a) the angular velocity of a body moving in the slightly elliptical orbit is:

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{C}{r^2} \approx \frac{C}{R_0^2} \cdot \left[1 + 2e \cdot \cos(\omega_h t - \varepsilon_h^0)\right] =$$
$$= \Omega_{\text{K}} \left[1 + 2e \cdot \cos(\omega_h t - \varepsilon_h^0)\right]. \qquad (9.2.30)$$

Substituting (9.2.30) into (9.2.29) we find that:

$$\frac{d\eta}{dt} \approx R_0(\dot{\varepsilon} - \Omega_{\text{K}}) = \frac{2eC}{R_0} \cos(\omega_h t - \varepsilon_h^0) . \qquad (9.2.31)$$

Integrating Eq. (9.2.31) and bearing in mind formula (9.2.27) we define:

$$\eta \approx \frac{2eC}{R_0} \cdot \frac{1}{\omega_h} \sin(\omega_h t - \varepsilon_h^0) = 2eR_0 \frac{\Omega_{\text{K}}}{\omega_h} \sin(\omega_h t - \varepsilon_h^0) =$$
$$= 2eR_0[1 + \frac{\Omega_{\text{P}}}{\omega_h}] \sin(\Omega_{\text{K}} t - \Omega_{\text{P}} t - \varepsilon_h^0) . \qquad (9.2.32)$$

From the comparison of Eqs (9.2.32) and (9.2.28) with regard to $\Omega_{\text{P}} << \omega_h$ it follows that the epicycle is an ellipse with the axis ratio 2:1 and the minor axis eR_0. Moreover, “the epicycle motion is retrograde” [9, 19]. The pericenter (point closest to the center of attraction) is reached when ξ is a minimum, that is, when:

$$\Omega_{\mathrm{K}} t - \Omega_{\mathrm{P}} t - \varepsilon_h^0 = 2\pi n, \quad n = 0, 1, 2, \ldots . \tag{9.2.33}$$

Assuming:

$$\varepsilon_{\mathrm{P}} = \Omega_{\mathrm{P}} t + \varepsilon_h^0 , \tag{9.2.34}$$

we obtain that the pericenter is reached when:

$$t = \frac{\varepsilon_{\mathrm{P}} + 2\pi n}{\Omega_{\mathrm{K}}} , \tag{9.2.35}$$

that is, according to (9.2.27) the pericenter has a precession with the angular velocity Ω_{P} [9, 19].

In a similar way (see (9.2.27)) Alfvén and Arrhenius introduced the following oscillation frequency:

$$\Omega_{\zeta} = \Omega_{\mathrm{K}} - \omega_z , \tag{9.2.36}$$

so that we can find that the axial oscillation equation (9.2.26b) takes the form:

$$z = iR_0 \sin(\Omega_{\mathrm{K}} t - \Omega_{\zeta} t - \varepsilon_z^0) , \tag{9.2.37}$$

where i is an inclination ($i << 1$). The angular velocity Ω_{ζ} determines the angle ε_{ζ} of the "ascending node" [96], that is, the point where the coordinate z becomes positive:

$$\varepsilon_{\zeta} = \Omega_{\zeta} t + \varepsilon_z^0 . \tag{9.2.38}$$

Let us note when an orbiting body moves in the gravitational field of a *non-rotating central body* the specific force of attraction (or the gravitational field strength) is equal to:

$$f_g = \frac{\gamma M}{r^2} , \tag{9.2.39}$$

where M is a mass of the central body, γ is the Newtonian gravitational constant. From Eq. (9.2.39) we establish directly that:

$$\frac{\partial f_g}{\partial r} = -\frac{2f_g}{r}. \qquad (9.2.40)$$

Substitution of (9.2.40) into the formulas (9.2.12) and (9.2.16) shows that:

$$\omega_h = \sqrt{\frac{3f_g(R_0)}{R_0} - \frac{2f_g(R_0)}{R_0}} = \sqrt{\frac{f_g(R_0)}{R_0}} \equiv \Omega_K; \qquad (9.2.41a)$$

$$\omega_z = \sqrt{\frac{2f_g(R_0)}{R_0} - \frac{f_g(R_0)}{R_0}} = \sqrt{\frac{f_g(R_0)}{R_0}} \equiv \Omega_K, \qquad (9.2.41b)$$

that is, $\omega_h = \omega_z = \Omega_K$ in the case of non-rotating (immovable) central body. This means that the identity (9.2.20) is true, and $\Omega_K = \sqrt{\gamma M / R_0^3}$ is the *Keplerian angular velocity* in accordance with (9.2.19) and (9.2.39).

The significance of (9.2.41a, b) is that the frequencies of radial and axial oscillations coincide with the fundamental angular velocity of circular motion when a body is moving in the *slightly elliptical orbit* in the central gravitational field of a non-rotating body. Consequently, in this case $\Omega_P = \Omega_\zeta = 0$, that is, there is no precession of the pericenter or the nodes. According to Eqs (9.2.28) and (9.2.32) the body moves along the epicycle [9, 19]:

$$\xi = -eR_0 \cos(\Omega_K t - \varepsilon_h^0); \qquad (9.2.42a)$$

$$\eta = 2eR_0 \sin(\Omega_K t - \varepsilon_h^0). \qquad (9.2.42b)$$

The center of the epicycle moves with *constant velocity* in the circle with the radius R_0. The motion in the epicycle takes place in the retrograde direction (see Fig.9.1). According to the condition (9.2.41b), the equation for the axial oscillations (9.2.37) becomes:

$$z = iR_0 \sin(\Omega_K t - \varepsilon_z^0). \qquad (9.2.43)$$

As a result, the orbit of the moving body in the gravitational field of the central body is an ellipse but its plane has the inclination i with the plane of an undisturbed circular motion.

Thus, at the transition from the circular to the slightly elliptical orbit, an additional *periodic force* acts on the body, as Alfvén and Arrhenius indicated [9, 19]. Indeed, when the body is moving along a circular (undisturbed) orbit with a constant velocity v_0 the specific gravitational attraction f_g is exactly balanced by the specific centrifugal force f_c (see Eq. (9.2.1)):

$$f_g = f_c = \frac{v_0^2}{R_0} = \Omega_{\mathrm{K}}^2 \cdot R_0 . \qquad (9.2.44a)$$

Since $C^2 = \gamma Ma \cdot (1 - e^2)$ the additional periodic force f_a can be found if first we calculate the specific centrifugal force f_c for a body moving in the disturbed (slightly elliptical) orbit following (9.2.5a), (9.2.21), (9.2.41a), and (9.2.39):

$$f_c = \dot{\varepsilon}^2 \cdot r = \frac{C^2}{r^3} = \frac{\gamma Ma \cdot (1 - e^2)}{r^3} = \frac{\gamma M}{r^2} \cdot \frac{a \cdot (1 - e^2)}{r} =$$

$$= f_g \cdot \left[1 + e \cos\left(\Omega_{\mathrm{K}} t - \varepsilon_h^0\right)\right], \qquad (9.2.44b)$$

whence, taking into account Eqs (9.2.2), (9.2.30), (9.2.41a), and Eq. (9.2.44a), we establish that, if $\dot{\varepsilon} \approx \Omega_{\mathrm{K}}$, then the specific periodic *radial* force is equal [73]:

$$f_a = f_c - f_g = f_g \cdot e \cos\left(\Omega_{\mathrm{K}} t - \varepsilon_h^0\right) = e \frac{\gamma M}{r^2} \cos\left(\Omega_{\mathrm{K}} t - \varepsilon_h^0\right) \approx$$

$$\approx e \Omega_{\mathrm{K}}^2 \cdot R_0 \cos\left(\Omega_{\mathrm{K}} t - \varepsilon_h^0\right). \qquad (9.2.45)$$

Here it seems natural to ask: *What is the nature of an additional radial force?* As shown in Sections 5.2–5.4 above, the proposed statistical theory of gravitating spheroidal bodies provides a possible answer to this question using formula (5.2.3) for calculating the gravitational field strength $\vec{a}$ of a forming spheroidal body (see relevant formulas (5.4.38) and (5.4.39)).

9.3. Newtonian prediction of the Alfvén–Arrhenius specific additional periodic force

In the remote zone II, particles move around a central cosmogonical body in the orbits with the Keplerian angular velocity (see, for example, (5.4.8) in Chapter 5) following Kepler's 3rd law [96, 158]. The particles, having almost identical specific orbital angular momentums, form exterior shells. During the process of a conglomeration of particles of gas-dust matter, their specific angular momentums are to be averaged thus forming bunches (or *protoplanetary embryos* [208, 250]) in the exterior shells (see formula (6.1.2) from Section 6.1 of Chapter 6). In other words, during the process of protoplanet origin, each particle in a gas-dust protoplanetary cloud (generally speaking, a swarm of those particles or planetesimals) has a chance to reach (or land on) the protoplanet whose specific angular momentum value is the same as one for the particle/planetesimal (see Chapter 6 and Ref. [2, 6, 21, 242–247]). Though it seems surprising, we show below that knowledge of the process of a conglomeration of particles based on the coincidence of their specific orbital angular momentums reveals the Alfvén–Arrhenius additional orbital periodic force [205].

Indeed, starting from the law of conservation of specific orbital angular momentum (6.2.32) (or (7.2.43)) in the central gravitational field for all particles of a protoplanetary embryo:

$$C = \dot{\varepsilon} r^2 = \sqrt{\gamma Ma(1-e^2)} = \text{const} \qquad (9.3.1)$$

and taking into account equation (9.2.21) of the orbit of a moving particle:

$$r = \frac{a(1-e^2)}{1+e\cdot\cos\varepsilon(t)}, \qquad (9.3.2)$$

we have:

$$\dot{\varepsilon}r^2 = \sqrt{\gamma Mr(1+e\cdot\cos\varepsilon(t))}, \qquad (9.3.3)$$

whence:

$$\dot{\varepsilon}^2 r^3 = \gamma M(1+e\cdot\cos\varepsilon(t)), \qquad (9.3.4)$$

where e is an eccentricity of the elliptical orbit. Like Eq. (5.4.20) from Section 5.4, rewriting Eq. (9.3.4) we obtain again [205]:

$$f_c = \dot{\varepsilon}^2 r = \frac{\gamma M}{r^2} + e\frac{\gamma M}{r^2}\cdot\cos\varepsilon(t) = f_g + f_a, \qquad (9.3.5)$$

which entirely confirms the obtained Eqs (5.4.24) and (5.4.38) in Section 5.4 (as well as Eq. (9.2.2) for a slightly elliptical orbit in the previous Section 9.2). Taking into account Eqs (9.2.2) and (9.3.5) we also find additionally:

$$f_a = e\frac{\gamma M}{r^2}\cdot\cos\varepsilon(t), \qquad (9.3.6)$$

which verifies Eq. (9.2.45) completely [205].

Now let us consider the forming l-protoplanetary embryo moving around the core of a spheroidal cloud (as a central cosmogonical body) in orbit with the l-Keplerian angular velocity in accordance with the formula (5.4.8):

$$\Omega_{\mathrm{K}l} = \sqrt{\frac{\gamma M}{a_l^3}}, \qquad (9.3.7a)$$

then:

$$\varepsilon_l(t) = \Omega_{\mathrm{K}l}\, t - \varepsilon_l^0, \quad \varepsilon_l^0 = \varepsilon_l(0). \qquad (9.3.7b)$$

Taking into account Eq. (9.3.7b) the specific additional periodic Alfvén–Arrhenius force (9.3.6) acting on l-protoplanetary embryo (or protoplanet) becomes:

$$f_{a_l} = e_l \frac{\gamma M}{r_l^2} \cdot \cos(\Omega_{\text{K}l}\, t - \varepsilon_l^0), \quad l = 1,...,N . \qquad (9.3.8)$$

Comparing Eq. (9.3.8) with Eq. (5.4.40b) we conclude that formula (9.3.8) corresponds well to a spectral component $\vec{f}_a^{(l)}$ in the expansion (5.4.40a) [205].

9.4. The regular and wave gravitational potentials arising under the orbital motion of a gravitating body in the theory of retarded gravitational potentials

The consideration in Sections 5.4, 9.2, 9.3 points to a *wave gravitational potential* $\delta\varphi_g^{(\text{II})}$ (5.4.37a) arising in a remote zone II under the orbital motion of a gravitating body around a central body (the core of a spheroidal cloud). We are now going to investigate the wave gravitational field origin from the point of view of the theory of retarded potentials [100].

First, let us consider the body of mass m (planet) moving in an orbit in the gravitational field of a central body (star) of mass M. Relative to an observer on the planet m, the central body M moves along an elliptical orbit around the planet being in focus. We will now study the *variable gravitational field* in the presence of an orbitally moving mass. As we know from field theory [100], in this case, the Poisson equation (2.4.1) is generalized by means of the D'Alembert equation (see Eq. (4.2.33)):

$$\nabla^2 \varphi_g - \frac{1}{c_g} \cdot \frac{\partial^2 \varphi_g}{\partial t^2} = 4\pi\gamma\rho, \qquad (9.4.1)$$

where

c_g is the speed of gravitational interactions,

$\gamma = 6.67 \cdot 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$ is the Newtonian gravitational constant,

φ_g is a gravitational field potential, and

ρ is a body mass density.

Its solution, for arbitrary mass distribution with a mass density $\rho(\vec{r},t)$, is defined by a formula of the *retarded potentials* of gravitational field induced by a moving body [100, 205]:

$$\varphi_g = -\gamma \int_V \frac{\rho\left(\vec{r}\,',t-R/c_g\right)}{R} dV', \tag{9.4.2}$$

where $R=\left|\vec{R}\right|$ is a distance from a volume element to the "observation point" of measurement of the gravitational potential, and dV is an element of volume, that is, $\vec{R}=\vec{r}-\vec{r}\,', dV'=dx'dy'dz', \vec{r}\,'=(x',y',z')$. The field created by moving masses can be decomposed into monochromatic waves [100]. The potential of a separate *monochromatic* field component has the form $\varphi_\omega e^{-\mathrm{i}\omega t}$, where φ_ω is a Fourier component of the gravitational potential, $\mathrm{i}=\sqrt{-1}$. According to the equation of D'Alembert (9.4.1), the mass density ρ can also be subjected to spectral decomposition. The Fourier transform of the mass density function with a delayed argument is, therefore, expressed through the Fourier component of mass density ρ_ω [205]:

$$\int_{-\infty}^{\infty} \rho\left(\vec{r},t-\frac{R}{c_g}\right) e^{\mathrm{i}\omega t} dt = \int_{-\infty}^{\infty} \rho(\vec{r},\tau) e^{\mathrm{i}\omega(\tau+\frac{R}{c_g})} d\tau = \rho_\omega e^{\frac{\mathrm{i}\omega R}{c_g}} . \tag{9.4.3}$$

Using (9.4.2), (9.4.3) we can obtain [100, 205]:

$$\varphi_\omega = -\gamma \int_V \rho_\omega \frac{e^{\mathrm{i}\omega R/c_g}}{R} dV' = -\gamma \int_{-\infty}^{\infty} \int_V \frac{\rho}{R} e^{\mathrm{i}\omega(t+R/c_g)} dV' dt \tag{9.4.4}$$

In a remote zone II, we can consider the moving central body as a material point M relative to the distant planet m, so that the mass density ρ at the large distance can be approximated by δ - function:

$$\rho = M\delta[\vec{r}\,' - \vec{r}_0(t)]\,, \qquad (9.4.5)$$

where $\vec{r}_0(t)$ is a radius-vector of moving mass which is a function of time. Substituting (9.4.5) into (9.4.4) and then integrating with respect to dV' (leading to replace $\vec{r}\,'$ by $\vec{r}_0(t)$) we obtain [100]:

$$\varphi_\omega = -\gamma M \int_{-\infty}^{\infty} \frac{1}{R(t)} e^{\mathrm{i}\omega[t + R(t)/c_g]} dt\,, \qquad (9.4.6)$$

where $R(t)$ is a distance from the moving mass to the observation point. A formula similar to (9.4.6) can also be written when the spectral decomposition of the mass density contains a discrete series of frequencies. Indeed, in the case of *periodic* (with a period $T = 2\pi / \Omega_{\mathrm{K}}$) orbital motion of a mass point, the spectral decomposition of the gravitational field contains only the frequencies of the form $n\Omega_{\mathrm{K}}$, so that the corresponding components of the gravitational potential have the form [100, 205]:

$$\varphi_n = -\frac{\gamma M}{T} \int_0^T \frac{e^{\mathrm{i}n\Omega_{\mathrm{K}}[t + R(t)/c_g]}}{R(t)} dt\,, \qquad (9.4.7)$$

where $\Omega_{\mathrm{K}} = 2\pi/T$. In field theory [100 §70], an approach to calculate the radiation accompanying the elliptic motion of two particles attracted by the Coulomb law is known. Taking into account equation (9.3.2) of the elliptical orbit (with an eccentricity $e < 1$) in polar coordinates, the spectral components of the gravitational potential (9.4.7) can be rewritten using the *parametric* representation of the dependence R on t [158, 205]:

$$R = a(1 - e\cos\xi)\,; \qquad (9.4.8a)$$

$$t = \sqrt{\frac{a^3}{\gamma M}} \cdot (\xi - e \sin \xi)\,, \qquad (9.4.8b)$$

whence $T = 2\pi\sqrt{a^3/\gamma M}$ or $\Omega_K = \sqrt{\gamma M / a^3}$, $\xi \in [0, 2\pi]$. In other words, a full circle in the elliptic orbit corresponds to a change in the parameter ξ from 0 to 2π (at the moment $t = 0$, a gravitating body is in the perihelion). So, substituting Eqs (9.4.8a, b) into Eq. (9.4.7) we obtain:

$$\varphi_n = -\frac{\gamma M \Omega_K}{2\pi} \cdot \int_0^{2\pi} \frac{e^{\mathrm{i}n[\xi - e\sin\xi + \frac{\Omega_K}{c_g} a(1-e\cos\xi)]}}{a(1-e\cos\xi)} \cdot \frac{1}{\Omega_K}(1-e\cos\xi)d\xi =$$

$$= -\frac{\gamma M}{2\pi a}\int_0^{2\pi} e^{\mathrm{i}n[\xi - e\sin\xi + \frac{\Omega_K a}{c_g}(1-e\cos\xi)]} d\xi\,. \qquad (9.4.9)$$

The integral in Eq. (9.4.9) should be calculated taking into account the well-known formula of the theory of Bessel functions [330]:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(n\xi - p\sin\xi)} d\xi = \frac{1}{\pi}\int_0^{\pi} \cos(n\xi - p\sin\xi) d\xi = J_n(p)\,, \qquad (9.4.10)$$

where $J_n(p)$ is the Bessel function of integer order n. Moreover, $p = ne$ in Eq. (9.4.9). So, to find the integral (9.4.9), we can apply the method of integration by parts using Eq. (9.4.10) [205, 331]:

$$\varphi_n = -\frac{\gamma M}{2\pi a}\int_0^{2\pi} e^{\mathrm{i}n[\xi - e\sin\xi + \frac{\Omega_K a}{c_g}(1-e\cos\xi)]} d\xi =$$

$$= -\frac{\gamma M}{2\pi a}\left\{ e^{\frac{\mathrm{i}n\Omega_K a}{c_g}(1-e\cos\xi)} \int e^{\mathrm{i}[n\xi - en\sin\xi]} d\xi \Big|_0^{2\pi} - \right.$$

$$\left. -\int_0^{2\pi} \frac{-(-\mathrm{i}\,n\Omega_K\, a)e\sin\xi}{c_g} e^{\frac{\mathrm{i}n\Omega_K a}{c_g}(1-e\cos\xi)} \int e^{\mathrm{i}[n\xi' - en\sin\xi']} d\xi' d\xi \right\} =$$

$$= -\frac{\gamma M}{a}\left\{ J_n(en)e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}\, a(1-e)}{c_g}} - \mathrm{i}\frac{ean\,\Omega_{\mathrm{K}}}{2\pi c_g}\int_0^{2\pi} e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}\, a}{c_g}(1-e\cos\xi)} \times \right.$$

$$\left. \times (\int e^{\mathrm{i}n[\xi - e\sin\xi]} d\xi)\sin\xi d\xi \right\}. \qquad (9.4.11)$$

Under the condition of the very high speed of the gravitational interaction:

$$n\Omega_{\mathrm{K}}\, a/c_g \ll 1, \qquad (9.4.12)$$

the spectral representation of the gravitational potential φ_g based on damping monochromatic waves (9.4.11) has the form [205, 331]:

$$\varphi_g = \sum_{n=-\infty}^{\infty} \varphi_n e^{-\mathrm{i}n\Omega_{\mathrm{K}} t} = -\frac{\gamma M}{a}\left(J_0(0) + \sum_{n=1}^{\infty} J_n(en)(e^{-\mathrm{i}n\Omega_{\mathrm{K}} t} + e^{\mathrm{i}n\Omega_{\mathrm{K}} t}) \right) =$$

$$= -\frac{\gamma M}{a}\left(1 + 2\sum_{n=1}^{\infty} J_n(en)\cos(n\Omega_{\mathrm{K}}\, t) \right). \qquad (9.4.13)$$

This spectral representation (9.4.13) of the gravitational potential is obtained here under the condition of the infinitely high speed of the gravitational interaction (9.4.12) and the property $J_n(en) = J_{-n}(-en)$ which follows from the definition (9.4.10) of the Bessel integral of integer order n for $p = ne$. The comparison of Eq. (9.4.13) with Eq. (5.4.36) (or Eq. (5.4.35) in general) shows that $C_n(t) = 2J_n(en)$. The expression (9.4.13) is, therefore, a special case of formula (5.4.35). Thus, the obtained spectral representation (9.4.13) completely corresponds to the spectral expansion (5.4.36) derived within the framework of the statistical theory of gravitating spheroidal bodies (see Chapter 5).

Now we will obtain a spectral representation of the gravitational potential φ_g in the case of the *finite speed* c_g of propagation of gravitational interactions. For this purpose, we use the Fourier series expansion in the Bessel functions of the second exponential function belonging to the subintegral expression (9.4.9):

$$e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}(1-e\cos\xi)}=e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}e^{\frac{-\mathrm{i}n\Omega_{\mathrm{K}}ea}{c_g}\cos\xi}. \quad (9.4.14)$$

Decomposing the function (9.4.14) in a Fourier series [330 p. 384] we obtain:

$$e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}e^{\frac{-\mathrm{i}n\Omega_{\mathrm{K}}ea}{c_g}\cos\xi}=e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\left[\cos(\frac{n\Omega_{\mathrm{K}}ea}{c_g}\cos\xi)-\mathrm{i}\sin(\frac{n\Omega_{\mathrm{K}}ea}{c_g}\cos\xi)\right]=$$

$$=e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\left[J_0(\frac{n\Omega_{\mathrm{K}}ea}{c_g})+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\cos 2m\xi-\right.$$

$$\left.-\mathrm{i}\left(-2\sum_{m'=1}^{\infty}(-1)^{m'}J_{2m'-1}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\cos(2m'-1)\xi\right)\right]. \quad (9.4.15)$$

Then substituting (9.4.15) into (9.4.11) we obtain [331]:

$$\varphi_n=-\frac{\gamma M}{2\pi a}\int_0^{2\pi}e^{\mathrm{i}n[\xi-e\sin\xi]}e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\left\{J_0(\frac{n\Omega_{\mathrm{K}}ea}{c_g})+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\cos 2m\xi d\xi\right\}-$$

$$-\frac{2\mathrm{i}\gamma M}{2\pi a}\int_0^{2\pi}e^{\mathrm{i}n[\xi-e\sin\xi]}e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\cdot\sum_{m'=1}^{\infty}(-1)^{m'}J_{2m'-1}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\cos(2m'-1)\xi d\xi=$$

$$=-\frac{\gamma M}{a}e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}J_0(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\frac{1}{2\pi}\int_0^{2\pi}e^{\mathrm{i}n[\xi-e\sin\xi]}d\xi-\frac{2\gamma M}{a}e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\times$$

$$\times\sum_{m=1}^{\infty}(-1)^m J_{2m}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{\mathrm{i}n[\xi-e\sin\xi]}\cos 2m\xi d\xi-\mathrm{i}\frac{2\gamma M}{a}e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}}a}{c_g}}\times$$

$$\times\sum_{m=1}^{\infty}(-1)^m J_{2m-1}(\frac{n\Omega_{\mathrm{K}}ea}{c_g})\int_0^{2\pi}e^{\mathrm{i}[n\xi-en\sin\xi]}\cos(2m-1)\xi d\xi. \quad (9.4.16)$$

To find the spectral components of the gravitational potential (9.4.16), it is necessary to calculate the following three integrals:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n[\xi - e\sin\xi]}d\xi = \frac{1}{\pi}\int_0^{\pi}\cos(n\xi - en\sin\xi)d\xi = J_n(en); \qquad (9.4.17a)$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n[\xi - e\sin\xi]}\cos(2m\xi)d\xi = \frac{1}{\pi}\int_0^{\pi}\cos(n\xi - en\sin\xi)\cos(2m\xi)d\xi =$$

$$= \frac{1}{\pi}\int_0^{\pi}\{\cos(n\xi)\cos(en\sin\xi) + \sin(n\xi)\sin(en\sin\xi)\}\cos(2m\xi)d\xi =$$

$$= \frac{1}{\pi}\int_0^{\pi}\{[\cos(n\xi)\cos(2m\xi)]\cos(en\sin\xi) + [\sin(n\xi)\cos(2m\xi)]\sin(en\sin\xi)\}d\xi =$$

$$= \frac{1}{2\pi}\int_0^{\pi}\cos(n-2m)\xi\cos(en\sin\xi)d\xi + \frac{1}{2\pi}\int_0^{\pi}\cos(n+2m)\xi\cos(en\sin\xi)d\xi +$$

$$+ \frac{1}{2\pi}\int_0^{\pi}\sin(n-2m)\xi\sin(en\sin\xi)d\xi + \frac{1}{2\pi}\int_0^{\pi}\sin(n+2m)\xi\sin(en\sin\xi)d\xi =$$

$$= \left| J_n(z) = \frac{1}{\pi}\int_0^{\pi}[\cos(z\sin\theta)\cos n\theta + \sin(z\sin\theta)\sin n\theta]d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)d\theta \right| =$$

$$= \frac{1}{2\pi}\int_0^{\pi}\cos([n-2m]\xi - en\sin\xi)d\xi + \frac{1}{2\pi}\int_0^{\pi}\cos([n+2m]\xi - en\sin\xi)d\xi =$$

$$= \frac{1}{2}[J_{n-2m}(en) + J_{n+2m}(en)]; \qquad (9.4.17b)$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}[n\xi - en\sin\xi]}\cos(2m-1)\xi d\xi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}[n\xi - en\sin\xi]}\frac{e^{\mathrm{i}(2m-1)\xi} + e^{-\mathrm{i}(2m-1)\xi}}{2}d\xi =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} e^{\mathrm{i}[(n+2m-1)\xi - en\sin\xi]}d\xi + \frac{1}{4\pi}\int_0^{2\pi} e^{\mathrm{i}[(n-2m+1)\xi - en\sin\xi]}d\xi =$$

$$= \frac{1}{2\pi}\int_0^{\pi}\cos([n+2m-1]\xi - en\sin\xi)d\xi +$$

$$+ \frac{1}{2\pi}\int_0^{\pi}\cos([n-2m+1]\xi - en\sin\xi)d\xi =$$

$$= \frac{1}{2}[J_{n+2m-1}(en) + J_{n-2m+1}(en)]. \qquad (9.4.17c)$$

Finally, we substitute (9.4.17a), (9.4.17b), and (9.4.17c) into Eq. (9.4.16) [205, 331]:

$$\varphi_n = -\frac{\gamma M}{a} e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} J_0\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_n(en) -$$

$$-\frac{2\gamma M}{a} e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} \sum_{m=1}^{\infty} (-1)^m J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) \frac{1}{2}[J_{n+2m}(en) + J_{n-2m}(en)] -$$

$$-\frac{\mathrm{i}2\gamma M}{a} \cdot e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} \sum_{m=1}^{\infty} (-1)^m J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) \frac{1}{2}[J_{n+2m-1}(en) + J_{n-2m+1}(en)] \cdot \quad (9.4.18)$$

Using the property $(-1)^m = \mathrm{i}^{2m}, \mathrm{i}^{2m+1} = \mathrm{i}^{2m+2-1} = -\mathrm{i}^{2m-1}$ let us simplify Eq. (9.4.18):

$$\varphi_n = -\frac{\gamma M}{a} e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} \{J_0\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_n(en) + \sum_{m=1}^{\infty} (-1)^m J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+2m}(en) +$$

$$+\sum_{m=1}^{\infty} (-1)^m J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n-2m}(en) + \mathrm{i}\left(\sum_{m=1}^{\infty} (-1)^m J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+2m-1}(en) +\right.$$

$$\left.+\sum_{m=1}^{\infty} (-1)^m J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n-(2m-1)}(en)\right)\} =$$

$$= -\frac{\gamma M}{a} e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} [J_0\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_n(en) + \sum_{m=1}^{\infty} \{\mathrm{i}^{2m} J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+2m}(en) +$$

$$+\mathrm{i}^{2m+1} J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+2m-1}(en)\} + \sum_{m=1}^{\infty} \{\mathrm{i}^{2m} J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n-2m}(en) +$$

$$+\mathrm{i}^{2m+1} J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n-(2m-1)}(en)\}] =$$

$$= -\frac{\gamma M}{a} e^{\frac{\mathrm{i} n\Omega_\mathrm{K} a}{c_g}} [J_0\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_n(en) + \sum_{m=1}^{\infty} \{\mathrm{i}^{2m} J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+2m}(en) -$$

$$-\mathrm{i}^{2m-1} J_{2m-1}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n+(2m-1)}(en)\} + \sum_{m=1}^{\infty} \{\mathrm{i}^{2m} J_{2m}\left(\frac{n\Omega_\mathrm{K} ea}{c_g}\right) J_{n-2m}(en) -$$

$$-\mathrm{i}^{2m-1} J_{2m-1}(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_{n-(2m-1)}(en)\}] \qquad (9.4.19)$$

Let us note that in the right-hand part of equality (9.4.19), the even ($n'=2m$) and odd ($n'=2m-1$) terms are grouped in two sums separately, and therefore it is advisable to unite them:

$$\varphi_n = -\frac{\gamma M}{a} e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}} a}{c_g}} [J_0(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_n(en) + \sum_{n'=1}^{\infty} (-1)^{n'} \mathrm{i}^{n'} J_{n'}(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_{n+n'}(en) +$$

$$+\sum_{n'=1}^{\infty} (-1)^{n'} \mathrm{i}^{n'} J_{n'}(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_{n-n'}(en)] = -\frac{\gamma M}{a} e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}} a}{c_g}} [J_0(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_n(en) +$$

$$+\sum_{n=1}^{\infty} (-\mathrm{i})^{n'} J_{n'}(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) [J_{n+n'}(en) + J_{n-n'}(en)]\,. \qquad (9.4.20)$$

So, according to the derivations (9.4.11)–(9.4.20), the spectral components of the gravitational potential with regard to the finite speed c_g of gravitational waves have the form [205]:

$$\varphi_n = -\frac{\gamma M}{a} \{J_0(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) J_n(en) + \sum_{s=1}^{\infty} (-\mathrm{i})^s J_s(\frac{n\Omega_{\mathrm{K}}\, ea}{c_g}) \times$$

$$\times [J_{n+s}(en) + J_{n-s}(en)]\} e^{\frac{\mathrm{i}n\Omega_{\mathrm{K}} a}{c_g}}\,. \qquad (9.4.21)$$

Using (9.4.21) we can estimate the first spectral components of the gravitational potential:

$$\varphi_0 = -\frac{\gamma M}{a} \{J_0(0) J_0(0) + \sum_{s=1}^{\infty} (-\mathrm{i})^s J_s(0)[J_s(0) + J_{-s}(0)]\} \cdot 1 =$$

$$= -\frac{\gamma M}{a} \{1 \cdot 1 + \sum_{s=1}^{\infty} (-\mathrm{i})^s\, 0 \cdot [0 + (-1)^s \cdot 0]\} = -\frac{\gamma M}{a}, \qquad (9.4.22)$$

because $J_0(0)=1$ and $J_s(0)=0$ if $s \neq 0$;

$$\varphi_1 = -\frac{\gamma M}{a} \{J_0(\frac{\Omega_{\mathrm{K}}\, ea}{c_g}) J_1(e) + \sum_{s=1}^{\infty} (-\mathrm{i})^s J_s(\frac{\Omega_{\mathrm{K}}\, ea}{c_g}) [J_{1+s}(e) + J_{1-s}(e)]\} e^{\frac{\mathrm{i}\Omega_{\mathrm{K}} a}{c_g}} =$$

$$= -\frac{\gamma M}{a}\{J_0(\frac{\Omega_K ea}{c_g})J_1(e) - \mathrm{i}\, J_1(\frac{\Omega_K ea}{c_g})[J_2(e) + J_0(e)] -$$

$$-J_2(\frac{\Omega_K ea}{c_g})[J_3(e) + J_1(e)]...\}e^{\frac{\mathrm{i}\Omega_K a}{c_g}} \qquad (9.4.23)$$

We restrict ourselves in formula (9.4.23) by the first two terms of the series:

$$\varphi_1 \approx -\frac{\gamma M}{a}\{J_0(\frac{\Omega_K ea}{c_g})J_1(e) - \mathrm{i}J_1(\frac{\Omega_K ea}{c_g})[J_2(e)+J_0(e)]\}e^{\frac{\mathrm{i}\Omega_K a}{c_g}} . \qquad (9.4.24a)$$

Analogously, using (9.4.21) we can estimate the spectral components φ_2 and φ_3 limiting ourselves to the first two terms of the series:

$$\varphi_2 = -\frac{\gamma M}{a}\{J_0(\frac{2\Omega_K ea}{c_g})J_2(2e) + \sum_{s=1}^{\infty}(-\mathrm{i})^s J_s(\frac{2\Omega_K ea}{c_g})[J_{2+s}(2e) + J_{2-s}(2e)]\}e^{\frac{\mathrm{i}2\Omega_K a}{c_g}} \approx$$

$$\approx -\frac{\gamma M}{a}\{J_0(\frac{2\Omega_K ea}{c_g})J_2(2e) - \mathrm{i}J_1(\frac{2\Omega_K ea}{c_g})[J_3(2e) + J_1(2e)]\}e^{\frac{\mathrm{i}2\Omega_K a}{c_g}} ; \qquad (9.4.24b)$$

$$\varphi_3 \approx -\frac{\gamma M}{a}\{J_0(\frac{3\Omega_K ea}{c_g})J_3(3e) - \mathrm{i}J_1(\frac{3\Omega_K ea}{c_g})[J_4(3e) + J_2(3e)]\}e^{\frac{\mathrm{i}3\Omega_K a}{c_g}} . \qquad (9.4.24c)$$

Because $e < 1$ and $\Omega_K ea / c_g << 1$, for the calculation:

$$J_0(e), J_0(\frac{\Omega_K ea}{c_g}), J_0(\frac{2\Omega_K ea}{c_g}), J_0(\frac{3\Omega_K ea}{c_g}) ,$$

$$J_1(e), J_1(\frac{\Omega_K ea}{c_g}), J_1(\frac{2\Omega_K ea}{c_g}), J_1(\frac{3\Omega_K ea}{c_g}) , J_2(e), J_2(2e), J_2(3e), \ldots,$$

we can use the expansion in a series of the Bessel function of integer order:

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r+n)!}\left(\frac{z}{2}\right)^{2r} \qquad (9.4.25a)$$

and in the particular case when $n = 0$ [330]:

$$J_0(z)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{z}{2}\right)^{2r}. \qquad (9.4.25b)$$

Using (9.4.25b) we estimate $J_0(x)$ at points $x=e,\quad e\Omega_K a/c_g,...:$

$$J_0(e)=1-\left(\frac{e}{2}\right)^2+\frac{1}{4}\left(\frac{e}{2}\right)^4-\frac{1}{36}\left(\frac{e}{2}\right)^6+...; \qquad (9.4.26a)$$

$$J_0\left(\frac{\Omega_K ea}{c_g}\right)=1-\left(\frac{\Omega_K ea}{2c_g}\right)^2+\frac{1}{4}\left(\frac{\Omega_K ea}{2c_g}\right)^4-\frac{1}{36}\left(\frac{\Omega_K ea}{2c_g}\right)^6+..., \qquad (9.4.26b)$$

in accordance with (9.4.25a), we calculate $J_1(x)$ at points $x=e,\quad e\Omega_K a/c_g,\quad ...:$

$$J_1(e)=\frac{e}{2}\left[1-\frac{1}{2}\left(\frac{e}{2}\right)^2+\frac{1}{2\cdot 3!}\left(\frac{e}{2}\right)^4-\frac{1}{3!4!}\left(\frac{e}{2}\right)^6+...\right]; \qquad (9.4.27a)$$

$$J_1\left(\frac{\Omega_K ea}{c_g}\right)=\frac{\Omega_K ea}{2c_g}\left[1-\frac{1}{2}\left(\frac{\Omega_K ea}{2c_g}\right)^2+\frac{1}{12}\left(\frac{\Omega_K ea}{2c_g}\right)^4-\frac{1}{144}\left(\frac{\Omega_K ea}{2c_g}\right)^6+...\right] \qquad (9.4.27b)$$

and then $J_2(x)$ at points $x=e,\quad e\Omega_K a/c_g,\quad 2e,\quad 2e\Omega_K a/c_g,...:$

$$J_2(e)=\left(\frac{e}{2}\right)^2\left[\frac{1}{2}-\frac{1}{6}\left(\frac{e}{2}\right)^2+\frac{1}{48}\left(\frac{e}{2}\right)^4-\frac{1}{720}\left(\frac{e}{2}\right)^6+...\right]; \qquad (9.4.28a)$$

$$J_2\left(\frac{\Omega_K ea}{c_g}\right)=\left(\frac{\Omega_K ea}{2c_g}\right)^2\left[\frac{1}{2}-\frac{1}{6}\left(\frac{\Omega_K ea}{2c_g}\right)^2+\frac{1}{48}\left(\frac{\Omega_K ea}{2c_g}\right)^4-\frac{1}{720}\left(\frac{\Omega_K ea}{2c_g}\right)^6+...\right]; \qquad (9.4.28b)$$

as well as $J_3(x)$ and $J_4(x)$ at points $x=3e$:

$$J_3(3e)=\left(\frac{3e}{2}\right)^3\left[\frac{1}{6}-\frac{1}{24}\left(\frac{3e}{2}\right)^2+\frac{1}{120}\left(\frac{3e}{2}\right)^4-\frac{1}{4320}\left(\frac{3e}{2}\right)^6+...\right]; \qquad (9.4.29a)$$

$$J_4(3e)=\left(\frac{3e}{2}\right)^4\left[\frac{1}{24}-\frac{1}{120}\left(\frac{3e}{2}\right)^2+\frac{1}{1440}\left(\frac{3e}{2}\right)^4-\frac{1}{30240}\left(\frac{3e}{2}\right)^6+...\right]. \qquad (9.4.29b)$$

Then, limiting ourselves in Eqs (9.4.26a, b)–(9.4.29a, b) to terms of the second order of smallness and substituting the corresponding expressions in Eqs (9.4.24a)–(9.4.24c) we obtain:

$$\varphi_1 = -\frac{\gamma M}{a}\left\{\left(1-\left(\frac{\Omega_K ea}{2c_g}\right)^2\right)\cdot\frac{e}{2}\left(1-\frac{1}{2}\left(\frac{e}{2}\right)^2\right)-\mathrm{i}\frac{\Omega_K ea}{2c_g}\left(1-\frac{1}{2}\left(\frac{\Omega_K ea}{2c_g}\right)^2\right)\times\right.$$

$$\left.\times\left[\left(\frac{e}{2}\right)^2\left(\frac{1}{2}-\frac{1}{6}\left(\frac{e}{2}\right)^2\right)+1-\left(\frac{e}{2}\right)^2\right]\right\}e^{\frac{\mathrm{i}\Omega_K a}{c_g}} \approx$$

$$\approx -\frac{\gamma M}{a}\left\{\frac{e}{2}+\frac{e\,\Omega_K\,a}{\mathrm{i}\,2c_g}\right\}e^{\frac{\mathrm{i}\Omega_K a}{c_g}}; \qquad (9.4.30a)$$

$$\varphi_2 = -\frac{\gamma M}{a}e^{\frac{\mathrm{i}2\Omega_K a}{c_g}}\left\{\left(1-\left(\frac{\Omega_K ea}{c_g}\right)^2\right)\cdot e^2\left(\frac{1}{2}-\frac{1}{6}e^2\right)-\mathrm{i}\frac{\Omega_K ea}{c_g}\left(1-\frac{1}{2}\left(\frac{\Omega_K ea}{c_g}\right)^2\right)\times\right.$$

$$\left.\times\left[e^3\left(\frac{1}{6}-\frac{1}{24}e^2\right)+e\left(1-\frac{1}{2}e^2\right)\right]\right\}\approx -\frac{\gamma M}{a}\left\{\frac{e^2}{2}+\frac{e^2\,\Omega_K\,a}{\mathrm{i}c_g}\right\}e^{\frac{\mathrm{i}2\Omega_K a}{c_g}}; \qquad (9.4.30b)$$

$$\varphi_3 = -\frac{\gamma M}{a}\left\{\left(1-\left(\frac{3\Omega_K ea}{2c_g}\right)^2\right)\cdot\left(\frac{3e}{2}\right)^3\left(\frac{1}{6}-\frac{1}{24}\left(\frac{3e}{2}\right)^2\right)-\mathrm{i}\frac{3\Omega_K ea}{2c_g}\left(1-\frac{1}{2}\left(\frac{3\Omega_K ea}{2c_g}\right)^2\right)\times\right.$$

$$\left.\times\left[\left(\frac{3e}{2}\right)^4\frac{1}{24}-\ldots\left(\frac{3e}{2}\right)^2\frac{1}{2}-\frac{1}{6}\left(\frac{3e}{2}\right)^4+\ldots\right]\right\}e^{\frac{\mathrm{i}3\Omega_K a}{c_g}} \approx$$

$$\approx -\frac{\gamma M}{a}\left\{\frac{9e^3}{16}+\frac{27e^3\,\Omega_K\,a}{\mathrm{i}16c_g}\right\}e^{\frac{\mathrm{i}3\Omega_K a}{c_g}}. \qquad (9.4.30c)$$

Let us estimate the spectral component of the gravitational potential (9.4.21) in the case of negative integers, that is, for $\forall(-n)\in\mathbf{Z}$. Indeed, according to the property of Bessel functions we have [330]:

$$J_{-n}(z) = (-1)^n J_n(z). \qquad (9.4.31)$$

Taking into account formula (9.4.10) and property (9.4.31), we find that:

$$J_s\left(-\frac{\Omega_K ea}{c_g}\right)=J_{-s}\left(\frac{\Omega_K ea}{c_g}\right)=(-1)^s J_s\left(\frac{\Omega_K ea}{c_g}\right), \qquad (9.4.32a)$$

whence it follows that the above-mentioned (relative to Eq. (9.4.13)) identity is valid:

$$J_{-n}(-ne)=J_n(ne). \qquad (9.4.32b)$$

Using properties (9.4.32a) and (9.4.32b) we can write the formula for φ_{-n} based on Eq. (9.4.21):

$$\varphi_{-n}=-\frac{\gamma M}{a}\{J_0\left(-\frac{n\Omega_K ea}{c_g}\right)J_{-n}(-en)+$$

$$+\sum_{s=1}^{\infty}(-\mathrm{i})^s J_s\left(-\frac{n\Omega_K ea}{c_g}\right)[J_{-n+s}(-en)+J_{-n-s}(-en)]\}e^{-\frac{\mathrm{i}n\Omega_K a}{c_g}}=$$

$$=-\frac{\gamma M}{a}\{J_0\left(\frac{n\Omega_K ea}{c_g}\right)J_n(en)+$$

$$+\sum_{s=1}^{\infty}\mathrm{i}^s J_s\left(\frac{n\Omega_K ea}{c_g}\right)[J_{n+s}(en)+J_{n-s}(en)]\}e^{-\frac{\mathrm{i}n\Omega_K a}{c_g}}=$$

$$=-\frac{\gamma M}{a}\Bigg(\{J_0\left(\frac{n\Omega_K ea}{c_g}\right)J_n(en)+$$

$$+\sum_{s=1}^{\infty}(-\mathrm{i})^s J_s\left(\frac{n\Omega_K ea}{c_g}\right)[J_{n+s}(en)+J_{n-s}(en)]\}e^{\frac{\mathrm{i}n\Omega_K a}{c_g}}\Bigg)^*=\varphi_n{}^*, \qquad (9.4.33)$$

where * is the symbol of complex conjugation. So, according to formula (9.4.33) and Eqs (9.4.30a)–(9.4.30c) the following relationships hold:

$$\varphi_{-1}=\varphi_1{}^*\approx-\frac{\gamma M}{a}\left\{\frac{e}{2}-\frac{e\,\Omega_K\, a}{\mathrm{i}\,2c_g}\right\}e^{-\frac{\mathrm{i}\Omega_K a}{c_g}}; \qquad (9.4.34a)$$

$$\varphi_{-2} \approx -\frac{\gamma M}{a}\left\{\frac{e^2}{2}-\frac{e^2\,\Omega_{\mathrm{K}}\,a}{\mathrm{i}\,c_g}\right\}e^{-\frac{\mathrm{i}2\Omega_{\mathrm{K}}a}{c_g}}; \tag{9.4.34b}$$

$$\varphi_{-3} \approx -\frac{\gamma M}{a}\left\{\frac{9e^3}{16}-\frac{27e^3\,\Omega_{\mathrm{K}}\,a}{\mathrm{i}16c_g}\right\}e^{-\frac{\mathrm{i}3\Omega_{\mathrm{K}}a}{c_g}}. \tag{9.4.34c}$$

Taking into account (9.4.22), (9.4.30a)–(9.4.30c), and (9.4.34a)–(9.4.34c) we obtain the expression for the retarded gravitational potential based on the first four monochromatic waves [331]:

$$\varphi_g \approx \sum_{n=-3}^{3}\varphi_n e^{-\mathrm{i}n\Omega_{\mathrm{K}}t} = -\frac{\gamma M}{a}\left\{\frac{1}{2}\cdot\frac{9e^3}{8}-\frac{9e^3\cdot 3\Omega_{\mathrm{K}}\,a}{2\mathrm{i}\cdot 8c_g}\right\}e^{\mathrm{i}(3\Omega_{\mathrm{K}}t-3\Omega_{\mathrm{K}}\frac{a}{c_g})}-$$

$$-\frac{\gamma M}{a}\left\{\frac{e^2}{2}-\frac{e^2\cdot 2\Omega_{\mathrm{K}}a}{2\mathrm{i}\cdot c_g}\right\}e^{\mathrm{i}(2\Omega_{\mathrm{K}}t-2\Omega_{\mathrm{K}}\frac{a}{c_g})}-\frac{\gamma M}{a}\left\{\frac{e}{2}-\frac{e\Omega_{\mathrm{K}}a}{2\mathrm{i}\cdot c_g}\right\}e^{\mathrm{i}(\Omega_{\mathrm{K}}t-\Omega_{\mathrm{K}}\frac{a}{c_g})}-$$

$$-\frac{\gamma M}{a}\quad -\frac{\gamma M}{a}\left\{\frac{e}{2}+\frac{e\Omega_{\mathrm{K}}a}{2\mathrm{i}\cdot c_g}\right\}e^{-\mathrm{i}(\Omega_{\mathrm{K}}t-\Omega_{\mathrm{K}}\frac{a}{c_g})}-\frac{\gamma M}{a}\left\{\frac{e^2}{2}-\frac{e^2\cdot 2\Omega_{\mathrm{K}}a}{2\mathrm{i}\cdot c_g}\right\}e^{-\mathrm{i}(2\Omega_{\mathrm{K}}t-2\Omega_{\mathrm{K}}\frac{a}{c_g})}-$$

$$-\frac{\gamma M}{a}\left\{\frac{1}{2}\cdot\frac{9e^3}{8}+\frac{9e^3}{2\mathrm{i}}\cdot\frac{3\Omega_{\mathrm{K}}\,a}{8c_g}\right\}e^{-\mathrm{i}(3\Omega_{\mathrm{K}}t-3\Omega_{\mathrm{K}}\frac{a}{c_g})}. \tag{9.4.35}$$

Grouping the conjugate spectral components (according to (9.4.33)) into pairs and then using the Euler formula in Eq. (9.4.35) we obtain [331]:

$$\varphi_g \approx -\frac{\gamma M}{a}-\frac{\gamma M}{a}e\cos(\Omega_{\mathrm{K}}t-\Omega_{\mathrm{K}}\frac{a}{c_g})+$$

$$+\frac{\gamma M}{a}\frac{\Omega_{\mathrm{K}}a}{c_g}e\sin(\Omega_{\mathrm{K}}t-\Omega_{\mathrm{K}}\frac{a}{c_g})-\frac{\gamma M}{a}e^2\cos(2\Omega_{\mathrm{K}}t-2\Omega_{\mathrm{K}}\frac{a}{c_g})+$$

$$+\frac{\gamma M}{a}\frac{2\Omega_{\mathrm{K}}a}{c_g}e^2\sin(2\Omega_{\mathrm{K}}t-2\Omega_{\mathrm{K}}\frac{a}{c_g})-$$

$$-\frac{\gamma M}{a}\frac{9}{8}e^3\cos(3\Omega_{\mathrm{K}}t-3\Omega_{\mathrm{K}}\frac{a}{c_g})+\frac{\gamma M}{a}\frac{9}{8}\frac{3\Omega_{\mathrm{K}}a}{c_g}e^3\sin(3\Omega_{\mathrm{K}}t-3\Omega_{\mathrm{K}}\frac{a}{c_g})=$$

$$= -\frac{\gamma M}{a} - e\frac{\gamma M}{a}[\cos(\Omega_K t - \Omega_K \frac{a}{c_g}) - \frac{\Omega_K a}{c_g}\sin(\Omega_K t - \Omega_K \frac{a}{c_g})] -$$

$$-e^2 \frac{\gamma M}{a}[\cos(2\Omega_K t - 2\Omega_K \frac{a}{c_g}) -$$

$$-\frac{2\Omega_K a}{c_g}\sin(2\Omega_K t - 2\Omega_K \frac{a}{c_g})] - \frac{9}{8}e^3 \frac{\gamma M}{a}[\cos(3\Omega_K t - 3\Omega_K \frac{a}{c_g}) -$$

$$-\frac{3\Omega_K a}{c_g}\sin(3\Omega_K t - 3\Omega_K \frac{a}{c_g})]. \quad (9.4.36)$$

As seen in the right-hand side of Eq. (9.4.36), the sum of two harmonic oscillations of the same frequency is written in the square brackets. Therefore (according to the formula of the addition of two harmonic oscillations) the resulting oscillation can be found by the parallelogram rule [206] so that we have [331]:

$$\varphi_g = -\frac{\gamma M}{a} - e\frac{\gamma M}{a}\sqrt{1+[\Omega_K a/c_g]^2}\cos(\Omega_K t + \psi_1) -$$

$$-e^2 \frac{\gamma M}{a}\sqrt{1+[2\Omega_K a/c_g]^2}\cos(2\Omega_K t + \psi_2) -$$

$$-e^3 \frac{\gamma M}{a}\frac{9}{8}\sqrt{1+[3\Omega_K a/c_g]^2}\cos(3\Omega_K t + \psi_3) - ..., \quad (9.4.37)$$

where:

$$\psi_n = \arctan\frac{-\sin(n\Omega_K a/c_g) + (n\Omega_K a/c_g)\cos(n\Omega_K a/c_g)}{\cos(n\Omega_K a/c_g) + (n\Omega_K a/c_g)\sin(n\Omega_K a/c_g)} =$$

$$= \arctan\frac{n\Omega_K a/c_g - \tan(n\Omega_K a/c_g)}{(n\Omega_K a/c_g)\tan(n\Omega_K a/c_g) + 1},$$

$n = 1, 2, 3, ...$. As can be seen from (9.4.37), this series is a special case of formula (5.4.35) or (5.4.36) obtained within the framework of the statistical theory of gravitating spheroidal bodies (see Section 5.4 of Chapter 5). By comparing Eq. (9.4.37) and Eq. (5.4.36) we find that the wave gravitational potential $\delta\varphi_g^{(\mathrm{II})}$ from the point of view of the theory of retarded gravitational potentials is expressed by the formula:

$$\delta\varphi_g^{(\mathrm{II})} = -e\frac{\gamma M}{a}\sqrt{1+[\Omega_{\mathrm{K}}\, a/c_g]^2}\cos(\Omega_{\mathrm{K}}\, t+\psi_1) -$$

$$-e^2\frac{\gamma M}{a}\sqrt{1+[2\Omega_{\mathrm{K}}\, a/c_g]^2}\cos(2\Omega_{\mathrm{K}}\, t+\psi_2) -$$

$$-e^3\frac{\gamma M}{a}\frac{9}{8}\sqrt{1+[3\Omega_{\mathrm{K}}\, a/c_g]^2}\cos(3\Omega_{\mathrm{K}}\, t+\psi_3) - ..., \qquad (9.4.38)$$

moreover, $C_1 = e\sqrt{1+[\Omega_{\mathrm{K}}\, a/c_g]^2}$, $C_2 = e^2\sqrt{1+[2\Omega_{\mathrm{K}}\, a/c_g]^2}$, $C_3 = e^3\frac{9}{8}\sqrt{1+[3\Omega_{\mathrm{K}}\, a/c_g]^2}$,... According to (9.4.38) as well as (5.4.37a), the wave gravitational potential $\delta\varphi_g^{(\mathrm{II})}$ arises in the remote zone II under the orbital motion of a body around the central gravitating body (the core of a spheroidal cloud).

9.5. Oscillations of gravitational field strength acting on planets: toward the nature of Alfvén–Arrhenius oscillations from the point of view of the statistical theory

Indeed, within the framework of the statistical theory of gravitating spheroidal bodies the gravitational potential of a spheroidal body φ_g in the case of its mechanical *quasi-equilibrium* (following formula (5.4.35)) is equal:

$$\varphi_g\Big|_{\mathrm{quasiequil}} = -\frac{\gamma M}{r}\mathrm{erf}\left(r\sqrt{\alpha_s/2}\right)\left[1+\sum_{n=1}^{\infty}C_n(t)\cos(n\omega_s t+\psi_n)\right], \qquad (9.5.1)$$

where ω_s is an average integral value of main circular frequency (5.3.26b) of the radial oscillations inside a forming core of a centrally symmetric (*slowly* rotating) spheroidal body (a central cosmogonical body). For large r ($r\to\infty$) the error function $\operatorname{erf}\left(r\sqrt{\alpha_s/2}\right)\to 1$. In a remote zone II Eq. (9.5.1), therefore, becomes [205]:

$$\varphi_g^{(\mathrm{II})}\Big|_{\mathrm{quasiequil}} = \varphi_g^{(\mathrm{II})}\Big|_{\mathrm{equil}} + \delta\varphi_g^{(\mathrm{II})} =$$

$$= -\frac{\gamma M}{r} - \frac{\gamma M}{r}\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t + \psi_n)\,, \qquad (9.5.2)$$

whence, in accordance with the formula (5.4.37a), we have:

$$\varphi_g^{(\mathrm{II})}\Big|_{\mathrm{equil}} = \overline{\varphi_g^{(\mathrm{II})}} = -\frac{\gamma M}{r}, \qquad (9.5.3a)$$

$$\delta\varphi_g^{(\mathrm{II})} = -\frac{\gamma M}{r}\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t + \psi_n)\,. \qquad (9.5.3b)$$

According to Eq. (9.5.2) the specific force of gravity in a remote zone II of a quasi-equilibrium gravitational field has the form (5.4.38):

$$\vec{f}_g = -\operatorname{grad}\varphi_g^{(\mathrm{II})} = -\frac{\gamma M}{r^2}\cdot\frac{\vec{r}}{r}\Big|_{\mathrm{quasiequil}} =$$

$$= -\frac{\gamma M\left[1 + \sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t + \psi_n)\right]}{r^2}\cdot\frac{\vec{r}}{r} = \vec{f}_g\Big|_{\mathrm{equil}} + \vec{f}_a\,, \qquad (9.5.4)$$

that is:

$$\vec{f}_g\Big|_{\mathrm{equil}} = -\operatorname{grad}\overline{\varphi_g^{(\mathrm{II})}} = -\frac{\gamma M}{r^2}\cdot\frac{\vec{r}}{r}, \qquad (9.5.5a)$$

$$\vec{f}_a = -\operatorname{grad}\delta\varphi_g^{(\mathrm{II})} = -\frac{\gamma M}{r^2}\sum_{n=1}^{\infty} C_n(t)\cos(n\omega_s t + \psi_n)\cdot\frac{\vec{r}}{r}\,. \qquad (9.5.5b)$$

Evidently, formula (9.5.5b) generalizes (9.2.45) or (9.3.8), so that the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a$ causing the radial and axial oscillations (which modify an initial circular orbit of protoplanets (see Sections 9.2 and 9.3)) can be calculated using this relation (9.5.5b) [205, 328].

Moreover, comparing Eq. (9.5.5b) (or (5.4.37b)) with Eq. (9.3.8) we find approximately that:

$$e_l \cdot \cos(\Omega_{\mathrm{K}l}\, t - \varepsilon_l^0) \propto C_{n_l}(t) \cos(n_l \omega_s t + \psi_{n_l}), \qquad (9.5.6)$$

that is, according to Eqs (5.4.40a, b) the moving l-protoplanet undergoes the action of a spectral component of the Alfvén–Arrhenius additional periodic force, mainly that spectral component whose frequency $n_l \omega_s$ is close to the l-Keplerian angular velocity $\Omega_{\mathrm{K}l}$ (this is the *first* scenario). There exists the *second* scenario based on the orbital motion of l-protoplanet in a fast oscillating wave gravitational field of the central gravitating body (the core of a spheroidal body) [158, 328].

As mentioned in Section 9.2, the amplitude (9.2.28) of small orbital forced oscillations of l-planet (protoplanet) depends on the amplitude of a constraining periodic force (9.2.45) of Alfvén–Arrhenius. In particular, Eq. (9.2.45) shows that the amplitude of the specific additional periodic force of Alfvén–Arrhenius $f_a(t)$ is equal to:

$$f_0 = e_l R_{0l}\, \Omega_{\mathrm{K}l}^2, \qquad (9.5.7)$$

where $a = R_{0l}$. Indeed, it is well known in mechanical system theory [206] that the amplitude f_0 of a constraining specific periodic force depends on the amplitude x_0 of small forced vibrations:

$$f_0 = x_0 \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\,\delta^2\,\Omega^2}\,, \qquad (9.5.8)$$

where

ω_0 is a circular frequency of the eigen (free) oscillations of system,
δ is a damping factor, and
Ω is a circular frequency of the forced oscillations.

If $\Omega >> \omega_0$ in Eq. (9.5.8) then $f_0 = x_0\Omega^2$ [206] which corresponds to Eq. (9.5.7) when $\Omega = \Omega_{\mathrm{K}l}$ and $x_0 = e_l R_{0l}$. Let us also note that if $\Omega \approx \omega_0$ (the resonance action) in Eq. (9.5.8) then $f_0 = 2\delta x_0 \Omega$ [206] which can correspond to Eq. (9.5.7), too, at $\Omega = \Omega_{\mathrm{K}l}$, $2\delta = e_l \Omega_{\mathrm{K}l}$ and $x_0 = R_{0l}$. So, the *stationary* forced oscillations are also harmonic with the same circular frequency Ω:

$$x = x_0 \cos(\Omega t + \varepsilon), \tag{9.5.9}$$

that is fully confirmed by Eqs (9.2.28) and (9.2.45) in Section 9.2.

Thus, according to the simple first scenario based on the theory of retarded gravitational potentials, if $\Omega_{\mathrm{K}l} \approx n_l \omega_s$ then a selective action of the central cosmogonical body on l-protoplanet can occur. The dynamical system "central cosmogonical body–l-protoplanet" is, therefore, a singular selective filter with the resonance frequency $n_l \omega_s$. Considering each l-protoplanet as a selective receiver of a respective n_l-spectral component of the oscillation part of the gravitational field strength of a central cosmogonical body, we conclude that spectral components of the oscillating gravitational strength with multiple frequencies $n_k \omega_s$, $n_l \omega_s$, $n_m \omega_s$,... act permanently on planets moving around a central cosmogonical body in their orbits with the Keplerian angular velocities $\Omega_{\mathrm{K}k} \approx n_k \omega_s$, $\Omega_{\mathrm{K}l} \approx n_l \omega_s$, $\Omega_{\mathrm{K}m} \approx n_m \omega_s$,... like periodic forces of an anchoring mechanism.

On the whole, the protoplanetary system can be considered as a complex dynamical system supporting steady dynamical states of its subsystems (orbital motion of protoplanets) through the oscillation components of the strength gravitational field of a central cosmogonical body (or specific additional periodic force of Alfvén–Arrhenius). Similar to an oscillating circuit of the selective receiver, each protoplanet undergoes the greatest effect of such harmonics of oscillating gravitational force (see Eq. (9.5.5b)) whose circular frequencies correspond to the orbital angular velocity Ω_{Kl}.

However, the condition of the *infinitesimal* closeness of these frequencies is not always accessible within the framework of the assumption that GCF $\mathrm{G}(t)$ is a periodic function (5.3.25) only (see Section 5.3). Indeed, the quasi-equilibrium state of a compressible gravitating spheroidal body when $\overline{\mathrm{G}(t)} = \mathrm{G}_s = \mathrm{const}$ admits a more general model of *quasiperiodic function* [119]. So, in the case of mechanical *quasi-equilibrium* an oscillating behavior of the GCF, generally speaking, is described by a quasiperiodic function $\mathrm{G}(t) = \mathrm{G}(\omega_1 t,...,\omega_r t)$. Since the quasiperiodic function has r main periods $T_i = 2\pi/\omega_i$ (when $\omega_i, i = 1,...,r$ are the *main frequencies*) then the condition (5.3.19b) also applies (see Theorem 5.1). In the simplest case, if a quasiperiodic function is the sum of periodic functions:

$$\mathrm{G}(t) = \mathrm{G}(\omega_1 t,...,\omega_r t) = \sum_{i=1}^{r} \mathrm{G}_i(\omega_i t) \ , \qquad (9.5.10)$$

then its Fourier spectrum consists of components with frequencies $\omega_1,...,\omega_r$ and their harmonics:

$$m_1\omega_1,...,m_r\omega_r \, , \qquad (9.5.11)$$

where $m_1,...,m_r$ are positive integers, that is, $m_i \in \mathbf{N}$. Moreover, if the quasiperiodic function includes the products of periodic (in particular, harmonic) functions $\mathrm{G}_i(\omega_i t)\mathrm{G}_j(\omega_j t)$ then its Fourier spectrum contains spectral components with main frequencies $|\omega_i - \omega_j|$ and $|\omega_i + \omega_j|$ as well as their harmonics. In general, the Fourier spectrum of quasiperiodic function *nonlinearly* depending on periodic functions $G_i(\omega_i t), i=1,...,r,$ contains spectral components with all frequencies of the kind [119]:

$$\left| m_1\omega_1 + m_2\omega_2 + ... + m_r\omega_r \right|, \qquad (9.5.12)$$

where $m_1,...,m_r$ are arbitrary positive integers, besides the ratio ω_i / ω_j could be a *rational or irrational* number. In the last (irrational) case, as we know from number theory, these sums $\left| m_i\omega_i + m_j\omega_j \right|$ form an point that is dense everywhere, set on the real frequency axis among them for $r=2$. That is why any real positive number (for example, the value of l-orbital angular velocity Ω_{Kl}) is *infinitesimally close* to some sum $\left| n_l\omega_{s1} + m_l\omega_{s2} \right|$, where ω_{s1} and ω_{s2} are *the main frequencies* (in particular, the main circular frequencies of the oscillations of gravitational field strength into a forming protoplanetary system).

Thus, if we consider GCF $G(t)$ as a quasiperiodic function satisfying condition (9.5.10) for $r=2$, that is, $G(t) = G_1(\omega_{s1}t) + G_2(\omega_{s2}t)$, then, following the derivation of Eqs (5.4.30)-(5.4.34a, b) for each $G_i(\omega_{si}t), i=1,2$, we obtain the generalization of Eq. (9.5.6):

$$e_l \cdot \cos(\Omega_{Kl}\, t - \varepsilon_l^0) \propto A_{n_l}(t)\cos(n_l\omega_{s1}t + \psi_{n_l}) + \\ + A_{m_l}(t)\cos(m_l\omega_{s2}t + \psi_{m_l})\,. \qquad (9.5.13)$$

Moreover, these main circular frequencies ω_{s1} and ω_{s2} have to be very close with each other in accordance with physical reasoning (because of beating) [206], that is, the moving l-protoplanet undergoes the action of a superposition of two spectral components of the Alfvén–Arrhenius additional periodic force, namely, of those spectral components with frequencies close to the l-Keplerian angular velocity.

To obtain the infinitesimal closeness of these frequencies we can consider the quasiperiodic function *nonlinearly* depending on two periodic functions $\mathrm{G}_i(\omega_{si}t) = \mathrm{G}_{si}[1 + a_i \sin\omega_{si}t],\, a_i << \mathrm{G}_{si},\, i = 1, 2$, in the form of their product:

$$\begin{aligned}
&\mathrm{G}(t) = \mathrm{G}_1(\omega_{s1}t)\,\mathrm{G}_2(\omega_{s2}t) = \\
&= \mathrm{G}_{s1}\mathrm{G}_{s2}[1 + a_1\sin\omega_{s1}t + a_2\sin\omega_{s2}t + a_1a_2\sin\omega_{s1}t\sin\omega_{s2}t] = \\
&= \mathrm{G}_s[1 + a_1\sin\omega_{s1}t + a_2\sin\omega_{s2}t + (a_1a_2/2)\cos|\omega_{s1} - \omega_{s2}|t - \\
&- (a_1a_2/2)\cos|\omega_{s1} + \omega_{s2}|t],
\end{aligned} \tag{9.5.14}$$

where $\mathrm{G}_s = \mathrm{G}_{s1}\mathrm{G}_{s2}$. Taking into account Eq. (9.5.14) we can see that: $\overline{\mathrm{G}(t)} = \mathrm{G}_s = \mathrm{const}$, if the averaging is carried out on a multiple period. Using Eq. (9.5.14) let us calculate Eq. (5.4.30) in this case:

$$\begin{aligned}
&\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t) = \frac{1}{2}\cdot\mathrm{G}(t)\dot{\alpha}(t) + \dot{\mathrm{G}}(t)\alpha(t) = \\
&= \frac{1}{2}\cdot\mathrm{G_s}\,\dot{\alpha}(t) + \mathrm{G_s}\,a_1\sqrt{\alpha^2\omega_{s1}^2 + \dot{\alpha}^2/4}\cos(\omega_{\mathrm{s}1}t - \varphi_{s1}) + \\
&+ \mathrm{G_s}\,a_2\sqrt{\alpha^2\omega_{s2}^2 + \dot{\alpha}^2/4}\cos(\omega_{\mathrm{s}2}t - \varphi_{s2}) + \\
&+ \mathrm{G_s}(a_1a_2/2)\sqrt{\alpha^2|\omega_{\mathrm{s}1} - \omega_{\mathrm{s}2}|^2 + \dot{\alpha}^2/4}\cos(|\omega_{\mathrm{s}1} - \omega_{\mathrm{s}2}|t + \varphi_{s1-s2}) + \\
&+ \mathrm{G_s}(a_1a_2/2)\sqrt{\alpha^2|\omega_{\mathrm{s}1} + \omega_{\mathrm{s}2}|^2 + \dot{\alpha}^2/4}\cos(|\omega_{\mathrm{s}1} + \omega_{\mathrm{s}2}|t + \varphi_{s1+s2}),
\end{aligned}$$

whence:

$$\begin{aligned}
&\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t) = \\
&= \frac{1}{2}\cdot\mathrm{G_s}\,\dot{\alpha}[1 + a_1\sqrt{1 + (2\alpha\omega_{s1}/\dot{\alpha})^2}\cos(\omega_{\mathrm{s}1}t - \varphi_{s1}) +
\end{aligned}$$

$$+a_2\sqrt{1+(2\alpha\omega_{s2}/\dot{\alpha})^2}\cos(\omega_{s2}t-\varphi_{s2})+$$
$$+(a_1a_2/2)\sqrt{1+(2\alpha|\omega_{s1}-\omega_{s2}|/\dot{\alpha})^2}\cos(|\omega_{s1}-\omega_{s2}|t+\varphi_{s1-s2})+$$
$$+(a_1a_2/2)\sqrt{1+(2\alpha|\omega_{s1}+\omega_{s2}|/\dot{\alpha})^2}\cos(|\omega_{s1}+\omega_{s2}|t+\varphi_{s1+s2})], \quad (9.5.15)$$

where:

$$\varphi_{s1}=\arctan\frac{\dot{\alpha}}{2\alpha\omega_{1s}}; \quad \varphi_{s2}=\arctan\frac{\dot{\alpha}}{2\alpha\omega_{2s}};$$
$$\varphi_{s1-s2}=\arctan\frac{2\alpha|\omega_{s1}-\omega_{s2}|}{\dot{\alpha}}; \quad \varphi_{s1+s2}=\arctan\frac{2\alpha|\omega_{s1}+\omega_{s2}|}{\dot{\alpha}}.$$

Obviously, Eq. (9.5.15) recalls analogous Eq. (5.4.32) for the pure periodic function. Generalizing Eq. (9.5.6) we can, therefore, see that:

$$e_l\cdot\cos(\Omega_{Kl}t-\varepsilon_l^0)\propto a_1\sqrt{1+(2\omega_{s1}\alpha/\dot{\alpha})^2}\cos(\omega_{s1}t-\varphi_{s1})+$$
$$+a_2\sqrt{1+(2\omega_{s2}\alpha/\dot{\alpha})^2}\cos(\omega_{s2}t-\varphi_{s2})+$$
$$+(a_1a_2/2)\sqrt{1+(2|\omega_{s1}-\omega_{s2}|\alpha/\dot{\alpha})^2}\cos(|\omega_{s1}-\omega_{s2}|t+\varphi_{s1-s2})+$$
$$+(a_1a_2/2)\sqrt{1+(2|\omega_{s1}+\omega_{s2}|\alpha/\dot{\alpha})^2}\cos(|\omega_{s1}+\omega_{s2}|t+\varphi_{s1+s2}), \quad (9.5.16)$$

that is, some angular velocities can be close to the main circular frequencies ω_{s1} and ω_{s2} or their combinations $|\omega_{s1}-\omega_{s2}|$ and $|\omega_{s1}+\omega_{s2}|$.

In the more general case of arbitrary periodic functions $G_1(\omega_{s1}t)$ and $G_2(\omega_{s2}t)$ (when the quasiperiodic function is equal to their product) we can expand each of them in a Fourier series like Eq. (5.3.25):

$$G_i(\omega_{si}t)=G_{s_i}[1+\sum_{n_i=1}^{\infty}\sqrt{a_{n_i}^2+b_{n_i}^2}\sin(n_i\omega_{si}t+\varphi_{ni})], \quad i=1,2, \quad (9.5.17)$$

where:

$$a_{ni} = \frac{2}{T_i} \int_{-T_i/2}^{T_i/2} [\mathrm{G}_i(\omega_{si} t) / \mathrm{G}_{s_i}] \cos(n_i \omega_{s_i} t) dt;$$

$$b_{n_i} = \frac{2}{T_i} \int_{-T_i/2}^{T_i/2} [\mathrm{G}_i(\omega_{si} t) / \mathrm{G}_{s_i}] \sin(n_i \omega_{s_i} t) dt; \varphi_{n_i} = \arctan \frac{a_{n_i}}{b_{n_i}}; \;\; \omega_{si} = \frac{2\pi}{T_i}.$$

Then, according to Eq. (9.5.17), the quasiperiodic function as their product can be represented by the relation:

$$\mathrm{G}(t) = \mathrm{G}_1(\omega_{s1} t) \, \mathrm{G}_2(\omega_{s2} t) =$$

$$= \mathrm{G}_s [1 + \sum_{n_1=1}^{\infty} \sqrt{a_{n_1}^2 + b_{n_1}^2} \sin(n_1 \omega_{s1} t + \varphi_{n1}) + \sum_{n_2=1}^{\infty} \sqrt{a_{n_2}^2 + b_{n_2}^2} \sin(n_2 \omega_{s2} t + \varphi_{n2}) +$$

$$+ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sqrt{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)} \sin(n_1 \omega_{s1} t + \varphi_{n1}) \sin(n_2 \omega_{s2} t + \varphi_{n2})] \qquad (9.5.18)$$

The last term in the right-hand part of Eq. (9.5.18) can be calculated as follows:

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sqrt{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)} \sin(n_1 \omega_{s1} t + \varphi_{n1}) \sin(n_2 \omega_{s2} t + \varphi_{n2})] =$$

$$= \tfrac{1}{2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sqrt{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)} \cos(\left| n_1 \omega_{s1} - n_2 \omega_{s2} \right| t + \varphi_{n1} - \varphi_{n2}) +$$

$$+ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sqrt{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)} \cos(\left| n_1 \omega_{s1} + n_2 \omega_{s2} \right| t + \varphi_{n1} + \varphi_{n2}). \qquad (9.5.19)$$

Let us note that for Eq. (9.5.19) calculating the multiple ($r = 2$) Fourier series [191] can be used:

$$\mathbf{c}(\boldsymbol{\omega}) = \sum_{\mathbf{n}} c_{\mathbf{n}} \, e^{\mathrm{i} \mathbf{n} \cdot \boldsymbol{\omega}} ,$$

where $\mathbf{n} \cdot \boldsymbol{\omega} = n_1 \omega_{s1} + n_2 \omega_{s2} + ... + n_r \omega_{sr}$, in particular, in the given case $c_{\mathbf{n}} = c_{n_1} c_{n_2} = \sqrt{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)}$, $\mathbf{n} = (n_1, n_2)$. Substituting Eq. (9.5.19) into Eq. (9.5.18) we then find the derivative:

$$\dot{\mathrm{G}}(t) = \mathrm{G_s}\,\omega_{\mathrm{s}1}\sum_{n_1=1}^{\infty} n_1 c_{n1}\cos\left(n_1\omega_{\mathrm{s}1}t + \varphi_{n1}\right) +$$

$$+\,\mathrm{G_s}\,\omega_{\mathrm{s}2}\sum_{n_2=1}^{\infty} n_2 c_{n2}\cos\left(n_2\omega_{\mathrm{s}2}t + \varphi_{n2}\right) +$$

$$-(\mathrm{G_s}/2)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} c_{n1}c_{n2}\left|n_1\omega_{\mathrm{s}1} - n_2\omega_{\mathrm{s}2}\right|\sin\left(\left|n_1\omega_{\mathrm{s}1} - n_2\omega_{\mathrm{s}2}\right|t + \varphi_{n1} - \varphi_{n2}\right) +$$

$$-(\mathrm{G_s}/2)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} c_{n1}c_{n2}\left|n_1\omega_{\mathrm{s}1} + n_2\omega_{\mathrm{s}2}\right|\sin\left(\left|n_1\omega_{\mathrm{s}1} + n_2\omega_{\mathrm{s}2}\right|t + \varphi_{n1} + \varphi_{n2}\right). \quad (9.5.20)$$

Therefore, following the derivation of Eq. (5.4.30)-(5.4.32) and using Eq. (9.5.18)-(9.5.20) we can see that:

$$\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t) = \frac{1}{2}\cdot\mathrm{G}(t)\dot{\alpha}(t) + \dot{\mathrm{G}}(t)\alpha(t) =$$

$$= (\mathrm{G_s}\,\dot{\alpha}/2)[1 + \sum_{n_1=1}^{\infty} c_{n1}\sqrt{1 + (2\alpha n_1\omega_{\mathrm{s}1}/\dot{\alpha})^2}\cos(n_1\omega_{\mathrm{s}1}t + \psi_{n1}) +$$

$$+\sum_{n_2=1}^{\infty} c_{n2}\sqrt{1 + (2\alpha n_2\omega_{\mathrm{s}2}/\dot{\alpha})^2}\cos(n_2\omega_{\mathrm{s}2}t + \psi_{n2}) +$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\tfrac{1}{2}c_{n1}c_{n2}\sqrt{1 + (2\alpha\left|n_1\omega_{\mathrm{s}1} - n_2\omega_{\mathrm{s}2}\right|/\dot{\alpha})^2}\cos\left(\left|n_1\omega_{\mathrm{s}1} - n_2\omega_{\mathrm{s}2}\right|t + \psi_{n_1\omega_{\mathrm{s}1} - n_2\omega_{\mathrm{s}2}}\right) +$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\tfrac{1}{2}c_{n1}c_{n2}\sqrt{1 + \left(\frac{2\alpha\left|n_1\omega_{\mathrm{s}1} + n_2\omega_{\mathrm{s}2}\right|}{\dot{\alpha}}\right)^2}\cos\left(\left|n_1\omega_{\mathrm{s}1} + n_2\omega_{\mathrm{s}2}\right|t + \psi_{n_1\omega_{\mathrm{s}1} + n_2\omega_{\mathrm{s}2}}\right). \quad (9.5.21)$$

As in (9.5.16), we can see analogously that the angular velocities $\Omega_{\mathrm{K}l}, l = 1,\cdots,N$, can be close to the multiply circular frequencies $n_1(l)\omega_{s1}$ and $n_2(l)\omega_{s2}$, where $n_1(l), n_2(l) \in \mathbf{N}$, as well as combinations of these frequencies $\left|n_1(l)\omega_{\mathrm{s}1} - n_2(l)\omega_{\mathrm{s}2}\right|$ and $\left|n_1(l)\omega_{\mathrm{s}1} + n_2(l)\omega_{\mathrm{s}2}\right|$.

Thus, though the quasi-equilibrium state of a compressible gravitating spheroidal body implies both a model of a periodic function and a quasiperiodic function for an oscillation behavior of the GCF description, the general condition $\overline{\mathrm{G}(t)} = \mathrm{G}_s = \mathrm{const}$ (or $\overline{\dot{\mathrm{G}}(t)} = 0$) is *common* for these models. In this connection, the average of the quasi-equilibrium gravitational potential (9.5.1) (or (9.5.2)) means that

$$\overline{\varphi_g\big|_{\text{quasiequil}}(r,t)} = \varphi_g\big|_{\text{equil}}(r). \qquad (9.5.22)$$

In this meaning, formula (5.4.29) in Section 5.4 generalizes not only Eq. (9.5.5b) (or (5.4.35)) in that case if $\mathrm{G}(t)$ as well as $\dot{\mathrm{G}}(t)$ is a periodic function in accordance with the expansion in the Fourier series (5.3.27a), but it includes the more complex cases when $\mathrm{G}(t)$ as well as $\dot{\mathrm{G}}(t)$ are quasiperiodic functions of the kind (9.5.14), (9.5.18)), and (9.5.20). So, the deviation of each moving l-protoplanet away from its initial circular orbit $|\vec{r}| = R_{0l}$ can occur under the action of the specific additional force $\vec{f}_a$ caused by *quasiperiodic* fluctuations (based on the multiple Fourier series) of the GCF derivative $\dot{\mathrm{G}}(t)$ of a central body (the star).

The quasiperiodic (or periodic) fluctuations of GCF of the central stellar body (as the core of a spheroidal body) mean that a star with a stellar corona *undergoes small compression pulsations* which induce the radial and axial oscillations of the orbital motion of protoplanets (planets). Thus, according to the proposed statistical theory, the *quasi-equilibrium* gravitational field of a central stellar body and, as a consequence, the Alfvén–Arrhenius additional force (modifying forms of circular planetary orbits to elliptical ones) is caused by fluctuations of the GCF $\mathrm{G}(t)$ relative to its stabilization value $\mathrm{G}_s = \mathrm{const}$.

Using Eqs (9.3.8) and (9.5.5b) we can estimate the absolute values of the pulsations of the Alfvén–Arrhenius specific force relative to the specific force of gravity acting on each l-protoplanet, for which let us find the ratio:

$$\frac{\left|\vec{f}_{a_l}\right|}{\left|\vec{f}_{g_l}\right|} = \frac{\left| e_l \dfrac{\gamma M}{r_l^2} \cdot \cos(\Omega_{\mathrm{K}l}\, t - \varepsilon_l^0) \right|}{\left| \dfrac{\gamma M}{r_l^2} \right|} \le e_l, \quad l = 1,\dots, N \,. \qquad (9.5.23)$$

The action of an additional force $\vec{f}_{a_l}$ is similar to Hooke's spring force which affects free oscillations of the body. Due to dissipation, these oscillations are damped gradually, so that they need support through the periodic (quasiperiodic) impact of the additional force $\vec{f}_{a_l}$ by analogy with the *principle of an anchoring mechanism* in a clock. The frequency of action of the additional force is $\Omega_{\mathrm{K}l}$ for the l-protoplanet. As mentioned above, there are the main circular frequencies ω_{s1}, ω_{s2}, ω_{s3},... of the quasiperiodic pulsations of GCF of a central stellar body (including the rotational angular velocity Ω_s of exterior media of the stellar corona). The Keplerian angular velocity of motion $\Omega_{\mathrm{K}l}$ of l-protoplanet is, therefore, equal to $n_i(l)\omega_{si}$ where ω_{si} is some main circular frequency of pulsations of the central stellar body.

For example, according to the first scenario the Sun (including the Solar corona) has two main circular frequencies of inner oscillations ω_{s1}, ω_{s2}, at least supporting stable motion planets in the Solar system, that is, the so-called the main planetary circular frequencies: $\omega_{s1} = 4 \cdot 10^{-10}$ Hz and $\omega_{s2} = 5.1 \cdot 10^{-8}$ Hz. This means that there were some *stable stages* in the process of compression of the proto-Sun and the formation of planets ($N = 9$) from a protoplanetary cloud. In particular, multiple frequencies $2\omega_{s1} \approx \Omega_{\mathrm{K}9}$, $3\omega_{s1} \approx \Omega_{\mathrm{K}8}$,

$6\omega_{s1} \approx \Omega_{K7}$, $17\omega_{s1} \approx \Omega_{K6}$, and $42\omega_{s1} \approx \Omega_{K5}$ correspond to the Keplerian angular velocities of the orbital movement of Pluto, Neptune, Uranus, Saturn, and Jupiter respectively (see Table 9.2), that is, to the stage of the Jupiter group planet formation. The next stage was connected with the Earth group planet formation, therefore multiply frequencies $2\omega_{s2} \approx \Omega_{K4}$, $4\omega_{s2} \approx \Omega_{K3}$, $6\omega_{s2} \approx \Omega_{K2}$ and $16\omega_{s2} \approx \Omega_{K1}$ correspond to the Keplerian angular velocities of the orbital motion of Mars, Earth, Venus, and Mercury (see Table 9.2).

According to Eq. (9.2.30) in Section 9.2, we can estimate the real part of the generalized circular frequency (5.3.5b) of radial oscillations for l-planet moving in slightly elliptical orbit:

$$\mathrm{Re}\,\delta\varpi_h^{(l)} = \dot{\varepsilon}_l - \Omega_{Kl} = 2e_l\Omega_{Kl}\cos(\Omega_{Kl}t - \varepsilon_{hl}^0), \qquad (9.5.24)$$

that is, the amplitude of the real part of the generalized circular frequency of oscillations of angular velocity is

$$\delta\varpi_l^0 = \max \mathrm{Re}\,\delta\varpi_h^{(l)} = 2e_l\Omega_{Kl}, \quad l = 1,2,...,N. \qquad (9.5.25)$$

The amplitudes (9.5.25) of the real parts of the generalized circular frequencies of angular velocity fluctuations for the Solar system planets are estimated in Table 9.2, according to which:

$\Omega_{K9} \in [2\omega_{s1} - \delta\varpi_9^0, 2\omega_{s1} + \delta\varpi_9^0]$, $\Omega_{K8} \in [3\omega_{s1} - \delta\varpi_8^0, 3\omega_{s1} + \delta\varpi_8^0]$, $\Omega_{K7} \in [6\omega_{s1} - \delta\varpi_7^0, 6\omega_{s1} + \delta\varpi_7^0]$, $\Omega_{K6} \in [17\omega_{s1} - \delta\varpi_6^0, 17\omega_{s1} + \delta\varpi_6^0]$, $\Omega_{K5} \in [42\omega_{s1} - \delta\varpi_5^0, 42\omega_{s1} + \delta\varpi_5^0]$ as well as $\Omega_{K4} \in [2\omega_{s2} - \delta\varpi_4^0, 2\omega_{s2} + \delta\varpi_4^0]$, $\Omega_{K3} \in [4\omega_{s2} - \delta\varpi_3^0, 4\omega_{s2} + \delta\varpi_3^0]$, $\Omega_{K2} \notin [6\omega_{s2} - \delta\varpi_2^0, 6\omega_{s2} + \delta\varpi_2^0]$, $\Omega_{K1} \in [16\omega_{s2} - \delta\varpi_1^0, 16\omega_{s2} + \delta\varpi_1^0]$, that is, only the planet Venus does not belong to the frequency interval.

Using Eqs (9.5.23) and (9.5.25) the comparative analysis of some orbital characteristics of planets for the Solar system is realized in Table 9.2. Here the values of amplitudes f_a^0 of the specific force of Alfvén–Arrhenius (9.3.8) are estimated for the planets of the Solar system. Taking into account that 1AU is equal to the distance from the Sun to the Earth (the major semi-axis of its orbit is $a_{\text{Earth}} = 1.495983 \cdot 10^{11}\,\text{m}$), the dependencies of amplitudes of specific additional periodic force f_a^0 on the planetary distances $d = a / a_{\text{Earth}}$ are presented in Fig. 9.2.

Table 9.2 and Fig. 9.2 show that the most significant values of the amplitude of specific additional periodic force occur for Mercury ($f_a^0 = 0.8135531\,\text{Gal}$) and Mars ($f_a^0 = 0.02384768\,\text{Gal}$). The amplitudes of additional periodic forces $F_{a_l}^0 = m_l f_{a_l}^0$ acting on the Solar system planets (with the masses $m_l, l = 1,2,...,9$) are also estimated in Table 9.2.

Table 9.2. The comparative analysis of orbital characteristics of the Solar system planets

Solar system planets	Observation data				Estimation of values		
	Kepler's period T_K, in 24hrs	**Kepler's angular velocity Ω_K, s^{-1}**	**Major semi-axis of orbit a, m**	**Orbital eccentricity e**	**Amplitude of the real part of the generalized circular frequency of oscillations $\delta\varpi_l^0$, Hz**	**Amplitude of specific additional periodic force f_a^0, m/s^2**	**Amplitude of additional periodic force F_a^0, N**
Mercury	87.97	$8.266688\cdot10^{-7}$	$5.790923\cdot10^{10}$	0.20563593	$3.39985615\cdot10^{-7}$	$8.135531\cdot10^{-3}$	$2.7091315\cdot10^{21}$
Venus	224.7	$3.236406\cdot10^{-7}$	$1.082089\cdot10^{11}$	0.0068	$4.40151216\cdot10^{-9}$	$7.704870\cdot10^{-5}$	$3.75034547\cdot10^{20}$
Earth	365.26	$1.990967\cdot10^{-7}$	$1.495983\cdot10^{11}$	0.01671123	$6.6543014919\cdot10^{-9}$	$9.906895\cdot10^{-5}$	$5.9169891\cdot10^{20}$
Mars	686.69	$1.059025\cdot10^{-7}$	$2.279438\cdot10^{11}$	0.0933941	$1.9781337351\cdot10^{-8}$	$2.384768\cdot10^{-4}$	$1.5306582\cdot10^{20}$
Jupiter	4331.98	$1.678724\cdot10^{-8}$	$7.785472\cdot10^{11}$	0.048775	$1.637595262\cdot10^{-9}$	$1.067604\cdot10^{-5}$	$2.02\ 69529\cdot10^{22}$
Saturn	10760.6	$6.758204\cdot10^{-9}$	$1.433449\cdot10^{12}$	0.05572322	$7.531777766\cdot10^{-10}$	$3.597942\cdot10^{-6}$	$2.04528611\cdot10^{21}$
Uranus	30685.5	$2.369916\cdot10^{-9}$	$2.876679\cdot10^{12}$	0.04440559	$2.104750365\cdot10^{-10}$	$7.119298\cdot10^{-7}$	$6.18182884\cdot10^{19}$
Neptune	60191.2	$1.208184\cdot10^{-9}$	$4.503444\cdot10^{12}$	0.01121427	$2.709780317\cdot10^{-11}$	$7.336072\cdot10^{-8}$	$7.51433855\cdot10^{18}$
Pluto	90474.9	$8.03782\cdot10^{-10}$	$5.906460\cdot10^{12}$	0.24880766	$3.999742371\cdot10^{-10}$	$9.462181\cdot10^{-7}$	$1.23292218\cdot10^{16}$

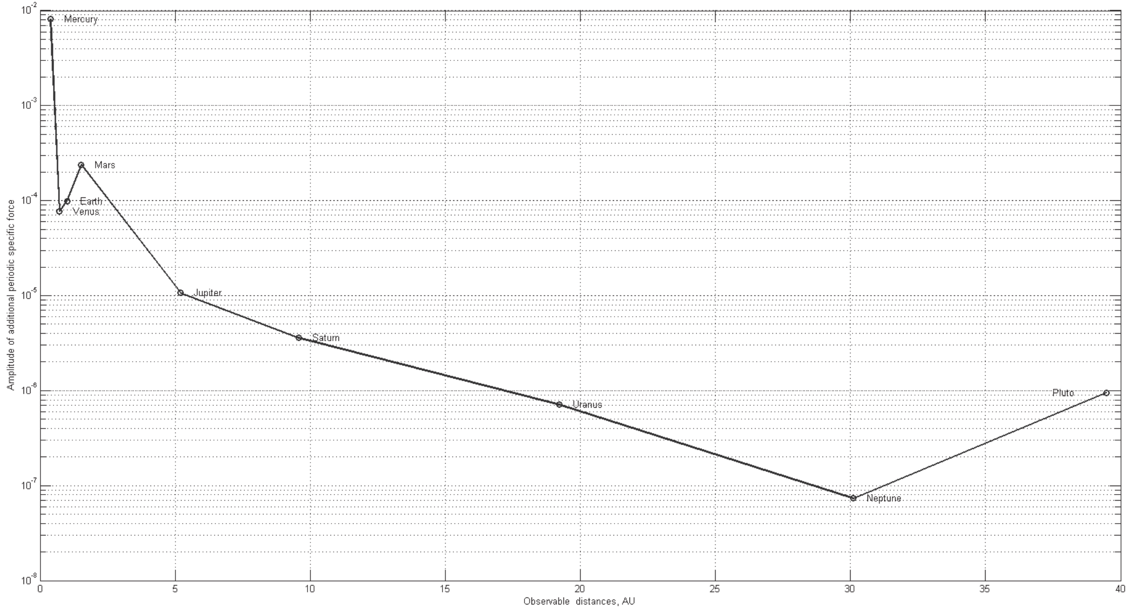


Figure 9.2. The plot of dependence of the amplitudes of specific additional periodic force f_a^0, m/s ^{2} on the planetary distances d, AU

9.6. Axial and radial oscillations of the orbital motion in the gravitational field of a rotating and gravitating ellipsoid-like central body

Let us now estimate the frequencies of the radial and axial oscillations of a rotating and gravitating *ellipsoid-like* central cosmogonical body (the star) from the point of view of the statistical theory [16, 73, 77]. As noted in Section 9.2 (see Eqs (9.2.41a, b)), the motion in a gravitational field of an almost immovable (slowly rotating) spheroidal body is degenerate in the sense that $\omega_h = \omega_z = \Omega_K$. However, the axial rotation of a spheroidal body creates a *flattening* of its core, that is, the gravitational field of a rotating spheroidal body deviates from the centrally symmetric field of a sphere-like body.

Indeed, since the gravitational potential in the remote zone II of a uniformly rotating spheroidal body is described by the expression (7.1.24) in spherical coordinates [72, 73]:

$$\varphi_g(r,\theta)|_{r>>r_*} = -\sqrt{\frac{2\alpha}{\pi}} \cdot \frac{\gamma M}{r\sqrt{1-\varepsilon_0{}^2 \sin^2\theta}} \times$$

$$\times \int_0^{r\sqrt{1-\varepsilon_0{}^2\sin^2\theta}} \exp[-\alpha r'^2/2]dr'|_{r>>r_*} \approx -\frac{\gamma M}{r\sqrt{1-\varepsilon_0^2\sin^2\theta}} =$$

$$= -\frac{\gamma M}{r}\left(1+\frac{\varepsilon_0^2}{2}\sin^2\theta\right) = -\frac{\gamma M}{r}\left(1+\frac{\varepsilon_0^2\cdot r_*^2}{r^2}\right) = \varphi_g^*(r), \qquad (9.6.1)$$

the potential of the gravitational field of a rotating spheroidal body then deviates from the centrally symmetric field $1/r$ – potential of the sphere-like spheroidal body (see Eq. (7.3.6) in Section 7.3). Using the formula (9.6.1) we can estimate the gravitational field strength which is essentially a specific force of gravity [72, 73]:

$$\vec{f}_g(r,\theta)\Big|_{r>>r_*} = -\nabla\varphi_g(r,\theta)\Big|_{r>>r_*} =$$

$$= -\frac{\gamma M}{r^2}\left(1+\frac{3\delta_0^2}{r^2}\right)\cdot \vec{e}_r = \vec{f}_g^{\,*}(r), \qquad (9.6.2)$$

where $\delta_0 = \varepsilon_0 r_*, \delta_0 > 0$. Indeed, as seen from (9.6.2) the gravitational field of an ellipsoid-like cosmogonical body consists of the $1/r^2$–field from a spherically symmetric body, on which is superimposed the field from the "equatorial bulge" [9, 19]. In this regard, the calculation of the derivative $\partial f_g^{\,*}/\partial r$ in accordance with (9.6.2) gives the following:

$$f_g^{\,*}(r)\Big|_{R_0} = f_g(R_0) + \frac{3\delta_0^2}{R_0^{\ 2}}\cdot f_g(R_0) = f_g(R_0)\cdot\left(1+\frac{3\delta_0^2}{R_0^{\ 2}}\right), \quad (9.6.3a)$$

$$\left.\frac{\partial f_g^{\,*}(r)}{\partial r}\right|_{R_0} = \left(1+\frac{3\delta_0^2}{R_0^{\ 2}}\right)\cdot\left.\frac{\partial f_g(r)}{\partial r}\right|_{R_0} - \frac{6\delta_0^2}{R_0^{\ 3}} f_g(R_0). \qquad (9.6.3b)$$

In view of Eqs (9.6.3a) and (9.6.3b)—as well as Eq. (9.2.40) from Section 9.2—the frequency of the radial orbital oscillations (9.2.12) of a body in the gravitational field of a rotating spheroidal body is equal:

$$\omega_h = \sqrt{\frac{3 f_g^{\,*}(R_0)}{R_0} + \left.\frac{\partial f_g^{\,*}(r)}{\partial r}\right|_{R_0}} =$$

$$= \sqrt{\frac{3 f_g(R_0)}{R_0} + \frac{9\delta_0^2}{R_0^3} f_g(R_0) + \left(1+\frac{3\delta_0^2}{R_0^{\ 2}}\right)\cdot\left.\frac{\partial f_g(r)}{\partial r}\right|_{R_0} - \frac{6\delta_0^2}{R_0^{\ 3}} f_g(R_0)} =$$

$$= \sqrt{\frac{3 f_g(R_0)}{R_0} + \frac{3\delta_0^2}{R_0^3} f_g(R_0) + \left(1+\frac{3\delta_0^2}{R_0^{\ 2}}\right)\cdot\left.\frac{\partial f_g(r)}{\partial r}\right|_{R_0}} =$$

$$= \sqrt{\frac{f_g(R_0)}{R_0} - \frac{3\delta_0^2}{R_0^3} f_g(R_0)}, \qquad (9.6.4)$$

while the frequency of the axial orbital oscillations (9.2.16) of a body in the gravitational field of a rotating spheroidal body is respectively:

$$\omega_z = \sqrt{-\frac{f_g^{\,*}(R_0)}{R_0} - \frac{\partial f_g^{\,*}(r)}{\partial r}\bigg|_{R_0}} =$$

$$= \sqrt{-\frac{f_g(R_0)}{R_0} - \frac{3\delta_0^2}{R_0^3} f_g(R_0) - \left(1+\frac{3\delta_0^2}{R_0^{\;2}}\right)\cdot\frac{\partial f_g(r)}{\partial r}\bigg|_{R_0} + \frac{6\delta_0^2}{R_0^{\;3}} f_g(R_0)} =$$

$$= \sqrt{-\frac{f_g(R_0)}{R_0} + \frac{3\delta_0^2}{R_0^3} f_g(R_0) - \left(1+\frac{3\delta_0^2}{R_0^{\;2}}\right)\cdot\frac{\partial f_g(r)}{\partial r}\bigg|_{R_0}} =$$

$$= \sqrt{\frac{f_g(R_0)}{R_0} + \frac{9\delta_0^2}{R_0^3} f_g(R_0)}. \qquad (9.6.5)$$

Comparing Eq. (9.2.19) (from Section 9.2) with Eqs (9.6.4) and (9.6.5) we find that in the case of a rotating flattened spheroidal body the Alfvén–Arrhenius inequalities are also true [19 p.348]:

$$\omega_z > \Omega_{\mathrm{K}} > \omega_h. \qquad (9.6.6)$$

Let us note that within the framework of the statistical theory of gravitating spheroidal bodies (see Section 5.3 of Chapter 5) according to Eq. (5.3.43) the analogous inequality is also true in the case of the relative mechanical equilibrium $(\widetilde{\mathrm{G}}(t) = \widetilde{\mathrm{G}}_s = \mathrm{const})$ of a rotating axially symmetric spheroidal body (ellipsoid-like cloud):

$$\omega_z > \omega_h. \qquad (9.6.7)$$

Moreover, when a gravitational field in an ellipsoid-like body becomes stable, an interference of the orthogonal radial and the axial oscillations leads to the rotation of the core of this spheroidal body. According to (9.6.7), the interferences of these orbital oscillations may be various at different latitudes of the core of the ellipsoid-like cloud modeled on a star (with stellar corona). That is why we can consider a uniform angular velocity of rotation Ω that it is relative to the chosen latitude only (for example, in the equatorial plane of a star). In particular, the Sun's sidereal rotation period is $T_{\rm Sun}=25.05$ days at the equator, $T_{\rm Sun}=25.38$ days at the 16° latitude and $T_{\rm Sun}=34.4$ days at the poles [283], that is, $\Omega_{0\rm Sun}=2.9030759\cdot 10^{-6}\ {\rm s}^{-1}$, $\Omega_{16^{\circ}\rm Sun}=2.8653291\cdot 10^{-6}\ {\rm s}^{-1}$, and $\Omega_{90^{\circ}\rm Sun}=2.1140131\cdot 10^{-6}\ {\rm s}^{-1}$ respectively.

Returning to Eqs (9.6.4) and (9.6.5) we can obtain similarly Eq. (9.2.20) showing that:

$$\omega_h^2+\omega_z^2=\frac{f_g(R_0)}{R_0}-3\left(\frac{\delta_0}{R_0}\right)^2\cdot\frac{f_g(R_0)}{R_0}+\frac{f_g(R_0)}{R_0}+9\left(\frac{\delta_0}{R_0}\right)^2\cdot\frac{f_g(R_0)}{R_0}=$$

$$=2\frac{f_g(R_0)}{R_0}+6\left(\frac{\delta_0}{R_0}\right)^2\cdot\frac{f_g(R_0)}{R_0}=$$

$$=2\frac{f_g(R_0)}{R_0}\cdot\left[1+3\left(\frac{\delta_0}{R_0}\right)^2\right]=2\Omega_{\rm K}^2\left[1+3\left(\frac{\delta_0}{R_0}\right)^2\right]=2\Omega_{\rm K}^2\left[1+\frac{3\varepsilon_0^2}{\alpha R_0^2}\right]. \quad (9.6.8)$$

By analogy with Eq. (9.2.20), if the expression in the right-hand side of Eq. (9.6.8) is assumed to be the double square of a new Kepler angular velocity, that is, the Kepler orbital frequency under the condition of action of the constraining force $\delta f_g=-3\delta_0^2\gamma M/r^4$ in accordance with Eq. (9.6.2):

$$\Omega_{\rm K}^*=\sqrt{\frac{f_{\rm g}^*(R_0)}{R_0}}=\sqrt{\frac{f_{\rm g}(R_0)}{R_0}\cdot\left[1+3\left(\frac{\delta_0}{R_0}\right)^2\right]}=$$

$$=\Omega_{\rm K}\cdot\sqrt{1+\frac{3\varepsilon_0^2}{\alpha R_0^2}}, \quad (9.6.9)$$

then formula (9.6.9) shows that the Keplerian angular velocity of a body's motion in orbit around a rotating ellipsoid-like cosmogonical body is *slightly increased* in comparison with the Keplerian angular velocity of orbital motion around a spherically symmetric cosmogonical body:

$$\delta\Omega = \Omega_{\mathrm{K}} \cdot \frac{\varepsilon_0}{R_0}\sqrt{\frac{3}{\alpha}} . \qquad (9.6.10)$$

Then, using (9.6.10), formula (9.6.9) takes the form:

$$\Omega_{\mathrm{K}}^{*2} = \Omega_{\mathrm{K}}^2 + \delta\Omega^2 . \qquad (9.6.11)$$

Consequently, taking into account the definition (9.6.9) and Eq. (9.6.8) we obtain:

$$\omega_h^2 + \omega_z^2 = 2\Omega_{\mathrm{K}}^{*2} . \qquad (9.6.12)$$

As shown in Section 7.3, under the action of the constraining force δf_g in accordance with Eq. (9.6.2) the orbit of a planet (for example, Mercury) in the vicinity of the core of a rotating spheroidal body becomes disturbed. On the other hand, the planet Mercury is moving in a *precessing elliptic orbit* (7.3.29) with the orbital angular velocity (9.6.9) (since there is a modulating multiplier η (7.3.32) of the phase in equation (7.3.31) of the "disturbed" ellipse with the origin of polar coordinates in its focus). Using Eqs (9.6.2), (9.6.4), (9.6.5), (9.6.8), and (9.6.10) the comparative analysis of some orbital characteristics of planets for the Solar system is carried out in Table 9.3 (see also Figs. 9.3–9.5).

Table 9.3. The comparative analysis of the orbital motion characteristics of planets for the Solar system

Solar system planets	**Observable distances d, AU**	**Keplerian angular velocity Ω_K, s^{-1}**	**Deviation of Keplerian angular velocity $\delta\Omega$, s^{-1}**	**Amplitude of constraining specific force $\left\|\delta\vec{f}_g\right\|$, m/s^2**	**Ratio of frequencies of orbital oscillations ω_z/ω_h-1**	**Angular velocity of orbital pericenter motion Ω_P, s^{-1}**	**Angular velocity of ascending node motion Ω_ζ, s^{-1}**
Mercury	0.3871	$8.265508\cdot10^{-7}$	$9.2848\cdot10^{-14}$	$4.9922\cdot10^{-16}$	$2.5236757\cdot10^{-14}$	$5.2148655\cdot10^{-14}$	$-1.5644596\cdot10^{-13}$
Venus	0.7233	$3.235912\cdot10^{-7}$	$1.9453\cdot10^{-14}$	$4.0948\cdot10^{-17}$	$7.2277507\cdot10^{-15}$	$5.8470913\cdot10^{-15}$	$-1.75412740\cdot10^{-14}$
Earth	1.0000	$1.990679\cdot10^{-7}$	$8.6561\cdot10^{-15}$	$1.1209\cdot10^{-17}$	$3.7816020\cdot10^{-15}$	$1.8819889\cdot10^{-15}$	$-5.64596669\cdot10^{-15}$
Mars	1.5237	$1.058399\cdot10^{-7}$	$3.0205\cdot10^{-15}$	$2.0796\cdot10^{-18}$	$1.6288218\cdot10^{-15}$	$4.3098583\cdot10^{-16}$	$-1.29295749\cdot10^{-15}$
Jupiter	5.2042	$1.676733\cdot10^{-8}$	$1.4010\cdot10^{-16}$	$1.5281\cdot10^{-20}$	$1.3962373\cdot10^{-16}$	$5.8527929\cdot10^{-17}$	$-1.75583787\cdot10^{-16}$
Saturn	9.5819	$6.711473\cdot10^{-9}$	$3.0457\cdot10^{-17}$	$1.3297\cdot10^{-21}$	$4.1187425\cdot10^{-17}$	$6.9107072\cdot10^{-17}$	$-2.07321217\cdot10^{-16}$
Uranus	19.2291	$2.360772\cdot10^{-9}$	$5.3384\cdot10^{-18}$	$8.1982\cdot10^{-23}$	$1.0226955\cdot10^{-17}$	$6.0358770\cdot10^{-18}$	$-1.81076309\cdot10^{-17}$
Neptune	30.1032	$1.205241\cdot10^{-9}$	$1.7409\cdot10^{-18}$	$1.3649\cdot10^{-23}$	$4.1729156\cdot10^{-18}$	$1.2573422\cdot10^{-18}$	$-3.77202669\cdot10^{-18}$
Pluto	39.4817	$8.02417\cdot10^{-10}$	$8.8374\cdot10^{-19}$	$4.6129\cdot10^{-24}$	$2.4259095\cdot10^{-18}$	$4.8664776\cdot10^{-18}$	$-1.45994327\cdot10^{-17}$

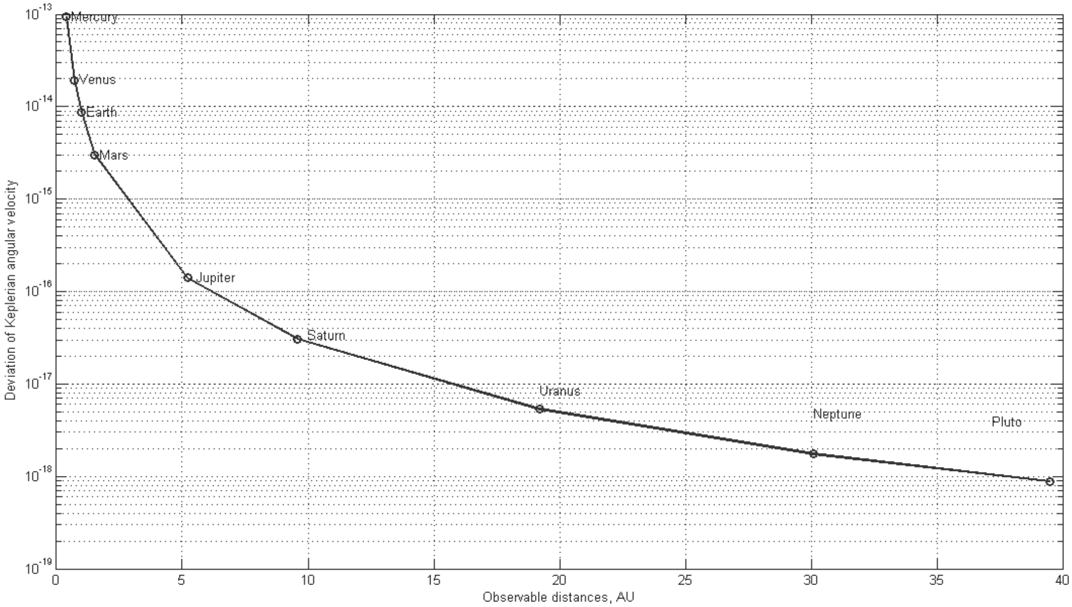


Figure 9.3. The plot of dependence of the deviation of Keplerian angular velocity $\delta\Omega$, s^{-1} on the planetary distances d, AU

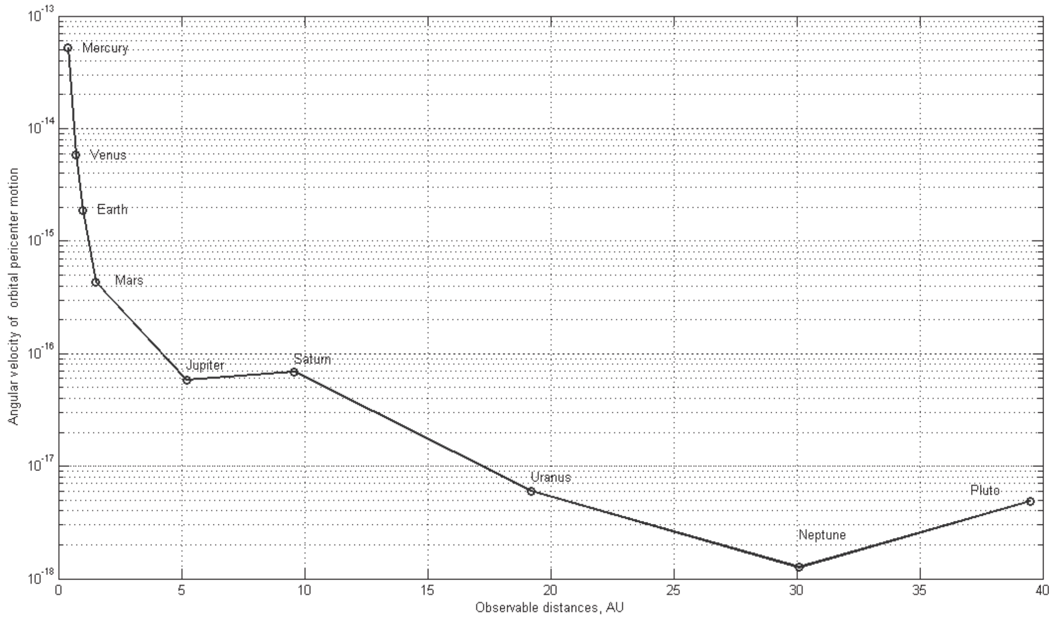


Figure 9.4. The plot of dependence of the angular velocity of orbital pericenter motion Ω_P , s^{-1} on the planetary distances d , AU

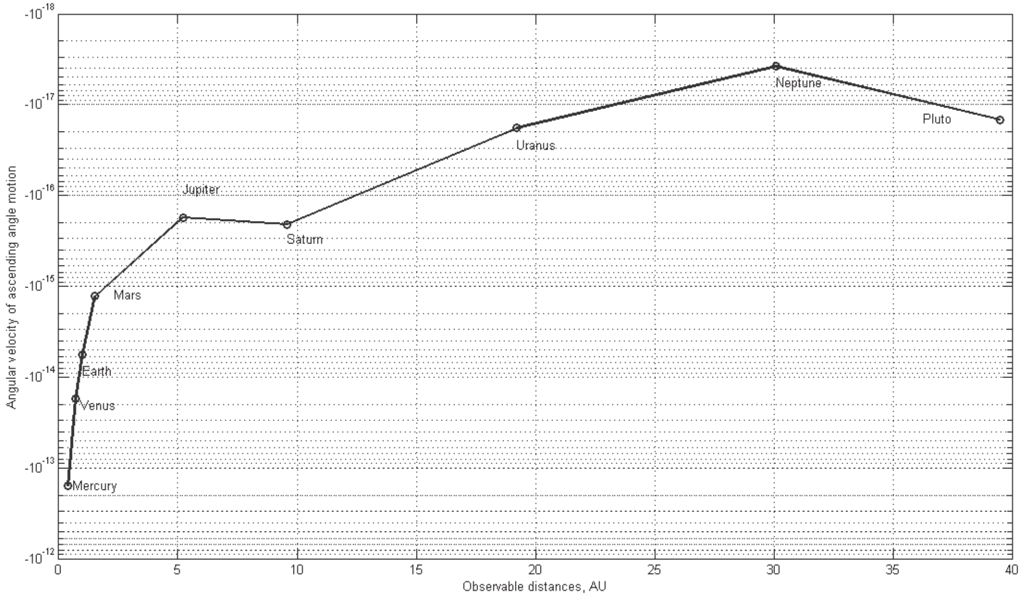


Figure 9.5. The plot of dependence of the angular velocity of ascending node motion Ω_ζ ,s^{-1} on the planetary distances d , AU

According to Eqs (9.2.27) and (9.2.36), as well as the Alfvén–Arrhenius' inequalities (9.6.6), this means that in the remote zone of the gravitational field of a rotating ellipsoid-like body the pericenter moves with the angular velocity:

$$\Omega_{\mathrm{P}} = \Omega_{\mathrm{K}} - \omega_h > 0\,, \qquad (9.6.13)$$

that is, in the *prograde* direction, whereas the nodes move with the angular velocity:

$$\Omega_{\zeta} = \Omega_{\mathrm{K}} - \omega_z < 0\,, \qquad (9.6.14)$$

that is, in the *retrograde* direction [19 p. 348]. The calculated values of angular velocities Ω_{P} and Ω_{ζ} following Eqs (9.6.13) and (9.6.14) for planets of the Solar system are placed in Table 9.3 and also presented in the form of Figures 9.4 and 9.5. Comparing Table 9.2 with Table 9.3 we can see that the specific additional periodic force $\left|\vec{f}_a\right|$ in $1.6296 \cdot 10^{13}$ times greater than the specific constraining force $\left|\delta\vec{f}_g\right|$ acting on Mercury; the analogous specific periodic force $\left|\vec{f}_a\right|$ in $1.1467 \cdot 10^{14}$ times greater than the respective specific constraining force $\left|\delta\vec{f}_g\right|$ acting on Mars. Consideration of Figures 9.4 and 9.5 also reveals a singular point on the plots of dependencies of Ω_{P} and Ω_{ζ} on the planetary distances d in the vicinity of Saturn.

Further, taking into account Eqs (9.6.12), (9.6.13), and (9.6.14) we obtain:

$$(\Omega_{\mathrm{K}} - \Omega_{\mathrm{P}})^2 + (\Omega_{\mathrm{K}} - \Omega_{\zeta})^2 = 2\Omega^{*2}_{\mathrm{K}}\,, \qquad (9.6.15)$$

whence it follows using (9.6.11) that:

$$\Omega_{\mathrm{P}} + \Omega_{\zeta} = \frac{\Omega_{\mathrm{P}}^2 + \Omega_{\zeta}^2 - 2\delta\Omega^2}{2\Omega_{\mathrm{K}}}\,. \qquad (9.6.16)$$

Since the right-hand term is very small in Eq. (9.6.16) it can be neglected so that we estimate in a first approximation:

$$\Omega_{\mathrm{P}} \approx -\Omega_{\zeta} . \tag{9.6.17}$$

Introducing (9.6.17) into Eq. (9.6.16) we obtain in a second approximation:

$$\Omega_{\mathrm{P}} = -\Omega_{\zeta} + \Delta\omega, \tag{9.6.18a}$$

where:

$$\Delta\omega = \frac{\Omega_{\mathrm{P}}^2}{\Omega_{\mathrm{K}}} - \frac{3\varepsilon_0^2 \Omega_{\mathrm{K}}}{\alpha R_0^2} . \tag{9.6.18b}$$

Let us note that the Alfvén–Arrhenius estimation of $\Delta\omega$ is $\Delta\omega = \Omega_{\mathrm{P}}^2 / \Omega_{\mathrm{K}}$ [9, 19]. Nevertheless, as Alfvén and Arrhenius indicated this formula gives a result in good agreement with the elaborate treatment by the usual methods in the case of small inclinations [9, 19 pp. 348–349]. Due to the fact that both the values α and ε_0^2 in (9.6.8) and (9.6.18b) have a statistical meaning, we can conclude that the radial and the axial oscillations modifying the initially circular orbit are determined by the statistical nature of processes of cosmogonical body formation.

Conclusion and comments

In this chapter, we have investigated the stability of planetary orbits based on the statistical theory of gravitating spheroidal bodies [16, 45–73]. Using the obtained universal stellar law (USL) [75, 76] and the modification of the USL connecting the temperature, size, and mass of a star (see Table 9.1) we show in Section 9.1 that knowledge of some orbital characteristics of multi-planet extrasolar systems refines the knowledge of the parameters of the stars based on the combination of Kepler's 3rd law and the universal stellar law (3KL-USL) [79].

Indeed, as shown in Section 9.1, the combined 3KL-USL law (9.1.18) connects among themselves both the mechanical values (the Keplerian angular velocity Ω_K and the major semi-axis a of a planetary orbit) and the statistical (thermodynamic) values (the parameter of gravitational condensation α and the temperature T). This means that the stability of the mechanical values (belonging to the left-hand part of Eq. (9.1.18)) directly depends on the statistical regularity of the right-hand part of the 3KL–USL equation (9.1.18). This relation points to a possibility of the presence of *statistical oscillations* of the motion of planets in orbit, that is, oscillations of the major semi-axis a and the orbital angular velocity Ω_K of rotation of planets around stars as well as bodies around planets. Such a conclusion is confirmed by the existence of the Alfvén–Arrhenius radial and axial orbital oscillations of planets and satellites [9, 19], as wel as by the experiments of V. Janibekov on board the orbital station "Mir." This section also noted that the stability of the parameters of planetary orbits is determined by a *constancy of the specific entropy* (9.1.19b) in conformity with the principles of self-organization in complex systems [79]. Therefore, the proposed 3KL–USL explains the stability of planetary orbits in extrasolar systems.

In this context, Section 9.2 investigates the additional periodic force f_a (9.2.45) causing the radial and axial orbital oscillations (which modify initial circular orbits of bodies) based on the approach of Alfvén and Arrhenius [9, 19]. A prediction of the Alfvén–Arrhenius specific additional periodic force within the framework of the Newtonian theory of gravity was considered in Section 9.3. As shown in Section 9.4, from the point of view of the theory of retarded potentials, the *wave gravitational potential* $\delta\varphi_g^{(\mathrm{II})}$ (9.3.38) and the Alfvén–Arrhenius specific additional periodic force $\vec{f}_a = -\operatorname{grad}\delta\varphi_g^{(\mathrm{II})}$ arise in a remote zone II of the gravitational field under the orbital motion of a body around a central gravitating body [205, 331]. The obtained spectral representations (9.4.13) and (9.4.37) correspond completely to the spectral expansion (5.4.36) derived in the statistical theory of gravitating spheroidal bodies (see Chapter 5). Thus, the proposed statistical *theory of the formation* of planetary systems, pointing to the regular and wave gravitational potentials origin, is confirmed by the *theories of existence* (Newtonian and retarded potentials ones) [205, 331].

Indeed, as shown in the previous Sections 5.2–5.4 and the current Section 9.5, within the framework of the statistical theory of gravitating spheroidal bodies the nature of the Alfvén–Arrhenius additional periodic force $\vec{f}_a$ (9.5.5b) is caused by the *wave gravitational potential* $\delta\varphi_g$ (9.5.3b) origin due to the periodic (or quasiperiodic) fluctuations of the derivative of the gravitational compression function (GCF) $\dot{\mathrm{G}}(t)$ of a central body (the star) in accordance with Eqs (5.3.27a), (5.4.29), (5.4.34a, b), (5.4.35), and (5.4.37a, b). Concretely, using the statistical theory of gravitating spheroidal bodies we affirm the following:

– the temporal deviation of GCF $\mathrm{G}(t)$ (or $\widetilde{\mathrm{G}}(t)$) of a central cosmogonical body (a star) in *quasi-equilibrium* state (in a vicinity of its equilibrium value $\mathrm{G_s} = \mathrm{const}$) leads to the origin of the additional periodic force $\vec{f}_a$ modifying forms of the circular orbits of moving bodies (planets and satellites) to the *slightly elliptical orbits* (see Table 9.2 and Fig. 9.2);

– in special cases, when $\dot{\mathrm{G}}(t) < 0$ (or $\dot{\widetilde{\mathrm{G}}}(t) < 0$), that is, $(\delta\varpi)^2 < 0$, the additional periodic force $\vec{f}_a$ becomes *oriented opposite* the gravitational force $\vec{f}_g$ following Eqs (5.3.15b), (5.3.40b), (5.4.39), and (9.2.45).

This means that the *principle of an anchoring mechanism* is realized in planetary systems. In particular, Section 9.5 finds that the additional periodic force $\vec{f}_a$ is similar to Hooke's force which affects free oscillations of a body in orbit. Due to dissipation, these oscillations are damped gradually, so that they need support through the periodic impact of the additional force $\vec{f}_a$ by analogy with the principle of an anchoring mechanism in a clock. The frequency of action of the additional periodic force $\vec{f}_{a_l}$ is $\Omega_{\mathrm{K}l}$ for the l-protoplanet. As mentioned in Section 9.5, there are the main circular frequencies ω_{s1}, ω_{s2}, ω_{s3},... of the quasiperiodic pulsations of GCF of a central stellar body (including the stellar corona). The Keplerian angular velocity of motion $\Omega_{\mathrm{K}l}$ of l-protoplanet is, therefore, equal to $n_i(l)\omega_{si}$ where ω_{si} is some main circular frequency of pulsations of the central stellar body.

Section 9.6 also justifies that the spatial deviation of the gravitational potential (9.6.1) of an ellipsoid-like rotating cosmogonical body from the centrally symmetric field $1/r$ – gravitational potential (of a sphere-like spheroidal body) implies different values of the radial ω_h and the axial ω_z orbital oscillations in accordance with (9.6.6) (even in the case of relative mechanical equilibrium $\widetilde{\mathrm{G}}(t) = \widetilde{\mathrm{G}}_s = \mathrm{const}$). An interference of these orbital oscillations leads, therefore, as a rule to a nonuniform rotation of the stellar layers at different latitudes of a star. As shown in Sections 7.3 and 9.6, under the action of the constraining force δf_g (9.6.2), a planet (for example, Mercury) in the vicinity of an ellipsoid-like rotating cosmogonical body is moving in the *precessing elliptic orbit* (7.3.29) with orbital angular velocity (9.6.9) (see data in Table 9.3 and Figs. 9.3-9.5).

References

1. Jeans, J. *Astronomy and Cosmogony*. Cambridge: University Press, 1929.
2. Safronov, V.S. *Evolution of Protoplanetary Cloud and Formation of Earth and Planets*. Moscow: Nauka, 1969 [reprinted by NASA Tech. Transl. F-677, Washington, D.C., 1972].
3. Mayor, M. and Queloz, D. A Jupiter-mass companion to a solar-type star. *Nature*, 1995, vol. 378, pp. 355–359.
4. Kempton, E. First exoplanet found around a Sun-like star. *Nature*, 2019, vol. 575, pp. 43–44.
5. Schmidt, O.Yu. The problem of the origin of the Earth and planets. In book: *Proceedings of the 1st Meeting on Cosmogony*, April 16–19, 1951. Moscow: Publishing House of the USSR Academy of Sciences, 1951, pp. 10–32 [in Russian].
6. Schmidt, O.Yu. *Origin of Earth and Planets*. Moscow: Publishing House of the USSR Academy of Sciences, 1962 [in Russian].
7. ter Haar, D. and Cameron, A.G.W. Historical review of theories of the origin of the solar system. In: *Origin of the Solar system*, edited by R. Jastrow and A.G.W. Cameron. New York: Academic Press, 1963, pp. 1–37.
8. Nieto, M.M. *The Titius–Bode Law of Planetary Distances: its History and Theory*. Oxford, New York: Pergamon, 1972.
9. Alfvén, H. and Arrhenius, G. *Evolution of the Solar System*. Washington: Sci. and tech. inform. office of NASA, 1976.

10. Cameron, A.G.W. Origin of the solar system. *Annual Review of Astronomy and Astrophysics*, 1988, vol. 26, pp. 441–472.
11. Brush, S.G. Theories of the origin of the solar system 1956–1985. *Review of Modern Physics*, 1990, vol. 62(1), pp. 43–112.
12. Vityazev, A.V., Pechernikova, G.V., and Safronov, V.S. *Terrestrial Planets: Origin and Early Evolution*. Moscow: Nauka, 1990 [in Russian].
13. Lissauer, J.J. Planet formation. *Annual Review of Astronomy and Astrophysics*, 1993, vol. 31, pp. 129–174.
14. Woolfson, M.M. *The Origin and Evolution of the Solar System*. Bristol and Philadelphia: Institute of Physics Publishing, 2000.
15. De Oliveira Neto, M., Maia, L.A., and Carneiro, S. An alternative theoretical approach to describe planetary systems through a Schrödinger-type diffusion equation. *Chaos, Solitons & Fractals*, 2004, vol. 21, pp. 21–28.
16. Krot, A.M. A statistical approach to investigate the formation of the solar system. *Chaos, Solitons & Fractals*, 2009, vol. 41(3), pp.1481–1500; doi:10.1016/j.chaos.2008.06.014
17. Birkeland, O.K. Sur l'origine des planetes et de leurs satellites. *Akad. Sci.* (*Paris*)*, Compt. Rend.*, 1912, vol. 155, pp. 892–895.
18. Berlage, H.P., Jr. The disc theory of the origin of the solar system. *Proc. Koninkl. Ned. Akad. Wetenschap.* (*Amsterdam*), 1948, vol. 51, pp. 796–806.
19. Alfvén, H. and Arrhenius, G. Structure and evolutionary history of the Solar System. I. *Astrophysics and Space Science*, 1970, vol.8, pp. 338–421.

20. Alfvén, H. and Arrhenius, G. Origin and evolution of the Solar System. II. *Astrophysics and Space Science*, 1970, vol. 9, pp. 3–33.
21. Schmidt, O.Yu. Meteorite theory of the origin of Earth and planets. *Dokl. of the USSR Academy of Sciences*, 1944, vol. 45(6), pp. 245–249 [in Russian].
22. Gurevich, L.E. and Lebedinsky, A.I. Gravitational condensation of a dust cloud. *Dokl. of the USSR Academy of Sciences*, 1950, vol. 74 (4), pp. 673–676 [in Russian].
23. Gurevich, L.E. and Lebedinsky, A.I. On the planet formation. I. Gravitational condensation; II. Planetary distance law and rotation of planets; III. Structure of initial cloud and separation of planets by outer and inner ones. *Izv. of the USSR Academy of Sciences*, Ser. Fiz., 1950, vol. 14(6), pp. 765–775, pp.776–789, pp. 790–799 [in Russian].
24. Woolfson, M.M. A capture theory of the origin of the solar system. *Proc. Roy. Soc.* (*London*), 1964, vol. A282, pp. 485–507.
25. Dole, S.H. Computer simulation of the formation of planetary systems. *Icarus*, 1970, vol. 13, pp. 494–508.
26. von Weizsäcker, C.F. Uber die Enstehung des Planetensystems. *Z. fur Astrophy*s., 1943, Bd. 22, pp. 319–355.
27. von Weizsäcker, C.F. Zur Kosmogonie. *Z. fur Astrophys.*, 1947, Bd. 24, pp. 181–206.
28. Kuiper, G.P. The law of planetary and satellite distances. *Astrophys. J.*, 1949, vol. 109, pp. 308–313.
29. Kuiper, G.P. On the origin of the solar system. In: *Astrophysics*, edited by J.A. Hynek. New York, 1951, ch.8, p. 357.
30. Hoyle F. On the origin of the solar nebula. *Quart. J. R. Astron. Soc*., 1960, vol. 1, pp. 28–55.

31. Hoyle, F. Formation of the planets. In: *Origin of the Solar system*, edited by R. Jastrow and A.G.W. Cameron. New York: Academic Press, 1963, p. 63.
32. ter Haar D. Some remarks on solar nebula types theories of the origin of the solar system. In: *Origin of the Solar System*. Paris: CNRS, 1972, pp. 71–79.
33. Nakano, T. Origin of the solar system. *Progr. Theor. Phys*., 1970, vol. 44, pp. 77–98.
34. Nelson, E. Derivation of the Schrödinger equation from Newtonian mechanics. *Physical Review*, 1966, vol. 150(4), pp. 1079–1085.
35. Nelson, E. *Quantum Fluctuations*. Princeton (NJ): Princeton University Press, 1985.
36. Nottale L. *Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity*. Singapore: World Scientific, 1993. – 311p.
37. Nottale, L., El Naschie, M.S., Athel, S., and Ord, G. Fractal space-time and Cantorian geometry in quantum mechanics. *Chaos, Solitons & Fractals*, 1996, vol. 7(6) [a special issue].
38. Ord, G.N. Classical particles and the Dirac equation with an electromagnetic force. *Chaos, Solitons & Fractals*, 1997, vol. 8, pp. 727–741.
39. De Oliveira Neto, M. *Ciência e Cultura* (*J Brazil Assoc Advance Sci*), 1996, vol. 48, p. 166.
40. Agnese, A.G. and Festa, R. Clues to discretization on the cosmic scale. *Phys. Lett. A*, 1997, vol. 227, pp. 165–171.
41. *Quantum Mechanics, Diffusion and Chaotic Fractals*, edited by M.S. El Naschie, E. Rössler, and I. Prigogine. Oxford: Pergamon Press, 1995.
42. El Naschie, M.S. Quantum gravity from descriptive set theory. *Chaos, Solitons & Fractals*, 2004, vol. 19, pp. 1339–1344.

43. Sidharth, E.G. *The Chaotic Universe*. New York: Nova Science, 2001.
44. Zharkov, V.N., Trubitsyn, V.P., and Samsonenko, L.V. Physics of the Earth and Planets: Figures and Interior Structures. Moscow: Nauka, 1971 [in Russian].
45. Krot, A.M. The statistical model of gravitational interaction of particles. *Usp. Sovr. Radioelektron.* (special issue "Cosmic Radiophysics", Moscow, Russia), 1996, vol. 8, pp. 66–81 [in Russian].
46. Krot, A.M. Use of the statistical model of gravity for analysis of nonhomogeneity in earth surface. *Proceedings of the SPIE's 13th Annual International Symposium "AeroSense"*, Orlando, Florida, USA, April 5-9, 1999. Orlando: SPIE Press, 1999, vol. 3710, pp. 1248–1259; https://doi.org/10.1117/12.357004
47. Krot, A.M. Statistical description of gravitational field: a new approach. *Proceedings of the SPIE's 14th Annual International Symposium "AeroSense"*, Orlando, Florida, USA, April 24–28, 2000. Orlando: SPIE Press, 2000, vol. 4038, pp. 1318-1329; https://doi.org/10.1117/12.396216
48. Krot, A.M. Gravidynamical equations for a weakly gravitating spheroidal body. *Proceedings of the SPIE's 15th Annual International Symposium "AeroSense"*, Orlando, Florida, USA, April 16-20, 2001. Orlando: SPIE Press, 2001, vol. 4394, pp. 1271–1282; https://doi.org/10.1117/
49. Krot AM. The model of the slow-gravitating spheroidal body based on a statistical approach. In: *Proceedings of the 2nd World Conference on Mathematics & Computers in Physics* (MCP 2000), Vouliagmeni, Athens, Greece; July 10-15, 2000, pp. 5181–5188.
50. Krot, A.M. Development of gravidynamical equations for a weakly gravitating body in the vicinity of absolute

zero temperature. *Proceedings of the 53rd International Astronautical Congress (IAC) – The 2nd World Space Congress-2002*, Houston, Texas, USA, October 10-19, 2002/ Houston, 2002; Preprint IAC-02-J.P.01.

51. Krot, A.M. A quantum mechanical approach to the description of a gravitating body. *Meeting Abstracts of the 34th Scientific Assembly of the Committee on Space Research (COSPAR) – The 2nd World Space Congress-2002*, Houston, Texas, USA, October 10-19, 2002, abstract id.12.
52. Krot, A.M. The kinetic equations for a rotating and gravitating spheroidal body. *Geophysical Research Abstracts of the EGS-AGU-EUG Joint Assembly*, Nice, France, April 6–11, 2003, vol.5, EAE03-A-05568.
53. Krot, A.M. The equations of movement of rotating and gravitating spheroidal body. *Proceedings of the* 54th *International Astronautical Congress (IAC)*, Bremen, Germany, September 29-October 3, 2003/ Bremen, 2003, pp. 2361–2371 (Published Online: 29 Nov 2012; https://doi.org/10.2514/6.IAC-03-J.1.08).
54. Krot, A.M. The statistical model of rotating and gravitating spheroidal body from the point of view of general relativity. *Meeting Abstracts of the 35th COSPAR Scientific Assembly*, Paris, France, July 18–25, 2004, Abstract-Nr.: COSPAR 04-A-00162.
55. Krot, A.M. A statistical model of beginning rotation of gravitating body. *Geophysical Research Abstracts of the 2nd EGU General Assembly*, Vienna, Austria, April 24–29, 2005, vol.7, EGU05-A-04550, SRef-ID: 1607-7962/gra/.
56. Krot, A.M. The statistical approach to exploring the formation of the Solar system. *Geophysical Research Abstracts of the EGU General Assembly*, Vienna,

Austria, April 2–7, 2006, vol. 8, EGU06-A-00216, SRef-ID: 1607-7962/gra/.
57. Krot, A.M. Non-equilibrium distribution functions for the beginning rotation of a cosmological body. *Geophysical Research Abstracts of the EGU General Assembly*, Vienna, Austria, April 2-7, 2006, vol.8, EGU06-A-00219.
58. Krot, A.M. The statistical model of origin and evolution planets of the Solar system and planetary satellites. *Planetary Research Abstracts of European Planetary Science Congress*, Berlin, Germany, September 18-22, 2006, ESPC2006-A-00014, p.14.
59. Krot, A.M. On the principal difficulties and ways to their solution in the theory of gravitational condensation of infinitely distributed dust substance. *Book of Abstracts of the XXIV IUGG General Assembly, GS002 Symposium "Gravity Field"*, Perugia, Italy, July 2–13, 2007, Abstract GS002-3598, pp. 143–144.
60. Krot, A.M. The equation for the evolution of the distribution function of a gas-dust proto-planetary cloud and its application for Solar system formation modeling. *Geophysical Research Abstracts of the EGU General Assembly*, Vienna, Austria, April 16-20, 2007, vol.9, EGU07-A-00271.
61. Krot, A.M. Investigation of spherical waves propagation in a slow-flowing gravitational compressed spheroidal body. *Geophysical Research Abstracts of the EGU General Assembly*, Vienna, Austria, April 13-18, 2008, vol.10, EGU2008-A-00423.
62. Krot, A.M. A gravitational potential finding for rotating cosmological body in the context of proto-planetary dynamics problem solving. *Planetary Research Abstracts of 3rd European Planetary Science Congress*, Münster, Germany, September 22–26, 2008, vol.3,

EPSC2008-A-00077; http://meetings.copernicus.org/epsc2008, p.794
63. Krot, A.M. Becoming angular momentum density flow through nonlinear mass transfer into a gravitating spheroidal body. *Geophysical Research Abstracts of the EGU General Assembly*, Vienna, Austria, April 19–24, 2009, vol.11, EGU2009-A-453.
64. Krot, A.M. A slow-flowing process of initial gravitational condensation of a protoplanetary cloud from the point of view of quantum mechanical interaction of particles. *EPSC Abstracts of 4th European Planetary Science Congress*, Potsdam, Germany, September 13–18, 2009, vol. 4, EPSC2009-97.
65. Krot, A.M. On the principal difficulties and ways to their solution in the theory of gravitational condensation of infinitely distributed dust substance. In: *Observing our Changing Earth*, edited by M.G. Sideris. Berlin, Heidelberg, etc.: Springer, 2009, vol.133, pp. 283–292.
66. Krot, A.M. Self-organization processes in a slow-flowing gravitational compressible cosmological body. In: *Topics on Chaotic Systems*, edited by C.H. Skiadas, I. Dimotikalis, C.Skiadas. Singapore, New Jersey, etc.: World Scientific, 2009, pp. 190–198.
67. Krot, A. M. Development of a quantum mechanical approach to the description of initial gravitational interactions using the statistical theory of spheroidal bodies. *Proceedings of the ESF-FWF Conference in Partnership with LFUI "Putting our Solar System in Context: Origin, Dynamical and Physical Evolution of Multiple Planet Systems"*, Universitatszentrum Obergurgl, Austria, April 25–30, 2010.
68. Krot, A.M. A quantum mechanical approach to the description of initial gravitational interactions based on

the statistical theory of spheroidal bodies. *Nonlinear Science Letters A*, 2010, vol. 1(4), pp. 329–369.
69. Krot, A.M. A model of forming planetary orbits in the Solar system based on the statistical theory of spheroidal bodies. *Book of Abstracts of Chaotic Modeling and Simulation International Conference (CHAOS 2011)*, Chania, Crete, Greece, May 31–June 3, 2011, pp. 69–70.
70. Krot, A.M. Models of the planet's origin and distribution of planetary distances and orbits in the Solar System based on the statistical theory of spheroidal bodies. *EPSC Abstracts of the European Planetary Science Congress (EPSC)-DPS Joint Meeting*, La Cité Internationale des Congrès, Nantes Métropole, Nantes, France, October 2-7, 2011, vol. 6, EPSC–DPS2011-466.
71. Krot, A.M. A nonlinear Schrödinger-like equation in the statistical theory of spheroidal bodies. *Chaotic Modeling and Simulation* (*CMSIM*), 2012, vol. 2(1), pp. 67–80.
72. Krot, A.M. A model of forming planets and distribution of planetary distances and orbits in the Solar system based on the statistical theory of spheroidal bodies. In: *Solar System: Structure, Formation, and Exploration*, edited by M. de Rossi. New York: Nova Science, 2012, ch. 9, pp. 201–264.
73. Krot, A.M. *Statistical Theory of Formation of Gravitating Cosmogonical Bodies*. Minsk: Bel. Navuka, 2012 [in Russian].
74. Krot, A.M. An explanation of the forms of planetary orbits with regards to Alfvén's oscillating force and estimation of the angular displacement of the perihelion of Mercury based on the statistical theory of gravitating cosmogonical bodies. *Proceedings of the IV Congress of Physicists of Belarus*, Minsk, April 24–26, 2013. Minsk: Kovcheg, 2013, pp. 43–44 [in Russian].

75. Krot, A.M. On the universal stellar law. *Meeting Abstracts of the 40th COSPAR Scientific Assembly*, Moscow, Russia, August 2-10, 2014, E1.14-0032-14.
76. Krot, A.M. On the universal stellar law for extrasolar systems. *Planetary and Space Science*, 2014, vol.101C, pp. 12–26; http://dx.doi.org/10.1016/j.pss.2014.05.002.
77. Krot, A.M. Development of the generalized nonlinear Schrödinger equation of rotating cosmogonical body formation. In: *Complex Systems: Theory and Applications*, edited by R. Martinez. New York: Nova Science Publishers, 2017, ch. 3, pp. 49–94.
78. Krot, A.M. Derivation and investigation of the generalized nonlinear Schrödinger equation of cosmogonical body forming. In: *Mathematical and Numerical Aspects of Dynamical System Analysis*, edited by J. Awrejcewicz, M. Kazmierczak, J. Mrozowski, P. Olejnik. Łódź: ARSA Druk i Reklama, 2017, pp. 277–288.
79. Krot, A.M. An explanation of the stability of extrasolar systems based on the universal stellar law. *Chaotic Modeling and Simulation* (*CMSIM*), 2017, vol. 7(4), pp. 513–529.
80. Newton, I. *Principia Mathematica Philosophia Naturalis*, 1st ed. London: Streater, 1686 (reprinted by Berkeley (CA): California: University of California Press, 1934).
81. Einstein, A. *The Meaning of Relativity*, 1st ed. Princeton, New York: Princeton University Press, 1921.
82. Turok, N.J. Global texture as the origin of cosmic structure. In: *Proceedings of the Nobel Symposium no.79 "The Birth and Early Evolution of our Universe"*, edited by J.S. Nilsson, B. Gustafsson and B.-S. Skagerstam. World Scientific/ Physica Scripta, 1991.

83. Spergel, D.N. and Turok, N.J. Textures and cosmic structure. *Scientific American*, 1992, vol. 266(3), pp. 52–59.
84. Halliwell, J.J. Quantum cosmology and the creation of the Universe. *Scientific American*, 1991, December, pp.76–85.
85. Guth, A.H. Fluctuations in the new inflationary universe. *Physical Review Letters*, 1982, vol. 49(15), pp. 1110–1113.
86. Guth, A.H. and Steinhardt, P.J. The inflationary Universe. *Scientific American*, 1984, vol. 250(5), pp. 116–129.
87. Cen, R.Y., Ostriker, J.P., Spergel, D.N., and Turok, N.J. A hydrodynamic approach to cosmology: Texture-seeded CDM and HDM cosmogonies. *Astrophysical Journal*, 1991, vol. 383(1), pp. 1–18.
88. Durrer, R., Gangui, A. and Sakellariadou, M. Doppler peaks in the angular power spectrum of the cosmic microwave background: A fingerprint of topological defects. *Phys. Rev. Lett.*, 1996, vol.76, pp. 579–582.
89. Boylan-Kolchin, M., Springel, V., White, S.D.M., Jenkins, A., and Lemson, G. Resolving cosmic structure formation with the Millennium-II simulation. *Mon. Not. Roy. Astron. Soc.*, 2009, vol. 398, pp. 1150–1166.
90. Springel, V., White, S.D.M., Jenkins, A., et al. Simulations of the formation, evolution and clustering of galaxies and quasars. *Nature*, 2005, vol. 435, pp. 629–636.
91. Einasto, J., Einasto, M., Tago, E., et al. Superclusters of galaxies from the 2dF redshift survey. I. The catalogue. *Astronomy & Astrophysics*, 2007, vol. 462(2), pp. 811–825.

92. Steinhardt, P.J. Inflation debate: Is the theory at the heart of modern cosmology deeply flawed? *Scientific American*, 2011, vol. 304(4), pp. 36–43.
93. Jeans, J. *Problems of Astronomy and Stellar Dynamics*. Cambridge: University Press, 1919.
94. Landau, L.D. and Lifschitz, E.M. *Fluid Mechanics*. Oxford: Pergamon, 1959.
95. Sretenskyi, L.N. *Theory of Newtonian Potential*. Moscow–Leningrad: Gostekhizdat, 1946 [in Russian].
96. Moulton, F.R. *An Introduction to Celestial Mechanics*. New York, London: Macmillan Co., 1914.
97. Grushinskyi, N.P. *Principles of Gravimetric.* Nauka: Moscow, 1983 [in Russian].
98. Dive, P. Attraction des ellipsoides homogènes et réciproque d'un théoreme de Newton. *Bulletin de la Société Mathématique de France*, 1931, t. LIX.
99. Duboshin, N.G. *Celestial Mechanics: Main Tasks and Methods*. Moscow: Nauka, 1968 [in Russian].
100. Landau, L.D. and Lifschitz, E.M. *Classical Theory of Fields*. Reading (MA): Addison–Wesley Publishing, Co., 1951.
101. Weinberg, S. *The First Three Minutes: A Contemporary View on the Origin of the Universe*. Cambridge, Massachusetts, 1977.
102. Einstein, A. Kosmologiche Betrachtungen zur allgemeinen Relativitätstheorie. *Sitzungsber. d. Berl. Akad.*, 1917, Bd. 1, p. 142.
103. Chernin, A.D. Cosmic vacuum. *Usp. Fiz. Nauk* , 2001, vol.171(11), pp. 1153–1175 [in Russian].
104. Weinberg, S. The cosmological constant problem. *Reviews of Modern Physics*, 1989, vol. 61, pp. 1– 23.
105. Poincaré, H. *Lecons sur les Hypothèses Cosmogoniques*, §74. Paris, 1911.

106. Chandrasekhar, S. *An Introduction to the Study of Stellar Structure*. Cambridge: University Press, 1939.
107. Dvorak, R. and Lhotka, C. *Celestial Dynamics: Chaoticity and Dynamics of Celestial Systems*. Hamburg, Weinheim: Wiley-VCH, 2013.
108. Glansdorff, P. and Prigogine, I. *Thermodynamic Theory of Structure, Stability and Fluctuations*. London, 1971.
109. Eddington, A.S. On the radiative equilibrium of the stars. *Mon. Not. Roy. Astron. Soc.*, 1916, vol. 84(7), pp. 525–528.
110. Landau, L.D. and Lifschitz, E.M. *Statistical Physics*, part 1. Reading (MA): Addison–Wesley Publishing Co., 1955.
111. Loytsyanskyi, L.G. *Mechanics of Fluid and Gas*. Moscow: Nauka, 1973 [in Russian].
112. Chandrasekhar, S. The gravitational instability of an infinite homogeneous medium when a Coriolis acceleration is acting. *Vistas in Astronomy*, 1955, vol. 1, p. 344.
113. Bel, N. and Schatzman, E. On the gravitational instability of a medium in nonuniform rotation. *Rev. Mod. Phys*., 1958, vol. 30, pp. 1015–1023.
114. Safronov, V.S. On the formation and evolution of protoplanetary dust bunches. *Cosmogony Issues*, 1960, vol. 7, pp. 121–141 [in Russian].
115. Genkin, I.L. and Safronov, V.S. Instability of rotating and gravitating system with radial disturbances. *Astrophysical Journal*, 1975, vol. 52(2), pp. 306–315.
116. Toomre, A. On the gravitation stability of a disk of stars. *Astrophys. J.*, 1964, vol. 139, pp. 1217–1238.
117. Goldreich, P. and Lynden-Bell, D. Spiral arms and sheared gravitational instabilities. *Mon. Not. Roy. Astron. Soc*., 1965, vol. 130, pp. 125–158.

118. Goldreich, P. and Ward, W.R. The formation of planetesimals. *Astrophys. J.*, 1973, vol. 183, pp. 1051–1062.
119. Bergé, P., Pomeau, Y. et Vidal, C. *L'ordre dans le chaos: Vers une approche déterministe de la turbulence.* Paris: Hermann, 1988.
120. Rayleigh. On the dynamics of revolving fluids. *Sci. Papers*, vol. 6. Cambridge: University Press, 1916, p. 197.
121. Taylor, G.I. Stability of a viscous liquid contained between two rotating cylinders. *Philos. Trans. Roy. Soc. London*, ser. A, 1923, vol. 223, pp. 289–293.
122. Chandrasechar, S. The stability of viscous flow between rotating cylinders. *Proc. Roy. Soc. London,* ser. A, 1958, vol. 246, p. 301.
123. Donnelly, R.J. Experiments on the stability of viscous flow between rotating cylinders. I. Torque measurements. *Proc. Roy. Soc. London*, ser. A, 1958, vol. 246, p. 312.
124. Donnelly, R.J. and Fultz, D. Experiments on the stability of viscous flow between rotating cylinders. II. Visual observations. *Proc. Roy. Soc. London*, ser. A, 1960, vol. 258, p. 101.
125. Chandrasekhar, S. *The Principles of Stellar Dynamics*. Chicago: University of Chicago Press, 1943.
126. von Weizsäcker, C.F. Rotation kosmisher Gasmassen. *Z. Naturforsch*., 1948, Bd. 3a, p. 524.
127. Safronov, V.S. and Ruskol, E.L. On the hypothesis of turbulence in a protoplanetary cloud. *Cosmogony Issues*, 1957, vol. 5, p.22 [in Russian].
128. Korn, G.A. and Korn, T.M. *Mathematical Handbook for Scientists and Engineers*, 2nd ed. Mc Graw–Hill Book Co., 1968.

129. von Lichtenstein, L. *Gleichgewichtsfiguren Rotierender Flüssigkeiten*. Berlin: Verlag von Julius Springer, 1933.
130. Jardetzky, W.S. *Theories of Figures of Celestial Bodies*. Mineola, New York: Dover Publications, Inc., 2005.
131. Laplace, P. S. *Oeuvres Complètes*. Paris: Gauthier-Villars, 1895.
132. Lifschitz, E.M. On the gravitational stability of the expanding Universe. *Zh. Eksp. Teor. Fiz.*, 1946, vol. 16, pp. 587–602 [reprinted by *J. Phys.*, 1946, vol.10, p. 116].
133. Bonnor, W.B. Jeans' formula for gravitational instability. *Monthly Notices Roy. Astron. Soc.*, 1957, vol. 117, p. 104.
134. Nicolis, G. and Prigogine, I. *Self-organization in Nonequilibrium Systems: From Dissipative Structures to Order through Fluctuation*. New York: John Willey & Sons, 1977.
135. Nicolis, G. and Prigogine, I. *Exploring Complexity: An Introduction*. New York: W.H. Freeman and Co., 1989.
136. Kikoin, A.K. and Kikoin, I.K. *Molecular Physics*. Nauka: Moscow, 1976 [in Russian].
137. *Proceedings of the Symposium on the Origin of the Solar System*, edited by H. Reeves. Paris: Edition du Centre National de la Recherche Scientifique, 1972.
138. Simon, R. Gravitational instability in a gaseous medium in non-uniform rotation. *Ann. d' Astrophys.*, 1962, vol. 25, p. 405.
139. Weindenschilling, S. Can gravitational instability form planetesimals? *Icarus*, 1995, vol. 116, pp. 433–435.
140. Ledoux, P. On the radial pulsation of gaseous stars. *Astrophys. J.*, 1945, vol. 102, pp.143–153.
141. Sanders, D.B., Solomon, P.M., and Scoville, N.Z. Giant molecular clouds in the Galaxy. I. The axisymmetric

distribution of H_2. *Astrophys. J.*, 1984, vol. 276, pp. 182–203.
142. Tscharnuter, W.M. 1D, 2D and 3D collapse of interstellar clouds. *Space Sci. Rev.*, 1980, vol. 27, pp. 235–246.
143. Larson, B. The formation of solar-type stars. In: *Protostars and Planets*, edited by T. Gehrels. Tucson: University of Arizona Press, 1978, pp. 53–70.
144. Bodenheimer, P. and Black, D.C. Numerical calculations of protostellar hydrodynamic collapse. In: *Protostars and Planets*, edited by T. Gehrels. Tucson: University of Arizona Press, 1978, pp. 288–327.
145. Boss, A.P. Protostellar formation in rotating interstellar clouds. IV. Nonisothermal collapse. *Astrophys. J.*, 1984, vol.277, pp. 768–782.
146. Norman, M. and Wilson, J. The fragmentation of isothermal rings and stars formation. *Astrophys. J.*, 1978, vol. 224(2), pp. 497–511.
147. Safronov, V.S. and Ruzmaikina, T.V. On the transfer of angular momentum and accumulation of solids in a solar nebula. In: *Protostars and Planets*, edited by T. Gehrels. Tucson: University of Arizona Press, 1978, pp. 623–644.
148. Lyttleton, R.A. *The Stability of Rotating Liquid Masses*. Cambridge: University Press, 1953.
149. Shakura, N. and Syunyaev, R. Black holes in binary systems: Observational appearance. *Astronomy & Astrophysics*, 1973, vol. 24(2), pp. 327–335.
150. Lynden-Bell, D. and Pringle, J.E. The evolution of viscous disks and the origin of nebular variables. *Mon. Not. Roy. Astron. Soc.*, 1974, vol. 168, pp. 603–637.
151. Cassen, P.M. and Summers, A. Models of the formation of the solar nebula. *Icarus*, 1983, vol. 53(1), pp. 26–40.

152. Safronov, V.S. and Ruzmaikina, T.V. Formation of the solar nebula and the planets. In: *Protostars and Planets* II, edited by D.C. Black and M.S. Matthews. Tucson: University of Arizona Press, 1985, pp. 959–980.
153. Lin, D.N.C. and Papaloizou, J. C. B. On the structure and evolution of the primordial solar nebula. *Mon. Not. Roy. Astron. Soc*., 1980, vol. 191, pp. 37–48.
154. Lin, D.N.C. and Papaloizou, J. C. B. On the dynamical origin of the solar system. In: *Protostars and planets* II, edited by D.C. Black and M.S. Matthews. Tucson: University of Arizona Press, 1985, pp. 981–1072.
155. Parenago, P.P. *Lectures on Stellar Astronomy*. Moscow: Gosizdat, 1954 [in Russian].
156. Ogorodnikoff, K.F. On the kinematics of the Metagalaxy. *Cosmogony Issues*, vol. I. Moscow: Publishing House of the USSR Academy of Sciences, 1952 [in Russian].
157. Oort, J.H. Absorption and density distribution to the galactic system. *Bull. of the Astr. Inst. of the Netherlands*, 1938, vol. 8, p. 308.
158. Landau, L.D. and Lifschitz, E.M. *Mechanics*, vol.I. Moscow: Nauka, 1973 [in Russian].
159. Lifschitz, E.M. and Pitaevsky, L.P. *Physical Kinetics*. Moscow: Nauka, 1979 [in Russian].
160. Jeans, J. *The Dynamical Theory of Gases*. London: Dover Publ., 1954.
161. Prigogine, I. and Severne, G. On the statistical mechanics of a gravitational plasma, *Physica*. 1966, vol. 32, p. 1376.
162. Marochnik, L. S. *Soviet Astron. AJ*, 1968, vol. 11, p. 873 [in Russian].
163. Pomeau, Y. Time reversal symmetry of fluctuations. *J. de Phys*. (Paris), 1982, vol. 43, p. 859.

164. Bashkirov, A. G. and Vityazev, A. V. Screening of gravitational interactions and its astrophysical manifestations. *Astrophysical Journal*, 1998, vol. 497 (April 10), pp.10–15.
165. Chabanol, M.L., Corson, F., and Pomeau, Y. Statistical mechanics of point particles with a gravitational interaction. *Europhys. Lett*., 2000, vol. 50, pp. 148.
166. Pomeau, Y. Statistical Mechanics of Gravitational Plasmas: the Effect of Close Interactions. Tucson, AZ: University of Arizona, Department of Mathematics, USA (June 4, 2011).
167. Zamora, D. J. Rocca1, M. C. Plastino, A., and Ferri, G. L. Gravitational Boltzmann–Gibbs Classical Distribution (October 9, 2017).
168. Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: John Wiley & Sons, 1972.
169. Misner, C.W., Thorne, K.S., and Wheeler, J.A. *Gravitation*. San Francisco: W.H. Freeman and Co., 1973.
170. Hawking, S.W. Particle creation by black holes. *Commun. Math. Phys*., 1975, vol. 43(3), pp. 199–220.
171. Brans, C. and Dicke, R.H. Mach's principle and relativistic theory of gravitation. *Physical Review*, 1961, vol. 124(3), pp. 925–935.
172. Logunov, A.A. and Mestvirishvili, M.A. *Fundamentals of the Relativistic Theory of Gravity*. Moscow: Moscow State University, 1985 [in Russian].
173. Logunov, A.A., Mestvirishvili, M.A., and Chugreev, Yu.V. The mass of graviton and the development of the Friedman universe. Moscow: Moscow State University, 1987 [in Russian].
174. Logunov, A.A. and Mestvirishvili, M.A. *Relativistic Theory of Gravity*. Moscow: Nauka, 1989 [in Russian].

175. Nottale, L. Scale relativity, fractal space-time, and quantum mechanics. *Chaos, Solitons & Fractals*, 1994, vol. 4, pp. 361–388.
176. Nottale, L. Scale-relativity and quantization of the universe: I. Theoretical framework. *Astronomy & Astrophysics*, 1997, vol. 327, pp. 867–889.
177. Koshkin, N.I. and Shirkevich, M.G. *A Handbook of Elementary Physics*. Moscow: Nauka, 1965 [in Russian].
178. Darwin, G.H. On the mechanical conditions of a swarm of meteorites, and on theories of cosmogony. *Phil. Trans. Roy. Soc.*, 1889, vol. 180A, p. 169.
179. Schuster. *British Association Report*, 1883, p. 428.
180. Roche. Essai sur la constitution et l'origine du Système solaire. *Acad. de Montpellier*, *Sciences*, 1873, vol. VIII, p. 235.
181. Tolman, R.C. *Relativity, Thermodynamics and Cosmology*. Oxford: Clarendon Press, 1934.
182. Zeldovich, Ya.B. and Novikov, I.D. *Structure and Evolution of the Universe*. Moscow: Nauka, 1975 [in Russian].
183. Eddington, A.S. *The Mathematical Theory of Relativity*, 2nd ed. Cambridge: Cambridge University Press, 1924.
184. Oppenheimer, J.R. and Snyder, H. On Unlimited Gravitational Compression. *Physical Review*, 1939, vol. 56, p. 455.
185. Krot, A.M. Development of the statistical theory of forming cosmogonical bodies to explain the stability of the orbital movements of planets and the forms of planetary orbits. *Proceedings of the 2nd International Conference and Exhibition on Satellite & Space Missions*, Berlin, Germany, July 21–23, 2016/ *Journal of Aeronautics & Aerospace Engineering*, 2017, vol.6(2), p. 45;

https://dx.doi.org/10.4172/2168-9792.C1.013.
186. Stacey, F.D. *Physics of the Earth*. New York, London, etc.: John Wiley & Sons, 1969.
187. Schrödinger, E. An undulatory theory of the mechanics of atoms and molecules. *Physical Review*, 1926, vol. 28(6), pp. 1049–1070.
188. Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., and Morris, H.C. *Solitons and Nonlinear Wave Equations*. London: Academic Press, 1984.
189. Filippov, A.T. *The Versatile Soliton*. Modern Birkhauser Classics, 2000.
190. van Kampen, N.G. *Stochastic Processes in Physics and Chemistry*. Amsterdam: North-Holland, 1984.
191. Haken, H. *Advanced Synergetics: Instability Hierarchies of Self-Organizing Systems and Devices*. Berlin etc.: Springer-Verlag, 1983.
192. Tikhonov, A.N. and Samarsky, A.A. *Equations of Mathematical Physics*. Moscow: Publishing House of Moscow State University, 1999 [in Russian].
193. Ebeling, W. *Origin of Structures at Irreversible Processes: An Introduction to the Theory of Dissipative Structures*. Rostock, 1977.
194. Prigogine, I. Rediscovery of time. *Problems of Philosophy*, 1989, no. 8, pp. 3–19 [in Russian].
195. Simakov, K.V. An essay on the history of "Rediscovery of time". *Bulletin of the Russian Academy of Sciences*, 1995, vol. 65(6), pp. 502–512 [in Russian].
196. Prigogine, I. *From the Being to Becoming – Time and Complexity in the Physical Sciences*. San Francisco: W.H. Freeman & Co., 1980.
197. Silin, A.A. On the nature of time. *Bulletin of the Russian Academy of Sciences*, 1995, vol. 65(2), pp. 140–148 [in Russian].

198. Silin, A.A. Entropy, probability, information. *Bulletin of the Russian Academy of Sciences*, 1994, vol. 64(6), pp. 490–496 [in Russian].
199. Correspondence of A. Einstein with M. Besso (1903–1955). In: *Einstein's collection*. Moscow: Nauka, 1980 [in Russian].
200. Nottale, L., Schumacher, G., and Gay, J. Scale relativity and quantization of the solar system. *Astronomy & Astrophysics*, 1997, vol. 322, pp. 1018–1025.
201. Nottale, L., Schumacher, G., and Lefèvre, E.T. Scale-relativity and quantization of exoplanet orbital semi-major axes. *Astronomy&Astrophysics*, 2000, vol. 361, pp. 379–387.
202. Haken, H. and Wolf, H.C. The physics of atoms and quanta. Introduction to experiments and theory; translated by W.D. Brewer. Berlin: Springer–Verlag, 1994.
203. Davydov, A.S. *Quantum Mechanics*. Moscow: Nauka, 1973 [in Russian].
204. Bohm, A. *Quantum Mechanics: Foundation and Applications*. New York: Springer–Verlag, 1993.
205. Krot, A.M. On the theory of wave gravitational field arising under the orbital motion of a gravitating body. *Proceedings of the 15th International Conference* "*Dynamical Systems–Theory and Applications*" (DSTA-2019), Łódź, Poland, December 2–5, 2019.
206. Javorsky, B.M. and Detlaf, A.A. *A Handbook of Physics for Engineers and Students*. Moscow: PhysMathGiz, 1963 [in Russian].
207. Maeder, A. *Physics, Formation and Evolution of Rotating Stars*. Berlin, Heidelberg: Springer-Verlag, 2009.
208. O'Brien, D.P., Morbidelli, A., and Levinson, H.F. Terrestrial planet formation with strong dynamical friction. *Icarus*, 2006, vol. 184, pp. 39–58.

209. Krot, A.M. The stability of planetary orbits in extrasolar systems based on the spectral representation of an additional periodic force. *Proceedings of the 4th International Conference and Exhibition on "Satellite & Space Missions"*, June 18–20, 2018, Rome, Italy/ Journal of Aeronautics & Aerospace Engineering, 2018, vol.7 (Suppl.).
210. Einstein, A. *Investigations on the Theory of the Brownian Movement*, translated by A.D. Cowper. London: Methuen and Co., 1926.
211. von Smoluchowski, M. *Abhandlungen über die Brownsche Bewegung und verwandte Erscheinungen*. Leipzig: Akademische Verlagsgesellschaft, 1923.
212. Chandrasekhar, S., Uhlenbeck, G.E., and Ornstein, L.S. In: *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax. New York: Dover Publications, 1954.
213. Agop, M., Matsuzawa, H., Oprea, I., Vlad, R., Sandu, C., and Buzea, C.Gh. Some implications of the gravitomagnetic field in fractal space-time theory. *Australian Journal of Physics*, 2000, vol. 53, pp. 217–230.
214. Krot, A.M. Analysis of dynamical states of cosmogonical body formation based on the generalized nonlinear Schrödinger-like equation. *Chaotic Modeling and Simulation* (*CMSIM*), 2019, vol. 9(2), pp. 95–107.
215. Zakharov, V.E. and Shabat, A.B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics J.E.T.P.*, 1972, vol. 34(1), pp. 62–69 [a translation of *Zh. Eksp. Teor. Fiz.*, 1971, vol. 61(July), pp. 118–134].
216. Takhtadjan, L.A. and Faddeev, L.D. *Hamiltonian Approach to Theory of Solitons*. Moscow: Nauka, 1986 [in Russian].

217. Couairon, A. and Mysyrowicz, A. Femtosecond filamentation in transparent media. *Physics Reports*, 2007, vol. 441, pp. 47–189.
218. Lorenz, E. N. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 1963, vol. 20, pp. 130–141.
219. Baldin, V.A., Krot, A.M., and Minervina, H.B. The development of model for boundary layers past a concave wall with usage of nonlinear dynamics methods. *Advances in Space Research*, 2006, vol. 37(3), pp. 501–506.
220. Baldin, V.A., Krot, A.M., and Minervina, H.B. The attractor model for finite length Görtler whirlwinds. *Electromagnetic Waves and Electronic Systems*, 2006, vol.11(2/3), pp. 33-40.
221. Andereck, C.D. et al. Flow regimes in a circular Couette system with independently rotating cylinders. *J. Fluid Mech.*, 1986, vol. 164, pp. 155–183.
222. Görtler, H. Dreidimensionales zur stabilitätstheorie laminarer grenzschichten. *ZAMM*, 1955, Bd. 35, pp. 326–364.
223. Ord, G.N. Fractal space-time and the statistical mechanics of random walks. *Chaos, Solitons & Fractals*, 1996, vol. 7, pp. 821–843.
224. Ord, G.N. and Deakin, A.S. Random walks, continuum limits, and Schrödinger's equation. *Phys. Rev. A* 1996, vol. 54, pp. 3772–3778.
225. Brandt, S. and Dahmen, H.D. *The Picture Book of Quantum Mechanics*. New York, Berlin, etc.: Springer-Verlag, 1994.
226. Kepler, J. *Astronomia Nova*. 1609.
227. Kant, I. *Allgemeine Naturgeschichte und Theorie des Himmels*. Königsberg and Leipzig: Johann Friedrich Petersen, 1755.

228. Laplace, P.S.M. *Traité de Méchanique Céleste.* Paris: Duprat J.B.M., 1789 (*Celestial Mechanics*, reprinted by Chelsea Publ. Co., New York, 1966).
229. Blagg, M.A. On a suggested substitute for Bode's law. *Mon. Not. Roy. Astron. Soc.*, 1913, vol. 73, pp. 414–422.
230. Richardson, D.E. Distances of planets from the Sun and of satellites from their primaires in the satellite systems of Jupiter, Saturn, and Uranus. *Pop. Astron*., 1945, vol. 53, pp. 14–26.
231. Lovis, C., Segransan, D., Mayor, M., et al. The HARPS search for southern extra-solar planets – XXVII. Up to seven planets orbiting HD 10180: probing the architecture of low-mass planetary systems. *Astronomy & Astrophysics*, 2011, vol. 528(April). – id.A112.
232. *New Light on Dark Stars: Red Dwarfs, Low-Mass Stars, Brown Dwarfs*, edited by I.N. Reid, S.L. Hawley, 2nd ed. Berlin, Heidelberg: Springer, 2005, p. 495.
233. Agnese, A.G. and Festa, R. Discretizing μ-Andromedae planetary system. 1999, vol. 2. Available from: <astro-ph/9910534>.
234. Arp, H. Seeing red: redshifts, cosmology and academic science. Montreal: Apeiron, 1998, p. 219.
235. De Oliveira Neto, M. Using the dimensionless Newton gravity constant $\overline{\alpha}_G$ to estimate planetary orbits. *Chaos, Solitons & Fractals*, 2005, vol. 24, pp. 19–27.
236. Laskar, J. A numerical experiment on the chaotic behaviour of the Solar System. *Nature*, 1989, vol. 338, pp. 237–238.
237. Sussman, G.J. and Wisdom, J. Chaotic evolution of the Solar System. *Science*, 1992, vol. 257, pp. 56–62.
238. De Oliveira Neto, M. Pythagoras' celestial spheres in the context of a simple model for quantization of planetary orbits. *Chaos, Solitons & Fractals*, 2006, vol. 30, pp. 399–406.

239. Pintr, P., Peřinová, V., and Lukš, A. Allowed planetary orbits in the solar system. *Chaos, Solitons & Fractals*, 2008, vol. 36, pp. 1273–1282.
240. Kenyon, S.J. Planet formation in the outer Solar System. *Publ. of the Astronomical Society of the Pacific*, 2002, vol. 114, pp. 265–283.
241. *Astrobiology: Future Perspectives*, edited by P. Ehrenfreund et al. Kluwer Academic Publishers, 2004, pp. 80, 267.
242. Weidenschilling, S.J. Dust to planetesimals – Settling and coagulation in the solar nebula. *Icarus*, 1980, vol.44, pp. 172–189.
243. Hayashi, C., Nakazawa, K., and Nakagawa, Y. Formation of the solar system. In: *Protostars and Planets* II, edited by D.C. Black and M.S. Matthews. Tucson: University of Arizona Press, 1985, pp. 1155–1176.
244. Weidenschilling, S.J. Formation of planetesimals and accretion of the terrestrial planets. *Space Sci. Rev*., 2000, vol. 92, pp. 295–310.
245. Ward, W.R. On planetesimal formation: The role of collective particle behavior. In: *Origin of the Earth and Moon*, edited by R.M. Canup and K. Righter. Tucson: University of Arizona Press, 2000, pp. 75–84.
246. Youdin, A.N. and Shu, F.H. Planetesimal formation by gravitational instability. *Astrophys. J.*, 2002, vol. 580, pp. 494–505.
247. Weidenschilling, S. J. and Cuzzi, J. N. Formation of planetesimals in the solar nebula. In: *Protostars and Planets* III, edited by E. Levy and J. Lunine. Tucson: University of Arizona Press, 1993, pp. 1031–1060.
248. Wurm, G., Blum, J., and Colwell, J.E. Note: A new mechanism relevant to the formation of planetesimals in the solar nebula. *Icarus*, 2001, vol. 151, pp. 318–321.

249. Kouchi, A., Kudo, T., Nakano, H., Arakawa, M., Watanabe, N., Sirono, S., Higa, M., and Maeno, N. Rapid growth of asteroids due to very sticky interstellar organic grains. *Astrophys. J.*, 2002, vol. 566, pp. L121-L12.
250. Greenberg, R., Hartmann, W.K., Chapman, C.R., and Wacker, J.F. Planetesimals to planets – Numerical simulation of collisional evolution. *Icarus*, 1978, vol. 35, pp. 1–26.
251. Wetherill, G. W. and Stewart, G. R. Accumulation of a swarm of small planetesimals. *Icarus*, 1989, vol. 77, pp. 330–357.
252. Kokubo, E. and Ida, S. On runaway growth of planetesimals. *Icarus*, 1996, vol. 123, pp. 180–191.
253. Weidenschilling, S. J., Spaute, D., Davis, D. R., Marzari, F., and Ohtsuki K. Accretional evolution of a planetesimal swarm. *Icarus*, 1997, vol. 128, pp. 429–455.
254. Makino, J., Fukushige, T., Funato, Y., and Kokubo, E. On the mass distribution of planetesimals in the early runaway stage. *New Astron*., 1998, vol. 3, pp. 411–417.
255. Kortenkamp, S. J., Wetherill, G. W., and Inaba, S. Runaway growth of planetary embryos facilitated by massive bodies in a protoplanetary disk. *Science*, 2001, vol. 293, pp. 1127–1129.
256. Kley, W., D'Angelo, G., and Henning, T. Three-dimensional simulations of a planet embedded in a protoplanetary disk. *Astrophys. J*., 2001, vol. 547, p. 457.
257. Laskar, J. On the spacing of planetary systems. *Phys. Rev. Letters*, 2000, vol. 84(15), pp. 3240–3243.
258. Murray, C.D. and Dermott, S.F. *Solar System Dynamics*. Cambridge: University Press of Cambridge, 1999, p. 592

259. Lynch, P. On the significance of the Titius-Bode law for the distribution of the planets. *Monthly Notice of the Royal Astronomical Society*, 2003, vol. 341(4), pp. 1174–1178.
260. Poveda, A. and Lara, P. The exo-planetary system of 55 Cancri and the Titius–Bode law. *Revista Mexicana de Astronomia y Astrofisica*, 2008, vol. 44, pp. 243–246.
261. Lara, P., Poveda, A., and Allen, C. On the structural law of exoplanetary systems. *AIP Conference Proceedings*, 2012, vol.1479, p. 2356; https://doi.org/10.1063/1.4756667
262. Flores-Gutierrez, J.D. and Garcia-Guerra, C. A variant of the Titius–Bode law. *Revista Mexicana de Astronomia y Astrofisica*, 2011, vol. 47, pp. 173–184.
263. Bailey, J.M. The Moon may be a former planet. *Nature*, 1969, vol. 223, pp. 251–253.
264. Hartmann, W.K. and Davis, D.R. Satellite-sized planetesimals and lunar origin. *Icarus*, 1975, vol. 24(4), pp. 504–515.
265. *Origin of the Moon*, edited by W.K. Hartmann, R.J. Phillips, and G.J. Taylor. Houston: Lunar and Planetary Institute, 1986.
266. Benz, W., Slattery, W.L., and Cameron, A.G.W. The origin of the Moon and the single-impact hypothesis. 1 and 2. *Icarus*, 1986, vol. 66(3), pp. 515–535 and *Icarus*, 1987, vol. 71(1), pp. 30–45.
267. Cameron, A.G.W. The origin of the Moon and the single impact hypothesis. 5. *Icarus*, 1997, vol. 126(1), pp. 126–137.
268. Canup, R.M. and Asphaug, E. Origin of the Moon in a giant impact near the end of the Earth's formation. *Nature*, 2001, vol. 412 (6848), pp. 708–712.
269. Canup, R.M. Simulations of a late lunar-forming impact. *Icarus*, 2004, vol. 168(2), pp. 433–456.

270. *Astrophysics Update: Topical and Timely Reviews on Astrophysics*, edited by J. W. Mason. Springer, 2004.
271. *Chaos and Stability in Planetary Systems*, edited by R. Dvorak and J. Kurths. Berlin, Heidelberg: Springer–Verlag, 2005.
272. Wetherill, G.W. Formation of the terrestrial planets: Numerical calculations, in: *The Continent, Crust and its Mineral Deposits*, Geol. Assoc. of Canada, 1980, pp. 3–24.
273. Levison, H.F., Lissauer, J.J., and Duncan, M.J. Modeling the diversity of outer planetary systems. *Astron. J.*, 1998, vol. 116, pp. 1998–2014.
274. Chambers, J.E. and Wetherill, G.W. Making the terrestrial planets: N-body integrations of planetary embryos in three dimensions. *Icarus*, 1998, vol. 136, pp. 304–327.
275. Chambers, J. E. Making More Terrestrial Planets. *Icarus*, 2001, vol. 152, pp. 205–224.
276. Lee, M. H. A survey of numerical solutions to the coagulation equation. *Journal of Physics A*, 2001, vol. 34, pp.1021.
277. Raymond, S.N., Quinn, T., and Lunine, J.I. Making other earths: Dynamical simulations of terrestrial planet formation and water delivery. *Icarus*, 2004, vol. 168, pp. 1–17.
278. Boss, A.P. Evolution of the solar nebula. I. Nonaxisymmetric structure during nebula formation. *Astrophys. J.*, 1989, vol. 345, pp. 554–571.
279. Lubow, S. H., Seibert, M., and Artymowicz, P. Disk accretion onto high-mass planets. *Astrophys. J.*, 1999, vol. 526 (2), pp. 1001–1012.
280. D’Angelo, G., Henning, T., and Kley, W. Nested-grid calculations of disk-planet interaction. *Astronomy & Astrophysics*, 2002, vol. 385, pp. 647–670.

281. Papaloizou, J.C.B. and Nelson, R. P. The interaction of a giant planet with a disc with MHD turbulence – I; The initial turbulent disc models – II and The interaction of the planet with the disc – III. *Mon. Not. Roy. Astron. Soc*., 2003, pp. 339, 983 and 993.
282. *Planet Formation: Theory, Observations, and Experiments*, edited by H. Klahr and W.Brandner. Cambridge, 2006.
283. The Sun. http://en.wikipedia.org/wiki/Sun
284. Bendat, J.S. and Piersol, A.G. *Random Data: Analysis and Measurement Procedures*. New York, Chichester, etc.: John Wiley&Sons, 1986.
285. Schneider, J. 2019. http://www.obspm.fr/encycl/catalog.html.
286. Léger, A., Rouan, D., Schneider, J., et al. Transiting exoplanets from the CoRoT space mission VIII. CoRoT-7b: the first Super-Earth with measured radius. *Astronomy & Astrophysics*, 2009, vol. 506 (1), pp. 287–302.
287. Queloz, D., Bouchy, F., Moutou, C., et al. The CoRoT-7 planetary system: two orbiting super-Earths. *Astronomy & Astrophysics*, 2009, vol. 506(1), pp. 303–319.
288. Charbonneau, D., Berta, Z.K., Irwin, J., et al. A super-Earth transiting a nearby low-mass star. *Nature*, 2009, vol. 462, pp. 891–894.
289. Mayor, M., Bonfils, X., Forveille, T., et al. The HARPS search for southern extra-solar planets – XVIII. An Earth-mass planet in the GJ 581 planetary system. *Astronomy & Astrophysics*, 2009, vol. 507, pp. 487–494.
290. Borucki, W.J. & the Kepler Team, ArXiv e-prints, 2010.
291. Mayor, M., Pepe, F., Queloz, D., et al. *The Messenger*, 2003, vol. 114, pp. 20.
292. Santos, N.C., Bouchy, F., Mayor, M., Pepe, F., Queloz, D. et al. The HARPS survey for southern extra-solar

planets – II. A 14 Earth-masses exoplanet around μ Arae. *Astronomy & Astrophysics*, 2004, vol. 426(1), pp. L19–L23.

293. Pepe, F., Correia, A.C., Mayor, M., et al. The HARPS search for southern extra-solar planets – VIII. μ Arae, a system with four planets. *Astronomy & Astrophysics*, 2007, vol. 462(2), pp. 769–776.
294. Lovis, C., Mayor, M., Pepe, F., et al. An extrasolar planetary system with three Neptune-mass planets. *Nature*, 2006, vol. 441(7091), pp. 305–309.
295. Udry, S., Mayor, M., Benz, W., et al. The HARPS search for southern extra-solar planets – V. A 14 Earth-masses planet orbiting HD 4308. *Astronomy & Astrophysics*, 2006, vol. 447(1), pp. 361–367.
296. Mayor, M., Udry, S., Lovis, C., et al. The HARPS search for southern extra-solar planets. XIII. A planetary system with 3 super-Earths (4.2, 6.9, and 9.2 $M_\oplus$). *Astronomy & Astrophysics*, 2009, vol. 493(2), pp. 639–644.
297. Bouchy, F., Mayor, M., Lovis, C., et al. The HARPS search for southern extra-solar planets. – XVII. Super-Earth and Neptune-mass planets in multiple planet systems HD 47 186 and HD 181 433. *Astronomy & Astrophysics*, 2009, vol. 496(2), pp. 527–531.
298. Greenberger, D.M. Quantization in the large. *Foundations of Physics*, 1983, vol. 13(9), pp. 903–951.
299. Nottale, L. Scale-relativity and quantization of extra-solar planetary systems. *Astronomy & Astrophysics*, 1996, vol. 315, pp. L09–L12.
300. Nottale, L. Scale-relativity: from quantum mechanics to chaotic dynamics. *Chaos, Solitons & Fractals*, 1995, vol. 6, pp. 399–410.

301. Gradshtein, I.S. and Ryzhik, I.M. *Tables of Integrals, Sums, Series and Products*. Moscow: Nauka, 1971 [in Russian].
302. *Albert Einstein and Gravitation Theory*. Moscow: Mir, 1979 [in Russian].
303. Einstein, A. Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsber. preuss. Akad. Wiss*., 1915, Bd. 47(2), pp. 831–839.
304. Schwarzschild, K. On gravitational field of point mass in Einstein' theory. *Sitzungsber. d. Berl. Akad*., 1916, pp. 189–197.
305. Krot, A.M. *Discrete Models of Dynamical Systems Based on Polynomial Algebra*. Minsk: Nauka i tekhnika, 1990 [in Russian].
306. Odenwald, S. How thick is the solar corona? 1997, http://www.astronomycafe.net/qadir/q1612.html 1997.
307. Clemence, G.M. The relativity effect in planetary motions. *Rev. Mod. Phys*., 1947, vol. 19, pp. 361–364.
308. Pintr, P., Peřinová, V., Lukš, A., and Pathak, A. Statistical and regression analyses of detected extrasolar systems. *Planetary and Space Science*, 2013, vol. 75(1), pp. 37–45.
309. Butler, R. P., Marcy, G.W., Fischer, D.A., et al. Evidence for multiple companions to υ Andromedae. *Astrophysical Journal*, 1999, vol. 526, pp. 916–927.
310. Marcy, G. W., Butler, R.P., Fischer, D.A. et al. A pair of resonant planets orbiting GJ 876. *Astrophysical Journal*, 2001, vol. 556, pp. 296–301.
311. Wittenmyer, R.A., Endl, M., Cochran, W.D. Long-period objects in the extrasolar planetary systems 47 Ursae Majoris and 14 Herculis. *Astrophysical Journal*, 2007, vol. 654, pp. 625–632.
312. Döllinger, M.P., Hatzes, A.P., Pasquini, L., Guenzer, E.W., Hartman, M., Girardi, L., and Esposito, M.

Discovery of a planet around the K giant star 4 Ursae Majoris. *Astronomy & Astrophysics*, 2007, vol. 472(2), pp. 649–652.
313. Sato, B., Izumiura, H., Toyota, E., et al. Planetary companions around three intermediate-mass G and K Giants: 18 Delphini, ξ Aquilae, and HD81688. *PASJ: Publ. Astron. Soc. Japan*, 2008, vol. 60, pp. 539–550.
314. Borucki, J. W., et al. *Science*, 2010, vol. 327, p. 977.
315. Borucki, J. W., et al. *Astrophysical Journal*, 2011, vol. 728, p. 117.
316. Kunitomo, M., Ikoma, M., Sato, B., et al. Planet engulfment by 1.5–3 M_ red giants. *Astrophysical Journal*, 2011, vol. 737(2), pp. 66–73.
317. Flores-Gutiérrez, J.D. and Garcia-Guerra, C. A variant of the Titius–Bode law. *Revista Mexicana de Astronomia y Astrofisica*, 2011, vol. 47, pp. 173–184.
318. Fressin, F., Torres, G., Rowe, J.F., et al. Two Earth-sized planets orbiting Kepler-20. *Nature*, 2011, vol. 482, pp. 195–198.
319. Strömgren, B. Die Theorie der Sterninnern und die Entwicklung der Sterne. *Ergebnisse der Exakten Naturwissenschafften*, 1937, vol. 16, p. 465.
320. Fowler, R.H. *Statistical Mechanics*, Ch. XIV-XVII, 2nd ed. Cambridge: University Press, 1936.
321. Allen, C.W. *Astrophysical Quantities*, 3rd Ed. London: University of London, Athlone Press, 1973.
322. *Prentice-Hall Earth Science*. Englewood Cliffs, NJ: Prentice Hall, 1987, p. 73.
323. Sun. In: *Compton's Encyclopedia*. Chicago: Compton's, 1996, p. 708.
324. Clark, S. *Stars and Atoms*. New York: Oxford University Press, 1995, p. 106.
325. Soper, D. *Corona*. University of Oregon, 1999.

326. Temperature of the Sun's Corona. In: *The Physics Factbook*™, edited by G. Elert. Fair Use website; http://hypertextbook.com/facts/2000/CCoraThomas.shtml.
327. Townes, C.H., Wishnow, E. H., Hale, D. D. S. and Walp, B.A. Systematic change with time in the size of Betelgeuse. *Astrophysical Journal Letters*, 2009, vol. 697(2), pp. L127-L128.
328. Krot, A.M. The combined Kepler 3rd law with universal stellar law: Derivation and application to extrasolar systems investigation. 3rd International Conference and Exhibition on «Satellite & Space Missions», May 11-13, 2017, Barcelona, Spain / Journal of Aeronautics & Aerospace Engineering, 2017, vol.6(2), p. 72.
329. Alfvén, H. and Fälthammar, C.-G. *Cosmic Electrodynamics: Fundamental Principles*, 2nd ed. Oxford: Clarendon Press, 1963.
330. Angot, A. *Compléments de Mathématiques. A l'Usage des Ingénieurs de l'Electrotechnique et des Télecommunications*. Paris, 1957.
331. Krot, A.M. On the wave solutions of the generalized nonlinear Schrödinger-like equation of formation of a cosmogonical body. In: *Understanding the Schrödinger Equation: Some [Non]Linear Perspectives*, edited by V.A. Simpao and H.C. Little. New York: Nova Science Publishers, 2020, Ch.4, pp. 93–133.